Some results on Operator Semigroups and Applications to Evolution Problems

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Abstract

In this thesis we address certain questions arising in the functional analytic study of dynamical systems and differential equations.

First, we discuss the operator theoretic counterparts of the central ergodic theoretical notions of strong and weak mixing. These concepts correspond to particular types of asymptotic behaviour of operator semigroups in the weak operator topology, called weak and almost weak stability. Using functional analytic tools and methods from ergodic theory, we describe various features of (almost) weakly stable semigroups. In particular, we show that (in the Baire category sense) typical elements in certain natural spaces of semigroups are almost weakly but not weakly stable, thus we carry over classical theorems of Halmos and Rohlin for measure preserving transformations to the Hilbert space operator setting.

Further, we illustrate operator semigroup methods and results on a class of telegraph systems with various boundary conditions. We study both linear and nonlinear boundary value problems. The stability of linear telegraph systems is discussed by applying theorems from the previous chapters. For the existence of solutions, we are particularly interested in time-dependent boundary conditions, since this case has little been investigated so far. The operator semigroup techniques applied to the case of Lipschitz continuous nonlinearities are combined with estimates from the theory of monotone operators to yield well-posedness and the regularity of the solutions, also in the case of dynamic boundary conditions.

Preface

The last decades have seen a remarkable development of abstract methods for various kinds of dynamical systems and functional analytic approaches now provide sufficiently general structures for the study of differential equations. On the other hand, the rich variety of differential equations makes the existence of a unified theory extremely unlikely and specific equations that are of interest because of applications but do not neatly fit in any major abstract framework are often investigated with classical, concrete or even *ad hoc* techniques. Therefore, while extending general theories it is equally important to study specific problems; the aim of this thesis is to contribute to the abstract as well as to the concrete domain.

The mathematical field our investigations here originate in is the theory of linear and nonlinear operator semigroups. The theory is large and well-developed by now; for an introduction as well as for further information we recommend the books [Arendt et al., 2001], [Engel and Nagel, 2000], [Goldstein, 1985], [Ito and Kappel, 2002], [Lax, 2002], [Moroşanu, 1988] and for the functional analytic background [Rudin, 1973].

The thesis is organised as follows. The introduction in the first chapter briefly reviews relevant parts from the theory of linear and nonlinear operator semigroups. In the next two chapters we study weakly and almost weakly stable C_0-semigroups; the results here are largely based on the papers [Eisner et al., 2007], [Eisner and Serény, 2008] and [Eisner and Serény, 2007]. Finally, in Chapters 4 and 5 we investigate a class of telegraph systems by extensively using linear and nonlinear semigroup techniques; the discussion here is based on [Moroşanu and Serény, 2006] and [Serény, 2007].

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Contents

Chapter 1

Introduction

1.1 Background

Numerous initial value problems can be written in the form of a (non-autonomous) abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t) = A(t)u(t), \quad t > 0, \\ u(0) = x, \end{cases} \tag{NACP}$$

where $(A(t))_{t>0}$ is a given family of operators on a Banach space X, $x \in X$ is given and we look for a solution $u : \mathbb{R}_+ \to X$. In (NACP) the family $(A(t))_{t>0}$ represents the differential equation and the boundary conditions, while x represents the initial data. Under certain conditions some of which we will discuss, (NACP) gives rise to a family of operators $(U(t,s))_{t\geq s\geq 0}$ on X such that the solution $u(t)$ belonging to the initial data $u(s) = x$ is given by $u(t) = U(t,s)x$. The family $(U(t,s))_{t\geq s\geq 0}$ satisfies

$$U(t,s) = U(t,r)U(r,s), \quad t \geq r \geq s \geq 0$$

and is called the evolution family associated with $(A(t))_{t>0}$.

Now the operator family $(U(t,s))_{t\geq s}$ stands in for the solutions of the original differential equation and questions concerning the well-posedness of the problem or regularity and asymptotic behaviour of the solutions directly translate to questions about $(U(t,s))_{t\geq s}$. In most cases, since there is no known analytic solution to the underlying differential equation, we have no explicit description of $(U(t,s))_{t\geq s}$ and the dominant part of the abstract theory of evolution families is concerned with the problem of describing the behaviour of $(U(t,s))_{t\geq s}$ using information on $(A(t))_{t>0}$ only.

A notable special case of (NACP) arises if the operator family $(A(t))_{t>0}$ in (NACP) does not actually depend on time and our problem turns into the autonomous Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t), \quad t > 0, \\ \quad u(0) = x. \end{cases} \tag{ACP}$$

In this case $U(t, s)$ depends only on $t - s$ and if we let $T(t) = U(s + t, s)$ we obtain a family $(T(t))_{t\geq 0}$ satisfying

$$T(t + s) = T(t)T(s) \quad t, s \geq 0;$$

such an operator family is called an operator semigroup. Briefly, operator semigroups represent the time evolution of an autonomous deterministic system and apart from the connection with abstract differential equations they also arise in the study of continuous or measure-preserving dynamical systems.

1.2 Linear Cauchy problems

In this section we gather some basic features of linear Cauchy problems; all proofs along with a great deal of additional information can be found for instance in [Engel and Nagel, 2000].

Definition 1.2.1. Let X denote a Banach space and let $(T(t))_{t\geq 0}$ be a family of bounded linear operators on X. The family $(T(t))_{t\geq 0}$ is called a C_0-semigroup on X if $T(0) = I$, the identity on X, $T(t)T(s) = T(t + s)$ $(t, s \geq 0)$ and the mapping $t \mapsto T(t)x$ $(t \geq 0)$ is continuous for all $x \in X$.

Definition 1.2.2. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. The generator A of $(T(t))_{t\geq 0}$ is the (not necessarily bounded) linear operator on X defined by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \quad (x \in D(A)),$$

where $D(A)$ is the domain of A given naturally by

$$D(A) = \left\{ x \in X \;\middle|\; \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

The generator of a C_0-semigroup is densely defined, closed and determines the semigroup uniquely. The resolvent of the generator is sometimes called the resolvent of the semigroup and is denoted by $R(\lambda, A) = (\lambda I - A)^{-1}$, for all $\lambda \in \mathbb{C}$ where the operator $\lambda I - A$ is (continuously) invertible. Among all closed, densely defined linear operators, a resolvent condition characterizes generators.

Theorem 1.2.3 (Miyadera–Phillips)**.** *Let A be a linear operator on a Banach space X. Then A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq M$ for some $M \in \mathbb{R}$ and all $t \geq 0$ if and only if A is densely defined, closed and for all $\lambda > 0$ the resolvent $R(\lambda, A)$ exists and $\|(\lambda R(\lambda, A))^n\| \leq M$ $(n \in \mathbb{N})$.*

The following theorem connects the generator property to well-posedness of Cauchy problems.

Theorem 1.2.4. *Let the linear operator A generate the C_0-semigroup $(T(t))_{t\geq 0}$ on the Banach space X. Then the associated abstract Cauchy problem*

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t), \quad t > 0, \\ \quad u(0) = x \end{cases} \tag{ACP}$$

has a unique classical solution for all initial data $x \in D(A)$, that is, there is a unique function $u \in C^1(\mathbb{R}_+, X), u(t) \in D(A)$ $(t \geq 0)$ satisfying (ACP). This function is given by $u(t) = T(t)x$.

A kind of converse also holds.

Theorem 1.2.5. *If A is a closed linear operator on X with non-empty resolvent set, and (ACP) has a unique classical solution for all $x \in D(A)$, then A is the generator of a C_0-semigroup.*

We now combine a special case of Theorem 1.2.3 with Theorem 1.2.4.

Theorem 1.2.6. *Let A be a densely defined linear operator on a Hilbert space H satisfying $\operatorname{Re}\langle Ax, x\rangle \leq 0$ for all $x \in D(A)$ and $\operatorname{range}(\lambda I - A) = H$ for some $\lambda > 0$. Then A is the generator of a contractive C_0-semigroup, thus the conclusion of Theorem 1.2.4 holds with $\|u(t)\| \leq \|x\|$ for $t \geq 0$.*

1.3 Asymptotic behaviour

Questions about the long term behaviour of the solution of the abstract Cauchy problem carry over to questions about the long term (i.e. asymptotic) behaviour of the corresponding operator semigroup, that is, the (non)existence of $\lim_{t\to\infty} T(t)$, where the limit can be understood in various ways. Given that C_0-semigroups live on Banach spaces, at least three concepts of limit, corresponding to the uniform, strong and weak topologies of the operator algebra come to mind, the main aim being to deduce asymptotic properties of the semigroup from information about the generator. A principal type of asymptotic behaviour is stability, i.e. convergence to zero.

Uniform and strong stability (convergence to zero in the uniform and strong operator topology, respectively) have thoroughly been investigated and are by now well understood; the theory is comprehensively presented in [van Neerven, 1996] and [Arendt et al., 2001]. We recall some results that we shall use later; the first one is a characterization of uniformly exponentially stable semigroups.

Theorem 1.3.1. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space. The resolvent $R(\lambda, A)$ exists for all $\operatorname{Re}\lambda > 0$ and $\sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A)\| < \infty$, if and only if $(T(t))_{t\geq 0}$ is uniformly exponentially stable, that is, there is an $\varepsilon > 0$ such that $\lim_{t\to\infty} e^{\varepsilon t}\|T(t)\| = 0$.*

The second one is a well-known sufficient (but not necessary) condition for strong stability.

Theorem 1.3.2 (Arendt, Batty, Lyubich, Vũ)**.** *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup with generator A on a Banach space X. If the adjoint of A has no eigenvalues on the imaginary axis and $\sigma(A) \cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is strongly stable, that is $\lim_{t\to\infty} T(t)x = 0$ for all $x \in X$.*

However, weak asymptotic properties of C_0-semigroups have so far mostly eluded our mathematical grasp; we devote Chapter 2 to their study.

1.4 Nonlinear Cauchy problems

It was an amazing discovery of [Komura, 1967] that the assumption on the operator A being linear in Theorem 1.2.6 can be omitted.

Definition 1.4.1. Let H denote a real Hilbert space. A subset of $H \times H$ is called a multivalued operator on H. If A is a multivalued operator on H, then its domain $D(A)$ is defined as $D(A) = \{\, x \in H \mid \exists y \in H \text{ such that } (x, y) \in A \,\}$. Further, for any $x \in H$ let Ax denote the set $Ax = \{\, y \in H \mid (x, y) \in A \,\}$.

Definition 1.4.2. A multivalued operator A on a real Hilbert space H is said to be a monotone operator on H if for all pairs $(x, y), (u, v) \in A$ we have $\langle x - u, y - v\rangle \geq 0$.

Definition 1.4.3. A monotone operator A on a real Hilbert space H is called maximal monotone, if for any monotone operator B on H with $A \subseteq B$ we have $A = B$.

Theorem 1.4.4. *Let A be a maximal monotone operator on a Hilbert space H, let $\tau > 0$, $x \in D(A)$ and let $f \in W^{1,1}([0, \tau], H)$. Then the monotone abstract Cauchy problem*

$$\begin{cases} \dfrac{d}{dt}u(t) + Au(t) \ni f(t), \quad 0 < t < \tau \\ u(0) = x \end{cases} \tag{MACP}$$

admits a unique strong solution for all initial data $x \in D(A)$, that is, there is a unique function $u \in W^{1,\infty}([0, \tau], H)$, $u(t) \in D(A)$ satisfying (MACP).

Theorem 1.4.4 can be extended to Banach spaces and to non-autonomous monotone Cauchy problems in several ways, see [Ito and Kappel, 2002]. We include here a theorem of [Tătaru, 1991] that we shall use in Chapter 4 where we study a time-dependent monotone problem.

Theorem 1.4.5 (D. Tătaru)**.** *Let H be a Hilbert space and let $A(t) : D(A(t)) \subset H \to H$, $t \in [0, \tau]$, be a family of maximal monotone operators such that*

$$\begin{aligned} -\langle x - y, A(t)x - A(s)y\rangle_H \leq M\|x - y\|_H^2 + \\ |t - s||g(t) - g(s)| \left(1 + \|x\|_H^2 + \|y\|_H^2 + \|A(t)x\|_H^2 + \|A(s)y\|_H^2\right) \end{aligned} \tag{1.1}$$

for all $t, s \in [0, \tau]$, $x \in D(A(t))$, $y \in D(A(s))$, where M is a constant and g is a function of bounded variation on $[0, \tau]$. Then, for each $u_0 \in D(A(0))$ there is a unique function $u \in W^{1,\infty}([0, \tau], H)$ satisfying

$$\begin{cases} u'(t) + A(t)u(t) \ni 0 & \text{for almost all } t \in [0, \tau] \\ u(0) = u_0. \end{cases} \tag{1.2}$$

For results on the asymptotic properties of nonlinear operator semigroups we refer the reader to [Moroşanu, 1988] and [Hokkanen and Moroşanu, 2002b].

Chapter 2

Weak stability

In this chapter we investigate weak asymptotic properties of bounded C_0-semigroups. Not only does the concept of weakly convergent semigroups arise directly from the underlying Banach (or Hilbert) space structure but it is also the parallel notion of strongly mixing measurable dynamical systems [Krengel, 1985]. Despite its being a perfectly natural concept, there is no hope to obtain a (spectral) characterization [Katok and Hasselblatt, 1995]. Therefore, we first set out to introduce and study the related notion of almost weak stability, more yielding to our tools, and proceed to explore its connection to weak stability.

2.1 Almost weak stability

Let us begin with a definition of weakly stable C_0-semigroups.

Definition 2.1.1. The C_0-semigroup $(T(t))_{t\geq 0}$ on the Banach space X is called weakly stable, if $\lim_{t\to\infty}\langle T(t)x, y\rangle = 0$ for all $x \in X$ and $y \in X'$, where X' denotes the dual Banach space of X.

Roughly speaking, a semigroup is called almost weakly stable if it tends to zero weakly as time tends to infinity, except for time values in a set of zero density.

Definition 2.1.2. Let λ denote the Lebesgue measure on $\mathbb{R}$. The (asymptotic) density of a measurable set $M \subset \mathbb{R}_+$ is given by

$$d(M) = \lim_{t\to\infty} \frac{1}{t}\lambda([0, t] \cap M),$$

whenever the limit exists.

The following discussion, leading up to Definition 2.1.8, aims for a formal definition of almost weak stability. For a Banach space X let $\mathcal{L}_\sigma(X)$ denote the algebra of bounded linear operators on X endowed with the weak operator topology. Throughout this chapter, our basic assumption is that the C_0-semigroup under investigation is relatively weakly compact. Equivalent definitions are given in the following lemma; for a proof see [Engel and Nagel, 2000, Corollary A.5].

Lemma 2.1.3. *Let $\mathcal{T}$ be a set of bounded linear operators on the Banach space X. Then the following assertions are equivalent.*

(i) *$\mathcal{T}$ is relatively compact in $\mathcal{L}_\sigma(X)$.*

(ii) *$\{Tx : T \in \mathcal{T}\}$ is relatively weakly compact in X for all $x \in X$.*

(iii) *$\mathcal{T}$ is bounded, and $\{Tx : T \in \mathcal{T}\}$ is relatively weakly compact in X for all x in some dense subset of X.*

For instance, norm bounded sets of operators on reflexive Banach spaces are relatively weakly compact.

Definition 2.1.4. A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is called relatively weakly compact if the set $\{T(t) \mid t \geq 0\}$ satisfies one of the equivalent conditions in Lemma 2.1.3.

Weakly stable semigroups are bounded, hence on reflexive spaces they are also relatively weakly compact. In turn, relatively weakly compact semigroups are mean ergodic [Engel and Nagel, 2000, Sec. V.4].

Proposition 2.1.5. *Let $(T(t))_{t\geq 0}$ be a relatively weakly compact semigroup on a Banach space X. Then it is mean ergodic, i.e.*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t T(s)x\,ds = Px \qquad \text{for all } x \in X,$$

where $P \in \mathcal{L}(X)$ is a projection onto $\mathrm{Fix}(T) = \bigcap_{t\geq 0} \mathrm{Fix}(T(t))$, the so-called ergodic projection.

Now, if $(T(t))_{t\geq 0}$ is relatively weakly compact on the Banach space X, then the Jacobs-Glicksberg-de Leeuw decomposition theorem [Krengel, 1985, Sec. 2.4] yields that X is the direct sum of the two $(T(t))_{t\geq 0}-$invariant subspaces

$$\begin{aligned} X_r &= \overline{\mathrm{lin}}\big\{x \in D(A) \mid Ax = i\lambda x \text{ for some } \lambda \in \mathbb{R}\big\},\\ X_s &= \big\{x \in X \mid 0 \text{ is a weak accumulation point of } \{T(t)x \mid t \geq 0\}\big\}. \end{aligned}$$

Specifically, the generator has no eigenvalue on the imaginary axis if and only if zero is a weak accumulation point of the orbit $\{T(t)x \mid t \geq 0\}$ for every $x \in X$. The following theorem gives a more detailed description of the asymptotic behaviour of the orbits in this case.

Theorem 2.1.6. *Let $(T(t))_{t\geq 0}$ be a relatively weakly compact C_0-semigroup on a Banach space X with generator A. Then the following assertions are equivalent.*

(i) $0 \in \overline{\{T(t)x : t \geq 0\}}^{\sigma}$ *for every* $x \in X$.

(i') $0 \in \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma}$.

(ii) For every $x \in X$ *there exists a sequence* $\{t_n\}_{n=1}^{\infty}$ *with* $t_n \to \infty$ *such that* $T(t_n)x \xrightarrow{\sigma} 0$

(iii) For every $x \in X$ *there exists a set* $M \subset \mathbb{R}_+$ *with density one such that* $T(t)x \xrightarrow{\sigma} 0$, *as* $t \in M$, $t \to \infty$.

(iv) $\frac{1}{t}\int\limits_0^t |\langle T(s)x, y\rangle|\, ds \underset{t\to\infty}{\longrightarrow} 0$ *for all* $x \in X$, $y \in X'$.

(v) $\lim\limits_{a\to 0+} a \int\limits_{-\infty}^{\infty} |\langle R(a+is, A)x, y\rangle|^2\, ds = 0$ *for all* $x \in X$, $y \in X'$.

(vi) $\lim\limits_{a\to 0+} aR(a+is, A)x = 0$ *for all* $x \in X$ *and* $s \in \mathbb{R}$.

(vii) $P\sigma(A) \cap i\mathbb{R} = \emptyset$, *i.e.,* A *has no purely imaginary eigenvalues.*

If, in addition, X' *is separable, then the conditions above are also equivalent to*

(ii) There exists a sequence* $\{t_n\}_{n=1}^{\infty}$ *with* $t_n \to \infty$ *such that* $T(t_n) \xrightarrow{\sigma} 0$.

(iii) There exists a set* $M \subset \mathbb{R}_+$ *with density one such that* $T(t) \xrightarrow{\sigma} 0$, $t \in M$ *and* $t \to \infty$.

For the proof we need the following elementary lemma [Petersen, 1983, Lemma 6.2].

Lemma 2.1.7. *Let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be continuous and bounded. The following assertions are equivalent.*

(i) $\frac{1}{t}\int\limits_0^t f(s)\, ds \to 0$ *as* $t \to \infty$;

(ii) There exists a set $M \subset \mathbb{R}_+$ *with density one such that* $f(t) \to 0$, $t \in M$ *and* $t \to \infty$.

Proof of Theorem 2.1.6. The proof of the implication (i') $\Rightarrow$ (i) is straightforward. The implication (i) $\Rightarrow$ (ii) holds since in Banach spaces weak compactness and weak sequential compactness coincide (by the Eberlein-Šmulian theorem, e.g. [Dunford and Schwartz, 1958, Thm. V.6.1].

If (vii) does not hold, then (ii) can not be true by the spectral mapping theorem [Engel and Nagel, 2000, Theorem V.3.7] for the point spectrum, hence (ii) $\Rightarrow$ (vii).

The implication (vii) $\Rightarrow$ (i') is the main consequence of the Jacobs-Glicksberg-de Leeuw theorem and follows from the construction in its proof [Engel and Nagel, 2000, p. 313]. This proves the equivalences (i) $\Leftrightarrow$ (i') $\Leftrightarrow$ (ii) $\Leftrightarrow$ (vii).

(vi) $\Leftrightarrow$ (vii): Since the semigroup $(T(t))_{t\geq 0}$ is mean ergodic and bounded, the decomposition $X = \ker A \oplus \overline{\operatorname{rg} A}$ holds (see [Engel and Nagel, 2000], Lemma V.4.4). This implies by the mean ergodic theorem (see [Arendt et al., 2001, Cor. 4.3.2]) that the limit

$$Px = \lim_{a\to 0+} aR(a, A)x$$

exists for all $x \in X$ with a projection P onto $\ker A$. Therefore, $0 \notin P\sigma(A)$ if and only if $P = 0$. Now take $s \in \mathbb{R}$. The semigroup $(\mathrm{e}^{ist}T(t))_{t\geq 0}$ is also relatively weakly compact and hence mean ergodic. Repeating the argument for this semigroup we obtain (vi) $\Leftrightarrow$ (vii).

(i') $\Rightarrow$ (iii): Let $S = \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma} \subseteq \mathcal{L}(X)$ which is a compact semi-topological semigroup if considered with the usual multiplication and the weak operator topology. By (i) we have $0 \in S$. Define the operators $\tilde{T}(t) : C(S) \to C(S)$ by

$$(\tilde{T}(t)f)(R) = f(T(t)R), \quad f \in C(S),\ R \in S.$$

By [Nagel, 1986, Lemma B-II.3.2], $(\tilde{T}(t))_{t\geq 0}$ is a C_0-semigroup on $C(S)$.

The set $\{f(T(t)\,\cdot) : t \geq 0\}$ is relatively weakly compact in $C(S)$ for every $f \in C(S)$. It means that every orbit $\{\tilde{T}(t)f : t \geq 0\}$ is relatively weakly compact, and, by Lemma 2.1.3, $(\tilde{T}(t))_{t\geq 0}$ is a relatively weakly compact semigroup.

Denote the mean ergodic projection of $(\tilde{T}(t))_{t\geq 0}$ by $\tilde{P}$. We have $\operatorname{Fix}(\tilde{T}) = \bigcap_{t\geq 0} \operatorname{Fix}(\tilde{T}(t)) = \langle \mathbf{1} \rangle$. Indeed, for $f \in \operatorname{Fix}(\tilde{T})$ one has $f(T(t)I) = f(I)$ for all $t \geq 0$ and therefore f should be constant. Hence $\tilde{P}f$ is constant for every $f \in C(S)$. By the definition of the ergodic projection

$$(\tilde{P}f)(0) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \tilde{T}(s)f(0)\, ds = f(0). \tag{2.1}$$

Thus we have

$$(\tilde{P}f)(R) = f(0) \cdot \mathbf{1}, \qquad f \in C(S),\ R \in S. \tag{2.2}$$

Now take $x \in X$. By [Dunford and Schwartz, 1958, p. 434], the weak topology on the orbit $\{T(t)x : t \geq 0\}$ is metrisable and coincides with the topology induced by a sequence $\{y_n\}_{n=1}^{\infty} \subset X' \setminus \{0\}$. Consider $f_{x,n} \in C(S)$ defined by

$$f_{x,n}(R) = |\langle Rx, \tfrac{y_n}{\|y_n\|} \rangle|, \qquad R \in S,$$

and $f_x \in C(S)$ defined by

$$f_x(R) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} f_{x,n}(R), \qquad R \in S.$$

By (2.2) we obtain

$$0 = \lim_{t\to\infty} \frac{1}{t} \int_0^t \tilde{T}(s) f_{x,y}(I)\, ds = \lim_{t\to\infty} \frac{1}{t} \int_0^t f_x(T(s))\, ds.$$

Lemma 2.1.7 applied to the continuous and bounded function $\mathbb{R}_+ \ni t \mapsto f(T(t)I)$ yields a set $M \subset \mathbb{R}$ with density one such that

$$\lim_{t\to\infty, t\in M} f_x(T(t)) = 0.$$

By the definition of f_x and the fact that the weak topology on the orbit is induced by $\{y_n\}_{n=1}^{\infty}$ we have in particular that

$$\lim_{t\to\infty, t\in M} T(t)x = 0$$

weakly, and this proves (iii).

(iii) $\Rightarrow$ (iv) follows directly from Lemma 2.1.7.

(iv) $\Rightarrow$ (vii) holds by the spectral mapping theorem for the point spectrum.

(iv) $\Leftrightarrow$ (v): Clearly, the semigroup $(T(t))_{t\geq 0}$ is bounded. Take $x \in X$, $y \in X'$ and let $a > 0$. By the Plancherel theorem applied to the function $t \mapsto \mathrm{e}^{-at}\langle T(t)x, y\rangle$ we have

$$\int_{-\infty}^{\infty} |\langle R(a+is, A)x, y\rangle|^2\, ds = 2\pi \int_0^{\infty} \mathrm{e}^{-2at} |\langle T(t)x, y\rangle|^2\, dt.$$

We obtain by the equivalence of Abel and Cesàro limits (see for instance [Hardy, 1949, p. 136])

$$\begin{aligned} \lim_{a\to 0+} a \int_{-\infty}^{\infty} |\langle R(a+is, A)x, y\rangle|^2\, ds &= 2\pi \lim_{a\to 0+} a \int_0^{\infty} \mathrm{e}^{-2at} |\langle T(s)x, y\rangle|^2\, ds \\ &= \pi \lim_{t\to\infty} \frac{1}{t} \int_0^t |\langle T(s)x, y\rangle|^2\, ds. \qquad (2.3) \end{aligned}$$

Note that for a bounded continuous not identically zero function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $C = \sup f(\mathbb{R}_+)$ we have

$$\left(\frac{1}{Ct} \int_0^t f^2(s)\, ds\right)^2 \leq \left(\frac{1}{t} \int_0^t f(s)\, ds\right)^2 \leq \frac{1}{t} \int_0^t f^2(s)\, ds,$$

which together with (2.3) gives the equivalence of (iv) and (v).

For the additional part of the theorem suppose that X' is separable. Then so is X, and we can take dense subsets $\{x_n \neq 0 : \ n \in \mathbb{N}\} \subseteq X$ and $\{y_m \neq 0 : \ m \in \mathbb{N}\} \subseteq X'$. Consider the functions

$$f_{n,m} : S \to \mathbb{R}, \qquad f_{n,m}(R) = \left|\left\langle R\tfrac{x_n}{\|x_n\|}, \tfrac{y_m}{\|y_m\|}\right\rangle\right|, \quad n, m \in \mathbb{N},$$

which are continuous and uniformly bounded in $n, m \in \mathbb{N}$. Define the function

$$f : S \to \mathbb{R}, \qquad f(R) = \sum_{n,m\in\mathbb{N}} \frac{1}{2^{n+m}} f_{n,m}(R).$$

Clearly $f \in C(S)$. Thus, as in the proof of the implication (i') $\Rightarrow$ (iv), i.e., using (2.1) we obtain

$$\frac{1}{t}\int_0^t f(T(s)I)\, ds \underset{t\to\infty}{\longrightarrow} 0.$$

Hence, applying Lemma 2.1.7 to the continuous and bounded function $\mathbb{R}_+ \ni t \mapsto f(T(t)I)$ gives the existence of a set M with density one such that

$$\lim_{t\to\infty, t\in M} f(T(t)) = 0.$$

In particular, $|\langle T(t)x_n, y_m\rangle| \to 0$ for all $n, m \in \mathbb{N}$ as $t \in M$, $t \to \infty$, which, together with the boundedness of $(T(t))_{t\geq 0}$, proves the implication (i') $\Rightarrow$ (iii*). The implications (iii*) $\Rightarrow$ (ii*) $\Rightarrow$ (ii') are straightforward, hence the proof is complete. □

The above theorem shows that starting from "no purely imaginary eigenvalues of the generator", one arrives at properties like (iii) concerning the asymptotic behaviour of the semigroup. This justifies the name we have coined.

Definition 2.1.8. A relatively weakly compact C_0-semigroup is called almost weakly stable if it satisfies one of the equivalent conditions in Theorem 2.1.6.

Historical remark 2.1.9. Theorem 2.1.6 and especially the implication (vii) $\Rightarrow$ (iii) has a long history. It goes back to ergodic theory and von Neumann's spectral mixing theorem for flows, see [Halmos, 1956, page 39]. There has been a great deal of attempts to generalize these ideas to operators on Banach spaces, see [Nagel, 1974], [Jones and Lin, 1976], [Jones and Lin, 1980], [Krengel, 1985, pp. 108–110] among others. The implication (vii) $\Rightarrow$ (i) appears also in [Ruess and Summers, 1992]. The conditions (i), (iii) and (iv) were studied by [Hiai, 1978] also for strongly measurable semigroups. He related it to the discrete case as well.

Remark 2.1.10. The conditions (iii) and (iii*) show that all the orbits $t \mapsto T(t)x$ converge weakly to zero as $t \to \infty$ for t in a large set. In general, however, it may happen that this large set is not the whole $\mathbb{R}_+$, i.e., $(T(t))_{t\geq 0}$ is not weakly stable (for examples, see Section 2.3). This is an essential difference to strong stability: for a bounded semigroup $(T(t))_{t\geq 0}$ the convergence $\|T(t_n)x\| \to 0$ for a sequence $t_n \to \infty$ already implies $\lim_{t\to\infty} \|T(t)x\| = 0$.

Remark 2.1.11. Even if there is obviously no notion of "almost strong stability", it is still remarkable that the strong version of condition (v) already yields the strong stability of a bounded C_0-semigroup on a Hilbert space. More precisely, a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space H with generator A is strongly stable if and only if

$$\lim_{a\to 0+} a\int_{-\infty}^{\infty} \|R(a+is,A)x\|^2\,ds = 0$$

holds for every $x \in H$ [Tomilov, 2001, p. 108–110]. We note that the proofs of the equivalence (i') $\Leftrightarrow$ (v) and of [Tomilov, 2001, Theorem 3.1] are analogous.

2.2 Weak stability

One of the central components of our proof in the previous section is that a relatively weakly compact C_0-semigroup on a Banach space X induces the Jacobs-Glicksberg-de Leeuw decomposition $X = X_r \oplus X_s$. Although orbits in X_s do not generally converge (weakly) to zero, in the particular case of contractive semigroups on Hilbert spaces one can detach the subspace of all weakly stable orbits and characterise its complement.

Theorem 2.2.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space H and define*

$$W = \left\{x \in H : \lim_{t\to\infty} \langle T(t)x, x\rangle = 0\right\}.$$

Then W is a closed subspace of H, W and $W^{\perp}$ are $(T(t))_{t\geq 0}$-invariant, the restricted semigroup $(T(t)_{|W})_{t\geq 0}$ is weakly stable on W and $(T(t)_{|_{W^{\perp}}})_{t\geq 0}$ is unitary on $W^{\perp}$.

For the proof we refer the reader to [Luo et al., 1999, Theorem 3.18, p. 122], or see [Foguel, 1963] for the analogous discrete case. In the following propositions we state some immediate consequences of the above decomposition.

Proposition 2.2.2. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space H and let $x \in H$. Then the following assertions hold.*

(i) $\lim_{t\to\infty} T(t)x = 0$ *weakly if and only if* $\lim_{t\to\infty}\langle T(t)x, x\rangle = 0$.

(ii) If $(T(t))_{t\geq 0}$ is completely non-unitary, i.e., if there is no reducing subspace on which it is unitary, then $(T(t))_{t\geq 0}$ is weakly stable.

Proposition 2.2.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space H. Then $(T(t)_{|_{W^{\perp}}})_{t\geq 0}$ has no weakly stable orbit, hence the spectral measures of its generator are non-Rajchman.*

(For the definition of Rajchman measures and a brief discussion see Example 2.3.2.)

We now turn to sufficient conditions for weak stability proved partly in [Chill and Tomilov, 2003]. It is based on the behaviour of the resolvent $R(\cdot, A)$ of the generator and uses the *pseudo-spectral bound of* A (also called *abscissa of uniform boundedness of the resolvent*)

$$s_0(A) = \inf\big\{a \in \mathbb{R} : R(\lambda, A) \text{ is bounded on } \{\lambda : \operatorname{Re}\lambda > a\}\big\}.$$

Theorem 2.2.4. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A satisfying $s_0(A) \leq 0$. Further, let $x \in X$ and $y \in X'$ be fixed. Consider the following assertions:*

(i)
$$\int_0^1 \int_{-\infty}^{\infty} |\langle R^2(a+is, A)x, y\rangle| \, ds \, da < \infty.$$

(ii)
$$\lim_{a\to 0+} a \int_{-\infty}^{\infty} |\langle R^2(a+is, A)x, y\rangle| \, ds = 0.$$

(iii)
$$\lim_{t\to\infty} \langle T(t)x, y\rangle = 0$$

Then (i) ⇒ (ii) ⇒ (iii). In particular, if (i) or (ii) holds for all $x \in X$ and $y \in X'$, then $(T(t))_{t\geq 0}$ is weakly stable.

Proof. First we show that (i) implies (ii). Assume that (i) holds. From the theory of Hardy spaces we know that the function $f : (0,1) \mapsto \mathbb{R}_+$ defined by

$$f(a) = \int_{-\infty}^{\infty} |\langle R^2(a+is, A)x, y\rangle| \, ds$$

is monotone decreasing for $a > 0$ (see [Rosenblum and Rovnyak, 1994] for the theory of Hardy spaces). Assume now that (ii) is not true. Then there exists a monotonic decreasing null sequence $\{a_n\}_{n=1}^{\infty}$ such that

$$a_n f(a_n) \geq c \tag{2.4}$$

holds for some $c > 0$ and all $n \in N$.

Take now any $n, m \in \mathbb{N}$ such that $a_n \leq \frac{a_m}{2}$. By (2.4) and the monotonicity of f we have

$$\int_{a_n}^{a_m} f(a) \, da \geq \sum_{k=m}^{n-1} (a_k - a_{k+1}) f(a_k) \geq \frac{c}{a_n}(a_m - a_n) = c\left(\frac{a_m}{a_n} - 1\right) \geq c$$

holds. This contradicts (i) and the implication (i) ⇒ (ii) is proved. It remains to show that (ii) implies (iii). By (ii) we have for every $a > 0$

$$\int_{-\infty}^{\infty} |\langle R^2(a+is, A)x, y\rangle| \, ds < \infty.$$

Moreover, condition $s_0(A) \leq 0$ implies that the function $\lambda \mapsto \langle R^2(\lambda, A)x, y\rangle$ is bounded on every half-plane $\{\lambda : \text{Re}\ \lambda \geq a\}$. Therefore, it belongs to the Hardy space $H^1(\{\lambda : \text{Re}\,\lambda > a\})$ and

$$\int_{-\infty}^{\infty} |\langle R^2(a+is, A)x, y\rangle|\, ds < \infty$$

holds for all $a > 0$. This allows us to represent the semigroup as the inverse Laplace transform for all $a > \max\{0, \omega_0(T)\}$, where $\omega_0(T)$ is the growth bound of $(T(t))_{t\geq 0}$. Indeed, from [Kaashoek and Lunel, 1994] it follows that

$$\langle T(t)x, y\rangle = \frac{1}{2\pi t} \int_{-\infty}^{\infty} \mathrm{e}^{(a+is)t} \langle R^2(a+is, A)x, y\rangle\, ds. \tag{2.5}$$

A standard application of Cauchy's theorem extends the validity of (2.5) to all $a > 0$. We now take $t = \frac{1}{a}$ to obtain

$$|\langle T(t)x, y\rangle| \leq a \int_{-\infty}^{\infty} |\langle R^2(a+is, A)x, y\rangle|\, ds \to 0$$

as $a \to 0+$, so $t = \frac{1}{a} \to \infty$. □

The implication (ii) ⇒ (iii) is stated in [Chill and Tomilov, 2003]. They also show that the strong analogues of (i) and (ii) both imply strong stability of the semigroup. Note that the relation (i) ⇒ (ii) is also valid for the strong case by the same arguments.

We conclude this section with the following remarkable fact about weak stability. By Theorem 2.1.6 one has almost weak stability under quite general assumptions. As we will see in the next section, almost weak stability does not imply weak stability (in fact, the difference between these two concepts is fundamental, see Chapter 3). In particular, this means that weak convergence of the semigroup to zero along some sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \to \infty$ does not in general imply weak stability. However, once the sequence $\{t_n\}_{n=1}^{\infty}$ is relatively dense, i.e., there exists a number $\ell > 0$ such that every sub-interval of $\mathbb{R}_+$ of length ℓ intersects $\{t_n : n \in \mathbb{N}\}$, one does obtain weak stability.

Theorem 2.2.5. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Suppose that $\lim_{n\to\infty} T(t_n) = 0$ weakly for some relatively dense sequence $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$. Then $(T(t))_{t\geq 0}$ is weakly stable.*

Proof. Without loss of generality assume that $\{t_n\}_{n=1}^{\infty}$ is monotone increasing and set $\ell = \sup_{n\in\mathbb{N}}(t_{n+1} - t_n)$, which is finite by assumption. Since every C_0-semigroup is bounded on compact time intervals, and $(T(t_n))_{n\in\mathbb{N}}$ is weakly converging, hence bounded, we obtain that the semigroup $(T(t))_{t\geq 0}$ is bounded.

Fix $x \in X$, $y \in X'$. For $t \in [t_n, t_{n+1}]$ we have

$$\langle T(t)x, y\rangle = \langle T(t-t_n)x, T'(t_n)y\rangle,$$

where $(T'(t))_{t\geq 0}$ is the adjoint semigroup. We note that by assumption $T'(t_n)y \to 0$ in the weak*-topology.

Further, the set $K_x = \{T(s)x : \ 0 \leq s \leq \ell\}$ is compact in X and $T(t-t_n)x \in K_x$ for every $n \in \mathbb{N}$. Since pointwise convergence is equivalent to the uniform convergence on compact sets (see, e.g., [Engel and Nagel, 2000], Prop. A.3), we see that $\langle T(t)x, y\rangle \to 0$. $\square$

Note that by taking $t_n = n$ in Theorem 2.2.5 we obtain that $(T(t))_{t\geq 0}$ is weakly stable if and only if $T(n) \to 0$ weakly as $n \to \infty$, $n \in \mathbb{N}$. This gives a connection between weak stability of discrete and continuous semigroups.

2.3 Examples

In this section we discuss concrete and abstract examples of almost weakly but not weakly stable semigroups. The first example indicates how one can construct almost weakly but not weakly stable semigroups using dynamical systems arising in ergodic theory.

Example 2.3.1. A measurable measure-preserving semiflow $(\varphi_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{M}, \mu)$ is called *strongly mixing* if $\lim\limits_{t\to\infty} \mu(\varphi_t^{-1}(A) \cap B) = \mu(A)\mu(B)$ for any two measurable sets $A, B \in \mathcal{M}$. The semiflow $(\varphi_t)_{t\geq 0}$ is called *weakly mixing* if for all $A, B \in \mathcal{M}$ we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |\mu(\varphi_s^{-1}(A) \cap B) - \mu(A)\mu(B)|\, ds = 0.$$

These concepts play an essential role in ergodic theory, and we refer to the monographs [Cornfeld et al., 1982], [Krengel, 1985], [Petersen, 1983], or [Halmos, 1956] for further information. Clearly, strong mixing implies weak mixing, but the converse implication does not hold in general. However, examples of weakly but not strongly mixing semiflows are not easy to construct; see [Lind, 1975] for an example and [Petersen, 1983, p. 209] for a method of constructing such semiflows.

The semiflow $(\varphi_t)_{t\geq 0}$ on $(\Omega, \mathcal{M}, \mu)$ induces a semigroup of isometries $(T(t))_{t\geq 0}$ on each of the Banach spaces $X = L^p(\Omega, \mu)$ $(1 \leq p < \infty)$ by defining

$$(T(t)f)(\omega) = f(\varphi_t(\omega)), \quad \omega \in \Omega, f \in L^p(\Omega, \mu).$$

This semigroup is strongly continuous (see [Krengel, 1985, Thm. 6.13]) and relatively weakly compact. It is well-known (see, e.g., [Halmos, 1956, pp. 37–38]) that

$$(\varphi_t)_{t\geq 0} \text{ is strongly mixing}$$

$$\Longleftrightarrow$$

$$\lim_{t\to\infty} \langle T(t)f, g\rangle = \langle Pf, g\rangle \text{ for all } f \in X,\ g \in X',$$

and

$$(\varphi_t)_{t\geq 0} \text{ is weakly mixing}$$
$$\Longleftrightarrow$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t |\langle T(s)f,g\rangle - \langle Pf,g\rangle|\,ds = 0 \text{ for all } f\in X,\ g\in X',$$

where P is the projection onto $\mathrm{Fix}(T)$ given by $Pf = \int_\Omega f\,d\mu\cdot\mathbf{1}$ for all $f\in X$. Note that in both cases $\mathrm{Fix}(T)=\langle\mathbf{1}\rangle$ holds.

Take now any semiflow $(\varphi_t)_{t\geq 0}$ which is weakly but not strongly mixing. Observe that $X = X_0\oplus\langle\mathbf{1}\rangle$, where

$$X_0 = \left\{f\in X\ :\ \int_\Omega f\,d\mu = 0\right\}$$

is closed and $(T(t))_{t\geq 0}$-invariant. We denote the restriction of $(T(t))_{t\geq 0}$ to X_0 by $(T_0(t))_{t\geq 0}$ and its generator by A_0. The semigroup $(T_0(t))_{t\geq 0}$ is still relatively weakly compact and, since $P\sigma(A)\cap i\mathbb{R}=\emptyset$, it is almost weakly stable. On the other hand, $(T_0(t))_{t\geq 0}$ is not weakly stable since $(\varphi_t)_{t\geq 0}$ is not strongly mixing.

We can also look at this example from a different perspective. If $(\varphi_t)_{t\in\mathbb{R}}$ is even a measure preserving *flow*, it induces a C_0-group $(T(t))_{t\in\mathbb{R}}$ of unitary operators on the Hilbert space $L^2(\Omega,\mu)$. Hence we can apply the spectral theorem and obtain for each $x\in H$ a measure ν_x on $\mathbb{R}$ such that

$$\langle T(t)x,x\rangle = \int_{\mathbb{R}} \mathrm{e}^{itr}\,d\nu_x(r)\quad \text{for all } t\geq 0.$$

Thus $\langle T(t)x,x\rangle$ becomes the Fourier transform of the measure ν_x. In the next example we classify these measures according to the behaviour of their Fourier transform at infinity.

Example 2.3.2. Let us consider the Hilbert space $H = L^2(\mathbb{R},\mu)$, where μ is a finite positive Borel measure, and the operator A on H is the multiplication operator

$$Af(r) = irf(r),\quad r\in\mathbb{R},$$

on its maximal domain. Then A generates the unitary group $(T(t)f)(r) = \mathrm{e}^{itr}f(r)$. Since Hilbert spaces are reflexive, $(T(t))_{t\geq 0}$ is relatively weakly compact.

Clearly, $\sigma(A)\subseteq i\mathbb{R}$ and $ir\in i\mathbb{R}$ is an eigenvalue of A if and only if $\mu(\{r\})>0$. Hence, if $\mu(\{r\})=0$ for all $r\in\mathbb{R}$, then A has no eigenvalues and the Jacobs-Glicksberg-de Leeuw decomposition yields that $(T(t))_{t\geq 0}$ is almost weakly stable.

For $f,g\in H$ we have $\langle T(t)f,g\rangle = \int_{\mathbb{R}}\mathrm{e}^{itr}f(r)\overline{g}(r)\,d\mu$. In particular, by taking $f=g=\mathbf{1}$ we obtain $\langle T(t)\mathbf{1},\mathbf{1}\rangle = \int_{\mathbb{R}}\mathrm{e}^{itr}\,d\mu = \mathcal{F}\mu(t)$, the Fourier transform of μ. On the other hand, $\lim_{t\to\infty}\mathcal{F}\mu(t)=0$ implies $\lim_{t\to\infty}\langle T(t)f,g\rangle = 0$ for all $f,g\in H$, therefore

$$(T(t))_{t\geq 0} \text{ is weakly stable} \iff \lim_{t\to\infty}\mathcal{F}\mu(t)=0.$$

Note that since for unitary groups weak stability as $t \to \infty$ coincides with weak stability as $t \to -\infty$, the property above is equivalent to

$$\lim_{|t|\to\infty} \mathcal{F}\mu(t) = 0.$$

In harmonic analysis, this property of the measure μ got its own name. Indeed, μ is called *Rajchman* if its Fourier transform vanishes at infinity. We refer to [Lyons, 1985] and [Lyons, 1995] for a brief historical overview on these measures and their properties.

We note that absolutely continuous measures are always Rajchman by the Riemann-Lebesgue lemma and all Rajchman measures are continuous by Wiener's theorem. However, there are continuous measures which are not Rajchman and Rajchman measures which are not absolutely continuous (see [Lyons, 1995]). It is now a consequence of the considerations above that each continuous non-Rajchman measure gives rise to an almost weakly but not weakly stable unitary group. In [Engel and Nagel, 2000, p. 316] an example of a unitary group with bounded generator is given, for which the corresponding spectral measures are not Rajchman.

Next, we give an example of a positive semigroup on a Banach lattice which is almost weakly stable but not weakly stable.

Example 2.3.3. As in [Nagel, 1986, p. 206] we start from a flow on $\mathbb{C}\backslash\{0\}$ with the following properties:

1) The orbits starting in z with $|z| \neq 1$ spiral towards the unit circle Γ;

2) 1 is the fixed point of φ and $\Gamma\backslash\{1\}$ is a homoclinic orbit, i.e., $\lim\limits_{t\to-\infty} \varphi_t(z) = \lim\limits_{t\to\infty} \varphi_t(z) = 1$ for every $z \in \Gamma$.

A concrete example comes from the differential equation in polar coordinates $(r, \omega) = (r(t), \omega(t))$:

$$\begin{cases} \dot{r} &= 1 - r, \\ \dot{\omega} &= 1 + (r^2 - 2r\cos\omega). \end{cases}$$

Take $x_0 \in \mathbb{C}$ with $0 < |x_0| < 1$ and denote by $S_{x_0} = \{\varphi_t(x_0) : t \geq 0\}$ the orbit starting from x_0. Then $S = S_{x_0} \cup \Gamma$ is compact for the usual topology of $\mathbb{C}$.

We define a multiplication on S as follows. For $x = \varphi_t(x_0)$ and $y = \varphi_s(x_0)$ we put

$$xy = \varphi_{t+s}(x_0).$$

For $x \in \Gamma$, $x = \lim_{n\to\infty} x_n$, $x_n = \varphi_{t_n}(x_0) \in S_{x_0}$ and $y = \varphi_s(x_0) \in S_{x_0}$, we define $xy = yx = \lim_{n\to\infty} x_n y$. Note that by $|x_n y - \varphi_s(x)| = |\varphi_s(x_n) - \varphi_s(x)| \leq C|x_n - x| \underset{n\to\infty}{\longrightarrow} 0$ the definition is correct and satisfies

$$xy = \varphi_s(x).$$

For $x, y \in \Gamma$ we define $xy = 1$. This multiplication on S is separately continuous and makes S a semi-topological semigroup (see [Engel and Nagel, 2000, Sec. V.2]).

Consider now the Banach space $X = C(S)$. The set

$$\{f(s\,\cdot) : s \in S\} \subset C(S)$$

is relatively weakly compact for every $f \in C(S)$. By definition of the multiplication on S this implies that

$$\{f(\varphi_t(\cdot)) : t \geq 0\}$$

is relatively weakly compact in $C(S)$. Consider the semigroup induced by the flow, i.e.,

$$(T(t)f)(x) = f(\varphi_t(x)), \qquad f \in C(S),\, x \in S.$$

By the above, each orbit $\{T(t)f : t \geq 0\}$ is relatively weakly compact in $C(S)$ and hence, by Lemma 2.1.3, $(T(t))_{t\geq 0}$ is weakly compact. Note that the strong continuity of $(T(t))_{t\geq 0}$ follows, as shown in [Nagel, 1986], Lemma B-II.3.2, from the separate continuity of the flow. Furthermore, the semigroup $(T(t))_{t\geq 0}$ is isometric.

Next, we take $X_0 = \{f \in C(S) : f(1) = 0\}$ and identify it with the Banach lattice $C_0(S \setminus \{1\})$. Then both subspaces in the decomposition $C(S) = X_0 \oplus \langle \mathbf{1} \rangle$ are invariant under $(T(t))_{t\geq 0}$. Denote by $(T_0(t))_{t\geq 0}$ the restricted semigroup to X_0 and by A_0 its generator. The semigroup $(T_0(t))_{t\geq 0}$ is still relatively weakly compact.

Since $\operatorname{Fix}(T_0) = \bigcap_{t\geq 0} \operatorname{Fix}(T_0(t)) = \{0\}$, we have that $0 \notin P\sigma(A_0)$. Moreover, $P\sigma(A_0) \cap i\mathbb{R} = \emptyset$ holds, which implies by the Jacobs-Glicksberg-de Leeuw theorem that $(T_0(t))_{t\geq 0}$ is almost weakly stable.

To see that $(T_0(t))_{t\geq 0}$ is not weakly stable it is enough to consider $\delta_{x_0} \in X_0'$. Since

$$\langle T_0(t)f, \delta_{x_0} \rangle = f(\varphi(t, x_0)),\ f \in X_0,$$

$f(\Gamma)$ always belongs to the closure of $\{\langle T_0(t)f, \delta_{x_0}\rangle : t \geq 0\}$ and hence the semigroup $(T_0(t))_{t\geq 0}$ can not be weakly stable.

Let us summarise the above as follows.

Proposition 2.3.4. *There exist a locally compact space Ω and a positive, relatively weakly compact C_0-semigroup of isometries on $C_0(\Omega)$ which is almost weakly but not weakly stable.*

This enables us to answer a question of [Emelyanov, 2005] in the negative. Consider the discrete semigroup $(T(n))_{n\in\mathbb{N}} = (T(1)^n)_{n\in\mathbb{N}}$ from Proposition 2.3.4. By a result of [Jones and Lin, 1980], we know that zero belongs to the weak closure of each of the orbits, whereas Theorem 2.2.5 shows that this semigroup is not weakly stable. The semigroup is positive and isometric on the Banach lattice $C_0(\Omega)$.

Moreover, Proposition 2.3.4 becomes particularly interesting in view of the following results; for details and discussion see [Chill and Tomilov, 2007].

Theorem 2.3.5 ([Groh and Neubrander, 1981, Thm. 3.2], [Chill and Tomilov, 2007, Thm. 7.7]). *For a bounded, positive, mean ergodic C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X with generator $(A, D(A))$, the following assertions hold.*

(i) If $X \cong L^1(\Omega, \mu)$, then $P\sigma(A) \cap i\mathbb{R} = \emptyset$ is equivalent to the strong stability *of $(T(t))_{t\geq 0}$.*

(ii) If $X \cong C(K)$, K compact, then $P\sigma(A) \cap i\mathbb{R} = \emptyset$ is equivalent to the uniform exponential stability *of $(T(t))_{t\geq 0}$.*

Example 2.3.3 shows that we can not drop the assumption on the existence of a unit element in X in Theorem 2.3.5 (ii).

In the next chapter we shall see that the examples above represent the usual situation. Indeed, in a sense to be made precise, typical isometric semigroups and typical unitary groups are almost weakly but not weakly stable.

2.4 Individual stability and local resolvent

In this section, we restrict our attention to single orbits and present results implying

$$\lim_{t\to\infty} \langle T(t)x, y\rangle = 0 \quad \text{for some given } x \text{ and } y.$$

The tool will be the *bounded local resolvent* $R(\lambda)x_0$ which exists by definition if the function $\rho(A) \ni \lambda \mapsto R(\lambda, A)x_0$ admits a bounded, holomorphic extension $R(\lambda)x_0$ to the whole right half-plane $\{\lambda : \operatorname{Re}\lambda > 0\}$. This we assume in the following.

Clearly, if we suppose that for *all* $x_0 \in X$ the local resolvent $R(\lambda)x_0$ is bounded, analyticity and the principle of uniform boundedness yield the boundedness of the operator resolvent $R(\lambda, A)$ on $\{\lambda : \operatorname{Re}\lambda > 0\}$, hence uniform exponential stability for semigroups on Hilbert spaces and (at least) strong stability for semigroups on reflexive Banach spaces (see Theorems 1.3.1 and 1.3.2). The reasonable questions therefore address the individual stability of a single orbit in terms of the local resolvent of one single element $x_0 \in X$.

Without any differentiability or boundedness assumption on the semigroup it is necessary to do some initial smoothing on x_0 in order to have stability in any sense. However, if the Banach space X has nice geometric properties, even strong stability can be derived. Huang and van Neerven [Huang and van Neerven, 1999] proved that if the Banach space is B-convex or has the analytic Radon-Nikodým property, then the existence of a bounded local resolvent $R(\lambda)x_0$ on $\{\lambda : \operatorname{Re}\lambda > 0\}$ already implies *strong convergence* $T(t)R(\mu, A)^\alpha x_0 \to 0$ as $t \to +\infty$ for any $\alpha > 1$. (Here μ is greater than the growth bound $\omega_0(A)$, thus $R(\mu, A)$ is a sectorial operator admitting fractional powers.) Actually, if X has Fourier type $p > 1$, then we can take $\alpha > 1/p$, and if the semigroup is eventually differentiable and $p = 2$, then no smoothing is needed, i.e. $\alpha \geq 0$ is allowed, see [Huang, 1999].

In general, without any additional assumptions on the space or on the regularity of the semigroup one can only deduce weak individual stability. The following result is due to [Huang and van Neerven, 1999].

Theorem 2.4.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A. Let $x_0 \in X$ and suppose that the function $\lambda \mapsto R(\lambda, A)x_0$ has a bounded holomorphic extension to $\{\lambda : \operatorname{Re}\lambda > 0\}$. Then*

(i) $\lim_{t\to\infty}\langle T(t)x_0, y\rangle = 0$ *for all* $y \in D((A')^2)$.

(ii) $\lim_{t\to\infty} T(t)x_0 = 0$ *weakly, provided that the semigroup $(T(t))_{t\geq 0}$ is uniformly bounded.*

Tauberian theorems are among the primary tools to deduce information on the asymptotic behaviour of the semigroup from properties of the resolvent, and they have extensively been used to obtain strong and weak stability. We refer the reader to the monograph [Arendt et al., 2001] and also to [Chill, 1998]. As illustration we include here a proof for part (i) based on Ingham's Tauberian theorem.

Theorem 2.4.2 (Ingham)**.** *Let $f : \mathbb{R}_+ \to \mathbb{C}$ be bounded and uniformly continuous and suppose that the Laplace transform $\hat{f}$ of f has a locally integrable boundary function on the imaginary axis (that is, there exists $h \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C})$ such that $\lim_{a\to 0+} \hat{f}(a + i\cdot) = h$ in the distributional sense). Then $\lim_{t\to\infty} f(t) = 0$.*

For proofs and a detailed treatment see [Arendt et al., 2001, Sect. 4].

Proof. [Proof of Theorem 2.4.1, part (ii)] By assumption the operator resolvent $R(\lambda, A)x_0$ and the local resolvent $R(\lambda)x_0$ coincide on the right halfplane. So for a fixed $y \in X'$, the function $\lambda \mapsto \langle R(\lambda)x_0, y\rangle$ is the Laplace transform of the function $t \mapsto \langle T(t)x_0, y\rangle$ on the right half-plane. Since $(T(t))_{t\geq 0}$ is uniformly bounded, the weak orbit $t \mapsto \langle T(t)x_0, y\rangle$ is bounded and uniformly continuous, so we can apply Ingham's theorem to obtain $\lim_{t\to\infty}\langle T(t)x_0, y\rangle = 0$. □

For the proof of Theorem 2.4.1 part (i) one could also use Ingham's Theorem, and check the assumptions of the theorem along the lines of [Batty et al., 2000]. Actually, in [Batty et al., 2000] a powerful functional calculus method is developed, which among other yields the proof of the more general Theorem 2.4.3 below. To prove part (i) we nevertheless choose a different, fairly elementary way (see [Eisner and Farkas, 2007]).

Proof. [Proof of Theorem 2.4.1 part (i)] By $\lambda \mapsto R(\lambda)x_0$ we denote the holomorphic continuation of $\lambda \mapsto R(\lambda, A)x_0$ to the half-plane $\{\lambda : \operatorname{Re}\lambda > 0\}$. The uniqueness theorem for holomorphic functions and the resolvent identity imply

$$\begin{aligned}R(\delta+is)x_0 &= R(a+is,A)x_0 + (a-\delta)R(a+is,A)R(\delta+is)x_0\\ &= R(a+is,A)x_0 + (a-\delta)R^2(a+is,A)x_0 + (a-\delta)^2R^2(a+is,A)R(\delta+is)x_0.\end{aligned}$$

For all $y \in D(A'^2)$ we have

$$\begin{aligned}&2\pi \mathrm{e}^{-\delta t}\langle T(t)x_0, y\rangle =\\ &\qquad \int_{-\infty}^{\infty} \mathrm{e}^{ist}\langle R(\delta+is)x_0, y\rangle\, ds = \int_{-\infty}^{\infty} \mathrm{e}^{ist}\langle R(a+is,A)x_0, y\rangle\, ds\\ &\qquad\quad + (a-\delta)\int_{-\infty}^{\infty} \mathrm{e}^{ist}\langle R^2(a+is,A)x_0, y\rangle\, ds + (a-\delta)^2\, ds\cdot\\ &\qquad\qquad\cdot \int_{-\infty}^{\infty} \mathrm{e}^{ist}\langle R^2(a+is,A)R(\delta+is)x_0, y\rangle\, ds.\end{aligned}$$

Indeed, for $a > \omega_0(T)$ the first equality follows from representing the semigroup as the inverse Laplace transform of the resolvent and the Cauchy theorem extends this representation for all $a > 0$. The functions $f_\delta(s) = \langle R^2(a+is,A)R(\delta+is)x_0, y\rangle$ form a relatively compact subset of $L^1(\mathbb{R})$, because

$$\begin{aligned}|f_\delta(s)| &= |\langle R^2(a+is,A)R(\delta+is)x_0, y\rangle| =\\ &= |\langle R(\delta+is)x_0, R^2(a+is,A')y\rangle| \le M\|R^2(a+is,A')y\|,\end{aligned}$$

and the function on the right hand side lies in $L^1(\mathbb{R})$, so the family f_δ is uniformly integrable (and bounded), thus relatively compact. By compactness we find a sequence $\delta_n \to 0$ such that $\lim_{n\to\infty} f_{\delta_n} = f$ in $L^1(\mathbb{R})$. By substituting δ_n in the above equality and letting $n \to \infty$ we obtain

$$\begin{aligned}2\pi\langle T(t)x_0, y\rangle &= \int_{-\infty}^{\infty} \mathrm{e}^{ist}\langle R(a+is,A)x_0, y\rangle\, ds\\ &+ a\int_{-\infty}^{\infty} \mathrm{e}^{ist}\langle R^2(a+is,A)x_0, y\rangle\, ds + a^2\int_{-\infty}^{\infty} \mathrm{e}^{ist} f(s)\, ds = I_1(t) + I_2(t) + I_3(t).\end{aligned}$$

It is easy to deal with the last term I_3. The function f lies in $L^1(\mathbb{R})$, so by the Riemann-Lebesgue Lemma its Fourier transform vanishes at infinity, i.e., $\lim_{t\to\infty} I_3(t) = 0$. Since $y \in D((A')^2)$, we can integrate by parts in I_1 to obtain

$$I_1(t) = \int_{-\infty}^{\infty} \mathrm{e}^{ist}\langle x_0, R(a+is,A')y\rangle\, ds = \frac{1}{t}\int_{-\infty}^{\infty} \mathrm{e}^{ist}\langle x_0, R^2(a+is,A')y\rangle\, ds.$$

The last integral is absolutely convergent, because using the resolvent identity for $R(\lambda, A')$, one can show that for $y \in D(A'^2)$ and $a > \omega_0(T)$ fixed, $\|R^2(a+is, A')y\| = O((a^2+s^2)^{-1})$ holds. Hence

$$|I_1(t)| \leq \frac{1}{t}\int_{-\infty}^{\infty} \|x_0\| \cdot \|R^2(a+is, A')y\|\, ds \to 0 \qquad \text{as } t \to \infty.$$

Concerning I_2 we observe that $\langle x_0, R^2(a+i\cdot, A')y\rangle \in L^1(\mathbb{R})$, so once again by the Riemann-Lebesgue Lemma we have

$$I_2(t) = a\int_{-\infty}^{\infty} \mathrm{e}^{ist}\langle x_0, R^2(a+is, A')y\rangle\, ds \to 0 \qquad \text{as } t \to \infty,$$

and the proof is complete. □

Actually, Huang and van Neerven proved the following more general theorem.

Theorem 2.4.3 ([Huang and van Neerven, 1999]). *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A. Assume that the bounded local resolvent exists for $x_0 \in X$. Then $\lim_{t\to\infty} T(t)(\lambda_0 - A)^{-\beta}x_0 = 0$ weakly for all $\beta > 1$ and $\lambda_0 > \omega_0(T)$.*

Under a special positivity condition one can take $\beta = 1$ in Theorem 2.4.3.

Theorem 2.4.4 ([van Neerven, 2002]). *Suppose that X is an ordered Banach space with weakly closed normal cone C. If for some $x_0 \in X$ the function $\lambda \mapsto R(\lambda, A)x_0$ has a bounded holomorphic extension to $\{\lambda : \operatorname{Re}\lambda > 0\}$ and $T(t)x_0 \in C$ for all sufficiently large t, then $\lim_{t\to\infty} T(t)R(\mu, A)x_0 = 0$ weakly for all $\mu \in \rho(A)$.*

The eventual positivity assumption above cannot be omitted. Indeed, [van Neerven, 2002] proved that the existence of a bounded local resolvent $R(\lambda)x_0$ in general implies $\|T(t)R(\mu, A)x_0\| = O(1+t)$, and [Batty, 2003] showed this to be optimal, whereas weak convergence of $T(t)R(\mu, A)x_0$ to zero would imply $\|T(t)R(\mu, A)x_0\| = O(1)$.

2.5 Cogenerator of contractive semigroups

Let $(T(t))_{t\geq 0}$ be a contractive C_0-semigroup on a Hilbert space H. The cogenerator of $(T(t))_{t\geq 0}$ is defined as the (negative) Cayley-transform of the infinitesimal generator A of $(T(t))_{t\geq 0}$, i.e.,

$$G = -(I + A)(I - A)^{-1} = I - 2R(1, A).$$

It is easy to see that the cogenerator is a contraction, see [Sz.-Nagy and Foiaş, 1970, Sections III.8–9] for details. The semigroup can be recovered using a functional calculus; moreover, many important properties of the semigroup can directly be read off from its cogenerator. Namely, the semigroup consists of normal, self-adjoint, isometric or unitary operators if and only if the cogenerator is normal, self-adjoint, isometric or unitary, respectively. The asymptotic behaviour of the semigroup can also be characterised with the help of the cogenerator.

Theorem 2.5.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions and let G be its cogenerator. Then*
$$\lim_{t\to\infty} \|T(t)x\| = \lim_{n\to\infty} \|G^n x\|.$$
In particular, the semigroup is strongly stable if and only if G is strongly stable.

Motivated by the above theorem we ask the following.

Question 2.5.2. Is the analogue of Theorem 2.5.1 true for weak stability? Or, generally speaking, is there a connection between the weak stability of a contractive semigroup and the weak stability of its cogenerator?

Note that the function $z \mapsto -\frac{1+z}{1-z}$ maps the imaginary axis onto the unit circle, so by the spectral mapping theorem for the point spectrum (see [Engel and Nagel, 2000, Theorem IV.3.7]), we have that

$$P\sigma(A) \cap i\mathbb{R} = \emptyset \quad \text{if and only if} \quad P\sigma(G) \cap \{z : |z| = 1\} = \emptyset.$$

Hence by a result of [Jones and Lin, 1980] we obtain the following.

Proposition 2.5.3. *A contractive semigroup $(T(t))_{t\geq 0}$ on a Hilbert space H is almost weakly stable if and only if its cogenerator G is "almost weakly stable", i.e., when zero belongs to the weak closure of each orbit $\{G^n x : n \in \mathbb{N}\}$, $x \in H$.*

This again connects the asymptotic behaviour of $(T(t))_{t\geq 0}$ to the behaviour of the powers of a single operator.

Chapter 3

Category theorems on weakly stable semigroups

Having studied weak stability and almost weak stability in the previous chapter, our aim now is to show that (for operators on separable infinite-dimensional Hilbert spaces) these concepts differ fundamentally. Strictly speaking, we show that the sets of almost weakly and weakly stable operators have different Baire category in each of three complete metric spaces comprised by all unitary, isometric and contractive operators respectively (see Theorems 3.1.6, 3.2.5 and 3.3.3). The set of all weakly stable operators is of the first category, while the set of all almost weakly stable operators is residual in each of these spaces. In this sense, a typical operator is almost weakly but not weakly stable. These results are the operator theoretic counterpart of classical category theorems of Halmos and Rohlin from ergodic theory, see [Halmos, 1956, pp. 77–80], or the original papers [Halmos, 1944] and [Rohlin, 1948].

In this chapter, statements and proofs are formulated for discrete operator semigroups; see Remark 3.3.4 nevertheless.

3.1 Unitary operators

First of all, we formally define the concept of almost weak stability for power bounded operators on a Hilbert space.

Definition 3.1.1. A power bounded operator T on a Hilbert space is called almost weakly stable if zero is a weak accumulation point of every orbit $\{T^n x : n \in \mathbb{N}\}$.

By the Jacobs–Glicksberg–de Leeuw theorem, this is equivalent to the property "no eigenvalues on the unit circle".

Let $\mathcal{U}$ denote the set of all unitary operators on the separable infinite-dimensional Hilbert space H. We observe that any unitary operator can be approximated by periodic ones.

Definition 3.1.2. We say that an operator $P \in \mathcal{L}(H)$ is periodic with period $n \in \mathbb{N}$ if $P^n = I$ and $P^m \neq I$ for all $1 \leq m < n$.

Proposition 3.1.3. *For any given $n \in \mathbb{N}$, $\varepsilon > 0$ and $U \in \mathcal{U}$ there is a periodic operator $P \in \mathcal{U}$ with period at least n such that $\|P - U\| < \varepsilon$.*

Proof. By the spectral theorem H is isomorphic to $L^2(\Omega, \mu)$ for some locally compact space Ω and finite measure μ and U is unitary equivalent to a multiplication operator $\tilde{U}$ with

$$(\tilde{U}f)(\omega) = \varphi(\omega)f(\omega), \quad \forall \omega \in \Omega,$$

for some measurable $\varphi : \Omega \to \Gamma = \{z \in \mathbb{C} \mid |z| = 1\}$.

We approximate the operator $\tilde{U}$ as follows. Consider the set

$$\Gamma_N = \{e^{2\pi i \frac{p}{q}} \mid p, q \in \mathbb{N} \text{ relatively prime }, \ q > N\},$$

which is dense in Γ. Take a finite set $\{\alpha_j\}_{j=1}^n \subset \Gamma_N$ such that $\arg(\alpha_{j-1}) < \arg(\alpha_j)$ and $|\alpha_j - \alpha_{j-1}| < \varepsilon$ hold for all $2 \leq j \leq n$. Define

$$\psi(\omega) = \alpha_{j-1}, \quad \omega \in \varphi^{-1}(\{z \in \Gamma \mid \arg(\alpha_{j-1}) \leq \arg(z) \leq \arg(\alpha_j)\}).$$

Let $\tilde{P}$ denote the operator of multiplication with ψ. The operator $\tilde{P}$ is periodic with period greater than N. Moreover,

$$\|\tilde{U} - \tilde{P}\| = \sup_{\omega \in \Omega} |\varphi(\omega) - \psi(\omega)| \leq \varepsilon$$

holds and the proposition is proved. □

Lemma 3.1.4. *Let H be a separable infinite-dimensional Hilbert space. Then there exists a sequence $(T_n)_{n=1}^\infty$ of almost weakly stable unitary operators satisfying $\lim_{n\to\infty} \|T_n - I\| = 0$.*

Proof. Without loss of generality let us take the model Hilbert space $H = L^2(\mathbb{R}, \lambda)$, where λ is the Lebesgue measure. For $n \in \mathbb{N}$ we define T_n by

$$(T_n f)(s) = e^{\frac{iq(s)}{n}} f(s), \quad s \in \mathbb{R}, \quad f \in L^2(\mathbb{R}, \lambda),$$

where $q : \mathbb{R} \to [0, 1]$ is a fixed strictly monotone function. Then all the operators T_n are almost weakly stable by the theorem of Jacobs–Glicksberg–de Leeuw and we have

$$\|T_n - I\| = \sup_{s \in \mathbb{R}} |e^{\frac{iq(s)}{n}} - 1| \leq |e^{\frac{i}{n}} - 1| \to 0, \quad n \to \infty.$$

□

We now introduce the appropriate topology. We say that an operator sequence $\{T_n\} \subset \mathcal{L}(H)$ converges to $T \in \mathcal{L}(H)$ in the strong*-topology if $T_n \to T$ and $T_n^* \to T^*$ strongly (for details see [Takesaki, 1979, p. 68]). From now on, we consider the space $\mathcal{U}$ of all unitary operators on H endowed with the strong*-topology. Observe that $\mathcal{U}$ is a complete metric space with respect to the metric given by

$$d(U,V) = \sum_{j=1}^{\infty} \frac{\|Ux_j - Vx_j\| + \|U^*x_j - V^*x_j\|}{2^j\|x_j\|} \quad \text{for } U, V \in \mathcal{U},$$

where $\{x_j\}_{j=1}^{\infty}$ is a fixed dense subset of H. Further, by $\mathcal{S}_{\mathcal{U}}$ we denote the set of all weakly stable unitary operators on H and by $\mathcal{W}_{\mathcal{U}}$ we denote the set of all almost weakly stable unitary operators on H.

We now show the following density property for $\mathcal{W}_{\mathcal{U}}$.

Proposition 3.1.5. *The set $\mathcal{W}_{\mathcal{U}}$ of all almost weakly stable unitary operators is dense in $\mathcal{U}$.*

Proof. By Proposition 3.1.3 it is enough to approximate periodic unitary operators by almost weakly stable unitary operators. Let U be a periodic unitary operator and let N be its period. Take $\varepsilon > 0$, $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in H$. We have to find an almost weakly stable unitary operator T with $\|Ux_j - Tx_j\| < \varepsilon$ and $\|U^*x_j - T^*x_j\| < \varepsilon$ for all $j = 1, \ldots, n$.

By $U^N = I$ and the spectral theorem, $\sigma(U) \subset \left\{1, e^{\frac{2\pi i}{N}}, \ldots, e^{\frac{2\pi(N-1)i}{N}}\right\}$ and the orthogonal decomposition

$$H = \ker(I - U) \oplus \ker(e^{\frac{2\pi i}{N}} I - U) \oplus \ldots \oplus \ker(e^{\frac{2\pi(N-1)i}{N}} I - U) \tag{3.1}$$

holds.

Assume first that $x_1, \ldots, x_n$ are orthogonal eigenvectors of U. In order to use Lemma 3.1.4 we first construct a periodic unitary operator S which satisfies $Ux_j = Sx_j$ for all $j = 1, \ldots, n$ and whose maximal eigenspaces are infinite-dimensional. For this purpose define the n-dimensional U- and U^*-invariant subspace $H_0 = \text{lin}\{x_j\}_{j=1}^n$ and the operator S_0 on H_0 as the restriction of U to H_0. Decompose H as an orthogonal sum

$$H = \bigoplus_{k=0}^{\infty} H_k \quad \text{with } \dim H_k = \dim H_0 \quad \text{for all } k \in \mathbb{N}.$$

For each k, let P_k denote an isomorphism from H_k to H_0. Now define $S_k = P_k^{-1} U P_k$ on each H_k as a copy of $U|_{H_0}$ and consider $S = \bigoplus_{k=0}^{\infty} S_k$ on H.

The operator S is unitary and periodic with period being a divisor of N. So a decomposition analogous to (3.1) is valid for S. Moreover, $Ux_j = Sx_j$ and $U^*x_j = S^*x_j$ hold for all $j = 1, \dots, n$ and the maximal eigenspaces of S are infinite dimensional. Denote by F_j the maximal eigenspace of S containing x_j and by λ_j the corresponding eigenvalue. By Lemma 3.1.4 for every $j = 1, \dots, n$ there exists an almost weakly stable unitary operator T_j on F_j satisfying $\|T_j - S_{|F_j}\| = \|T_j - \lambda_j I\| < \varepsilon$. Finally, we define the desired operator T as T_j on F_j for every $j = 1 \dots, n$ and extend it linearly to H.

Now, let $x_1, \dots, x_n \in H$ be arbitrary and take an orthonormal basis of eigenvalues $\{y_k\}_{k=1}^{\infty}$. Then there exists $K \in \mathbb{N}$ such that $x_j = \sum_{k=1}^{K} a_{jk} y_k + o_j$ with $\|o_j\| < \frac{\varepsilon}{4}$ for every $j = 1, \dots, n$. By the arguments above applied to $y_1, \dots, y_K$ there is an almost weakly stable unitary operator T with $\|Uy_k - Ty_k\| < \frac{\varepsilon}{4KM}$ and $\|U^* y_k - T^* y_k\| < \frac{\varepsilon}{4KM}$ for $M = \max_{k=1,\dots,K, j=1,\dots,n} |a_{jk}|$ and every $k = 1, \dots, K$. Therefore we obtain

$$\|Ux_j - Tx_j\| \le \sum_{k=1}^{K} |a_{jk}| \|Uy_k - Ty_k\| + 2\|o_j\| < \varepsilon$$

for every $j = 1, \dots, n$. Analogously, $\|U^* x_j - T^* x_j\| < \varepsilon$ holds for every $j = 1, \dots, n$, and the proposition is proved. □

We can now prove the following category theorem for weakly and almost weakly stable unitary operators. To do so we extend the argument used in the proof of the corresponding category theorems for flows in ergodic theory (see [Halmos, 1956, pp. 77–80]).

Theorem 3.1.6. *The set $\mathcal{S}_{\mathcal{U}}$ of weakly stable operators is of first category and the set $\mathcal{W}_{\mathcal{U}}$ of almost weakly stable operators is residual in $\mathcal{U}$.*

Proof. First we prove that $\mathcal{S}$ is of first category in $\mathcal{U}$. Fix $x \in H$ with $\|x\| = 1$ and consider

$$M_k = \left\{ U \in \mathcal{U} : \ |\langle U^k x, x \rangle| \le \frac{1}{2} \right\}.$$

Note that all the sets M_k are closed. Let $U \in \mathcal{U}$ be weakly stable. Then there exists $n \in \mathbb{N}$ such that $U \in M_k$ for all $k \ge n$, i.e., $U \in \cap_{k \ge n} M_k$. So we obtain

$$\mathcal{S}_{\mathcal{U}} \subset \bigcup_{n=1}^{\infty} N_n, \tag{3.2}$$

where we put $N_n = \cap_{k \ge n} M_k$. Since all the sets N_n are closed, it remains to show that $\mathcal{U} \setminus N_n$ is dense for every n.

Fix $n \in \mathbb{N}$ and let U be a periodic unitary operator. Then $U \notin M_k$ for some $k \ge n$ and therefore $U \notin N_n$. Since by Proposition 3.1.3 periodic unitary operators are dense in $\mathcal{U}$, $\mathcal{S}$ is of first category. To show that $\mathcal{W}_{\mathcal{U}}$ is residual we take a dense subspace $D = \{x_j\}_{j=1}^{\infty}$ of H and define

$$W_{jkn} = \left\{ U \in \mathcal{U} : \ | \langle U^n x_j, x_j \rangle | < \frac{1}{k} \right\}.$$

All these sets are open. Therefore the sets $W_{jk} = \cup_{n=1}^{\infty} W_{jkn}$ are also open. We show that

$$\mathcal{W}_{\mathcal{U}} = \bigcap_{j,k=1}^{\infty} W_{jk} \tag{3.3}$$

holds.

The inclusion "$\subset$" follows from the definition of almost weak stability. To prove the converse inclusion we take $U \notin \mathcal{W}_{\mathcal{U}}$ and $n \in \mathbb{N}$. Then there exists $x \in H$ with $\|x\| = 1$ and $\varphi \in \mathbb{R}$ such that $Ux = e^{i\varphi}x$. Therefore $|\langle U^n x, x\rangle| = 1$. Take $x_j \in D$ with $\|x_j - x\| \leq \frac{1}{4}$. Then

$$\begin{aligned} |\langle U^n x_j, x_j\rangle| &= |\langle U^n(x-x_j), x-x_j\rangle + \langle U^n x, x\rangle - \langle U^n x, x-x_j\rangle - \langle U^n(x-x_j), x\rangle| \\ &\geq 1 - \|x-x_j\|^2 - 2\|x-x_j\| > \frac{1}{3}. \end{aligned}$$

So $U \notin W_{j3}$ which implies $U \notin \cap_{j,k=1}^{\infty} W_{jk}$. Therefore (3.3) holds. So $\mathcal{W}_{\mathcal{U}}$ is residual as a countable intersection of open dense sets. □

3.2 Isometries

In this section we consider the space $\mathcal{I}$ of all isometries on H endowed with the strong topology and prove analogous category results as in the previous section. We again assume H to be separable and infinite-dimensional. Note that $\mathcal{I}$ is a complete metric space with respect to the metric given by the formula

$$d(T,S) = \sum_{j=1}^{\infty} \frac{\|Tx_j - Sx_j\|}{2^j \|x_j\|} \quad \text{for } T, S \in \mathcal{I},$$

where $\{x_j\}_{j=1}^{\infty}$ is a fixed dense subset of H. Further, we denote by $\mathcal{S}_{\mathcal{I}}$ the set of all weakly stable isometries on H and by $\mathcal{W}_{\mathcal{I}}$ the set of all almost weakly stable isometries on H.

The basis of our results in this section is the following classical theorem on Hilbert spaces isometries [Sz.-Nagy and Foiaş, 1970, Theorem 1.1] .

Theorem 3.2.1 (Wold decomposition). *Let V be an isometry on a Hilbert space H. Then H can be decomposed into an orthogonal sum $H = H_0 \oplus H_1$ of V-invariant subspaces such that the restriction of V on H_0 is unitary and the restriction of V on H_1 is a unilateral shift, i.e. there exists a subspace $Y \subset H_1$ with $V^nY \perp V^mY$ for all $n \neq m$, $n, m \in \mathbb{N}$, such that $H_1 = \oplus_{n=1}^{\infty} V^nY$ holds.*

As a first application of the Wold decomposition we obtain the density result for periodic operators in $\mathcal{I}$. (Note that periodic isometries are unitary.) But first we need the following easy lemma, see also [Peller, 1981].

Lemma 3.2.2. *Let Y be a Hilbert space and let R be the right shift on $H = l^2(\mathbb{N}, Y)$. Then there exists a sequence $(T_n)_{n=1}^{\infty}$ of periodic unitary operators on H converging strongly to R.*

Proof. We define the operators T_n by

$$T_n(x_1, x_2, \ldots, x_n, \ldots) = (x_n, x_1, x_2, \ldots, x_{n-1}, x_{n+1}, \ldots).$$

Every T_n is unitary and has period n. Moreover, for an arbitrary $x = (x_1, x_2, \ldots) \in H$ we have

$$\|T_n x - Rx\|^2 = \|x_n\|^2 + \sum_{k=n}^{\infty} \|x_{k+1} - x_k\|^2 \underset{n\to\infty}{\longrightarrow} 0,$$

and the lemma is proved. □

Proposition 3.2.3. *The set of all periodic operators is dense in $\mathcal{I}$.*

Proof. Let V be an isometry on H. Then by Theorem 3.2.1 the orthogonal decomposition $H = H_0 \oplus H_1$ holds, where the restriction V_0 on H_0 is unitary and the space H_1 is unitarily equivalent to $l^2(\mathbb{N}, Y)$. The restriction V_1 of V on H_1 corresponds by this equivalence to the right shift operator on $l^2(\mathbb{N}, Y)$. By Proposition 3.1.3 and Lemma 3.2.2 we can approximate both operators V_0 and V_1 by unitary periodic ones and the assertion follows. □

As a consequence of the Wold decomposition we obtain the following density result for almost weakly stable operators in $\mathcal{I}$.

Proposition 3.2.4. *The set $\mathcal{W}_{\mathcal{I}}$ of almost weakly stable isometries is dense in $\mathcal{I}$.*

Proof. Let V be an isometry on H, H_0, H_1 the orthogonal subspaces from Theorem 3.2.1 and V_0 and V_1 the corresponding restrictions of V. By Lemma 3.2.2 the operator V_1 can be approximated by unitary operators on H_1. The assertion now follows from Proposition 3.1.5. □

Using the same idea as in the proof of Theorem 3.1.6 one obtains with the help of Propositions 3.2.3 and 3.2.4 the following category result for weakly and almost weakly stable isometries.

Theorem 3.2.5. *The set $\mathcal{S}_{\mathcal{I}}$ of all weakly stable isometries is of first category and the set $\mathcal{W}_{\mathcal{I}}$ of all almost weakly stable isometries is residual in $\mathcal{I}$.*

3.3 Contractions

We now extend the category results in the previous sections to the case of contractive operators. Let $\mathcal{C}$ denote the set of all contractions on H endowed with the weak operator topology. Note that $\mathcal{C}$ is a complete metric space with respect to the metric given by

$$d(T,S) = \sum_{i,j=1}^{\infty} \frac{|\langle Tx_i, x_j\rangle - \langle Sx_i, x_j\rangle|}{2^{i+j}\|x_i\|\|x_j\|} \quad \text{for} \quad T,S \in \mathcal{C},$$

where $\{x_j\}_{j=1}^{\infty}$ is a fixed dense subset of H. By [Takesaki, 1979, p. 99], the set of all unitary operators is dense in $\mathcal{C}$ (see also [Peller, 1981] for a much stronger assertion). Combining this fact with Propositions 3.1.3 and 3.1.5 we gain the following.

Proposition 3.3.1. *The set of all periodic unitary operators and the set of all almost weakly stable unitary operators are both dense in $\mathcal{C}$.*

The next well-known property is a key for the further results (cf. [Halmos, 1967, p. 14]).

Lemma 3.3.2. *Let $(T_n)_{n=1}^{\infty}$ be a sequence of contractions on a Hilbert space H converging weakly to an isometry S. Then $T_n \to S$ strongly.*

Proof. For each $x \in H$ we have

$$\begin{aligned} \|T_n x - Sx\|^2 &= \langle T_n x - Sx, T_n x - Sx\rangle = \|Sx\|^2 + \|T_n x\|^2 - 2Re\,\langle T_n x, Sx\rangle \\ &\leq 2\,\langle Sx, Sx\rangle - 2Re\,\langle T_n x, Sx\rangle = 2\mathrm{Re}\,\langle (S-T_n)x, Sx\rangle \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

and the lemma is proved. □

We now state the category result for contractions. We note that its proof differs from the corresponding proofs in the previous sections.

Theorem 3.3.3. *The set $\mathcal{S}_{\mathcal{C}}$ of all weakly stable contractions is of first category and the set $\mathcal{W}_{\mathcal{C}}$ of all almost weakly stable contractions is residual in $\mathcal{C}$.*

Proof. To prove the first statement we fix $x \in X$, $\|x\| = 1$, and define as before the sets

$$N_n = \left\{ T \in \mathcal{C} : \ |\langle T^k x, x\rangle| \leq \frac{1}{2} \text{ for all } k \geq n \right\}.$$

Let $T \in \mathcal{C}$ be weakly stable. Then there exists $n \in \mathbb{N}$ such that $T \in N_n$, and we obtain

$$\mathcal{S}_{\mathcal{C}} \subset \bigcup_{n=1}^{\infty} N_n. \tag{3.4}$$

It remains to show that the sets N_n are nowhere dense. Fix $n \in \mathbb{N}$ and let U be a periodic unitary operator. We show that U does not belong to the closure of N_n. Assume the opposite, i.e., that there exists a sequence $\{T_k\}_{k\in\mathbb{N}} \subset N_n$ satisfying $T_k \to U$ weakly. Then, by Lemma 3.3.2, $T_k \to U$ strongly and therefore $U \in N_n$ by the definition of N_n. This contradicts the periodicity of U. By the density of the set of unitary periodic operators in $\mathcal{C}$ we obtain that N_n is nowhere dense and therefore $\mathcal{S}_{\mathcal{C}}$ is of first category.

To show the residuality of $\mathcal{W}$ we again take a dense subset $D = \{x_j\}_{j=1}^{\infty}$ of H and define

$$W_{jk} = \left\{ T \in \mathcal{C} : \ |\langle T^n x_j, x_j\rangle| < \frac{1}{k} \text{ for some } n \in \mathbb{N} \right\}.$$

As in the proof of Theorem 3.1.6 the equality

$$\mathcal{W}_{\mathcal{C}} = \bigcap_{j,k=1}^{\infty} W_{jk} \tag{3.5}$$

holds.

Fix $j, k \in \mathbb{N}$. We have to show that the complement W_{jk}^c of W_{jk} is nowhere dense. We note that

$$W_{jk}^c = \left\{ T \in \mathcal{C} : \ |\langle T^n x_j, x_j\rangle| \geq \frac{1}{k} \text{ for all } n \in \mathbb{N} \right\}.$$

Let U be a unitary almost weakly stable operator. Assume that there exists a sequence $\{T_m\}_{m=1}^{\infty} \subset W_{ijk}^c$ satisfying $T_m \to U$ weakly. Then, by Lemma 3.3.2, $T_m \to U$ strongly and therefore $U \in W_{jk}^c$. This contradicts the almost weak stability of U. Therefore the set of all unitary almost weakly stable operators does not intersect the closure of W_{jk}^c. By Proposition 3.3.1 all sets W_{jk}^c are nowhere dense and therefore $\mathcal{W}_{\mathcal{C}}$ is residual. □

Remark 3.3.4. While the statements above carry over almost obviously to unitary and isometric C_0-semigroups, it is currently unknown to us whether the analogous claim for contractive C_0-semigroups holds. In any case, contractive semigroups do not comprise a complete metric space with the natural metric, as the following example shows.

Example 3.3.5. [Eisner and Serény, 2007] Consider $X = l^p$, $1 \leq p \leq \infty$, and for $x = (x_1, x_2, \dots) \in X$, $n \in \mathbb{N}$ let us define the operator A_n by

$$A_n x = (x_{n+1}, x_{n+2}, \dots, x_{2n}, x_1, x_2, \dots, x_n, x_{2n+1}, x_{2n+2}, \dots);$$

that is, the operator A_n swaps the first n and the second n coordinates of x and leaves the other coordinates unchanged. Then $\|A_n\| \leq 1$ implies that A_n generates a C_0-semigroup $(T_n(t))_{t\geq 0}$ satisfying $\|T_n(t)\| \leq e^t$ for every $n \in \mathbb{N}$ and $t \geq 0$. Clearly, $A_n^2 = I$ for every

$n \in \mathbb{N}$ and the operators A_n converge weakly to zero as $n \to \infty$. Therefore

$$\begin{aligned} T_n(t) &= \sum_{k=0}^{\infty} \frac{t^k A_n^k}{k!} = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} A_n + \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} I \\ &= \frac{e^t - e^{-t}}{2} A_n + \frac{e^t + e^{-t}}{2} I \\ &\xrightarrow{\sigma} \frac{e^t + e^{-t}}{2} I, \end{aligned}$$

and the convergence is uniform on compact time intervals. We see that the limit does not satisfy the semigroup law.

By rescaling we obtain a sequence of contractive semigroups $(T_n(t))_{t\geq 0}$ on X converging weakly and uniformly on compact time intervals to a family of operators which is not a semigroup (and, by the way, the bounded generators converge weakly to $-I$, which is itself a generator).

Chapter 4

Applications to the telegraph system: linear methods

4.1 The telegraph system

In this and the next chapter we study a particular system of partial differential equations, the telegraph system. Although our boundary conditions (and hence the resulting abstract system) will be time-dependent, we shall manage to apply pieces of operator semigroup theory. The telegraph equations can be written as

$$\begin{cases} u_t(x,t) + v_x(x,t) + r(u(x,t)) = f_1(x,t), & x \in I,\ t > 0, \\ v_t(x,t) + u_x(x,t) + g(v(x,t)) = f_2(x,t), & x \in I,\ t > 0, \end{cases} \tag{TS}$$

where r, g, f_1, f_2 are given functions, I is an interval on the real line, and we look for the unknown functions u and v; subscripts denote differentiation.

As it is usual for partial differential equations, a large part of the motivation comes from models that describe processes in physics. Originally, the telegraph system models the spread of electricity in a wire with distributed parameters, but phenomena in different fields of physics often exhibit common features and in the mathematical formulation we obtain the same equations; so, specifically, the telegraph system arises in acoustics and mechanics as well; see [Tikhonov and Samarskii, 1963, page 189] for details and [Moroşanu, 1988, page 326] for references to applications in hydraulics.

In the context of electrical phenomena the functions r, g correspond to resistance and leakage, respectively; see [Feynman et al., 1970, Chapter 24], [Vágó, 2003] for more information on the physical background and for the derivation of the equations.

The boundary conditions associated with (TS) depend on the particular process we study. [Brayton, 1967] and [Cooke and Krumme, 1968] consider $I = (0, 1)$ and a pair of long wires connected to each other by linear or nonlinear resistances at the ends; this leads to algebraic boundary conditions of the form

$$\begin{cases} -u(0,t) = \beta_0(v(0,t)), \\ \quad u(1,t) = \beta_1(v(1,t)), \end{cases} \tag{BC1}$$

where the functions β_0, β_1 model the terminating resistances. We often use the general form

$$\begin{pmatrix} -u(0,t) \\ u(1,t) \end{pmatrix} \in \beta \begin{pmatrix} v(0,t) \\ v(1,t) \end{pmatrix}, \qquad \beta \subset \mathbb{R}^2 \times \mathbb{R}^2, \tag{BC2}$$

and we allow the behaviour of the resistances to depend on time:

$$\begin{pmatrix} -u(0,t) \\ u(1,t) \end{pmatrix} \in \beta \left(t, \begin{pmatrix} v(0,t) \\ v(1,t) \end{pmatrix} \right). \tag{BC3}$$

Furthermore, one can add a capacitor to the circuit; then we obtain so-called dynamic boundary conditions

$$\begin{cases} -u(0,t) = \beta_0(v(0,t)), \\ \quad u(1,t) = cv_t(1,t) + \beta_1(v(1,t)), \end{cases} \tag{BC4}$$

which we shall deal with in Section 5.3. We can also consider several cables simultaneously, which are connected so that they comprise a network (to model *inter alia* electrical circuits [Marinov and Neittaanmäki, 1991], power grids [Vágó, 2003] and neural networks [Keener and Sneyd, 1998]). In this case, we have functions $u, v : [0,1] \times \mathbb{R}_+ \to \mathbb{R}^m$, where m is the number of cables and a boundary condition of the form

$$\begin{pmatrix} -u(0,t) \\ u(1,t) \end{pmatrix} \in L \begin{pmatrix} v(0,t) \\ v(1,t) \end{pmatrix}, \tag{BC5}$$

where $L \subset \mathbb{R}^{2m} \times \mathbb{R}^{2m}$ is a relation, encoding Kirchoff laws.

To simplify notation for the following discussion, we write (BC3) as

$$\begin{pmatrix} -u(0,t) \\ u(1,t) \end{pmatrix} = \beta(t) \begin{pmatrix} v(0,t) \\ v(1,t) \end{pmatrix} + \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}, \tag{BC}$$

and we fix the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \tag{IC}$$

Here $\beta(t)$ is a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ for each $t \in [0,\tau]$, hence we have a mapping $\beta : [0,\tau] \times \mathbb{R}^2 \to \mathbb{R}^2$. Further, we denote $\alpha = (\alpha_1, \alpha_2)^T : [0,\tau] \to \mathbb{R}^2$. In Sections 4.2 - 4.3 we impose fairly weak conditions on β (see (βC) on page 36) and we prove that (TS), (BC), (IC) has a unique solution provided that $r, g : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions. In Sections 5.1 - 5.2 we assume monotonicity on r, g and β.

Owing to its practical significance, several variants of the telegraph system have been examined; a detailed discussion of the autonomous case along with additional references can be found in [Moroşanu, 1988, Chapter III, Section 4]. The results here generalize [Hokkanen and Moroşanu, 2002a, Theorem 5.1] to the case of Lipschitz continuous r, g and discontinuous inhomogenities f_1, f_2 on the one hand, and parts of [Moroşanu, 1988, Theorem 4.2] to time-dependent boundary conditions on the other hand. A motive in studying higher regularity is to facilitate the asymptotic analysis of singularly perturbed telegraph systems; for, smoothness of the solutions is essential if one tries to validate a formal asymptotic expansion; see [Barbu and Moroşanu, 2007].

4.2 The linear case

The general solution of the homogenous linear telegraph system

$$\begin{cases} u_t(x,t) + v_x(x,t) = 0, \\ v_t(x,t) + u_x(x,t) = 0, \end{cases} \tag{LTS}$$

for $(x,t) \in [0,1] \times [0,\tau]$ is explicitly given by the d'Alembert formulae

$$u(t,x) = \varphi(x-t) + \psi(x+t), \quad v(t,x) = \varphi(x-t) - \psi(x+t), \tag{4.2.1}$$

where $\varphi : [-\tau, 1] \to \mathbb{R}, \psi : [0, \tau+1] \to \mathbb{R}$ are arbitrary (smooth) functions. The initial condition (IC) is equivalent to

$$\varphi(t) = \frac{1}{2}\left(u_0(t) + v_0(t)\right), \psi(t) = \frac{1}{2}\left(u_0(t) - v_0(t)\right) \quad (0 \le t \le 1) \tag{4.2.2}$$

and a simple calculation shows that (BC) is equivalent to

$$\begin{pmatrix} \varphi(-t) \\ -\psi(1+t) \end{pmatrix} = (\beta(t) + \mathrm{id})^{-1} \left(\begin{pmatrix} -2\psi(t) \\ 2\varphi(1-t) \end{pmatrix} - \alpha(t) \right) + \begin{pmatrix} \psi(t) \\ -\varphi(1-t) \end{pmatrix} \tag{4.2.3}$$

for all $0 \le t \le \tau$. That is, given the data u_0, v_0, β, α in (IC) and (BC), the equation (4.2.2) defines φ and ψ on $[0,1]$, (4.2.3) extends them to $[-\tau, 1]$ and $[0, \tau+1]$, respectively, and finally (4.2.1) produces u and v. If the initial and boundary data are smooth and satisfy the compatibility conditions, then (u,v) is a classical solution to (TS), (BC), (IC); but (4.2.1), (4.2.2), (4.2.3) make sense and yield (u,v) under weaker assumptions on u_0, v_0, β, α; in this context we say that (u,v) is a *generalized solution* of (TS), (BC), (IC).

In any case, to perform the extension in (4.2.3) we require that $(\beta(t)+\mathrm{id})^{-1}$ be uniformly bounded, uniformly Lipschitz continuous and jointly measurable on $0 \le t \le \tau$, that is

$$\begin{cases} \beta(t) + \mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is invertible for all } 0 \le t \le \tau; \\ \text{there are constants } C \text{ and } L \text{ such that} \\ \qquad \|(\beta(t)+\mathrm{id})^{-1} a\|_{\mathbb{R}^2} \le C(1 + \|a\|_{\mathbb{R}^2}) \text{ and} \\ \qquad \|(\beta(t)+\mathrm{id})^{-1} a - (\beta(t)+\mathrm{id})^{-1} b\|_{\mathbb{R}^2} \le L\|a-b\|_{\mathbb{R}^2} \\ \text{for all } a, b \in \mathbb{R}^2, 0 \le t \le \tau; \\ \text{the function } (t,a) \mapsto (\beta(t)+\mathrm{id})^{-1} a \text{ is measurable.} \end{cases} \tag{βC}$$

We note that (βC) are satisfied in a number of classical situations, for instance if each $\beta(t)$ is maximal monotone and $t \mapsto (\beta(t)+\mathrm{id})^{-1}a$ is bounded measurable for all $a \in \mathbb{R}^2$ (cf. [Hokkanen and Moroşanu, 2002a]) or if β is a continuous matrix-valued function and no $\beta(t)$ has the eigenvalue one.

In view of these remarks, we obtain the following lemma.

Lemma 4.2.1. Let $\tau > 0$ be fixed and suppose that for some $1 \le p \le \infty$, $u_0, v_0 \in L^p[0,1]$, $\alpha \in L^p([0,\tau], \mathbb{R}^2)$ and $\beta : [0,\tau] \to \mathbb{R}^2$ satisfies (βC). Then (LTS), (BC), (IC) has a unique generalized solution $(u,v) \in L^\infty([0,\tau], L^p[0,1])^2$. In fact, $(u,v) \in C([0,\tau], L^p[0,1])^2$, unless $p = \infty$. Furthermore, the solution depends Lipschitz continuously on α, that is there is a constant C_1 such that

$$\|(u,v) - (\tilde{u},\tilde{v})\|_{L^\infty([0,\tau],L^p[0,1])^2} \le C_1 \|\alpha - \tilde{\alpha}\|_{L^p([0,\tau],\mathbb{R}^2)},$$

where $(\tilde{u},\tilde{v})$ is the solution of (LTS), (BC), (IC) with $\tilde{\alpha} \in L^p([0,\tau],\mathbb{R}^2)$ in place of α.

D'Alembert's formulae imply that continuous data and compatibility conditions produce continuous solutions. Let us consider the compatibility condition of order zero:

$$\begin{pmatrix} -u_0(0) \\ u_0(1) \end{pmatrix} = \beta(0) \begin{pmatrix} v_0(0) \\ v_0(1) \end{pmatrix} + \begin{pmatrix} \alpha_1(0) \\ \alpha_2(0) \end{pmatrix}. \tag{4.2.4}$$

The last lemma of this section will be used later to obtain continuous solutions of (TS).

Lemma 4.2.2. Let $\tau > 0$ be fixed and suppose that $u_0, v_0 \in C[0,1]$, $\alpha \in C([0,\tau],\mathbb{R}^2)$, the function $(t,a) \mapsto (\beta(t)+\mathrm{id})^{-1}a$ is continuous, and (4.2.4) holds. Then (LTS), (BC), (IC) has a unique generalized solution $(u,v) \in C([0,\tau] \times [0,1])$.

Example 4.2.3. As an example, let us consider a linear telegraph system with space variable x having values in the whole of $\mathbb{R}$. The initial value problem

$$\begin{cases} u_t(x,t) + v_x(x,t) + ru(x,t) = 0, \quad x \in \mathbb{R},\ t > 0, \\ v_t(x,t) + u_x(x,t) + gv(x,t) = 0, \quad x \in \mathbb{R},\ t > 0, \\ \qquad u(x,0) = u_0(x),\ v(x,0) = v_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{4.2.5}$$

where $r, g \in \mathbb{C}$, $u_0, v_0 \in L^2(\mathbb{R})$ can be written in the abstract form

$$\begin{cases} \dfrac{d}{dt} w(t) = Bw(t), \\ \quad w(0) = w_0 \end{cases}$$

on the space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, with $w = (u, v)$, $w_0 = (u_0, v_0)$ and

$$B = -\begin{pmatrix} r & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & g \end{pmatrix}, \quad D(B) = H^1(\mathbb{R}) \times H^1(\mathbb{R}).$$

By taking Fourier transforms we see that the operator B is unitarily equivalent to the multiplication operator

$$M = -\begin{pmatrix} r & iy \\ iy & g \end{pmatrix}$$

on the space $L^2(\mathbb{R}, dy) \times L^2(\mathbb{R}, dy)$ of Fourier transforms, which is the generator of a C_0-semigroup. For a $\lambda \in \mathbb{C}$ we formally calculate

$$R(\lambda, M) = (\lambda - M)^{-1} = \frac{1}{(\lambda + r)(\lambda + g) + y^2} \begin{pmatrix} \lambda + g & -iy \\ -iy & \lambda + r \end{pmatrix}.$$

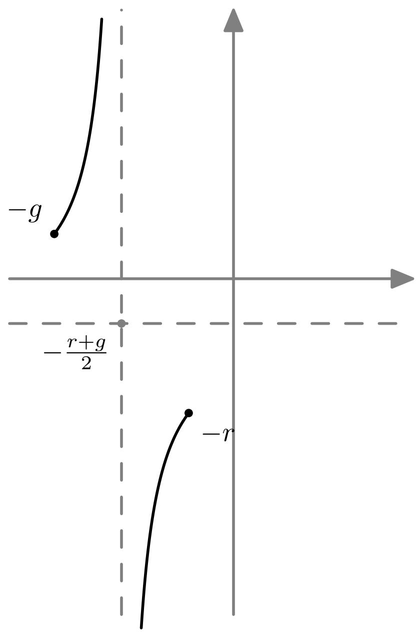

Figure 4.1: The spectrum of M for $\mathrm{Re}\, r, \mathrm{Re}\, g > 0$

Now, whenever $(\lambda+r)(\lambda+g)+y^2 \neq 0$ for all $y \in \mathbb{R}$, then $(\lambda+r)(\lambda+g)+y^2$ is bounded away from zero and $R(\lambda, M)$ is a bounded operator on $L^2(\mathbb{R}, dy) \times L^2(\mathbb{R}, dy)$. From $(\lambda+r)(\lambda+g)+y^2 = 0$ we get

$$\lambda = -\frac{r+g}{2} \pm \sqrt{\left(\frac{r-g}{2}\right)^2 - y^2}$$

and any $\lambda \in \mathbb{C}$ of this form is in $\sigma(M) = \sigma(B)$ (see the figure). We infer that

1. if either $\operatorname{Re} r < 0$ or $\operatorname{Re} g < 0$, then the generated semigroup is not stable;

2. if $\operatorname{Re} r = 0$ and $\operatorname{Re} g = 0$, then $R(\lambda, A)$ is holomorphic for $\operatorname{Re}\lambda > 0$ and since the semigroup is clearly contractive, it is weakly stable by Theorem 2.4.1 (ii) (one could also apply Theorem 2.2.4, because condition (i) of the theorem is satisfied);

3. if either $\operatorname{Re} r = 0, \operatorname{Re} g > 0$ or $\operatorname{Re} r > 0, \operatorname{Re} g = 0$, then (since $-r$ and $-g$ are no eigenvalues) the semigroup is strongly stable by Theorem 1.3.2;

4. if $\operatorname{Re} r > 0, \operatorname{Re} g > 0$, then (since the resolvent $R(\lambda, M)$ is uniformly bounded for $\operatorname{Re} \lambda > 0$) the semigroup is uniformly exponentially stable by Theorem 1.3.1.

Example 4.2.4. Now let us consider the following linear telegraph system with space variable $x \in (0,1)$.

$$\begin{cases} u_t(x,t) + v_x(x,t) + ru(x,t) = 0, & x \in (0,1),\ t > 0, \\ v_t(x,t) + u_x(x,t) + gv(x,t) = 0, & x \in (0,1),\ t > 0, \\ u(0,t) + a_0 v(0,t) = 0, & t > 0, \\ u(1,t) - b_0 v(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in (0,1), \end{cases} \tag{4.2.6}$$

where $r, g \in \mathbb{C}$, $u_0, v_0 \in L^2([0,1])$; since we mainly investigate monotone boundary conditions (see Chapter 5), we assume $a_0, b_0 \geq 0$. The well-posedness of (4.2.6) follows from e.g. the bounded perturbation theorem [Engel and Nagel, 2000, Theorem III.1.10] but will be proved later in broader generality (Corollary 4.3.7). Let us now examine the asymptotic behaviour of its solutions.

As before, the system admits the abstract form

$$\begin{cases} \dfrac{d}{dt} w(t) = Bw(t), \\ w(0) = w_0 \end{cases}$$

on the space $L^2([0,1]) \times L^2([0,1])$, with $w = (u,v)$, $w_0 = (u_0, v_0)$ and

$$B = -\begin{pmatrix} r & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & g \end{pmatrix},$$

$$D(B) = \left\{ (w_1, w_2) \in H^1([0,1]) \times H^1([0,1]) \;\middle|\; \begin{array}{c} w_1(0) + a_0 w_2(0) = 0, \\ w_1(1) - b_0 w_2(1) = 0 \end{array} \right\}.$$

In order to get information on the spectrum of B we have to examine the solutions $w = (w_1, w_2) \in D(B)$ of the equation

$$(\lambda - B)w = z,$$

which is equivalent to the system of ordinary differential equations given by

$$\begin{aligned} \frac{\partial}{\partial x} w_1 &= z_2 - (\lambda + g) w_2, \\ \frac{\partial}{\partial x} w_2 &= z_1 - (\lambda + r) w_1, \end{aligned} \tag{4.2.7}$$

with the conditions that

$$\begin{aligned} w_1(0) + a_0 w_2(0) &= 0, \\ w_1(1) - b_0 w_2(1) &= 0\,. \end{aligned} \tag{4.2.8}$$

To simplify notation, let us put $a = \sqrt{\lambda + g}$, $b = \sqrt{\lambda + r}$. (We shall see that we can choose any of the square roots.) Now we consider two cases, the first being the case $ab = 0$. If $b = 0$ then $\lambda = -r$ and by solving (4.2.7), (4.2.8) we see that the system has a unique solution in $D(B)$ if and only if $r - g \neq a_0 + b_0$. On the other hand, if $r - g = a_0 + b_0$ then (4.2.7), (4.2.8) with $z_1 \equiv 0, z_2 \equiv 0$ admits a nontrivial solution; so we obtain that $-r \in \sigma(B) \Leftrightarrow r - g = a_0 + b_0$, and in this case $-r$ is an eigenvalue. If $a = 0$, then similarly we get that $\lambda = -g$ is in the spectrum if and only if it is an eigenvalue if and only if $a_0 = b_0 = 0$ or $r - g = \frac{a_0 + b_0}{a_0 b_0}$. Now we may assume $ab \neq 0$. In this case the solution of (4.2.7) is given by

$$\begin{aligned} w_1(x) = {} & \frac{1}{2ab} \big(ab(e^{-xab} + e^{xab}) w_1(0) + a^2 (e^{-xab} - e^{xab}) w_2(0) \big) \\ & + \int_0^x \frac{1}{2ab} \big(ab(e^{-(x-y)ab} + e^{(x-y)ab}) z_1(y) + a^2 (e^{-(x-y)ab} - e^{(x-y)ab}) z_2(y) \big)\, dy, \end{aligned} \tag{4.2.9}$$

$$\begin{aligned} w_2(x) = {} & \frac{1}{2ab} \big(b^2 (e^{-xab} - e^{xab}) w_1(0) + ab(e^{-xab} + e^{xab}) w_2(0) \big) \\ & + \int_0^x \frac{1}{2ab} \big(b^2 (e^{-(x-y)ab} - e^{(x-y)ab}) z_1(y) + ab(e^{-(x-y)ab} + e^{(x-y)ab}) z_2(y) \big)\, dy. \end{aligned} \tag{4.2.10}$$

By (4.2.8) we obtain

$$0 = w_1(1) - b_0 w_2(1) = \frac{1}{2ab} \Big(\big(e^{-ab}(a - bb_0)(a - ba_0) - e^{ab}(a + bb_0)(a + ba_0)\big)\, w_2(0) \\ + \big(ae^{-ab} - bb_0 e^{-ab}\big) \int_0^1 (bz_1(y) + az_2(y)) e^{yab}\, dy \\ + \big(ae^{ab} + bb_0 e^{ab}\big) \int_0^1 (bz_1(y) - az_2(y)) e^{-yab}\, dy \Big).$$

Hence we see that

$$\lambda \in \sigma(B) \iff e^{-ab}(a - bb_0)(a - ba_0) - e^{ab}(a + bb_0)(a + ba_0) = 0.$$

Now, the cases $a + bb_0 = 0$ or $ba_0 + a = 0$ give no root here, so the condition can equivalently be written in the form

$$\frac{a - bb_0}{a - ba_0} \frac{a + bb_0}{a + ba_0} = e^{2ab}. \tag{4.2.11}$$

If $a_0 = b_0 = 0$, this equation turns into

$$e^{2\sqrt{\lambda+r}\sqrt{\lambda+g}} = 1,$$

the solutions of which are given by

$$\lambda = \frac{-(r + g) \pm \sqrt{(r - g)^2 - 4k^2\pi^2}}{2} \quad (k \in \mathbb{Z}).$$

The rightmost solution is at $k = 0$, that is, $\lambda = -r$ or $\lambda = -g$, whichever has greater real part. If $a_0 \neq 0$ or $b_0 \neq 0$, then for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \operatorname{Re}(-r)$, $\operatorname{Re}\lambda > \operatorname{Re}(-g)$ we have $|a - bb_0| \leq |a + bb_0|$, $|a - ba_0| \leq |a + ba_0|$ where at least one of the inequalities is strict, but $|e^{2ab}| > 1$, so (4.2.11) has no roots in the region $\{\lambda \mid \operatorname{Re}\lambda > \operatorname{Re}(-r), \operatorname{Re}(-g)\}$.

To simplify further computation, we now assume $r = g$. Then $a = b$ and (4.2.11) attains the form

$$\frac{(1 - b_0)(1 - a_0)}{(1 + b_0)(1 + a_0)} = e^{2a^2}, \tag{4.2.12}$$

which yields the solutions

$$\lambda = \frac{1}{2} \log \left| \frac{(1 - b_0)(1 - a_0)}{(1 + b_0)(1 + a_0)} \right| - g + i \arg \frac{(1 - b_0)(1 - a_0)}{(1 + b_0)(1 + a_0)}.$$

Using (4.2.9), (4.2.10) it is easily seen that the resolvent of B is bounded on any closed right halfplane disjoint from $\sigma(B)$, hence we come to the following conclusions:

1. if $a_0 = b_0 = 0$, then r and g are decisive:
   (a) if $\operatorname{Re} r \le 0$ or $\operatorname{Re} g \le 0$ then the generated semigroup is not (even almost weakly) stable, since we have an eigenvalue on the closed right halfplane;
   (b) if $\operatorname{Re} r > 0$ and $\operatorname{Re} g > 0$ then the generated semigroup is uniformly exponentially stable by Theorem 1.3.1;
2. if $r - g = a_0 + b_0$ and $a_0 b_0 \neq 1$, then r is decisive:
   (a) if $\operatorname{Re} r \le 0$ then the generated semigroup is not stable;
   (b) if $\operatorname{Re} r > 0$ then the generated semigroup is uniformly exponentially stable;
3. if $r - g = \frac{a_0+b_0}{a_0 b_0}$ and $a_0 b_0 \neq 1$,then g is decisive, similarly to 2;
4. if $r - g = a_0 + b_0$ and $a_0 b_0 = 1$ then r and g are decisive as in 1;
5. further, if none of the above conditions hold and $\operatorname{Re} r < 0$, $\operatorname{Re} g < 0$, then we have uniform exponential stability;
6. finally, if none of the conditions in 1,2,3,4 hold and $r = g$, then the number $-\frac{1}{2}\log\left|\frac{(1-b_0)(1-a_0)}{(1+b_0)(1+a_0)}\right| + g$ is decisive in the above sense.

Example 4.2.5. For the sake of completeness, we briefly discuss the linear telegraph system with space variable in $\mathbb{R}_+$, that is,

$$\begin{cases} u_t(x,t) + v_x(x,t) + ru(x,t) = 0, & x > 0,\ t > 0, \\ v_t(x,t) + u_x(x,t) + gv(x,t) = 0, & x > 0,\ t > 0, \\ u(0,t) + a_0 v(0,t) = 0, & t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x > 0 \end{cases} \tag{4.2.13}$$

(where the parameters have the same meaning as in Example 4.2.4), or in abstract form

$$\begin{cases} \dfrac{d}{dt} w(t) = Bw(t), \\ w(0) = w_0 \end{cases}$$

on the space $L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$, with $w = (u, v)$, $w_0 = (u_0, v_0)$ and

$$B = -\begin{pmatrix} r & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & g \end{pmatrix},$$
$$D(B) = \left\{ (w_1, w_2) \in H^1(\mathbb{R}_+) \times H^1(\mathbb{R}_+) \mid w_1(0) + a_0 w_2(0) = 0 \right\}.$$

As in Example 4.2.4 it is easily checked that $-g, -r \in \sigma(B)$ (but not eigenvalues). Otherwise $a \neq 0$, $b \neq 0$ and the solutions w_1, w_2 are given by (4.2.7). If $\operatorname{Re} ab \neq 0$ we may choose the square roots such that $\operatorname{Re} ab > 0$ and in this case we see that for w_1, w_2 to be in $L^2(\mathbb{R}_+)$ it is necessary and sufficient that

$$bw_1(0) - aw_2(0) + b\int_0^\infty e^{-yab} z_1(y)\, dy - a\int_0^\infty e^{-yab} z_2(y)\, dy = 0$$

Combining this with $w_1(0) + a_0 w_2(0) = 0$ we obtain an eigenvalue $\lambda_0 = \frac{a_0^2 r - g}{1 - a_0^2}$ (unless $a_0 = 1$). The case $\operatorname{Re} ab = 0$ is completely analogous to the situation of Example 4.2.3, so if λ_0 is not dominating, the asymptotic behaviour of this equation is similar to that in Example 4.2.3.

4.3 Lipschitz continuous perturbations

In this chapter we apply C_0-semigroup arguments to obtain solutions of the linear telegraph system with Lipschitz continuous perturbations. The results of this section will, in turn, make it possible to derive the well-posedness of the monotonely perturbed telegraph system (see Chapter 5). We present the case $x \in (0,1)$ because this is when our method is interesting and brings new results; but of course the statements of this chapter apply equally well in the case $x \in \mathbb{R}$.

Throughout this section we assume that $r, g : \mathbb{R} \to \mathbb{R}$ in (TS) are Lipschitz continuous: there is a constant C_2 such that

$$|r(a) - r(b)| \le C_2|a - b|, \quad |g(a) - g(b)| \le C_2|a - b| \tag{4.3.1}$$

for all $a, b \in \mathbb{R}$.

We examine (TS) in the space of continuous functions. Continuity allows us to decompose the system and then treat our problem as a Lipschitz continuous perturbation of a linear autonomous equation. With this method, we are able to consider inhomogenities f_1, f_2 in $L^1([0,\tau], L^\infty[0,1])$.

We split the problem (TS), (BC), (IC) into the following two parts:

$$\begin{cases} k_t(x,t) + l_x(x,t) = 0, \\ l_t(x,t) + k_x(x,t) = 0, \\ \begin{pmatrix} -k(0,t) \\ k(1,t) \end{pmatrix} = \beta(t)\begin{pmatrix} l(0,t) \\ l(1,t) \end{pmatrix} + \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} - \begin{pmatrix} -\tilde{u}(0,t) \\ \tilde{u}(1,t) \end{pmatrix}, \\ k(x,0) = u_0(x),\ l(x,0) = v_0(x), \end{cases} \tag{4.3.2}$$

and

$$\begin{cases} \tilde{u}_t(x,t) + \tilde{v}_x(x,t) + r(\tilde{u}(x,t) + k(x,t)) = f_1(x,t), \\ \tilde{v}_t(x,t) + \tilde{u}_x(x,t) + g(\tilde{v}(x,t) + l(x,t)) = f_2(x,t), \\ \tilde{v}(0,t) = \tilde{v}(1,t) = 0, \\ \tilde{u}(x,0) = \tilde{v}(x,0) = 0. \end{cases} \tag{4.3.3}$$

Observe that the solution $(u(x,t), v(x,t))$ of the original problem is given by $u(x,t) = \tilde{u}(x,t) + k(x,t)$, $v(x,t) = \tilde{v}(x,t) + l(x,t)$, thus all we have to do is solve (4.3.2) and (4.3.3) simultaneously. Were $\tilde{u}$ given, (4.3.2) would assume the form (LTS), (BC), (IC), and Lemma 4.2.1 would provide us with a solution; hence we focus on (4.3.3) and use (4.3.2) as a sub-problem.

To begin with, we convert the equations to an autonomous linear Cauchy problem with Lipschitz continuous perturbation on the suitable Banach space $Y_0 = \{y = (y_1, y_2) \in (C[0,1])^2 \mid y_2(0) = y_2(1) = 0\}$, which we endow with the norm $\|y\| = \sup|y_1| + \sup|y_2|$. We define the linear operator $A : D(A) \to Y_0$ by

$$A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -\begin{pmatrix} y_2' \\ y_1' \end{pmatrix}, \tag{4.3.4}$$

where the prime denotes differentiation, on its natural domain

$$D(A) = \{(y_1, y_2) \in Y_0 \mid y_1'(0) = y_1'(1) = 0\}.$$

The operator A shall represent the linear part of (4.3.3), therefore we include a lemma on A being the generator of a C_0-semigroup, compare with [Hokkanen and Moroşanu, 2002a, Lemma 5.1]. Although it is not demanding to construct the semigroup right away, we will find it useful to define it on a larger space first.

Lemma 4.3.1. The operator $\tilde{A} : D(\tilde{A}) \to (L^1[0,1])^2$, $\tilde{A}(y_1, y_2) = -(y_2', y_1')$ with domain $D(\tilde{A}) = \{(y_1, y_2) \in (W^{1,1}[0,1])^2 \mid y_2(0) = y_1(0) = 0\}$ generates a C_0-semigroup $(\tilde{T}(t))_{t\geq 0}$ on $(L^1[0,1])^2$.

The lemma is verified by an adaptation of a standard example in semigroup theory, cf. for instance [Engel and Nagel, 2000, Section II.2.10].

Lemma 4.3.2. The linear subspace $Y_0 \subset (L^1[0,1])^2$ is invariant under the semigroup operators $\tilde{T}(t)$. The restrictions $T(t) = \tilde{T}(t)_{|Y_0}$ form a C_0-semigroup on Y_0 with generator $(A, D(A))$.

Proof. It is easy to see that Y_0 is invariant and the restricted semigroup $(T(t))_{t\geq 0}$ is strongly continuous with respect to the supremum norm of Y_0. Therefore the generator of $(T(t))_{t\geq 0}$ is the part of $\tilde{A}$ in Y_0, which is indeed A; see, for instance [Tanabe, 1997, Section 7.1]. □

We go on to deal with the nonlinear part of (4.3.3). For a function $w \in C([0,\tau], Y_0)$, let $F : C([0,\tau], Y_0) \mapsto L^1\left([0,\tau], (L^\infty[0,1])^2\right)$ be defined by

$$F(w)(t)(x) = \begin{pmatrix} -r(\tilde{u}(x,t) + k(x,t)) \\ -g(\tilde{v}(x,t) + l(x,t)) \end{pmatrix} + \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix}, \tag{4.3.5}$$

where $(\tilde{u}(x,t), \tilde{v}(x,t)) = w(t)(x)$ $(0 \le x \le 1,\ 0 \le t \le \tau)$ and $(k(x,t), l(x,t))$ is the solution of (4.3.2) with $\tilde{u}(x,t)$ given by the first coordinate of $w(t)(x)$. By virtue of Lemma 4.2.1 and by assumption (4.3.1), F is well-defined and there is a constant K such that F fulfils the Lipschitz condition

$$\|F(w) - F(z)\|_{L^\infty([0,t],(L^\infty[0,1])^2)} \le K\|w - z\|_{C([0,t],Y_0)} \tag{4.3.6}$$

for all $w, z \in C([0,t], Y_0)$, $0 \le t \le \tau$.

All this notation renders (4.3.2), (4.3.3) into a semilinear abstract Cauchy problem

$$\begin{cases} w'(t) = Aw(t) + F(w)(t) \\ w(0) = 0. \end{cases} \tag{SACP}$$

We shall now invoke a standard fixed point argument in $C([0,\tau], Y_0)$ to solve (SACP). However, $F(w)(t)$ is not necessarily in Y_0. Consequently, in order to gain a weak solution of (SACP) we have to use the semigroup $(\tilde{T}(t))_{t\ge}$ on the larger space $(L^1[0,1])^2$ rather than $(T(t))_{t\ge 0}$ on Y_0 and we will see that weak solutions do indeed take values in Y_0. The next lemma and the subsequent Theorem 4.3.5 admit a general character, but to make our exposition self-contained, we state and prove their specific version; see nevertheless Remark 4.3.11.

Lemma 4.3.3. Let the semigroup $(\tilde{T}(t))_{t\ge 0}$, and the mapping F be as above and take $w \in C([0,\tau], Y_0)$. Then the function

$$t \mapsto \int_0^t \tilde{T}(t-s)F(w)(s)\,ds$$

is in $C([0,\tau], Y_0)$.

Proof. Let h be any function in $L^1([0,\tau], (L^\infty[0,1])^2)$; firstly, we show that $\int_0^t \tilde{T}(t-s)h(s)\,ds \in Y_0$ for all $0 \le t \le \tau$. Choose a sequence h_n in $C^1([0,\tau], (L^\infty[0,1])^2)$ such that $h_n \to h$ in $L^1([0,\tau], (L^\infty[0,1])^2)$. There is a constant M such that $\|\tilde{T}(t)\|_{\mathcal{B}(L^\infty[0,1]^2)} \le M$ hence we obtain

$$\int_0^t \tilde{T}(t-s)h_n(s)\,ds \overset{n\to\infty}{\longrightarrow} \int_0^t \tilde{T}(t-s)h(s)\,ds \quad \text{in } (L^\infty[0,1])^2. \tag{4.3.7}$$

A standard argument (see for example [Goldstein, 1985, Chapter 2, Section 1.3]) gives $\int_0^t \tilde{T}(t-s)h_n(s)\,ds \in D(\tilde{A})$, therefore $\int_0^t \tilde{T}(t-s)h_n(s)\,ds \in Y_0$ as well, and as a consequence of (4.3.7) we derive $\int_0^t \tilde{T}(t-s)h(s)\,ds \in Y_0$. Furthermore,

$$
\begin{aligned}
&\left\| \int_0^{t+\delta} \tilde{T}(t+\delta-s)h(s)\,ds - \int_0^t \tilde{T}(t-s)h(s)\,ds \right\|_{(L^\infty[0,1])^2} \\
&\qquad = \left\| \int_0^{t+\delta} \tilde{T}(s)h(t+\delta-s)\,ds - \int_0^t \tilde{T}(s)h(t-s)\,ds \right\|_{(L^\infty[0,1])^2} \\
&\quad = \left\| \int_t^{t+\delta} \tilde{T}(s)h(t+\delta-s)\,ds + \int_0^t \tilde{T}(s)\left(h(t+\delta-s)-h(t-s)\right)\,ds \right\|_{(L^\infty[0,1])^2} \\
&\qquad \le M \int_0^\delta \|h(s)\|_{(L^\infty[0,1])^2}\,ds + M\int_0^t \|h(\delta+s)-h(s)\|_{(L^\infty[0,1])^2}\,ds \overset{\delta\to 0+}{\longrightarrow} 0,
\end{aligned}
$$

so the function $t \mapsto \int_0^t \tilde{T}(t-s)h(s)$ is continuous from the right and by a similar computation it is continuous from the left. The choice $h(s) = F(w)(s)$ yields the statement of the lemma. □

Now we follow the usual route to solving semilinear abstract Cauchy problems and define the operator $\mathcal{I} : C([0,\tau], Y_0) \to C([0,\tau], Y_0)$ by

$$\mathcal{I}(w)(t) = \int_0^t \tilde{T}(t-s)F(w)(s)\,ds \quad (w \in C([0,\tau], Y_0),\, 0 \le t \le \tau).$$

Actually, we need Lemma 4.3.3 to guarantee that $\mathcal{I}$ maps $C([0,\tau], Y_0)$ into itself.

Definition 4.3.4. We say that a function $w \in C([0,\tau], Y_0)$ is a *weak solution* of (SACP) if

$$w(t) = \int_0^t \tilde{T}(t-s)F(w)(s)\,ds \quad (0 \le t \le \tau),$$

that is w is a fixed point of $\mathcal{I}$.

The discussion above now easily yields the existence and uniqueness of weak solutions.

Theorem 4.3.5. Suppose that u_0, v_0 are in $L^\infty[0,1]$, f_1, f_2 are in $L^1([0,\tau], L^\infty[0,1])$, $\alpha \in L^\infty([0,\tau], \mathbb{R}^2)$, $\beta : [0,\tau] \to \mathbb{R}^2$ satisfies (βC) and (4.3.1) holds. Let A and F be defined as in (4.3.4) and (4.3.5), respectively. Then (SACP) has a unique weak solution.

Proof. The inequality (4.3.6) implies that $\mathcal{I}^n$, the nth iterate of $\mathcal{I}$, is a contraction for sufficiently large n, thus the Banach fixed point theorem can be applied to $\mathcal{I}^n$ acting on the space $\{w \in C([0,\tau], Y_0) \mid w(0) = 0\}$; see [Goldstein, 1985, Theorem 2.5] for the details. □

As we have noted before, if systems (4.3.2) and (4.3.3) admit smooth solutions (k, l) and $(\tilde{u}, \tilde{v})$ simultaneously, then a smooth solution of the original system (TS), (BC), (IC) is given by $u = k + \tilde{u}, v = l + \tilde{v}$. Generally speaking, we have the following definition.

Definition 4.3.6. We say that (u, v) is a *generalized solution* of (TS), (BC), (IC), if $u = k + \tilde{u}, v = l + \tilde{v}$, where (k, l) is a generalized solution of (4.3.2) and $w = (\tilde{u}, \tilde{v})$ is a weak solution of (SACP).

In order to see that classical solutions are generalized solutions, suppose that $u, v \in C^1([0, 1] \times [0, \tau])$ is an arbitrary solution of (TS), (BC), (IC). Given the system of equations, we produce $(\tilde{u}, \tilde{v})$ and (k, l) as described above. It is readily verified that the pair $(u-k, v-l)$ determines a weak solution of (SACP), and by uniqueness of weak solutions to (SACP) we obtain $u - k = \tilde{u}$, $v - l = \tilde{v}$; hence $u = k + \tilde{u}$, $v = l + \tilde{v}$ so (u, v) is indeed a generalized solution.

Corollary 4.3.7. Let $u_0, v_0 \in C[0, 1]$, $f_1, f_2 \in L^1([0, \tau], L^\infty[0, 1])$,$\alpha \in C([0, \tau], \mathbb{R}^2)$. Suppose $\beta : [0, \tau] \to \mathbb{R}^2$ satisfies (βC), $(t, a) \mapsto (\mathrm{id}+\beta(t))^{-1}a$ is continuous, (4.2.4) and (4.3.1) hold. Then (TS), (BC), (IC) has a unique generalized solution $(u, v) \in C([0, 1] \times [0, \tau])^2$.

Proof. By Theorem 4.3.5, (SACP) has a unique weak solution, hence (4.3.3) has a unique continuous solution $(\tilde{u}, \tilde{v})$. By Lemma 4.2.2, the unique solution (k, l) of (4.3.2) is also continuous, therefore $(\tilde{u} + k, \tilde{v} + l)$ is the unique continuous generalized solution of (TS), (BC), (IC). $\square$

Lemma 4.3.8. Suppose that F maps $C^1([0, \tau], Y_0)$ into $W^{1,1}([0, \tau], (L^\infty[0, 1])^2)$ Lipschitz continuously and $F(w)(0) \in Y_0$ whenever $w \in C^1([0, \tau], Y_0)$, $w(0) = 0$. Then the unique weak solution of (SACP) is in $C^1([0, \tau], Y_0)$.

Proof. By an argument similar to the proof of Lemma 4.3.3 we see that our assumptions imply that $\mathcal{I}$ maps the space $\{w \in C^1([0, \tau], Y_0) \mid w(0) = 0, w'(0) = F(w)(0)\}$ into itself, and, as in Theorem 4.3.5, $\mathcal{I}^n$ is a contraction for sufficiently large n, thus the Banach fixed point theorem provides us with a fixed point of $\mathcal{I}$ in $C^1([0, \tau], Y_0)$. $\square$

To obtain continuously differentiable solutions of our original problem, we also need the first order compatibility condition:

$$\begin{pmatrix} v_0'(0) + r(u_0(0)) - f_1(0, 0) \\ -v_0'(1) - r(u_0(1)) + f_1(1, 0) \end{pmatrix} = \left(\frac{d}{dt}\beta(0)\right) \begin{pmatrix} v_0(0) \\ v_0(1) \end{pmatrix} + \left\langle \beta'(0) \begin{pmatrix} v_0(0) \\ v_0(1) \end{pmatrix}, \begin{pmatrix} -u_0'(0) - g(v_0(0)) + f_2(0, 0) \\ -u_0'(1) - g(v_0(1)) + f_2(1, 0) \end{pmatrix} \right\rangle_{\mathbb{R}^2} + \begin{pmatrix} \alpha_1'(0) \\ \alpha_2'(0) \end{pmatrix}, \quad (4.3.8)$$

where β' denotes the derivative of β with respect to the second variable.

Corollary 4.3.9. Let $u_0, v_0 \in C^1[0, 1]$, $f_1, f_2 \in W^{1,1}([0, \tau], L^\infty[0, 1])$ and α in $C^1([0, \tau], \mathbb{R}^2)$. Suppose $\beta : [0, \tau] \to \mathbb{R}^2$ satisfies (βC), $(t, a) \mapsto (\mathrm{id}+\beta(t))^{-1}a$ is continuously differentiable, r and g are continuously differentiable, (4.2.4), (4.3.8) and (4.3.1) hold. Then the unique generalized solution (u, v) of (TS), (BC), (IC) is in $C^1([0, \tau], C[0, 1])$. Furthermore, if $f_1, f_2 \in W^{1,1}([0, \tau], C[0, 1])$, then $u, v \in C^1([0, 1] \times [0, \tau])$.

Remark 4.3.10. By an analogous procedure, in the case of k times differentiable data and the corresponding compatibility conditions, we have k times differentiable solutions.

Remark 4.3.11. Assume that a linear operator B generates a C_0-semigroup $(S(t))_{t\geq 0}$ on a Banach space X_0 and let F_0 denote the associated Favard space of order zero, see [Engel and Nagel, 2000, Section II.5] for a definition. If $f \in L^1(\mathbb{R}, F_0)$ then the function

$$z(t) = \int_0^t S(t-s)f(s)\,ds$$

is in $C(\mathbb{R}_+, X_0)$ and if $f \in W^{1,1}(\mathbb{R}, F_0)$, $f(0) \in X_0$, then $z \in C^1(\mathbb{R}, X_0)$; a concise proof is provided in [Engel and Nagel, 2000, Section VI.7]. Observe that the Favard space corresponding to our Banach space Y_0 and semigroup $(T(t))_{t\geq 0}$ is $(L^\infty[0,1])^2$.

Remark 4.3.12. The method demonstrated above depends heavily on using the space of continuous functions. Indeed, our line of argument is partly based on the fact that point evaluations are (Lipschitz) continuous functionals on $C[0,1]$, which, to be sure, does not remain valid if we replace $C[0,1]$ by $L^p[0,1]$.

Chapter 5

The telegraph system: monotonicity methods

In this chapter we drop the Lipschitz condition on r and g, and impose monotonicity conditions instead. Building on the conclusions of the previous chapter, we extend the the well-posedness results to telegraph systems with monotone perturbations.

In Section 5.1 we apply nonlinear semigroup theory and take a quick route through Theorem 1.4.5. This, however, does not bring the degree of generality we can achieve by direct computation that we carry out in Section 5.2, using standard estimates. In Section 5.3 we investigate dynamic boundary conditions.

5.1 Monotone perturbations, a first approach

Let us consider the following assumptions.

$$\begin{cases} r, g : \mathbb{R} \to \mathbb{R} \text{ are everywhere defined maximal} \\ \quad \text{monotone functions;} \\ \beta(t) : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is single-valued maximal monotone and} \\ \|\beta(t)a - \beta(s)a\|_{\mathbb{R}^2} \leq C_1 |t-s| \|a\|_{\mathbb{R}^2} \\ \langle \beta(t)a - \beta(t)b, a-b \rangle_{\mathbb{R}^2} \geq \delta \|a-b\|^2_{\mathbb{R}^2} \\ \text{ for some constants } \delta,\ C_1 > 0 \text{ and all } t, s \in [0,\tau],\ a, b \in \mathbb{R}^2. \end{cases} \tag{5.1.1}$$

Applying Theorem 1.4.5 we shall show the existence of a classical solution to our system in the Hilbert space $(L^2[0,1])^2$.

Theorem 5.1.1. Let $\tau >$ be fixed. We suppose that $u_0, v_0 \in H^1[0,1]$, $\alpha \in W^{1,\infty}([0,\tau], \mathbb{R}^2)$, $f_1, f_2 \in W^{1,\infty}([0,\tau], L^2[0,1])$ and (4.2.4), (5.1.1) hold. Then (TS), (BC), (IC) has a unique solution $(u,v) \in W^{1,\infty}([0,\tau], L^2[0,1])^2$.

Proof. For each $t \in [0, \tau]$, let us define the operator

$$A(t)y = \begin{pmatrix} y_2' \\ y_1' \end{pmatrix} - \begin{pmatrix} f_1(\cdot, t) \\ f_2(\cdot, t) \end{pmatrix}$$

on the domain

$$D(A(t)) = \left\{ y = (y_1, y_2) \in \left(H^1[0,1]\right)^2 \;\middle|\; \begin{pmatrix} -y_1(0) \\ y_1(1) \end{pmatrix} = \beta(t) \begin{pmatrix} y_2(0) \\ y_2(1) \end{pmatrix} + \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} \right\}$$

and the operator

$$By = \left\{ z \in \left(L^2[0,1]\right)^2 \;\middle|\; z(\xi) \in \begin{pmatrix} r(y_1(\xi)) \\ g(y_2(\xi)) \end{pmatrix} \text{ for almost all } \xi \in [0,1] \right\}$$

on

$$\begin{aligned} D(B) = \Big\{ y \in \left(L^2[0,1]\right)^2 \;\Big|\; \exists z \in \left(L^2[0,1]\right)^2 \text{ such that} \\ z(\xi) \in \begin{pmatrix} r(y_1(\xi)) \\ g(y_2(\xi)) \end{pmatrix} \text{ for almost all } \xi \in [0,1] \Big\}. \end{aligned}$$

With this notation, we rewrite (TS), (BC), (IC) as a Cauchy problem on $H = (L^2[0,1])^2$:

$$\begin{cases} w'(t) = (A(t) + B)w(t), \\ w(0) = w_0, \end{cases} \tag{5.1.2}$$

where $w(t) = (u(\cdot, t), v(\cdot, t))$, $w_0 = (u_0, v_0)$. Using the fact that $\beta(t)$ is maximal monotone, it is easy to show that $A(t)$ is maximal monotone for every $t \in [0, \tau]$, cf. [Moroşanu, 1988, Lemma III.4.1]. By virtue of [Moroşanu, 1988, Lemma III.4.4] we obtain that the operator $A(t) + B$ with domain $D(A(t))$ is also maximal monotone. Our assumptions $u_0, v_0 \in H^1[0,1]$ and (4.2.4) guarantee that $w_0 \in D(A(0))$, so in order to apply Theorem 1.4.5 it remains to verify condition (1.1) for $A(t) + B$. If we take $x \in D(A(t)), y \in D(A(s))$ then we have

$$\begin{aligned} &- \langle x - y, (A(t) + B)x - (A(s) + B)y \rangle_H \\ &\quad = -\langle x - y, A(t)x - A(s)y \rangle_H - \langle x - y, Bx - By \rangle_H \leq -\langle x - y, A(t)x - A(s)y \rangle_H \\ &\qquad = -\int_0^1 (x_1 - y_1)(x_2' - y_2') + (x_2 - y_2)(x_1' - y_1')\, d\lambda \\ &\quad + \int_0^1 (x_1 - y_1)\left(f_1(\cdot, t) - f_1(\cdot, s)\right) + (x_2 - y_2)\left(f_2(\cdot, t) - f_2(\cdot, s)\right)\, d\lambda. \end{aligned} \tag{5.1.3}$$

For the second integral in (5.1.3) the Cauchy inequality yields

$$\int_0^1 (x_1-y_1)\left(f_1(\cdot,t)-f_1(\cdot,s)\right)+(x_2-y_2)\left(f_2(\cdot,t)-f_2(\cdot,s)\right)\,d\lambda$$
$$\le \|x_1-y_1\|_{L^2[0,1]}\|f_1(\cdot,t)-f_1(\cdot,s)\|_{L^2[0,1]}+\|x_2-y_2\|_{L^2[0,1]}\|f_2(\cdot,t)-f_2(\cdot,s)\|_{L^2[0,1]}$$
$$\le C_2|t-s|\|x-y\|_H \le \frac{1}{2}C_2^2|t-s|^2+\frac{1}{2}\|x-y\|_H^2.$$

For the first integral in (5.1.3) we calculate

$$-\int_0^1 (x_1-y_1)(x_2'-y_2')+(x_2-y_2)(x_1'-y_1')\,d\lambda$$
$$=\left\langle \begin{pmatrix}-x_1(0)\\ x_1(1)\end{pmatrix}-\begin{pmatrix}-y_1(0)\\ y_1(1)\end{pmatrix}, \begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}-\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}\right\rangle_{\mathbb{R}^2}$$
$$=\left\langle \beta(t)\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}+\begin{pmatrix}\alpha_1(t)\\ \alpha_2(t)\end{pmatrix}-\beta(s)\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}-\begin{pmatrix}\alpha_1(s)\\ \alpha_2(s)\end{pmatrix}, \begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}-\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}\right\rangle_{\mathbb{R}^2}=$$

$$=\left\langle \beta(t)\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}-\beta(t)\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}, \begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}-\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}\right\rangle_{\mathbb{R}^2}$$
$$+\left\langle \beta(t)\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}-\beta(s)\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}, \begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}-\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}\right\rangle_{\mathbb{R}^2}$$
$$+\left\langle \begin{pmatrix}\alpha_1(t)\\ \alpha_2(t)\end{pmatrix}-\begin{pmatrix}\alpha_1(s)\\ \alpha_2(s)\end{pmatrix}, \begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}-\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}\right\rangle_{\mathbb{R}^2}\le$$

$$\le -\delta\left\|\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}-\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}\right\|_{\mathbb{R}^2}^2+C_1|t-s|\left\|\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}\right\|_{\mathbb{R}^2}\left\|\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}-\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}\right\|_{\mathbb{R}^2}$$
$$+C_3|t-s|\left\|\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}-\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}\right\|_{\mathbb{R}^2}.$$

On account of

$$C_1|t-s|\left\|\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}\right\|_{\mathbb{R}^2}\left\|\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}-\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}\right\|_{\mathbb{R}^2}$$
$$\le \frac{1}{2\delta}C_1^2|t-s|^2\left\|\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}\right\|_{\mathbb{R}^2}+\frac{\delta}{2}\left\|\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}-\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}\right\|_{\mathbb{R}^2}$$

and

$$C_3|t-s|\left\|\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}-\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}\right\|_{\mathbb{R}^2}\le \frac{1}{2\delta}C_3^2|t-s|^2+\frac{\delta}{2}\left\|\begin{pmatrix}x_2(0)\\ x_2(1)\end{pmatrix}-\begin{pmatrix}y_2(0)\\ y_2(1)\end{pmatrix}\right\|_{\mathbb{R}^2}^2$$

we infer

$$
\begin{aligned}
&-\int_0^1 (x_1 - y_1)(x_2' - y_2') + (x_2 - y_2)(x_1' - y_1')\, d\lambda \\
&\qquad \leq \frac{1}{2\delta}(C_1^2 + C_3^2)|t-s|^2 \left(1 + \left\| \begin{pmatrix} y_2(0) \\ y_2(1) \end{pmatrix} \right\|_{\mathbb{R}^2}^2 \right) \leq C_4 |t-s|^2 (1 + \|y\|_H^2 + \|A(s)y\|_H^2).
\end{aligned}
$$

Bringing together all the inequalities above we deduce

$$
\begin{aligned}
&-\langle x - y, (A(t)+B)x - (A(s)+B)y \rangle_H \\
&\qquad\qquad\qquad \leq \frac{1}{2}\|x-y\|^2 + C_5|t-s|^2(1 + \|y\|_H^2 + \|A(s)y\|_H^2),
\end{aligned}
$$

and this shows that condition (1.1) is satisfied, where we choose $M = \frac{1}{2}$, $g(t) = C_5 t$. $\square$

5.2 Existence and uniqueness of strong solutions

In this section we prove that under monotonicity assumptions the system in (TS), (BC), (IC) has a unique solution. Our methods in this section are different from those used in Section 5.1 and allow us to relax the assumption we made on the inhomogenities f_1, f_2. More precisely, we impose the following conditions.

$$
\begin{cases}
\beta(t) : \mathbb{R}^2 \to \mathbb{R}^2 \\
\|\beta(t)a - \beta(s)a\|_{\mathbb{R}^2} \leq L|t-s|(1 + \|a\|_{\mathbb{R}^2}) \\
\langle \beta(t)a - \beta(t)b, a-b \rangle_{\mathbb{R}^2} \geq \delta \|a-b\|_{\mathbb{R}^2}^2
\end{cases} \tag{5.2.1}
$$

for some constants $\delta,\ L > 0$ and all $t, s \in [0,\tau],\ a, b \in \mathbb{R}^2$.

$$
r, g : \mathbb{R} \to \mathbb{R} \text{ monotone increasing} \tag{5.2.2}
$$

For each $\lambda > 0$ let r_λ and g_λ denote the Yosida-approximants of r and g, respectively (see [Moroşanu, 1988, Theorem 1.3] for the basic properties of the Yosida-approximation). Then, in order to gain information on the system in (TS), (BC), (IC), we examine the approximative system

$$
\begin{cases}
u_{\lambda t}(x,t) + v_{\lambda x}(x,t) + r_\lambda(u_\lambda(x,t)) = f_1(x,t), \\
v_{\lambda t}(x,t) + u_{\lambda x}(x,t) + g_\lambda(v_\lambda(x,t)) = f_2(x,t), \\
\qquad\qquad \begin{pmatrix} -u_\lambda(0,t) \\ u_\lambda(1,t) \end{pmatrix} = \beta(t) \begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix} \\
u_\lambda(x,0) = u_0(x),\ v_\lambda(x,0) = v_0(x)
\end{cases} \tag{TS_λ}
$$

To perform the approximation procedure it is convenient to assume the following regularity conditions, which we will later relax.

$$f_1, f_2 \in W^{1,1}([0,\tau], C[0,1]), \quad u_0, v_0 \in C^1[0,1], \quad r_\lambda, g_\lambda \in W^{1,\infty}(\mathbb{R}, \mathbb{R}) \tag{5.2.3}$$

We also require that the compatibility condition

$$\begin{pmatrix} -u_0(0) \\ u_0(1) \end{pmatrix} = \beta(0) \begin{pmatrix} v_0(0) \\ v_0(1) \end{pmatrix} \tag{5.2.4}$$

should hold. Our first step is to obtain well-posedness for the approximative systems.

Lemma 5.2.1. For each $\lambda > 0$, the system in (TS_λ) has a unique solution

$$u_\lambda, v_\lambda \in W^{1,\infty}([0,\tau], C[0,1])\,.$$

Proof. Since r_λ and g_λ are globally Lipschitz continuous on $\mathbb{R}$, the arguments of Section 4.3, in particular Corollary 4.3.9 apply; with no first-order compatibility condition we obtain solutions in $W^{1,\infty}([0,\tau], C[0,1])$. □

By verifying the standard estimates (cf. [Moroşanu, 1988, Theorem 2.1]) we now show that u_λ and v_λ converge as $\lambda \to 0+$, and that the limit solves the original system in (TS), (BC), (IC). Throughout, we use the symbol C_n, where n is a positive integer, to denote constants not dependent on t or λ. Further, H shall denote the Hilbert space $(L^2[0,1])^2$.

Lemma 5.2.2. There exist u^*, v^* in $W^{1,\infty}([0,\tau], C[0,1])$ and $f^*_{1\lambda}, f^*_{2\lambda}$ in $L^\infty([0,\tau], C[0,1])$ for all $\lambda > 0$, such that

$$\begin{cases} u^*_t(x,t) + v^*_x(x,t) + r_\lambda(u^*(x,t)) = f^*_{1\lambda}(x,t), \\ v^*_t(x,t) + u^*_x(x,t) + g_\lambda(v^*(x,t)) = f^*_{2\lambda}(x,t), \\ \qquad\qquad \begin{pmatrix} -u^*(0,t) \\ u^*(1,t) \end{pmatrix} = \beta(t) \begin{pmatrix} v^*(0,t) \\ v^*(1,t) \end{pmatrix} \end{cases} \tag{5.2.5}$$

and the sets of functions $\{ f^*_{1\lambda}(\cdot,t) \mid \lambda > 0,\ t \in [0,\tau] \}$, $\{ f^*_{2\lambda}(\cdot,t) \mid \lambda > 0,\ t \in [0,\tau] \}$ are bounded in $L^2[0,1]$.

Lemma 5.2.3. For each $\lambda > 0$, let u_λ, v_λ be the solution of (TS_λ). Then the sets $\{ u_\lambda(\cdot,t) \mid \lambda > 0,\ t \in [0,\tau] \}$, $\{ v_\lambda(\cdot,t) \mid \lambda > 0,\ t \in [0,\tau] \}$ are bounded in $L^2[0,1]$.

Proof. We compute

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}(\cdot,t)-\begin{pmatrix}u^*\\ v^*\end{pmatrix}(\cdot,t)\right\|_H^2\\
&\qquad=\left\langle\begin{pmatrix}u_{\lambda t}\\ v_{\lambda t}\end{pmatrix}(\cdot,t)-\begin{pmatrix}u^*_t\\ v^*_t\end{pmatrix}(\cdot,t),\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}-\begin{pmatrix}u^*\\ v^*\end{pmatrix}(\cdot,t)\right\rangle_H\\
&\qquad=-\left\langle\begin{pmatrix}v_{\lambda x}\\ u_{\lambda x}\end{pmatrix}(\cdot,t)-\begin{pmatrix}v^*_x\\ u^*_x\end{pmatrix}(\cdot,t),\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}(\cdot,t)-\begin{pmatrix}u^*\\ v^*\end{pmatrix}(\cdot,t)\right\rangle_H\\
&-\left\langle\begin{pmatrix}r_\lambda(u_\lambda)\\ g_\lambda(v_\lambda)\end{pmatrix}-\begin{pmatrix}r_\lambda(u^*)\\ g_\lambda(v^*)\end{pmatrix},\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}-\begin{pmatrix}u^*\\ v^*\end{pmatrix}\right\rangle_H+\left\langle\begin{pmatrix}f_1\\ f_2\end{pmatrix}-\begin{pmatrix}f^*_{1\lambda}\\ f^*_{2\lambda}\end{pmatrix},\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}-\begin{pmatrix}u^*\\ v^*\end{pmatrix}\right\rangle_H\\
&\qquad\le-\left\langle\begin{pmatrix}-u_\lambda(0,t)\\ u_\lambda(1,t)\end{pmatrix}-\begin{pmatrix}u^*(0,t)\\ u^*(1,t)\end{pmatrix},\begin{pmatrix}v_\lambda(0,t)\\ v_\lambda(1,t)\end{pmatrix}-\begin{pmatrix}v^*(0,t)\\ v^*(1,t)\end{pmatrix}\right\rangle_{\mathbb{R}^2}\\
&\qquad\quad+\left\|\begin{pmatrix}f_1\\ f_2\end{pmatrix}(\cdot,t)-\begin{pmatrix}f^*_{1\lambda}\\ f^*_{2\lambda}\end{pmatrix}\right\|_H\cdot\left\|\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}(\cdot,t)-\begin{pmatrix}u^*\\ v^*\end{pmatrix}(\cdot,t)\right\|_H\\
&\qquad=-\left\langle\beta(t)\begin{pmatrix}v_\lambda(0,t)\\ v_\lambda(1,t)\end{pmatrix}-\beta(t)\begin{pmatrix}v^*(0,t)\\ v^*(1,t)\end{pmatrix},\begin{pmatrix}v_\lambda(0,t)\\ v_\lambda(1,t)\end{pmatrix}-\begin{pmatrix}v^*(0,t)\\ v^*(1,t)\end{pmatrix}\right\rangle_{\mathbb{R}^2}\\
&\qquad\quad+\left\|\begin{pmatrix}f_1\\ f_2\end{pmatrix}(\cdot,t)-\begin{pmatrix}f^*_{1\lambda}\\ f^*_{2\lambda}\end{pmatrix}\right\|_H\cdot\left\|\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}(\cdot,t)-\begin{pmatrix}u^*\\ v^*\end{pmatrix}(\cdot,t)\right\|_H\\
&\le-\delta\left\|\begin{pmatrix}v_\lambda(0,t)\\ v_\lambda(1,t)\end{pmatrix}-\begin{pmatrix}v^*(0,t)\\ v^*(1,t)\end{pmatrix}\right\|_{\mathbb{R}^2}^2+\left\|\begin{pmatrix}f_1\\ f_2\end{pmatrix}(\cdot,t)-\begin{pmatrix}f^*_{1\lambda}\\ f^*_{2\lambda}\end{pmatrix}\right\|_H\cdot\left\|\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}(\cdot,t)-\begin{pmatrix}u^*\\ v^*\end{pmatrix}(\cdot,t)\right\|_H.
\end{aligned}$$

By integration we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}(\cdot,t)-\begin{pmatrix}u^*\\ v^*\end{pmatrix}(\cdot,t)\right\|_H^2+\delta\int_0^t\left\|\begin{pmatrix}v_\lambda(0,s)\\ v_\lambda(1,s)\end{pmatrix}-\begin{pmatrix}v^*(0,s)\\ v^*(1,s)\end{pmatrix}\right\|_{\mathbb{R}^2}^2 ds\\
&\qquad\le\frac{1}{2}\left\|\begin{pmatrix}u_0\\ v_0\end{pmatrix}-\begin{pmatrix}u^*\\ v^*\end{pmatrix}(\cdot,0)\right\|_H^2+\\
&\qquad\int_0^t\left\|\begin{pmatrix}f_1\\ f_2\end{pmatrix}(\cdot,s)-\begin{pmatrix}f^*_{1\lambda}\\ f^*_{2\lambda}\end{pmatrix}(\cdot,s)\right\|_H\cdot\left\|\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}(\cdot,s)-\begin{pmatrix}u^*\\ v^*\end{pmatrix}(\cdot,s)\right\|_H ds. \qquad (5.2.6)
\end{aligned}$$

A variant of Gronwall's lemma (see [Moroşanu, 1988, Chapter I, Lemma 2.1]) yields that

$$\left\|\begin{pmatrix}u_\lambda\\ v_\lambda\end{pmatrix}(\cdot,t)-\begin{pmatrix}u^*\\ v^*\end{pmatrix}(\cdot,t)\right\|_H^2\le C_1+\int_0^t\left\|\begin{pmatrix}f_1\\ f_2\end{pmatrix}(\cdot,s)-\begin{pmatrix}f^*_{1\lambda}\\ f^*_{2\lambda}\end{pmatrix}(\cdot,s)\right\|_H ds$$

and by Lemma 5.2.2

$$\int_0^t\left\|\begin{pmatrix}f_1\\ f_2\end{pmatrix}(\cdot,s)-\begin{pmatrix}f^*_{1\lambda}\\ f^*_{2\lambda}\end{pmatrix}(\cdot,s)\right\|_H ds\le C_2,$$

hence the statement follows. $\square$

Lemma 5.2.4. The sets of boundary functions $\{v_\lambda(0,\cdot) \mid \lambda > 0\}$, $\{v_\lambda(1,\cdot) \mid \lambda > 0\}$ are bounded in $L^2[0,\tau]$.

Proof. The assertion follows directly from the estimate in (5.2.6) and the result of the previous lemma. $\square$

Lemma 5.2.5. For each $\lambda > 0$, let u_λ, v_λ be the solution of (TS_λ). Then the sets $\{u_{\lambda t}(\cdot,t) \mid \lambda > 0,\, t \in [0,\tau]\}$, $\{v_{\lambda t}(\cdot,t) \mid \lambda > 0,\, t \in [0,\tau]\}$ are bounded in $L^2[0,1]$.

Proof. Let us fix $h > 0$. A similar computation as in Lemma 5.2.3 gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix}(\cdot,t+h) - \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix}(\cdot,t)\right\|_H^2 \\
&\qquad \le -\left\langle \begin{pmatrix} -u_\lambda(0,t+h) \\ u_\lambda(1,t+h) \end{pmatrix} - \begin{pmatrix} -u_\lambda(0,t) \\ u_\lambda(1,t) \end{pmatrix}, \begin{pmatrix} v_\lambda(0,t+h) \\ v_\lambda(1,t+h) \end{pmatrix} - \begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix} \right\rangle_{R^2} \\
&\qquad\quad + \left\|\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\cdot,t+h) - \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\cdot,t)\right\|_H \left\|\begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix}(\cdot,t+h) - \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix}(\cdot,t)\right\|_H .
\end{aligned} \tag{5.2.7}$$

Using the boundary conditions in (BC) and the Lipschitz condition in (5.2.1) we have

$$\begin{aligned}
&-\left\langle \begin{pmatrix} -u_\lambda(0,t+h) \\ u_\lambda(1,t+h) \end{pmatrix} - \begin{pmatrix} u_\lambda(0,t) \\ u_\lambda(1,t) \end{pmatrix}, \begin{pmatrix} v_\lambda(0,t+h) \\ v_\lambda(1,t+h) \end{pmatrix} - \begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix} \right\rangle_{\mathbb{R}^2} \\
&\quad = -\left\langle \beta(t+h)\begin{pmatrix} v_\lambda(0,t+h) \\ v_\lambda(1,t+h) \end{pmatrix} - \beta(t)\begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix}, \begin{pmatrix} v_\lambda(0,t+h) \\ v_\lambda(1,t+h) \end{pmatrix} - \begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix} \right\rangle_{\mathbb{R}^2} \\
&\quad \le -\delta \left\|\begin{pmatrix} v_\lambda(0,t+h) \\ v_\lambda(1,t+h) \end{pmatrix} - \begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix}\right\|_{\mathbb{R}^2}^2 \\
&\qquad - \left\langle \beta(t+h)\begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix} - \beta(t)\begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix}, \begin{pmatrix} v_\lambda(0,t+h) \\ v_\lambda(1,t+h) \end{pmatrix} - \begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix} \right\rangle_{\mathbb{R}^2} \\
&\quad \le -\frac{\delta}{2}\left\|\begin{pmatrix} v_\lambda(0,t+h) \\ v_\lambda(1,t+h) \end{pmatrix} - \begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix}\right\|_{\mathbb{R}^2}^2 + \frac{2}{\delta}\left\|\beta(t+h)\begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix} - \beta(t)\begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix}\right\|_{\mathbb{R}^2}^2 \\
&\quad \le \frac{2}{\delta}L^2h^2\left(\left\|\begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t) \end{pmatrix}\right\| + 1\right)^2 .
\end{aligned} \tag{5.2.8}$$

Integrating the inequality in (5.2.7) and using the estimates in (5.2.8) and Lemma 5.2.4

we obtain

$$
\begin{aligned}
\frac{1}{2}\left\|\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,t+h)-\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,t)\right\|_H^2 &\le \frac{1}{2}\left\|\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,h)-\begin{pmatrix}u_0\\v_0\end{pmatrix}(\cdot)\right\|_H^2\\
&\quad+\frac{4}{\delta}L^2h^2\int_0^t\left\|\begin{pmatrix}v_\lambda(0,s)\\v_\lambda(1,s)\end{pmatrix}\right\|^2+1\,ds\\
&\quad+\int_0^t\left\|\begin{pmatrix}f_1\\f_2\end{pmatrix}(\cdot,s+h)-\begin{pmatrix}f_1\\f_2\end{pmatrix}(\cdot,s)\right\|_H\left\|\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,s+h)-\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,s)\right\|_H\\
&\le \frac{1}{2}\left\|\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,h)-\begin{pmatrix}u_0\\v_0\end{pmatrix}(\cdot)\right\|_H^2+\frac{1}{2}C_3h^2\\
&\quad+\int_0^t\left\|\begin{pmatrix}f_1\\f_2\end{pmatrix}(\cdot,s+h)-\begin{pmatrix}f_1\\f_2\end{pmatrix}(\cdot,s)\right\|_H\left\|\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,s+h)-\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,s)\right\|_H.
\end{aligned}
$$

By virtue of [Moroşanu, 1988, Chapter I, Lemma 2.1]

$$
\begin{aligned}
\left\|\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,t+h)-\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,t)\right\|_H &\le \sqrt{\left\|\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,h)-\begin{pmatrix}u_0\\v_0\end{pmatrix}(\cdot)\right\|_H^2+C_3h^2}\\
&\quad+\int_0^t\left\|\begin{pmatrix}f_1\\f_2\end{pmatrix}(\cdot,s+h)-\begin{pmatrix}f_1\\f_2\end{pmatrix}(\cdot,s)\right\|_H.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left\|\begin{pmatrix}u_{\lambda t}\\v_{\lambda t}\end{pmatrix}(\cdot,t)\right\|_H &= \lim_{h\to 0}\frac{1}{h}\left\|\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,t+h)-\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,t)\right\|_H\\
&\le \sqrt{\left\|\begin{pmatrix}u_{\lambda t}\\v_{\lambda t}\end{pmatrix}(\cdot,0)\right\|_H^2+C_3}+\int_0^t\left\|\begin{pmatrix}f_{1t}\\f_{2t}\end{pmatrix}(\cdot,s)\right\|_H ds\le C_4,
\end{aligned}
$$

as stated. □

We now follow the line of arguments from [Moroşanu, 1988, Chapter III, Lemma 4.2] in order to enhance Lemma 5.2.3 and prove the boundedness of our approximative solutions in the supremum norm.

Lemma 5.2.6. For each $\lambda > 0$, let u_λ, v_λ be the solution of (TS_λ). Then the sets $\{u_{\lambda t}(\cdot,t) \mid \lambda > 0,\, t\in[0,\tau]\}$, $\{v_{\lambda t}(\cdot,t) \mid \lambda > 0,\, t\in[0,\tau]\}$ are bounded in $C[0,1]$.

Proof. Let us consider the function $u_\lambda(x,t)-u^*(x,t)+\frac{v_{\lambda x}(x,t)}{|v_{\lambda x}(x,t)|}$, where we take $\frac{v_{\lambda x}(x,t)}{|v_{\lambda x}(x,t)|}=0$, whenever $v_{\lambda x}(x,t)=0$. Then

$$
\begin{aligned}
&\left\langle u_{\lambda t}(\cdot,t)+v_{\lambda x}(\cdot,t)+r_\lambda(u_\lambda(\cdot,t)),u_\lambda(\cdot,t)-u^*(\cdot,t)+\frac{v_{\lambda x}(\cdot,t)}{|v_{\lambda x}(\cdot,t)|}\right\rangle_{L^2[0,1]}\\
&\qquad=\left\langle f_1(\cdot,t),u_\lambda(\cdot,t)-u^*(\cdot,t)+\frac{v_{\lambda x}(\cdot,t)}{|v_{\lambda x}(\cdot,t)|}\right\rangle_{L^2[0,1]}
\end{aligned}
$$

which leads to

$$\int_0^1 |v_{\lambda x}(x,t)|\,dx = -\int_0^1 (v_{\lambda x}(x,t) - v_x^*(x,t))(u_\lambda(x,t) - u^*(x,t))\,dx$$
$$-\int_0^1 v_x^*(x,t)\cdot(u_\lambda(x,t) - u^*(x,t))\,dx$$
$$+\int_0^1 (f_1(x,t) - u_{\lambda t}(x,t))\left(u_\lambda(x,t) - u^*(x,t) - \frac{v_{\lambda x}(x,t)}{|v_{\lambda x}(x,t)|}\right)$$
$$-\int_0^1 r_\lambda(u_\lambda(x,t))\left(u_\lambda(x,t) - u^*(x,t) + \frac{v_{\lambda x}(x,t)}{|v_{\lambda x}(x,t)|}\right) =$$

$$= -\int_0^1 (v_{\lambda x}(x,t) - v_x^*(x,t))(u_\lambda(x,t) - u^*(x,t))\,dx - \int_0^1 v_x^*(x,t)\cdot(u_\lambda(x,t) - u^*(x,t))\,dx$$
$$+\int_0^1 (f_1(x,t) - u_{\lambda t}(x,t))\left(u_\lambda(x,t) - u^*(x,t) - \frac{v_{\lambda x}(x,t)}{|v_{\lambda x}(x,t)|}\right)$$
$$-\int_0^1 \left(r_\lambda(u_\lambda(x,t)) - r_\lambda\left(u^*(x,t) - \frac{v_{\lambda x}(x,t)}{|v_{\lambda x}(x,t)|}\right)\right)\left(u_\lambda(x,t) - u^*(x,t) + \frac{v_{\lambda x}(x,t)}{|v_{\lambda x}(x,t)|}\right) ds$$
$$-\int_0^1 r_\lambda\left(u^*(x,t) - \frac{v_{\lambda x}(x,t)}{|v_{\lambda x}(x,t)|}\right)\left(u_\lambda(x,t) - u^*(x,t) + \frac{v_{\lambda x}(x,t)}{|v_{\lambda x}(x,t)|}\right)$$
$$\leq -\int_0^1 (v_{\lambda x}(x,t) - v_x^*(x,t))(u_\lambda(x,t) - u^*(x,t))\,dx$$
$$+\|v_x^*(\cdot,t)\|_{L^2[0,1]}\|u_\lambda(\cdot,t) - u^*(\cdot,t)\|_{L^2[0,1]}$$
$$+\left(\left\|r_\lambda\left(u^*(\cdot,t) - \frac{v_{\lambda x}(\cdot,t)}{|v_{\lambda x}(\cdot,t)|}\right)\right\|_{L^2[0,1]} + \|f_1(\cdot,t) - u_{\lambda t}(\cdot,t)\|_{L^2[0,1]}\right)$$
$$\cdot\left(\left\|u_\lambda(\cdot,t) - u^*(\cdot,t) - \frac{v_{\lambda x}(\cdot,t)}{|v_{\lambda x}(\cdot,t)|}\right\|_{L^2[0,1]}\right)$$

Here

$$\left\|r_\lambda\left(u^*(\cdot,t) - \frac{v_{\lambda x}(\cdot,t)}{|v_{\lambda x}(\cdot,t)|}\right)\right\|_{L^2[0,1]} \leq \left\|r\left(u^*(\cdot,t) - \frac{v_{\lambda x}(\cdot,t)}{|v_{\lambda x}(\cdot,t)|}\right)\right\|_{L^2[0,1]}$$
$$\leq \left\|r\left(u^*(\cdot,t) - \frac{v_{\lambda x}(\cdot,t)}{|v_{\lambda x}(\cdot,t)|}\right)\right\|_{L^\infty[0,1]} \leq C_5$$

and recall from Lemmas 5.2.3, 5.2.5 that the L^2-norms of $u_\lambda(\cdot,t)$, $v_\lambda(\cdot,t)$ are bounded. So, to sum up,

$$\int_0^1 |v_{\lambda x}(x,t)|\,dx \leq -\int_0^1 (v_{\lambda x}(x,t) - v_x^*(x,t))(u_\lambda(x,t) - u^*(x,t))\,dx + C_6.$$

Since similar inequalities are valid for $\int_0^1 |u_{\lambda x}|$ we infer

$$\begin{aligned}
\int_0^1 |v_{\lambda x}(x,t)|\,dx &+ \int_0^1 |u_{\lambda x}(x,t)|\,dx \\
&\le -\int_0^1 (v_{\lambda x}(x,t) - v_x^*(x,t))(u_\lambda(x,t) - u^*(x,t))\,dx \\
&\quad -\int_0^1 (u_{\lambda x}(x,t) - u_x^*(x,t))(v_\lambda(x,t) - v^*(x,t))\,dx + C_7 \\
= -\left\langle \beta(t)\begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t)\end{pmatrix} \right. &\left. - \beta(t)\begin{pmatrix} v^*(0,t) \\ v^*(1,t)\end{pmatrix}, \begin{pmatrix} v_\lambda(0,t) \\ v_\lambda(1,t)\end{pmatrix} - \begin{pmatrix} v^*(0,t) \\ v^*(1,t)\end{pmatrix} \right\rangle_{\mathbb{R}^2} + C_7 \le C_7\,. \qquad (5.2.9)
\end{aligned}$$

The identity

$$u_\lambda(x,t) = \int_0^1 y u_{\lambda x}(y,t) + u_\lambda(y,t)\,dy - \int_x^1 u_{\lambda x}(y,t)\,dy$$

together with the estimate in (5.2.9) implies

$$|u_\lambda(x,t)| = \int_0^1 |u_{\lambda x}(y,t)| + |u_\lambda(y,t)|\,dy - \int_0^1 |u_{\lambda x}(y,t)|\,dy \le C_8$$

and a similar estimate holds for $v_\lambda(x,t)$, thus the proof is complete. □

Theorem 5.2.7. Let us consider the system in (TS), (BC), (IC) and assume that

$$f_1, f_2 \in W^{1,1}([0,\tau], L^2[0,1]), \quad u_0, v_0 \in H^1[0,1], \quad r_\lambda, g_\lambda \in C^1(\mathbb{R},\mathbb{R})$$

and the conditions in (5.2.1), (5.2.2), (5.2.4) hold. Then (TS), (BC), (IC) has a unique solution $u, v \in W^{1,\infty}([0,\tau], L^2[0,1]) \cap L^\infty([0,\tau], H^1[0,1])$.

Proof. First, suppose that (5.2.3) holds. By the previous lemma, $u_\lambda(x,t)$ is uniformly bounded in $[0,1]\times[0,\tau]$ for $\lambda > 0$, hence

$$\|r_\lambda(u_\lambda(\cdot,t))\|_{L^2[0,1]} \le \|r(u_\lambda(\cdot,t))\|_{L^\infty[0,1]} \le C_9.$$

A simple calculation (as in the time-independent case, see [Moroşanu, 1988, Chapter I, Theorem 2.1]) gives

$$\frac{1}{2}\frac{d}{dt}\left\| \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix}(\cdot,t) - \begin{pmatrix} u_\mu \\ v_\mu \end{pmatrix}(\cdot,t) \right\|_H^2 \le C_{10}(\lambda+\mu),$$

for all $\lambda, \mu > 0$. Consequently, there are $u, v \in W^{1,\infty}([0,\tau], L^2[0,1])$ such that $\lim_{\lambda\to 0} u_\lambda(\cdot,t) = u(\cdot,t)$, $\lim_{\lambda\to 0} v_\lambda(\cdot,t) = v(\cdot,t)$ in $L^2[0,1]$ uniformly for $t \in$

$[0,\tau]$. It is easy to see that u and v satisfy the equations in (TS). Furthermore, $u \in L^\infty([0,1] \times [0,\tau])$ implies $r(u) \in L^\infty([0,1] \times [0,\tau])$, and we have $u_t \in L^\infty([0,\tau], L^2[0,1])$, so the first equation in (TS) leads to $v_x \in L^\infty([0,\tau], L^2[0,1])$, that is, $v \in L^\infty([0,\tau], H^1[0,1])$. Likewise, $u \in L^\infty([0,\tau], H^1[0,1])$. Now, for a fixed $t \in [0,\tau]$, maximal monotonicity of $\beta(t)$ on $\mathbb{R}^2$ implies that the operator given by

$$A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2' \\ y_1' \end{pmatrix}, \quad (y_1, y_2) \in H$$

on the domain

$$D(A) = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \left(H^1[0,1]\right)^2 \;\middle|\; \begin{pmatrix} -y_1(0) \\ y_1(1) \end{pmatrix} = \beta(t) \begin{pmatrix} y_2(0) \\ y_2(1) \end{pmatrix} \right\}$$

is maximal monotone, therefore demiclosed, on H (see [Moroşanu, 1988, Chapter I, Proposition 1.1]). Observe that there is a sequence $\lambda_n \to 0$ such that

$$\lim_{n\to\infty} u_{\lambda_n x}(\cdot,t) = u_x(\cdot,t), \lim_{n\to\infty} v_{\lambda_n x}(\cdot,t) = v_x(\cdot,t)$$

weakly in $L^2[0,1]$, hence it follows from the demiclosedness of A that u and v satisfy (BC).

Any data $f_1, f_2 \in W^{1,1}([0,\tau], L^2[0,1])$, $u_0, v_0 \in H^1[0,1]$ can be approximated by data satisfying (5.2.3) and a standard argument similar to those in Lemma 5.2.3 and Lemma 5.2.5 extends the existence result to the general case. The equations in (IC) are obviously satisfied, and the uniqueness of the solution is a straightforward consequence of the monotonicity assumptions. □

Remark 5.2.8. Using the estimate in (5.2.8), it would be easy to verify that the boundary functions $v(0,\cdot)$, $v(1,\cdot)$ are Lipschitz continuous.

5.3 Dynamic boundary conditions

In this section we study the telegraph system in (TS) endowed with the dynamic boundary conditions

$$\begin{cases} -u(0,t) = \beta_1(t, v(0,t)) \\ \quad u(1,t) = c(t)v_t(1,t) + \beta_2(t, v(1,t)) \end{cases} \tag{DBC}$$

and the initial conditions in (IC). On the data in the boundary conditions we impose the following assumptions.

$$\begin{cases} \beta_1(t,a),\ \beta_2(t,a) : [0,\tau] \times \mathbb{R} \to \mathbb{R} \\ |\beta_1(t,a) - \beta_1(s,a)| + |\beta_2(t,a) - \beta_2(s,a)| \le L|t-s|(1+|a|) \\ (\beta_1(t,a) - \beta_1(t,b))(a-b) \ge \delta|a-b|^2 \\ (\beta_2(t,a) - \beta_2(t,b))(a-b) \ge 0 \\ c : [0,\tau] \to \mathbb{R} \text{ Lipschitz continuous},\ c(t) > 0 \end{cases} \tag{5.3.1}$$

for some constants δ, $L > 0$ and all $t, s \in [0, \tau]$, $a, b \in \mathbb{R}$. We employ analogous reasoning as in the previous section, the main difference being that instead of $H = (L^2[0,1])^2$ we consider a new Hilbert space $K = (L^2[0,1])^2 \times \mathbb{R}$, and instead of the pair

$$\begin{pmatrix} u(\cdot, t) \\ v(\cdot, t) \end{pmatrix}$$

we use the triplet

$$\begin{pmatrix} u(\cdot, t) \\ v(\cdot, t) \\ \sqrt{c(t)} v(1, t) \end{pmatrix}.$$

Suppose that the regularity conditions in (5.2.3) hold. Then the approximative system with dynamic boundary conditions

$$\begin{cases} u_{\lambda t}(x,t) + v_{\lambda x}(x,t) + r_\lambda(u_\lambda(x,t)) = f_1(x,t), \\ v_{\lambda t}(x,t) + u_{\lambda x}(x,t) + g_\lambda(v_\lambda(x,t)) = f_2(x,t), \\ \qquad -u_\lambda(0,t) = \beta_1(t, v_\lambda(0,t)) \\ \qquad u_\lambda(1,t) = c(t) v_{\lambda t}(1,t) + \beta_2(t, v_\lambda(1,t)) \\ u_\lambda(x,0) = u_0(x), \ v_\lambda(x,0) = v_0(x) \end{cases} \tag{DTS_λ}$$

has a unique solution $u_\lambda, v_\lambda \in W^{1,\infty}([0,\tau], C[0,1])$. Our first step is to adapt Lemma 5.2.2 to the new situation.

Lemma 5.3.1. There exist u^*, v^* in $W^{1,\infty}([0,\tau], C[0,1])$ and $f^*_{1\lambda}, f^*_{2\lambda}$ in $L^\infty([0,\tau], C[0,1])$ for all $\lambda > 0$, such that

$$\begin{cases} u^*_t(x,t) + v^*_x(x,t) + r_\lambda(u^*(x,t)) = f^*_{1\lambda}(x,t), \\ v^*_t(x,t) + u^*_x(x,t) + g_\lambda(v^*(x,t)) = f^*_{2\lambda}(x,t), \\ \qquad -u^*(0,t) = \beta_1(t, v^*(0,t)) \\ \qquad u^*(1,t) = c(t) v^*_t(1,t) + \beta_2(t, v^*(1,t)) \end{cases} \tag{5.3.2}$$

and the sets of functions $\{ f^*_{1\lambda}(\cdot, t) \mid \lambda > 0, \, t \in [0,\tau] \}$, $\{ f^*_{2\lambda}(\cdot, t) \mid \lambda > 0, \, t \in [0,\tau] \}$ are bounded in $L^2[0,1]$.

Lemma 5.3.2. For each $\lambda > 0$, let u_λ, v_λ be the solution of (DTS_λ). Then the sets $\{ u_\lambda(\cdot, t) \mid \lambda > 0, \, t \in [0,\tau] \}$, $\{ v_\lambda(\cdot, t) \mid \lambda > 0, \, t \in [0,\tau] \}$ are bounded in $L^2[0,1]$ and the functions $v_\lambda(1, \cdot)$ are uniformly bounded in $[0,\tau]$.

Proof. We have

$$\frac{1}{2}\frac{d}{dt}\left\|\begin{pmatrix} u_\lambda(\cdot,t) \\ v_\lambda(\cdot,t) \\ \sqrt{c(t)}v_\lambda(1,t)\end{pmatrix} - \begin{pmatrix} u^*(\cdot,t) \\ v^*(\cdot,t) \\ \sqrt{c(t)}v^*(1,t)\end{pmatrix}\right\|_K^2$$
$$\le -c(t)\left(v_{\lambda t}(1,t)-v_t^*(1,t)\right)\left(v_\lambda(1,t)-v^*(1,t)\right)+\left(\left(\sqrt{c(t)}\right)'(v_\lambda(1,t)-v^*(1,t))\right.$$
$$\left.+\sqrt{c(t)}(v_{\lambda t}(1,t)-v_t^*(1,t))\right)\sqrt{c(t)}(v_\lambda(1,t)-v^*(1,t))$$
$$+\left\|\begin{pmatrix} f_1 \\ f_2\end{pmatrix}(\cdot,t)-\begin{pmatrix} f_{1\lambda}^* \\ f_{2\lambda}^*\end{pmatrix}(\cdot,t)\right\|_H\left\|\begin{pmatrix} u_\lambda \\ v_\lambda\end{pmatrix}(\cdot,t)-\begin{pmatrix} u^* \\ v^*\end{pmatrix}(\cdot,t)\right\|_H$$
$$\le \frac{1}{2}\frac{c'(t)}{c(t)}c(t)(v_\lambda(1,t)-v^*(1,t))^2+$$
$$\left\|\begin{pmatrix} f_1 \\ f_2\end{pmatrix}(\cdot,t)-\begin{pmatrix} f_{1\lambda}^* \\ f_{2\lambda}^*\end{pmatrix}(\cdot,t)\right\|_H\left\|\begin{pmatrix} u_\lambda \\ v_\lambda\end{pmatrix}(\cdot,t)-\begin{pmatrix} u^* \\ v^*\end{pmatrix}(\cdot,t)\right\|_H$$

Since

$$\left|\frac{c'(t)}{c(t)}\right|\le C_{11},$$

Gronwall's lemma gives the result. □

Lemma 5.3.3. For each $\lambda>0$, let u_λ, v_λ be the solution of (DTS_λ). Then the sets $\{u_{\lambda t}(\cdot,t)\ |\ \lambda>0,\ t\in[0,\tau]\}$, $\{v_{\lambda t}(\cdot,t)\ |\ \lambda>0,\ t\in[0,\tau]\}$ are bounded in $L^2[0,1]$ and the functions $v_{\lambda t}(1,\cdot)$ are uniformly bounded in $[0,\tau]$.

Proof. We fix $h>0$ and proceed as in Lemma 5.2.5.

$$\frac{1}{2}\frac{d}{dt}\left\|\begin{pmatrix} u_\lambda(\cdot,t+h) \\ v_\lambda(\cdot,t+h) \\ \sqrt{c(t)}v_\lambda(1,t+h)\end{pmatrix} - \begin{pmatrix} u_\lambda(\cdot,t) \\ v_\lambda(\cdot,t) \\ \sqrt{c(t)}v_\lambda(1,t)\end{pmatrix}\right\|_K^2\le$$
$$\le -\delta(v_\lambda(0,t+h)-v_\lambda(0,t))^2-(\beta_1(t+h,v_\lambda(0,t))-\beta_1(t,v_\lambda(0,t)))(v_\lambda(0,t+h)-v_\lambda(0,t))$$
$$-c(t)(v_{\lambda t}(1,t+h)-v_{\lambda t}(1,t))(v_\lambda(1,t+h)-v_\lambda(1,t))$$
$$-(c(t+h)-c(t))v_{\lambda t}(1,t+h)(v_\lambda(1,t+h)-v_\lambda(1,t))$$
$$+c(t)(v_{\lambda t}(1,t+h)-v_{\lambda t}(1,t))(v_\lambda(1,t+h)-v_\lambda(1,t))+\frac{1}{2}c'(t)(v_\lambda(1,t+h)-v_\lambda(1,t))^2$$
$$+\left\|\begin{pmatrix} f_1 \\ f_2\end{pmatrix}(\cdot,t+h)-\begin{pmatrix} f_1 \\ f_2\end{pmatrix}(\cdot,t)\right\|_H\left\|\begin{pmatrix} u_\lambda \\ v_\lambda\end{pmatrix}(\cdot,t+h)-\begin{pmatrix} u_\lambda \\ v_\lambda\end{pmatrix}(\cdot,t)\right\|_H$$

$$\leq C_{12}h^2L^2\left((|v_\lambda(0,t)|+1)^2+(|v_\lambda(1,t)|+1)^2\right)+(c(t+h)-c(t))^2(v_{\lambda t}(1,t+h))^2$$
$$+\frac{1}{2}\left(\frac{c'(t)}{c(t)}+\frac{2}{c(t)}\right)c(t)(v_\lambda(1,t+h)-v_\lambda(1,t))^2$$
$$+\left\|\begin{pmatrix}f_1\\f_2\end{pmatrix}(\cdot,s+h)-\begin{pmatrix}f_1\\f_2\end{pmatrix}(\cdot,s)\right\|_H\left\|\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,s+h)-\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix}(\cdot,s)\right\|_H.$$

Let us divide by h^2, let h tend to zero, and integrate. Then we arrive at

$$\left\|\begin{pmatrix}u_{\lambda t}(\cdot,t)\\v_{\lambda t}(\cdot,t)\\\sqrt{c(t)}v_{\lambda t}(1,t)\end{pmatrix}\right\|_K^2\leq\left\|\begin{pmatrix}u_{\lambda t}(\cdot,0)\\v_{\lambda t}(\cdot,0)\\\sqrt{c(0)}v_{\lambda t}(1,0)\end{pmatrix}\right\|_K^2$$
$$+C_{13}L^2\int_0^t(|v_\lambda(0,s)|+1)^2+(v_\lambda(1,s)|+1)^2\,ds+\int_0^t\left(\frac{c'(s)}{c(s)}\right)^2c(s)^2(v_{\lambda t}(1,s))^2$$
$$+c(s)(v_{\lambda t}(1,s))^2\,ds+\int_0^t\left\|\begin{pmatrix}f_{1t}\\f_{2t}\end{pmatrix}(\cdot,s)\right\|_H\left\|\begin{pmatrix}u_{\lambda t}\\v_{\lambda t}\end{pmatrix}(\cdot,s)\right\|_H\,ds$$
$$\leq C_{14}+C_{15}\int_0^t\left\|\begin{pmatrix}u_{\lambda t}\\v_{\lambda t}\end{pmatrix}(\cdot,s)\right\|_H^2+c(t)v_{\lambda t}(1,s)^2\,ds$$

Once again by Gronwall's lemma

$$\left\|\begin{pmatrix}u_{\lambda t}(\cdot,t)\\v_{\lambda t}(\cdot,t)\\\sqrt{c(t)}v_{\lambda t}(1,t)\end{pmatrix}\right\|_K^2\leq C_{16},$$

as stated. □

Theorem 5.3.4. Let us consider the system in (TS), (DBC), (IC) and assume that

$$f_1,f_2\in W^{1,1}([0,\tau],L^2[0,1]),\quad u_0,v_0\in H^1[0,1],$$
$$r_\lambda,g_\lambda\in C^1(\mathbb{R},\mathbb{R}),\quad -u_0(0)=\beta_1(0,v_0(0))$$

and the conditions in (5.3.1), (5.2.2) hold. Then (TS), (DBC), (IC) has a unique solution $u,v\in W^{1,\infty}([0,\tau],L^2[0,1])\cap L^\infty([0,\tau],H^1[0,1])$.

Proof. In view of Lemmas 5.3.2, 5.3.3, the reasoning in the proof of Theorem 5.2.7 applies. □

Example 5.3.5. Let us consider a network represented by a finite graph $G = (\mathtt{V}, \mathtt{E})$, where $\mathtt{V} = \{\mathtt{v}_1, \mathtt{v}_2, \ldots, \mathtt{v}_n\}$ is the set of vertices and $\mathtt{E} = \{\mathtt{e}_1, \mathtt{e}_2, \ldots, \mathtt{e}_m\}$ is the set of edges. We assume that G is connected and all vertices have degree at least two. To each edge we assign a copy of the interval $[0, 1]$ (in other words, we parametrize the edges) and we identify endpoints corresponding to the same vertex in the obvious way. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$ we write

$$\varphi_{ij}^{+} = \begin{cases} 1 & \text{if } \mathtt{e}_j(0) = \mathtt{v}_i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \varphi_{ij}^{-} = \begin{cases} 1 & \text{if } \mathtt{e}_j(1) = \mathtt{v}_i \\ 0 & \text{otherwise} \end{cases}$$

so $\Phi = (\varphi_{ij}) = (\varphi_{ij}^{+}) - (\varphi_{ij}^{-})$ gives the incidence matrix of G.

The following system of equations describes a telegraph-like process on the network, where continuity and Kirchoff-type conditions are imposed at the vertices.

$$\begin{cases} u_{jt}(x,t) + v_{jx}(x,t) + r_j(u_j(x,t)) = 0, \quad 0 < x < 1,\ t > 0,\ j = 1, \ldots, m, \\ v_{jt}(x,t) + u_{jx}(x,t) + g_j(v_j(x,t)) = 0, \quad 0 < x < 1,\ t > 0, j = 1, \ldots, m, \\ \qquad v_j(\mathtt{v}_i, t) = v_l(\mathtt{v}_i, t), \quad t \geq 0,\ j, l \in \Gamma(\mathtt{v}_i), i = 1, \ldots, n \\ \qquad \displaystyle\sum_{j=1}^{m} \varphi_{ij} u_j(\mathtt{v}_i, t) = -c(t) \left(\sum_{j=1}^{m} (\varphi_{ij}^{+} + \varphi_{ij}^{-}) v_{jt}(\mathtt{v}, t) \right),\ t \geq 0,\ i = 1, \ldots, n \\ \qquad u_j(0,x) = u_{j0}(x), v_j(0,x) = v_{j0}(x), \quad 0 < x < 1, j = 1, \ldots, m, \end{cases}$$

where $r_j, g_j : \mathbb{R} \to \mathbb{R}$ are given monotone increasing functions, $c : \mathbb{R}_+ \to \mathbb{R}$ is a given positive function, $\Gamma(\mathtt{v}_i)$ denotes the set of all indices of edges having an endpoint at $\mathtt{v}_i$, and $u_j, v_j : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ are the unknown functions (representing current and voltage along the edges). Let us write $u = (u_j)_{j=1}^{n}, v = (v_j)_{j=1}^{n}$. It is easily seen that the boundary conditions above can be written as

$$\begin{pmatrix} -u(0,t) \\ u(1,t) \end{pmatrix} \in c(t) \cdot \begin{pmatrix} v_t(0,t) \\ v_t(1,t) \end{pmatrix} + L \begin{pmatrix} v(0,t) \\ v(1,t) \end{pmatrix}, \tag{5.3.3}$$

where $L \subset \mathbb{R}^{2m} \times \mathbb{R}^{2m}$ is a monotone operator (representing Kirchoff laws), thus the arguments of this section can be applied.

Bibliography

[Arendt et al., 2001] Arendt, W., Batty, C. J. K., Hieber, M., and Neubrander, F. (2001). *Vector-valued Laplace Transforms and Cauchy Problems*, volume 96 of *Monographs in Mathematics.* Birkhäuser, Basel.

[Barbu and Moroşanu, 2007] Barbu, L. and Moroşanu, G. (2007). *Singularly Perturbed Boundary-Value Problems*, volume 156 of *International Series of Numerical Mathematics.* Birkhäuser, Basel.

[Batty, 2003] Batty, C. J. K. (2003). Bounded Laplace transforms, primitives and semigroup orbits. *Arch. Math. (Basel)*, 81:72–81.

[Batty et al., 2000] Batty, C. J. K., Chill, R., and van Neerven, J. (2000). Asymptotic behaviour of C_0-semigroups with bounded local resolvents. *Math. Nachr.*, 219:65–83.

[Brayton, 1967] Brayton, R. (1967). Nonlinear oscillations in a distributed network. *Quarterly of Applied Mathematics*, 24(4):289–301.

[Chill, 1998] Chill, R. (1998). Tauberian theorems for vector-valued Fourier and Laplace transforms. *Stud. Math.*, 128(1):55–69.

[Chill and Tomilov, 2003] Chill, R. and Tomilov, Y. (2003). Stability of C_0-semigroups and geometry of Banach spaces. *Math. Proc. Cambridge Phil. Soc.*, 135:493–511.

[Chill and Tomilov, 2007] Chill, R. and Tomilov, Y. (2007). Stability of operator semigroups: ideas and results. *Banach Center Publications*, 75:71–109.

[Cooke and Krumme, 1968] Cooke, K. and Krumme, D. (1968). Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations. *J. Math. Anal. App.*, 24:372–387.

[Cornfeld et al., 1982] Cornfeld, I. P., Fomin, S. V., and Sinai, Y. G. (1982). *Ergodic Theory*, volume 245 of *Grundlehren der mathematischen Wissenschaften.* Springer.

[Dunford and Schwartz, 1958] Dunford, N. and Schwartz, J. T. (1958). *Linear Operators*, volume I. Interscience Publishers, Inc., New York.

[Eisner and Farkas, 2007] Eisner, T. and Farkas, B. (2007). Weak stability for orbits of C_0-semigroups on Banach spaces. In Amann, H., Arendt, W., Hieber, M., Neubrander, F., Nicaise, S., and von Below, J., editors, *Functional Analysis and Evolution Equations. The Günter Lumer Volume*, pages 201–208. Birkhäuser.

[Eisner et al., 2007] Eisner, T., Farkas, B., Nagel, R., and Serény, A. (2007). Weakly and almost weakly stable C_0-semigroups. *Int. J. Dynamical Systems and Differential Equations*, 1(1):44–57.

[Eisner and Serény, 2007] Eisner, T. and Serény, A. (2007). On a weak analogoue of the Trotter–Kato theorem. Submitted to Archiv der Mathematik.

[Eisner and Serény, 2008] Eisner, T. and Serény, A. (2008). Category theorems for stable operators on Hilbert spaces. *Acta Sci. Math. (Szeged)*, 74:259–270.

[Emelyanov, 2005] Emelyanov, E. Y. (2005). Some open questions on positive operators in Banach lattices. *Vladikavkaz. Mat. Zh.*, 7(4):17–21.

[Engel and Nagel, 2000] Engel, K.-J. and Nagel, R. (2000). *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[Feynman et al., 1970] Feynman, R., Leighton, R., and Sands, M. (1970). *The Feynman lectures on Physics*, volume II. Addison-Wesley.

[Foguel, 1963] Foguel, S. R. (1963). Powers of a contraction in Hilbert space. *Pacific J. Math.*, 13:551–562.

[Goldstein, 1985] Goldstein, J. (1985). *Semigroups of Linear Operators and Applications.* Oxford Mathematical Monographs. Oxford University Press, Oxford.

[Groh and Neubrander, 1981] Groh, U. and Neubrander, F. (1981). Stabilität stark-stetiger, positiver Operatorhalbgruppen auf C^*-Algebren. *Math. Ann.*, 256:509–516.

[Halmos, 1944] Halmos, P. R. (1944). In general a measure preserving transformation is mixing. *Ann. of Math.*, 45:786–792.

[Halmos, 1956] Halmos, P. R. (1956). *Lectures on Ergodic Theory.* Chelsea Publishing Co., New York.

[Halmos, 1967] Halmos, P. R. (1967). *A Hilbert Space Problem Book.* D. Van Nostrand Co., Inc., Princeton.

[Hardy, 1949] Hardy, G. H. (1949). *Divergent Series.* Clarendon Press, Oxford.

[Hiai, 1978] Hiai, F. (1978). Weakly mixing properties of semigroups of linear operators. *Kodai Math. J.*, 1:376–393.

[Hokkanen and Moroşanu, 2002a] Hokkanen, V.-M. and Moroşanu, G. (2002a). Existence and regularity for a class of nonlinear hyperbolic boundary value problems. *J. Math. Anal. Appl.*, 266:432–450.

[Hokkanen and Moroşanu, 2002b] Hokkanen, V.-M. and Moroşanu, G. (2002b). *Functional methods in differential equations*, volume 432 of *Research notes in mathematics*. Chapman & Hall/CRC.

[Huang, 1999] Huang, S.-Z. (1999). A local version of Gearhart's theorem. *Semigroup Forum*, 58:323–335.

[Huang and van Neerven, 1999] Huang, S.-Z. and van Neerven, J. (1999). B-convexity, the analyitc Radon-Nikodym property, and individual stability of C_0-semigroups. *J. Math. Anal. Appl.*, 231:1–20.

[Ito and Kappel, 2002] Ito, K. and Kappel, F. (2002). *Evolution Equations and Approximations*, volume 61 of *Advances in Mathematics for Applied Sciences*. World Scientific Publishing Co.

[Jones and Lin, 1976] Jones, L. K. and Lin, M. (1976). Ergodic theorems of weak mixing type. *Proc. Amer. Math. Soc.*, 57:50–52.

[Jones and Lin, 1980] Jones, L. K. and Lin, M. (1980). Unimodular eigenvalues and weak mixing. *J. Funct. Anal.*, 35:42–48.

[Kaashoek and Lunel, 1994] Kaashoek, M. A. and Lunel, S. M. V. (1994). An integrability condition on the resolvent for hyperbolicity of the semigroup. *J. Diff. Eq.*, 112:374–406.

[Katok and Hasselblatt, 1995] Katok, A. and Hasselblatt, B. (1995). *Introduction to the Modern Theory of Dynamical Systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge.

[Keener and Sneyd, 1998] Keener, J. and Sneyd, J. (1998). *Mathematical Physiology*, volume 8 of *Interdisciplinary Applied Mathematics*. Springer.

[Komura, 1967] Komura, Y. (1967). Nonlinear semigroups in Hilbert spaces. *J. Math. Soc. Japan*, 19:508–520.

[Krengel, 1985] Krengel, U. (1985). *Ergodic Theorems*. de Gruyter Studies in Mathematics. de Gruyter, Berlin.

[Lax, 2002] Lax, P. (2002). *Functional Analysis*. Wiley.

[Lind, 1975] Lind, D. A. (1975)). A counterexample to a conjecture of Hopf. *Duke Math J.*, 42:755–757.

[Luo et al., 1999] Luo, Z.-H., Guo, B.-Z., and Morgul, O. (1999). *Stability and stabilization of infinite dimensional systems with applications*. Springer, London.

[Lyons, 1985] Lyons, R. (1985). Fourier-Stieltjes coefficients and asymptotic distribution modulo 1. *Annals Math.*, 122:155–170.

[Lyons, 1995] Lyons, R. (1995). Seventy years of Rajchman measures. *J. Fourier Anal. Appl. Kahane Special Issue*, pages 363–377.

[Marinov and Neittaanmäki, 1991] Marinov, C. and Neittaanmäki, P. (1991). *Mathematical Models in Electrical Circuits: Theory and Applications*, volume 66 of *Mathematics and its Applications*. Kluwer, Dordrecht.

[Moroşanu, 1988] Moroşanu, G. (1988). *Nonlinear Evolution Equations and Applications*. Editura Academiei and Reidel, Bucharest and Dordrecht.

[Moroşanu and Serény, 2006] Moroşanu, G. and Serény, A. (2006). A telegraph system with time-dependent boundary conditons. *Math. Sci. Res. J.*, 10(7):177–187.

[Nagel, 1974] Nagel, R. (1974). Ergodic and mixing properties of linear operators. *Proc. Roy. Irish Acad. Sect. A.*, 74:245–261.

[Nagel, 1986] Nagel, R., editor (1986). *One-parameter Semigroups of Positive Operators*, volume 1184 of *Lecture Notes in Mathematics*. Springer, Berlin.

[Peller, 1981] Peller, V. V. (1981). Estimates of operator polynomials in the space L^p with respect to the multiplicative norm. *J. Math. Sci.*, 16:1139–1149.

[Petersen, 1983] Petersen, K. (1983). *Ergodic Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press.

[Rohlin, 1948] Rohlin, V. A. (1948). A "general" measure-preserving transformation is not mixing. *Doklady Akad. Nauk SSSR*, 60:349–351.

[Rosenblum and Rovnyak, 1994] Rosenblum, M. and Rovnyak, J. (1994). *Topics in Hardy classes and Univalent Functions*. Birkhäuser, Basel.

[Rudin, 1973] Rudin, W. (1973). *Functional Analysis*. McGraw-Hill.

[Ruess and Summers, 1992] Ruess, W. M. and Summers, W. H. (1992). Weak asymptotic almost periodicity for semigroups of operators. *J. Math. Anal. Appl.*, 164(1):242–262.

[Serény, 2007] Serény, A. (2007). Well-posedness for a semilinear hyperbolic system with time-dependent boundary conditions. Preprint.

[Sz.-Nagy and Foiaş, 1970] Sz.-Nagy, B. and Foiaş, C. (1970). *Harmonic Analysis of Operators on Hilbert Space.* North-Holland Publ. Comp, Akadémiai Kiadó, Amsterdam, Budapest.

[Takesaki, 1979] Takesaki, M. (1979). *Theory of Operator Algebras*, volume I. Springer Verlag.

[Tanabe, 1997] Tanabe, H. (1997). *Functional Analytic Methods for Partial Differential Equations*, volume 204 of *Monographs and textbooks in pure and applied mathematics.* Marcel Dekker, New York.

[Tătaru, 1991] Tătaru, D. (1991). Time dependent m-accretive operators generating differential evolutions. *Diff. Int. Eq.*, 4(1):137–150.

[Tikhonov and Samarskii, 1963] Tikhonov, A. and Samarskii, A. (1963). *Equations of Mathematical Physics.* Dover Publications, Inc., New York.

[Tomilov, 2001] Tomilov, Y. (2001). A resolvent approach to stability of operator semigroups. *J. Operator Th.*, 46:63–98.

[Vágó, 2003] Vágó, I. (2003). *Theory of Transmission Line Systems.* Akadémiai Kiadó, Budapest.

[van Neerven, 1996] van Neerven, J. (1996). *The Asymptotic Behaviour of Semigroups of Linear Operators*, volume 88 of *Operator Theory Advances and Applications.* Birkhäuser, Basel.

[van Neerven, 2002] van Neerven, J. (2002). On individual stability of C_0-semigroups. *Proc. Amer. Math. Soc.*, 130(8):2325–2333.

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