ITERATIVE PROCESSES FOR SOLVING NONLINEAR OPERATOR EQUATIONS

By

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To my parents Seth and Omphile Boikanyo,

and my siblings

Thapelo, Motsei, Job, Gorata, Ofentse,

Mmoloki and Akanyang.

Ka lorato...

Oganeditse

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Abstract

As a method of approximating zeros of a given maximal monotone operator A in a real Hilbert space, the proximal point algorithm (PPA) which was initiated by B. Martinet (1979) was considered in a more general setting by R. T. Rockafellar (1976), who proved that it converges weakly to a solution of $0 \in Ax$ when the sequence of errors is summable in norm. After O. Güler (1991) showed that the PPA fails in general to converge strongly, modifications of the PPA, among them the inexact 'Halpern-type' iterative process which was introduced by H. K. Xu (2002) and the regularization method of N. Lehdili and A. Moudafi (1996), were obtained in order to enforce strong convergence, still under the summability condition on errors. Definitely this condition is too strong from a computational point of view. We obtain in this thesis other strong convergence results associated with these methods as well as their generalizations under the general condition that errors converge to zero in norm. These results are proved under new sets of conditions on the control parameters involved, which are either weaker than the ones previously used by other authors or are distinct alternative sets of conditions. Other strongly convergent sequences of proximal iterates, such as the method of alternating resolvents and the viscosity approximation method are also constructed. Some illustrations on how these methods can be used to approximate minimum values and/or minimizers of certain convex functionals are given. Apart from addressing the two important problems in the theory of proximal point algorithms – that of strong convergence instead of weak convergence and the one concerning acceptable errors – the results presented in this thesis improve, generalize and refine many existing results in the literature.

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Introduction

Iterative processes have been widely used as a successful tool in finding/approximating solutions of nonlinear operator equations $-$ find an $x \in D(A)$ such that $0 \in Ax$. The problem of finding zeros of nonlinear operators is strongly connected to the theory of fixed points. In many cases, solutions of such operator equations happens to be fixed points of certain operators. Indeed, for a map T whose set of fixed points is nonvoid, the operator equation $x = Tx$ is clearly equivalent to $0 = (I - T)x$. In fact, the map $A := I - T$ is maximal monotone whenever T is nonexpansive, in which case, zeroes of A coincide with fixed points of the resolvent operator of A. Since the problem of finding (and/or approximating) solutions of operator equations can be reformulated as that of finding fixed points of certain operators, one may turn to the theory of fixed points to search for tools that can serve as solution techniques for solving such operator equations.

A fixed point theorem will serve as a solution technique if apart from giving us some information on the existence and possibly the uniqueness of the fixed point, it also provides a method (usually iterative) for finding such fixed point(s). Moreover, it will be appreciated if it has the ability to provide us with extra information regarding the rate of convergence of the iterative process used to approximate the fixed point. To cite as an example, the classical fixed point theorem of Banach has all the aforementioned properties. Indeed, given any strict contraction T taking values from a complete metric space X into itself (with Lipschitz constant $a \in (0, 1)$), one generates a sequence (x_n) defined by $x_{n+1} = T^n x_0$, the so called Picard iteration, which converges strongly to the unique fixed point of T for all $x_0 \in X$. Even more, we know from the same theorem that (x_n) converges to the fixed point of T at least as fast as the terms of the geometric series (whose ratio is a) does. In other words, the convergence is linear.

Unlike strict contractions, nonexpansive mappings behave differently – they may not have fixed points at all, nor the fixed point may not be unique if it does exist. Also, the sequence generated by the Picard iterative process may not converge if strict contractions are not involved. On the other hand, among the many fixed point theorems that exist in the literature, there are only a few of them which resembles the characteristics possessed by the Banach fixed point theorem. In fact, almost all fixed point theorems particularly for nonexpasive mappings are existence theorems. Unfortunately, this fact limits one in constructing fixed points of nonlinear operators such as nonexpansive mappings by means of iterative processes provided by the fixed point theorems. Therefore, other means of constructing iterative methods, such as modifying the Picard iterative process to generate sequences that converge to the fixed point of a given nonexpansive map have to be sought.

For a nonexpansive mapping T, it was R. W. Mann (1953) who first constructed an algorithm, now commonly known as the Mann's iteration, which converges weakly to the fixed point of T. Although the scheme of Mann fails in general to converge strongly, its discovery was a major milestone in finding fixed points of nonexpansive mappings iteratively, since it overcame the difficulty (of failure to converge, even in the weak topology of the underlying space) created by the Picard iterative process. Besides Mann's iterative process, many mathematicians have studied other iterative processes such as the proximal point algorithm (PPA), the regularization method and Halpern's iterative process for solving nonlinear operator equations. They investigated the convergence of such iterative processes and in some cases gave the rate of convergence of such methods. Among them, the work of B. Halpern (1967), F. E. Browder (1967), B. Martinet (1970), S. Ishikawa (1974), R. T. Rockafellar (1976), O. Güler (1991), N. Lehdili and A. Moudafi (1996), M. V. Solodov and B. F. Svaiter (2000), and H. K. Xu (2002), is worth mentioning.

Other methods for solving nonlinear operator equations have been shown to be strongly connected with the above mentioned methods. For instance, in 1992, J. Eckstein and D. P. Bertsekas showed by means of an operator called a "splitting operator" that the Doughlus-Rachford splitting method for finding a zero of the sum of two operators is a special case of the PPA. They observed that applications of the Doughlus-Rachford splitting method, such as the alternating direction method of multipliers for convex programming decomposition, are also special cases of the PPA, an observation which allows the unification and generalization of a variety of convex programming algorithms. The so called support point algorithm is also a special case of the proximal point algorithm, as was shown by H. H. Bauscheke et al. (2005) [3]. Worth pointing out is the fact that not only does the PPA contain several algorithms as special cases, but it is also a powerful and versatile solution technique for solving variational inequalities and many problems in convex optimization such as convex minimizations and convex-concave mini-max (saddle-point) problems.

The observations above form enough basis for us to carry out in this thesis further investigations on different types of proximal point methods. In particular, several sets of control conditions (which are weaker than those previously studied by other authors) are introduced, and strong convergence results associated with them are proved. Also being introduced are different kinds of iterative processes that generate sequences which always converge strongly. Our interest in strong convergence, rather than weak convergence is partly motivated by the results of O. Güler (1991), which revealed that compared to weak convergence, strong convergence has a positive bearing on the rate of convergence of the PPA. All the methods considered in this thesis are inexact, and since the errors indicate how far away one is from the exact solution, we shall investigate strong convergence of such methods under the general condition that the sequence of errors tends to zero in norm. In this way, the two main problems (that of strong convergence instead of weak convergence and that of acceptable errors) related to proximal point methods are effectively addressed. The results of this thesis, which are contained in Chapter 3 - 6 improve, generalize and refine many existing results in the literature.

Chapter 1

Some Iterative Processes: Historical Comments

In the sequel, H will be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced Hilbertian norm $\| \cdot \|$. Recall that an operator $A : D(A) \subset H \to 2^H$ is said to be monotone if it satisfies the monotonicity property

$$\langle x - x', y - y' \rangle \geq 0, \quad \forall\, (x, y), (x', y') \in G(A).$$

That is, its graph $G(A) = \{(x, y) \in H \times H : x \in D(A), y \in Ax\}$ is a monotone subset of $H \times H$. More often, we shall write $(x, y) \in A$ to mean $(x, y) \in G(A)$. If an operator is maximal with respect to this monotonicity property, then it is said to be maximal monotone. In other words, an operator A is called maximal monotone if in addition to being monotone, its graph is not properly contained in the graph of any other monotone operator. In nonlinear analysis and convex optimization, an important and perhaps interesting topic is to find zeros of maximal monotone operators. Indeed, many problems that involve convexity can be formulated as finding zeroes of maximal monotone operators. Such problems include, but are not limited to convex minimization, variational inequalities and concave-convex mini-max problems.

The aim of this chapter is to give a brief overview of some iterative processes that can be applied to solve nonlinear operator equations of monotone type. In the next chapters, these processes will be studied further. Notably, the so called prox-Tikhonov method of Xu [55], which was formerly introduced by Lehdili and Moudafi [32], will be developed even further, and it will be shown later that such a method is better placed to be applied in approximating minimum values of convex functionals.

1.1 Mann's iterative process

Let K be a subset of a real Hilbert space H, with the scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. We begin by reminding the reader that an operator $T : K \to K$ is said to be compact if $T(K)$ is relatively compact. If the operator T has the property that existence of a strongly convergent subsequence (x_{n_k}) of (x_n) follows from the fact that (x_n) is bounded in K with $(I - T)x_n$ being strongly convergent, then such an operator T is called demicompact. Recall also that an operator T is said to be a Lipschitzian one if there exists a constant $a \geq 0$ such that

$$\|Tx - Ty\| \leq a \|x - y\| , \tag{1.1}$$

holds true for all $x, y \in K$. Of course Lipschitzian maps are necessarily continuous. The constant a is called the Lipschitz constant for T if it happens to be the smallest constant such that (1.1) holds true, and it is usually denoted by L. The map T is said to be a strict contraction if $L < 1$, and it is termed nonexpansive if $L = 1$. A typical example of a nonexpansive map is an anticlockwise rotation in $H := \mathbb{R}^2$ about the origin of the closed unit ball $B_r(0, 1) = \{x \in H \mid \|x\| \leq 1\}$ through an angle of, say, $\pi/2$. Note that such a map, say, $T : B_r(0, 1) \to B_r(0, 1)$ has the origin as the unique fixed point. Clearly, starting at any $x_0 \in B_r(0, 1) \setminus \{0\}$, the sequence $(T^n x_0)$ generated by the Picard iterative process fails to converge in this case, see for example, [18]. Instead of taking the original map (an anticlockwise rotation $T : B_r(0, 1) \to_r (0, 1)$), Krasnosel'skiĭ [31] considered the average mapping $F : B_r(0, 1) \to B_r(0, 1)$, where $F := (I + T)/2$ with I being the identity transformation, and showed that starting at any point $x_0 \in B_r(0, 1)$, the sequence of iterates $(F^n x_0)$ generated from the Picard iterative process converges strongly to the unique fixed point of F. (Note that the average mapping F is nonexpansive whenever T is so, and its set of fixed points coincides with that of T). More generally, if T is a nonexpansive and compact operator that maps a closed, bounded and convex subset K of a uniformly convex Banach space X, then for any given $x_0 \in K$, the sequence (x_n) defined by

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n), \quad \text{for all } n \geq 0, \tag{1.2}$$

converges strongly to a fixed point of T. The iterative process above is thus called the Krasnosel'skiĭ's iterative process. The result of Krasnosel'skiĭ holds true even if X is a strictly convex Banach space, as shown by Edelstein [20]. Schaefer's [46] idea was to replace the average mapping in (1.2) by a more general one, say, $V : K \to K$, where $V := \lambda I + (1 - \lambda)T$, for $\lambda \in (0, 1)$, thus obtaining the following generalized iterative process: Given any $x_0 \in K$, generate a sequence (x_n) recursively by the rule

$$x_{n+1} = \lambda x_n + (1 - \lambda)Tx_n, \quad \text{for all } n \geq 0. \tag{1.3}$$

Still in a uniformly convex Banach space X, Schaefer showed that when T is again a compact and nonexpansive mapping of a closed, bounded and convex subset K of X, the sequence (x_n) defined by (1.3) converges strongly to a fixed point of T. The requirement that T be compact can be weakened to demicompactness, and still derive the convergence of (x_n), see Petryshyn [42]. When the demicompactness restriction on T is removed, then the sequence generated by the generalized Krasnosel'skiĭ iterative process (1.3) converges at least weakly, see for example, [6] and the references therein for details. Actually, strong convergence may fail as shown by Genel and Lindenstrauss [21].

It should be pointed out that the iterative process (1.3) is in fact a special case of the iterative process of Mann, which is defined in the following way: Given any $x_0 \in K$, and a nonexpansive map $T : K \to K$, a sequence (x_n) is generated recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \text{for all } n \geq 0, \tag{1.4}$$

where K is a closed convex subset of a Banach space X, and the parameter sequence (α_n) belong to the interval $(0, 1)$, with the conditions[1]

$$(C1) \lim_{n\to\infty} \alpha_n = 0, \quad \text{and} \quad (C2) \sum_{n=0}^{\infty} \alpha_n = \infty$$

being satisfied. The sequence of Mann's iterates is known to converge only weakly in general, the same counterexample of Genel and Lindenstrauss [21] applies. Reich [43] extended the result of Mann [34], which was initially proved in Hilbert spaces by Mann himself, to uniformly convex Banach spaces whose norms are Frechet differentiable.

At this point, it is important to remember that we are looking for fixed points of a certain resolvent operator of a given maximal monotone operator, which by the way is nonexpansive. In other words, we want to solve the following set valued equation

$$\text{find an } x \in D(A) \text{ such that } 0 \in A(x), \tag{1.5}$$

where $A : D(A) \subset H \to 2^H$ is a maximal monotone operator. So, in order to generate a sequence that is weakly convergent to the point of the set $A^{-1}(0)$, it becomes a natural attempt to replace the nonexpansive operator T in (1.4) by the resolvent operator of A, $(I + \beta A)^{-1} : H \to H$, for $\beta > 0$. More generally, we may replace T by a sequence of resolvent operators and hope to generate a sequence that is weakly convergent to the point of the set $A^{-1}(0)$. In that case, the sequence (x_n) generated by the iterative process of Mann type becomes

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(I + \beta_n A)^{-1}(x_n) + e_n, \quad \text{for all } n \geq 0, \tag{1.6}$$

[1] see the appendix section for a collection of control conditions

where $x_0 \in H$ is chosen arbitrarily, $(\beta_n) \subset (0,\infty)$, and (e_n) is regarded as a sequence of computational errors. Indeed, the weak convergence to an element of the set $A^{-1}(0)$ was confirmed independently by Xu [54], and Kamimura and Takahashi [28] in the case when the error sequence (e_n) is summable, $\beta_n \to \infty$ and $(\alpha_n) \subset (0,1)$ satisfy the control conditions (C1) and (C2). (For the sake of brevity, we shall employ the notation $J_\beta^A := (I+\beta A)^{-1}$, however, when no confusion will arise, we shall write J_β instead of J_β^A). Note that for each $n \geq 0$, the operator $(I+\beta_n A)^{-1}$ has the whole H as its domain, is single valued and nonexpansive.

Concerning the existence of solutions of A, it is worth mentioning that there is at least one if A satisfies the coercivity condition, (see (2.4) below).

We wish to point out that there exists in the literature some modifications of the Mann iteration which were introduced in order to enforce such a scheme to converge strongly. One example that comes to our mind is the so called CQ Method which was proposed by Nakajo and Takahashi [40], and in Hilbert space setting, it is defined in the following way:

$$\begin{cases} x_0 \in K \text{ is chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in K \,:\, \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in K \,:\, \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.7}$$

where P_C denotes the metric projection from H onto the closed convex subset C of H. Strong convergence results associated with this method can be found in [40]. The reader may also consult [30, 37] for extensions of this method. Another modification of (1.4) that yields strong convergence results can be found in [29].

1.2 Halpern's iterative process

Let T be a nonexpansive map of a real Hilbert space H into itself, and suppose that there exists a bounded closed convex subset C of H mapped by T into itself. For an arbitrary (but fixed) $u \in C$ and $t \in (0,1)$, the map $T_t : C \to C$ defined by the rule $x \mapsto tu + (1-t)Tx$ is a strict contraction with Lipschitz constant $(1-t)$. It then follows from the Banach contraction principle that T_t has a unique fixed point in C, say z_t, that is,

$$z_t = tu + (1-t)Tz_t. \tag{1.8}$$

Note that here z_t depend on both u and t. Most importantly, the fixed point set of T is nonvoid, by Browder's fixed point theorem. One can prove that, as $t \to 0^+$, the path z_t given by the above formula converges strongly to the fixed point of T that is closest to u. This result of Browder [16] has been widely used in the theory of fixed points and extended in many different directions by several researchers. Following Browder's iterative method, where fixed points of a nonexpansive map T were sought by means of an implicit scheme (1.8), Halpern [24] initiated the study of a strongly convergent (explicit) iterative process

$$x_{n+1} = t_n T x_n, \tag{1.9}$$

for an arbitrary point x_0 of a closed unit ball $B_r(0,1) = \{x \in H \mid \|x\| \leq 1\}$, where $t_n \in (0,1)$ for all $n \geq 0$ and T is a nonexpansive map of $B_r(0,1)$ into itself. Halpern showed in a Hilbert space setting that if the set of fixed points of T is nonempty and the sequence of parameters (t_n) is chosen in such a way that they meet the requirements (i) $t_n \to 1$ as $n \to \infty$, and (ii) $\prod_{n=0}^{\infty} t_n = 0$, then (x_n) generated by (1.9) always converge strongly to the fixed point of T. Such a fixed point is specified among others as the unique element in the fixed point set $F(T) = \{x \in H \mid Tx = x\}$ of T that is of minimum norm. In fact, conditions (i) and (ii) are necessary for (1.9) to converge strongly to the fixed point of T as shown in [24] already. One may view the scheme of Halpern as a modification of the Picard iteration for nonexpansive maps developed in order to enforce strong convergence of the later. Introducing the parameters (t_n) into the Picard iterative process alters the $(n+1)$th iterate of such a scheme from being the value of a nonexpansive map T evaluated at the previous iterate to being that of a strict contraction $t_n T$ with Lipschitz constant t_n, again evaluated at the n^{th} iterate.

In light of the above discussion, given any arbitrary (but fixed) points $u, x_0 \in H$, one may generate an explicit iterative process (x_n) that approximates fixed points of a nonexpansive mapping $T : H \to H$ by the rule

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \tag{1.10}$$

where $\alpha_n \in (0,1)$ for all $n \geq 0$. Still in a Hilbert space setting, strong convergence of algorithm (1.10) to the metric projection of u on the fixed point set $F(T)$ was proved by Lions [33] under the control conditions (C1), (C2) and

$$(C3)\ \lim_{n\to\infty} \frac{(\alpha_{n+1} - \alpha_n)}{\alpha_{n+1}^2} = 0.$$

Our preference in algorithm (1.10) over (1.9) rests in its ability to reveal how the approximated fixed points are dependent on the given/initial data. Unfortunately, Lions' result excludes the natural choice $\alpha_n = n^{-1}$. This was overcome in 1992 by Wittmann [53] who showed strong convergence of (x_n) under the control conditions (C1), (C2), and

$$(C4)\ \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

In 2002, Xu [54] studied Algorithm 1.10 extensively. First, he showed that in a Banach space setting, (x_n) still maintains its strong convergence on removing the square in the denominator of (C3), thereby improving Lions' result twofold. The conditions used were (C1), (C2), and

$$(C5)\ \lim_{n\to\infty} \frac{(\alpha_{n+1} - \alpha_n)}{\alpha_{n+1}} = 0, \quad \text{or equivalently,} \quad \lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1.$$

He then showed that the conditions (C3) and (C4) are not comparable, and did the same for (C4) and (C5). Xu then observed that Halpern actually showed that the conditions (C1) and (C2) are necessary to have strong convergence to the metric projection of u on the set $F(T)$. This provided a partial answer to Reich's question: Concerning (α_n), what are the necessary and sufficient conditions for (x_n) to converge strongly? To the best of our knowledge, the other part of the question concerning sufficiency remains open. However, in a recent paper of Suzuki [50], it is shown that if the nonexpansive mapping T in (1.10) is of the form $T := \lambda S + (1-\lambda)I$ (with $\lambda \in (0,1)$, S a nonexpansive mapping and I the identity operator), then the conditions (C1) and (C2) are not only necessary for (x_n) to converge strongly, but they are also sufficient. In fact, for any fixed $u, x_0 \in H$, Suzuki showed strong convergence of the iterative process

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(\lambda S x_n + (1-\lambda)x_n), \quad \text{for all } n \geq 0, \tag{1.11}$$

in Banach spaces. The same result was also obtained by Chidume and Chidume [19] independently.

It is worth mentioning that strong convergence results can still be obtained if one replaces the nonexpansive map in the algorithms discussed in this section by a sequence of nonexpansive mappings. Of particular interest to us is an instance whereby the resolvent operator of a maximal monotone operator is brought into picture. When such an operator is involved, the resulting algorithm is termed "*the proximal point algorithm*". We shall discuss it in detail in the next section.

1.3 The proximal point algorithm

In 1970, B. Martinet [36] propounded an effective algorithm for solving the set valued equation: $0 \in A(x)$, where A is an operator of the form

$$A(x) = \begin{cases} T(x) + N_C(x), & \text{if} \quad x \in C, \\ \emptyset, & \text{if} \quad x \notin C, \end{cases}$$

for a singlevalued, monotone and hemicontinuous operator $T : C \to H$ defined on a closed and convex subset C of a real Hilbert space H, and $N_C(x)$ is the normal cone to C at x (that is, $N_C(x) = \{w \in H \mid \langle x - u, w\rangle \geq 0, \forall\, u \in C\}$). By a hemicontinuous operator, we mean one that is continuous along each line segment in H with respect to the weak topology. Rockafellar [44] proved that the operator A defined above is maximal monotone. The algorithm of Martinet is based on the result due to Minty, which states that for each $z \in H$ and $\beta > 0$, there is a unique $u \in H$ such that $z \in (I + \beta A)u$ (for an arbitrary maximal monotone operator $A : D(A) \subset H \to 2^H$), and it is defined as

$$x_{n+1} = (I + \beta A)^{-1}(x_n), \quad \text{for all } n \geq 0,$$

where $x_0 \in H$ is given. Martinet then showed that all the weak accumulation points of (x_n) belong to $A^{-1}(0)$ (if it is nonempty), that is, they solve problem (1.5). However, a close inspection at the proof provided by Martinet reveals that (x_n) satisfies the two conditions of Opial's lemma (see Lemma 2.2.6 below), hence the whole sequence (x_n) converges weakly to the point of $A^{-1}(0)$. It is worth pointing out that the set $A^{-1}(0)$ is closed and convex, and it coincides with the set of fixed points of the resolvent operator of A. In the present case, we however note that the solutions of problem (1.5) coincides with the solutions of the following variational inequality:

$$\text{find an } x \in C \text{ such that } \langle x - u, Tx\rangle \leq 0, \text{ for all } u \in C.$$

The main idea behind Martinet's method is to replace the original problem (1.5) by a sequence of regularized problems

$$\text{find an } x \in D(A) \text{ such that } 0 \in A(x) + \beta^{-1}(x - x_n), \tag{1.12}$$

so that at each step, problem (1.12) has a unique solution $x := x_{n+1}$.

The algorithm of Martinet inherited the name "proximal point algorithm," due to its firm connection with the *proximal mapping*, $J_\beta(x) = x_\beta = \arg\min\{\varphi(z) + \|z - x\|^2/2\beta : z \in H\}$, introduced in 1965 by J. J. Moreau. After the PPA was introduced, Rockafellar systematically developed it by considering an arbitrary maximal monotone operator A :

$D(A) \subset H \to 2^H$ and allowing β to vary as n does. More precisely, he generated a sequence (x_n) by the approximate rule

$$x_{n+1} \approx J_{\beta_n} x_n, \quad \text{for all } n \geq 0, \tag{1.13}$$

where $x_0 \in H$ is given, and showed that such a scheme converges weakly to the point of the set $A^{-1}(0)$ (if it is not empty), provided that β_n is bounded below away from zero and the criterion for the approximate computation of x_n is given by

$$\|x_{n+1} - J_{\beta_n} x_n\| \leq \delta_n \quad \text{with} \quad \sum_{n=0}^{\infty} \delta_n < \infty.$$

He also proved strong convergence if in addition, the operator A^{-1} is Lipschitz continuous at zero (with modulus $a \geq 0$), that is, $A^{-1}(0) = \{y\}$, and for some $\tau > 0$, $\|z - y\| \leq a \|z'\|$ whenever $(z, z') \in G(A)$ and $\|z'\| \leq \tau$.

The proximal point algorithm (PPA) is also important in convex optimization as evidenced from the fact that many problems that involve convexity such as convex minimizations and convex-concave mini-max (saddle-point) problems can be formulated as finding zeros of maximal monotone operators. In particular, the subdifferential operator $\partial\varphi : H \to H$ defined by

$$\partial\varphi(x) = \{w \in H \mid \varphi(x) - \varphi(v) \leq \langle w, x - v\rangle, \quad \forall\, v \in H\}$$

of a proper, convex and lower semicontinuous function (lsc) $\varphi : H \to (-\infty, +\infty]$, is a maximal monotone operator and a point $p \in H$ minimizes φ if and only if $p \in D(\partial\varphi)$, and $0 \in \partial\varphi(p)$. Therefore, in this case, the proximal point algorithm in exact form generates a sequence (x_n) by taking the $(n+1)$th iterate to be the minimizer of the function $\psi_n : H \to (-\infty, +\infty]$, for

$$\psi_n(x) = \varphi(x) + \frac{1}{2\beta_n}\|x - x_n\|^2, \quad \text{where } \beta_n > 0.$$

The above example and the the one concerning variational inequalities show that the PPA has natural applications in nonlinear analysis and convex optimization. We wish to point out that it was shown by Y. Censor and S. A. Zenois (1992) that the quadratic additive term appearing above can be replaced by more general D-functions which resembles (but are not strictly) distance functions. They characterized the properties of such D-functions which when used in the proximal minimization algorithm preserve its convergence. It was further shown by J. Eckstein (1993) that for every Bregman function (a strictly convex differentiable function that induces the distance measure or a D-function on the Euclidean space) there exists a "nonlinear" version of the PPA.

Note that neither Martinet nor Rockafellar could characterize the point to which the proximal point algorithm converges to, and unlike Halpern's iterative process, the proximal point algorithm does not converge strongly in general. Indeed, Güler [23] constructed an example showing that Rockafellar's algorithm (1.13), with equality taken instead of $\approx$, does not converge strongly, in general[2]. In particular, he showed that there exists a proper, closed convex function φ in ℓ^2 such that given any bounded sequence of positive real numbers (β_n), there exist a starting point $x_0 \in D(\varphi)$ for which the PPA converges weakly, but not strongly to a minimizing point of φ. Güler showed in the same paper that the convergence rate of the PPA is governed by the type of convergence involved (weak or strong convergence). More precisely, he gave the following estimates for the rate of convergence: $\varphi(x_n) - \inf_{x\in H} \varphi(x) = O(\sigma_n^{-1})$ in the case when (x_n) converges weakly, and $\varphi(x_n) - \inf_{x\in H} \varphi(x) = o(\sigma_n^{-1})$ in the case when (x_n) converges strongly. Here $\sigma_n = \sum_{k=0}^{n} \beta_k$.

Since weak convergence is not enough for an efficient algorithm and the PPA does not converge strongly in general, much of research have been devoted to either constructing new algorithms which will always converge strongly, or at least modify Rockafellar's algorithm in such a way that strong convergence is guaranteed. One such construction have been obtained by Solodov and Svaiter [47]. In an attempt to obtain strong convergence, Solodov and Svaiter proposed an algorithm which generates a sequence (x_n) satisfying

$$x_{n+1} = P_{H_n \cap W_n} x_0, \quad \text{for all} \quad n \geq 0, \quad \text{where}$$

(a) $x_0 \in H$ is arbitrary and $(y_n, v_n) \in H \times H$ is an inexact solution of the inclusion:

$$0 \in A(x) + \mu_n (x - x_n),$$

with $\mu_n > 0$ and tolerance $\sigma \in [0, 1)$, that is,

$$v_n \in A(y_n), \quad v_n + \mu_n(y_n - x_n) = e_n, \quad \text{and} \quad \|e_n\| \leq \sigma \max\{\|v_n\|, \mu_n\|y_n - x_n\|\};$$

(b) $P_{H_n \cap W_n}$ is the projection of H onto $H_n \cap W_n$ where

$$H_n := \{z \in H : \langle z - y_n, v_n\rangle \leq 0\} \quad \text{and} \quad W_n := \{z \in H : \langle z - x_n, x_0 - x_n\rangle \leq 0\}.$$

It was proved in [47] that if the sequence (μ_n) is bounded from above, then the sequence (x_n) constructed above converges strongly to $P_{A^{-1}(0)} x_0$. Though their algorithm is strongly convergent, it needs more computing time since it requires at each iterate, to calculate a projection, a task which may not always be easy. Xu's idea was to construct a less

[2]Another example in which the PPA fails to convergence strongly in general may be found in [3]

time consuming algorithm which still converge strongly. In view of Halpern's algorithm, Xu [54] proposed the following algorithm

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)(I+\beta_n A)^{-1}(x_n) + e_n, \quad \text{for all} \quad n \geq 0, \tag{1.14}$$

where $x_0 \in H$ is given, and showed that algorithm (1.14) converge strongly provided that $(\|e_n\|) \in \ell^1$ and the sequences (α_n), (β_n) of real numbers are chosen appropriately. (Here (e_n) is the sequence of computational errors). In fact, the same result was proved independently by Kamimura and Takahashi [28]. It seems that strong convergence is still ensured even if x_0 is replaced by any arbitrary point u of H (not necessarily the starting point of the PPA). What is not yet clear is whether or not the result will still hold if one were to take the error sequence outside ℓ^1, for example by taking $(\|e_n\|) \in \ell^p$ for $1 < p < 2$.

In connection with algorithm (1.11), He et al. [25] showed in Banach space settings that if the nonexpansive map S in (1.11) is replaced by the resolvent, J_{β_n}, of an m-accretive operator, then the resulting sequence converges strongly under the control conditions (C1), (C2), and the condition

$$(C6) \ \lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0,$$

with β_n bounded from below away from zero. Under the condition (C6), it is not clear if strong convergence of the sequence generated by algorithm (1.14) is guaranteed.

Recently, Takahashi [51] studied the proximal point algorithm in a Banach space by the viscosity approximation method, where the $(n+1)$th iterate was given as

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)(I+\beta_n A)^{-1}(x_n), \tag{1.15}$$

where $f : C \to C$ is a strict contraction (a-contraction with $0 < a < 1$) defined on a nonempty closed convex subset C of a reflexive Banach space X, and $A : D(A) \subset X \to 2^X$ is an m-accretive operator. It is clear that in a Hilbert space setting, we can generalize the result under algorithm (1.14) by the viscosity approximation method even when one takes into account the error terms in ℓ^1, see Section 3.3. Exploring the case when f is nonexpansive leads to several interesting results, some of which guarantees strong convergence of the modified PPA. We have discussed this case in Section 3.3.

1.4 The regularization method

Another algorithm used for solving the set valued equation $0 \in Ax$ is the so-called Tikhonov method which generates a sequence (x_n) by the rule

$$x_n = (I+\mu_n A)^{-1}(0), \quad \text{for all } n \geq 0, \tag{1.16}$$

where the sequence of positive real numbers (μ_n) is required to tend to infinity. Contrary to the PPA which generates a sequence that generally converge only weakly, and the limit of which is not characterized, the Tikhonov regularization method does not only yield a strongly convergent sequence (x_n), but it also provide us with the extra information on the strong limit of (x_n), namely that it is of minimum norm in the solution set $A^{-1}(0)$. It is worth noting that this favorable feature is also possessed by Halpern's iterative process. Actually, for any $u \in H$, we know from the result of Bruck [17] that $(I+\lambda A)^{-1}u \to P_F u$, as $\lambda \to \infty$. In fact the same result was also proved by Moroşanu [38] independently.

In the year 1996, Lehdili and Moudafi [32] combined the regularization method of Tikhonov with the proximal point algorithm to obtain a sequence (x_n) defined as follows: for any fixed $x_0 \in H$, generate the $(n+1)$th iterate of (x_n) by

$$x_{n+1} = (I + \beta_n A_n)^{-1}(x_n), \quad \text{for all } n \geq 0, \tag{1.17}$$

where $A_n : D(A_n) = D(A) \subset H \to 2^H$ is the operator defined by $x \mapsto (\mu_n I + A)x$, with $\mu_n > 0$ for all $n \geq 0$, and $(\beta_n) \subset (0, \infty)$. The operator A_n is usually regarded as a Tikhonov regularization of A, and it is strongly monotone (and hence coercive). The algorithm (1.17), otherwise known as the prox-Tikhonov method according to the terminology of its inventors, thus amounts to replacing the maximal monotone operator A by a sequence of coercive operators A_n, and it converges to the zero of A rather than to the (unique) fixed point of the operator $J_{\beta_n}^{A_n}$. Note that the equation (1.17) can be written in the following equivalent form:

$$x_{n+1} = (I + \lambda_n A)^{-1}(\gamma_n x_n), \quad \text{for all } n \geq 0, \tag{1.18}$$

where $\lambda_n = \beta_n(1+\beta_n\mu_n)^{-1}$, and $\gamma_n = (1+\beta_n\mu_n)^{-1}$. Motivated by the work of Lehdili and Moudafi, Xu [55] proposed the following regularization method for the proximal point algorithm which essentially includes algorithm (1.17) as a special case, (as noticed by Xu himself): for any fixed $x_0, u \in H$, generate a sequence (x_n) iteratively by

$$x_{n+1} = J_{\beta_n}(\alpha_n u + (1-\alpha_n)x_n + e_n), \quad \text{for all } n \geq 0. \tag{1.19}$$

Strong convergence of the sequence (x_n) defined by (1.19) to the metric projection of u onto the fixed point set $A^{-1}(0)$ was shown in [55] under the control conditions which appear as a combination of α_n and β_n. More precisely, the conditions used were

$$(C7)\ \sum_{n=0}^{\infty}\left|1 - \frac{\alpha_n\beta_{n+1}}{\alpha_{n+1}\beta_n}\right| < \infty \quad \text{or,} \quad (C8)\ \lim_{n\to\infty}\frac{1}{\alpha_n}\left(1 - \frac{\alpha_n\beta_{n+1}}{\alpha_{n+1}\beta_n}\right) = 0.$$

Note that for $\beta_n \to \infty$, the natural choices of $\alpha_n = n^{-1}$ and $\beta_n = n$, fails under both conditions. In fact, for any choice of α_n and β_n, condition (C7) is impossible to achieve as shall be shown in Chapter 3, (see Remark 3.4.1). In another result of Xu, Theorem 3.3 [55], it is shown that for summable errors, strong convergence is still maintained under the conditions (C1), (C2), (C4), and β_n bounded (from above and from below away from zero) with (C9) (as defined below) being satisfied. Song and Yang [48] established strong convergence of the prox-Tikhonov algorithm (1.19) when the errors are summable, (C1), (C2), (C4) being satisfied, and the following condition on β_n imposed: β_n is bounded from below away from zero with either

$$(C9) \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \quad \text{or} \quad (C9)' \sum_{n=0}^{\infty} \frac{|\beta_{n+1} - \beta_n|}{\beta_{n+1}} < \infty.$$

They remarked that their result (Theorem 2 [48]) contains Theorem 3.3 [55] as a special case. Although this seems to be the case at first glance, it turns out that the two theorems are equivalent. In fact, the condition (C9)′ on β_n is equivalent to (C9) and β_n bounded from below away from zero. Obviously, from this equivalence follows the equivalence of the two theorems. This equivalence is not so obvious and it is discussed in Lemma 2.1.2 below. The result of Xu discussed above was improved significantly by Wang [52], who showed that for summable errors, strong convergence of the sequence generated by algorithm (1.19) is preserved when one assumes that the parameter sequence (α_n) satisfies only the conditions (C1) and (C2), and $\beta_n \in (0, \infty)$ is bounded from below and from above with the condition (C6) being satisfied.

Chapter 2

Preliminaries

In order to make this thesis as self contained as possible, we shall collect in this chapter some results which will be useful in proving the main results of the subsequent chapters.

2.1 Some sequences of real numbers

We begin with a Lemma which is due to Xu [54].

Lemma 2.1.1 (Xu [54]). *Let (s_n) be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-a_n)s_n + a_n b_n + c_n, \quad n \geq 0,$$

where (a_n), (b_n), (c_n) satisfy the conditions: (i) $(a_n) \subset [0,1]$, *with* $\sum_{n=0}^{\infty} a_n = \infty$, (ii) $\limsup_{n\to\infty} b_n \leq 0$, *and* (iii) $c_n \geq 0$ *for all* $n \geq 0$ *with* $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $\lim_{n\to\infty} s_n = 0$.

We next show that any sequence of positive real numbers satisfying the condition of (C9)′ is bounded (with the lower bound being strictly positive).

Lemma 2.1.2. *For any sequence (b_n) of positive real numbers, the following conditions are equivalent:* (i) $\sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty$ *and* $0 < \liminf_{n\to\infty} b_n$ $(= \lim_{n\to\infty} b_n)$, (ii) $\sum_{n=0}^{\infty} \frac{|b_{n+1}-b_n|}{b_n} < \infty$, *and* (iii) $\sum_{n=0}^{\infty} \frac{|b_{n+1}-b_n|}{b_{n+1}} < \infty$.

Proof. First, it is easily seen that (i) ⇒ (ii), and (i) ⇒ (iii). Now let us prove that (ii) ⇒ (i). For this, it suffices to show that there exist constants $m, M > 0$ such that $m \leq b_n \leq M$ for all $n = 0, 1 \ldots$

From (ii), there exists a sequence $(a_n) \subset \mathbb{R}$, such that $\sum_{n=0}^{\infty} |a_n| < \infty$, and

$$\frac{b_{n+1} - b_n}{b_n} = a_n \quad \Leftrightarrow \quad \frac{b_{n+1}}{b_n} = 1 + a_n, \quad n = 0, 1, \ldots$$

Note that in particular, $\lim_{n\to\infty} a_n = 0$. Therefore, we may assume without any loss of generality that $|a_n| < 1$ for all n. Then by simple induction, we have

$$\frac{b_n}{b_0} = \prod_{k=0}^{n-1}(1+a_k). \tag{2.1}$$

Since $1 + x \leq \exp(x)$ for all $x \geq 0$, it follows from (2.1) that

$$\frac{b_n}{b_0} = \prod_{k=0}^{n-1}(1+a_k) \leq \prod_{k=0}^{n-1}(1+|a_k|) \leq \exp\left(\sum_{k=0}^{n-1}|a_k|\right) \leq \exp\left(\sum_{k=0}^{\infty}|a_k|\right) =: M_0 < \infty. \tag{2.2}$$

On the other hand,

$$\sum_{k=0}^{\infty}|a_k| < \infty \quad \Leftrightarrow \quad \prod_{k=0}^{\infty}(1-|a_k|) > 0,$$

and again from (2.1) we obtain

$$\frac{b_n}{b_0} = \prod_{k=0}^{n-1}(1+a_k) \geq \prod_{k=0}^{n-1}(1-|a_k|) \geq \prod_{k=0}^{\infty}(1-|a_k|) =: m_0 > 0. \tag{2.3}$$

The conclusion then follows from (2.2) and (2.3). Replacing b_n by b_n^{-1} in (ii), one readily gets (iii), showing that (iii) $\Rightarrow$ (i) as desired. □

2.2 Some tools from functional analysis

Throughout this thesis, shall employ the following notations: given a sequence $(x_n)_{n\in\mathbb{N}_0}$ in a Banach Space X, $\mathbb{N}_0 = \{0, 1, \dots\}$, (or (x_n) in short), and a point $x \in X$, $x_n \to x$ (respectively, $x_n \rightharpoonup x$) means that (x_n) converges strongly (resp. weakly) to x.

Lemma 2.2.1 (Suzuki [49]). *Let (x_n) and (y_n) be bounded sequences in a real Banach space and let (ρ_n) be a sequence in $(0,1)$, with $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < 1$. Suppose $x_{n+1} = \rho_n y_n + (1-\rho_n)x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.2.2 (Eberlein and Šmulyan). *A Banach space X is reflexive if and only if any bounded sequence in X contains at least one subsequence that converges weakly in X.*

In the sequel, we will consider a real Hilbert space H, which is a typical example of a reflexive Banach space. A map $T : H \to H$ is called firmly nonexpansive if for any $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle.$$

Obviously, firmly nonexpansive mappings are monotone and nonexpansive. Such mappings are characterized by

Lemma 2.2.3 (Goebel and Kirk [22])**.** *A map $T : H \to H$ is firmly nonexpansive if and only if $2T - I$ (where I is the identity map) is nonexpansive.*

Perhaps the projection operator and the resolvent of a maximal monotone operator are well known and widely used examples of firmly nonexpansive maps. For a nonempty closed and convex subset C of H, the metric projection (nearest point mapping) $P_C : H \to C$ is defined as follows: Given $x \in H$, $P_C x$ is the unique point in C having the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| .$$

Just like firmly nonexpansive mappings, projections have a nice characterization:

Lemma 2.2.4. *Assume that C is a nonempty closed and convex subset of H. Let $x \in H$ and $y \in C$ be given. Then $y = P_C x$ if and only if the inequality*

$$\langle x - y, z - y\rangle \leq 0, \quad \text{for all } z \in C,$$

holds true.

In fact, from this inequality characterizing projections, one can further derive the aforementioned fact that the projection operator is firmly nonexpansive.

Most of the analysis of this thesis will depend on the the following identity

Lemma 2.2.5 (Resolvent Identity)**.** *For any $\beta, \gamma > 0$, and $x \in H$, the identity*

$$J_\beta x = J_\gamma \left(\frac{\gamma}{\beta} x + \left(1 - \frac{\gamma}{\beta}\right) J_\beta x \right)$$

holds true, where $J_\beta := (I + \beta A)^{-1}$ for a maximal monotone operator $A : D(A) \subset H \to 2^H$.

Proof. (For the sake of completeness, we provide the proof of this Lemma). Set $y := J_\beta x$. Then using the definition of the resolvent, we have

$$y = J_\beta x \;\Leftrightarrow y + \gamma A y \ni \frac{\gamma}{\beta} x + \left(1 - \frac{\gamma}{\beta}\right) y \;\Leftrightarrow\; y = J_\gamma \left(\frac{\gamma}{\beta} x + \left(1 - \frac{\gamma}{\beta}\right) y \right).$$

This completes the proof of the resolvent identity. □

Given any sequence (x_n), we shall denote its weak ω-limit set by $\omega_{\mathtt{w}}((x_n))$, that is,

$$\omega_{\mathtt{w}}((x_n)) := \{x \in H \mid x_{n_k} \rightharpoonup x \text{ for some subsequence } (x_{n_k}) \text{ of } (x_n)\} .$$

Here "$\rightharpoonup$" denotes weak convergence. We shall prove weak convergence results with the aid of the following lemma, whose proof can be found in many functional analysis books, (see, e.g., [39, p. 5]).

Lemma 2.2.6 (Z. Opial)**.** *Let C be a nonempty subset of H. Assume that the sequence (x_n) satisfies the conditions (a) $\lim_{n\to\infty}\|x_n - q\| = \rho(q)$ exists for all $q \in C$, and (b) any weak cluster point of (x_n) belongs to C (that is $\omega_{\mathtt{w}}((x_n)) \subset C$). Then, there exists a point $p \in C$ such that (x_n) converges weakly to p.*

More often, we shall make use of the following inequality which is usually referred to as the subdifferential inequality. Its proof is immediate.

Lemma 2.2.7 (Subdifferential Inequality)**.** *For all $x, y \in H$, we have*

$$\|x+y\|^2 \le \|y\|^2 + 2\langle x, x+y\rangle.$$

The next lemma will also be useful in proving our main results. Its proof can be found in [39, p. 20].

Lemma 2.2.8. *Any maximal monotone operator $A : D(A) \subset H \to 2^H$ satisfies the demicloseness principle. In other words, given any two sequences (x_n) and (y_n) satisfying $x_n \to x$ and $y_n \rightharpoonup y$ with $(x_n, y_n) \in A$, then $(x, y) \in A$.*

Recall that an operator $A : D(A) \subset H \to 2^H$ is said to be coercive if it satisfies the following condition

$$\lim_{\|\xi\|\to\infty,\ (\xi,\eta)\in A} \frac{\langle \eta, \xi - v_0\rangle}{\|\xi\|} = \infty, \tag{2.4}$$

for some $v_0 \in H$.

When A is the subdifferential, coercivity of A is equivalent to the conditions given in the following result.

Proposition 2.2.9. [14] *Let $\varphi : H \to (-\infty, +\infty]$ be a proper, convex, lower semi-continuous function, and let $A = \partial\varphi$, the subdifferential of φ. Then the following conditions are equivalent;*

(i) $\displaystyle\lim_{\|\xi\|\to\infty,\ (\xi,\eta)\in A} \frac{\varphi(\xi)}{\|\xi\|} = \infty$;

(ii) *A is coercive;*

(iii) *$R(A) = H$ and A^{-1} is bounded.*

Proof. (This result is a combination of well known facts, with the statement and proof that are convenient to us). For $x_0 \in D(A)$,

$$\varphi(x) \le \varphi(x_0) + \langle y, x - x_0\rangle \quad \text{for all} \quad (x, y) \in A, \tag{2.5}$$

which shows that (i)⇒(ii).

A coercive implies that $R(A) = H$ (see e.g., [39, p. 26]), and it can be easily seen that A^{-1} is bounded. Otherwise there will exist $(x_n, y_n) \in A$ such that $\{\|y_n\|\}$ is bounded and $\|x_n\| \to \infty$, and so

$$\frac{\langle y_n, x_n - x_0 \rangle}{\|x_n\|} \leq \|y_n\| \cdot \frac{\|x_n\| + \|x_0\|}{\|x_n\|} \leq C,$$

which contradicts the coercivity of A. This shows that (ii)⇒(iii).

Now, assume that (iii) hold. Let $r > 0$. For all $z \in H$, $\|z\| \leq r$, there exists $v \in D(A)$ such that

$$z \in A(v) \quad \|v\| \leq M, \tag{2.6}$$

where $M > 0$ (depending on r). Since φ is bounded below by an affine function, see for example Theorem 1.8 [39], we have from (2.6),

$$\begin{aligned} \varphi(u) &\geq \varphi(v) + \langle u - v, z \rangle \\ &\geq -C_1\|v\| - C_2 + \langle u, z \rangle - Mr \\ &\geq -C_1 M - C_2 + \langle u, z \rangle - Mr, \end{aligned} \tag{2.7}$$

for all $u \in D(\varphi)$. It follows from (2.7) that for all $u \in D(\varphi)$

$$\langle u, z \rangle \leq \varphi(u) + Mr + C_3 \quad \text{for all} \quad z \quad \text{with} \quad \|z\| \leq r,$$

and therefore

$$r\|u\| \leq \varphi(u) + Mr + C_3 \quad \text{for all} \quad u \in D(\varphi),$$

which implies

$$\liminf_{\|u\| \to \infty,\, u \in D(\varphi)} \frac{\varphi(u)}{\|u\|} = \infty.$$

□

We remark that coercivity of $A = \partial\varphi$ is stronger than the condition

$$\lim_{\|\xi\| \to \infty} \varphi(\xi) = \infty. \tag{2.8}$$

Chapter 3

Two Parameter Proximal Point Algorithms

We kick start this chapter by giving some remarks to the effect that the sequence generated by the original approximate proximal point algorithm is bounded under certain assumptions. It is note-worthy that R. T. Rockafellar (1976) assumed summability for the error sequence to derive weak convergence of the PPA in its initial form (see (3.1) below), and this restrictive condition has been extensively used to derive either weak or strong convergence results associated with different versions of the PPA. As shown by Rockafellar, the sequence generated by the approximate proximal point algorithm in its initial form may fail to converge if the summability condition on errors is replaced by the weaker condition that the errors tends to zero in norm.

For the error sequence (e_n) with $(\|e_n\|) \notin \ell^1$, we construct a sequence of parameters $(\alpha_n) \subset (0,1)$ satisfying the conditions $\alpha_n \to 0$ and $\|e_n\|/\alpha_n \to 0$, and then demonstrates that under this condition on errors (and of course some additional assumptions), the sequence generated by an algorithm of Halpern's type (see Section 3.2 below) preserves its convergence properties, namely convergence in the strong topology to the limit that is characterized as the projection of a given element of H to the set $A^{-1}(0)$. This construction thus offers a solution to the long standing problem of whether it is within the realms of possibility to relax the summability condition on the errors by taking a sequence that only converge to zero in norm when the PPA under consideration is of Halpern's type.

It should be noted that the construction mentioned above hinders one to freely choose the sequence (α_n) – only those α_n's in the interval $(0,1)$ that depend on the sequence of errors are allowed. The limitation in the freedom of choice of the α_n's can be overcome by assuming that the error sequence (e_n) is in ℓ^p for $1 \le p < 2$. Although this condition covers non-summable errors which converge to zero in norm, it is stronger than the one provided by the above mentioned construction.

Our main interest lies in covering a wide range of errors as possible. For this reason, we shall always assume that the error sequence satisfies the condition $\|e_n\|/\alpha_n \to 0$. This general condition on errors will be exploited further to derive several strong convergence results associated with other modified inexact proximal point methods such as the regularization method and the viscosity approximation method. Such results are proved under relatively new sets of assumptions on the two control parameters involved, which are weaker than those that have been used before by other authors. It is in this chapter that the relationship between Halpern's type proximal point algorithm and the regularization method is exposed. The connection between the two proximal methods avails the convenience of shifting from one method to the other when analyzing the behavior of the trajectories generated from either method.

3.1 Rockafellar's proximal point algorithm

In this section, we give some remarks concerning the initial inexact proximal point algorithm. We show under certain conditions that the sequence (x_n) generated by Rockafellar's algorithm

$$x_{n+1} = (I + \beta_n A)^{-1} x_n + e_n, \quad n \geq 0, \tag{3.1}$$

for an arbitrary (but fixed) $x_0 \in H$, is bounded. Here $A : D(A) \subset H \to 2^H$ is a maximal monotone operator, (e_n) is the error sequence, and $(\beta_n) \subset (0, \infty)$. In the first result, we show that if $(\beta_n) \notin \ell^1$, then the assumption $A^{-1}(0) \neq \emptyset$ is necessary and sufficient for (x_n) to be bounded. Note that Rockafellar [45] derived the same conclusion in the case when (β_n) is assumed to be bounded from below away from zero. Since the non-summability condition on (β_n) is weaker than the condition used by Rockafellar, Theorem 3.1.1 below may be regarded as a refinement of the above mentioned result of Rockafellar. The proof given in [45] is a constructive one, whereas the one given below is direct.

Theorem 3.1.1. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator. Assume that $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\beta_n \in (0, \infty)$ with $\sum_{n=0}^{\infty} \beta_n = \infty$. Given any $x_0 \in H$, the sequence (x_n) generated by (3.1) is bounded if and only if $F := A^{-1}(0)$ is nonempty.*

Proof. Let $x_0 \in H$ be such that the sequence (x_n) generated by (3.1) is bounded. Denote $u_n := x_n - e_{n-1}$ and let $(x, y) \in A$. Then we have from (3.1)

$$u_{n+1} - x + \beta_n (A u_{n+1} - y) + \beta_n y \ni u_n - x + e_{n-1}.$$

Multiplying this inclusion relation scalarly by $u_{n+1} - x$ yields,

$$\begin{aligned} \|u_{n+1} - x\|^2 + \beta_n \langle y, u_{n+1} - x\rangle &\leq \langle u_n - x, u_{n+1} - x\rangle + \langle e_{n-1}, u_{n+1} - x\rangle \\ &\leq \frac{1}{2}\|u_n - x\|^2 + \frac{1}{2}\|u_{n+1} - x\|^2 + K\|e_{n-1}\|, \end{aligned}$$

where the first inequality follows from the monotonicity of A. Therefore,

$$\frac{1}{2}\|u_{n+1} - x\|^2 + \beta_n \langle y, u_{n+1} - x\rangle \leq \frac{1}{2}\|u_n - x\|^2 + K\|e_{n-1}\|.$$

By summing from $n = 1$ to $n = N$, we get,

$$\left\langle y, \frac{\sum_{n=1}^{N} \beta_n u_{n+1}}{\sum_{n=1}^{N} \beta_n} - x \right\rangle \leq \frac{\|u_1 - x\|^2 + 2K \sum_{n=0}^{N} \|e_n\|}{2\sum_{n=1}^{N} \beta_n}\,. \tag{3.2}$$

Since (u_n) is bounded, so is (w_n), where

$$w_n := \left(\sum_{k=1}^{n} \beta_k\right)^{-1} \sum_{k=1}^{n} \beta_k u_{k+1}. \tag{3.3}$$

Let p be a weak cluster point of (w_n). Then passing to the limit in (3.2), we obtain,

$$\langle y, p - x\rangle \leq 0 \tag{3.4}$$

for all $(x, y) \in A$ since $\sum_{n=0}^{\infty} \beta_n = \infty$. By (3.4) and the maximality of A, it follows that $(p, 0) \in A$, which implies that $F \neq \emptyset$.

Conversely, if $F \neq \emptyset$, then for any $p \in F$, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \|e_n\|,$$

and therefore, by summing from $n = 0$ to $n = N$, we get (for $x_0 \in H$)

$$\|x_{N+1} - p\| \leq \|x_0 - p\| + \sum_{n=0}^{N} \|e_n\| \leq \|x_0 - p\| + \sum_{n=0}^{\infty} \|e_n\| < \infty.$$

This completes the proof of the theorem. □

Note that if $F \neq \emptyset$, and $\sum_{n=0}^{\infty} \beta_n = \infty$, then the average w_n defined by (3.3) converges weakly to some point $p \in F$, (see [39, p. 139]). If we assume more, that is, $\sum_{n=0}^{\infty} \beta_n^2 = \infty$, then the sequence (x_n) itself converges weakly to some point $p \in F$, (see [39, p. 142]).

In the case when $A = \partial\varphi$ where $\varphi : H \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous, the weaker additional condition $\sum_{n=0}^{\infty} \beta_n = \infty$ is enough to ensure weak convergence of (x_n), (see again [39, p. 142]).

We know that if A is coercive, then its range $R(A) = H$, (see for example [39, p. 26]), thus $F \neq \emptyset$. We now show that if (e_n) is bounded and (β_n) is bounded below away from zero, then the sequence (x_n) generated by (3.1) is bounded, provided A is assumed to be coercive.

Theorem 3.1.2. *Assume that $A : D(A) \subset H \to 2^H$ is coercive and maximal monotone. Let $\|e_n\| \leq C$ and $\beta_n \geq \varepsilon > 0$ for $n \geq 0$, where C and ε are given constants. Then for any given $x_0 \in H$, the sequence (x_n) generated by (3.1) is bounded.*

Proof. The proof is essentially done in, [39, p. 152]. We just adapt the old proof to the present framework.

Since A is coercive, the set $F := A^{-1}(0)$ is nonempty. Now setting $u_n := x_n - e_{n-1}$, equation (3.1) becomes

$$u_n + e_{n-1} \in u_{n+1} + \beta_n A u_{n+1}, \quad \text{for all} \quad n \geq 1,$$

which implies that

$$\|u_{n+1} - p\|^2 \leq \langle u_n - p, u_{n+1} - p\rangle + \langle e_{n-1}, u_{n+1} - p\rangle, \quad \text{for all} \quad n \geq 1,$$

for every $p \in F$. Therefore,

$$\|u_{n+1}\| \leq \|u_n\| + C + 2dist(0, F), \quad \text{for all} \quad n \geq 1. \tag{3.5}$$

Denote $C_1 := C + 2\|v_0\|$. By (2.4), there exists $K > 0$ such that

$$(\xi, \eta) \in A, \quad \|\xi\| > K \quad \text{implies} \quad \frac{(\eta, \xi - v_0)}{\|\xi - v_0\|} \geq \frac{C_1}{\varepsilon}. \tag{3.6}$$

Suppose that there exists k such that $\|u_{k+1}\| > K$. Then multiplying

$$u_k - v_0 + e_{k-1} \in u_{k+1} - v_0 + \beta_k A u_{k+1}$$

by $(u_{k+1} - v_0)/\|u_{k+1} - v_0\|$, where v_0 is the vector associated with the coercivity of A, and making use of (3.6), we get,

$$\|u_{k+1} - v_0\| + C_1 \leq \|u_k - v_0\| + \|e_{k-1}\| \leq \|u_k\| + \|v_0\| + C,$$

or

$$\|u_{k+1}\| \leq \|u_{k+1} - v_0\| + \|v_0\| \leq \|u_k\| + 2\|v_0\| + C - C_1,$$

which implies that

$$\|u_{k+1}\| \leq \|u_k\|.$$

Therefore, for all $n \geq 1$, we have

$$\|u_{n+1}\| \leq \max\{K + C + 2dist(0, F), \|u_n\|\}. \tag{3.7}$$

Setting $\rho_n = \max\{K + C + 2dist(0, F), \|u_n\|\}$, we deduce from (3.7) that $\rho_{n+1} \leq \rho_n$, for all $n \geq 1$. Hence

$$\|x_n\| \leq \|u_n\| + C \leq \rho_n + C \leq \max\{K + C + 2dist(0, F), \|x_1 - e_0\|\} + C,$$

for all $n \geq 1$, showing that (x_n) is bounded. □

Remark 3.1.1. Note that for a maximal monotone operator A, which is also strongly monotone (hence coercive), if $\beta_n \geq \varepsilon > 0$ for all $n \geq 0$, and $\sum_{n=0}^{\infty} \|e_n\| < \infty$, then for any given starting point $x_0 \in H$, the sequence (x_n) generated by algorithm (3.1) converges strongly to $p = A^{-1}(0)$. Indeed, if we denote $u_n := x_n - e_{n-1}$, and multiply

$$u_n - p + e_{n-1} \in u_{n+1} - p + \beta_n A u_{n+1}$$

scalarly by $(u_{n+1} - p)/\|u_{n+1} - p\|$, we get,

$$(1 + c\beta_n)\|u_{n+1} - p\| \quad \leq \quad \|u_n - p\| + \|e_{n-1}\|, \tag{3.8}$$

where c is the strong monotonicity constant of A. Therefore,

$$\varepsilon c \sum_{n=1}^{\infty} \|u_{n+1} - p\| \leq c \sum_{n=1}^{\infty} \beta_n \|u_{n+1} - p\| \leq \|u_1 - p\| + \sum_{n=0}^{\infty} \|e_n\| < \infty.$$

Clearly, this implies that $\|u_n - p\| \to 0$, and consequently affirms the strong convergence of (x_n) to p as claimed.

When $\beta_n \to \infty$, norm convergence of (e_n) to zero is enough to guarantee strong convergence of the sequence (x_n) to p. This follows easily on dividing (3.8) by $(1 + c\beta_n)$ and passing to the limit in the resulting inequality.

Remark 3.1.2. If in addition to the assumptions of Remark (3.1.1), $e_n = 0$ for all $n \geq 0$, then the sequence (x_n) does not only converge strongly to the unique point (of) $A^{-1}(0)$, but it does so at least as fast as the linear rate with coefficient $(1 + c\beta_n)^{-1} < 1$. The speed of convergence is improved to superlinearity if (β_n) tends to ∞ as n does[1].

In the case of the subdifferential, Theorem 3.1.2 can be proved under the weaker coercivity condition (2.8). More precisely, we have:

[1]The conclusions of Remark 3.1.2 also appear in an earlier paper of Rockafellar [45]. However, the arguments used there are quite different from ours.

Theorem 3.1.3. *Assume* $A = \partial\varphi$*, where* $\varphi : H \to (-\infty, +\infty]$ *is a proper, convex and lower semicontinuous function satisfying condition* (2.8)*. Let* $\sum_{n=0}^{\infty} \|e_n\|^2 < \infty$ *and* $\beta_n \geq \varepsilon > 0$ *for a given constant* ε *and all* $n \geq 0$*. Then for any given* $x_0 \in H$*, the sequence* (x_n) *generated by* (3.1) *is bounded.*

Proof. According to Theorem 1.10 of [39], there exists a point $p \in D(\varphi)$ such that $\varphi(p) = \inf_{x\in H} \varphi(x)$, that is, $F := A^{-1}(0)$ is nonempty. Denote $u_n := x_n - e_{n-1}$. Since

$$u_n - u_{n+1} + e_{n-1} \in \beta_n A u_{n+1},$$

we have from the definition of the subdifferential

$$\begin{aligned} 2\beta_n(\varphi(u_{n+1}) - \varphi(u_n)) &\leq 2\langle e_{n-1}, u_{n+1} - u_n\rangle - 2\|u_{n+1} - u_n\|^2 \\ &\leq \|e_{n-1}\|^2 - \|u_{n+1} - u_n\|^2 \\ &\leq \|e_{n-1}\|^2, \end{aligned}$$

for all $n \geq 1$. Therefore,

$$\varphi(u_{n+1}) \leq \varphi(u_n) + \frac{1}{2\varepsilon}\|e_{n-1}\|^2, \quad \text{for all} \quad n \geq 1.$$

By summing, we have

$$\varphi(u_{n+1}) \leq \varphi(u_1) + \frac{1}{2\varepsilon}\sum_{j=0}^{n-1}\|e_j\|^2 \leq \varphi(u_1) + \frac{1}{2\varepsilon}\sum_{j=0}^{\infty}\|e_j\|^2 < \infty.$$

It follows from (2.8) that (u_n) is bounded, and so is (x_n), since $e_n \to 0$. ☐

Remark 3.1.3. According to Theorem 3.6 of [39], we have under the assumptions of Theorem 3.1.3,

$$\varphi(u_{n+1}) = \varphi(x_{n+1} - e_n) \to \inf_{x\in H} \varphi(x).$$

This implies that every weak limit point of (x_n) belongs to $F := A^{-1}(0)$ (the set of all minimum points of φ). Therefore if F is singleton (which happens if, for example, φ is strictly convex), then (x_n) converges weakly to the (unique) minimum point, say p, of φ. If in addition, we assume that the "level sets" $\{v \in H : \varphi(v) \leq \lambda\}$, $\lambda \in \mathbf{R}$, are compact (which happens in many cases), then we have strong convergence: $\|x_n - p\| \to 0$.

3.2 Inexact Halpern-type proximal point algorithm

In the present section we consider the following algorithm which is a slight modification of Algorithm 5.1 of [54].

Algorithm 3.2.1. Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator.
Step 1. Choose $x_0, u \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose a regularization parameter $\beta_n > 0$ and compute

$$y_n = (I + \beta_n A)^{-1}(x_n) + e_n. \tag{3.9}$$

Step 3. For each $n \geq 0$, choose the relaxation parameter $\alpha_n \in (0,1)$ and compute the $(n+1)$th iterate:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n + e'_n, \tag{3.10}$$

where (e_n) and (e'_n) can be interpreted as sequences of computational errors. Here e_n is considered to be the "main error" while e'_n is a "smaller error". More precisely, we assume that there exists $K > 0$ such that $\|e'_n\| \leq K\|e_n\|$, for all $n \geq 0$.

Note that if the sequences (e_n) and (x_n) generated by the above algorithm are bounded, then $F := A^{-1}(0)$ is nonempty, provided $\beta_n \to \infty$.

Indeed, if we denote $u_n := y_n - e_n$, then by (3.10), (y_n) is bounded, and so is (u_n). Hence there exists a subsequence (u_{n_k}) which converges weakly to some $p \in H$. From (3.9) and (3.10), we derive

$$u_n + \beta_n A u_n \ni \alpha_{n-1} u + (1 - \alpha_{n-1})(u_{n-1} + e_{n-1}) + e'_{n-1}$$

which is equivalent to

$$A u_n \ni \frac{1}{\beta_n}(\alpha_{n-1} u - u_n + (1 - \alpha_{n-1})(u_{n-1} + e_{n-1}) + e'_{n-1}). \tag{3.11}$$

Since A is demiclosed, $u_{n_k} \rightharpoonup p$ and the right hand side of (3.11) converges strongly to zero, it follows that $(p, 0) \in A$, hence $F \neq \emptyset$. We have therefore proved that:

Proposition 3.2.1. *Assume that $A : D(A) \subset H \to 2^H$ is maximal monotone, and $\beta_n \to \infty$. If for any given $x_0, u \in H$, the sequences (e_n) and (x_n) defined by Algorithm 3.2.1 are bounded, then $F := A^{-1}(0)$ is nonempty.*

On the other hand, for a coercive operator A, it is immediate that the set $A^{-1}(0)$ is nonempty. If in addition, we assume that the sequence (e_n) is bounded and the sequence (β_n) is bounded below away from zero, then we can show that the sequence (x_n) generated by Algorithm 3.2.1 is itself bounded. We state this fact more formally in the following theorem whose proof is similar to the proof of Theorem 3.1.2.

Theorem 3.2.2. *Assume that $A : D(A) \subset H \to 2^H$ is maximal monotone and coercive. Let $\|e_n\| \leq C$ and $\beta_n \geq \varepsilon > 0$ for $n \geq 0$, where C and ε are given constants. Then for any given $x_0, u \in H$, the sequence (x_n) generated by Algorithm 3.2.1 is bounded.*

Taking $u = x_0$ and $e'_n = 0$ for all $n \geq 0$, the above algorithm reduces to Algorithm 5.1 [54]. On proving strong convergence in this case, the error sequence in Theorem 5.1 [54] is required to be summable, a condition too strong for computational purposes. We now address the question of whether or not can the summability of (e_n) be replaced by a weaker condition and still get strong convergence results. The result that we prove next is an extension and improvement of Theorem 5.1 [54].

Theorem 3.2.3. *Assume that*

(i) *Either* $(\|e_n\|) \in \ell^1$, $\alpha_n \in (0,1)$ *with* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(ii) *Or* $(\|e_n\|) \in \ell^p \backslash \ell^1$, $p \in (1,2)$, $\alpha_n \in (0,1)$ *with* $\alpha_n \geq \varepsilon \|e_n\|^{2-p}$ *for some* $\varepsilon > 0$*, and* $\alpha_n \to 0$.

If $A : D(A) \subset H \to 2^H$ *is maximal monotone with* $F := A^{-1}(0) \neq \emptyset$*, and* $\beta_n \to \infty$*, then the sequence* (x_n) *generated by Algorithm 3.2.1 converges strongly to* $P_F u$*, the projection of* u *on* F*.*

Proof. ([2]The first part of the proof is analogous to the proof of Theorem 5.1 [54]).

(i). Case $(\|e_n\|) \in \ell^1$, and $\alpha_n \in (0,1)$ satisfying, $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. We divide the proof into steps.

Step 1: Note that for any $p \in F$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|u - p\| + (1-\alpha_n)\|y_n - p\| + \|e'_n\| \\ &\leq \alpha_n \|u - p\| + (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\|e_n\| + \|e'_n\|, \end{aligned}$$

which implies that

$$\|x_{n+1} - p\| \leq \prod_{k=0}^{n}(1-\alpha_k)\|x_0 - p\| + \left[1 - \prod_{k=0}^{n}(1-\alpha_k)\right]\|u - p\| + \sum_{k=0}^{n}(\|e_k\| + \|e'_k\|),$$

showing that (x_n) is bounded, and so is (y_n).

Step 2: Denote $q := P_F u$. We want to show that $\limsup_{n\to\infty}\langle u - q, x_n - q\rangle \leq 0$. Take a subsequence (x_{n_k}) of (x_n) such that

$$\limsup_{n\to\infty} \langle u - q, x_n - q\rangle = \lim_{k\to\infty}\langle u - q, x_{n_k} - q\rangle.$$

Since (x_n) is bounded, (x_{n_k}) converges weakly on a subsequence, again denoted by (x_{n_k}), to some x_∞. Then it follows that

$$\limsup_{n\to\infty} \langle u - q, x_n - q\rangle = \langle u - q, x_\infty - q\rangle.$$

[2]The ideas involved in proving the boundedness of (x_n) were borrowed from the proof of Lemma 2.1.1

In view of Lemma 2.2.4, it remains to show that $x_\infty \in F$. Note that

$$\|x_{n+1} - y_n\| \leq \alpha_n \|u - y_n\| + \|e'_n\| \leq M\alpha_n + \|e'_n\| \to 0,$$

which implies that $y_{n_k-1} - e_{n_k-1} \rightharpoonup x_\infty$. On the other hand, since

$$A(y_{n_k-1} - e_{n_k-1}) \ni \frac{1}{\beta_{n_k-1}}(x_{n_k-1} - y_{n_k-1} + e_{n_k-1}) \to 0,$$

and because A is demiclosed, we have $x_\infty \in F$.

Step 3: Now we show that (x_n) converges strongly to $q = P_F u$. Applying the subdifferential inequality, we get

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n(u - q + e'_n) + (1-\alpha_n)(y_n - q + e'_n)\|^2 \\
&\leq (1-\alpha_n)\|y_n - q + e'_n\|^2 + 2\alpha_n\langle u - q + e'_n, x_{n+1} - q\rangle \\
&\leq (1-\alpha_n)(\|x_n - q\| + \|e_n\| + \|e'_n\|)^2 + 2\alpha_n\langle u - q + e'_n, x_{n+1} - q\rangle \\
&\leq (1-\alpha_n)\|x_n - q\|^2 + \alpha_n b_n + c_n,
\end{aligned}$$

where $c_n = (1+K)\|e_n\|(2\|x_n - q\| + (1+K)\|e_n\|)$ with $\sum_{n=0}^\infty c_n < \infty$ and from Step 2, $\limsup_{n\to\infty} b_n \leq 0$, where $b_n = 2\langle u - q + e'_n, x_{n+1} - q\rangle$. Hence it follows from Lemma 2.1.1 that $x_n \to q$.

(ii). Suppose that $\sum_{n=0}^\infty \|e_n\| = \infty$ and $\sum_{n=0}^\infty \|e_n\|^p < \infty$, for some $p \in (1,2)$. Denote $z_n := y_n - e_n$ and $\tilde{e}_n := (1-\alpha_n)e_n + e'_n$. Then we have from (3.9) and (3.10),

$$z_n = (I + \beta_n A)^{-1}(x_n) \quad \text{and} \quad x_{n+1} = \alpha_n u + (1-\alpha_n)z_n + \tilde{e}_n. \tag{3.12}$$

Let $p \in F$. Then, we have from the subdifferential inequality

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)^2\|z_n - p + \tilde{e}_n\|^2 + 2\alpha_n\langle u - p + \tilde{e}_n, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)^2(\|x_n - p\| + \|\tilde{e}_n\|)^2 + 2\alpha_n\langle u - p + \tilde{e}_n, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)^2(\|x_n - p\| + (1+K)\|e_n\|)^2 + 2M\alpha_n\|x_{n+1} - p\|,
\end{aligned}$$

where $M > 0$ is a constant such that $\|x_0 - p\| \leq M$, and $\|u - p\| + \|\tilde{e}_n\| \leq M$ for all $n \geq 0$.

Assume that $\|e_n\|$ is small enough for all $n \geq 0$, otherwise one can consider Algorithm 3.2.1 for $n \geq N$, with $x_0 := x_N$. We want to prove that for $C := 2M$, we have

$$\|x_n - p\| \leq C, \tag{3.13}$$

for all $n \geq 0$. Inequality (3.13) is clearly true for $n = 0$. Assume that it is true for some $n \geq 0$. Then, we have from the previous estimate,

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)^2(C + (1+K)\|e_n\|)^2 + 2M\alpha_n\|x_{n+1} - p\|.$$

Therefore,

$$(\|x_{n+1}-p\|-M\alpha_n)^2 \le M^2\alpha_n^2+(1-\alpha_n)^2(C+(1+K)\|e_n\|)^2,$$

which implies that

$$\|x_{n+1}-p\| \le M\alpha_n+\sqrt{M^2\alpha_n^2+(1-\alpha_n)^2(C+(1+K)\|e_n\|)^2}. \tag{3.14}$$

Now let us prove that

$$M\alpha_n+\sqrt{M^2\alpha_n^2+(1-\alpha_n)^2(C+(1+K)\|e_n\|)^2} \le C,$$

or equivalently,

$$M^2\alpha_n^2+(1-\alpha_n)^2(C+(1+K)\|e_n\|)^2 \le C^2-2MC\alpha_n+M^2\alpha_n^2,$$

or equivalently,

$$(1-\alpha_n)(C+(1+K)\|e_n\|)^2 \le C^2. \tag{3.15}$$

Since $\alpha_n \ge \varepsilon\|e_n\|^{2-p}$, to prove (3.15), it suffices to show that

$$(1-\varepsilon\|e_n\|^{2-p})(C+(1+K)\|e_n\|)^2 \le C^2,$$

or equivalently,

$$-\varepsilon C^2\|e_n\|^{2-p}+2C(1+K)\|e_n\|-2C(1+K)\varepsilon\|e_n\|^{3-p}+(1+K)^2\|e_n\|^2-(1+K)^2\varepsilon\|e_n\|^{4-p} \le 0,$$

or equivalently,

$$-\varepsilon C^2+2C(1+K)\|e_n\|^{p-1}-2C(1+K)\varepsilon\|e_n\|+(1+K)^2\|e_n\|^p-(1+K)^2\varepsilon\|e_n\|^2 \le 0,$$

which holds true because $\|e_n\|$ is small and $p>1$. Therefore from (3.14) and (3.15) we see that (3.13) holds true for $n+1$.

Step 2: As in Step 2 of the first part, we take a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \rightharpoonup x_\infty$ as $k\to\infty$, and

$$\limsup_{n\to\infty}\langle u-q, x_n-q\rangle = \lim_{k\to\infty}\langle u-q, x_{n_k}-q\rangle = \langle u-q, x_\infty-q\rangle.$$

On the other hand, for any $p\in F$, we have from (3.12)

$$\begin{aligned}\|x_{n+1}-z_n\| &\le \alpha_n(\|u-z_n\|)+\|\tilde{e}_n\| \le \alpha_n(\|u-p\|+\|z_n-p\|)+\|\tilde{e}_n\| \\ &\le \alpha_n(\|u-p\|+\|x_n-p\|)+\|\tilde{e}_n\| \le \alpha_n M+\|\tilde{e}_n\| \to 0,\end{aligned}$$

which implies that

$$z_{n_k-1} = (I + \beta_{n_k-1}A)^{-1}(x_{n_k-1}) \rightharpoonup x_\infty.$$

Therefore $(x_\infty, 0) \in A$, which implies that $x_\infty \in F$.

Step 3: Finally, we show that (x_n) converges strongly to $q = P_F u$. We have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq (1-\alpha_n)\|z_n - q + \tilde{e}_n\|^2 + 2\alpha_n\langle u - q + \tilde{e}_n, x_{n+1} - q\rangle \\
&\leq (1-\alpha_n)(\|x_n - q\| + \|\tilde{e}_n\|)^2 + 2\alpha_n\langle u - q + \tilde{e}_n, x_{n+1} - q\rangle \\
&\leq (1-\alpha_n)(\|x_n - q\| + (1+K)\|e_n\|)^2 + \frac{\alpha_n}{2}b_n \\
&= (1-\alpha_n)\|x_n - q\|^2 + (1+K)^2\|e_n\|^2 + \frac{\alpha_n}{2}b_n \\
&+ 2(1-\alpha_n)\left[\left(\sqrt{\frac{\varepsilon}{2}}\|x_n - q\|\|e_n\|^{1-\frac{p}{2}}\right)\left(\sqrt{\frac{2}{\varepsilon}}(1+K)\|e_n\|^{\frac{p}{2}}\right)\right] \\
&\leq (1-\alpha_n)\left(1 + \frac{\varepsilon}{2}\|e_n\|^{2-p}\right)\|x_n - q\|^2 + \frac{\alpha_n}{2}b_n + \frac{2}{\varepsilon}(1+K)^2\|e_n\|^p \\
&+ (1+K)^2\|e_n\|^2, \qquad (3.16)
\end{aligned}$$

where $b_n = 4\langle u - q + \tilde{e}_n, x_{n+1} - q\rangle$ with $\limsup_{n\to\infty} b_n \leq 0$. Set $a_n := \alpha_n/2$. Then $\sum_{n=0}^{\infty} a_n = \infty$, $a_n \to 0$ and

$$\alpha_n = a_n + \frac{1}{2}\alpha_n \geq a_n + \frac{\varepsilon}{2}\|e_n\|^{2-p}.$$

Therefore, we have from (3.16)

$$\|x_{n+1} - q\|^2 \leq (1 - a_n)\|x_n - q\|^2 + a_n b_n + c_n,$$

where $c_n = (1+K)^2(\|e_n\|^2 + 2\varepsilon^{-1}\|e_n\|^p)$ with $\sum_{n=0}^{\infty} c_n < \infty$ because $1 < p < 2$ and $\sum_{n=0}^{\infty}\|e_n\|^p < \infty$ implies that $\sum_{n=0}^{\infty}\|e_n\|^2 < \infty$. Hence it follows from Lemma 2.1.1 that (x_n) converges strongly to $q = P_F u$. □

Example 3.2.1. Let $\|e_n\| = (n+1)^{-2/3}$, $p = 5/3$, $\alpha_n = \|e_n\|^{2-p} = (n+1)^{-1/3}$ and $e'_n = 0$ for all $n \geq 0$. This case is not covered in Theorem 5.1 [54]. However according to Theorem 3.2.3, (x_n) as defined in Algorithm 3.2.1 converges strongly to $P_F u$, if $\beta_n \to \infty$.

While Algorithm 3.2.1 takes into account all the possible errors at each step, its disadvantage is that it might not be so convenient to work with. There is therefore a need to present it in a simplified version. To this end, let us note that from (3.9) and (3.10), we can rewrite the $(n+1)$th iterate as

$$x_{n+1} = \alpha_n u + (1-\alpha_n)y_n + e_n^*, \qquad (3.17)$$

where $e_n^* = e'_n + (1-\alpha_n)e_n$. Moreover, if $(\|e'_n\|), (\|e_n\|) \in \ell^p$, for $1 \leq p < \infty$, then $(\|e_n^*\|) \in \ell^p$ also. We may therefore assume that for each n, the computational errors occur only in the $(n+1)$th iterate. Thus, the algorithm under consideration becomes

Algorithm 3.2.2. Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator.
Step 1. Choose $x_0, u \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose a regularization parameter $\beta_n > 0$ and the relaxation parameter $\alpha_n \in (0, 1)$, then compute the $(n+1)$th iterate:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{\beta_n} x_n + e_n, \tag{3.18}$$

where (e_n) is a sequences of computational errors.

From now on, we will always assume that our algorithms have already been converted into a form similar to the one given above.

Although the requirement that the error sequence (e_n) is in ℓ^p for $1 \leq p < 2$ is weaker than summability condition (E1) (see below), it is still restrictive. The next result shows that one can further relax the condition on (e_n) without having to temper with the convergence properties of (x_n). More precisely, we assume that for non-summable errors, the sequence (e_n) satisfy condition (E2) defined below:

$$(E1) \sum_{n=0}^{\infty} \|e_n\| < \infty, \quad \text{or} \quad (E2) \lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0.$$

We point out that these two conditions are generally different. For example, $\|e_n\| = n^{-1}$ and $\alpha_n = 1/\sqrt{n}$ satisfy (E2) but not (E1), while $\|e_n\| = n^{-2}$ and $\alpha_n = n^{-1} + (-1)^n (n+1)^{-1}$ satisfy (E1) but fails to satisfy (E2). The advantage of using condition (E1) (and/or the condition $(e_n) \in \ell^p$ for $1 \leq p < 2$) over (E2) is that it allows one to choose freely the sequence of parameters (α_n). Despite that (E2) does not allow us such freedom, it is still a good condition since it covers the errors that are not summable, as shown in the above example. In fact, having any sequence of errors (e_n), converging strongly to zero, one can always construct the PPA that is strongly convergent by constructing a sequence of parameters (α_n) in such a way that condition (E2) is satisfied. The resulting α_n's depend on (e_n), but this is acceptable from the numerical point of view.

Theorem 3.2.4. *Assume that the conditions $\alpha_n \in (0, 1)$ with (C1), (C2), and either (E1) or (E2) are satisfied. If $A : D(A) \subset H \to 2^H$ is maximal monotone with $F := A^{-1}(0) \neq \emptyset$, and $\beta_n \to \infty$, then the sequence (x_n) generated by Algorithm 3.2.2 converges strongly to $P_F u$, the projection of u on F.*

Proof. We have already shown in the proof of Theorem 3.2.3 that if the error sequence (e_n) satisfies condition (E1), then (x_n) is bounded. In a similar way, we show that (x_n) is bounded when the error sequence satisfies condition (E2). Note that since $(\|e_n\| / \alpha_n)$ is bounded, there exists a positive constant K such that for $p \in F$,

$$\sup_{n \in \mathbb{N}_0} \left\{ \|u - p\| + \frac{\|e_n\|}{\alpha_n} \right\} \leq K.$$

From (3.18) and the fact that the resolvent operator is nonexpansive, we have for $p \in F$,

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)(J_{\beta_n}x_n - p) + \alpha_n(u - p + e_n/\alpha_n)\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \left[\|u-p\| + \frac{\|e_n\|}{\alpha_n}\right] \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n K. \end{aligned}$$

Applying induction, we get

$$\|x_{n+1} - p\| \leq \|x_0 - p\| \prod_{k=0}^{n}(1-\alpha_k) + K\left[1 - \prod_{k=0}^{n}(1-\alpha_k)\right],$$

showing that (x_n) is bounded.

A close inspection at the proof of Theorem 3.2.3 reveals that

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle \leq 0.$$

It remains to show that (x_n) converges strongly to $P_F u$. Note that from the subdifferential inequality, we have

$$\begin{aligned} \|x_{n+1} - P_F u\|^2 &= \|(1-\alpha_n)(J_{\beta_n}x_n - P_F u) + \alpha_n(u - P_F u + e_n/\alpha_n)\|^2 \\ &\leq (1-\alpha_n)\|x_n - P_F u\|^2 + 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, x_{n+1} - P_F u\right\rangle, \end{aligned}$$

so that for $\|e_n\|/\alpha_n \to 0$, we get at once via Lemma 2.1.1 the strong convergence of (x_n) to $P_F u$. In the case when (E1) is satisfied, we derive from subdifferential inequality

$$\|x_{n+1} - P_F u\|^2 \leq (1-\alpha_n)\|x_n - P_F u\|^2 + 2\alpha_n\langle u - P_F u, x_{n+1} - P_F u\rangle + M\|e_n\|,$$

for some positive constant M. Again in this case, Lemma 2.1.1 implies that $x_n \to P_F u$ as desired. □

Note that if $\|e_n\|/\alpha_n \to 0$ and $\sum_{n=0}^{\infty}\|e_n\| = \infty$, then automatically $\sum_{n=0}^{\infty}\alpha_n = \infty$. This shows that assumption (E2) covers the case when the error sequence is not summable. More precisely, we have

Corollary 3.2.5. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator with $F := A^{-1}(0) \neq \emptyset$. If $\|e_n\| \to 0$ and $\sum_{n=0}^{\infty}\|e_n\| = \infty$, then one can choose an appropriate sequence $(\alpha_n) \subset (0,1)$ such that the sequence (x_n) generated by Algorithm 3.2.2 converges strongly to $P_F u$, the projection of u on F, provided that $\beta_n \to \infty$.*

Proof. One can take, for example, $\alpha_n = \sqrt{\|e_n\|}$ if $e_n \neq 0$ and n is large enough, and $\alpha_n = 1/(n+2)$ otherwise. Obviously, $\alpha_n \in (0,1)$ for all $n \geq 0$, the conditions (C1), (C2) and (E2) of Theorem 3.2.4 are satisfied. This concludes the proof. □

We point out that practically every sequence (e_n) converging strongly to zero is good to obtain strong convergence of (x_n). Indeed, according to Theorem 3.2.4, if $\sum_{n=0}^{\infty} \|e_n\| < \infty$, then we can choose freely $\alpha_n \in (0,1)$ with $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Otherwise (i.e., if $\sum_{n=0}^{\infty} \|e_n\| = \infty$), we can choose for example (α_n) as in the proof of the above corollary to obtain strong convergence of (x_n). (Of course this choice is not unique). This conclusion is extremely important from the computational point of view. Note that in fact, in the case when condition (E2) of Theorem 3.2.4 is satisfied, which is not covered by the existing related results, we can build a good PPA by choosing appropriate regularization parameters α_n to keep the strong convergence of (x_n) under the general condition $\|e_n\| \to 0$. These α_n's depend on errors, but this is acceptable from the numerical point of view. On the other hand, Rockafellar showed that the original proximal point algorithm in its approximate form need not converge − even when H is one-dimensional − when one assumes that the sequence of errors (e_n) satisfies the weaker condition $\|e_n\| \to 0$ instead of the summability condition.

It is worth pointing out that in the particular case when A is the subdifferential of a proper, convex, lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, our result offers a reliable algorithm generating a sequence which approximates a minimum point of φ, provided that the error sequence converges to zero in norm. Indeed, in this case any point of $A^{-1}(0)$, in particular $P_F u$, is a minimum point of φ. We discuss this case further in Chapter 6.

We have found in Theorem 3.2.4 a more general condition on errors which guarantees strong convergence of (x_n) generated by Algorithm 3.2.2 under the assumption $\beta_n \to \infty$ (and of course the necessary conditions on α_n required for Algorithm 3.2.2 to converge strongly). This brings us to the following question: can one design a proximal point algorithm by choosing an appropriate sequence of regularization parameters (α_n) such that strong convergence of (x_n) is preserved, for $\|e_n\| \to 0$ and β_n bounded? One possible solution to this question is given in Theorem 3.2.7. Other affirmative answers to this question are given in Section 3.4.

Now let the mapping $h : H \to H$ be defined by $x \mapsto tu + (1-t)J_c x + e(t)$ for $c > 0$, $u \in H$ and $t \in (0,1)$, where $e = e(t)$ is a given function defined on $(0,1)$. For any fixed t (and c, u), one can easily check that the map h is a contraction with Lipschitz constant $1-t$. The Banach contraction principle asserts that h has a unique fixed point, say, z_t. That is,

$$z_t = tu + (1-t)J_c z_t + e(t) \quad \text{for} \quad c > 0 \quad \text{and} \quad u \in H. \tag{3.19}$$

In fact z_t depends on u and c as well. The main tool used in proving Theorem 3.2.7 is the following result

Theorem 3.2.6. *Take any $c > 0$ and $u \in H$, and assume that*

$$t^{-1}\|e(t)\| \to 0 \quad \text{as} \quad t \to 0^+. \tag{3.20}$$

If $F := A^{-1}(0) \neq \emptyset$, where $A : D(A) \subset H \to 2^H$ is a maximal monotone operator, then (z_t) defined in (3.19) converges strongly as $t \to 0^+$ to the point of F nearest to u, denoted by $P_F u$. Moreover, this limit is attained uniformly with respect to $c \geq \delta$ for every $\delta > 0$.

Proof. For every $p \in F$, we have from the subdifferential inequality (see Lemma 2.2.7 above)

$$\|z_t - p\|^2 \leq (1-t)^2\|z_t - p\|^2 + 2t\langle u - p + t^{-1}e(t), z_t - p\rangle.$$

In other words,

$$(2-t)\|z_t - p\|^2 \leq 2\langle u - p + t^{-1}e(t), z_t - p\rangle. \tag{3.21}$$

This shows that (z_t) is bounded as $t \to 0^+$. Now setting

$$v_t := (1-t)^{-1}(z_t - tu - e(t)) = J_c z_t,$$

we see that (v_t) is also bounded as $t \to 0^+$, and the weak ω-limit sets of (z_t) and (v_t) (as $t \to 0^+$) coincide, that is, $\omega_{\mathtt{w}}((z_t)) = \omega_{\mathtt{w}}((v_t))$. Since

$$Av_t \ni c^{-1}(z_t - v_t) \to 0 \quad \text{as } t \to 0^+,$$

we have $\omega_{\mathtt{w}}((z_t)) \subset F$. By (3.20) and (3.21) with $p = P_F u$, we get

$$\limsup_{t \to 0^+} \|z_t - P_F u\|^2 \leq 0,$$

which shows that $z_t \to P_F u$ as $t \to 0^+$. Obviously, the above limit is attained uniformly with respect to $c \geq \delta$ for every $\delta > 0$. □

Remark 3.2.1. Theorem 3.2.6 is an extension of Theorem 3.1 in [55], since v_t converges strongly to $P_F u$ (as $t \to 0^+$) if and only if z_t does. We note that Theorem 3.1 in [55] contains a mistake, since the strong limit of v_t (as $t \to 0^+$) is not attained uniformly for $c > 0$ (but for $c \geq \delta$ for every $\delta > 0$).

We now prove a strong convergence result satisfying similar conditions to those of Lions [33]. One of the conditions

$$(C3)' \lim_{n \to \infty} \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2} = 0,$$

is weaker than Lions' condition (C3) in the case when α_n is decreasing.

Theorem 3.2.7. *Assume that* $A : D(A) \subset H \to 2^H$ *is a maximal monotone operator and* $F := A^{-1}(0) \neq \emptyset$. *For any fixed* $u, x_0 \in H$, *let* (x_n) *be the sequence generated by Algorithm 3.2.2 with the conditions:* (i) $\alpha_n \in (0,1)$, *(C1), (C2) and (C3)′*, (ii) *either (E1) or (E2), and* (iii) $\beta_n \in (0,\infty)$ *with (C6)′* $\lim_{n\to\infty} \beta_n = \beta$ *for some* $\beta > 0$, *being satisfied. Then* (x_n) *converges strongly to* $P_F u$, *the projection of* u *on* F.

Proof. We already know from the proof of Theorem 3.2.4 that (x_n) is bounded. For each n, let z_n be the unique fixed point of the contraction $x \mapsto \alpha_n u + (1-\alpha_n) J_\beta x$. According to Theorem 3.2.6, $z_n \to P_F u$ as $n \to \infty$. Therefore it is enough to show that $\|x_n - z_n\| \to 0$ as $n \to \infty$. For this purpose, we estimate $\|x_{n+1} - z_{n+1}\|$ as follows:

$$\|x_{n+1} - z_{n+1}\| \leq \|x_{n+1} - z_n\| + \|z_n - z_{n+1}\|. \tag{3.22}$$

Noting that $z_n = \alpha_n u + (1-\alpha_n) J_\beta z_n$ and the fact that J_β is nonexpansive for all $\beta > 0$, we get

$$\begin{aligned}
\|x_{n+1} - z_n\| &\leq (1-\alpha_n)\|J_{\beta_n} x_n - J_\beta z_n\| + \|e_n\| \\
&\leq (1-\alpha_n)\|J_{\beta_n} x_n - J_{\beta_n} z_n\| + \|J_{\beta_n} z_n - J_\beta z_n\| + \|e_n\| \\
&\leq (1-\alpha_n)\|x_n - z_n\| + \frac{|\beta - \beta_n|}{\beta}\|z_n - J_\beta z_n\| + \|e_n\| \\
&\leq (1-\alpha_n)\|x_n - z_n\| + \alpha_n \frac{|\beta - \beta_n|}{\beta}\|u - J_\beta z_n\| + \|e_n\|, \qquad (3.23)
\end{aligned}$$

where the third inequality follows from the application of the resolvent identity. On the other hand, we compare z_n and z_{n+1} as follows

$$\begin{aligned}
\|z_n - z_{n+1}\| &= \|(\alpha_n - \alpha_{n+1})(u - J_\beta z_{n+1}) + (1-\alpha_n)(J_\beta z_n - J_\beta z_{n+1})\| \\
&\leq |\alpha_n - \alpha_{n+1}|\|u - J_\beta z_{n+1}\| + (1-\alpha_n)\|z_n - z_{n+1}\|,
\end{aligned}$$

which gives

$$\|z_n - z_{n+1}\| \leq \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} K, \tag{3.24}$$

where K is a positive constant such that $\|u - J_\beta z_n\| \leq K$ for all n. Combining (3.22), (3.23) and (3.24), we either get

$$\|x_{n+1} - z_{n+1}\| \leq (1-\alpha_n)\|x_n - z_n\| + \alpha_n \left[\frac{|\beta - \beta_n|}{\beta} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2}\right] K + \|e_n\|,$$

in the case when (e_n) satisfies condition (E1), or

$$\|x_{n+1} - z_{n+1}\| \leq (1-\alpha_n)\|x_n - z_n\| + \alpha_n \left\{\left[\frac{|\beta - \beta_n|}{\beta} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2}\right] K + \frac{\|e_n\|}{\alpha_n}\right\},$$

for the case $\|e_n\|/\alpha_n \to 0$. In either case, Lemma 2.1.1 gives the required conclusion. □

Remark 3.2.2. For $\beta > 0$ and $\beta_n = \beta + (-1)^n/(n+1)$, the condition (C6)′ is satisfied, whereas (C9) is not, showing that our condition on β_n is weaker than the one used in the following theorem due to Xu [55]. On the other hand, the sequences $\alpha_n = n^{-3/4}$ and $\alpha_n = 1/\ln n$ satisfy condition (i) of Theorem 3.2.7. Since (C3) and (C3)′ are not comparable to (C4) (see Remark 3.1 [54]), Theorem 3.2.7 is new.

Theorem 3.2.8 (Xu [55])**.** *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let (x_n) be the sequence generated by Algorithm 3.2.2 with the conditions:* (i) *$\alpha_n \in (0,1)$, (C1), (C2) and (C4),* (ii) *$\beta_n \in (0,\infty)$ with $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $0 < \liminf_{n\to\infty} \beta_n \ (= \lim_{n\to\infty} \beta_n)$, being satisfied. If $\sum_{n=0}^{\infty} \|e_n\| < \infty$, then (x_n) converges strongly to $P_F u$, the projection of u on F.*

Remark 3.2.3. Although it appears from Lemma 2.1.1 and inequality (3.24) that

$$\sum_{n=0}^{\infty} \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} < \infty$$

can be a possible assumption on α_n, there is no sequence $(\alpha_n) \subset (0,1)$ satisfying (C1) and this condition. Indeed, if this condition is satisfied, then Lemma 2.1.2 would imply that α_n is bounded below away from zero, contradicting (C1).

3.3 Viscosity approximation methods

In a recent paper of Takahashi [51], a strong convergence theorem of a modified proximal point method was proved for resolvents of accretive operators in Banach spaces by the so called viscosity approximation method without taking into account the error terms. For this method, the $(n+1)$th iterate was given as

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)(I + \beta_n A)^{-1}(x_n),$$

where $\alpha_n \in (0,1)$, $\beta_n \in (0,\infty)$, $f : C \to C$ is a strict contraction (a-contraction with $0 < a < 1$) defined on a nonempty closed convex subset C of a reflexive Banach space X, and $A : D(A) \subset X \to 2^X$ is an m-accretive operator. In a Hilbert space setting, an analogue of the above mentioned theorem can also be proved even when one takes into account the error terms. Indeed, having a sequence (x_n) conforming to the iterative process

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)(I + \beta_n A)^{-1}(x_n) + e_n, \tag{3.25}$$

one proves the following result whose proof relies on the ideas contained in the proof of Theorem 4.2 [51].

Theorem 3.3.1. *Let $f : H \to H$ be a strict contraction with Lipschitz constant $a \in (0,1)$, and let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator with $F := A^{-1}(0) \neq \emptyset$. Assume that the conditions: $\alpha_n \in (0,1)$ with (C1) and (C2), either (E1) or (E2), and $\beta_n \to \infty$ are satisfied. Then given any $x_0 \in H$, the sequence (x_n) generated by Algorithm 3.25 converges strongly to the unique fixed point z of $P_F \circ f$, that is $z = P_F f(z)$.*

Proof. Fix $p \in A^{-1}(0)$ and set $M = \max\{\|x_0 - p\|, (1-a)^{-1}\|f(p) - p\|\}$. We show by induction that for any $n \geq 0$,

$$\|x_n - p\| \leq M + \sum_{k=0}^{n-1} \|e_k\|. \tag{3.26}$$

For $n = 0$, (3.26) is clearly true. Assume that (3.26) holds for some $n \geq 0$. We show that it also holds for $n+1$. For $p \in F$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|J_{\beta_n} x_n - p\| + \|e_n\| \\
&\leq \alpha_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) + (1-\alpha_n)\|x_n - p\| + \|e_n\| \\
&\leq (1 - \alpha_n(1-a))\|x_n - p\| + \alpha_n \|f(p) - p\| + \|e_n\| \\
&= (1 - \alpha_n(1-a))\|x_n - p\| + \alpha_n(1-a)\frac{1}{1-a}\|f(p) - p\| + \|e_n\| \\
&\leq (1 - \alpha_n(1-a))\left[M + \sum_{k=0}^{n-1}\|e_k\|\right] + \alpha_n(1-a)M + \|e_n\| \\
&\leq M + \sum_{k=0}^{n}\|e_k\|.
\end{aligned}$$

Now assume that $(\|e_n\|/\alpha_n)$ is bounded. Then there exists a constant C such that

$$\sup_{n \in \mathbb{N}_0} \left\{ \|x_0 - p\| + \frac{1}{1-a}\left(\|f(p) - p\| + \frac{\|e_n\|}{\alpha_n} \right) \right\} \leq C.$$

We show by induction that for any $n \geq 0$,

$$\|x_n - p\| \leq C. \tag{3.27}$$

Obviously this inequality is satisfied for $n = 0$. Assume that it also holds true for some $n \geq 0$. Then from

$$\|x_{n+1} - p\| \leq (1 - \alpha_n(1-a))\|x_n - p\| + \alpha_n(1-a)\left\{ \frac{1}{1-a}\left(\|f(p) - p\| + \frac{\|e_n\|}{\alpha_n} \right) \right\},$$

we can readily conclude that (3.27) holds true for $n+1$.

Therefore, we can find a subsequence (x_{n_k}) of (x_n) converging weakly to some v, such that

$$\limsup_{n\to\infty}\langle f(z)-z, x_n - z\rangle = \lim_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle = \langle f(z)-z, v-z\rangle.$$

So it only remains to show that $v\in F$. For this purpose, we note that

$$\|x_{n+1}-(I+\beta_n A)^{-1}(x_n)\| \le \alpha_n\|f(x_n)-(I+\beta_n A)^{-1}(x_n)\| + \|e_n\| \to 0.$$

As in the proof of Theorem 3.2.3, we deduce that $v\in F$, hence

$$\limsup_{n\to\infty}\langle f(z)-z, x_n-z\rangle \le 0.$$

Finally, we show that $\|x_n - z\|\to 0$. Applying the subdifferential inequality, we have,

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\alpha_n)^2\|J_{\beta_n}x_n - z + e_n\|^2 + 2\alpha_n\langle f(x_n)-z+e_n, x_{n+1}-z\rangle\\ &\le (1-\alpha_n)^2(\|x_n-z\|^2 + \|e_n\|(\|e_n\|+2\|x_n-z\|)) +\\ &\quad 2\alpha_n\langle f(x_n)-f(z), x_{n+1}-z\rangle + 2\alpha_n\langle f(z)-z+e_n, x_{n+1}-z\rangle.\end{aligned}$$

Since $(\|e_n\|)$ and (x_n) are bounded, for all $n\in\mathbb{N}_0$, we have $\|e_n\|+2\|x_n-z\|\le K$ for some positive constant K. Therefore,

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\alpha_n)^2\|x_n-z\|^2 + K\|e_n\| + 2\alpha_n\langle f(z)-z+e_n, x_{n+1}-z\rangle +\\ &\quad 2a\alpha_n\|x_n-z\|\|x_{n+1}-z\|\\ &\le (1-\alpha_n)^2\|x_n-z\|^2 + K\|e_n\| + 2\alpha_n\langle f(z)-z+e_n, x_{n+1}-z\rangle +\\ &\quad a\alpha_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2),\end{aligned}$$

which implies that

$$\begin{aligned}(1-a\alpha_n)\|x_{n+1}-z\|^2 &\le (1-2\alpha_n+a\alpha_n)\|x_n-z\|^2 + K\|e_n\| + \alpha_n^2\|x_n-z\|^2 +\\ &\quad 2\alpha_n\langle f(z)-z+e_n, x_{n+1}-z\rangle,\end{aligned}$$

or equivalently,

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \left(1-\frac{2(1-a)\alpha_n}{1-a\alpha_n}\right)\|x_n-z\|^2 + \frac{K\|e_n\|}{1-a\alpha_n} +\\ &\quad \frac{2(1-a)\alpha_n}{1-a\alpha_n}\left(\frac{\alpha_n M'}{2(1-a)} + \frac{1}{1-a}\langle f(z)-z+e_n, x_{n+1}-z\rangle\right)\\ &\le (1-a_n)\|x_n-z\|^2 + a_n b_n + c_n,\end{aligned}$$

where $M'\ge \sup_{n\in\mathbb{N}_0}\|x_n-z\|^2$, and

$$c_n = \frac{K\|e_n\|}{1-a},\ a_n = \frac{2(1-a)\alpha_n}{1-a\alpha_n},\ b_n = \frac{\alpha_n M'}{2(1-a)} + \frac{1}{1-a}\langle f(z)-z+e_n, x_{n+1}-z\rangle,$$

with $\sum_{n=0}^{\infty} a_n = \infty$ and $\lim_{n\to\infty} a_n = 0$. Hence from Lemma 2.1.1, we have $\|x_n - z\| \to 0$.

Note that in the case when $\|e_n\|/\alpha_n \to 0$, then the subdifferential inequality yields

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2\|J_{\beta_n}x_n - z\|^2 + 2\alpha_n\langle f(x_n) - z + e_n/\alpha_n, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle \\ &+ 2\alpha_n\langle f(z) - z + e_n/\alpha_n, x_{n+1} - z\rangle.\end{aligned}$$

Employing similar computations as above, we arrive at

$$\|x_{n+1} - z\|^2 \leq (1-a_n)\|x_n - z\|^2 + a_n b'_n,$$

where

$$a_n = \frac{2(1-a)\alpha_n}{1-a\alpha_n}, \quad \text{and} \quad b'_n = \frac{\alpha_n C'}{2(1-a)} + \frac{1}{1-a}\left\langle f(z) - z + \frac{e_n}{\alpha_n}, x_{n+1} - z\right\rangle,$$

for some $C' > 0$. Again the conclusion follows on applying Lemma 2.1.1. □

Let us note that Theorem 3.3.1 is a generalization of Theorem 5.1 [54]. Motivated by Takahashi's result, or rather Theorem 3.3.1, we explore the case when $f : H \to H$ is a nonexpansive map. In that case, we expect to generate a sequence which converges weakly to the point of $A^{-1}(0)$. An interesting case occurs when f is the projection mapping, in which case the generated sequence is strongly convergent (of course under some appropriate assumptions). Strictly speaking, our interest lies in the case when $f := f_n$ is any sequence of nonexpansive maps, particularly when such a sequence (f_n) is taken to be the sequence of resolvent operators. This case allows us to construct a strongly convergent sequence by choosing an appropriate sequence of regularization parameters, say (λ_n).

To motivate our discussion, we begin by showing that the trajectory of the PPA given by the following algorithm that starts at any given point x_0 approaches a zero of A (that is, a fixed point of J_{λ_n} for all $\lambda_n > 0$) if λ_n grows large without bound as n does. Such a zero is characterized among others by the property that it is closest to u in the solution set $A^{-1}(0)$.

Algorithm 3.3.1. Let $f_n : H \to H$ be a sequence of nonexpansive maps, and let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator.
Step 1. Choose $x_0, u \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose the regularization parameter $\lambda_n > 0$ and the relaxation parameter $\alpha_n \in (0,1)$. Then compute the $(n+1)$th iterate:

$$x_{n+1} = \alpha_n(I + \lambda_n A)^{-1}u + (1-\alpha_n)f_n(x_n) + e_n,$$

where (e_n) is a sequence of computational errors.

Theorem 3.3.2. *Assume that either (E1) or (E2) is satisfied. Let* $f_n : H \to H$ *be a sequence of nonexpansive maps, and* $A : D(A) \subset H \to 2^H$ *be maximal monotone with* $\emptyset \neq F := A^{-1}(0) \subset \bigcap_n F(f_n)$. *If* $\alpha_n \in (0,1)$ *with* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lambda_n \in (0,\infty)$ *and* $\lambda_n \to \infty$, *then given any* $x_0, u \in H$, *the sequence* (x_n) *generated by Algorithm 3.3.1 converges strongly to* $P_F u$, *the projection of* u *on* F. *Here* $F(f_n)$ *denotes the set of all fixed points of* f_n.

Proof. Note that (x_n) is bounded[3]. Setting $q = P_F u$, we have from the subdifferential inequality

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(f_n(x_n) - q + e_n) + \alpha_n((I+\lambda_n A)^{-1}u - q + e_n)\|^2 \\
&\leq (1-\alpha_n)\|f_n(x_n) - q + e_n\|^2 + 2\alpha_n\langle (I+\lambda_n A)^{-1}u - q + e_n, x_{n+1} - q\rangle \\
&\leq (1-\alpha_n)(\|x_n - q\| + \|e_n\|)^2 + M\alpha_n\|(I+\lambda_n A)^{-1}u - q + e_n\| \\
&\leq (1-\alpha_n)\|x_n - q\|^2 + M\alpha_n\|(I+\lambda_n A)^{-1}u - q + e_n\| + K\|e_n\|,
\end{aligned}
$$

for some positive constants K and M. Hence by Lemma 2.1.1, we derive $\|x_n - q\| \to 0$.

In a similar manner, we can show that (x_n) converges strongly to the metric projection of u on F when (e_n) satisfies condition (E2). □

In the case when $f_n = (I + \beta_n B)^{-1}$, we can show that if (e_n) is bounded and (β_n) is bounded below away from zero, then the sequence (x_n) generated by Algorithm 3.3.1 with $f_n = (I + \beta_n B)^{-1}$ is bounded, provided B is assumed to be coercive. (See Section 3.1).

Theorem 3.3.3. *Assume that* $A : D(A) \subset H \to 2^H$, $B : D(B) \subset H \to 2^H$ *are maximal monotone operators and* B *is coercive with* $\emptyset \neq F := A^{-1}(0) \subset B^{-1}(0)$. *Let* $\|e_n\| \leq C$ *and* $\beta_n \geq \varepsilon > 0$ *for* $n \geq 0$, *where* C *and* ε *are given constants. Then for any given* $x_0, u \in H$, *the sequence* (x_n) *generated by Algorithm 3.3.1 with* $f_n = (I + \beta_n B)^{-1}$ *is bounded.*

As a consequence of the above result, we have

Corollary 3.3.4. *Assume that* $A : D(A) \subset H \to 2^H$ *is maximal monotone and coercive. Let* $\|e_n\| \leq C$ *and* $\beta_n \geq \varepsilon > 0$ *for* $n \geq 0$, *where* C *and* ε *are given constants. Then for any given* $x_0, u \in H$, *the sequence* (x_n) *generated by Algorithm 3.3.1 with* $f_n = (I + \beta_n A)^{-1}$ *is bounded.*

We now discuss in detail the following relaxed algorithm.

Algorithm 3.3.2. Let $f_n : H \to H$ be a sequence of nonexpansive maps, and let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator.

[3] This fact can be derived directly from the proof of Theorem 3.2.3 by noting that the resolvent operator is nonexpansive.

Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose the regularization parameter $\beta_n > 0$ and the relaxation parameter $\alpha_n \in (0,1)$. Then compute the $(n+1)$th iterate:

$$x_{n+1} = \alpha_n f_n(x_n) + (1-\alpha_n)(I+\beta_n A)^{-1}(x_n) + e_n,$$

where (e_n) is a sequence of computational errors.

In the next theorem we generalize Theorem 5.2 [54]. Note that if $f_n = 0$ for all $n \geq 0$, we are in the case (3.9), (3.10) with $u = 0$.

Claim: If $p \in \bigcap_n F(f_n)$ and $p \in F := A^{-1}(0)$, then (x_n) is bounded, provided that (e_n) satisfies condition (E1). Indeed,

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f_n(x_n) - p\| + (1-\alpha_n)\|J_{\beta_n} x_n - p\| + \|e_n\| \\ &\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| + \|e_n\| \\ &= \|x_n - p\| + \|e_n\|, \end{aligned}$$

which implies that

$$\|x_{n+1} - p\| - \sum_{k=0}^{n} \|e_k\| \leq \|x_n - p\| - \sum_{k=0}^{n-1} \|e_k\|.$$

Therefore, the sequence $(\|x_n - p\|)$ is convergent, hence (x_n) is bounded.

If in addition, $\bigcap_n F(f_n) \supset F$, then $(\|x_n - p\|)$ converges for all $p \in F$. Moreover, if $\alpha_n \to 0$ and $\beta_n \to \infty$, then $\omega_{\mathtt{w}}((x_n)) \subset F$, so that Opial's lemma guarantees the weak convergence of (x_n) to a point of F.

We have thus proved the following result.

Theorem 3.3.5. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator with $\emptyset \neq A^{-1}(0) \subset \bigcap_n F(f_n)$, where $f_n : H \to H$ is a sequence of nonexpansive maps, and $F(f_n)$ is the fixed point set of f_n. Assume that $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\alpha_n \in (0,1)$ with $\alpha_n \to 0$, and $\beta_n \to \infty$. Then given any $x_0 \in H$, the sequence (x_n) generated by Algorithm 3.3.2 converges weakly to some point $q \in A^{-1}(0)$.*

The following remarks are now in order.

1). If $f_n := f = I$ for all $n \geq 0$, then $F(f) \supset F$, and hence Algorithm 3.3.2 reduces to Algorithm 5.2 of [54].

2). Note that the Yosida approximation of $A : D(A) \subset H \to 2^H$, $A_\lambda : H \to H$, is nonexpansive for $\lambda = 1$. In this case, if $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\beta_n \to \infty$, either condition (E1) or (E2) is satisfied, and $F = A^{-1}(0) \neq \emptyset$, then we again get weak convergence for the sequence generated by Algorithm 3.3.2 with $f_n = A_1$ for all $n \geq 0$. However, this result is weaker than Theorem 3.3.5.

Indeed, for $q \in F$, we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n(A_1(x_n) - A_1(q) - q) + (1-\alpha_n)(J_{\beta_n}x_n - q) + e_n\| \\ &\leq \alpha_n\|x_n - q\| + \alpha_n\|q\| + (1-\alpha_n)\|x_n - q\| + \|e_n\| \\ &= \|x_n - q\| + \alpha_n\|q\| + \|e_n\|, \end{aligned}$$

which implies that

$$\|x_{n+1} - q\| - \sum_{k=0}^{n}(\alpha_k\|q\| + \|e_k\|) \leq \|x_n - q\| - \sum_{k=0}^{n-1}(\alpha_k\|q\| + \|e_k\|),$$

showing that $\lim_{n\to\infty}\|x_n - q\| = \rho(q)$ for all $q \in F$. Therefore (x_n) is bounded.

In the case when $(\|e_n\|/\alpha_n)$ is bounded, we have for each $q \in F$

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n(A_1(x_n) - A_1(q) - q + e_n/\alpha_n) + (1-\alpha_n)(J_{\beta_n}x_n - q)\| \\ &\leq \alpha_n\|x_n - q\| + \alpha_n\left\{\|q\| + \frac{\|e_n\|}{\alpha_n}\right\} + (1-\alpha_n)\|x_n - q\| \\ &\leq \|x_n - q\| + \alpha_n M, \end{aligned}$$

for some positive constant M. Again we derive in a similar way that (x_n) is bounded.

Moreover,

$$\begin{aligned} \|x_{n+1} - J_{\beta_n}x_n\| &\leq \alpha_n\|A_1(x_n) - A_1(q)\| + \alpha_n\|q\| + \alpha_n\|q - J_{\beta_n}x_n\| + \|e_n\| \\ &\leq 2\alpha_n\|x_n - q\| + \alpha_n\|q\| + \|e_n\| \to 0. \end{aligned}$$

Consequently, if $x_{n_k} \rightharpoonup x_\infty$, then $x_\infty \in A^{-1}(0)$. Hence by Opial's lemma, there exists $p \in F$ such that $x_n \rightharpoonup p$.

It is easy to see that this result holds for any sequence of nonexpansive maps, hence the following theorem.

Theorem 3.3.6. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator with $F := A^{-1}(0) \neq \emptyset$, and $f_n : H \to H$ be a sequence of nonexpansive maps. Assume that $\alpha_n \in (0,1)$, with $\sum_{n=0}^{\infty}\alpha_n < \infty$, either (E1) or (E2) holds, and $\beta_n \to \infty$. Then given any $x_0 \in H$, the sequence (x_n) generated by Algorithm 3.3.2 converges weakly to some point $q \in F$.*

3). **Special Case:** $f_n := f = P_F$, where $F = A^{-1}(0) \neq \emptyset$. In this case we have strong convergence.

Theorem 3.3.7. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator with $F := A^{-1}(0) \neq \emptyset$. Assume that $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\alpha_n \in (0,1)$, $\alpha_n \to 0$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\beta_n \to \infty$. Then for every $x_0 \in H$, the sequence (x_n) generated by Algorithm 3.3.2 with $f_n = P_F$ (for all $n \geq 0$) converges strongly to some point $q \in F$.*

Proof. We first show that $v_n = P_F x_n$ is strongly convergent (to some $q \in F$). Note that

$$\begin{aligned} \|x_{n+m} - v_n\| &\leq \alpha_{n+m-1}\|x_{n+m-1} - v_n\| + (1-\alpha_{n+m-1})\|x_{n+m-1} - v_n\| + \|e_{n+m-1}\| \\ &= \|x_{n+m-1} - v_n\| + \|e_{n+m-1}\| \\ &\leq \|x_n - v_n\| + \sum_{k=n}^{n+m-1} \|e_k\|, \end{aligned} \tag{3.28}$$

which implies that

$$\|x_{n+m} - v_{n+m}\| \leq \|x_n - v_n\| + \sum_{k=n}^{n+m-1} \|e_k\|.$$

In particular, $(\|x_n - v_n\|)$ is convergent. By the parallelogram law applied to $v_{n+m} - x_{n+m}$ and $v_n - x_{n+m}$,

$$\|v_{n+m} - v_n\|^2 + \|2x_{n+m} - (v_n + v_{n+m})\|^2 = 2(\|x_{n+m} - v_{n+m}\|^2 + \|x_{n+m} - v_n\|^2).$$

Therefore, using (3.28), we have

$$\begin{aligned} \|v_{n+m} - v_n\|^2 + 4\|x_{n+m} - v_{n+m}\|^2 &\leq 2\left(\|x_{n+m} - v_{n+m}\|^2 + \|x_{n+m} - v_n\|^2\right) \\ &\leq 2\|x_{n+m} - v_{n+m}\|^2 + 2\left(\|x_n - v_n\| + \sum_{k=n}^{\infty} \|e_k\|\right)^2, \end{aligned}$$

which implies that

$$\|v_{n+m} - v_n\|^2 \leq -2\|x_{n+m} - v_{n+m}\|^2 + 2\left(\|x_n - v_n\| + \sum_{k=n}^{\infty} \|e_k\|\right)^2.$$

Thus (v_n) is Cauchy, hence converges strongly to some $q \in F$. In addition, for some positive constants K and M, we have from the subdifferential inequality

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1-\alpha_n)^2\|J_{\beta_n} x_n - q + e_n\|^2 + 2\alpha_n\langle P_F x_n - q + e_n, x_{n+1} - q\rangle \\ &\leq (1-\alpha_n)(\|x_n - q\| + \|e_n\|)^2 + M\alpha_n\|P_F x_n - q + e_n\| \\ &\leq (1-\alpha_n)\|x_n - q\|^2 + M\alpha_n\|P_F x_n - q + e_n\| + K\|e_n\|. \end{aligned}$$

Hence by Lemma 2.1.1, we see that (x_n) converges strongly to q. □

4). If $f_n = f = (I + \lambda A)^{-1}$ for all $n \geq 0$ and $\lambda > 0$, then $F(f) = F := A^{-1}(0)$, and again we obtain weak convergence of (x_n) under the assumptions of Theorem 3.3.5.

5). If $f_n = (I + \lambda_n A)^{-1}$, for $\lambda_n > 0$, then $F(f_n) = F := A^{-1}(0)$ for all $n \geq 0$, so we have the following algorithm:

Algorithm 3.3.3. Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator.
Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose the regularization parameters $\beta_n, \lambda_n > 0$ and the relaxation parameter $\alpha_n \in (0, 1)$. Then compute the $(n + 1)$th iterate:

$$x_{n+1} = \alpha_n (I + \lambda_n A)^{-1}(x_n) + (1 - \alpha_n)(I + \beta_n A)^{-1}(x_n) + e_n,$$

where (e_n) is a sequence of computational errors.

Note that for $q \in F := A^{-1}(0)$, we have

$$\|x_{n+1} - q\| \leq \|x_n - q\| + \|e_n\|.$$

So if $\sum_{n=0}^{\infty} \|e_n\| < \infty$ we deduce that $\lim_{n\to\infty} \|x_n - q\| = \rho(q)$ for all $q \in F$. Since $\omega_{\mathtt{w}}((x_n)) \subset F$ for $\beta_n \to \infty$ and $\alpha_n \to 0$, Opial's lemma guarantees the weak convergence of (x_n) to a point of F.

Having in mind that $(I + \lambda A)^{-1}x \to P_F x \quad \text{as} \quad \lambda \to \infty$ (see for example, [39, Theorem 1.3, p. 21]), it is expected that the sequence (x_n) generated by the above algorithm converges strongly, if both $\lambda_n, \beta_n \to \infty$. However, it turns out that only the assumption $\lambda_n \to \infty$ is enough to guarantee strong convergence. Our aim now is to construct a sequence of parameters (λ_n) such that for very large n, the corresponding sequence (x_n) as given by the algorithm in question converges strongly to a point of $F := A^{-1}(0)$. We then have the following modified algorithm.

Algorithm 3.3.4. Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator.
Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose the regularization parameter $\beta_n > 0$ and compute

$$y_n = (I + \beta_n A)^{-1}(x_n).$$

Step 3. Choose another regularization parameter λ_n large enough such that

$$\|(I + \lambda_n A)^{-1}(x_n) - P_F x_n\| < \frac{1}{n}, \quad \text{and compute} \quad z_n = (I + \lambda_n A)^{-1}(x_n). \tag{3.29}$$

Step 4. For $n \geq 0$, choose a relaxation parameter $\alpha_n \in (0, 1)$ and compute the $(n + 1)$th iterate:

$$x_{n+1} = \alpha_n z_n + (1 - \alpha_n) y_n + e_n,$$

where (e_n) is a sequence of computational errors.

Theorem 3.3.8. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator with $\emptyset \neq F := A^{-1}(0)$. Assume that the conditions $\alpha_n \in (0,1)$ with (C1), (C2) and (E1) are satisfied. Then given any $x_0 \in H$, the sequence (x_n) constructed by the above algorithm converges strongly to some $p \in F$.*

Proof. Note that for any $q \in F$, the sequence $(\|x_n - q\|)$ is convergent, (and hence bounded), since

$$\|x_{n+1} - q\| \leq \|x_n - q\| + \|e_n\|.$$

Now denote $v_n := P_F x_n$. As in Theorem 3.3.7, we derive strong convergence of (v_n) to some point $p \in F$. Moreover, for some positive constants K and M, the subdifferential inequality gives

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(y_n - p + e_n) + \alpha_n(z_n - p + e_n)\|^2 \\
&\leq (1-\alpha_n)\|y_n - p + e_n\|^2 + 2\alpha_n\langle z_n - p + e_n, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)(\|x_n - p\| + \|e_n\|)^2 + 2\alpha_n\langle z_n - v_n, x_{n+1} - p\rangle + \\
&\quad\; 2\alpha_n\langle v_n - p + e_n, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + M\alpha_n(\|z_n - v_n\| + \|v_n - p + e_n\|) + K\|e_n\|.
\end{aligned}$$

Hence by Lemma 2.1.1, we have $\|x_n - p\| \to 0$. □

We observe that in proving the above result, we only required β_n to be positive, so we can actually replace y_n in the above algorithm by any nonexpansive map f (and hence by a sequence of nonexpansive maps (f_n)) satisfying the condition $F \subset F(f)$ (and $F \subset \bigcap_n F(f_n)$, respectively). Therefore we can generalize at once the above algorithm and result in the following way:

Algorithm 3.3.5. Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator and $f_n : H \to H$ be a sequence of nonexpansive maps.
Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \geq 0$, compute

$$y_n = f_n(x_n).$$

Step 3. Choose a regularization parameter λ_n large enough such that

$$\|(I + \lambda_n A)^{-1}(x_n) - P_F x_n\| < \frac{1}{n}, \quad \text{and compute} \quad z_n = (I + \lambda_n A)^{-1}(x_n).$$

Step 4. For $n \geq 0$, choose a relaxation parameter $\alpha_n \in (0,1)$ and compute the $(n+1)$th iterate:

$$x_{n+1} = \alpha_n z_n + (1-\alpha_n)y_n + e_n,$$

where (e_n) is a sequence of computational errors.

Theorem 3.3.9. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator with $\emptyset \neq F := A^{-1}(0) \subset \bigcap_n F(f_n)$, where $f_n : H \to H$ is a sequence of nonexpansive maps, and $F(f_n)$ is the fixed point set of f_n. Assume that the conditions $\alpha_n \in (0,1)$ with (C1), (C2) and (E1) are satisfied. Then given any $x_0 \in H$, the sequence (x_n) constructed by Algorithm 3.3.5 converges strongly to some $p \in F$.*

6). If in Algorithm 3.3.2 we take $f_n = (I + \lambda_n B)^{-1}$, for $\lambda_n > 0$, then we have the following algorithm:

Algorithm 3.3.6. Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators.
Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose the regularization parameters $\beta_n, \lambda_n > 0$ and the relaxation parameter $\alpha_n \in (0,1)$. Then compute the $(n+1)$th iterate:

$$x_{n+1} = \alpha_n (I + \lambda_n B)^{-1}(x_n) + (1 - \alpha_n)(I + \beta_n A)^{-1}(x_n) + e_n,$$

where (e_n) is a sequence of computational errors.

Theorem 3.3.10. *Assume that $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ are maximal monotone operators with $\emptyset \neq F := A^{-1}(0) = B^{-1}(0)$. If $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\alpha_n \in (0,1)$, with $\alpha_n \to 0$, and $\beta_n \to \infty$, then for any given $x_0 \in H$, the sequence (x_n) generated by Algorithm 3.3.6 converges weakly to some $q \in F$.*

Proof. For $p \in F$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|J^B_{\lambda_n} x_n - p\| + (1 - \alpha_n)\|J^A_{\beta_n} x_n - p\| + \|e_n\| \\ &\leq \|x_n - p\| + \|e_n\|, \end{aligned}$$

where $J^A_{\beta_n} = (I + \beta_n A)^{-1}$, and $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$. The above estimate implies that $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F$. Moreover,

$$\begin{aligned} \|x_{n+1} - J^A_{\beta_n} x_n\| &= \|\alpha_n (J^B_{\lambda_n} x_n - J^A_{\beta_n} x_n) + e_n\| \\ &\leq \alpha_n (\|J^B_{\lambda_n} x_n - p\| + \|J^A_{\beta_n} x_n - p\|) + \|e_n\| \\ &\leq 2\alpha_n \|x_n - p\| + \|e_n\| \to 0. \end{aligned}$$

Consequently, $x_\infty \in F$ if $x_{n_k} \rightharpoonup x_\infty$. Hence by Opial's lemma, there exists a point, say $q \in F$ such that $x_n \rightharpoonup q$. □

3.4 The regularization method

We devote this section to demonstrate the strong convergence of the prox-Tikhonov method, formerly introduced by Lehdili and Moudafi [32] and later developed by Xu [55], under different sets of conditions on the parameters α_n and β_n. Such conditions will allow choices such as $\alpha_n = n^{-1}$ and $\beta_n = n$, and they are weaker than those previously studied by other authors, so the results contained in this section can be viewed as significant improvements and refinements of previously known results. Theorem 3.4.2 deals with the conditions

$$\text{either} \quad (C10) \sum_{n=1}^{\infty} \left| \frac{\alpha_{n-1}}{\beta_n} - \frac{\alpha_n}{\beta_{n+1}} \right| < \infty \quad \text{or,} \quad (C11) \lim_{n\to\infty} \left(1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right) = 0,$$

and Theorem 3.4.3 is concerned with the conditions

$$\text{either} \quad (C12) \lim_{n\to\infty} \frac{(\alpha_n - \alpha_{n-1})}{\alpha_{n-1}\beta_n} = 0 \quad \text{or,} \quad (C13) \sum_{n=1}^{\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} < \infty,$$

and

$$(C14) \lim_{n\to\infty} \left(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right) = 0.$$

In particular, our results provide an answer to the question we asked earlier: Can one design a proximal point algorithm by choosing appropriate regularization parameters α_n such that strong convergence of (x_n) is preserved, for $\|e_n\| \to 0$ and β_n bounded? Of course, for constant β_n, (C10) reduces to (C4) and (C11) reduces to (C5). Most importantly, if (C8)$'$ holds, then so does (C11).

We remark that if $A : D(A) \subset H \to 2^H$ is the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, then our convergence results provide sequences which converge strongly to the minimum point of φ nearest to u. In addition, we shall show that the regularization method is ideal in providing convergence rate estimates for a sequence converging to $\inf \varphi$ (see Section 6.1 of Chapter 6).

Before giving our convergence results, we begin by showing that there is a strong connection between the regularization method defined by (1.19) and the inexact Halpern-type Algorithm 3.2.2 discussed in Section 3.2. In fact, these two iterative methods are equivalent. Indeed, setting

$$v_n := \frac{x_n - \alpha_{n-1}u - e_{n-1}}{1 - \alpha_{n-1}}, \tag{3.30}$$

we see that Algorithm 3.2.2 can be reformulated as

Algorithm 3.4.1. Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator.
Step 1. Choose $v_1, u \in H$ arbitrarily.
Step 2. For each $n \geq 1$, choose a regularization parameter $\beta_n > 0$ and the relaxation parameter $\alpha_n \in (0, 1)$, then compute the $(n+1)$th iterate:

$$v_{n+1} = J_{\beta_n}(\alpha_{n-1}u + (1 - \alpha_{n-1})v_n + e_{n-1}), \tag{3.31}$$

where (e_n) is a sequences of computational errors.

It is worth pointing out that, for $\alpha_n \to 0$ and $e_n \to 0$, the equations (3.18) and (3.31) are equivalent, that is, (v_n) converges if and only if (x_n) does. Therefore, results already proved in Section 3.2 concerning the inexact Halpern-type algorithm remains valid if one uses the regularization method described by (3.31) instead of Algorithm 3.2.2. Likewise, results of this section are carried over to the algorithm of Halpern's type. It is not hard to see that (3.31) can be put in the form (1.19), with α_{n-1}, β_n, and e_{n-1} instead of α_n, β_n, and e_n. If we consider (3.31) instead of (1.19), then conditions (C7) and (C8) take the form (C7)$'$ and (C8)$'$, respectively. (See the table containing these control condition in the appendix section).

We now give a result similar to Theorem 3.2 of Xu [55]. In the next result, if we consider equation (1.19) instead of (3.18), then we can prove the same result with (C8)$'$ being replaced by (C8). In that case, the result extends Theorem 3.2 [55] to a larger class of errors which include those that are non-summable and still converge to zero in norm. Moreover, we can show that Theorem 3.2 [55] fails to hold under the condition (C7), see Remark 3.4.1 and Example 3.4.1 below.

Theorem 3.4.1. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let (x_n) be the sequence generated by Algorithm 3.2.2, where* (i) *$\alpha_n \in (0,1)$, with (C1), and (C2),* (ii) *either (E1) or (E2) and* (iii) *$\beta_n \in (0,\infty)$ with $\liminf_{n\to\infty} \beta_n > 0$, $\beta_{n+1} \geq \alpha_n\beta_n$ and (C8)$'$ being satisfied. Then (x_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. For each fixed n, let y_n be the unique fixed point of the contraction $x \mapsto \alpha_{n-1}u + (1-\alpha_{n-1})J_{\beta_n}x$. Then according to Theorem 3.2.6, $y_n \to P_F u$ as $n \to \infty$. Set

$$v_n := \frac{x_n - \alpha_{n-1}u - e_{n-1}}{1-\alpha_{n-1}} \quad \text{and} \quad w_n := \frac{y_n - \alpha_{n-1}u}{1-\alpha_{n-1}}. \tag{3.32}$$

As a consequence of the boundedness of (x_n) and (y_n) (see Theorem 3.2.4), the sequences (v_n) and (w_n) are bounded. Also by virtue of (3.32), $w_n \to P_F u$ as $n \to \infty$. It follows from (3.18) and the definition of y_n that

$$v_{n+1} = J_{\beta_n}((1-\alpha_{n-1})v_n + \alpha_{n-1}u + e_{n-1}) \quad \text{and} \quad w_n = J_{\beta_n}((1-\alpha_{n-1})w_n + \alpha_{n-1}u).$$

Using the nonexpansivity of the resolvent, we estimate $\|v_{n+1} - w_{n+1}\|$ as follows:

$$\begin{aligned} \|v_{n+1} - w_{n+1}\| &\leq \|v_{n+1} - w_n\| + \|w_{n+1} - w_n\| \\ &\leq (1-\alpha_{n-1})\|v_n - w_n\| + \|w_{n+1} - w_n\| + \|e_{n-1}\|. \end{aligned} \tag{3.33}$$

Now using the resolvent identity and the nonexpansivity of the resolvent, we can estimate $\|w_{n+1} - w_n\|$ as follows:

$$\begin{aligned}\|w_{n+1} - w_n\| &= \left\| J_{\beta_n}\left(\frac{\beta_n}{\beta_{n+1}}((1-\alpha_n)w_{n+1} + \alpha_n u) + \left(1 - \frac{\beta_n}{\beta_{n+1}}\right) w_{n+1}\right)\right. \\ &- J_{\beta_n}((1-\alpha_{n-1})w_n + \alpha_{n-1}u)\| \\ &\leq \left(1 - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right)\|w_{n+1} - w_n\| + \left|\alpha_{n-1} - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right| K,\end{aligned}$$

which gives

$$\|w_{n+1} - w_n\| \leq \left|1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right| K, \tag{3.34}$$

for some positive constant K. Combining (3.33) and (3.34) we get

$$\|v_{n+1} - w_{n+1}\| \leq (1-\alpha_{n-1})\|v_n - w_n\| + \left|1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right| K + \|e_{n-1}\|. \tag{3.35}$$

Hence from Lemma 2.1.1, we see that $\|v_n - w_n\| \to 0$, and the proof is complete. □

Remark 3.4.1. In view of Lemma 2.1.1 and (3.35), it is tempting to infer that the theorem is still valid under the condition (C7)′. However we show that this condition is impossible to attain for any sequences (β_n) and (α_n) satisfying the conditions of the above theorem. To this end, we assume that (C7)′ holds true. Denote $b_n := \alpha_{n-1}/\beta_n$. Then

$$\sum_{n=1}^{\infty}\left|1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right| < \infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty}\frac{|b_{n+1} - b_n|}{b_{n+1}} < \infty.$$

Therefore, it follows from Lemma 2.1.2 that

$$\liminf_{n\to\infty}\frac{\alpha_{n-1}}{\beta_n} = \liminf_{n\to\infty} b_n > 0,$$

which implies that $\beta_n \to 0$ (since $\alpha_n \to 0$). This is a contradiction as β_n is bounded below away from zero.

However, if we allow $\beta_n \to 0$, then Theorem 3.2.6 is no longer applicable. Indeed, from $w_n = J_{\beta_n}((1-\alpha_{n-1})w_n + \alpha_{n-1}u)$, we have

$$\frac{\alpha_{n-1}}{\beta_n}(u - w_n) \in Aw_n. \tag{3.36}$$

From the above inclusion relation, we can not derive $\omega_{\mathtt{w}}((w_n)) \subset F := A^{-1}(0)$, even if w_n is strongly convergent (since by (3.34), $\sum_{n=1}^{\infty}\|w_{n+1} - w_n\| < \infty$) because α_{n-1}/β_n may not necessarily converge to zero. Therefore, in this case (x_n) is still strongly convergent (according to (3.35)) but we can not derive that its limit is in F. In fact, its limit need not be in F. We give an example to that effect.

Example 3.4.1. Let $\beta_n = 1/n$ and $\alpha_n = 1/(n+2)$ for $n \geq 1$. Then we have

$$1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n} = \frac{1}{(n+1)^2} =: a_n, \quad \text{for all} \quad n \geq 1, \quad \text{and} \quad \frac{\beta_{n+1}}{\alpha_n} \to 1 \quad \text{as } n \to \infty.$$

Clearly the condition $\beta_{n+1} \geq \alpha_n\beta_n$ for all $n \geq 1$ is fulfilled. Let $H = \mathbb{R}$, and let the sequence $(e_n) \subset \mathbb{R}$ satisfy either the condition $\sum_{n=0}^{\infty} |e_n| < \infty$ or $|e_n|/\alpha_n \to 0$, (for example, $|e_n| = (n+2)^{-2}$ or $|e_n| = 1/(n \ln n)$ for $n \geq 2$ with $\sum_{n=2}^{\infty} |e_n| = \infty$, respectively), and let $A : D(A) = [0, \infty) \subset \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$Ax = \begin{cases} ax, & \text{if} \quad x > 0, \\ (-\infty, 0], & \text{if} \quad x = 0, \\ \emptyset, & \text{if} \quad x < 0, \end{cases}$$

for some $a > 0$. Then, if $u > 0$, we have for sufficiently large n, $\alpha_{n-1}u + e_{n-1} > 0$ and

$$\begin{aligned} 0 < w_n &= J_{\beta_n}((1-\alpha_{n-1})w_n + \alpha_{n-1}u + e_{n-1}) \\ &= \frac{1}{1+\beta_n a}((1-\alpha_{n-1})w_n + \alpha_{n-1}u + e_{n-1}), \end{aligned}$$

which implies that $w_n \to w_\infty := (1+a)^{-1}u \notin F = \{0\}$. Hence $x_n \to w_\infty \notin F$. The same conclusion is true if $u < 0$.

The argument given above shows that if β_n is bounded away from zero in Theorem 3.2 of [55], then the condition (C7) is impossible to achieve. Also the above example shows that the result may not hold if $\beta_n \to 0$.

We now give an example to show the applicability of Theorem 3.4.1.

Example 3.4.2. Choose $\beta_n = \beta_0 > 0$ for all n, $\alpha_n = (n+1)^{-1/2}$ and $\|e_n\| = 1/(n+1)$ for all $n \geq 0$.

Theorem 3.4.2. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let (x_n) be the sequence generated by Algorithm 3.2.2 where conditions* (i) *and* (ii) *of Theorem 3.4.1 are fulfilled. If $\beta_n \in (0, \infty)$ is increasing and either (C10) or (C11) is satisfied, then (x_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. We know that (x_n) (and hence (v_n)) is bounded, see Theorem 3.2.4.

Claim: $\limsup_{n\to\infty}\langle u - P_F u, x_n - P_F u\rangle \leq 0$.

Let (x_{n_k}) be a subsequence of (x_n) converging weakly to some x_∞, such that

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle = \lim_{k\to\infty} \langle u - P_F u, x_{n_k} - P_F u\rangle = \langle u - P_F u, x_\infty - P_F u\rangle.$$

To prove the claim, we only need to show that $x_\infty \in F$, or more generally $\omega_{\mathtt{w}}((x_n)) \subset F$. In view of the inclusion relation

$$\frac{v_{n+1} - v_n}{\beta_n} + Av_{n+1} \ni \frac{\alpha_{n-1}}{\beta_n}(u - v_n) + \frac{1}{\beta_n} e_{n-1}, \tag{3.37}$$

it will be enough if we could show that

$$\frac{\|v_{n+1} - v_n\|}{\beta_n} \to 0. \tag{3.38}$$

Note that (3.38) is already satisfied if (β_n) is unbounded. Henceforth, we shall assume that (β_n) is bounded. Observe that from equation (3.31) and the resolvent identity, we have

$$v_{n+2} = J_{\beta_n}\left(\frac{\beta_n}{\beta_{n+1}}((1-\alpha_n)v_{n+1} + \alpha_n u + e_n) + \left(1 - \frac{\beta_n}{\beta_{n+1}}\right) v_{n+2}\right).$$

Therefore, using the boundedness of $(\|e_n\|/\alpha_n)$ and (v_n), and the nonexpansivity of J_{β_n}, one can compare v_{n+2} and v_{n+1} as follows

$$\begin{aligned}
\|v_{n+2} - v_{n+1}\| &\leq \left\|\left(1 - \frac{\beta_n}{\beta_{n+1}}\right)(v_{n+2} - v_{n+1}) + \left(1 - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right)(v_{n+1} - v_n)\right. \\
&+ \left.\left(\alpha_{n-1} - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right)\left(v_n - u - \frac{e_{n-1}}{\alpha_{n-1}}\right) + \frac{\alpha_n\beta_n}{\beta_{n+1}}\left(\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right)\right\| \\
&\leq \left(1 - \frac{\beta_n}{\beta_{n+1}}\right)\|v_{n+2} - v_{n+1}\| + \left(1 - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right)\|v_{n+1} - v_n\| \\
&+ \left|\alpha_{n-1} - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right| K + \frac{\alpha_n\beta_n}{\beta_{n+1}}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{\|v_{n+2} - v_{n+1}\|}{\beta_{n+1}} &\leq \left(1 - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right)\frac{\|v_{n+1} - v_n\|}{\beta_n} + \left|\frac{\alpha_{n-1}}{\beta_n} - \frac{\alpha_n}{\beta_{n+1}}\right| K + \frac{\alpha_n}{\beta_{n+1}}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\| \\
&\leq \left(1 - \frac{\alpha_n\beta_0}{\beta_{n+1}}\right)\frac{\|v_{n+1} - v_n\|}{\beta_n} + \left|\frac{\alpha_{n-1}}{\beta_n} - \frac{\alpha_n}{\beta_{n+1}}\right| K + \frac{\alpha_n}{\beta_{n+1}}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|,
\end{aligned}$$

for some $K > 0$. Similarly, for the case $\sum_{n=0}^\infty \|e_n\| < \infty$, we have

$$\begin{aligned}
\|v_{n+2} - v_{n+1}\| &\leq \left\|\left(1 - \frac{\beta_n}{\beta_{n+1}}\right)(v_{n+2} - v_{n+1}) + \left(1 - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right)(v_{n+1} - v_n)\right. \\
&+ \left.\left(\alpha_{n-1} - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right)(v_n - u) + \left(\frac{\beta_n}{\beta_{n+1}} e_n - e_{n-1}\right)\right\| \\
&\leq \left(1 - \frac{\beta_n}{\beta_{n+1}}\right)\|v_{n+2} - v_{n+1}\| + \left(1 - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right)\|v_{n+1} - v_n\| \\
&+ \left|\alpha_{n-1} - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right| K' + \left\|\frac{\beta_n}{\beta_{n+1}} e_n - e_{n-1}\right\|,
\end{aligned}$$

which implies that

$$\frac{\|v_{n+2}-v_{n+1}\|}{\beta_{n+1}} \leq \left(1-\frac{\alpha_n\beta_0}{\beta_{n+1}}\right)\frac{\|v_{n+1}-v_n\|}{\beta_n}+\left|\frac{\alpha_{n-1}}{\beta_n}-\frac{\alpha_n}{\beta_{n+1}}\right|K'+\frac{1}{\beta_0}\left(\|e_n\|+\|e_{n-1}\|\right),$$

for some positive constant K'. Let us observe that

$$\left|\frac{\alpha_{n-1}}{\beta_n}-\frac{\alpha_n}{\beta_{n+1}}\right| = \frac{\alpha_n\beta_0}{\beta_{n+1}}\left|\frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}-1\right|\frac{1}{\beta_0}.$$

Denote $a_n := \alpha_n\beta_0/\beta_{n+1}$. Since (α_n) satisfies $\alpha_n \in (0,1)$, $\alpha_n \to 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, so do (a_n). Therefore, from Lemma 2.1.1, we derive (3.38). Moreover, (3.37) implies that $\omega_{\mathtt{w}}((v_n)) \subset F$, and from (3.30), we derive $\omega_{\mathtt{w}}((v_n)) = \omega_{\mathtt{w}}((x_n))$, hence the claim.

Finally we show that (x_n) converges strongly to $P_F u$. We have from the subdifferential inequality

$$\|x_{n+1}-P_Fu\|^2 \leq (1-\alpha_n)\|x_n-P_Fu\|^2+2\alpha_n\left\langle u-P_Fu+\frac{e_n}{\alpha_n}, x_{n+1}-P_Fu\right\rangle. \tag{3.39}$$

In the case when $\|e_n\|/\alpha_n \to 0$, inequality (3.39) implies by Lemma 2.1.1 that $x_n \to P_F u$. If $\sum_{n=0}^{\infty}\|e_n\| < \infty$, then we derive from inequality (3.39)

$$\|x_{n+1}-P_Fu\|^2 \leq (1-\alpha_n)\|x_n-P_Fu\|^2+2\alpha_n\langle u-P_Fu, x_{n+1}-P_Fu\rangle + C\|e_n\|,$$

for some $C > 0$, and Lemma 2.1.1 again implies that $x_n \to P_F u$ as desired. □

Remark 3.4.2. The condition (C10) is weaker than the conditions (C4) and (C9) if $\beta_n \geq \delta$ for all n and for some $\delta > 0$. Indeed,

$$\begin{aligned}\left|\frac{\alpha_{n-1}}{\beta_n}-\frac{\alpha_n}{\beta_{n+1}}\right| &\leq \frac{1}{\beta_n}|\alpha_{n-1}-\alpha_n|+\alpha_n\left|\frac{1}{\beta_n}-\frac{1}{\beta_{n+1}}\right| \\ &\leq \frac{1}{\delta}\left[|\alpha_{n-1}-\alpha_n|+\frac{|\beta_{n+1}-\beta_n|}{\delta}\right].\end{aligned}$$

Note that if $\beta_n = n^2$ for $n \geq 1$, then (C10) holds true for any choice of $\alpha_n \in (0,1)$.

Remark 3.4.3. Observe that (C11) is satisfied for $\beta_n = n$ and $\alpha_n = (n+1)^{-1}$, whereas the condition (C8)′ of Theorem 3.2.8 fails. Also, (C11) works if β_n is constant and α_n taken as before but (C8)′ fails. In addition, any sequences (α_n) and (β_n) satisfying condition (C8)′ also satisfy (C11). This shows that the assumption (C11) is weaker than (C8)′.

Although the condition (C10) is weaker than (C4) and (C9) if $\liminf_{n\to\infty}\beta_n > 0$, our result is restricted only to those β_n's which are increasing. The next result is designed to cater for those β_n's who does not satisfy this restrictive condition. It is actually an extension and improvement of Theorem 3.2.8 above. Our proof differs from those given in [48] and [55], and it relies on the equivalence of the equations (1.19) and (3.18). Note that it was observed in [48] that a gap exists in the proof of Theorem 3.2.8. We remark here that our method of transforming equation (3.31) into equation (3.18) is an alternative way of solving this gap as can be seen from the proof of Theorem 3.4.3 below.

Theorem 3.4.3. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let the sequence (x_n) be generated by Algorithm 3.2.2 with the following conditions being satisfied:* (i) $\alpha_n \in (0,1)$, *(C1), (C2),* (ii) *either (E1) or (E2),* (iii) $\liminf_{n\to\infty} \beta_n > 0$, *and (C14). If either (C12) or (C13) hold, then (x_n) (and hence (v_n)) converges strongly to $P_F u$, the projection of u on F.*

Proof. We know from the proof of Theorem 3.2.4 that (x_n) is bounded. For $\|e_n\|/\alpha_n \to 0$, we have, (by the resolvent identity and the nonexpansivity of the resolvent),

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1-\alpha_{n-1}) \left\| J_{\beta_n} x_n - J_{\beta_n}\left(\frac{\beta_n}{\beta_{n-1}} x_{n-1} + \left(1 - \frac{\beta_n}{\beta_{n-1}}\right) J_{\beta_{n-1}} x_{n-1}\right)\right\| \\
&+ |\alpha_n - \alpha_{n-1}| \left\| u - J_{\beta_n} x_n + \frac{e_n}{\alpha_n}\right\| + \alpha_{n-1} \left\| \frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\| \\
&\leq (1-\alpha_{n-1}) \left\| \frac{\beta_n}{\beta_{n-1}} (x_n - x_{n-1}) + \left(1 - \frac{\beta_n}{\beta_{n-1}}\right)(x_n - J_{\beta_{n-1}} x_{n-1})\right\| \\
&+ |\alpha_n - \alpha_{n-1}| \left\| u - J_{\beta_n} x_n + \frac{e_n}{\alpha_n}\right\| + \alpha_{n-1} \left\| \frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\| \\
&\leq (1-\alpha_{n-1}) \frac{\beta_n}{\beta_{n-1}} \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|u - J_{\beta_n} x_n + e_n/\alpha_n\| \\
&+ \alpha_{n-1} \left|1 - \frac{\beta_n}{\beta_{n-1}}\right| \left\| u - J_{\beta_{n-1}} x_{n-1} + \frac{e_{n-1}}{\alpha_{n-1}}\right\| + \alpha_{n-1} \left\| \frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|,
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\beta_n} &\leq (1-\alpha_{n-1}) \frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \alpha_{n-1} \left| \frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| K \\
&+ \frac{\alpha_{n-1}}{\beta_n} \left\| \frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\| + K \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}, \qquad (3.40)
\end{aligned}$$

for some positive constant K. From Lemma 2.1.1 and inequality (3.40), we have

$$\frac{\|x_{n+1} - x_n\|}{\beta_n} \to 0 \quad \Leftrightarrow \quad \frac{\|v_{n+1} - v_n\|}{\beta_n} \to 0.$$

Hence we can derive (see (3.37) above), $\omega_{\mathtt{w}}((x_n)) = \omega_{\mathtt{w}}((v_n)) \subset F$. Consequently, we have

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle \leq 0. \qquad (3.41)$$

Note that for some positive constant C, $|\beta_{n+1}^{-1} - \beta_n^{-1}| \le C$ (since $\liminf_{n\to\infty} \beta_n > 0$) and

$$x_{n+1} - x_n = (\alpha_n - \alpha_{n-1})(u - J_{\beta_n} x_n) + (e_n - e_{n-1}) + (1 - \alpha_{n-1})(J_{\beta_n} x_n - J_{\beta_{n-1}} x_{n-1}),$$

so that in the case when $\sum_{n=1}^{\infty} \|e_n\| < \infty$, we again get inequality (3.41) on applying similar arguments as above.

As in the proof of Theorem 3.4.2, we derive strong convergence of (x_n) to $P_F u$. □

Remark 3.4.4. Clearly (C14) is weaker than the conditions

$$(C15)\ \sum_{n=1}^{\infty} \left| \frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right| < \infty \quad \text{and} \quad (C16)\ \lim_{n\to\infty} \frac{1}{\alpha_n} \left(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right) = 0,$$

both of which hold true if $\alpha_n = n^{-1}$ and $\beta_n = n$ while (C6) fails for this choice of β_n. However, both (C6) and (C14) hold if $\beta_n = \ln n$. We point out that the first inequality in Remark 3.4.2 suggests that for $\liminf_{n\to\infty} \beta_n > 0$, the condition (C10) is weaker than (C4) and (C15). Also, the condition (C15) is weaker than (C9) whenever $\liminf_{n\to\infty} \beta_n > 0$ holds. But the condition that β_n is increasing is stronger than the assumption $\liminf_{n\to\infty} \beta_n > 0$, so there are cases in which the following corollary is applicable and Theorem 3.4.2 is not. We remark that both (C4) and (C5) are not satisfied by

$$\alpha_n = \begin{cases} 1/n, & \text{if } n \text{ is odd,} \\ 1/(2n), & \text{if } n \text{ is even.} \end{cases}$$

This choice of α_n however fulfills the assumptions of (C13) for the case when $\beta_n = n$ and (C12) for any $\beta_n \to \infty$.

Corollary 3.4.4. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let the sequence (x_n) be generated by Algorithm 3.2.2, where $\alpha_n \in (0,1)$ and $\beta_n \in (0,\infty)$, with the conditions* (i) *and* (ii) *taken as in Theorem 3.4.3, and $\liminf_{n\to\infty} \beta_n > 0$ with either (C15) or (C16). If either (C12) or (C13) hold, then (x_n) (and hence (v_n)) converges strongly to $P_F u$.*

The following corollary is an extension of Theorem 3.2.8.

Corollary 3.4.5. *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let the sequence (x_n) be generated by Algorithm 3.2.2, where $\alpha_n \in (0,1)$ and $\beta_n \in (0,\infty)$, with the conditions* (i) *and* (ii) *taken as in Theorem 3.4.3, and* (iii) $\liminf_{n\to\infty} \beta_n > 0$ *and either* (C9) *or* (C17). *If either (C12) or (C13) hold, then (x_n)* (hence (v_n)) *converges strongly to $P_F u$.*

We give an example to show that the conditions of (iii) are different.

Example 3.4.3. Let $\alpha_n = (n+2)^{-1/4}$ and $\beta_n = 2(n+1)(n+2)^{-1}$ for all $n \ge 0$. Then α_n and β_n satisfy both conditions of *(iii)* while $\beta_n = (n+1)$ and α_n as above satisfy only (C17).

Chapter 4

Multi Parameter Proximal Point Algorithms

The aim of this chapter is to analyze the convergence properties of two distinct proximal point methods with multiple parameters, under different sets of assumptions on these parameters. The algorithm discussed in Section 4.1 was proposed by Y. Yao and M. A. Noor (2008), and it is in fact a generalization of the one studied before by T. Suzuki (2007) and C. E. Chidume and C. O. Chidume (2006) independently. Note that in their papers, Suzuki [50] and Chidume and Chidume [19] showed that the control conditions (C1) and (C2) on the sequence of parameters (α_n) are necessary and sufficient for such an algorithm to converge strongly. Although we are able to address the two important problems related to the proximal point algorithm − that of strong convergence (instead of weak convergence) and that of acceptable errors − with Yao and Noor's algorithm, it is not convenient in estimating the convergence rate of a sequence that approximates minimum values of convex functionals, let alone finding minimizers of such functionals under fairly mild conditions. An ideal algorithm which is in fact a generalization of the regularization method discussed in Section 3.4 was thus sought in [10] mainly for the purpose of applications in convex optimization. It turns out that for a particular case, the two algorithms discussed in this chapter are equivalent. This special case reduces the two algorithms to the inexact Halpern-type proximal point algorithm and/or the regularization method already discussed in Chapter 3. Some results of this chapter improve the corresponding results of Chapter 3, while some are not comparable to the results of the previous chapter.

4.1 A generalized proximal point algorithm

In this section, we discuss strong convergence of (x_n) generated by the following algorithm which was previously studied by Yao and Noor [56] under different sets of assumptions on (α_n), (β_n), and (λ_n). Given any fixed $u, x_0 \in H$, the sequence (x_n) is generated by

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \gamma_n J_{\beta_n} x_n + e_n, \quad n \geq 0, \tag{4.1}$$

where $\alpha_n \in (0,1)$, $\lambda_n, \gamma_n \in [0,1]$ with $\alpha_n + \lambda_n + \gamma_n = 1$ for all $n \geq 0$, $\beta_n \in (0, \infty)$, and (e_n) is a sequence of computational errors.

As we have already seen in the previous chapter, before we can start talking about convergence of a sequence generated from the proximal point method, it is imperative to first check the boundedness of such a sequence. Hence we begin by proving the following lemma.

Lemma 4.1.1. *Let $\beta_n \in (0,\infty)$, $\alpha_n \in (0,1)$, and $\lambda_n, \gamma_n \in [0,1]$ with $\alpha_n + \lambda_n + \gamma_n = 1$ for all $n \geq 0$. Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator with $F := A^{-1}(0) \neq \emptyset$, and either $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $(\|e_n\|/\alpha_n)$ is bounded. Then for any fixed $u, x_0 \in H$, the sequence (x_n) defined by (4.1) is bounded.*

Proof. The proof is essentially done in Section 3.2. We repeat it for the sake of the reader's convenience.

If $\sum_{n=0}^{\infty} \|e_n\| < \infty$, then one shows by induction that, for any $p \in F$ and $n \geq 0$,

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \|u - p\|\} + \sum_{k=0}^{n-1} \|e_k\|. \tag{4.2}$$

Hence (x_n) is bounded.

Now assume that $(\|e_n\|/\alpha_n)$ is bounded. Then, there exists a positive constant M such that

$$\|u - p\| + \frac{\|e_n\|}{\alpha_n} \leq M,$$

for any $p \in F$ and all $n \geq 0$. Without loss of generality, we assume such a constant is such that $\|x_0 - p\| \leq 2M := C$. We show by induction that, for all $n \geq 0$,

$$\|x_n - p\| \leq C.$$

Using (4.1), and the subdifferential inequality, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(u - p + e_n/\alpha_n) + \lambda_n(x_n - p) + \gamma_n(J_{\beta_n} x_n - p)\|^2 \\
&\leq \|\lambda_n(x_n - p) + \gamma_n(J_{\beta_n} x_n - p)\|^2 + 2\alpha_n \langle u - p + e_n/\alpha_n, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2M\alpha_n \|x_{n+1} - p\|.
\end{aligned}$$

If $\|x_n - p\| \le C$ for some $n \ge 0$, then the last estimate gives

$$\|x_{n+1} - p\|^2 \le (1-\alpha_n)^2 C^2 + 2M\alpha_n \|x_{n+1} - p\|.$$

Hence,

$$(\|x_{n+1} - p\| - M\alpha_n)^2 \le M^2\alpha_n^2 + (1-\alpha_n)^2 C^2,$$

which yields

$$\|x_{n+1} - p\| \le M\alpha_n + \sqrt{M^2\alpha_n^2 + (1-\alpha_n)^2 C^2}.$$

Since the inequality

$$M\alpha_n + \sqrt{M^2\alpha_n^2 + (1-\alpha_n)^2 C^2} \le C$$

holds true, we conclude that (x_n) is bounded. □

Note that the proof of Lemma 4.1.1 can be derived from the proof of Theorem 3.2.4 but only under the additional assumption $\sum_{n=0}^{\infty} \alpha_n = \infty$. Such assumption on α_n is necessary for the sequence generated by the modified proximal point algorithm to converge strongly, and it shall be used in the sequel.

Theorem 4.1.2. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. Fix $u, x_0 \in H$, and let (x_n) be the sequence generated by algorithm (4.1) with the conditions:* (i) *$\alpha_n \in (0,1)$ with (C1) and (C2),* (ii) *either (E1) or (E2),* (iii) *$\lambda_n, \gamma_n \in [0,1]$, $\alpha_n + \lambda_n + \gamma_n = 1$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$, and* (iv) *$\beta_n \in (0,\infty)$ with $\liminf_{n\to\infty} \beta_n > 0$ and (C14) being satisfied. If, in addition (C13) holds, then (x_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. We know from Lemma 4.1.1 that (x_n) is bounded. Denote

$$v_n := \frac{x_{n+1} - \alpha_n u - \lambda_n x_n - e_n}{\gamma_n}. \tag{4.3}$$

Note that (v_n) is bounded since (x_n) is bounded and for $\alpha_n, \lambda_n \to 0$, we see that the weak ω-limit sets of (x_n) and (v_n) coincide, that is, $\omega_{\mathtt{w}}((x_n)) = \omega_{\mathtt{w}}((v_n))$. Moreover, we have from (4.1), and (4.3),

$$Av_n \ni \frac{x_n - x_{n+1} + \alpha_n(u - x_n) + e_n}{\beta_n \gamma_n}. \tag{4.4}$$

Our aim is to prove that the relation $\omega_{\mathtt{w}}((x_n)) \subset F$ holds, from which we can establish the inequality

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle \le 0. \tag{4.5}$$

Indeed, for some subsequence (x_{n_k}) of (x_n) converging weakly to some $x_\infty \in F$, we have

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle = \lim_{k\to\infty} \langle u - P_F u, x_{n_k} - P_F u\rangle = \langle u - P_F u, x_\infty - P_F u\rangle \le 0.$$

In view of (4.4), it would be enough if we could show that

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_n} = 0. \tag{4.6}$$

For this purpose, we first compare x_{n+2} and x_{n+1} as follows

$$\begin{aligned} x_{n+2} - x_{n+1} &= (1-\alpha_n)(J_{\beta_{n+1}} x_{n+1} - J_{\beta_n} x_n) + \lambda_{n+1}(x_{n+1} - J_{\beta_{n+1}} x_{n+1}) \\ &+ \lambda_n (J_{\beta_n} x_n - x_n) + (\alpha_{n+1} - \alpha_n)(u - J_{\beta_{n+1}} x_{n+1} + e_{n+1}/\alpha_{n+1}) \\ &+ \alpha_n (e_{n+1}/\alpha_{n+1} - e_n/\alpha_n). \end{aligned}$$

Using the resolvent identity and the fact that the resolvent operator is nonexpansive, we get

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\le (1-\alpha_n)\left\| J_{\beta_{n+1}} x_{n+1} - J_{\beta_{n+1}}\left(\frac{\beta_{n+1}}{\beta_n} x_n + \left(1 - \frac{\beta_{n+1}}{\beta_n}\right) J_{\beta_n} x_n\right)\right\| \\ &+ \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| + (\lambda_n + \lambda_{n+1})K + |\alpha_{n+1} - \alpha_n| M \\ &\le (1-\alpha_n)\left\| \frac{\beta_{n+1}}{\beta_n}(x_{n+1} - x_n) + \left(1 - \frac{\beta_{n+1}}{\beta_n}\right)(x_{n+1} - J_{\beta_n} x_n)\right\| \\ &+ \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| + (\lambda_n + \lambda_{n+1})K + |\alpha_{n+1} - \alpha_n| M \\ &\le (1-\alpha_n)\frac{\beta_{n+1}}{\beta_n}\|x_{n+1} - x_n\| + \left|1 - \frac{\beta_{n+1}}{\beta_n}\right| \|x_{n+1} - J_{\beta_n} x_n\| \\ &+ \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| + (\lambda_n + \lambda_{n+1})K + |\alpha_{n+1} - \alpha_n| M, \end{aligned} \tag{4.7}$$

for some positive constants K and M. Note that we have from (4.1)

$$\|x_{n+1} - J_{\beta_n} x_n\| \le \alpha_n \|u - J_{\beta_n} x_n + e_n/\alpha_n\| + \lambda_n \|x_n - J_{\beta_n} x_n\|,$$

which together with (4.7) yields

$$\begin{aligned} \frac{\|x_{n+2} - x_{n+1}\|}{\beta_{n+1}} &\le (1-\alpha_n)\frac{\|x_{n+1} - x_n\|}{\beta_n} + \alpha_n \left|\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right| M + \frac{\alpha_n}{\beta_{n+1}}\left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| \\ &+ \frac{|\alpha_{n+1} - \alpha_n|}{\beta_{n+1}} M + (2\lambda_n + \lambda_{n+1})K', \end{aligned}$$

for some $K' > 0$. Equation (4.6) then follows from Lemma 2.1.1 and this last estimate.

The application of the subdifferential inequality to (4.1) yields

$$\|x_{n+1} - P_F u\|^2 \le (1-\alpha_n)\|x_n - P_F u\|^2 + 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, x_{n+1} - P_F u \right\rangle.$$

Therefore from Lemma 2.1.1, inequality (4.5) and conditions (i) and (E2) of the theorem, we derive $x_n \to P_F u$. The proof is similar in the case when (E1) is satisfied. □

Remark 4.1.1. Since λ_n is summable and (x_n) is bounded, the term $\lambda_n(x_n - J_{\beta_n}x_n)$ can be regarded as the error term in the case when the sequence of errors is also summable. In that case, algorithm (4.1) assumes the form $x_{n+1} = \alpha_n u + (1-\alpha_n)J_{\beta_n}x_n + E_n$, and therefore can be analyzed in the same way we did in Section 3.2. In particular, we reobtain Theorem 3.4.3 from Theorem 4.1.2, and other results of Section 3.2 and Section 3.4 can also be obtained in this way. From this point of view, Theorem 4.1.2 is already known. For the case when (e_n) satisfies condition (E2), we cannot absorb the term $\lambda_n(x_n - J_{\beta_n}x_n)$ into the error sequence, otherwise it will mean that the quotient $\|E_n\|/\alpha_n = \|\lambda_n(x_n - J_{\beta_n}x_n) + e_n\|/\alpha_n$ must always converge to zero. This is not the case as one can check by taking the sequences defined by $\lambda_n = n^{-2}$ and $\|e_n\| = \alpha_n^2$, with $\alpha_n = n^{-1} + (-1)^n(n+1)^{-1}$. In this case, our theorem remains valid, thus Theorem 4.1.2 is 'really' new. We mention here that the upper bound of $\|x_n\|$ is independent of the coefficients (α_n), (λ_n) and (γ_n). Even though the above quotient does not always converge to zero (as the above example shows), $\|E_n\|$ does, so we may choose another sequence of parameters, say (α_n^*), to reobtain the condition $\|E_n\|/\alpha_n^* \to 0$. Of course such α_n^*'s result in another proximal point algorithm of Halpern type − the form mentioned earlier − in which (E_n) is the new error sequence, and again we may analyze the resulting algorithm as in Section 3.2 and Section 3.4. The above theoretical observations illustrate the significance of the condition (E2).

Remark 4.1.2. Note that the series condition on (λ_n) can be relaxed, but at the expense of stronger assumptions on (β_n). For instance, we may assume that (λ_n) and (β_n) satisfy

$$(C19)\ \sum_{n=0}^{\infty} \frac{|\lambda_{n+1} - \lambda_n|}{\beta_{n+1}} < \infty,$$

and (β_n) increasing, as the following theorem shows.

Theorem 4.1.3. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. Fix $u, x_0 \in H$, and let (x_n) be the sequence generated by algorithm* (4.1) *with the conditions:* (i) *and* (ii) *of Theorem 4.1.2,* (iii) $\lambda_n, \gamma_n \in [0,1]$, $\alpha_n + \lambda_n + \gamma_n = 1$ *with* $\lim_{n\to\infty} \lambda_n = 0$, *and* (iv) $\beta_n \in (0,\infty)$ *with $\beta_n \le \beta_{n+1}$ for all $n \ge 0$ being satisfied. If, in addition, (C13) and (C19) hold, then (x_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. According to Lemma 4.1.1, (x_n) is bounded. Like in the proof of Theorem 4.1.2, we have $\omega_{\mathtt{w}}((x_n)) = \omega_{\mathtt{w}}((v_n))$, where (v_n) is defined by (4.3). In order to derive $\omega_{\mathtt{w}}((x_n)) \subset F$, it would suffice if we could prove that (4.6) holds. For this purpose, we compare x_{n+2} and x_{n+1} as follows:

$$\begin{aligned} x_{n+2} - x_{n+1} &= \gamma_n(J_{\beta_{n+1}}x_{n+1} - J_{\beta_n}x_n) + (\lambda_{n+1} - \lambda_n)(x_{n+1} - J_{\beta_{n+1}}x_{n+1}) \\ &+ \lambda_n(x_{n+1} - x_n) + (\alpha_{n+1} - \alpha_n)(u - J_{\beta_{n+1}}x_{n+1} + e_{n+1}/\alpha_{n+1}) \\ &+ \alpha_n(e_{n+1}/\alpha_{n+1} - e_n/\alpha_n). \end{aligned}$$

Using the resolvent identity and the fact that the resolvent is nonexpansive, we get

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\le \gamma_n \left\| J_{\beta_{n+1}}x_{n+1} - J_{\beta_{n+1}}\left(\frac{\beta_{n+1}}{\beta_n}x_n + \left(1 - \frac{\beta_{n+1}}{\beta_n}\right) J_{\beta_n}x_n\right)\right\| \\ &+ \alpha_n\left\|\frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| + \lambda_n\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|K + |\alpha_{n+1} - \alpha_n|L \\ &\le \gamma_n\left\|\frac{\beta_{n+1}}{\beta_n}(x_{n+1} - x_n) + \left(1 - \frac{\beta_{n+1}}{\beta_n}\right)(x_{n+1} - J_{\beta_n}x_n)\right\| \\ &+ \alpha_n\left\|\frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| + \lambda_n\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|K + |\alpha_{n+1} - \alpha_n|L, \end{aligned}$$

which implies that

$$\begin{aligned} \frac{\|x_{n+2} - x_{n+1}\|}{\beta_{n+1}} &\le (1-\alpha_n)\frac{\|x_{n+1} - x_n\|}{\beta_n} + \left(\frac{1}{\beta_n} - \frac{1}{\beta_{n+1}}\right)K + \frac{\alpha_n}{\beta_{n+1}}\left\|\frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| \\ &+ \frac{|\lambda_{n+1} - \lambda_n|}{\beta_{n+1}}K + \frac{|\alpha_{n+1} - \alpha_n|}{\beta_{n+1}}L, \end{aligned}$$

for some positive constants K and L. On the other hand, (β_n) increasing implies

$$\frac{1}{\beta_n} \text{ is convergent} \quad \Leftrightarrow \quad \sum_{n=0}^{\infty}\left(\frac{1}{\beta_n} - \frac{1}{\beta_{n+1}}\right) < \infty. \tag{4.8}$$

Therefore, from this fact and Lemma 2.1.1, we get (4.6). The rest of the proof is similar to that of Theorem 4.1.2. ☐

Theorem 4.1.4. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. Fix $u, x_0 \in H$, and let (x_n) be the sequence generated by algorithm (4.1) with the conditions:* (i) *and* (ii) *of Theorem 4.1.2,* (iii) $\lambda_n, \gamma_n \in [0,1]$, $\alpha_n + \lambda_n + \gamma_n = 1$ *with* $\lim_{n\to\infty}\lambda_n = 0$*, and* (iv) $\beta_n \in (0,\infty)$ *with* $\lim_{n\to\infty}\beta_n = \infty$*, being satisfied. Then (x_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. Observe that for $\beta_n \to \infty$ and (x_n) bounded, passing to the limit in (4.4) immediately yields $\omega_{\mathtt{w}}((x_n)) = \omega_{\mathtt{w}}((v_n)) \subset F$. Again we derive strong convergence of (x_n) to $P_F u$ in a similar way as in the proof of Theorem 4.1.2. ☐

Remark 4.1.3. Theorem 4.1.4 contains Theorem 3.2.4 as a special case.

In [10, Theorem 4], we proved the following theorem (whose proof we shall omit) which is an extension of the result of Yao and Noor [56, Theorem 3.3] to general errors. Compared with the previous results of this section, strong convergence of the sequence (x_n) generated by algorithm (4.1) is still derived when (λ_n) does not converge to zero.

Theorem 4.1.5. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. Fix $u, x_0 \in H$, and let (x_n) be the sequence generated by algorithm (4.1) with the conditions:* (i) *and* (ii) *of Theorem 4.1.2,* (iii) $\gamma_n \in (0,1)$*, and* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$*,* (iv) $\beta_n \in (0,\infty)$ *with* $\liminf_{n\to\infty} \beta_n > 0$ *and (C6) being satisfied. Then (x_n) converges strongly to $P_F u$, the projection of u on F.*

As it may have been pointed out before, for unbounded (β_n), the condition (C6) excludes the natural choice $\beta_n = n$ for all $n \in \mathbb{N}_0$. The above result therefore brings us to the following question: Does Theorem 4.1.5 remains true if (λ_n) is only bounded from above away from 1 and/or (β_n) satisfies weaker conditions which include choices such as $\beta_n = n$ for all $n \in \mathbb{N}_0$? This question is addressed in the following result

Theorem 4.1.6. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. Fix $u, x_0 \in H$, and let (x_n) be the sequence generated by algorithm* (4.1) *with the conditions:* (i) $\alpha_n \in (0,1)$ *with (C1) and (C2),* (ii) *either (E1) or (E2),* (iii) $\lambda_n, \gamma_n \in [0,1]$, $\alpha_n + \lambda_n + \gamma_n = 1$ *with* $\liminf_{n\to\infty} \gamma_n > 0$*, and* (iv) $\beta_n \in (0,\infty)$ *with* $\liminf_{n\to\infty} \beta_n > 0$ *and* $(C18)$ *being satisfied. Then (x_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. From Lemma 4.1.1, we know that (x_n) is bounded. Denote

$$y_n := T_n x_n + \mu_n(u - x_n) + \sigma_n,$$

where $T_n = 2J_{\beta_n} - I$, $\mu_n = 2\alpha_n/\gamma_n$ and $\sigma_n = 2e_n/\gamma_n$. Obviously, the sequence (y_n) is bounded (since (x_n) is so), and from the definition of T_n, (4.1) can be written as

$$\begin{aligned} x_{n+1} &= \alpha_n u + \lambda_n x_n + \frac{\gamma_n}{2} x_n + \frac{\gamma_n}{2} T_n x_n + e_n \\ &= \left(1 - \frac{\gamma_n}{2}\right) x_n + \frac{\gamma_n}{2}\left(T_n x_n + \frac{2\alpha_n}{\gamma_n}(u - x_n) + \frac{2e_n}{\gamma_n}\right) \\ &= \left(1 - \frac{\gamma_n}{2}\right) x_n + \frac{\gamma_n}{2} y_n. \end{aligned}$$

Since T_n is nonexpansive, (see Lemma 2.2.3), we have for some positive constant M

$$\begin{aligned}
\|y_{n+1}-y_n\| &\leq \|T_{n+1}x_{n+1}-T_nx_n\|+\mu_{n+1}\|u-x_{n+1}\|+\mu_n\|u-x_n\|+\|\sigma_{n+1}-\sigma_n\| \\
&\leq \|T_{n+1}x_{n+1}-T_{n+1}x_n\|+\|T_{n+1}x_n-T_nx_n\|+(\mu_{n+1}+\mu_n)M \\
&+ \|\sigma_{n+1}-\sigma_n\| \\
&\leq \|x_{n+1}-x_n\|+2\left\|J_{\beta_{n+1}}x_n-J_{\beta_n}x_n\right\|+(\mu_{n+1}+\mu_n)M+\|\sigma_{n+1}-\sigma_n\| \\
&= \|x_{n+1}-x_n\|+2\left\|J_{\beta_{n+1}}x_n-J_{\beta_{n+1}}\left(\frac{\beta_{n+1}}{\beta_n}x_n+\left(1-\frac{\beta_{n+1}}{\beta_n}\right)J_{\beta_n}x_n\right)\right\| \\
&+ (\mu_{n+1}+\mu_n)M+\|\sigma_{n+1}-\sigma_n\| \\
&\leq \|x_{n+1}-x_n\|+2\left|1-\frac{\beta_{n+1}}{\beta_n}\right|\|x_n-J_{\beta_n}x_n\|+(\mu_{n+1}+\mu_n)M \\
&+ \|\sigma_{n+1}-\sigma_n\|, \qquad (4.9)
\end{aligned}$$

where equality follows from the application of the resolvent identity. Rearranging terms of (4.9) and passing to the limit as $n\to\infty$, we get

$$\limsup_{n\to\infty}\{\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\} \leq 0.$$

Therefore applying Lemma 2.2.1, we get

$$\lim_{n\to\infty}\|x_n-y_n\|=0,$$

which implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (4.10)$$

Note that since $\liminf_{n\to\infty}\gamma_n>0$, there exists $\delta\in[0,1)$ such that $\lambda_n\leq\delta$ for all $n\in\mathbb{N}_0$. Then from (4.1), we see that

$$\begin{aligned}
\|x_{n+1}-J_{\beta_n}x_n\| &\leq \alpha_n\|u-J_{\beta_n}x_n+e_n/\alpha_n\|+\lambda_n\|x_n-J_{\beta_n}x_n\| \\
&\leq \alpha_n\|u-J_{\beta_n}x_n+e_n/\alpha_n\|+\delta(\|x_n-x_{n+1}\|+\|x_{n+1}-J_{\beta_n}x_n\|),
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|x_{n+1}-J_{\beta_n}x_n\|=0. \qquad (4.11)$$

On the other hand, we observe that if $\beta>0$ is the greatest lower bound of (β_n), then the application of the resolvent identity yields

$$\begin{aligned}
\|J_{\beta_n}x_n-J_\beta x_n\| &\leq \left\|\left(1-\frac{\beta}{\beta_n}\right)(J_{\beta_n}x_n-x_n)\right\| \\
&\leq \|J_{\beta_n}x_n-x_{n+1}\|+\|x_{n+1}-x_n\|.
\end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality, and noticing (4.10) and (4.11), we have

$$\lim_{n\to\infty} \|J_{\beta_n}x_n - J_\beta x_n\| = 0. \tag{4.12}$$

Moreover, from (4.10), (4.11) and (4.12), we get

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - J_\beta x_n\| &\leq \limsup_{n\to\infty} \left(\|x_n - x_{n+1}\| + \|x_{n+1} - J_{\beta_n}x_n\| + \|J_{\beta_n}x_n - J_\beta x_n\|\right) \\ &= 0. \end{aligned} \tag{4.13}$$

Now let (x_{n_k}) be a subsequence of (x_n) converging weakly to some z. Then for some positive constant K,

$$\begin{aligned} 2\langle x_{n_k} - J_\beta z, z - J_\beta z\rangle &= \|x_{n_k} - J_\beta z\|^2 + \|z - J_\beta z\|^2 - \|x_{n_k} - z\|^2 \\ &\leq (\|x_{n_k} - J_\beta x_{n_k}\| + \|x_{n_k} - z\|)^2 + \|z - J_\beta z\|^2 - \|x_{n_k} - z\|^2 \\ &\leq K\|x_{n_k} - J_\beta x_{n_k}\| + \|z - J_\beta z\|^2. \end{aligned}$$

Passing to the limit in the above inequality as $k \to \infty$, and noticing (4.13), we arrive at $z \in A^{-1}(0)$. Hence for some subsequence (x_{n_j}) of (x_n) converging weakly to some point x_∞, say, we have

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle = \lim_{j\to\infty} \langle u - P_F u, x_{n_j} - P_F u\rangle = \langle u - P_F u, x_\infty - P_F u\rangle \leq 0.$$

Finally, from Lemma 2.2.7 and equation (4.1), we have

$$\begin{aligned} \|x_{n+1} - P_F u\|^2 &\leq (\lambda_n \|x_n - P_F u\| + \gamma_n \|J_{\beta_n}x_n - P_F u\|)^2 \\ &+ 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, x_{n+1} - P_F u \right\rangle \\ &\leq (1 - \alpha_n)\|x_n - P_F u\|^2 + 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, x_{n+1} - P_F u \right\rangle. \end{aligned}$$

Therefore, from Lemma 2.1.1 we derive strong convergence of (x_n) to $P_F u$. In the case when the error sequence (e_n) satisfies condition (E1), then we get from Lemma 2.2.7 and equation (4.1)

$$\|x_{n+1} - P_F u\|^2 \leq (1 - \alpha_n)\|x_n - P_F u\|^2 + 2\alpha_n \langle u - P_F u, x_{n+1} - P_F u\rangle + \|e_n\| C,$$

for some $C > 0$. As before, strong convergence of (x_n) to $P_F u$ can be derived. □

Remark 4.1.4. Unlike in Theorem 4.1.5, we do not require the sequence (γ_n) to be bounded above away from one. In addition, we have used the weaker condition (C18) instead of (C6). Therefore, Theorem 4.1.6 is a significant improvement of Theorem 4.1.5.

Compared with Theorems 4.1.2 and 4.1.3 above, the main advance in Theorem 4.1.6 is that strong convergence of the sequence (x_n) is proved under weaker assumptions on both (α_n) and (λ_n). However, the condition (C18) of the above result is stronger than the assumption (C14) used in Theorems 4.1.2 and 4.1.3. (For the comparison of the conditions (C14) and (C18), see Remark 4.1.5 below). We also note that for the case when $\lambda_n = 0$ for all $n \in \mathbb{N}_0$, the condition $\liminf_{n\to\infty} \gamma_n > 0$ is automatically satisfied, since the sequence (α_n) converges to zero. In this particular case, our result properly contains Theorem 4 [52] which required the error sequence (e_n) to be summable in norm, and (β_n) to be bounded both from above and from below away from zero, with the condition (C6) being satisfied.

Remark 4.1.5. Note that for the sequence (β_n) satisfying $\beta_n \geq \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}_0$,

$$\left|\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right| = \frac{1}{\beta_{n+1}}\left|1 - \frac{\beta_{n+1}}{\beta_n}\right| \leq \frac{1}{\varepsilon}\left|1 - \frac{\beta_{n+1}}{\beta_n}\right|$$

implying that the condition (C14) is weaker than the condition (C18) of the preceding theorem. Indeed, one can verify that the sequence

$$\beta_n = \begin{cases} 2n \text{ if } n \text{ is odd,} \\ 3n \text{ if } n \text{ is even} \end{cases}$$

satisfies (C14) but not (C18). These two conditions are however equivalent if (β_n) is bounded both from below away from zero and from above.

4.2 A generalized regularization method

In this section, we suggest and analyze a new iterative process for solving problem (1.5) which is in fact a generalization of Algorithm 3.4.1. It is defined as follows: Given any fixed $u, v_1 \in H$, generate a sequence (v_n) iteratively by the rule

$$v_{n+1} = J_{\beta_n}(\alpha_{n-1}u + \lambda_{n-1}v_n + \gamma_{n-1}Tv_n + e_{n-1}), \quad \text{for all} \quad n \geq 1, \tag{4.14}$$

where $T : H \to H$ is a nonexpansive map, $\alpha_n \in (0, 1)$, $\lambda_n, \gamma_n \in [0, 1]$ with $\alpha_n+\lambda_n+\gamma_n = 1$, $\beta_n \in (0, \infty)$, and (e_n) is a sequence of computational errors. It is clear that for $\lambda_n = 0$ for all $n \geq 1$, algorithm (4.14) collapses to

$$v_{n+1} = J_{\beta_n}(\alpha_{n-1}u + (1 - \alpha_{n-1})Tv_n + e_{n-1}), \quad \text{for all} \quad n \geq 1, \tag{4.15}$$

which is of the same form as the regularization method proposed by Xu [55], and discussed in Chapter 3. In fact, for $T = I$ (the identity operator), this special case corresponds to the case when $\lambda_n = 0$ for all $n \geq 1$ in algorithm (4.1). Note that the equivalence of such algorithms was discussed in Section 3.4.

Concerning the boundedness of the sequence (v_n) defined by (4.14) above, we have the following lemma

Lemma 4.2.1. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $T : H \to H$ is a nonexpansive map, with $\emptyset \neq A^{-1}(0) \subset F(T)$, where $F(T)$ is the fixed point set of T. Let $\beta_n \in (0, \infty)$, $\alpha_n \in (0, 1)$, and $\lambda_n, \gamma_n \in [0, 1]$ with $\alpha_n + \lambda_n + \gamma_n = 1$ for all $n \geq 1$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$. If either $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $(\|e_n\|/\alpha_n)$ is bounded, then for any fixed $u, v_1 \in H$, the sequence (v_n) defined by (4.14) is bounded.*

Proof. Assume that $(\|e_n\|/\alpha_n)$ is bounded. Then, there exists $M > 0$ such that

$$\|u - p\| + \frac{\|e_n\|}{\alpha_n} \leq M,$$

for some $p \in F$ and all $n \geq 0$. Using (4.14), we see that

$$\begin{aligned} \|v_{n+1} - p\| &\leq \|\alpha_{n-1}(u - p + e_{n-1}/\alpha_{n-1}) + \lambda_{n-1}(v_n - p) + \gamma_{n-1}(Tv_n - p)\| \\ &\leq \alpha_{n-1}\left[\|u - p\| + \frac{\|e_{n-1}\|}{\alpha_{n-1}}\right] + \lambda_{n-1}\|v_n - p\| + \gamma_{n-1}\|Tv_n - p\| \\ &\leq \alpha_{n-1}M + (1 - \alpha_{n-1})\|v_n - p\|, \end{aligned}$$

where the first two inequalities follow from the fact that J_{β_n} and T are nonexpansive. By induction, we have

$$\|v_{n+1} - p\| \leq M\left[1 - \prod_{k=0}^{n-1}(1 - \alpha_k)\right] + \|v_1 - p\| \prod_{k=0}^{n-1}(1 - \alpha_k),$$

showing that (v_n) is bounded.

In the case when condition (E1) is satisfied, then we have for any $p \in F$,

$$\begin{aligned} \|v_{n+1} - p\| &\leq \|\alpha_{n-1}(u - p) + \lambda_{n-1}(v_n - p) + \gamma_{n-1}(Tv_n - p) + e_{n-1}\| \\ &\leq \alpha_{n-1}\|u - p\| + (1 - \alpha_{n-1})\|v_n - p\| + \|e_{n-1}\|, \end{aligned}$$

which implies that

$$\|v_{n+1} - p\| \leq \left[1 - \prod_{k=0}^{n-1}(1 - \alpha_k)\right]\|u - p\| + \|v_1 - p\| \prod_{k=0}^{n-1}(1 - \alpha_k) + \sum_{k=0}^{n-1}\|e_k\|.$$

This shows that (v_n) is bounded. □

Let $T : H \to H$ be a nonexpansive map, and let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator. Fix $n \in \mathbb{N}_0$, and define a map $f_n : H \to H$ by the rule $x \mapsto J_{\beta_n}(\alpha_n u + \lambda_n x + \gamma_n Tx + e_n)$, where $\beta_n > 0$, (α_n), (λ_n) and (γ_n) are real sequences in $(0, 1)$

such that $\alpha_n + \lambda_n + \gamma_n = 1$, and $u, e_n \in H$ are given. Then one can easily check that f_n is a contraction. Therefore it follows from the Banach contraction principle that f_n has a unique fixed point z_n, say. In other words,

$$z_n = J_{\beta_n}(\alpha_n u + \lambda_n z_n + \gamma_n T z_n + e_n), \quad n \geq 0. \tag{4.16}$$

We prove the convergence result associated with the sequence (z_n).

Lemma 4.2.2. *Let $\beta_n \in (0, \infty)$, and $\alpha_n, \lambda_n, \gamma_n \in (0, 1)$ with $\alpha_n + \lambda_n + \gamma_n = 1$ for all $n \in \mathbb{N}_0$, and $\lim_{n\to\infty} \alpha_n = 0$. Assume that $\emptyset \neq A^{-1}(0) =: F \subset F(T)$, where $F(T)$ is the set of fixed points of the nonexpansive map $T : H \to H$, and either (E1) or (E2) is satisfied. Then for any fixed $u \in H$, the sequence (z_n) generated by (4.16) converges strongly to $P_F u$, the projection of u on F.*

Proof. To show that (z_n) is bounded, we first note that if $(\|e_n\|/\alpha_n)$ is bounded, then there exists a positive constant C such that

$$\sup_{n\in\mathbb{N}_0} \left(\|u - p\| + \frac{\|e_n\|}{\alpha_n} \right) \leq C.$$

For every $p \in F$, we have from (4.16)

$$\begin{aligned} \|z_n - p\| &\leq \|\alpha_n(u - p + e_n/\alpha_n) + \lambda_n(z_n - p) + \gamma_n(Tz_n - p)\| \\ &\leq \alpha_n \left(\|u - p\| + \frac{\|e_n\|}{\alpha_n} \right) + \lambda_n \|z_n - p\| + \gamma_n \|z_n - p\| \\ &\leq (1 - \alpha_n) \|z_n - p\| + \alpha_n C, \end{aligned}$$

where the first two inequalities follow from the fact that J_{β_n} and T are nonexpansive. The last estimate clearly shows that (z_n) is bounded.

Let $\omega_{\mathtt{w}}((z_n))$ be the weak $\omega-$limit set of (z_n). That is,

$$\omega_{\mathtt{w}}((z_n)) = \{y \in H \mid z_{n_k} \rightharpoonup y \text{ for some subsequence } (z_{n_k}) \text{ of } (z_n)\}.$$

We claim that $\omega_{\mathtt{w}}((z_n)) \subset F$. Let (z_{n_j}) be a subsequence of (z_n) converging weakly to some z_∞. Since (λ_{n_j}) is bounded, it has a convergent subsequence, again denoted by (λ_{n_j}). There are two possibilities here: either $\lambda_{n_j} \to 1$, or $\lambda_{n_j} \to \theta \in [0, 1)$. In the first case, we derive from

$$Az_{n_j} \ni \frac{\alpha_{n_j} u + (\lambda_{n_j} - 1) z_{n_j} + \gamma_{n_j} T z_{n_j} + e_{n_j}}{\beta_{n_j}} \to 0, \quad \text{as} \quad j \to \infty, \tag{4.17}$$

that $z_\infty \in F$. In the second case, we note that from (4.16), we have

$$(1 - \lambda_n)\langle z_n - Tz_n, z_n - p\rangle + \beta_n\langle Az_n, z_n - p\rangle = \alpha_n\langle u - Tz_n + e_n/\alpha_n, z_n - p\rangle,$$

where $p \in F$. Using the monotonicity of A, we have for some $M > 0$

$$\begin{aligned}\alpha_n M &\geq 2(1-\lambda_n)\langle z_n - Tz_n, z_n - p\rangle \\ &= (1-\lambda_n)(\|z_n - Tz_n\|^2 + \|z_n - p\|^2 - \|Tz_n - Tp\|^2) \\ &\geq (1-\lambda_n)\,\|z_n - Tz_n\|^2 .\end{aligned}$$

Passing to the limit in the above estimate, with $n = n_j$, we get

$$\lim_{j\to\infty} \left\|z_{n_j} - Tz_{n_j}\right\| = 0.$$

Again from (4.17), we derive $z_\infty \in F$, showing that $\omega_{\mathtt{w}}((z_n)) \subset F$. Therefore, there exists a subsequence (z_{n_k}) of (z_n) converging weakly to $z \in F$ such that

$$\limsup_{n\to\infty} \langle u - P_F u, z_n - P_F u\rangle = \lim_{k\to\infty} \langle u - P_F u, z_{n_k} - P_F u\rangle = \langle u - P_F u, z - P_F u\rangle \leq 0.$$

On the other hand,

$$\begin{aligned}\|z_n - P_F u\|^2 &\leq \alpha_n^2 \left(\|u - P_F u\| + \frac{\|e_n\|}{\alpha_n}\right)^2 + (\lambda_n \|z_n - P_F u\| + \gamma_n \|Tz_n - P_F u\|)^2 \\ &+ 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, \lambda_n(z_n - P_F u) + \gamma_n(Tz_n - P_F u)\right\rangle \\ &\leq (1-\alpha_n)^2 \|z_n - P_F u\|^2 + \alpha_n^2 \left(\|u - P_F u\| + \frac{\|e_n\|}{\alpha_n}\right)^2 \\ &+ 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, (1-\alpha_n)(z_n - P_F u) + \gamma_n(Tz_n - z_n)\right\rangle ,\end{aligned}$$

where the second inequality follows from the nonexpansivity of T. Hence for some positive constant C^*,

$$(2-\alpha_n)\,\|z_n - P_F u\|^2 \leq \alpha_n C^* + 2\left\langle u - P_F u + \frac{e_n}{\alpha_n}, (z_n - P_F u) + \gamma_n(Tz_n - z_n)\right\rangle .$$

Passing to the limit in the above inequality, we deduce strong convergence of (z_n) to $P_F u$ as claimed. The proof of this result is easier in the case when $(\|e_n\|) \in \ell^1$, we therefore omit it. □

We note that Lemma 4.2.2 above contains Theorem 3.2.6 as a special case. Using the above result, we can prove the following

Theorem 4.2.3. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $\emptyset \neq A^{-1}(0) =: F \subset F(T)$, where $T : H \to H$ is a nonexpansive map, and $F(T)$ is the set of fixed points of T. Fix $u, v_1 \in H$, and let (v_n) be the sequence generated by algorithm*

(4.14) *with the conditions:* (i) $\alpha_n \in (0,1)$, $\lambda_n, \gamma_n \in [0,1]$, $\alpha_n + \lambda_n + \gamma_n = 1$ *with (C1) and (C2),* (ii) *either (E1) or (E2),* (iii) $\beta_n \in (0,\infty)$ *with* $\liminf_{n\to\infty} \beta_n > 0$,

$$(C8)' \lim_{n\to\infty} \frac{1}{\alpha_{n-1}}\left(1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right) = 0, \quad \text{and} \quad (C20) \lim_{n\to\infty} \frac{1}{\alpha_{n-1}\alpha_n}\left(\gamma_n - \frac{\gamma_{n-1}\beta_{n+1}}{\beta_n}\right) = 0,$$

being satisfied. Then (v_n) *converges strongly to* $P_F u$*, the projection of* u *on* F*.*

Proof. According to Lemma 4.2.1, the sequence (v_n) is bounded. Setting

$$w_n = J_{\beta_n}\left(\alpha_{n-1}u + \lambda_{n-1}w_n + \gamma_{n-1}Tw_n\right), \tag{4.18}$$

we see from Lemma 4.2.2 that $w_n \to P_F u$. Now it follows from (4.14) that

$$\begin{aligned} \|v_{n+1} - w_{n+1}\| &\leq \|v_{n+1} - w_n\| + \|w_n - w_{n+1}\| \\ &\leq \lambda_{n-1}\|v_n - w_n\| + \gamma_{n-1}\|Tv_n - Tw_n\| + \|e_{n-1}\| + \|w_n - w_{n+1}\| \\ &\leq (1-\alpha_{n-1})\|v_n - w_n\| + \|e_{n-1}\| + \|w_n - w_{n+1}\|, \end{aligned} \tag{4.19}$$

where the last inequality comes form the nonexpansivity of the map T. Using the resolvent identity, we note that (4.18) can be written as

$$w_n = J_\varepsilon\left(\frac{\varepsilon}{\beta_n}(\alpha_{n-1}u + \lambda_{n-1}w_n + \gamma_{n-1}Tw_n) + \left(1 - \frac{\varepsilon}{\beta_n}\right)w_n\right),$$

where $\varepsilon > 0$ is the greatest lower bound of (β_n). This together with the fact that the resolvent operator J_ε is nonexpansive gives

$$\begin{aligned} \|w_{n+1} - w_n\| &\leq \left(1 - \frac{\varepsilon}{\beta_{n+1}}\right)\|w_{n+1} - w_n\| + \frac{\varepsilon\lambda_n}{\beta_{n+1}}\|w_{n+1} - w_n\| \\ &+ \frac{\varepsilon\gamma_n}{\beta_{n+1}}\|Tw_{n+1} - Tw_n\| + \left|\frac{\varepsilon\alpha_n}{\beta_{n+1}} - \frac{\varepsilon\alpha_{n-1}}{\beta_n}\right|\|u - w_n\| \\ &+ \left|\frac{\varepsilon\gamma_n}{\beta_{n+1}} - \frac{\varepsilon\gamma_{n-1}}{\beta_n}\right|\|Tw_n - w_n\| \\ &\leq \left(1 - \frac{\varepsilon\alpha_n}{\beta_{n+1}}\right)\|w_{n+1} - w_n\| + \left|\frac{\varepsilon\alpha_n}{\beta_{n+1}} - \frac{\varepsilon\alpha_{n-1}}{\beta_n}\right|K \\ &+ \left|\frac{\varepsilon\gamma_n}{\beta_{n+1}} - \frac{\varepsilon\gamma_{n-1}}{\beta_n}\right|M, \end{aligned} \tag{4.20}$$

for some positive constants K and M. This last estimate reduces to

$$\|w_{n+1} - w_n\| \leq \left|1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right|K + \left|\frac{\gamma_n}{\alpha_n} - \frac{\gamma_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right|M. \tag{4.21}$$

Using this last inequality in (4.19) we arrive at

$$\begin{aligned} \|v_{n+1} - w_{n+1}\| &\leq (1-\alpha_{n-1})\|v_n - w_n\| + \|e_{n-1}\| + \left|1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right|K \\ &+ \frac{1}{\alpha_n}\left|\gamma_n - \frac{\gamma_{n-1}\beta_{n+1}}{\beta_n}\right|M. \end{aligned}$$

Therefore by Lemma 2.1.1, we derive $\|v_n - w_n\| \to 0$, which in turn implies that the sequence (v_n) converges strongly to $P_F u$. $\square$

Example 4.2.1. Clearly, the sequences (α_n), (β_n) and (γ_n) defined by $\alpha_n = 1/\sqrt{n+1}$, $\beta_n = 1 + n^{-1}$ and $\gamma_n = 1/(n+1)$ for $n \geq 2$ satisfy the conditions (C8)′ and (C20).

Remark 4.2.1. A result similar to the above theorem was proved in Section 3.4 for $T = I$, the identity operator, and under the additional assumption that $\beta_{n+1} \geq \alpha_n \beta_n$. Therefore, Theorem 4.2.3 is a generalization and improvement of Theorem 4 [9]. Note that Theorem 3.2 [55] which is similar in nature to Theorem 4 [9] can also be generalized in the same way.

Other strong convergence results associated with the sequence (v_n) can be established under weaker assumptions on (α_n) and (β_n) such as

$$(C10)\ \sum_{n=1}^{\infty} \left| \frac{\alpha_n}{\beta_{n+1}} - \frac{\alpha_{n-1}}{\beta_n} \right| < \infty, \quad \text{or} \quad (C11)\ \lim_{n\to\infty} \left(1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n \beta_n} \right) = 0,$$

discussed in the previous chapter. In fact, we modify the arguments contained in Section 3.4, and by so doing, we are able to make some improvements to the results of that section. In particular, the condition that the sequence on (β_n) is increasing in Theorem 3.4.2 can be dropped completely, and for (α_n) decreasing, (C11) can be replaced with a slightly better condition (C21). Obviously, the condition (C21) is weaker than (C8)′. Thus, we refine Theorem 3.4.2, (in the case when (α_n) is monotonically decreasing), and also Theorem 3.4.1, (see Theorem 4.2.7 below). In the next theorem, we shall use similar conditions to (C10) and (C21) on (γ_n) for the general case when this sequence is not identically zero for all $n \geq 1$ to derive strong convergence of (v_n). These conditions are

$$(C22)\ \sum_{n=1}^{\infty} \left| \frac{\gamma_n}{\beta_{n+1}} - \frac{\gamma_{n-1}}{\beta_n} \right| < \infty, \quad \text{and} \quad (C23)\ \lim_{n\to\infty} \frac{1}{\alpha_{n-1}} \left(\frac{\gamma_n}{\beta_{n+1}} - \frac{\gamma_{n-1}}{\beta_n} \right) = 0.$$

Theorem 4.2.4. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator and $T : H \to H$ a nonexpansive map with $\emptyset \neq A^{-1}(0) =: F \subset F(T)$, where $F(T)$ is the set of all fixed points of T. For any fixed $u, v_1 \in H$, let the sequence (v_n) be generated by algorithm (4.14), where $\alpha_n \in (0,1)$, $\lambda_n, \gamma_n \in [0,1]$ with $\alpha_n + \lambda_n + \gamma_n = 1$, and $\beta_n \in (0,\infty)$. Assume that (α_n), (β_n), (γ_n) and (e_n) satisfy* (i) *(C1), (C2),* (ii) *either (E1) or (E2),* (iii) $\liminf_{n\to\infty} \beta_n > 0$, *with either (C10) or (C21) and* (iv) *either (C22) or (C23). If either $\lambda_n \to 0$ and (β_n) is bounded, or $\gamma_n \to 0$, then (v_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. We have shown in Lemma 4.2.1 that (v_n) is bounded. The next step is to show that the relation $\omega_{\mathtt{w}}((v_n)) \subset F$ holds. This will suffice to guarantee

$$\limsup_{n\to\infty} \langle u - P_F u, v_n - P_F u\rangle \le 0.$$

Indeed, for some subsequence of (v_n) converging weakly to some v_∞, we have

$$\limsup_{n\to\infty} \langle u - P_F u, v_n - P_F u\rangle = \lim_{k\to\infty} \langle u - P_F u, v_{n_k} - P_F u\rangle = \langle u - P_F u, v_\infty - P_F u\rangle.$$

Note that the resolvent identity allows us to write (4.14) as

$$v_{n+1} = J_{\beta_{n+1}}\left(\frac{\beta_{n+1}}{\beta_n}(\alpha_{n-1}u + \lambda_{n-1}v_n + \gamma_{n-1}Tv_n + e_{n-1}) + \left(1 - \frac{\beta_{n+1}}{\beta_n}\right)v_{n+1}\right).$$

For the case when $\|e_n\|/\alpha_n \to 0$, we have, (for some positive constants K and M),

$$\begin{aligned}
\|v_{n+2} - v_{n+1}\| &\le \left\| \frac{\gamma_{n-1}\beta_{n+1}}{\beta_n}(Tv_{n+1} - Tv_n) + \frac{\lambda_{n-1}\beta_{n+1}}{\beta_n}(v_{n+1} - v_n) \right.\\
&+ \frac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\left(\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right) + \left(\gamma_n - \frac{\gamma_{n-1}\beta_{n+1}}{\beta_n}\right)(Tv_{n+1} - v_{n+1})\\
&+ \left.\left(\alpha_n - \frac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\right)\left(u - v_{n+1} + \frac{e_n}{\alpha_n}\right)\right\|\\
&\le \frac{\beta_{n+1}}{\beta_n}(1-\alpha_{n-1})\|v_{n+1} - v_n\| + \frac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|\\
&+ \left|\gamma_n - \frac{\gamma_{n-1}\beta_{n+1}}{\beta_n}\right| K + \left|\alpha_n - \frac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\right| M, \qquad (4.22)
\end{aligned}$$

where the first inequality follows from the fact that the resolvent operator is nonexpansive. Estimate (4.22) implies that

$$\begin{aligned}
\frac{\|v_{n+2} - v_{n+1}\|}{\beta_{n+1}} &\le (1-\alpha_{n-1})\frac{\|v_{n+1} - v_n\|}{\beta_n} + \frac{\alpha_{n-1}}{\beta_n}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|\\
&+ \left|\frac{\gamma_n}{\beta_{n+1}} - \frac{\gamma_{n-1}}{\beta_n}\right| K + \left|\frac{\alpha_n}{\beta_{n+1}} - \frac{\alpha_{n-1}}{\beta_n}\right| M.
\end{aligned}$$

So if either (C10) or (C21), and either one of the conditions (C22) or (C23) is fulfilled, then we have by Lemma 2.1.1

$$\frac{\|v_{n+1} - v_n\|}{\beta_n} \to 0. \qquad (4.23)$$

Note that from (4.14) we have

$$\frac{(v_{n+1} - v_n)}{\beta_n} + Av_{n+1} \ni \frac{\alpha_{n-1}}{\beta_n}\left(u - v_n + \frac{e_{n-1}}{\alpha_{n-1}}\right) + \frac{\gamma_{n-1}}{\beta_n}(Tv_n - v_n),$$

so that for the case when $\gamma_n \to 0$, we derive $\omega_{\mathtt{w}}((v_n)) \subset F$.

Now we assume that $\lambda_n \to 0$ and (β_n) is bounded. Then, we have again from (4.14)

$$\begin{aligned}(v_{n+1} - Tv_{n+1}) + \beta_n Av_{n+1} &\ni \alpha_{n-1}(u - Tv_n + e_{n-1}/\alpha_{n-1}) + \lambda_{n-1}(v_n - Tv_n)\\ &+ (Tv_n - Tv_{n+1}),\end{aligned}$$

which (together with the monotonicity of A, the boundedness of (v_n) and $(\|e_n\|/\alpha_n)$, and the fact that T is nonexpansive) implies that

$$\begin{aligned}C(\alpha_{n-1} + \lambda_{n-1} + \|v_n - v_{n+1}\|) &\geq 2\langle v_{n+1} - Tv_{n+1}, v_{n+1} - p\rangle + 2\beta_n\langle Av_{n+1}, v_{n+1} - p\rangle\\ &\geq \|v_{n+1} - Tv_{n+1}\|^2 + \|v_{n+1} - p\|^2 - \|Tv_{n+1} - p\|^2\\ &\geq \|v_{n+1} - Tv_{n+1}\|^2, \qquad (4.24)\end{aligned}$$

for some $C > 0$, where p is any point of F. Passing to the limit in (4.24) and using (4.23) with (β_n) bounded, we get

$$\|v_{n+1} - Tv_{n+1}\| \to 0.$$

Moreover, using again (4.23), we have

$$\begin{aligned}\|v_{n+1} - Tv_n\| &\leq \|v_{n+1} - Tv_{n+1}\| + \|Tv_{n+1} - Tv_n\|\\ &\leq \|v_{n+1} - Tv_{n+1}\| + \|v_{n+1} - v_n\| \to 0. \qquad (4.25)\end{aligned}$$

On the other hand, from (4.14), we have

$$(v_{n+1} - Tv_n) + \beta_n Av_{n+1} \ni \alpha_{n-1}(u - Tv_n + e_{n-1}/\alpha_{n-1}) + \lambda_{n-1}(v_n - Tv_n).$$

Since β_n is bounded, this inclusion relation together with (4.25) imply $\omega_{\mathtt{w}}((v_n)) \subset F$. Note that the proof can be done similarly for the case when (E1) is satisfied.

Finally, we establish strong convergence of (v_n) to $P_F u$.

Since both T and the resolvent operator are nonexpansive, we have from (4.14)

$$\begin{aligned}\|v_{n+1} - P_F u\|^2 &\leq \|\alpha_{n-1}(u - P_F u + e_{n-1}/\alpha_{n-1}) + \lambda_{n-1}(v_n - P_F u) + \gamma_{n-1}(Tv_n - P_F u)\|^2\\ &\leq (1 - \alpha_{n-1})\|v_n - P_F u\|^2 + \alpha_{n-1}^2\|u - P_F u + e_{n-1}/\alpha_{n-1}\|^2\\ &+ 2\alpha_{n-1}\langle u - P_F u + e_{n-1}/\alpha_{n-1}, \lambda_{n-1}(v_n - P_F u) + \gamma_{n-1}(Tv_n - P_F u)\rangle.\end{aligned}$$

Therefore if either $\gamma_n \to 0$ or $\lambda_n \to 0$ and (β_n) is bounded, then by Lemma 2.1.1, we derive $v_n \to P_F u$. The proof is similar for the case when (E1) is satisfied. □

When (β_n) is bounded, (both from above and from below away from zero), condition (C21) reduces to

$$(C21)' \lim_{n\to\infty}\left(\frac{\alpha_n}{\alpha_{n-1}} - \frac{\beta_{n+1}}{\beta_n}\right) = 0.$$

Therefore, in the case when $\lambda_n = 0$ for all $n \geq 1$, we have

Theorem 4.2.5. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator and $T : H \to H$ a nonexpansive map with $\emptyset \neq A^{-1}(0) =: F \subset F(T)$, where $F(T)$ is the set of all fixed points of T. For any fixed $u, v_1 \in H$, let the sequence (v_n) be generated by algorithm (4.15), where $\alpha_n \in (0,1)$ and $\beta_n \in (0,\infty)$. Assume that (α_n), (β_n) and (e_n) satisfy* (i) *(C1), (C2),* (ii) *either (E1) or (E2),* (iii) $\liminf_{n\to\infty}\beta_n > 0$, *with (C9), and either (C10) or (C21)′. Then (v_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. We know from the proof of lemma 4.2.1 that (v_n) is bounded. Moreover, the resolvent identity enables us to write equation (4.15) as

$$v_{n+1} = J_{\beta_{n+1}}\left(\frac{\beta_{n+1}}{\beta_n}((1-\alpha_{n-1})Tv_n + \alpha_{n-1}u + e_{n-1}) + \left(1 - \frac{\beta_{n+1}}{\beta_n}\right)v_{n+1}\right).$$

Therefore, for $\|e_n\|/\alpha_n \to 0$, we have from the nonexpansivity of the resolvent,

$$\begin{aligned}
\|v_{n+2} - v_{n+1}\| &\leq \left\| \frac{\beta_{n+1}}{\beta_n}(1-\alpha_{n-1})(Tv_{n+1} - Tv_n) + \left(1 - \frac{\beta_{n+1}}{\beta_n}\right)(Tv_{n+1} - v_{n+1}) \right. \\
&+ \left. \left(\alpha_n - \frac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\right)\left(u - Tv_{n+1} + \frac{e_n}{\alpha_n}\right) + \frac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\left(\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right)\right\| \\
&\leq \frac{\beta_{n+1}}{\beta_n}(1-\alpha_{n-1})\|v_{n+1} - v_n\| + \left|\alpha_n - \frac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\right| \left\|u - Tv_{n+1} + \frac{e_n}{\alpha_n}\right\| \\
&+ \left|1 - \frac{\beta_{n+1}}{\beta_n}\right| \|Tv_{n+1} - v_{n+1}\| + \frac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|, \qquad (4.26)
\end{aligned}$$

so that for some positive constants K and M, we have

$$\begin{aligned}
\frac{\|v_{n+2} - v_{n+1}\|}{\beta_{n+1}} &\leq (1-\alpha_{n-1})\frac{\|v_{n+1} - v_n\|}{\beta_n} + \left|\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right| K + \left|\frac{\alpha_n}{\beta_{n+1}} - \frac{\alpha_{n-1}}{\beta_n}\right| M \\
&+ \frac{\alpha_{n-1}}{\beta_n}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|.
\end{aligned}$$

So, if condition (C9) and either (C10) or (C21)′ are fulfilled, then from Lemma 2.1.1, we have

$$\frac{\|v_{n+1} - v_n\|}{\beta_n} \to 0 \quad \Leftrightarrow \quad \|v_{n+1} - v_n\| \to 0. \qquad (4.27)$$

Now proceeding in a similar way as in the proof of Theorem 4.2.4 (with $\lambda_n = 0$ for all $n \geq 1$), we derive strong convergence of (v_n) to $P_F u$. The proof is similar for the case when (E1) is satisfied. □

Remark 4.2.2. Note that condition (C21) is weaker than (C11) (in the case when (α_n) is decreasing), and also (C21)$'$. Indeed, the sequences (α_n) and (β_n) defined by $\alpha_n = n^{-1}$ and $\beta_n = n!$ satisfy (C21) but not (C11) and (C21)$'$. Also, if

$$\beta_n = (n-1)!\beta_1 \Leftrightarrow \beta_{n+1} = n\beta_n, \quad \text{and} \quad \alpha_n = \frac{1}{n} + (-1)^n \frac{1}{n+1},$$

then (C21) is satisfied but not (C21)$'$ and (C11). Of course both (C11) and (C21)$'$ are satisfied for $\alpha_n = n^{-1}$ and $\beta_n = n$.

Remark 4.2.3. We observe that the condition that β_n be bounded in Theorems 4.2.4 and 4.2.5 is superfluous if T is linear. Of course this is the case also when H is finite dimensional. In these two cases, Theorem 4.2.5 holds under the more general condition

$$\sum_{n=0}^{\infty} \left| \frac{1}{\beta_{n+1}} - \frac{1}{\beta_n} \right| < \infty$$

instead of condition (C9). In fact, it still holds even for any $\beta_n \to \infty$ as shown in the following corollary. Such conditions on β_n are not comparable in general. Indeed, for any (β_n) bounded below away from zero, one can check that

$$\sum_{n=0}^{\infty} \left| \frac{1}{\beta_{n+1}} - \frac{1}{\beta_n} \right| = \infty \quad \text{for} \quad \beta_n = \begin{cases} 2n, & \text{if} \quad n \quad \text{is odd,} \\ n+1, & \text{if} \quad n \quad \text{is even.} \end{cases}$$

However, for increasing (β_n), the condition (4.8) is implied by $\beta_n \to \infty$.

Corollary 4.2.6. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator and $T : H \to H$ a nonexpansive map with $\emptyset \neq A^{-1}(0) =: F \subset F(T)$, where $F(T)$ is the set of all fixed points of T. For any fixed $u, v_1 \in H$, let the sequence (v_n) be generated by algorithm (4.15), where $\alpha_n \in (0,1)$ and $\beta_n \in (0,\infty)$. Assume that (α_n), (β_n) and (e_n) satisfy conditions* (i) *and* (ii) *of Theorem 4.2.5, and* (iii) *$\beta_n \to \infty$. If T is linear, then (v_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. We note that from (4.15), we have

$$\frac{(v_{n+1} - Tv_n)}{\beta_n} + Av_{n+1} \ni \frac{\alpha_{n-1}}{\beta_n}(u - Tv_n) + \frac{e_{n-1}}{\beta_n}.$$

Since $\beta_n \to \infty$, this inclusion relation gives $\omega_{\mathtt{w}}((v_n)) \subset F$. Since T is linear, $\omega_{\mathtt{w}}((v_n)) = \omega_{\mathtt{w}}((Tv_n))$. The rest of the proof is similar to that of Theorem 4.2.5. □

Since (C21) is weaker than (C8)$'$, the following result improves Theorem 3.4.1.

Theorem 4.2.7. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator with $\emptyset \neq A^{-1}(0) =: F$. For any fixed $u, v_1 \in H$, let the sequence (v_n) be generated by algorithm (4.15) with $T = I$ (the identity operator), where $\alpha_n \in (0,1)$ and $\beta_n \in (0,\infty)$. Assume that (α_n), and (e_n) satisfy conditions* (i) *and* (ii) *of Theorem 4.2.5. If $\liminf_{n\to\infty} \beta_n > 0$, and either one of the conditions (C10) or (C21) is met, then (v_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. Assume $\|e_n\|/\alpha_n \to 0$. We know that (v_n) is bounded, see Theorem 4.2.4.

Claim: $\limsup_{n\to\infty}\langle u - P_F u, v_n - P_F u\rangle \leq 0$.

Let (v_{n_k}) be a subsequence of (v_n) converging weakly to some v_∞, such that

$$\limsup_{n\to\infty} \langle u - P_F u, v_n - P_F u\rangle = \lim_{k\to\infty} \langle u - P_F u, v_{n_k} - P_F u\rangle = \langle u - P_F u, v_\infty - P_F u\rangle.$$

To prove the claim, we only need to show that $v_\infty \in F$, or more generally $\omega_{\mathtt{w}}((v_n)) \subset F$. From equation (4.26) with $T = I$, we have for some $M > 0$

$$\frac{\|v_{n+2} - v_{n+1}\|}{\beta_{n+1}} \leq (1 - \alpha_{n-1})\frac{\|v_{n+1} - v_n\|}{\beta_n} + \left|\frac{\alpha_n}{\beta_{n+1}} - \frac{\alpha_{n-1}}{\beta_n}\right| M + \frac{\alpha_{n-1}}{\beta_n}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|,$$

so that if either one of the conditions (C10) or (C21) is fulfilled, then we establish

$$\frac{\|v_{n+1} - v_n\|}{\beta_n} \to 0.$$

Moreover, we get from (4.15) with $T = I$

$$\frac{v_{n+1} - v_n}{\beta_n} + Av_{n+1} \ni \frac{\alpha_{n-1}}{\beta_n}(u - v_n) + \frac{1}{\beta_n}e_{n-1},$$

which implies that $\omega_{\mathtt{w}}((v_n)) \subset F$, hence the claim.

Again proceeding in a similar way as in the proof of Theorem 4.2.4 (with $\gamma_n = 0$ for all $n \geq 1$), we derive strong convergence of (v_n) to $P_F u$. □

Chapter 5

The method of alternating resolvents

Early in the 1930s, von Neumann showed that given any two closed subspaces K_1 and K_2 of a real Hilbert space H, the sequence of alternating projections

$$H \ni x_0 \mapsto x_1 = P_{K_1}x_0 \mapsto x_2 = P_{K_2}x_1 \mapsto x_3 = P_{K_1}x_2 \mapsto x_4 = P_{K_2}x_3 \mapsto \cdots ,$$

converges strongly to the point in the intersection of K_1 and K_2 that is nearest to the starting point x_0. (For the proof of this result, see e.g., [5]). Three decades later, Bregman [13] showed that for two arbitrary closed convex sets K_1 and K_2 with nonempty intersection, the sequence (x_n) generated by the method of alternating projections converges weakly to some point in $K_1 \cap K_2$. The question on whether or not (x_n) converge strongly was recently settled by H. Hundal [26], who constructed an example in ℓ^2 showing that for any starting point $x_0 \in \ell^2$, there exists a hyperplane K_1 and a cone K_2 such that $K_1 \cap K_2 = \{0\}$ and the sequence of alternating projections (x_n) converges weakly to zero, but not strongly.

In a recent paper of Bauschke et al. [4], it was shown that for maximal monotone operators $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$, the sequence generated from the method of alternating (or composition of) resolvents

$$H \ni x_0 \mapsto x_1 = J_\lambda^A x_0 \mapsto x_2 = J_\lambda^B x_1 \mapsto x_3 = J_\lambda^A x_2 \mapsto x_4 = J_\lambda^B x_3 \mapsto \cdots ,$$

for $\lambda > 0$, converges weakly to a point of $\operatorname{Fix} J_\lambda^A J_\lambda^B$ – the fixed point set of the composition $J_\lambda^A J_\lambda^B$ – provided that this set is not empty. The method of alternating resolvents given above is a natural extension of the method of alternating projections, since projection operators coincide with resolvent operators of normal cones. In this chapter, we shall investigate the convergence of the sequence generated by the inexact method of alternating resolvents

$$\begin{aligned} x_{2n+1} &= J^A_{\beta_n}(x_{2n}+e_n), \quad \text{for } n=0,1,\dots, \\ x_{2n} &= J^B_{\gamma_n}(x_{2n-1}+e'_n), \quad \text{for } n=1,2,\dots, \end{aligned}$$

where $x_0 \in H$ is a given starting point, (β_n) and (γ_n) are sequences of positive real numbers, and (e_n) and (e'_n) are sequences of computational errors. More precisely, we shall show that under the summability condition on $(\|e_n\|)$ and $(\|e'_n\|)$, the sequence of alternating resolvents defined above is weakly convergent to a point of $A^{-1}(0) \cap B^{-1}(0)$ provided that this set is not empty, and that both (β_n) and (γ_n) are bounded from below away from zero. In order to obtain strong convergence results for general maximal monotone operators A and B, a modification (following the idea from the case of a single maximal monotone operator (cf., Section 3.4)) of this method is carried out, see Section 5.2 below. With such a modification, the summability condition on the error sequences $(\|e_n\|)$ and $(\|e'_n\|)$ is also relaxed. The main results of this chapter are Theorem 5.2.1 and Theorem 5.2.2.

5.1 Some remarks

We begin this section by showing that whenever A (respectively, B) is strongly (and maximal) monotone and (β_n) (respectively, (γ_n)) is bounded from below away from zero, with the error sequences (e_n) and (e'_n) being bounded, then the sequence (x_n) generated by

$$x_{2n+1} = J^A_{\beta_n}(x_{2n}+e_n), \quad \text{for } n=0,1,\dots, \tag{5.1}$$

$$x_{2n} = J^B_{\gamma_n}(x_{2n-1}+e'_n), \quad \text{for } n=1,2,\dots, \tag{5.2}$$

is bounded. In fact, we prove this result for coercive operators of which strongly monotone operators are particular cases.

Proposition 5.1.1. *Let $F := A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$, where $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ are maximal monotone operators. Assume that the error sequences (e_n) and (e'_n) are bounded. If either* (i) *A is coercive and (β_n) is bounded from below away from zero, or* (ii) *B is coercive and (γ_n) is bounded from below away from zero, then the sequence (x_n) generated by (5.1) and (5.2) is bounded.*

Proof. (The proof of this result is essentially borrowed from the proof of Theorem 3.5 [39, p. 152]). Fix $p \in F$, and let C^* be a positive constant such that

$$\|e_n\| + \|e'_n\| \le C^*, \quad \text{for all } n \ge 0.$$

Then we have from (5.2), and the fact that the resolvent operator is nonexpansive

$$\begin{aligned} \|x_{2n}-p\| &\leq \|x_{2n-1}-p+e'_n\| \\ &\leq \|x_{2n-1}-p\|+\|e'_n\|\,, \quad \text{for all } n\geq 1. \end{aligned} \tag{5.3}$$

Similarly, from (5.1) and the above estimate, we have

$$\begin{aligned} \|x_{2n+1}-p\| &\leq \|x_{2n}-p\|+\|e_n\| \\ &\leq \|x_{2n-1}-p\|+\|e_n\|+\|e'_n\|\,, \quad \text{for all } n\geq 1, \end{aligned}$$

which implies that

$$\|x_{2n+1}\| \leq \|x_{2n-1}\|+2\|p\|+C^*, \quad \text{for all } n\geq 1.$$

Now let v_0 be the vector associated with the coercivity of A. Denote $C_1 := C^*+2(\|p\|+\|v_0\|)$. Then by (2.4) there exists a constant $K^*>0$ such that

$$(\xi,\eta)\in A, \quad \|\xi\|>K^* \quad \text{implies} \quad \frac{\langle \eta, \xi-v_0\rangle}{\|\xi-v_0\|} \geq \frac{C_1}{\varepsilon}\,, \tag{5.4}$$

where $\varepsilon>0$ is the greatest lower bound of (β_n). If $\|x_{2n+1}\|\leq K^*$ for all $n\geq 0$, then there is nothing to do. So we assume that there is an index k such that $\|x_{2k+1}\|>K^*$. Then multiplying

$$x_{2k}-v_0+e_k \in x_{2k+1}-v_0+\beta_k A x_{2k+1}$$

by the unit vector $(x_{2k+1}-v_0)/\|x_{2k+1}-v_0\|$, and making use of (5.4), we get,

$$\|x_{2k+1}-v_0\|+C_1 \leq \|x_{2k}-v_0\|+\|e_k\| \leq \|x_{2k}\|+\|v_0\|+\|e_k\|\,,$$

which implies that

$$\|x_{2k+1}\| \leq \|x_{2k+1}-v_0\|+\|v_0\| \leq \|x_{2k}\|+2\|v_0\|+\|e_k\|-C_1\,.$$

On the other hand, from (5.3) we derive

$$\|x_{2n}\| \leq \|x_{2n-1}\|+2\|p\|+\|e'_n\|\,, \quad \text{for all } n\geq 1,$$

so that

$$\begin{aligned} \|x_{2k+1}\| &\leq \|x_{2k-1}\|+2\|v_0\|+2\|p\|+\|e'_k\|+\|e_k\|-C_1 \\ &\leq \|x_{2k-1}\|+2(\|v_0\|+\|p\|)+C^*-C_1 \\ &= \|x_{2k-1}\|\,. \end{aligned}$$

Therefore, we have for each $n \geq 1$

$$\|x_{2n+1}\| \leq \max\{K^* + C^* + 2\|p\|, \|x_{2n-1}\|\}. \tag{5.5}$$

Setting $\rho_n = \max\{K^* + C^* + 2\|p\|, \|x_{2n-1}\|\}$, we deduce from (5.5) that the sequence (ρ_n) is decreasing. Hence

$$\|x_{2n+1}\| \leq \rho_n \leq \max\{K^* + C^* + 2\|p\|, \|x_1\|\}, \quad \text{for all } n \geq 1,$$

showing that the subsequence (x_{2n+1}) of (x_n) is bounded, and so is the subsequence (x_{2n}) of (x_n) (cf. (5.3)). Hence the sequence (x_n) itself is bounded. The proof of this result when B is coercive and (γ_n) bounded from below away from zero is an analogue of the above given proof. □

We now give a convergence result associated with the sequence generated from the method of alternating resolvents.

Theorem 5.1.2. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $F := A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Let (x_n) be the sequence generated by (5.1) and (5.2), where $\beta_n, \gamma_n \in (0, \infty)$. Assume that $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\sum_{n=1}^{\infty} \|e'_n\| < \infty$. If both (β_n) and (γ_n) are bounded from below away from zero, then (x_n) converges weakly to some point of F for any given $x_0 \in H$.*

Proof. It is worth pointing out that

$$x_{2n+1} + \beta_n A x_{2n+1} \ni x_{2n} + e_n, \quad \text{for } n = 0, 1, \ldots, \tag{5.6}$$

$$x_{2n} + \gamma_n B x_{2n} \ni x_{2n-1} + e'_n, \quad \text{for } n = 1, 2, \ldots, \tag{5.7}$$

are equivalent forms of equations (5.1) and (5.2) respectively. Let us first note that if the set F is nonempty, then the sequence $(\|x_n - p\|)$ is convergent for any $p \in F$, hence (x_n) is bounded. Indeed, for any $p \in F$, we have from (5.2)

$$\|x_{2n} - p\| \leq \|x_{2n-1} - p\| + \|e'_n\|. \tag{5.8}$$

Similarly, from (5.1) we derive

$$\|x_{2n+1} - p\| \leq \|x_{2n} - p\| + \|e_n\|,$$

which together with (5.8) implies that

$$\|x_{2n+1} - p\| - \sum_{k=0}^{n}(\|e'_k\| + \|e_k\|) \leq \|x_{2n-1} - p\| - \sum_{k=0}^{n-1}(\|e'_k\| + \|e_k\|).$$

This shows that the sequence $(\|x_{2n+1}-p\|)$ is convergent. Similarly, $(\|x_{2n}-p\|)$ is convergent, with the same limit (see above). Consequently, the sequence $(\|x_n-p\|)$ is convergent, as claimed.

Now subtracting x_{2n} from both sides of (5.6) (resp. x_{2n-1} from both sides of (5.7)) and multiplying the resulting inclusion relation scalarly by $x_{2n+1}-p$ (resp. by $x_{2n}-p$) for some $p \in F$, we get upon the use of the monotonicity of A (resp. of B)

$$\langle x_{2n+1}-x_{2n}, x_{2n+1}-p\rangle \leq \langle e_n, x_{2n+1}-p\rangle, \quad \text{and} \quad \langle x_{2n}-x_{2n-1}, x_{2n}-p\rangle \leq \langle e_n', x_{2n}-p\rangle,$$

respectively. Equivalently, we have, for some positive constant C,

$$\|x_{2n+1}-x_{2n}\|^2 + \|x_{2n+1}-p\|^2 - \|x_{2n}-p\|^2 \leq C\|e_n\|,$$

and

$$\|x_{2n}-x_{2n-1}\|^2 + \|x_{2n}-p\|^2 - \|x_{2n-1}-p\|^2 \leq C\|e_n'\|,$$

respectively. Adding these last two inequalities, and passing to the limit in the resulting inequality, we arrive at

$$\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0. \tag{5.9}$$

Notice that from (5.6) and the fact that (β_n) is bounded from below away from zero, we have

$$Ax_{2n+1} \ni \frac{x_{2n}-x_{2n+1}+e_n}{\beta_n} \to 0, \quad \text{as} \quad n\to\infty,$$

which implies that $\omega_{\mathtt{w}}((x_{2n+1})) \subset A^{-1}(0)$. Similarly, we derive from (5.7) the relation $\omega_{\mathtt{w}}((x_{2n})) \subset B^{-1}(0)$. Therefore by Opial's lemma, there exists $v \in A^{-1}(0)$ and $w \in B^{-1}(0)$ such that $x_{2n+1} \rightharpoonup v$ and $x_{2n} \rightharpoonup w$.

We finally show that $v \equiv w$. Indeed, for all $z \in H$, we have

$$\langle v-w, z\rangle = \lim_{n\to\infty}\langle x_{2n+1}-w, z\rangle = \lim_{n\to\infty}\langle x_{2n+1}-x_{2n}, z\rangle + \lim_{n\to\infty}\langle x_{2n}-w, z\rangle = 0.$$

Consequently, $x_n \rightharpoonup v \in A^{-1}(0)\cap B^{-1}(0)$. □

Remark 5.1.1. If, in addition to the assumptions of Theorem 5.1.2, either A or B is strongly monotone, then we have strong convergence to the unique element of F. Assume without loss of generality that B is strongly monotone with monotonicity constant c. Then, since (x_n) is bounded, we have from (5.7)

$$\langle x_{2n}-x_{2n-1}, x_{2n}-p\rangle + \gamma_n\langle Bx_{2n}, x_{2n}-p\rangle \leq M\|e_n'\|,$$

for some $M > 0$, where p is the unique point of $B^{-1}(0)$, $p \in A^{-1}(0)$. Using the strong monotonicity of B, we have

$$\|x_{2n} - x_{2n-1}\|^2 + \|x_{2n} - p\|^2 - \|x_{2n-1} - p\|^2 + 2c\gamma_n\|x_{2n} - p\|^2 \le 2M\|e'_n\|. \tag{5.10}$$

On the other hand, since

$$\|x_{2n+1} - x_{2n}\|^2 + \|x_{2n+1} - p\|^2 - \|x_{2n} - p\|^2 \le C\|e_n\|, \tag{5.11}$$

for some positive constant C, we have

$$\|x_{2n+1} - x_{2n}\|^2 + 2c\gamma_n\|x_{2n} - p\|^2 \le \|x_{2n-1} - p\|^2 - \|x_{2n+1} - p\|^2 + C\|e_n\| + 2M\|e'_n\|.$$

Summing this last inequality from $n = 1$ to $n = \infty$, and using the fact that γ_n is bounded from below away from zero, one derives the strong convergence of (x_n) to p.

Remark 5.1.2. When $\gamma_n \to \infty$ (in the case when B is strongly monotone) or $\beta_n \to \infty$ (in the case when A is strongly monotone), norm convergence of the error sequences (e_n) to zero and boundedness of (e'_n) (respectively, norm convergence of (e'_n) to zero and boundedness of (e_n)) are enough to guarantee strong convergence of (x_n) to the unique point $p \in F$. Indeed, we have from (5.10),

$$\|x_{2n} - p\|^2 \le \frac{1}{1 + 2c\gamma_n}(\|x_{2n-1} - p\|^2 + 2M\|e'_n\|) \le \frac{K}{1 + 2c\gamma_n},$$

for some $K > 0$, where the last inequality follows from the fact that the sequences (x_n) and (e'_n) are bounded. (For boundedness of (x_n) consult Proposition 5.1.1 above). Therefore, passing to the limit in the above estimate, we see that (x_{2n}) is strongly convergent to p. On the other hand, passing to the limit in (5.11), we also derive strong convergence of (x_{2n+1}) to p. Consequently, the whole sequence (x_n) is strongly convergent to p, as claimed. The other case is proved analogously.

Remark 5.1.3. In the case when (e_n) and (e'_n) are zero for every n, then the sequence (x_n) converges to p at least at a linear rate, whenever both (β_n) and (γ_n) are bounded from below away from zero. In the event that B is strongly monotone and $\gamma_n \to \infty$ (resp. A is strongly monotone and $\beta_n \to \infty$), the rate of convergence is improved to superlinearity. These facts follow from the two inequalities

$$\frac{\|x_{2n} - p\|}{\|x_{2n-2} - p\|} \le \frac{1}{\sqrt{1 + 2c\gamma_n}}, \quad \text{and} \quad \frac{\|x_{2n+1} - p\|}{\|x_{2n-1} - p\|} \le \frac{1}{\sqrt{1 + 2c\gamma_n}},$$

which are a result of combining inequalities (5.10) and (5.11) with $e_n = 0 = e'_n$ for all $n \ge 0$.

Remark 5.1.4. Strong convergence can also be obtained if in addition to the assumptions of Theorem 5.1.2, either one of the resolvents $J_1^A := (I + A)^{-1}$ or $J_1^B := (I + B)^{-1}$ is compact. Indeed, if for instance the former resolvent is compact, then we may write (with the help of the resolvent identity) equation (5.1) as

$$x_{2n+1} = J_1^A z_n, \quad \text{where} \quad z_n = \frac{1}{\beta_n}(x_{2n} + e_n) + \left(1 - \frac{1}{\beta_n}\right) x_{2n+1}.$$

Since (z_n) is bounded and J_1^A is compact, there is at least one strongly convergent subsequence, say (x_{2n_k+1}), of (x_{2n+1}). Let q be the limit of (x_{2n_k+1}). Then from (5.9), we derive $x_{2n_k} \to q$. Therefore, a subsequence (x_{n_k}) of (x_n) converges strongly to q. We note that $q \in F$, since $\omega_{\mathtt{w}}((x_n)) \subset F$. On the other hand, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$, so this limit is zero for $p = q$, i.e., (x_n) converges strongly to q.

Remark 5.1.5. We have shown in Theorem 5.1.2 that if the sequences $(\|e_n\|)$ and $(\|e'_n\|)$ are summable, and the parameters β_n and γ_n are bounded away from zero, then the sequence generated from (5.1) and (5.2) converges weakly to a point of $F := A^{-1}(0) \cap B^{-1}(0)$, provided that this set is not empty. Note that weak convergence of (x_n) (with $\beta_n = \gamma_n = \lambda$ for some $\lambda > 0$) was also shown in the paper of Bauschke et. al [4] where the set F was replaced by a generally larger set Fix $J_\lambda^A J_\lambda^B$. However, in the particular case when A and B are specialized to subdifferentials of indicator functions of closed and convex sets K_1 and K_2 (that is, normal cones to K_1 and K_2, respectively), then $p \in K_1 \cap K_2$ (equivalently, a point p belongs to the set F) if and only if p is a fixed point of the composition mapping $P_{K_2}P_{K_1}$. Indeed, for $A = \partial I_{K_1}$, then $p \in A^{-1}(0) \Leftrightarrow p = J_\beta^A p = P_{K_1}p$ for some $\beta > 0$. Similarly, for $B = \partial I_{K_2}$, we have $p \in B^{-1}(0) \Leftrightarrow p = P_{K_2}p$. Therefore, $p \in A^{-1}(0) \cap B^{-1}(0)$ implies that $p = P_{K_2}P_{K_1}p$. Conversely, suppose that there exists a point $p \in H$ such that $p = P_{K_2}P_{K_1}p$. Then it is obvious that $p \in K_2$. It only remains to show that $p \in K_1$. For this purpose, we derive from the inequality characterizing projections

$$\langle p - P_{K_1}p, p - v\rangle \le 0 \quad \text{for all } v \in K_2, \quad \text{and} \quad \langle p - P_{K_1}p, w - P_{K_1}p\rangle \le 0 \quad \text{for all } w \in K_1.$$

Therefore, if $K_1 \cap K_2$ is not empty, then taking any point $y \in K_1 \cap K_2$ in place of v and w in the above inequalities, we readily establish that $p = P_{K_1}p$. Hence $p \in K_1$ as desired.

It is worth pointing out that the above remark shows that if $K_1 \cap K_2 \neq \emptyset$, then the sets $K_1 \cap K_2$ and Fix $P_{K_2}P_{K_1}$ coincide. In the case when $K_1 \cap K_2 = \emptyset$, then either one of the following three cases may occur: (a) Fix $P_{K_2}P_{K_1} = K_2$, or (b) Fix $P_{K_2}P_{K_1} = \emptyset$, or (c) $\emptyset \neq$ Fix $P_{K_2}P_{K_1} \subsetneq K_2$. Indeed, taking K_1 and K_2 to be two parallel (and distinct) lines in $\mathbb{R}^2$, then one can easily check that for any point $x \in K_2$, we have $P_{K_2}P_{K_1}x = x$, and the intersection of K_1 and K_2 is the empty set. This verifies case (a). An easy way to see the validity of case (b) is to consider the two closed and convex sets $K_1 = \{x \times y \in \mathbb{R}^2 \mid x > 0 \text{ with } xy \geq 1\}$ and $K_2 = \{x \times y \in \mathbb{R}^2 \mid x < 0 \text{ with } -xy \geq 1\}$. We leave it to the reader to verify the other case.

The above discussion shows that the set Fix $P_{K_2}P_{K_1}$ is in general larger than the set $K_1 \cap K_2$. One can easily prove that the set Fix $P_{K_2}P_{K_1}$ contains at least one element if in addition to the convexity and closedness of K_1 and K_2, either one of K_1 or K_2 is bounded. We state this fact more formally in the next proposition.

Proposition 5.1.3. *Let K_1 and K_2 be two nonempty, disjoint, closed, convex subsets of a real Hilbert space H. Assume that either K_1 or K_2 is bounded. Then* Fix $P_{K_2}P_{K_1} \neq \emptyset$.

5.2 Strong convergence results

Since the sequence generated by the method of alternating resolvents (equations (5.1) and (5.2)) is in general only weakly convergent, even in the case of the method of alternating projections (see [26]), we modify this method in order to enforce strong convergence. Thus we define the sequence (x_n) iteratively by

$$x_{2n+1} = J^A_{\beta_n}(\alpha_n u + (1-\alpha_n)x_{2n} + e_n), \quad \text{for } n = 0, 1, \dots, \tag{5.12}$$

$$x_{2n} = J^B_{\gamma_n}(x_{2n-1} + e'_n), \quad \text{for } n = 1, 2, \dots, \tag{5.13}$$

where the points $x_0, u \in H$ are given and $e'_0 = 0$ by convention. The idea is borrowed from the regularization method for the case of a single maximal monotone operator (see [32, 55]). Besides producing strongly convergent sequences, such a modification works well in the case when the error sequences (e_n) and (e'_n) are not summable, see Remark 5.2.1 below. Note also that from (5.12) and (5.13) with $A = \partial I_{K_1}$ for $K_1 = H$, we reobtain the old modified proximal point algorithm (introduced by Xu [54] and Kamimura and Takahashi [28]) for $x_n := x_{2n}$. Similarly, if $B = \partial I_{K_2}$ where $K_2 = H$, we reobtain the regularization method [55] which is in fact equivalent [8] to the old modified proximal point algorithm for $x_n := x_{2n+1}$. Moreover, from (5.12) and (5.13) with $A = \partial I_{K_1}$ and $B = \partial I_{K_2}$ where K_1 and K_2 are nonempty, convex and closed subsets of H, we recover the method of alternating projections. With our modification, these algorithms become strongly convergent (under suitable assumptions) as the following theorems show.

Theorem 5.2.1. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$. For arbitrary but fixed vectors $x_0, u \in H$, let (x_n) be the sequence generated by (5.12) and (5.13), where $\alpha_n \in (0,1)$ and $\beta_n, \gamma_n \in (0, \infty)$. Assume that* (i) $\lim_{n\to\infty} \alpha_n = 0$ *with $\sum_{n=0}^{\infty} \alpha_n = \infty$, and either*

$\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$ *or* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*, and that* (ii) *both* (β_n) *and* (γ_n) *are bounded from below away from zero, with*

$$\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$$

If the error sequences (e_n) *and* (e'_n) *satisfy either one of the following conditions,*

(a) $\sum_{n=0}^{\infty} \|e_n\| < \infty$*, and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$*;*

(b) $\sum_{n=0}^{\infty} \|e_n\| < \infty$*, and* $\|e'_n\|/\alpha_n \to 0$*, with* $\sum_{n=1}^{\infty} \|e'_{n+1} - e'_n\| < \infty$*;*

(c) $\|e_n\|/\alpha_n \to 0$*, and* $\|e'_n\|/\alpha_n \to 0$*, with* $\sum_{n=1}^{\infty} \|e'_{n+1} - e'_n\| < \infty$*;*

(d) $\|e_n\|/\alpha_n \to 0$*, and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$*,*

then (x_n) *converges strongly to the point of* F *nearest to* u*.*

Proof. We shall prove the theorem only when the error sequences (e_n) and (e'_n) satisfy either condition (c) or (d). The reader will find it easy to adapt the proof given below to the other two cases.

The first step is to show that (x_n) is bounded. Take any $p \in F$. Then from (5.12) and (5.13), we have

$$\begin{aligned} \|x_{2n+1} - p\| &\le \|\alpha_n(u - p + e_n/\alpha_n) + (1-\alpha_n)(x_{2n} - p)\| \\ &\le \alpha_n \left\{ \|u - p\| + \frac{\|e_n\|}{\alpha_n} \right\} + (1-\alpha_n)\|x_{2n} - p\| \\ &\le \alpha_n C + (1-\alpha_n)\|x_{2n-1} - p\| + \|e'_n\|, \end{aligned} \tag{5.14}$$

where the first inequality follows from the fact that the resolvent operator is nonexpansive, and C is a positive constant such that

$$\sup\left\{ \|u - p\| + \frac{\|e_n\|}{\alpha_n} \mid n \ge 0 \right\} \le C.$$

Such a constant exists because the sequence $(\|e_n\|/\alpha_n)$ is bounded. Applying induction to (5.14), yields

$$\|x_{2n+1} - p\| \le C\left[1 - \prod_{k=0}^{n}(1-\alpha_k)\right] + \|x_0 - p\| \prod_{k=0}^{n}(1-\alpha_k) + \sum_{k=1}^{n} \|e'_k\|.$$

Therefore, under condition (d), the subsequence (x_{2n+1}) of (x_n) is bounded, and so is the subsequence (x_{2n}). Consequently (x_n) is bounded.

In the case when both $(\|e_n\|/\alpha_n)$ and $(\|e'_n\|/\alpha_n)$ are bounded, we have from equations (5.12) and (5.13)

$$\begin{aligned}
\|x_{2n+1}-p\| &\leq \alpha_n\left\{\|u-p\|+\frac{\|e_n\|}{\alpha_n}\right\}+(1-\alpha_n)\|x_{2n}-p\| \\
&\leq \alpha_n\left\{\|u-p\|+\frac{\|e_n\|}{\alpha_n}\right\}+(1-\alpha_n)\left(\|x_{2n-1}-p\|+\alpha_n\frac{\|e'_n\|}{\alpha_n}\right) \\
&\leq \alpha_n C^*+(1-\alpha_n)\|x_{2n-1}-p\|, \qquad (5.15)
\end{aligned}$$

where C^* is a positive constant such that

$$\sup\left\{\|u-p\|+\frac{\|e_n\|}{\alpha_n}+\frac{\|e'_n\|}{\alpha_n}\mid n\geq 0\right\}\leq C^*.$$

Applying induction to (5.15), we again derive that (x_n) is bounded.

Next we show that the weak $\omega-$limit set of (x_n) is contained in F, that is, $\omega_{\mathtt{w}}((x_n))\subset F$. Using the resolvent identity, we write (5.12) as

$$x_{2n+1}=J^A_{\beta_{n+1}}\left(\frac{\beta_{n+1}}{\beta_n}(\alpha_n u+(1-\alpha_n)x_{2n}+e_n)+\left(1-\frac{\beta_{n+1}}{\beta_n}\right)x_{2n+1}\right).$$

Therefore, by the nonexpansivity of the resolvent operator, we have

$$\begin{aligned}
\|x_{2n+3}-x_{2n+1}\| &\leq \left\|\frac{\beta_{n+1}}{\beta_n}(1-\alpha_n)(x_{2n+2}-x_{2n})+\left(1-\frac{\beta_{n+1}}{\beta_n}\right)(x_{2n+2}-x_{2n+1})\right. \\
&+ \left.\left(\alpha_{n+1}-\frac{\beta_{n+1}\alpha_n}{\beta_n}\right)\left(u-x_{2n+2}+\frac{e_{n+1}}{\alpha_{n+1}}\right)+\frac{\beta_{n+1}\alpha_n}{\beta_n}\left(\frac{e_{n+1}}{\alpha_{n+1}}-\frac{e_n}{\alpha_n}\right)\right\| \\
&\leq \frac{\beta_{n+1}}{\beta_n}(1-\alpha_n)\|x_{2n+2}-x_{2n}\|+\left|1-\frac{\beta_{n+1}}{\beta_n}\right|K \\
&+ \left|\alpha_{n+1}-\frac{\beta_{n+1}\alpha_n}{\beta_n}\right|K+\frac{\beta_{n+1}\alpha_n}{\beta_n}\left\|\frac{e_{n+1}}{\alpha_{n+1}}-\frac{e_n}{\alpha_n}\right\|, \qquad (5.16)
\end{aligned}$$

where K is a positive constant such that

$$\sup\left\{\left\|u-x_{2n+2}+\frac{e_{n+1}}{\alpha_{n+1}}\right\|+\|x_{2n+2}-x_{2n+1}\|\mid n\geq 0\right\}\leq K.$$

We now estimate $\|x_{2n+2}-x_{2n}\|$ as follows:

$$\begin{aligned}
\|x_{2n+2}-x_{2n}\| &= \left\|J^B_{\gamma_n}\left(\frac{\gamma_n}{\gamma_{n+1}}(x_{2n+1}+e'_{n+1})+\left(1-\frac{\gamma_n}{\gamma_{n+1}}\right)x_{2n+2}\right)-J^B_{\gamma_n}(x_{2n-1}+e'_n)\right\| \\
&\leq \left\|(x_{2n+1}-x_{2n-1})+\left(1-\frac{\gamma_n}{\gamma_{n+1}}\right)(x_{2n+2}-x_{2n+1}-e'_{n+1})+(e'_{n+1}-e'_n)\right\| \\
&\leq \|x_{2n+1}-x_{2n-1}\|+\left|1-\frac{\gamma_n}{\gamma_{n+1}}\right|M+\|e'_{n+1}-e'_n\|,
\end{aligned}$$

for some positive constant M. Therefore

$$\begin{aligned}\frac{\|x_{2n+3}-x_{2n+1}\|}{\beta_{n+1}} &\leq (1-\alpha_n)\frac{\|x_{2n+1}-x_{2n-1}\|}{\beta_n}+\left|\frac{1}{\beta_{n+1}}-\frac{1}{\beta_n}\right|K+\left|1-\frac{\gamma_n}{\gamma_{n+1}}\right|M' \\ &+ \left|\frac{\alpha_{n+1}}{\beta_{n+1}}-\frac{\alpha_n}{\beta_n}\right|K+\frac{\alpha_n}{\beta_n}\left\|\frac{e_{n+1}}{\alpha_{n+1}}-\frac{e_n}{\alpha_n}\right\|+\frac{\|e'_{n+1}-e'_n\|}{\beta_n},\end{aligned}$$

for some positive constant M'. Hence by Lemma 2.1.1 (and Lemma 2.1.2), we derive

$$\frac{\|x_{2n+1}-x_{2n-1}\|}{\beta_n}\to 0 \quad\Leftrightarrow\quad \|x_{2n+1}-x_{2n-1}\|\to 0.$$

As a matter of fact, $\|x_{2n+2}-x_{2n}\|\to 0$ as well. Now, multiplying the inclusion relation

$$x_{2n+1}-x_{2n+2}+\beta_n A x_{2n+1}\ni \alpha_n(u-x_{2n}+e_n/\alpha_n)+x_{2n}-x_{2n+2},$$

scalarly by $x_{2n+1}-p$ (where $p\in F$) and using the monotonicity of A, we get

$$\langle x_{2n+1}-x_{2n+2},x_{2n+1}-p\rangle\leq \alpha_n K'+L\|x_{2n}-x_{2n+2}\|, \tag{5.17}$$

for some positive constants K' and L. Similarly, multiplying the inclusion relation

$$x_{2n+2}-x_{2n+1}+\gamma_{n+1}Bx_{2n+2}\ni e'_{n+1}$$

scalarly by $x_{2n+2}-p$ and using the monotonicity of B, we arrive at

$$\langle x_{2n+2}-x_{2n+1},x_{2n+2}-p\rangle\leq L'\|e'_{n+1}\|, \tag{5.18}$$

for some constant $L'>0$. Adding the inequalities (5.17) and (5.18) and passing to the limit in the resulting inequality yields

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{5.19}$$

Therefore passing to the limit in

$$Ax_{2n+1}\ni\frac{\alpha_n(u-x_{2n})+x_{2n}-x_{2n+1}+e_n}{\beta_n}, \tag{5.20}$$

we see that $\omega_{\mathtt{w}}((x_{2n+1}))\subset A^{-1}(0)$. Similarly, we derive $\omega_{\mathtt{w}}((x_{2n}))\subset B^{-1}(0)$. It then follows from these two inclusion relations and (5.19) that $\omega_{\mathtt{w}}((x_n))\subset F=A^{-1}(0)\cap B^{-1}(0)$. This suffices to deduce that

$$\limsup_{n\to\infty}\langle u-P_F u,x_n-P_F u\rangle\leq 0. \tag{5.21}$$

We remark that the set F is closed and convex − being the intersection of two closed and convex sets, therefore there exists a unique point $q \in F$ such that $q = P_F u$. From (5.12) and (5.13), we have for $\|e_n\|/\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \|e'_n\| < \infty$,

$$\begin{aligned}
\|x_{2n+1} - q\|^2 &\leq \|(1-\alpha_n)(x_{2n} - q) + \alpha_n(u - q + e_n/\alpha_n)\|^2 \\
&= (1-\alpha_n)^2 \|x_{2n} - q\|^2 + \alpha_n^2 \|u - q + e_n/\alpha_n\|^2 \\
&+ 2\alpha_n(1-\alpha_n) \langle u - q + e_n/\alpha_n, x_{2n} - q \rangle \\
&\leq (1-\alpha_n)\|x_{2n-1} - q\|^2 + C'\|e'_n\| \\
&+ \alpha_n \left[\alpha_n C' + 2(1-\alpha_n) \left\langle u - q + \frac{e_n}{\alpha_n}, x_{2n} - q \right\rangle \right],
\end{aligned}$$

where C' is a positive constant such that

$$\sup \left\{ \left\| u - q + \frac{e_n}{\alpha_n} \right\|^2 + \|e'_n\| + 2\|x_{2n-1} - q\| \mid n \geq 1 \right\} \leq C',$$

or

$$\begin{aligned}
\|x_{2n+1} - q\|^2 &\leq (1-\alpha_n)\|x_{2n-1} - q\|^2 + \alpha_n \left[C' \left(\alpha_n + \frac{\|e'_n\|}{\alpha_n} \right) + \right. \\
&+ \left. 2(1-\alpha_n) \left\langle u - q + \frac{e_n}{\alpha_n}, x_{2n} - q \right\rangle \right],
\end{aligned}$$

in the case when $\|e_n\|/\alpha_n \to 0$ and $\|e'_n\|/\alpha_n \to 0$. Hence by Lemma 2.1.1, we have (in both cases) $\|x_{2n+1} - q\| \to 0$, as $n \to \infty$. Moreover, since $x_{2n+1} - x_{2n} \to 0$, we also have $\|x_{2n} - q\| \to 0$. Consequently, $x_n \to q$ as $n \to \infty$. □

Example 5.2.1. The sequences

$$\alpha_n = \begin{cases} \dfrac{2}{3n}, & \text{if } n \text{ is even} \\ \dfrac{1}{2n}, & \text{if } n \text{ is odd,} \end{cases} \qquad \beta_n = \begin{cases} \dfrac{2(n+1)}{3n}, & \text{if } n \text{ is odd} \\ \dfrac{n+1}{2n}, & \text{if } n \text{ is even} \end{cases}$$

satisfy the conditions

$$\sum_{n=0}^{\infty} \left| \frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n} \right| < \infty, \quad \text{and} \quad \lim_{n\to\infty} \left(\frac{\alpha_{n+1}}{\alpha_n} - \frac{\beta_{n+1}}{\beta_n} \right) = 0,$$

but α_n defined above does not satisfy neither $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$ nor $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, (as assumed in Theorem 5.2.1).

The modification carried out in (5.12) and (5.13) is not the only possible one that can be made. Alternatively, we could have opted to modify (5.2) instead of (5.1), we however do not carry out such modification as it yields similar results to the ones obtained from (5.12) and (5.13). The other possibility is to modify both (5.1) and (5.2), in which case the sequence (x_n) is generated via the algorithm

$$x_{2n+1} = J^A_{\beta_n}(\alpha_n u + (1-\alpha_n)x_{2n} + e_n), \quad \text{for } n = 0, 1, \ldots, \tag{5.22}$$

$$x_{2n} = J^B_{\gamma_n}(\alpha_n u + (1-\alpha_n)x_{2n-1} + e'_n), \quad \text{for } n = 1, 2, \ldots, \tag{5.23}$$

for given $x_0, u \in H$. A consideration of (5.22) and (5.23) instead of (5.12) and (5.13) enables one to dispense with the assumption $\sum_{n=1}^{\infty} \|e'_{n+1} - e'_n\| < \infty$ appearing in condition (c) and (d) of Theorem 5.2.1.

Theorem 5.2.2. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$. Fix $x_0, u \in H$, and let (x_n) be the sequence generated by (5.22) and (5.23), where $\alpha_n \in (0,1)$ and $\beta_n, \gamma_n \in (0, \infty)$. Assume that* (i) $\lim_{n\to\infty} \alpha_n = 0$ *with* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and either* $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$ *or* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, *and that* (ii) *both* (β_n) *and* (γ_n) *are bounded from below away from zero, with*

$$\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$$

If the error sequences (e_n) and (e'_n) satisfy either one of the following conditions,

(a) $\sum_{n=0}^{\infty} \|e_n\| < \infty$, *and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$;

(b) $\sum_{n=0}^{\infty} \|e_n\| < \infty$, *and* $\|e'_n\|/\alpha_n \to 0$;

(c) $\|e_n\|/\alpha_n \to 0$, *and* $\|e'_n\|/\alpha_n \to 0$;

(d) $\|e_n\|/\alpha_n \to 0$, *and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$,

then (x_n) converges strongly to the point of F nearest to u.

Proof. We prove the result only for the case when condition (c) is satisfied.

As before, we begin by showing that (x_n) is bounded. As we have already seen in the proof of the previous theorem

$$\|x_{2n+1} - p\| \leq \alpha_n \left\{ \|u - p\| + \frac{\|e_n\|}{\alpha_n} \right\} + (1-\alpha_n)\|x_{2n} - p\|.$$

Similarly, we have

$$\|x_{2n} - p\| \leq \alpha_n \left\{ \|u - p\| + \frac{\|e'_n\|}{\alpha_n} \right\} + (1-\alpha_n)\|x_{2n-1} - p\|.$$

Therefore

$$\|x_{2n+1} - p\| \leq \alpha_n C^* + (1-\alpha_n)\|x_{2n-1} - p\|,$$

where C^* is a positive constant such that

$$\sup_{n\geq 0}\left\{2\|u-p\| + \frac{\|e_n\|}{\alpha_n} + \frac{\|e'_n\|}{\alpha_n}\right\} \leq C^*.$$

The boundedness of (x_n) follows easily from the above inequality. Dividing (5.16) by $\beta_{n+1}\gamma_{n+1}$, and noting that (γ_n) is bounded from below away from zero, we have

$$\begin{aligned} \frac{\|x_{2n+3} - x_{2n+1}\|}{\beta_{n+1}\gamma_{n+1}} &\leq (1-\alpha_n)\frac{\|x_{2n+2} - x_{2n}\|}{\beta_n\gamma_{n+1}} + \left|\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right| K' \\ &+ \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right| K' + \frac{\alpha_n}{\beta_n\gamma_{n+1}}\left\|\frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\|, \end{aligned} \tag{5.24}$$

for some $K' > 0$. On the other hand, a similar computation to (5.16) yields

$$\begin{aligned} \|x_{2n+2} - x_{2n}\| &\leq \frac{\gamma_{n+1}}{\gamma_n}(1-\alpha_n)\|x_{2n+1} - x_{2n-1}\| + \left|1 - \frac{\gamma_{n+1}}{\gamma_n}\right| K^* \\ &+ \left|\alpha_{n+1} - \frac{\gamma_{n+1}\alpha_n}{\gamma_n}\right| K^* + \frac{\gamma_{n+1}\alpha_n}{\gamma_n}\left\|\frac{e'_{n+1}}{\alpha_{n+1}} - \frac{e'_n}{\alpha_n}\right\|, \end{aligned} \tag{5.25}$$

for some positive constant K^*.

Now dividing (5.25) by $\beta_n\gamma_{n+1}$, and noting that (β_n) is bounded from below away from zero, we have

$$\begin{aligned} \frac{\|x_{2n+2} - x_{2n}\|}{\beta_n\gamma_{n+1}} &\leq (1-\alpha_n)\frac{\|x_{2n+1} - x_{2n-1}\|}{\beta_n\gamma_n} + \left|\frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n}\right| L \\ &+ \left|\frac{\alpha_{n+1}}{\gamma_{n+1}} - \frac{\alpha_n}{\gamma_n}\right| L + \frac{\alpha_n}{\beta_n\gamma_n}\left\|\frac{e'_{n+1}}{\alpha_{n+1}} - \frac{e'_n}{\alpha_n}\right\|, \end{aligned}$$

for some positive constant L. This last inequality together with (5.24) gives

$$\begin{aligned} \frac{\|x_{2n+3} - x_{2n+1}\|}{\beta_{n+1}\gamma_{n+1}} &\leq (1-\alpha_n)\frac{\|x_{2n+1} - x_{2n-1}\|}{\beta_n\gamma_n} + \left|\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right| K' + \left|\frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n}\right| L \\ &+ \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right| K' + \left|\frac{\alpha_{n+1}}{\gamma_{n+1}} - \frac{\alpha_n}{\gamma_n}\right| L + \frac{\alpha_n}{\beta_n\gamma_{n+1}}\left\|\frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| \\ &+ \frac{\alpha_n}{\beta_n\gamma_n}\left\|\frac{e'_{n+1}}{\alpha_{n+1}} - \frac{e'_n}{\alpha_n}\right\|, \end{aligned}$$

Hence by Lemma 2.1.1, we derive

$$\frac{\|x_{2n+1} - x_{2n-1}\|}{\beta_n\gamma_n} \to 0 \quad \Leftrightarrow \quad \|x_{2n+1} - x_{2n-1}\| \to 0,$$

where the equivalence is due to the fact that both (β_n) and (γ_n) are convergent.

Multiplying the inclusion relation

$$x_{2n+2} - x_{2n+1} + \gamma_{n+1} B x_{2n+2} \ni \alpha_{n+1}(u - x_{2n+1} + e'_{n+1}/\alpha_{n+1})$$

scalarly by $x_{2n+2} - p$ and using the monotonicity of B, we arrive at

$$\langle x_{2n+2} - x_{2n+1}, x_{2n+2} - p \rangle \leq \alpha_{n+1} L^*,$$

for some constant $L^* > 0$. We have already shown in (5.17) that

$$\langle x_{2n+1} - x_{2n+2}, x_{2n+1} - p \rangle \leq \alpha_n K' + L\|x_{2n} - x_{2n+2}\|,$$

so that adding these two inequalities and passing to the limit in the resulting inequality, we arrive at

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Therefore as in (5.20) we derive the inclusion relations $\omega_{\mathtt{w}}((x_{2n+1})) \subset A^{-1}(0)$ and $\omega_{\mathtt{w}}((x_{2n})) \subset B^{-1}(0)$. Since $x_{n+1} - x_n \to 0$, these inclusions imply that $\omega_{\mathtt{w}}((x_n)) \subset F$, and hence

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u \rangle \leq 0.$$

Finally, we show that $x_n \to P_F u$. From (5.22) we have

$$\begin{aligned} \|x_{2n+1} - P_F u\|^2 &\leq \|(1-\alpha_n)(x_{2n} - P_F u) + \alpha_n(u - P_F u + e_n/\alpha_n)\|^2 \\ &= (1-\alpha_n)^2 \|x_{2n} - P_F u\|^2 + \alpha_n^2 \|u - P_F u + e_n/\alpha_n\|^2 \\ &+ 2\alpha_n(1-\alpha_n) \langle u - P_F u + e_n/\alpha_n, x_{2n} - P_F u \rangle . \end{aligned} \tag{5.26}$$

Similarly, we have from (5.23)

$$\begin{aligned} \|x_{2n} - P_F u\|^2 &\leq (1-\alpha_n)^2 \|x_{2n-1} - P_F u\|^2 + \alpha_n^2 \|u - P_F u + e'_n/\alpha_n\|^2 \\ &+ 2\alpha_n(1-\alpha_n) \langle u - P_F u + e'_n/\alpha_n, x_{2n-1} - P_F u \rangle , \end{aligned}$$

which together with (5.26) gives

$$\|x_{2n+1} - P_F u\|^2 \leq (1-\alpha_n)\|x_{2n-1} - P_F u\|^2 + \alpha_n b_n,$$

where

$$b_n = \alpha_n C_1 + 2(1-\alpha_n) \left[\left\langle u - P_F u + \frac{e_n}{\alpha_n}, x_{2n} - P_F u \right\rangle + \left\langle u - P_F u + \frac{e'_n}{\alpha_n}, x_{2n-1} - P_F u \right\rangle \right],$$

for some positive constant C_1. Therefore by Lemma 2.1.1, we derive $x_{2n+1} \to P_F u$. Since $x_{n+1} - x_n \to 0$, we deduce strong convergence of (x_n) to $P_F u$. □

Perhaps we should conclude this section by briefly explaining how the algorithms presented in this section work under the error condition(s) $\|e_n\|/\alpha_n \to 0$ (and/or $\|e'_n\|/\alpha_n \to 0$) of Theorem 5.2.1 (Theorem 5.2.2). The condition $\sum_{n=0}^{\infty}\|e_n\| < \infty$ needs no further elaboration as it has been widely used by several authors.

Remark 5.2.1. The non summability condition on the sequence of computational errors appearing in Theorem 5.2.1 was discussed in Section 3.2. As explained in that section, such a condition renders the algorithm in question applicable in approximating zeroes of maximal monotone operators for every sequence of errors converging to zero in norm. In the current setting − where two nonlinear operators are involved − for sequences of summable errors the algorithm works as expected, that is, one chooses the sequence (α_n) independent of the errors involved and execute the algorithm as usual. However, as soon as the sequence of computational errors associated with either one of the operators is norm convergent to zero and satisfy the condition $\sum_{n=0}^{\infty}\|e_n\| = \infty$, then an appropriate construction of the sequence of parameters (α_n) is carried out. Such a construction always depends on the error sequence (e_n) as it must meet the condition $\|e_n\|/\alpha_n \to 0$ for Theorem 5.2.1 to be applicable. The process of finding common zeroes of two maximal monotone operators by means of iterative processes thus entails dividing the error sequences into two classes − the class of errors satisfying the condition $\sum_{n=0}^{\infty}\|e_n\| < \infty$ and the ones that satisfy the condition $\sum_{n=0}^{\infty}\|e_n\| = \infty$ − and constructing an algorithm in accordance with the rules introduced and discussed in this section.

Chapter 6

Some Applications

In this chapter, we illustrate how the results of the previous chapters can: (a) be used to approximate minimizers of convex functionals, and also (b) be applied to solve variational inequality problems. In particular, we shall show that when the sequence of errors does not satisfy the summability condition, then the pool of items from which the sequence of parameters (α_n) can be chosen in order to construct sequences that approximate minimum values of convex functionals can be increased. A typical construction of such a sequence can be obtained by choosing the parameters α_n's in such a way that $\alpha_n = \sqrt{\|e_n\|}$ for $e_n \neq 0$ if n stays large, and $\alpha_n = 1/2n$ otherwise. We also state and prove some convergence results for the particular case of subdifferential operators, under weaker conditions on the control parameters.

6.1 Minimizers of convex functionals

The subdifferential of a proper and convex function $\varphi : H \to (-\infty, +\infty]$ is the operator (possibly multivalued) $\partial\varphi : H \to H$ defined by

$$\partial\varphi(x) = \{w \in H \mid \varphi(x) - \varphi(v) \leq \langle w, x - v\rangle, \quad \forall\, v \in H\}.$$

From this definition, we see that a pair $(x, y) \in \partial\varphi$ if and only if the point x minimizes the function $\psi : H \to (-\infty, +\infty]$ defined by the rule $z \mapsto \varphi(z) - \langle z, y\rangle$. In particular, a point $x \in H$ is a minimizer of φ if and only if $x \in D(\partial\varphi)$ and $0 \in \partial\varphi(x)$. In the case when the function $\varphi : H \to (-\infty, +\infty]$ is proper, convex and lower semicontinuous, then its subdifferential $\partial\varphi =: A$, is a maximal monotone operator, and therefore algorithm (4.14) reduces to

$$v_{n+1} = \arg\,\min_{x\in H}\varphi_n(x),$$

where

$$\varphi_n(x) = \varphi(x) + \frac{1}{2\beta_n}\|x - \alpha_{n-1}u - \lambda_{n-1}v_n - \gamma_{n-1}Tv_n - e_{n-1}\|^2.$$

Thus, under the assumptions of Theorems 4.2.4, 4.2.5 and 4.2.7, (v_n) converges strongly to the minimizer of φ nearest to u. In fact, in this case of the subdifferential, we can derive weak convergence of the sequence (v_n) under fairly mild conditions. Even more, we are able to give convergence rate estimates for the residual $\varphi(w_{k+1}) - \varphi(z)$, where φ is a proper, convex and lower semicontinuous function, z is an arbitrary point of H, and

$$w_n = \sigma_n^{-1}\sum_{k=1}^{n}\beta_k v_{k+1}, \quad \text{with} \quad \sigma_n = \sum_{k=1}^{n}\beta_k. \tag{6.1}$$

In general, if a sequence (v_n) converges strongly (resp. weakly) to a point, say p, then the sequence of its weighted means with positive weights (β_n) defined by (6.1) also converges strongly (resp. weakly) to the same limit p, provided $\sigma_n \to \infty$.

We remark that the function φ_n is somewhat favorable compared to the original function φ as it preserve all the properties of φ, and even more, it is always coercive having a unique minimizer v_{n+1} due to the quadratic term added to φ. Concerning points of the sequence (v_n) generated by algorithm (4.14), it is worth noting that since $D(\partial\varphi) \subset D(\varphi)$, for any given starting point $v_0 \in H$, all of the v_n's (for $n \geq 1$) are elements of the domain of φ. This favorable feature which is not possessed by algorithm (4.1) allows us to approximate minimum points of the functional φ. It should however be mentioned that this property is shared by both algorithm (4.14) and algorithm (4.1) in the case when λ_n is identically zero for all n and T is the identity operator, since for this case the two algorithms are equivalent.

The first result of this section is concerned with the convergence of $\varphi(w_{k+1})$.

Theorem 6.1.1. *Let $A = \partial\varphi$ and $\emptyset \neq A^{-1}(0) \subset F(T)$ where $\varphi : H \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function, and $F(T)$ is the set of all fixed points of the nonexpansive map $T : H \to H$. For any fixed $u, v_1 \in H$, let (v_n) be the sequence generated by algorithm (4.14) and (w_n) be as in (6.1).*

- *If (e_n/α_n) is bounded, then for some $M > 0$, we have*

$$\varphi(w_n) - \varphi(z) \leq \frac{\|v_1 - z\|^2 + M\sum_{k=0}^{n-1}(\alpha_k + \gamma_k)}{2\sigma_n}, \quad \text{for all} \quad z \in H. \tag{6.2}$$

- *If $\sum_{k=0}^{\infty}\|e_k\| < \infty$, then for some $K > 0$, the following estimate holds*

$$\varphi(w_n) - \varphi(z) \leq \frac{\|v_1 - z\|^2 + K\sum_{k=0}^{n-1}(\alpha_k + \gamma_k + \|e_k\|)}{2\sigma_n}, \quad \text{for all} \quad z \in H. \tag{6.3}$$

If in addition, $\sigma_n^{-1}\sum_{k=0}^{n-1}(\alpha_k+\gamma_k)\to 0$ *as* $n\to\infty$*, then* $\varphi(w_n)\to\inf_{y\in H}\varphi(y)$*.*

Proof. We prove only estimate (6.2). The proof of the other estimate is similar. Note that for $A=\partial\varphi$, we have from (4.14),

$$\alpha_{k-1}(u-v_k+e_{k-1}/\alpha_{k-1})+\gamma_{k-1}(Tv_k-v_k)+(v_k-v_{k+1})\in\beta_k\partial\varphi(v_{k+1}),$$

and since both (e_k/α_k) and (v_k) are bounded, we have for all $z\in H$

$$\begin{aligned}2\beta_k(\varphi(v_{k+1})-\varphi(z)) &\le 2\langle v_k-v_{k+1},v_{k+1}-z\rangle+\gamma_{k-1}\langle Tv_k-v_k,v_{k+1}-z\rangle\\ &+ 2\alpha_{k-1}\langle u-v_k+e_{k-1}/\alpha_{k-1},v_{k+1}-z\rangle\\ &\le (\|v_k-z\|^2-\|v_{k+1}-v_k\|^2-\|v_{k+1}-z\|^2)\\ &+ M(\gamma_{k-1}+\alpha_{k-1}),\end{aligned}\tag{6.4}$$

for some $M>0$. Summing (6.4) from $k=1$ to $k=n$, and rearranging terms, we get

$$\varphi(z)+\frac{\|v_1-z\|^2+M\sum_{k=1}^{n}(1-\lambda_{k-1})}{2\sigma_n}\ge\frac{\sum_{k=1}^{n}\beta_k\varphi(v_{k+1})}{\sigma_n}\ge\varphi\left(\frac{\sum_{k=1}^{n}\beta_k v_{k+1}}{\sigma_n}\right).\tag{6.5}$$

Therefore (6.2) follows from (6.5). The final assertion of the theorem is obvious. □

Theorem 6.1.2. *Let* $A=\partial\varphi$ *and* $\emptyset\ne A^{-1}(0)\subset F(T)$ *where* $\varphi:H\to(-\infty,+\infty]$ *is a proper, convex and lower semicontinuous function, and* $F(T)$ *is the set of all fixed points of the nonexpansive map* $T:H\to H$*. For any fixed* $u,v_1\in H$*, let* (v_n) *be the sequence generated by algorithm (4.14) with* $\liminf_{n\to\infty}\beta_n>0$ *and* $\sum_{k=0}^{\infty}(\alpha_k+\gamma_k)\beta_{k+1}^{-1}<\infty$ *being satisfied. If either* $(\|e_n\|/\alpha_n)$ *is bounded, or* $\sum_{k=0}^{\infty}\|e_k\|<\infty$*, then* $\varphi(v_n)\to\inf_{y\in H}\varphi(y)$ *as* $n\to\infty$*. Moreover, if either* φ *has a unique minimizer, or* (β_n) *is bounded, then* (v_n) *converges weakly to the minimizer of* φ*.*

Proof. We prove the result only for the case when $(\|e_n\|/\alpha_n)$ is bounded. The proof of the other case is similar. Note that from (6.4) (with $z:=v_n$), we have

$$\varphi(v_{n+1})-\frac{M}{2}\sum_{k=0}^{n-1}\frac{(\alpha_k+\gamma_k)}{\beta_{k+1}}\le\varphi(v_n)-\frac{M}{2}\sum_{k=0}^{n-2}\frac{(\alpha_k+\gamma_k)}{\beta_{k+1}},$$

and since

$$\varphi(v_{n+1})-\frac{M}{2}\sum_{k=0}^{n-1}\frac{(\alpha_k+\gamma_k)}{\beta_{k+1}}\ge\inf_{y\in H}\varphi(y)-\frac{M}{2}\sum_{k=0}^{\infty}\frac{(\alpha_k+\gamma_k)}{\beta_{k+1}}>-\infty,$$

we conclude that $\lim_{n\to\infty}\varphi(v_n)$ exists and it is finite. Again we note from (6.4), (with $z:=v_n$), that

$$\left(\frac{\|v_{n+1}-v_n\|}{\beta_n}\right)^2\le\frac{1}{\beta_n}\left[\varphi(v_n)-\varphi(v_{n+1})+\frac{M(\alpha_{n-1}+\gamma_{n-1})}{2\beta_n}\right]\to 0.\tag{6.6}$$

Moreover, for any $z \in H$, we have from (6.4)

$$\varphi(v_{n+1}) - \varphi(z) \leq \left\langle \frac{v_n - v_{n+1}}{\beta_n}, v_{k+1} - z \right\rangle + \frac{M(\alpha_{n-1} + \gamma_{n-1})}{2\beta_n} \to 0.$$

In particular,

$$\lim_{n\to\infty} \varphi(v_n) = \inf_{y\in H} \varphi(y). \tag{6.7}$$

We have from (4.14),

$$\partial\varphi(v_{k+1}) \ni \frac{\alpha_{k-1}}{\beta_k}(u - v_k + \frac{e_{k-1}}{\alpha_{k-1}}) + \frac{\gamma_{k-1}}{\beta_k}(Tv_k - v_k) + \frac{(v_k - v_{k+1})}{\beta_k} \to 0.$$

Since $\partial\varphi$ is demiclosed, (see Lemma 2.2.8), for a subsequence (v_{n_k}) of (v_n) converging weakly to some v_∞, we conclude that $v_\infty \in F$. Thus $\omega_{\mathtt{w}}((v_n)) \subset F$. The conclusion follows if φ has a unique minimizer, that is, F is a singleton. Now assume that (β_n) is bounded. Then, for any $p \in F$, we have from (6.4)

$$\|v_{n+1} - p\|^2 - M\sum_{k=0}^{n-1}(\alpha_k + \gamma_k) \leq \|v_n - p\|^2 - M\sum_{k=0}^{n-2}(\alpha_k + \gamma_k).$$

On the other hand,

$$\|v_{n+1} - p\|^2 - M\sum_{k=0}^{n-1}(\alpha_k + \gamma_k) \geq -M\sum_{k=0}^{\infty}(\alpha_k + \gamma_k) > -\infty.$$

These two inequalities imply that $\lim_{n\to\infty}\|v_n - p\|$ exists and it is finite. Therefore, by Opial's lemma, there exists $q \in F$ such that $v_n \rightharpoonup q$. □

Obviously the above discussion is carried over to the regularization method of Section 3.4. In particular, we have the following corollaries.

Corollary 6.1.3. *Let $A = \partial\varphi$ and $A^{-1}(0) \neq \emptyset$ where $\varphi : H \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function. For any fixed $u, v_1 \in H$, let (v_n) be the sequence generated by algorithm (3.31) and (w_n) be as in (6.1).*

- *If $\sum_{k=1}^{\infty}\|e_{k-1}\| < \infty$, then for some $K > 0$, the following estimate holds*

$$\varphi(w_n) - \varphi(z) \leq \frac{\|v_1 - z\|^2 + K\sum_{k=1}^{n}(\alpha_{k-1} + \|e_{k-1}\|)}{2\sigma_n}, \quad \text{for all} \quad z \in H.$$

- *If (e_n/α_n) is bounded, then for some $M > 0$, we have*

$$\varphi(w_n) - \varphi(z) \leq \frac{\|v_1 - z\|^2 + M\sum_{k=1}^{n}\alpha_{k-1}}{2\sigma_n}, \quad \text{for all} \quad z \in H.$$

If in addition, $\sigma_n^{-1}\sum_{k=1}^{n}\alpha_{k-1}\to 0$ *as* $n\to\infty$*, then* $\varphi(w_n)\to\inf_{y\in H}\varphi(y)$.

Corollary 6.1.4. *Let* $A=\partial\varphi$ *and* $\emptyset\neq A^{-1}(0)$ *where* $\varphi: H\to(-\infty,+\infty]$ *is a proper, convex and lower semicontinuous function. For any fixed* $u, v_1\in H$*, let* (v_n) *be the sequence generated by algorithm (3.31) with* $\liminf_{n\to\infty}\beta_n>0$ *and* $\sum_{k=0}^{\infty}\alpha_k\beta_{k+1}^{-1}<\infty$ *being satisfied. If either* $(\|e_n\|/\alpha_n)$ *is bounded, or* $\sum_{k=0}^{\infty}\|e_k\|<\infty$*, then* $\varphi(v_n)\to\inf_{y\in H}\varphi(y)$ *as* $n\to\infty$*. Moreover, if either* φ *has a unique minimizer, or* (β_n) *is bounded, then* (v_n) *converges weakly to the minimizer of* φ.

We conclude this section by discussing the case when two subdifferential operators are involved.

Let $f,h: H\to(-\infty,+\infty]$ be two proper, convex and lower semicontinuous functions. For $A=\partial f$ and $B=\partial h$, we note that if both β_n and γ_n are bounded below away from zero, and the sequences $(\|e_n\|)$ and $(\|e_n'\|)$ are summable, then the sequence (x_n) generated by (5.1) and (5.2) converges weakly to an element of $A^{-1}(0)\cap B^{-1}(0)$, provided this intersection is nonempty. In addition, $\lim_{n\to\infty}f(x_{2n+1})=\inf_{x\in H}f(x)$, thus the subsequence (x_{2n+1}) of (x_n) approximates the minimum value of the convex functional $f: H\to(-\infty,+\infty]$. Similarly, the subsequence (x_{2n}) is used to approximate the minimum value of the convex functional $h: H\to(-\infty,+\infty]$. Under the assumptions of Theorem 5.2.1 (respectively, Theorem 5.2.2), the sequence generated by (5.12) and (5.13) (respectively, by (5.22) and (5.23)) converges strongly to the common minimizer of f and g which is closest to u. Furthermore, we have

Theorem 6.1.5. *Assume that* $f,h: H\to(-\infty,+\infty]$ *are two proper, convex and lower semicontinuous functions, with* $A^{-1}(0)\cap B^{-1}(0)=: F\neq\emptyset$*, where* $A=\partial f$ *and* $B=\partial h$*. For any fixed* $x_0,u\in H$*, let the sequence* (x_n) *be defined iteratively by (5.22) and (5.23), where* $\alpha_n\in(0,1)$ *and* $\beta_n\in(0,\infty)$*. Assume also that the error sequences* (e_n) *and* (e_n') *satisfy either one of the following conditions:*

(a) $\sum_{n=0}^{\infty}\|e_n\|<\infty$*, and* $\sum_{n=1}^{\infty}\|e_n'\|<\infty$*;*

(b) $\sum_{n=0}^{\infty}\|e_n\|<\infty$*, and* $(\|e_n'\|/\alpha_n)$ *is bounded;*

(c) *both* $(\|e_n\|/\alpha_n)$*, and* $(\|e_n'\|/\alpha_n)$ *are bounded;*

(d) $(\|e_n\|/\alpha_n)$ *is bounded, and* $\sum_{n=1}^{\infty}\|e_n'\|<\infty$.

If (β_n) *and* (γ_n) *are both increasing, and* $\lim_{n\to\infty}\alpha_n=0$*, with* $\sum_{n=0}^{\infty}\alpha_n\beta_n^{-1}<\infty$*, and* $\sum_{n=0}^{\infty}\alpha_n\gamma_n^{-1}<\infty$*, then*

$$\lim_{n\to\infty}f(x_{2n+1})=\inf_{x\in H}f(x),\quad \text{and}\quad \lim_{n\to\infty}h(x_{2n})=\inf_{y\in H}h(y). \tag{6.8}$$

The limits in (6.8) also hold true if both (β_n) and (γ_n) are bounded from below away from zero, (β_n) (or (γ_n)) is also bounded from above, and $\sum_{n=0}^{\infty}\alpha_n < \infty$. In this case, the sequence (x_n) is weakly convergent to some point in F.

Proof. We prove this result only for the case when condition (c) holds. The boundedness of (x_n) has already been shown in the proof of Theorem 5.2.2. Since

$$\beta_n \partial f(x_{2n+1}) \ \ni \ \alpha_n(u - x_{2n} + e_n/\alpha_n) + x_{2n} - x_{2n+1}, \tag{6.9}$$

for all $z \in D(f)$, we have from the subdifferential inequality

$$\begin{aligned} 2\beta_n(f(x_{2n+1}) - f(z)) &\le 2\langle x_{2n} - x_{2n+1}, x_{2n+1} - z\rangle + 2\alpha_n\langle u - x_{2n} + e_n/\alpha_n, x_{2n+1} - z\rangle \\ &\le \|x_{2n} - z\|^2 - \|x_{2n+1} - z\|^2 - \|x_{2n} - x_{2n+1}\|^2 + \alpha_n M_z, \end{aligned}$$

where M_z is a positive constant such that

$$\sup\left\{\left\|u - x_{2n} + \frac{e_n}{\alpha_n}\right\|^2 + \|x_{2n+1} - z\|^2 \mid n \ge 0\right\} \le M_z.$$

In particular, for any $v \in F$,

$$\|x_{2n} - x_{2n+1}\|^2 \ \le \ \|x_{2n} - v\|^2 - \|x_{2n+1} - v\|^2 + \alpha_n M_v.$$

Similarly, starting with

$$\gamma_n \partial h(x_{2n}) \ \ni \ \alpha_n(u - x_{2n-1} + e'_n/\alpha_n) + x_{2n-1} - x_{2n}, \tag{6.10}$$

we derive

$$\|x_{2n} - x_{2n-1}\|^2 \ \le \ \|x_{2n-1} - v\|^2 - \|x_{2n} - v\|^2 + \alpha_n K_v,$$

for some positive constant K_v, which together with the previous estimate yields

$$\|x_{2n} - x_{2n+1}\|^2 \ \le \ \|x_{2n-1} - v\|^2 - \|x_{2n+1} - v\|^2 + \alpha_n C_v, \tag{6.11}$$

for some positive constant C_v. Dividing (6.11) by β_n^2 and using the fact that (β_n) is increasing, we have upon summing from $n = 1$ to $n = \infty$

$$\sum_{n=1}^{\infty}\left(\frac{\|x_{2n} - x_{2n+1}\|}{\beta_n}\right)^2 < \infty \quad \Rightarrow \quad \lim_{n\to\infty}\frac{\|x_{2n} - x_{2n+1}\|}{\beta_n} = 0. \tag{6.12}$$

Therefore, passing to the limit in

$$f(x_{2n+1}) - f(z) \ \le \ \left\langle \frac{x_{2n} - x_{2n+1}}{\beta_n}, x_{2n+1} - z\right\rangle + \frac{\alpha_n M_z}{2\beta_n},$$

we get

$$\limsup_{n\to\infty} \left(f(x_{2n+1}) - f(z)\right) \leq 0, \quad \text{for all } z \in D(f).$$

This verifies the first part of (6.8). Similarly, passing to the limit in

$$h(x_{2n}) - h(w) \quad \leq \quad \left\langle \frac{x_{2n-1} - x_{2n}}{\gamma_n}, x_{2n} - w \right\rangle + \frac{\alpha_n K_w}{2\gamma_n},$$

we get

$$\limsup_{n\to\infty} \left(h(x_{2n}) - h(w)\right) \leq 0, \quad \text{for all } w \in D(h),$$

verifying the second part of (6.8).

To prove the last part of the theorem, assume that (β_n) is bounded, plus the other conditions specified in the statement. Then

$$\lim_{n\to\infty} \frac{\|x_{2n} - x_{2n+1}\|}{\beta_n} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \|x_{2n} - x_{2n+1}\| = 0. \tag{6.13}$$

Moreover, we have from (6.11)

$$\|x_{2n+1} - v\|^2 - C_v \sum_{k=0}^{n} \alpha_k \leq \ \|x_{2n-1} - v\|^2 - C_v \sum_{k=0}^{n-1} \alpha_k,$$

showings that for all $v \in F$, the sequence $(\|x_{2n+1} - v\|)$ is convergent. Hence $(\|x_{2n} - v\|)$ is also convergent. On the other hand, from (6.9) and (6.10) we derive $\omega_{\mathtt{w}}((x_{2n+1})) \subset A^{-1}(0)$ and $\omega_{\mathtt{w}}((x_{2n})) \subset B^{-1}(0)$, respectively. These two inclusions together with (6.13) imply that $\omega_{\mathtt{w}}((x_n)) \subset A^{-1}(0) \cap B^{-1}(0)$. Since the limit of the sequence $(\|x_n - v\|)$ exists for all $v \in F$, we have via Opial's lemma $x_n \rightharpoonup y$ for some $y \in F$. The proof is similar for the case when (γ_n) is bounded. □

Remark 6.1.1. Recall [39, Prop. 2.7, pg. 110] that for a proper, convex and lower semi continuous function $\varphi : H \to (-\infty, +\infty]$, the operator $(I + \partial\varphi)^{-1}$ is compact if and only if the set

$$K_{C,\varphi} := \{x \in H \mid \|x\| \leq C, \text{ and } \varphi(x) \leq C\}$$

is compact for every $C > 0$. Note that the set $K_{C,\varphi}$ is compact if and only if the set

$$M_{D,\varphi} := \{x \in H \mid \|x\|^2 + \varphi(x) \leq D\}$$

is compact for every $D > 0$. Therefore, using similar arguments to those contained in Remark 5.1.4, one can show that for $A = \partial f$ and $B = \partial h$, where $f, h : H \to (-\infty, +\infty]$ are two proper, convex and lower semicontinuous functions, the sequence (x_n) generated by (5.22) and (5.23) converges strongly to a common minimizer of f and h provided that both (β_n) and (γ_n) are bounded from below away from zero, either $M_{D,f}$ or $M_{D,h}$ is compact for all $D > 0$, $\sum_{n=0}^{\infty} \alpha_n < \infty$, and either (β_n) or (γ_n) is bounded from above.

6.2 An application to variational inequalities

Let $K \subset H$ be a nonempty, convex, closed set and let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator. Consider the problem

$$\text{Find } u \in K \cap D(A) \text{ such that there exists } z \in Au : \langle z, v-u \rangle \geq 0 \ \forall v \in K. \tag{6.14}$$

This is a variational inequality. It is easy to see that (6.14) is equivalent to the inclusion relation

$$0 \in Au + \partial I_K(u), \tag{6.15}$$

where I_K is the indicator function of K. If the set $F := K \cap A^{-1}(0)$ is assumed to be nonempty, then the method of alternating resolvents described in the previous chapter can be used to approximate points of F (which are solutions to (6.14)).

Example 6.2.1. Let Ω be an open and bounded subset of the Euclidean space $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Let $\beta : D(\beta) \subset \mathbb{R} \to 2^{\mathbb{R}}$ be a maximal monotone mapping, i.e., β is the subdifferential of a proper, convex, lower semicontinuous function $j : \mathbb{R} \to (-\infty, +\infty]$. Consider the boundary value problem

$$(BVP) \begin{cases} -\Delta u = f, \text{ in } \Omega, \\ -\frac{\partial u}{\partial \nu} \in \beta(u), \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega, \end{cases}$$

where $f \in L^2(\Omega)$ is a given function, and $\frac{\partial u}{\partial \nu}$ denotes the unit outward normal derivative of u. Let H denote the space $L^2(\Omega)$ equipped with its usual scalar product and norm. Define $A : D(A) \subset H \to H$,

$$Av = -\Delta v - f,$$
$$D(A) = \{v \in H^2(\Omega) \mid -\tfrac{\partial v}{\partial \nu} \in \beta(v), \text{ a.a } x \in \partial\Omega\},$$

where $H^2(\Omega)$ is the usual Sobolev space. Operator A is maximal monotone, being the subdifferential of the functional $\Phi : H \to (-\infty, +\infty]$

$$\Phi(v) = \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{2}|\nabla v|^2 - fv\right) dx + \int_{\partial\Omega} j(v)\, d\sigma, & \text{if } \ v \in H^1(\Omega),\ j(v) \in L^1(\partial\Omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

which is proper, convex and lower semicontinuous. For details see, e.g., [2, Proposition 2.9, p. 62]). Let us assume that (BVP) has at least a solution $u \in H^2(\Omega)$. This means that $F := K \cap A^{-1}(0)$ is nonempty, where K denotes the cone $\{v \in H : v(x) \geq 0$ for $a.a.\ x \in$

$\Omega\}$. It is easy to imagine cases when this situation happens, and in general F is not a singleton. We consider a sequence (v_n) generated by algorithm (5.1), (5.2), where A is the operator just defined above, and $B = \partial I_K$. Clearly, for all $\gamma > 0$, $J_\gamma^B v = P_K v = v^+ := max\{v, 0\}$. For simplicity, we assume that $\beta_n = \gamma_n = 1$ for all n. Therefore, algorithm (5.1), (5.2) reads

$$v_{2n+1} = J_1^A(v_{2n} + e_n), \quad \text{for } n = 0, 1, \dots, \tag{6.16}$$

$$v_{2n} = (v_{2n-1} + e_n')^+, \quad \text{for } n = 1, 2, \dots, \tag{6.17}$$

where v_0 is a given starting function. It is easy to check that J_1^A is a compact operator, so (v_n) is strongly convergent in $H = L^2(\Omega)$ to a point of $F := K \cap A^{-1}(0)$, whenever (e_n), (e_n') are summable in norm (cf. Theorem 5.1.2 and Remark 5.1.4). In fact, the sequence $y_n := v_{2n+1}$ generated by the difference equation

$$y_{n+1} = J_1^A(y_n^+ + e_n),\ n = 0, 1, \dots,$$

with y_0 a given starting function, converges strongly in $H^2(\Omega)$ to a solution of (BVP), under the summability condition on (e_n). Here we have incorporated the error e_n' into e_n; this is possible because P_K is a nonexpansive operator. In other words, (y_n) is a solution of the difference equation

$$\begin{cases} y_{n+1} - \Delta y_{n+1} = f + y_n^+ + e_n, \text{ in } \Omega, \\ -\frac{\partial y_{n+1}}{\partial \nu} \in \beta(y_{n+1}), \text{ on } \partial\Omega, \end{cases}$$

which allows us to derive strong convergence in $H^2(\Omega)$ for (y_n). Note that further convergence results for this particular example may be obtained from our abstract results above, (see Chapter 5, Section 5.2).

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Appendix: Control Conditions

In the table below, we collect different conditions which are used in this thesis.

(C1) $\lim_{n\to\infty} \alpha_n = 0$	(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$
(C3) $\lim_{n\to\infty} \frac{(\alpha_{n+1}-\alpha_n)}{\alpha_{n+1}^2} = 0$	(C4) $\sum_{n=0}^{\infty} \lvert\alpha_{n+1}-\alpha_n\rvert < \infty$
(C5) $\lim_{n\to\infty} \frac{(\alpha_{n+1}-\alpha_n)}{\alpha_{n+1}} = 0$	(C6) $\lim_{n\to\infty} (\beta_{n+1}-\beta_n) = 0$
(C7) $\sum_{n=0}^{\infty} \left\lvert 1-\frac{\alpha_n\beta_{n+1}}{\alpha_{n+1}\beta_n}\right\rvert < \infty$	(C8) $\lim_{n\to\infty} \frac{1}{\alpha_n}\left(1-\frac{\alpha_n\beta_{n+1}}{\alpha_{n+1}\beta_n}\right) = 0$
(C7)′ $\sum_{n=1}^{\infty} \left\lvert 1-\frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right\rvert < \infty$	(C8)′ $\lim_{n\to\infty} \frac{1}{\alpha_{n-1}}\left(1-\frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right) = 0$
(C9) $\sum_{n=0}^{\infty} \lvert\beta_{n+1}-\beta_n\rvert < \infty$	(C9)′ $\sum_{n=0}^{\infty} \frac{\lvert\beta_{n+1}-\beta_n\rvert}{\beta_{n+1}} < \infty$
(C10) $\sum_{n=1}^{\infty} \left\lvert \frac{\alpha_{n-1}}{\beta_n}-\frac{\alpha_n}{\beta_{n+1}}\right\rvert < \infty$	(C11) $\lim_{n\to\infty} \left(1-\frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right) = 0$
(C12) $\lim_{n\to\infty} \frac{(\alpha_n-\alpha_{n-1})}{\alpha_{n-1}\beta_n} = 0$	(C13) $\sum_{n=1}^{\infty} \frac{\lvert\alpha_n-\alpha_{n-1}\rvert}{\beta_n} < \infty$
(C14) $\lim_{n\to\infty} \left(\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right) = 0$	(C15) $\sum_{n=1}^{\infty} \left\lvert \frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right\rvert < \infty$
(C16) $\lim_{n\to\infty} \frac{1}{\alpha_n}\left(\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right) = 0$	(C17) $\lim_{n\to\infty} \frac{1}{\alpha_n}\left(1-\frac{\beta_n}{\beta_{n+1}}\right) = 0$
(C18) $\lim_{n\to\infty} \frac{(\beta_{n+1}-\beta_n)}{\beta_n} = 0$	(C19) $\sum_{n=1}^{\infty} \frac{\lvert\lambda_n-\lambda_{n-1}\rvert}{\beta_n} < \infty$
(C20) $\lim_{n\to\infty} \frac{1}{\alpha_{n-1}\alpha_n}\left(\gamma_n-\frac{\gamma_{n-1}\beta_{n+1}}{\beta_n}\right) = 0$	(C21) $\lim_{n\to\infty} \frac{1}{\beta_{n+1}}\left(\frac{\alpha_n}{\alpha_{n-1}}-\frac{\beta_{n+1}}{\beta_n}\right) = 0$
(C22) $\sum_{n=1}^{\infty} \left\lvert \frac{\gamma_n}{\beta_{n+1}}-\frac{\gamma_{n-1}}{\beta_n}\right\rvert < \infty$	(C23) $\lim_{n\to\infty} \frac{1}{\alpha_{n-1}}\left(\frac{\gamma_n}{\beta_{n+1}}-\frac{\gamma_{n-1}}{\beta_n}\right) = 0$