On Variational and Hemivariational Inequalities

Author:
Osman E. Ali

Supervisor:
Alexandru Kristály

Submitted to
Central European University
Department of Mathematics and its Applications

In partial fulfillment for the requirement of Master of Science
Budapest, Hungary

May 11, 2012

Contents

CHAPTER 1

Introduction

The theory of variational inequalities were first introduced in 1960's, in relation with the notion of subdifferential in convex analysis. The pioneer result in this theory is the theorem of Hartman-Stampacchia [9] which asserts that if X is a finite dimensional Banach space, $K \subset X$ is compact and convex, $F\colon K \to X^*$ is continuous, then there exists $u \in K$ such that

$$\langle F(u), u - v \rangle \geq 0, \quad \forall v \in K. \tag{1.1}$$

They also proved a necessary and sufficient condition for the existence of solutions to (1.1) when K is only closed and convex under certain formulation 3.8. Variational inequalities are a powerful tools for the study of Optimization and Equilibrium problems, Operation research, and other field of studies.

However, hemivariational inequalities have been introduced and investigated by P. D. Panagiotopoulos about two decades ago. Hemivariational inequality problems are arising in many field of studies, such as in Mechanics, Engineering, and Economics in connection to nonconvex energy functionals. The basic development of the problem can be framed as; Let X be a real Banach space, $a(\cdot, \cdot)$ is a bilinear form in $X \times X$, $J(\cdot)$ is a locally lipschitz functional on X and $f \in X^*$. Then find $u \in X$ such that

$$a(u, v - u) + J^0(u, v - u) \geq (f, v - u), \quad \forall v \in X$$

where $J^0(\cdot,\cdot)$ is the generalized directional derivative in the sense of Clarke and $J(\cdot)$ is generally nonsmooth function.

There are many interesting application problems solved via hemivariational inequalities. In particular, consider the nonconvex problem in [27] as follows: We consider an open, bounded, connected subset Ω of $\mathbb{R}^3$ referred to a fixed Cartesian coordinate system $0x_1x_2x_3$ and we formulate the equation

$$-\triangle u = f \quad in \quad \Omega. \tag{1.2}$$

Here u represents the temperature in the case of heat conduction, whereas in hydraulics and electrostatics the pressure and the electric potential, respectively. We denote further by Γ the boundary of Ω and we assume that Γ is sufficiently smooth. If $n = \{n_i\}$ denotes the outward unit normal to Γ then $\frac{\partial u}{\partial n}$ is the flux of heat, fluid or electricity through Γ for the aforementioned classes of problems.

We may consider the interior and the boundary semipermeability problems. For the first class of problems the classical boundary condition.

$$u = 0 \quad on \quad \Gamma. \tag{1.3}$$

is assumed to be hold.

For the second class also we find a function u such that (1.2) is satisfied together with the boundary condition

$$-\frac{\partial u}{\partial n} \in \partial j(x,u) \quad on \quad \Gamma_1 \subset \Gamma \quad and \quad u = 0 \quad on \quad \Gamma \setminus \Gamma_1. \tag{1.4}$$

$j(x,\cdot)$ is locally Lipschitz function and ∂ denotes the generalized gradient. Note that, if $q = \{q_i\}$ denotes the heat flux vector and $k > 0$ is the coefficient of thermal conductivity of the material. We may write by Fourier's law that $q_i n_i = -k\partial u/\partial n$.

Let us introduce the notations

$$a(u,v) = \int_\Omega \triangledown u \cdot \triangledown v dx,$$

and

$$(f,u) = \int_\Omega f u dx.$$

We may ask in addition that u is constrained to belong to a convex bounded closed set $K \subset V$ due to some technical reasons, e.g., constraints for the temperature or the pressure of the fluid, etc.

The hemivariational inequalities correspond to the following two classes of

problems written up as: Let for the first class $V = H_0^1(\Omega)$ and $\bar{\bar{f}} \in L^2(\Omega)$; for the second class $V = \{v : v \in H^1(\Omega), \text{v} = 0 \text{ on } \Gamma \setminus \Gamma_1\}$ and $f \in L^2(\Omega)$. Then from the Green-Gauss theorem applied to (1.2), with the definition of (1.4), we are leading to the following hemivariational inequality.
(P) Find $u \in K$ such that,

$$a(u, v-u) + \int_{\Gamma_1} j^0(x, u(x); v(x) - u(x)) d\Gamma \geq (f, v-u) \quad \forall v \in K. \tag{1.5}$$

The existence of the solutions for the problems (P) follows by Theorem 3.35.

In this thesis we will explore the Hartman-Stampacchia theorems in finite and infinite dimensional spaces, generalizing these results framed in hemivariational inequalities, and weakening the assumption of K to be closed and convex. We also discuss the notion of monotone operator, coercievity and present most important existence and uniqueness results using concepts, such as KKM principle [10].
This thesis has two main chapters. Chapter one covers the basic definitions and properties of nonsmooth analysis. Chapter two is devoted to existence and uniqueness of solutions to (1.1) in finite and infinite dimensional spaces. Moreover, we extend these results in hemivarational setting. An the end we provide an abstract result which appears as variational-hemivariational inequality problem.

CHAPTER 2

Basics of Nonsmooth Analysis

In this section, we shall define the basic notion of classical derivative and generalized directional derivatives.

2.1 Classical Derivatives and Their Properties

Definition 2.1 (Gâteaux Derivative)**.** *Let X and Y be Banach spaces, and let $f\colon U \subset X \to Y$ be a map whose domain $D(f) = U$ is an open subset of X. The directional derivative of f at $u \in U$ in the direction $h \in X$ is given by*

$$f'(u;h) = \lim_{t \to 0} \frac{f(u+th) - f(u)}{t}, \tag{2.1}$$

provided that the limit exists. If $f'(u;h)$ exists for every $h \in X$, and if the mapping $D_G f(u) : X \to Y$ defined by

$$D_G f(u)h = f'(u;h).$$

is linear and continuous, then we say that f is Gâteaux differentiable at u, and we call $D_G f(u)$ the Gâteaux derivative of f at u.

Definition 2.2 (Fréchet Derivative)**.** *Let X and Y be Banach spaces, and let $f\colon U \subset X \to Y$ be a map whose domain $D(f) = U$ is an open subset of X. Then f is called Fréchet differentiable at $u \in U$ if and only if a linear* *and continuous mapping $A: X \to Y$ exists such that*

$$\lim_{\|h\| \to 0} \frac{\|f(u+h) - f(u) - Ah\|}{\|h\|} = 0, \tag{2.2}$$

or equivalently

$$f(u+h) - f(u) = Ah + o(\|h\|), (h \to 0).$$

If such a mapping A exists, then we call $D_F f(u) = A$ or (simply $f'(u) = A$) the Fréchet derivative of f at u.

The two differentiability notions are not equivalent even on finite dimensions, which can be easily verified that Fréchet differentiability at u implies continuity at u, but Gâteaux differentiability does not imply continuity. In addition, Fréchet differentiability implies Gâteaux differentiability but the converse is not always true.
For instance, consider the function $f(x) = |x|$ at $x = 0$, it has a Gâteaux derivative but not Fréchet derivative at this point. The following corollary is immediate from the above definitions.

Corollary 2.3. *Let X and Y be Banach spaces, and let $f: U \subset X \to Y$. Then the relations between Gâteaux and Fréchet derivative hold:*
(i) If f is Fréchet differentiable at $u \in U$, then f is Gâteaux-differentiable at u.
(ii) If f is Gâteaux differentiable in a neighborhood of v and $D_G f$ is continuous at v , then f is Fréchet-differentiable at v and $f'(v) = D_G f(v)$.

Definition 2.4. *Let X be a Banach space and $f: X \to \mathbb{R}$. We say f is Lipschitz of constant $K > 0$ near a point $x \in X$, if for some $\epsilon > 0$ we have*

$$|f(y) - f(z)| \leq K\|y - z\|, \quad \forall y, z \in B(x; \epsilon). \tag{2.3}$$

It is not always true that functions having Lipschitz property near a point is differentiable. For example, $f(x) = |x|$ in $\mathbb{R}$ is Lipschitz near $x = 0$, But not differentiable at this point in the classical sense.

Theorem 2.5. *The function f is Fréchet differentiable at x_0 if and only if for all $y \in \mathbb{R}^n$.*

$$\lim_{t \downarrow 0, y \to h} \frac{f(x_0 + ty) - f(x_0)}{t} = f'(x_0)h.$$

Theorem 2.6. *A Lipschitz function around a point x_0 is Fréchet differentiable at x_0 if and only if it is Gâteaux differentiable at x_0.*

Proof. If f is Lipschitz around x_0 with Lipschitz constant $K > 0$, then for any $y \in X$

$$|\frac{f(x_0 + tz) - F(x_0)}{t} - \frac{f(x_0 + ty) - F(x_0)}{t}|,$$

$$= |\frac{f(x_0 + tz) - f(x_0 + ty)}{t}| \leq K\|z - y\|.$$

for all z contained in a small neighborhood of y, and $t > 0$ sufficiently small. Therefore, the two differentiability notions coincide. □

The following theorem has significant importance in application involving Lipschitz functions, The proof can be found in many real analysis books.

Theorem 2.7 (Rademacher's Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $f: \Omega \longrightarrow \mathbb{R}$ be Lipschitz on Ω, then f is differentiable at almost every point in Ω (in the sense of Lebesgue measure).*

Definition 2.8. *The function f is strictly differentiable at x_0 if*

$$\lim_{h \to 0, x \to x_0} \frac{f(x + h) - f(x) - f'(x_0)h}{\|h\|} = 0. \tag{2.4}$$

Immediate from the definition, if f is strict differentiable at x_0, then f is Lipschitz around x_0 and also sufficient for Fréchet differentiability.

Theorem 2.9. *The function f is strictly differentiable at x_0 if and only if for all $h \in \mathbb{R}^n$*

$$\lim_{y \to h, x \to x_0, t \downarrow 0} \frac{f(x + ty) - f(x)}{t} = f'(x_0)h. \tag{2.5}$$

The following example will show the strict differentiability and Fréchet differentiability are not always the same.

Example 2.10. *Let the function $f: \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} x^2 sin(\frac{1}{x}) & , x \neq 0 \\ 0 & , x = 0. \end{cases}$$

The function f has a classical derivative at $x_0 = 0$ with $f'(x_0) = 0$. Moreover, f is Lipschitz around $x_0 = 0$ but f is not strictly differentiable at $x_0 = 0$. To show this consider the sequence $\{x_k\}$ and $\{t_k\}$ defined as

$$x_k = \frac{1}{2k\pi + \frac{\pi}{2}}, \quad t_k = \frac{1}{2k\pi - \frac{\pi}{2}} - \frac{1}{2k\pi + \frac{\pi}{2}}.$$

We can easily see that both sequence converge to zero. Let us set $h = 1$ and evaluate

$$\lim_{k\to\infty} \frac{f(x_k + t_k h) - f(x_k)}{t_k}$$

$$= -\lim_{k\to\infty} \frac{8k^2\pi^2 + \frac{\pi^2}{2}}{\pi(4k^2\pi^2 - \frac{\pi^2}{4})} = -\frac{2}{\pi}.$$

which is not equal to the Fréchet derivative at $x_0 = 0$.
Generally it is not easy to check strict differentiability of a function for this reason the following theorem provides a sufficient condition for strict differentiability.

Theorem 2.11. *If f is continuously Gâteaux differentiable at x_0, then f strict differentiable at x_0.*

Proof. Let $x \in \mathbb{R}^n$ with $\|x - x_0\|$ sufficiently small and $z \in \mathbb{R}^n$ be given. Then for small $t > 0$ the function $\phi_{x,z}(t) = f(x + tz)$ is continuous and differentiable. Applying the mean value theorem we have that

$$\frac{f(x + tz) - f(x)}{t} = \frac{\phi_{x,z}(t) - \phi_{x,z}(0)}{t} = \phi'_{x,z}(\alpha) = zf'(x + \alpha z),$$

where $\alpha \in (0, t)$.
Since the gradient is continuous at x_0, we obtain

$$\lim_{t\downarrow 0, z\to h, x\to x_0} \frac{f(x + tz) - f(x)}{t} = f'(x_0)h.$$

holds for all $h \in \mathbb{R}^n$. By definition 2.8 the function f is strict differentiable at x_0. □

2.2 Convex Functions

Convex functions have many important differential properties such as Lipschitz property, existence of one sided directional derivative (subgradients), monotonicity of the gradient for smooth convex functions, and local optimizers are also global in this class of functions, etc. In this section we shall see important results which can be used in the later sections as well.

Definition 2.12. *Let X be a real Banach space and K be subset of X. The set K is said to be convex if*

$$\lambda x + (1 - \lambda)y \in K,$$

for all x and y in K and $0 < \lambda < 1$.

All linear subspace of X, empty set and X are convex sets. Moreover a set $K \subset X$ is a cone, for $x \in K$ and for all $\lambda \geq 0$ whenever $\lambda x \in K$, and K is convex cone if $x + y \in K$ for $x \in K$ and $y \in K$.

Definition 2.13. *A real valued function $f \colon K \to \mathbb{R}$ is convex on K if for each $x, y \in K$ and $0 \leq \lambda \leq 1$*

$$f(\lambda x + (1-\lambda)y \leq \lambda f(x) + (1-\lambda)f(y). \tag{2.6}$$

A function f is concave if and only if $-f$ is convex.

we can extend f to all of X by $\bar{f}$ and setting $\bar{f}(x) = f(x)$ for $x \in K$ and $\bar{f}(x) = \infty$ for $x \notin K$. The convex indicator function $I_K \colon X \to \bar{\mathbb{R}}$ for any convex set K is defined by

$$I_K(x) = \begin{cases} 0 & , x \in K \\ \infty & , x \notin K. \end{cases} \tag{2.7}$$

Another useful notion is the *epigraph* of a convex function $f \colon X \to \bar{\mathbb{R}}$ which is convex and defined as

$$\{(x, \lambda) : f(x) \leq \lambda, \quad \lambda \in \mathbb{R}, \quad x \in X\}.$$

An equivalent definition of convexity is: A function $f \colon X \to \bar{\mathbb{R}}$ is convex if its epigraph is a convex subset of $X \times \mathbb{R}$.

Proposition 2.14. *If f is a convex function on U that is bounded above on a neighborhood of some point in U, then for any x in U, f is Lipschitz near x.*

The above proposition implies that Theorem 2.5 also holds for convex functions on a bounded neighborhood. This in turn shows that Fréchet and Gâteaux differentiability is equivalent.

Definition 2.15. *Let $f \colon X \to \mathbb{R} \cup \infty$ be a convex function. An element $\eta \in X^*$ is said to be a subgradient of the convex function f at a point $x \in X$. Provided that for any $y \in X$,*

$$f(y) - f(x) \geq \langle \eta, y - x \rangle, \quad \forall y \in X. \tag{2.8}$$

The set of all subgradient of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$. If the function f is differentiable then this set reduces to a singleton set $\{\bigtriangledown f\}$.
The following examples verifies subdifferential of convex function.

Example 2.16. *Let $f(x) = |x|$ in the real line $\mathbb{R}$, then*

$$\partial f(x) = \begin{cases} -1, & x < 0, \\ [-1,1], & x = 0, \\ 1, & x > 0. \end{cases}$$

Example 2.17. *let $f\colon \mathbb{R} \to \bar{\mathbb{R}}$, $f(x) = \bar{f}(x) + |x|$, where $\bar{f}(x)$ is given by $b > 0$*

$$\bar{f}(x) = \begin{cases} \frac{3bx}{a}(\frac{x}{a} - 1), & x \leq 0, \\ 0, & 0 \leq x \leq a, \\ \infty, & x > a. \end{cases}$$

then

$$\partial f(x) = \begin{cases} \frac{3b}{a}(\frac{2x}{a} - 1) - 1, & x < 0, \\ \left[\frac{-3b}{a} - 1, 1\right], & x = 0, \\ 1, & 0 < x < a \\ [1, \infty], & x = a \\ \emptyset, & x > a. \end{cases}$$

The following theorem assures the existence of subgradient of a convex function.

Theorem 2.18. *Let $f\colon U \longrightarrow \mathbb{R}$, $U \subset \mathbb{R}^n$ be an open convex set, then f is convex on U if and only if for each $x_0 \in U$ there exists a vector $\eta \in \mathbb{R}^n$ such that*

$$f(x) - f(x_0) \geq \eta \cdot (x - x_0), \quad \forall x \in U.$$

Proof. Let f be a convex function. Then the epigraph of f is a convex set. For each $x_0 \in U$, $(x_0, f(x_0))$ is a point of the boundary of the convex set $epif$. Then from convex analysis there exists a vector $(v, v_0) \neq (0,0)$ such that

$$v \cdot x + v_0\alpha \geq v \cdot x_0 + v_0 f(x_0),$$

for each $(x, \alpha) \in epif$. If $v_0 = 0$ then $v \cdot (x - x_0) \geq 0, \quad \forall x \in U$ which implies $v = 0$. U being open set, therefore we have the absurd result $(v, v_0) = (0,0)$. If $v_0 < 0$, then it is possible to take α sufficiently large in order to have

$$v \cdot x + v_0\alpha < v \cdot x_0 + v_0 f(x_0),$$

which is a contradiction.
Hence, $v_0 > 0$, choose $\alpha = f(x)$ and $\eta = \frac{-v}{v_0}$ which gives the desired inequality.

Conversely, let $x_1, x_2 \in U$ and $\lambda \in [0,1]$. For each $x_0 \in U$, there exists a vector $\eta \in \mathbb{R}^n$ such that

$$f(x_1) - f(x_0) \geq \eta \cdot (x_1 - x_0),$$

and

$$f(x_2) - f(x_0) \geq \eta \cdot (x_2 - x_0).$$

multiplying these inequalities by λ and $1-\lambda$, respectively and summing up we obtain

$$\lambda f(x_1) + (1-\lambda) f(x_2) - f(x_0) \geq [\lambda x_1 + (1-\lambda)x_2 - x_0],$$

taking $x_0 = \lambda x_1 + (1-\lambda)x_2$ convexity of f follows. □

Theorem 2.19. *Let f be a convex function, and let x be a point where f is finite. Then η is a subgradient of f at x if and only if*

$$f'(x; y) \geq (\eta, x), \quad \forall y \in X. \tag{2.9}$$

Proof. Put $z = x + \lambda y$. Then using the subgradient inequality and plugging z in to the inequality we obtain

$$[f(x + \lambda y) - f(x)]/\lambda \geq (\eta, y),$$

for every y and $\lambda > 0$. This implies (2.9). The converse holds trivially from the definition. □

2.3 Generalized Directional Derivatives

In this section we present the known Clarke generalized gradient which has a wide application in real life problems for Lipschitz functions and the generalized directional derivative for nonconvex functions. For the later case the limit of the difference quotient discussed in the previous section does not exist, so the simplest replacement of the limit is by upper and lower limits.

Definition 2.20 (Clarke generalized gradient)**.** *The generalized directional derivative of $f\colon X \to \mathbb{R}$ at x in the direction of h, denoted by $f^0(x; h)$ is defined by*

$$f^0(x; h) = \limsup_{t \downarrow 0, y \to x} \frac{f(y + th) - f(y)}{t} \tag{2.10}$$

where y is a vector in X and t is a positive scalar.

Definition 2.21. *The generalized gradient of a locally Lipschitz functional $f\colon X \longrightarrow \mathbb{R}$ at a point x is a subset of X^* defined by*

$$\partial f(x) = \{z \in X^* : f^0(x;h) \geq \langle z, h\rangle, \quad \forall h \in X\}.$$

Proposition 2.22. *Let f be a Lipschitz function of constant K near x . Then:*
i) The function $h \to f^0(x;h)$ is finite, positive homogeneous, and sub-additive on X, and satisfies

$$|f^0(x;h)| \leq K\|h\|;$$

ii)$f^0(x;h)$ is upper semicontinuous as a function of (x,h), and as a function of h alone, is Lipschitz of constant K on X.
iii) $f^0(x;h) = (-f)^0(x;h)$.

Proof. The absolute value of the difference quotient in the definition of $f^0(x;h)$ is bounded by $K\|h\|$ when y is sufficiently close to x and t sufficiently near 0. It follows that $|f^0(x;h)|$ admits the same upper bound. The fact that $f^0(x;\lambda h) = \lambda f^0(x;h)$ for any $\lambda \geq 0$ (positive homogeneity) is immediate, so let us turn now to subadditivity. With all upper limit below understood to be taken as $y \to x$ and $t \downarrow 0$, we calculate :

$$\begin{aligned} f^0(x;h+d) &= \limsup_{t\downarrow 0, y\to x} \frac{f(y+th+td)-f(y)}{t}, \\ &\leq \limsup_{t\downarrow 0, y\to x} \frac{f(y+th+td)-f(y+td)}{t} + \limsup_{t\downarrow 0, y\to x} \frac{f(y+td)-f(y)}{t}. \end{aligned}$$

We conclude from the above inequality that $f^0(x;h+d) \leq f^0(x;h)+f^0(x;d)$, which implies (i).
Consider an arbitrary sequence $\{x_i\}$ and $\{h_i\}$ converges to x and h respectively. For each i, by definition of the upper limit there exists y_i in X and $t_i > 0$ such that

$$\|y_i - x_i\| + t_i < \frac{1}{i},$$

$$f^0(x_i;h_i) - \frac{1}{i} \leq \frac{f(y_i+t_ih)-f(y_i)}{t_i}$$

$$= \frac{f(y_i+t_ih)-f(y_i)}{t_i} + \frac{f(y_i+t_ih_i)-f(y_i+t_ih_i)}{t_i}.$$

Note that the last term is bounded in magnitude by $K\|h_i - h\|$ then taking the upper limits ($i \to \infty$), we obtain

$$\limsup_{i\to\infty} f^0(x_i;h_i) \leq f^0(x;h),$$

which shows the upper semicontinuity.
Let any h and d in X be given. Then

$$f(y+th) - f(y) \leq f(y+td) - f(y) + Kt\|h-d\|.$$

For any y near x and $t \to 0$, dividing by t and taking the upper limit as $y \to x$, gives

$$f^0(x;h) \leq f^0(x;d) + K\|h-d\|$$

This also holds when we interchange the role of h and d. Hence (ii) holds. Finally it remains to show (iii);

$$\begin{aligned} f^0(x;-h) &= \limsup_{t\downarrow 0, y\to x} \frac{f(y-th)-f(y)}{t} \\ &= \limsup_{t\downarrow 0, z\to x} \frac{(-f)(z+th)-(-f)(y)}{t}, \quad \text{where} \quad z := y - th \\ &= (-f)^0(x;h) \end{aligned}$$

□

Proposition 2.23. *Let f be a Lipschitz function of constant K near x. Then*
a) $\partial f(x)$ is a nonempty, convex, weak-compact subset of X, and $\|\eta\| \leq K$ for every $\eta \in \partial f(x)$.*
b) For every h in X we have

$$f^0(x;h) = max\{(\eta,h) : \eta \in \partial f(x)\};$$

c) $\eta \in \partial f(x)$ if and only if $f^0(x;h) \geq (\eta,h), \forall h \in X$;
d) If $\{x_i\}$ and $\{\eta_i\}$ are sequences in X and X^* such that $\eta_i \in \partial f(x_i)$ for each i, and if x_i converges to x and η is a weak cluster point of the sequence $\{\eta_i\}$, then we have $\eta_i \in \partial f(x)$.*

Theorem 2.24. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function at a point $x \in \mathbb{R}^n$. Then*

$$\partial f(x) = conv\{\eta \in \mathbb{R}^n : \nabla f(x_i) \to \eta, x_i \to x, \quad \text{and } f \text{ is differentiable at} \quad x_i\}.$$

where conv denotes the convex hull of the set.

For nonconvex functions we define the following generalized directional derivatives as follows:
Let $f\colon \mathbb{R}^n \to \bar{\mathbb{R}}$ and $x_0 \in \mathbb{R}^n$ where f is finite, then the upper and lower *Dini*

directional derivative at the point x_0 in the direction of $h \in \mathbb{R}^n$, is defined as

$$f_U^{'}(x_0;h) = \limsup_{t \downarrow 0} \frac{f(x_0+th)-f(x_0)}{t}, \tag{2.11}$$

$$f_L^{'}(x_0;h) = \liminf_{t \downarrow 0} \frac{f(x_0+th)-f(x_0)}{t}. \tag{2.12}$$

Clearly, from the above definition we have that $f_L^{'}(x_0;h) \leq f_U^{'}(x_0;h)$.
In the case of equality we get the classical directional derivative $f'(x_0,h)$. To assure the uniformly convergence of directional derivative we consider the topological structure of the space and define the upper and lower Hadamard directional derivative as follows:

$$f_U^{+}(x_0;h) = \limsup_{t \downarrow 0, y \to h} \frac{f(x_0+ty)-f(x_0)}{t}, \tag{2.13}$$

$$f_L^{+}(x_0;h) = \liminf_{t \downarrow 0, y \to h} \frac{f(x_0+ty)-f(x_0)}{t}. \tag{2.14}$$

From the definition we have that $f_U^{+}(x_0,\cdot)$ is upper semicontinuous and $f_L^{+}(x_0,\cdot)$ is lower semicontinuous with respect to the direction vector. Moreover, we have the inequality

$$f_L^{+}(x_0,\cdot) \leq f_L^{'}(x_0,\cdot) \leq f_U^{'}(x_0,\cdot) \leq f_U^{+}(x_0,\cdot).$$

When the function f is uniformly directional differentiable at the point x_0 then $f_L^{+}(x_0,\cdot) = f_U^{+}(x_0,\cdot)$.
In addition, for strict differentiability as we discussed earlier we have also a generalized directional derivative defined by

$$f_U^{*}(x_0;h) = \limsup_{t \downarrow 0, y \to h, x \to x_0} \frac{f(x+ty)-f(x)}{t}, \tag{2.15}$$

$$f_L^{*}(x_0;h) = \liminf_{t \downarrow 0, y \to h, x \to x_0} \frac{f(x+ty)-f(x)}{t}. \tag{2.16}$$

We can verify that $f_U^{*}(x_0,\cdot)$ and $f_L^{*}(x_0,\cdot)$ are upper semicontinuous and lower semicontinuous, respectively. Moreover, their comparison can be written as

$$f_L^{*}(x_0,\cdot) \leq f_L^{+}(x_0,\cdot) \leq f_L^{'}(x_0,\cdot) \leq f_u^{'}(x_0,\cdot) \leq f_u^{+}(x_0,\cdot) \leq f_U^{*}(x_0,\cdot)$$

If the function f is Lipschitz around x_0, then the generalized directional derivative $f_U^{*}(x_0;h)$ reduces to exactly the Clarke generalized derivative.

This in turn has the useful property of convexity (sublinear) proved in Proposition 2.22. Generally there are many ways to construct convex generalized directional derivatives using complicated convergence concepts. For our discussion we only consider the already mentioned generalized derivatives and including the *Rockafellar upper sub derivative* which is defined as

$$f^{\nearrow}(x_0;h) = \limsup_{t\downarrow 0, x\to x_0} inf_{z\to h}\frac{f(x+tz)-f(x)}{t}$$
$$= \sup_{\epsilon>0}\limsup_{t\downarrow 0, x\to x_0} inf_{z;\|z-h\|<\epsilon}\frac{f(x+tz)-f(x)}{t}. \tag{2.17}$$

Finally, the following theorem together with the convexity of Clarke generalized derivative will put an end our discussion for the construction of convex generalized derivatives which replace the gradient of a function.

Theorem 2.25. *The directional derivatives $f_U^*(x_0,\cdot)$ and $f^{\nearrow}(x_0,\cdot)$ are convex.*

Proof. First let us consider the case for $f_U^*(x_0,\cdot)$. Let $h_1,h_2\in\mathbb{R}^n$ and $\{x_k\}\subset\mathbb{R}^n$, $\{t_k\}$ is a sequence of positive numbers and $\{z_k\}\subset\mathbb{R}^n$ such that $x_k\to x_0$, $t_k\downarrow 0$, $z_k\to h_1+h_2$. We can decompose $\{z_k\}$ as $z_k=u_k+v_k$, and $u_k\to h_1$, $v_k\to h_2$ for each k. Then we have

$$\begin{aligned} f_U^*(x_0;h_1+h_2) &= \lim_{k\to\infty}\frac{f(x_k+t_kz_k)-f(x_k)}{t_k}\\ &= \limsup_{k\to\infty}\frac{f(x_k+t_k(u_k+v_k))-f(x_k)}{t_k}\\ &\le \limsup_{k\to\infty}\frac{f(x_k+t_ku_k+t_kv_k))-f(x_k+t_kv_k)}{t_k}+\limsup_{k\to\infty}\frac{f(x_k+t_kv_k))-f(x_k)}{t_k}\\ &\le f_U^*(x_0;h_1)+f_U^*(x_0;h_2)\end{aligned}$$

This implies $f_U^*(x_0,\cdot)$ is subadditive and since it is positive homogeneous, convexity holds.
Finally, let us show the convexity of $f^{\nearrow}(x_0,\cdot)$. Let $h_1,h_2\in\mathbb{R}^n$, $\epsilon>0$ and $\{x_k\}\subset\mathbb{R}^n$, $\{t_k\}\subset(0,\infty)$ such that $x_k\to x_0$, $t_k\downarrow 0$.
By definition of $f^{\nearrow}(x_0,h_2)$, there exists a sequence $\{v_k\}\subset B_{\frac{\epsilon}{2}}(h_2)$ such that

$$f^{\nearrow}(x_0,h_2)\ge\limsup_{k\to\infty}\frac{f(x_k+t_kv_k))-f(x_k)}{t_k}.$$

Moreover, since $x_k + t_k v_k \to x_0$, by definition of $f^{\nearrow}(x_0, h_1)$ again we have a sequence $\{u_k\} \subset B_{\frac{\epsilon}{2}}(h_1)$ such that

$$f^{\nearrow}(x_0, h_1) \geq \limsup_{k\to\infty} \frac{f((x_k + t_k v_k) + t_k u_k) - f(x_k + t_k v_k)}{t_k}$$

Setting $h_k = u_k + v_k$ for $k \in \mathbb{N}$, we get a sequence $\{h_k\} \subset B_\epsilon(h_1, h_2)$

$$\limsup_{k\to\infty} \frac{f(x_k + t_k h_k) - f(x_k)}{t_k}$$

$$\leq \limsup_{k\to\infty} \frac{f((x_k + t_k v_k) + t_k u_k) - f(x_k + t_k v_k)}{t_k} + \limsup_{k\to\infty} \frac{f(x_k + t_k v_k)) - f(x_k)}{t_k}$$

$$\leq f^{\nearrow}(x_0, h_1) + f^{\nearrow}(x_0, h_2).$$

This condition holds for any $\epsilon > 0$, and since $\{x_k\} \to x_0$, thus we have

$$f^{\nearrow}(x_0, h_1 + h_2) = sup_{\epsilon>0} \limsup_{t\downarrow 0, x\to x_0} inf_{z;\|z-h_1+h_2\|<\epsilon} \frac{f(x+tz) - f(x)}{t}$$

$$\leq f^{\nearrow}(x_0, h_1) + f^{\nearrow}(x_0, h_2).$$

Hence convexity is assured. □

Note that;

$$f^{\nearrow}(x_0, \cdot) \leq f^0(x_0, \cdot) \leq f_U^*(x_0, \cdot).$$

Equality holds whenever f is lipschitz near X_0.

CHAPTER 3

Variational Inequalities

This section is devoted to variational inequalities both in $\mathbb{R}^n$ and in Hilbert spaces. Indeed, we will discuss also on hemivariational inequalities, variational-hemivariational inequality problems and related results.

3.1 Variational Inequalities in Finite Dimensional Spaces

3.1.1 Definition and Examples

The finite dimensional variational inequality problem, denoted by $(VI(F,K))$, is to determine a vector $u \in K \subset \mathbb{R}^n$ such that

$$\langle F(u), v-u\rangle \geq 0, \quad \forall v \in K,$$

where F is a given continuous function from a convex closed set K to $\mathbb{R}^n$.

Example 3.1. *Let f be a smooth function defined on a closed convex set $K \subset \mathbb{R}^n$ and let $x_0 \in K$ such that*

$$f(x_0) = \min_{x \in K} f(x).$$

Since K is convex, $tx + (1-t)x_0 \in K$, for $0 \leq t \leq 1$. The function $\varphi(x) = f(x_0 + t(x - x_0))$ attains its minimum at $t = 0$. Thus we have that

$$\varphi'(0) = \bigtriangledown f(x_0)(x - x_0) \geq 0, \quad \forall x \in K.$$

Hence the point x_0 satisfies the variational inequality problem,

$$\bigtriangledown f(x_0)(x - x_0) \geq 0, \quad \forall x \in K.$$

Example 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ and ψ be a given function on $\bar{\Omega} = \Omega \cup \partial\Omega$ satisfying*

$$\max_{\Omega} \psi \geq 0, \quad \text{and} \quad \psi \leq 0 \quad \text{on} \quad \partial\Omega.$$

Define

$$K = \{v \in C^1(\bar{\Omega}) : v \geq \psi \quad \text{in} \quad \Omega \quad \text{and} \quad v = 0 \quad \text{on} \quad \partial\Omega\},$$

a convex set of functions which is non empty. We look for a function $u \in K$ of the least area given by

$$\int_{\Omega} \sqrt{1 + |\bigtriangledown u|^2} dx = \min_{v \in K} \int_{\Omega} \sqrt{1 + |\bigtriangledown v|^2} dx.$$

Then the associated variational inequality is: Find a function $u \in K$ such that

$$\int_{\Omega} \frac{\bigtriangledown u \bigtriangledown (v - u)}{\sqrt{1 + |\bigtriangledown u|^2}} dx \geq 0, \quad \forall v \in K.$$

Definition 3.3. *Let X be a metric space and a mapping $f \colon X \to X$ is a contraction mapping if*

$$d(f(x), f(y)) \leq \alpha d(x, y), \qquad x, y \in X$$

. for some α , $0 \leq \alpha < 1$. When $\alpha = 1$ the mapping is called nonexpansive.

3.1.2 Basic Properties and Theorems

Theorem 3.4. *Let K be a closed convex set of a Hilbert space H. Then $y = P_K x$ (the projection of x on K) if and only if*

$$y \in K : (y, z - y) \geq (x, z - y), \quad \forall z \in K.$$

Proof. Let $x \in H$ and $y = P_K x$. Since K is convex we have $y + t(z - y) \in K$ for any $z \in K$, $0 \leq t < 1$, then the function $\varphi(t) = \|x - y - t(z - y)\|^2$ attains its minimum at $t = 0$. Therefore $\varphi'(0) \geq 0$, and gives

$$(x - y, z - y) \leq 0,$$

which can be written as

$$(y, z - y) \geq (x, z - y), \quad \forall z \in K.$$

Conversely,

$$y \in K\colon (y, z - y) \geq (x, z - y), \quad \text{for all} \quad z \in K.$$

can be rewritten as

$$0 \leq (y - x, (z - x) + (x - y)) \leq -\|x - y\|^2 + (y - x, z - x).$$

We obtain

$$\|y - x\|^2 \leq (y - x, z - x) \leq \|y - x\| \|z - x\|,$$

which gives

$$\|y - x\| \leq \|z - x\| \quad \text{for} \quad z \in K.$$

□

Corollary 3.5. *Let K be a closed convex set of a Hilbert space H. Then the operator P_K is nonexpansive, that is $\|P_K x - P_K y\| \leq \|x - y\|$, for $x, y \in K$.*

Theorem 3.6 (Brouwer)**.** *Let $K \subset \mathbb{R}^n$ be a compact and convex and let $F\colon K \longrightarrow K$ be continuous. Then F admits a fixed point.*

Proof. Let Σ be a closed ball in $\mathbb{R}^n$ such that $K \subset \Sigma$. From the above corollary we can see that P_K is continuous, hence the mapping

$$F \circ P_K\colon \Sigma \to K \subset \Sigma$$

is a continuous mapping of Σ to itself. It admits a fixed point x by the closed ball version of Brouwer's theorem. Namely $(F \circ P_K)(x) = x$, which gives $F(x) = x$. □

The dual $(\mathbb{R}^n)'$ of $\mathbb{R}^n$ is the space of all linear forms

$$a\colon \mathbb{R}^n \to \mathbb{R}, \quad x \to \langle a, x \rangle$$

and the bilinear mapping

$$(\mathbb{R}^n)' \times \mathbb{R}^n \to \mathbb{R}, \quad a, x \to \langle a, x \rangle$$

is refereed to the pairing. We always assume that

$$\langle a, x \rangle = (\pi a, x), \quad a \in (\mathbb{R}^n)', \quad x \in \mathbb{R}^n$$

where $\pi\colon (\mathbb{R}^n)' \to \mathbb{R}^n$ is the identification and $(\cdot, \cdot)$ is scalar product on $\mathbb{R}^n$.

Theorem 3.7. *Let $K \subset \mathbb{R}^n$ be compact and convex and let $F\colon K \longrightarrow (\mathbb{R}^n)'$ be continuous. Then there exists $x \in K$ such that*

$$\langle F(x), y - x\rangle \geq 0, \quad \forall y \in K. \tag{3.1}$$

Proof. It is equivalent to prove that there exists $x \in K$ such that

$$(x, y - x) \geq (x - \pi F(x), y - x), \quad \forall y \in K.$$

The mapping

$$P_K(I - \pi F) : K \longrightarrow K$$

is continuous since P_K and $(I - \pi F)$ are continuous. Hence by Brouwer's theorem it admits a fixed point $x \in K$. That is $x = P_K(I - \pi F)x$. Consequently, by applying Theorem 3.4 we obtain

$$(x, y - x) \geq (x - \pi F(x), y - x), \qquad \forall y \in K$$

□

If K is unbounded, the problem does not always have a solution.
For example, the case when $K = \mathbb{R}$ and $F(x) = e^x$.
Given a convex set K, we set $K_R = K \cap \Sigma_R$, where Σ_R is the closed ball of radius R and center $0 \in \mathbb{R}^n$. We have that there exists at least one

$$x_R \in K_R : (F(x_R), y - x_R) \geq 0, \qquad \forall y \in \mathbb{R}^n. \tag{3.2}$$

Theorem 3.8. *Let $K \subset \mathbb{R}^n$ be closed and convex and $F\colon K \longrightarrow (\mathbb{R}^n)'$ be continuous. A necessary and sufficient condition that there exists a solution to* (3.1) *is that there exists an $R > 0$ such that a solution $x_R \in K_R$ of* (3.2) *satisfies $|x_R| < R$.*

Proof. It is obvious that if there exist a solution x to (3.1), then x is a solution to (3.2) whenever $|x| < R$. Suppose that $x_R \in K_R$ satisfies the condition in the theorem. Then x_R is also a solution to (3.1). Since $|x_R| < R$, given $y \in K$, $w = x_R + \varepsilon(y - x_R) \in K_R$, for $\varepsilon > 0$ sufficiently small. Consequently

$$x_R \in K_R \subset K : 0 \leq \langle F(x_R), w - x_R\rangle = \varepsilon\langle F(x_R, y - x_R\rangle, \quad \text{for} \quad y \in K.$$

which means that x_R is a solution to (3.1). □

Theorem 3.9 (Existence under Coercivity)**.** *Let $F\colon K \to (\mathbb{R}^n)'$ satisfy*

$$\frac{\langle F(x) - F(x_0), x - x_0\rangle}{|x - x_0|} \to +\infty, \quad \textit{as} \quad |x| \to +\infty, x \in K, \tag{3.3}$$

for some $x_0 \in K$. Then there exists a solution to (3.1).

Proof. Choose $M > |F(x_0)|$ and $R > |x_0|$ such that

$$\langle F(x) - F(x_0), x - x_0 \rangle \geq M|x - x_0|, \quad |x| \geq R, \quad x \in K.$$

we obtain

$$\begin{aligned}\langle F(x), x - x_0 \rangle &\geq M|x - x_0| + \langle F(x_0), x - x_0 \rangle \\ &\geq M|x - x_0| - |F(x_0)||x - x_0| \\ &\geq (M - |F(x_0)|)(|x| - |x_0|) > 0, \quad \text{for} \quad |x| = R.\end{aligned}$$

Let $x_R \in K_R$ be the solution of (3.2). Then

$$\langle F(x_R), x_R - x_0 \rangle = -\langle F(x_R, x_0 - x_R \rangle \leq 0.$$

Thus $|x| \neq R$, hence $|x| < R$ as required together with Theorem 3.7. □

Definition 3.10. *The mapping $F: K \to (\mathbb{R}^n)'$ is called monotone if*

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0 \quad \textit{for all} \quad x_1, x_2 \in K,$$

and strictly monotone

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle > 0, \quad \textit{for all} \quad x_1, x_2 \in K, \quad x_1 \neq x_2.$$

A uniqueness result is presented in the subsequent theorem.

Theorem 3.11. *Let $F: K \longrightarrow (R^n)'$ be a strictly monotone function, then the solution of the variational problem (3.1) is unique, if it exists.*

Proof. Suppose $x_1$and x_2 be solutions of problem (3.1) with $x_1 \neq x_2$, then the following hold:

$$\langle F(x_1), y - x_1 \rangle \geq 0, \quad \forall y \in K,$$

$$\langle F(x_2), y - x_2 \rangle \geq 0, \quad \forall y \in K.$$

Substituting x_1 for y in the first and x_2 for y in the second inequality and adding the resulting inequality resulted in

$$\langle F(x_1) - F(x_2), x_2 - x_1 \rangle \geq 0$$

, Contradiction to strict monotonicity. Hence $x_1 = x_2$. □

3.2 Variational Inequalities in Hilbert Spaces

Let H be a real Hilbert space and H' be its dual, $(\cdot,\cdot)$ denotes the inner product on H and $\|\cdot\|$ is its norm. Let $a\colon H \times H \to \mathbb{R}$ be a linear continuous form. A linear and continuous mapping $A\colon H \to H'$ determine a bilinear form via the pairing $a(u,v) = \langle Au, v\rangle$.

Definition 3.12. *The bilinear form $a(u,v)$ is coercive on H if there exists $\alpha > 0$, such that*

$$a(v,v) \geq \alpha\|v\|^2, \qquad \textit{for} \quad v \in H.$$

Next we discuss the existence and uniqueness of solution of the problem: Let $K \subset H$ be a closed and convex set and $f \in H'$, Find

$$u \in K\colon a(u, v-u) \geq \langle f, v-u\rangle, \quad \text{for all} \quad v \in K. \tag{3.4}$$

Theorem 3.13. *Let $a(u,v)$ be a continuous, symmetric, bilinear form on H, $K \subset H$ closed and convex set and $f \in H'$. Then there exists a unique solution to (3.4). In addition the mapping $f \to u$ is Lipschitz, that is if u_1 and u_2 are solutions to* (3.4) *corresponding to $f_1, f_2 \in H'$, then*

$$\|u_1 - u_2\| \leq (\frac{1}{\alpha})\|f_1 - f_2\|_{H'}.$$

Proof. Suppose there exist $u_1, u_2 \in H$ solution to (3.4):

$$u_i \in K : a(u_i, v-u_i) \geq \langle f_i, v-u_i\rangle, \quad \forall v \in K, i = 1,2.$$

It follows that

$$a(u_1 - u_2, u_1 - u_2) \leq \langle f_1 - f_2, u_1 - u_2\rangle.$$

From coercivity of a, we have that

$$\alpha\|u_1 - u_2\|^2 \leq \langle f_1 - f_2, u_1 - u_2\rangle \leq \|f_1 - f_2\| \cdot \|u_1 - u_2\|$$

which proves the Lipschitz condition. It remains to show existence of u. Define a functional

$$F(u) = a(u,u) - 2\langle f, u\rangle, \quad u \in H.$$

Let $d = \inf_K F(u)$ and since

$$F(u) \geq \alpha\|u\|^2 - 2\|f\| \cdot \|u\| \geq \alpha\|u\|^2 - (\frac{1}{\alpha})\|f\|^2 - \alpha\|u\|^2 \geq -(\frac{1}{\alpha})\|f\|^2.$$

we have that

$$d \geq \frac{1}{\alpha}\|f\|^2 > -\infty.$$

Let u_n be a minimizing sequence of F in K:

$$u_n \in K : d \leq F(u_n) \leq d + \frac{1}{n}$$

Since K is convex and applying parallelogram law,

$$\begin{aligned}
\alpha\|u_n - u_m\|^2 &\leq a(u_n - u_m, u_n - u_m),\\
&= 2a(u_n, u_n) + 2a(u_m, u_m) - 4a(\frac{1}{2}(u_n + u_m), \frac{1}{2}(u_n + u_m)),\\
&= 2F(u_n) + 2F(u_m) - 4F(\frac{1}{2}(u_n + u_m)),\\
&\leq 2[\frac{1}{n} + \frac{1}{m}].
\end{aligned}$$

Hence, the sequence u_n is cauchy. Since K is closed there exist an element $u \in K$ such that $u_n \to u$ in H and $F(u_n) \to u$, therefore $F(u) = d$.
Now for any $v \in K$, $u + \varepsilon(v - u) \in K, \quad 0 \leq \varepsilon \leq 1$ and

$$F(u + \varepsilon(v - u)) \geq F(u).$$

Then

$$(d/d\varepsilon) = F(u + \varepsilon(v - u)) \mid_{\varepsilon=0} \geq 0.$$

which gives

$$2\varepsilon a(u, v - u) + \varepsilon^2 a(v - u, v - u) - 2\varepsilon\langle f, v - u\rangle \geq 0$$

Equivalently,

$$a(u, v - u) \geq \langle f, v - u\rangle - \frac{1}{2}\varepsilon a(v - u, v - u), \quad \forall\varepsilon \quad , \quad 0 \leq \varepsilon \leq 1.$$

Setting $\varepsilon = 0$, u becomes a solution for Problem (3.4). □

3.2.1 Sobolev Spaces

Let $\alpha = (\alpha_1, \cdots, \alpha_N)$ with nonnegative integers $\alpha_1, \cdots, \alpha_N$ be a multi-index and denote its order by $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$. Set $D_i = \frac{\partial}{\partial} x_i, i = 1, \cdots, N$ and $D^\alpha u = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$ with $D^0 = u$. Let Ω be a domain in $\mathbb{R}^N$ with

$N \geq 1$. Then $w \in L^1_{Loc}(\Omega)$ is called the α^{th} weak or generalized derivative of $u \in L^1_{Loc}(\Omega)$ if and only if

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega w \varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

holds, Where $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω. The generalized derivative w denoted by $w = D^\alpha u$ is unique up to a change of values of w on a set of Lebesgue measure zero.

Definition 3.14. *Let $1 \leq p \leq \infty$ and $m = 0, 1, 2, \ldots$. The Sobolev space $W^{m,p}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$, which have generalized derivatives up to order m such that $D^\alpha u \in L^p(\Omega)$ for all $\alpha \colon |\alpha| \leq m$. For $m = 0$ we set $W^{0,p}(\Omega) = L^p(\Omega)$.*

With the corresponding norm is given by

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

the space $W^{m,p}(\Omega)$ is a real Banach space.

Definition 3.15. *$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.*

We have also a special case when $m = 2$

$$W^{m,2}(\Omega) = H^m(\Omega),$$
$$W_0^{m,2}(\Omega) = H_0^m(\Omega).$$

3.2.2 Variational Inequalities for Monotone Operators

In this section we extend the results that we have discussed before and include more existence and uniqueness results in general spaces.

Definition 3.16. *The mapping $A \colon K \longrightarrow H'$ is continuous on finite dimensional subspace if any finite dimensional subspace $M \subset H$, the restriction of A to $K \cap M$ is continuous, namely, if*

$$A \colon K \cap M \to H'.$$

is weakly continuous.

Theorem 3.17. *Let K be a closed, bounded, and convex subset of H and $A\colon K \longrightarrow H'$ be monotone and continuous on finite dimensional subspace. Then there exists*

$$u \in K : \langle Au, v-u \rangle \geq 0, \quad forall \quad v \in K$$

Note that, if A is strictly monotone then the solution u is unique.

Lemma 3.18 (Minty)**.** *Let K be a closed convex subset of H and let $A\colon K \longrightarrow H'$ be monotone and continuous on finite dimensional subspace. Then u satisfies*

$$u \in K : \langle Au, v-u \rangle \geq 0, \quad \forall v \in K.$$

if and only if it satisfies

$$u \in K : \langle Av, v-u \rangle \geq 0, \quad \forall v \in K.$$

Proof. Assume

$$u \in K : \langle Au, v-u \rangle \geq 0, \quad \forall v \in K.$$

holds. Then from monotonicity of A, we have that

$$0 \leq \langle Av - Au, v-u \rangle = \langle Av, v-u \rangle - \langle Au, v-u \rangle, \quad \text{for} \quad u, v \in K$$

Thus,

$$u \in K : 0 \leq \langle Au, v-u \rangle \leq \langle Av, v-u \rangle, \quad \forall v \in K$$

Conversely, let $w \in K$, and set for $0 \leq t \leq 1$, $v = u + t(w-u) \in K$, since K is convex. From our hypothesis

$$\langle A(u + t(w-u)), t(w-u) \rangle \geq 0,$$

or, equivalently,

$$\langle A(u + t(w-u)), w-u \rangle \geq 0, \quad \forall w \in K.$$

Since A is weakly continuous on the intersection of K with the finite dimension subspace spanned by u and w, we may allow $t \to 0$ to obtain

$$\langle Au, w-u \rangle \geq 0, \quad \forall w \in K.$$

□

Proof of the theorem. Let $M \subset H$ be a finite dimensional subspace of H of dimension $N < \infty$. We may assume with out loss of generality that $0 \in K$. Define

$$j : M \to H$$

be the injection map and

$$j^{'} : H^{'} \to M^{'}$$

be its dual. The pairing between $M^{'}$ and $M, \langle \cdot, \cdot \rangle_M$ is chosen, so that

$$\langle f, jx \rangle = \langle f, j^{'} x \rangle_M, \quad \text{whenever} \quad x \in M, f \in H^{'}.$$

We set $K_M = K \cap M \equiv K \cap jM$ and consider the mapping $j^{'}Aj : K_M \longrightarrow M^{'}$. Here K_M is a compact convex subset of M and $j^{'}Aj$ is continuous by hypothesis from K in to $M^{'}$. Hence, there exists an element $u_M \in K_M$ such that

$$\langle j^{'} Aju_M, v - u_M \rangle_M \geq 0, \quad \forall v \in K_M.$$

since $ju_M = u_M$ and $jv_M = v$.

$$\langle Au_M, v - u_M \rangle \geq 0, \quad \forall v \in K_M.$$

By Minty's lemma

$$\langle Av, v - u_M \rangle \geq 0, \quad \forall v \in K_M.$$

At this point, we define

$$S(v) = \{u \in K : \langle Av, v - u \rangle \geq 0\}.$$

S(v) is weakly closed for each $v \in K$. Moreover since K is bounded, K is weakly compact. Consequently $\bigcap_{v \in K} S(v)$ is closed subset of K, is weakly compact . To conclude that it is nonempty, we employ the finite intersection property. Let $\{v_1, \cdots, v_m\} \subset K$. We claim that

$$S(v_1) \cap S(v_2) \cap \cdots \cap S(v_m) \neq \emptyset.$$

Let M be the finite dimensional subspace of X spanned by $\{v_1, \cdots, v_m\}$ and define $K_M = K \cap M$ as before. According to the argument given earlier, there is an element $u_M \in K_M$ such that

$$\langle Av, v - u_M \rangle \geq 0, \quad \forall v \in K_M.$$

In particular,

$$\langle Av_i, v_i - u_M \rangle \geq 0, \quad \forall i = 1, \cdots, m.$$

Thus $u_M \in S(v_i), i = 1, \cdots, m$. Hence for any finite collection $v_1, \cdots, v_m$, the inequality holds.
Therefore there exists an element $u \in \bigcap_{v \in K} S(v)$, which means

$$u \in K : \langle Av, v - u \rangle \geq 0, \quad \forall v \in K.$$

By the Minty's lemma again, we obtain

$$u \in K : \langle Au, v - u \rangle \geq 0, \quad \forall v \in K.$$

Corollary 3.19. *Let H be a Hilbert space and $K \subset H$ be nonempty, closed, bounded, and convex set. Suppose that $F \colon K \longrightarrow K$ is nonexpansive. Then F possesses a nonempty closed convex subset $M \subset K$ of fixed points.*

Proof. It is enough to observe that we may take $H = H^{'}$ and the pairing $(\cdot, \cdot)$ with the scalar product in H. Now if F is nonexpansive, $I - F$ is monotone so we may apply Theorem 3.17. Any solution to the variational inequality for $I - F$ is a fixed point for F. □

Theorem 3.20. *Let K be a closed convex subset of H and let $A \colon K \longrightarrow H^{'}$ be a monotone and continuous on finite dimensional subspace. A necessary and sufficient condition that there exists a solution to the variational inequality*

$$u \in K : \langle Au, v - u \rangle \geq 0, \quad \forall v \in K.$$

is that there exists an $R > 0$ such that at least one solution of the variational inequality

$$u_R \in K_R : \langle Au_R, v - u_R \rangle \geq 0, \quad \forall v \in K_R.$$

$$K_R = K \cap \{v : \|v\| \leq R\}.$$

satisfies the inequality

$$\|u_R\| < R.$$

Corollary 3.21. *Let $K \subset H$ be a nonempty, closed, and convex set and $A \colon K \longrightarrow H^{'}$ be monotone, coercive and continuous on finite dimensional subspace. Then there exists*

$$u \in K : \langle Au, v - u \rangle \geq 0, \quad \forall v \in K.$$

The proof of the above Theorem 3.20 and Corollary 3.21 are analogous to those in the case of finite dimension.

3.3 Hemivariational Inequalities

In this section we explore the general formulation of hemivariational inequalities. These type of problems are a generalization of the classical variational inequality which arises in the variational formulation of Engineering, Mechanical and Economic problems whenever nonconvex energy functionals are involved.

The basic form of the problem may be developed as in the following form: suppose that X is a Banach space $a(\cdot,\cdot)$ is a bilinear form on $X \times X$, $j(\cdot)$ a locally Lipschitz functional on X and $f \in X^*$, then we seek a $u \in X$ such as to satisfy

$$a(u, v-u) + j^0(u, v-u) \geq (f, v-u), \quad \forall v \in X.$$

Where $j^0(\cdot,\cdot)$ is the generalized derivative in the sense of Clarke.

3.3.1 Coercieve Hemivaritional Inequalities

In this section we deal with the common and simplest type of hemivariational inequalities in the case of one dimensional nonconvex superpotentials which was first studied by P. D. Panagiotopoulos concerning the existence of solutions.
Let X be a real Hilbert space with the property that

$$X \subset L^2(\Omega) \subset X^*$$

where Ω is an open bounded subset of $\mathbb{R}^n$, and the injections are continuous and dense. Denote $(\cdot,\cdot)_{L^2}$ the $L^2(\Omega)$ inner product and duality pairing, by $\|\cdot\|$ the norm of X and by $|\cdot|_{L^2}$ for the $L^2(\Omega)$ norm.

Moreover, let $L\colon X \longrightarrow L^2(\Omega)$, $Lu = \hat{u}$, $\hat{u} \in \mathbb{R}$ be a linear continuous mapping. Further assume that $l \in X^*$ that

$$L\colon X \to L^2(\Omega)$$

is compact and

$$\tilde{X} = \{v \in X : \hat{v} \in L^\infty(\Omega)\}.$$

is dense in X for the X-norm. It is also assumed that $a(\cdot,\cdot)\colon X \times X \to \mathbb{R}$ is a symmetric continuous bilinear form which is coercive.
Suppose that $\beta\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that $\beta \in L^\infty_{loc}(\mathbb{R})$, i.e. An

essentially bounded function on any bounded interval of $\mathbb{R}$. For any $\rho > 0$ and $\xi \in \mathbb{R}$, let us define

$$\bar{\beta}_\rho(\xi) = \operatorname{ess\,inf}_{|\xi_1-\xi|\le\rho} \beta(\xi_1), \quad and \quad \bar{\bar{\beta}}_\rho(\xi) = \operatorname{ess\,inf}_{|\xi_1-\xi|\le\rho} \beta(\xi_1).$$

Monotonicity properties of $\rho \to \bar{\beta}_\rho(\xi)$ and $\rho \to \bar{\bar{\beta}}_\rho(\xi)$ implies that the limits as $\rho \to 0_+$ exists. Therefore one may write as

$$\bar{\beta}_\rho(\xi) = \operatorname{ess\,inf}_{\rho\to 0_+} \beta(\xi_1), \quad and \quad \bar{\bar{\beta}}_\rho(\xi) = \operatorname{ess\,inf}_{\rho\to 0_+} \beta(\xi_1).$$

and define the multivalued function

$$\tilde{\beta}(\xi) = [\bar{\beta}(\xi), \bar{\bar{\beta}}(\xi)].$$

If $\beta(\xi_{\pm 0})$ exists for every $\xi \in \mathbb{R}$, then we can apply a result proved by Chang: A locally Lipschitz function j can be determined up to an additive constant by the relation

$$\tilde{\beta}(\xi) = \partial j(\xi).$$

Now we formulate the following coercive hemivariational inequality problem (P^C): Find $u \in X$ such that

$$a(u, v-u) + \int_\Omega j^0(\hat{u}, \hat{v} - \hat{u}) d\Omega \ge (l, v-u), \quad \forall v \in X. \tag{3.5}$$

An element $u \in X$ is said to be a solution of (P^C) if there exists $\chi \in L^1(\Omega)$ with $L^*\chi \in X^*$,(L^* denotes the transpose operator of L) such that

$$a(u, v) + (L^*\chi, v) = (l, v) \quad \forall v \in X.$$

and

$$\chi(x) \in \partial j(u(x)) \quad a.e \quad on \quad \Omega.$$

and where

$$(L^*\chi, v) = \int_\Omega \chi L v d\Omega = \int_\Omega \chi \hat{v} d\Omega, \quad if \quad v \in \tilde{X}.$$

Therefore, an element $u \in X$ is said to be a solution of (P^C) if there exists $\chi \in L^1(\Omega)$ such that

$$a(u, v) + \int_\Omega \chi \hat{v} d\Omega = (l, v), \quad \forall v \in \tilde{X}.$$

and

$$\chi(x) \in \partial j(u(x)) \quad a.e \quad on \quad \Omega.$$

holds. In order to define a regularized problem P_ε^C we consider the mollifier

$$p \in C_0^\infty(-1,1), \quad p \geq 0, \quad with \quad \int_{-\infty}^{\infty} p(\xi)d\xi = 1.$$

and let

$$\beta_\varepsilon = p_\varepsilon \star \beta \quad with \quad p_\varepsilon(\xi) = \frac{1}{\varepsilon}p(\frac{\xi}{\epsilon}), \quad 0 < \varepsilon < 1.$$

The regularized problem P_ε^C can be formulated as: Find $u_\varepsilon \in X$ with $\beta_\varepsilon(\hat{u_\varepsilon}) \in L^1(\Omega)$, such as to satisfy the variational equality

$$a(u_\varepsilon, v) + \int_\Omega \beta_\varepsilon(\hat{u_\varepsilon})\hat{v}d\Omega = (l, v), \quad \forall v \in \bar{X}.$$

To define the corresponding finite dimensional problem $P_{\varepsilon n}^C$, we consider a Galerkin basis of $\bar{X}$ in X and let X_n be the resulting n-dimensional subspace. The problem becomes
Problem $P_{\varepsilon n}^C$: Find $\hat{u}_{\epsilon n} \in X_n$ such as to satisfy the variational equality

$$a(\hat{u}_{\varepsilon n}, v) + \int_\Omega \beta_\varepsilon(\hat{u}_{\varepsilon n})\hat{v}d\Omega = (l, v), \forall v \in \bar{X_n}. \tag{3.6}$$

Now we assume that there exists $\xi \in \mathbb{R}^+$ such that

$$\operatorname{ess\,sup}_{(-\infty,-\xi)} \beta(\xi)_1 \leq 0 \leq \operatorname{ess\,inf}_{(\xi,\infty)} \beta(\xi_1). \tag{3.7}$$

Roughly speaking we may say that the graph $(\xi, \beta(\xi))$ ultimately increases. We state existence results based on the following lemmas though some of the proofs are not written up here. For further detail refer to [27].

Lemma 3.22. *Suppose that* (3.7) *holds. Then we can determine* $\rho_1 > 0$, $\rho_2 > 0$ *such that for every* $\hat{u}_{\varepsilon n} \in X_n$

$$\int_\Omega \beta_\varepsilon(\hat{u}_{\varepsilon n})\hat{v}d\Omega \geq -\rho_1\rho_2 mes\Omega. \tag{3.8}$$

Lemma 3.23. *The problem* $P_{\varepsilon n}^C$ *has at least one solution* $\hat{u}_{\varepsilon n} \in X_n$.

Proof. Equation (3.6) can be written in the form $(\Lambda(\hat{u}_{\varepsilon n}), v) = 0 \quad \forall v \in X_n$ and we have the estimate from the coercivity and from equation (3.8)

$$(\Lambda(\hat{u}_{\varepsilon n}), v) \geq c\|\hat{u}_{\varepsilon n}\|^2 - \rho_1\rho_2 mes\Omega - c_1\|\hat{u}_{\varepsilon n}\|, c, c_1 > 0.$$

By applying Brouwer's fixed point theorem, we obtain a solution $\hat{u}_{\varepsilon n}$ with $\|\hat{u}_{\varepsilon n}\| \leq c$, where c is independent of ε and n. ☐

Lemma 3.24. *The sequence $\{\beta_\varepsilon(\hat{u}_{\varepsilon n})\}$ is weakly precompact in $L^1(\Omega)$.*

Lemma 3.25. *The problem P^C has at least one solution.*

Proof. From lemma 3.30, we have that $\|\hat{u}_{\varepsilon n}\| < c$, where c is independent of ε and n. Thus as $\varepsilon \to 0$, $n \to \infty$ (by considering subsequences if necessary) we may write

$$u_{\varepsilon n} \to u, \quad \text{weakly in} \quad X.$$

and from compactness of L.

$$\hat{u}_{\varepsilon n} \to \hat{u}, \quad \text{strongly in} L^2(\Omega).$$

and thus

$$\hat{u}_{\varepsilon n} \to \hat{u}, \quad a.e. \quad on \quad \Omega.$$

Moreover, due to lemma 3.24, we can write

$$\beta_\varepsilon(\hat{u}_{\varepsilon n}) \to \chi, \quad \text{weakly in} \quad L^1(\Omega).$$

From our previous assumptions and properties of the Galerkin basis we can pass to the limit $\varepsilon \to 0$, $n \to \infty$, and we obtain

$$a(u,v) + \int_\Omega \chi \hat{v} d\Omega = (l,v), \quad \forall v \in \tilde{X}.$$

from which it follows that a linear functional

$$(L^*\chi, v) = \int_\Omega \chi \hat{v} d\Omega, \quad \forall v \in \tilde{X}.$$

can be uniquely extended to the whole space as $L^*\chi \in \tilde{X}$. Thus the above can be written in the form

$$a(u,v) + (L^*\chi, v) = (l,v), \quad \forall v \in X.$$

In order to complete the proof, we need to show

$$\chi \in \bar{\beta}(\hat{u}) = \partial j(\hat{u}) \quad a.e. \quad on \quad \Omega.$$

From Egoroff's theorem, we can find that for any $\alpha > 0$, and determine $\omega \subset \Omega$ with $mes\omega < \alpha$ such that for any $\mu > 0$ and for $\varepsilon < \varepsilon_0 < \mu/2$ and $n > n_0 > 2/\mu$. We have

$$|\hat{u}_{\epsilon n} - \hat{u}| < \frac{\mu}{2}, \quad \text{for all} \quad x \in \Omega - \omega.$$

From (3.7), (3.6) we obtain

$$\beta_\varepsilon(\hat{u}_{\varepsilon n}) \le \operatorname{ess\,sup}_{|\hat{u}_{\varepsilon n}-\xi|\le\varepsilon} \beta(\xi) \le \operatorname{ess\,sup}_{|\hat{u}_{\varepsilon n}-\xi|<\frac{\mu}{2}} \beta(\xi) \le \operatorname{ess\,sup}_{|\hat{u}-\xi|\le\mu} \beta(\xi) = \bar{\bar{\beta}}_\mu(\hat{u}),$$

where $\bar{\bar{\beta}}_\mu$ was previously defined. Analogously we prove the inequality

$$\bar{\beta}_\mu(\hat{u}) = \operatorname{ess\,inf}_{|\hat{u}-\xi|\le\mu} \beta(\xi) \le \beta_\varepsilon(\hat{u}_{\varepsilon n}).$$

We take now $\tau \ge 0$ a.e. on $\Omega - \omega$ with $\tau \in L^\infty(\Omega - \omega)$, and we obtain from the above inequality

$$\int_{\Omega-\omega} \bar{\beta}_\mu(\hat{u})\tau d\Omega \le \int_{\Omega-\omega} \bar{\beta}_\varepsilon(\hat{u}_{\varepsilon n}\tau d\Omega \le \int_{\Omega-\omega} \bar{\bar{\beta}}_\mu(\hat{u})\tau d\Omega.$$

Taking the limit as $\varepsilon \to 0$ and $n \to \infty$ we obtain that

$$\int_{\Omega-\omega} \bar{\beta}_\mu(\hat{u})\tau d\Omega \le \int_{\Omega-\omega} \chi\tau d\Omega \le \int_{\Omega-\omega} \bar{\bar{\beta}}_\mu(\hat{u})\tau d\Omega.$$

Since τ is arbitrary, we have that

$$\chi \in [\bar{\beta}(\hat{u}), \bar{\bar{\beta}}(\hat{u})] = \tilde{\beta}(\hat{u}), \quad a.e. \quad on \quad \Omega - \omega.$$

where $mes\omega < \alpha$, for α as small as possible, so the result follows. □

3.3.2 Existence Theorems on Hemivariational Inequalities

In this subsection we extend the results of Hartman and Stampacchia to hemivariational inequalities. Before we discuss the results we need to define the basic ingredients and state the basic assumptions as follows.

Definition 3.26 (Carathéodory Function)**.** *Let $\Omega \subset \mathbb{R}^n, n \ge 1$ be a nonempty measurable set, and $f\colon \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}, m \ge 1$. The function f is called a Carathéodory function if the following two conditions are satisfied:*
(i) $x \to f(x,s)$ is continuous in Ω for all $s \in \mathbb{R}^m$;
(ii)$s \to f(x,s)$ is continuous on $\mathbb{R}^m$ for a.e. $x \in \Omega$.

Definition 3.27. *A normed linear space X is called reflexive if the canonical embedding $j : X \to X^{**}$ is surjective: $j(X) = X^{**}$.*

Definition 3.28. *The operator $A : K \to X^*$ is w^* demicontinuous for $K \subseteq X$ if for any sequence $\{u_n\} \subset K$ converging to u, the sequence $\{Au_n\}$ converges to Au for the w^* topology in X^*.*

Definition 3.29. *The operator $A: K \to X^*$ is continuous on finite dimensional subspace of K, if for any finite dimensional space $F \subset X$, which intersects with K, the operator $A|_{K \cap F}$ is demicontinuous , that is $\{Au_n\}$ converges weakly to Au in X^* for each sequence $\{u_n\} \subset K \cap F$ which converges to u.*

The following information is useful in order state and proof theorems, lemmas and corollaries in this section. Let X be a real Banach space and let $T: X \longrightarrow L^p(\Omega, \mathbb{R}^k)$ be a linear and continuous operator , where $1 \leq p < \infty, k \geq 1$ and Ω is a bounded open set in $\mathbb{R}^N$, $K \subseteq X$. Define an operator $A: K \to X^*$ and a function $j = j(x, y): \Omega \times \mathbb{R}^k \to \mathbb{R}$. j assumed to be a Carathéodory function which is locally Lipschitz with respect to the second variable and satisfies the following assumption:
There exists $h_1 \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R})$ and $h_2 \in L^{\infty}(\Omega, \mathbb{R})$ such that

$$(j) \qquad |z| \leq h_1(x) + h_2(x)|y|^{p-1},$$

for a.e. $x \in \Omega$, every $y \in \mathbb{R}^k$ and $z \in \partial j(x, y)$. Denote $Tu = \hat{u}, u \in X$. Our aim is to study the problem
Find $u \in K$ such that, for every $v \in K$,

$$(P) \qquad \langle Au, v - u \rangle + \int_{\Omega} j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0. \qquad (3.9)$$

Theorem 3.30. *Let K be a compact and convex subset of an infinite dimensional Banach space X and let j satisfies condition (j). If the operator $A: K \longrightarrow X^*$ is w^* demicontinuous, then the problem (P) has at least a solution.*

The above theorem has an equivalent finite dimensional formulation as follows:

Corollary 3.31. *Let X be a finite dimensional Banach space and let K be a compact and convex subset of X. If the assumption $(j)$is satisfied and if $A: K \longrightarrow X^*$ is continuous operator, then the problem (P) has at least a solution.*

Remark 3.32. *In reflexive Banach space the following hold:*
a)The w^* demicontinuity and demicontinuity are the same;
b) a demicontinuous operator $A: K \longrightarrow X^*$ is continuous on finite dimensional subspace of K;
c) the condition of w^* demicontinuity on the operator $A: K \longrightarrow X^*$ in Theorem 3.30 may be replaced by the equivalently assumption:

(A-1) the mapping $K \ni u \longrightarrow \langle Au, v\rangle$ is weakly upper semi continuous, for each $v \in X$.
d) If A is w^* demicontinuous , $\{u_n\} \subset K$ and $u_n \to u$, then

$$\lim_{n\to\infty} \langle Au, u_n\rangle = \langle Au, u_n\rangle.$$

The basic input to prove the above theorem and corollary is the following auxiliary result.

Lemma 3.33. *i) If condition (j) is satisfied and V_1, V_2 are nonempty subsets of X, then the mapping $V_1 \times V_2 \to \mathbb{R}$ defined by*

$$(u, v) \longrightarrow \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x))dx$$

is upper semi-continuous.
ii) In addition, if $T\colon X \longrightarrow L^p(\Omega, \mathbb{R}^k)$ is a linear compact operator, then the above mapping is weakly upper semi-continuous.

Proof. (i) Let $\{(u_m, v_m)\}_{m\in\mathbb{N}} \subset V_1 \times V_2$ be a sequence converging to $(u, v) \in V_1 \times V_2$, as $m \to \infty$. Since $T\colon X \longrightarrow L^p(\Omega, \mathbb{R}^k)$ is continuous, it follows that

$$\hat{u}_m \to \hat{u}, \quad \hat{v}_m \to \hat{v} \quad in \quad L^p(\Omega, \mathbb{R}^k), \quad as \quad m \to \infty$$

There exists a subsequence $\{(\hat{u}_n, \hat{v}_n)\}$ of the sequence $\{(\hat{u}_m, \hat{v}_m)\}$ such that

$$\limsup_{m\to\infty} \int_\Omega j^0(x, \hat{u}_m(x); \hat{v}_m(x))dx = \lim_{n\to\infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}_n(x))dx.$$

By Proposition 4.11 of [13], one may suppose that there exist two functions $\hat{u}_0$, $\hat{v}_0$ in $L^p(\Omega; \mathbb{R}^+)$ and of two subsequences $\{(\hat{u}_n)\}$ and $\{(\hat{v}_n)\}$ denoted again by the same symbols and such that:

$$|\hat{u}_n(x)| \le \hat{u}_0(x), \qquad |\hat{v}_n(x)| \le \hat{v}_0(x)$$

$$\hat{u}_n(x) \longrightarrow \hat{u}(x), \qquad \hat{v}_n(x) \longrightarrow \hat{v}(x), \quad as \quad n \to \infty$$

for a.e. $x \in \Omega$. On the other hand, for each x where condition (j) holds and for each $y, h \in \mathbb{R}^k$, there exists $z \in \partial j(x, y)$ such that

$$j^0(x, y; h) = \langle z, h\rangle = max\{\langle w, h\rangle : w \in \partial j(x, y)\}.$$

Therefore

$$|j^0(x, y; h)| \le |z||h| \le (h_1(x) + h_2(x)|y|^{p-1}) \cdot |h|.$$

Denoting, $F(x) = (h_1(x) + h_2(x)|\hat{u}_0|^{p-1}) \cdot |\hat{v}_0|$, we have that

$$|j^0(x, \hat{u}_n(x); \hat{v}_n(x))| \leq F(x),$$

for all $n \in \mathbb{N}$ and for a.e. $x \in \Omega$. From Hölder's inequality and condition (j) for the function h_1 and h_2 it follows that $F \in L^1(\Omega, \mathbb{R})$. Applying Fatou's lemma yields

$$\lim_{n\to\infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}_n(x))dx \leq \int_\Omega \limsup_{n\to\infty} j^0(x, \hat{u}_n(x); \hat{v}_n(x))dx.$$

By the upper semicontinuity of the mapping $j^0(x, \cdot; \cdot)$. we get

$$\limsup_{n\to\infty} j^0(x, \hat{u}_n(x); \hat{v}_n(x)) \leq j^0(x, \hat{u}(x); \hat{v}(x)),$$

for a.e. $x \in \Omega$. Since

$$\hat{u}_n(x) \longrightarrow \hat{u}(x), \quad \hat{v}_n(x) \longrightarrow \hat{v}(x), \quad as \quad n \to \infty,$$

for a.e. $x \in \Omega$. Hence

$$\limsup_{m\to\infty} \int_\Omega j^0(x, \hat{u}_m(x); \hat{v}_m(x))dx \leq \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x))dx,$$

which proves part (a) of our lemma.
(b) Let $\{(u_m, v_m)\}_{m\in\mathbb{N}} \subset V_1 \times V_2$ be now a sequence weakly converging to $\{u, v\} \in V_1 \times V_2$ as $m \to \infty$. Thus $u_m \rightharpoonup u, v_m \rightharpoonup v$ weakly as $m \to \infty$. Since $T \colon X \longrightarrow L^p(\Omega, \mathbb{R}^k)$ is a linear compact operator, it follows that

$$\hat{u}_m \longrightarrow \hat{u}, \quad \hat{v}_m \longrightarrow \hat{v} \quad in \quad L^p(\Omega, \mathbb{R}^k).$$

□

Proof of corollary.
Let us assume by contradiction, for every $u \in K$, there is some $v = v_u \in K$ such that

$$\langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx < 0.$$

for every $v \in K$, Put

$$N(v) = \{u \in K : \langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx < 0\}.$$

For any fixed $v \in K$, the mapping $K \longrightarrow \mathbb{R}$ defined by

$$u \longmapsto \langle Au, v-u \rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx.$$

is upper semicontinuous, by Lemma 3.33 and continuity of A. Thus by the definition of the upper semicontinuity ,$N(v)$ is an open set. Our initial assumption implies that $\{N(v); v \in K\}$ is a covering of K. Hence by compactness of K there exist $v_1, \cdots, v_n \in K$ such that

$$K \subset \bigcup_{j=1}^{n} N(v_j).$$

Let $\rho_j(u)$ be the distance from u to $K \setminus N(v_j)$. Then ρ_j is a Lipschitz map which vanishes outside $N(v_j)$ and the functionals

$$\psi_j(u) = \frac{\rho_j(u)}{\sum_{i=1}^{n} \rho_i}$$

define a partition of the unity subordinated to the covering $\{\rho_1, \cdots, \rho_n\}$. Moreover, the mapping

$$p(u) = \sum_{j=1}^{n} \psi_j(u) v_j.$$

is continuous and maps K into itself, because of the convexity of K. Thus by Brouwer's fixed point theorem , there exists u_0 in the convex closed hull of $\{v_1, \cdots, v_1\}$ such that $P(u_0) = u_0$. Define

$$q(u) = \langle Au, P(u) - u \rangle + \int_\Omega j^0(x, \hat{u}(x); P(\hat{u})(x) - \hat{u}(x))dx$$

The convexity of the map $j^0(\hat{u}; \cdot)$ and the fact that $\sum_{j=1}^{n} \psi_j(u) = 1$ implies

$$q(u) \leq \sum_{j=1}^{n} \psi_j(u) \langle Au, v_j - u \rangle + \sum_{j=1}^{n} \psi_j(u) \int_\Omega j^0(x, \hat{u}(x); \hat{v}_j(x) - \hat{u}(x))dx.$$

For arbitrary $u \in K$, there are only two possibilities : if $u \notin N(v_j)$, then $\psi_i(u) = 0$. On the other hand, for all $1 \leq j \leq n$. That is, there exists at least such an indice such that $u \in N(v_j)$, we have $\psi_j(u) > 0$. By definition of $N(v_j)$, $q(u) < 0$ for every $u \in K$, but $q(u) = 0$ which is a contradiction.

Proof of the Theorem

We need the following Lemma to prove our theorem. Let F be an arbitrary finite dimensional subspace of X which intersect with K. Let $i_{K \cap F}$ be the canonical injection of $K \cap F$ in to K and i_F^* be the adjoint of the canonical injection i_F^* of F into X. Then

Lemma 3.34. *The operator*

$$B\colon K\cap F \longrightarrow F^*, \quad B = i_F^* A i_{K\cap F}$$

is continuous.

Proof. For any $v \in K$. set

$$S(v) = \left\{ u \in K : \langle Au, v-u\rangle + \int_\Omega j^0(x,\hat{u}(x); P(\hat{u})(x) - \hat{u}(x))dx \right\}.$$

We need to show the following two conditions;
(I) $S(v)$ is a closed set.
Since $v \in S(v), S(v) \neq \emptyset$. Let $\{u_n\} \subset S(v)$ be an arbitrary sequence which converges to u as $n \to \infty$. We need to prove that $u \in S(v)$, by the part (a) of Lemma 3.33 we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}[\langle Au_n, v-u_n\rangle + \int_\Omega j^0(x,\hat{u}_n(x);\hat{v}(x)-\hat{u}_n(x))]dx, \\ &= \lim_{n\to\infty}\langle Au_n, v-u_n\rangle + \limsup_{n\to\infty}\int_\Omega j^0(x,\hat{u}_n(x);\hat{v}(x)-\hat{u}_n(x))dx, \\ &\leq \langle Au, v-u\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))dx. \end{aligned}$$

which is equivalent to $u \in S(v)$.
(II) The family $\{S(v); v \in K\}$ has finite intersection property
Let $\{v_1, \cdots, v_n\}$ be an arbitrary finite subset of K, and let F be the linear space spanned by this family. Applying Corollary 3.31 to the operator B defined in lemma 3.34, we find $u \in K \cap F$ such that $u \in \bigcap_{j=1}^n S(v_j)$, which means that the family of closed sets $\{S(v); v \in K\}$ has the finite intersection property. But the set K is compact hence $\bigcap_{v\in K} S(v) \neq \emptyset$, which implies the problem (P) has at least one solution. □

When we weaken the assumption and considering K to be a closed, bounded and convex set, then the existence result is assured by extra assumption on the operator A and T.

Theorem 3.35. *Let X be a reflexive infinite dimensional Banach space and let $T\colon X \longrightarrow L^p(\Omega,\mathbb{R}^k)$ be a linear and compact operator. Assume K is closed, bounded and convex subset of X and $A\colon K \longrightarrow X^*$ is monotone and continuous on finite dimensional subspaces of K. If j satisfies the condition (j) then the problem (P) has at least one solution.*

Proof. Let F be an arbitrary finite dimensional subspace of X, which intersect with K. Consider the canonical injection $i_{K\cap F}\colon K\cap F \longrightarrow K$ and $i_F\colon F\longrightarrow X$ and let $i_F^*\colon X^*\longrightarrow F^*$ be the adjoint of i_F, applying Corollary 3.31 to the continuous operator $B = i_F^* A i_{K\cap F}$, we find some u_F in the compact set $K\cap F$, such that for every $v\in K\cap F$,

$$\langle i_F^* A i_{K\cap F} u_F, v-u_F\rangle + \int_\Omega j^0(x,\hat{u}_F(x);\hat{v}(x)-\hat{u}_F(x))dx \geq 0.$$

But

$$0\leq \langle Av - Au_F, v-u_F\rangle = \langle Av, v-u_F\rangle - \langle Au_F, v-u_F\rangle.$$

Now we can verify that $\langle i_F^* A i_{K\cap F} u_F, v-u_F\rangle = \langle Au_F, v-u_F\rangle$, and from the above results we have that

$$\langle Av, v-u_F\rangle + \int_\Omega j^0(x,\hat{u}_F(x);\hat{v}(x)-\hat{u}_F(x))dx \geq 0,$$

for any $v\in K\cap F$. The set K is weakly closed as a closed convex set, moreover it is weakly compact since it is bounded and X is a Banach space. Now for every $v\in K$, define

$$M(v) = \left\{ u\in K : \langle Av, v-u\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))dx \geq 0 \right\}. \tag{3.10}$$

The set $M(v)$ is weakly closed by the part (b) of lemma 3.33 and also it is weakly sequentially dense. Now we need to show that the set $M = \bigcap_{v\in K} M(v)\subset K$ is nonempty. To prove this it suffices to show that

$$\bigcap_{j=1}^{n} M(v_j)\neq\emptyset. \tag{3.11}$$

for any $v_1,\cdots,v_n\in K$. Let F be the finite dimensional linear space spanned by $\{v_1,\cdots,v_n\}$. Hence by (3.10) there exists $u_F\in F$ such that for every $v\in K\cap F$

$$\langle Av, v-u_F\rangle + \int_\Omega j^0(x,\hat{u}_F(x);\hat{v}(x)-\hat{u}_F(x))dx \geq 0. \tag{3.12}$$

Thus $u_F\in M(v_j)$ for every $1\leq j\leq n$, which implies (3.11). Consequently it follows that $M\neq\emptyset$. Therefore there is some $u\in K$ such that for every $v\in K$

$$\langle Av, v-u\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))dx \geq 0. \tag{3.13}$$

Next we shall prove by (3.13) that u is a solution of problem (P). Fix $w \in K$ and $\lambda \in (0,1)$. Putting $v = (1-\lambda)u + \lambda w \in k$ in (3.13) we obtain

$$\langle A((1-\lambda)u + \lambda w), \lambda(w-u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \lambda(\hat{w}-\hat{u})(x)dx \geq 0. \quad (3.14)$$

since $j^0(x,\hat{u};\lambda\hat{v}) = \lambda j^0(x,\hat{u};\hat{v})$, for any $\lambda > 0$, (3.14) can be written as

$$\langle A((1-\lambda)u + \lambda w), (w-u)\rangle + \int_\Omega j^0(x, \hat{u}(x); (\hat{w}-\hat{u})(x)dx \geq 0. \quad (3.15)$$

Let F be the vector space spanned by u and w. Taking into account the demicontinuity of the operator $A_{K\cap F}$ and passing to the limit in (3.15) as $\lambda \to 0$, we have that u is a solution to (P). ☐

Theorem 3.36. *Consider the same hypothesis as in Theorem 3.35 without the assumption of boundedness of K. Then the necessary and sufficient condition for existence of solution for (P) is that there exists $R > 0$ with the property that at least one solution of the problem*

$$(P-2) \qquad \begin{cases} u_R \in K \cap \{u \in X; \|u\| \leq R\}; \\ \langle Au_R, v - u_R\rangle + \int_\Omega j^0, (x, \hat{u}_R(x); \hat{v} - \hat{u}_R(x))dx \geq 0 \\ \textit{for every} \quad v \in K \quad \textit{with} \quad \|v\| \leq R. \end{cases}$$

satisfies the inequality $\|u_R\| < R$.

Proof. Observe that the set $K \cap \{u \in X; \|u\| \leq R\}$ is a closed, bounded and convex in X. Moreover, from Theorem 3.35 it follows that problem $(P-2)$ has at least one solution for any fixed $R > 0$. Which asserts the necessary condition.
On the other hand, let us suppose there exists a solution u_R of $(P-2)$ with $\|u_R\| < \mathbb{R}$. We prove that u_R is solution of (P). For any fixed $v \in K$ we choose $\varepsilon > 0$ small enough so that $w = u_R + \varepsilon(v - u_R)$ satisfies $\|w\| < R$. Hence,

$$\langle Au_R, \varepsilon(v-u_R)\rangle + \int_\Omega j^0(x, \hat{u}_R(x); \varepsilon(\hat{v} - \hat{u}_R)(x)dx \geq 0.$$

Due to the positive homogeneity of the map $v \longmapsto j^0(u;v)$, the conclusion follows. ☐

3.3.3 Basic Elements of Critical Point Theory

In this section we introduce the basic elements of critical point theory for nonsmooth functionals and present some related results.
Let $I\colon X \longrightarrow \mathbb{R}\cup\{\infty\}$ satisfy the structural hypothesis:

$$\begin{aligned}(H) \quad I = \Phi+\Psi, \quad \text{with} \quad \Phi\colon X\to\mathbb{R} \quad \text{locally lipschitz}, \quad \text{and}, \\ \Psi\colon X \longrightarrow \mathbb{R}\cup\{\infty\}, \text{convex, lower semicontinuous, and proper} \quad (i.e. \neq\infty), \\ \text{where X is a real Banach space.}\end{aligned} \tag{3.16}$$

Definition 3.37. *An element $u\in X$ is called a critical point of the functional $I\colon X \longrightarrow \mathbb{R}\cup\{\infty\}$ satisfying (H) if*

$$\Phi^0(u; v-u)+\Psi(v)-\Psi(u)\geq 0, \quad \forall v\in X.$$

The above definition can be equivalently expressed as follows.

Proposition 3.38. *An element $u\in X$ is a critical point of the functional $I\colon X \longrightarrow \mathbb{R}\cup\{\infty\}$ satisfying (H) if and only if $u\in D(\partial\Psi)$ and*

$$0\in\partial\Phi(u)+\partial\Psi(u),$$

where the notations $\partial\Phi(u)$ and $\partial\Psi(u)$ stands for the generalized gradient of Φ at u and the subdifferential in the sense of convex analysis of Ψ at u respectively, and $D(\partial\Psi)=\{x\in X : \partial\Psi(x)\neq\emptyset\}$.

Proof. Assume that $u\in X$ is a critical point. Then it satisfies the relation in the above definition which is equivalent to

$$\Phi^0(u; w)+\Psi(w+u)-\Psi(u)\geq 0, \quad \forall w\in X.$$

It follows that 0 is a minimum point of the convex function

$$w\longmapsto \Phi^0(u; w)+\Psi(w+u)-\Psi(u).$$

so $u\in D(\partial\Psi)$ and by using the subdifferential calculus of convex functions,

$$0\in\partial(\Phi^0(u;\cdot)+\Psi(\cdot+u)-\Psi(u))(0)=\partial(\Phi^0(u;\cdot))(0)+\partial\Psi(u)=\partial\Phi(u)+\partial\Psi(u).$$

Conversely, there exists $\zeta\in\partial\Phi(u)$ and $\eta\in\partial\Psi(u)$ such that $0=\zeta+\eta$ in X. By definition of the corresponding generalized gradients, we obtain

$$\Phi^0(u; v-u)+\Psi(v)-\Psi(u)\geq\langle\zeta, v-u\rangle+\langle\eta, v-u\rangle=\langle\zeta+\eta, v-u\rangle=0,$$

for all $v\in X$. □

Corollary 3.39. *Let $\Phi\colon X \longrightarrow \mathbb{R}$ be a locally Lipschitz function, and let K be a nonempty, closed, convex subset of X. Let I_K be the indicator function of K. Then $u \in K$ is a critical point of $\Phi + I_K$ if and only if $u \in K$ and $0 \in \partial\Phi(u) + N_K(u)$ where $N_K(u) = \{\eta \in X^* : \langle \eta, v-u\rangle \leq 0, \forall v \in K\}$ is the normal cone of K at u.*

The proof is similar to the proof of proposition 3.38 replacing Ψ by I_K.

Example 3.40. *Every local minimum $u \in X$ of a nonsmooth functional $I\colon X \longrightarrow \mathbb{R}\cup\{\infty\}$ satisfying (H) with $I(u) < +\infty$ is a critical point. Indeed, if $u \in X$ with $I(u) < +\infty$ is a local minimum of I then from convexity of Ψ for any $v \in X$ and a small $t > 0$ we have*

$$0 \leq I(u + t(v-u)) - I(u) \leq \Phi(u + t(v-u)) - \Phi(u) + t(\Psi(v) - \Psi(u)).$$

Dividing by t and letting $t \to 0^+$, we deduce that u satisfies the definition.

Definition 3.41. *The functional $I : X \longrightarrow \mathbb{R}\cup\{\infty\}$ with (H) is said to satisfy the Palais-Smale condition (for short,(PS)) if every sequence$(u_n) \subset X$ such that $(I(u_n))$ is bounded in $\mathbb{R}$ and*

$$\Phi^0(u_n; v-u_n) + \Psi(v) - \Psi(u_n) \geq -\epsilon_n\|v-u_n\|, \quad \forall v \in X,$$

for a sequence (ϵ_n) with $\epsilon_n \downarrow 0$ contains a strong convergent subsequence.

Lemma 3.42. *Let $\chi : X \longrightarrow \mathbb{R}\cup\{\infty\}$ be a lower semicontinuous, convex function with $\chi(0) = 0$. If*

$$\chi(x) \geq -\|x\|, \quad \forall x \in X.$$

Then $z \in X^*$ exists such that $\|z\|_{X^*} \leq 1$ and

$$\chi(x) \geq \langle z, x\rangle, \quad \forall x \in X.$$

Proof. Consider the following convex subsets A and B of $X \times \mathbb{R}$:

$$A = \{(x,t) \in X \times \mathbb{R} : \|x\| < -t\} \quad \textit{and} \quad B = \{(x,t) \in X \times \mathbb{R} : \chi(x) \leq t\}.$$

Note that: A is an open set and due to the condition $\chi(x) \geq -\|x\|$, one has $A \cap B = \emptyset$. A well known separation result yields the existence of $\alpha, \beta \in \mathbb{R}$ and $\omega \in X^*$ such that $(\omega, \alpha) \neq (0,0)$

$$\langle \omega, x\rangle - \alpha t \geq \beta, \quad \forall (x,t) \in A,$$

and

$$\langle \omega, x\rangle - \alpha t \leq \beta, \quad \forall (x,t) \in B.$$

We see that $\beta = 0$ because $(0,0) \in \bar{A} \cap B$, set $t = -\|x\|$ in the first inequality above. It follows that $\langle \omega, x\rangle \geq -\alpha\|x\|, \forall x \in X$, which implies $\alpha > 0$ and $\|\omega\|_{X^*} \leq \alpha$. Set $z = \alpha^{-1}\omega$ using $t = \chi(x)$, we deduce that

$$\langle z, x\rangle \leq \chi(x), \forall x \in X,$$

as $\|\omega\|_{X^*} \leq \alpha$, thus we obtain $\|z\|_{X^*} \leq 1$. □

Theorem 3.43. *Assume that the function $I = \Phi + \Psi : X \longrightarrow \mathbb{R} \cup \{\infty\}$ satisfies hypothesis (H), is bounded from below, and verifies the (PS) condition. Then there exists $x \in X$ such that $I(u) = \inf_X I \in \mathbb{R}$ and u is a critical point of I.*

Proof. Denote $m = inf_X I \in \mathbb{R}$. There exists a minimizing sequence $(u_n) \subset X$ such that

$$I(u_n) < m + \epsilon_n^2,$$

for a sequence (ϵ_n) of positive numbers,with $\epsilon \downarrow 0$. Applying Ekeland's variational principle to the function I, a sequence $(v_n) \subset X$ exists such that

$$I(v_n) < m + \epsilon_n^2,$$

and

$$I(v) \geq I(v_n) - \epsilon_n\|v_n - v\|, \quad \forall v \in X, \forall n \in \mathbb{N}.$$

Setting $v = (1-t)v_n + tw$ in the above inequality , for arbitrary $0 < t < 1$ and $w \in X$, we obtain

$$\Phi((1-t)v_n+tw)+\Psi(((1-t)v_n+tw) \geq \Phi(v_n)+\Psi(v_n)-\epsilon_n t\|v_n-v\|, \quad \forall w \in X.$$

The convexity of $\Psi : X \longrightarrow \mathbb{R} \cup \{\infty\}$ yields

$$\Phi((1-t)v_n+tw)-t\Psi(v_n)+t\Psi(w) \geq \Phi(v_n)-\epsilon_n t\|v_n-v\|, \quad \forall w \in X, \forall t \in (0,1)$$

Dividing by t and letting $t \downarrow 0$ we deduce that for all $w \in X$, we obtain

$$\Phi^0(v_n; w - v_n) + \Psi(w) - \Psi(v_n)$$
$$\geq \limsup_{t\downarrow 0} \frac{1}{t}(\Phi(v_n + t(w - v_n)) - \Phi(v_n) + \Psi(v_n) \geq -\epsilon_n\|w - v_n\|.$$

On the other hand, we have $\Phi(v_n) + \Psi(v_n) \to \quad m \quad as \quad n \longrightarrow \infty$. Then the (PS) condition implies that along a relabelled subsequence $u_n \to u$ in X, for some $u \in X$. The lower semicontinuity of I yields

$$I(u) \leq \liminf_{n\to\infty} I(V_n) \leq m,$$

so $I(u) = m$ and u satisfies definition of critical point. □

3.3.4 Variational-Hemivariational Inequality Problems with Lack of Convexity

In this section we present an abstract result in connection with the well known KKM principle.

Let X be a real Banach space, (S,μ) be a finite positive measure space, $A\colon X\to X^*$ an operator. We assume a compact mapping $\gamma\colon X\to L^p(S;\mathbb{R}^m)$, and q be the conjugate of p. If $\Phi\colon X\to\mathbb{R}$ is locally lipschitz functional. Let $j\colon S\times\mathbb{R}^m\to\mathbb{R}$ be a function such that for any $y\in\mathbb{R}^m$ the mapping $j(\cdot,y)\colon S\to\mathbb{R}$ is measurable.
In the following condition we assumed at least one holds; either there exists $k\in L^q(S,\mathbb{R}$ such that

$$|j(x,y_1)-j(x,y_2)|\le k(x)|y_1-y_2|,\quad \forall\in S,\forall y_1,y_2\in\mathbb{R}^m, \tag{3.17}$$

or the mapping $j(x,\cdot)$ is locally lipschitz, $\forall x\in S$, and there exists $C>0$ such that

$$|z|\le C(1+|y|^{p-1}),\quad \forall y_1,y_2\in\mathbb{R}^m,\quad \forall z\in\partial j(x,y). \tag{3.18}$$

Let $K\subset X$ be a nonempty closed and convex, $f\in X^*$ and $\Psi\colon X\to\mathbb{R}\cup\{+\infty\}$ convex, lower semicontinuous functional such that

$$D(\Psi)\cap K\neq\emptyset. \tag{3.19}$$

Now consider the problem: Find $u\in K$

$$\begin{aligned}\langle Au-f,v-u\rangle+\Psi(v)-\Psi(u)&\\+\int_S j^0(x,\gamma(u(x));\gamma(v(x)-u(x))d\mu\ge 0,\quad\forall v\in K.&\end{aligned} \tag{3.20}$$

consider the two practical cases as follows:
(i) $T=\Omega$, $\mu=dx$, $X=W^{1,q}(\Omega,\mathbb{R}^m)$ and $\gamma\colon X\to L^p(\Omega,\mathbb{R}^m)$, is sobolev embedding operator;
(ii) $T=\partial\Omega$, $\mu=d\sigma$, $X=W^{1,p}(\Omega,\mathbb{R}^m)$, and $\gamma=i\circ\eta$, where $\eta\colon X\to W^{1-\frac{1}{p},p}(\partial\Omega,\mathbb{R}^m)$ is the trace operator and $i\colon W^{1-\frac{1}{p},p}(\partial\Omega,\mathbb{R}^m)\to L^p(\partial\Omega,\mathbb{R}^m)$ is embedding operator.

Lemma 3.44. *Let $K\subset X$ be nonempty,closed, bounded, and convex, $\Psi\colon X\to\mathbb{R}\cup\{+\infty\}$ convex, lower semicontinuous functional such that*(3.19) *holds. Consider a Banach Space Y such that $L\colon X\to Y$ be linear and compact,* *and $J\colon Y \to \mathbb{R}$ be locally lipschitz function. In addition suppose that the mapping $K \ni v \mapsto \langle Av, v-u\rangle$ is weakly lower semicontinuous, for every $u \in k$. Then for every $f \in X^*$, there exists $u \in K$ such that*

$$\langle Au - f, v-u\rangle + \Psi(v) - \Psi(u) + J^0(L(u); L(v-u)) \geq 0, \quad \forall v \in K. \tag{3.21}$$

The proof is based on the Knaster-Kuratowski-Mazurkiewicz (in short, KKM) principle. Let E be a vector space, and $A \subset E$ is called *finitely closed* if its intersection with any finite dimensional linear variety $L \subset E$ is closed in the Euclidean topology of L. Let X be an arbitrary subspace of E.

A function $G\colon X \to 2^E$ is called a *KKM-mapping* if

$$conv\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i),$$

for any finite set $\{x_1, \cdots, x_n\} \subset X$.

The KKM Principle Let E be a vector space, X be an arbitrary subspace of E, and $G\colon X \to 2^E$ KKM-mapping such that $G(x)$ is finitely closed for any $x \in X$. Then the family $\{G(x)\}_{x\in X}$ has the finite intersection property.

Proof. Assume by contradiction, let $x_1, \cdots, x_n \in X$ be such that $\bigcap_{i=1}^n G(x_i) = \emptyset$. Let L be a linear manifold spanned by the set $\{x_1, \cdots, x_n\}$, thus

$$conv\{x_1, \cdots, x_n\} \subset L.$$

Let d be the Euclidean metric on L, since $L \cap G(x_i)$ is closed in L, it follows that $d(x, L\cap G(x_i)) = 0$ if and only if $x \in L \cap G(x_i)$. Now define $\lambda\colon conv\{x_1, \cdots, x_n\} \to \mathbb{R}$ by

$$\lambda(u) = \sum_{i=1}^{n} d(u, L \cap G(x_i), \forall u \in conv\{x_1, \cdots, x_n\}.$$

From our assumption, we obtain

$$\bigcap_{i=1}^{n} L \cap G(x_i) = \emptyset.$$

Which implies $\lambda(u) \neq 0$, for any $u \in conv\{x_1, \cdots, x_n\}$. Then we can define a continuous function

$$f\colon conv\{x_1, \cdots, x_n\} \to conv\{x_1, \cdots, x_n\}$$

by

$$f(u) = \frac{1}{\lambda(u)} \sum_{i=1}^{n} d(u, L \cap G(x_i) x_i$$

By Brouwer's theorem we assured the existence of a fixed point $u_0 \in conv\{x_1, \cdots, x_n\}$ of f. Set

$$I = \{i : d(u_0, L \cap G(x_i)) \neq 0\}.$$

Then u_0 will not belongs to $\bigcup_{i \in I} G(x_i)$. On the other hand

$$u_0 = f(u_0) \in conv\{x_i : i \in I\} \subset \bigcup_{i \in I} G(x_i).$$

which is a contradiction, hence the proof holds. □

Proof of Lemma 3.44 Define the set-valued mapping $G \colon K \cap D(\Psi) \to 2^X$ by

$$G(x) = \{v \in K \cap D(\Psi) : \langle Av - f, v - x\rangle$$
$$-J^0(L(v); L(x) - L(v)) + \Psi(v) - \Psi(x) \leq 0\}.$$

We claim that $G(x)$ is weakly closed. Indeed, if $G(x) \ni v_n \rightharpoonup v$ then

$$\langle Av, v - x\rangle \leq \liminf_{n \to \infty} \langle Av_n, v_n - x\rangle,$$

and

$$\Psi(v) \leq \liminf_{n \to \infty} \Psi(v_n).$$

In addition, $L(v_n) \to L(v)$ and by upper semicontinuity of J^0, we also have

$$\limsup_{n \to \infty} J^0(L(v_n; L(x - v_n)) \leq J^0 L(L(v); L(x - v)).$$

Therefore,

$$-J^0(L(v); L(x - v)) \leq \liminf_{n \to \infty}(-J^0(L(v_n); L(x - v_n)).$$

If $v_n \in G(x)$ and $v_n \rightharpoonup v$, then

$$\langle Av - f, v - x\rangle - J^0(L(v); L(x - v)) + \Psi(v) - \Psi(x)$$
$$\leq \liminf\{\langle Av_n - f, v - x\rangle$$
$$-J^0(L(v_n); L(x - v_n)) + \Psi(v) - \Psi(x)\} \leq 0,$$

which implies $v \in G(x)$. Since K is bounded, it follows that $G(x)$ is weakly compact. This shows that

$$\bigcap_{x \in K \cap D(\Psi)} G(x) \neq \emptyset,$$

provided the the family $\{G(x) : x \in K \cap D(\Psi)\}$ has the finite intersection property. By using the KKM principle after showing G is a KKM-mapping. Suppose by contradiction that there exists $x_1, \cdots, x_n \in K \cap D(\Psi)$ and $y_0 \in conv\{x_1, \cdots, x_n\}$ such that $y0 \notin \bigcup_{i=1}^{n} G(x_i)$. Then

$$\langle Ay_0 - f, y_0 - x_i \rangle + \Psi(y_0) - \Psi(x_i) - J^0(L(y_0; L(x_i - y_0)) > 0,$$

for all $i = 1, \cdots, n$. Therefore, $x_i \in \Delta$, $\forall i \in \{1, \cdots, n\}$, where

$$\Delta := \{x \in X; \langle Ay_0 - f, y_0 - x_i \rangle + \Psi(y_0) - \Psi(x_i) - J^0(L(y_0; L(x_i - y_0)) > 0\}.$$

The set Δ is convex and hence $y_0 \in \Delta$, which is a contradiction. Therefore,

$$\bigcap_{x \in K \cap D(\Psi)} G(x) \neq \emptyset.$$

This gives an element $u \in K \cap D(\Psi)$ such that;

$$\langle Au - f, v - u \rangle + \Psi(v) - \Psi(u) - J^0(L(u; L(v - u)) > 0, \forall v \in K \cap D(\Psi).$$

The conclusion follows.

Remark 3.45.

- *Using the hypothesis in Lemma 3.44 Motreanu and Rădulescu [3] proved the existence of at least one solution to the problem* (3.20) *for the case when* $Y = L^p(S; \mathbb{R}^m)$, $L = \gamma$, *and* J *is defined as*

$$J(u) = \int_S j(x, u(x)) d\mu$$

and when K *is unbounded they also proved existence with coercievity condition. Moreover, for monotone and hemicontinuous operators the problem* (3.21) *has a solution using the result due to Mosco's theorem [6].*

- *Many scholars working in the varational-hemivariational inequality problems discovered related results by employing different approaches. For example, recently using the principle of symmetric criticality (which* *states that it is enough to study the existence of critical points of a given function on a certain subspace, not on the whole space) will be applied for functions satisfying* (3.16) *studied on certain unbounded strip [1] and existence results on unbounded domains with smooth boundary [12].*

- *The theory of variational and hemivariational inequalities are active area of research and could solve many open problems in mechanics and engineering. Interested readers are motivated to refer monographs of P. D. Panagiotopoulos [20],[19], and D. Motreanu and P. D. Panagiotopoulos [2], and related materials.*

Bibliography

[1] A. Kristály, Cs. Varga, V. Varga, A nonsmooth principle of symmetric criticality and variational-hemivariational inequalities, Journal of Mathematical Analysis and Applications, 325(2007), no. 2, 975-986.

[2] D. Motreanu, P.D. Panagiotopoulos, Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities, Kluwer Academic Publishers, Dordrecht, 1999.

[3] D. Goeleven, D. Motreanu and P. D. Panagiotopoulos, Multiple Solutions for a Class of Hemivariational Inequalities involving Periodic Energy Functionals, Math. Methods Appl. Sci. 20, 547-568, 1997.

[4] F.H. Clarke, Optimization and Nonsmooth Analysis, Classical in Applied Mathematics, 5, Society for Industrial and Applied mathematics (SIAM) PA, 1990.

[5] F.H. Clarke, Non Smooth Analysis and Control Theory, Springer-Verlag, New York, Berlin Heidelberg, 1998.

[6] F. Gazzola and V. D. Rădulescu, A Nonsmooth Critical Point Theory Approach to Some Nonlinear Elliptic Equations in $\mathbb{R}^n$, Differential Integral Equations, Vol. 13, 47-60, 2000.

[7] Giorgio Giorgi,A. Guerraggio, and J. Thierfelder, Mathematics of Optimization: Smooth and Nonsmooth Case, Elsevier B.V, Netherlands, 2004.

[8] Hartman G.J. and Stampacchia G., On Some Nonlinear Elliptic Differential Equations, Acta Math.115 , 271-310, 1966.

[9] Hartman, G.J.and Stampacchia, G. On Some Nonlinear Elliptic Differential Functional Equations, Acta Math. 115, pp. 153âĂŞ188, 1996.

[10] I. Ekeland and R. Temam, Convex Analysis and Variational Problems, Stud- ies in Mathematics and its Applications, 1, North-Holland Publishing Co., Amsterdam-Oxford, 1976.

[11] Ivar Ekeland, Nonconvex Minimization Problems, Bull. Amer. Math. Soc. (N.S.) Volume 1, Number 3, 443-474, 1979.

[12] Ildikó-Ilona Mezei, Lia Săplăcan, Existence Results and Application for General Variational-Hemivariational Inequalities on Unbounded Domains, Electronic Journal of Differential Equations,Vol. No. 48, 2009.

[13] Kavian. O, Introduction a la théorie des points critiques et applications aux problém s elliptiques, Springer-Verlag, Paris, Berlin, Heidelberg, New York, London, Tokyo, Hong Kong, Barcelona, Budapest, 1993.

[14] K.C. Chang, Variational Methods for Nondifferentiable Functionals and Their Applications to Partial Differential Equations, J. Math. Anal. Appl., 80, 102-129, 1981.

[15] Kinderlehrer D. and Stampacchia G., An Introduction to Variational Inequalities, Academic Press, New York, 1980.

[16] Kinderlehrer D. and Stampacchia G, An Introduction to Variational Inequalities and Their Applications, Classical in Applied mathematics, 31, Society for Industrial and Applied Mathematics (SIAM) PA, 2000.

[17] Michel Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications, Birkhauser, 1996.

[18] Nicolas H., Sandor K. and Siegfried S., Handbook of Generalized Convexity and Generalized Monotonicity, Springer Science + Business Media, Inc, 2005.

[19] P.D. Panagiotopoulos, Inequality Problems in Mechanics and Applications: Convex and Nonconvex Energy Functionals, Birkhäuser Verlag, Basel, 1985.

[20] P.D. Panagiotopoulos, Hemivariational Inequalities, Applications in Mechanics and Engineering, Springer-Verlag, Berlin, 1993.

[21] P.D. Panagiotopoulos, Micheal F. and Vicentiu R., Existence Theorems of Hartman-Stampacchia Type for Hemivariational Inequalities and Applications, Kluwer Academic Publisher, Netherlands, 1999.

[22] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman Publishing Inc., 1985.

[23] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1790.

[24] R.T. Rockafellar , Generalized Directional Derivatives and Subgradients of Nonconvex Functions, Canadian J. of Math., 32, 257-280, 1980.

[25] Siegfried Carl, Vy Khoi Le, Dumitru Motreanu, Nonsmooth Variational Problems and Their Inequalities Comparison Principles and Applications, Springer Science + Business Media, LLC, 2007.

[26] V.F. Demyanov, The Rise of Nonsmooth Analysis: Its main tools, Cybernetics and Systems Analysis, Vol. 38, No. 4, 2002.

[27] Z. Naniewicz, P.D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications, Marcel Dekker, Inc.New York, Basel, Hong Kong, 1995.