Problems From Extremal Combinatorics

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Contents

Chapter 1

Introduction

First a short introduction is provided regarding the different problems the thesis discusses. Standard notations are used, for convenience these are defined in Chapter 2.

1.1 On the minimal length of the longest trail in a fixed edge-density graph

One typical problem of extremal graph and hypergraph theory is the Turán-type question: Given a certain forbidden structure or family of structures, how large can a graph (hypergraph) be, i.e. how many edges (hyperedges) might it contain on n vertices so that it does not contain any of the forbidden structures as substructure. The popularity of this type of problems traces back to Turán's famous article [49] about the maximal number

of edges of a graph on n vertices without a complete graph on k vertices K_k as subgraph, and the extremal graph's exact structure.

Theorem 1.1.1 (Turán's Theorem)**.** *A graph G on n vertices not containing a K_{k+1} as a subgraph can have at most as many edges as the complete, evenly distributed k-partite graph on n vertices, $T(n,k)$, and in case it has this many then it is isomorphic to $T(n,k)$.*

Later the general question for the graph case was more or less solved by the theorems of Erdős, Stone and Simonovits ([21] and [20]), determining that the extremal configuration can be characterized by the chromatic number. This gave a solution for all excluded configurations which have a chromatic number larger than 2:

Theorem 1.1.2 (Erdős, Stone, Simonovits)**.** *If $\mathcal{F}$ is a family of graphs for which the minimal chromatic number amongst the members of the family is χ, and G is a graph on n vertices not containing any member of the family $\mathcal{F}$ as a subgraph, then it has at most $(1 - \frac{1}{\chi-1})\binom{n}{2} + o(n^2)$ edges.*

This theorem does not settle the case when the chromatic number is 2, the excluded graph is bipartite. For excluded bipartite graphs there is no general, all-including theory, but in a lot of cases the question is solved as well. One such theorem is the theorem of Erdős and Gallai [18], who determined the maximal size and the configuration of a graph on n vertices not containing a path of $k - 1$ edges:

Theorem 1.1.3 (Erdős, Gallai). *If G is a graph on n vertices, and it does not contain a path of length $k-1$ then it cannot have more than $\frac{n(k-2)}{2}$ edges.*

They also proved that the extremal configuration is the graph consisting of disjoint complete graphs on $k-1$ vertices.

Chapter 3 introduces a similar result but with the twist that the forbidden structure is an l-long trail.

1.2 A Turán-type extremal question on 3-graphs

In general, Turán-type questions have proven to be much harder in the general setting than in the graph case. Knowing Turán's theorem one of the logical things to ask is the same question for 3 (k)-uniform graphs. Unfortunately, this proves to be a tough nut to crack, as the question of the excluded complete 3-graph on 4 points is a famous open problem since Turán's original paper [49], where he conjectured that the graph could have at most $\frac{5}{9}n^3 + o(n^3)$ edges, and gave such a construction. Since that several non-isomorphic constructions have been created with the same amount of edges, but the conjecture could not be proven or disproven. If Turán's conjecture is true then the fact that there are several different extremal families with the conjectured size might show why the question is so hard compared with the graph case.

The best known upper bound for the size of 3-uniform hypergraphs without a complete subhypergraph on 4 vertices is from Chung and Lu [14], and it gives an upper bound of $\frac{3+\sqrt{17}}{2}n^3 + o(n^3)$.

Another famous open problem is the question of the excluded graph $K_3(3,4)$, which has 3 out of the possible 4 edges on 4 vertices. This can be rephrased as to determine the size of the largest 3-graph not containing a C_3 in any of its vertices' neighborhood graphs. The best known bounds on the size of the extremal graph gives the number of edges between $\frac{2}{7}\binom{n}{3} + o(n^3)$ and $\frac{1}{3}\binom{n}{3} + o(n^3)$, given correspondingly in [24] and [15].

Chapter 4 describes a similar problem, the extremal behavior of k-fan-free 3-uniform hypergraphs.

1.3 Almost intersecting families

One of the basic results about extremal set families is due to Erdős, Ko and Rado [19] and states that if $2k \leq n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then the size of $\mathcal{F}$ is at most $\binom{n-1}{k-1}$, furthermore if $2k < n$, then equality holds if and only if $\mathcal{F}$ is a trivially intersecting family. The non-uniform version (i.e. when sets in $\mathcal{F}$ do not necessarily have equal size) of the above theorem is also due to Erdős, Ko and Rado. However, it is rather an easy exercise to prove that any intersecting family $\mathcal{G} \subseteq 2^{[n]}$ can be extended to an intersecting family $\mathcal{F}'$ of size 2^{n-1} and there exists no intersecting family of larger size.

These theorems attracted the attention of many researchers. Several gen-

eralizations have been proved, the intersecting condition has been relaxed or strengthened in many ways. One relaxation is to allow some fixed number of disjoint pairs formed by members of the family $\mathcal{F}$ [1],[6],[23].

In Chapter 5 such families $\mathcal{F} \subseteq 2^{[n]}$ are considered that for any $F \in \mathcal{F}$ there are at most a fixed number of sets (l) disjoint from F.

1.4 Cross-Sperner families

One of the first theorems in the area of extremal set families is that of Sperner [46], stating that if we consider a family $\mathcal{F} \subseteq 2^{[n]}$ such that no set $F \in \mathcal{F}$ can contain any other $F' \in \mathcal{F}$, then the number of sets in $\mathcal{F}$ is at most $\binom{n}{\lfloor n/2 \rfloor}$ and equality holds if and only if $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$. Families satisfying the assumption of Sperner's theorem are called *Sperner families* or *antichains*. There have been many generalizations and extensions both to the theorem of Sperner and to the Erdős-Ko-Rado theorem (two excellent but not really recent surveys are [16] and [17]). One such generalization is the following: a pair $(\mathcal{F}, \mathcal{G})$ of families is said to be *cross-intersecting* if for any $F \in \mathcal{F}, G \in \mathcal{G}$: $F \cap G \neq \emptyset$. Cross-intersecting pairs of families have been investigated for quite a while and attracted the attention of many researchers ([5], [25], [26], [27], [40], [41], [39]).

Chapter 6 deals with the analogous generalization of Sperner families to cross-Sperner families that has not been considered in the literature.

1.5 $(\leq l)$-almost properties

Chapter 7 contains a more general collection of similarly defined problems on finite set systems. The definition is inspired by, and contains as a sub-problem, some well-researched problems. One of them is the Sperner-property [46] which has been described above. Others are the almost intersecting property [28] (which includes the intersecting property as a sub-case), Katona's r-fork problem ([10], [48], [31]), and the intersecting Sperner property [44], [36].

Chapter 2

Notations and definitions

A collection of notations and definitions used in the dissertation is provided here for the reader.

2.1 Basic notations

Definitions of standard notations will not be repeated within the chapters using them, but naturally more specific definitions and notations are defined at their logical place as well.

Notations through the dissertation are the standard notations unless stated otherwise. The base set is $\{1, \ldots, n\} = [n]$, $n \in \mathbb{N}$. The collection of all possible subsets of $[n]$ is $2^{[n]}$. The size of a set S is denoted by $|S|$. The complement of a set F is marked with $\overline{F}$. The letters k and l will be typically used as parameters.

The notation $(f(x) = O(g(x)))$ stands for: $|f(x)| \leq c \cdot g(x)$ for some constant $c \in \Re$ as $x \to \infty$. Similarly, the notation $f(x) = o(g(x))$ stands for $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$, and the notation $f(x) = \Omega(g(x))$ stands for $f(x) \geq c \cdot g(x)$ for some constant $c \in \Re$ as $x \to \infty$.

The notation $\lceil x \rceil$ stands for the smallest integer m with $m \geq x$, and similarly $\lfloor x \rfloor$ stands for the largest integer m' with $m' \leq x$.

$G = (V, E)$ is a graph with vertex set V and edge set E. $V(G)$ means the vertex set of graph G, $E(G)$ the edge set of G and $e(G) = |E(G)|$ the number of edges in G. When the identity of the graph in question is clear it might be left out of the notation. For subsets the notation $e(V)$ stands for the number of edges spanned on the vertex set V and $e(V_1, V_2)$ is the number of edges between the disjoint vertex sets V_1, V_2. The same notations stand for hypergraphs as well. The shorthand notation $v \in \mathscr{H}$ ($E \in \mathscr{H}$) will be used for $v \in V(\mathscr{H})$ and $E \in E(\mathscr{H})$ when the meaning of the notation is clear.

A path of length l in a graph is a sequence of edges $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, ... , $e_l = \{v_l, v_{l+1}\}$ where the vertices $v_i, i = 1, \ldots, l+1$ are all different. A trail of length l is defined the same way, but only the edges need to be different, not the vertices. A cycle of size l in a graph is a sequence of edges $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, ... , $e_l = \{v_l, v_1\}$ where the vertices $v_i, i = 1, \ldots, l$ are all different.

K_l stands for the complete graph on l vertices, and $K_{a,b}$ for the complete bipartite graph with vertex set sizes a, b for the two parts of the graph.

The neighborhood graph $\mathscr{H}_v$ for a 3-uniform hypergraph $\mathscr{H}$ with respect to its vertex v is:

Definition 2.1.1 (Neighborhood graph). $H_v = (V(\mathscr{H}) - \{v\}, \{E - \{v\} : v \in E \in E(\mathscr{H})\})$.

In general the neighborhood of a vertex v in the hypergraph $\mathscr{H}(V, E)$ is $N(v) = \{p \in V : \exists e \in E : \{p\} \cup \{v\} \subseteq E\}$.

A notation for convenience is the following.

Definition 2.1.2. $\mathcal{D}_{\mathscr{F}}(F) = \{G \in \mathscr{F} : F \cap G = \emptyset\}$.

Some useful definitions for hypergraphs follow.

Definition 2.1.3. *A k-uniform hypergraph is a set system $\mathscr{H} \subseteq 2^{[n]}$ where all edges have size k: $\forall F \in \mathscr{H} : |F| = k$.*

Definition 2.1.4. *A chain of length k in $2^{[n]}$ is a collection of sets $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subseteq [n]$. The set of all chains in $2^{[n]}$ is denoted by $\mathscr{C} = \mathscr{C}_n$. For any $C \in \mathscr{C}$ the complementing chain is $\overline{C} = \{\overline{S} | S \in C\}$. $(C, \overline{C})$ is a complementing pair of chains.*

Definition 2.1.5 (Upset). *A family $\mathscr{F} \subseteq 2^{[n]}$ is an upset if for any $F \in \mathscr{F}, G \supset F$ implies $G \in \mathscr{F}$.*

Definition 2.1.6. *Let $\mathscr{U}(F) = \{G \in 2^{[n]} : F \subseteq G\}$ and $\mathscr{L}(F) = \{G \in 2^{[n]} : G \subseteq F\}$ be the upper and lower set of F.*

Definition 2.1.7 (Sperner property). *A family $\mathscr{F}$ has the Sperner property, or simply $\mathscr{F}$ is Sperner, if for any distinct $F_1, F_2 \in \mathscr{F}$, F_1 and F_2 are incomparable: $F_1 \nsubseteq F_2$ and $F_2 \nsubseteq F_1$.*

Definition 2.1.8 (Intersecting property). *A family $\mathscr{F}$ has the intersecting property, or simply $\mathscr{F}$ is intersecting if for any $F_1, F_2 \in \mathscr{F}$: $F_1 \cap F_2 \neq \emptyset$.*

Definition 2.1.9 (Comparability graph). *For any family $\mathscr{F}$ of sets, the comparability graph $G(\mathscr{F})$ is the graph with vertex set $V(G) = \mathscr{F}$ and edge set $E(G) = \{(F_1, F_2) : (F_1 \subsetneq F_2) \vee (F_2 \subsetneq F_1)\}$.*

2.2 Definitions specific to the thesis

One of the constructions studied in the dissertation is the k-fan.

Definition 2.2.1 (k-fan). *Let a k-fan to be the 3-uniform hypergraph on $k + 2$ vertices which contains k edges, $E_1, E_2, \ldots, E_k$ such that $|\cap_{i=1}^{k} E_i| = 1$ and $|E_i \cap E_j| = 2$ if and only if $|j - i| = 1$.*

The following definitions are used to generalize the intersecting property.

Definition 2.2.2 (Cross-intersecting family). *The pairs of sets $(A_i, B_i)_{i=1}^{m}$ form a cross-intersecting family if for any $1 \leq i, j \leq m$: $A_i \cap B_j = \emptyset$ if and only if $i = j$.*

Definition 2.2.3 (($\leq l$)-almost intersecting family). *$\mathscr{F}$ is a ($\leq l$)-almost*

intersecting family if for every $F \in \mathscr{F}$:

$$|\{G \in \mathscr{F} : F \cap G = \emptyset\}| \leq l.$$

Definition 2.2.4 (*l*-almost intersecting family). *$\mathscr{F}$ is an l-almost intersecting family if for every $F \in \mathscr{F}$:*

$$|\{G \in \mathscr{F} : F \cap G = \emptyset\}| = l.$$

Definition 2.2.5 (*l*-almost (($\leq$l)-almost) cross-intersecting pair of families). *The pair of families $(\mathscr{F}, \mathscr{G})$ is l-almost cross-intersecting (($\leq l$)-almost cross-intersecting) if $|\mathcal{D}_{\mathscr{F}}(G)| = l$ ($|\mathcal{D}_{\mathscr{F}}(G)| \leq l$) and $|\mathcal{D}_{\mathscr{G}}(F)| = l$ ($|\mathcal{D}_{\mathscr{G}}| \leq l$) for all $F \in \mathscr{F}$ and $G \in \mathscr{G}$.*

Definition 2.2.6. *A family $\mathscr{F}$ is (p, q)-chain intersecting if $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$, $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_q$ with $A_i, B_j \in \mathscr{F}$ implies $A_p \cap B_q \neq \emptyset$.*

Definition 2.2.7 (Cross-Sperner family). *A pair of families $(\mathscr{F}, \mathscr{G})$ is cross-Sperner if there is no pair of sets (F, G) with $F \in \mathscr{F}, G \in \mathscr{G}$ and $F \subseteq G$ or $G \subseteq F$.*

Definition 2.2.8 (cross-Sperner k-tuples of families). *$\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k$ is cross-Sperner if for any $1 \leq i < j \leq k$ there is no pair $F_i \in \mathscr{F}_i$ and $F_j \in \mathscr{F}_j$ that $F_i \subseteq F_j$ or $F_j \subseteq F_i$.*

Definition 2.2.9 (($\leq l$)-almost intersecting Sperner family). *A family $\mathscr{F}$ is*

($\leq l$)*-almost intersecting Sperner if for any* $F \in \mathscr{F}$ *the following equation holds:* $|\mathscr{U}(F)| + |\mathscr{L}(F)| + |\mathscr{L}(\overline{F})| \leq l + 2$.

Definition 2.2.10 (($\leq l$)-almost unrelated family)**.** *The family* $\mathscr{F} \subseteq 2^{[n]}$ *is* ($\leq l$)*-almost unrelated if for any* $F \in \mathscr{F}$*:* $|\mathscr{U}(F)| + |\mathscr{L}(F)| + |\mathscr{L}(\overline{F}) \cup \mathscr{U}(\overline{F})| \leq l + 2$.

The next configuration implies an interesting extremal question on its own, and will be used for other investigations as well.

Definition 2.2.11 (*r*-fork)**.** *An* r*-fork is a family of* $r + 1$ *distinct sets* $F_0, F_1, \ldots, F_r$ *where* $F_0 \subsetneq F_i$ *for all* $i = 1, \ldots, r$.

Definition 2.2.12 (Hamming distance)**.** *The Hamming-distance of two subsets of an n-element set is the size of the symmetric difference of the two sets. For* $F, G \in \mathscr{F}$ *the distance is:* $d(F, G) = |F - G| + |G - F|$.

Chapter 3

On the minimal length of the longest trail in a fixed edge-density graph

3.1 Background and main theorem

Paths have been studied extensively by extremal combinatorics, as they are one of the most natural notions in a graph. Some of these results are the famous Erdős-Gallai theorem and the Faudree-Schelp theorem regarding the existence of an l-long path:

Theorem 3.1.1 (Erdős, Gallai [18]). *If G is a graph on n vertices, and it does not contain a path of length l then it cannot have more than $n(l-1)/2$ edges. Equality holds if and only if G is a disjoint union of multiple copies*

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of K_l.

Theorem 3.1.2 (Faudree, Schelp [22]). *If G is a graph of order $n \geq 5$ with $d(u) + d(v) > n$ for each pair of distinct nonadjacent vertices u and v, then there is a path of length i between any two distinct vertices for every $4 \leq i \leq n - 1$.*

Inspired by the results above, and not having found any similar result for trails, the following problem is investigated in the next chapter: find the graph on n vertices with a fixed average degree k that has the longest trail. The following theorem will be proved about connected graphs.

Theorem 3.1.3. *A connected graph G on n vertices with an average degree at least $k = 2e/n \geq 12.5$ contains either a trail of length at least $(\lceil k \rceil + 1)k/2$ or $n \leq \lceil k \rceil + 2$.*

It is easy to see that for a graph with multiple components the longest trail is contained in one of the components, thus the theorem implies the structure for the not connected case as well. The theorem is the best possible for the length of the longest trail without adding extra constraints on the size of n. Indeed, a graph can be easily constructed where the average degree is close to k and has about $(\lceil k \rceil + 1)k/2$ edges by taking a complete graph on $\lceil k \rceil + 1$ vertices and deleting edges until the needed average degree is reached. If k is large enough then the proof yields additionally that the connected components of G cannot be larger than $\lceil k \rceil + 2$, so the extremal configurations should look like disconnected components of size $\lceil k \rceil + 2$ or

$\lceil k \rceil + 1$. Good candidates are those graph that are obtained from a complete graph on even number of vertices (thus having odd degree) by deleting some cycles, but due to the flexibility of this construction in most cases there will not be a unique extremal configuration.

The method of the proof is to find a large Eulerian circuit in a subgraph of G and to construct from this a long enough trail in G. A similar problem, finding spanning circuits have been studied extensively, see [13].

3.2 Proof of theorem 3.1.3

During the proof four graphs are used. These are the original graph G and three subgraphs: a forest H, the Eulerian graph G' which remains after the deletion of the edges of H, and the component A with the highest average degree in G', or one of these, if there are more with the same average degree.

3.2.1 Initial setup

It can be supposed that the degree of each vertex in G is strictly larger than $k/2$, as by deleting every vertex with $d(v) \leq k/2$ the average degree of the graph does not decrease (new parameters marked by $'$), as

$$\bar{d}' = \frac{2e'}{n'} = \frac{2e - 2d(x)}{n - 1} \geq \frac{2e - k}{n - 1} = \frac{kn - k}{n - 1} = k = \bar{d}.$$

It can be supposed also that G has odd degree vertices, otherwise G is an Eulerian graph, all edges are contained in an Eulerian circuit, thus there is a trail of length at least $e = kn/2 \geq (\lceil k \rceil + 1)k/2$.

Proposition 3.2.1. *Any graph G is an edge-disjoint union of an Eulerian subgraph G' (not necessarily connected) and a sub-forest H of G.*

Proof: The following algorithm deletes the edges of a sub-forest of G to obtain an Eulerian graph. Take a pairing on the odd degree vertices of G and connect the paired vertices by shortest paths. By taking the collection of those edges which are used in an odd number of paths, a subgraph H of G is formed where the parity of the degrees of the vertices is the same as in G. Eliminate cycles one by one in H by taking a cycle C and dropping its edges out of H (and replace H with the new graph). Continue until there are no cycles left in H, thus at the end H is a forest, so H has at most $n - 1$ edges. An Eulerian graph G' is obtained by deleting the edges of H from G. $\square$

Take those sub-forests of G which have the smallest number of edges and G minus the edges of the forest is an Euler graph. Choose H out of these to be such that the number of the connected components of G' is minimal. The edges of H will be sometimes referred to as deleted or dropped edges. The

edge-density of G' is strictly larger than $k-2$, as

$$\bar{d}' = \frac{2e'}{n} \geq \frac{2(e-(n-1))}{n} = \frac{2e+2}{n} - 2 = k - 2 + \frac{2}{n}.$$

Let A be the component of G' with the largest average degree. A contains more than $k-1$ vertices as the average degree is more than $k-2$ in the whole graph, so in A as well. A case analysis follows by limiting the size and the edge-density of A.

Proposition 3.2.2. *If $\bar{d}_A \geq k$ or $|V(A)| \geq \lceil k \rceil + 4$, then the statement of the theorem holds for $k \geq 10$.*

Proof: If the average degree of A, $\bar{d}_A$, is at least k, then either A has strictly more than $k+1$ vertices or it is exactly K_{k+1} and k is an even integer. In both cases there is an Eulerian component with at least $k(\lceil k \rceil + 1)/2$ edges. If A has at least $\lceil k \rceil + 4$ vertices, then there is an Eulerian trail of length longer than

$$\frac{(k-2)(\lceil k \rceil + 4)}{2} = \frac{(\lceil k \rceil + 1)k}{2} + \frac{k}{2} - 4 - (\lceil k \rceil - k).$$

This gives a trail of the needed length for $k \geq 10$. □

A has $\lceil k \rceil + x$ vertices, where x is between -1 and 3. G, and consequently G', has $\lceil k \rceil + x + y$ vertices, where the sum is at least $\lceil k \rceil + 1$ as $\bar{d}_G = k$.

3.2.2 Analysis of a disconnected Euler-graph

Proposition 3.2.3. *There is a trail of the required length provided $k - 2 < \bar{d}_A \leq k - 1$, $k - 1 < |V(A)| \leq \lceil k \rceil + 3$, $k \geq 64/6$, and there are at least 3 vertices in $V(G) - V(A)$.*

Proof: A lower estimate on the number of edges of G' outside A follows. There were $(\lceil k \rceil + x + y)k/2$ edges in G, at most $\lceil k \rceil + x + y - 1$ were dropped out, and in A there are at most $(\lceil k \rceil + x)(k - 1)/2$ edges, thus at least the following amount of edges remain in $G' - A$

$$\frac{(\lceil k \rceil + x + y)k}{2} - \frac{(\lceil k \rceil + x)(k - 1)}{2} - (\lceil k \rceil + x + y - 1) = \frac{(k - 2)y + 2 - \lceil k \rceil - x}{2} .$$

As $y \geq 3$ and $x \leq 3$, a lower bound on the average degree of $G' - A$ is

$$\bar{d}_{G'-A} \geq k - 2 - \frac{\lceil k \rceil + x - 2}{y} \geq k - 2 - \frac{k + 1 + 3 - 2}{3} = \frac{2k - 8}{3}$$

This implies that there are at least $\bar{d}_{G'-A} + 1 \geq (2k - 5)/3$ vertices outside A. As $k \geq 12.5 > 8.5$ this is an improvement. Repeating the previous average degree argument yields an average degree

$$\bar{d}_{G'-A} \geq k - 2 - \frac{\lceil k \rceil + x - 2}{(2k - 5)/3} > k - \frac{7}{2} - 3\frac{x + 3/2}{2k - 5} .$$

A new estimate is obtained by using the bounds of x and k

$$\bar{d}_{G'-A} \geq k - \frac{7}{2} - 3\frac{x+3/2}{2k-5} \geq k - \frac{7}{2} - 3\frac{9/2}{20} > k - 5\,.$$

Therefore there is a component B in G' that is disjoint of A. B has at least the upper average degree and at least $\lceil k \rceil - 4$ vertices, so its edges form an Eulerian trail of length at least: $(k-5)(\lceil k \rceil - 4)/2$. The circuit in A, a connecting path from H and, if necessary for the connecting path, more edges from some other components form together a trail longer than

$$\frac{(\lceil k \rceil - 1)(k-2)}{2} + \frac{(\lceil k \rceil - 4)(k-5)}{2} + 1 = \lceil k \rceil k - \frac{7}{2}\lceil k \rceil - \frac{5}{2}k + 12\,.$$

This is larger than $(\lceil k \rceil + 1)k/2$ if $k \geq 64/6$.   □

Proposition 3.2.4. *There is a trail of the required length provided $k - 1 < \bar{d}_A \leq k$, $\lceil k \rceil + 1 \leq |V(A)| \leq \lceil k \rceil + 3$, $|V(G) - V(A)| \geq 1$ and $\lceil k \rceil \geq 12$.*

Proof: The size of A is at least $\lceil k \rceil$ because of the average degree. If the size of A is at least $\lceil k \rceil + 2$, then there is an Eulerian trail strictly longer than

$$\frac{(\lceil k \rceil + 2)(k-1)}{2} = \frac{k(\lceil k \rceil + 1)}{2} + \frac{k - \lceil k \rceil}{2} - 1 > \frac{k(\lceil k \rceil + 1)}{2} - 2\,.$$

As there were edges outside A in G, there is a trail in G of length at least two, which is outside A and ends in A due to the minimal degree condition.

The union of this trail and the Eulerian trail of A yields a desired trail in G. $\qquad\square$

Proposition 3.2.5. *There is a trail of the required length provided $k - 1 < \bar{d}_A \leq k$, $k - 1 < |V(A)| \leq \lceil k \rceil + 1$, $|V(G) - V(A)| \geq 4$ and $\lceil k \rceil \geq 12$.*

Proof: The remaining case is when the size of A is at most $\lceil k \rceil + 1$. The same process as before is used to estimate the number of edges outside A. In A there are at most $(\lceil k \rceil + x)k/2$ edges, at most $\lceil k \rceil + x + y - 1$ edges were dropped out, so $|E(G' - A)|$ is at least

$$\frac{(\lceil k \rceil + x + y)k}{2} - \frac{(\lceil k \rceil + x)k}{2} - (\lceil k \rceil + x + y - 1) = \frac{(k - 2)y + 2 - 2\lceil k \rceil - 2x}{2} \, .$$

As before the average degree is counted as

$$\bar{d}_{G'-A} \geq k - 2 - 2\frac{\lceil k \rceil + x - 1}{y} \, .$$

$$\bar{d}_{G'-A} \geq k - 2 - 2\frac{\lceil k \rceil}{4} \geq \frac{\lceil k \rceil}{2} - 3 \, .$$

This gives $y \geq \lceil k \rceil/2 - 2$, which applied again in the inequality above gives

$$\bar{d}_{G'-A} \geq k - 2 - 2\frac{\lceil k \rceil}{\frac{\lceil k \rceil}{2} - 2} = k - 6 - \frac{8}{\frac{\lceil k \rceil}{2} - 2} \geq k - 8 \, .$$

Again, this yields an estimate for $y \geq \lceil k \rceil - 7$, another bound for the degree

is obtained by repetition

$$\bar{d}_{G'-A} \geq k - 2 - 2\frac{\lceil k \rceil}{\lceil k \rceil - 7} = k - 4 - \frac{14}{\lceil k \rceil - 7} \geq k - 7 \,.$$

Iterating the process once again, $y \geq \lceil k \rceil - 6$ and

$$\bar{d}_{G'-A} \geq k - 2 - 2\frac{\lceil k \rceil}{\lceil k \rceil - 6} = k - 4 - \frac{12}{\lceil k \rceil - 6} \geq k - 6 \,.$$

This estimate for the average degree and the subsequent estimation $y \geq \lceil k \rceil - 5$ gives a trail in G with length longer than

$$\frac{\lceil k \rceil (k - 1)}{2} + \frac{(k - 6)(\lceil k \rceil - 5)}{2} + 1 = \frac{2\lceil k \rceil k - 5k - 7\lceil k \rceil + 32}{2} \,.$$

As $\lceil k \rceil \geq 12$, this is already longer than the trail needed. $\square$

Proposition 3.2.6. *If $\lceil k \rceil \geq 12$ then $y = |V(G) - V(A)| \geq 4$, $y = 0$ or 3.1.3 is already proven.*

   Proof: Suppose $y = 3$. Either there are no edges in $G' - A$ or there is a C_3 component. As the minimum degree of G is larger than $k/2$, the number of edges in H is larger than $3(k/2 - 2)$. In G' there are no edges between A and the vertices outside.

   Suppose there is a C_3 in $G' - A$. If outside A there is a C_3 then in G every vertex of A is connected to only one of the vertices outside. Otherwise

if any vertex of A is connected to two such vertices, then these three vertices would form a triangle (C_3) where two edges are in H. By switching them with the third edge a smaller H is obtained, but H was minimal.

The same argument shows that there are no edges going between vertices which are in G and are connected to the same vertex outside A. If there is an edge between vertices connected to different vertices outside, then these form a C_4 in G. Switching between the deleted and the kept edges of this C_4 does not increase the number of edges of H. This operation extends A to contain more vertices. Switching the edges only deletes one edge from A, leaving an Eulerian trail, thus A is still a connected component, and now there are less connected components in G'. This contradicts the definition of H.

Therefore it can be assumed that in A there are no edges in the neighborhood of the three vertices outside A. This gives an empty graph in A of size larger than $\binom{\lceil 3(k/2-2)\rceil}{2}$. An estimate for the number of edges in A is

$$\binom{\lceil k\rceil + x}{2} \geq \frac{1}{2}(\lceil k\rceil + x)(k-1) + \binom{\lceil 3(\frac{k}{2}-2)\rceil}{2}.$$

The left side is larger than

$$\binom{\lceil k\rceil + x}{2} + \binom{3(\frac{k}{2}-2)}{2} - (\lceil k\rceil + 1).$$

But there are too many edges in the component already, as

$$\binom{3(\frac{k}{2} - 2)}{2} - (\lceil k \rceil + 1) \geq \frac{(3k - 12)(3k - 14) - 8k - 16}{8} \geq 0 \text{ if } k \geq 8 \,.$$

A slightly better estimate is obtained if there is no edge outside A. At most $\lceil k \rceil + x + 3 - 1 \leq \lceil k \rceil + 3$ edges are in H, and not less than $3k/2$, so

$$3\frac{k}{2} \leq \lceil k \rceil + 3 \Rightarrow k \leq 8 \,.$$

Therefore the case $y = 3$ cannot happen.

If $y \leq 2$, there are no edges outside A as there are not enough vertices to support an Eulerian graph. Note that for any path $v_1 v_2 v_3$ of length two in H, the edge $v_1 v_3$ is not in G. Otherwise $v_1 v_2$ and $v_2 v_3$ could be switched to $v_1 v_3$ and a smaller H would be obtained, but H is the smallest already. Thus if there is at least one vertex outside A, then there is an empty subgraph in G of size more than $k/2$.

For $y = 2$ observe that the two vertices outside A, P_1 and P_2, cannot share more than one common neighbor in G. Otherwise, there is a C_4 in H, but H is a forest. As the neighborhood of P_1 (and of P_2) in G cannot contain any edge, there are two large (more than $k/2 - 1$ vertices) empty subgraphs in G which share at most one vertex. Similarly, using the minimality of H it can be observed that there are no two edges on the union of the neighbors of P_1 and P_2 which do not share a common vertex. Otherwise there would

CHAPTER 3. THE LONGEST TRAIL

be a C_6, from which 4 edges would be in H. Switching the edges of the C_6 would give a smaller H, but it is minimal. This implies that $N(P_1) \cup N(P_2)$ has at most $|N(P_1) \cup N(P_2)| - 1$ edges in G. Then G contains a big empty subgraph, and G satisfies

$$\binom{\lceil k \rceil + x + 2}{2} \geq \frac{1}{2}(\lceil k \rceil + x + 2)k + \binom{2\lceil \frac{k}{2} \rceil - 1}{2}.$$

Using $x \leq 3$ and $\lceil k \rceil \geq 12$ in the upper inequality gives a contradiction, so the graph G could not hold these many edges.

For $y = 1$ (with the bound on the size of A) G is already a subgraph of $K_{\lceil k \rceil + 2}$. Nevertheless, a similar elementary argument also yields a contradiction with a larger bound on k. Again, there is a big empty subgraph in G. Denote the vertex outside A by P. As before the neighborhood of P does not hold any edge, and a similar computation of the number of edges yields

$$\binom{\lceil k \rceil + x + 1}{2} \geq \frac{1}{2}(\lceil k \rceil + x + 1)k + \binom{\deg P}{2} + \lceil k \rceil + x - \deg P.$$

Solving this with $x \leq 3$ and $\lceil k \rceil - k \leq 1$ gives the inequality

$$2\lceil k \rceil + 10 \geq \deg P(\deg P - 3) \Rightarrow \deg P \leq \frac{3 + \sqrt{49 + 8\lceil k \rceil}}{2}.$$

As $\deg P \geq \lceil k \rceil/2$, the condition above can be satified only if $\lceil k \rceil \leq 16$. This is again a contradiction if k is large enough. $\qquad \square$

3.2.3 Analysis if the connected Euler-graph covers all of G

Proposition 3.2.7. *Let A and G have the same vertex set and $k = 2e/n \geq 12.5$. If in G there is no trail of length at least $(\lceil k \rceil + 1)k/2$, then $n \leq \lceil k \rceil + 2$.*

Proof: The size of $V(A)$ is either $\lceil k \rceil + 2$ or $\lceil k \rceil + 3$. Otherwise there is a long trail by Proposition 3.2.2, or G is a subgraph of $K_{\lceil k \rceil + 1}$. It will be shown that knowing the size of the graph limits the number of edges of H. If there are $\lceil k \rceil + 3$ vertices, then the question is whether during the process no more than k edges are deleted

$$\frac{(\lceil k \rceil + 3)k}{2} - \frac{(\lceil k \rceil + 1)k}{2} = k \, .$$

Suppose that at least $\lceil k \rceil$ edges are deleted. This means that H consists of at most three components, covering nearly the whole graph. As in A there are no edges between vertices of the same component of H, an inequality for the amount of edges follows (here formulated for the worst case, when H has three roughly equal-sized components)

$$\binom{\lceil k \rceil + 3}{2} \geq \frac{(\lceil k \rceil + 3)k}{2} - \lceil k \rceil + 3\binom{\frac{\lceil k \rceil + 3}{3}}{2} \, .$$

Solving this inequality gives $k < 12.5$, a contradiction. This proves that

there is a long enough trail in G.

Unfortunately it might happen that the size of $V(A)$ is $\lceil k \rceil + 2$, and there is no trail of length $(\lceil k \rceil + 1)k/2$. An example is obtained by taking a large complete graph on an even number of vertices and deleting a cycle of length $n - 1$. It is easy to check that this graph does not contain a trail of length $(\lceil k \rceil + 1)k/2$. Many examples can be constructed similarly.

As a conclusion, the densest graph with a given maximal trail length is a nearly complete graph. $\qquad\qquad\square$

3.3 Consequences and open cases

One consequence of the theorem is that given an upper bound on the length of the maximal trail for large enough base set, the graph with the most edges should consist of disconnected components which all have at most $\lceil k \rceil + 2$ vertices (where k comes from the theorem) and all components have edge-density about k.

The problem of smaller edge-densities (or shorter longest trails) remains open for now, as the amount of possible configurations makes the checking of every candidate nearly impossible.

Chapter 4

A Turán-type extremal question on 3-graphs

4.1 Introduction

The neighborhood graph H_v for a 3-uniform hypergraph H with respect to its vertex v is:

Definition 4.1.1. $H_v = (V(H) - \{v\}, \{E - \{v\} : v \in E \in E(H)\})$.

Probably the most famous question which can be easily reformulated to a question avoiding a neighborhood graph is the size of a maximal triple system (3-uniform hypergraph) with any 4 vertices spanning at most 2 hyperedges. The rephrased question is to determine the size of the largest 3-graph not containing a C_3 in any of its vertices' neighborhood graphs. This question

is studied extensively but the exact size remains unknown (for the latest results, see [45], [47]).

Definition 4.1.2 (*k*-fan). *Let a k-fan to be the 3-uniform hypergraph on* $k + 2$ *vertices which contains k edges,* $E_1, E_2, \ldots E_k$ *such that* $|\cap_{i=1}^{k} E_i| = 1$ *and* $|E_i \cap E_j| = 2$ *if and only if* $|j - i| = 1$.

The following theorem establishes an upper bound on the size of the extremal graph not containing a *k*-fan and an infinite family of hypergraphs reaching this bound is given as well.

Theorem 4.1.1. *Any 3-uniform hypergraph H not containing a k-fan has at most* $\binom{n}{2}(k-1)/3$ *many edges. There are infinitely many pairs* (n, k) *where there is an extremal graph* $G(n, k)$ *on n vertices containing* $\binom{n}{2}(k-1)/3$ *3-edges and not containing any k-fan as a subhypergraph.*

The condition is equivalent to demanding that none of the neighborhood graphs of H contain a path of length k (measured as the number of edges in the path). During the proof we will use the Erdős-Gallai Theorem [18] as stated in THeorem 3.1.1.

4.2 Proof of the theorem

The first step is to prove the upper bound for these graphs. Since H does not contain any *k*-fan, one can see that for any vertex v H_v does not contain P_k, a path of length k. Therefore by the Erdős-Gallai theorem [18] there is

an upper bound on the degree of v as $\deg(v) \leq (n-1)(k-1)/2$. This bound is reached only when the graph is the disjoint union of complete K_ks. The implication is that the upper bound for the number of edges of H is:

$$\sum_{e \in E(H)} 1 = \sum_{v \in V(H)} d(v)/3 \leq n(n-1)(k-1)/6 = \binom{n}{2}(k-1)/3.$$

In order to have such amount of edges the 2-graph H_v for every vertex should be the disjoint union of complete K_ks. Let us define $H(m,k)$ in the following way: First, take $PG(m,k)$, a finite m-dimensional projective space of order k. $PG(m,k)$ exists if k is a prime power. The space has $n = (k^{m+1} - 1)/(k - 1)$ points, these will form the base set of $H(m,k)$. For every point there are $(k^m - 1)/(k - 1)$ lines going through this point, partitioning the whole space without the point to sets of size k. Define the 3-graph on the vertices by putting on every line a complete 3-uniform hypergraph (on $k + 1$ vertices):

$$E(H(m,k)) = \{(P, Q, R) \in PG(m,k) | PQR \text{ collinear}\}$$

It is easy to see that for any vertex v the 2-graph $H(m,k)_v$ is the disjoint union of K_ks, as each and every line defines a K_k. Therefore $H(m,k)$ is a graph without a k-fan. For a given k an infinite, though sparse family of extremal k-fan-free 3-graphs is obtained this way as the dimension of the projective space increases.

Finally, to see that the number of edges really gives our bound, the number of the edges of $H(m, k)$ count out as:

$$|E(H(m,k))| = \frac{d(v)n}{3} = \frac{k^m - 1}{k - 1}\binom{k}{2}\frac{n}{3} = \frac{(k^{m+1} - k)n}{6} = \frac{(n-1)(k-1)n}{6}.$$

4.3 Additional notes

The extremal hypergraph above is not unique in general, as there might be multiple projective spaces for a given q. For now it is not known if there is a projective plane with not prime power order. As it is obvious from the construction method, the transformation can be used the opposite way as well, so a finite projective space can be built from a 3-graph which satisfies the conditions of the theorem with equality. Unfortunately finding such 3-graphs does not seem to be easier at all than finding appropriate projective spaces.

Chapter 5

Almost intersecting families

5.1 Introduction

The following chapter proposes a possible way to generalize the Erdös-Ko-Rado theorem [19] about the maximal size of an intersecting set system in $2^{[n]}$ and the maximal size of an intersecting k-uniform set system in $\binom{[n]}{k}$. The results are due to joint research with D. Gerbner, N. Lemons, C. Palmer and B. Patkós [28], and additionally with D. Pálvölgyi and all the others [30].

Theorem 5.1.1 (k-uniform Erdős-Ko-Rado [19]). *If $n \geq 2k$ and $\mathscr{F} \subseteq \binom{[n]}{k}$ where any pair of sets from $\mathscr{F}$ intersects, then $|\mathscr{F}| \leq \binom{n-1}{k-1}$. If $n > 2k$ then equality hold if and only if $\mathscr{F}$ is a family of all k-element subsets containing a common vertex.*

Theorem 5.1.2 (Nonuniform Erdős-Ko-Rado). *An intersecting family $\mathscr{F} \subseteq \binom{[n]}{k}$ has size at most 2^{n-1}. Any intersecting family on n vertices can be*

extended to this size.

These theorems have been generalized many ways, and similar theorems were found in different settings. One way of relaxing the intersecting condition is to allow a number of exceptions to the intersection rule, that is, to allow some non-intersecting pairs of sets ([1],[6],[23]).

In this chapter the intersecting property is relaxed by letting each set have (at most) l disjoint sets in the system. This property is called $(\leq)l$-almost intersecting property. First we define the set of disjoint elements from a set F in $\mathscr{F}$ for convenience.

Definition 5.1.1. $\mathcal{D}_{\mathscr{F}}(F) = \{G \in \mathscr{F} : F \cap G = \emptyset\}$.

Definition 5.1.2 $((\leq l)$-almost intersecting family$)$**.** *$\mathscr{F}$ is a $(\leq l)$-almost intersecting family if for every $F \in \mathscr{F}$:*

$$|\mathcal{D}_{\mathscr{F}}(F)| \leq l\,.$$

Definition 5.1.3 $(l$-almost intersecting family$)$**.** *$\mathscr{F}$ is an (l)-almost intersecting family if for every $F \in \mathscr{F}$:*

$$|\mathcal{D}_{\mathscr{F}}(F)| = l\,.$$

Obviously every l-almost intersecting family is a $(\leq l)$-almost intersecting family, and the opposite is not true. $l = 0$ gives back the original intersection property. The 2 definitions above together with the possibilities of observing

the uniform or the non-uniform case give rise to four distinct problems. The extremal families differ significantly. This is the most significant in the case of the uniform ($\leq l$)-almost intersecting and the uniform l-almost intersecting properties, where the latter seems to have a maximal size which does not depend on the size of the base set, if certain size conditions are satisfied.

The next two sections deal with the l-almost intersecting case for the uniform and the non-uniform families. The two sections after consist of the same families regarding the ($\leq l$)-almost intersecting case.

5.2 k-uniform l-almost intersecting families

5.2.1 k-uniform 1-almost intersecting families

In order to investigate the extremal question the notion of a *cross-intersecting family* is important to note.

Definition 5.2.1 (Cross-intersecting family)**.** *The pairs of sets $(A_i, B_i)_{i=1}^m$ form a cross-intersecting family if for any $1 \leq i, j \leq m$ $A_i \cap B_j = \emptyset$ if and only if $i = j$.*

The following is the main theorem about cross-intersecting families. It is due to Bollobás [7].

Theorem 5.2.1 (Bollobás [7])**.** *If the pairs $(A_i, B_i)_{i=1}^m$ form a cross-intersecting family then*

$$\sum_{i=1}^m \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1 \, .$$

In particular, if $|A_i| \leq k$ and $|B_i| \leq l$ for all $1 \leq i \leq m$ then $m \leq \binom{k+l}{k}$, equality holds if and only if the pairs give all possible partitions to size k and l of some $(k+l)$-set X.

This theorem gives the size of the maximal k-uniform 1-almost intersecting family for arbitrary k.

Corollary 5.2.1. *If $\mathscr{F}$ is a k-uniform 1-almost intersecting family, then $|\mathscr{F}| \leq \binom{2k}{k}$ and equality holds if and only if $\mathscr{F} = \binom{X}{k}$ where $|X| = 2k$.*

Proof: For $F_i \in \mathscr{F}$ let $A_i = F_i$ and let B_i be the only set in $\mathscr{F}$ which is disjoint from F_i. Then the pairs (A_i, B_i) form a cross-intersecting family, and we are done by Theorem 5.2.1. $\square$

5.2.2 Conjecture and best known bound on l-almost intersecting k-uniform families

The exact upper bound is not known for general l and k, the conjectured value and family is described below.

Conjecture 5.2.1. *For any k there exists $l_0 = l_0(k)$ such that if $l \geq l_0$ and $\mathscr{F}$ is a k-uniform l-almost intersecting family, then $|\mathscr{F}| \leq (l+1)\binom{2k-2}{k-1}$.*

If this is true then the bound is sharp as the following example shows:

$$\left\{ F \cup \{i\} \ : \ F \in \binom{[2k-2]}{k-1}, i \in \{2k-1, 2k, \ldots, 2k+l-1\} \right\}.$$

The following theorem gives an upper bound for the size of a k-uniform l-almost intersecting family, which is independent from the size of the ground set and is close to the conjectured bound.

Theorem 5.2.2. *Let $\mathscr{F}$ be an l-almost intersecting family of k-sets. Then $|\mathscr{F}| \leq (2l-1)\binom{2k}{k}$.*

The proof of the theorem is a special case of a more general theorem. To state this the idea of l-almost cross-intersecting families needs to be defined. This is a similar generalization of the cross-intersection property (see for ex. [7]) as the l-almost intersecting property is the generalization of the intersecting property.

Definition 5.2.2 (l-almost cross-intersecting pair of families)**.** *The pair of families $(\mathscr{F}, \mathscr{G})$ is l-almost cross-intersecting $((\leq l)$-almost cross-intersecting) if $|\mathcal{D}_{\mathscr{F}}(G)| = l$ $(|\mathcal{D}_{\mathscr{F}}(G)| \leq l)$ and $|\mathcal{D}_{\mathscr{G}}(F)| = l$ $(|\mathcal{D}_{\mathscr{G}}| \leq l)$ for all $F \in \mathscr{F}$ and $G \in \mathscr{G}$.*

An easy observation is that for any l-almost cross-intersecting pair $\mathscr{F}, \mathscr{G}$ $|\mathscr{F}| = |\mathscr{G}|$ if $l \geq 1$. This follows from double-counting the number of disjoint pairs: $|\{(F,G) : F \in \mathscr{F}, G \in \mathscr{G}, F \cap G = \emptyset\}| = l|\mathscr{F}| = l|\mathscr{G}|$. If a family $\mathscr{F}$ is l-almost intersecting, then the pair $(\mathscr{F}, \mathscr{F})$ is l-almost cross-intersecting. This observation is used to gain the theorem above from the following one.

Theorem 5.2.3. *Let $(\mathscr{F}, \mathscr{G})$ be a pair of l-almost cross-intersecting families, where $\mathscr{F} \subseteq \binom{[n]}{r}$ and $\mathscr{F} \subseteq \binom{[n]}{k}$. Then $|\mathscr{F}| = |\mathscr{G}| \leq (2l-1)\binom{k+r}{k}$.*

Proof: Let B be the bipartite graph with vertex classes $\mathscr{F}$ and $\mathscr{G}$ and edge-set $E(B) = \{(F, G) : F \in \mathscr{F}, G \in \mathscr{G}, F \cap G = \emptyset\}$. As $(\mathscr{F}, \mathscr{G})$ l-almost cross-intersecting B is l-regular. so there exists a complete matching M on its edges with $|M| = |\mathscr{F}| = |\mathscr{G}|$. Let the graph Γ be defined on the edges of this matching: $V(\Gamma) = M$ and $E(\Gamma) = \{((F, G)(F', G')) : F' \cap G = \emptyset$ and/or $F \cap G' = \emptyset\}$. As $(\mathscr{F}, \mathscr{G})$ is l-almost cross-intersecting, the maximum degree in Γ is at most $2(l - 1)$, therefore its vertices can be properly colored with at most $2l - 1$ colors (actually, with at most $2l - 2$ colors as well by Brooks' theorem [11], provided Γ is not complete and is not an odd cycle). For any color class C in the proper coloring the vertices (F, G) belonging to C form a cross-intersecting family. This is due to the fact that (F, G), (F', G') not connected in Γ means $F \cap G' \neq \emptyset$ and $F' \cap G \neq \emptyset$. Thus by Bollobás' theorem regarding cross-intersecting families their number is at most $\binom{k+r}{k}$ and the theorem follows by adding up these for each color class. □

Proof of Theorem 5.2.2: Apply the previous theorem with $\mathscr{F} = \mathscr{G}$. □

5.2.3 l-almost intersecting graphs

The conjecture above is proven for $k = 2$, that is, when the (2-uniform) hypergraph is a graph. The theorem and its proof is stated in the language of graphs.

Theorem 5.2.4. *If G is an l-almost intersecting graph with no isolated vertices and $l \geq 1$, then G has at most $2l + 2$ edges and the unique extremal*

graph is $K_{2,l+1}$, provided that $l \neq 1, 3, 5, 6$. For $l = 1, 3, 6$ the unique extremal graphs are K_4, K_5, K_6. For $l = 5$ there are two extremal graphs: $K_{2,6}$ and the complement of a matching on 6 vertices.

Proof: It can be supposed that G is connected. Otherwise there would be a cut C_1, C_2 of $V(G)$ with $e(C_1, C_2) = 0$, $e(C_1), e(C_2) \neq 0$. By the l-almost intersecting property $e(C_1), e(C_2) \leq l$, and so $e(G) = e(C_1) + e(C_2) + e(C_1, C_2) \leq 2l$. Let e be $e(G)$ and $n = |V(G)|$. As there are exactly l disjoint edges from any edge (u, v), the degrees of the vertices u, v need to satisfy the condition $d(u) + d(v) - 1 = e - l$. Thus for any vertex its neighbors have the same degree. There are two main cases.

Case 1. G is d-regular.

Clearly $e = 2d + l - 1$ and $dn = 2e = 2(2d + l - 1)$, so $(n - 4)d = 2l - 2$. Of course $d \leq n - 1$, so:

$$2l - 2 = (n - 4)d \geq (d - 3)d = (d - \frac{3}{2})^2 - \frac{9}{4} \Rightarrow d \leq \frac{3}{2} + \sqrt{2l + \frac{1}{4}} \,.$$

This gives an upper bound on the number of edges of G:

$$e = 2d + l - 1 \leq l + 2\sqrt{2l + \frac{1}{4}} + 2 < 2l + 2 \text{ if } l > 8 \,.$$

If $d = n - 1$ then $G = K_n$ and every edge is disjoint from $\binom{n-2}{2}$ other edges. By the bound on l above $\binom{n-2}{2} = l \leq 8$, thus $n \leq 6$. By checking the possibilities it can be seen that K_4, K_5, K_6 contain more edges than $2l + 2$

for the respective ls.

If $d \leq n - 2$ then the previous counting argument gives us $e \leq l + 2\sqrt{2l - 1} + 1 < 2l + 2$ if $l > 5$. For $l = 5$ there is an equality here which gives the other extremal graph.

For $l < 5$ the only meaningful integer solution of the equation $2l - 2 = d(d - 2)$, $l = 1$ and $d = 2$ characterizes $K_{2,2}$. So it can be supposed that $d \leq n - 3$. The calculation from before leads to $e \leq l + 2\sqrt{2l - \frac{7}{4}} < 2l + 2$.

Case 2. G is not regular, then the degrees give a 2-partition of the vertices such that there are no edges within a partition, so G is bipartite.

Let the two sides of G be $|A| = a$ and $|B| = b$, the degrees d_A and d_B. It can be supposed that $a \leq b$ and then from $e = ad_A = bd_B$ follows that $d_A \geq d_B$. As $e = l + d_A + d_B - 1$:

$$e \leq 2d_A + l - 1 = \frac{2e}{a} + l - 1 \Rightarrow e \leq (l - 1)\frac{a}{a - 2} \text{ if } a > 2.$$

For $a \geq 4$ this is smaller than $2l + 2$. For $a = 2$ it is easy to see that $G = K_{2,l+1}$.

For $a = 3$ the equation on the number of edges is:

$$bd_B = e = l + d_B - 1 + \frac{bd_B}{3}, \text{ so } \frac{2bd_B}{3} = l + d_B - 1.$$

Thus

$$e = bd_B = \frac{3}{2}(l + d_B - 1) \leq \frac{3}{2}(l + 2) = \frac{3l}{2} + 3 < 2l + 2 \text{ if } l \geq 3.$$

If $l \leq 2$ then by the previous argument $e \leq (l-1)3 < 2l + 2$. The case $a = 1$ implies $l = 0$. $\qquad\square$

5.3 Non-uniform l-almost intersecting families

The following definition and proposition will prove useful for investigating non-uniform l-almost intersecting families.

Definition 5.3.1. *For any family $\mathscr{F}$ of sets, the comparability graph $G(\mathscr{F})$ is the graph with vertex set $V(G) = \mathscr{F}$ and edge set $E(G) = \{(F_1, F_2) : (F_1 \subsetneq F_2) \vee (F_2 \subsetneq F_1)\}$.*

Proposition 5.3.1. *If $\mathscr{F}$ is an l-almost intersecting family then all connected components of $G(\mathscr{F})$ have size at most l.*

Proof: Within a component of $G(\mathscr{F})$ the set $D_{\mathscr{F}}$ is the same, as for any $F_1 \subseteq F_2$ the collections of disjoint sets are the same: $D_{\mathscr{F}}(F_1) = D_{\mathscr{F}}(F_2)$. This is due to the fact that $D_{\mathscr{F}}(F_2) \subseteq D_{\mathscr{F}}(F_1)$ and for any $F \in \mathscr{F}$ $|D_{\mathscr{F}}(F)| = l$. Therefore if any component C would have more than l vertices, then for $F \in C$ any set $H \in D_{\mathscr{F}}(F)$ is disjoint to all elements of C, which is a

contradiction with the *l*-almost intersecting property. □

An interesting related configuration is the *r*-fork.

Definition 5.3.2 (*r*-fork)**.** *An r-fork is a family of $r + 1$ distinct sets $F_0, F_1, \ldots, F_r$ where $F_0 \subsetneq F_i$ for all $i = 1, \ldots, r$.*

The maximal size of a family not containing an *r*-fork is a much researched question, one of the related theorems is the following one.

Theorem 5.3.1 (De Bonis, Katona [10])**.** *If a family $\mathscr{F} \subseteq 2^{[n]}$ does not contain an r-fork, then $|\mathscr{F}| \leq (1 + \frac{2r}{n} + O(\frac{1}{n^2}))\binom{n}{\lfloor n/2 \rfloor}$*

As a special case an *l*-almost intersecting family is also *l*-fork-free by Proposition 5.3.1

Corollary 5.3.1. *If $\mathscr{F}$ is an l-almost intersecting family, then*

$$|\mathscr{F}| \leq (1 + \frac{2l}{n} + O(\frac{1}{n^2}))\binom{n}{n/2}.$$

This bound is asymptotically tight for $l = 1$. For $l = 2$ it is twice as big as the real bound as it will be shown by Theorem 5.3.4. For larger *l*s it is probably even further from the real bound, but no results are known.

5.3.1 Non-uniform 1-almost intersecting families

Let us settle the case $l = 1$.

Theorem 5.3.2. *If $\mathscr{F} \subseteq 2^n$ is a 1-almost intersecting family, then*

$$|\mathscr{F}| \leq \begin{cases} \binom{n}{n/2} & \text{if } n \text{ is even} \\ 2\binom{n-1}{\lfloor n/2 \rfloor - 1} & \text{if } n \text{ is odd,} \end{cases}$$

and equality holds if and only if $\mathscr{F} = \binom{[n]}{n/2}$ provided n is even, or $\mathscr{F} = \{F \in \binom{[n]}{\lfloor n/2 \rfloor} : x \in F\} \cup \{F \in \binom{[n]}{\lceil n/2 \rceil} : x \notin F\}$ for some fixed $x \in [n]$ provided n is odd and $n \geq 5$, or $\mathscr{F}$ consists of two arbitrary disjoint sets in case $n = 3$.

Proof: Let $\mathscr{F}$ be an 1-almost intersecting family which is of maximal size. Then by Proposition 5.3.1 we know that $\mathscr{F}$ is Sperner, therefore if n is even Sperner's theorem [46] gives the needed result.

Suppose n is odd. $\mathscr{F}$ consists of pairs of disjoint sets due to the 1-almost intersecting property. Let $\mathscr{F}' \subsetneq \mathscr{F}$ be the family which consists of the smaller sets from each pair, choosing arbitrarily if they have the same size, and let $\mathscr{F}''$ be the rest of the family, $\mathscr{F}'' = \mathscr{F} - \mathscr{F}'$. Now $\mathscr{F}'$ is intersecting, Sperner, and $2|F| \leq n$ for any $F \in \mathscr{F}'$. The following theorem of Bollobás [8] (only stated for odd n) deals with such families.

Theorem 5.3.3 (Bollobás [8]). *If n is odd and $\mathscr{F} \subseteq 2^{[n]}$ is an intersecting Sperner family with $2|F| \leq n$ for all $F \in \mathscr{F}$, then*

$$|\mathscr{F}| \leq \binom{n-1}{\lfloor n/2 \rfloor - 1},$$

and equality holds if and only if $\mathscr{F} = \{F \in \binom{[n]}{\lfloor n/2 \rfloor}) : x \in F\}$ for some fixed

$x \in [n]$.

As $|\mathscr{F}'| = |\mathscr{F}''|$ it follows that $|\mathscr{F}| \leq 2\binom{n-1}{\lfloor n/2\rfloor - 1}$ and if equality is true then $\mathscr{F}' = \{F \in \binom{[n]}{\lfloor n/2\rfloor : x \in F}\}$ for some fixed $x \in [n]$. If $\mathscr{F}'' = \overline{\mathscr{F}'} = \{\overline{F'} : F' \in \mathscr{F}'\}$ then the theorem is proven. Otherwise there is a set $F'' \in \mathscr{F}''$ with $|F''| \leq n/2$. Let $\mathscr{F}^+$ be the family obtained from $\mathscr{F}'$ by switching F' and F'', where $F' \in \mathscr{F}'$ is the unique set in $\mathscr{F}'$ disjoint from F'': $\mathscr{F}^+ = (\mathscr{F}' - \{F'\}) \cup \{F''\}$. the family $\mathscr{F}^+$ satisfies the conditions of Theorem 5.3.3 so $\mathscr{F}^+ = \{F \in \binom{[n]}{\lfloor n/2\rfloor} : y \in F\}$ for some fixed element $y \in [n]$, but neither $x \neq y$ nor $x = y$ is possible. $\qquad\square$

5.3.2 Non-uniform 2-almost intersecting families

The following constructions are 2-almost intersecting families for a base set of even and odd size. They will turn out to be best possible.

Construction 5.3.1. *If $n = 2k + 2$, then let $\mathscr{G}_1, \mathscr{G}_2$ be a partition of $\binom{[2k]}{k}$ so that $G \in \mathscr{G}_1$ if and only if $\overline{G} \in \mathscr{G}_2$. The following family is 2-almost intersecting and has size $2\binom{2k}{k} = (\frac{1}{2} + o(1))\binom{n}{n/2}$*

$$\mathscr{G} = \mathscr{G}_1 \cup \{G \cup \{2k+1\} : G \in \mathscr{G}_1\} \cup \mathscr{G}_2 \cup \{G \cup \{2k+2\} : G \in \mathscr{G}_2\}.$$

If $n = 2k+1$ then there is a similar construction. Let $\mathscr{G}_1 = \{G \in \binom{[2k-1]}{k-1} : x \in G\}$ and $\mathscr{G}_2 = \{G \in \binom{[2k-1]}{k} : x \notin G\}$ for some fixed $x \in [n]$. The following

family is 2-almost intersecting and has size $4\binom{2k-2}{k-1} = (\frac{1}{2} + o(1))\binom{n}{n/2}$:

$$\mathscr{G} = \mathscr{G}_1 \cup \{G \cup \{2k\} : G \in \mathscr{G}_1\} \cup \mathscr{G}_2 \cup \{G \cup \{2k+1\} : G \in \mathscr{G}_2\}.$$

Let $\mathscr{F} \subsetneq 2^{[n]}$ be a 2-almost intersecting family. Let $\mathscr{F}_2^U = \{F \in \mathscr{F} : \exists F' \in \mathscr{F}, F' \subsetneq F\}$, $\mathscr{F}_2^L = \{F \in \mathscr{F} : \exists F' \in \mathscr{F}, F \subsetneq F'\}$ and $\mathscr{F}_1 = \mathscr{F} - \mathscr{F}_2^U - \mathscr{F}_2^L$.

Proposition 5.3.2. *If* $F \in \mathscr{F}_1$ *then for any* $G \supseteq \overline{F} : G \notin \mathscr{F}$.

Proof: If there was a G like this in $\mathscr{F}$ then the two sets in $\mathscr{F}$ disjoint from G would be subsets of F, so $F \notin \mathscr{F}_1$, a contradiction. $\qquad\square$

Proposition 5.3.3. $\mathscr{F}_2^L \cap \mathscr{F}_2^U = \emptyset$

Proof: For a set $F \in \mathscr{F}_2^L \cap \mathscr{F}_2^U$ the component in the comparability graph $G(\mathscr{F})$ which holds F would have size at least 3, which is a contradiction with Proposition 5.3.1. $\qquad\square$

Proposition 5.3.4. *For any* $F \in \mathscr{F}_2^U$ $(F \in \mathscr{F}_2^L)$ *there exists exactly one set* $F' \in \mathscr{F}_2^L$ $(F' \in \mathscr{F}_2^U)$, *such that* $F \supsetneq F'$ $(F \subsetneq F')$.

Proof: This is implied by the special case $l = 2$ of Proposition 5.3.1. $\qquad\square$

Corollary 5.3.2. *If* $F \in \mathscr{F}_2^L$ *then for any* $G \supseteq \overline{F} : G \notin \mathscr{F}$.

Proof: Such a G would contain the two sets in $\mathscr{F}$ disjoint from $F' \in \mathscr{F}$, $F \subsetneq F'$. $\qquad\square$

Proposition 5.3.5. *For any 2-almost intersecting family $\mathscr{F} \subsetneq 2^{[n]}$ there exists another family $\mathscr{G}$ with $|\mathscr{F}| = |\mathscr{G}|$ such that*

(1) $G \in \mathscr{G}_2^U$ implies $\overline{G} \in \mathscr{G}_2^U$,

(2) for any pair $G_1, G_2 \in \mathscr{G}$, $G_1 \subsetneq G_2$ implies $|G_2 - G_1| = 1$.

Proof: If $\mathscr{F}_2^U \cup \mathscr{F}_2^L = \emptyset$ then $\mathscr{F}$ already satisfies **(1)** and **(2)**. Otherwise let $F_1, F_2 \in \mathscr{F}$, $F_1 \subsetneq F_2$. There are two distinct sets $F', F'' \in \mathscr{F}$ which are disjoint from F_2, and therefore from F_1 as well. All other sets of $\mathscr{F}$ intersect F_1. By replacing F_1 with any $G \notin \mathscr{F}$ satisfying $F_1 \subsetneq G \subsetneq F_2$, $|F_2 - G| = 1$ the 2-almost intersecting property is retained. By repeating this operation a family satisfying **(2)** is obtained.

If **(1)** is not satisfied by the new family, then we have $F', F'' \neq \overline{F_2}$. Then either of these sets can be replaced with $\overline{F_2}$ and the new family still is 2-almost intersecting. By repeating this operation combined with the previous step after finite many steps a family satisfying **(1)** and **(2)** is obtained. $\square$

Definition 5.3.3. *A maximal 2-almost intersecting family will be called good if it satisfies (1) and (2) from the above proposition.*

If $\mathscr{G}$ is good, then for any $G \in \mathscr{G}_2^U$ there are G_1, G_2 such that $G_1 \subsetneq G$, $G_2 \subsetneq \overline{G}$ and $|G - G_1| = |\overline{G} - G_2| = 1$. Let $\{x\} = G - G_1$ and $\{y\} = \overline{G} - G_2$, and let $G^* = G - \{x\} \cup \{y\} = G_1 \cup \{y\}$.

Proposition 5.3.6. *Let $\mathscr{G} \subseteq 2^{[n]}$ be a good 2-almost intersecting family. Then for any $G \in \mathscr{G}_2^U$ the sets G_1, G_2, G^* defined as above satisfy the following:*

(i) $G^* \notin \mathscr{G}$,

(ii) $\overline{G^*} \notin \mathscr{G}$,

(iii) the only set in $\mathscr{G} \cup \overline{\mathscr{G}}$ contained in G^* is G_1,

(iv) the only set in $\mathscr{G} \cup \overline{\mathscr{G}}$ containing G^* is $\overline{G_2}$,

(v) $\{G^* : G \in \mathscr{G}_2^U\}$ is a Sperner family.

Proof: The properties **(i)** and **(ii)** follow from Proposition 5.3.4 used for G_1 and G_2. If there would be a set $G' \in \mathscr{G}$ contradicting **(iii)** then there would be three sets in $\mathscr{G}$ disjoint from G_2. If $G' \in \overline{\mathscr{G}}$ would contradict **(iii)** then $\overline{G'}$ would contradict Proposition 5.3.4 for G_2. The statement **(iv)** follows the same way using G_1 instead of G_2. The last property, **(v)** follows from the face that $G^{*'} \subsetneq G^*$ implies that G_2 and G_1' are disjoint and therefore there are three sets in $\mathscr{G}$ disjoint from G_2. $\square$

Lemma 5.3.1. *Let $\mathscr{G} \subseteq 2^{[n]}$ be a good 2-almost intersecting family. Then the following LYM-type inequality holds:*

$$\sum_{G \in \mathscr{G}} \frac{2}{\binom{n}{|G|}} - \sum_{G \in \mathscr{G}_1} \frac{2}{(n-|G|)\binom{n}{|G|}}$$

$$- \sum_{G \in \mathscr{G}_2^L} \left(\frac{2}{(n-|G|)\binom{n}{|G|}} + \frac{1}{(n-|G|)(n-|G|-1)\binom{n}{|G|}} \right)$$

$$- \sum_{G \in \mathscr{G}_2^U} \left(\frac{1}{(n-|G|)\binom{n}{|G|}} + \frac{1}{|G|\binom{n}{|G|}} - \frac{1}{|G|(n-|G|)\binom{n}{|G|}} \right) \leq 1.$$

Proof: First note that $\emptyset \notin \mathscr{G}$ as otherwise there would be only one other set in $\mathscr{G}$ due to $\emptyset$ being self-disjoint, and the other set would not have two disjoint elements. The proof follows a LYM-type argument [7] of double counting maximal chains and sets from $\mathscr{G}$ on the chains. Consider the following pairs: $(G, \mathscr{C})$ where $G \in \mathscr{G}$ and $\mathscr{C}$ is a maximal chain in $2^{[n]}$ and $G \in \mathscr{C}$. For a set G there are exactly $|G|!(n - |G|)!$ different maximal chains containing G, thus the number of pairs is:

$$\sum_{G \in \mathscr{G}} |G|!(n - |G|)! \,.$$

On the other hand every chain $\mathscr{C}$ contains at most 2 sets from $\mathscr{G}$ by Proposition 5.3.4. So the number of such pairs is $n! + c_2 - c_0$ where c_i is the number of maximal chains containing i sets from $\mathscr{G}$. From the same proposition and from Proposition 5.3.6 we have:

$$c_2 = \sum_{G \in \mathscr{G}_2^L} |G|!(n - |G| - 1)!.$$

In order to obtain a lower bound on c_0 consider the set $S_{\overline{G}}$ of the chains which contain $\overline{G}$ for some fixed set $G \in \mathscr{G}_1 \cup \mathscr{G}_2^L$. By Proposition 5.3.2 and Corollary 5.3.2 there is no $G' \in \mathscr{G}, G' \supseteq \overline{G}$. A k-subset of $\overline{G}$ is contained in $|G|!k!(n - |G| - k)!$ chains that go through $\overline{G}$.

Thus, if $G \in \mathscr{G}_1$, as the empty set is not in $\mathscr{G}$ and there are exactly 2 sets in $\mathscr{G}$ which are disjoint from G, there are at least $(n - |G| - 2)(n - |G| - 1)!|G|!$

chains in $S_{\overline{G}}$ that do not contain any set from $\mathscr{G}$.

If $G \in \mathscr{G}_2^L$, then the 2 sets in $\mathscr{G}$ contained in $\overline{G}$ have size $(n - |G| - 1)$ and $(n - |G| - 2)$ and thus the number of chains in $S_{\overline{G}}$ that avoid $\mathscr{G}$ is $|G|!((n - |G|)! - (n - |G| - 1)! - (n - |G| - 2)!)$. By definition there are no 2 sets G, G' in $\mathscr{G}_1 \cup \mathscr{G}_2^L$ with $G \subsetneq G'$, thus we have $S_{\overline{G}} \cap S_{\overline{G'}} = \emptyset$ for any distinct $G, G' \in \mathscr{G}_1 \cup \mathscr{G}_2^L$.

Finally consider the sets $G \in \mathscr{G}_2^U$. By Proposition 5.3.6, using the same notations, any chain containing G^* but neither G_1 nor $\overline{G_2}$ avoids $\mathscr{G} \cup \overline{\mathscr{G}}$. So these chains are different from all those in $\cup_{G \in \mathscr{G}_1 \cup \mathscr{G}_2^L} S_{\overline{G}}$, and they do not contain any set from $\mathscr{G}$. By Proposition 5.3.6 **(v)** $S_{G'*} \cap S_{G''*} = \emptyset$ for any $G', G'' \in \mathscr{G}_2^U$. For a fixed $G \in \mathscr{G}_2^U$ the number of new chains is:

$$(|G|! - (|G| - 1)!)((n - |G|)! - (n - |G| - 1)!).$$

These observations yield the following inequality:

$$
\begin{aligned}
\sum_{G \in \mathscr{G}} |G|!(n - |G|)! \leq\ & n! + \sum_{G \in \mathscr{G}_2^L} |G|!(n - |G| - 1)! - \\
& \sum_{G \in \mathscr{G}_1} (n - |G| - 2) \cdot |G|!(n - |G| - 1)! - \\
& \sum_{G \in \mathscr{G}_2^L} |G|! \left((n - |G|)! - (n - |G| - 1)! - (n - |G| - 2)!\right) - \\
& \sum_{G \in \mathscr{G}_2^U} (|G|! - (|G| - 1)!)((n - |G|)! - (n - |G| - 1)!).
\end{aligned}
$$

Rearranging and dividing by $n!$ yield the statement of the lemma. $\qquad\square$

As the main term in Lemma 5.3.1 is the first term $\sum_{G \in \mathscr{G}} \frac{2}{\binom{n}{|G|}}$, and the other terms are negligible compared to it, Construction 5.3.1 is asymptotically best. The next theorem shows that it is best not only asymptotically.

Theorem 5.3.4. *Let $\mathscr{F} \subseteq 2^{[n]}$ be a 2-almost intersecting family, then*

$$
|\mathscr{F}| \leq
\begin{cases}
2\binom{n-2}{\frac{n-2}{2}} & \text{if } n \text{ is even} \\
4\binom{n-3}{\lfloor n/2 \rfloor - 2} & \text{if } n \text{ is odd},
\end{cases}
$$

and this bound is best possible as shown by Construction 5.3.1.

It can be assumed that $\mathscr{F}$ is good. By definition and the previous observations, the sets in $\mathscr{F}_2^U \cup \mathscr{F}_2^L$ come in pairs with sizes differing by one. Consider the summands in Lemma 5.3.1. Let a_k be the sum of all summands from sets in $\mathscr{F}_1$ with size k and let b_k be the sum of the summands from pairs in $\mathscr{F}_2^L \cup \mathscr{F}_2^U$ where the size of the smaller set is k. If $m = \min\{a_k, \frac{b_k}{2} : 1 \leq k \leq n-1\}$ then $|\mathscr{F}| \leq \frac{1}{m}$. Let $a'_k = n!a_k = 2k!(n-k)! - 2k!(n-k-1)!$ and

$$
\begin{aligned}
b'_k &= n!b_k = 2k!(n-k)! + 2(k+1)!(n-k-1)! - 2k!(n-k-1)! \\
&\quad - k!(n-k-2)! - (k+1)!(n-k-2)! - k!(n-k-1)! + k!(n-k-2)!
\end{aligned}
$$

Observe that the minimum of a'_k is at $k = \lfloor n/2 \rfloor$:

$$
\Delta a'_k = a'_{k+1} - a'_k = 2k!(n-k-2)![(k+1)(n-k-2) - (n-k-1)^2]
$$

Here the expression in the brackets is quadratic, and one of its roots is between $n - 3$ and $n - 2$ and the other is between $\lceil n/2 \rceil - 1$ and $\lceil n/2 \rceil$.

For b'_k the place of the minimum depends on the parity of n. For even n the minimum of b'_k is $b'_{n/2}$, while if n is odd then $b'_{\lfloor n/2 \rfloor} = b'_{\lceil n/2 \rceil}$. This is shown by a similar calculation for:

$$
\begin{aligned}
\Delta b'_k &= b'_{k+1} - b'_k = k!(n - k - 3)! \times \\
&\quad [2(k + 1)(k + 2)(n - k - 2) - 2(n - k)(n - k - 1)(n - k - 2) - \\
&\quad (k + 1)(3n - 2k - 4) + (3n - 2k - 2)(n - k - 2)].
\end{aligned}
$$

Finally, by substituting, $a_{\lfloor n/2 \rfloor} > \frac{1}{2} b_{\lfloor n/2 \rfloor}$ and the theorem follows as the size of the families in Construction 5.3.1 is exactly $\frac{2}{b_{\lfloor n/2 \rfloor}}$.　　　$\square$

A related open question is the following one: What is the maximal size of a family which is Sperner and 2-almost intersecting (or l-almost intersecting, $l \geq 2$)?

The following construction is a Sperner l-almost intersecting family:

Construction 5.3.2. *Let $\mathscr{F}$ be an optimal 1-almost intersecting family on $[n - l - 1]$. These are known from Theorem 5.3.2. Then*

$$
\mathscr{G} = \mathscr{F} \times \binom{[n - l, n]}{1} = \{F \cup \{x\} : F \in \mathscr{F}, x \in [n - l, n]\}
$$

is a Sperner l-almost intersecting family of size $(\frac{l+1}{2^{l+1}} + o(1)) \binom{n}{\lfloor n/2 \rfloor}$.

This construction together with Corollary 5.3.1 proves the following corol-

lary:

Corollary 5.3.3. *Let $\mathscr{F} \subseteq 2^{[n]}$ be an optimal l-almost intersecting Sperner family. Then*

$$|\mathscr{F}| = \Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right).$$

5.4 $(\leq l)$-almost intersecting uniform families

5.4.1 k-uniform $(\leq l)$-almost intersecting families

The main result regarding k-uniform $\leq l$-almost intersecting families is the following proposition. It states that if n is large enough then regarding the largest family we do not gain anything by relaxing this way the intersecting property from the Erdős-Ko-Rado theorem [19].

Proposition 5.4.1. *For any $k, l \in \mathbb{N}$ there exists $n_0 = n_0(k, l)$ such that if $n \geq n_0$ and $\mathscr{F} \subseteq \binom{[n]}{k}$ is an $(\leq l)$-almost intersecting family, then $|\mathscr{F}| \leq \binom{n-1}{k-1}$ with equality if and only if $\mathscr{F}$ is the family of all k-sets containing a fixed element of $[n]$.*

Proof: If $\mathscr{F}$ is intersecting, then by the Erdős-Ko-Rado theorem the proposition stands. If $F_1, F_2 \in \mathscr{F}$ are disjoint, then any $F \in \mathscr{F} - (\mathcal{D}_{\mathscr{F}}(F_1) \cup \mathcal{D}_{\mathscr{F}}(F_2))$ should meet both F_1 and F_2 and thus $|\mathscr{F}| \leq k^2 \binom{n-2}{k-2} + 2l$ which is smaller than $\binom{n-1}{k-1}$ if n is large enough. $\qquad \square$

This argument gives $n_0(k, l) \leq O(k^3 + kl)$. By a theorem of Hilton and Milner [34] stating that the largest possible size of a non-trivially intersecting

family is $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$, a better bound of $n_0(k,l) = O(k^2 l)$ can be obtained if $l = o(k)$. Determining the exact values remains an open problem.

If $l \geq \binom{m}{k}$ for some m, then $\binom{\lfloor k+m \rfloor}{k}$ is $(\leq l)$-intersecting. If $l \geq \binom{n-k-1}{k-1}$, then the trivially intersecting family $\mathscr{F} = \{f \in \binom{[n]}{k} : 1 \in F\}$ is not maximal.

5.4.2 Uniform (≤ 1)-almost intersecting families

The following theorem determines $n_0(k,1)$.

Theorem 5.4.1. *If $k \geq 3$, $n \geq 2k+2$ and $\mathscr{F} \subseteq \binom{[n]}{k}$ is an (≤ 1)-almost intersecting family, then $|\mathscr{F}| \leq \binom{n-1}{k-1}$ with equality if and only if $\mathscr{F}$ is the family of all k-sets containing a fixed element of $[n]$. For $k = 2$ and $n \geq 7$ the same statement is true.*

Proof: Note that if $k = 2$ then either $\mathscr{F}$ is intersecting, and the theorem is a corollary of the Erdős-Ko-Rado theorem [19], or there are two disjoint edges $e_1 \cap e_2 = \emptyset$. All other edges meet both, so they are subsets of $e_1 \cup e_2$, therefore $|\mathscr{F}| \leq \binom{4}{2} = 6$. This proves the case $k = 2$.

For $k \geq 3$ Katona's cycle method [35] will be used.

Definition 5.4.1. *A set $S \subseteq [n]$ is an* interval *in cyclic permutation Π of $[n]$ if $S = \{\Pi(i), \Pi(i+1), \ldots, \Pi(i+|S|-1)\}$ for some i, where addition is modulo n.*

Definition 5.4.2. *Two sets $S, T \subseteq [n]$ are* separated *in a cyclic permutation Π if there are disjoint intervals S', T' in Π with $S \subseteq S'$, $T \subseteq T'$ and $S' \cup T' = [n]$.*

Let $\mathscr{F}_i = \{F \in \mathscr{F} : |\mathcal{D}_{\mathscr{F}}(F)| = i\}$, where $i = 0, 1$. *Weighted objects* are defined the following way:

- Type 1 object: For $F \in \mathscr{F}_0$ and Π such a cyclic permutation of $[n]$ that F is an interval in Π, the *type 1 object* (F, Π) has weight $\frac{1}{k}$.

- Type 2 object: For $F, F' \in \mathscr{F}_1$ and Π such that Π separates F, F' the *type 2 object* $(\{F, F'\}, \Pi)$ has weight $\frac{2}{n}$.

Lemma 5.4.1. *If $2k \leq n$ and $\mathscr{F} \subseteq \binom{[n]}{k}$ is a (≤ 1)-almost intersecting family, then for a cyclic permutation Π of $[n]$ the sum of weights of all objects having Π in their second coordinate is at most 1.*

Proof: Take a cyclic permutation Π. For simplicity objects will be referred to as their first coordinates, the second will be Π for the proof. Let a *milestone* mark every pair of consecutive elements $\Pi(i), \Pi(i+1)$, laying between them. Define *separating milestones* as:

- For a type 1 object the separating milestones are the 2 milestones laying between the endpoints of the interval and its complement.

- For a type 2 object $\{S, T\}$ the separating milestones are those laying between the endpoints of S' and T' of an arbitrary separating interval pair.

A separating milestone belongs to its object. The main observation is the fact that any milestone can belong to at most one object. Otherwise, if a

milestone belongs to two type 1 objects, then to be different they must lie on the two sides of the milestone, but then they do not intersect each other as $2k \leq n$, this contradicts the definition of a type 1 object. If type 1 object F and type 2 object $\{S, T\}$ would share a milestone then either $F \subseteq S'$ or $F \subseteq T'$, thus either $F \cap T = \emptyset$ or $F \cap S = \emptyset$. This again contradicts the definition of a type 1 object. Finally, if the milestone would be shared by two type 2 objects, $\{S_1, T_1\}$ and $\{S_2, T_2\}$ then either $S_1 \subseteq S'_2$ or $S_2 \subseteq S'_1$, giving $S_1 \cap T_2 = \emptyset$ or $S_2 \cap T_1 = \emptyset$, again a contradiction, now with the (≤ 1)-almost intersecting property.

As all objects have 2 separating milestones, the lemma follows if there are only type 2 objects. If F is a type 1 object then F *crosses* the milestones between any two of its elements. For any object O the interval F crosses at least one of the separating milestones of O as $F \in \mathscr{F}_0$. As F is an interval of size k it can cross at most $(k-1)$ milestones, so in this case there are at most k objects with second coordinate Π and as $\frac{1}{k} \geq \frac{2}{n}$ the lemma is proven. $\square$

Let α be the number of cyclic permutations in which the disjoint k-sets S and T are separated. Obviously α is independent from the choice of S, T. Lemma 5.4.1 implies the following inequality:

$$|\mathscr{F}_0| k! (n-k)! \frac{1}{k} + |\mathscr{F}_1| \alpha \frac{1}{n} \leq (n-1)!$$

Thus it is enough to show that

$$k!(n-k)!\frac{1}{k} < \alpha\frac{1}{n} .$$

Consider the family $\mathscr{G} = \binom{[2k]}{k}$ and count the type 2 objects $(\{G_1, G_2\}, \Pi)$ with $G_1, G_2 \in \mathscr{G}$ and separated in Π. The number of these objects is $\frac{1}{2}\binom{2k}{k}\alpha$. At the same time this number is $k(n-1)!$ as any cyclic permutation Π of $[n]$ contains exactly k separated pairs from $\mathscr{G}$. Thus

$$\alpha = \frac{2k(n-1)!}{\binom{2k}{k}} .$$

By substituting this to the previous inequality it changes to

$$k!(n-k)!\frac{1}{k} < \frac{2k(n-1)!}{n\binom{2k}{k}} .$$

This is equivalent to

$$\frac{(2k)!}{k!k^2} < \frac{2(n-1)!}{(n-k)!n} .$$

This holds if $2k+2 \leq n$ provided $k \geq 5$. For $k < 5$ the inequality above is true except for the following cases: $k = 3, n = 8$ or 9 and $k = 4, n = 10$. For these cases the weight of the type 2 objects $(\{F, F'\}, \Pi)$ need to be changed to $\frac{1}{k}$ when F or F' (or both) is an interval in Π. Lemma 5.4.1 holds with the modified weights as well for these special cases. This finishes the proof of Theorem 5.4.1. $\square$

The bound $n_0(k, 1) = 2k + 2$ is best possible as the family $\binom{[2k]}{k}$ shows. This is a (≤ 1)-almost intersecting family with size $\binom{2k}{k} > \binom{2k+1-1}{k-1}$.

The following proposition settles the question for $n = 2k + 1$. It will be used for the non-uniform (≤ 2)-almost intersecting case.

Proposition 5.4.2. *If $\mathscr{F} \subseteq \binom{[2k+1]}{k}$ is an (≤ 1)-almost intersecting family, then $|\mathscr{F}| \leq \binom{2k}{k}$ and equality holds if and only if $\mathscr{F}$ is the family of all k-sets not containing a fixed element of $[2k + 1]$.*

Proof: The proof is a double counting argument on (F, G), $F \in \mathscr{F}$, $G \notin \mathscr{F}$. The number of such pairs is at least $k|\mathscr{F}|$ as for any $F \in \mathscr{F}$ there are $k + 1$ disjoint sets of size k, and at most one of them is in $\mathscr{F}$. On the other hand the number of such sets is at most $(\binom{2k+1}{k} - |\mathscr{F}|)(k + 1)$. This gives us $(\binom{2k+1}{k} - |\mathscr{F}|)(k + 1) \geq k|\mathscr{F}|$, which yields the stated bound.

In order to characterize the case of equality note that all lower and upper bounds in the argument hold if and only if $\mathscr{F}$ is an 1-almost intersecting family. The proof is finished by Corollary 5.2.1. □

5.5 Non-uniform $(\leq l)$-almost intersecting families

The conjecture for the general case is the following one:

Conjecture 5.5.1. *For any positive integer $l \geq 2$ there exists an $n_0 = n_0(l)$*

such that if $n \geq n_0$ and $\mathscr{F} \subseteq 2^{[n]}$ is an $(\leq l)$-almost intersecting family, then

$$|\mathscr{F}| \leq \begin{cases} \sum_{i=n/2}^{n} \binom{n}{i} & \text{if } n \text{ is even} \\ \binom{n-1}{\lfloor n/2 \rfloor} + \sum_{i=\lceil n/2 \rceil}^{n} \binom{n}{i} & \text{if } n \text{ is odd,} \end{cases}$$

and equality holds if and only if $\mathscr{F}$ is the family of sets of size at least $\frac{n}{2}$ and (if n is odd) the sets of size $\lfloor \frac{n}{2} \rfloor$ not containing a fixed element of $[n]$.

Definition 5.5.1 (Upset)**.** *A family $\mathscr{F} \subseteq 2^{[n]}$ is an upset if for any $F \in \mathscr{F}, G \supset F$ implies $G \in \mathscr{F}$.*

For any $(\leq l)$-almost intersecting family $\mathscr{F}$ there is another such family $\mathscr{F}'$ with the same size, but $\mathscr{F}'$ is an upset. Indeed, for any $F \in \mathscr{F}, G \notin \mathscr{F}, F \subsetneq G$ the family $\mathscr{F}' = \mathscr{F} - \{F\} \cup \{G\}$ is $(\leq l)$-almost intersecting if $\mathscr{F}$ is as well. Noting this it is enough to consider upsets to prove the upper bound in Conjecture 5.5.1. Uniqueness for upsets also implies uniqueness for arbitrary families, as for any maximal family $\mathscr{F}$ we can obtain an upset of the same size by the operation above, so this must be the one in the conjecture, but then the family before the last operation in not $(\leq l)$-almost intersecting. Indeed, if $\mathscr{F}'$ is the family before the last operation, that is , $\mathscr{F}' - \{F\} \cup \{G\} = \mathscr{F}''$ where $\mathscr{F}''$ is the family in the conjecture, $G \notin \mathscr{F}'$, $F \in \mathscr{F}', F \subsetneq G$ then $\mathcal{D}_{\mathscr{F}'}(F) \geq \lfloor \frac{n-1}{2} \rfloor$, a contradiction.

5.5.1 Non-uniform (≤ 1)-almost intersecting families

The following lemma will be used to settle the cases $l = 1, 2$ and might be useful for the general conjecture as well.

Lemma 5.5.1. *Let l be a positive integer. If $\mathscr{F}$ is an $(\leq l)$-almost intersecting upset where the size m of the smallest set is less than $\frac{n-l}{2}$, then there is an $(\leq l)$-almost intersecting upset $\mathscr{F}'$ with $|\mathscr{F}| < |\mathscr{F}'|$ that the size of the smallest set is $m + 1$. If $l \geq 2$ then for $m = \frac{n-l}{2}$ there is such a family $\mathscr{F}'$, with $|\mathscr{F}| \leq |\mathscr{F}'|$.*

Proof: Let $\mathscr{F}_i = \{F \in \mathscr{F} : |F| = i\}$, and define the bipartite graph $G(V_1, V_2, E)$ as $V_1 = \mathscr{F}_m$, $V_2 = \{G \in \binom{[n]}{n-m-1} - \mathscr{F}_{n-m-1} : \exists F \in \mathscr{F}_m, F \cap G = \emptyset\}$, $E = \{(F, G) : F \cap G = \emptyset\}$. Clearly $d(G) \leq m + 1$ for any $G \in V_2$. For any $F \in \mathscr{F}_m = V_1$ the degree $d(F) \geq n - m - l + 1$ as there are $n - m$ sets of size $n - m - 1$ disjoint from F and at most $l - 1$ of them belong to $\mathscr{F}$ as if any is in $\mathscr{F}$ then $\overline{F} \in \mathscr{F}$. It follows that $|V_1| \leq |V_2|$ and if $m < \frac{n-l}{2}$ then $|V_1| < |V_2|$.

Take the family $\mathscr{F}' = \mathscr{F} - V_1 \cup V_2$. Then $|\mathscr{F}'| \geq |\mathscr{F}|$ and if $m < \frac{n-l}{2}$ then this is a strict inequality, so it is enough to show that $\mathscr{F}'$ is $(\leq l)$-almost intersecting. As sets in V_2 have size $n - m - 1$ and the minimum set size in $\mathscr{F}'$ is $m + 1$, then for any $G \in V_2 : |\mathscr{D}_{\mathscr{F}'}(G)| \leq 1$. For any set $F \in \mathscr{F} \cap \mathscr{F}', |F| \geq m+2$: $\mathscr{D}_{\mathscr{F}'}(F) \subseteq \mathscr{D}_{\mathscr{F}}(F)$, so the $(\leq l)$-almost intersecting property still holds. Finally, for a set $|F| = m + 1$, $F \in \mathscr{F} \cap \mathscr{F}'$, the set $\mathscr{D}_{\mathscr{F}'}(F) = \mathscr{D}_{\mathscr{F}}(F)$ if there was no $F' \subseteq F$ with $F' \in \mathscr{F}$. If there is such an

F' then $\mathscr{D}_{\mathscr{F}'}(F) - \mathscr{D}_{\mathscr{F}}(F) = \{\overline{F}\}$, so $|\mathscr{D}_{\mathscr{F}'}(F)| \leq |\mathscr{D}_{\mathscr{F}}(F)| + 1$, but as $\overline{F'} \in \mathscr{F}$ we have $|\mathscr{D}_{\mathscr{F}}(F)| < |\mathscr{D}_{\mathscr{F}}(F')| \leq l$. □

Theorem 5.5.1. *If $n \geq 2$ and $\mathscr{F} \subseteq 2^{[n]}$ is an (≤ 1)-almost intersecting family, then*

$$|\mathscr{F}| \leq \begin{cases} \sum_{i=n/2}^{n} \binom{n}{i} & \text{if } n \text{ is even} \\ \binom{n-1}{\lfloor n/2 \rfloor} + \sum_{i=\lceil n/2 \rceil}^{n} \binom{n}{i} & \text{if } n \text{ is odd,} \end{cases}$$

and equality holds if and only if $\mathscr{F}$ is the family of sets of size at least $n/2$ and (if n is odd) the sets of size $\lfloor n/2 \rfloor$ containing a fixed element of $[n]$.

Proof: Let $\mathscr{F}$ be an (≤ 1)-almost intersecting family of maximal size, it can be assumed that it is an upset. By Lemma 5.5.1 the size of the smallest sets in $\mathscr{F}$ is at least $\lceil \frac{n-1}{2} \rceil$. For even n this implies the theorem as $\mathscr{F} \subseteq \{F \subseteq [n] : |F| \geq \frac{n}{2}\}$.

If $n = 2k + 1$ then by Lemma 5.5.1 $\mathscr{F} \subseteq \{F \subseteq [n] : |F| \geq k\}$. Then $\binom{[2k+1]}{k+1} \subseteq \mathscr{F}$, as for $G \in \binom{[2k+1]}{k+1} \mathcal{D}_{\mathscr{F}}(G) \leq 1$ so due to the maximality of $\mathscr{F}$ G would not be in $\mathscr{F}$ only because $\overline{G} \in \mathscr{F}$ and some $G' \in \mathscr{F}$, $G' \subsetneq G$, but this contradicts that $\mathscr{F}$ is an upset.

As $\binom{[2k+1]}{k+1} \subseteq \mathscr{F}$ for every $F \in \mathscr{F}_k$ $\overline{F} \in \mathscr{F}$ so $\mathscr{F}_k$ needs to be an intersecting family. By the Erdős-Ko-Rado theorem [19] the proof is finished. □

Theorem 5.4.1 can be derived from a result of Bernáth and Gerbner [4] as well.

Definition 5.5.2. *A family $\mathscr{F}$ is* (p,q)-chain intersecting *if $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq$*

A_p, $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_q$ *with* $A_i, B_j \in \mathscr{F}$ *implies* $A_p \cap B_q \neq \emptyset$.

As the $(1, 2)$-chain intersecting property is equivalent to the condition that $\mathcal{D}_\mathscr{F}(F)$ is Sperner for any $F \in \mathscr{F}$, the following theorem implies Theorem 5.5.1. The authors also give a uniqueness proof.

Theorem 5.5.2 (Bernáth, Gerbner [4])**.** *If* $\mathscr{F} \subseteq 2^{[n]}$ *is* (p, q)-*chain intersecting, then*

$$|\mathscr{F}| \leq \begin{cases} \sum_{i=(n-p-q+3)/2}^{n} \binom{n}{i} & \text{if } n - p - q \text{ is odd} \\ \binom{n-1}{\lfloor (n-p-q+3)/2 \rfloor - 1} + \sum_{i=\lceil (n-p-q+3)/2 \rceil}^{n} \binom{n}{i} & \text{if } n - p - q \text{ is even.} \end{cases}$$

5.5.2   Non-uniform (≤ 2)-almost intersecting families

Here and in the next chapter Lovász's version of the Kruskal-Katona theorem will be used repeatedly.

Theorem 5.5.3 (Lovász [42])**.** *Let* $\mathscr{F} \subseteq \binom{[n]}{k}$ *and let* x *be the real number defined by* $|\mathscr{F}| = \binom{x}{k}$. *Then* $|\delta(\mathscr{F})| \geq \binom{x}{k-1}$, *where* $\delta(\mathscr{F}) = \{G \in \binom{[n]}{k-1} | \exists F \in \mathscr{F}, G \subsetneq F\}$

The following theorem shows that Conjecture 5.5.1 is true for $l = 2$.

Theorem 5.5.4. *For* $n \geq 2$ *and* $\mathscr{F} \subseteq 2^{[n]}$ *is an* (≤ 2)-*almost intersecting family, then*

$$|\mathscr{F}| \leq \begin{cases} \sum_{i=n/2}^{n} \binom{n}{i} & \text{if } n \text{ is even} \\ \binom{n-1}{\lfloor n/2 \rfloor} + \sum_{i=\lceil n/2 \rceil}^{n} \binom{n}{i} & \text{if } n \text{ is odd,} \end{cases}$$

and equality holds if and only if $\mathscr{F}$ is the family of sets of size at least $n/2$ and (if n is odd) the sets of size $\lfloor n/2 \rfloor$ not containing a fixed element of $[n]$.

Proof: Consider the case $n = 2k + 1$. It can be assumed that $\mathscr{F}$ is an upset and by Lemma 5.5.1 all sets have size at least k, also that $|\mathscr{F}|$ is maximal. Therefore all sets $|F| \geq k+2$ belong to $\mathscr{F}$ as for these $\mathcal{D}_{\mathscr{F}}(F) = \emptyset$. If $\mathscr{F}$ is maximal then $\binom{[2k+1]}{k+1} \subseteq \mathscr{F}$, as for any $G \in \binom{[2k+1]}{k+1}$, $\mathcal{D}_{\mathscr{F}}(G) \leq 1$, so it might not be in $\mathscr{F}$ only because $\overline{G} \in \mathscr{F}$ and $\mathcal{D}_{\mathscr{F}}(\overline{G}) = 2$. But as $\mathscr{F}$ is an upset, $G \in \mathscr{F}$, a contradiction.

Consider $\mathscr{F}_k = \mathscr{F} \cap \binom{[2k+1]}{k}$. As before, for any $F \in \mathscr{F}_k$ $\overline{F} \in \mathscr{F}$, so $\mathscr{F}_k$ is (≤ 1)-almost intersecting and by Proposition 5.4.2 the proof is ready (for odd n).

If $n = 2k$ then again $\mathscr{F}$ is an upset and all sets have size at least $k-1$. By Lemma 5.5.1 there is an upset $\mathscr{F}'$ with size $|\mathscr{F}|$, and with smallest set size at least k. This proves the bound in this case. For the uniqueness suppose $\mathscr{F}_{k-1} \neq \emptyset$ and let $\mathscr{B}_k = \{B \in \binom{[2k]}{k} - \mathscr{F}_k\}$, then from the proof of Lemma 5.5.1 $|\mathscr{F}_{k-1}| = |\{\binom{[2k]}{k} - \mathscr{F}_k\}|$ because $\mathscr{F}$ is the largest (≤ 2)-almost intersecting family, and $\mathscr{F}_{k-1} = \delta \overline{\mathscr{B}_k} = \{F \in \binom{[2k]}{k-1} : \exists B \in \mathscr{B}_k \text{ that } (F \subseteq \overline{B})\}$.

By Lovász's version [42] of the Kruskal-Katona shadow theorem (Theorem 5.5.3) if $\overline{\mathscr{B}_k} = \binom{x}{k}$ for a real x then $\delta \overline{\mathscr{B}_k} \geq \binom{x}{k-1}$. As $\binom{x}{k} \leq \binom{x}{k-1}$ if $x < 2k - 1$, the size $|\mathscr{F}_{k-1}| = |\mathscr{B}_k| \geq \binom{2k-1}{k-1}$. To finish the proof note that $\mathscr{F}_{k-1}$ is intersecting. Indeed, if $F, F' \in \mathscr{F}_{k-1}$, $F \cap F' = \emptyset$ then $|\mathcal{D}_{\mathscr{F}}(F)| \geq |\{G : F' \subseteq G \subseteq \overline{F}\}| = 4$ as $\mathscr{F}$ is an upset, a contradiction. By the Erdős-Ko-Rado theorem [19] $|\mathscr{F}_{k-1}| \leq \binom{2k-1}{k-2} < \binom{2k-1}{k-1}$. This is a

contradiction again, which concludes the proof. $\qquad\square$

For the general case one can obtain a better upper bound than the one from Lemma 5.5.1 by using Theorem 5.5.2, as an $(\leq l)$-almost intersecting family is $(1, p)$-chain intersecting as well for $p = \lceil \log_2(l + 1) \rceil$.

Chapter 6

Cross-Sperner families

6.1 Introduction

This chapter contains results inspired by two theorems mentioned previously, Bollobás's cross-intersecting theorem [7] and Sperner's theorem [46]. The results are due to joint research with D. Gerbner, N. Lemons, C. Palmer and B. Patkós [29].

The definition of a cross-intersecting pair of family is:

Definition 6.1.1 (Cross-intersecting family)**.** *The pairs of sets $(A_i, B_i)_{i=1}^m$ form a cross-intersecting family if for any $1 \leq i, j \leq m$ $A_i \cap B_j = \emptyset$ if and only if $i = j$.*

The question of cross-intersecting sets is one which attracted many researchers ([5], [25], [26], [27], [40], [41], [39]). This chapter investigates the analogous question of a cross-Sperner family.

Definition 6.1.2 (Cross-Sperner family). *A pair of families* $(\mathscr{F}, \mathscr{G})$ *is cross-Sperner if there is no pair of sets* (F, G) *with* $F \in \mathscr{F}, G \in \mathscr{G}$ *and* $F \subseteq G$ *or* $G \subseteq F$.

The size of the pair of families $(\mathscr{F}, \mathscr{G})$ can be measured multiple ways. The questions considered below are the sum of the sizes of the families, $|\mathscr{F}| + |\mathscr{G}|$ and the product of those, $|\mathscr{F}||\mathscr{G}|$.

Also, the cross-Sperner property for $(\mathscr{F}, \mathscr{G})$ is equivalent to $(\mathscr{F}, \overline{\mathscr{G}})$ being cross-intersecting and cross-co-intersecting, where the latter means that the complements of the sets from the families are cross-intersecting.

6.2 Additively maximal cross-Sperner families

Clearly the trivial family $\mathscr{F} = \emptyset, \mathscr{G} = 2^{[n]}$ satisfies the cross-Sperner property, and due to definition $\mathscr{F} \cap \mathscr{G} = \emptyset$, so this is a maximal possible family as well. From now on both families are assumed to be non-empty.

The problem can be reformulated into an isoparametric problem. Let $G_n = (V_n, E_n)$ be a graph with $V_n = 2^{[n]}$ and $E_n = \{(F, G) : F, G \in V_n, F \subsetneq G \text{ or } G \subsetneq F\}$. Then the size of the largest cross-Sperner pair of families, $\max\{|\mathscr{F}| + |\mathscr{G}|\} = 2^n - c(G_n)$, where $c(G_n)$ is the vertex connectivity of G_n.

Also, if $F(n, m)$ is defined as:

$$F(n, m) = \max_{\mathscr{G} \subseteq 2^{[n]}} \{|\mathscr{G}| : \exists \mathscr{F} \subseteq 2^{[n]} \text{ with } |\mathscr{F}| = m, (\mathscr{F}, \mathscr{G}) \text{ cross-Sperner}\},$$

then with the usual notation $N_{G_n}(U)$ for the neighborhood of $U \subseteq V_n$ in G_n,

$$F(n, m) = 2^n - m - \min\{|N_{G_n}(\mathscr{F})| : \mathscr{F} \subseteq V_n, |\mathscr{F}| = m\}.$$

The main theorem is the following one:

Theorem 6.2.1. *There is an integer n_0 that for any $n \geq n_0$ and $\mathscr{F}, \mathscr{G} \subsetneq 2^{[n]}$ if $(\mathscr{F}, \mathscr{G})$ is cross-Sperner, then*

$$|\mathscr{F}| + |\mathscr{G}| \leq F(n, 1) + 1 = 2^n - 2^{\lceil \frac{n}{2} \rceil} - 2^{\lfloor \frac{n}{2} \rfloor} + 2,$$

and equality holds if and only if one of $\mathscr{F}$ or $\mathscr{G}$ consists of exactly one set S of size $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ and the other family consists of all possible sets (all sets not containing S and not part of S).

Again Lovász's version [42] of the Kruskal-Katona shadow theorem (Theorem 5.5.3) will prove useful in proving the key lemma during the proof of Theorem 6.2.1.

Before proving Theorem 6.2.1 the following observations need to be made:

Note that for any $F \in 2^{[n]}$ we have $N_{G_n}(F) = 2^{|F|} + 2^{n-|F|} - 2$, which is

minimal if $|F| = \lfloor \frac{n}{2} \rfloor$. This implies:

$$F(n,1) = 2^n - 2^{\lceil \frac{n}{2} \rceil} - 2^{\lfloor \frac{n}{2} \rfloor} + 1$$

Proposition 6.2.1. *If a cross-Sperner pair $(\mathscr{F}, \mathscr{G})$ maximizes $|\mathscr{F}| + |\mathscr{G}|$ then both families are* interval-closed, *i.e. for any $F_1 \subseteq F \subseteq F_2$, $F_i \in \mathscr{F}$ $(i = 1, 2)$ implies $F \in \mathscr{F}$.*

Proof: Obviously if F, F_1, F_2 are as above then F can be added to the family as any subset of F is a subset of F_2 and any superset of F is a superset of F_1. $\qquad\square$

Let F_0 and G_0 be sets of minimal size in $\mathscr{F}$ and $\mathscr{G}$, which are cross-Sperner.

Proposition 6.2.2. *If $|F_0| + |G_0| < \lceil \frac{n}{2} \rceil - 1$, then $|\mathscr{F}| + |\mathscr{G}| < F(n,1)$.*

Proof: No set which contains $F_0 \cup G_0$ can be a member in either of the families. $\qquad\square$

Proof of Theorem 6.2.1: As $(\mathscr{F}, \mathscr{G})$ is cross-Sperner if and only if $(\overline{\mathscr{F}}, \overline{\mathscr{G}})$ is as well, and by the previous proposition, it can be assumed that the size of one of the minimal sets, F_0, is $m = |F_0| \geq \lfloor \frac{n}{4} \rfloor$. Subsets of F_0 are not in $\mathscr{G}$ due to definition and not in $\mathscr{F}$ due to the minimality of F_0. Let $\mathscr{F}^* = \{F \in \mathscr{F} : F_0 \subsetneq F\}$, the collection of sets covering F_0. For any $F^* \in \mathscr{F}^*$ let

$$B(F^*) = \{F^* - F_0' : F_0' \subseteq F_0, |F^* - F_0'| < m\}.$$

For any two distinct sets $F_1^*, F_2^* \in \mathscr{F}^*$ the families $B(F_1^*)$ and $B(F_2^*)$ are disjoint, as F_1^* and F_2^* differ outside F_0. By definition no set in $\mathscr{B} = \cup_{F^* \in \mathscr{F}^*} B(F^*)$ is a subset of F_0. Therefore $\mathscr{B} \cap \mathscr{F} = \emptyset$ due to the sizes and $\mathscr{B} \cap \mathscr{G} = \emptyset$ due to the cross-Sperner property. It is enough to show that $|\mathscr{F}^*| < |\mathscr{B}|$ in order to finish the proof. This is due to the fact that no subset of F_0 is in $\mathscr{F} \cup \mathscr{G}$ and no superset of F_0 is in $\mathscr{G}$ either. So by showing $|\mathscr{F}^*| < |\mathscr{B}|$ we obtain a lower bound on the number of sets not in the union of $\mathscr{F}$ and $\mathscr{G}$ by the number of sub- and supersets of F_0:

$$2^n - |\mathscr{F} \cup \mathscr{G}| \geq |\mathscr{B}| + 2^m - 1 > |\mathscr{F}^*| + 2^m - 1 \geq 2^{|F_0|} + 2^{n-|F_0|} - 2 \geq F(n,1).$$

Note the following properties:

- $|B(F^*)| = \sum_{i=|F^*-F_0|+1}^{m} \binom{m}{i}$,

- $\mathscr{F}^{**} = \{F^* - F_0 : F^* \in \mathscr{F}\}$ is downward closed as $\mathscr{F}$ and $\mathscr{F}^*$ are interval-closed,

- $|\mathscr{F}^{**}| = |\mathscr{F}^*|$.

Noting the points above, the following lemma finishes the proof of Theorem 6.2.1 by choosing $\mathscr{A} = \mathscr{F}^{**}$, $k = m$ and $n' = n - |F_0|$.

Lemma 6.2.1. *Let $\emptyset \neq \mathscr{A} \subseteq 2^{[n']}$ be a downward closed family and $k \geq \frac{n'}{3}$. Then if n' is large enough the following holds:*

$$|\mathscr{A}| < \sum_{A \in \mathscr{A}} \sum_{i=|A|+1}^{k} \binom{k}{i} \tag{6.1}$$

Proof: Let $a_i = |\{A \in \mathscr{A} : |A| = i\}|$ and $w(j) = \sum_{i=j+1}^{k} \binom{k}{i}$. Then the statement can be reformulated in the following way:

$$\sum_{j=0}^{n'} a_j < \sum_{j=0}^{n'} a_j w(j).$$

Let x be defined by $a_{k-1} = \binom{x}{k-1}$. By Lovász's theorem [42] (Theorem 5.5.3) if $j < k - 1$ then $a_j \geq \binom{x}{j}$. When in the previous inequality a_j is replaced by $\binom{x}{j}$ the left hand-side decreases by $a_j - \binom{x}{j}$ and the right hand-side decreases by $(a_j - \binom{x}{j})w(j)$, which is larger.

For $j > k-1$ by the same theorem $a_j \leq \binom{x}{j}$. By replacing a_j with $\binom{x}{j}$ the left hand-side increases and the right hand-side does not change as $w(j) = 0$ for $j \geq k$. So it is enough to see that

$$\sum_{j=0}^{n'} \binom{x}{j} < \sum_{j=0}^{n'} \binom{x}{j} w(j). \tag{6.2}$$

First we prove this for the special case where $x = n'$. In this case the left hand-side is $2^{n'}$ while the right hand-side is monotone increasing in k, so it is enough to prove for the smallest one, $k = \lceil \frac{n'}{3} \rceil$.

$$\sum_{j=0}^{n'} \binom{n'}{j} w(j) > \binom{n'}{j} \binom{k}{j+1} \geq \binom{n'}{j} \binom{n'/3}{j+1}.$$

Let $j = \alpha n'$ for some $0 \leq \alpha \leq \frac{1}{3}$. By Stirling's formula

$$\binom{n'}{j}\binom{n'/3}{j+1} = \binom{n'}{\alpha n'}\binom{n'/3}{\alpha n'+1} =$$
$$= \Theta\left(\frac{1}{n'}\left(\frac{1}{\alpha^{2\alpha}(1-\alpha)^{1-\alpha}3^{1/3}(1/3-\alpha)^{1/3-\alpha}}\right)^{n'}\right).$$

The value of this fraction is larger than 2 to a suitable α, for example for $\alpha = 2/9$. This proves 6.2 for $x = n'$ which is large enough.

For an arbitrary x let $c = \binom{x}{k-1}/\binom{n'}{k-1}$. If $j > k-1$ then $c > \binom{x}{j}/\binom{n'}{j}$, while if $j < k-1$ then $c < \binom{x}{j}/\binom{n'}{j}$. Based on the previous case the following inequality is true:

$$\sum_{j=0}^{n'} c\binom{n'}{j} < \sum_{j=0}^{n'} c\binom{n'}{j}w(j).$$

By replacing $c\binom{n'}{j}$ with $\binom{x}{j}$ the following might happen:

- If $j > k-1$ then the left hand-side decreases and the right hand side does not change,

- If $j = k-1$ then nothing changes due to the definition of c,

- If $j < k-1$ both sides increase, but the right hand-side increases more as $w(j) \geq 1$ for $0 \leq j \leq k-1$.

So the inequality holds after this operation, and now it is the same as 6.2. This proves the lemma. □

As we have seen, Theorem 6.2.1 follows from the theorem, so the proof of the theorem is also finished. □

One might wonder whether a similar theorem is true if the definition of a cross-Sperner family is changed so that sets can belong to both families:

Definition 6.2.1 (Alternate cross-Sperner families)**.** $\mathscr{F}$ *and* $\mathscr{G}$ *form an alternate cross-Sperner family if for any* $F \in \mathscr{F}$, $G \in \mathscr{F}$ *neither* $F \subsetneq G$ *nor* $G \subsetneq F$ *occur.*

For the additive case Theorem 6.2.1 still holds, basically with the same proof. Let $\mathscr{C} = \mathscr{F} \cap \mathscr{G} \neq \emptyset$, and $D(\mathscr{C}) = \{C - C' : C, C' \in \mathscr{C}\}$. Then $D(\mathscr{C})$ is disjoint from both $\mathscr{F}$ and $\mathscr{G}$, and by a result of Marica and Schönheim [43] $|D(\mathscr{C})| \geq |\mathscr{C}|$. The proof of Theorem 6.2.1 gives $|\mathscr{F}| + |\mathscr{G}| \leq F(n, 1) + 2$.

6.3 Multiplicatively maximal cross-Sperner families

Here the question is how large can $|\mathscr{F}||\mathscr{G}|$ get if $(\mathscr{F}, \mathscr{G})$ is a cross-Sperner family. The main result is the following theorem.

Theorem 6.3.1. *If* $n \geq 2$ *and* $(\mathscr{F}, \mathscr{G})$ *is cross-Sperner with* $\mathscr{F}, \mathscr{G} \subseteq 2^{[n]}$ *then the following inequality holds:*

$$|\mathscr{F}||\mathscr{G}| \leq 2^{2n-4}.$$

This bound is best possible as shown by $\mathscr{F} = \{F \in 2^{[n]} : 1 \in F, n \notin F\}$ *and* $\mathscr{G} = \{G \in 2^{[n]} : n \in G, 1 \notin G\}.$

The main tool of the proof is a special case of the Four Functions Theorem of Ahlswede and Daykin [2]. The following notation will be used in the statement: for any pair of families $\mathscr{A}$ and $\mathscr{B}$ let $\mathscr{A} \vee \mathscr{B} = \{A \cap B : A \in \mathscr{A}, B \in \mathscr{B}\}$ and $\mathscr{A} \wedge \mathscr{B} = \{A \cup B : A \in \mathscr{A}, B \in \mathscr{B}\}$.

Theorem 6.3.2 (Ahlswede-Daykin, [2]). *For any pair of families $\mathscr{A}, \mathscr{B}$*

$$|\mathscr{A}||\mathscr{B}| \leq |\mathscr{A} \wedge \mathscr{B}||\mathscr{A} \vee \mathscr{B}|.$$

The following lemma will be used in the proof.

Lemma 6.3.1. *If $(\mathscr{F}, \mathscr{G})$ is a pair of cross-Sperner families, then the families $\mathscr{F}, \mathscr{G}, \mathscr{F} \wedge \mathscr{G}, \mathscr{F} \vee \mathscr{G}$ are pairwise disjoint.*

Proof: $\mathscr{F}$ and $\mathscr{G}$ are disjoint as they are cross-Sperner. They are both disjoint from $\mathscr{F} \vee \mathscr{G}$ and $\mathscr{F} \wedge \mathscr{G}$ again because of the cross-Sperner property. Finally, $\mathscr{F} \vee \mathscr{G}$ and $\mathscr{F} \wedge \mathscr{G}$ are disjoint, as $F_1 \cap G_1 = F_2 \cup G_2$ would imply $F_2 \subseteq G_1$. □

Proof of Theorem 6.3.1: Clearly if $|\mathscr{F}| + |\mathscr{G}| \leq 2^{n-1}$, then the statement of the theorem holds. If $|\mathscr{F}| + |\mathscr{G}| > 2^{n-1}$, then by the previous lemma $|\mathscr{F} \wedge \mathscr{G}| + |\mathscr{F} \vee \mathscr{G}| \leq 2^{n-1}$ and by Theorem 6.3.2

$$|\mathscr{F}||\mathscr{G}| \leq |\mathscr{F} \wedge \mathscr{G}||\mathscr{F} \vee \mathscr{G}| \leq 2^{2n-4}.$$

This proves the theorem. □

Corollary 6.3.1. *For $n \geq 2$ $F(n, 2^{n-2}) = 2^{n-2}$.*

6.4 Open problems

One natural way to generalize the cross-Sperner property is to extend it to k-tuples of families instead of pairs.

Definition 6.4.1 (cross-Sperner k-tuples of families)**.** $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k$ *is cross-Sperner if for any $1 \leq i < j \leq k$ there is no pair $F_i \in \mathscr{F}_i$ and $F_j \in \mathscr{F}_j$ that $F_i \subseteq F_j$ or $F_j \subseteq F_i$.*

For the additive case the trivial family ($\mathscr{F}_1 = 2^{[n]}$, all other families are empty) is still the best, so it is best to consider only non-empty families. A natural conjecture in this case is that the best possible k-tuple of non-empty families would be the one where all except one family contains a single set. Here these families would form a Sperner family. Therefore the following function could prove useful (as it is a parallel of $F(n, m)$ from the proof of Theorem 6.2.1).

$$F^*(n, m) = \max\{|\mathscr{G}| : \mathscr{G} \subseteq 2^{[n]}, \exists \mathscr{F} \subseteq 2^{[n]} \text{ with } |\mathscr{F}| = m,$$
$$(\mathscr{F}, \mathscr{G}) \text{ cross-Sperner, } \mathscr{F} \text{ Sperner}\}.$$

For the multiplicative case the following statement is conjectured to be true.

Conjecture 6.4.1. *If $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k \subseteq 2^{[n]}$ for a k-tuple of cross-Sperner*

families, then

$$\prod_{i=1}^{k} |\mathscr{F}_i| \leq 2^{k(n-l)},$$

where l is the least positive integer with $\binom{l}{\lfloor l/2 \rfloor} \geq k$.

The following construction is optimal if the conjecture is true.

Construction 6.4.1. *Let l be as in the conjecture, then take a Sperner family $\mathscr{S} = \{S_1, S_2, \ldots, S_l\} \subseteq 2^{[l]}$. Let $\mathscr{F}_i = \{F \subseteq [n] : F \cap [l] = S_i\}$.*

Chapter 7

$(\leq l)$-almost properties

7.1   Introduction

This chapter attempts to extend the $(\leq l)$-almost intersecting, l-almost and $(\leq l)$-almost Sperner, l-fork-free properties into a common point of view and analyze similar questions. For this aim the following families need to be defined.

Definition 7.1.1. *Let* $\mathscr{U}(F) = \{G \in 2^{[n]} : F \subseteq G\}$ *and* $\mathscr{L}(F) = \{G \in 2^{[n]} : G \subseteq F\}$ *be the upper and lower set of* F.

Many extremal questions can be reformulated to a question of the maximal possible family with limited size intersections with the upper and lower sets. The two most studied properties are probably the intersecting property

and the Sperner property, which are equivalent to:

$$\forall F \in \mathscr{F} : \mathscr{L}(\overline{F}) \cap \mathscr{F} = \emptyset \qquad \text{Intersecting property,}$$

$$\forall F \in \mathscr{F} : (\mathscr{L}(F) \cup \mathscr{U}(F)) \cap \mathscr{F} = \{F\} \qquad \text{Sperner property.}$$

Another famous extremal question which can be easily expressed in this language is the r-fork ([10], [48], [31]). As a reminder, an r-fork is the configuration of $(r + 1)$ distinct sets $F, G_1, G_2, \ldots, G_r$ with $F \subseteq G_i$ for $\forall i \in \{1, 2, \ldots, r\}$. A family $\mathscr{F}$ is r-fork-free if and only if $\forall F \in \mathscr{F} : |\mathscr{U}(F) \cap \mathscr{F}| \leq r$.

The idea in this chapter is to investigate properties of the form $c_1|\mathscr{F} \cap \mathscr{U}(L)| + c_2|\mathscr{F} \cap \mathscr{L}(F)| + c_3|\mathscr{F} \cap \mathscr{U}(\overline{F})| + c_4|\mathscr{F} \cap \mathscr{L}(\overline{F})| \leq l$, where c_i are either 0 or 1. As mentioned above many extremal properties can be stated in this form.

First of all, an easy observation is that $c_1 = c_2 = c_3 = 0$, $c_4 = 1$, that is, $|\mathscr{L}(\overline{F}) \cap \mathscr{F}| \leq l$ is the ($\leq l$)-almost intersecting property.

For any property defined with c_1, c_2, c_3, c_4, for any family $\mathscr{F}$ satisfying this property there is a family of equal size satisfying the property defined with $c_1' = c_2, c_2' = c_1, c_3' = c_4, c_4' = c_3$. This family consists of the complements of the sets in $\mathscr{F}$: $\overline{\mathscr{F}} = \{\overline{F} : F \in \mathscr{F}\}$. Therefore it is enough to investigate only one of the two properties above to describe the extremal behavior of both.

Considering this it is easy to see that there are two significantly different properties when exactly one of the four c_i-s is nonzero. These correspond to

the l-fork-free property ($|\mathscr{U}(F) \cap \mathscr{F}| \leq l+1$) and the ($\leq l$)-almost intersecting property ($|\mathscr{L}(\overline{F}) \cap \mathscr{F}| \leq l$).

Similarly, there are three properties where two of the coefficients are nonzero. One of them, when $|\mathscr{U}(F) \cap \mathscr{F}| + |\mathscr{L}(F) \cap \mathscr{F}| \leq l+2$ is the ($\leq l$)-almost Sperner property, which is defined similarly to the ($\leq l$)-almost intersecting property.

If there is only one coefficient which is zero, then there are again two properties which can be investigated. The case of $|\mathscr{U}(F) \cap \mathscr{F}| + |\mathscr{L}(F) \cap \mathscr{F}| + |\mathscr{L}(\overline{F}) \cap \mathscr{F}| \leq l+2$ is the ($\leq l$)-almost generalization of the intersecting Sperner property.

Finally, in case all the coefficients are ones then the property is $|\mathscr{U}(L) \cap \mathscr{F}| + |\mathscr{L}(F) \cap \mathscr{F}| + |\mathscr{U}(\overline{F}) \cap \mathscr{F}| + |\mathscr{L}(\overline{F}) \cap \mathscr{F}| \leq l+2$. This will be called ($\leq l$)-almost unrelated property.

7.2 2-almost and (≤ 2)-almost Sperner families

The main theorem in this section is about 2-almost and (≤ 2)-almost Sperner families. This is again the result of joint work with D. Gerbner, N. Lemons, C. Palmer and B. Patkós [30].

Theorem 7.2.1. *If $\mathscr{F} \subseteq 2^{[n]}$ is 2-almost Sperner or (≤ 2)-almost Sperner, then $|\mathscr{F}| \leq 2\binom{n-1}{\lfloor n/2 \rfloor}$. The only families with this size are isomorphic to*

$\mathscr{F}_1 = \binom{[n-1]}{\lfloor n/2 \rfloor} \cup \{F \cup \{n\} : F \in \binom{[n-1]}{\lfloor n/2 \rfloor}\}$ *when n is odd. If n is even then $\mathscr{F}_1$,*

$\mathscr{F}_2 = \overline{\mathscr{F}_1}$ *and $\mathscr{F}_3 = \binom{[n]}{n/2}$ are the only extremal families.*

The bound stated in the theorem was first proved by Katona and Tarján [38] and recently reproved by Katona [37]. Katona's proof will be sketched (but some tedious calculations omitted) as the argument will use it heavily to determine the extremal families. Before starting the proof the context of both previous papers will be sketched and some terminology introduced.

The *comparability graph* $G(P)$ of a poset P is a directed graph with vertex set P and (p, q) is an edge if and only if $p \prec_P q$. The *components* of a family $\mathscr{F} \subseteq 2^{[n]}$ are the subfamilies of which the corresponding vertices form an undirected component of $G(P_\mathscr{F})$, where $P_\mathscr{F}$ is the subposet of the Boolean poset induced by $\mathscr{F}$. The poset P contains another poset Q if $G(Q)$ is a (not necessarily induced) subgraph of $G(P)$ in the directed sense. For any set $\mathscr{P}$ of posets $La(n, \mathscr{P})$ denotes the maximum size that a family $\mathscr{F} \subseteq 2^{[n]}$ can have such that $P_\mathscr{F}$ does not contain any $P \in \mathscr{P}$.

Proof: First follows a sketch of Katona's proof for the bounds [37]. If $\mathscr{F}$ is (≤ 2)-almost Sperner, then any connected component in the comparability graph $G(P)$ is either an isolated vertex or an edge. Let there be α_1 isolated vertices and α_2 edges. Then $|\mathscr{F}| = \alpha_1 + 2\alpha_2$. Let $c(P(1, a))$ $(c(P(2, a_1, a_2)))$ denote the number of full chains going through a one element component (two element component) where the size of the set in the component is a

(a_1, a_2). Katona [37] showed that

$$c(P(1, a)) \geq \lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil! c(P(2, a_1, a_2)) \geq n \lfloor \frac{n-1}{2} \rfloor! \lceil \frac{n-1}{2} \rceil!, \tag{7.1}$$

where equality holds if and only if $a = \lfloor \frac{n}{2} \rfloor$ or $a = \lceil \frac{n}{2} \rceil$ and $a_1 = a_2 - 1 = \lfloor \frac{n-1}{2} \rfloor$ or $a_1 = a_2 - 1 = \lceil \frac{n-1}{2} \rceil$. Let us count the pairs $(C, \mathscr{C})$ where C is a connected component of $\mathscr{F}$ and $\mathscr{C}$ is a full chain in $[n]$. As any full chain can meet at most one component, by (7.1)

$$\alpha_1 \lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil! + 2\alpha_2 \frac{n}{2} \lfloor \frac{n-1}{2} \rfloor! \lceil \frac{n-1}{2} \rceil! \leq n!. \tag{7.2}$$

As $\frac{n}{2} \lfloor \frac{n-1}{2} \rfloor! \lceil \frac{n-1}{2} \rceil! \leq \lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil!$ with equality if and only if n is even, it follows that

$$(\alpha_1 + 2\alpha_2) \frac{n}{2} \lfloor \frac{n-1}{2} \rfloor! \lceil \frac{n-1}{2} \rceil! \leq n!. \tag{7.3}$$

Rearranging gives the bound on $|\mathscr{F}| = \alpha_1 + 2\alpha_2$.

Note that to prove that a particular 2-almost Sperner (or (≤ 2)-almost Sperner) family is not maximal it is enough to show that there is at least one full chain that does not meet $\mathscr{F}$ and thus strict inequality holds in (7.3). Also, by the above computations if $\mathscr{F}$ is of maximum size, then the connected components of $\mathscr{F}$ must be of the following types:

- if n is even, isolated points with corresponding sets of size $\frac{n}{2}$, pairs of sets with sizes $\frac{n}{2} - 1$ and $\frac{n}{2}$ or $\frac{n}{2}$ and $\frac{n}{2} + 1$,

- if n is odd, pairs of sets with sizes $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$.

The following two claims reduce the problem to the situation when $\mathscr{F}$ contains only one type of components, provided n is even.

Lemma 7.2.1. *If n is even and $\mathscr{F}$ contains both isolated sets and pairs of sets as connected components, then $\mathscr{F}$ is not of maximum size.*

Proof: Let F be an isolated set of $\mathscr{F}$ and $H_1 \subsetneq H_2$ be a pair of sets in $\mathscr{F}$ such that $d(F, H_1) + d(F, H_2)$ is minimal (where d denotes the Hamming distance of F and H_i). As $|F| = \frac{n}{2}$ and, by changing to $\overline{\mathscr{F}}$ if necessary, it can be assumed that $|H_1| = |H_2| - 1 = \frac{n}{2}$. First assume that $d(F, H_1) + d(F, H_2) = 5$, which is the minimum possible. Let $C = F \cap H_1, \{x\} = F - H_2$ and $\{z\} = H_2 - H_1$ and consider any full chain $\mathscr{C}$ containing $A = C \cup \{z\}$ and $B = C \cup \{x, z\}$. By the assumption on the sizes of sets in $\mathscr{F}$, $\mathscr{F} \cap \mathscr{C} \subseteq \{A, B\}$, but A is a proper subset of H_2 different from H_1, thus $A \notin \mathscr{F}$ and B is a proper superset of F, thus $B \notin \mathscr{F}$. This proves $\mathscr{F} \cap \mathscr{C} = \emptyset$ which shows that (7.3) cannot hold with equality and thus $\mathscr{F}$ cannot be of maximum size.

Assume now that $d(F, H_1) + d(F, H_2) > 5$. Consider two cases: if $H_2 - H_1 \not\subseteq F$, then fix an arbitrary $f \in F - H_1$ and write $\{h\} = H_2 - H_1$. Consider a full chain containing $G_1 = (F - \{f\}) \cup \{h\}$ and $G_2 = F \cup \{h\}$. Again, by the assumption on the sizes of sets in $\mathscr{F}$, $\mathscr{F} \cap \mathscr{C} \subseteq \{G_1, G_2\}$, but G_2 is a proper superset of F, thus $G_2 \notin \mathscr{F}$, while G_1 is closer to the pair (H_1, H_2) than F (in Hamming distance) and also if G_1 was contained in a pair component of $\mathscr{F}$, then its distance from F would be 5, thus $G_1 \notin \mathscr{F}$. So there exists a chain $\mathscr{C}$ so that $\mathscr{F} \cap \mathscr{C} = \emptyset$, thus (7.3) cannot hold with equality and thus $\mathscr{F}$ cannot be of maximum size. Similarly, in case $H_2 - H_1 \in F$ then let $\{f\} = H_2 - H_1$

and fix an arbitrary $h \in H_1 - F$. Then with the same notations as above we again found a full chain not intersecting $\mathscr{F}$, so $\mathscr{F}$ cannot be of maximal size. $\qquad\square$

Lemma 7.2.2. *If n is even and $\mathscr{F}$ contains pairs of sets as connected components with different set sizes, then $\mathscr{F}$ is not of maximum size.*

Proof: Suppose not and pick two components $F_1 \subsetneq F_2$ and $G_1 \subsetneq G_2$ of $\mathscr{F}$ such that the sizes of the sets are different (and thus, by the optimality of $\mathscr{F}$, $\frac{n}{2} = |F_1| + 1 = |F_2| = |G_1| = |G_2| - 1$) and $d(F_2, G_1)$ is minimal. Assume first that $d(F_2, G_1) = 2$ and let x be the single element of $F_2 - F_1$ and let u be the single element of $F_2 - G_1$. As $d(F_2, G_1) = 2$ and $F_1 \not\subseteq G_1$ one obtains $x \neq u$ and $x \in G_1$. Consider any full chain $\mathscr{C}$ going through $G_1 - \{x\}$ and $(G_1 - \{x\}) \cup \{u\}$. $\mathscr{C}$ cannot contain any set from $\mathscr{F}$ as its members with size at most $\frac{n}{2} - 1$ are proper subsets of G_1 and those with size at least $\frac{n}{2}$ are proper supersets of F_1 different from F_2, therefore these chains show that (7.3) cannot hold with equality, which contradicts $\mathscr{F}$ being of maximum size.

Now suppose $d(F_2, G_1) > 2$ and let us fix $x \in F_1 - G_2$ and $y \in G_1 - F_2$. Consider any full chain $\mathscr{C}$ going through $A = F_2 \setminus \{x\}, B = A \cup \{y\}, C = B \cup \{x\} = F_2 \cup \{y\}$. By the maximality of $\mathscr{F}$ we have $\mathscr{C} \cap \mathscr{F} \subseteq \{A, B, C\}$, but A is a subset of F_2 different from F_1, C is a superset of F_2 and thus A and C cannot be in $\mathscr{F}$. While $B \in \mathscr{F}$ would contradict the minimality of $d(F_2, G_1)$. Again the existence of such $\mathscr{C}$ shows that (7.3) cannot hold with equality, which contradicts $\mathscr{F}$ being of maximum size. $\qquad\square$

Lemma 7.2.3. *If all components of $\mathscr{F}$ are pairs of the same type but for two pairs $F_1 \subsetneq F_2$ and $H_1 \subsetneq H_2$ of sets in $\mathscr{F}$ we have $F_2 - F_1 \neq H_2 - H_1$, then $\mathscr{F}$ is not of maximum size.*

Proof: Again, pick the two pairs such that the sum $d(H_1, F_1) + d(H_1, F_2) + d(H_2, F_1) + d(H_2, F_2)$ is minimal. This sum is at least 12 by the assumption $F_2 - F_1 \neq H_2 - H_1$. First assume that this sum is 12. Then let h be the only element of $H_2 - H_1$, f be the only element of $F_1 - H_1$ and let $C = F_1 \cap H_1$. Consider any full chain $\mathscr{C}$ containing $A = C \cup \{h\}$ and $B = C \cup \{h, f\}$. By the assumption on the sizes of sets in $\mathscr{F}$, we know that $\mathscr{F} \cap \mathscr{C} \subseteq \{A, B\}$, but A is a proper subset of H_2 different from H_1, thus $A \notin \mathscr{F}$ and B is a proper superset of F_1 different from F_2, thus $B \notin \mathscr{F}$. This proves that $\mathscr{F} \cap \mathscr{C} = \emptyset$ which shows that (7.3) cannot hold with equality and thus $\mathscr{F}$ cannot be of maximum size.

Assume now that the sum mentioned above is strictly larger than 12. Let h be an element of $H_1 - F_1$ and if the only element x of $F_2 - F_1$ is contained in H_1, then let $h = x$. Furthermore, let f be an element of $F_1 - H_2$. Consider a full chain $\mathscr{C}$ containing $G_1 = H_2 - \{h\}$ and $G_2 = (H_2 - \{h\}) \cup \{f\}$. Again, by the assumption on the sizes of sets in $\mathscr{F}$, $\mathscr{F} \cap \mathscr{C} \subseteq \{G_1, G_2\}$. We have $H_1 \neq G_1 \subsetneq H_2$, thus $G_1 \notin \mathscr{F}$. Finally, observe that $G_2 \notin \mathscr{F}$ as the component (G', G_2) containing G_2 would be closer to the pair F_1, F_2 and as $x \notin G_2$ we would still have that $F_2 - F_1 \neq G_2 - G'$. $\square$

Lemma 7.2.3 completes the proof of Theorem 7.2.1. $\square$

7.3 ($\leq l$)-almost intersecting Sperner families

Here the property which is investigated is $|\mathscr{U}(F) \cap \mathscr{F}| + |\mathscr{L}(F) \cap \mathscr{F}| + |\mathscr{L}(\overline{F}) \cap \mathscr{F}| \leq l + 2$. This property will be called ($\leq l$)-almost intersecting Sperner property. For $l = 0$, the smallest possible parameter (as $\{F\} = \mathscr{U}(F) \cap \mathscr{L}(F)$), this is equivalent to $\mathscr{F}$ being intersecting and Sperner. This problem has been solved by Milner in [44], later Katona gave a simple proof [36].

Theorem 7.3.1 (Milner). *An intersecting Sperner family on an n-element set has at most $\binom{n}{\lceil \frac{n+1}{2} \rceil}$ members.*

A bound on the size of the maximal ($\leq l$)-almost intersecting Sperner family gives immediately a bound for the size of the maximal ($\leq l+1$)-almost intersecting family through the following observation. Recall that any maximal ($\leq l$)-almost intersecting family is an upset, so for any $F \in \mathscr{F}$ and $F \subseteq G$ it follows that $G \in \mathscr{F}$.

Proposition 7.3.1. *Let $\mathscr{G}$ be a maximal ($\leq l$)-almost intersecting family on n vertices. Then there exists a ($\leq l$-1)-almost intersecting Sperner family $\mathscr{F}$ on $(n-1)$ vertices with size at least $|\mathscr{G}| - 2^{n-1}$.*

Proof: $\mathscr{G}$ contains two type of sets: complementing pairs and sets whose complement is not in $\mathscr{G}$. Therefore if the number of the complementing pairs is a then $|\mathscr{G}| \leq a + 2^{n-1}$. It is very easy to see that due to the maximality of $\mathscr{G}$ this is an equality, otherwise $\mathscr{G}$ could be extended by adding extra elements

from those complementing pairs where neither set from the pair is in $\mathscr{G}$. Let the subfamily $\mathscr{G}_0$ be the family of the complementing pairs from $\mathscr{G}$.

First note that $\mathscr{G}_0$ is interval-closed, for any $F_1, F_2 \in \mathscr{G}_0$ and any $F_1 \subseteq G \subseteq F_2$ it follows that $G \in \mathscr{G}_0$.

Next note that the ($\leq l$)-almost intersecting property of $\mathscr{G}$ implies for $\mathscr{G}_0$ that for any $F \in \mathscr{G}_0$, $|\mathscr{L}(F) \cap \mathscr{G}_0| - 1 + |\mathscr{U}(F) \cap \mathscr{G}_0| - 1 \leq l - 1$. It is easy to see this for those sets F for which $\mathscr{L}(F) \cap \mathscr{G}_0$ or $\mathscr{U}(F) \cap \mathscr{G}_0$ consists of F only, as it follows straight from the ($\leq l$)-almost intersecting property and the fact that $\mathscr{G}_0$ is complement-closed. Let's suppose that $|\mathscr{L}(F) \cap \mathscr{G}_0| - 1 + |\mathscr{U}(F) \cap \mathscr{G}_0| - 1 \geq l$ for some $F \in \mathscr{G}_0$. There is $F_L \subsetneq F \subsetneq F_U$, $F_L, F_U \in \mathscr{G}_0$, where F_U and F_L are the ends of a maximal interval in $\mathscr{G}_0$. Suppose for simplicity that $|\mathscr{L}(F) \cap \mathscr{G}_0| - 1 \geq l/2$. Then $|\mathscr{L}(F_U) \cap \mathscr{G}_0| - 1 \geq l$. This is so as if $x \in F_U - F \neq \emptyset$ then for any set G in $\mathscr{L}(F) \cap \mathscr{G}_0 - \{F\}$ the set $G \cup \{x\}$ is in $(\mathscr{L}(F_U) \cap \mathscr{G}_0) - (\mathscr{L}(F) \cap \mathscr{G}_0)$, so $l - 1 \geq |\mathscr{L}(F_U) \cap \mathscr{G}_0| - 1 \geq 2(|\mathscr{L}(F) \cap \mathscr{G}_0| - 1) \geq l$, which is a contradiction. The other case can be seen with the same argument by taking the complement of every set and noting that $\mathscr{G}_0$ is complement-closed.

The next step is to reduce the family to half of its size. This is performed by selecting an arbitrary vertex v from the base set, and retaining the sets not containing v for the new family $\mathscr{F}$. As $\mathscr{G}_0$ is complement-closed this drops from every complementing pair exactly one of the sets, so $2|\mathscr{F}| = |\mathscr{G}_0|$. The family $\mathscr{F}$ is ($\leq l$-1)-almost intersecting Sperner, as for any $F \in \mathscr{F}$ the number of sets containing F and disjoint from F equals the number of sets

containing F in $\mathcal{G}_0$: $|\mathcal{U}(F) \cap \mathcal{F}| + |\mathcal{L}(\overline{F}) \cap \mathcal{F}| = |\mathcal{U}(F) \cap \mathcal{G}_0|$. Obviously $|\mathcal{L}(F) \cap \mathcal{F}| = |\mathcal{L}(F) \cap \mathcal{G}_0|$, which implies the property. $\qquad\square$

Theorem 7.3.2. *A (≤ 1)-almost intersecting Sperner family on $n = 2k$ vertices can have at most $|\mathcal{F}| \leq 2\binom{n-1}{\lfloor n/2 \rfloor}$ many edges. Any (≤ 1)-almost intersecting Sperner family with $|\mathcal{F}| = 2\binom{n-1}{\lfloor n/2 \rfloor}$ is isomorphic to $\mathcal{F}_1 = \binom{[n-1]}{\lfloor n/2 \rfloor} \cup \{F \cup \{n\} : F \in \binom{[n-1]}{\lfloor n/2 \rfloor}\}$ or $\mathcal{F}_2 = \binom{[n]}{n/2}$.*

Proof: Any (≤ 1)-almost intersecting Sperner family is 2-fork-free by definition, as the conditions exclude a 2-fork and other configurations as well. Thus it is enough to prove that the families above are (≤ 1)-almost intersecting Sperner, as Theorem 7.2.1 already proves that the extremal families cannot be larger. It is easy to see that $\mathcal{F}_1$ is (≤ 1)-almost intersecting Sperner, as any $F \in \mathcal{F}_1$ is compared with exactly one other set from $\mathcal{F}_1$, and is disjoint from all other sets. It is also easy to see that $\mathcal{F}_2$ also satisfies the property. The only step remaining is to show that $\overline{\mathcal{F}_1}$ is not (≤ 1)-almost intersecting Sperner, this is again easy to see from the fact that for any $F \in \overline{\mathcal{F}_1}$ there are $n/2$ disjoint sets in $\overline{\mathcal{F}_1}$. $\qquad\square$

For the case of an odd n the unique extremal family given by Theorem 7.2.1 is not applicable as the family contains disjoint pairs of sets. Our conjecture is that the extremal family is the extremal Sperner family: $\binom{[n]}{\lceil n/2 \rceil}$.

7.4   ($\leq l$)-almost unrelated families

Definition 7.4.1 (($\leq l$)-almost unrelated family). *The family $\mathscr{F} \subseteq 2^{[n]}$ is ($\leq l$)-almost unrelated if for any $F \in \mathscr{F}$: $|\mathscr{U}(F) \cap \mathscr{F}| + |\mathscr{L}(F) \cap \mathscr{F}| + |\mathscr{L}(\overline{F}) \cup \mathscr{U}(\overline{F}) \cap \mathscr{F}| \leq l + 2$.*

Theorem 7.4.1. *Any (≤ 1)-almost unrelated family has size at most $\binom{n}{n/2}$. The extremal families have only sets with size $n/2$ if n is even and $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ if n is odd.*

   Proof: Let $\mathscr{F}$ be a (≤ 1)-almost unrelated family. The method used to show the theorem will be using the LYM inequality by double-counting the intersections of $\mathscr{F}$ and the family of all maximal chains in 2^n, which will be noted as $\mathscr{C}$. For any $C \in \mathscr{C}$ the complementing chain is $\overline{C} = \{\overline{S} | S \in C\}$. $(C, \overline{C})$ is a complementing pair of chains. For any $F \in \mathscr{F}$ there are exactly $|F|!(n - |F|)!$ pairs of complementing maximal chains containing F. For any pair of chains $(C, \overline{C})$ they contain at most 2 sets from $\mathscr{F}$ due to the (≤ 1)-almost unrelated property. There are exactly $n!/2$ maximal complementing chain pairs. Therefore the number of pairs $((F, (C, \overline{C})) | F \in C \cup \overline{C})$ is:

$$\sum_{F \in \mathscr{F}} |F|!(n - |F|)! = |\{(F, (C, \overline{C})) | F \in C \cup \overline{C}\}| \leq 2n!/2$$

Dividing by 2 yields the usual LYM inequality:

$$\sum_{F \in \mathscr{F}} \frac{1}{\binom{n}{|F|}} \leq 1$$

This proves that any extremal (≤ 1)-almost unrelated family contains at most $\binom{n}{n/2}$ sets, and the extremal family has only sets with size $\lceil n/2 \rceil$ if n is even and $\lfloor n \rfloor$ or $\lceil n \rceil$ if n is odd. $\qquad \square$

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I hereby declare that Vajk Szecsi took an equal part in the following joint papers contain material shared with the thesis:

1. Almost Intersecting Families of Sets (submitted)
2. Cross-Sperner Families (Studia Sci. Math. Hungarica 49 (2012) 44-51)
3. Almost Cross-Intersecting and Almost Cross-Sperner Pairs of Families of Sets (Graphs and Combinatorics, to appear)

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