Generalizations of Classical Theorems in Extremal Set Theory

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Contents

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2 Terminology

In this section we introduce the most important terminology for the thesis. We denote the set $\{1, 2, \ldots, n\}$ by $[n]$. We will use the word collection or family when referring to a set whose elements are sets. We denote the set of its r-elements subset of $[n]$ by $\binom{[n]}{r}$. For a set X, we denote the power set of X by 2^X. A collection $\mathcal{A}$ is said to be intersecting if every two sets $A \in \mathcal{A}$ share at least one element. Two or more collections are said to be cross-intersecting if for every two collections $\mathcal{A}$ and $\mathcal{B}$ and for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, A and B have at least one common point. A chain refers to a collection of sets $\{A_1, A_2, \ldots, A_t\}$ such that $A_i \subset A_{i+1}$. A chain of subsets of $[n]$ is said to be maximal if it contains a set of all $n+1$ possible cardinalities. A collection $\mathcal{A}$ is said to be Sperner or an antichain if there do not exist elements $A, B \in \mathcal{A}$ such that $A \subset B$.

Two important objects we will use are permutations and cyclic permutations. By a permutation of X we simply mean a word of length $|X|$ with each element of X occurring exactly once. We will use π as our notation for a permutation in this thesis. Formally, a cyclic permutation is an equivalence class of permutations under the equivalence relation on words that $a \sim b$ if a=xy and b=yx for some words x, y where the product is concatenation. Intuitively, we think of a cyclic permutation of X simply as the elements of X arranged in a circle. We will say a set A is an interval in the cyclic permutation σ if its elements occur consecutively in some word in the equivalence class. Informally, A is an interval if it forms an arc viewing σ as the elements arranged in a circle.

We will also need some basic concepts in graph theory. A clique is a set of vertices for which there exists an edge joining each pair. An independent set is a set of vertices with no edge between any pair. The complete graph on k vertices is the graph on k vertices containing every edge. A bipartite graph is a graph in which the vertices can be partitioned into two classes, such that the only edges join vertices of distinct classes. A bipartite graph containing every possible edge is called a complete bipartite graph. A blown-up complete graph is a graph consisting multiple independent sets of vertices with possibly different sizes and every possible edge between two different independent sets.

Finally we introduce the basic concepts of order theory. A partially ordered set (or poset) is a pair $(S, \leq)$ where $\leq$ is a relation on S which such that: (1) for all $x \in S$, $x \leq x$ (2) $x \leq y$ and $y \leq x$ implies $x = y$ (3) $x \leq y$ and $y \leq z$ implies $x \leq z$. Observe that $(2^{[n]}, \subset)$ forms a partially ordered set; we will sometimes refer to this set as the boolean lattice. The containment graph of a partially ordered set is the graph whose vertices are the elements of the set and with an edge between any two related elements. The Hasse diagram of a partially ordered set is the graph whose vertices are the elements of the partially ordered set with an edge between x and y if $x \leq y$ and there is no z such that $x \leq z \leq y$.

3 Introduction

The purpose of this thesis is to discuss some ideas for generalizing classical results in extremal set theory. We emphasize that the majority of the ideas are joint work with David Malec at UW Madison. To be specific everything except the last two research sections of the thesis was done in collaboration. Any errors, on the other hand, are likely my own and should be interpreted as such.

The first few sections of the thesis will discuss some alternate proofs of Hilton's generalization of the Erdős-Ko-Rado theorem. We then further generalize Hlton's theorem in multiple directions. In particular we optimize over more general classes of functions of the set sizes, and we consider less restrictive cross-intersection problems. Next we consider some very easy proofs of the Erdős-Ko-Rado theorem and Hilton's theorem for the special case where r divides n. The next short section is a discussion of some attempts to generalize Sperner's theorem and intersection theorems to an infinite setting. The following section introduces a generalization of chains of subsets allowing us to generalize Sperner's theorem. The final section discusses a new variant of multipart Sperner problems and a slight simplification of an old result.

We begin with a brief review of the classical extremal set theory theorems which are relevant to our thesis. We do not intend to provide an extensive survey of the field. Such surveys already exist, and we refer the reader to the excellent books of Anderson[1], Bollobás[3], and Engel[6]. For reference, and to motivate the rest of the thesis, we will recall the results of Erdős-Ko-Rado[14] and Sperner[15]. We begin with Lubell's elegant proof[13] of Sperner's theorem,

Theorem 1. *Let $\mathcal{A}$ be an antichain of subsets of $[n]$, then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

and this bound is sharp.

Proof. The collection of all subsets of size $\lfloor \frac{n}{2} \rfloor$ is an antichain and acheives the bound. We double count pairs $(\mathcal{C}, A)$ where $\mathcal{C}$ is a maximal chain of subsets, $A \in \mathcal{A}$, and A occurs in the chain $\mathcal{C}$. First observe for a fixed set A there are exactly $|A|!(n-|A|)!$. We first add each element of A one at a time; there are $|A|!$ ways to do this. Next we can add the the remaining $n - |A|$ elements one a time in $(n - |A|)!$ ways. It follows the number of pairs $(\mathcal{C}, A)$ is exactly,

$$\sum_{A \in \mathcal{A}} |A|!(n - |A|)!.$$

Now, fix a maximal chain $\mathcal{C}$ and observe that, by the antichain property, there may be at most one $A \in \mathcal{A}$ in $\mathcal{C}$. Since the total number of maximal chains is $n!$, it follows that the number of pairs $(\mathcal{C}, A)$ is at most $n!$. Thus we have,

$$\sum_{A \in \mathcal{A}} |A|!(n - |A|)! \leq n!.$$

and dividing through gives,

$$\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1.$$

Since $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is a largest binomial coefficient we may replace A with $\lfloor \frac{n}{2} \rfloor$ in the left hand side of the inequality. Then, multiplying through gives,

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

□

We now present Katona's[10] elegant proof of the Erdős-Ko-Rado theorem using cyclic permutations,

Theorem 2. *Let $\mathcal{A}$ be an intersecting collection of r element subsets of $[n]$, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1},$$

and this bound is sharp.

Proof. By taking all r element subsets containing a particular element the bound is achieved. To prove the bound we will count pairs (σ, A) where $A \in \mathcal{A}$, σ is a cyclic permutation of $[n]$ and A is an interval in σ. First observe for a fixed set A there are exactly $r!(n-r)!$ cyclic permutations containing it as an interval. We can first order the elements of A in $r!$ ways; this fixes the starting end ending position where we may place the remaining

$n-r$ elements. Thus, since there are $(n-r)!$ permutations of $(n-r)$ objects we have a total of $r!(n-r)!$ cyclic permutations containing A. It follows that the number of pairs (σ, A) is exactly $|\mathcal{A}|r!(n-r)!$. If, on the other hand, we first fix a cyclic permutation σ then there may be at most r elements of $\mathcal{A}$ forming intervals in it. To see this, fix the first interval I, and observe that all other intervals must intersect this one. If we number the positions of σ used by I $1, 2, \ldots, r$; then we see for each position $i, i \geq 2$ we may have at most one of the following two intervals: the one starting at i, or the one ending at $i-1$. Thus, in addition to the first interval we may have at most $r-1$ more intervals, for a total of r. Now, since the total number of cyclic permutations is $(n-1)!$, we have that the number of pairs (σ, A) is at most $r(n-1)!$. Hence, it follows,

$$|\mathcal{A}|r!(n-r)! \leq r(n-1)!,$$

and thus,

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

□

Hilton[9] introduced the following generalization of the Erdős-Ko-Rado theorem,

Theorem 3. *Let* $\{\mathcal{A}_i\}, i = 1, 2, \ldots, k$ *be a cross-intersecting collection of* r

element subsets of an n element set where $2r \leq n$. Then,

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \max\left(\binom{n}{r}, k\binom{n-1}{r-1}\right).$$

Note that we are allowing the $\mathcal{A}_i$ to be identical collections. The bound is obtained by either taking $\mathcal{A}_1$ to be all r elements subsets and every other $\mathcal{A}_i$ to be empty, In this case the cross-intersecting condition is met trivially; or we take each $\mathcal{A}_i$ to be all sets containing the same fixed element.. Which choice is optimal depends on the relationship between k, r, and n. The standard Erdős-Ko-Rado theorem is recovered easily: Let $\mathcal{A}$ be an intersecting family of r element subsets. Choose $\mathcal{A}_i = \mathcal{A}$ for all i in Hilton's theorem and assume $k \geq \frac{n}{r}$. Then Hilton's bound gives,

$$\sum_{i=1}^{k} |\mathcal{A}_i| = k|\mathcal{A}| \leq k\binom{n-1}{r-1},$$

and the Erdős-Ko-Rado bound follows upon dividing by k. Hilton's proof of this theorem was fairly long and made use of the Kruskal-Katona Theorem. In the following sections, we will present a two different proofs using cyclic permutations. We acknowledge that Peter Borg has an earlier proof of Hilton's theorem using cyclic permutations, and we encourage the reader to read his article [5] (also see [4] for Kruskal-Katona based proof). More generally he has proved a Hilton-type theorem for signed sets. However, the proofs here take a largely different approach and lead to generalizations in several directions not yet considered. We now begin our study of Hilton's theorem.

4 A proof of Hilton's theorem

We recall the statement which we will prove,

Theorem 4. *Let* $\{\mathcal{A}_i\}, i = 1, 2, \dots, k$ *be a cross-intersecting collection of* r *element subsets of an* n *element set where* $2r \leq n$. *Then,*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \max\left(\binom{n}{r}, k\binom{n-1}{r-1}\right).$$

The approach of the proof will be to count triples $(A, \mathcal{A}_i, \sigma)$ where $A \in \mathcal{A}_i$ and A is an interval in the cyclic permutation σ in two different ways. First we establish a bound on the number of pairs $(A, \mathcal{A}_i)$ such that $A \in \mathcal{A}_i$ where A is compatible with a fixed cyclic permutation σ. This will follow from the three lemmas below. Then we count the triples in a different order by considering how many cyclic permutations contain each A in some fixed $\mathcal{A}_i$.

Lemma 1. *Let* $A_1, A_2, \dots, A_t$ *be intervals of size* h *in a cyclic permutation* σ. *Then* $|\bigcap_{i=1}^{t} A_i| \leq max(0, h - t + 1)$.

Proof. If $\bigcap_{i=1}^{t} A_i = \varnothing$ then we are done. Therefore, we may take $x \in \bigcap_{i=1}^{t} A_i$. Reindex the A_i so that $i < j$ implies x occurs later in A_i than A_j. Formally, we insist that any string in the equivalence class σ containing the elements of A_i consecutively x occurs closer to the right end of A_i than it does to the right end of A_j in any string containing the elements A_j consecutively. Then we have for any j that $|A_1| = h$ and $|\bigcap_{i=1}^{j} A_i| - |\bigcap_{i=1}^{j+1} A_i| \geq 1$, and it follows

$|\bigcap_{i=1}^{t} A_i| \leq h - t + 1.$ □

We now introduce some notation for simplicity. For a collection $\mathcal{A}$ and a cyclic permutation σ of $[n]$ let $\mathcal{A}_1^\sigma$ denote the set of those $A \in \mathcal{A}$ which are intervals in σ.

Lemma 2. *Let σ be a cyclic permutation of $[n]$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be cross-intersecting collections of r element subsets of $[n]$ with $2r \leq n$. Assume $\mathcal{A}_1^\sigma$ and $\mathcal{A}_2^\sigma$ are nonempty, then $|\mathcal{A}_1^\sigma| + |\mathcal{A}_2^\sigma| \leq 2r$.*

Proof. For each $A \in \mathcal{A}_1^\sigma$ let A' denote those positions of σ for a which an interval of length r beginning there intersects A, then $|A'| = 2r - 1$ for all A. Applying the previous lemma we have,

$$|\mathcal{A}_2^\sigma| \leq |\bigcap_{A \in \mathcal{A}_1^\sigma} A'| \leq \max(0, 2r - 1 - |\mathcal{A}_1^\sigma| + 1).$$

By the assumption that $\mathcal{A}_2^\sigma$ is nonempty, it follows $|\mathcal{A}_1^\sigma| + |\mathcal{A}_2^\sigma| \leq 2r$. □

We are now ready to prove a bound on the number of pairs $(A, \mathcal{A}_i)$ with $A \in \mathcal{A}_i$ an interval in a fixed cyclic permutation σ. To this end we introduce some more notation. Let $I = \{i : \text{some } A \in \mathcal{A}_i \text{ is an interval in } \sigma\}$. Also let i^* denote an index i for which $|\mathcal{A}_i^\sigma|$ is maximal.

Lemma 3. *Fix a cyclic permutation σ, let $\mathcal{A}_1 \ldots \mathcal{A}_k$ be pairwise cross-intersecting collections of r element subsets of an n element base set with $2r \leq n$, then $\sum_{i=1}^{k} |\mathcal{A}_i^\sigma| \leq \max(n, |I|r) \leq \max(n, kr)$.*

Proof. If $|I| = 1$ then n is an upper bound trivially. Hence, we will assume $|I| \geq 2$.

$$\begin{aligned}
\sum_{i=1}^{k} |\mathcal{A}_i^\sigma| &= |\mathcal{A}_{i^*}^\sigma| + \sum_{I \setminus \{i^*\}} |\mathcal{A}_i^\sigma| \\
&\leq |\mathcal{A}_{i^*}^\sigma| + \sum_{I \setminus \{i^*\}} (2r - |\mathcal{A}_{i^*}^\sigma|) \\
&= |\mathcal{A}_{i^*}^\sigma| + (|I|-1)r + (|I|-1)(r - |\mathcal{A}_{i^*}^\sigma|) \\
&= |I|r + (|I|-2)(r - |\mathcal{A}_{i^*}^\sigma|).
\end{aligned}$$

Now if $|\mathcal{A}_{i^*}^\sigma| \leq r$ then by the definition of $\mathcal{A}_{i^*}^\sigma$ we have the bound $|I|r$ on the sum. Otherwise the second term is ≤ 0 (since $|I| \geq 2$), and so again $|I|r$ bounds the sum. □

We can now proceed with the proof of the theorem,

Proof. Let $f(A, i, \sigma) = 1$ if $A \in \mathcal{A}_i$ and A is an interval in σ and $f(A, i, \sigma) = 0$ otherwise. Then since each A is an interval in $r!(n-r)!$ cyclic permutations we have,

$$\sum_{i=1}^{k} \sum_{A} \sum_{\sigma} f(A, i, \sigma) = \sum_{i=1}^{k} \mid \mathcal{A}_i \mid r!(n-r)!.$$

Changing the order of summation and applying the third lemma above gives,

$$\sum_{\sigma} \sum_{i=1}^{k} \sum_{A} f(A, i, \sigma) \leq \sum_{\sigma} max(n, kr) \leq (n-1)!max(n, kr).$$

Dividing through by $r!(n-r)!$ gives the result. $\square$

5 Varied r_i Hilton

We will modify our proof of Hilton's theorem to give us an inequality which holds in the more general situation in which we allow $\mathcal{A}_i$ consist of r_i element subsets of $[n]$ rather than just a fixed size r.

Theorem 5. *Let $\mathcal{A}_i \in \binom{[n]}{r_i}, i = 1, 2, \ldots, k$ be cross-intersecting and assume $r_i + r_j \leq n$ for all $i \neq j$, then*

$$\sum_{i=1}^{k} \frac{|\mathcal{A}_i|}{\binom{n}{r_i}} \leq 1 \text{ if } n \geq \sum_{i=1}^{k} r_i,$$

$$\sum_{i=1}^{k} \frac{r_i}{\sum\limits_{j=1}^{k} r_j} \frac{|\mathcal{A}_i|}{\binom{n}{r_i}} \leq 1 \text{ if } n \leq \sum_{i=1}^{k} r_i.$$

Observe that the standard version of Hilton's theorem is recovered if we take all the r_i to be the same. We will again proceed with a sequence of lemmas. First we will need the following generalization of the lemma used to prove Hilton's theorem. As before, for each cyclic permutation σ, denote by $\mathcal{A}^\sigma$ the set of $A \in \mathcal{A}$ which are intervals in σ.

Lemma 4. *Let $\mathcal{A}_1 \in \binom{[n]}{r_1}$ and $\mathcal{A}_2 \in \binom{[n]}{r_2}$. Assume $r_1 + r_2 \leq n$, and that $\mathcal{A}_1$ and $\mathcal{A}_2$ are cross-intersecting. Suppose $\mathcal{A}_1^\sigma$ and $\mathcal{A}_2^\sigma$ are nonempty. Then we have $|\mathcal{A}_1^\sigma| + |\mathcal{A}_2^\sigma| \leq r_1 + r_2$.*

Proof. For each $A \in \mathcal{A}_1^\sigma$ let A' denote those positions of σ for a which an interval of length r_2 beginning there intersects A, then $|A'| = r_1 + r_2 - 1$ for all A. Applying the first lemma in the proof of Hilton's theorem we have,

$$|\mathcal{A}_2^\sigma| \leq |\bigcap_{A \in \mathcal{A}_1^\sigma} A'| \leq \max(0, r_1 + r_2 - 1 - |\mathcal{A}_1^\sigma| + 1).$$

By the assumption that $\mathcal{A}_2^\sigma$ is nonempty, it follows $|\mathcal{A}_1^\sigma| + |\mathcal{A}_2^\sigma| \leq r_1 + r_2$. $\square$

We now generalize the third lemma of our proof of Hilton's theorem to the nonuniform case. Again for a fixed cyclic permutation σ, define I to be the set of i for which there is a pair $(A, \mathcal{A}_i)$ such that $A \in \mathcal{A}_i$ and A is an interval in σ. We Define $\Delta^* = \min_i (r_i - |\mathcal{A}_i^\sigma|)$ and $i^* = \operatorname{argmin}_i (r_i - |\mathcal{A}_i^\sigma|)$. Observe that if we have $r_i = r$ for all i, then the definition of i^* coincides with our old definition.

Lemma 5. *Fix a cyclic permutation σ. Let $\mathcal{A}_1 \ldots \mathcal{A}_k$ be pairwise cross-intersecting collections of respectively r_i element subsets of an n element base set with $r_i + r_j \leq n$ for all $r_i \neq r_j$, then $\sum_{i=1}^{k} |\mathcal{A}_i^\sigma| \leq \max(n, \sum_I r_i)$.*

Proof. If $|I| = 1$ we immediately have the upper bound of n, so we may suppose that $|I| \geq 2$.

$$\begin{aligned}
\sum_{i=1}^{k} |\mathcal{A}_i^\sigma| &= |\mathcal{A}_{i^*}^\sigma| + \sum_{I\setminus\{i^*\}} |\mathcal{A}_i^\sigma| \\
&\leq |\mathcal{A}_{i^*}^\sigma| + \sum_{I\setminus\{i^*\}} (r_i + r_{i^*} - |\mathcal{A}_{i^*}^\sigma|) \\
&= r_{i^*} - \Delta^* + \sum_{I\setminus\{i^*\}} (r_i + \Delta^*) \\
&= r_{i^*} - \Delta^* + (|I|-1)\Delta^* + \sum_{I\setminus\{i^*\}} r_i \\
&= (|I|-2)\Delta^* + \sum_{I\setminus\{i^*\}} r_i.
\end{aligned}$$

Now if $\Delta^* \leq 0$ we have the bound by the inequality above. If, on the other hand, $\Delta^* \geq 0$, then $r_i \geq |\mathcal{A}_i^\sigma|$ for every i. Hence, by summing over all i we get $\sum_{i=1}^{k} |\mathcal{A}_i^\sigma| \leq \sum_{i \in I} r_i$. □

We are now ready to prove the theorem,

Proof. Let $f(A,i,\sigma) = 1$ if $A \in \mathcal{A}_i$ and A is an interval in σ and $f(A,i,\sigma) = 0$ otherwise. If $n \geq \sum_{i=1}^{k} r_i$, then we have by the previous lemma,

$$\sum_{\sigma} \sum_{i=1}^{k} \sum_{A} f(A,i,\sigma) \leq \sum_{\sigma} n = n!.$$

Summing in another order we have,

$$\sum_{i=1}^{k} \sum_{A} \sum_{\sigma} f(A,i,\sigma) = \sum_{i=1}^{k} \mid \mathcal{A}_i \mid r_i!(n-r_i)!.$$

After dividing through by $n!$ we have established the first inequality. Now suppose $n \leq \sum_{i=1}^{k} r_i$. Then,

$$\sum_{\sigma}\sum_{i=1}^{k}\sum_{A} f(A,i,\sigma) \leq \sum_{\sigma}\sum_{i=1}^{k} r_i = (n-1)!\sum_{i=1}^{k} r_i.$$

Summing in a different order we again get,

$$\sum_{i=1}^{k}\sum_{A}\sum_{\sigma} f(A,i,\sigma) = \sum_{i=1}^{k} \mid \mathcal{A}_i \mid r_i!(n-r_i)!.$$

After dividing through by $(n-1)!\sum_{i=1}^{k} r_i$ we have established the second inequality. $\square$

6 EKR and Hilton When r Divides n

Here we present very short proofs of the Erdős-Ko-Rado theorem and of Hilton's theorem in the case where r divides n. These proofs are likely known, but we have yet to come across them in the literature. The idea is to replace the notion of an interval in the cyclic permutation method with that of an anti-interval, defined as a set of r elements each occurring $\frac{n}{r}$ positions after the previous in σ. For example, consider the cyclic permutation 231546. This cyclic permutation has anti-intervals $214, 356, 25, 34, 16$ as well as the singletons, the complete interval, and the empty set.

Theorem 6. *Let $\mathcal{A}$ be an intersecting collection of r element subsets of an $[n]$ where $r < n$ and r divides n, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.*

Proof. We double count pairs (A, σ) where $A \in \mathcal{A}$ is an anti-interval in σ. Observe that for a fixed A there are $(r-1)!(n-r)!$ permutations for which A is an anti-interval. To see this first we place the elements of A; there are $(r-1)!$ distinguishable ways to do this. Next we have $(n-r)!$ ways to fill in the remaining elements. On the other hand each cyclic permutation may contain at most one anti-interval from $\mathcal{A}$. Thus, we have $|\mathcal{A}|(r-1)!(n-r)! \leq (n-1)!$, and dividing gives the bound. □

We now provide a similar proof for the r dividing n case of Hilton's theorem.

Theorem 7. *Let $\{\mathcal{A}_i\}$ be a cross-intersecting collection of r element subsets* *of an n element set such that $r < n$ and r divides n. Then,*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \max\left(\binom{n}{r}, k\binom{n-1}{r-1}\right)$$

Proof. Denote by $\mathcal{A}^\sigma$ the number of $A \in \mathcal{A}$ which are anti-intervals in σ. Let $f(A, i, \sigma) = 1$ if $A \in \mathcal{A}_i$ and A is an anti-interval in σ and $f(A, i, \sigma) = 0$ otherwise. By the same reasoning as in the above proof we have,

$$\sum_{i=1}^{k} \sum_{A} \sum_{\sigma} f(A, i, \sigma) = \sum_{i=1}^{k} \mid \mathcal{A}_i \mid (r-1)!(n-r)!$$

Observe that for any fixed σ we have,

$$\sum_{i=1}^{k} |\mathcal{A}_i^\sigma| \leq \max(n, k)$$

since if exactly one $\mathcal{A}_i^\sigma$ is nonempty, then n is an upper bound. If more than one $\mathcal{A}_i^\sigma$ is nonempty, then they must contain exactly the same set, and so we have an upper bound of k. Rearranging the sum gives,

$$\sum_{\sigma} \sum_{i=1}^{k} \sum_{A} f(A, i, \sigma) \leq \sum_{\sigma} max(n, kr) \leq (n-1)! max(n, k)$$

Dividing through by $(r-1)!(n-r)!$ gives the result. □

7 Another Proof of Hilton's Theorem

We recall the statement of Hilton's theorem,

Theorem 8. *Let $\{\mathcal{A}_i\}, i = 1, 2, \ldots, k$ be a cross-intersecting collection of r element subsets of an n element set where $2r \leq n$. Then,*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \max\left(\binom{n}{r}, k\binom{n-1}{r-1}\right).$$

We also recall a key lemma from our first proof of Hilton's theorem,

Lemma 6. *Let σ be a cyclic permutation of $[n]$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be cross-intersecting collections of r element subsets of $[n]$ with $2r \leq n$. Assume $\mathcal{A}_1^\sigma$ and $\mathcal{A}_2^\sigma$ are nonempty, then $|\mathcal{A}_1^\sigma| + |\mathcal{A}_2^\sigma| \leq 2r$.*

This proof of Hilton's theorem will differ from the previous in that we will replace the $\mathcal{A}_i$ with a new collections $\mathcal{A}_i'$ which are easier to work with in such a way that the sum of their cardinalities does not decrease. We then establish the bound for $\sum_{i=1}^{k} |\mathcal{A}_i'|$ which is, in turn, a bound for $\sum_{i=1}^{k} |\mathcal{A}_i|$. We now state this modification as a lemma,

Lemma 7. *Let $\mathcal{A}_i$, $i = 1, 2, \ldots, k$ be cross-intersecting collections of r elements subsets of $[n]$. Let $\mathcal{A}$ denote those sets A which occur in at least two $\mathcal{A}_i$ and $\mathcal{B}$ be the collection of those sets occurring in exactly one $\mathcal{A}_i$. We define the sets $\mathcal{A}_i'$ by $\mathcal{A}_1' = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A}_i' = \mathcal{A}$ for $1 < i \leq k$. Then $\mathcal{A}_i'$ are cross-intersecting and $\sum_{i=1}^{k} |\mathcal{A}_i| \leq \sum_{i=1}^{k} |\mathcal{A}_i'|$.*

Proof. First, we may add all sets A occurring in at least two $\mathcal{A}_i$ to every $\mathcal{A}_i$ without violating the cross-intersecting property. This is clear since, for all i, every $B \in \mathcal{A}_i$ must intersect A since A lies in a some $\mathcal{A}_j$, $j \neq i$. Next we may move all sets A which occur in exactly one $\mathcal{A}_i$ to $\mathcal{A}_1$ since every set A either lies in $0, 1$, or k of the $\mathcal{A}_i$ and we already have that the sets in just one $\mathcal{A}_i$ intersect with the ones in all k. □

Now proving the theorem amounts to bounding the size of $k|\mathcal{A}| + |\mathcal{B}|$. The proof will again use cyclic permutations.

Proof. We introduce some terminology we will need. Let T_1 be the set of those cyclic permutations, σ, on $[n]$ for which there is $A \in \mathcal{A}$ forming an interval in σ. Let T_2 be the set of cyclic permutation, σ, for which there is no $A \in \mathcal{A}$ forming an interval in σ. Let

$$T_1 = \{\sigma \text{ such that some } A \in \mathcal{A} \text{ is an interval in } \sigma\}, \tag{1}$$

$$T_2 = \{\sigma \text{ such that no } A \in \mathcal{A} \text{ is an interval in } \sigma\}, \tag{2}$$

$$N_1(\mathcal{A}) = \{(A, \sigma) \text{ such that } A \in \mathcal{A},\ \sigma \in T_1,\ \text{and } A \text{ is an interval in } \sigma\}, \tag{3}$$

$$N_1(\mathcal{B}) = \{(A, \sigma) \text{ such that } A \in \mathcal{B},\ \sigma \in T_1,\ \text{and } A \text{ is an interval in } \sigma\}, \tag{4}$$

$$N_2(\mathcal{A}) = \{(A, \sigma) \text{ such that } A \in \mathcal{A},\ \sigma \in T_2,\ \text{and } A \text{ is an interval in } \sigma\}, \tag{5}$$

$$N_2(\mathcal{B}) = \{(A, \sigma) \text{ such that } A \in \mathcal{B},\ \sigma \in T_2,\ \text{and } A \text{ is an interval in } \sigma\}. \tag{6}$$

We now list some simple relationships between $|\mathcal{A}|, |\mathcal{B}|$, and the above quantities:

$$|T_1| + |T_2| = (n-1)!, \tag{7}$$

$$N_1(\mathcal{A}) = |\mathcal{A}| r!(n-r)!, \tag{8}$$

$$N_1(\mathcal{B}) + N_2(\mathcal{B}) \leq |\mathcal{B}| r!(n-r)!, \tag{9}$$

$$N_2(\mathcal{B}) \leq n|T_1|, \tag{10}$$

$$2N_1(\mathcal{A}) + N_1(\mathcal{B}) \leq 2r|T_1|, \tag{11}$$

where the last inequality follows from the first lemma applied to the cross intersecting sets $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A}$. We now use these facts to bound $k|\mathcal{A}| + |\mathcal{B}|$,

$$\begin{aligned}
k|\mathcal{A}| + |\mathcal{B}| &= \frac{1}{r!(n-r)!}(kN_1(\mathcal{A}) + N_1(\mathcal{B}) + N_2(\mathcal{B})) \text{ by (8) and (9)} && (12)\\
&= \frac{k}{2r!(n-r)!}(2N_1(\mathcal{A}) + \frac{2}{k}N_1(\mathcal{B}) + \frac{2}{k}N_2(\mathcal{B})) && (13)\\
&\leq \frac{k}{2r!(n-r)!}(2N_1(\mathcal{A}) + N_1(\mathcal{B}) + \frac{2}{k}N_2(\mathcal{B})) \text{ since } k \geq 2 && (14)\\
&\leq \frac{k}{2r!(n-r)!}(2r|T_1| + \frac{2}{k}N_2(\mathcal{B})) \text{ by (11)} && (15)\\
&\leq \frac{k}{r!(n-r)!}(r|T_1| + \frac{n}{k}|T_2|) \text{ by (10)} && (16)\\
&\leq \frac{k}{r!(n-r)!} \max(r, \frac{n}{k})(|T_1| + |T_2|) && (17)\\
&= \frac{k}{r!(n-r)!} \max(r, \frac{n}{k})(n-1)! \text{ by (7)} && (18)\\
&= \max\left(\binom{n}{r}, k\binom{n-1}{r-1}\right). && (19)
\end{aligned}$$

$\square$

8 A Weighted Version of Hilton's theorem

We will prove the following weighted version of Hilton's theorem:

Theorem 9. *Let* $\{\mathcal{A}_i\}, i = 1, 2, \ldots, k$ *be a cross-intersecting collection of* r *element subsets of an* n *element set. Let* w_i *be positive real numbers. Let* $w_{max} = \max_i w_i$. *Let* $W = \sum_{i=1}^{k} w_i$. *Then,*

$$\sum_{i=1}^{k} w_i |\mathcal{A}_i| \leq \max\left(w_{max}\binom{n}{r}, W\binom{n-1}{r-1} \right).$$

Hilton's theorem is recovered by taking all weights equal to one. We will use a natural extension of our lemma from our proof of Hilton's theorm. For simplicity we reindex so that $w_1 = w_{max}$.

Lemma 8. *Let* $\mathcal{A}_i$, $i = 1, 2, \ldots, k$ *be cross-intersecting collections of* r *elements subsets of* $[n]$. *Let* $\mathcal{A}$ *denote those sets* A *which occur in at least two* $\mathcal{A}_i$ *and* $\mathcal{B}$ *be the collection of those sets occurring in exactly one* $\mathcal{A}_i$. *We define the sets* $\mathcal{A}_i'$ *by* $\mathcal{A}_1' = \mathcal{A} \cup \mathcal{B}$ *and* $\mathcal{A}_i' = \mathcal{A}$ *for* $1 < i \leq k$. *Then* $\mathcal{A}_i'$ *are cross-intersecting and* $\sum_{i=1}^{k} w_i |\mathcal{A}_i| \leq \sum_{i=1}^{k} w_i |\mathcal{A}_i'|$.

Proof. By the same reasoning as in the unweighted version, the cross-intersecting property is preserved if we copy the A's occurring in at least two $\mathcal{A}_i$ to all of them and move all the A's only occurring once to $\mathcal{A}_1$. Now we just note that since the weights are positive the weighted sum can only increase by copying sets. Furthermore, $w_1 = w_{max}$ implies the sum can not decrease by moving sets to $\mathcal{A}_1$. $\square$

Now, all that remains is to bound the sum $W|\mathcal{A}| + w_1|\mathcal{B}|$. We will first assume that $W \geq 2w_1$ and then deal with the remaining simple case separately,

$$
\begin{aligned}
W|\mathcal{A}| + w_1|\mathcal{B}| &= \frac{1}{r!(n-r)!}(WN_1(\mathcal{A}) + w_1N_1(\mathcal{B}) + w_1N_2(\mathcal{B})) && (20)\\
&= \frac{W}{2r!(n-r)!}(2N_1(\mathcal{A}) + \frac{2w_1}{W}N_1(\mathcal{B}) + \frac{2w_1}{W}N_2(\mathcal{B})) && (21)\\
&\leq \frac{W}{2r!(n-r)!}(2N_1(\mathcal{A}) + w_1N_1(\mathcal{B}) + \frac{2w_1}{W}N_2(\mathcal{B})) && (22)\\
&\leq \frac{W}{2r!(n-r)!}(2r|T_1| + \frac{2w_1}{W}N_2(\mathcal{B})) && (23)\\
&\leq \frac{W}{r!(n-r)!}(r|T_1| + \frac{w_1 n}{W}|T_2|) && (24)\\
&\leq \frac{W}{r!(n-r)!}\max(r, \frac{w_1 n}{W})(|T_1| + |T_2|) && (25)\\
&= \frac{W}{r!(n-r)!}\max(r, \frac{w_1 n}{W})(n-1)! && (26)\\
&= \max\left(w_1\binom{n}{r}, W\binom{n-1}{r-1}\right). && (27)
\end{aligned}
$$

Now suppose $W \leq 2w_1$. Then we have,

$$
\begin{aligned}
W|\mathcal{A}| + w_1|\mathcal{B}| &= w_1(\frac{W}{w_1}|\mathcal{A}| + |\mathcal{B}|)) && (28)\\
&\leq w_1(2|\mathcal{A}| + |\mathcal{B}|) && (29)\\
&\leq w_1 \max\left(\binom{n}{r}, 2\binom{n-1}{r-1}\right) \text{ by Hilton's theorem} && (30)\\
&= w_1\binom{n}{r} && (31)
\end{aligned}
$$

In fact if we assume the sum of the weights is convergent, then the same proof implies the following infinite version:

Theorem 10. *Let* $\{\mathcal{A}_i\}, i = 1, 2, \dots$ *be a cross-intersecting collection of* r *element subsets of an* n *element set. Let* w_i *be positive real numbers. Let* $w_{max} = \max_i w_i$*. Let* $W = \sum_{i=1}^{k} w_i$*. Then,*

$$\sum_{i=1}^{\infty} w_i |\mathcal{A}_i| \leq \max\left(w_{max}\binom{n}{r}, W\binom{n-1}{r-1}\right).$$

9 The Cross-intersection Graph

We will generalize the notion of a cross-intersecting family of $\mathcal{A}_i, i = 1, 2, \ldots, k$ by no removing the assumption that for all (i, j), $\mathcal{A}_i$ and $\mathcal{A}_j$ are cross intersecting. Rather, we define a graph on k vertices, representing the collections, and add edges for those pairs of $\mathcal{A}_i$ we wish to be cross intersecting. We call this graph the cross-intersection graph, G. The standard theorem corresponds to the case where G is the complete graph.

The first theorem we prove may be thought of as a defect version of Hilton's theorem. We will allow l of the $\mathcal{A}_i$ to not cross-intersect with each other, but we insist that every other possible cross-intersection occurs. We state the theorem in terms of the cross-intersection graph.

Theorem 11. *Let the cross-intersection graph G to be a complete bipartite graph with classes of size l and $k - l$; furthermore, we add all edges among the vertices in the $k - l$ class forming a clique. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \max \left(l\binom{n}{r}, k\binom{n-1}{r-1} \right).$$

Again, we will introduce a lemma which allows us to push the sets around,

Lemma 9. *Let $\mathcal{A}_i$ be cross-intersecting with respect G. Index the $\mathcal{A}_i$ so that the first l collections correspond to the independent partition of the bipartite graph. Let $\mathcal{A}$ denote the collection of sets A occurring in at least two $\mathcal{A}_i, i > l$ or occurring at least once for $i \leq l$ and at least once for $i > l$. Let $\mathcal{B}$ be the* *collection of all remaining sets. Define the collections* $\mathcal{A}_i'$ *by* $\mathcal{A}_i' = \mathcal{A} \cup \mathcal{B}$ *for* $i = 1, 2, \ldots, l$ *and* $\mathcal{A}_i' = \mathcal{A}$ *for* $i = l+1, l+2, \ldots, k$. *Then* $\mathcal{A}_i'$ *are cross-intersecting with respect to* G *and* $\sum_{i=1}^{k} |\mathcal{A}_i| \leq \sum_{i=1}^{k} |\mathcal{A}_i'|$.

Proof. As before, we first copy all sets A occurring in $\mathcal{A}$ to all collections $\mathcal{A}_i$. By the choice of $\mathcal{A}$ we can not violate the cross-intersecting property by doing this since, otherwise, we would have had two $\mathcal{A}_i$ with an edge between them but not cross-intersecting to begin with. Next we move every set occurring in exactly one $\mathcal{A}_i$ to $\mathcal{A}_1$. As before, we do not violate the cross-intersecting property since all remaining A, not contained in the first l $\mathcal{A}_i$ already occur in all $\mathcal{A}_i$; hence, we would have already had a violation. Finally, we copy every set A found in one of the first l $\mathcal{A}_i$ to all of the first l $\mathcal{A}_i$. No violation can occur because there is no edges between these collections in G. $\square$

Proving the theorem now amounts to determining an upper bound for

$k|\mathcal{A}| + l|\mathcal{B}|$. We proceed as in our proof of Hilton's theorem,

$$\begin{aligned}
k|\mathcal{A}| + l|\mathcal{B}| &= \frac{1}{r!(n-r)!}(kN_1(\mathcal{A}) + lN_1(\mathcal{B}) + lN_2(\mathcal{B})) && (32)\\
&= \frac{k}{2r!(n-r)!}(2N_1(\mathcal{A}) + \frac{2l}{k}N_1(\mathcal{B}) + \frac{2l}{k}N_2(\mathcal{B})) && (33)\\
&\leq \frac{k}{2r!(n-r)!}(2N_1(\mathcal{A}) + lN_1(\mathcal{B}) + \frac{2l}{k}N_2(\mathcal{B})) && (34)\\
&\leq \frac{k}{2r!(n-r)!}(2r|T_1| + \frac{2l}{k}N_2(\mathcal{B})) && (35)\\
&\leq \frac{k}{r!(n-r)!}(r|T_1| + \frac{nl}{k}|T_2|) && (36)\\
&\leq \frac{k}{r!(n-r)!}\max(r, \frac{nl}{k})(|T_1| + |T_2|) && (37)\\
&= \frac{k}{r!(n-r)!}\max(r, \frac{nl}{k})(n-1)! && (38)\\
&= \max\left(l\binom{n}{r}, k\binom{n-1}{r-1}\right) && (39)
\end{aligned}$$

The standard version of Hilton's theorem is recovered by take $l = 1$. We note that if we instead to G to be a blown-up complete graph, where each vertex is blown-up by possibly a different amount, then the same exact proof goes through. We merely have treat the biggest component in the blown-up graph like the independent set above. This direction of research can be continued allowing one to obtain results for other types of graphs. For example it is not too hard to prove the obvious bounds hold for cycles and paths. However, this direction of research would be more interesting if we could find a more general result involving a parameter of G like the independence number. For example, we could try to prove that for a tree T, we should always take the independence number of T copies of full $\mathcal{A}_i$s or every $\mathcal{A}_i$ should be a star.

10 Infinite Analogues for Classical Theorems

We will discuss some of our attempts to extend some classical theorems to the infinite setting. The first question one might ask is does there exist an uncountable antichain consisting of subsets of the integers? This exercise is interesting to think about, but it turns out there are indeed such collections. For example we may the collection $\mathcal{A}$ consisting of every subset of the positive integers union its complement in the positive integers negated. For example $\{2,4,6,8,\ldots,-1,-3,-5,\ldots\} \in\mathcal{A}$. This collection is uncountable since it contains the powerset of an infinite countable set. Furthermore, it is an antichain since containment with respect to the positive part implies reverse containment with respect to the negative part and vice versa. This example suggests that cardinality is not the right property to study if we are looking for infinite analogues. Instead, we will see that other notions of measure might be more interesting.

We will think of our subsets of the integers as infinite binary strings. Furthermore we will identify these strings with the numbers in the interval $(0,1)$ considered in base 2. Now we can think of $\mathcal{A}$ as a collection of strings, and if we let $A = .a_1a_2a_3\ldots$ and $B = b_1b_2b_3\ldots$ then $A \subset B$ just means $a_i \leq b_i$ for every i. Now, strictly speaking there is not a $1-1$ correspondence due to the ambiguity of strings ending with repeating $1's$, but there are only countable such strings and hence this does not contribute to measure calculations. Then our question is the following: Suppose $\mathcal{A}$ is an antichain (and measurable), then what is the best upper bound for $\mu\mathcal{A}$ where μ is the Lebesgue measure? We have yet to succeed n answering this question, but we can say something about a related problem. Recall the following elementary theorem in extremal set theory,

Theorem 12. *Suppose $\mathcal{A} \subset 2^{[n]}$ is intersecting. Then $|\mathcal{A}| \leq 2^{n-1}$ and this is sharp.*

Proof. The family of subsets of $[n]$ containing the element 1 achieves the bound. Now, partition the subsets of $[n]$ into pairs consisting of a set and its complement. At most one set from each pair can be in an intersecting collection. Hence we have a bound of 2^{n-1}. $\square$

Now we will prove a measure theoretic analogue of this result,

Theorem 13. *Let $\mathcal{A}$ be intersecting; that is, when we compare any two binary strings in $\mathcal{A}$, there is a common 1 in some position. Then $\mu\mathcal{A} \leq \frac{1}{2}$ and this bound is sharp.*

Proof. First observe that if we let $\mathcal{A}$ be the set of strings with 1 in the first position, then the bound is achieved. This is obvious since then $\mathcal{A}$ is just the second half of the interval. Now let $\mathcal{A}$ be an arbitrary intersecting collection. Consider the function defined by $f(x) = 1 - x$. This function is bijective and measure preserving as a reflection across a point. Thus $\mu\mathcal{A} = \mu f(\mathcal{A})$. Now, observe that f flips all the bits of the string it is applied to. Since A and $f(A)$ are disjoint for all $A \in \mathcal{A}$, we know that we never have $f(A) \in \mathcal{A}$. It follows that $\mathcal{A} \cap f(\mathcal{A}) = \varnothing$. Then, by disjointness we have,

$$\mu\mathcal{A} + \mu\mathcal{A} = \mu\mathcal{A} + \mu f(\mathcal{A}) = \mu(\mathcal{A} \cup f(\mathcal{A})) \leq 1.$$

Dividing through by 2 gives the bound. $\square$

11 A Generalization of Sperner's Theorem

We will prove the following extension of Sperner's theorem,

Theorem 14. *Let $\mathcal{A}$ be a collection of subsets of $[n]$ such that for all collections of three distinct elements of $\mathcal{A}$, $\{A, B, C\}$, there does not exist $x \in A$ and $y \in [n] \setminus (A \setminus \{x\})$ such that $(A \setminus \{x\}) \cup \{y\}$, B, and C lie in a chain. Then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Before proving the theorem we will explain it in more detail. In words the restricted configuration is having 3 sets in $\mathcal{A}$ such that optionally swapping out one element of one of the sets for another gives us 3 (possibly non-distinct) sets in a chain. To clarify we allow the case where $x = y$. We also allow $(A \setminus \{x\}) \cup \{y\} = B$ or C.

We call the operation of creating $(A \setminus \{x\}) \cup \{y\}$ from A a swap. Observe that the theorem is indeed a generalization of Sperner's: If $\mathcal{A}$ is an antichain then a swap can lead to, at worst, two sets (possibly identical) in a chain. On the other hand, the collection $\{\{1, 2, 3\}, \{1, 2\}, \{5, 6\}\}$ is not an antichain but satisfies the conditions of the theorem.

To prove the theorem we will introduce a generalization of the key object in Lubell's proof of Sperner's theorem: the maximal chain. Recall that a maximal chain of subsets of $[n]$ is a collection of distinct subsets $\{A_0 = \varnothing, A_1, \ldots, A_n = [n]\}$ where $A_i \subset A_{i+1}$ for all i. That is, a maximal chain is a chain of length $n + 1$. We will consider a variation on this object which we call a "thick chain". Let $x, y \in [n]$, then take a maximal chain

$\{A_0 = \varnothing, A_1, \dots, A_{n-2} = [n] \setminus \{x, y\}\}$ in $[n] \setminus \{x, y\}$. We call the following collection of distinct subsets a thick chain:

$\{\varnothing, \{x\}, \{y\}, A_1 \cup \{x\}, A_1 \cup \{y\}, A_2 \cup \{x\}, A_2 \cup \{y\}, \dots, A_{n-2} \cup \{x\}, A_{n-2} \cup \{y\}, [n]\}$.

We are now ready to prove the theorem using Lubell's double counting approach but with thick chains playing the role of chains.

Proof. If $\mathcal{A}$ contains $\varnothing$ or $[n]$, then the remaining sets must form an antichain. Now a full level of the $2^{[n]}$ is prohibited since then we could create a collection of 3 sets in a chain by performing a swap operation to get two identical subsets and $\varnothing$ in a chain. Then by the equality characterization of Sperner's theorem we have strictly smaller than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ sets excluding $\varnothing$.

Now, we assume $\varnothing$ and $[n]$ are not contained in $\mathcal{A}$ and count pairs $(\mathcal{C}, A)$ where $A \in \mathcal{A}$ and $\mathcal{C}$ is a thick chain containing A. First, note that a thick chain contains at most 2 elements of $\mathcal{A}$. To see this, suppose we had 3 elements of $\mathcal{A}$ in $\mathcal{C}$. Let $\{x\}$ and $\{y\}$ be the cardinality one sets in $\mathcal{C}$. If the 3 sets form a chain we have a contradiction so we may assume two of them contain x and the third contains y. Swapping y for x in the third set creates 3 sets in a chain, a contradiction.

Observe that there are $\frac{n!}{2}$ distinct thick chains on $[n]$. To see this note that we may choose x and y in $\binom{n}{2}$ ways, and since the A_i form a maximal chain in an $n-2$ element set, there are $(n-2)!$ factorial ways to choose them. Multiplying these numbers gives us $\frac{n!}{2}$. Thus, we see the number of pairs $(\mathcal{C}, A)$ is at most $2\frac{n!}{2} = n!$.

If, on the other hand, we fix $A \in \mathcal{A}$, we see that $|A|!(n-|A|)!$ thick chains contain it. First we must select an interval of sets from $\varnothing$ to A; there are $|A|!$ ways to do this. Now we choose the one element set not contained in A; there are $n-|A|$ such choices. So far we have determined all sets in $\mathcal{C}$ of cardinality at most $|A|$. Finally, we can add the remaining $(n-|A|-1)$ elements in $(n-|A|-1)!$ ways. Multiplying gives $|A|!(n-|A|)!$ thick chains containing A. Thus we have the number of pairs $(\mathcal{C}, A)$ is equal to $\sum\limits_{A\in\mathcal{A}} |A|!(n-|A|)!$. Dividing gives the LYM-type inequality:

$$\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$$

Thus we have, $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. $\square$

The proof above generalizes to prove an analogue of Erdős's[7] $k+1$-chain generalization of Sperner's theorem.

Theorem 15. *Let $\mathcal{A}$ be a collection of subsets of $[n]$ such that for all collections of $k+2$ distinct elements of $\mathcal{A}$, $\{A_1, A_2, \ldots, A_{k+2}\}$, there does not exist $x \in A_1$ and $y \in [n] \setminus (A_1 \setminus \{x\})$ such that $(A_1 \setminus \{x\}) \cup \{y\}$, $A_2, \ldots, A_{k+1}$, and A_{k+2} lie in a chain. Then $|\mathcal{A}|$ is at most the sum of the k largest binomial coefficients.*

Proof. If $\varnothing$ or $[n]$ are in $\mathcal{A}$ we are done by the equality characterization of Erdős theorem. Suppose not, and again consider pairs $(\mathcal{C}, A)$ where $A \in \mathcal{A}$ is contained in $\mathcal{C}$. There are at most $2k$ sets contained on the thick chain, or else, by the pigeonhole principle, we could perform a swap and create $k+2$ sets contained in a chain. Thus the number of pairs $(\mathcal{C}, A)$ is at most $2k$ times the number of thick chains, $\frac{n!}{2}$, that is $kn!$. Counting in the other order we again get, $\sum_{A\in\mathcal{A}} |A|!(n-|A|)!$. Dividing gives the LYM-type inequality,

$$\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq k$$

which in turn gives us that $|\mathcal{A}|$ is at most the sum of the k largest binomial coefficients. □

It seems plausible that one could get the same Sperner and Erdős type bounds for natural generalizations involving more swapping. We could define even "thicker" chains and do similar counting. The barrier to continuing in this direction is the fact we needed to deal with the case of $\mathcal{A}$ containing $\varnothing$ or $[n]$ in an ad hoc way not contained in the counting argument. Nonetheless, we can obtain an LYM-type inequality for the nonempty elements of $\mathcal{A}$ which are not too large, and the analogous Sperner type bounds. To this end we extend the notion of a thick chain to a "t-thick chain". t-thick chains will be truncated consisting of subsets of $[n]$ with sizes in the interval $[1, n-t+1]$. Our original notion of thick chain corresponds to a 2-thick chain if we forget about $\varnothing$ and $[n]$. We are now ready to give the formal definition. Let $x_1, x_2, \ldots, x_t$ be distinct elements of $[n]$. Let $\{A_0 = \varnothing, A_1, \ldots, A_{n-t}\}$ be a maxiamal chain in $[n] \setminus \{x_1, x_2, \ldots, x_t\}$. Then a t-thick chain is a collection of the following form:

$\{A_0 \cup \{x_0\}, A_1 \cup \{x_0\}, \ldots, A_{n-t} \cup \{x_0\}, A_0 \cup \{x_1\}, A_1 \cup \{x_1\}, \ldots, A_{n-t} \cup \{x_1\}, \ldots, A_0 \cup \{x_t\}, A_1 \cup \{x_t\}, \ldots, A_{n-t} \cup \{x_t\}\}$. We now state our limited extension of the above theorem applying only when $\mathcal{A}$ consists of sets in the range $[1, n-t+1]$:

Theorem 16. *Let $\mathcal{A}$ be a collection of subsets of $[n]$ consisting of elements whose sizes lie in the interval $[1, n-t+1]$. Furthermore, suppose that for all $t+1$ distinct elements $A_1, A_2, \ldots, A_{t+1} \in \mathcal{A}$, there does not exist $x_i \in A_i$ and $y_i \in [n] \setminus (A_i \setminus \{x_i\})$ for $i = 1, 2, \ldots, t-1$ such that $(A_i \setminus \{x_i\}) \cup \{y_i\}$, $i = 1, 2, \ldots, t-1$, A_t, and A_{t+1} lie in a chain. Then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. We will double count pairs $(A, \mathcal{C})$ where $A \in \mathcal{A}$, $\mathcal{C}$ is a t-thick chain and $A \in \mathcal{C}$. First, observe that in total there are $\dfrac{n!}{t!}$ t-thick chains. First we can choose the x_i in $\binom{n}{t}$ ways, and then we can add the remaining $n-t$ elements one at a time in $(n-t)!$ ways. Multiplying gives $\dfrac{n!}{t!}$.

We may have at most t elements of $\mathcal{A}$ in any t-thick chain since otherwise, by the pigeonhole principle we would have some two of them containing the x_i. Then we could perform $t-1$ swaps to create $t+1$ sets in a chain. Thus, by fixing $\mathcal{C}$ first we have an upper bound on the number of pairs $(A, \mathcal{C})$ of $t\dfrac{n!}{t!} = \dfrac{n!}{(t-1)!}$.

We now first fix some $A \in \mathcal{A}$ and consider how many $\mathcal{C}$ contain it. First, we have a chain of $|A|!$ sets up to A. Next we choose the $t-1$ singleton sets of the t-thick chain which are not subsets of A. There are $\binom{n-|A|}{t-1}$ ways to do this. All sets in the t-thick chain of size at most $|A|$ have now been determined. Finally, we can add the remaining $n-|A|-t+1$ elements to complete the t-thick chain; we can do this in $(n-|A|-t+1)!$ ways. Hence, we see that the number of $(A, \mathcal{C})$ pairs is exactly $\sum\limits_{A\in\mathcal{A}} |A|!\binom{n-|A|}{t-1}(n-|A|-t+1)!$. After simplifying, we get,

$$\sum_{A\in\mathcal{A}} \frac{|A|!(n-|A|)!}{(t-1)!} \leq \frac{n!}{(t-1)!}$$

Or equivalently,

$$\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1.$$

Thus we have, $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. □

In the same manner, one can prove a $k+1$ chain version, but we omit this proof.

To conclude we return to our original simple definition of thick chains and consider another extremal problem. Although this bound, unlike the ones above, is probably not sharp. We have yet to find other research on this problem, but Katona mentioned he thinks Frankl has proved the following (or something better),

Theorem 17. *Let $\mathcal{A}$ be such that for all $A, B \in \mathcal{A}$ we have $|B \setminus A| \geq 2$, then $|\mathcal{A}| \leq \frac{1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. If $\varnothing$ or $[n]$ are in $\mathcal{A}$ the problem is trivial. Suppose not and count pairs $(\mathcal{C}, A)$. The proof is exactly as in the first theorem of the section except that now a thick chain may contain at most one $A \in \mathcal{A}$ rather than 2. Carrying through the proof we get the LYM-type inequality,

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq \frac{1}{2}$$

Thus, we get a bound $|\mathcal{A}| \leq \frac{1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor}$. $\square$

Observe that if instead of considering the constraint $|B \setminus A| \geq 2$ we consider $|B \setminus A| \geq 1$, then we have exactly the conditions of Sperner's theorem.

12 An Inequality for a Skew k-part Sperner System

First, we recall what it means for a collection $\mathcal{F}$ of subsets of $[n]$ to be k-part Sperner. Let $[n] = X_1 \cup X_2 \cup \cdots \cup X_k$ be partition (k-coloring) of $[n]$. Then $\mathcal{F}$ is said to be k-part Sperner if for all $F, G \in \mathcal{F}$ we do not have $G \setminus F \subset X_i$. We recall without proof the famous 2-part Sperner theorem of Katona[11] and Kleitman[12]:

Theorem 18. *Let $\mathcal{F}$ be 2-part Sperner, then $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \lfloor}$.*

Unfortunately, things become more complicated in more part Sperner families and similar bounds do not hold. Nonetheless, it is possible to determine some LYM-type inequalities for such families. For a k-part Sperner family $\mathcal{F}$ and $F \in \mathcal{F}$ we let $F_i = F \cap X_i$. Furthermore, for all such families we assume, $n_1 \leq n_2 \leq \cdots \leq n_k$. In [2] the following inequality was proved,

Theorem 19. *Let $\mathcal{F}$ be k-part Sperner, then*

$$\sum_{F \in \mathcal{F}} \frac{1}{\prod_{i=1}^{k} \binom{n_i}{|F_i|}} \leq \prod_{i=2}^{k} (n_i + 1).$$

The proof given in [2] used a reduction to the classical LYM-inequality. In proving results about the convex hull of k-part Sperner families, Péter Erdős and Katona [8] inroduced an extension of Lubell's permutation method to k-tuples of permutations. We will show that the idea of Péter Erdős and

Katona, with one additional insight can be used to give the above LYM-type inequality for k-part Sperner systems.

Proof. Consider k-tuples $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ where π_i is a permutation of X_i. $F \in \mathcal{F}$ is said to be compatible with π if the first $|F_i|$ elements of π_i are the elements of F_i in some order. We will count pairs (π, F), where $F \in \mathcal{F}$ is compatible with π, in two different ways. If we fix the set F first then we must have the elements of F_i at the beginning of each π. Thus the number of pairs (π, F) is equal to,

$$\sum_{F\in\mathcal{F}} \prod_{i=1}^{k} |F_i|!(n_i - |F_i|)!$$

Now, fix a k-tuple of permutations π. Observe that there each $F \in \mathcal{F}$ may be identified with the k-tuple of its cooresponding F_i, $(F_1, F_2, \ldots, F_k)$. By the pigeonhole principle, if we have more than $\prod_{i=2}^{k}(n_i+1)$, $F \in \mathcal{F}$ compatible with π then some two $k-1$ tuples $(F_2, \ldots, F_k)$ coincide. However, this can not happen because if two compatible F end in the same $k-1$ tuple then the first coordinate must also coincide by the k-part Sperner property. Thus, the number of F compatible with a given π is at most $\prod_{i=2}^{k}(n_i+1)$. It follow that the number of pairs (π, F) is at most $\prod_{i=1}^{k} n_i! \prod_{i=2}^{k}(n_i+1)$. Dividing though by $\prod_{i=1}^{k} n_i!$ gives,

$$\sum_{F\in\mathcal{F}} \frac{1}{\prod_{i=1}^{k} \binom{n_i}{|F_i|}} \leq \prod_{i=2}^{k}(n_i+1)$$

□

After seeing this inequality it is natural to wonder what conditions we would need to put on $\mathcal{F}$ to replace the right hand side of the the inequality with a 1. After all, the asymmetry with respect to the n_i is somewhat unpleasing. In the following theorem we remedy this asymmetry, but unfortunately it is at the cost of introducing asymmetry to the conditions on $\mathcal{F}$. Given a partition, $[n] = X_1 \cup X_2 \cup \cdots \cup X_k$, we say that $\mathcal{F}$ is skew-k-part Sperner if there is no $F, G \in \mathcal{F}$ and $t \in [n]$ such that $F_i = G_i$ for $i < t$ and $F_t \subset G_t$ and $F_t \neq G_t$.

Theorem 20. *Let $\mathcal{F}$ be skew-k-part Sperner, then*

$$\sum_{F \in \mathcal{F}} \frac{1}{\prod_{i=1}^{k} \binom{n_i}{|F_i|}} \leq 1.$$

Proof. We will again double count pairs (π, F). By first fixing $F \in \mathcal{F}$ we see that as before the number of pairs is equal to,

$$\sum_{F \in \mathcal{F}} \prod_{i=1}^{k} |F_i|!(n_i - |F_i|)!$$

Fix a k-tuple of permutations π. There is at most one $F \in \mathcal{F}$ compatible with π. To see this suppose F and G are compatible with π and and again identify F and G with their k-tuples, $(F_1, F_2, \ldots, F_k)$ and $(G_1, G_2, \ldots, G_k)$. Let t be the smallest i for which $F_i \neq G_i$. Then, $F_t \subset G_t$ or $G_t \subset F_t$, and, hence, we have violated the skew-k-part Sperner property. It follows that the number of pairs (π, F) is at most $\prod_{i=1}^{k} n_i!$. Dividing through by $\prod_{i=1}^{k} n_i!$ gives the inequality. $\square$

The above inequality is the LYM-type inequality for the following poset: Take the Hasse diagram of the poset B_{n_1}. Replace each vertex in the diagram with a copy of the Hasse diagrame for B_{n_2}. Continue in this way until we replace each vertex with a copy of the Hasse diagram of B_{n_k}.

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