Modelling Firm-Product Level Trade: A Multi-Dimensional Random Effects Panel Data Approach

by

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Preface

Parts of chapter 3-5 have been published in the Pus, Matyas and Hornok [2013] working paper, and will be presented at the 19th Panel Data Conference in London, July 4-5. Chapters 3-4 are joint work with Laszlo Matyas and Cecilia Hornok. Chapter 1, 2, 5, 6 and 7 are solely my own work.

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Abstract

This thesis deals with the problems of formalizing econometric models on firm-product level trade data sets, or similar economic flows. A multi-dimensional random effects panel data approach is adopted. Several models are introduced taking into account different types of specific effects, interactions and cross correlations. The respective covariance matrixes are derived, as well as procedures to estimate the unknown variance and covariance components, in order to make the Feasible Generalized Least Squares estimation operational. Whenever possible, the spectral decomposition of the covariance matrixes is also provided to make the estimation procedure simpler to implement. Both balanced and unbalanced data sets are considered.

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Chapter 1

Introduction

The mass-media is overwhelmed with the reports about how big data revolution will transform all the aspects of economy. Many new large socio-economic data sets appeared which led to a vast interest in the use of multi-dimensional panel data regression methods. Panel data is most likely to be one of the most popular data structures, as compared to cross section data and aggregated time series data; it has the large number of observations, the possibility to separate within group variation and between group variation. With the rapid grows of the data it became crucial to derive appropriate estimation methods for the higher dimensional panel data sets, to turn from "usual" two dimensional (ij) case to tree- (ijt) and four- dimensional (ijst) cases. These data structures are frequently used to analyze different types of economic flows, like capital flows (FDI) or trade relationships. While due to recent research three-dimensional panel data, mostly related to macro trade, and other macroeconomic flows, is now better understood (Matyas and Balazsi [2012], Matyas, Hornok and Pus [2012]), in higher dimensions, with potentially extremely large number of observations, the models and, even the simplest, estimation methods can quickly become very complex.

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In this thesis I focus my attention on four-dimensional panel data case in a trade context, specifically on estimation methods for different types of linear model specifications. These four dimensions consist of firm or disaggregated sector (called in short firm), product (or, again sector, called in short product), destination country and time period. As the number of observations usually in these cases is (very) large, a fixed effects approach would mean the explicit or implicit inclusion into the model of tens of thousands of additional parameters and dummy variables. This would look in fact very much like a case of over-fitting, mostly "destroying" the explanatory power of any other variables. And also, in these higher dimension, different fixed effects specifications can substantially change the estimation results, so instead, in this thesis, I introduce and analyze several appropriate random effects model specifications.

This thesis begins with an introduction to the topic and review of the relevant literature. In Chapter 3 the baseline model for this four-dimensional panel data approach is introduced, followed by the description of sample size and sample structure. Then 7 different model specifications are considered, which in fact are different structures of disturbance term. And the main assumptions on the error components for the models are presented. Described models are mostly generalizations of the "usual"error components panel data models or approaches used in multilevel modeling, which are widely applied in trade.

In Chapter 4 proper estimation methods for the balanced case are derived for the models introduced in Chapter 3. The most efficient way to estimate them is through the Feasible GLS estimator. Firstly, I derive covariance matrixes in Section 4.1. Given the four-dimensional set up, sample size can become very large, and inverse matrix needed for FGLS estimation can not be easily calculated, thus spectral decomposition is presented

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for each covariance matrix. In Section 4.2 estimation of unknown variance components of respective covariance matrixes are derived using identifying equations and Within transformation which cancels out specific effects (Matyas and Balazsi [2012]). So the FGLS estimation can be made. In the case of trade models, and in general for most of the flow type data, assumption that random effects are pairwise uncorrelated may be too restrictive as it does not allow for any type of cross correlation. Thus in Sections 4.3 and 4.4 I relax this assumption and derive covariance matrixes and estimation of variance components for the models with cross correlation.

In Chapter 5 I present how the procedure will change if the data in hand is unbalanced. The unbalanced data structure is very general in this case, as most of the trade data sets are presented with no-observation holes in it. Introducing a range of new notations and different index sets makes derivations for these case quite complex but feasible. I present covariance matrixes in Section 5.1, estimation of variance components in Section 5.2 and same for the case with cross correlation in Sections 5.3 and 5.4, respectively. Then I discuss available data sources and present potential applications in Chapter 6 and conclude in Chapter 7.

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Chapter 2

Review of Relevant Literature

Analysis of panel data has always attracted a lot of attention of researchers in econometrics. Work by Wallace and Hussain [1969], Nerlove [1971a, b], Fuller and Battese [1974], Wansbeek and Kapteyn [1989], Baltagi [1985], Baltagi et al. [2008] and others provide comprehensive description of the estimation methods both for balanced and unbalanced two-dimensional panel data cases, especially error components models. However, the fast grows of multi-dimensional panel data sets was not accompanied with the same grows of methods to analyze them. The literature on multi-dimensional panel data sets developed from the one that describes the "usual" two-way set up. Most of the models were based on the direct generalization of the "usual" fixed effects and error components models. Three-dimensional panel data is now better understood, due to the research conducted in recent years. One of the first researches in this area was done by Matyas [1997], who presented a correct representation of gravity model in a form of triple-index model with countries and time effects, and estimated it using OLS. Three-dimensional fixed effects panel data models were studied by Matyas and Balazsi [2012]. They derived the appropriate Within transformations which eliminate fixed effects from the model in order to make Within

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estimation operational. Matyas, Hornok and Pus [2012] derived appropriate estimation methods for the balanced and unbalanced cases for several random effect specifications for tree-dimensional trade panel data sets.

However, as was already mentioned previously, micro trade relationships are better described in a four-dimensional panel data framework. Bekes and Murakozi [2012] study the length of export spells for Hungary in firm-product-destination level data with one observation in the dataset being the export of a product j by firm i to country k in year t. The same structure of dataset was use by Berthou and Fontagne [2009] to test hypothesis of effects of trade liberalization (see also Corcos et al. [2012], Gorg et al. [2010], and Defer and Toubal [2007]). Arkolakis and Muendler [2009] develop a model of firm-product heterogeneity in a four dimensional setup, and show the potential falsity of fixed effects approach, as different fixed effects specifications change the estimation results. Thus it is crucial to develop literature on estimation methods of higher dimensional panel data sets.

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Chapter 3

The Model Specifications Considered

In this thesis I introduce different types of random effects model specifications suited for this four-dimensional panel data approach, derive proper estimation methods for each of them, and analyze their properties under different data structures.

The baseline model to be considered is

$$y_{ijst} = \beta' x_{IS,t} + u_{ijst}$$

where x are the explanatory variables of the model, β are the the unknown parameters, u are the idiosyncratic disturbance terms (which are assumed to be uncorrelated with the explanatory variables x), and IS is the time-invariant Index Set (it can be: ijs, ij, is, js or just have a single index for the explanatory variables, and can be different for different for each one of them).

It is important to be specific about the sample size and sample structure. First of all, the baseline sample structure - let me call it the balanced one - is when $i = 1, \dots, N^{(1)}$, $j = 1, \dots, N^{(2)}$, $s = 1, \dots, N^{(3)}$ and $t = 1, \dots, T, \forall i, s, j$. Note here that a "real" balanced sample would mean, as in Davis [2002], that $N^{(1)} = N^{(2)} = N^{(3)}$, but in our case this makes little sense, as essentially we are dealing here with micro trade data. Typically i stands for

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firms, j for products (or disaggregated sectors like ISIC level 1, 2 or 3 classifications), and s for the trade destination countries. The unbalanced data structure considered in this thesis is quite general as most of these data sets present themselves in "granular" form (looking like an Emmental cheese with high density of missing data bubbles). Individual time series may not only be of different length, but may also have no-observation holes in it. This can only be handled by the introduction of different index sets, referring to different groups of observations, which makes the analytical treatment of these cases quite complex.

As I am using here a random effects approach, the different model specifications are characterized by different structures of the disturbance terms u_{ijst}. In fact Moulton [1990] already pointed out more than two decades ago how important these structures are from a practical point of view. For each of them I derive the covariance matrix of the model, and then proper estimators for its variance and covariance components in order to be able to use the Feasible GLS (FGLS) estimator to estimate the unknown parameters of the model.

The simplest model considered is a straight generalization of the "usual" error components panel data model (see, for example, Matyas [1997], and Baltagi et al. [2008])

$$u_{ijst} = \mu_i + \gamma_j + \alpha_s + \lambda_t + \varepsilon_{ijst} \tag{1}$$

where the error components are pair-wise uncorrelated, have zero expected values, and their variances are

$$E(\mu_i \mu_{i'}) = \begin{cases} \sigma_\mu^2 & \text{if } i = i' \\ 0 & \text{otherwise} \end{cases} \qquad E(\gamma_j \gamma_{j'}) = \begin{cases} \sigma_\gamma^2 & \text{if } j = j' \\ 0 & \text{otherwise} \end{cases}$$

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$$E(\alpha_s \alpha_{s'}) = \begin{cases} \sigma_\alpha^2 & \text{if } s = s' \\ 0 & \text{otherwise} \end{cases} \qquad E(\lambda_t \lambda_{t'}) = \begin{cases} \sigma_\lambda^2 & \text{if } t = t' \\ 0 & \text{otherwise} \end{cases}$$

The next model to be considered is

$$u_{ijst} = \mu_{ijs} + \varepsilon_{ijst} \tag{2}$$

$$E(\mu_{ijs}\mu_{i'j's'}) = \begin{cases} \sigma_\mu^2 & \text{if } i = i', j = j' \text{ and } s = s' \\ 0 & \text{otherwise} \end{cases}$$

This is the "usual" panel data model with random individual effects, where the individual effects correspond to the (ijs) triplets. An extended version of this model is

$$u_{ijst} = \mu_{ijs} + \lambda_t + \varepsilon_{ijst} \tag{3}$$

where

$$E(\lambda_t \lambda_{t'}) = \begin{cases} \sigma_\lambda^2 & \text{if } t = t' \\ 0 & \text{otherwise} \end{cases}$$

The next model to be considered is with pair-wise interaction effects

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \varepsilon_{ijst} \tag{4}$$

where $\mu_{ij}^{(1)}$, $\mu_{is}^{(2)}$, $\mu_{js}^{(3)}$ and ε_{ijst} are pair-wise uncorrelated, have zero expected value, and

$$E(\mu_{ij}\mu_{i'j'}) = \begin{cases} \sigma_\mu^{(1)^2} & \text{if } i = i', \text{ and } j = j' \\ 0 & \text{otherwise} \end{cases}$$

$$E(\mu_{is}\mu_{i's'}) = \begin{cases} \sigma_\mu^{(2)^2} & \text{if } i = i', \text{ and } s = s' \\ 0 & \text{otherwise} \end{cases}$$

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$$E(\mu_{js}\mu_{j's'}) = \begin{cases} \sigma_\mu^{(3)^2} & \text{if } j = j', \text{ and } s = s' \\ 0 & \text{otherwise} \end{cases}$$

The next model is the extension of model (4) with a time effect

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \lambda_t + \varepsilon_{ijst} \tag{5}$$

Another form of heterogeneity is to use individual-time-varying effects. This in fact is the generalization of the approach used in multilevel modeling (see for example, Snijders and Boske [1999], Ebbes, Bockenholt and Wedel [2004], Hubler [2006] or Gelman [2006]). In this case model (2) is extended with pair-wise split individual specific time effects

$$u_{ijst} = \mu_{ijs} + v_{it}^{(1)} + v_{jt}^{(2)} + v_{st}^{(3)} + \varepsilon_{ijst} \tag{6}$$

where $v_{it}^{(1)}$, $v_{jt}^{(2)}$, $v_{st}^{(3)}$ and ε_{ijst} are pair-wise uncorrelated, have zero expected value, and

$$E(v_{it}v_{i't'}) = \begin{cases} \sigma_v^{(1)^2} & \text{if } i = i', \text{ and } t = t' \\ 0 & \text{otherwise} \end{cases}$$

$$E(v_{jt}v_{j't'}) = \begin{cases} \sigma_v^{(2)^2} & \text{if } j = j', \text{ and } t = t' \\ 0 & \text{otherwise} \end{cases}$$

$$E(v_{st}v_{s't'}) = \begin{cases} \sigma_v^{(3)^2} & \text{if } s = s', \text{ and } t = t' \\ 0 & \text{otherwise} \end{cases}$$

Finally, the last model to be considered is an all-encompassing model with

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + v_{it}^{(1)} + v_{jt}^{(2)} + v_{st}^{(3)} + \varepsilon_{ijst} \tag{7}$$

In some cases it is important to deal with cross-correlations as well. The cross-

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correlations to be considered are for models (2), (3) and (6)

$$E(\mu_{ijs}\mu_{i'j's'}) = \begin{cases} \sigma_\mu^2 & i = i', j = j' \text{ and } s = s' \\ \rho_{(1)} & i \neq i', j = j' and s = s' \\ \rho_{(2)} & i = i', j \neq j' and s = s' \\ \rho_{(3)} & i = i', j = j' and s \neq s' \\ 0 & \text{otherwise} \end{cases}$$

and for models (4), (5) and (7)

$$E(\mu_{ij}\mu_{i'j'}) = \begin{cases} \sigma_\mu^{(1)^2} & i = i' \text{ and } j = j' \\ \rho_{(1)}^{(1)} & i \neq i' \text{ and } j = j' \\ \rho_{(2)}^{(1)} & i = i' \text{ and } j \neq j' \\ 0 & \text{otherwise} \end{cases}$$

$$E(\mu_{is}\mu_{i's'}) = \begin{cases} \sigma_\mu^{(2)^2} & i = i' \text{ and } s = s' \\ \rho_{(1)}^{(2)} & i \neq i' \text{ and } s = s' \\ \rho_{(2)}^{(2)} & i = i' \text{ and } s \neq s' \\ 0 & \text{otherwise} \end{cases}$$

and

$$E(\mu_{js}\mu_{j's'}) = \begin{cases} \sigma_\mu^{(3)^2} & j = j' \text{ and } s = s' \\ \rho_{(1)}^{(3)} & j \neq j' \text{ and } s = s' \\ \rho_{(2)}^{(3)} & j = j' \text{ and } s \neq s' \\ 0 & \text{otherwise} \end{cases}$$

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Chapter 4

Covariance Matrixes and the Estimation of the Variance Components

As it is well known, the most efficient way to estimate the models introduced in Chapter 3 is through the Feasible GLS estimator. First, starting with the balanced case, I need to derive the covariance matrix of each model (1) - (7). However, given the four dimensions, the sample size can become very large quite quickly, meaning that the inverse of the covariance matrix (needed to perform a the FGLS estimation) frequently cannot easily be calculated in practice. To overcome this problem I also derive the spectral decomposition for each covariance matrix, which makes the inverse operation much more easy to perform. Then, I estimate the unknown variance and covariance components of the respective covariance matrixes in order to make the FGLS operational.

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4.1 Covariance Matrixes of the Different Models

As the four dimensional setup makes the matrix algebra a bit more complex, some new notations are needed to be introduced upfront. Let me make the following definitions:

$$B_i = I_{N^{(1)}} \otimes \frac{J_{N^{(2)}N^{(3)}T}}{N^{(2)}N^{(3)}T}$$

$$B_j = \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}} \otimes \frac{J_{N^{(3)}T}}{N^{(3)}T}$$

$$B_s = \frac{J_{N^{(1)}N^{(2)}}}{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}} \otimes \frac{J_T}{T}$$

$$B_t = \frac{J_{N^{(1)}N^{(2)}N^{(3)}}}{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T$$

$$B_{ij} = I_{N^{(1)}N^{(2)}} \otimes \frac{J_{N^{(3)}T}}{N^{(3)}T}$$

$$B_{is} = I_{N^{(1)}} \otimes \frac{J_{N^{(2)}}}{N^{(2)}} \otimes I_{N^{(3)}} \otimes \frac{J_T}{T}$$

$$B_{js} = \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes \frac{J_T}{T}$$

$$B_{it} = I_{N^{(1)}} \otimes \frac{J_{N^{(2)}N^{(3)}}}{N^{(2)}N^{(3)}} \otimes I_T$$

$$B_{jt} = \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}} \frac{J_{N^{(3)}}}{N^{(3)}} \otimes I_T$$

$$B_{st} = \frac{J_{N^{(1)}N^{(2)}}}{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}T}$$

$$B_{ijs} = I_{N^{(1)}N^{(2)}N^{(3)}} \otimes \frac{J_T}{T}$$

$$B_{ijt} = I_{N^{(1)}N^{(2)}} \otimes \frac{J_{N^{(3)}}}{N^{(3)}} \otimes I_T$$

$$B_{jst} = \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}T}$$

$$B_{ist} = I_{N^{(1)}} \otimes \frac{J_{N^{(2)}}}{N^{(2)}}$$

$$J = \frac{J_{N^{(1)}N^{(2)}N^{(3)}T}}{N^{(1)}N^{(2)}N^{(3)}T}$$

$$I = I_{N^{(1)}N^{(2)}N^{(3)}T}$$

$$B_i^* = I_{N^{(1)}} \otimes \frac{J_{N^{(2)}N^{(3)}}}{N^{(2)}N^{(3)}}$$

$$B_j^* = \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}} \otimes \frac{J_{N^{(3)}}}{N^{(3)}}$$

$$B_s^* = \frac{J_{N^{(1)}N^{(2)}}}{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}}$$

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$$B^*_{ij} = I_{N^{(1)}N^{(2)}} \otimes \frac{J_{N^{(3)}}}{N^{(3)}}$$

$$B^*_{is} = I_{N^{(1)}} \otimes \frac{J_{N^{(2)}}}{N^{(2)}} \otimes I_{N^{(3)}}$$

$$B^*_{js} = \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}}$$

$$J^* = \frac{J_{N^{(1)}N^{(2)}N^{(3)}}}{N^{(1)}N^{(2)}N^{(3)}}$$

$$I^* = I_{N^{(1)}N^{(2)}N^{(3)}}$$

Model (1)

To derive the covariance matrix of model (1) I start from composite disturbance term

$$u_{ijst} = \mu_i + \gamma_j + \alpha_s + \lambda_t + \epsilon_{ijst}$$

For all T observations I get

$$u_{ijs} = \mu_i \otimes l_T + \gamma_j \otimes l_T + \alpha_s \otimes l_T + \lambda + \epsilon_{ijs}$$

$$E[u_{ijs}u'_{ijs}] = E[(\mu_i \otimes l_T)(\mu_i \otimes l_T)'] + E[(\gamma_j \otimes l_T)(\gamma_j \otimes l_T)'] +$$

$$+ E[(\alpha_s \otimes l_T)(\alpha_s \otimes l_T)'] + E[\lambda\lambda'] + E[\epsilon_{ijs}\epsilon'_{ijs}] =$$

$$= \sigma^2_\mu J_T + \sigma^2_\gamma J_T + \sigma^2_\alpha J_T + \sigma^2_\lambda I_T + \sigma^2_\epsilon I_T$$

Continuing this building up of the observations for the s index and then the j and i indexes as well, I get

$$u_{ij} = \mu_i \otimes l_T \otimes l_{N^{(3)}} + \gamma_j \otimes l_T \otimes l_{N^{(3)}} + \alpha \otimes l_T + l_{N^{(3)}} \otimes \lambda + \epsilon_{ij}$$

$$E[u_{ij}u'_{ij}] = \sigma^2_\mu J_{N^{(3)}T} + \sigma^2_\gamma J_{N^{(3)}T} + \sigma^2_\alpha I_{N^{(3)}} \otimes J_T + \sigma^2_\lambda J_{N^{(3)}} \otimes I_T + \sigma^2_\epsilon I_{N^{(3)}T}$$

$$u_i = \mu_i \otimes l_T \otimes l_{N^{(3)}} \otimes l_{N^{(2)}} + \gamma \otimes l_T \otimes l_{N^{(3)}} + l_{N^{(2)}} \otimes \alpha \otimes l_T + l_{N^{(2)}} \otimes l_{N^{(3)}} \otimes \lambda + \epsilon_i$$

$$E[u_iu'_i] = \sigma^2_\mu J_{N^{(2)}N^{(3)}T} + \sigma^2_\gamma I_{N^{(2)}} \otimes J_{N^{(3)}T} + \sigma^2_\alpha J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T +$$

$$+ \sigma^2_\lambda J_{N^{(2)}N^{(3)}} \otimes I_T + \sigma^2_\epsilon I_{N^{(2)}N^{(3)}T}$$

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So this gives finally

$$u = \mu \otimes l_T \otimes l_{N^{(3)}} \otimes l_{N^{(2)}} + l_{N^{(1)}} \otimes \gamma \otimes l_T \otimes l_{N^{(3)}} + l_{N^{(1)}} \otimes l_{N^{(2)}} \otimes \alpha \otimes l_T$$

$$+ l_{N^{(1)}} \otimes l_{N^{(2)}} \otimes l_{N^{(3)}} \otimes \lambda + \epsilon$$

$$E[uu'] = \sigma_\mu^2 I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} + \sigma_\gamma^2 J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}T} + \sigma_\alpha^2 J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T +$$

$$+ \sigma_\lambda^2 J_{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T} = \Omega$$

where l is the vector of ones (all elements being 1) with its size in the index, J is the matrix of ones with its size in the index and I is the identity matrix, with its size in the index, and μ, α, γ, λ and ϵ are the vectors containing the elements of μ_i, γ_j, α_s, λ_t, and ϵ_{ijst} respectively.

Like in the usual panel data case let me work out the spectral decomposition of this matrix to simplify the inverse needed for the FGLS. Using the notation

$$C_{11} = B_i - J$$

$$C_{12} = B_j - J$$

$$C_{13} = B_s - J$$

$$C_{14} = B_t - J$$

$$W_1 = I - C_{11} - C_{12} - C_{13} - C_{14} - J$$

we get

$$\Omega = N^{(2)}N^{(3)}T\sigma_\mu^2 (C_{11} + J) + N^{(1)}N^{(3)}T\sigma_\gamma^2 (C_{12} + J) + N^{(2)}N^{(3)}T\sigma_\alpha^2 (C_{13} + J) +$$

$$+ N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 (C_{14} + J) + +\sigma_\epsilon^2 (W_1 + C_{11} + C_{12} + C_{13} + C_{14} + J) =$$

$$= \left(N^{(2)}N^{(3)}T\sigma_\mu^2 + N^{(1)}N^{(3)}T\sigma_\gamma^2 + N^{(2)}N^{(3)}T\sigma_\alpha^2 + N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2\right) J +$$

$$+ \left(N^{(2)}N^{(3)}T\sigma_\mu^2 + \sigma_\epsilon^2\right) C_{11} + \left(N^{(1)}N^{(3)}T\sigma_\gamma^2 + \sigma_\epsilon^2\right) C_{12} +$$

$$+ \left(N^{(2)}N^{(3)}T\sigma_\alpha^2 + \sigma_\epsilon^2\right) C_{13} + \left(N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2\right) C_{14} + \sigma_\epsilon^2 W_1$$

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Now using

$$\theta_1 = \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}T\sigma_\mu^2 + \sigma_\epsilon^2}$$

$$\theta_2 = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(3)}T\sigma_\gamma^2 + \sigma_\epsilon^2}$$

$$\theta_3 = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}T\sigma_\alpha^2 + \sigma_\epsilon^2}$$

$$\theta_4 = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2}$$

$$\theta_5 = \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}T\sigma_\mu^2 + N^{(1)}N^{(3)}T\sigma_\gamma^2 + N^{(1)}N^{(2)}T\sigma_\alpha^2 + N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2}$$

I get for the inverse of the covariance matrix

$$\sigma_\epsilon^2\Omega^{-1} = \theta_1 C_{11} + \theta_2 C_{12} + \theta_3 C_{13} + \theta_4 C_{14} + \theta_5 J + W_1 =$$

$$= I - (1-\theta_1)\,B_i - (1-\theta_2)\,B_j - (1-\theta_3)\,B_s - (1-\theta_4)\,B_t +$$

$$+ (3 - \theta_1 - \theta_2 - \theta_3 - \theta_4 + \theta_5)\,J$$

Now this model is suited to deal with purely cross sectional data as well, that is when $T = 1$. In this case

$$E[uu'] = \sigma_\mu^2 I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}} + \sigma_\gamma^2 J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}} + \sigma_\alpha^2 J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}} +$$

$$+ \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}} = \Omega$$

with

$$C_{11}^* = B_i^* - J^*$$

$$C_{12}^* = B_j^* - J^*$$

$$C_{13}^* = B_s^* - J^*$$

$$W_1^* = I^* - C_{11}^* - C_{12}^* - C_{13}^* - J^*$$

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and I get

$$\Omega = N^{(2)}N^{(3)}\sigma_\mu^2\left(C_{11}^* + J^*\right) + N^{(1)}N^{(3)}\sigma_\gamma^2\left(C_{12}^* + J^*\right) + N^{(2)}N^{(3)}\sigma_\alpha^2\left(C_{13}^* + J^*\right) +$$

$$+ \sigma_\epsilon^2\left(W_1^* + C_{11}^* + C_{12}^* + C_{13}^* + J^*\right) =$$

$$= \left(N^{(2)}N^{(3)}\sigma_\mu^2 + N^{(1)}N^{(3)}\sigma_\gamma^2 + N^{(2)}N^{(3)}\sigma_\alpha^2 + \sigma_\epsilon^2\right)J^* + \left(N^{(2)}N^{(3)}\sigma_\mu^2 + \sigma_\epsilon^2\right)C_{11}^* +$$

$$+ \left(N^{(1)}N^{(3)}\sigma_\gamma^2 + \sigma_\epsilon^2\right)C_{12}^* + \left(N^{(2)}N^{(3)}\sigma_\alpha^2 + \sigma_\epsilon^2\right)C_{13}^* + \sigma_\epsilon^2 W_1^*$$

Proceeding like in the panel data case above, with the notation

$$\theta_1^* = \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}\sigma_\mu^2 + \sigma_\epsilon^2}$$

$$\theta_2^* = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(3)}\sigma_\gamma^2 + \sigma_\epsilon^2}$$

$$\theta_3^* = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}\sigma_\alpha^2 + \sigma_\epsilon^2}$$

$$\theta_4^* = \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}\sigma_\mu^2 + N^{(1)}N^{(3)}\sigma_\gamma^2 + N^{(1)}N^{(2)}\sigma_\alpha^2 + \sigma_\epsilon^2}$$

I get

$$\sigma_\epsilon^2\Omega^{-1} = \theta_1^* C_{11}^* + \theta_2^* C_{12}^* + \theta_3^* C_{13}^* + \theta_4^* J^* + W_1^* =$$

$$= I - \left(1 - \theta_1^*\right)B_i^* - \left(1 - \theta_2^*\right)B_j^* - \left(1 - \theta_3^*\right)B_s^* + \left(2 - \theta_1^* - \theta_2^* - \theta_3^* + \theta_4^*\right)J^*$$

Model (2)

Proceeding likewise for model (2) I first build up the covariance matrix

$$u_{ijst} = \mu_{ijs} + \epsilon_{ijst}$$

$$u_{ijs} = \mu_{ijs} \otimes l_T + \epsilon_{ijs}$$

$$E[u_{ijs}u'_{ijs}] = \sigma_\mu^2 J_T + \sigma_\epsilon^2 I_T$$

$$u_{ij} = \mu_{ij} \otimes l_T + \epsilon_{ij}$$

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$$E[u_{ij}u'_{ij}] = \sigma^2_\mu I_{N^{(3)}} J_T + \sigma^2_\epsilon I_{N^{(3)}T}$$

$$u_i = \mu_i \otimes l_T + \epsilon_i$$

$$E[u_i u'_i] = \sigma^2_\mu I_{N^{(2)}N^{(3)}} \otimes J_T + \sigma^2_\epsilon I_{N^{(2)}N^{(3)}T}$$

$$u = \mu \otimes l_T + \epsilon$$

$$E[uu'] = \sigma^2_\mu I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \sigma^2_\epsilon I_{N^{(1)}N^{(2)}N^{(3)}T} = \Omega$$

Then using

$$C_2 = B_{ijs} - J$$

$$W_2 = I - B_{ijs}$$

I get for the covariance matrix

$$\Omega = T\sigma^2_\mu (C_2 + J) + \sigma^2_\epsilon (W_2 + C_2 + J) = \left(T\sigma^2_\mu + \sigma^2_\epsilon\right) J + \left(T\sigma^2_\mu + \sigma^2_\epsilon\right) C_2 + \sigma^2_\epsilon W_2$$

and for the spectral decomposition

$$\sigma^2_\epsilon \Omega^{-1} = \theta J + \theta C_2 + W_2 = I - (1-\theta) B_{ijs}$$

with

$$\theta = \frac{\sigma^2_\epsilon}{T\sigma^2_\mu + \sigma^2_\epsilon}$$

The GLS estimator then is

$$\hat{\beta}_{GLS} = \left[X' \left(I - (1-\theta)B_{ijs}\right) X\right]^{-1} X' \left(I - (1-\theta)B_{ijs}\right) y$$

The GLS estimator is in fact an OLS estimator on the transformed model, where all the variable of the model are transformed like

$$\tilde{y}_{ijst} = y_{ijst} - (1-\theta) \sum_{t=1}^{T} \frac{1}{T} y_{ijst}$$

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Model (3)

Proceeding in the same way as above for model (3) I get

$$u_{ijst} = \mu_{ijs} + \lambda_t + \epsilon_{ijst}$$

$$E[uu'] = \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \sigma_\lambda^2 J_{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T} = \Omega$$

and so

$$\Omega = T\sigma_\mu^2 \left(C_{32} + J\right) + N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 \left(C_{31} + J\right) + \sigma_\epsilon^2 \left(W_3 + C_{31} + C_{32} + J\right) =$$

$$= \left(T\sigma_\mu^2 + N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2\right) J + \left(T\sigma_\mu^2 + \sigma_\epsilon^2\right) C_{32} + \left(N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2\right) C_{31} + \sigma_\epsilon^2 W_3$$

with

$$C_{31} = B_t - J$$

$$C_{32} = B_{ijs} - J$$

$$W_3 = I - C_{31} - C_{32} - J$$

For the spectral decomposition I get

$$\sigma_\epsilon^2 \Omega^{-1} = \theta_1 J + \theta_2 C_{32} + \theta_3 C_{31} + W_3 = I - \left(1-\theta_2\right) B_{ijs} - \left(1-\theta_3\right) B_t + \left(1 + \theta_1 - \theta_2 - \theta_3\right) J$$

with

$$\theta_1 = \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2}$$

$$\theta_2 = \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + \sigma_\epsilon^2}$$

$$\theta_3 = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2}$$

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Model (4)

For model (4) I get

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \epsilon_{ijst}$$

$$E[uu'] = \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T +$$
$$+ \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}$$

and

$$\Omega = N^{(3)}T\sigma_\mu^{(1)^2}\left(C_{41} + B_i + B_j - J\right) + N^{(2)}T\sigma_\mu^{(2)^2}\left(C_{42} + B_i + B_s - J\right) +$$
$$+ N^{(1)}T\sigma_\mu^{(3)^2}\left(C_{43} + B_j + B_s - J\right) + \sigma_\epsilon^2\left(W_4 + B_i + B_j + B_s + C41 + C_{42} + C_{43} - 2J\right)$$
$$= \sigma_\epsilon^2 W_4 + + \left(N^{(3)}T\sigma_\mu^{(1)^2} + \sigma_\epsilon^2\right) C_{41} + \left(N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) C_{42} +$$
$$+ \left(N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) C_{43} + \left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) B_i +$$
$$+ \left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) B_j + \left(N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) B_s -$$
$$- \left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + 2\sigma_\epsilon^2\right) J$$

with

$$C_{41} = B_{ij} - B_i - B_j + J$$
$$C_{42} = B_{is} - B_i - B_s + J$$
$$C_{43} = B_{js} - B_j - B_s + J$$
$$W_4 = I - B_i - B_j - B_s - C_{41} - C_{42} - C_{43} + 2J$$

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The spectral decomposition now is

$$\sigma_\epsilon^2 \Omega^{-1} = W_4 + \theta_1 C_{41} + \theta_2 C_{42} + \theta_3 C_{43} + \theta_4 B_i + \theta_5 B_j + \theta_6 B_s - \theta_7 J =$$
$$= I - (1-\theta_1) B_{ij} - (1-\theta_2) B_{is} - (1-\theta_3) B_{js} + (1 - \theta_1 - \theta_2 + \theta_4) B_i +$$
$$+ (1 - \theta_1 - \theta_3 + \theta_5) B_j + (1 - \theta_2 - \theta_3 + \theta_6) B_s - (1 - \theta_1 - \theta_2 - \theta_3 + \theta_7) J$$

with

$$\theta_1 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + \sigma_\epsilon^2}$$

$$\theta_2 = \frac{\sigma_\epsilon^2}{N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_3 = \frac{\sigma_\epsilon^2}{N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_4 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_5 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_6 = \frac{\sigma_\epsilon^2}{N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_7 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + 2\sigma_\epsilon^2}$$

This model seems to be the perfect choice when one is dealing with cross sectional data. In this case

$$E[uu'] = \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}} + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} +$$

$$+ \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}} = \Omega$$

with the notation

$$C_{41}^* = B_{ij}^* - B_i^* - B_j^* + J^*$$

$$C_{42}^* = B_{is}^* - B_i^* - B_s^* + J^*$$

$$C_{43}^* = B_{js}^* - B_j^* - B_s^* + J^*$$

$$W_4^* = I^* - B_i^* - B_j^* - B_s^* - C_{41}^* - C_{42}^* - C_{43}^* + 2J^*$$

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I get

$$
\begin{aligned}
\Omega = & N^{(3)}\sigma_\mu^{(1)^2}\left(C_{41}^* + B_i^* + B_j^* - J^*\right) + N^{(2)}\sigma_\mu^{(2)^2}\left(C_{42}^* + B_i^* + B_s^* - J^*\right) + \\
& + N^{(1)}\sigma_\mu^{(3)^2}\left(C_{43}^* + B_j^* + B_s^* - J^*\right) + \\
& + \sigma_\epsilon^2\left(W_4^* + B_i^* + B_j^* + B_s^* + C_{41}^* + C_{42}^* + C_{43}^* - 2J^*\right) \\
= & \sigma_\epsilon^2 W_4^* + \left(N^{(3)}\sigma_\mu^{(1)^2} + \sigma_\epsilon^2\right)C_{41}^* + \left(N^{(2)}\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right)C_{42}^* + \\
& + \left(N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right)C_{43}^* + \left(N^{(3)}\sigma_\mu^{(1)^2} + N^{(2)}\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right)B_i^* + \\
& + \left(N^{(3)}\sigma_\mu^{(1)^2} + N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right)B_j^* + \left(N^{(2)}\sigma_\mu^{(2)^2} + N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right)B_s^* - \\
& - \left(N^{(3)}\sigma_\mu^{(1)^2} + N^{(2)}\sigma_\mu^{(2)^2} + N^{(1)}\sigma_\mu^{(3)^2} + 2\sigma_\epsilon^2\right)J^*
\end{aligned}
$$

Introducing a similar notation than earlier

$$
\begin{aligned}
\theta_1^* &= \frac{\sigma_\epsilon^2}{N^{(3)}\sigma_\mu^{(1)^2} + \sigma_\epsilon^2} \\
\theta_2^* &= \frac{\sigma_\epsilon^2}{N^{(2)}\sigma_\mu^{(2)^2} + \sigma_\epsilon^2} \\
\theta_3^* &= \frac{\sigma_\epsilon^2}{N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2} \\
\theta_4^* &= \frac{\sigma_\epsilon^2}{N^{(3)}\sigma_\mu^{(1)^2} + N^{(2)}\sigma_\mu^{(2)^2} + \sigma_\epsilon^2} \\
\theta_5^* &= \frac{\sigma_\epsilon^2}{N^{(3)}\sigma_\mu^{(1)^2} + N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2} \\
\theta_6^* &= \frac{\sigma_\epsilon^2}{N^{(2)}\sigma_\mu^{(2)^2} + N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2} \\
\theta_7^* &= \frac{\sigma_\epsilon^2}{N^{(3)}\sigma_\mu^{(1)^2} + N^{(2)}\sigma_\mu^{(2)^2} + N^{(1)}\sigma_\mu^{(3)^2} + 2\sigma_\epsilon^2}
\end{aligned}
$$

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I get

$$\sigma_\epsilon^2\Omega^{-1} = W_4^* + \theta_1^* C_{41}^* + \theta_2^* C_{42}^* + \theta_3^* C_{43}^* + \theta_4^* B_i^* + \theta_5^* B_j^* + \theta_6^* B_s^* - \theta_7^* J^* =$$

$$= I^* - (1-\theta_1^*) B_{ij}^* - (1-\theta_2^*) B_{is}^* - (1-\theta_3^*) B_{js}^* + (1-\theta_1^* - \theta_2^* + \theta_4^*) B_i^* +$$

$$+ (1-\theta_1^* - \theta_3^* + \theta_5^*) B_j^* + (1-\theta_2^* - \theta_3^* + \theta_6^*) B_s^* -$$

$$- (1-\theta_1^* - \theta_2^* - \theta_3^* + \theta_7^*) J^*$$

Model (5)

For model (5) I get

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \lambda_t + \epsilon_{ijst}$$

$$E[uu'] = \Omega = \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T$$

$$+ \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T + \sigma_\lambda^2 J_{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}$$

This leads to

$$\Omega = N^{(3)}T\sigma_\mu^{(1)^2}(C_{51} + B_i + B_j) + N^{(2)}T\sigma_\mu^{(2)^2}(C_{52} + B_i + B_s) + N^{(1)}T\sigma_\mu^{(3)^2}(C_{53} + B_s + B_j) +$$

$$+ N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 C_{54} + \sigma_\epsilon^2 (W_5 + C_{51} + C_{52} + C_{53} + C_{54} + B_i + B_j + B_s) =$$

$$= \sigma_\epsilon^2 W_5 + \left(N^{(3)}T\sigma_\mu^{(1)^2} + \sigma_\epsilon^2\right) C_{51} + \left(N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) C_{52} + \left(N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) C_{53} +$$

$$+ \left(N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2\right) C_{54} + \left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) B_i +$$

$$+ \left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) B_j + \left(N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) B_s$$

with

$$C_{51} = B_{ij} - B_i - B_j$$

$$C_{52} = B_{is} - B_s - B_i$$

$$C_{53} = B_{js} - B_s - B_j$$

$$C_{54} = B_t$$

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$$W_5 = I - C_{51} - C_{52} - C_{53} - C_{54} - B_i - B_j - B_s$$

and the spectral decomposition is

$$\sigma_\epsilon^2 \Omega^{-1} = W_5 + \theta_1 C_{51} + \theta_2 C_{52} + \theta_3 C_{53} + \theta_4 C_{54} + \theta_5 B_i + \theta_6 B_j + \theta_7 B_s =$$

$$= I - (1-\theta_1) B_{ij} - (1-\theta_2) B_{is} - (1-\theta_3) B_{js} - (1-\theta_4) B_t + (1-\theta_1-\theta_2+\theta_5) B_i +$$

$$+ (1-\theta_1-\theta_3+\theta_6) B_j + (1-\theta_2-\theta_3+\theta_7) B_s$$

with

$$\theta_1 = \frac{\sigma_\epsilon^2}{N^{(3)} T \sigma_\mu^{(1)^2} + \sigma_\epsilon^2}$$

$$\theta_2 = \frac{\sigma_\epsilon^2}{N^{(2)} T \sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_3 = \frac{\sigma_\epsilon^2}{N^{(1)} T \sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_4 = \frac{\sigma_\epsilon^2}{N^{(1)} N^{(2)} N^{(3)} \sigma_\lambda^2 + \sigma_\epsilon^2}$$

$$\theta_5 = \frac{\sigma_\epsilon^2}{N^{(3)} T \sigma_\mu^{(1)^2} + N^{(2)} T \sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_6 = \frac{\sigma_\epsilon^2}{N^{(3)} T \sigma_\mu^{(1)^2} + N^{(1)} T \sigma_\mu^{(3)^2} + + \sigma_\epsilon^2}$$

$$\theta_7 = \frac{\sigma_\epsilon^2}{N^{(2)} T \sigma_\mu^{(2)^2} + N^{(1)} T \sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

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Model (6)

For model (6) I get

$$u_{ijst} = \mu_{ijs} + v_{it}^{(1)} + v_{jt}^{(2)} + v_{st}^{(3)} + \epsilon_{ijst}$$

$$E[uu'] = \Omega = \sigma_\mu^2 I_{N^{(1)} N^{(2)} N^{(3)}} \otimes J_T + \sigma_v^{(1)^2} I_{N^{(1)}} \otimes J_{N^{(2)} N^{(3)}} \otimes I_T +$$

$$+ \sigma_v^{(2)^2} J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}} \otimes I_T + \sigma_v^{(3)^2} J_{N^{(1)} N^{(2)}} \otimes I_{N^{(3)} T} + \sigma_\epsilon^2 I_{N^{(1)} N^{(2)} N^{(3)} T}$$

leading to

$$\Omega = T\sigma_\mu^2 \left(C_{61} + B_i + B_j + B_s - J\right) + N^{(2)}N^{(3)}\sigma_v^{(1)^2} \left(C_{62} + B_i + B_t - J\right) +$$

$$+ N^{(1)}N^{(3)}\sigma_v^{(2)^2} \left(C_{63} + B_j + B_t - J\right) + N^{(1)}N^{(2)}\sigma_v^{(3)^2} \left(C_{64} + B_s + B_t - J\right) +$$

$$+ \sigma_\epsilon^2 \left(W_6 + C_{61} + C_{62} + C_{63} + C_{64} + B_i + B_j + B_s + B_t - 2J\right) =$$

$$= \left(T\sigma_\mu^2 + \sigma_\epsilon^2\right) C_{61} + \left(N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2\right) C_{62} + \left(N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2\right) C_{63} +$$

$$+ \left(N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) C_{64} + \left(T\sigma_\mu^2 + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2\right) B_i +$$

$$+ \left(T\sigma_\mu^2 + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2\right) B_j + \left(T\sigma_\mu^2 + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) B_s +$$

$$+ \left(N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) B_t -$$

$$- \left(T\sigma_\mu^2 + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + 2\sigma_\epsilon^2\right) J + \sigma_\epsilon^2 W_6$$

with

$$C_{61} = B_{ijs} - B_i - B_j - B_s + J$$

$$C_{62} = B_{it} - B_i - B_t + J$$

$$C_{63} = B_{jt} - B_j - B_t + J$$

$$C_{64} = B_{st} - B_s - B_t + J$$

$$W_6 = I - C_{61} - C_{62} - C_{63} - C_{64} - B_i - B_j - B_s - B_t + 2J$$

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The spectral decomposition now is

$$\sigma_\epsilon^2\Omega^{-1} = W_6 + \theta_1 C_{61} + \theta_2 C_{62} + \theta_3 C_{63} + \theta_4 C_{64} + \theta_5 B_i + \theta_6 B_j + \theta_7 B_s + \theta_8 B_t - \theta_9 J =$$

$$= I - (1-\theta_1) B_{ijs} - (1-\theta_2) B_{it} - (1-\theta_3) B_{jt} - (1-\theta_4) B_{st} + (1-\theta_1-\theta_2+\theta_5) B_i +$$

$$+ (1-\theta_1-\theta_3+\theta_6) B_j + (1-\theta_1-\theta_4+\theta_7) B_s + (2-\theta_2-\theta_3-\theta_4+\theta_8) B_t -$$

$$- (2-\theta_1-\theta_2-\theta_3-\theta_4+\theta_9) J$$

with

$$\theta_1 = \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + \sigma_\epsilon^2}$$

$$\theta_2 = \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2}$$

$$\theta_3 = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_4 = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_5 = \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2}$$

$$\theta_6 = \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_7 = \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_8 = \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_9 = \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + 2\sigma_\epsilon^2}$$

Model (7)

Finally, for model (7) I get

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + v_{it}^{(1)} + v_{jt}^{(2)} + v_{st}^{(3)} + \epsilon_{ijst}$$

$$E[uu'] = \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T +$$

$$+ \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T + \sigma_v^{(1)^2} I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}} \otimes I_T +$$

$$+ \sigma_v^{(2)^2} J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}} \otimes I_T + \sigma_v^{(3)^2} J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}T} + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}$$

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leading to

$$\Omega = N^{(3)}T\sigma_\mu^{(1)^2}\left(C_{71} + B_i + B_j - J\right) + N^{(2)}T\sigma_\mu^{(2)^2}\left(C_{72} + B_i + B_s - J\right) +$$

$$+ N^{(1)}T\sigma_\mu^{(3)^2}\left(C_{73} + B_j + B_s - J\right) + N^{(2)}N^{(3)}\sigma_v^{(1)^2}\left(C_{74} + B_i + B_t - J\right) +$$

$$+ N^{(1)}N^{(3)}\sigma_v^{(2)^2}\left(C_{75} + B_j + B_t - J\right) + N^{(1)}N^{(2)}\sigma_v^{(3)^2}\left(C_{76} + B_s + B_t - J\right) +$$

$$+ \sigma_\epsilon^2\left(W_7 + C_{71} + C_{72} + C_{73} + C_{75} + C_{76} + B_i + B_j + B_s + B_t - 3J\right) =$$

$$= \sigma_\epsilon^2 W_7 + C_{71}\left(N^{(3)}T\sigma_\mu^{(1)^2} + \sigma_\epsilon^2\right) + C_{72}\left(N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) + C_{73}\left(N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) +$$

$$+ C_{74}\left(N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2\right) + C_{75}\left(N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2\right) +$$

$$+ C_{76}\left(N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) + B_i\left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2\right) +$$

$$+ B_j\left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2\right) +$$

$$+ B_s\left(N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right)$$

$$+ B_t\left(N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) -$$

$$- J\Big(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + N^{(2)}N^{(3)}\sigma_v^{(1)^2} +$$

$$+ N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + 3\sigma_\epsilon^2\Big)$$

with

$$C_{71} = B_{ij} - B_i - B_j + J$$

$$C_{72} = B_{is} - B_i - B_s + J$$

$$C_{73} = B_{js} - B_j - B_s + J$$

$$C_{74} = B_{it} - B_i - B_t + J$$

$$C_{75} = B_{jt} - B_j - B_t + J$$

$$C_{76} = B_{st} - B_s - B_t + J$$

$$W_7 = I - C_{71} - C_{72} - C_{73} - C_{74} - C_{75} - C_{76} - B_i - B_j - B_s - B_t + 3J$$

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The spectral decomposition now is

$$\sigma_\epsilon^2\Omega^{-1} = W_7 + \theta_1 C_{71} + \theta_2 C_{72} + \theta_3 C73 + \theta_4 C_{74} + \theta_5 C_{75} + \theta_6 C_{76} + \theta_7 B_i + \theta_8 B_j +$$

$$+ \theta_9 B_s + \theta_{10} B_t - \theta_{11} J =$$

$$= I - (1-\theta_1) B_{ij} - (1-\theta_2) B_{is} - (1-\theta_3) B_{js} - (1-\theta_4) B_{it} - (1-\theta_5) B_{jt} -$$

$$- (1-\theta_6) B_{st} + (2-\theta_1-\theta_2-\theta_4+\theta_7) B_i + (2-\theta_1-\theta_3-\theta_5+\theta_8) B_j +$$

$$+ (2-\theta_2-\theta_3-\theta_6+\theta_9) B_s + (2-\theta_4-\theta_5-\theta_6+\theta_{10}) B_t -$$

$$- (3-\theta_1-\theta_2-\theta_3-\theta_4-\theta_5-\theta_6+\theta_{11}) J$$

with

$$\theta_1 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + \sigma_\epsilon^2}$$

$$\theta_2 = \frac{\sigma_\epsilon^2}{N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_3 = \frac{\sigma_\epsilon^2}{N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_4 = \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2}$$

$$\theta_5 = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_6 = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_7 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2}$$

$$\theta_8 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_9 = \frac{\sigma_\epsilon^2}{N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_{10} = \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2}$$

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$$\theta_{11} = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + A}$$

$$A = N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + 3\sigma_\epsilon^2$$

4.2 Estimation of the Variance Components

In order to make the GLS estimator feasible I need to estimate the variance components of the different models. Given the four dimensions this is quite tedious and unfortunately there is no way to get around it. This is done below for all models in two steps. First, using the appropriate Within transformation for each model, which cancels out the specific effects (see Matyas and Balazsi [2012]), identifying equations are derived for the unknown variance components. Then, using these identifying equations, estimators for the variance components are derived one by one.

Model (1)

The Within transformation that cancels out the specific effects for this model is

$$u_{ijst} - \bar{u}_i - \bar{u}_j - \bar{u}_s - \bar{u}_t + 3\bar{u} = \epsilon_{ijst} - \bar{\epsilon}_i - \bar{\epsilon}_j - \bar{\epsilon}_s - \bar{\epsilon}_t + 3\bar{\epsilon}$$

which leads to the following identifying equations

$$E\left[(u_{ijst} - \bar{u}_i - \bar{u}_j - \bar{u}_s - \bar{u}_t + 3\bar{u})^2\right] = \sigma_\epsilon^2 \frac{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)} - N^{(2)} - N^{(3)} - T + 3}{N^{(1)}N^{(2)}N^{(3)}T}$$

$$E\left[u_{ijst}^2\right] = E\left[(\mu_i + \gamma_j + \alpha_s + \lambda_t + \epsilon_{ijst})^2\right] = \sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\lambda^2 + \sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{T}\sum_{t=1}^{T} u_{ijst}\right)^2\right] = E\left[\left(\frac{1}{T}\sum_{t=1}^{T} (\mu_i + \gamma_j + \alpha_s + \lambda_t + \epsilon_{ijst})\right)^2\right] =$$

$$= E\left[\mu_i^2\right] + E\left[\gamma_j^2\right] + E\left[\alpha_s^2\right] + \frac{1}{T^2}E\left[\sum_{t=1}^{T}\lambda_t^2\right] + \frac{1}{T^2}E\left[\sum_{t=1}^{T}\epsilon_{ijst}^2\right] =$$

$$= \sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\alpha^2 + \frac{1}{T}\sigma_\lambda^2 + \frac{1}{T}\sigma_\epsilon^2$$

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$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^2+\sigma_\gamma^2+\sigma_\alpha^2+\sigma_\lambda^2+\frac{1}{N^{(1)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\sigma_\mu^2+\frac{1}{N^{(2)}}\sigma_\gamma^2+\sigma_\alpha^2+\sigma_\lambda^2+\frac{1}{N^{(2)}}\sigma_\epsilon^2$$

So the appropriate estimators for the variance components are

$$\hat{\sigma}_\epsilon^2=\frac{N^{(1)}N^{(2)}N^{(3)}T}{N^{(1)}N^{(2)}N^{(3)}T-N^{(1)}-N^{(2)}-N^{(3)}-T+3}\hat{u}'_{within}\hat{u}_{within}$$

$$\hat{\sigma}_\mu^2=\frac{1}{\left(N^{(1)}-1\right)N^{(2)}N^{(3)}T}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\left(\sum_i^{N^{(1)}}\hat{u}_{ijst}^2-\frac{1}{N^{(1)}}\left(\sum_i^{N^{(1)}}\hat{u}_{ijst}\right)^2\right)-\hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_\gamma^2=\frac{1}{N^{(1)}\left(N^{(2)}-1\right)N^{(3)}T}\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\sum_t^{T}\left(\sum_j^{N^{(2)}}\hat{u}_{ijst}^2-\frac{1}{N^{(2)}}\left(\sum_j^{N^{(2)}}\hat{u}_{ijst}\right)^2\right)-\hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_\lambda^2=\frac{1}{N^{(1)}N^{(2)}N^{(3)}(T-1)}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\left(\sum_t^{T}\hat{u}_{ijst}^2-\frac{1}{T}\left(\sum_t^{T}\hat{u}_{ijst}\right)^2\right)-\hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_\alpha^2=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\hat{u}_{ijst}^2-\hat{\sigma}_\mu^2-\hat{\sigma}_\gamma^2-\hat{\sigma}_\lambda^2-\hat{\sigma}_\epsilon^2$$

where the "within" index means that it is the residual obtained from the appropriate Within estimation of the model.

For cross sectional data only, the Within transformation that cancels out the specific effects for this model is

$$u_{ijs}-\bar{u}_i-\bar{u}_j-\bar{u}_s+2\bar{u}=\epsilon_{ijs}-\bar{\epsilon}_i-\bar{\epsilon}_j-\bar{\epsilon}_s+2\bar{\epsilon}$$

which leads to the following identification equations

$$E\left[\left(u_{ijs}-\bar{u}_i-\bar{u}_j-\bar{u}_s+2\bar{u}\right)^2\right]=\sigma_\epsilon^2\frac{N^{(1)}N^{(2)}N^{(3)}-N^{(1)}-N^{(2)}-N^{(3)}+2}{N^{(1)}N^{(2)}N^{(3)}}$$

$$E\left[u_{ijs}^2\right]=\sigma_\mu^2+\sigma_\gamma^2+\sigma_\alpha^2+\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijs}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^2+\sigma_\gamma^2+\sigma_\alpha^2+\sigma_\lambda^2+\frac{1}{N^{(1)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{i=1}^{N^{(2)}}u_{ijs}\right)^2\right]=\sigma_\mu^2+\frac{1}{N^{(2)}}\sigma_\gamma^2+\sigma_\alpha^2+\sigma_\lambda^2+\frac{1}{N^{(2)}}\sigma_\epsilon^2$$

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Thus the estimators for the variance components are

$$\hat{\sigma}_\epsilon^2 = \frac{N^{(1)}N^{(2)}N^{(3)}}{N^{(1)}N^{(2)}N^{(3)} - N^{(1)} - N^{(2)} - N^{(3)} + 2}\hat{u}'_{within}\hat{u}_{within}$$

$$\hat{\sigma}_\mu^2 = \frac{1}{\left(N^{(1)}-1\right)N^{(2)}N^{(3)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\left(\sum_i^{N^{(1)}}\hat{u}_{ijs}^2 - \frac{1}{N^{(1)}}\left(\sum_i^{N^{(1)}}\hat{u}_{ijs}\right)^2\right) - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_\gamma^2 = \frac{1}{N^{(1)}\left(N^{(2)}-1\right)N^{(3)}}\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\left(\sum_j^{N^{(2)}}\hat{u}_{ijs}^2 - \frac{1}{N^{(2)}}\left(\sum_j^{N^{(2)}}\hat{u}_{ijs}\right)^2\right) - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_\alpha^2 = \frac{1}{N^{(1)}N^{(2)}N^{(3)}}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\hat{u}_{ijs}^2 - \hat{\sigma}_\mu^2 - \hat{\sigma}_\gamma^2 - \hat{\sigma}_\epsilon^2$$

Model (2)

The Within transformation now is

$$u_{ijst} - \bar{u}_{ijs} = \epsilon_{ijst} - \bar{\epsilon}_{ijs}$$

The identifying equations are

$$E\left[(u_{ijst} - \bar{u}_{ijs})^2\right] = E\left[(\epsilon_{ijst} - \bar{\epsilon}_{ijs})^2\right] = E\left[\epsilon_{ijst}^2 - 2\epsilon_{ijst}\bar{\epsilon}_{ijs} + \bar{\epsilon}_{ijs}^2\right] =$$

$$= \sigma_\epsilon^2 - \frac{2}{T}\sigma_\epsilon^2 + \frac{1}{T}\sigma_\epsilon^2 = \sigma_\epsilon^2\frac{T-1}{T}$$

$$E\left[u^{\star 2}_{ijst}\right] = E\left[(\mu_{ij} + \epsilon_{ijst})^2\right] = E\left[\mu_{ij}^2\right] + E\left[\epsilon_{ijst}^2\right] = \sigma_\mu^2 + \sigma_\epsilon^2$$

So the estimators for the variance components are

$$\hat{\sigma}_\epsilon^2 = \frac{T}{T-1}\hat{u}'_{within}\hat{u}_{within}$$

$$\hat{\sigma}_\mu^2 = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\hat{u}_{ijst}^2 - \hat{\sigma}_\epsilon^2$$

Model (3)

The Within transformation is

$$u_{ijst} - \bar{u}_{ijs} - \bar{u}_t + \bar{u} = \epsilon_{ijst} - \bar{\epsilon}_{ijs} - \bar{\epsilon}_t + \bar{\epsilon}$$

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which leads to the following identifying equations

$$E\left[(u_{ijst}-\bar{u}_{ijs}-\bar{u}_t+\bar{u})^2\right]=E\left[(\epsilon_{ijst}-\bar{\epsilon}_{ijs}-\bar{\epsilon}_t+\bar{\epsilon})^2\right]=$$

$$=E\left[\epsilon_{ijst}^2\right]+E\left[\bar{\epsilon}_{ijs}^2\right]+E\left[\bar{\epsilon}_t^2\right]+E\left[\bar{\epsilon}^2\right]-2E\left[\epsilon_{ijst}\bar{\epsilon}_{ijs}\right]-2E\left[\epsilon_{ijst}\bar{\epsilon}_t\right]+$$

$$+2E\left[\epsilon_{ijst}\bar{\epsilon}\right]+2E\left[\bar{\epsilon}_{ijs}\bar{\epsilon}_t\right]-2E\left[\bar{\epsilon}_{ijs}\bar{\epsilon}\right]-2E\left[\bar{\epsilon}_t\bar{\epsilon}\right]=$$

$$=\sigma_\epsilon^2+\frac{1}{T}\sigma_\epsilon^2+\frac{1}{N^{(1)}N^{(2)}N^{(3)}}\sigma_\epsilon^2+\frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sigma_\epsilon^2-\frac{2}{T}\sigma_\epsilon^2-\frac{2}{N^{(1)}N^{(2)}N^{(3)}}\sigma_\epsilon^2+$$
$$+\frac{2}{N^{(1)}N^{(2)}N^{(3)}T}\sigma_\epsilon^2+\frac{2}{N^{(1)}N^{(2)}N^{(3)}T}\sigma_\epsilon^2-\frac{2}{N^{(1)}N^{(2)}N^{(3)}T}\sigma_\epsilon^2-$$
$$-\frac{2}{N^{(1)}N^{(2)}N^{(3)}T}\sigma_\epsilon^2=\sigma_\epsilon^2\frac{\left(N^{(1)}N^{(2)}N^{(3)}-1\right)(T-1)}{N^{(1)}N^{(2)}N^{(3)}T}$$

The estimators for the variance components now are

$$\hat{\sigma}_\epsilon^2=\frac{N^{(1)}N^{(2)}N^{(3)}T}{(N^{(1)}N^{(2)}N^{(3)}-1)(T-1)}\hat{u}_{within}'\hat{u}_{within}$$
$$\hat{\sigma}_\mu^2=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T(T-1)}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\left(\left(\sum_t^T\hat{u}_{ijst}\right)^2-\sum_t^T\hat{u}_{ijst}^2\right)$$
$$\hat{\sigma}_\lambda^2=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^T\hat{u}_{ijst}^2-\hat{\sigma}_\mu^2-\hat{\sigma}_\epsilon^2$$

Model (4)

The Within transformation now is

$$u_{ijst}-\bar{u}_{ij}-\bar{u}_{is}-\bar{u}_{js}+\bar{u}_i+\bar{u}_j+\bar{u}_s-\bar{u}$$

The identifying equations are

$$E\left[(u_{ijst}-\bar{u}_{ij}-\bar{u}_{is}-\bar{u}_{js}+\bar{u}_i+\bar{u}_j+\bar{u}_s-\bar{u})^2\right]=$$
$$=\sigma_\epsilon^2\frac{N^{(1)}N^{(2)}N^{(3)}T-N^{(1)}N^{(2)}-N^{(1)}N^{(3)}-N^{(2)}N^{(3)}+N^{(1)}+N^{(2)}+N^{(3)}-1}{N^{(1)}N^{(2)}N^{(3)}T}$$
$$E\left[u_{ijst}^2\right]=\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}+\sigma_\epsilon^2$$

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$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}}\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}+\frac{1}{N^{(1)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(2)}}\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(2)}}\sigma_\epsilon^2$$

The estimators of the variance components now are

$$\hat{\sigma}_\epsilon^2=\frac{N^{(1)}N^{(2)}N^{(3)}T}{N^{(1)}N^{(2)}N^{(3)}T-N^{(1)}N^{(2)}-N^{(1)}N^{(3)}-N^{(2)}N^{(3)}+N^{(1)}+N^{(2)}+N^{(3)}-1}\hat{u}_{with}'\hat{u}_{with}$$

$$\hat{\sigma}_\mu^{(3)^2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\left(\left(\sum_i^{N^{(1)}}\hat{u}_{ijst}\right)^2-\sum_i^{N^{(1)}}\hat{u}_{ijst}^2\right)$$

$$\hat{\sigma}_\mu^{(2)^2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\sum_t^{T}\left(\left(\sum_j^{N^{(2)}}\hat{u}_{ijst}\right)^2-\sum_j^{N^{(2)}}\hat{u}_{ijst}^2\right)$$

$$\hat{\sigma}_\mu^{(1)^2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\hat{u}_{ijst}^2-\hat{\sigma}_\mu^{(2)^2}-\hat{\sigma}_\mu^{(3)^2}-\hat{\sigma}_\epsilon^2$$

For cross sectional data only, the Within transformation now is

$$u_{ijs}-\bar{u}_{ij}-\bar{u}_{is}-\bar{u}_{js}+\bar{u}_i+\bar{u}_j+\bar{u}_s-\bar{u}$$

The identifying equations are

$$E\left[\left(u_{ijs}-\bar{u}_{ij}-\bar{u}_{is}-\bar{u}_{js}+\bar{u}_i+\bar{u}_j+\bar{u}_s-\bar{u}\right)^2\right]=$$
$$=\sigma_\epsilon^2\frac{N^{(1)}N^{(2)}N^{(3)}-N^{(1)}N^{(2)}-N^{(1)}N^{(3)}-N^{(2)}N^{(3)}+N^{(1)}+N^{(2)}+N^{(3)}-1}{N^{(1)}N^{(2)}N^{(3)}}$$

$$E\left[u_{ijs}^2\right]=\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}+\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijs}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}}\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}+\frac{1}{N^{(1)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{i=1}^{N^{(2)}}u_{ijs}\right)^2\right]=\frac{1}{N^{(2)}}\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(2)}}\sigma_\epsilon^2$$

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And so the estimators of variance components now are

$$\hat{\sigma}_{\epsilon}^2 = \frac{N^{(1)}N^{(2)}N^{(3)}}{N^{(1)}N^{(2)}N^{(3)} - N^{(1)}N^{(2)} - N^{(1)}N^{(3)} - N^{(2)}N^{(3)} + N^{(1)} + N^{(2)} + N^{(3)} - 1} \times$$
$$\times\ \hat{u}'_{within}\hat{u}_{within}$$

$$\hat{\sigma}_{\mu}^{(3)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}\left(N^{(1)}-1\right)} \sum_{j}^{N^{(2)}} \sum_{s}^{N^{(3)}} \left(\left(\sum_{i}^{N^{(1)}} \hat{u}_{ijs} \right)^2 - \sum_{i}^{N^{(1)}} \hat{u}_{ijs}^2 \right)$$

$$\hat{\sigma}_{\mu}^{(2)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}\left(N^{(2)}-1\right)} \sum_{i}^{N^{(1)}} \sum_{s}^{N^{(3)}} \left(\left(\sum_{j}^{N^{(2)}} \hat{u}_{ijs} \right)^2 - \sum_{j}^{N^{(2)}} \hat{u}_{ijs}^2 \right)$$

$$\hat{\sigma}_{\mu}^{(1)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}} \sum_{i}^{N^{(1)}} \sum_{j}^{N^{(2)}} \sum_{s}^{N^{(3)}} \hat{u}_{ijs}^2 - \hat{\sigma}_{\mu}^{(2)^2} - \hat{\sigma}_{\mu}^{(3)^2} - \hat{\sigma}_{\epsilon}^2$$

Model (5)

The Within transformation now is

$$u_{ijst} - \bar{u}_{ij} - \bar{u}_{is} - \bar{u}_{js} - \bar{u}_t + \bar{u}_i + \bar{u}_j + \bar{u}_s$$

The identifying equations are

$$E\left[\left(u_{ijst} - \bar{u}_{ij} - \bar{u}_{is} - \bar{u}_{js} - \bar{u}_t + \bar{u}_i + \bar{u}_j + \bar{u}_s\right)^2\right] =$$
$$= \sigma_{\epsilon}^2 \frac{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)}N^{(2)} - N^{(1)}N^{(3)} - N^{(2)}N^{(3)} - T + N^{(1)} + N^{(2)} + N^{(3)}}{N^{(1)}N^{(2)}N^{(3)}T}$$

$$E\left[u_{ijst}^2\right] = \sigma_{\mu}^{(1)^2} + \sigma_{\mu}^{(2)^2} + \sigma_{\mu}^{(3)^2} + \sigma_{\lambda}^2 + \sigma_{\epsilon}^2$$

$$E\left[\left(\frac{1}{T}\sum_{t=1}^{T} u_{ijst}\right)^2\right] = \sigma_{\mu}^{(1)^2} + \sigma_{\mu}^{(2)^2} + \sigma_{\mu}^{(3)^2} + \frac{1}{T}\sigma_{\lambda}^2 + \frac{1}{T}\sigma_{\epsilon}^2$$

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(1)}}\sigma_{\mu}^{(1)^2} + \frac{1}{N^{(1)}}\sigma_{\mu}^{(2)^2} + \sigma_{\mu}^{(3)^2} + \sigma_{\lambda}^2 + \frac{1}{N^{(1)}}\sigma_{\epsilon}^2$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(2)}}\sigma_{\mu}^{(1)^2} + \sigma_{\mu}^{(2)^2} + \frac{1}{N^{(2)}}\sigma_{\mu}^{(3)^2} + \sigma_{\lambda}^2 + \frac{1}{N^{(2)}}\sigma_{\epsilon}^2$$

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The estimators of the variance components now are

$$\hat{\sigma}^2_\epsilon = \frac{N^{(1)}N^{(2)}N^{(3)}T}{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)}N^{(2)} - N^{(1)}N^{(3)} - N^{(2)}N^{(3)} - T + N^{(1)} + N^{(2)} + N^{(3)}}\hat{u}'_{with}\hat{u}_{with}$$

$$\hat{\sigma}^2_\lambda = \frac{1}{N^{(1)}N^{(2)}N^{(3)}(T-1)}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\left(\sum_t^T \hat{u}^2_{ijst} - \frac{1}{T}\left(\sum_t^T \hat{u}_{ijst}\right)^2\right) - \hat{\sigma}^2_\epsilon$$

$$\hat{\sigma}^{(3)^2}_\mu = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^T\left(\left(\sum_i^{N^{(1)}}\hat{u}_{ijst}\right)^2 - \sum_i^{N^{(1)}}\hat{u}^2_{ijst}\right) - \hat{\sigma}^2_\lambda$$

$$\hat{\sigma}^{(2)^2}_\mu = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\sum_t^T\left(\left(\sum_j^{N^{(2)}}\hat{u}_{ijst}\right)^2 - \sum_j^{N^{(2)}}\hat{u}^2_{ijst}\right) - \hat{\sigma}^2_\lambda$$

$$\hat{\sigma}^{(1)^2}_\mu = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^T \hat{u}^2_{ijst} - \hat{\sigma}^{(2)^2}_\mu - \hat{\sigma}^{(3)^2}_\mu - \hat{\sigma}^2_\lambda - \hat{\sigma}^2_\epsilon$$

Model (6)

The Within transformation now is

$$u_{ijst} - \bar{u}_{ijs} - \bar{u}_{it} - \bar{u}_{jt} - \bar{u}_{st} + \bar{u}_i + \bar{u}_j + \bar{u}_s + 2\bar{u}_t - 2\bar{u}$$

and the identifying equations are

$$E\left[\left(u_{ijst} - \bar{u}_{ijs} - \bar{u}_{it} - \bar{u}_{jt} - \bar{u}_{st} + \bar{u}_i + \bar{u}_j + \bar{u}_s + 2\bar{u}_t - 2\bar{u}\right)^2\right] =$$
$$= \sigma^2_\epsilon \frac{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)}N^{(2)}N^{(3)} - N^{(1)}T - N^{(2)}T - N^{(3)}T + N^{(1)} + N^{(2)} + N^{(3)} + 2T - 2}{N^{(1)}N^{(2)}N^{(3)}T}$$

$$E\left[u^2_{ijst}\right] = \sigma^2_\mu + \sigma^{(1)^2}_v + \sigma^{(2)^2}_v + \sigma^{(3)^2}_v + \sigma^2_\epsilon$$

$$E\left[\left(\frac{1}{T}\sum_{t=1}^T u_{ijst}\right)^2\right] = \sigma^2_\mu + \frac{1}{T}\sigma^{(1)^2}_v + \frac{1}{T}\sigma^{(2)^2}_v + \frac{1}{T}\sigma^{(3)^2}_v + \frac{1}{T}\sigma^2_\epsilon$$

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(1)}}\sigma^2_\mu + \frac{1}{N^{(1)}}\sigma^{(1)^2}_v + \sigma^{(2)^2}_v + \sigma^{(3)^2}_v + \frac{1}{N^{(1)}}\sigma^2_\epsilon$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(2)}}\sigma^2_\mu + \sigma^{(1)^2}_v + \frac{1}{N^{(2)}}\sigma^{(2)^2}_v + \sigma^{(3)^2}_v + \frac{1}{N^{(2)}}\sigma^2_\epsilon$$

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The estimators of the variance components now are

$$\hat{\sigma}_{\epsilon}^{2}=\frac{N^{(1)}N^{(2)}N^{(3)}T}{N^{(1)}N^{(2)}N^{(3)}T-N^{(1)}N^{(2)}N^{(3)}-N^{(1)}T-N^{(2)}T-N^{(3)}T+A}\hat{u}'_{within}\hat{u}_{within}$$

with $A=N^{(1)}+N^{(2)}+N^{(3)}+2T-2$

$$\hat{\sigma}_{\mu}^{2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T(T-1)}\sum_{i}^{N^{(1)}}\sum_{j}^{N^{(2)}}\sum_{s}^{N^{(3)}}\left(\left(\sum_{t}^{T}\hat{u}_{ijst}\right)^{2}-\sum_{t}^{T}\hat{u}_{ijst}^{2}\right)$$

$$\hat{\sigma}_{v}^{(1)^{2}}=\frac{1}{\left(N^{(1)}-1\right)N^{(2)}N^{(3)}T}\sum_{j}^{N^{(2)}}\sum_{s}^{N^{(3)}}\sum_{t}^{T}\left(\sum_{i}^{N^{(1)}}\hat{u}_{ijst}^{2}-\frac{1}{N^{(1)}}\left(\sum_{i}^{N^{(1)}}\hat{u}_{ijst}\right)^{2}\right)-\hat{\sigma}_{\mu}^{2}-\hat{\sigma}_{\epsilon}^{2}$$

$$\hat{\sigma}_{v}^{(2)^{2}}=\frac{1}{N^{(1)}\left(N^{(2)}-1\right)N^{(3)}T}\sum_{i}^{N^{(1)}}\sum_{s}^{N^{(3)}}\sum_{t}^{T}\left(\sum_{j}^{N^{(2)}}\hat{u}_{ijst}^{2}-\frac{1}{N^{(2)}}\left(\sum_{j}^{N^{(2)}}\hat{u}_{ijst}\right)^{2}\right)-\hat{\sigma}_{\mu}^{2}-\hat{\sigma}_{\epsilon}^{2}$$

$$\hat{\sigma}_{v}^{(3)^{2}}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_{i}^{N^{(1)}}\sum_{j}^{N^{(2)}}\sum_{s}^{N^{(3)}}\sum_{t}^{T}\hat{u}_{ijst}^{2}-\hat{\sigma}_{\mu}^{2}-\hat{\sigma}_{v}^{(1)^{2}}-\hat{\sigma}_{v}^{(2)^{2}}-\hat{\sigma}_{\epsilon}^{2}$$

Model (7)

The Within transformation for this last model is

$$u_{ijst}-\bar{u}_{ij}-\bar{u}_{is}-\bar{u}_{js}-\bar{u}_{it}-\bar{u}_{jt}-\bar{u}_{st}+2\bar{u}_{i}+2\bar{u}_{j}+2\bar{u}_{s}+2\bar{u}_{t}-3\bar{u}$$

The identifying equations are

$$E\left[\left(u_{ijst}-\bar{u}_{ij}-\bar{u}_{is}-\bar{u}_{js}-\bar{u}_{it}-\bar{u}_{jt}-\bar{u}_{st}+2\bar{u}_{i}+2\bar{u}_{j}+2\bar{u}_{s}+2\bar{u}_{t}-3\bar{u}\right)^{2}\right]=$$

$$=\sigma_{\epsilon}^{2}\frac{N^{(1)}N^{(2)}N^{(3)}T-N^{(1)}N^{(2)}-N^{(1)}N^{(3)}-N^{(2)}N^{(3)}-N^{(1)}T-N^{(2)}T-N^{(3)}T+A}{N^{(1)}N^{(2)}N^{(3)}T}$$

with $A=2N^{(1)}+2N^{(2)}+2N^{(3)}+2T-3$

$$E\left[u_{ijst}^{2}\right]=\sigma_{\mu}^{(1)^{2}}+\sigma_{\mu}^{(2)^{2}}+\sigma_{\mu}^{(3)^{2}}+\sigma_{v}^{(1)^{2}}+\sigma_{v}^{(2)^{2}}+\sigma_{v}^{(3)^{2}}+\sigma_{\epsilon}^{2}$$

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^{2}\right]=\frac{1}{N^{(1)}}\sigma_{\mu}^{(1)^{2}}+\frac{1}{N^{(1)}}\sigma_{\mu}^{(2)^{2}}+\sigma_{\mu}^{(3)^{2}}+\frac{1}{N^{(1)}}\sigma_{v}^{(1)^{2}}+\sigma_{v}^{(2)^{2}}+\sigma_{v}^{(3)^{2}}+\frac{1}{N^{(1)}}\sigma_{\epsilon}^{2}$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^{2}\right]=\frac{1}{N^{(2)}}\sigma_{\mu}^{(1)^{2}}+\sigma_{\mu}^{(2)^{2}}+\frac{1}{N^{(2)}}\sigma_{\mu}^{(3)^{2}}+\sigma_{v}^{(1)^{2}}+\frac{1}{N^{(2)}}\sigma_{v}^{(2)^{2}}+\sigma_{v}^{(3)^{2}}+\frac{1}{N^{(2)}}\sigma_{\epsilon}^{2}$$

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$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\sigma_\mu^{(1)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(3)^2}+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\frac{1}{N^{(3)}}\sigma_v^{(3)^2}+\frac{1}{N^{(3)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}N^{(2)}}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}N^{(2)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(1)}}\sigma_v^{(1)^2}+$$

$$+\frac{1}{N^{(2)}}\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\frac{1}{N^{(1)}N^{(2)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}N^{(3)}}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(1)}}\sigma_v^{(1)^2}+$$

$$+\sigma_v^{(2)^2}+\frac{1}{N^{(3)}}\sigma_v^{(3)^2}+\frac{1}{N^{(1)}N^{(3)}}\sigma_\epsilon^2$$

Finally, the estimators of the variance components are

$$\hat\sigma_\epsilon^2=\frac{N^{(1)}N^{(2)}N^{(3)}T}{N^{(1)}N^{(2)}N^{(3)}T-N^{(1)}N^{(2)}-N^{(1)}N^{(3)}-N^{(2)}N^{(3)}+A}\hat u'_{within}\hat u_{within}$$

$$\text{with } A=-N^{(1)}T-N^{(2)}T-N^{(3)}T+2N^{(1)}+2N^{(2)}+2N^{(3)}+2T-3$$

$$\hat\sigma_v^{(3)^2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)\left(N^{(2)}-1\right)}\times$$

$$\times\sum_s^{N^{(3)}}\sum_t^{T}\left(\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\hat u_{ijst}^2-\sum_i^{N^{(1)}}\left(\sum_j^{N^{(2)}}\hat u_{ijst}\right)^2-\sum_j^{N^{(2)}}\left(\sum_i^{N^{(1)}}\hat u_{ijst}\right)^2+\left(\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\hat u_{ijst}\right)^2\right)$$

$$\hat\sigma_v^{(2)^2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)\left(N^{(3)}-1\right)}\times$$

$$\times\sum_j^{N^{(2)}}\sum_t^{T}\left(\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\hat u_{ijst}^2-\sum_i^{N^{(1)}}\left(\sum_s^{N^{(3)}}\hat u_{ijst}\right)^2-\sum_s^{N^{(3)}}\left(\sum_i^{N^{(1)}}\hat u_{ijst}\right)^2+\left(\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\hat u_{ijst}\right)^2\right)$$

$$\hat\sigma_v^{(1)^2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)\left(N^{(3)}-1\right)}\times$$

$$\times\sum_i^{N^{(2)}}\sum_t^{T}\left(\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\hat u_{ijst}^2-\sum_j^{N^{(2)}}\left(\sum_s^{N^{(3)}}\hat u_{ijst}\right)^2-\sum_s^{N^{(3)}}\left(\sum_j^{N^{(2)}}\hat u_{ijst}\right)^2+\left(\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\hat u_{ijst}\right)^2\right)$$

$$\hat\sigma_\mu^{(3)^2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\left(\left(\sum_i^{N^{(1)}}\hat u_{ijst}\right)^2-\sum_i^{N^{(1)}}\hat u_{ijst}^2\right)-\hat\sigma_v^{(2)^2}-\hat\sigma_v^{(3)^2}$$

$$\hat\sigma_\mu^{(2)^2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\sum_t^{T}\left(\left(\sum_j^{N^{(2)}}\hat u_{ijst}\right)^2-\sum_j^{N^{(2)}}\hat u_{ijst}^2\right)-\hat\sigma_v^{(1)^2}-\hat\sigma_v^{(3)^2}$$

$$\hat\sigma_\mu^{(1)^2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\hat u_{ijst}^2-\hat\sigma_\mu^{(2)^2}-\hat\sigma_\mu^{(3)^2}-\hat\sigma_v^{(1)^2}-\hat\sigma_v^{(2)^2}-\hat\sigma_v^{(3)^2}-\hat\sigma_\epsilon^2$$

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4.3 Covariance Matrixes of the Models with Cross Correlations

Models (2), (3) and (6)

Both for models (2), (3) and (6) I have

$$E[\mu_{ij}\mu_{ij}'] = \sigma_\mu^2 I_{N^{(3)}} \otimes J_T + \rho_{(3)} \left(J_{N^{(3)}T} - I_{N^{(3)}} \otimes J_T\right)$$

$$E[\mu_i\mu_i'] = \sigma_\mu^2 I_{N^{(2)}N^{(3)}} \otimes J_T + \rho_{(3)} \left(I_{N^{(2)}} \otimes J_{N^{(3)}T} - I_{N^{(2)}N^{(3)}} \otimes J_T\right) +$$
$$+ \rho_{(2)} \left(\left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes \left(I_{N^{(3)}} \otimes J_T\right)\right)$$

$$E[\mu\mu'] = \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \rho_{(3)} \left(I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} - I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T\right) +$$
$$+ \rho_{(2)} \left(I_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes \left(I_{N^{(3)}} \otimes J_T\right)\right) + \rho_{(1)} \left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}N^{(3)}} \otimes J_T\right)$$

Thus, the covariance matrix of model (2) takes the form

$$E[uu'] = \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \rho_{(3)} \left(I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} - I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T\right) +$$
$$+ \rho_{(2)} \left(I_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes \left(I_{N^{(3)}} \otimes J_T\right)\right) + \rho_{(1)} \left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}N^{(3)}} \otimes J_T\right) +$$
$$+ \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}$$

and the covariance matrix of model (3) looks like

$$E[uu'] = \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \rho_{(3)} \left(I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} - I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T\right) +$$
$$+ \rho_{(2)} \left(I_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes \left(I_{N^{(3)}} \otimes J_T\right)\right) + \rho_{(1)} \left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}N^{(3)}} \otimes J_T\right) +$$
$$+ \sigma_\lambda^2 J_{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}$$

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Finally, the covariance matrix of model (6) is

$$E[uu'] = \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \rho_{(3)} \left(I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} - I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T\right) +$$
$$+ \rho_{(2)} \left(I_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes \left(I_{N^{(3)}} \otimes J_T\right)\right) + \rho_{(1)} \left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}N^{(3)}} \otimes J_T\right) +$$
$$+ \sigma_v^{(1)^2} I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}} \otimes I_T + \sigma_v^{(2)^2} J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}} \otimes I_T +$$
$$+ \sigma_v^{(3)^2} J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}T} + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}$$

Models (4), (5) and (7)

The covariance matrixes of models (4), (5) and (7) are slightly more complicated as there more variance components to take into account. The covariance matrix of model (4) now is

$$E[uu'] = \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \rho_{(2)}^{(1)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T}\right) +$$
$$+ \rho_{(1)}^{(1)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}} \otimes J_{N^{(3)}T}\right)\right) + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T +$$
$$+ \rho_{(2)}^{(2)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) +$$
$$+ \rho_{(1)}^{(2)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) + \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T +$$
$$+ \rho_{(2)}^{(3)} \left(J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}T} - J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T\right) +$$
$$+ \rho_{(1)}^{(3)} \left(J_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes I_{N^{(3)}} \otimes J_T\right) + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}$$

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For model (5) I get

$$E[uu'] = \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \rho_{(2)}^{(1)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T}\right) +$$

$$+ \rho_{(1)}^{(1)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}} \otimes J_{N^{(3)}T}\right)\right) + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T +$$

$$+ \rho_{(2)}^{(2)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) +$$

$$+ \rho_{(1)}^{(2)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) +$$

$$+ \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T + \rho_{(2)}^{(3)} \left(J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}T} - J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T\right) +$$

$$+ \rho_{(1)}^{(3)} \left(J_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes I_{N^{(3)}} \otimes J_T\right) + \sigma_\lambda^2 J_{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}$$

Finally, for model (7) I get the covariance matrix

$$E[uu'] = \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \rho_{(2)}^{(1)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T}\right) +$$

$$+ \rho_{(1)}^{(1)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}} \otimes J_{N^{(3)}T}\right)\right) + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T +$$

$$+ \rho_{(2)}^{(2)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) +$$

$$+ \rho_{(1)}^{(2)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) + \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T +$$

$$+ \rho_{(2)}^{(3)} \left(J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}T} - J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T\right) +$$

$$+ \rho_{(1)}^{(3)} \left(J_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes I_{N^{(3)}} \otimes J_T\right) + \sigma_v^{(1)^2} I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}} \otimes I_T +$$

$$+ \sigma_v^{(2)^2} J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}} \otimes I_T + \sigma_v^{(3)^2} J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}T} + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}$$

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4.4 Estimation of the Variance Components and Cross Correlations

Models (2) and (3)

The estimation of the variance components for models (2) and (3) does not change in this case, but of course the cross correlation coefficients need to be estimated. For model (3) the identifying equation are

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^2+\frac{1}{N^{(1)}}\sigma_\epsilon^2+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(1)}$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(2)}}\sigma_\mu^2+\frac{1}{N^{(2)}}\sigma_\epsilon^2+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(2)}$$

$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(3)}}\sigma_\mu^2+\frac{1}{N^{(3)}}\sigma_\epsilon^2+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(3)}$$

So I get

$$\hat{\rho}_{(1)}=\frac{1}{\left(N^{(1)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{i=1}^{N^{(1)}}\hat{u}_{ijst}\right)^2-\frac{1}{N^{(1)}-1}\hat{\sigma}_\mu^2-\frac{1}{N^{(1)}-1}\hat{\sigma}_\epsilon^2$$

$$\hat{\rho}_{(2)}=\frac{1}{\left(N^{(2)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2-\frac{1}{N^{(2)}-1}\hat{\sigma}_\mu^2-\frac{1}{N^{(2)}-1}\hat{\sigma}_\epsilon^2$$

$$\hat{\rho}_{(3)}=\frac{1}{\left(N^{(3)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\left(\sum_{s=1}^{N^{(3)}}\hat{u}_{ijst}\right)^2-\frac{1}{N^{(3)}-1}\hat{\sigma}_\mu^2-\frac{1}{N^{(3)}-1}\hat{\sigma}_\epsilon^2$$

Turning our attention to model (3) now

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^2+\sigma_\lambda^2+\frac{1}{N^{(1)}}\sigma_\epsilon^2+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(1)}$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(2)}}\sigma_\mu^2+\sigma_\lambda^2+\frac{1}{N^{(2)}}\sigma_\epsilon^2+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(2)}$$

$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(3)}}\sigma_\mu^2+\sigma_\lambda^2+\frac{1}{N^{(3)}}\sigma_\epsilon^2+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(3)}$$

and so

$$\hat{\rho}_{(1)}=\frac{1}{\left(N^{(1)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\times$$
$$\times\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{i=1}^{N^{(1)}}\hat{u}_{ijst}\right)^2-\frac{1}{N^{(1)}-1}\hat{\sigma}_\mu^2-\frac{N^{(1)}}{N^{(1)}-1}\hat{\sigma}_\lambda^2-\frac{1}{N^{(1)}-1}\hat{\sigma}_\epsilon^2$$

$$\hat{\rho}_{(2)}=\frac{1}{\left(N^{(2)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\times$$
$$\times\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2-\frac{1}{N^{(2)}-1}\hat{\sigma}_\mu^2-\frac{N^{(2)}}{N^{(2)}-1}\hat{\sigma}_\lambda^2-\frac{1}{N^{(2)}-1}\hat{\sigma}_\epsilon^2$$

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$$\hat{\rho}_{(3)} = \frac{1}{\left(N^{(3)}-1\right) N^{(1)}N^{(2)}N^{(3)}T} \times$$
$$\times \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{t=1}^{T} \left(\sum_{s=1}^{N^{(3)}} \hat{u}_{ijst} \right)^2 - \frac{1}{N^{(3)}-1}\hat{\sigma}_\mu^2 - \frac{N^{(3)}}{N^{(3)}-1}\hat{\sigma}_\lambda^2 - \frac{1}{N^{(3)}-1}\hat{\sigma}_\epsilon^2$$

Model (6)

In the case of model (6) the estimation of the variance components of ϵ and μ remain unchanged (i.e., are as in the case of the model without cross correlation); otherwise

$$E\left[u_{ijst}^2\right] = \sigma_\mu^2 + \sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \sigma_\epsilon^2$$
$$E\left[\left(\frac{1}{N^{(1)}} \sum_{i=1}^{N^{(1)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(1)}}\sigma_\mu^2 + \frac{1}{N^{(1)}}\sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \frac{1}{N^{(1)}}\sigma_\epsilon^2 + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(1)}$$
$$E\left[\left(\frac{1}{N^{(2)}} \sum_{j=1}^{N^{(2)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(2)}}\sigma_\mu^2 + \sigma_v^{(1)^2} + \frac{1}{N^{(2)}}\sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \frac{1}{N^{(2)}}\sigma_\epsilon^2 + \frac{N^{(2)}-1}{N^{(2)}}\rho_{(2)}$$
$$E\left[\left(\frac{1}{N^{(3)}} \sum_{s=1}^{N^{(3)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(3)}}\sigma_\mu^2 + \sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \frac{1}{N^{(3)}}\sigma_v^{(3)^2} + \frac{1}{N^{(3)}}\sigma_\epsilon^2 + \frac{N^{(3)}-1}{N^{(3)}}\rho_{(3)}$$
$$E\left[\left(\frac{1}{N^{(1)}T} \sum_{i=1}^{N^{(1)}} \sum_{t=1}^{T} u_{ijst}\right)^2\right] = \frac{1}{N^{(1)}}\sigma_\mu^2 + \frac{1}{N^{(1)}T}\sigma_v^{(1)^2} + \frac{1}{T}\sigma_v^{(2)^2} + \frac{1}{T}\sigma_v^{(3)^2} +$$
$$+ \frac{1}{N^{(1)}T}\sigma_\epsilon^2 + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(1)}$$
$$E\left[\left(\frac{1}{N^{(2)}T} \sum_{j=1}^{N^{(2)}} \sum_{t=1}^{T} u_{ijst}\right)^2\right] = \frac{1}{N^{(2)}}\sigma_\mu^2 + \frac{1}{T}\sigma_v^{(1)^2} + \frac{1}{N^{(2)}T}\sigma_v^{(2)^2} + \frac{1}{T}\sigma_v^{(3)^2} + \frac{1}{N^{(2)}T}\sigma_\epsilon^2 +$$
$$+ \frac{N^{(2)}-1}{N^{(2)}}\rho_{(2)}$$

So I get for the estimation of the cross correlations and the variance components

$$\hat{\rho}_{(1)} = \frac{1}{(T-1)\left(N^{(1)}-1\right) N^{(1)}N^{(2)}N^{(3)}T} \times$$
$$\times \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \left[\left(\sum_{i=1}^{N^{(1)}} \sum_{t=1}^{T} \hat{u}_{ijst} \right)^2 - \sum_{t=1}^{T} \left(\sum_{i=1}^{N^{(1)}} \hat{u}_{ijst} \right)^2 \right] - \frac{1}{N^{(1)}-1}\hat{\sigma}_\mu^2$$

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$$\hat{\rho}_{(2)} = \frac{1}{(T-1)\left(N^{(2)}-1\right)N^{(1)}N^{(2)}N^{(3)}T} \times$$
$$\times \sum_{i=1}^{N^{(1)}} \sum_{s=1}^{N^{(3)}} \left[\left(\sum_{j=1}^{N^{(2)}} \sum_{t=1}^{T} \hat{u}_{ijst} \right)^2 - \sum_{t=1}^{T} \left(\sum_{j=1}^{N^{(2)}} \hat{u}_{ijst} \right)^2 \right] - \frac{1}{N^{(2)}-1} \hat{\sigma}_{\mu}^2$$

$$\hat{\sigma}_v^{(1)^2} = \frac{1}{\left(N^{(1)}-1\right)N^{(1)}N^{(2)}N^{(3)}T} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \sum_{t}^{T} \left(N^{(1)} \sum_{i=1}^{N^{(1)}} \hat{u}_{ijst}^2 - \left(\sum_{i=1}^{N^{(1)}} \hat{u}_{ijst} \right)^2 \right) + \hat{\rho}_{(1)}$$

$$\hat{\sigma}_v^{(2)^2} = \frac{1}{\left(N^{(2)}-1\right)N^{(1)}N^{(2)}N^{(3)}T} \sum_{i=1}^{N^{(1)}} \sum_{s=1}^{N^{(3)}} \sum_{t}^{T} \left(N^{(2)} \sum_{j=1}^{N^{(2)}} \hat{u}_{ijst}^2 - \left(\sum_{j=1}^{N^{(2)}} \hat{u}_{ijst} \right)^2 \right) + \hat{\rho}_{(2)}$$

$$\hat{\sigma}_v^{(3)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \hat{u}_{ijst}^2 - \hat{\sigma}_{\mu}^2 - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(2)^2} - \hat{\sigma}_{\epsilon}^2$$

$$\hat{\rho_{(3)}} = \frac{1}{\left(N^{(3)}-1\right)N^{(1)}N^{(2)}N^{(3)}T} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{t}^{T} \left(\sum_{s=1}^{N^{(3)}} \hat{u}_{ijst} \right)^2 - \frac{1}{N^{(3)}-1} \hat{\sigma}_{\mu}^2 - \frac{N^{(3)}}{N^{(3)}-1} \hat{\sigma}_v^{(1)^2} -$$
$$- \frac{N^{(3)}}{N^{(3)}-1} \hat{\sigma}_v^{(2)^2} - \frac{1}{N^{(3)}-1} \hat{\sigma}_v^{(3)^2} - \frac{1}{N^{(3)}-1} \hat{\sigma}_{\epsilon}^2$$

Model (4)

For model (4) the Within transformation remains as for the model without cross correlation, so the estimation of the variance of ϵ is exactly as in section 3.2. Overall, the following identifying equations can be derived

$$E\left[u_{ijst}^2\right] = \sigma_{\mu}^{(1)^2} + \sigma_{\mu}^{(2)^2} + \sigma_{\mu}^{(3)^2} + \sigma_{\epsilon}^2$$

$$E\left[\left(\frac{1}{N^{(1)}} \sum_{i=1}^{N^{(1)}} u_{ijst} \right)^2 \right] = \frac{1}{N^{(1)}} \sigma_{\mu}^{(1)^2} + \frac{1}{N^{(1)}} \sigma_{\mu}^{(2)^2} + \sigma_{\mu}^{(3)^2} + \frac{1}{N^{(1)}} \sigma_{\epsilon}^2 + \frac{N^{(1)}-1}{N^{(1)}} \rho_{(1)}^{(1)} +$$
$$+ \frac{N^{(1)}-1}{N^{(1)}} \rho_{(1)}^{(2)}$$

$$E\left[\left(\frac{1}{N^{(2)}} \sum_{j=1}^{N^{(2)}} u_{ijst} \right)^2 \right] = \frac{1}{N^{(2)}} \sigma_{\mu}^{(1)^2} + \sigma_{\mu}^{(2)^2} + \frac{1}{N^{(2)}} \sigma_{\mu}^{(3)^2} + \frac{1}{N^{(2)}} \sigma_{\epsilon}^2 + \frac{N^{(2)}-1}{N^{(2)}} \rho_{(2)}^{(1)} +$$
$$+ \frac{N^{(2)}-1}{N^{(2)}} \rho_{(1)}^{(3)}$$

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$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\sigma_\mu^{(1)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(3)}}\sigma_\epsilon^2+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(2)}^{(2)}+$$
$$+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(2)}^{(3)}$$
$$E\left[\left(\frac{1}{N^{(1)}N^{(2)}}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}N^{(2)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(1)}N^{(2)}}\sigma_\epsilon^2+$$
$$+\frac{N^{(1)}-1}{N^{(1)}N^{(2)}}\rho_{(1)}^{(1)}+\frac{N^{(1)}-1}{N^{(1)}N^{(2)}}\rho_{(1)}^{(2)}+\frac{N^{(2)}-1}{N^{(2)}N^{(1)}}\rho_{(2)}^{(1)}+\frac{N^{(2)}-1}{N^{(2)}N^{(1)}}\rho_{(1)}^{(3)}$$
$$E\left[\left(\frac{1}{N^{(1)}N^{(3)}}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(1)}N^{(3)}}\sigma_\epsilon^2+$$
$$+\frac{N^{(1)}-1}{N^{(1)}N^{(3)}}\rho_{(1)}^{(1)}+\frac{N^{(1)}-1}{N^{(1)}N^{(3)}}\rho_{(1)}^{(2)}+\frac{N^{(3)}-1}{N^{(1)}N^{(3)}}\rho_{(2)}^{(2)}+\frac{N^{(3)}-1}{N^{(1)}N^{(3)}}\rho_{(2)}^{(3)}$$
$$E\left[\left(\frac{1}{N^{(2)}N^{(3)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(2)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}N^{(3)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(2)}N^{(3)}}\sigma_\epsilon^2+$$
$$+\frac{N^{(2)}-1}{N^{(2)}N^{(3)}}\rho_{(2)}^{(1)}+\frac{N^{(2)}-1}{N^{(2)}N^{(3)}}\rho_{(1)}^{(3)}+\frac{N^{(3)}-1}{N^{(2)}N^{(3)}}\rho_{(2)}^{(2)}+\frac{N^{(3)}-1}{N^{(2)}N^{(3)}}\rho_{(2)}^{(3)}$$
$$E\left[\left(\frac{1}{N^{(1)}N^{(2)}N^{(3)}}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}N^{(2)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}N^{(3)}}\sigma_\mu^{(3)^2}+$$
$$+\frac{1}{N^{(1)}N^{(2)}N^{(3)}}\sigma_\epsilon^2+\frac{N^{(1)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(1)}^{(1)}+\frac{N^{(1)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(1)}^{(2)}+\frac{N^{(2)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(2)}^{(1)}+$$
$$+\frac{N^{(2)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(1)}^{(3)}+\frac{N^{(3)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(2)}^{(2)}+\frac{N^{(3)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(2)}^{(3)}$$

Altogether I have 8 identifying equations but unfortunately 9 unknown variance components and correlation coefficients. These cannot be estimated without further restrictions on the parameters. Let me impose the additional assumption that $\sigma_\mu^{(1)^2}=\sigma_\mu^{(2)^2}=\sigma_\mu^{(3)^2}=\sigma_\mu^2$. Under this assumption I need to estimate only 7 unknown parameters. From the first identifying equation

$$\hat{\sigma}_\mu^2=\frac{1}{3N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\hat{u}_{ijst}^2-\frac{1}{3}\hat{\sigma}_\epsilon^2$$

Note however, that the above identifying equations $5-8$ are, unfortunately, linear combinations of the equations 2 – 5. This means that I need to impose further restrictions on

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the model.

Let us assume, in addition, that

$$E[\mu^{(1)}_{ij}\mu^{(1)}_{i'j'}] = \begin{cases} \sigma^2_\mu & i = i' \text{ and } j = j' \\ \rho_{(1)} & i = i' \text{ and } j \neq j' \\ \rho_{(2)} & i \neq i' \text{ and } j = j' \\ 0 & i \neq i' \text{ and } j \neq j' \end{cases}$$

$$E[\mu^{(2)}_{is}\mu^{(2)}_{i's'}] = \begin{cases} \sigma^2_\mu & i = i' \text{ and } s = s' \\ \rho_{(1)} & i = i' \text{ and } s \neq s' \\ \rho_{(3)} & i \neq i' \text{ and } s = s' \\ 0 & i \neq i' \text{ and } j \neq j' \end{cases}$$

$$E[\mu^{(3)}_{js}\mu^{(3)}_{j's'}] = \begin{cases} \sigma^2_\mu & j = j' \text{ and } s = s' \\ \rho_{(2)} & j = j' \text{ and } s \neq s' \\ \rho_{(3)} & j \neq j' \text{ and } s = s' \\ 0 & i \neq i' \text{ and } j \neq j' \end{cases}$$

Now, the identifying equations take the form

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}} u_{ijst}\right)^2\right] = \frac{2+N^{(1)}}{N^{(1)}}\sigma^2_\mu + \frac{1}{N^{(1)}}\sigma^2_\epsilon + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(2)} + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}} u_{ijst}\right)^2\right] = \frac{2+N^{(2)}}{N^{(2)}}\sigma^2_\mu + \frac{1}{N^{(2)}}\sigma^2_\epsilon + \frac{N^{(2)}-1}{N^{(2)}}\rho_{(1)} + \frac{N^{(2)}-1}{N^{(2)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}} u_{ijst}\right)^2\right] = \frac{2+N^{(3)}}{N^{(3)}}\sigma^2_\mu + \frac{1}{N^{(3)}}\sigma^2_\epsilon + \frac{N^{(3)}-1}{N^{(3)}}\rho_{(1)} + \frac{N^{(3)}-1}{N^{(3)}}\rho_{(2)}$$

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So I get the following estimators for the cross correlations

$$\hat{\rho}_{(1)} = \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)} \sum_{i=1}^{N^{(1)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \left(\sum_{j=1}^{N^{(2)}} \hat{u}_{ijst} \right)^2 +$$
$$+ \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{t=1}^{T} \left(\sum_{s=1}^{N^{(3)}} \hat{u}_{ijst} \right)^2 -$$
$$- \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \left(\sum_{i=1}^{N^{(1)}} \hat{u}_{ijst} \right)^2 -$$
$$- \hat{\sigma}_{\mu}^2 \frac{1}{2} \left(\frac{N^{(3)}+2}{N^{(3)}-1} + \frac{N^{(2)}+2}{N^{(2)}-1} - \frac{N^{(1)}+2}{N^{(1)}-1} \right) -$$
$$- \hat{\sigma}_{\epsilon}^2 \frac{1}{2} \left(\frac{1}{N^{(3)}-1} + \frac{1}{N^{(2)}-1} - \frac{1}{N^{(1)}-1} \right)$$
$$\hat{\rho}_{(2)} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{t=1}^{T} \left(\sum_{s=1}^{N^{(3)}} \hat{u}_{ijst} \right)^2 - \hat{\sigma}_{\mu}^2 \frac{N^{(3)}+2}{N^{(3)}-1} - \hat{\sigma}_{\epsilon}^2 \frac{1}{N^{(3)}-1} - \hat{\rho}_{(1)}$$
$$\hat{\rho}_{(3)} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)} \sum_{i=1}^{N^{(1)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \left(\sum_{j=1}^{N^{(2)}} \hat{u}_{ijst} \right)^2 - \hat{\sigma}_{\mu}^2 \frac{N^{(2)}+2}{N^{(2)}-1} - \hat{\sigma}_{\epsilon}^2 \frac{1}{N^{(2)}-1} - \hat{\rho}_{(1)}$$

Model (5)

Like for the previous model, the Within transformation is still as for the model without cross correlation, so the estimation of the variance of ϵ and that of λ remains as in section 3.2. Making the same assumption as above for model (4), the identifying equations now are

$$E[u_{ijst}^2] = 3\sigma_{\mu}^2 + \sigma_{\lambda}^2 + \sigma_{\epsilon}^2$$

$$E\left[\left(\frac{1}{N^{(1)}} \sum_{i=1}^{N^{(1)}} u_{ijst}\right)^2\right] = \frac{2+N^{(1)}}{N^{(1)}} \sigma_{\mu}^2 + \sigma_{\lambda}^2 + \frac{1}{N^{(1)}} \sigma_{\epsilon}^2 + \frac{N^{(1)}-1}{N^{(1)}} \rho_{(2)} + \frac{N^{(1)}-1}{N^{(1)}} \rho_{(3)}$$
$$E\left[\left(\frac{1}{N^{(2)}} \sum_{j=1}^{N^{(2)}} u_{ijst}\right)^2\right] = \frac{2+N^{(2)}}{N^{(2)}} \sigma_{\mu}^2 + \sigma_{\lambda}^2 + \frac{1}{N^{(2)}} \sigma_{\epsilon}^2 + \frac{N^{(2)}-1}{N^{(2)}} \rho_{(1)} + \frac{N^{(2)}-1}{N^{(2)}} \rho_{(3)}$$
$$E\left[\left(\frac{1}{N^{(3)}} \sum_{s=1}^{N^{(3)}} u_{ijst}\right)^2\right] = \frac{2+N^{(3)}}{N^{(3)}} \sigma_{\mu}^2 + \sigma_{\lambda}^2 + \frac{1}{N^{(3)}} \sigma_{\epsilon}^2 + \frac{N^{(3)}-1}{N^{(3)}} \rho_{(1)} + \frac{N^{(3)}-1}{N^{(3)}} \rho_{(2)}$$

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And so this leads to

$$\hat{\sigma}_{\mu}^{2}=\frac{1}{3N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\hat{u}_{ijst}^{2}-\frac{1}{3}\hat{\sigma}_{\lambda}^{2}-\frac{1}{3}\hat{\sigma}_{\epsilon}^{2}$$

$$\hat{\rho}_{(1)}=\frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^{2}+$$

$$+\frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\left(\sum_{s=1}^{N^{(3)}}\hat{u}_{ijst}\right)^{2}-$$

$$-\frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{i=1}^{N^{(1)}}\hat{u}_{ijst}\right)^{2}-$$

$$-\hat{\sigma}_{\mu}^{2}\frac{1}{2}\left(\frac{N^{(3)}+2}{N^{(3)}-1}+\frac{N^{(2)}+2}{N^{(2)}-1}-\frac{N^{(1)}+2}{N^{(1)}-1}\right)-\hat{\sigma}_{\lambda}^{2}\frac{1}{2}\left(\frac{N^{(3)}}{N^{(3)}-1}+\frac{N^{(2)}}{N^{(2)}-1}-\frac{N^{(1)}}{N^{(1)}-1}\right)-$$

$$-\hat{\sigma}_{\epsilon}^{2}\frac{1}{2}\left(\frac{1}{N^{(3)}-1}+\frac{1}{N^{(2)}-1}-\frac{1}{N^{(1)}-1}\right)$$

$$\hat{\rho}_{(2)}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\left(\sum_{s=1}^{N^{(3)}}\hat{u}_{ijst}\right)^{2}-\hat{\sigma}_{\mu}^{2}\frac{N^{(3)}+2}{N^{(3)}-1}-\hat{\sigma}_{\lambda}^{2}\frac{N^{(3)}}{N^{(3)}-1}-$$

$$-\hat{\sigma}_{\epsilon}^{2}\frac{1}{N^{(3)}-1}-\hat{\rho}_{(1)}$$

$$\hat{\rho}_{(3)}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^{2}-\hat{\sigma}_{\mu}^{2}\frac{N^{(2)}+2}{N^{(2)}-1}-\hat{\sigma}_{\lambda}^{2}\frac{N^{(2)}}{N^{(2)}-1}-$$

$$-\hat{\sigma}_{\epsilon}^{2}\frac{1}{N^{(2)}-1}-\hat{\rho}_{(1)}$$

Model (7)

Finally, for model (7), making the same additional parameter restrictions as for models (4) and (5) I get the following identifying equations

$$E[u_{ijst}^{2}]=3\sigma_{\mu}^{2}+\sigma_{v}^{(1)^{2}}+\sigma_{v}^{(2)^{2}}+\sigma_{v}^{(3)^{2}}+\sigma_{\epsilon}^{2}$$

$$E\left[\left(\frac{1}{T}\sum_{t=1}^{T}u_{ijst}\right)^{2}\right]=3\sigma_{\mu}^{2}+\frac{1}{T}\sigma_{v}^{(1)^{2}}+\frac{1}{T}\sigma_{v}^{(2)^{2}}+\frac{1}{T}\sigma_{v}^{(3)^{2}}+\frac{1}{T}\sigma_{\epsilon}^{2}$$

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$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^2\right]=\frac{2+N^{(1)}}{N^{(1)}}\sigma_\mu^2+\frac{1}{N^{(1)}}\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\frac{1}{N^{(1)}}\sigma_\epsilon^2+$$

$$+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(2)}+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{2+N^{(2)}}{N^{(2)}}\sigma_\mu^2+\sigma_v^{(1)^2}+\frac{1}{N^{(2)}}\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\frac{1}{N^{(2)}}\sigma_\epsilon^2+$$

$$+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(1)}+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{2+N^{(3)}}{N^{(3)}}\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\frac{1}{N^{(3)}}\sigma_v^{(3)^2}+\frac{1}{N^{(3)}}\sigma_\epsilon^2+$$

$$+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(1)}+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(2)}$$

$$E\left[\left(\frac{1}{N^{(1)}T}\sum_{i=1}^{N^{(1)}}\sum_{t=1}^{T}u_{ijst}\right)^2\right]=\frac{2+N^{(1)}}{N^{(1)}}\sigma_\mu^2+\frac{1}{N^{(1)}T}\sigma_v^{(1)^2}+\frac{1}{T}\sigma_v^{(2)^2}+\frac{1}{T}\sigma_v^{(3)^2}+$$

$$+\frac{1}{N^{(1)}T}\sigma_\epsilon^2+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(2)}+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(2)}T}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}u_{ijst}\right)^2\right]=\frac{2+N^{(2)}}{N^{(2)}}\sigma_\mu^2+\frac{1}{T}\sigma_v^{(1)^2}+\frac{1}{N^{(2)}T}\sigma_v^{(2)^2}+\frac{1}{T}\sigma_v^{(3)^2}+$$

$$+\frac{1}{N^{(2)}T}\sigma_\epsilon^2+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(1)}+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(3)}$$

which lead to the following estimators

$$\hat{\sigma}_\mu^2=\frac{1}{3N^{(1)}N^{(2)}N^{(3)}T(T-1)}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\left(\left(\sum_{t=1}^{T}\hat{u}_{ijst}\right)^2-\sum_{t=1}^{T}\hat{u}_{ijst}^2\right)$$

$$\hat{\sigma}_v^{(1)^2}=\frac{1}{\left(N^{(1)}-1\right)N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\hat{u}_{ijst}^2-$$

$$-\frac{1}{N^{(1)}\left(N^{(1)}-1\right)N^{(2)}N^{(3)}(T-1)}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{i=1}^{N^{(1)}}\hat{u}_{ijst}\right)^2+$$

$$+\frac{1}{N^{(1)}\left(N^{(1)}-1\right)N^{(2)}N^{(3)}T(T-1)}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\left(\sum_{i=1}^{N^{(1)}}\sum_{t=1}^{T}\hat{u}_{ijst}\right)^2-\frac{3N^{(1)}}{N^{(1)}-1}\hat{\sigma}_\mu^2-\hat{\sigma}_\epsilon^2$$

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$$\hat{\sigma}_v^{(2)^2} = \frac{1}{N^{(1)}\left(N^{(2)}-1\right)N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\hat{u}_{ijst}^2 -$$

$$-\frac{1}{N^{(1)}N^{(2)}\left(N^{(2)}-1\right)N^{(3)}(T-1)}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2 +$$

$$+\frac{1}{N^{(1)}N^{(2)}\left(N^{(2)}-1\right)N^{(3)}T(T-1)}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\left(\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\hat{u}_{ijst}\right)^2 - \frac{3N^{(2)}}{N^{(2)}-1}\hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_v^{(3)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\hat{u}_{ijst}^2 - 3\hat{\sigma}_\mu^2 - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(2)^2} - \hat{\sigma}_\epsilon^2$$

$$\hat{\rho}_{(1)} = \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2 +$$

$$+\frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\left(\sum_{s=1}^{N^{(3)}}\hat{u}_{ijst}\right)^2 -$$

$$-\frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{i=1}^{N^{(1)}}\hat{u}_{ijst}\right)^2 -$$

$$-\hat{\sigma}_\mu^2\frac{1}{2}\left(\frac{N^{(3)}+2}{N^{(3)}-1}+\frac{N^{(2)}+2}{N^{(2)}-1}-\frac{N^{(1)}+2}{N^{(1)}-1}\right) - \hat{\sigma}_\epsilon^2\frac{1}{2}\left(\frac{1}{N^{(3)}-1}+\frac{1}{N^{(2)}-1}-\frac{1}{N^{(1)}-1}\right) -$$

$$-\hat{\sigma}_v^{(1)^2}\frac{2N^{(1)}-1}{2N^{(1)}} - \hat{\sigma}_v^{(2)^2}\frac{1}{2N^{(2)}} - \hat{\sigma}_v^{(3)^2}\frac{1}{2N^{(3)}}$$

$$\hat{\rho}_{(2)} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\left(\sum_{s=1}^{N^{(3)}}\hat{u}_{ijst}\right)^2 - \hat{\sigma}_\mu^2\frac{N^{(3)}+2}{N^{(3)}-1} -$$

$$-\hat{\sigma}_\epsilon^2\frac{1}{N^{(3)}-1} - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(2)^2} - \frac{1}{N^{(3)}}\hat{\sigma}_v^{(3)^2} - \hat{\rho}_{(1)}$$

$$\hat{\rho}_{(3)} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2 - \hat{\sigma}_\mu^2\frac{N^{(2)}+2}{N^{(2)}-1} -$$

$$-\hat{\sigma}_\epsilon^2\frac{1}{N^{(2)}-1} - \hat{\sigma}_v^{(1)^2} - \frac{1}{N^{(2)}}\hat{\sigma}_v^{(2)^2} - \hat{\sigma}_v^{(3)^2} - \hat{\rho}_{(1)}$$

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Chapter 5

Unbalanced Data

In order to be able formalize the nature of the data here, some new notations need to be introduced. Let be:

- Z_{ijs} the set of time periods when firm i sells product j to country s, with T_{ijs} now being the number of elements in Z_{ijs}, $Z_{ijs} = \left\{ z_{ijs}^{1}, ..., z_{ijs}^{T_{ijs}} \right\}$;
- $Z_{js}^{(1)}$ the set of time periods when any firm sells product j to country s, with $T_{js}^{(1)}$ being the number of elements in $Z_{js}^{(1)}$;
- $Z_{is}^{(2)}$ the set of time periods when firm i sells any product to country s, with $T_{is}^{(2)}$ being the number of elements in $Z_{is}^{(2)}$;
- $Z_{ij}^{(3)}$ the set of time periods when firm i sells product j to any country, with $T_{ij}^{(3)}$ being the number of elements in $Z_{ij}^{(3)}$;
- $Z_{i}^{(1)}$ the set of time periods when firm i sells any product to any country, with $T_{i}^{(1)}$ being the number of elements in $Z_{i}^{(1)}$;
- $Z_{j}^{(2)}$ the set of time periods when any firm sells product j to any country, with $T_{j}^{(2)}$ being the number of elements in $Z_{j}^{(2)}$;

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- $Z_s^{(3)}$ the set of time periods when any firm sells any product to country s, with $T_s^{(3)}$ being number of elements in $Z_s^{(3)}$;

- $Q_{jst}^{(1)}$ the set of firms that sell product j to country s at period of time t, with $N_{jst}^{(1)}$ being the number of elements in $Q_{jst}^{(1)}$;

- $Q_{ist}^{(2)}$ the set of products that firm i sells to country s at period t, with $N_{ist}^{(2)}$ being the number of elements in $Q_{ist}^{(2)}$;

- $Q_{ijt}^{(3)}$ the set of countries to which firm i sells product j at time t, with $N_{ijt}^{(3)}$ being the number of elements in $Q_{ijt}^{(3)}$;

- $Q_{ij}^{(3)}$ the set of countries to which firm i sells product j at any time, with $N_{ij}^{(3)}$ being the number of elements in $Q_{ij}^{(3)}$;

- $Q_{st}^{(1)}$ the set of firms that sell any product to country s at period of time t, with $N_{st}^{(1)}$ being the number of elements in $Q_{st}^{(1)}$;

- $Q_{jt}^{(1)}$ the set of firms that sell product j to any country at period of time t, with $N_{jt}^{(1)}$ being the number of elements in $Q_{jt}^{(1)}$;

- $Q_{it}^{(2)}$ the set of products that firm i sells to any country at period t, with $N_{it}^{(2)}$ being the number of elements in $Q_{it}^{(2)}$;

- $Q_{i}^{(2)}$ the set of products that firm i sells to any country at any time, with $N_{i}^{(2)}$ being the number of elements in $Q_{i}^{(2)}$;

- $Q_{i}^{(3)}$ the set of countries to which firm i sells any product at any time, with $N_{i}^{(3)}$ being the number of elements in $Q_{i}^{(3)}$;

- $Q_{j}^{(3)}$ the set of countries to which any firm sells product j at any time, with $N_{j}^{(3)}$ being the number of elements in $Q_{j}^{(3)}$;

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- $Q^{(1)}$ the set of firms that at least sell a product to any country at any times, with $N^{(1)}$ being the number of elements in $Q^{(1)}$;

- $Q^{(2)}$ the set of products being sold by any firm to any country at any times, with $N^{(2)}$ being the number of elements in $Q^{(2)}$;

- $Q^{(3)}$ the set of countries to which any product has been sold by any firm at any times, with $N^{(3)}$ being the number of elements in $Q^{(3)}$.

5.1 Covariance Matrixes of the Different Models

Model (1)

For this model I have

$$u_{ijst} = \mu_i + \gamma_j + \alpha_s + \lambda_t + \epsilon_{ijst}$$

So I can build up the covariance matrix in the following way

$$u_{ijs} = \mu_i \otimes l_{T_{ijs}} + \gamma_j \otimes l_{T_{ijs}} + \alpha_s \otimes l_{T_{ijs}} + \lambda_{Z_{ijs}} + \epsilon_{ijs}$$

$$E\left[u_{ijs}{u'}_{ijs}\right] = \sigma_\mu^2 J_{T_{ijs}} + \sigma_\gamma^2 J_{T_{ijs}} + \sigma_\alpha^2 J_{T_{ijs}} + \sigma_\lambda^2 I_{T_{ijs}} + \sigma_\epsilon^2 I_{T_{ijs}}$$

$$u_{ij} = \mu_i \otimes l_{\sum_s T_{ijs}} + \gamma_j \otimes l_{\sum_s T_{ijs}} + \tilde{\alpha}_{ij} + \tilde{\lambda}_{ij} + \epsilon_{ij}$$

$$E\left[u_{ij}{u'}_{ij}\right] = \sigma_\mu^2 J_{\sum_s T_{ijs}} + \sigma_\gamma^2 J_{\sum_s T_{ijs}} + \sigma_\alpha^2 A_{ij} + \sigma_\lambda^2 D_{ij} + \sigma_\epsilon^2 I_{\sum_s T_{ijs}}$$

$$u_i = \mu_i \otimes l_{\sum_j \sum_s T_{ijs}} + \tilde{\gamma}_i + \tilde{\alpha}_i + \tilde{\lambda}_i + \epsilon_i$$

$$E\left[u_i{u'}_i\right] = \sigma_\mu^2 J_{\sum_j \sum_s T_{ijs}} + \sigma_\gamma^2 B_i + \sigma_\alpha^2 F_{i,i} + \sigma_\lambda^2 D_i + \sigma_\epsilon^2 I_{\sum_j \sum_s T_{ijs}}$$

$$u = \tilde{\mu} + \tilde{\gamma} + \tilde{\alpha} + \tilde{\lambda} + \epsilon$$

$$E\left[uu'\right] = \sigma_\mu^2 C + \sigma_\gamma^2 B + \sigma_\alpha^2 F + \sigma_\lambda^2 D + \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

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where

$$\tilde{\mu}' = \left(\underbrace{\mu_1 \dots \mu_1}_{\sum_j \sum_s T_{1js} \text{ times}} \quad \underbrace{\mu_2 \dots \mu_2}_{\sum_j \sum_s T_{2js} \text{ times}} \quad \cdots \quad \underbrace{\mu_N^{(1)} \dots \mu_N^{(1)}}_{\sum_j \sum_s T_{N^{(1)}js} \text{ times}} \right)$$

$$C = \begin{pmatrix} J_{\sum_j \sum_s T_{1js}} & 0 & \dots & 0 \\ 0 & J_{\sum_j \sum_s T_{2js}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\sum_j \sum_s T_{N^{(1)}js}} \end{pmatrix}$$

$$\tilde{\gamma}_i' = \left(\underbrace{\gamma_1 \cdots \gamma_1}_{\sum_s T_{i1s} \text{ times}} \quad \underbrace{\gamma_2 \cdots \gamma_2}_{\sum_s T_{i2s} \text{ times}} \quad \cdots \quad \underbrace{\gamma_N^{(2)} \cdots \gamma_N^{(2)}}_{\sum_s T_{iN^{(2)}s} \text{ times}} \right)$$

$$\tilde{\gamma}' = \left(\tilde{\gamma}_1', \tilde{\gamma}_2', ..., \tilde{\gamma}_{N^{(1)}}' \right)$$

$$B_i = \begin{pmatrix} J_{\sum_s T_{i1s}} & 0 & \dots & 0 \\ 0 & J_{\sum_s T_{i2s}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\sum_s T_{iN^{(2)}s}} \end{pmatrix}, \quad B = \begin{pmatrix} P_{1,1} & P_{1,2} & \dots & P_{1,N^{(1)}} \\ P_{2,1} & P_{2,2} & \dots & P_{2,N^{(1)}} \\ \vdots & \vdots & \ddots & \vdots \\ P_{N^{(1)},1} & P_{N^{(1)},2} & \dots & P_{N^{(1)},N^{(1)}} \end{pmatrix}$$

where

$$P_{i,p} = \begin{pmatrix} J_{\left(\sum_s T_{i1s} \times \sum_s T_{p1s}\right)} & 0 & \dots & 0 \\ 0 & J_{\left(\sum_s T_{i2s} \times \sum_s T_{p2s}\right)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\left(\sum_s T_{iN^{(2)}s} \times \sum_s T_{pN^{(2)}s}\right)} \end{pmatrix}$$

$$\tilde{\alpha}_{ij}' = \left(\underbrace{\alpha_1 \dots \alpha_1}_{T_{ij1} \text{times}} \quad \underbrace{\alpha_2 \dots \alpha_2}_{T_{ji2} \text{times}} \quad \cdots \quad \underbrace{\alpha_{N^{(3)}} \dots \alpha_{N^{(3)}}}_{T_{ijN^{(3)}} \text{ times}} \right)$$

$$\tilde{\alpha}_i' = \left(\tilde{\alpha}_{i1}', \tilde{\alpha}_{i2}', ..., \tilde{\alpha}_{iN^{(2)}}' \right), \quad \tilde{\alpha}' = \left(\tilde{\alpha}_1', \tilde{\alpha}_2', ..., \tilde{\alpha}_{N^{(1)}}' \right)$$

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$$A_{ij} = \begin{pmatrix} J_{T_{ij1}} & 0 & \dots & 0 \\ 0 & J_{T_{ij2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{T_{ijN^{(3)}}} \end{pmatrix}$$

$$K_{i,p}^{j,l} = \begin{pmatrix} J_{\left(T_{ij1} \times T_{pl1}\right)} & 0 & \dots & 0 \\ 0 & J_{\left(T_{ij2} \times T_{pl2}\right)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\left(T_{ijN^{(3)}} \times T_{plN^{(3)}}\right)} \end{pmatrix}$$

$$F_{i,p} = \begin{pmatrix} K_{i,p}^{1,1} & K_{i,p}^{1,2} & \dots & K_{i,p}^{1,N^{(2)}} \\ K_{i,p}^{2,1} & K_{i,p}^{2,2} & \dots & K_{i,p}^{2,N^{(2)}} \\ \vdots & \vdots & \ddots & \vdots \\ K_{i,p}^{N^{(2)},1} & K_{i,p}^{N^{(2)},2} & \dots & K_{i,p}^{N^{(2)},N^{(2)}} \end{pmatrix}, \quad F = \begin{pmatrix} F_{1,1} & F_{1,2} & \dots & F_{1,N^{(1)}} \\ F_{2,1} & F_{2,2} & \dots & F_{2,N^{(1)}} \\ \vdots & \vdots & \ddots & \vdots \\ F_{N^{(1)},1} & F_{N^{(1)},2} & \dots & F_{N^{(1)},N^{(1)}} \end{pmatrix}$$

Let me denote by $\lambda_{Z_{ijs}}$ the vector of length T_{ijs} of time effects associated with time periods from the set Z_{ijs}. Then $\tilde{\lambda}'_{ij} = \left(\lambda_{Z_{ij1}}, ..., \lambda_{Z_{ijN^{(3)}}}\right)$ and

$$\tilde{\lambda}'_i = \left(\tilde{\lambda}'_{i1}\tilde{\lambda}'_{i2}, ..., \tilde{\lambda}'_{iN^{(2)}}\right)$$

$$\tilde{\lambda}' = \left(\tilde{\lambda}'_1, \tilde{\lambda}'_2, ..., \tilde{\lambda}'_{N^{(1)}}\right)$$

Now about the $M_{T_{ijs} \times T_{lpr}}$ matrix of size $(T_{ijs} \times T_{lpr})$. Element of n-th row and k-th column is

$$\{m\}_{nk} = \begin{cases} 1 & \text{if } z_{ijs}^n = z_{lpr}^k \\ 0 & \text{otherwise} \end{cases}$$

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Further,

$$E_s = \begin{pmatrix} M_{T_{ij1} \times T_{ijs}} \\ M_{T_{ij2} \times T_{ijs}} \\ \dots \\ M_{T_{ijN^{(3)}} \times T_{ijs}} \end{pmatrix}, \quad D_{ij} = (E_1, E_2, ..., E_{N^{(3)}})$$

$$E_{js} = \begin{pmatrix} M_{T_{i11} \times T_{ijs}} \\ M_{T_{i12} \times T_{ijs}} \\ \dots \\ M_{T_{iN^{(2)}N^{(3)}} \times T_{ijs}} \end{pmatrix}, \quad D_i = (E_{11}, E_{12}, ..., E_{N^{(2)}N^{(3)}})$$

$$E_{ijs} = \begin{pmatrix} M_{T_{111} \times T_{ijs}} \\ M_{T_{112} \times T_{ijs}} \\ \dots \\ M_{T_{N^{(1)}N^{(2)}N^{(3)}} \times T_{ijs}} \end{pmatrix}, \quad D = (E_{111}, E_{112}, ..., E_{N^{(1)}N^{(2)}N^{(3)}})$$

Model (2)

This is a slightly simpler case than model (1) above. I have

$$u_{ijst} = \mu_{ijs} + \epsilon_{ijst}$$

So I can build up the covariance matrix in the usual way

$$u_{ijs} = \mu_{ijs} \otimes l_{T_{ijs}} + \epsilon_{ijst}$$

$$E\left[u_{ijs}u'_{ijs}\right] = \sigma_\mu^2 J_{T_{ijs}} + \sigma_\epsilon^2 I_{T_{ijs}}$$

$$u_{ij} = \tilde{\mu}_{ij} + \epsilon_{ij}$$

$$E\left[u_{ij}u'_{ij}\right] = \sigma_\mu^2 A_{ij} + \sigma_\epsilon^2 I_{\sum_s T_{ijs}}$$

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$$u_i = \tilde{\mu}_i + \epsilon_i$$

$$E\left[u_i u_i'\right] = \sigma_\mu^2 A_i + \sigma_\epsilon^2 I_{\sum_j \sum_s T_{ijs}}$$

$$u = \tilde{\mu} + \epsilon$$

$$E\left[u u'\right] = \sigma_\mu^2 A + \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

where

$$\tilde{\mu}_{ij}' = \left(\underbrace{\mu_{ij1} \ldots \mu_{ij1}}_{T_{ij1}\,\text{times}} \;\; \underbrace{\mu_{ij2} \ldots \mu_{ij2}}_{T_{ij2}\,\text{times}} \;\; \cdots \;\; \underbrace{\mu_{ijN^{(3)}} \ldots \mu_{ijN^{(3)}}}_{T_{ijN^{(3)}}\,\text{times}} \right)$$

$$\tilde{\mu}_i' = \left(\tilde{\mu}_{i1}', \tilde{\mu}_{i2}', ..., \tilde{\mu}_{i3}'\right), \quad \tilde{\mu}' = \left(\tilde{\mu}_1', \tilde{\mu}_2', ..., \tilde{\mu}_3'\right)$$

$$A_i = \begin{pmatrix} A_{i1} & 0 & \ldots & 0 \\ 0 & A_{i2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{iN^{(2)}} \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & 0 & \ldots & 0 \\ 0 & A_{12} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{N^{(1)}N^{(2)}} \end{pmatrix}$$

Model (3)

Now I have

$$u_{ijst} = \mu_{ijs} + \lambda_t + \epsilon_{ijst}$$

I have already derived the covariance matrix of μ_{ijs} in model (2) and λ_t in model (2). Thus for model (3) covariance matrix takes the form

$$E\left[u u'\right] = \sigma_\mu^2 A + \sigma_\lambda^2 D + \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

Model (4)

The composition of the disturbance term now is

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \epsilon_{ijst}$$

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So I have

$$
\begin{aligned}
u_{ijs} &= \mu^{(1)}_{ij} \otimes l_{T_{ijs}} + \mu^{(2)}_{is} \otimes l_{T_{ijs}} + \mu^{(3)}_{js} \otimes l_{T_{ijs}} + \epsilon_{ijs} \\
E\left[u_{ijs}u'_{ijs}\right] &= \sigma^{(1)^2}_{\mu} J_{T_{ijs}} + \sigma^{(2)^2}_{\mu} J_{T_{ijs}} + \sigma^{(3)^2}_{\mu} J_{T_{ijs}} + \sigma^2_{\epsilon} I_{T_{ijs}} \\
u_{ij} &= \mu^{(1)}_{ij} \otimes l_{\sum_s T_{ijs}} + \tilde{\mu}^{(2)}_{ij} + \tilde{\mu}^{(3)}_{ij} + \epsilon_{ij} \\
E\left[u_{ij}u'_{ij}\right] &= \sigma^{(1)^2}_{\mu} J_{\sum_s T_{ijs}} + \sigma^{(2)^2}_{\mu} A_{ij} + \sigma^{(3)^2}_{\mu} A_{ij} + \sigma^2_{\epsilon} I_{\sum_s T_{ijs}} \\
u_{i} &= \tilde{\mu}^{(1)}_{i} + \tilde{\mu}^{(2)}_{i} + \tilde{\mu}^{(3)}_{i} + \epsilon_{i} \\
E\left[u_{i}u'_{i}\right] &= \sigma^{(1)^2}_{\mu} B_{i} + \sigma^{(2)^2}_{\mu} F_{i,i} + \sigma^{(3)^2}_{\mu} A_{i} + \sigma^2_{\epsilon} I_{\sum_j \sum_s T_{ijs}} \\
u &= \tilde{\mu}^{(1)} + \tilde{\mu}^{(2)} + \tilde{\mu}^{(3)} + \epsilon \\
E\left[uu'\right] &= \sigma^{(1)^2}_{\mu} G + \sigma^{(2)^2}_{\mu} L + \sigma^{(3)^2}_{\mu} M + \sigma^2_{\epsilon} I_{\sum_i \sum_j \sum_s T_{ijs}}
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{\mu}^{(2)}_{ij} &= \left(\underbrace{\mu^{(2)}_{i1} \dots \mu^{(2)}_{i1}}_{T_{ij1}\text{times}} \;\; \underbrace{\mu^{(2)}_{i2} \dots \mu^{(2)}_{i2}}_{T_{ij2}\text{times}} \;\; \cdots \;\; \underbrace{\mu^{(2)}_{iN^{(3)}} \dots \mu^{(2)}_{iN^{(3)}}}_{T_{ijN^{(3)}}\text{times}} \right) \\
\tilde{\mu}^{(2)}_{i} &= \left(\tilde{\mu}^{(2)}_{i1}, \tilde{\mu}^{(2)}_{i2}, ..., \tilde{\mu}^{(2)}_{iN^{(2)}} \right) \\
\tilde{\mu}^{(2)} &= \left(\tilde{\mu}^{(2)}_{1}, \tilde{\mu}^{(2)}_{2}, ..., \tilde{\mu}^{(2)}_{N^{(1)}} \right)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mu}^{(3)}_{ij} &= \left(\underbrace{\mu^{(3)}_{j1} \dots \mu^{(3)}_{j1}}_{T_{ij1}\text{ times}} \;\; \underbrace{\mu^{(3)}_{j2} \dots \mu^{(3)}_{j2}}_{T_{ij2}\text{ times}} \;\; \cdots \;\; \underbrace{\mu^{(3)}_{jN^{(3)}} \dots \mu^{(3)}_{jN^{(3)}}}_{T_{ijN^{(3)}}\text{ times}} \right) \\
\tilde{\mu}^{(3)}_{i} &= \left(\tilde{\mu}^{(3)}_{i1}, \tilde{\mu}^{(3)}_{i2}, ..., \tilde{\mu}^{(3)}_{iN^{(2)}} \right) \\
\tilde{\mu}^{(3)} &= \left(\tilde{\mu}^{(3)}_{1}, \tilde{\mu}^{(3)}_{2}, ..., \tilde{\mu}^{(3)}_{N^{(1)}} \right)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mu}^{(1)}_{i} &= \left(\underbrace{\mu^{(1)}_{i1} \dots \mu^{(1)}_{i1}}_{\sum_s T_{i1s}\text{ times}} \;\; \underbrace{\mu^{(1)}_{i2} \dots \mu^{(1)}_{i2}}_{\sum_s T_{i2s}\text{ times}} \;\; \cdots \;\; \underbrace{\mu^{(1)}_{iN^{(2)}} \dots \mu^{(1)}_{iN^{(2)}}}_{\sum_s T_{iN^{(2)}s}\text{ times}} \right) \\
\tilde{\mu}^{(1)} &= \left(\tilde{\mu}^{(1)}_{1}, \tilde{\mu}^{(1)}_{2}, ..., \tilde{\mu}^{(1)}_{N^{(1)}} \right)
\end{aligned}
$$

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and

$$G = \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{N^{(1)}} \end{pmatrix}, \quad L = \begin{pmatrix} F_{1,1} & 0 & \dots & 0 \\ 0 & F_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F_{N^{(1)},N^{(1)}} \end{pmatrix}$$

$$Z_{i,p} = \begin{pmatrix} J_{(T_{i11} \times T_{p11})} & 0 & \dots & 0 \\ 0 & J_{(T_{i12} \times T_{p12})} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\left(T_{iN^{(2)}N^{(3)}} \times T_{pN^{(2)}N^{(3)}}\right)} \end{pmatrix}$$

$$M = \begin{pmatrix} Z_{1,1} & Z_{1,2} & \dots & Z_{1,N^{(1)}} \\ Z_{2,1} & Z_{2,2} & \dots & Z_{2,N^{(1)}} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N^{(1)},1} & Z_{N^{(1)},2} & \dots & Z_{N^{(1)},N^{(1)}} \end{pmatrix}$$

Model (5)

The disturbance term is now structured as

$$u_{ijst} = \mu^{(1)}_{ij} + \mu^{(2)}_{is} + \mu^{(3)}_{js} + \lambda_t + \epsilon_{ijst}$$

Given that I have already derived the covariance matrix of $\mu^{(1)}_{ij}$, $\mu^{(2)}_{is}$, and $\mu^{(3)}_{js}$ in model (4) and λ_t in model (1), for covariance matrix of model (5) I get

$$E\left[uu'\right] = \sigma^{(1)^2}_{\mu} G + \sigma^{(2)^2}_{\mu} L + \sigma^{(3)^2}_{\mu} M + \sigma^2_{\lambda} D + \sigma^2_{\epsilon} I_{\sum_i \sum_j \sum_s T_{ijs}}$$

Model (6)

The disturbance term now is

$$u_{ijst} = \mu_{ijs} + v^{(1)}_{it} + v^{(2)}_{jt} + v^{(3)}_{st} + \epsilon_{ijst}$$

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so I get for the covariance matrix

$$u_{ijs} = \mu_{ijs} \otimes l_{T_{ijs}} + v^{(1)}_{iZ_{ijs}} + v^{(2)}_{jZ_{ijs}} + v^{(3)}_{sZ_{ijs}} + \epsilon_{ijs}$$

$$E\left[u_{ijs}u'_{ijs}\right] = \sigma^2_\mu J_{T_{ijs}} + \sigma^{(1)^2}_v I_{T_{ijs}} + \sigma^{(2)^2}_v I_{T_{ijs}} + \sigma^{(3)^2}_v I_{T_{ijs}} + \sigma^2_\epsilon I_{T_{ijs}}$$

$$u_{ij} = \tilde{\mu}_{ij} + \tilde{v}^{(1)}_{ij} + \tilde{v}^{(2)}_{ij} + \tilde{v}^{(3)}_{ij} + \epsilon_{ij}$$

$$E\left[u_{ij}u'_{ij}\right] = \sigma^2_\mu A_{ij} + \sigma^{(1)^2}_v D_{ij} + \sigma^{(2)^2}_v D_{ij} + \sigma^{(3)^2}_v I_{\sum_s T_{ijs}} + \sigma^2_\epsilon I_{\sum_s T_{ijs}}$$

$$u_i = \tilde{\mu}_i + \tilde{v}^{(1)}_i + \tilde{v}^{(2)}_i + \tilde{v}^{(3)}_i + \epsilon_i$$

$$E\left[u_i u'_i\right] = \sigma^2_\mu A_i + \sigma^{(1)^2}_v D_i + \sigma^{(2)^2}_v R^1_i + \sigma^{(3)^2}_v N_{i,i} + \sigma^2_\epsilon I_{\sum_j \sum_s T_{ijs}}$$

$$u = \tilde{\mu} + \tilde{v}^{(1)} + \tilde{v}^{(2)} + \tilde{v}^{(3)} + \epsilon$$

$$E\left[uu'\right] = \sigma^2_\mu A + \sigma^{(1)^2}_v H + \sigma^{(2)^2}_v Q + \sigma^{(3)^2}_v N + \sigma^2_\epsilon I_{\sum_i \sum_j \sum_s T_{ijs}}$$

Denoting by

$v^{(1)}_{iZ_{ijs}}$ the vector of length T_{ijs} of the individual-time varying effects associated with the firm i and time periods from the set Z_{ijs},

$v^{(2)}_{jZ_{ijs}}$ the vector of length T_{ijs} of the individual-time varying effects associated with the product j and time periods from the set Z_{ijs}, and

$v^{(3)}_{sZ_{ijs}}$ the vector of length T_{ijs} of the individual-time varying effects associated with the country s and time periods from the set Z_{ijs}

I get

$$\tilde{v}^{(1)}_{ij} = \left(v^{(1)}_{iZ_{ij1}}, ..., v^{(1)}_{iZ_{ijN^{(3)}}}\right), \ \tilde{v}^{(1)}_i = \left(\tilde{v}^{(1)}_{i1}, \tilde{v}^{(1)}_{i2}, ..., \tilde{v}^{(1)}_{iN^{(2)}}\right)$$

$$\tilde{v}^{(1)} = \left(\tilde{v}^{(1)}_1, \tilde{v}^{(1)}_2, ..., \tilde{v}^{(1)}_{N^{(1)}}\right)$$

$$\tilde{v}^{(2)}_{ij} = \left(v^{(2)}_{jZ_{ij1}}, ..., v^{(2)}_{jZ_{ijN^{(3)}}}\right), \ \tilde{v}^{(2)}_i = \left(\tilde{v}^{(2)}_{i1}, \tilde{v}^{(2)}_{i2}, ..., \tilde{v}^{(2)}_{iN^{(2)}}\right)$$

$$\tilde{v}^{(2)} = \left(\tilde{v}^{(2)}_1, \tilde{v}^{(2)}_2, ..., \tilde{v}^{(2)}_{N^{(1)}}\right)$$

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$$\tilde{v}_{ij}^{(3)} = \left(v_{1Z_{ij1}}^{(3)}, ..., v_{N^{(3)}Z_{ijN^{(3)}}}^{(3)}\right), \ \tilde{v}_{i}^{(3)} = \left(\tilde{v}_{i1}^{(3)}, \tilde{v}_{i2}^{(3)}, ..., \tilde{v}_{iN^{(2)}}^{(3)}\right)$$

$$\tilde{v}^{(3)} = \left(\tilde{v}_{1}^{(3)}, \tilde{v}_{2}^{(3)}, ..., \tilde{v}_{N^{(1)}}^{(3)}\right)$$

and

$$H = \begin{pmatrix} D_l & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_N^{(1)} \end{pmatrix}$$

$$E_s^l = \begin{pmatrix} M_{T_{ij1} \times T_{ljs}} \\ M_{T_{ij2} \times T_{ljs}} \\ \dots \\ M_{T_{ijN^{(3)}} \times T_{ljs}} \end{pmatrix}, \quad D_{ij}^l = \left(E_1^l, E_2^l, ..., E_{N^{(3)}}^l\right)$$

$$R_i^l = \begin{pmatrix} D_{i1}^l & 0 & \dots & 0 \\ 0 & D_{i2}^l & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_{iN^{(2)}}^l \end{pmatrix}, \quad Q = \begin{pmatrix} R_1^1 & R_1^2 & \dots & R_1^{N^{(1)}} \\ R_2^1 & R_2^2 & \dots & R_2^{N^{(1)}} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N^{(1)}}^1 & R_{N^{(1)}}^2 & \dots & R_{N^{(1)}}^{N^{(1)}} \end{pmatrix}$$

$$S_{i,s}^{j,l} = \begin{pmatrix} M_{T_{ij1} \times T_{sl1}} & 0 & \dots & 0 \\ 0 & M_{T_{ij2} \times T_{sl2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{T_{ijN^{(3)}} \times T_{slN^{(3)}}} \end{pmatrix}$$

$$N_{i,s} = \begin{pmatrix} S_{i,s}^{1,1} & S_{i,s}^{1,2} & \dots & S_{i,s}^{1,N^{(2)}} \\ S_{i,s}^{2,1} & S_{i,s}^{2,2} & \dots & S_{i,s}^{2,N^{(2)}} \\ \vdots & \vdots & \ddots & \vdots \\ S_{i,s}^{N^{(2)},1} & S_{i,s}^{N^{(2)},2} & \dots & S_{i,s}^{N^{(2)},N^{(2)}} \end{pmatrix}$$

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$$N = \begin{pmatrix} N_{1,1} & N_{1,2} & \dots & N_{1,N^{(1)}} \\ N_{2,1} & N_{2,2} & \dots & N_{2,N^{(1)}} \\ \vdots & \vdots & \ddots & \vdots \\ N_{N^{(1)},1} & N_{N^{(1)},2} & \dots & N_{N^{(1)},N^{(1)}} \end{pmatrix}$$

Model (7)

The disturbance term now is

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + v_{it}^{(1)} + v_{jt}^{(2)} + v_{st}^{(3)} + \epsilon_{ijst}$$

Fortunately,I have already derived the covariance matrix of $\mu_{ij}^{(1)}$, $\mu_{is}^{(2)}$, $\mu_{js}^{(3)}$, $v_{it}^{(1)}$, $v_{jt}^{(2)}$, and $v_{st}^{(3)}$ previously, thus for model (7) the covariance matrix takes the form

$$E\left[uu'\right] = \sigma_{\mu}^{(1)^2} G + \sigma_{\mu}^{(2)^2} L + \sigma_{\mu}^{(3)^2} M + \sigma_{v}^{(1)^2} H + \sigma_{v}^{(2)^2} Q + \sigma_{v}^{(3)^2} N + \sigma_{\epsilon}^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

5.2 Estimation of the Variance Components

Just like in the balanced case, the estimation of the variance components is carried out in two steps. First, some identifying equations are presented, then; based on these, estimators for the different variance components are derived. But first, some additional notation need to be introduced. Let denote

$$\tilde{A} = \frac{1}{T} \sum_{i \in Q^{(1)}} \sum_{j \in Q_i^{(2)}} \sum_{s \in Q_{ij}^{(3)}} \sum_{t \in Z_{ijs}} \hat{u}_{ijst}^2$$

$$\tilde{B} = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{N_j^{(3)}} \sum_{s \in Q_j^{(3)}} \frac{1}{T_{js}^{(1)}} \sum_{t \in Z_{js}^{(1)}} \left(\frac{1}{N_{jst}^{(1)}} \sum_{i \in Q_{jst}^{(1)}} \hat{u}_{ijst} \right)^2$$

$$\tilde{C} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(3)}} \sum_{s \in Q_i^{(3)}} \frac{1}{T_{is}^{(2)}} \sum_{t \in Z_{is}^{(2)}} \left(\frac{1}{N_{ist}^{(2)}} \sum_{j \in Q_{ist}^{(2)}} \hat{u}_{ijst} \right)^2$$

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$$\tilde{D} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{T_{ij}^{(3)}} \sum_{t \in Z_{ij}^{(3)}} \left(\frac{1}{N_{ijt}^{(3)}} \sum_{s \in Q_{ijt}^{(3)}} \hat{u}_{ijst} \right)^2$$

$$\tilde{E} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{N_{ij}^{(3)}} \sum_{s \in Q_{ij}^{(3)}} \left(\frac{1}{T_{ijs}} \sum_{t \in Z_{ijs}} \hat{u}_{ijst} \right)^2$$

and

$$b = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{N_j^{(3)}} \sum_{s \in Q_j^{(3)}} \frac{1}{T_{js}^{(1)}} \sum_{t \in Z_{js}^{(1)}} \frac{1}{N_{jst}^{(1)}}$$

$$c = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(3)}} \sum_{s \in Q_i^{(3)}} \frac{1}{T_{is}^{(2)}} \sum_{t \in Z_{is}^{(2)}} \frac{1}{N_{ist}^{(2)}}$$

$$d = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{T_{ij}^{(3)}} \sum_{t \in Z_{ij}^{(3)}} \frac{1}{N_{ijt}^{(3)}}$$

$$e = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{N_{ij}^{(3)}} \sum_{s \in Q_{ij}^{(3)}} \frac{1}{T_{ijs}}$$

Model (1)

The identifying equations are the following

$$E\left[u_{ijst}^2\right] = \sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\lambda^2 + \sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{N_j^{(3)}} \sum_{s \in Q_j^{(3)}} \frac{1}{T_{js}^{(1)}} \sum_{t \in Z_{js}^{(1)}} \left(\frac{1}{N_{jst}^{(1)}} \sum_{i \in Q_{jst}^{(1)}} u_{ijst} \right)^2 \right] = b\left(\sigma_\mu^2 + \sigma_\epsilon^2\right) + \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\lambda^2$$

$$E\left[\frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(3)}} \sum_{s \in Q_i^{(3)}} \frac{1}{T_{is}^{(2)}} \sum_{t \in Z_{is}^{(2)}} \left(\frac{1}{N_{ist}^{(2)}} \sum_{j \in Q_{ist}^{(2)}} u_{ijst} \right)^2 \right] = c\left(\sigma_\gamma^2 + \sigma_\epsilon^2\right) + \sigma_\mu^2 + \sigma_\alpha^2 + \sigma_\lambda^2$$

$$E\left[\frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{T_{ij}^{(3)}} \sum_{t \in Z_{ij}^{(3)}} \left(\frac{1}{N_{ijt}^{(3)}} \sum_{s \in Q_{ijt}^{(3)}} u_{ijst} \right)^2 \right] = d\left(\sigma_\alpha^2 + \sigma_\epsilon^2\right) + \sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\lambda^2$$

$$E\left[\frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{N_{ij}^{(3)}} \sum_{s \in Q_{ij}^{(3)}} \left(\frac{1}{T_{ijs}} \sum_{t \in Z_{ijs}} u_{ijst} \right)^2 \right] = e\left(\sigma_\lambda^2 + \sigma_\epsilon^2\right) + \sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\alpha^2$$

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These leads to the following estimators

$$\hat{\sigma}_{\epsilon}^{2}=\frac{1}{3}\left(\frac{\tilde{B}}{b-1}+\frac{\tilde{C}}{c-1}+\frac{\tilde{D}}{d-1}+\frac{\tilde{E}}{e-1}\right)-\frac{\tilde{A}}{3}\left(\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{d-1}+\frac{1}{e-1}+1\right)$$

$$\hat{\sigma}_{\lambda}^{2}=\frac{\tilde{A}-\tilde{E}}{1-e}-\hat{\sigma}_{\epsilon}^{2}$$

$$\hat{\sigma}_{\alpha}^{2}=\frac{\tilde{A}-\tilde{D}}{1-d}-\hat{\sigma}_{\epsilon}^{2}$$

$$\hat{\sigma}_{\gamma}^{2}=\frac{\tilde{A}-\tilde{C}}{1-c}-\hat{\sigma}_{\epsilon}^{2}$$

$$\hat{\sigma}_{\mu}^{2}=\frac{\tilde{A}-\tilde{B}}{1-b}-\hat{\sigma}_{\epsilon}^{2}$$

Model (2)

Using the identifying equations

$$E\left[u_{ijst}^{2}\right]=\sigma_{\mu}^{2}+\sigma_{\epsilon}^{2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_{i}^{(2)}}\sum_{j\in Q_{i}^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^{2}\right]=\sigma_{\mu}^{2}+e\sigma_{\epsilon}^{2}$$

the estimators of variance components are

$$\hat{\sigma}_{\mu}^{2}=\frac{\tilde{E}-e\tilde{A}}{1-e}$$

$$\hat{\sigma}_{\epsilon}^{2}=\tilde{A}-\hat{\sigma}_{\mu}^{2}$$

Model (3)

The identifying equations are

$$E\left[u_{ijst}^{2}\right]=\sigma_{\mu}^{2}+\sigma_{\lambda}^{2}+\sigma_{\epsilon}^{2}$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_{j}^{(3)}}\sum_{s\in Q_{j}^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^{2}\right]=b\left(\sigma_{\mu}^{2}+\sigma_{\epsilon}^{2}\right)+\sigma_{\lambda}^{2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_{i}^{(2)}}\sum_{j\in Q_{i}^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^{2}\right]=e\left(\sigma_{\lambda}^{2}+\sigma_{\epsilon}^{2}\right)+\sigma_{\mu}^{2}$$

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And so the estimators are

$$\hat{\sigma}_\mu^2 = \frac{\tilde{E} - e\tilde{A}}{1-e}$$
$$\hat{\sigma}_\lambda^2 = \frac{\tilde{B} - b\tilde{A}}{1-b}$$
$$\hat{\sigma}_\epsilon^2 = \tilde{A} - \hat{\sigma}_\mu^2 - \hat{\sigma}_\lambda^2$$

Model (4)

The identifying equations in this case are simply

$$E\left[u_{ijst}^2\right] = \sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_\epsilon^2$$
$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right] = b\left(\sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^{(3)^2}$$
$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right] = c\left(\sigma_\mu^{(1)^2} + \sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^{(2)^2}$$
$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right] = d\left(\sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^{(1)^2}$$

And the estimators are

$$\hat{\sigma}_\mu^{(1)^2} = \frac{\tilde{D} - d\tilde{A}}{1-d}$$
$$\hat{\sigma}_\mu^{(2)^2} = \frac{\tilde{C} - c\tilde{A}}{1-c}$$
$$\hat{\sigma}_\mu^{(3)^2} = \frac{\tilde{B} - b\tilde{A}}{1-b}$$
$$\hat{\sigma}_\epsilon^2 = \tilde{A} - \hat{\sigma}_\mu^{(1)^2} - \hat{\sigma}_\mu^{(2)^2} - \hat{\sigma}_\mu^{(3)^2}$$

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Model (5)

The identifying equations now are

$$E\left[u_{ijst}^2\right] = \sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_\lambda^2 + \sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right] =$$

$$= b\left(\sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^{(3)^2} + \sigma_\lambda^2$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right] =$$

$$= c\left(\sigma_\mu^{(1)^2} + \sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^{(2)^2} + \sigma_\lambda^2$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right] =$$

$$= d\left(\sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^{(1)^2} + \sigma_\lambda^2$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right] =$$

$$= e\left(\sigma_\lambda^2 + \sigma_\epsilon^2\right) + \sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2}$$

The variance components' estimators now are

$$\hat{\sigma}_\lambda^2 = \frac{1}{3}\left(\frac{\tilde{B}}{1-b} + \frac{\tilde{C}}{1-c} + \frac{\tilde{D}}{1-d} - \frac{\tilde{E}}{1-e}\right) - \frac{\tilde{A}}{3}\left(\frac{b}{1-b} + \frac{c}{1-c} + \frac{d}{1-d} - \frac{e}{1-e}\right)$$

$$\hat{\sigma}_\mu^{(1)^2} = \frac{\tilde{D} - d\tilde{A}}{1-d} - \hat{\sigma}_\lambda^2$$

$$\hat{\sigma}_\mu^{(2)^2} = \frac{\tilde{C} - c\tilde{A}}{1-c} - \hat{\sigma}_\lambda^2$$

$$\hat{\sigma}_\mu^{(3)^2} = \frac{\tilde{B} - b\tilde{A}}{1-b} - \hat{\sigma}_\lambda^2$$

$$\hat{\sigma}_\epsilon^2 = \tilde{A} - \hat{\sigma}_\mu^{(1)^2} - \hat{\sigma}_\mu^{(2)^2} - \hat{\sigma}_\mu^{(3)^2} - \hat{\sigma}_\lambda^2$$

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Model (6)

The identifying equations for this model are

$$E\left[u_{ijst}^2\right] = \sigma_\mu^2 + \sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right] =$$

$$= b\left(\sigma_\mu^2 + \sigma_v^{(1)^2} + \sigma_\epsilon^2\right) + \sigma_v^{(2)^2} + \sigma_v^{(3)^2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right] =$$

$$= c\left(\sigma_\mu^2 + \sigma_v^{(2)^2} + \sigma_\epsilon^2\right) + \sigma_v^{(1)^2} + \sigma_v^{(3)^2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right] =$$

$$= d\left(\sigma_\mu^2 + \sigma_v^{(3)^2} + \sigma_\epsilon^2\right) + \sigma_v^{(1)^2} + \sigma_v^{(2)^2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right] =$$

$$= e\left(\sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^2$$

And the estimators are

$$\hat{\sigma}_\mu^2 = \frac{\tilde{E} - e\tilde{A}}{1-e}$$

$$\hat{\sigma}_\epsilon^2 = \tilde{A}\left(1 + \frac{b}{2(1-b)} + \frac{c}{2(1-c)} + \frac{d}{2(1-d)}\right) - \frac{1}{2}\left(\frac{\tilde{B}}{1-b} + \frac{\tilde{C}}{1-c} + \frac{\tilde{D}}{1-d}\right) - \hat{\sigma}_\mu^2$$

$$\hat{\sigma}_v^{(1)^2} = \frac{\tilde{A} - \tilde{B}}{1-b} - \hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_v^{(2)^2} = \frac{\tilde{A} - \tilde{C}}{1-c} - \hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_v^{(3)^2} = \tilde{A} - \hat{\sigma}_\mu^2 - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(2)^2} - \hat{\sigma}_\epsilon^2$$

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Model(7)

For this last model I need to introduce some additional notations:

$$\tilde{F} = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \left(\frac{1}{N_{st}^{(1)}} \sum_{i \in Q_{st}^{(1)}} \frac{1}{N_{ist}^{(2)}} \sum_{j \in Q_{ist}^{(2)}} \hat{u}_{ijst} \right)^2$$

$$\tilde{G} = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \left(\frac{1}{N_{jt}^{(1)}} \sum_{i \in Q_{jt}^{(1)}} \frac{1}{N_{ijt}^{(3)}} \sum_{s \in Q_{ijt}^{(3)}} \hat{u}_{ijst} \right)^2$$

$$f_1 = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \frac{1}{N_{st}^{(1)^2}} \sum_{i \in Q_{st}^{(1)}} \frac{1}{N_{ist}^{(2)}}$$

$$f_2 = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \frac{1}{N_{st}^{(1)}}$$

$$f_3 = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \frac{1}{N_{st}^{(1)^2}} \left(\sum_{i \in Q_{st}^{(1)}} \frac{1}{N_{ist}^{(2)}} \left(1 + \sum_{l \in Q_{st}^{(1)}, l \neq i} \frac{1}{N_{lst}^{(2)}} I_{il}^{st} \right) \right)$$

$$f = \frac{1 - f_2 - f_3}{f_1}$$

$$g_1 = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \frac{1}{N_{jt}^{(1)^2}} \sum_{i \in Q_{jt}^{(1)}} \frac{1}{N_{ijt}^{(3)}}$$

$$g_2 = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \frac{1}{N_{jt}^{(1)^2}} \left(\sum_{i \in Q_{jt}^{(1)}} \frac{1}{N_{ijt}^{(3)}} \left(1 + \sum_{p \in Q_{jt}^{(1)}} \frac{1}{N_{pjt}^{(3)}} J_{ip}^{jt} \right) \right)$$

$$g_3 = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \frac{1}{N_{jt}^{(1)}}$$

$$g = \frac{1 - g_2 - g_3}{g_1}$$

where I_{il}^{st} is the number of common products that firm i and l sell to country s at time t, and J_{ip}^{jt} is the number of common countries to which firms i and p sell product j at time t.

Turning now our attention to the identifying equations

$$E\left[u_{ijst}^2\right] = \sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \sigma_\epsilon^2$$

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$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right]=$$

$$=b\left(\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\sigma_v^{(1)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^{(3)^2}+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$

$$=c\left(\sigma_\mu^{(1)^2}+\sigma_\mu^{(3)^2}+\sigma_v^{(2)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^{(2)^2}+\sigma_v^{(1)^2}+\sigma_v^{(3)^2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$=d\left(\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}+\sigma_v^{(3)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^{(1)^2}+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right]=$$

$$=e\left(\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}$$

$$E\left[\frac{1}{N^{(3)}}\sum_{s\in Q^{(3)}}\frac{1}{T_s^{(3)}}\sum_{t\in Z_s^{(3)}}\left(\frac{1}{N_{st}^{(1)}}\sum_{i\in Q_{st}^{(1)}}\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$

$$=f_1\left(\sigma_\mu^{(1)^2}+\sigma_\epsilon^2\right)+f_2\left(\sigma_\mu^{(2)^2}+\sigma_v^{(1)^2}\right)+f_3\left(\sigma_\mu^{(3)^2}+\sigma_v^{(2)^2}\right)+\sigma_v^{(3)^2}$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{T_j^{(2)}}\sum_{t\in Z_j^{(2)}}\left(\frac{1}{N_{jt}^{(1)}}\sum_{i\in Q_{jt}^{(1)}}\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$=g_1\left(\sigma_\mu^{(2)^2}+\sigma_\epsilon^2\right)+g_2\left(\sigma_\mu^{(3)^2}+\sigma_v^{(3)^2}\right)+g_3\left(\sigma_\mu^{(1)^2}+\sigma_v^{(1)^2}\right)+\sigma_v^{(2)^2}$$

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So the estimators of variance components of the last model are

$$\hat{\sigma}_\epsilon^2=\frac{\tilde{A}}{2}\left(1+\frac{b}{1-b}+\frac{c}{1-c}+\frac{d}{1-d}-\frac{e}{1-e}\right)-\frac{1}{2}\left(\frac{\tilde{B}}{1-b}+\frac{\tilde{C}}{1-c}+\frac{\tilde{D}}{1-d}-\frac{\tilde{E}}{1-e}\right)$$

$$\hat{\sigma}_v^{(3)^2}=\frac{1}{1+f}\left(\frac{\tilde{E}-e\tilde{A}}{1-e}-\frac{\tilde{D}-d\tilde{A}}{1-d}-\frac{f_2}{f_1}\cdot\frac{\tilde{C}-c\tilde{A}}{1-c}-\frac{f_3}{f_1}\cdot\frac{\tilde{B}-b\tilde{A}}{1-b}+\frac{\tilde{F}}{f_1}-2\hat{\sigma}_\epsilon^2\right)$$

$$\hat{\sigma}_v^{(2)^2}=\frac{1}{1+g}\left(\frac{\tilde{E}-e\tilde{A}}{1-e}-\frac{\tilde{C}-c\tilde{A}}{1-c}-\frac{g_2}{g_1}\cdot\frac{\tilde{B}-b\tilde{A}}{1-b}-\frac{g_3}{g_1}\cdot\frac{\tilde{D}-d\tilde{A}}{1-d}+\frac{\tilde{G}}{g_1}-2\hat{\sigma}_\epsilon^2\right)$$

$$\hat{\sigma}_v^{(1)^2} = \frac{\tilde{E} - e\tilde{A}}{1-e} - \hat{\sigma}_\epsilon^2 - \hat{\sigma}_v^{(3)^2} - \hat{\sigma}_v^{(2)^2}$$
$$\hat{\sigma}_\mu^{(1)^2} = \frac{\tilde{D} - d\tilde{A}}{1-d} - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(2)^2}$$
$$\hat{\sigma}_\mu^{(2)^2} = \frac{\tilde{C} - c\tilde{A}}{1-c} - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(3)^2}$$
$$\hat{\sigma}_\mu^{(3)^2} = \tilde{A} - \hat{\sigma}_\mu^{(1)^2} - \hat{\sigma}_\mu^{(2)^2} - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(2)^2} - \hat{\sigma}_v^{(3)^2} - \hat{\sigma}_\epsilon^2$$

5.3 Covariance Matrixes of the Models with Cross Correlation

Let us turn now our attention to the models with cross correlations. For models (2), (3) and (6) I have

$$E[\mu_{ij}\mu'_{ij}] = \sigma_\mu^2 A_{ij} + \rho_{(3)}\left(J_{N^{(3)}T} - A_{ij}\right)$$
$$E[\mu_i\mu'_i] = \sigma_\mu^2 A_i + \rho_{(3)}\left(B_i - A_i\right) + \rho_{(2)}\left(F_{i,i} - A_i\right)$$
$$E[\mu\mu'] = \sigma_\mu^2 A + \rho_{(3)}\left(G - A\right) + \rho_{(2)}\left(L - A\right) + \rho_{(1)}\left(M - A\right)$$

Thus the covariance matrix of model (2) takes the form

$$E[uu'] = \sigma_\mu^2 A + \rho_{(3)}\left(G - A\right) + \rho_{(2)}\left(L - A\right) + \rho_{(1)}\left(M - A\right) + \sigma_\epsilon^2 I_{\sum_i\sum_j\sum_s T_{ijs}}$$

and the covariance matrix of model (3) looks like

$$E\left[uu'\right] = \sigma_\mu^2 A + \rho_{(3)}\left(G - A\right) + \rho_{(2)}\left(L - A\right) + \rho_{(1)}\left(M - A\right) + \sigma_\lambda^2 D + \sigma_\epsilon^2 I_{\sum_i\sum_j\sum_s T_{ijs}}$$

Finally, the covariance matrix of model (6) is

$$E\left[uu'\right] = \sigma_\mu^2 A + \rho_{(3)}\left(G - A\right) + \rho_{(2)}\left(L - A\right) + \rho_{(1)}\left(M - A\right) +$$
$$+ \sigma_v^{(1)^2} H + \sigma_v^{(2)^2} Q + \sigma_v^{(3)^2} N + \sigma_\epsilon^2 I_{\sum_i\sum_j\sum_s T_{ijs}}$$

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In the case of models (4), (5), and (7) I have for $\mu_{ij}^{(1)}$

$$E[\tilde{\mu_i}^{(1)}\tilde{\mu_i}^{(1)\prime}] = {\sigma_\mu^{(1)}}^2 B_i + \rho_{(2)}^{(1)} \left(J_{\sum_j \sum_s T_{ijs}} - B_i\right)$$

$$E[\tilde{\mu}^{(1)}\tilde{\mu}^{(1)\prime}] = {\sigma_\mu^{(1)}}^2 G + \rho_{(2)}^{(1)} (C - G) + \rho_{(1)}^{(1)} (B - G)$$

for $\mu_{is}^{(2)}$

$$E[\tilde{\mu}_i^{(2)}\tilde{\mu}_i^{(2)\prime}] = {\sigma_\mu^{(2)}}^2 F_{i,i} + \rho_{(2)}^{(2)} \left(J_{\sum_j \sum_s T_{ijs}} - F_{i,i}\right)$$

$$E[\tilde{\mu}^{(2)}\tilde{\mu}^{(2)\prime}] = {\sigma_\mu^{(2)}}^2 L + \rho_{(2)}^{(2)} (C - L) + \rho_{(1)}^{(2)} (F - L)$$

and for $\mu_{js}^{(3)}$

$$E[\tilde{\mu}_{ij}^{(3)}\tilde{\mu}_{ij}^{(3)\prime}] = {\sigma_\mu^{(3)}}^2 A_{ij} + \rho_{(2)}^{(3)} \left(J_{\sum_s T_{ijs}} - A_{ij}\right)$$

$$E[\tilde{\mu}_i^{(3)}\tilde{\mu}_i^{(3)\prime}] = {\sigma_\mu^{(3)}}^2 A_i + \rho_{(2)}^{(3)} (B_i - A_i) + \rho_{(1)}^{(3)} (F_{i,i} - A_i)$$

$$E[\tilde{\mu}^{(3)}\tilde{\mu}^{(3)\prime}] = {\sigma_\mu^{(3)}}^2 M + \rho_{(2)}^{(3)} (B - M) + \rho_{(1)}^{(3)} (F - M)$$

So the covariance matrix of model (4) now is

$$E[uu'] = {\sigma_\mu^{(1)}}^2 G + \rho_{(2)}^{(1)} (C - G) + \rho_{(1)}^{(1)} (B - G) + {\sigma_\mu^{(2)}}^2 L + \rho_{(2)}^{(2)} (C - L) + \rho_{(1)}^{(2)} (F - L) +$$
$$+ {\sigma_\mu^{(3)}}^2 M \rho_{(2)}^{(3)} (B - M) + \rho_{(1)}^{(3)} (F - M) + \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

for model (5) I get

$$E[uu'] = {\sigma_\mu^{(1)}}^2 G + \rho_{(2)}^{(1)} (C - G) + \rho_{(1)}^{(1)} (B - G) + {\sigma_\mu^{(2)}}^2 L + \rho_{(2)}^{(2)} (C - L) + \rho_{(1)}^{(2)} (F - L) +$$
$$+ {\sigma_\mu^{(3)}}^2 M + \rho_{(2)}^{(3)} (B - M) + \rho_{(1)}^{(3)} (F - M) + \sigma_\lambda^2 D + \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

and, finally, for model (7) I get the following covariance matrix

$$E[uu'] = {\sigma_\mu^{(1)}}^2 G + \rho_{(2)}^{(1)} (C - G) + \rho_{(1)}^{(1)} (B - G) + {\sigma_\mu^{(2)}}^2 L + \rho_{(2)}^{(2)} (C - L) + \rho_{(1)}^{(2)} (F - L) +$$
$$+ {\sigma_\mu^{(3)}}^2 M + \rho_{(2)}^{(3)} (B - M) + \rho_{(1)}^{(3)} (F - M) + \sigma_v^{(1)^2} H + \sigma_v^{(2)^2} Q + \sigma_v^{(3)^2} N +$$
$$+ \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

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5.4 Estimation of the Variance Components and Cross Correlations

Again, some additional notations need to be introduced:

$$\tilde{K} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \left(\frac{1}{N_{it}^{(2)}} \sum_{j \in Q_{it}^{(2)}} \frac{1}{N_{ijt}^{(3)}} \sum_{s \in Q_{ijt}^{(3)}} \hat{u}_{ijst} \right)^2$$

$$k_1 = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \frac{1}{N_{it}^{(2)^2}} \sum_{j \in Q_{it}^{(2)}} \frac{1}{N_{ijt}^{(3)}}$$

$$k_2 = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \frac{1}{N_{it}^{(2)}}$$

$$k_3 = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \frac{1}{N_{it}^{(2)^2}} \left(\sum_{j \in Q_{it}^{(2)}} \frac{1}{N_{ijt}^{(3)}} \left(1 + \sum_{k \in Q_{it}^{(2)}, k \neq j} \frac{1}{N_{ikt}^{(3)}} S_{jk}^{it} \right) \right)$$

$$b^{(1)} = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{N_j^{(3)}} \sum_{s \in Q_j^{(3)}} \frac{1}{T_{js}^{(1)}} \sum_{t \in Z_{js}^{(1)}} \frac{N_{jst}^{(1)} - 1}{N_{jst}^{(1)}}$$

$$c^{(1)} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(3)}} \sum_{s \in Q_i^{(3)}} \frac{1}{T_{is}^{(2)}} \sum_{t \in Z_{is}^{(2)}} \frac{N_{ist}^{(2)} - 1}{N_{ist}^{(2)}}$$

$$d^{(1)} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{T_{ij}^{(3)}} \sum_{t \in Z_{ij}^{(3)}} \frac{N_{ijt}^{(3)} - 1}{N_{ijt}^{(3)}}$$

$$f^{(1)} = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \frac{1}{N_{st}^{(1)^2}} \sum_{i \in Q_{st}^{(1)}} \sum_{l \in Q_{st}^{(1)}, l \neq i} \frac{1}{N_{ist}^{(2)}} \frac{1}{N_{lst}^{(2)}} I_{il}^{st}$$

$$f^{(2)} = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \frac{1}{N_{st}^{(1)^2}} \sum_{i \in Q_{st}^{(1)}} \frac{N_{ist}^{(2)} - 1}{N_{ist}^{(2)}}$$

$$g^{(1)} = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \frac{1}{N_{jt}^{(1)^2}} \sum_{i \in Q_{jt}^{(1)}} \sum_{p \in Q_{jt}^{(1)}, p \neq i} \frac{1}{N_{ijt}^{(3)}} \frac{1}{N_{pjt}^{(3)}} J_{ip}^{jt}$$

$$g^{(3)} = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \frac{1}{N_{jt}^{(1)^2}} \sum_{i \in Q_{jt}^{(1)}} \frac{N_{ijt}^{(3)} - 1}{N_{ijt}^{(3)}}$$

$$k^{(2)} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \frac{1}{N_{it}^{(2)^2}} \sum_{j \in Q_{it}^{(2)}} \sum_{k \in Q_{it}^{(2)}, k \neq j} \frac{1}{N_{ijt}^{(3)}} \frac{1}{N_{ikt}^{(3)}} S_{jk}^{it}$$

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$$k^{(3)} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \frac{1}{N_{it}^{(2)^2}} \sum_{j \in Q_{it}^{(2)}} \frac{N_{ijt}^{(3)} - 1}{N_{ijt}^{(3)}}$$

where S_{jk}^{it} is the number of countries to which firm i sells both products j and k at time t.

Model (2)

The estimation of variance components of ϵ and μ remain as in the model without cross correlations, but the cross correlation coefficients themselves need to be estimated. In this case the identifying equations are

$$E\left[\frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{N_j^{(3)}} \sum_{s \in Q_j^{(3)}} \frac{1}{T_{js}^{(1)}} \sum_{t \in Z_{js}^{(1)}} \left(\frac{1}{N_{jst}^{(1)}} \sum_{i \in Q_{jst}^{(1)}} u_{ijst}\right)^2\right] = b\left(\sigma_\mu^2 + \sigma_\epsilon^2\right) + b^{(1)}\rho_{(1)}$$

$$E\left[\frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(3)}} \sum_{s \in Q_i^{(3)}} \frac{1}{T_{is}^{(2)}} \sum_{t \in Z_{is}^{(2)}} \left(\frac{1}{N_{ist}^{(2)}} \sum_{j \in Q_{ist}^{(2)}} u_{ijst}\right)^2\right] = c\left(\sigma_\gamma^2 + \sigma_\epsilon^2\right) + c^{(1)}\rho_{(2)}$$

$$E\left[\frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{T_{ij}^{(3)}} \sum_{t \in Z_{ij}^{(3)}} \left(\frac{1}{N_{ijt}^{(3)}} \sum_{s \in Q_{ijt}^{(3)}} u_{ijst}\right)^2\right] = d\left(\sigma_\alpha^2 + \sigma_\epsilon^2\right) + d^{(1)}\rho_{(3)}$$

So I get

$$\hat{\rho}_{(1)} = \frac{1}{b^{(1)}}\left[\tilde{B} - b\left(\hat{\sigma}_\mu^2 + \hat{\sigma}_\epsilon^2\right)\right]$$
$$\hat{\rho}_{(2)} = \frac{1}{c^{(1)}}\left[\tilde{C} - c\left(\hat{\sigma}_\mu^2 + \hat{\sigma}_\epsilon^2\right)\right]$$
$$\hat{\rho}_{(3)} = \frac{1}{d^{(1)}}\left[\tilde{D} - d\left(\hat{\sigma}_\mu^2 + \hat{\sigma}_\epsilon^2\right)\right]$$

Model (3)

The estimation of the variance of μ remains the same, as above; however it changes for other variance components. Now the identifying equations are

$$E\left[u_{ijst}^2\right] = \sigma_\mu^2 + \sigma_\lambda^2 + \sigma_\epsilon^2$$

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$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right]=b\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\lambda^2+b^{(1)}\rho_{(1)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=c\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\lambda^2+c^{(1)}\rho_{(2)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=d\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\lambda^2+d^{(1)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(3)}}\sum_{s\in Q^{(3)}}\frac{1}{T_s^{(3)}}\sum_{t\in Z_s^{(3)}}\left(\frac{1}{N_{st}^{(1)}}\sum_{i\in Q_{st}^{(1)}}\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$

$$=f_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\lambda^2+f^{(1)}\rho_{(1)}+f^{(2)}\rho_{(2)}$$

and thus

$$\hat{\sigma}_\epsilon^2=\frac{1}{f_1-1-(b-1)\frac{f^{(1)}}{b^{(1)}}-(c-1)\frac{f^{(2)}}{c^{(1)}}}\left[\tilde{A}\left(\frac{f^{(1)}}{b^{(1)}}+\frac{f^{(2)}}{c^{(1)}}-1\right)-\frac{f^{(1)}}{b^{(1)}}\tilde{B}-\frac{f^{(2)}}{c^{(1)}}\tilde{C}+\tilde{F}\right]-\hat{\sigma}_\mu^2$$

$$\hat{\sigma}_\lambda^2=\tilde{A}-\hat{\sigma}_\mu^2-\hat{\sigma}_\epsilon^2$$

$$\hat{\rho}_{(1)}=\frac{1}{b^{(1)}}\left[\tilde{B}-b\left(\hat{\sigma}_\mu^2+\hat{\sigma}_\epsilon^2\right)-\hat{\sigma}_\lambda^2\right]$$

$$\hat{\rho}_{(2)}=\frac{1}{c^{(1)}}\left[\tilde{C}-c\left(\hat{\sigma}_\mu^2+\hat{\sigma}_\epsilon^2\right)-\hat{\sigma}_\lambda^2\right]$$

$$\hat{\rho}_{(3)}=\frac{1}{d^{(1)}}\left[\tilde{D}-d\left(\hat{\sigma}_\mu^2+\hat{\sigma}_\epsilon^2\right)-\hat{\sigma}_\lambda^2\right]$$

Model (6)

Again the estimation of the variance of μ remains unchanged, but I need, of course, to estimate all the remaining variance components and cross-correlations. The identifying equations now are

$$E\left[u_{ijst}^2\right]=\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\sigma_\epsilon^2$$

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$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right]=$$

$$=b\left(\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_\epsilon^2\right)+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+b^{(1)}\rho_{(1)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$

$$=c\left(\sigma_\mu^2+\sigma_v^{(2)^2}+\sigma_\epsilon^2\right)+\sigma_v^{(1)^2}+\sigma_v^{(3)^2}+c^{(1)}\rho_{(2)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$=d\left(\sigma_\mu^2+\sigma_v^{(3)^2}+\sigma_\epsilon^2\right)+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+d^{(1)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(3)}}\sum_{s\in Q^{(3)}}\frac{1}{T_s^{(3)}}\sum_{t\in Z_s^{(3)}}\left(\frac{1}{N_{st}^{(1)}}\sum_{i\in Q_{st}^{(1)}}\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$

$$=f_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+f_2\sigma_v^{(1)^2}+f_3\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+f^{(1)}\rho_{(1)}+f^{(2)}\rho_{(2)}$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{T_j^{(2)}}\sum_{t\in Z_j^{(2)}}\left(\frac{1}{N_{jt}^{(1)}}\sum_{i\in Q_{jt}^{(1)}}\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$=g_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+g_2\sigma_v^{(3)^2}+g_3\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+g^{(1)}\rho_{(1)}+g^{(3)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{T_i^{(1)}}\sum_{t\in Z_i^{(1)}}\left(\frac{1}{N_{it}^{(2)}}\sum_{j\in Q_{it}^{(2)}}\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$=k_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+k_2\sigma_v^{(2)^2}+k_3\sigma_v^{(3)^2}+\sigma_v^{(1)^2}+k^{(2)}\rho_{(2)}+k^{(3)}\rho_{(3)}$$

In order to get estimators of variance components and cross-correlations one needs to solve the following system of 7 equations for 7 unknowns: $\hat{\sigma}_v^{(1)^2}$, $\hat{\sigma}_v^{(2)^2}$, $\hat{\sigma}^2_{v^{(3)}}$, $\hat{\sigma}_\epsilon^2$, $\hat{\rho}_{(1)}$, $\hat{\rho}_{(2)}$, $\hat{\rho}_{(3)}$

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$$\hat{\sigma}_v^{(1)^2} + \hat{\sigma}_v^{(2)^2} + \hat{\sigma}_{v(3)}^2 + \hat{\sigma}_\epsilon^2 = \tilde{A} - \hat{\sigma}_\mu^2$$

$$b\left(\hat{\sigma}_v^{(1)^2} + \hat{\sigma}_\epsilon^2\right) + \hat{\sigma}_v^{(2)^2} + \hat{\sigma}_{v(3)}^2 + b^{(1)}\hat{\rho}_{(1)} = \tilde{B} - b\hat{\sigma}_\mu^2$$

$$c\left(\hat{\sigma}_v^{(2)^2} + \hat{\sigma}_\epsilon^2\right) + \hat{\sigma}_v^{(1)^2} + \hat{\sigma}_{v(3)}^2 + c^{(1)}\hat{\rho}_{(2)} = \tilde{C} - c\hat{\sigma}_\mu^2$$

$$d\left(\hat{\sigma}_{v(3)}^2 + \hat{\sigma}_\epsilon^2\right) + \hat{\sigma}_v^{(1)^2} + \hat{\sigma}_v^{(2)^2} + d^{(1)}\hat{\rho}_{(3)} = \tilde{D} - d\hat{\sigma}_\mu^2$$

$$f_1\hat{\sigma}_\epsilon^2 + f_2\hat{\sigma}_v^{(1)^2} + f_3\hat{\sigma}_v^{(2)^2} + \hat{\sigma}_{v(3)}^2 + f^{(1)}\hat{\rho}_{(1)} + f^{(2)}\hat{\rho}_{(2)} = \tilde{F} - f_1\hat{\sigma}_\mu^2$$

$$g_1\hat{\sigma}_\epsilon^2 + g_2\hat{\sigma}_{v(3)}^2 + g_3\hat{\sigma}_v^{(1)^2} + \hat{\sigma}_v^{(2)^2} + g^{(1)}\hat{\rho}_{(1)} + g^{(3)}\hat{\rho}_{(3)} = \tilde{G} - g_1\hat{\sigma}_\mu^2$$

$$k_1\hat{\sigma}_\epsilon^2 + k_2\hat{\sigma}_v^{(2)^2} + k_3\hat{\sigma}_{v(3)}^2 + \hat{\sigma}_v^{(1)^2} + k^{(2)}\hat{\rho}_{(2)} + k^{(3)}\hat{\rho}_{(3)} = \tilde{K} - k_1\hat{\sigma}_\mu^2$$

It is hard to solve this system in its general form. However, it can easily be solved for given data set.

Model (4)

For models (4), (5) and (7) I make the same assumptions for the variance components of $\mu_{ij}^{(1)}$, $\mu_{is}^{(2)}$, $\mu_{js}^{(3)}$ and for covariance parameters as in the balanced case. Now I have the following identifying equations

$$E\left[u_{ijst}^2\right] = 3\sigma_\mu^2 + \sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right] = b\left(2\sigma_\mu^2 + \sigma_\epsilon^2\right) + \sigma_\mu^2 + b^{(1)}\rho_{(1)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right] = c\left(2\sigma_\mu^2 + \sigma_\epsilon^2\right) + \sigma_\mu^2 + c^{(1)}\rho_{(2)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right] = d\left(2\sigma_\mu^2 + \sigma_\epsilon^2\right) + \sigma_\mu^2 + d^{(1)}\rho_{(3)}$$

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$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right]=3\sigma_\mu^2+e\sigma_\epsilon^2$$

Thus

$$\begin{aligned}
\hat{\sigma}_\epsilon^2 &= \frac{\tilde{A}-\tilde{E}}{1-e}\\
\hat{\sigma}_\mu^2 &= \frac{1}{3}\left(\tilde{A}-\hat{\sigma}_\epsilon^2\right)\\
\hat{\rho}_{(1)} &= \frac{1}{b^{(1)}}\left[\tilde{B}-(2b+1)\,\hat{\sigma}_\mu^2-b\hat{\sigma}_\epsilon^2\right]\\
\hat{\rho}_{(2)} &= \frac{1}{c^{(1)}}\left[\tilde{C}-(2c+1)\,\hat{\sigma}_\mu^2-c\hat{\sigma}_\epsilon^2\right]\\
\hat{\rho}_{(3)} &= \frac{1}{d^{(1)}}\left[\tilde{D}-(2d+1)\,\hat{\sigma}_\mu^2-d\hat{\sigma}_\epsilon^2\right]
\end{aligned}$$

Model (5)

From the identifying equations

$$E\left[u_{ijst}^2\right]=3\sigma_\mu^2+\sigma_\lambda^2+\sigma_\epsilon^2$$

$$\begin{aligned}
&E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right]=\\
&\quad= b\left(2\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_\lambda^2+b^{(1)}\rho_{(1)}\\
&E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=\\
&\quad= c\left(2\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_\lambda^2+c^{(1)}\rho_{(2)}\\
&E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=\\
&\quad= d\left(2\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_\lambda^2+d^{(1)}\rho_{(3)}
\end{aligned}$$

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$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right]=$$

$$=3\sigma_\mu^2+e\left(\sigma_\lambda^2+\sigma_\epsilon^2\right)$$

I get

$$\hat{\sigma}_\mu^2=\frac{1}{3}\cdot\frac{\tilde{E}-e\tilde{A}}{1-e}$$

$$\hat{\sigma}_\epsilon^2=\frac{1}{f_1-1-f_1^{(1)}\frac{b-1}{b^{(1)}}-f_1^{(2)}\frac{c-1}{c^{(1)}}-f_1^{(3)}\frac{d-1}{d^{(1)}}}\left[\tilde{A}\left(\frac{f_1^{(1)}}{b^{(1)}}+\frac{f_1^{(2)}}{c^{(1)}}+\frac{f_1^{(3)}}{d^{(1)}}-1\right)-\frac{f_1^{(1)}}{b^{(1)}}\tilde{B}-\right.$$
$$-\frac{f_1^{(2)}}{c^{(1)}}\tilde{C}-\frac{f_1^{(3)}}{d^{(1)}}\tilde{D}+\tilde{F}-\hat{\sigma}_\mu^2\Bigg((f_1+f_2+f_3-3+2(b-1)\frac{f_1^{(1)}}{b^{(1)}}+2(c-1)\frac{f_1^{(c)}}{c^{(1)}}+$$
$$\left.+2(d-1)\frac{f_1^{(3)}}{d^{(1)}}\Bigg)\right]$$

$$\hat{\sigma}_\lambda^2=\tilde{A}-3\hat{\sigma}_\mu^2-\hat{\sigma}_\epsilon^2$$

$$\hat{\rho}_{(1)}=\frac{1}{b^{(1)}}\left(\tilde{B}-(2b+1)\hat{\sigma}_\mu^2-\hat{\sigma}_\lambda^2-b\hat{\sigma}_\epsilon^2\right)$$

$$\hat{\rho}_{(2)}=\frac{1}{c^{(1)}}\left(\tilde{C}-(2c+1)\hat{\sigma}_\mu^2-\hat{\sigma}_\lambda^2-c\hat{\sigma}_\epsilon^2\right)$$

$$\hat{\rho}_{(3)}=\frac{1}{d^{(1)}}\left(\tilde{D}-(2d+1)\hat{\sigma}_\mu^2-\hat{\sigma}_\lambda^2-d\hat{\sigma}_\epsilon^2\right)$$

Model (7)

As above, from

$$E\left[u_{ijst}^2\right]=2\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right]=$$

$$=b\left(2\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+b^{(1)}\rho_{(1)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$

$$=c\left(2\sigma_\mu^2+\sigma_v^{(2)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_v^{(3)^2}+c^{(1)}\rho_{(2)}$$

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$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$= d\left(2\sigma_\mu^2+\sigma_v^{(3)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+d^{(1)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(3)}}\sum_{s\in Q^{(3)}}\frac{1}{T_s^{(3)}}\sum_{t\in Z_s^{(3)}}\left(\frac{1}{N_{st}^{(1)}}\sum_{i\in Q_{st}^{(1)}}\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$

$$= f_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+f_2\left(\sigma_\mu^2+\sigma_v^{(1)^2}\right)+f_3\left(\sigma_\mu^2+\sigma_v^{(2)^2}\right)+\sigma_v^{(3)^2}+$$

$$+f_1^{(1)}\rho_{(1)}+f_1^{(2)}\rho_{(2)}+f_1^{(3)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{T_j^{(2)}}\sum_{t\in Z_j^{(2)}}\left(\frac{1}{N_{jt}^{(1)}}\sum_{i\in Q_{jt}^{(1)}}\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$= g_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+g_2\left(\sigma_\mu^2+\sigma_v^{(3)^2}\right)+g_3\left(\sigma_\mu^2+\sigma_v^{(1)^2}\right)+\sigma_v^{(2)^2}+$$

$$+g_1^{(1)}\rho_{(1)}+g_1^{(2)}\rho_{(2)}+g_1^{(3)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{T_i^{(1)}}\sum_{t\in Z_i^{(1)}}\left(\frac{1}{N_{it}^{(2)}}\sum_{j\in Q_{it}^{(2)}}\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$= k_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+k_2\left(\sigma_\mu^2+\sigma_v^{(2)^2}\right)+k_3\left(\sigma_\mu^2+\sigma_v^{(3)^2}\right)+\sigma_v^{(1)^2}+$$

$$+k_1^{(1)}\rho_{(1)}+k_1^{(2)}\rho_{(2)}+k_1^{(3)}\rho_{(3)}$$

Here again, as in the case of the previous model, it is hard to get estimators of variance components in general form. However, it can easily be done for a given data set. One needs to solve the following system:

$$\hat{\sigma}_v^{(1)^2}+\hat{\sigma}_v^{(2)^2}+\hat{\sigma}_{v(3)}^2+\hat{\sigma}_\epsilon^2=\tilde{A}-2\hat{\sigma}_\mu^2$$

$$b\left(\hat{\sigma}_v^{(1)^2}+\hat{\sigma}_\epsilon^2\right)+\hat{\sigma}_v^{(2)^2}+\hat{\sigma}_{v(3)}^2+b^{(1)}\hat{\rho}_{(1)}=\tilde{B}-(2b+1)\hat{\sigma}_\mu^2$$

$$c\left(\hat{\sigma}_v^{(2)^2}+\hat{\sigma}_\epsilon^2\right)+\hat{\sigma}_v^{(1)^2}+\hat{\sigma}_{v(3)}^2+c^{(1)}\hat{\rho}_{(2)}=\tilde{C}-(2c+1)\hat{\sigma}_\mu^2$$

$$d\left(\hat{\sigma}_{v(3)}^2+\hat{\sigma}_\epsilon^2\right)+\hat{\sigma}_v^{(1)^2}+\hat{\sigma}_v^{(2)^2}+d^{(1)}\hat{\rho}_{(3)}=\tilde{D}-(2d+1)\hat{\sigma}_\mu^2$$

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$$f_1\hat{\sigma}_\epsilon^2 + f_2\hat{\sigma}_v^{(1)^2} + f_3\hat{\sigma}_v^{(2)^2} + \hat{\sigma}_{v^{(3)}}^2 + f_1^{(1)}\hat{\rho}_{(1)} + f_1^{(2)}\hat{\rho}_{(2)} + f_1^{(3)}\hat{\rho}_{(3)} = \tilde{F} - (f_1 + f_2 + f_3)\,\hat{\sigma}_\mu^2$$

$$g_1\hat{\sigma}_\epsilon^2 + g_2\hat{\sigma}_{v^{(3)}}^2 + g_3\hat{\sigma}_v^{(1)^2} + \hat{\sigma}_v^{(2)^2} + g_1^{(1)}\hat{\rho}_{(1)} + g_1^{(2)}\hat{\rho}_{(2)} + g_1^{(3)}\hat{\rho}_{(3)} = \tilde{G} - (g_1 + g_2 + g_3)\,\hat{\sigma}_\mu^2$$

$$k_1\hat{\sigma}_\epsilon^2 + k_2\hat{\sigma}_v^{(2)^2} + k_3\hat{\sigma}_{v^{(3)}}^2 + \hat{\sigma}_v^{(1)^2} + k_1^{(1)}\hat{\rho}_{(1)} + k_1^{(2)}\hat{\rho}_{(2)} + k_1^{(3)}\hat{\rho}_{(3)} = \tilde{K} - (k_1 + k_2 + k_3)\,\hat{\sigma}_\mu^2$$

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Chapter 6

Available Data Sources and Potential Applications

Even though in the last decade many new data sets became available, it is still hard to find a database which consists all the necessary information to make analysis on micro trade, especially in our set up. Probably, the most useful database is customs statistics, where data is collected from custom declarations. Usually it contains information on import and export flows at the firm, origin country and product level. For example, Bekes and Murakozi [2012], Gorg et al. [2010] used data from Hungarian Customs Statistics for their firm-product-destination level dataset; data coming from the French Customs Office was used by Corcos et al. [2012], Berthou and Fontagne [2009]. Another good data source is survey and census, which are held constantly by the research statistical centres and ministries. One of the examples is EIIG (Echanges Internationaux Intra-Groupe) database which comes from a survey conducted in 1999 by the French Ministry of Industry's SESSI. This database documents the firm's yearly imports and exports by origin country and industry code. However, these databases usually don't contain

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firm characteristics, thus it is necessary to link the data with another sources, like firm level balance sheet and income statement data from tax authority or another database. Usually, those databases don't contain firm names, however, they do have identification numbers (SIREN), which allows to merge the data from two databases and thereby to match exports with financial information. The process might seem difficult, but there is no easier way so far.

The next question which arises is potential applications of this thesis. Having the data in hand I have all necessary tools to be able to implement derived models using standard econometric/statistical software packages like Stata or Matlab. Higher dimensional panel data sets, however, can become very large, very quickly. Even for smaller data sets, the number of observations can be in the magnitude of $10^4 - 10^5$, but can easily reach 10^6. This is translated into huge computational resource requirements, both in hard drive capacity and CPU time. Typically, the models and methods presented for the balanced case (when closed form spectral decompositions are available) can be used when the data is of the size 10^5. However, when the data set becomes bigger or unbalanced methods and models need to be used, resource requirements can be forbidding. The main difficulty is that in the case of the GLS estimator the covariance matrix of the model, which has the size of the overall number of observations, needs to be inverted. If we run out of computing power the solution can be to use OLS instead of GLS. The OLS estimator is still consistent, although not optimal, for the models introduced in this thesis. But then the covariance matrix of the OLS estimator needs to be properly adjusted using the covariance matrix of each model and the estimated variance components derived above, in order to get the appropriate standard errors of the estimated parameters. Another way to ease computing power requirements is to use lower level programming languages like C+, etc., but this

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requires serious code writing skills and additional resources.

The importance of the model can be seen analysing previous literature on micro trade. For example, Berman et al. [2012] identify the effect of changes in export sales on domestic sales using firm-specific structure of exports by destination and by product. They run two-stage-least-squares estimator in a two-dimensional (firm-year) panel data set up and correct covariance matrix for heteroscedasticity. They include sector-year dummies to capture sector-specific business cycle. In order to illuminate the problem of endogeneity, they use instruments that are independent from firm-specific shocks. However, it is clear that four-dimensional set up will describe the data better in this case. Moreover, it will allow to account not only for firm-specific unobserved characteristics, but also for product and destination characteristics, which are clearly present in the data. Also estimation of sector-year dummies will absorb explanatory power and eat up degrees of freedom. Thus from my point of view random effects estimation of four dimensional panel data will produce the best estimation results in the second stage for the given model.

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Chapter 7

Conclusion

In this thesis I derived several four-dimensional random effects panel data models, suited to deal with economic flow type data like trade or FDI. I presented appropriate Feasible GLS estimation method which was done in two steps. Firstly, I derived covariance matrixes for each model specification, and to ease computation of inverse matrixes needed for FGLS, I presented spectral decomposition for each matrix. Then I obtained estimation of variance components for respective covariance matrixes. Later was presented an important extension. I relaxed assumption that random effects are pairwise uncorrelated. It is too restrictive in the case of trade models, and in general for most of the flow type data as it does not allow for any type of cross correlation. Two step derivations were repeated with the relaxed assumption. Finally, the most important case was presented - unbalanced data case. Unfortunately, most of the trade data is unbalanced in its nature. Moreover, individual time series may not only be different in length, but also have "holes". For example, given firm may export certain product to a country at time t, stop exporting at time $t+1$ and start again at time $t+2$. Dealing with this data structure turned out to be difficult, but possible. A range of new notations were introduced in order to derive

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FGLS estimation method.

It is worth mentioning that content of this thesis can be easily used in standard statistical software packages to implement derived models. However, implementing it one should be cautious about the huge computational resources needed to process the data, due to the very large data sets. The main problem is that one of the steps of GLS estimation is inverting of covariance matrix. I eliminated this problem in the balanced case presenting spectral decomposition, however, it is very costly or even unfeasible for unbalanced set up. One of the solutions can be to use OLS estimator instead, which is still consistent but not optimal for given models, but then the covariance matrix should be adjusted. Another solution is to use lower level programming languages like C+, etc.

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