# On distribution of prime numbers

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# INTRODUCTION

The main goal of this paper is to describe the result of Goldston-Pintz-Yildirim [1], obtained by this group of authors in 2005, about the small gaps between prime numbers and makes this result more available to the reader. Namely, the following result will be described:

$$\Delta_1 = \liminf_{n\to\infty}\left(\frac{p_{n+1}-p_n}{\log p_n}\right) = 0, \tag{1}$$

where $p_n$ is the $n$-th prime. In 2006 Motohashi [2] found easier proof of (1), and the final chapter of this paper will be based on his variant of proof.

Result (1) is the best known estimation of gap between two consecutive primes for today, but the smallest gaps of such type are generally believed to be 2, as predicted by the Twin Prime Conjecture. From this position the gap between (1) and general belief is still infinity. From the other side, the history of researches towards Twin Prime Conjecture says that result of type (1) is the breakthrough in this area. In general case, let us define

$$\Delta_v = \liminf_{n\to\infty}\left(\frac{p_{n+v}-p_n}{\log p_n}\right). \tag{2}$$

First nontrivial unconditional result was obtained by Erdös in 1940 using Brun's sieve:

$$\Delta_1 < 1.$$

After appearing the Bombieri-Vinogradov theorem in 1965, Bombieri and Davenport made a breakthrough:

$$\Delta_v \le v - \frac{1}{2}$$

Then Huxley made a several essential refinements and the last one (1984) was:

$$\Delta_v \le v - \frac{5}{8} + O\left(\frac{1}{v}\right) \quad \text{and} \quad \Delta_1 \le 0{,}43494\ldots$$

In 1988 Majer applied his own new method and got a solid result:

$$\Delta_1 \le e^{-\gamma}\cdot 0{,}4425\ldots = 0{,}2484\ldots,$$

where $\gamma$ is Euler's constant . In 2005 Goldston and Yildirim proved that

$$\Delta_v \le \left(\sqrt{v} - \frac{1}{2}\right)^2.$$

And finally in the same year Goldston, Pintz and Yildirim showed that

$$\Delta_v \leq \left(\sqrt{v}-1\right)^2 \text{ and } \quad \Delta_1 = 0.$$

The principal idea which made the Goldston-Pintz-Yildirim work is the introduction of parameter $l$ in the Selberg sieve with weight $\mu(d)\left(\log\frac{R}{d}\right)^{k+l}$(see Chapter 6 of this paper). Appearance of additional parameter $l$ plays the crucial role in several estimations in the proof.

Despite the complexity of problem, authors of [1] believe that mathematical society is not so far to show that

$$d_1 = \lim_{n\to\infty} \inf(p_{n+1} - p_n) < \infty. \tag{3}$$

The hypothesis stated in form (3) is called Bounded Gap Conjecture. Belief in inequality (3) has strong basis. Suppose we have the following inequality:

$$\sum_{q\leq x^\theta} \max_{(a,q)=1} \max_{y\leq x} \left|\psi(y;q,a) - \frac{y}{\phi(q)}\right| \ll x(\log x)^{-A}. \tag{4}$$

Statement which says that for a large $x$ with any positive $\theta < \frac{1}{2}$ inequality de (4) holds is called Bombieri-Vinogradov theorem. Statement which says that in (4) we can have $\theta < 1$ is known as Elliot-Halberstam Conjecture. Under this conjecture Goldston-Pintz-Yildirim proved in [1] that

$$d_1 \leq 16.$$

This paper has 6 chapters. Chapter 1 is consisted of basic definitions, lemmas and theorems about the distribution of prime numbers, knowledge of which are required for understanding the next chapters. The theorems about a logarithmic order of Riemann zeta function in chapter 1 are necessary for estimations in Chapter 6. Chapters 2, 3, 4 are mainly auxiliary tools for proof of Bombieri-Vinogradov theorem, but also contain wide applicable and powerful methods. Chapters 5 and 6 contain complete proofs of Bombieri-Vinogradov and Goldston-Pintz-Yildirim theorems up to similar cases.

## Chapter 1

# BASIC TOOLS IN ANALYTIC NUMBER THEORY

### 1.1 Basic arithmetic functions and their properties

**Definition 1.1.1.** A nonzero function $f: \mathbb{N} \to \mathbb{C}$ is multiplicative if $f(mn) = f(m)f(n)$ for all coprime $n$ and $m$. If equality holds for all pairs of $m$ and $n$, then $f$ is completely multiplicative.

**Definition 1.1.2.** Möbius function is a function of the form:

$$\mu(n) = \begin{cases} 1 & if\ n = 1, \\ (-1)^r & if\ n\ is\ the\ product\ of\ r\ distinct\ primes, \\ 0 & otherwise. \end{cases}$$

**Note 1.1.3.** Möbius function is multiplicative.

**Lemma 1.1.4.** Suppose that $f$ and $g$ are multiplicative functions. Then

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}),$$

is also multiplicative.

**Lemma 1.1.5.** Next equality is the famous property of Möbius function:

$$\sum_{d|n} \mu(n) = \begin{cases} 1 & if\ n = 1 \\ 0 & otherwise. \end{cases}$$

*Proof.* Suppose that $n$ is squarefree and $n = p_1 p_2 \cdots p_k$, and $m = p_1 p_2 \cdots p_{k-1}$. Then

$$\sum_{d|n} \mu(d) = \sum_{d|m} \mu(d) + \sum_{d|m} \mu(dp_k) = \sum_{d|m} \mu(d) + \sum_{d|m} (-\mu(d)) = 0. \blacksquare$$

**Lemma 1.1.6** (Möbius inversion formula). Suppose that $f: \mathbb{N} \to \mathbb{C}$ and define $F(n) = \sum_{d|n} f(d)$, then $f = F * \mu$ and $f$ is multiplicative if $F$ is multiplicative.

*Proof.* If $f = F * \mu$ then

$$\sum_{d|n} F(d)\mu(\frac{n}{d}) = \sum_{d|n} \sum_{k|d} f(k)\mu(\frac{n}{d}) = \sum_{k|n} \sum_{\substack{d|n \\ k|d}} f(k)\, \mu\left(\frac{n}{d}\right) = \sum_{k|n} \sum_{m|(\frac{n}{k})} f(k)\mu(\frac{n}{km}).$$

If $k \neq n$ then sum over $m$ is zero, so first claim is proved. Multiplicativity of $\mu$ and Lemma 1.1.4 give us the second claim. ■

**Lemma 1.1.7** (Abel summation formula). Suppose that $f(x)$ is continuously differentiable on $[a,b]$ and $a_n$ are complex numbers. Then

$$\sum_{a<n\leq b} a_n f(n) = A(b)f(b) - \int_a^b A(x)f'(x)dx,$$

Where $A(x) = \sum_{a<n\leq x} a_n$.

*Proof.* We can apply the Stieltjes integration by parts on the sum above:

$$\sum_{a<n\leq b} a_n f(n) = \int_{a^+}^{b^+} f(x)dA(x) = f(x)A(x)|_a^b - \int_a^b A(x)df(x),$$

And the result follows. ■

**Corollary 1.1.8.** There is a constant $c$ such that

$$\sum_{n\leq x} \frac{1}{n} = \log x + c + O(x^{-1}).$$

*Proof.* This famous result can be obtained by applying Abel formula with $f(n) = \frac{1}{n}$ and $a_n = 1$ for all $n \leq x$. ■

**Definition 1.1.9.** The following function is called a von Mangoldt's function:

$$\Lambda(n) = \begin{cases} \log p & \textit{if } n \textit{ is a power of prime } p, \\ 0 & \textit{otherwise.} \end{cases}$$

**Note 1.1.10.** Von Mangoldt's function is multiplicative.

**Theorem 1.1.11 (Mertens).** There is an absolute constant B such that

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + B + O((\log x)^{-1}).$$

Proof of this classical result can be found by reader in [3], page 12.

### 1.2 Dirichlet series and Perron's formula

**Definition 1.2.1.** Suppose that $a_n$ are complex numbers and $s = \sigma + it$ is a complex variable. Then a series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s},$$

is called Dirichlet series. The finite sum of the above form is called Dirichlet polynomial.

**Lemma 1.2.2.** Suppose that $s_0 = \sigma_0 + it_0$ and the series $\sum_{n=1}^{\infty} a_n n^{-s_0}$ converges. Then the sum $\sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly on the compact subsets of the half-plane $Re(s) > \sigma_0$ and the sum function $f(s)$ is holomorphic in that half-plane.

**Definition 1.2.3.** Abscissa of convergence of the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is the number

$$\inf \left\{ Re(s): \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges} \right\}$$

Abscissa of absolute convergence of Dirichlet series is called the abscissa of convergence of $\sum_{n=1}^{\infty} |a_n| n^{-s}$.

**Lemma 1.2.4.** Suppose that $a_n$ and $b_n$ are complex numbers and the following Dirichlet series are converge in $Re(s) > \sigma_0$:

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad and \quad g(s) = \sum_{n=1}^{\infty} b_n n^{-s}.$$

If $f(s) = g(s)$ for $Re(s) > \sigma_0$, then $a_n = b_n$ for all $n$.

**Lemma 1.2.5** (Multiplication of Dirichlet series). Suppose that the following Dirichlet series are converge absolutely in $Re(s) > \sigma_0$:

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad and \quad g(s) = \sum_{n=1}^{\infty} b_n n^{-s}.$$

Then the Dirichlet series

$$h(s) = \sum_{n=1}^{\infty} c_n n^{-s}$$

with $c_n = \sum_{uv=n} a_u b_v$ is also absolutely convergent in $Re(s) > \sigma_0$ and $h(s) = f(s)g(s)$.

Proofs of Lemmas 1.2.2, 1.2.4 and 1.2.5 can be found in [3] pp. 14-17.

**Lemma 1.2.6** (Perron's formula). Suppose that $a > 0$. Then

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{u^s}{s}ds = \begin{cases} 1 + O(u^a(T(\log u)^{-1}) & if\ u > 1, \\ \frac{1}{2} + O(aT^{-1}) & if\ u = 1, \\ O(u^a(T|\log u|)^{-1}) & if\ 0 < u < 1. \end{cases}$$

This variant of Perron's formula can be obtained by using the contour integration.

**Corollary 1.2.7.** Suppose that $\sigma_a$ is the abscissa of absolute convergence of $\sum_{n=1}^{\infty} a_n n^{-s}$, and $x$ is not integer. Also let $a > \sigma_a$ and $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Then

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} f(s)x^s s^{-1} ds + O\left(\frac{x^a}{T}\sum_{n=1}^{\infty}\frac{|a_n|n^{-a}}{\left|\log(\frac{x}{n})\right|}\right).$$

*Proof.* Since $x$ is not integer, we have

$$\sum_{n\le x} a_n = \sum_{n=1}^{\infty} a_n I_n,$$

where $I_n$ is 1, if $n < x$ and $I_n$ is zero, when $n > x$. Let $u = \frac{x}{n}$, then by Lemma 1.2.6:

$$\sum_{n=1}^{\infty} a_n I_n = \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\left(\frac{x}{n}\right)^s\frac{ds}{s} + O(\left(\frac{x}{n}\right)^a\left(T\left|\log\left(\frac{x}{n}\right)\right|)^{-1}\right) =$$

$$= \frac{1}{2\pi i}\int_{a-iT}^{a+iT} x^s \sum_{n=1}^{\infty} a_n n^{-s}\frac{ds}{s} + O\left(\frac{x^a}{T}\sum_{n=1}^{\infty}\frac{|a_n|n^{-a}}{\left|\log(\frac{x}{n})\right|}\right),$$

and Corollary is proved. ■



### 1.3 Divisor functions and Euler function.

In this section we consider several standard estimates for divisor function and for sum of reciprocal Euler functions.

**Definition 1.3.1.** The divisor function $d(n)$ denotes the number of different divisors of integer $n$.

**Definition 1.3.2.** The generalized divisor function $\tau_k(n)$ denotes the number of ways, in which integer $n$ can be written as a product of $k$ integers.

**Note 1.3.3.** It is clear that $d(n) = \tau_2(n)$ .

**Lemma 1.3.4** (Elementary bound for $d(n)$). For every integer n we have:

$$d(n) \leq 2\sqrt{n}.$$

*Proof.* Let $n = d_1 d_2$. If $d_1 < \sqrt{n}$ then $d_2 > \sqrt{n}$ , reverse is also true, hence

$$d(n) = \sum_{d|n} 1 \leq 2 \sum_{d|\sqrt{n}} 1 \leq 2\sqrt{n}.$$

Lemma is proved. ■

**Theorem 1.3.5.** For $d(n)$ and any nonnegative integer $k$ we have:

$$\sum_{n\leq x} \frac{d(n)^k}{n} \leq \left(\sum_{n\leq x} \frac{1}{n}\right)^{2^k}.$$

*Proof.* We will prove it by induction. Case when $k = 0$ is trivial. Assume inequality above holds for $k > 0$. For $k + 1$ we can write:

$$\sum_{n\leq x} \frac{d(n)^{k+1}}{n} = \sum_{n\leq x} \frac{d(n)^k}{n} \sum_{m|n} 1 = \sum_{ml\leq x} \frac{d(ml)^k}{mk} \leq \sum_{ml\leq x} \frac{\big(d(m)d(l)\big)^k}{mk} \leq$$

$$\leq \sum_{\substack{m\leq x\\ l\leq x}} \frac{\big(d(m)d(l)\big)^k}{mk} = \sum_{m\leq x} \frac{d(m)^k}{m} \sum_{n\leq x} \frac{d(l)^k}{l} \leq \left(\sum_{n\leq x} \frac{1}{n}\right)^{2^k} \left(\sum_{n\leq x} \frac{1}{n}\right)^{2^k}.$$

And result follows. ■

**Theorem 1.3.6.** Suppose that $k$ is nonnegative integer. Then

$$\sum_{n\leq x} d(n)^k \leq x \left(\sum_{n\leq x} \frac{1}{n}\right)^{2^k-1}.$$

*Proof.* We will prove this theorem by induction again. Case when $n = 0$ is clear. Assume that $n > 0$ and do induction for $n + 1$:

$$\sum_{n\le x} d(n)^k = \sum_{ml\le x} d(mk)^k \le \sum_{ml\le x} d(m)^k d(l)^k = \sum_{m\le x} d(m)^k \sum_{h\le \frac{x}{m}} d(h)^k.$$

By induction hypothesis theorem:

$$\sum_{m\le x} d(m)^k \sum_{h\le \frac{x}{m}} d(h)^k \le \sum_{m\le x} d(m)^k \frac{x}{m}\left(\sum_{n\le x}\frac{1}{n}\right)^{2^k-1} = x\sum_{m\le x}\frac{d(m)^k}{m}\left(\sum_{n\le x}\frac{1}{n}\right)^{2^k-1}.$$

By previous Theorem:

$$x\sum_{m\le x}\frac{d(m)^k}{m}\left(\sum_{n\le x}\frac{1}{n}\right)^{2^k-1} \le x\left(\sum_{n\le x}\frac{1}{n}\right)^{2^k}\left(\sum_{n\le x}\frac{1}{n}\right)^{2^k-1}.$$

And statement of the Theorem follows. ■

**Definition 1.3.7.** Euler totient function (or just Euler function) $\phi(n)$ is the number of positive integers, which are less then $n$ and coprime with $n$.

**Lemma 1.3.8.** For $\phi(n)$ two following statements hold:

$a)$ $\phi(n)$ is multiplicative

$b)$ $\phi(p^a) = p^a - p^{a-1}$ for prime $p$ and $a \ge 1$.

The proof of Lemma 1.3.8 can be found in book of Apostol [9], page 28.

**Lemma 1.3.9.** For any $x \ge 1$ we have

$$\sum_{n\le x}\frac{1}{\phi(n)} \ll \log x.$$

*Proof.* It is easy to see that for a prime $p$ and any nonnegative multiplicative function $f(n)$:

$$\sum_{n\le x} f(n) \le \prod_{p\le x}\sum_{k=0} f(p^k).$$

Hence by Theorem 1.1.11 and Lemma 1.3.8:

$$\sum_{n\le x}\frac{1}{\phi(n)} \le \prod_{p\le x}\sum_{k=0}\frac{1}{\phi(p^k)} = \prod_{p\le x}\left(1+\sum_{k=1}^{\infty}\frac{1}{p^{k-1}(p-1)}\right) \le \exp\left(\sum_{k=1}^{\infty}\frac{1}{p^{k-1}(p-1)}\right) =$$

$$= \exp\left(\sum_{p\le x}\frac{1}{p} + O\left(\sum_{p\le x}\frac{1}{(p-1)^2}\right)\right) \ll \log x.$$

Proof of Theorem is finished. ■

**1.4 Prime number theorem and Riemann zeta function.**

**Definition 1.4.1.** Prime counting function $\pi(x)$ is the function counting number of primes in interval $[1, x]$.

**Definition 1.4.2.** Denote the Chebyshev function $\psi(x)$ as:

$$\psi(x) = \sum_{n\le x}\Lambda(n).$$

The next theorem describes a classical result in analytic number theory, and appears in many important theorems about the distribution of prime numbers.

**Theorem 1.4.3 (Prime number theorem).**

$$\lim_{x\to\infty}\frac{\pi(x)}{x/\ln x} = 1.$$

Proof of PNT can be found in most books on analytic number theory, for example in Titchmarsh [4], in section 3.

PNT has many alternative formulations, the next theorem is one of them.

**Theorem 1.4.4.** There exists an absolute constant $c > 0$ such that for $x \ge 2$

$$\psi(x) = x + O\left(x\exp\left(-c\sqrt{\log x}\right)\right).$$

**Definition 1.4.5.** For $Res > 1$ define the Riemann zeta function as:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

**Lemma 1.4.6.** For $Res > 1$ we have an Euler product for $\zeta(s)$:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p}\right)^{-1}.$$

**Note 1.4.7.** We deduce from Lemma 1.4.6, that $\zeta(s)$ doesn't have zeroes in $Res > 1$.

**Lemma 1.4.8.** For $\zeta(s)$ the following equality holds:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

*Proof.* We have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \sum_{n=1}^{\infty} \frac{1}{n^s} = 1,$$

by Lemma 1.1.5 and property of multiplication of Dirichlet series. ■

**Theorem 1.4.9.** The function $\zeta(s)$ can be analytically continued to a regular function for all values of $s$, except $s = 1$, where there is a simple pole with residue 1. Extended $\zeta(s)$ satisfies the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2} s\pi\right) \Gamma(1-s)\zeta(1-s).$$

Zeroes of $\zeta(s)$ in $0 \leq Res \leq 1$ are of special interest in analytic number theory. Existence a zero-free region of $\zeta(s)$ in $0 \leq Res \leq 1$ gives many powerful properties to $\zeta(s)$, so the general problem in studying such zeroes is an extension of zero-free region to the left of $Res = 1$. A part of original proof of PNT was the next result:

**Theorem 1.4.10.** There is a constant $A$ such that $\zeta(s)$ is not a zero for

$$\sigma \geq 1 - \frac{A}{\log t},$$

For $t > t_0 = const$.

**Theorem 1.4.11.** For any $\sigma > 1$ and absolute positive constant A the following inequality holds:

$$\left|\frac{1}{\zeta(s)}\right| \ll \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(\sigma) < \frac{A}{\sigma - 1}.$$

**Theorem 1.4.12.** $\zeta(s) = O(\log t)$ uniformly in the region

$$1 - \frac{A}{\log t} \leq \sigma \leq 2 \ (t > t_0).$$

**Theorem 1.4.13** In a region $\sigma \geq 1 - \frac{A}{\log t}$:

$$\frac{1}{\zeta(s)} = O(\log t).$$

Theorems 1.4.9-1.4.13 reader can find in book of Titchmarsh [4]: Theorem 1.4.9 on p. 13, Theorem 1.4.10 on p. 54, Theorem 1.4.11 on p. 45, Theorem 1.4.12 on p. 49, Theorem 1.4.13 on p. 60.

# Chapter 2

# DIRICHLET CHARACTERS AND PÓLYA-VINOGRADOV INEQUALITY

## 2.1 Group characters

**Definition 2.1.1.** Let G be a finite commutative group of order m. A group homomorphism $\chi$: $G \to \mathbb{C}^\times$ is called a character of G if $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in G$.

**Note 2.1.2.** It can be easily established that $\chi(e) = 1$ and that $\chi(x)$ is an $m$-th root of unity: $\chi(x)^m = \chi(x^m) = \chi(e) = 1$.

**Note 2.1.3.** Group characters $\chi(x)$ form a group $\hat{G}$ under pointwise multiplication: $(\chi_1\chi_2)(x) = \chi_1(x)\chi_2(x)$ for all $x \in G$. Trivial character is the identity of $\hat{G}$: $\chi_0(x) = 1$ for all $x \in G$, and complex-conjugate character $\bar{\chi}$ is inverse of $\chi$.

**Theorem 2.1.4.** $G$ and $\hat{G}$ are isomorphic to each other.

*Proof.* Suppose that $G$ is cyclic with generator $g$. Since every $\chi(x)$ is the $m$-th root of unity, $\chi(g)$ must be of the form: $\chi(g) = \exp(\frac{a}{m})$ for some integer $a$. If $x = g^y$ then $\chi(x) = \chi(g^y) = \exp(\frac{ay}{m})$. We can define $\chi_a = \exp(\frac{ay}{m})$ and see that all $\hat{G}$ is generated by $\chi_1$, which mean that $\hat{G}$ is cyclic of order $m$.

Now, let $G$ be arbitrary finite abelian group. $G$ can be represented as the direct product of cyclic groups, $G = C_1 \times C_2 \times \cdots \times C_k$. Consider $y = y_1 y_2 \cdots y_k, y_j \in C_j$. We can define $\chi \in \hat{G}$ by $\chi(x) = \chi_1(x_1)\chi_2(x_2)\cdots \chi_k(x_k)$ for $x = x_1 x_2 \cdots x_k \in G, x_k \in C_k$, here $\chi_j$ is the character in $\widehat{C_j}$ corresponding to $y_j$. So, we constructed now an isomorphism between $G$ and $\hat{G}$. Theorem is proved. ■

**Corollary 2.1.5.** Let $G$ be an abelian group of finite order and $x \in G$, $x \neq e$. Then there is a character $\chi \in \hat{G}$ such that $\chi(x) \neq 1$.

*Proof.* Let write $G$ as the direct product of cyclic groups, $G = C_1 \times C_2 \times \cdots \times C_k$ and $x = x_1 x_2 \cdots x_k$. Some $x_j$ is not identity, so let it be $x_1$. Let $g$ be the generator of $C_1$. Consider the character $\chi$ corresponding to $ge \cdots e$ under the isomorphism from the Theorem 2.1.1. Character $\chi \neq 1$. ■

**Lemma 2.1.6.** Let G be a finite abelian group and denote by $\chi_0$ the trivial character of G. Then the following two relations hold:

$$\sum_{x \in G} \chi(x) = \begin{cases} |G| \ if \ \chi = \chi_0, \\ 0 \ \ otherwise; \end{cases} \tag{2.1.1}$$

$$\sum_{\chi \in \hat{G}} \chi(x) = \begin{cases} |G| \ if \ x = e, \\ 0 \ \ otherwise. \end{cases} \tag{2.1.2}$$

*Proof.* Clearly, when $\chi$ is trivial, then (2.1.1) holds. Suppose now that $\chi$ is nontrivial and for some $x_0$, $\chi(x_0) \neq 1$. We can write

$$\chi(x_0) \sum_{x \in G} \chi(x) = \sum_{x \in G} \chi(x x_0) = \sum_{y \in x_0 G} \chi(y) = \sum_{y \in G} \chi(y).$$

Since $\chi(x_0) \neq 1$, the sum on the right side must be equal to 0. Thus (2.1.1) holds.

Clearly, when $x = e$, then (2.1.2) holds. So, suppose that $x \neq e$. There is a $\chi_0(x) \neq 1$ by Corollary 2.1.2. We can write

$$\chi_0(x) \sum_{\chi \in \hat{G}} \chi(x) = \sum_{\chi \in \hat{G}} \chi_0 \chi(x) = \sum_{\psi \in \chi_0 \hat{G}} \psi(x) = \sum_{\psi \in \hat{G}} \psi(x).$$

Again, sum on the right side must be equal to zero. This establishes (2.1.2). ■

## 2.2 Dirichlet Characters

Let $q \geq 1$ be an integer. Let $G_q = (\mathbb{Z}/q\mathbb{Z})^{\times}$ be the multiplicative group of $(\mathbb{Z}/q\mathbb{Z})$. Then $G_q$ is a cyclic group of order $\phi(q)$, where $\phi(q)$ is Euler's function. Let us introduce an extended function $\chi(n) = 0$ if $\gcd(n, q) > 1$, where $\chi$ is a character of $G_q$.

**Definition 2.2.1.** Extended function denoted above is a Dirichlet character modulo q, or just a Dirichlet character.

**Note 2.2.2.** Dirichle characters are not group homomorphisms anymore, but they preserve multiplicativity: $\chi(n)\chi(m) = \chi(nm)$ for all $n, m \in \mathbb{Z}$. Also we will denote the extension of trivial group character $\chi_0$ as principal character modulo q and preserve the notation $\chi_0$.

Next lemma is a straightforward consequence of Lemma 2.1.6

**Lemma 2.2.3.** If $\chi$ is a Dirichlet character modulo q, then

$$\sum_{n=1}^{q} \chi(n) = \begin{cases} \phi(q) & if\ \chi = \chi_0, \\ 0 & otherwise; \end{cases} \tag{2.2.1}$$

$$\sum_{\chi mod q} \chi(n) = \begin{cases} \phi(q)\ if\ n \equiv 1 (mod\ q), \\ 0 \quad otherwise, \end{cases} \tag{2.2.2}$$

where the sum on the right side is over all Dirichlet characters modulo q.

Let $\chi$ be a non-principal character modulo $q$, let $q_1$ be a proper divisor of q, and let $\chi_1$ be a non-principal character modulo $q_1$, also let $\chi_0$ be a principal character modulo q such that

$$\chi(n) = \chi_1(\mathrm{n})\chi_0(n) \quad \text{for all } \ n \in \mathbb{Z} \tag{2.2.3}$$

**Definition 2.2.2.** We say that $\chi_1$ induces $\chi$, if (2.2.3) holds.

**Definition 2.2.3.** Dirichle character $\chi$ is imprimitive, if we can find $\chi_1$ and $\chi_0$ as in (2.2.3), otherwise $\chi$ is called primitive.

**Note 2.2.2.** Principal characters are neither primitive, nor imprimitive.

**Definition 2.2.4.** Let $\chi$ be an imprimitive Dirichlet character modulo q. Conductor of $\chi$ is the least modulus $q^*$ such that there exists a (necessarily primitive) character $\chi^*$ modulo $q^*$, which induces $\chi$.

**Note 2.2.3.** If $\chi$ is primitive, we define its conductor to be equal to the modulus $q$, and if $\chi$ is principal, we define the conductor to be equal to 1.

### 2.3 Gaussian Sums

Let $a$ is an integer, and $\chi$ is a Dirichlet character modulo $q$. Let us introduce the sum

$$\tau(\chi, a) = \sum_{m\ modq} \chi(m)\, e\left(\frac{am}{q}\right) \tag{2.3.1}$$

Where the summation is over any complete system of residues modulo $q$.

**Lemma 2.3.1.** Let $\chi$ be a Dirichlet character modulo q and suppose that $\gcd(a, q) = 1$ or $\chi$ is primitive. Then

$$\tau(\chi, a) = \bar{\chi}(a)\tau(\chi, 1). \tag{2.3.2}$$

*Proof.* Suppose that $(a, q) = 1$. If $m$ runs through a complete system of residues modulo $q$, then so does $am$. Then

$$\tau(\chi, a) = \bar{\chi}(a) \sum_{m\, mod q} \chi(am)\, e\left(\frac{am}{q}\right) = \bar{\chi}(a) \sum_{n\, mod q} \chi(n)\, e\left(\frac{n}{q}\right) = \bar{\chi}(a)\tau(\chi, 1).$$

Now, suppose that $\chi$ is primitive and $(a, q) = k > 1$. Obviously that $\bar{\chi}(a) = 0$. Let $a = ka_1$, $q = kq_1$. There exists such an integer $b$ that $(b, q) = 1$, $b \equiv 1(mod\ q_1)$, $\chi(b) \neq 1$. Then

$$\chi(b)\tau(\chi, a) = \sum_{m\, mod q} \chi(bm)\, e\left(\frac{a_1 m}{q_1}\right) = \sum_{m\, mod q} \chi(bm)\, e\left(\frac{a_1 bm}{q_1}\right) = \tau(\chi, a).$$

It follows, that $\tau(\chi, a) = 0$, which establishes the claim of the lemma. ■

**Lemma 2.3.2.** Let $\chi$ be a Dirichlet character modulo q and $\chi^*$ its conductor modulo $q^*$. Then

$$\tau(\chi, 1) = \mu\left(\frac{q}{q^*}\right) \chi^*\left(\frac{q}{q^*}\right) \tau(\chi^*, 1) \qquad (2.3.3)$$

Moreover, if $\chi$ is primitive, then $|\tau(\chi, 1)| = \sqrt{q}$.

*Proof.* First, let $\chi$ be primitive. Summing (2.3.2) over all $a$ modulo $q$, we get

$$|\tau(\chi, 1)|^2 \sum_{a\ mod\ q} |\chi(a)|^2 = \sum_{a\ mod\ q} |\tau(\chi, a)|^2 =$$

$$\sum_{a= mod q} \sum_{m\, mod q} \chi(m) e(\frac{am}{q}) \sum_{n\, mod q} \bar{\chi}(n)\, e\left(-\frac{an}{q}\right) =$$

$$= \sum_{m\, mod q} \sum_{n\, mod q} \chi(m)\bar{\chi}(n) \sum_{a\, mod q} e\left(\frac{a(m-n)}{q}\right) \qquad (2.3.4)$$

We know that $\sum_{a\, mod q} \mathrm{e}\left(\frac{a(m-n)}{q}\right)$ is the sum of all group characters, corresponding to cyclic group of order $q$, so it is $q$ or 0 according as $m \equiv n(mod q)$ or not. Hence,

$$|\tau(\chi, 1)|^2 \sum_{a\ mod\ q} |\chi(a)|^2 = q \sum_{m\ mod\ q} |\chi(m)|^2,$$

thus, the second claim of lemma holds.

Now turn to general case. Using the properties of möbius function in Lemma 1.1.5, we can write the principal character $\chi_0$ modulo $q$ as $\chi_0(n) = \sum_{d|(n,q)} \mu(d)$. Thus,

$$\tau(\chi,1) = \sum_{m\ modq} \chi^*(m)\,\chi_0(m)\,e\left(\frac{m}{q}\right) = \sum_{m\ modq} \chi^*(m)\,e\left(\frac{m}{q}\right) \sum_{d|(m,q)} \mu(d)$$

We can change an order of summation in the last sum, so first we will over $d|q$ and then sum over $n\ modq$. But if $m = nd$ and $q = q_1 d \Leftrightarrow n\ modq = ld,\ \ l = 0,1,\dots\frac{q}{d} - 1\ \ \Leftrightarrow nd = n_1 dk + ld\ \ \Leftrightarrow n = n_1 k + l$. So, we can replace $n\ mod\ q$ by $n\ modn_1$ or $n\ mod\ q/d$:

$$\tau(\chi,1) = \sum_{d|q} \mu(d) \sum_{n\ mod\ q/d} \chi^*(nd)\,e\left(\frac{nd}{q}\right) =$$

$$= \sum_{d|q} \mu(d)\chi^*(d) \sum_{n\ mod\ q/d} \chi^*(n)\,e\left(\frac{nd}{q}\right).$$

Note that all terms in last sum will be zero if $(d, q^*) > 1$, so we may restrict the summation over d to the divisors of $q_0 = q/q^*$:

$$\tau(\chi,1) = \sum_{d|q_0} \mu(d)\chi^*(d) \sum_{n\ mod\ q/d} \chi^*(n)\,e\left(\frac{nd}{q}\right).$$

Now our aim is to represent $n\ moq\ q/d$ as $q^*v + u$. Let write $q = q_0 q^*$ and $\frac{q}{d} = \frac{q_0}{d} q^*$. Thus $v$ runs over a complete system of residues modulo $\frac{q_0}{d}$, and $u$ runs over a complete system of residues modulo $q^*$. We can write

$$\tau(\chi,1) = \sum_{d|q_0} \mu(d)\chi^*(d) \sum_{u\ mod\ q^*} \sum_{v\ mod\ q_0/d} \chi^*(q^*v + u)\,e\left(\frac{(q^*v + u)d}{q}\right) =$$

$$= \sum_{d|q_0} \mu(d)\chi^*(d) \sum_{u\ mod\ q^*} \chi^*(u) \exp(\frac{ud}{q}) \sum_{v\ mod\ q_0/d} e\left(\frac{vd}{q_0}\right).$$

Note that $\sum_{v\ mod\ q_0/d} e\left(\frac{vd}{q_0}\right)$ vanishes when $\frac{q_0}{d} > 1$, so $d = q_0$ and result follows. ■

### 2.4 The Pólya - Vinogradov inequality.

**Theorem 2.4.1. (Pólya – Vinogradov).** Suppose that $M$ and $N$ are positive integers and $\chi$ is a non-principal Dirichlet character modulo $q$. Then

$$\left|\sum_{M\le n\le M+N} \chi(n)\right| \le\ 2\sqrt{q} logq. \tag{2.4.1}$$

*Proof.* First, let $\chi$ be a primitive character. Then, by (2.3.2),

$$\tau(\bar{\chi},1)\sum_{M\le n\le M+N}\chi(n)=\sum_{M\le n\le M+N}\sum_{m\, mod q}\bar{\chi}(m)\, e\left(\frac{mn}{q}\right)=\sum_{m\, mod q}\bar{\chi}(m)\sum_{M\le n\le M+N}e\left(\frac{mn}{q}\right).$$

Applying Lemma 2.3.2 we can see that

$$\left|\sum_{M\le n\le M+N}\chi(n)\right|\le q^{-\frac{1}{2}}\sum_{m=1}^{q-1}\left|\sum_{M\le n\le M+N}e\left(\frac{mn}{q}\right)\right| \tag{2.4.2}$$

Modulus of $\sum_{M\le n\le M+N}e\left(\frac{mn}{q}\right)$ is equal to:

$$\left|\sum_{M\le n\le M+N}e\left(\frac{mn}{q}\right)\right|=\left|\frac{1-e(\frac{m(N+1)}{q})}{1-e(\frac{m}{q})}\right|\left|e(\frac{mM}{q})\right|=\left|\frac{1-e(\frac{m(N+1)}{q})}{1-e(\frac{m}{q})}\right|$$

Absolute value of $(1-e\left(\frac{m}{q}\right))$ is:

$$\left|1-e\left(\frac{m}{q}\right)\right|=\sqrt{(1-\cos(2\pi\left(\frac{m}{q}\right))^2+\sin(\pi\left(\frac{m}{q}\right))^2}=\sqrt{4\sin(\pi\left(\frac{m}{q}\right))^2}=2\left|\sin(\pi(\frac{m}{q})\right|.$$

Similarly we can calculate modulus of $(1-e\left(\frac{m(N+1)}{q}\right))$. Thus, the inequality (2.4.2) will transform to

$$\left|\sum_{M\le n\le M+N}\chi(n)\right|\le q^{1/2}\sum_{m=1}^{q-1}\csc(\pi\left(\frac{m}{q}\right).$$

Now we apply the inequality $\sin(\pi x)\ge 2x$ for $0<x\le 1/2$. When $q=2k$, we can write

$$\sum_{m=1}^{q-1}\csc(\pi\left(\frac{m}{q}\right)=\sum_{m=1}^{k-1}\csc(\pi\left(\frac{m}{q}\right)+\sum_{m=k+1}^{q-1}\csc(\pi\left(\frac{m}{q}\right)+1\le$$

$$\le q\sum_{m=1}^{k-1}\frac{1}{\mathrm{m}}+1\le q\sum_{m=1}^{k-1}\frac{2}{2\mathrm{m}-1}+1\le q\sum_{m=1}^{k-1}\log\left(\frac{2\mathrm{m}+1}{2\mathrm{m}-1}\right)+1=$$

$$=qloq(q-1)+1\le qlogq.$$

Here we used the fact, that $\log(x)$ is concave and $\log(x)\ge x-1$ for all $x>1$. When $q=2k+1$, we have

$$\sum_{m=1}^{q-1}\csc(\pi\left(\frac{m}{q}\right)\le q\sum_{m=1}^{k}\frac{1}{\mathrm{m}}\le q\sum_{m=1}^{k}\log\left(\frac{2\mathrm{m}+1}{2\mathrm{m}-1}\right)=qlogq.$$

This establishes the theorem for primitive characters.

If $\chi$ is induced by a primitive character $\chi^*$ modulo $r$, $r < q$, we have

$$\sum_{M \le n \le M+N} \chi(n) = \sum_{M \le n \le M+N} \chi^*(n) \sum_{d|(n,q)} \mu(d) = \sum_{d|q} \chi^*(d)\, \mu(d) \sum_{\frac{M}{d} \le m \le \frac{M+N}{d}} \chi^*(m).$$

The sum over m is bounded above by $r^{1/2} logr$, since $\chi^*$ is primitive. Thus,

$$\left| \sum_{M \le n \le M+N} \chi(n) \right| \le r^{\frac{1}{2}} logr \sum_{d|q} |\chi^*(d)| \le d\left(\frac{q}{r}\right) r^{\frac{1}{2}} logr,$$

where the terms with $(d, r) > 1$ are equal to 0. If we use that $d(n) \le 2\sqrt{n}$ from Lemma 1.3.4, then the result follows. ■

## Chapter 3

# AUXILIARY TOOLS IN ANALYTIC NUMBER THEORY

### 3.1 Some generalized counting functions and primes in arithmetic progressions.

**Definition 3.1.1.** For a Dirichlet character $\chi$ define

$$\psi(x,\chi) = \sum_{n\le x} \Lambda(n)\chi(n). \tag{3.1.1}$$

**Definition 3.2.2.** Define

$$\psi(x,q,a) = \sum_{\substack{n\le x \\ n\equiv a(modq)}} \Lambda(n).$$

**Lemma 3.1.1.** Suppose that $\chi$ is a character modulo $q$ induced by a primitive character $\chi^*$ modulo $r$, $1 < r < q$. Also let $\chi_0$ be a principal character modulo $q$. Then following three relations hold:

$$|\psi(x,\chi) - \psi(x,\chi^*)| \le (\log(qx))^2. \tag{3.1.2}$$

$$\psi(x,\chi_0) = x + x\exp(-const\sqrt{\log(x)}). \tag{3.1.3}$$

$$\psi(x,q,a) = \frac{1}{\phi(q)} \sum_{\chi\, modq} \bar{\chi}(a)\ \psi(x,\chi). \tag{3.1.4}$$

*Proof.* First we will prove (3.1.2). We have

$$|\psi(x,\chi) - \psi(x,\chi^*)| \le \sum_{\substack{n\le x \\ (n,q)>1}} \Lambda(n) \le \sum_{p|q} (\log(p)) \sum_{k:p^k\le x} 1 \le \log(q)\log(x) \le (\log(qx))^2,$$

which establishes (3.1.2). Using the Theorem 1.4.4 and (3.1.2) we can write for $\psi(x,\chi_0)$:

$$\psi(x,\chi_0) = \psi(x) + O((\log(x))^2) = x + x\exp(-const\sqrt{\log(x)}),$$

which proves (3.1.3). For right part of (3.1.4) we have

$$\frac{1}{\phi(q)} \sum_{\chi\, modq} \bar{\chi}(a)\ \psi(x,\chi) = \frac{1}{\phi(q)} \sum_{\chi\, modq} \bar{\chi}(a) \sum_{n\le x} \Lambda(n)\chi(n) =$$

$$\frac{1}{\phi(q)} \sum_{n\le x} \Lambda(n) \sum_{\chi\, modq} \bar{\chi}(a)\chi(n).$$

If denote the $\bar{r}$ as inverse of $r$ modulo $q$, and of we write $a$ as $lq + r$, then $\bar{\chi}(a) = \chi(\bar{r})$ and $\bar{\chi}(a)\chi(n) = \chi(n\bar{r})$. $\chi(n\bar{r}) \ne 0 \Leftrightarrow n \equiv r \equiv a(modq)$, and the right part of last equality will be equal to

$$\frac{1}{\phi(q)}\sum_{n\le x}\Lambda(n)\sum_{\chi\, mod q}\bar{\chi}(a)\chi(n)=\frac{1}{\phi(q)}\sum_{\substack{n\le x\\ n\equiv a(mod q)}}\Lambda(n)\,\phi(q),$$

and the result follows. ■

**Theorem 3.1.2 (Siegel-Walfisz).** Let $A$ be a real fixed constant and $(a,q)=1$, where $q\le(\log(x))^A$. Then exists a positive constant $C(A)$ depending only on $A$ such that

$$\psi(x,q,a)=\frac{x}{\phi(q)}+O(x\exp\left(-C(A)\sqrt{\log(x)}\right). \tag{3.1.5}$$

Theorem 3.1.2 describes the asymptotic law of distribution of prime numbers in arithmetic progressions with best known error term. More about Siegel-Walfisz theorem can be found in book [5] of Davenport.

## 3.2 Vaughan Identity

**Theorem 3.2.1 (Vaughan).** Suppose that $2\le U,V\le X$. Then

$$\sum_{U<n\le X}\Lambda(n)f(n)=\ \Sigma_1+\Sigma_2+\Sigma_3, \tag{3.2.1}$$

where

$$\Sigma_1=\sum_{m\le V}\sum_{U<mk\le X}\mu(m)(\log k)f(mk),\qquad \Sigma_2=\sum_{m\le UV}\sum_{U<mk\le X}a_m f(mk),$$

$$\Sigma_3=\sum_{m>U}\sum_{\substack{k>V\\ mk\le X}}\Lambda(m)b_k f(mk),$$

with coefficients $|a_m|\le\log(m)$ and $|b_k|\le d(k)$.

*Proof.* Let us write a trivial identity

$$\frac{-\zeta'(s)}{\zeta(s)}=L(s)-M(s)\zeta'(s)-L(s)M(s)\zeta(s)+(\frac{-\zeta'(s)}{\zeta(s)}-L(s))(1-M(s)\zeta(s)) \tag{3.2.2}$$

We can choose $L(s)$ and $M(s)$ to be arbitrary functions, in particular Dirichlet polynomials:

$$L(s)=\textstyle\sum_{n\le U}\Lambda(n)\,n^{-s}\ \text{ and }\ M(s)=\sum_{n\le V}\mu(n)n^{-s}.$$

Our task is to compare the coefficients of $n^{-s}$ in the Dirichlet series representations of the left and right sides of (3.2.2) to obtain an identity for $\Lambda(n)$. We have

$$\sum_{n=1}^{\infty}\Lambda(n)n^{-s}=\sum_{n\le U}\Lambda(n)n^{-s}-\sum_{n\le V}\mu(n)n^{-s}\sum_{n=1}^{\infty}(-lnn)n^{-s}-\sum_{n\le U}\Lambda(n)n^{-s}\sum_{n\le V}\mu(n)n^{-s}\sum_{n=1}^{\infty}n^{-s}$$

$$+\left(\sum_{n=1}^{\infty}\Lambda(n)n^{-s}-\sum_{n\le U}\Lambda(n)n^{-s}\right)\left(1-M(s)\zeta(s)\right)=$$

$$-\sum_{\substack{mk=n\\ m\le V}}\mu(m)(-lnk)n^{-s}-\sum_{\substack{uvk=n\\ u\le U,v\le V}}\Lambda(u)\mu(v)n^{-s}+\sum_{n>U}\Lambda(n)n^{-s}\left(\left(1-M(s)\zeta(s)\right)\right).$$

From Lemmas 1.4.8 and 1.1.5

$$\textstyle\sum_{n=1}^{\infty}\mu(n)n^{-s}=1/\zeta(s)\ \text{ and }\quad \sum_{d|n}\mu(d)=0,\ \text{ when } n>1.$$

Last summand from (3.2.2) can be written as

$$\sum_{n>U}\Lambda(n)\,n^{-s}\sum_{n=1}^{\infty}\Lambda(n)n^{-s}\sum_{n>V}\mu(n)n^{-s}=\sum_{n>U}\Lambda(n)\,n^{-s}\sum_{\substack{uv=n\\ u>V}}\mu(u)n^{-s}=$$

$$\sum_{n>U}\Lambda(n)\,n^{-s}\sum_{\substack{uv=n\\ u\le V,n>V}}-\mu(u)n^{-s}=\sum_{\substack{mk\\ m>u,k>V}}\Lambda(m)\,n^{-s}\Big(\sum_{\substack{uv=k\\ u\le V}}-\mu(u)\Big)n^{-s}.$$

So, for $n>u$ we obtain the following identity for $\Lambda(n)$:

$$\Lambda(n)=-\sum_{\substack{mk=n\\ m\le V}}\mu(m)(-lnk)-\sum_{\substack{uvk=n\\ u\le U,v\le V}}\Lambda(u)\mu(v)+\sum_{\substack{mk\\ m>u,k>V}}\Lambda(m)\,n^{-s}\Big(\sum_{\substack{uv=k\\ u\le V}}-\mu(u)\Big)n^{-s}.$$

Multiplying both sides by $f(n)$ and summing over $U<n\le X$, we obtain Vaughan identity with

$$a_m=\sum_{\substack{uv=m\\ u\le U,v\le V}}\Lambda(u)\mu(v),\qquad\qquad b_k=\sum_{\substack{uv=k\\ u\le V<uv}}\mu(u)$$

And, easy to see that $|a_m|\le\sum_{u/m}\Lambda(u)=\log(m)$ and $|b_k|\le d(k)$. ■

Vaughan identity was found by Vaughan in 1977, and can be used for decomposing sums over primes into double sums, which are easier to evaluate. One of the applications of the Vaughan identity is simplification of proof of Bombieri-Vinogradov's theorem; it will be demonstrated in Chapter 5 of this paper.

## Chapter 4

# ONE RESULT FROM THE LARGE SIEVE

### 4.1 Inequality, which is involving an exponential sum

**Theorem 4.1.1** Suppose that $a_{-N}, \dots, a_N$ – arbitrary complex numbers and

$$S(x) = \sum_{n=-N}^{N} a_n e(nx).$$

Moreover, let be $x_1, \dots, x_r$ – arbitrary real numbers and $\delta = \min_{j \neq k} \|x_j - x_k\|$, where operator $\|\cdot\|$ means distance to the closest integer. Then

$$\sum_{r=1}^{R} |S(x_r)|^2 \leq 2{,}2 \max(\delta^{-1}, 2N) \sum_{n=-N}^{N} |a_n|^2.$$

*Proof.* Let $\eta$ be an arbitrary real number, such that $0 < \eta \leq \left(\frac{1}{2}\right) \delta$, $b_n$ – real number, such that $b_{-n} = b_n$ and

$$\varphi(x) = \sum_{-\infty}^{\infty} b_n e(nx)$$

- arbitrary Fourier series, which contains only cosines, and for which series $\sum b_n^2$ converges, and function $\varphi(x) = 0$ when $\|x\| > \eta$. Also suppose, that $b_n \neq 0$ when $|n| \leq N$. Denote

$$T(x) = \sum_{-N}^{N} b_n^{-1} a_n e(nx).$$

Then $S(x)$ is convolution of functions $\varphi(x)$ and $T(x)$, it means

$$S(x) = \int_0^1 \varphi(y) T(x-y) dy \quad \text{or} \quad S(x) = \int_{-\eta}^{\eta} \varphi(y) T(x-y) dy.$$

After applying the Cauchy inequality we have

$$|S(x)|^2 \leq \left( \int_{-\eta}^{\eta} (\varphi(y))^2 \, dy \right) \left( \int_{-\eta}^{\eta} (|T(x-y)|)^2 dy \right) = \left( \int_{0}^{1} (\varphi(y))^2 \, dy \right) \left( \int_{x-\eta}^{x+\eta} (|T(z)|)^2 dz \right).$$

Let us replace now $x$ by $x_r$ and sum over all $r$. Intervals $(x_r - \eta, x_r + \eta)$ doesn't overlap $(mod\ 1)$ according to the definition of $\delta$ and assumption that $\eta \leq \left(\frac{1}{2}\right)\delta$. Thus,

$$\sum_{r=1}^{R} |S(x_r)|^2 \leq \left( \int_{0}^{1} (\varphi(y))^2 \, dy \right) \left( \int_{0}^{1} (|T(z)|)^2 dz \right) = \left( \sum_{-\infty}^{\infty} b_n^2 \right) \left( \sum_{-N}^{N} b_n^{-2} |a_n|^2 \right).$$

Now we can denote function $\varphi(x)$ by another way:

$$\varphi(x) = \begin{cases} \eta^{-1}(1 - \eta^{-1}\|x\|) & when\ \|x\| \leq 1 \\ 0 & otherwise. \end{cases}$$

Hence, coefficients $b_n$ can be calculated according to Fourier formulas:

$$b_n = \int_{0}^{1} \varphi(x) e(-nx) dx = 2\eta^{-1} \int_{0}^{\eta} (1 - \eta^{-1}x) \cos(2\pi n x) \, dx.$$

After integrating by parts we obtain that $b_n = (\frac{\sin(\pi n \eta)}{\pi n \eta})^2$. Moreover,

$$\sum_{-\infty}^{\infty} b_n^2 = \int_{0}^{1} (\varphi(x))^2 \, dx = 2 \int_{0}^{\eta} \eta^{-2} (1 - \eta^{-1}x)^2 dx = \frac{2}{3} \eta^{-1}.$$

So, previous inequality can be written as

$$\sum_{r=1}^{R} |S(x_r)|^2 \leq \frac{2}{3} \eta^{-1} \sum_{n=-N}^{N} \left( \frac{\pi n \eta}{\sin(\pi n \eta)} \right)^4 |a_n|^2.$$

Now suppose that $N\eta \leq 1/2$, hence coefficient near $|a_n|^2$ is maximal if $n = \pm N$, because function $x/\sin(x)$ is monotonically increasing when $0 \leq x \leq \pi/2$. Thus

$$\sum_{r=1}^{R} |S(x_r)|^2 \leq \frac{2}{3} \eta^{-1} \left( \frac{\pi n \eta}{\sin(\pi n \eta)} \right)^4 \sum_{n=-N}^{N} |a_n|^2.$$

The parameter $\eta$ has only two restrictions: $\eta \leq \left(\frac{1}{2}\right)\delta$ and $N\eta \leq 1/2$, in other cases it is still arbitrary, so denote $\pi n \eta = \theta$, then

$$\frac{2}{3}\eta^{-1}\left(\frac{\pi n\eta}{\sin(\pi n\eta)}\right)^4 = \frac{2}{3}\pi N\frac{\theta^3}{(\sin(\theta))^4},$$

Where $\theta$ is arbitrary number, satisfying two conditions $\theta \le \left(\frac{\pi}{2}\right)N\delta$ and $\theta \le \pi/2$.

Function $\theta^3/(\sin(\theta))^4$ is decreasing, when $\theta$ is increasing from 0 to $\theta_0$, where $\theta_0$ is the unique solution of equation $\tan\theta_0 = (\frac{4}{3})\theta_0$, such that $0 < \theta_0 < \pi/2$. If $\theta_0 \le \left(\frac{\pi}{2}\right)N\delta$ then denote $\theta = \theta_0$. Then

$$\frac{2}{3}\pi N\frac{\theta^3}{(\sin(\theta))^4} = \frac{2}{3}\pi N\frac{\theta_0^3}{(\sin(\theta_0))^4}.$$

If $\theta_0 > \left(\frac{\pi}{2}\right)N\delta$, then denote $\theta = \left(\frac{\pi}{2}\right)N\delta < \theta_0$, in this case

$$\frac{2}{3}\pi N\frac{\theta^3}{(\sin(\theta))^4} = \frac{4}{3}\delta^{-1}\left(\frac{\theta}{\sin(\theta)}\right)^4 < \frac{4}{3}\delta^{-1}\left(\frac{\theta_0}{\sin(\theta_0)}\right)^4.$$

After calculations we find that $\theta_0 = 0{,}8447$, it means that

$$\frac{2}{3}\pi N\frac{\theta_0^3}{(\sin(\theta_0))^4} < 2{,}2.$$

Theorem is proved. ■

**Lemma 4.1.2** Suppose that $M$ and $N$ are positive integers. Then

$$\sum_{q\le Q}\sum_{\substack{1\le b\le q\\(b,q)=1}}\left|\sum_{n=M+1}^{M+N}a_n e\left(\frac{bn}{q}\right)\right|^2 \le 2{,}2\max(N,Q^2)\sum_{n=M+1}^{M+N}|a_n|^2.$$

*Proof.* First we will replace variable of summation $n$ to $n'$, where $n'$ is running between integers $-L$ and $L$. Denote $L = \left(\frac{1}{2}\right)N$ or $\left(\frac{1}{2}\right)(N-1)$, if $N$ is even or odd, and $n = n' + M + {} +1 + N$. So, now $n'$ is running between $-N$ and $N$ or $N-1$, in the last case we can create extra zero coefficient.

Now we can apply Theorem 4.1.1. Numbers $x_1, \dots, x_r$ in this theorem will be rational numbers $a/q$, for which the smallest of possible denominators $q \le Q$. Now, if $a/q \ne a'/q'$ and we have

$$\left\|\frac{a}{q} - \frac{a'}{q'}\right\| \ge \frac{1}{qq'} \ge \frac{1}{Q^2}.$$

Hence, $\delta \ge Q^{-2}$ and lemma is proved. ■

### 4.2 Main lemma

**Lemma 4.2.1.** Suppose that $Q, M, N$ are positive integers. Then

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi^* mod q}\left|\sum_{n=M+1}^{M+N}a_n\chi(n)\right|^2 \ll (N+Q^2)\sum_{n=M+1}^{M+N}|a_n|^2,$$

Where $\chi^* mod q$ denotes a summation restricted to primitive characters.

*Proof.* When $\chi$ is primitive, from Lemmas 2.3.1 and 2.3.2 we can deduce that

$$q\left|\sum_{n=M+1}^{M+N}a_n\chi(n)\right|^2 = \left|\sum_{n=M+1}^{M+N}a_n\tau(\bar{\chi},n)\right|^2 = \left|\sum_{\substack{1\le b\le q\\(b,q)=1}}\bar{\chi}(b)S(\frac{b}{q})\right|^2,$$

Where $S(a)=\sum_{n=1}^{N}c_n e(an),\ \ |c_n|=|a_n|$. Hence

$$\frac{q}{\phi(q)}\sum_{\chi^* mod q}\left|\sum_{n=M+1}^{M+N}a_n\chi(n)\right|^2 \le \frac{1}{\phi(q)}\sum_{\chi\, mod q}\left|\sum_{\substack{1\le b\le q\\(b,q)=1}}\bar{\chi}(b)S\left(\frac{b}{q}\right)\right|^2 =$$

$$=\frac{1}{\phi(q)}\sum_{\substack{1\le b_1\le b_2\le q\\(b_1,b_2q)=1}}S(\frac{b_1}{q})\overline{S(\frac{b_2}{q})}\sum_{\chi\, mod q}\chi(b_2\bar{b}_1) = \sum_{\substack{1\le b\le q\\(b,q)=1}}\left|S\left(\frac{b}{q}\right)\right|^2,$$

Where $\bar{b}_1$ denotes the multiplicative inverse of $b_1$ modulo q. Thus, the result follows after applying the Lemma 4.1.2. ■

Lemma 4.2.1 plays an important role in estimating sums in Bombieri-Vinogradov theorem. More about large sieve reader can find in book of Davenport [5], p.151 and in Russian version of this book [6], p.147.

## Chapter 5

# BOMBIERI-VINOGRADOV THEOREM

### 5.1 Main theorem

Bombieri-Vinogradov theorem is extremely important result in analytic number theory, which was proved in 1965, and has various applications. Sometimes it is called just Bombieri theorem.

**Theorem 5.1.1 (Bombieri-Vinogradov).** Suppose that $Q$ is real number and $2 \leq Q \leq x$. Then for any fixed real number $A > 0$,

$$\sum_{q \leq Q} \max_{(a,q)=1} \max_{y \leq x} \left| \psi(y;q,a) - \frac{y}{\phi(q)} \right| \ll x(\log x)^{-A} + Qx^{\frac{1}{2}}(\log x)^5.$$

*Proof.* First define

$$\delta_\chi = \begin{cases} 1 & if\ \chi\ is\ principal, \\ 0 & otherwise. \end{cases}$$

By (3.1.4),

$$\psi(y;q,a) - \frac{y}{\phi(q)} = \frac{1}{\phi(q)} \sum_{\chi\ mod q} \bar{\chi}(a)\big(\psi(y,\chi) - \delta_\chi y\big), \tag{5.1.1}$$

hence

$$\max_{(a,q)=1} \left| \psi(y;q,a) - \frac{y}{\phi(q)} \right| \leq \frac{1}{\phi(q)} \sum_{\chi\ mod q} |\psi(y,\chi) - \delta_\chi y|.$$

Writing $\Sigma(x,Q)$ for the left side of (5.1.1), we find that

$$\Sigma(x,Q) \leq \sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{\chi\ mod q} \max_{y \leq x} |\psi(y,\chi) - \delta_\chi y| = \Sigma_0 + \Sigma_1,$$

where $\Sigma_0$ denotes the contribution from the principal characters and $\Sigma_1$ from all other characters. Lemma 1.3.9 says that

$$\sum_{n \leq z} (\phi(mn))^{-1} \leq (\phi(m))^{-1} \log z, \tag{5.1.2}$$

By (3.1.3) and (5.1.2) we have

$$\Sigma_0 \ll x\exp\left(-c_1\sqrt{\log x}\right)\sum_{q\le Q}\frac{1}{\phi(q)} \ll x(logx)^{-A},$$

Where $-c_1$ is the absolute constant. Let $\chi^*$ be the primitive character inducing $\chi$ modulo $q$. By (3.1.2),

$$\Sigma_1 \ll \sum_{q\le Q}\frac{1}{\phi(q)}\sum_{\chi\, modq}\max_{y\le x}|\psi(y,\chi^*)| + Q(\log x)^2.$$

Let $q = rq_1 \le Q$. Then we can estimate the sum over the characters by double sum, where the summation in first sum will be over all $r \le Q$, and summation in second will be over all primitive characters modulo $r$. We have

$$\Sigma_1 \ll \sum_{r\le Q}\sum_{\chi^* modr}\max_{y\le x}|\psi(y,\chi)|\sum_{q_1\le\frac{Q}{r}}\frac{1}{\phi(rq_1)} + Q(\log x)^2 \ll$$

$$\ll (\log x)\sum_{r\le Q}\frac{1}{\phi(r)}\sum_{\chi^*\, modr}\max_{y\le x}|\psi(y,\chi)| + Q(\log x)^2, \qquad (5.1.3)$$

where we have used (5.1.2) again. Now, our idea is to estimate the contribution from the "small" moduli $r$. For this we can represent $\psi(y;q,a)$ in (3.1.4) as in Siegel-Walfisz theorem and $\psi(y,\chi)$ as in (3.1.3) for principal characters. We have

$$\max_{y\le x}|\psi(y,\chi)| \ll x\exp\left(-c_2\sqrt{\log x}\right).$$

For all primitive characters to moduli $r \le (\log x)^{A+5} = Q_0$, say. Hence

$$\sum_{r\le Q_0}\frac{1}{\phi(r)}\sum_{\chi^*\, modr}\max_{y\le x}|\psi(y,\chi)| \ll xQ_0x\exp\left(-c_2\sqrt{\log x}\right) \ll x(\log x)^{-A-1}.$$

Combining this estimation and (5.1.3), we obtain

$$\Sigma_1 \ll (\log x)\Sigma_2 + x(\log x)^{-A} + Q(\log x)^2, \qquad (5.1.4)$$

where

$$\Sigma_2 = \sum_{Q_0<r\le Q}\frac{1}{\phi(r)}\sum_{\chi^*\, modr}\max_{y\le x}|\psi(y,\chi)|.$$

## 5.2 Decomposition into sums, using Vaughan identity

Let U be a parameter to regulate. We can apply (3.2.1) with $f(n) = \chi(n), X = y, U \leq x^{\frac{1}{2}}$:

$$\psi(y,\chi) = S_1(y,\chi) - S_2(y,\chi) - S_3(y,\chi) + \psi(U,\chi),$$

Where $S_j(y,\chi)$ is corresponding to the sum $\Sigma_j$ on the right side of Vaughan Identity. And,

$$\Sigma_2 \ll \Sigma_3 + \Sigma_4 + \Sigma_5 + QU,$$

Where

$$\Sigma_j = \sum_{Q_0<r\leq Q} \frac{1}{\phi(r)} \sum_{\chi^*\, mod r} \max_{y\leq x} |S_{j-2}(y,\chi)| \quad (j = 3,4,5)$$

Let estimate $\Sigma_3$ first. For $S_1(y,\chi)$ we can write inequality:

$$S_1(y,\chi) \leq \sum_{m\leq U} \left| \sum_{U<mk\leq y} (\log k)\chi(k) \right| \ll (\log y) \sum_{m\leq U} \left| \sum_{U<mk\leq y} \chi(k) \right|.$$

We can apply the Pólya-Vinogradov inequality for sum with characters, thus

$$\max_{y\leq x} |S_1(y,\chi)| \ll r^{\frac{1}{2}} U (\log x)^2.$$

And for $\Sigma_3$ we have:

$$\Sigma_3 \ll Q^{\frac{3}{2}} U (\log x)^2. \qquad (5.2.1)$$

Now we will use large sieve for estimating $\Sigma_5$. Suppose that $a_1, \ldots, a_M$ and $b_1, \ldots, b_M$ are complex numbers. By Lemma 4.2.1 and Cauchy's inequality:

$$\sum_{r\leq R} \frac{r}{\phi(r)} \sum_{\chi^* mod r} \left| \sum_{m=1}^{M} \sum_{k=1}^{K} a_m b_k \chi(mk) \right| \ll$$

$$\ll \left\{ \sum_{r\leq R} \frac{r}{\phi(r)} \sum_{\chi^* mod r} \left| \sum_{m=1}^{M} a_m \chi(m) \right|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{r\leq R} \frac{r}{\phi(r)} \sum_{\chi^* mod r} \left| \sum_{k=1}^{K} b_k \chi(k) \right|^2 \right\}^{\frac{1}{2}} \ll$$

$$\ll (M+R^2)^{1/2} (K+R^2)^{1/2} \left( \sum_{m=1}^{M} |a_m|^2 \right)^{1/2} \left( \sum_{k=1}^{K} |b_k|^2 \right)^{1/2}. \qquad (5.2.2)$$

Before applying (5.2.2) for bounding $\Sigma_5$, we would like to make summation in $S_3(y,\chi)$ more straightforward.

### 5.3 Main estimation

Interval $U < m \le xU^{-1}$ can be splitted into $O(\log x)$ subintervals $M < m \le M_1$ such that $M_1 \le 2M$. Then we can choose $M_1$ and $M$ such that

$$S_3(y,\chi) \ll (\log x)\left|\sum_{M<m\le M_1}\sum_{U<k\le \frac{y}{m}}\Lambda(m)b_k\chi(mk)\right|. \qquad (5.3.1)$$

Next, we apply Perron's formula (Corollary 1.2.7) with $a = (\log x)^{-1}$, $T = x^2$ and $u = \frac{y}{mk}$. We get

$$\sum_{M<m\le M_1}\sum_{U<k\le \frac{y}{m}}\Lambda(m)b_k\chi(mk) = \frac{1}{2\pi i}\int_{-T}^{T} S(\chi,t)\frac{y^{a+it}}{a+it}dt + O(\Delta), \qquad (5.3.2)$$

Where

$$S(\chi,t) = \sum_{M<m\le M_1}\sum_{U<k\le \frac{x}{M}}\Lambda(m)b_k\chi(mk)\,(mk)^{-a-it}, \qquad \Delta = \frac{y^a}{T}\sum_{M<m\le M_1}\sum_{k\le yM^{-1}}\frac{\Lambda(m)d(k)}{\left|\log(\frac{y}{mk})\right|}.$$

We can assume that $\{y\} = \frac{1}{2}$, then $\left|\log(\frac{y}{mk})\right| \ge \frac{1}{2}y^{-1}$. PNT and Theorem 1.3.6 (where we using just Dirichlet polynomial) give us

$$\Delta \ll x^{-1}\sum_{M<m\le M_1}\sum_{k\le yM^{-1}}\Lambda(m)d(k) \ll x^{-1}\sum_{M<m\le M_1}\Lambda(m)\sum_{k\le yM^{-1}}d(k) \ll$$

$$\ll x^{-1}\,yM^{-1}\log yM^{-1}\sum_{M<m\le M_1}\Lambda(m) \ll \log x.$$

Also clear that $\int_{-T}^{T}|a+it|^{-1}dt \ll \log x$. Returning to (5.3.2):

$$\sum_{M<m\le M_1}\sum_{U<k\le \frac{y}{m}}\Lambda(m)b_k\chi(mk) \ll (|S(x,t_0)|+1)\log x,$$

For some $|t_0| \le T$. Combining this inequality and (5.3.1), we get

$$\max_{y\le x}|S_3(y,\chi)| \ll (\log x)^2\,(\log x)^2(|S(x,t_0)|+1).$$

Hence,

$$\Sigma_5 \ll (\log x)^2\sum_{Q_0<r\le Q}\frac{1}{\phi(r)}\sum_{\chi^*\,mod r}|S(x,t_0)| + Q(\log x)^2. \qquad (5.3.3)$$

Now we can apply (5.2.2) for $\Sigma_5$, observing that

$$\sum_{M<m\le M_1} \Lambda(m)^2 m^{-2a} \le \sum_{M<m\le M_1} \Lambda(m)^2 \le \Lambda(M_1) \sum_{M<m\le M_1} \Lambda(m) \ll M \log x,$$

because of PNT and

$$\sum_{U<k\le xM^{-1}} d(k)^2\, k^{-2a} \le \sum_{U<k\le xM^{-1}} d(k)^2 \ll xM^{-1}(\log(xM^{-1}))^3 \le xM^{-1}(\log x)^3,$$

because of Theorem 1.3.6. Hence (5.2.2) yields

$$\sum_{r\le R} \frac{r}{\phi(r)} \sum_{\chi^*\, mod r} |S(x,t_0)| \ll (\log x)^2 \left(x + xRU^{-\frac{1}{2}} + x^{\frac{1}{2}}R^2\right).$$

From this inequality and (5.3.3) a new inequality can be derived for $\Sigma_5$ by using a dyadic decomposition:

$$\Sigma_5 \ll (\log x)^4 (xQ_0^{-1} + xU^{-\frac{1}{2}}\left(\log x) + x^{\frac{1}{2}}Q\right). \tag{5.3.4}$$

### 5.4 Completion of the proof

We can still regulate parameter $U$, now let $U = V \le x^{\frac{1}{3}}$. Then we can decompose $S_2(y,\chi)$ on two smaller sums, where in first sum $m \le U$ and in the second $U < m \le UV \le xU^{-1}$:

$$S_2(y,\chi) = \sum_{m\le U} \sum_{U<mk\le y} a_m \chi(mk) + \sum_{U<m\le xU^{-1}} \sum_{U<mk\le y} a_m \chi(mk) =$$

$$S_2{}'(y,\chi) + S_2{}''(y,\chi), \text{ say.}$$

We can estimate $S_2{}'(y,\chi)$ right now:

$$|S_2{}'(y,\chi)| \le (\log x) \sum_{m\le U} \left| \sum_{U<mk\le y} a_m \chi(k) \right|.$$

So , clearly $S_2{}'(y,\chi)$ can be bounded similarly to $S_1(y,\chi)$ , and $S_2{}''(y,\chi)$ similarly to $S_3(y,\chi)$, hence the contribution of $S_2{}'(y,\chi)$ and $S_2{}''(y,\chi)$ to $\Sigma_4$ can be estimated similarly to $\Sigma_3$ and to $\Sigma_5$ respectively. Omitting all calculations, we can write for $\Sigma_4$:

$$\Sigma_4 \ll (\log x)^4 \left(Q^{\frac{3}{2}}U + xQ_0^{-1} + xU^{-\frac{1}{2}} + x^{\frac{1}{2}}Q\right). \tag{5.4.1}$$

So finally $\Sigma(x,Q)$ can be estimated. For this we need to combine all bounds for $\Sigma_0 - \Sigma_5$, we get

$$\Sigma(x,Q) \le (\log x)^5 \left(Q^{\frac{3}{2}}U + xQ_0^{-1} + xU^{-\frac{1}{2}}(\log x) + x^{\frac{1}{2}}Q\right).$$

We can see that to satisfy the condition of theorem the next two inequalities must hold:

$$Q^{\frac{3}{2}}U \leq x^{\frac{1}{2}} \qquad \text{and} \qquad U^{-\frac{1}{2}}(\log x) \leq (\log x)^{-A-5} \ ,$$

Thus $(\log x)^{2A+12} \leq U \leq (\frac{x}{Q})^{1/2}$ and $Q \leq x(\log x)^{-4A-24}$, for which theorem is proved. Now suppose that $Q > x^{1/2}$. Consider the trivial bound for $\Sigma(x, Q)$:

$$\sum_{q\leq Q} \max_{(a,q)=1} \max_{y\leq x} \left|\psi(y; q, a) - \frac{y}{\phi(q)}\right| \ll \sum_{q\leq Q} \max_{(a,q)=1} \max_{y\leq x} |\Lambda(y)y| = \sum_{q\leq Q} \Lambda(x)x \leq Q\Lambda(x)x \leq$$

$$x^2 \log x,$$

which is better, then bound in the theorem for such $Q$. Bombieri-Vinogradov theorem is proved. ■

Proof of Bombieri-Vinogradov theorem introduced here is based on proof in [3], pp. 77-81. Alternative proof can be found in book of Huxley [7], pp.103-107

**Chapter 6**

# GOLDSTON-YILDIRIM-PINTZ THEOREM

## 6.1 Main theorem and preparations

**Theorem 6.1.1 (Goldston-Pintz-Yildirim).** Let $p_n$ be the $n$ –th prime. Then

$$\lim_{n\to\infty}\inf\left(\frac{p_{n+1}-p_n}{\log p_n}\right)=0.$$

*Proof*. First let define all parameters, which will play essential role in a proof. Let $N$ be an integer, which tends monotonically to infinity. Also define parameters $H, R, k, l$, where

$$H \ll \log N \ll \log R \le \log N,$$

and $k$ and $l$ are some given integer constants, where $k$ is large and $l$ is much smaller.

Let $\mathcal{H} = \{h_1, h_2, \dots, h_k\} \subseteq \{1, H\} \cap \mathbb{Z}$, where all $h_i$ are different. For a prime $p$ let $\Omega(p)$ be g the set of different residue classes among $-h(mod p), h \in \mathcal{H}$, and let write $n \in \Omega(p) \Leftrightarrow n(mod p) \in \Omega(p)$.

We will consider only those primes $p$, which are sufficiently large and $|\Omega(p)| = k < p$.

Next we extend $\Omega$ multiplicatively. Suppose that $d$ is square-free. Then we will write $n \in \Omega(d) \Leftrightarrow n \in \Omega(p)$ for all $p|d$, which is equivalent to condition $d|(n+h_1)(n+h_2)\dots(n+h_k)$.

Now we should define the weights, specific form of which will make our result to be true. First, let introduce functions

$$\lambda_R(d,a)=\begin{cases}0, & if\ d>R\\ \dfrac{1}{a!}\mu(d)\left(\log\dfrac{R}{d}\right)^a, & if\ d\le R\end{cases}$$

And

$$\Lambda_R(n,\mathcal{H},a)=\sum_{\substack{n\in\Omega(d)\\ d\le R}}\lambda_R(d,a)=\sum_{\substack{n\in\Omega(d)\\ d\le R}}\frac{1}{a!}\mu(d)\left(\log\frac{R}{d}\right)^a.$$

Also we will define auxiliary functions

$$\mathfrak{S}(\mathcal{H})=\prod_p\left(1-\frac{|\Omega(p)|}{p}\right)\left(1-\frac{1}{p}\right)^{-k}$$

And

$$w(n)=\begin{cases}\log n, & if\ is\ prime\\ 0, & otherwise.\end{cases}$$

Our main goal is to show that

$$\sum_{\substack{\mathcal{H}\subseteq[1,H]\\|\mathcal{H}|=k}}\sum_{N<n\le 2N}\left(\sum_{h\le H} w(n+h)-\log 3N\right)\lambda_R^2(n,\mathcal{H},k+l) \qquad (6.1.1)$$

is positive. When it is true then exists $n\in(N,2N]$ such that

$$\sum_{n\le H} w(n+h)-\log 3N>0.$$

That means that there exist two primes $p_r, p_{r+1}\in(N,2N+H]$ such that $p_{r+1}-p_r\le H$. If we will show that we can make $H=\varepsilon\log N$, where $\varepsilon$ is an arbitrary small number, then theorem will be proved. Positivity of sum in (6.1.1) follows from next three lemmas:

**Lemma 1.** Assume that $R\le N^{\frac{1}{2}}/(\log N)^c$ with some large $c>0$ depending on $k,l$. Then

$$\sum_{N<n\le 2N}\lambda_R^2(n,\mathcal{H},k+l)=\frac{\mathfrak{S}(\mathcal{H})}{(k+2l)!}\binom{2l}{l}N(\log R)^{k+2l}+O(N(\log N)^{k+2l-1}(\log\log N)^c).$$

Denote an error term above as $Err$.

**Lemma 2.** Assume, in addition to the above, that also $R\le N^{\sigma/2}$, where $\sigma$ is a parameter below $1/2$. Then

$$\sum_{N<n\le 2N}\lambda_R^2(n,\mathcal{H},k+l)\,w(n+h)$$

$$=\begin{cases}\dfrac{\mathfrak{S}(\mathcal{H}\cup\{h\})}{(k+2l)!}\dbinom{2l}{l}N(\log R)^{k+2l}+O(Err), & h\in\mathcal{H}\\[2ex] \dfrac{\mathfrak{S}(\mathcal{H})}{(k+2l+1)!}\dbinom{2(l+1)}{l+1}N(\log R)^{k+2l+1}+O(Err), & h\notin\mathcal{H}.\end{cases}$$

**Lemma 3 (Gallagher 1976).** Suppose that $H\to\infty$. Then

$$\sum_{\substack{\mathcal{H}\subseteq[1,H]\\|\mathcal{H}|=k}}\mathfrak{S}(\mathcal{H})=\big(1+O(1)\big)H^k,$$

where summation is over all ordered subsets of $[1,H]$ of length $k$.

Now we will combine all of this Lemmas to estimate sum in (6.1.1). Thus

$$\sum_{h\le H}\sum_{\substack{\mathcal{H}\subseteq[1,H]\\|\mathcal{H}|=k}}\sum_{N\le n\le 2N} w(n+h)\,\lambda_R^2(n,\mathcal{H},k+l)-\log 3N\sum_{\substack{\mathcal{H}\subseteq[1,H]\\|\mathcal{H}|=k}}\sum_{N<n\le 2N}\lambda_R^2(n,\mathcal{H},k+l)=$$

$$= \sum_{\substack{h\le H\\ h\notin\mathcal{H}}} \sum_{\substack{\mathcal{H}\subseteq[1,H]\\ |\mathcal{H}|=k}} \sum_{N\le n\le 2N} w(n+h)\,\lambda_R^2(n,\mathcal{H},k+l) + \sum_{\substack{h\le H\\ h\in\mathcal{H}}} \sum_{\substack{\mathcal{H}\subseteq[1,H]\\ |\mathcal{H}|=k}} \sum_{N\le n\le 2N} w(n+h)\,\lambda_R^2(n,\mathcal{H},k+l)$$

$$-\log 3N \sum_{\substack{\mathcal{H}\subseteq[1,H]\\ |\mathcal{H}|=k}} \sum_{N<n\le 2N} \lambda_R^2(n,\mathcal{H},k+l) =$$

$$= \sum_{\substack{h\le H\\ h\notin\mathcal{H}}} \sum_{\substack{\mathcal{H}\subseteq[1,H]\\ |\mathcal{H}|=k}} \frac{\mathfrak{S}(\mathcal{H}\cup\{h\})}{(k+2l)!}\binom{2l}{l} N(\log R)^{k+2l} + O(Err) +$$

$$+ \sum_{\substack{h\le H\\ h\in\mathcal{H}}} \sum_{\substack{\mathcal{H}\subseteq[1,H]\\ |\mathcal{H}|=k}} \frac{\mathfrak{S}(\mathcal{H})}{(k+2l+1)!}\binom{2(l+1)}{l+1} N(\log R)^{k+2l+1} + O(err) -$$

$$-\log 3N \sum_{\substack{\mathcal{H}\subseteq[1,H]\\ |\mathcal{H}|=k}} \frac{\mathfrak{S}(\mathcal{H})}{(k+2l)!}\binom{2l}{l} N(\log R)^{k+2l} + O(err) \sim$$

$$\sim \frac{H^{k+1}}{(k+2l)!}\binom{2l}{l} N(\log R)^{k+2l} + \frac{kH^k}{(k+2l+1)!}\binom{2(l+1)}{l+1} N(\log R)^{k+2l+1} -$$

$$-\log 3N \frac{H^k}{(k+2l)!}\binom{2l}{l} N(\log R)^{k+2l} =$$

$$\frac{H^k}{(k+2l)!}\binom{2l}{l} N(\log R)^{k+2l}\left(H + \frac{k}{k+2l+1}\frac{(2l+2)(2l+1)}{l+1}\log R - \log 3N\right). \quad (6.1.2)$$

We can choose in (6.1.2) $R = N^{\sigma/2}$, $l = [\sqrt{k}]$ and $H = \varepsilon \log N$. Then (6.1.2) will turn to

$$\frac{H^k}{(k+2l)!}\binom{2l}{l} N(\log R)^{k+2l} \log N\,(\varepsilon + 2\sigma + o(1) - 1 + o(1).$$

So with large $N$ we can choose $o(1)$ to be arbitrary small, and $\sigma$ to be arbitrary close to $1/2$, hence $\varepsilon$ can be chosen arbitrary small to make sum in (6.1.1) positive, and statement of the theorem follows. Now we will prove Lemma 1.

### 6.2 Proof of Lemma 1.

*Proof.* Let $k + l = a$. Then

$$\sum_{N<n\le 2N} \lambda_R^2(n,\mathcal{H},k+l) = \sum_{N\le n\le 2N}\left(\sum_{d:n\in\Omega(d)} \lambda_R(d,a)\right)^2 =$$

$$= \sum_{d_1 d_2} \lambda_R(d_1,a)\,\lambda_R(d_2,a) \sum_{\substack{N<n\le 2N \\ n\in\Omega(d_1), n\in\Omega(d_2)}} 1 = \sum_{d_1 d_2} \lambda_R(d_1,a)\,\lambda_R(d_2,a) \sum_{\substack{N<n\le 2N \\ n\in\Omega(\mathrm{LCM}(d_1,d_2))}} 1 =$$

$$= \sum_{d_1 d_2} \lambda_R(d_1,a)\,\lambda_R(d_2,a) \left|\Omega\left(\mathrm{LCM}(d_1,d_2)\right)\right| \left(\frac{N}{\mathrm{LCM}(d_1,d_2)} + O(1)\right) =$$

$$= N \sum_{d_1 d_2} \lambda_R(d_1,a)\,\lambda_R(d_2,a) \frac{\left|\Omega\left(\mathrm{LCM}(d_1,d_2)\right)\right|}{\mathrm{LCM}(d_1,d_2)} +$$

$$+ O\left(\sum_{d_1 d_2} \lambda_R(d_1,a)\,\lambda_R(d_2,a) \left|\Omega\left(\mathrm{LCM}(d_1,d_2)\right)\right|\right).$$

Since $\left|\Omega\left(\mathrm{LCM}(d_1,d_2)\right)\right| \le |\Omega(d_1)||\Omega(d_2)|$ and $|\Omega(d)| \le \tau_k(d)$, and Theorem 1.3.6 the error term in the sum above can be written as:

$$O\left(\sum_{d_1 d_2} \lambda_R(d_1,a)\,\lambda_R(d_2,a) \left|\Omega\left(\mathrm{LCM}(d_1,d_2)\right)\right|\right) \le \sum_{d\le R^2} (\tau_k(d)\lambda_R(d,a))^2 = O(R^2(\log R)^c).$$

Define $[d_1 d_2] = \mathrm{LCM}(d_1,d_2)$. Then the sum in Lemma 1 is $NT + O(R^2(\log R)^c)$, where

$$T = \sum_{d_1 d_2} \lambda_R(d_1,a)\,\lambda_R(d_2,a) \frac{|\Omega([d_1 d_2])|}{[d_1 d_2]}.$$

We can write $\lambda_R(d,a) = \frac{\mu(d)}{2\pi i} \int_{(1)} \left(\frac{R}{d}\right)^s \frac{ds}{s^{a+1}}$, so for $T$ we have:

$$T = \frac{1}{(2\pi i)^2} \int_{(1)} \int_{(1)} \left(\sum_{d_1,d_2} \frac{|\Omega([d_1 d_2])|}{[d_1 d_2] d_1^{s_1} d_2^{s_2}}\right) \frac{R^{s_1+s_2}}{(s_1 s_2)^{a+1}} ds_1 ds_2.$$

Define $F(s_1,s_2,\Omega) = \sum_{d_1,d_2} \frac{|\Omega([d_1 d_2])|\mu_1(d)\mu_2(d)}{[d_1 d_2] d_1^{s_1} d_2^{s_2}}$. Using properties of Riemann zeta function and multiplicativity of $\Omega(\mathrm{d})$ and $[d_1 d_2]$, we can write the Euler product for $F(s_1,s_2,\Omega)$:

$$F(s_1,s_2,\Omega) = \prod_p \left(1 - \frac{|\Omega(p)|}{p}\left(\frac{1}{p^{s_1}} + \frac{1}{p^{s_2}} - \frac{1}{p^{s_1+s_2}}\right)\right).$$

Also define our main auxiliary function in this proof:

$$G(s_1,s_2,\Omega) = F(s_1,s_2,\Omega)\left(\frac{\zeta(s_1+1)\zeta(s_2+1)}{\zeta(s_1+s_2+1)}\right)^k.$$

The Euler product for $G(s_1, s_2, \Omega)$ is:

$$\prod_p \left(1 - \frac{|\Omega(p)|}{p^{s_1+1}} + \frac{|\Omega(p)|}{p^{s_2+1}} + \frac{|\Omega(p)|}{p^{s_1+s_2+1}}\right)\left(1 - \frac{1}{p^{s_1+1}}\right)^{-k}\left(1 - \frac{1}{p^{s_2+1}}\right)^{-k}\left(1 - \frac{1}{p^{s_1+s_2+1}}\right)^{-k}.$$

We assumed earlier that $|\Omega(p)| = k$, hence the logarithm of the corresponding Euler factor is

$$\log\left(1 - \frac{\mathrm{k}}{p^{s_1+1}} + \frac{\mathrm{k}}{p^{s_2+1}} + \frac{\mathrm{k}}{p^{s_1+s_2+1}}\right) - k\log\left(1 - \frac{1}{p^{s_1+1}}\right) - k\log\left(1 - \frac{1}{p^{s_2+1}}\right) +$$

$$+k\log\left(1 - \frac{1}{p^{s_1+s_2+1}}\right), \qquad \text{for } Res_1, Res_2 \geq c_1 > -\frac{1}{2}.$$

This logarithmic factor is equal to $O(p^{-4c_1-2})$. Hence for $0 \geq c_1 \geq -\frac{1}{4}$ and $Res_1, Res_2 \geq c_1$ the sum of this logarithms converge absolutely, i.e. $G(s_1, s_2, \Omega)$ is holomorphic in this region. Moreover, we will note that $G(0,0,\Omega) = \mathfrak{S}(\mathcal{H})$. For bounding $G(s_1, s_2, \Omega)$ we will estimate its logarithmic sum:

$$\sum_{p \leq H}\left(\log\left(1 - \frac{|\Omega(p)|}{p^{s_1+1}} + \frac{|\Omega(p)|}{p^{s_2+1}} + \frac{|\Omega(p)|}{p^{s_1+s_2+1}}\right) - k\log\left(1 - \frac{1}{p^{s_1+1}}\right) - k\log\left(1 - \frac{1}{p^{s_2+1}}\right) +\right.$$

$$\left. + k\log\left(1 - \frac{1}{p^{s_1+s_2+1}}\right)\right) \ll O(1) + \sum_{p \leq H} \frac{1}{p^{2\min(Res_1, Res_2, 0)+1}} \leq$$

$$\leq H^{-2\sigma} \sum_{p \leq H} \frac{1}{p} \ll H^{-2\sigma} \log\log H \ll (\log N)^{-2\sigma} \log\log\log N,$$

Where $\sigma = \min(Res_1, Res_2, 0) \geq c_1 > -\frac{1}{4}$ and $\sum_{p \leq H} \frac{1}{p} \ll \log\log H$ by Merten's theorem 1.1.11.

Hence $G(s_1, s_2, \Omega) \ll \exp(O((\log N)^{-2\sigma}) \log\log\log N)$. So, again consider the T as

$$T = \frac{1}{(2\pi i)^2} \int_{(1)} \int_{(1)} \left(G(s_1, s_2, \Omega)\left(\frac{\zeta(s_1 + s_2 + 1)}{\zeta(s_1 + 1)\zeta(s_2 + 1)}\right)^k\right) \frac{R^{s_1+s_2}}{(s_1 s_2)^{a+1}} ds_1 ds_2.$$

Now shift a contour from $Res_1 = 1$ to $Res_1 = \frac{c_0}{\log U}$ and from $Res_2 = 1$ to $Res_2 = \frac{c_0}{2\log U}$, where $U = \exp(\sqrt{\log N})$. Also we truncate $s_1$- integral to $|Ims_1| \leq U$ and $s_2$- integral to $|Ims_2| \leq U/2$, and denote the results by $L_1$ and $L_2$ respectively. We didn't obtain any new poles while shifting, so we only need to calculate truncated parts. Thus

$$G(s_1, s_2, \Omega) \ll \exp(O((\log N)^{-2Res_1}) \log\log\log N) \ll \log\log N^{o(1)} \tag{6.2.1}$$

By Theorem 1.4.11 we have

$$\zeta(s_1 + s_2 + 1) \ll \sqrt{\log N}, \tag{6.2.2}$$

$$|\zeta(s_1+1)^{-1}| \ll \sqrt{\log N} \quad , \quad |\zeta(s_2+1)^{-1}| \ll \sqrt{\log N}. \tag{6.2.3}$$

$$|R^{s_1+s_2}| = |R^{\sigma_1+\sigma_2}| = \exp\left(\frac{(\sigma_{1+}\sigma_2)}{\log R}\right) = \exp(O\left(\frac{1}{\sqrt{\log N}}\right) O(\log N) \ll \exp\left(\sqrt{\log N}\right). \tag{6.2.4}$$

Combining (6.2.1)-(6.2.4) we have

$$\left|\left(G(s_1,s_2,\Omega)\left(\frac{\zeta(s_1+s_2+1)}{\zeta(s_1+1)\zeta(s_2+1)}\right)^k\right) R^{s_1+s_2}\right| \le \exp\left(c\sqrt{\log N}\right), \tag{6.2.5}$$

For some constant $c > 0$. So, we can write for truncated error:

$$\frac{1}{(2\pi i)^2} \iint\limits_{\substack{truncated\\ domain}} \left(G(s_1,s_2,\Omega)\left(\frac{\zeta(s_1+s_2+1)}{\zeta(s_1+1)\zeta(s_2+1)}\right)^k\right) \frac{R^{s_1+s_2}}{(s_1s_2)^{a+1}} ds_1 ds_2 \ll$$

$$\ll \exp\left(c\sqrt{\log N}\right) \iint\limits_{\substack{(x,y)\in\mathbb{R}^2\\ \max(|x|,|y|)=r>\frac{U}{2}}} \max(|x|,|y|)^{-a-1}\, dxdy \ll$$

$$\ll \exp\left(c\sqrt{\log N}\right) \int\limits_{U/2}^{\infty} r r^{-a-1} dr \ll \exp\left(c\sqrt{\log N}\right) U^{-a} \ll \exp(-c\log N), \tag{6.2.6}$$

for a sufficiently large $a$. Now let fix $s_2$ and shift a contour from $Res_1 = \frac{c_0}{\log U}$ to $Res_1 = \frac{-c_0}{\log U}$ and denote the result by $L_3$. For calculating the error after shifting, we should admit that $G(s_1,s_2,\Omega)$ is still holomorphic at the $L_3$, hence bounded, $\zeta(s_1+s_2+1), \zeta(s_1+1), \zeta(s_2+1)$ are the powers of $\log N$ by Theorem 1.4.12, term $(s_1s_2)^{-a-1}$ is also bounded here, and $R^{s_1+s_2}$ makes an error small:

$$|R^{s_1+s_2}| = R^{\sigma_1+\sigma_2} = R^{\frac{-c_0}{\log U}+\frac{-c_0}{2\log U}} = R^{\frac{-c_0}{\sqrt{\log N}}} = \exp\left(\frac{-c_0\log R}{\sqrt{\log N}}\right) = \exp\left(-c_1\sqrt{\log N}\right), \tag{6.2.7}$$

where $c_1$ is a positive constant. Similarly to the previous calculations the new error will be of $\exp\left(-c_1\sqrt{\log N}\right)$. Denoting

$$\left(G(s_1,s_2,\Omega)\left(\frac{\zeta(s_1+s_2+1)}{\zeta(s_1+1)\zeta(s_2+1)}\right)^k\right)\frac{R^{s_1+s_2}}{(s_1s_2)^{a+1}}$$

as $g(s_1,s_2)$ and applying (6.2.6) and (6.2.7) we can write for $T$:

$$T = \frac{1}{2\pi i}\int\limits_{L_2} \left(Res_{s_1=0}\left(g(s_1,s_2)\right) + Res_{s_1=-s_2}\left(g(s_1,s_2)\right)\right) ds_2 + O\exp\left(-c_1\sqrt{\log N}\right) \tag{6.2.8}$$

First, let calculate $Res_{s_1=-s_2}(g(s_1,s_2))$. For this we will create a circle $C(s_2)$ with center in $-s_2$ with sufficiently small radius. Then on $C(s_2)$:

$$Res_2 = \frac{c_0}{2\sqrt{\log N}} \quad , \quad Res_1 \geq \frac{-c_0}{2\sqrt{\log N}} - \frac{1}{\log N},$$

So $\sigma = \min(Res_1, Res_2, 0) \geq \frac{-c_2}{\sqrt{\log N}}$. Hence

$$G(s_1, s_2, \Omega) \leq (\log\log N)^{O(1)}. \tag{6.2.9}$$

Also on $C(s_2)$

$$\zeta(s_1 + s_2 + 1) \leq \log N. \tag{6.2.10}$$

Our task is to exclude the $s_1$ from further calculations of $Res_{s_1=-s_2}(g(s_1, s_2))$. Thus by Theorem 1.4.13 we have

$$(\zeta(s_1+1)s_1)^{-1}, (\zeta(s_2+1)s_2)^{-1} \ll \frac{\log(|s_2|+2)}{|s_2|+1}. \tag{6.2.11}$$

Hence, applying (6.2.9)-(6.2.11)

$$Res_{s_1=-s_2}(g(s_1, s_2)) \ll (\log\log N)^{O(1)} (\log N)^{k-1} \left(\frac{\log(|s_2|+2)}{|s_2|+1}\right)^{2k} \frac{1}{|s_2|^{2l+2}}. \tag{6.2.12}$$

To calculate the contribution of $Res_{s_1=-s_2}(g(s_1, s_2))$ in (6.2.8) we will write

$$\int_{L_2} \left(\frac{\log(|s_2|+2)}{|s_2|+1}\right)^{2k} \frac{1}{|s_2|^{2l+2}} ds_2 = \int_{\substack{Res_2 = \frac{c_0}{2\sqrt{\log N}} \\ Ims_2 \leq \frac{c_0}{2\sqrt{\log N}}}} \left(\frac{\log(|s_2|+2)}{|s_2|+1}\right)^{2k} \frac{1}{|s_2|^{2l+2}} ds_2 +$$

$$+ \int_{\substack{Res_2 = \frac{c_0}{2\sqrt{\log N}} \\ \frac{c_0}{2\sqrt{\log N}} < Ims_2 \leq 1}} \left(\frac{\log(|s_2|+2)}{|s_2|+1}\right)^{2k} \frac{1}{|s_2|^{2l+2}} ds_2 + \int_{\substack{Res_2 = \frac{c_0}{2\sqrt{\log N}} \\ Ims_2 > 1}} \left(\frac{\log(|s_2|+2)}{|s_2|+1}\right)^{2k} \frac{1}{|s_2|^{2l+2}} ds_2.$$

For the first integral in the sum above the following estimation will be true:

$$\int_{\substack{Res_2 = \frac{c_0}{2\sqrt{\log N}} \\ Ims_2 \leq \frac{c_0}{2\sqrt{\log N}}}} \left(\frac{\log(|s_2|+2)}{|s_2|+1}\right)^{2k} \frac{1}{|s_2|^{2l+2}} ds_2 \ll \int_{\substack{Res_2 = \frac{c_0}{2\sqrt{\log N}} \\ Ims_2 \leq \frac{c_0}{2\sqrt{\log N}}}} \frac{1}{|s_2|^{2l+2}} ds_2 \ll$$

$$\ll \frac{1}{\sqrt{\log N}} \left(\sqrt{\log N}\right)^{2l+2} = (\log N)^{l+\frac{1}{2}}. \tag{6.2.13}$$

For the second integral we have:

$$\int\limits_{\substack{Res_2=\frac{c_0}{2\sqrt{\log N}}\\ \frac{c_0}{2\sqrt{\log N}}<Ims_2\le 1}} \left(\frac{\log(|s_2|+2)}{|s_2|+1}\right)^{2k} \frac{1}{|s_2|^{2l+2}} ds_2 \ll \int\limits_{\substack{Res_2=\frac{c_0}{2\sqrt{\log N}}\\ \frac{c_0}{2\sqrt{\log N}}<Ims_2\le 1}} \frac{1}{|s_2|^{2l+2}} ds_2 \ll$$

$$\int\limits_{\frac{1}{\sqrt{\log N}}}^{1} \frac{dt}{t^{2l+2}} \ll \left(\sqrt{\log N}\right)^{2l+1} = (\log N)^{l+\frac{1}{2}}. \tag{6.2.14}$$

And the third integral can be bounded in the following way:

$$\int\limits_{\substack{Res_2=\frac{c_0}{2\sqrt{\log N}}\\ Ims_2>1}} \left(\frac{\log(|s_2|+2)}{|s_2|+1}\right)^{2k} \frac{1}{|s_2|^{2l+2}} ds_2 \ll \int\limits_{1}^{\infty} \frac{(\log(t+2))^{2k}}{t^{2k+2l+2}} dt \ll 1. \tag{6.2.15}$$

Combining (6.2.8) and (6.2.12)-(6.2.15) we have

$$T = \frac{1}{2\pi i} \int\limits_{L_2} Res_{s_1=0}\big(g(s_1,s_2)\big)\, ds_2 + (\log\log N)^{O(1)} (\log N)^{k+l-\frac{1}{2}}. \tag{6.2.16}$$

We will call $z(s_1,s_2)$ the following function:

$$z(s_1,s_2) = G(s_1,s_2,\Omega)\left(\frac{\zeta(s_1+s_2+1)(s_1+s_2)}{s_1\zeta(s_1+1)s_2\zeta(s_2+1)}\right)^k.$$

Note, that $z(s_1,s_2)$ is holomorphic around $(0,0)$ and $z(0,0) = G(0,0,\Omega) = \mathfrak{S}(\mathcal{H})$. Now shift the integration contour from $L_2 = \{Res_2 = \frac{c_0}{2\log U}, Ims_2 \le \frac{U}{2}\}$ to $L_4 = \{Res_2 = \frac{-c_0}{\log U}, Ims_2 \le \frac{U}{2}\}$. This way we get

$$T = Res_{s_2=0} Res_{s_1=0} z(s_1,s_2) \frac{R^{s_1+s_2}}{(s_1s_2)^{a+1}(s_1+s_2)^k} =$$

$$\frac{1}{2\pi i^2} \int\limits_{|s_2|=\rho} \int\limits_{|s_1|=2\rho} z(s_1,s_2) \frac{R^{s_1+s_2}}{(s_1s_2)^{a+1}(s_1+s_2)^k} ds_1 ds_2 + (\log\log N)^{O(1)} (\log N)^{k+l-\frac{1}{2}},$$

where $\rho$ is the positive small constant and error from shifting the contour can be calculated in similar way to (6.2.7), and is of order $O\big(\exp\big(-c_2\sqrt{\log N}\big)\big)$. Let change the variables in double integral above: $s_1 = s, s_2 = s\xi$. Then

$$\frac{1}{2\pi i^2} \int\limits_{|s_2|=\rho} \int\limits_{|s_1|=2\rho} z(s_1,s_2) \frac{R^{s_1+s_2}}{(s_1s_2)^{a+1}(s_1+s_2)^k} ds_1 ds_2 =$$

$$= \frac{1}{2\pi i^2} \int\limits_{|\xi|=2} \int\limits_{|s|=\rho} z(s, s\xi) \frac{R^{s+s\xi} s}{(ss\xi)^{a+1}(s+s\xi)^k} ds d\xi =$$

$$= \frac{1}{2\pi i^2} \int\limits_{|\xi|=2} \frac{1}{(1+\xi)^k \xi^{l+1}} \int\limits_{|s|=\rho} z(s, s\xi) \frac{R^{s(1+\xi)}}{s^{k+2l+1}} ds d\xi. \qquad (6.2.17)$$

To calculate the pole of $f(s) = \left(z(s, s\xi)\frac{R^{s(1+\xi)}}{s^{k+2l+1}}\right)$, we need to know the coefficient near $s^{-1}$ of Teylor series of $f(s)$ in neighborhood of zero, which has value $\frac{z(0,0)}{(k+2l)!}(1+\xi)^{k+2l}(\log R)^{k+2l} + \sum_{r=o}^{k+2l-1}(1+\xi)^r (\log R)^r f_r(\xi)$, where $|f_r(\xi)| \ll z(0,0) \ll (\log\log N^c)$. Substituting this value into (6.2.17) we get

$$\frac{1}{2\pi i^2} \int\limits_{|\xi|=2} \frac{1}{(1+\xi)^k \xi^{l+1}} \int\limits_{|s|=\rho} z(s, s\xi) \frac{R^{s(1+\xi)}}{s^{k+2l+1}} ds d\xi =$$

$$= \frac{1}{2\pi i} \int\limits_{|\xi|=2} \frac{(1+\xi)^{2l}}{\xi^{l+1}} \frac{z(0,0)}{(k+2l)!} (\log R)^{k+2l} d\xi + O(\log N^{k+2l-1} \log\log N^c) =$$

$$\frac{z(0,0)}{(k+2l)!} (\log R)^{k+2l} \frac{1}{2\pi i} \int\limits_{|\xi|=2} \frac{(1+\xi)^{2l}}{\xi^{l+1}} d\xi + O(\log N^{k+2l-1} \log\log N^c). \qquad (6.2.18)$$

We observe that $\frac{(1+\xi)^{2l}}{\xi^{l+1}} = \sum_{m=o}^{\infty}\binom{2l}{m}\xi^{m-l-1}$, hence the pole of this function and integral in (6.2.18) will be equal to $\binom{2l}{l}$. Hence

$$T = \frac{z(0,0)}{(k+2l)!}\binom{2l}{l}(\log R)^{k+2l} + O(\log N^{k+2l-1} \log\log N^c).$$

Remembering that the sum in Lemma 1 is $NT + O(R^2(\log R)^c)$, we finishing proof of Lemma 1.

■

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### 6.3 Proof of Lemma 2.

*Proof.* We should make remark, that the sum in Lemma 2 doesn't change if we replace $\mathcal{H}$ by $\mathcal{H}\backslash\{h\}$. This is because if $w(n+h)$ is not a zero then $n+h$ is prime and

$$\Lambda_R(n, \mathcal{H}, a) = \sum_{\substack{d|(n+h_1)(n+h_2)\dots(n+h_k) \\ d \le R}} \frac{1}{a!}\mu(d)\left(\log\frac{R}{d}\right)^a,$$

so factor $(n+h)$ doesn't make a contribution to the sum above. We can assume that $\mathcal{H}$ doesn't contain $h$. The sum in Lemma 2 is:

$$\sum_{N<n\le 2N} \lambda_R^2(n,\mathcal{H},k+l)\, w(n+h) = \sum_{d_1 d_2} \lambda_R(d_1,k+l)\,\lambda_R(d_2,k+l) \sum_{\substack{N<n\le 2N \\ n\in\Omega(d_1), n\in\Omega(d_2)}} w(n+h) =$$

$$= \sum_{d_1 d_2} \lambda_R(d_1,k+l)\,\lambda_R(d_2,k+l) \sum_{b\in\Omega([d_1,d_2]} \sum_{\substack{N\le n\le 2N \\ n\equiv b \bmod([d_1,d_2])}} w(n+h). \tag{6.3.1}$$

In (6.3.1) $[d_1,d_2]$ denotes the $LCM(d_1,d_2)$. The inner sum from above is

$$\sum_{\substack{N\le n\le 2N \\ n\equiv b \bmod([d_1,d_2])}} w(n+h) = \sum_{\substack{N+h\le \tilde{n}\le 2N+h \\ \tilde{n}\equiv b+h \bmod([d_1,d_2])}} w(\tilde{n}+h) = \sum_{\substack{N\le \tilde{n}\le 2N \\ \tilde{n}\equiv b+h \bmod([d_1,d_2])}} w(\tilde{n}) +$$

$$+O(h\log N) = \sum_{\substack{N\le \tilde{n}\le 2N \\ \tilde{n}\equiv b+h \bmod([d_1,d_2])}} w(\tilde{n}) + O((\log N)^2). \tag{6.3.2}$$

The errors after replacing the (6.3.2) in (6.3.1) will add up to

$$\sum_{d_1 d_2\le R} (\log R)^{2k+2l} |\Omega([d_1 d_2])| O((\log N)^2) =$$

$$O((\log N)^c) \sum_{d_1 d_2\le R} |\Omega([d_1 d_2])| = O((\log N)^c) \sum_{\substack{d\le R^2 \\ d-squarefree}} |\Omega(d)| \sum_{\substack{d_1 d_2\le R \\ [d_1 d_2]=d}} 1 =$$

$$= ((\log N)^c) \sum_{d\le R^2} \tau_k(d)\tau_3(d) = O(R^2(\log n)^c).$$

Main part in (6.3.1) will be equal to

$$\sum_{d_1 d_2} \lambda_R(d_1,k+l)\,\lambda_R(d_2,k+l) \sum_{\substack{b\in\Omega([d_1,d_2] \\ (b+h,[d_1,d_2])=1}} \sigma^*(N,b+h,[d_1,d_2]),$$

where $\sigma^* = \sum_{\substack{y<n\le 2y \\ n\equiv a(\bmod q)}} w(n)$. So, the sum in (6.3.1) will equal to

$$\sum_{d_1 d_2} \lambda_R(d_1,k+l)\,\lambda_R(d_2,k+l) \sum_{\substack{b\in\Omega([d_1,d_2] \\ (b+h,[d_1,d_2])=1}} \frac{N}{\varphi([d_1,d_2])}$$

plus an error term of size

$$\sum_{d_1 d_2} \lambda_R(d_1, k+l)\, \lambda_R(d_2, k+l) \sum_{\substack{b \in \Omega([d_1, d_2]) \\ (b+h, [d_1, d_2]) = 1}} \left| \sigma^*(N, b+h, [d_1, d_2]) - \frac{N}{\varphi([d_1, d_2])} \right| \le$$

$$\le (\log N)^{2k+2l} \sum_{d \le R^2} \tau_3(d) \sum_{\substack{b \in \Omega(d) \\ (b+h, d) = 1}} \left| \sigma^*(N, b+h, d) - \frac{N}{\varphi(d)} \right|. \qquad (6.3.3)$$

Now we will decompose sum in (6.3.3) into two sums, where in first sum $\tau_k(d) \le (\log N)^{A/3}$ and in the second $\tau_k(d) > (\log N)^{A/3}$ for $A$, which should be sufficiently large. For a first sum we have:

$$(\log N)^{2k+2l} \sum_{\substack{d \le R^2 \\ \tau_k(d) \le (\log N)^{\frac{A}{3}}}} \tau_3(d) \sum_{\substack{b \in \Omega(d) \\ (b+h, d) = 1}} \left| \sigma^*(N, b+h, d) - \frac{N}{\varphi(d)} \right| \le$$

$$\le (\log N)^{\frac{2A}{3}} \sum_{d \le R^2} \max_{\substack{b \bmod (d) \\ (b+h, d) = 1}} \left| \sigma^*(N, b+h, d) - \frac{N}{\varphi(d)} \right|. \qquad (6.3.4)$$

For estimating the (6.3.4) Bombieri-Vinogradov theorem can be applied. For $R^2 \le N^\sigma$ we have:

$$(\log N)^{\frac{2A}{3}} \sum_{d \le R^2} \max_{\substack{b \bmod (d) \\ (b+h, d) = 1}} \left| \sigma^*(N, b+h, d) - \frac{N}{\varphi(d)} \right| \ll (\log N)^{\frac{2A}{3}} \frac{N}{(\log N)^A} = \frac{N}{(\log N)^{\frac{A}{3}}}. \qquad (6.3.5)$$

For a second sum we can write:

$$(\log N)^{2k+2l} \sum_{\substack{d \le R^2 \\ \tau_k(d) > (\log N)^{\frac{A}{3}}}} \tau_3(d) \sum_{\substack{b \in \Omega(d) \\ (b+h, d) = 1}} \left| \sigma^*(N, b+h, d) - \frac{N}{\varphi(d)} \right| \le$$

$$O(\log N)\, (\log N)^{2k+2l} \sum_{d \le R^2} \frac{\tau_k(d) \tau_3(d) \Omega(d)}{(\log N)^{\frac{A}{3}} d} \ll (\log N)^{2k+2l+1-\frac{A}{3}} \sum_{d \le N} \frac{\tau_3(d)}{d} \ll$$

$$\ll \frac{N}{(\log N)^{\frac{A}{3}}}. \qquad (6.3.6)$$

It means that the sum in Lemma 2 is

$$NT^* + O\left( \frac{N}{(\log N)^{\frac{A}{3}}} \right), \qquad (6.3.7)$$

where

$$T^* = \sum_{d_1 d_2} \frac{\lambda_R(d_1, k+l)}{\varphi([d_1, d_2])} \lambda_R(d_2, k+l) \sum_{\substack{b \in \Omega([d_1,d_2]) \\ (b+h,[d_1,d_2])=1}} 1$$

The innermost sum is by Chinese theorem:

$$\sum_{\substack{b \in \Omega([d_1,d_2]) \\ (b+h,[d_1,d_2])=1}} 1 = \prod_{p|[d_1,d_2]} \sum_{\substack{b \in \Omega(p) \\ (b+h,p)=1}} 1 = \prod_{p|[d_1,d_2]} (|\Omega^+(p)| - 1),$$

where $|\Omega^+(p)|$ is the cardinality of the set $\{-h_{1,} -h_2, \dots, -h_k, -h \, mod p\}$. As before $|\Omega^+(p)| = = k + 1$. We can calculate $T^*$ similarly to $T$ in Lemma 1 and $T^*$ will have a following form:

$$T^* = \frac{1}{(2\pi i)^2} \int_{(1)} \int_{(1)} \prod_p \left(1 - \frac{|\Omega^+(p)| - 1}{p-1} \left(\frac{1}{p^{s_1}} + \frac{1}{p^{s_2}} - \frac{1}{p^{s_1+s_2}}\right)\right) \frac{R^{s_1+s_2}}{(s_1 s_2)^{k+l+1}} ds_1 ds_2.$$

Omitting the calculations we can write for $T^*$, when $h \notin \mathcal{H}$:

$$T^* = \frac{\mathfrak{S}(\mathcal{H} \cup \{h\})}{(k+2l)!} \binom{2l}{l} (\log R)^{k+2l} + O((\log N)^{k+2l-1} (\log \log N)^c).$$

And when $h \in \mathcal{H}$:

$$T^* = \frac{\mathfrak{S}(\mathcal{H})}{(k+2l+1)!} \binom{2l+2}{l+1} (\log R)^{k+2l+1} + O((\log N)^{k+2l} (\log \log N)^c).$$

Substituting the expression of $T^*$ into (6.3.7) we will obtain the result of Lemma 2. ■

This chapter is based on paper of Motohashi [2], who found easier proof in 2006, then the original proof of Goldston-Pintz-Yildirim, which reader can find in [1]. Proof of Gallagher Lemma can be found in [8].

# CONCLUSION

In this work we introduced the complete proof of Goldston-Pintz-Yildirim theorem – the best known result on estimating the small gaps between consecutive primes. Also various wide applicable inequalities were highlighted, such as Bombieri-Vinogradov theorem, Pólya-Vinogradov inequality, large sieve method for character sums. Correlation and utility of analytic tools were presented – we showed the Vaughan identity as a simplifier in Bombieri-Vinogradov theorem's proof, and Bombieri-Vinogradov result as crucial part of Goldston-Pintz-Yildirim theorem.

The flexibility and variety of tools in analytic number theory demonstrate us, that many areas in prime number theory experience rapid development, and breakthrough is possible not only in bounding the small gaps between primes, but also in estimating the number of primes in short intervals, in Goldbach conjecture and other topics related to distribution of primes.

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