# Decomposing $\omega$-fold coverings

by

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Budapest, 3 May 2013

Signature

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# Abstract

The main result of this thesis is the following: any family of the translates of the open (resp. closed) unit square $\mathcal{F}$ is $\omega$-decomposable over the points which are covered $\omega$-fold by $\mathcal{F}$.

To get this result we prove several one dimensional covering decomposition results and finally we construct some examples to examine the sharpness of our main result.

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# 1 Introduction

## 1.1 General introduction

Let $\lambda, \kappa$ be cardinals and $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $\mathcal{F}$ covers every point of $X$ at least $\kappa$ times. Can we decompose $\mathcal{F}$ into $\lambda$ many disjoint covers of $X$?

In this thesis we deal with a version of this question. By choosing the parameters: $X$, $\mathcal{F}$, $\lambda$, $\kappa$ this problem has a long-standing history. Let us briefly summarize some of the results.

**Finite covers:**

A well understood question in this context is when $X$ is a finite set and $\mathcal{F}$ is a family of hyperedges of a graph on $X$. Almost optimal solutions of the relevant problems have already been found long ago.

However if the cardinality of $X$ is not finite, then the answer is not so clear. Pach in [4] posed the above question in the following form: let $X$ be the plane, and let $P$ be a convex planar set, then can we find a finite number $\kappa$ such that if $\mathcal{F}$ is a set of translates of $P$ then we can decompose $\mathcal{F}$ into two covers of the plane?

Note that if $P$ is a polygon, then the affirmative answer has been found in [5] and [3]. However the related question for e.g. circles is a much harder one. We do not know the solution, but we note that a positive answer is claimed in a more than 100 page-long manuscript of Mani-Levitska and Pach, however this is still not published.

**Infinite covers:**

In this direction - $\kappa$ can be infinite - the first result where the underlying set $X$ had geometric properties (and so would be relevant to us) appeared in [1]:

**Theorem 1.1.1.** *(Aharoni, Hajnal, Milner) If $\kappa$ is a cardinal, $X$ is a linearly ordered set and $\mathcal{F}$ is a set of intervals such that each point of $X$ is covered by (at least) $\kappa$ many elements of $\mathcal{F}$, then $\mathcal{F}$ is the disjoint union of $\kappa$ many covers.*

After this result, a question of Pach whether any infinite-fold cover of the plane by axis-parallel rectangles can be decomposed into two disjoint subcovers, inspired the authors of [2] to start a systematic study of 'infinite-fold covering problems'(see the exact definition later) and achieved numerous results about them. E.g.:

**Theorem 1.1.2** ([2], Theorem 7.4)**.** *Let $\kappa > \omega$ be a cardinal and $\mathcal{F}$ a family of closed polygons in the plane such that each point of the plane is covered by at least $\kappa$-many elements of $\mathcal{F}$. Then $\mathcal{F}$ can be decomposed into $\kappa$ many disjoint covers of the plane.*

This theorem is not true for $\kappa = \omega$:

**Theorem 1.1.3** ([2], Theorem 7.2)**.** *There exists a countable family $\mathcal{F}$ of axis-parallel closed rectangles in $\mathbf{R}^2$ such that $\mathcal{F}$ is an $\omega$-fold cover of $\mathbf{R}^2$ without two disjoint subcovers.*

We note that using similar techniques the authors of [2], we can construct a family $\mathcal{F}$ of closed unit squares in $\mathbf{R}^2$ such that $\mathcal{F}$ is an $\omega$-fold cover of $\mathbf{R}^2$ without two disjoint subcovers (see Theorem 1.3.9).

However after Theorem 1.1.2 and Theorem 1.1.3 it was natural to ask the following question:

**Question 1.1.4** ([2], Problem 8.6.1.)**.** *Is it true that if each point of $\mathbf{R}^n$ is covered $\omega$-many times by $\mathcal{F}$, a set of translates of the closed unit cube, then $\mathcal{F}$ can be decomposed into two disjoint covers of $\mathbf{R}^n$?*

To answer for this question was the starting point of our investigations. The main result of this thesis answers this question in the affirmative for $n = 2$.

## 1.2 Notation, easy facts

### 1.2.1 Notation

- Let us denote by $\mathbf{R}$ the set of real numbers.
- Let us denote by $E$ the axis-parallel open unit square.
- Let us denote by $\mathcal{P}(X)$ the power set of $X$.

- Let $ord(x, \mathcal{F}) := |\{F : x \in F \in \mathcal{F}\}|$.
- Let $\kappa$ be an infinite cardinal. We say that

  $\mathcal{F}$ is a *$\kappa$-fold cover of $X$* if $\kappa \leq ord(x, \mathcal{F})$ for each $x \in X$.
- Let $\kappa$ be a cardinal. Then let $(\mathcal{F})_\kappa := \{x \in X : \kappa \leq ord(x, \mathcal{F})\}$.

  (Using this notation $Y \subseteq (\mathcal{F})_\kappa$ means that $\mathcal{F}$ is a $\kappa$-fold cover of $Y$.)

- Let $\kappa$ be a cardinal. Then $[X]^\kappa, [X]^{\leq\kappa}, [X]^{<\kappa}$ stand for the set of

  subsets of $X$ which have cardinalities $\kappa, \leq \kappa, < \kappa$ respectively.
- We say that $\mathcal{F}$ is *disjoint* if $F \cap G = \emptyset$ for all $F, G \in \mathcal{F}$.

**Definition 1.2.1.** Let $\kappa$ be a cardinal. We say that $\mathcal{F}$ is *$\kappa$-decomposable over $X$* if there is $\mathcal{G} \in [\mathcal{P}(\mathcal{F})]^\kappa$ disjoint s.t. each $G \in \mathcal{G}$ is a cover of $Y$.

**Remark.**

(1) Note that for a fixed infinite cardinal $\kappa$ the following statements are equivalent:

$(i)$ $\mathcal{F}$ is $\kappa$-decomposable over $X$;

$(ii)$ there is $\mathcal{H} \in [\mathcal{P}(\mathcal{F})]^\kappa$ disjoint with $X \subseteq (H)_\kappa$ for all $H \in \mathcal{H}$.

(2) Note also that for any map $c : \mathcal{F} \to \kappa$, $\{c^{-1}(\{j\}) : j < \kappa\} \in [\mathcal{P}(\mathcal{F})]^\kappa$ is disjoint. So to prove that $\mathcal{F}$ is $\kappa$-decomposable over $X$, it is enough to construct a surjection $c : \mathcal{F} \to \kappa$ such that $c^{-1}(\{j\})$ is a cover of $X$ for all $j < \kappa$.

- If $(X, \tau)$ is a topological space and $A \subseteq X$ then $\partial A$ denotes the boundary of $A$ and $int(A)$ denotes the interior of $A$.
- For $A \subseteq \mathbf{R}^2$ let us denote by $proj_x(A)$ ($proj_y(A)$) the projection of $A$ to the $x$ (*resp.* $y$) axis.
- For $C \subseteq \mathbf{R}^2$ we denote by $\mathcal{T}_C$ the set of the translates of $C$.
- For a function $f : X \to Y$ and $\mathcal{F} \subseteq \mathcal{P}(X)$ let $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$.
- We use $\cup^*$ to denote disjoint union.
- For $\langle s(0), s(1), ..., s(i)\rangle = s \in \omega^{<\omega}$ and $\langle t(0), t(1), ..., t(j)\rangle = t \in \omega^{<\omega}$ we denote $\langle s(0), ..., s(i), t(0), ..., t(j)\rangle \in \omega^{<\omega}$ by $s \frown t$.

### 1.2.2 Easy facts

**Lemma 1.2.2.** *Let $(X, \tau)$ be a topological space and $\mathcal{F} \subseteq \tau$. If $Y \subseteq X$ is $\sigma$-compact with $Y \subseteq (\mathcal{F})_\omega$ then $\mathcal{F}$ is $\omega$-decomposable over $Y$.*

*Proof.* Let $\{K_i : i \in \omega\}$ be an increasing sequence of compact sets with $\cup_{i\in\omega} K_i = Y$.

We define $\mathcal{F}_s$ inductively. For $s \in \omega$ let

$$\mathcal{F}_s \in [\mathcal{F} \setminus \cup_{i<s}\mathcal{F}_i]^{<\omega} \text{ with } K_s \subseteq \cup\mathcal{F}_s.$$

Choose $\varphi : \omega \to \omega \times \omega$ arbitrary bijection, then

$$\{\cup\{\mathcal{F}_u : \varphi(u) = \langle m, n\rangle, n \in \omega\} : m \in \omega\}$$

proves the lemma.

$\square$

**Lemma 1.2.3.** *For $Y \subseteq (\mathcal{F})_\omega$ the following is true:*

*If each $\mathcal{G} \subseteq \mathcal{F}$ with $Y \subseteq (\mathcal{G})_\omega$ is the disjoint union of $\mathcal{G}_1$ and $\mathcal{G}_2$ with $Y \subseteq (\mathcal{G}_1)_\omega \cap (\mathcal{G}_2)_\omega$, then $\mathcal{F}$ is $\omega$-decomposable over $Y$.*

*Proof.* Let $c_0 : \mathcal{F} \to 2$ be a coloring witnessing the condition for $\mathcal{G} = \mathcal{F}$, and for $0 < i \in \omega$ let

$$c_i : (c_{i-1})^{-1}(\{1\}) \to 2$$

witness the condition of the lemma for $(c_{i-1})^{-1}(\{1\})$. Then $\{(c_i)^{-1}(\{0\}) : i < \omega\}$ proves the lemma.

□

## 1.3 Our main results

### 1.3.1 One dimensional results

Let $\mathfrak{I}$ denote the set of open, nonempty (finite or infinite) intervals in $\mathbf{R}$.

The following theorem is a special case of Theorem 5.1. in [2].

**Theorem 1.3.1.**

*Let $\mathfrak{R} \in [\mathfrak{I}]^{\leq\omega}$. Then $\mathfrak{R}$ is $\omega$-decomposable over $(\mathfrak{R})_\omega$.*

First we will prove the following strengthening of Theorem 1.3.1:

**Theorem 1.3.2.**

*Let $\{\mathfrak{R}_n : n \in \omega\} \subseteq [\mathfrak{I}]^{\leq\omega}$. Then there is $c : \cup_{n\in\omega}\mathfrak{R}_n \to \omega$ such that for each $n \in \omega$ the following is true:*

$$(\mathfrak{R}_n)_\omega = \cap_{j\in\omega} \cup (\mathfrak{R}_n \cap c^{-1}(\{j\})).$$

Then we will prove Theorem 1.3.4, which is stronger result than Theorem 1.3.2 for certain $\{\mathfrak{R}_n : n \in \omega\} \subseteq [\mathfrak{I}]^{\leq\omega}$. It has stronger requirements on the coloring, however we only prove it for $\mathfrak{R}_n \in [\mathfrak{I}]^{\leq\omega}$ of special form.

**Definition 1.3.3.**

Let $\mathfrak{R} \in [\mathfrak{I}]^{\leq\omega}$, $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^{\omega}$ with $D \cap \cup\{\partial R : R \in \mathfrak{R}\} = \emptyset$.

For $p, q \in D$ let us define the following sets:

$$\mathfrak{R}_{p,q,0} = \{(a,b) \in \mathfrak{R} : a < p, b < q\},$$

$$\mathfrak{R}_{p,q,1} = \{(a,b) \in \mathfrak{R} : a > p, b < q\},$$

$$\mathfrak{R}_{p,q,2} = \{(a,b) \in \mathfrak{R} : a < p, b > q\},$$

$$\mathfrak{R}_{p,q,3} = \{(a,b) \in \mathfrak{R} : a > p, b > q\}.$$

**Theorem 1.3.4.**

*Let $\mathfrak{R} \in [\mathfrak{I}]^{\leq\omega}$ and $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^{\omega}$ with $D \cap \cup\{\partial R : R \in \mathfrak{R}\} = \emptyset$.*

*There is $c : \mathfrak{R} \to \omega$ such that for all $p, q \in D$ and $\varepsilon \in 4$ the following statements hold:*

*(i) if $|\mathfrak{R}_{p,q,\varepsilon}| = \omega$ then $|c^{-1}(\{0\}) \cap \mathfrak{R}_{p,q,\varepsilon}| = \omega$;*

*(ii) $(\mathfrak{R}_{p,q,\varepsilon})_\omega = \bigcap_{j\in\omega} \cup(c^{-1}(\{j\}) \cap \mathfrak{R}_{p,q,\varepsilon})$.*

**Remark.** Note that $\mathfrak{R} = \cup^*_{p,q\in D,\varepsilon\in 4}\mathfrak{R}_{p,q,\varepsilon}$, so the statement of Theorem 1.3.2 is a consequence of Theorem 1.3.4 for the countable family $\{\mathfrak{R}_{p,q,\varepsilon} : p, q \in D, \varepsilon \in 4\}$.

The following example shows that we can not expect that Theorem 1.3.4 to hold for all $\{\mathfrak{R}_n : n \in \omega\} \subseteq [\mathfrak{I}]^{\leq\omega}$. So this natural strengthening of Theorem 1.3.2 fails.

**Example 1.3.5.**

*There is $\{\mathfrak{R}_n : n \in \omega\} \subseteq [\mathfrak{I}]^{\leq\omega}$ such that we have no $c : \cup_{n\in\omega}\mathfrak{R}_n \to \omega$ with:*

*(i) if $|\mathfrak{R}_n| = \omega$ then $|c^{-1}(\{0\}) \cap \mathfrak{R}_n| = \omega$ for $n \in \omega$;*

*(ii) $(\mathfrak{R}_n)_\omega = \cap_{j\in\omega} \cup (\mathfrak{R}_n \cap c^{-1}(\{j\}))$ for $n \in \omega$.*

*Proof.*

Let $\varphi : \omega^{<\omega} \to \omega \setminus \{0\}$ be a bijection and for $s \in \omega^{<\omega}$ let $I_s \subseteq (0,1)$ be an open interval satisfying for $s, t \in \omega^{<\omega}$:

$\bullet_1$ $\overline{I_t} \subseteq I_s$ if $t \supseteq s$;

$\bullet_2$ $I_t \cap I_s = \emptyset$ if $s \not\subseteq t$ and $t \not\subseteq s$.

Let $\mathfrak{R}_0 = \{I_s : s \in \omega^{<\omega}\}$ and $\mathfrak{R}_n = \{I_{\varphi^{-1}(\{n\})^\frown i} : i \in \omega\}$ for $n \in \omega \setminus \{0\}$. Now we prove that either $(i)$ or $(ii)$ does not hold for $c : \cup_{n\in\omega}\mathfrak{R}_n \to \omega$.

We prove it by contradiction. Suppose $(ii)$ holds for $c$ and $\mathfrak{R}_n$ $(n \in \omega)$. Then we can choose $u \in \omega^\omega$ such that $c(I_{\langle u(0),u(1),\dots,u(j)\rangle}) = 0$ for all $j \in \omega$. Then using $\bullet_1$ and $\bullet_2$ we have $\cap_{j\in\omega} I_{\langle u(0),u(1),\dots,u(j)\rangle} \neq \emptyset$ and $\cap_{i\in\omega} I_{\langle u(0),u(1),\dots,u(j)\rangle}$ is covered by $I_s$ iff $s = \langle u(0), u(1), \dots, u(j)\rangle$ for some $j \in \omega$. This contradicts to $(i)$ with $c$ and $\mathfrak{R}_0$, hence we are done. $\square$

We will need the following one dimensional result in the proof of the two dimensional decomposition results:

**Lemma 1.3.6.** *Let* $\{a_n, b_n : n \in \omega\}, \{A_n, B_n : n \in \omega\} \subseteq \mathbf{R}$ *such that:*

*(i)* $a_n \neq a_m$, $a_n < b_n \neq b_m$, $A_n \neq A_m$, $A_n < B_n \neq B_m$ *for all* $n, m \in \omega$ *different;*

*(ii)* $(a_n < a_m < b_m < b_n$ *or* $A_n < A_m < B_m < B_n)$ *is false for all* $n, m \in \omega$*;*

*(iii)* $a_n < a_m \le b_n < b_m$ *iff* $A_n < A_m \le B_n < B_m$ *for all* $n, m \in \omega$*;*

*(iv)* $a_n < a_m$ *iff* $A_n < A_m$ *for all* $n, m \in \omega$.

*Suppose that the following holds with some* $J \subseteq \omega$*:*

$$\bigcap_{n\in J}(a_n, b_n) \cap \bigcap_{n\in\omega\setminus J}(\mathbf{R} \setminus (a_n, b_n)) \neq \emptyset.$$

*Then there exists* $J \supseteq J'$ *with* $|J \setminus J'| < 3$ *satisfying:*

$$\bigcap_{n\in J'}(A_n, B_n) \cap \bigcap_{n\in\omega\setminus J}(\mathbf{R} \setminus (A_n, B_n)) \neq \emptyset.$$

### 1.3.2 Two dimensional results

Our main theorem is the following:

**Theorem 1.3.7.** *Recall that $E$ is the axis-parallel open unit square.*

(1) *If $\mathcal{F} \subseteq \mathcal{T}_E$, then $\mathcal{F}$ is $\omega$-decomposable over $(\mathcal{F})_\omega$;*

(2) *If $\mathcal{F} \subseteq \mathcal{T}_{\overline{E}}$, then $\mathcal{F}$ is $\omega$-decomposable over $(\mathcal{F})_\omega$.*

**Remark.** Theorem 1.3.7 (2) gives the affirmative answer to Question 1.1.4 in the case of translates of the closed unit square of $\mathbf{R}^2$.

### 1.3.3 Constructions

We will also provide three constructions showing the sharpness of Theorem 1.3.7:

*Construction 1:*

Let $\mathcal{R}_\varepsilon$ (*resp.* $\mathcal{Q}_\varepsilon$) be the set of axis-parallel closed (resp. open) rectangles with side lengths between $1-\varepsilon$ and 1.

**Theorem 1.3.8.**

(1) *For any $\varepsilon > 0$ there is $\mathcal{R} \in [\mathcal{R}_\varepsilon]^\omega$ which is not 2-decomposable over $(\mathcal{R})_\omega$.*

(2) *For any $\varepsilon > 0$ there is $\mathcal{Q} \in [\mathcal{Q}_\varepsilon]^\omega$ which is not 2-decomposable over $(\mathcal{Q})_\omega$.*

*Construction 2:*

Let $\mathcal{S}_\varepsilon$ be the family of all sets of the form $t(\overline{E})$ where for $t$ is the composition of an arbitrary translation of $\mathbf{R}^2$ and a rotation of $\mathbf{R}^2$ with angle at most $\varepsilon$.

**Theorem 1.3.9.** *For any $\varepsilon > 0$ there is $\mathcal{S} \in [\mathcal{S}_\varepsilon]^\omega$ such that:*

(*i*) *$\mathcal{S}$ is an $\omega$-fold cover of $\mathbf{R}^2$ (i.e. $(\mathcal{S})_\omega = \mathbf{R}^2$);*

(*ii*) *$\mathcal{S}$ is not 2-decomposable over $(\mathcal{S})_\omega$.*

*Construction 3:*

Let $\mathcal{U}_\varepsilon$ be the set of all axis-parallel closed squares with side length between $1-\varepsilon$ and 1.

**Theorem 1.3.10.** *For any $\varepsilon > 0$ there is $\mathcal{U} \in [\mathcal{U}_\varepsilon]^\omega$ which can not be decomposed into 2 disjoint $\omega$-fold covers of $(\mathcal{U})_\omega$.*

**Remark.** Theorem 1.3.10 easily implies that $\mathcal{U}$ is not $\omega$-decomposable over $(\mathcal{U})_\omega$.

# 2 Proof of the one dimensional results

## 2.1 Proof of Theorem 1.3.2 and Theorem 1.3.4

### 2.1.1 Choosing a subset of $\mathfrak{R}$

**Definition 2.1.1.** For $R_1, R_2 \in \{<,>,=\}$ we denote by $T(R_1, R_2)$ the set of $\mathfrak{A} \in [\mathfrak{I}]^\omega$ which has an enumeration: $\mathfrak{A} = \{(a_n, b_n) : n \in \omega\}$ such that $a_n R_1 a_m \wedge b_n R_2 b_m$ for all $n < m$. For $R \in \{<,>,=\}$ let $T(R,.) = \cup_{Q\in\{<,>,=\}} T(R,Q)$, $T(.,R) = \cup_{Q\in\{<,>,=\}} T(Q,R)$ and $T(.,.) = \cup_{R\in\{<,>,=\}} T(R,.)$.

**Remark.** If $\mathfrak{A} \in T(.,.)$ then there is exactly one enumeration witnessing this.

We mention 3 easy claims without proof:

**Claim 2.1.2.** For each $\mathfrak{A} \in [\mathfrak{I}]^\omega$ there is $\mathfrak{B} \in [\mathfrak{A}]^\omega$ with $\mathfrak{B} \in T(.,.)$.

**Claim 2.1.3.** If $\mathfrak{A} \in T(.,.)$ and $\mathfrak{B} \in [\mathfrak{A}]^\omega$ then we have $(\mathfrak{A})_\omega = (\mathfrak{B})_\omega$.

**Claim 2.1.4.** If $\mathfrak{A} \in T(.,<)$ with $a \in \cap\mathfrak{A}$, $b \in \mathbf{R} \cup \{+\infty\}$ and $[a,b) \subseteq \cup\mathfrak{A}$ then $[a,b) \subseteq \cup\mathfrak{B}$ for each $\mathfrak{B} \in [\mathfrak{A}]^\omega$.

**Lemma 2.1.5.** *For $\mathfrak{R} \in [\mathfrak{I}]^{\leq\omega}$ with $(a,b) = \cup\mathfrak{R} \in \mathfrak{I}$ there exist pairwise disjoint*

$$\mathfrak{R}^0, \mathfrak{R}^1, \mathfrak{R}^2 \subseteq \mathfrak{R}$$

*satisfying:*

(*i*) *if there is $c \in (a,b)$ with $(a,c) \in \mathfrak{R}$ then $\mathfrak{R}^0 = \emptyset$;*

*if* $\mathfrak{R}^0 \neq \emptyset$ *then* $\mathfrak{R}^0 \in T(>,.)$;

(*ii*) *if there is* $c \in (a,b)$ *with* $(c,b) \in \mathfrak{R}$ *then* $\mathfrak{R}^1 = \emptyset$*;*

*if* $\mathfrak{R}^1 \neq \emptyset$ *then* $\mathfrak{R}^1 \in T(.,<)$*;*

(*iii*) $\mathfrak{R}^2 \neq \emptyset$ *and* $(\mathfrak{R}^2)_{10} = \emptyset$*;*

(*iv*) $(\cap\mathfrak{R}^i) \cap \cup\mathfrak{R}^2 \neq \emptyset$ *for* $i \in 2$ *(note that* $\cap\emptyset = \mathbf{R}$*);*

(*v*) $\cup(\mathfrak{Q}^0 \cup \mathfrak{Q}^1 \cup \mathfrak{R}^2) = (a,b)$ *for all* $\mathfrak{Q}^0 \in [\mathfrak{R}^0]^\omega$ *and* $\mathfrak{Q}^1 \in [\mathfrak{R}^1]^\omega$*;*

(*vi*) *for all* $R \in \mathfrak{R}$ *we have* $|\{Q \in \mathfrak{R}^2 : Q \subseteq R\}| \leq 4$.

*Proof.* For $x \in (a,b)$ let:

$$f(x) = sup\{d : x \in (c,d) \in \mathfrak{R}\}, \text{ and}$$

$$g(x) = inf\{c : x \in (c,d) \in \mathfrak{R}\}.$$

$(f^{(0)}(x) = x$ and $f^{(n)}(x) = \underbrace{f(f(...(}_{n}x))).)$

**Claim 2.1.6.** For any $x \in (a,b)$ the following statements are true:

($f$) there is $k \in \omega$ with $f^{(k)}(x) = b$ or $\lim_{n\to\infty} f^{(n)}(x) = b$;

($g$) there is $k \in \omega$ with $g^{(k)}(x) = a$ or $\lim_{n\to\infty} g^{(n)}(x) = a$.

*Proof of Claim 2.1.6.* We prove ($f$), the proof of ($g$) is similar.

We prove this by contradiction. Suppose there is no $k \in \omega$ with $f^{(k)}(x) = b$. Then, since

$$f^{(0)}(x) < f^{(1)}(x) < ... < b,$$

there is $y \in (a,b]$ with $\lim_{n\to\infty} f^{(n)}(x) = y$.

If $y < b$ then there is $(c,d) \in \mathfrak{R}$ with $y \in (c,d)$, so $y < d$. But there is $m \in \omega$ satisfying $c < f^{(m)}(x)$, so we have $y < d \leq f^{(m+1)}(x) < y$. Contradiction, hence $y = b$.

We are done with the proof of Claim 2.1.6. □

**Claim 2.1.7.** For any $x \in (a, b)$ the following statements are true:

($f$) if $f^{(n)}(x) \in (a, b)$ for all $n \in \omega$, then there is $\mathfrak{R}^{2,r} \subseteq \mathfrak{R}$ with $(\mathfrak{R}^{2,r})_5 = \emptyset$ and $\cup\mathfrak{R}^{2,r} \supseteq [x, b)$;

($g$) if $g^{(n)}(x) \in (a, b)$ for all $n \in \omega$ then there is $\mathfrak{R}^{2,l} \subseteq \mathfrak{R}$ with $(\mathfrak{R}^{2,l})_5 = \emptyset$ and $\cup\mathfrak{R}^{2,l} \supseteq (a, x]$.

**Remark.** In Claim 2.1.7 the $r, l$ superscript means that we choose these intervals going toward the right or left endpoint of $(a, b)$.

*Proof of Claim 2.1.7.* We prove ($f$), the proof of ($g$) is similar.

For $i \in \omega$ choose $(a_i^0, b_i^0), (a_i^1, b_i^1) \in \mathfrak{R}$ satisfying the following conditions:

(1) $(a_i^0, b_i^0) \ni f^{(i)}(x)$;

(2) $(a_i^1, b_i^1) \ni f^{(i+1)}(x)$;

(3) $(a_i^0, b_i^0) \cap (a_i^1, b_i^1) \neq \emptyset$.

We can choose $(a_i^1, b_i^1) \in \mathfrak{R}$ ($i \in \omega$) satisfying (2) (since $f^{(i+1)}(x) \in (a, b)$) and after this $(a_i^0, b_i^0) \in \mathfrak{R}$ satisfying (1) with $(a_i^1, b_i^1) \cap (a_i^0, b_i^0) \neq \emptyset$ for each $i \in \omega$, because of $f^{(i+1)}(x) = f(f^{(i)}(x))$. So the chosen interval system will satisfy $(1) - (3)$.

Let

$$\mathfrak{R}^{2,r} := \{(a_i^0, b_i^0) : i \in \omega\} \cup \{(a_{i+1}^1, b_{i+1}^1) : i \in \omega\}.$$

If $f^{(i)}(x) \in (c, d) \in \mathfrak{R}$ for $i > 0$ then $f^{(i-1)}(x) \leq c$ and $d \leq f^{(i+1)}(x)$ and if $x \in (c, d) \in \mathfrak{R}$ then $d \leq f^{(1)}(x)$ by the definition of $f$, so if $0 < i$ then $[f^{(i)}(x), f^{(i+1)}(x)]$ meets only $(a_{i-1}^1, b_{i-1}^1), (a_i^0, b_i^0), (a_i^1, b_i^1), (a_{i+1}^0, b_{i+1}^0)$ and only $(a_i^0, b_i^0), (a_i^1, b_i^1), (a_{i+1}^0, b_{i+1}^0)$ if $i = 0$, hence $(\mathfrak{R}^{2,r})_5 = \emptyset$.

$\cup\mathfrak{R}^{2,r} \supseteq [x, b)$ is true by the fact that $lim_{n\to\infty} f^{(n)}(x) = b$ by Claim 2.1.6 and that $[f^{(i)}(x), f^{(i+1)}(x)] \subseteq (a_i^0, b_i^0) \cup (a_i^1, b_i^1)$ for all $i \in \omega$.

We are done with the proof of Claim 2.1.7.

□

**Claim 2.1.8.** If $f^{(k)}(x) = b$ for $x \in (a, b)$ and $k \in \omega$ then there are

$$\mathfrak{R}^{2,r}, \mathfrak{R}^{1,r} \subseteq \mathfrak{R}$$

with the following properties:

$(A)$ $\mathfrak{R}^{2,r} \cap \mathfrak{R}^{1,r} = \emptyset$;

$(B)$ $(\mathfrak{R}^{2,r})_5 = \emptyset$;

$(C)$ if there is $c \in (a, b)$ with $(c, b) \in \mathfrak{R}$ then $\mathfrak{R}^{1,r} = \emptyset$, and

if $\mathfrak{R}^{1,r} \neq \emptyset$ then $\mathfrak{R}^{1,r} \in T(., <)$;

$(D)$ $\cup\mathfrak{R}^{2,r} \bigcap \cap\mathfrak{R}^{1,r} \neq \emptyset$;

$(E)$ $(\cup\mathfrak{B}) \cup (\cup\mathfrak{R}^{2,r}) \supseteq [x, b)$ for all $\mathfrak{B} \in [\mathfrak{R}^{1,r}]^{\omega}$;

$(F)$ $|\{Q \in \mathfrak{R}^{2,r} : Q \subseteq R\}| \leq 2$ for all $R \in \mathfrak{R}$.

*Proof of Claim 2.1.8.*

*Case* 1:

There is $c \in (a, b)$ with $(c, b) \in \mathfrak{R}$.

In this case $f^{(k-1)}(x) \in (c, b)$ (note that $k > 0$ as $f^{(0)}(x) = x \in (a, b)$) and we can choose intervals $\{(a_i^j, b_i^j) : i \in k-1, j \in 2\}$ for $\{f^{(i)}(x) : i \in k-1\}$ as in the proof of Claim 2.1.7. Let $\mathfrak{R}^{2,r} := \{(a_i^j, b_i^j) : i \in k-1, j \in 2\} \cup (c, b)$ and $\mathfrak{R}^{1,r} := \emptyset$. $(A)-(E)$ are trivially satisfied. $(F)$ is true by the fact that each interval in $\mathfrak{R}^{2,r}$ contains $f^{(i)}(x)$ for some $i \in k$. So if $R \in \mathfrak{R}$ contains an interval from $\mathfrak{R}^{2,r}$, it must contain $f^{(i)}(x)$ for some $i \in k$. But $|\{i \in \omega : f^{(i)}(x) \in R\}| \leq 1$ for all $R \in \mathfrak{R}$ and $x \in (a, b)$ by the definition of $f$. So $(a_0^0, b_0^0), (c, b)$ if $i = 0$, $(a_{i-1}^1, b_{i-1}^1)$ and $(a_i^0, b_i^0)$ if $0 < i < k-2$ and $(a_{k-2}^1, b_{k-2}^1)$, $(c, b)$ if $i = k-1$ are the only intervals from $\mathfrak{R}^{2,r}$ which can be contained in $R$.

*Case* 2:

There is no $c \in (a, b)$ with $(c, b) \in \mathfrak{R}$.

We can choose intervals $\{(a_i^j, b_i^j) : i \in k-1, j \in 2\}$ for $\{f^{(i)}(x) : 0 \leq i \leq k-1\}$ as in the proof of Claim 2.1.7 and let $\mathfrak{R}^{2,r} := \{(a_i^j, b_i^j) : i \in k-1, j \in 2\}$. Since $f^{(k)}(x) = b$ and we are not in *Case* 1, there exists $\mathfrak{A} = \{(c_n, d_n) : n \in \omega\} \subseteq \mathfrak{R}$ with:

- $\mathfrak{A} \cap \mathfrak{R}^{2,r} = \emptyset$;
- $\lim_{n\to\infty} d_n = b$;
- $f^{(k-1)}(x) \in (c_n, d_n)$ for all $n \in \omega$.

By Claim 2.1.2 there exists $\mathfrak{R}^{1,r} \subseteq \mathfrak{A}$ with $\mathfrak{R}^{1,r} \in T(., <)$. Then

$(A) - (C)$ of the lemma are trivially satisfied,

$(D)$ is true, since $f^{(k-1)}(x) \in \cap\mathfrak{R}^{1,r} \cap (a^1_{k-1}, b^1_{k-1})$,

$(E)$ is true by Claim 2.1.4,

$(F)$ is true similarly as in *Case* 1.

We are done with the proof of Claim 2.1.8.

□

**Claim 2.1.9.** If $g^{(k)}(x) = b$ for $x \in (a, b)$ and $k \in \omega$ then we can find

$$\mathfrak{R}^{2,l}, \mathfrak{R}^{0,l} \subseteq \mathfrak{R}$$

with the following properties:

$(A)$ $\mathfrak{R}^{2,l} \cap \mathfrak{R}^{0,l} = \emptyset$;

$(B)$ $(\mathfrak{R}^{2,l})_5 = \emptyset$;

$(C)$ if there is $d \in (a, b)$ with $(a, d) \in \mathfrak{R}$ then $\mathfrak{R}^{0,l} = \emptyset$, and
if $\mathfrak{R}^{0,l} \neq \emptyset$ then $\mathfrak{R}^{0,l} \in T(>, .)$;

$(D)$ $\cup\mathfrak{R}^{2,l} \bigcap \cap\mathfrak{R}^{0,l} \neq \emptyset$;

$(E)$ $(\cup\mathfrak{B}) \cup (\cup\mathfrak{R}^{2,l}) \supseteq (a, x]$ for all $\mathfrak{B} \in [\mathfrak{R}^{0,l}]^\omega$;

$(F)$ $|\{Q \in \mathfrak{R}^{2,l} : Q \subseteq R\}| \leq 2$ for all $R \in \mathfrak{R}$.

*Proof of Claim 2.1.9.* The proof is similar to the proof of Claim 2.1.8 and left to the reader.

□

Let's continue the proof of Lemma 2.1.5 by choosing arbitrary $x \in (a,b)$. Use Claim 2.1.6 first and then Claim 2.1.7 or Claim 2.1.8 together with Claim 2.1.9 to choose $\mathfrak{R}^{2,r}$, $\mathfrak{R}^{1,r}$, $\mathfrak{R}^{2,l}$ and $\mathfrak{R}^{0,l}$. Let

$$\mathfrak{R}^2 = \mathfrak{R}^{2,l} \cup \mathfrak{R}^{2,r},$$

and let

$$\mathfrak{R}^0 \in [\mathfrak{R}^{0,l}]^\omega, \mathfrak{R}^1 \in [\mathfrak{R}^{1,r}]^\omega$$

with $\mathfrak{R}^0 \cap \mathfrak{R}^1 = \emptyset$ if $|\mathfrak{R}^{0,l}| = |\mathfrak{R}^{1,r}| = \omega$ and let $\mathfrak{R}^0 = \mathfrak{R}^{0,l}, \mathfrak{R}^1 = \mathfrak{R}^{1,r}$ otherwise.

Now we want to prove that $(i) - (vi)$ of Lemma 2.1.5 are satisfied:

- $(i), (ii)$ of Lemma 2.1.5 are satisfied by Claim 2.1.8 $(C)$, Claim 2.1.9 $(C)$ and the fact that if $\mathfrak{A}$ is in $T(.,.)$ then any subset of cardinality $\omega$ is in the same class,
- $(iii)$ of Lemma 2.1.5 is true by Claim 2.1.8 $(B)$, $(D)$ and Claim 2.1.9 $(B)$, $(D)$ and Claim 2.1.7,
- $(iv)$ of Lemma 2.1.5 is true by Claim 2.1.8 $(D)$ and Claim 2.1.9 $(D)$,
- $(v)$ of Lemma 2.1.5 is true by Claim 2.1.8 $(E)$ and Claim 2.1.9 $(E)$,
- $(vi)$ of Lemma 2.1.5 is true by Claim 2.1.8 $(F)$ and Claim 2.1.9 $(F)$.

We are done with the proof of Lemma 2.1.5.

□

**Lemma 2.1.10.** *Let $\mathfrak{Q}, \mathfrak{Q}_n \in [\mathfrak{I}]^{\leq\omega}$ for $n \in \omega$. We can find $\mathfrak{K} \subseteq \mathfrak{Q}$ satisfying the following properties:*

(1) $\cup\mathfrak{K} = \mathfrak{Q}$ *and* $(\mathfrak{Q} \setminus \mathfrak{K})_\omega = (\mathfrak{Q})_\omega$*;*

(2) $(\mathfrak{Q}_n \setminus \mathfrak{K})_\omega = (\mathfrak{Q}_n)_\omega$ *for all* $n \in \omega$*.*

*Proof.* Let $\mathcal{D} = \{(a_k, b_k) : k \in |\mathcal{D}|\}$ be the set of components of $\cup\mathfrak{Q}$. For $j \in 3, k \in |\mathcal{D}|$ let $\mathfrak{T}_k^j$ be the set provided by Lemma 2.1.5 for $\{Q \in \mathfrak{Q} : Q \subseteq (a_k, b_k)\}$. For $k \in |\mathcal{D}|$, $j \in 2$ we know that $\mathfrak{T}_k^j \in T(.,.)$, so using Claim 2.1.3 we can find $\mathfrak{L}_k^j \subseteq \mathfrak{T}_k^j$ with:

(1) $(\mathfrak{Q} \setminus \mathfrak{L}_k^j)_\omega = (\mathfrak{Q})_\omega$ and $(\mathfrak{Q}_n \setminus \mathfrak{L}_k^j)_\omega = (\mathfrak{Q}_n)_\omega$ for $n \in \omega$;

(2) $\cup_{k\in|\mathcal{D}|,j\in 2}\ \mathfrak{L}_k^j \bigcup \cup_{k\in|\mathcal{D}|}\mathfrak{T}_k^2 = \cup\mathfrak{Q}$.

Note that since for $k \neq l \in |\mathcal{D}|$, $j \in 2$ $(\cup\mathfrak{L}_k^j) \cap (\cup\mathfrak{L}_l^j) = \emptyset$ and $(\mathfrak{T}_k^2)_{10} = \emptyset$ for $k \in |\mathcal{D}|$

$$\cup_{k\in|\mathcal{D}|,j\in 2}\mathfrak{L}_k^j \bigcup \cup_{k\in|\mathcal{D}|}\mathfrak{T}_k^2 = \mathfrak{K}$$

fulfills the requirements of the statement.

$\square$

**Corollary 2.1.11.** *For $\mathfrak{R} \in [\mathfrak{I}]^{\leq\omega}$ there exist $\mathfrak{R}_i \subseteq \mathfrak{R}$ pairwise disjoint for $i \in \omega$ with:*

*(i)* $\cup\mathfrak{R} = \cup\mathfrak{R}_0$;

*(ii)* $(\mathfrak{R})_\omega = (\mathfrak{R}_i)_\omega$ *for* $i > 0$.

*Proof.* By induction on $j$ choose $\mathfrak{R}_j \subseteq \mathfrak{R} \setminus \cup_{i<j}\mathfrak{R}_i$ satisfying $\cup\mathfrak{R}_j = \cup(\mathfrak{R} \setminus \cup_{i<j}\mathfrak{R}_i)$ and $((\mathfrak{R})_\omega =)(\mathfrak{R} \setminus \cup_{i<j}\mathfrak{R}_i)_\omega = (\mathfrak{R} \setminus \cup_{i\leq j}\mathfrak{R}_i)_\omega$ using Lemma 2.1.10.

$\square$

### 2.1.2 Proof of Theorem 1.3.2

*Proof of Theorem 1.3.2.* Let $\mathcal{A} = \{\mathfrak{R}_n : n \in \omega\}$ and let $\{\mathfrak{R}'_n : n \in \omega\}$ be an enumeration of $\mathcal{A}$ satisfying $|\{n : \mathfrak{R}'_n = A\}| = \omega$ for all $A \in \mathcal{A}$. In the $jth$ step we define sets $\mathfrak{K}_j$ by applying Lemma 2.1.10 for

$$(\mathfrak{R}'_j \setminus \bigcup_{i<j}\mathfrak{K}_i) = \mathfrak{Q} \text{ and } (\mathfrak{R}'_n \setminus \bigcup_{i<j}\mathfrak{K}_i) = \mathfrak{Q}_n\ (n \in \omega).$$

Let

$$c(K) = \begin{cases} l & \text{if } K \in \mathfrak{K}_j \text{ and } \mathfrak{R}'_j \text{ is the } l\text{th appearance of some } \mathfrak{R}_n, \\ 0 & \text{otherwise.} \end{cases}$$

This coloring proves the theorem.

We are done with the proof of Theorem 1.3.2.

$\square$

### 2.1.3 Proof of Theorem 1.3.4

**Claim 2.1.12.** If $\mathfrak{R} \in [\mathfrak{I}]^\omega$ and $(\mathfrak{R})_\omega = \emptyset$ then we can find $\mathfrak{R}^+ \in [\mathfrak{R}]^\omega$ that is either disjoint ($R \cap Q = \emptyset$ for all different $R, Q \in \mathfrak{R}^+$) or nested ($R \subseteq Q$ or $Q \subseteq R$ for all $R, Q \in \mathfrak{R}^+$).

*Proof.* Let $\{R_t : t \in \omega\}$ be an enumeration of $\mathfrak{R}$. Let $J_{R_0} = \{t \in \omega : \partial R_0 \cap R_t \neq \emptyset\}$. $|J_{R_0}| < \omega$ since $(\mathfrak{R})_\omega = \emptyset$. Let $t_1 = min\ \{\omega \setminus (J_{R_0} \cup \{0\})\}$. Define $J_{R_{t_1}}$ similarly and continue this process in $\omega$ steps.

Let $\mathfrak{R}^- = \{R_{t_s} : s \in \omega\}$. $\mathfrak{R}^- \in [\mathfrak{I}]^\omega$ and by our choice for all $A, B \in \mathfrak{R}^-$ the following is true: $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \emptyset$. Then by $\omega \to (\omega)^2_2$ we are done with the proof of Claim 2.1.12.

$\square$

**Definition 2.1.13.** For $\mathfrak{R} \in [\mathfrak{I}]^\omega$ let

$$(\mathfrak{R})' = \begin{cases} \{R \in \mathfrak{R} : R \cap (\mathfrak{R})_\omega \neq \emptyset\} & \text{if } (\mathfrak{R})_\omega \neq \emptyset, \\ \mathfrak{R} & \text{otherwise.} \end{cases}$$

We mention the following two claims without proof:

**Claim 2.1.14.** $|(\mathfrak{R})'| = \omega$ and $((\mathfrak{R})')_\omega = (\mathfrak{R})_\omega$ for all $\mathfrak{R} \in [\mathfrak{I}]^\omega$.

**Claim 2.1.15.** $\mathfrak{R}_{p,q,\varepsilon} \setminus \mathfrak{K} = \mathfrak{R}_{p,q,\varepsilon} \setminus \mathfrak{K}_{p,q,\varepsilon}$ for all $\mathfrak{R} \in [\mathfrak{I}]^{\leq\omega}$, $\mathfrak{K} \in [\mathfrak{I}]^{\leq\omega}$, $p, q \in D$, $\varepsilon \in 4$ with $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^\omega$ satisfying $D \cap \cup\{\partial R : R \in \mathfrak{R} \cup \mathfrak{K}\} = \emptyset$.

**Notation.**

For $\mathfrak{R} \in [\mathfrak{I}]^\omega$ and $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^\omega$ with $D \cap \cup\{\partial R : R \in \mathfrak{R}\} = \emptyset$ we will use the following notation:

$$\mathcal{A}(\mathfrak{R}, D) := \{(\mathfrak{R}_{p,q,\varepsilon})' : p, q \in D, \varepsilon \in 4, |\mathfrak{R}_{p,q,\varepsilon}| = \omega\}.$$

**Lemma 2.1.16.** *Let $\mathfrak{R} \in [\mathfrak{I}]^\omega$, $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^\omega$ with $D \cap \cup\{\partial R : R \in \mathfrak{R}\} = \emptyset$. Let $\mathfrak{L} \in \mathcal{A}(\mathfrak{R}, D)$ and $\{\mathfrak{R}_i : i \in \omega\} \subseteq \mathcal{A}(\mathfrak{R}, D)$. Then we can choose $\mathfrak{K} \subseteq \mathfrak{L}$ with $(\mathfrak{L})_\omega \subseteq \cup\mathfrak{K}$, and $|\mathfrak{K}| = \omega$ if $(\mathfrak{L})_\omega = \emptyset$, such that the following statements hold for all $i \in \omega$:*

(i) $|\mathfrak{R}_i \setminus \mathfrak{K}| = \omega$;

(ii) $(\mathfrak{R}_i \setminus \mathfrak{K})_\omega = (\mathfrak{R}_i)_\omega$.

*Proof of Lemma 2.1.16.*

$\cup\mathfrak{L}$ is an open subset of $\mathbf{R}$ so is the union of countably many open intervals. Let us denote by $\mathcal{C}$ the set of components of $\cup\mathfrak{L}$ and let $\gamma = |\mathcal{C}|$. Fix $\{C_k : k \in \gamma\}$, a 1-1 enumeration of the components. Then for $m \in 3, k \in \gamma$ let us denote by $\mathfrak{L}_k^m$ the subset of $\{R \in \mathfrak{L} : R \subseteq C_k\}$ indexed by $m$ in Lemma 2.1.5.

(An 'index dictionary' for the proof: $m \in 3$ will always refer to subsets that come from Lemma 2.1.5. $i \in \omega$ refers to the enumeration $\{\mathfrak{R}_i : i \in \omega\}$ and $k$ denotes the index of a component of $\cup\mathfrak{L}$.)

*Case* 1: $(\mathfrak{L})_\omega \neq \emptyset$

**Claim 2.1.17.** For $m \in 2, i \in \omega, k \in \gamma$ there are

$$\mathfrak{Q}_{k,i}^m \subseteq \mathfrak{L}_k^m \cap \mathfrak{R}_i \text{ and } \mathfrak{T}_k^m \subseteq \mathfrak{L}_k^m$$

satisfying:

(1) if $(m, i, k) \neq (m', i', k')$ then $\mathfrak{Q}_{k,i}^m \cap \mathfrak{Q}_{k',i'}^{m'} = \emptyset$;

(2) $\mathfrak{T}_k^m \cap \mathfrak{Q}_{k,i}^m = \emptyset$;

(3) $(\mathfrak{Q}_{k,i}^m)_\omega = (\mathfrak{T}_k^m \cap \mathfrak{R}_i)_\omega$;

(4) $(\mathfrak{T}_k^m)_\omega = (\mathfrak{L}_k^m)_\omega$.

*Proof of Claim 2.1.17.*

$\mathfrak{L}_k^m \in T(.,.)$ or empty for all $m \in 2$, $k \in \gamma$, and $\mathfrak{L}_k^m \cap \mathfrak{L}_{k'}^{m'} = \emptyset$ for all $(m, k) \neq (m', k')$ $(m, m' \in 2, k, k' \in \gamma)$, so by the fact that we can choose disjoint infinite subsets of countable many infinite subsets of $\omega$ and by Claim 2.1.3 we are done with the proof of Claim 2.1.17.

□

Now put

$$\mathfrak{K} = \bigcup_{m\in 2, k\in\gamma} \mathfrak{T}_k^m \cup \bigcup_{k\in\gamma} \mathfrak{L}_k^2.$$

and prove that $\mathfrak{K}$ fulfills the requirements of Lemma 2.1.16:

• the proof of $(\mathfrak{L})_\omega \subseteq \cup\mathfrak{K}$:

We know that $\cup\mathfrak{K} = \cup\mathfrak{L}$, since $\cup(\cup_{m\in 2}\mathfrak{T}_k^m \cup \mathfrak{L}_k^2) = \cup(\cup_{m\in 3}\mathfrak{L}_k^m)$ for all $k \in \gamma$ by $(v)$ of Lemma 2.1.5. As $(\mathfrak{L})_\omega \subseteq \cup\mathfrak{L}$, we are done with the proof of $(\mathfrak{L})_\omega \subseteq \cup\mathfrak{K}$.

• the proof of $(ii)$ of Lemma 2.1.16:

$(\mathfrak{R}_i \setminus \mathfrak{K})_\omega \subseteq (\mathfrak{R}_i)_\omega$ is obvious.

We prove the other direction by contradiction.

Assume that $x \in (\mathfrak{R}_i)_\omega \setminus (\mathfrak{R}_i \setminus \mathfrak{K})_\omega$. Clearly then $x \in (\mathfrak{R}_i \cap \mathfrak{K})_\omega$. By the structure of $\mathfrak{K}$ we know that there exists $k \in \gamma$ and $m \in 2$ with $x \in (\mathfrak{R}_i \cap \mathfrak{T}_k^m)_\omega$. We also know that

$\circ_1$ $(\mathfrak{R}_i \cap \mathfrak{T}_k^m)_\omega \subseteq (\mathfrak{R}_i \cap \mathfrak{L}_k^m)_\omega$ by the definition of $\mathfrak{T}_k^m$,

$\circ_2$ $(\mathfrak{R}_i \cap \mathfrak{L}_k^m)_\omega = (\mathfrak{Q}_{k,i}^m)_\omega$ by Claim 2.1.17 (3),

$\circ_3$ $(\mathfrak{Q}_{k,i}^m)_\omega \subseteq (\mathfrak{R}_i \setminus \mathfrak{K})_\omega$ by the definition of $\mathfrak{Q}_{k,i}^m$ and Claim 2.1.17 (2).

So $x \in (\mathfrak{R}_i \setminus \mathfrak{K})_\omega$ which is a contradiction.

We are done with the proof of $(ii)$ of Lemma 2.1.16.

• the proof of $(i)$ of Lemma 2.1.16:

We prove by contradiction. Suppose $|\mathfrak{R}_i \setminus \mathfrak{K}| < \omega$ for some $i \in \omega$.

*Case* A: $(\mathfrak{R}_i)_\omega \neq \emptyset$.

If $|\mathfrak{R}_i \setminus \mathfrak{K}| < \omega$ then there are $m \in 2$ and $k \in \gamma$ with $(\mathfrak{R}_i \cap \mathfrak{T}_k^m)_\omega \neq \emptyset$. Using $\circ_1$ and $\circ_2$ above we know that $(\mathfrak{R}_i \cap \mathfrak{T}_k^m)_\omega \subseteq (\mathfrak{R}_i \cap \mathfrak{L}_k^m)_\omega = (\mathfrak{Q}_{k,i}^m)_\omega \neq \emptyset$ so $|\mathfrak{Q}_{k,i}^m| = \omega$. But $\mathfrak{K} \cap \mathfrak{Q}_i^{m,k} = \emptyset$ by the definition of $\mathfrak{K}$ implying $|\mathfrak{R}_i \setminus \mathfrak{K}| = \omega$.

*Case* B: $(\mathfrak{R}_i)_\omega = \emptyset$.

Apply Claim 2.1.12 to $\mathfrak{R}_i$ to obtain $\mathfrak{R}_i^+$ that is nested or disjoint.

*Subcase* B1:

$\mathfrak{R}_i^+ \in [\mathfrak{R}_i]^\omega$ is nested.

In this subcase $|\mathfrak{R}_i^+ \setminus \cup_{k\in\gamma}\mathfrak{L}_k^2| = \omega$, since $(\cup_{k\in\gamma}\mathfrak{L}_k^2)_{10} = \emptyset$ and $\mathfrak{R}_i^+$ is nested. So if $|\mathfrak{R}_i^+ \setminus \mathfrak{K}| < \omega$ then $|\mathfrak{R}_i^+ \cap \cup_{m\in 2, k\in\gamma}\mathfrak{T}_k^m| = \omega$. As $\mathfrak{R}_i^+$ is nested we know that there is $k \in \gamma$ with $\cup\mathfrak{R}_i^+ \subseteq C_k$ and the sets $\cup\bigcup_{m\in 2}\mathfrak{T}_k^m$ are pairwise disjoint for $k \in \gamma$, so there are $m \in 2$ and $k \in \gamma$ with $|\mathfrak{R}_i^+ \cap \mathfrak{T}_k^m| = \omega$. But this is impossible since $(\mathfrak{R}_i^+ \cap \mathfrak{T}_k^m)_\omega \subseteq (\mathfrak{R}_i)_\omega = \emptyset$ (as we are in *Case* B), and we also know that $\mathfrak{R}_i^+ \cap \mathfrak{T}_k^m \in [\mathfrak{L}_k^m]^\omega$ implying $(\mathfrak{R}_i^+ \cap \mathfrak{T}_k^m)_\omega \neq \emptyset$ by Claim 2.1.3. Contradiction, hence $|\mathfrak{R}_i \setminus \mathfrak{K}| = \omega$ in *Subcase* B1.

*Subcase* B2:

$\mathfrak{R}_i^+ \in [\mathfrak{R}_i]^\omega$ is disjoint.

There is $\mathfrak{X} \in [\mathfrak{R_i}^+]^\omega$ such that the order type of the left endpoints of the intervals in $\mathfrak{X}$ is either $\omega$ or $\omega^*$. By symmetry we can assume that this order type is $\omega$. As it is enough to prove that $|\mathfrak{X} \setminus \mathfrak{K}| = \omega$, arguing indirectly, we can assume that $\mathfrak{X} \subseteq \mathfrak{K}$.

Let $\{(a_n, b_n) : n \in \omega\}$ be the enumeration of $\mathfrak{X}$ such that $n < v \in \omega$ implies $b_n \leq a_v$. Since $(a_n, b_n) \in \mathfrak{L} = (\mathfrak{R}_{p_\mathfrak{L}, q_\mathfrak{L}, \varepsilon_\mathfrak{L}})'$ (with some $p_\mathfrak{L}, q_\mathfrak{L} \in D$ and $\varepsilon_\mathfrak{L} \in 4$), where $(\mathfrak{L})_\omega \neq \emptyset$, there is $x_n \in (a_n, b_n) \cap (\mathfrak{L})_\omega$ for all $n \in \omega$. So we can find $\{(y_u^n, z_u^n) : n \in \omega, u \in \omega\} \in [\mathfrak{L}]^\omega$ such that $x_n \in (y_u^n, z_u^n)$ for all $n, u \in \omega$. Since $(\mathfrak{R}_i)_\omega = \emptyset$ we may assume that $\{(y_u^n, z_u^n) : n \in \omega, u \in \omega\} \cap \mathfrak{R}_i = \emptyset$.

Let $A := sup_{n\in\omega} a_n = sup_{n\in\omega} b_n$.

**Claim 2.1.18.** For $n \in \omega \setminus \{0\}$ and $u \in \omega$ we have $(y_u^n, z_u^n) \not\subseteq (b_0, A)$ or equivalently: $y_u^n < b_0$ or $A < z_u^n$.

*Proof of Claim 2.1.18.*

By contradiction. Suppose $(y_u^n, z_u^n) \subseteq (b_0, A)$ for some $n \in \omega\setminus\{0\}$, $u \in \omega$ and $\mathfrak{R}_i = (\mathfrak{R}_{p,q,\varepsilon})'$ for some $p, q \in D$ and $\varepsilon \in 4$. Recall that $y_n^u, z_n^u \notin D$ for all $u, n \in \omega$.

(1) If $\varepsilon = 0$ then $A \leq p, A \leq q$. So if $(y^n_u, z^n_u) \subseteq (b_0, A)$ then $(y^n_u, z^n_u) \in \mathfrak{R}_i$, which is impossible.

(2) If $\varepsilon = 1$ then $p < a_0, A \leq q$. So if $(y^n_u, z^n_u) \subseteq (b_0, A)$ then $(y^n_u, z^n_u) \in \mathfrak{R}_i$, which is impossible.

(3) If $\varepsilon = 2$ then $A \leq p, q < b_0$. So if $(y^n_u, z^n_u) \subseteq (b_0, A)$ then $(y^n_u, z^n_u) \in \mathfrak{R}_i$ $(n \in \omega \setminus \{0\})$, which is impossible.

(4) If $\varepsilon = 3$ then $p < a_0$, $q < b_0$. So if $(y^n_u, z^n_u) \subseteq (b_0, A)$ then $(y^n_u, z^n_u) \in \mathfrak{R}_i$, which is impossible.

We are done with Claim 2.1.18.

□

Note that by definition $x_n \in (a_n, b_n) \cap (y^n_u, z^n_u)$ for all $n, u \in \omega$. Let $\mathfrak{C} = \{(y^n_u, z^n_u) : A < z^n_u\}$ and $\mathfrak{D} = \{(y^n_u, z^n_u) : y^n_u < b_0\}$. Note that $\mathfrak{C} \cup \mathfrak{D} \supseteq \{(y^n_u, z^n_u) : n \in \omega \setminus \{0\}, u \in \omega\}$ by Claim 2.1.18 and $A \in \cap\mathfrak{C}$, $b_0 \in \cap\mathfrak{D}$. So there are $k_1, k_2 \in \gamma$ with $\cup((\mathfrak{X} \setminus \{(a_0, b_0)\}) \cup \mathfrak{C} \cup \mathfrak{D}) \subseteq C_{k_1} \cup C_{k_2}$. Suppose $|\{I \in \mathfrak{X} \setminus \{(a_0, b_0)\} : I \subseteq C_{k_1}\}| = \omega$. Since $C_{k_1}$ is an open interval, the order type of the left endpoints of the intervals in $\mathfrak{X}$ is $\omega$ and $\mathfrak{X}$ is disjoint, there is $N \in \omega$ such that $(a_n, b_n) \subseteq C_{k_1}$ for all $n > N$. By the fact that $\cap\mathfrak{L}^0_{k_1} \neq \emptyset$ and $\cap\mathfrak{L}^1_{k_1} \neq \emptyset$ and $\mathfrak{X}$ is disjoint, we have $|(\mathfrak{L}^0_{k_1} \cup \mathfrak{L}^1_{k_1}) \cap \mathfrak{X}| \leq 2$, and $\mathfrak{X} \setminus (\mathfrak{L}^0_{k_1} \cup \mathfrak{L}^1_{k_1}) \subseteq \mathfrak{L}^2_{k_1}$ as $\mathfrak{X} \subseteq \mathfrak{K}$. Consider $S = \{(x^{N+8}_u, y^{N+8}_u) : u \in \omega\}$. Each interval in $S$ contains the point $x_{N+8}$ and either $x^{N+8}_u < b_0$ or $A < y^{N+8}_u$. By this each of the intervals in $S$ contains $(a_1, b_1), (a_2, b_2), ..., (a_7, b_7)$ or $(a_{N+9}, b_{N+9}), ...$, so at least 5 intervals contained in $\mathfrak{L}^2_{k_1}$, which is impossible by Lemma 2.1.5 $(vi)$ since the elements of $S$ are in $\mathfrak{L}$.

*Case* 2: $(\mathfrak{L})_\omega = \emptyset$.

In this case we use without proof the following claim:

**Claim 2.1.19.** We can find $\mathfrak{K} \in [\mathfrak{L}]^\omega$ and $\{\mathfrak{U}_i : i \in \omega\}$ such that for all $i \in \omega$ the following statements are true:

(1) $\mathfrak{U}_i \subseteq \mathfrak{R}_i \cap \mathfrak{L}$;

(2) $\mathfrak{U}_i \cap \mathfrak{U}_j = \emptyset$ for $i \neq j \in \omega$;

(3) $\mathfrak{K} \cap \mathfrak{U}_i = \emptyset$;

(4) If $|\mathfrak{R}_i \cap \mathfrak{L}| = \omega$ then $|\mathfrak{U}_i| = \omega$.

For the $\mathfrak{K}$ obtained in Claim 2.1.19, $(i)$ of Lemma 2.1.16 is the only not trivially satisfied requirement:

if $(\mathfrak{R}_i)_\omega = \emptyset$ then if $|\mathfrak{R}_i \cap \mathfrak{L}| = \omega$, $\mathfrak{U}_i \in [\mathfrak{R}_i]^\omega$ and $|\mathfrak{U}_i \cap \mathfrak{K}| = \emptyset$.

We are done with the proof of Lemma 2.1.16.

□

We prove a claim, that we will use in the proof of Lemma 2.1.21.

**Claim 2.1.20.** Let $\mathfrak{Q} \in [\mathfrak{I}]^\omega$ and $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^\omega$ with $D \cap \cup\{\partial R : R \in \mathfrak{Q}\} = \emptyset$. Suppose $p, q \in D$ and $\varepsilon \in 4$ satisfy $(\mathfrak{Q}_{p,q,\varepsilon})' \in \mathcal{A}(\mathfrak{Q}, D)$. Let $\mathfrak{K} \subseteq \mathfrak{Q}$ be such that $|(\mathfrak{Q}_{p,q,\varepsilon})' \setminus \mathfrak{K}| = \omega$ and $((\mathfrak{Q}_{p,q,\varepsilon})' \setminus \mathfrak{K})_\omega = ((\mathfrak{Q}_{p,q,\varepsilon})')_\omega$. Then $((\mathfrak{Q} \setminus \mathfrak{K})_{p,q,\varepsilon})' \in \mathcal{A}(\mathfrak{Q} \setminus \mathfrak{K}, D)$.

*Proof.* We have to prove that $|(\mathfrak{Q} \setminus \mathfrak{K})_{p,q,\varepsilon}| = \omega$. We will use the fact that $\mathfrak{Q}_{p,q,\varepsilon} \setminus \mathfrak{K}_{p,q,\varepsilon} = (\mathfrak{Q} \setminus \mathfrak{K})_{p,q,\varepsilon}$ without mentioning.

If $((\mathfrak{Q}_{p,q,\varepsilon})')_\omega \neq \emptyset$, then using the assumption of the claim we have

$$\emptyset \neq ((\mathfrak{Q}_{p,q,\varepsilon})')_\omega = ((\mathfrak{Q}_{p,q,\varepsilon})' \setminus \mathfrak{K})_\omega \subseteq (\mathfrak{Q}_{p,q,\varepsilon} \setminus \mathfrak{K}_{p,q,\varepsilon})_\omega = ((\mathfrak{Q} \setminus \mathfrak{K})_{p,q,\varepsilon})_\omega$$

So in this case we are done.

If $((\mathfrak{Q}_{p,q,\varepsilon})')_\omega = \emptyset$, then $(\mathfrak{Q}_{p,q,\varepsilon})' = \mathfrak{Q}_{p,q,\varepsilon}$. So by the assumptions we know that

$$\omega = |(\mathfrak{Q}_{p,q,\varepsilon})' \setminus \mathfrak{K}| = |\mathfrak{Q}_{p,q,\varepsilon} \setminus \mathfrak{K}| \leq |\mathfrak{Q}_{p,q,\varepsilon} \setminus \mathfrak{K}_{p,q,\varepsilon}| = |(\mathfrak{Q} \setminus \mathfrak{K})_{p,q,\varepsilon}|.$$

So we are done with the proof of Claim 2.1.20.

□

**Lemma 2.1.21.** *Let* $\mathfrak{R} \in [\mathfrak{I}]^\omega$ *and* $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^\omega$ *with* $D \cap \cup\{\partial R : R \in \mathfrak{R}\} = \emptyset$. *Let* $\{\mathfrak{R}_i : i \in \omega\}$ *be an* $\omega$*-abundant enumeration of* $\mathcal{A}(\mathfrak{R}, D)$. *Then there exists* $\{\mathfrak{K}_i : i \in \omega\}$ *pairwise disjoint such that for all* $i, s \in \omega$ *the following statements hold:*

*(i)* $\mathfrak{K}_i \subseteq \mathfrak{R}_i$*;*

*(ii) if* $(\mathfrak{R}_i)_\omega = \emptyset$ *then* $|\mathfrak{K}_i| = \omega$*;*

*(iii)* $(\mathfrak{R}_i)_\omega \subseteq \cup\mathfrak{K}_i$*;*

*(iv)* $|\mathfrak{R}_s \setminus \cup_{l \le i}\mathfrak{K}_l| = \omega$ *and* $(\mathfrak{R}_s \setminus \cup_{l \le i}\mathfrak{K}_l)_\omega = (\mathfrak{R}_s)_\omega$.

*Proof.* For all $i \in \omega$ let $p(i), q(i) \in D$ and $\varepsilon(i) \in 4$ be such that $\mathfrak{R}_i = (\mathfrak{R}_{p(i),q(i),\varepsilon(i)})'$.

We choose $\mathfrak{K}_j$ by induction on $j \in \omega$:

Assume we have defined $\{\mathfrak{K}_l : l < j\}$ so that $(i) - (iv)$ of Lemma 2.1.21 are satisfied for all $s \in \omega$ and $i < j$. Let us use the following notation:

- $\mathfrak{R}^{(j)} = \mathfrak{R} \setminus \cup_{l<j}\mathfrak{K}_l$.

By the fact that $(iv)$ of Lemma 2.1.21 is satisfied with $i = j - 1$, $s \in \omega$ and Claim 2.1.20 we have $\{((\mathfrak{R}^{(j)})_{p(i),q(i),\varepsilon(i)})' : i \in \omega\} \subseteq \mathcal{A}(\mathfrak{R}^{(j)}, D)$.

Now we apply Lemma 2.1.16 with $\mathfrak{R}^{(j)}$, $\mathfrak{L} = ((\mathfrak{R}^{(j)})_{p(j),q(j),\varepsilon(j)})'$ and $\{((\mathfrak{R}^{(j)})_{p(i),q(i),\varepsilon(i)})' : i \in \omega\}$. Let $\mathfrak{K}_j$ be the $\mathfrak{K}$ provided by Lemma 2.1.16 with these settings.

We prove that $(i) - (iv)$ of Lemma 2.1.21 hold for $j$ and $s \in \omega$:

$(i)$ and $(ii)$ are trivially satisfied. We know that $(\mathfrak{R}_j)_\omega = ((\mathfrak{R}^{(j)})_{p(j),q(j),\varepsilon(j)})_\omega$ holds by $(iv)$ for $j - 1$, $((\mathfrak{R}^{(j)})_{p(j),q(j),\varepsilon(j)})_\omega = (((\mathfrak{R}^{(j)})_{p(j),q(j),\varepsilon(j)})')_\omega$ holds by Claim 2.1.14 and that $(((\mathfrak{R}^{(j)})_{p(j),q(j),\varepsilon(j)})')_\omega \subseteq \cup\mathfrak{K}_j$ holds by Lemma 2.1.16. By these we have that $(iii)$ is satisfied for $j$.

Finally by Lemma 2.1.16 we have $\omega = |((\mathfrak{R}^{(j)})_{p(s),q(s),\varepsilon(s)})' \setminus \mathfrak{K}_j| \le |(\mathfrak{R}^{(j)})_{p(s),q(s),\varepsilon(s)} \setminus \mathfrak{K}_j| = |\mathfrak{R}_s \setminus \cup_{l \le j}\mathfrak{K}_l|$ and $(((\mathfrak{R}^{(j)})_{p(s),q(s),\varepsilon(s)})' \setminus \mathfrak{K})_\omega = (((\mathfrak{R}^{(j)})_{p(s),q(s),\varepsilon(s)})')_\omega = ((\mathfrak{R}^{(j)})_{p(s),q(s),\varepsilon(s)})_\omega = (\mathfrak{R}_s \setminus \cup_{l<j}\mathfrak{K}_l)_\omega = (\mathfrak{R}_s)_\omega$. The last equality holds by $(iv)$ for $s$ and $i = j - 1$. So we have $(iv)$ for $j$.

So as $\{\mathfrak{K}_j : j \in \omega\}$ are pairwise disjoint by the construction, we are done with the proof of Lemma 2.1.21.

□

*Proof of Theorem 1.3.4.* We use the Lemma 2.1.21 to construct such colorings. Let $\{\mathfrak{R}_i : i \in \omega\}$ be an $\omega$-abundant enumeration of $\{(\mathfrak{R}_{p,q,\varepsilon})' : p, q \in D,\ \varepsilon \in 4, |(\mathfrak{R}_{p,q,\varepsilon})'| = \omega\}$ and $\{\mathfrak{K}_j : j \in \omega\}$ be the sets provided in Lemma 2.1.21.

Let:

$$c(K) = \begin{cases} l & \text{if } K \in \mathfrak{K}_j, (\mathfrak{R}_j)_\omega \neq \emptyset \text{ and} \\ & \mathfrak{R}_j \text{ is the } l\text{th appearance of } (\mathfrak{R}_{p,q,\varepsilon})' \text{ in } \{\mathfrak{R}_i : i \in \omega\} \text{ for } p,q \in D, \varepsilon \in 4, \\ 0 & \text{otherwise.} \end{cases}$$

Now Theorem 1.3.4 $(i)$ holds by Lemma 2.1.21 $(iii)$, and

Theorem 1.3.4 $(ii)$ holds by Lemma 2.1.21 $(ii)$.

We are done with the proof of Theorem 1.3.4.

□

As we would like to apply a theorem similar to Theorem 1.3.4 for bijective images of $\mathbf{R} \cup \{-\infty, +\infty\}$ we state it as a corollary:

**Corollary 2.1.22.**

*Let $\varphi : \mathbf{R} \cup \{-\infty, +\infty\} \to X$ be a bijection, $E \in [X]^\omega$ and $\mathfrak{Q} \subseteq \mathcal{P}(X)$ such that $\varphi^{-1}(\mathfrak{Q}) \in [\mathfrak{J}]^{\leq\omega}$ and $\varphi^{-1}(E) \cap \cup\{\partial R : R \in \varphi^{-1}(\mathfrak{Q})\} = \emptyset$ and let $\mathfrak{Q}_{p,q,\varepsilon} = \varphi^{-1}(\mathfrak{U})_{\varphi^{-1}(p),\varphi^{-1}(q),\varepsilon}$ for all $p, q \in E$ and $\varepsilon \in 4$.*

*There is $c : \mathfrak{Q} \to \omega$ such that the following statements hold for all $p, q \in E$ and $\varepsilon \in 4$:*

*$(i)$ if $|\mathfrak{Q}_{p,q,\varepsilon}| = \omega$ then $|(c)^{-1}(\{0\}) \cap \mathfrak{Q}_{p,q,\varepsilon}| = \omega$;*

*$(ii)$ $\bigcap_{j\in\omega}((c)^{-1}(\{j\}) \cap \mathfrak{Q}_{p,q,\varepsilon})_\omega = (\mathfrak{Q}_{p,q,\varepsilon})_\omega$.*

## 2.2 Proof of Lemma 1.3.6

*Proof of Lemma 1.3.6.* Let $J \subseteq \omega$ be such that there is

$$x \in \bigcap_{n \in J} (a_n, b_n) \cap \bigcap_{n \in \omega \setminus J} (\mathbf{R} \setminus (a_n, b_n)).$$

Let $J_1 = \{n \in \omega \setminus J : b_n \leq x\}$ and $J_2 = \{n \in \omega \setminus J, x \leq a_n\}$. So $\omega = J \cup^* J_1 \cup^* J_2$ and we know that:

(1) $sup_{n \in J} a_n (\leq x) \leq inf_{n \in J} b_n$,

(2) $sup_{n \in J_1} b_n (\leq x) \leq inf_{n \in J_2} a_n$,

(3) $sup_{n \in J_1} b_n (\leq x) \leq inf_{n \in J} b_n$,

(4) $sup_{n \in J} a_n (\leq x) \leq inf_{n \in J_2} a_n$.

using the conditions the same (certainly without $x$) follows for $A_n' s$ and $B_n' s$:

$(1)^{'}$ $sup_{n \in J} A_n \leq inf_{n \in J} B_n$ is true since otherwise there would be $n (\neq) m \in J$ such that $B_n < A_m$. But then by $(ii)$ and $(iii)$ $b_n < a_m$ would be true contradicting (1).

$(2)^{'}$, $(3)^{'}$, $(4)^{'}$ are true similarly.

Let $m = max\{sup_{n \in J_1} B_n, sup_{n \in J} A_n\}$ and $M = min\{inf_{n \in J} B_n, inf_{n \in J_2} A_n\}$

By $(1)^{'} - (4)^{'}$ we know that $m \leq M$ and let $y \in [m, M]$ arbitrary.

Let $I = \{n \in \omega : A_n = y \text{ or } B_n = y\}$. By $(i)$ $|I| \leq 2$. Let $J^{'} = J \setminus I$.

Then

$$y \in \bigcap_{n \in J^{'}} (A_n, B_n) \cap \bigcap_{n \in \omega \setminus J} (\mathbf{R} \setminus (A_n, B_n)).$$

□

Let $\varphi : \mathbf{R} \cup \{-\infty, +\infty\} \to X$ arbitrary bijection. Let us define $<_\varphi$ on $X$ in the following way. For $x, y \in X$ let

$$x <_\varphi y \textit{ iff } \varphi^{-1}(x) < \varphi^{-1}(y).$$

The following lemma is an immediate consequence of Lemma 1.3.6.

**Corollary 2.2.1.** *Let* $\varphi_1 : \mathbf{R} \cup \{-\infty, +\infty\} \to X$*,* $\varphi_2 : \mathbf{R} \cup \{-\infty, +\infty\} \to Y$ *be arbitrary bijections. Suppose that* $\{a_n, b_n : n \in \omega\} \subseteq X$*,* $\{A_n, B_n : n \in \omega\} \subseteq Y$ *are satisfying:*

*(i)* $a_n \neq a_m$*,* $a_n <_{\varphi_1} b_n \neq b_m$*,* $A_n \neq A_m$*,* $A_n <_{\varphi_2} B_n \neq B_m$ *for all* $n, m \in \omega$ *different;*

*(ii)* *(*$a_n <_{\varphi_1} a_m <_{\varphi_1} b_m <_{\varphi_1} b_n$ *or* $A_n <_{\varphi_2} A_m <_{\varphi_2} B_m <_{\varphi_2} B_n$*) is false for all*

$n, m \in \omega$*;*

*(iii)* $a_n <_{\varphi_1} a_m \leq_{\varphi_1} b_n <_{\varphi_1} b_m$ *iff* $A_n <_{\varphi_2} A_m \leq_{\varphi_2} B_n <_{\varphi_2} B_m$ *for all* $n, m \in \omega$*;*

*(iv)* $a_n <_{\varphi_1} a_m$ *iff* $A_n < A_m$ *for all* $n, m \in \omega$*.*

*Suppose that*

$$\bigcap_{n \in J} (a_n, b_n)_{<_{\varphi_1}} \cap \bigcap_{n \in \omega \setminus J} (X \setminus (a_n, b_n))_{<_{\varphi_1}} \neq \emptyset$$

*for some* $J \subseteq \omega$*.*

*Then there exists* $J \supseteq J'$ *with* $|J \setminus J'| < 3$ *with*

$$\bigcap_{n \in J'} (A_n, B_n)_{<_{\varphi_2}} \cap \bigcap_{n \in \omega \setminus J} (Y \setminus (A_n, B_n)_{<_{\varphi_2}}) \neq \emptyset.$$

# 3 The proof of the two dimensional results

In this chapter we prove Theorem 1.3.7. We fix for the rest of this chapter

$$\mathfrak{S} \in [\mathcal{T}_E]^\omega$$

with $\mathbf{o} \in int(\cap\mathfrak{S})$ containing an open square $T$ around the origin of side length $\varepsilon$. We denote the elements of $\mathfrak{S}$ by $S$.

## 3.1 Notation

Let us introduce some more notation.

• We will work with subsets of $\mathbf{R}^2$, and use boldface letters to denote points of $\mathbf{R}^2$, e.g. $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$. For $\lambda \in \mathbf{R}$ and $(x_1, x_2) = \mathbf{x} \in \mathbf{R}^2$ let us denote $(\lambda x_1, \lambda x_2) \in \mathbf{R}^2$ by $\lambda\mathbf{x}$. The closed (open, half closed) segment between $\mathbf{x}$ and $\mathbf{y}$ ($\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$) will be denoted by $[\mathbf{x}, \mathbf{y}]$ ($(\mathbf{x}, \mathbf{y})$, $[\mathbf{x}, \mathbf{y})$, $(\mathbf{x}, \mathbf{y}]$ respectively).

• The vertices of $E$ will be denoted by

$$(0,1) = \mathbf{v_1}(E),\ (1,1) = \mathbf{v_2}(E),\ (1,0) = \mathbf{v_3}(E),\ (0,0) = \mathbf{v_4}(E).$$

(the labeling of the vertices is clockwise oriented and it is *mod 4* (meaning that e.g. *4+1=1*)).

For $S \in \mathcal{T}_E$ let us denote by $\mathbf{v_i}(S)$ the translate of $\mathbf{v_i}(E)$.

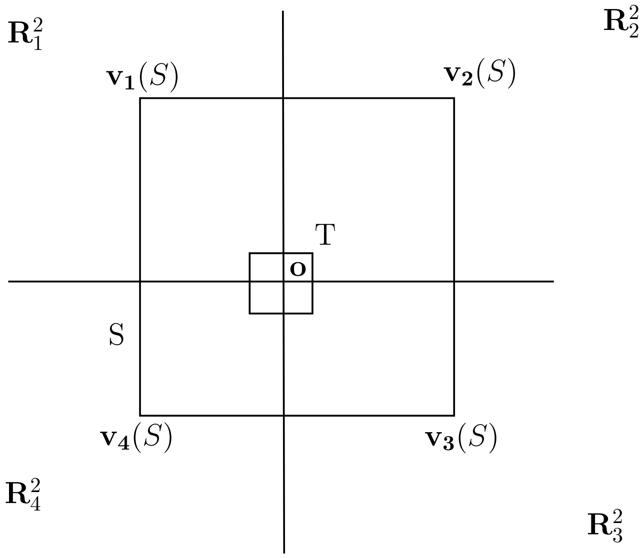


Figure 3.1: $\mathbf{o}$, $S$, $T$, $\mathbf{v_i}(S)$ and $\mathbf{R}^2_i$

- We use the following notation for the quadrants of $\mathbf{R}^2$:

$$\mathbf{R}^2_1 = \{(x,y) \in \mathbf{R}^2 : x \leq 0, 0 \leq y\}, \mathbf{R}^2_2 = \{(x,y) \in \mathbf{R}^2 : 0 \leq x, 0 \leq y\},$$

$$\mathbf{R}^2_3 = \{(x,y) \in \mathbf{R}^2 : 0 \leq x, y \leq 0\}, \mathbf{R}^2_4 = \{(x,y) \in \mathbf{R}^2 : x \leq 0, y \leq 0\}.$$

## 3.2 Limit squares

In this introductory technical part we introduce a new notion: the *limit square*. The definition and the main properties of the limit squares will help us to understand the geometry of $(\mathfrak{S})_\omega$.

**Definition 3.2.1.**

- Let $\{Q_i : i \in \omega\} \subseteq \mathcal{T}_E$. Then let

$$R = lim_{i\to\infty} Q_i \textit{ iff } \mathbf{v_4}(R) = lim_{i\to\infty} \mathbf{v_4}(Q_i) \textit{ for } R \in \mathcal{T}_E.$$

- For $\mathfrak{Q} \subseteq \mathcal{T}_E$ let $(\mathfrak{Q})_{lim} = \{R \in \mathcal{T}_E : R = lim_{i\to\infty} Q_i$ for some $\{Q_i : i \in \omega\} \in [\mathfrak{Q}]^\omega\}$, the set of *limit squares* of $\mathfrak{Q}$.

**Remark.**

1) Note that $R = lim_{i\to\infty} Q_i$ iff $(Q_i)_{i\in\omega}$ converges to $R$ in the Hausdorff metric.

2) In the sequel we will denote by $R$ the elements of $(\mathfrak{S})_{lim}$.

Let us start with some basic properties of the limit squares:

**Claim 3.2.2.** The following statements hold:

(1) $(\mathfrak{Q})_{lim} \neq \emptyset$ for $\mathfrak{Q} \in [\mathfrak{S}]^\omega$;

(2) $(\mathfrak{Q}_{lim})_{lim} = \mathfrak{Q}_{lim}$ for $\mathfrak{Q} \in [\mathfrak{S}]^\omega$;

(3) $\cap\mathfrak{S} \subseteq \cap\{\overline{R} : R \in (\mathfrak{S})_{lim}\}$;

(4) $int(\cap\mathfrak{S}) \subseteq \cap\{R : R \in (\mathfrak{S})_{lim}\}$.

*Proof.*

(1) follows by the fact that $\{\mathbf{v_4}(S) : S \in \mathfrak{Q}\}$ lies in a bounded part of the plane since each $S$ contains the origin.

(2) follows by the obvious analogue lemma for points (for the vertices).

(3) is true by an easy convergence argument.

(4) follows by (3).

□

The most important properties of limit squares of $\mathfrak{S}$ are summarized in the following theorem:

**Theorem 3.2.3.**

(1) $\cup\{R : R \in (\mathfrak{S})_{lim}\} = int\ \overline{(\mathfrak{S})_\omega}$;

(2) $\cup\{\overline{R} : R \in (\mathfrak{S})_{lim}\} = \overline{(\mathfrak{S})_\omega}$.

*Proof.* First we prove a lemma.

**Lemma 3.2.4.** *For $R \in (\mathfrak{S})_{lim}$ we have $R \subseteq (\mathfrak{S})_\omega$.*

*Proof.* Let $\mathbf{z} \in R \in (\mathfrak{S})_{lim}$. By the definition of a limit square, there is $\{Q_i : i \in \omega\} \in [\mathfrak{S}]^\omega$ with $R = lim_{i\to\infty} Q_i$. As $R$ is an open set (translate of the open unit square) there is $N(\mathbf{z})$ satisfying $\mathbf{z} \in Q_n$ for all $N(\mathbf{z}) < n$, hence $\mathbf{z} \in (\mathfrak{S})_\omega$. □

Using the fact that $R$ is open, we have the following corollary of Lemma 3.2.4, that proves $\subseteq$ in (1) and (2):

**Corollary 3.2.5.** *For* $R \in (\mathfrak{S})_{lim}$ *we have* $R \subseteq int(\mathfrak{S})_\omega$ *and* $\overline{R} \subseteq \overline{(\mathfrak{S})_\omega}$.

We prove $\supseteq$ in (2):

Choose $\mathbf{x} \in \overline{(\mathfrak{S})_\omega}$. By the definition of $\overline{(\mathfrak{S})_\omega}$ there are $\{\mathbf{x_i} : i \in \omega\} \subseteq (\mathfrak{S})_\omega$ and $\{Q_{i,j} : i, j \in \omega\} \in [\mathfrak{S}]^\omega$ satisfying $lim_{j\to\infty}\mathbf{x_i} = \mathbf{x}$ and $\mathbf{x_i} \in Q_{i,j}$ for all $i, j \in \omega$. Since $Q_{i,j}$ contains the origin for all $i, j \in \omega$, we can find $s \in \omega^\omega$ satisfying $lim_{i\to\infty}Q_{i,s(i)} = R$ with $R \in (\mathfrak{S})_{lim}$. We are done with $\supseteq$ in (2).

Now we prove a lemma.

**Lemma 3.2.6.** *If* $\mathbf{x} \in \overline{(\mathfrak{S})_\omega}$ *then* $(\mathbf{x}, \mathbf{o}] \subseteq int(\mathfrak{S})_\omega$.

*Proof.* By the above observation we know that there is $R \in (\mathfrak{S})_{lim}$ with $\mathbf{x} \in \overline{R}$. By Claim 3.2.2 (3) we have $(\mathbf{x}, \mathbf{o}] \subseteq R$, so by Corollary 3.2.5 we are done.

□

Finally the proof of $\supseteq$ in (1):

For $\mathbf{x} \in int(\mathfrak{S})_\omega$ there is $\lambda > 1$ with $\lambda\mathbf{x} \in \overline{(\mathfrak{S})_\omega}$. By Lemma 3.2.6 and the fact that $\mathbf{x} \in (\lambda\mathbf{x}, \mathbf{o}]$, we are done with $\supseteq$ in (1).

We are done with the proof of Theorem 3.2.3.

□

**Lemma 3.2.7.**

(1) $int\overline{(\mathfrak{S})_\omega} = int(\mathfrak{S})_\omega$;

(2) $\partial\overline{(\mathfrak{S})_\omega} = \partial(\mathfrak{S})_\omega$.

*Proof.* (2) is an easy consequence of (1). So let us prove (1).

$\supseteq$ is trivial, so pick any $\mathbf{x} \in int\overline{(\mathfrak{S})_\omega}$. Then there is $\lambda > 1$ with $\lambda\mathbf{x} \in \overline{(\mathfrak{S})_\omega}$. By Lemma 3.2.6 and the fact that $\mathbf{x} \in (\lambda\mathbf{x}, \mathbf{o}]$ we are done with Lemma 3.2.7. □

**Notation.** In the sequel we will denote $\partial(\mathfrak{S})_\omega(= \partial\overline{(\mathfrak{S})_\omega})$ by $\partial$.

## 3.3 Structure theorems

### 3.3.1 The structure of $\overline{(\mathfrak{S})_\omega}$

**Definition 3.3.1.**

1) We call $A \subseteq \mathbf{R}^2$ *horizontally/vertically convex*, if for all $[\mathbf{x}, \mathbf{y}]$ horizontal/vertical(resp.) segment with $\mathbf{x}, \mathbf{y} \in A$, $[\mathbf{x}, \mathbf{y}] \subseteq A$ holds.

2) We call $A \subseteq \mathbf{R}^2$ *star-like around* $\mathbf{x}$, if $\mathbf{x} \in A$ and $[\mathbf{x}, \mathbf{y}] \subseteq A$ holds for each $\mathbf{y} \in A$.

**Theorem 3.3.2.** *The following statements hold:*

*(1) If $l$ is a horizontal (vertical) line which intersects $\overline{(\mathfrak{S})_\omega}$, then $l \cap \overline{(\mathfrak{S})_\omega}$ is a horizontal (vertical, resp.) segment with length at least 1;*

*(2) $\overline{(\mathfrak{S})_\omega}$ is star-like around $\boldsymbol{o}$.*

*Proof.* The proof of (1):

We only prove that if $l$ is a horizontal line which intersects $\overline{(\mathfrak{S})_\omega}$, then $l \cap \overline{(\mathfrak{S})_\omega}$ is a horizontal segment with length at least 1. The proof of the other case is similar.

Suppose $l \cap \overline{(\mathfrak{S})_\omega} \neq \emptyset$ and let $\mathbf{x}, \mathbf{y} \in l \cap \overline{(\mathfrak{S})_\omega}$. We can choose $R^{\mathbf{x}}, R^{\mathbf{y}} \in (\mathfrak{S})_{lim}$ with $\mathbf{x} \in \overline{R^{\mathbf{x}}}$ and $\mathbf{y} \in \overline{R^{\mathbf{y}}}$ by Theorem 3.2.3 (2). We know that $\mathbf{o} \in \overline{R^{\mathbf{x}}} \cap \overline{R^{\mathbf{y}}}$ by Claim 3.2.2 (3) and we also know that the union of two intersecting translates of the closed unit square is horizontally convex. By this we have $[\mathbf{x}, \mathbf{y}] \subseteq \overline{R^{\mathbf{x}}} \cup \overline{R^{\mathbf{y}}}$. As $\overline{R^{\mathbf{x}}} \cup \overline{R^{\mathbf{y}}} \subseteq \overline{(\mathfrak{S})_\omega}$, we have that $l \cap \overline{(\mathfrak{S})_\omega}$ is a horizontal segment. As $l \cap \overline{R^{\mathbf{x}}} \subseteq l \cap \overline{(\mathfrak{S})_\omega}$, we have that the length of $l \cap \overline{(\mathfrak{S})_\omega}$ is at least 1.

The proof of (2):

Let $\mathbf{x} \in \overline{(\mathfrak{S})_\omega}$. By Theorem 3.2.3 (2) we can find $R^{\mathbf{x}} \in (\mathfrak{S})_{lim}$ with $\mathbf{x} \in \overline{R^{\mathbf{x}}}$. As $\mathbf{o} \in \overline{R^{\mathbf{x}}}$ by Claim 3.2.2 (4) and $R^{\mathbf{x}}$ is a translate of the open unit square, we have $[\mathbf{x}, \mathbf{o}] \subseteq \overline{R^{\mathbf{x}}}$. As $\overline{R^{\mathbf{x}}} \subseteq \overline{(\mathfrak{S})_\omega}$, we are done with (2).

We are done with the proof of Theorem 3.3.2.

$\square$

### 3.3.2 The structure of $\partial$

$S^1 = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\} \subseteq \mathbf{R}^2$ is the unit circle. First let us define $g : S^1 \to \mathbf{R}$ in the following way:

$$g(\mathbf{x}) := sup\{\lambda : \lambda\mathbf{x} \in \overline{(\mathfrak{S})_\omega}\}.$$

Then let $f : S^1 \to \mathbf{R}^2$ be defined by

$$f(\mathbf{x}) := g(\mathbf{x})\mathbf{x}.$$

**Theorem 3.3.3.** *$f$ is a homeomorphism between $S^1$ and $\partial$.*

*Proof.* First we prove that $\{f(\mathbf{x}) : \mathbf{x} \in S^1\} = \partial\overline{(\mathfrak{S})_\omega}$.

Note that for $\mathbf{x} \in S^1$ we have $f(\mathbf{x}) \in \partial$ by the definition of $f$, so $\{f(\mathbf{x}) : \mathbf{x} \in S^1\} \subseteq \partial$. To prove that $\{f(\mathbf{x}) : \mathbf{x} \in S^1\} \supseteq \partial$ first note that for $\mathbf{y} \in \partial$ we have $[\mathbf{o}, \mathbf{y}) \subseteq int(\mathfrak{S})_\omega$ by Lemma 3.2.6. By this fact there are no $0 < \lambda_1 \neq \lambda_2$ with $\lambda_1\mathbf{x}, \lambda_1\mathbf{x} \in \partial$, hence
$\{f(\mathbf{x}) : \mathbf{x} \in S^1\} \supseteq \partial$. We are done with $\{f(\mathbf{x}) : \mathbf{x} \in S^1\} = \partial$.

As a continuous bijection from compact to Hausdorff space is a homeomorphism, and $S^1 \subseteq \mathbf{R}^2$ is compact, $\partial \subseteq \mathbf{R}^2$ is Hausdorff, it is enough to prove that $f$ is continuous.

**Claim 3.3.4.** $f$ is continuous.

*Proof.* If $lim_{n\to\infty}\mathbf{x}_n = \mathbf{x}$ for $\mathbf{x}, \mathbf{x}_n \in S^1$, then we have $lim_{n\to\infty}f(\mathbf{x}_n) \in \partial$ since $f(\mathbf{x}_n) \in \partial$ and $\partial$ is closed. Hence we have $lim_{n\to\infty}f(\mathbf{x}_n) = f(\mathbf{x})$ by the fact that there are no $0 < \lambda_1 \neq \lambda_2$ with $\lambda_1\mathbf{x}, \lambda_1\mathbf{x} \in \partial\overline{(\mathfrak{S})_\omega}$.

□

We are done with Theorem 3.3.3.

□

**Notation**

• We know that $\partial$ is homeomorphic to $S^1$, so we can talk about the *clockwise orientation* of $\partial$. For $\mathbf{x}, \mathbf{y} \in \partial$ let us denote by

$$\partial(\mathbf{x}, \mathbf{y}), \partial[\mathbf{x}, \mathbf{y}], \partial[\mathbf{x}, \mathbf{y}), \partial(\mathbf{x}, \mathbf{y}]$$

the open, closed, half-closed *clockwise arc* of $\partial$ between $\mathbf{x}$ and $\mathbf{y}$. Note that $\partial(\mathbf{x}, \mathbf{y}), \partial[\mathbf{x}, \mathbf{y}], \partial[\mathbf{x}, \mathbf{y})$ and $\partial(\mathbf{x}, \mathbf{y}]$ are homeomorphic image of $(0,1), [0,1], [0,1)$ and $(0,1]$ respectively.

### 3.3.3 The structure of $S \cap \partial$ for $S \in \mathfrak{S}$

We divide $\mathfrak{S}$ into finitely many parts:

**Definition 3.3.5.**

(1) For $a \in \mathcal{P}(\{1,2,3,4\})$ let $\mathfrak{S}^a = \{S \in \mathfrak{S} : i \in a \Leftrightarrow \mathbf{v_i}(S) \notin \overline{(\mathfrak{S})_\omega}\}$.

(2) Let $\mathcal{E}_1 = [4]^1$;

$\mathcal{E}_{2,n} = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$ ($n$ refers to *neighbouring* vertices);

$\mathcal{E}_{2,o} = \{\{1,3\},\{2,4\}\}$ ($o$ refers to *opposite* vertices);

$\mathcal{E}_3 = [4]^3$.

**Remark.** As $S \in \mathfrak{S}$ is open, $S \subseteq int(\mathfrak{S})_\omega$ for all $S \in \mathfrak{S}^\emptyset$ and so $S \cap \partial = \emptyset$.

**Claim 3.3.6.** $\mathfrak{S}^{\{1,2,3,4\}} = \emptyset$.

*Proof.* Consider any $R \in (\mathfrak{S})_{lim}$ and $S \in \mathfrak{S}^{\{1,2,3,4\}}$. Since both $S$ and $R$ contains $T$, a vertex of $S$ must be contained in $\overline{R}$, hence $\mathfrak{S}^{\{1,2,3,4\}} = \emptyset$.

□

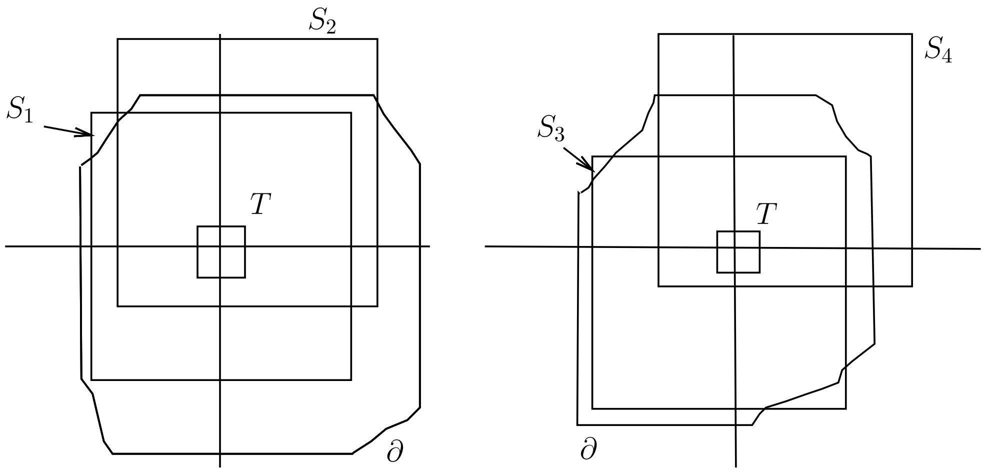


Figure 3.2: $S_1 \in \mathfrak{S}^{\{1\}}, S_2 \in \mathfrak{S}^{\{1,2\}}, S_3 \in \mathfrak{S}^{\{1,3\}}, S_4 \in \mathfrak{S}^{\{2,3,4\}}$

**Theorem 3.3.7.** *The following statements hold:*

(1) *For* $a \in \mathcal{E}_1 \cup \mathcal{E}_{2,n} \cup \mathcal{E}_3$ *and* $S \in \mathfrak{S}^a$

$$\partial \cap S := I(S)$$

*is an arc of* $\partial$*;*

(2) *For* $a \in \mathcal{E}_{2,o}$ *and* $S \in \mathfrak{S}^a$

$$\partial \cap S := I(S) \cup J(S)$$

*is the disjoint union of two arcs of* $\partial$*.*

*Proof.* We prove only (1) for $a = \{1\}$, the proof of other cases of $a \in \mathcal{E}_1 \cup \mathcal{E}_{2,n} \cup \mathcal{E}_3$ and the proof of (2) are similar.

Let $S \in \mathfrak{S}^{\{1\}}$, so $\mathbf{v}_1(S) \notin \overline{(\mathfrak{S})_\omega}$ and $\mathbf{v}_2(S), \mathbf{v}_3(S), \mathbf{v}_4(S) \in \overline{(\mathfrak{S})_\omega}$. By horizontal and vertical convexity of $\overline{(\mathfrak{S})_\omega}$ (see Theorem 3.3.2 (1)) we know that $[\mathbf{v}_2(S), \mathbf{v}_3(S)] \cup [\mathbf{v}_3(S), \mathbf{v}_4(S)] \subseteq \overline{(\mathfrak{S})_\omega}$.

We also know that the intersection of a horizontal or a vertical line with $\overline{(\mathfrak{S})_\omega}$ is (closed) segment. Let $\mathbf{x}$ be that endpoint of $l \cap \overline{(\mathfrak{S})_\omega}$, which is closer to $\mathbf{v}_1(S)$, where $[\mathbf{v}_1(S), \mathbf{v}_4(S)] \subseteq l$.

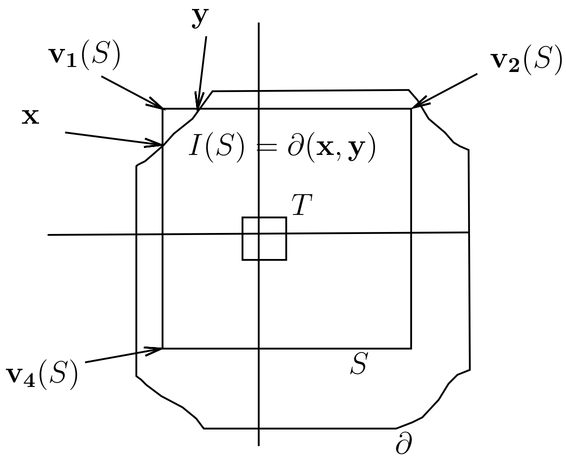


Figure 3.3: The definition of $I(S)$ for $S \in \mathfrak{S}^{\{1\}}$

And let $\mathbf{y}$ be that endpoint of $l \cap \overline{(\mathfrak{S})_\omega}$, which is closer to $\mathbf{v}_1(S)$, where $[\mathbf{v}_1(S), \mathbf{v}_2(S)] \subseteq l$.

Now we prove that $I(S) = \partial(\mathbf{x}, \mathbf{y})$:

We know that $f : S^1 \to \mathbf{R}^2$ defined by $f(\mathbf{x}) := g(\mathbf{x})\mathbf{x}$ (where $g : S^1 \to \mathbf{R}$ is defined by $g(\mathbf{x}) := sup\{\lambda : \lambda\mathbf{x} \in \overline{(\mathfrak{S})_\omega}\}$) is a homeomorphism between $S^1$ and $\partial$.

Using that $g(\mathbf{x}) > 0$ for $\mathbf{x} \in S^1$ and that $(\mathbf{x}, \mathbf{v}_1(S)] \cup [\mathbf{v}_1(S), \mathbf{y})$ is in the complement of $\overline{(\mathfrak{S})_\omega}$, we have $\partial(\mathbf{x}, \mathbf{y}) \subseteq S$. To prove the other direction, suppose there is $f(\mathbf{z}) \in S \cap \partial(\mathbf{x}, \mathbf{y})$ with some $\mathbf{z} \in S^1$. But then using that $f(\mathbf{z}) = g(\mathbf{z})\mathbf{z}$, there is $\lambda > g(\mathbf{z})$ with $\lambda\mathbf{z} \in (\mathbf{y}, \mathbf{v}_2(S)] \cup [\mathbf{v}_2(S), \mathbf{v}_3(S)] \cup [\mathbf{v}_3(S), \mathbf{v}_4(S)] \cup [\mathbf{v}_4(S), \mathbf{x})$ and also know that $\lambda\mathbf{z}$ is in the complement of $\overline{(\mathfrak{S})_\omega}$, since $\lambda > g(\mathbf{z})$. Which is a contradiction, since $(\mathbf{y}, \mathbf{v}_2(S)] \cup [\mathbf{v}_2(S), \mathbf{v}_3(S)] \cup [\mathbf{v}_3(S), \mathbf{v}_4(S)] \cup [\mathbf{v}_4(S), \mathbf{x}) \subseteq \overline{(\mathfrak{S})_\omega}$.

By this we know that $S \cap \partial = \partial(f(\mathbf{y}), f(\mathbf{x}))$.

We are done with Theorem 3.3.7.

$\square$

Now we state when does $I(S)$ determine $S$.

**Lemma 3.3.8.**

(1) *If* $a \in \mathcal{E}_1 \cup \mathcal{E}_{2,o} \cup \mathcal{E}_3$ *and* $S_1, S_2 \in \mathfrak{S}^a$ *are different, then* $I(S_1) \neq I(S_2)$.

*Proof.* First we prove the following claim:

**Claim 3.3.9.** If $\mathbf{v}_i(S) \in \overline{(\mathfrak{S})_\omega}$ and $\mathbf{v}_{i+1}(S) \notin \overline{(\mathfrak{S})_\omega}$ (or $\mathbf{v}_{i+1}(S) \in \overline{(\mathfrak{S})_\omega}$ and $\mathbf{v}_i(S) \notin \overline{(\mathfrak{S})_\omega}$) for some $S \in \mathfrak{S}$ and $i \in \{1,2,3,4\}$, then $(\mathbf{v}_i(S), \mathbf{v}_{i+1}(S)) \cap \overline{(\mathfrak{S})_\omega} \neq \emptyset$.

*Proof.* Suppose that $\mathbf{v}_i(S) \in \overline{(\mathfrak{S})_\omega}$, $\mathbf{v}_{i+1}(S) \notin \overline{(\mathfrak{S})_\omega}$ for some $S \in \mathfrak{S}$ and $i \in \{1,2,3,4\}$. Choose $R^{\mathbf{v}_i(S)} \in (\mathfrak{S})_{lim}$ with $\mathbf{v}_i(S) \in \overline{R^{\mathbf{v}_i(S)}}$ by (2) of Theorem 3.2.3. As $\mathbf{v}_i(S) \in int(\mathbf{R}_i^2)$, $\mathbf{v}_{i+1}(S) \in int(\mathbf{R}_{i+1}^2)$ (since $T \subseteq S$), $\overline{R^{\mathbf{v}_i(S)}}$ is axis-parallel and contains the origin we have $(\mathbf{v}_i(S), \mathbf{v}_{i+1}(S)) \cap \overline{(\mathfrak{S})_\omega} \neq \emptyset$. We are done.

□

The statement of the lemma follows by the fact that the endpoints of $I(S)$ determines $S$.

□

### 3.3.4 Summary

Let us briefly summarize in the following table, that we have achieved in the Structure theorems section:

| $a$ | Does $S \cap \partial$ determine $S$ for $S \in \mathfrak{S}^a$? | For $S \in \mathfrak{S}^a$ $S \cap \partial$ is ... |
|---|---|---|
| $\mathcal{E}_1 \cup \mathcal{E}_3$ | Yes. | an open arc. |
| $\mathcal{E}_{2,n}$ | No. | an open arc. |
| $\mathcal{E}_{2,o}$ | Yes. | union of two disjoint open arcs. |

## 3.4 Construction of the coloring

### 3.4.1 Notation, definitions

(1) For $a \subseteq \{1,2,3,4\}$ let

$$\partial_a = \partial \cap \cup_{i \in a} \mathbf{R}_i^2.$$

(2) We have $f : S^1 \to \partial$ homeomorphism, and let

$$f^{'} : S^1 \to \partial$$

be defined by $f^{'}((x_1, x_2)) = f((x_1, -x_2))$ for $\mathbf{x} = (x_1, x_2) \in S^1$.

(2.1.) For $a \in \mathcal{E}_1 \cup \mathcal{E}_{2,n} \cup \mathcal{E}_3$ let

$\bullet_1$ $f_a : S^1 \cap \cup_{i \in a} \mathbf{R}_i^2 \to \partial_a$ be the restriction of $f$ to $S^1 \cap \cup_{i \in a} \mathbf{R}_i^2$, and let $<_a$ be the pushforward (by $f_a$) of the clockwise ordering on $S^1 \cap \cup_{i \in a} \mathbf{R}_i^2$.

$\bullet_2$ Let $\mathcal{I}(a) = \{I(S) : S \in \mathfrak{S}^a\}$.

(2.2.) $\bullet_1$ Let $f^{'}_{\{1,3\}} : S^1 \cap \mathbf{R}_2^2 \to \partial_{\{3\}}$ be the restriction of $f^{'}$ to $S^1 \cap \mathbf{R}_2^2$;

$\bullet_2$ $\prec^{'}_{\{1,3\}}$ be the pushforward (by $f^{'}_{\{1,3\}}$) ordering on $\partial_{\{3\}}$; and

$\bullet_3$ $<_{\{1,3\}} := <_{\{1\}} \cup \prec^{'}_{\{1,3\}}$.

$\bullet_4$ Let $f^{'}_{\{2,4\}} : S^1 \cap \mathbf{R}_1^2 \to \partial_{\{4\}}$ be the restriction of $f^{'}$ to $S^1 \cap \mathbf{R}_1^2$;

$\bullet_5$ $\prec^{'}_{\{2,4\}}$ be the pushforward (by $f^{'}_{\{2,4\}}$) ordering on $\partial_{\{4\}}$; and

$\bullet_6$ $<_{\{2,4\}} := <_{\{2\}} \cup \prec^{'}_{\{2,4\}}$.

With the just introduced notations we state a lemma.

**Lemma 3.4.1.** *The following statements hold:*

$(i)_1$ *the endpoints of* $I(S)$ $(S \in \mathfrak{S}^a)$ *are in* $int(\cup_{i \in a} \mathbf{R}_i^2)$ *for* $a \in \mathcal{E}_1 \cup \mathcal{E}_3 \cup \mathcal{E}_{2,n}$,

$(i)_2$ *the endpoints of* $I(S)$ $(S \in \mathfrak{S}^a)$ *are in* $int(\mathbf{R}_1^2)$ *for* $a = \{1,3\}$ *and in* $int(\mathbf{R}_2^2)$ *for* $a = \{2,4\}$;

$(ii)$ *if* $(x_1, y_1), (x_2, y_2) \in \partial_{\{i\}}$ *and* $x_1 < x_2$, *then* $y_1 \leq y_2$ *if* $i = 1, 3$, *and* $y_1 \geq y_2$ *if* $i = 2, 4$.

*Proof.* We only prove $(i)_1$, the proof of $(i)_2$ is similar:

We know (see the proof of Theorem 3.3.7), that an endpoint of $I(S)$ is the endpoint of $[\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)] \cap \overline{(\mathfrak{S})_\omega}$, which is closer to $\mathbf{v_i}(S)$ if $\mathbf{v_i}(S) \notin \overline{(\mathfrak{S})_\omega}$ and $\mathbf{v_{i+1}}(S) \in \overline{(\mathfrak{S})_\omega}$, and the endpoint of $[\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)] \cap \overline{(\mathfrak{S})_\omega}$, which is closer to $\mathbf{v_{i+1}}(S)$ if $\mathbf{v_i}(S) \in \overline{(\mathfrak{S})_\omega}$ and $\mathbf{v_{i+1}}(S) \notin \overline{(\mathfrak{S})_\omega}$. W.l.o.g. we can assume that $\mathbf{v_i}(S) \notin \overline{(\mathfrak{S})_\omega}$ and $\mathbf{v_{i+1}}(S) \in \overline{(\mathfrak{S})_\omega}$. Choose $R \in (\mathfrak{S})_{lim}$ with $\mathbf{v_i}(S) \in \overline{R}$. As $T \subseteq R$, $R$ is axis parallel and $\overline{R} \cap [\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)] \subseteq \overline{(\mathfrak{S})_\omega}$, we have that the endpoint is in $int(\mathbf{R}_{i+1}^2)$. We are done with $(i)_1$.

Proof of $(ii)$:

We prove the statement for $i = 1$, the proofs of the other cases are similar. Choose $R \in (\mathfrak{S})_{lim}$ with $(x_1, y_1) \in \overline{R}$. As $T \subseteq R$ and $R$ is axis parallel and we have that $(x_2, y) \in R$ for all $y < y_1$. As $R \subseteq int((\mathfrak{S})_\omega)$ by Corollary 3.2.5, we are done with $(ii)$.

□

**Remark.** Note that as an easy consequence of Lemma 3.4.1 $(i)_1$ we have that the elements of $\mathcal{I}(a)$ are arcs of $\partial_a$ for $a \in \mathcal{E}_1 \cup \mathcal{E}_3 \cup \mathcal{E}_{2,n}$. As $(\partial_a, <_a)$ is isomorphic to $(\mathbf{R} \cup \{-\infty\} \cup \{+\infty\}, <)$ for $a \in \mathcal{E}_1 \cup \mathcal{E}_3 \cup \mathcal{E}_{2,n}$, by Definition 1.3.3 we have the definition of $\mathcal{I}(a)_{\mathbf{p},\mathbf{q},\varepsilon}$ for $\mathbf{p}, \mathbf{q} \in \partial_a$ and $\varepsilon \in 4$, where neither $\mathbf{p}$ nor $\mathbf{q}$ is an endpoint of an element of $\mathcal{I}(a)$.

(We will define $\mathcal{I}(a)$ as a subset of $\{I(S) : S \in \mathfrak{S}^a\}$ for $a \in \mathcal{E}_{2,o}$, so similarly by Lemma 3.4.1 $(i)_2$ and Definition 1.3.3 we have the definition of $\mathcal{I}(a)_{\mathbf{p},\mathbf{q},\varepsilon}$ for $\mathbf{p}, \mathbf{q} \in \partial_a$ and $\varepsilon \in 4$, where neither $\mathbf{p}$ nor $\mathbf{q}$ is an endpoint of any element of $\mathcal{I}(a)$.)

### 3.4.2 The statement

The aim of the following 3 sections is to prove the following theorem:

**Theorem 3.4.2.** *For* $a \subseteq \{1,2,3,4\}$ *there is* $d^a : \mathfrak{S}^a \to \omega$ *with:*

(*i*) $(\mathfrak{S}^a)_\omega \subseteq ((d^a)^{-1}(\{0\}))_\omega$;

(*ii*) $\partial \cap (\mathfrak{S}^a)_\omega \subseteq \cap_{j\in\omega}((d^a)^{-1}(\{j\}))_\omega$.

*Proof.* Before going into the details we sketch the proof:

1) we choose $D \in [\partial]^\omega$ with appropriate properties;

2) (with the help of Corollary 2.1.22) we construct $d^a$ for $a \in \mathcal{E}_{2,n}$;

3) we choose $\mathfrak{M}^a \subseteq \mathfrak{S}^a$ and define $\mathcal{I}(a) = \{I(S) : S \in \mathfrak{S}^a\}$ for $a \in \mathcal{E}_{2,o}$;

4) for $a \in \mathcal{E}_1 \cup \mathcal{E}_3 \cup \mathcal{E}_{2,o}$ we apply Corollary 2.1.22 with $\partial_a$, $\mathcal{I}(a)$ and $D \cap \partial_a$ getting $e^a : \mathcal{I}(a) \to \omega$, that satisfies (*i*) and (*ii*) of Corollary 2.1.22, and define $d^a(S) = e^a(I(S))$ for $S \in \mathfrak{S}^a$. And finally we prove that this coloring satisfies (*i*) and (*ii*) of Theorem 3.4.2.

Now we start the proof.

Let us introduce some notation. Let

- $X_M = max\{x :$ there is $y$ with $(x,y) \in \overline{(\mathfrak{S})_\omega}\}$;
- $X_m = min\{x :$ there is $y$ with $(x,y) \in \overline{(\mathfrak{S})_\omega}\}$;
- $Y_M = max\{y :$ there is $x$ with $(x,y) \in \overline{(\mathfrak{S})_\omega}\}$;
- $Y_m = min\{y :$ there is $x$ with $(x,y) \in \overline{(\mathfrak{S})_\omega}\}$.
- Let us denote by $\partial\mathcal{I}$ the set of all endpoints of $I(S)$ and $J(S)$ for all $S \in \mathfrak{S}$.

- Fix $D \in [\partial]^\omega$ for the rest of the proof such that:

($\circ$1) $(X_M, 0), (X_m, 0), (0, Y_M), (0, Y_m) \in D$ and $D \cap \partial\mathcal{I} = \emptyset$;

and for $i = 1,2,3,4$ :

($\circ$2) $proj_x(D \cap \partial_{\{i\}}) \setminus proj_x(\partial\mathcal{I} \cap \partial_{\{i\}}) \subseteq proj_x(\partial_{\{i\}})$ dense;

($\circ$3) $proj_y(D \cap \partial_{\{i\}}) \setminus proj_y(\partial\mathcal{I} \cap \partial_{\{i\}}) \subseteq proj_y(\partial_{\{i\}})$ dense.

We can choose such $D$ by Lemma 3.4.1 (*i*) and by $|\mathfrak{S}| = \omega$.

### 3.4.3 The construction of $d^a$ for $a \in \mathcal{E}_{2,n}$ (Step 2) of the strategy)

We construct $d^{\{1,2\}}$, the construction of $d^a$ for $a \in \mathcal{E}_{2,n}$ are similar by rotation.

So by the definition of $\mathfrak{S}^{\{1,2\}}$ we have $\mathbf{v}_1(S), \mathbf{v}_2(S) \notin \overline{(\mathfrak{S})_\omega}$ and $\mathbf{v}_3(S), \mathbf{v}_4(S) \in \overline{(\mathfrak{S})_\omega}$ for $S \in \mathfrak{S}^{\{1,2\}}$.

We know that $(0, Y_M) \in \overline{(\mathfrak{S})_\omega}$ and let $[\mathbf{x}, \mathbf{y}] \subseteq \overline{(\mathfrak{S})_\omega}$ be the horizontal segment through $(0, Y_M)$. We also know that $(v_1(S), u(S)) = \mathbf{v_1}(S) \in \mathbf{R}^2_1$, $(v_2(S), u(S)) = \mathbf{v_2}(S) \in \mathbf{R}^2_2$ and the length of $[\mathbf{x}, \mathbf{y}]$ is at least 1(by Theorem 3.3.2 (1)), so $u(S) > Y_M$ for all $S \in \mathfrak{S}^{\{1,2\}}$. This easily implies the following:

**Fact 3.4.3.** *If $R \in (\mathfrak{S}^{\{1,2\}})_{lim}$, then $[\mathbf{v_1}(R), \mathbf{v_2}(R)] \subseteq [\mathbf{x}, \mathbf{y}]$.*

**Lemma 3.4.4.** *For all $\mathbf{z} \in (\mathfrak{S}^{\{1,2\}})_\omega$ there is $\mathbf{z}^* \in [\boldsymbol{x}, \boldsymbol{y}]$ satisfying that $\mathbf{z} \in S$ iff $\mathbf{z}^* \in S$ for all but finitely many $S \in \mathfrak{S}^{\{1,2\}}$.*

*Proof.* If $\mathbf{z} = (z_1, z_2)$, then let $\mathbf{z}^* := (z_1, Y_M)$.

Using the fact that each $S \in \mathfrak{S}^{\{1,2\}}$ is axis-parallel, we know that if $\mathbf{z} \in S$ then $\mathbf{z}^* \in S$ for all $S \in \mathfrak{S}^{\{1,2\}}$. As $\mathbf{z} \in (\mathfrak{S}^{\{1,2\}})_\omega$, we have $|z_2 - Y_M| < 1$. Then we are done by the fact above. $\square$

Fix $\mathfrak{S}^{\{1,2\}+} \subseteq \mathfrak{S}^{\{1,2\}}$ with $I(S_1) \neq I(S_2)$ for $S_1, (\neq) S_2 \in \mathfrak{S}^{\{1,2\}+}$ and $\{I(S) : S \in \mathfrak{S}^{\{1,2\}+}\} = \{I(S) : S \in \mathfrak{S}^{\{1,2\}}\}$.

Apply Corollary 2.1.22 with $X = \partial_{\{1,2\}}, E = D \cap \partial_{\{1,2\}}, \mathfrak{Q} = \mathcal{I}(\{1,2\})$ and so we have $c^{\{1,2\}} : \mathcal{I}(\{1,2\}) \to \omega$ satisfying $(\mathcal{I}(\{1,2\}))_\omega \subseteq \cap_{j \in \omega}((c^{\{1,2\}})^{-1}(\{j\}))_\omega$ (($ii$) of Corollary 2.1.22 with $\mathbf{p} = (X_m, 0), \mathbf{q} = (X_M, 0), \varepsilon = 1$), and let $d^{\{1,2\}}(S) := c^{\{1,2\}}(I(S))$ for $S \in \mathfrak{S}^{\{1,2\}+}$.

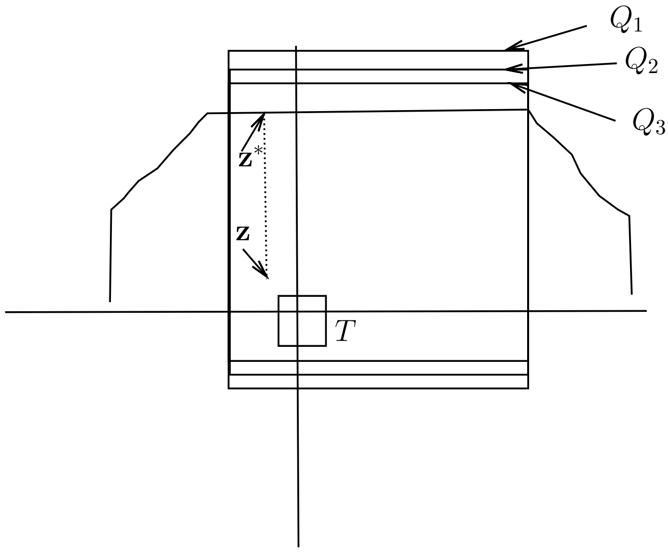


Figure 3.4: The definition of $\mathbf{z}^*$ and $Q_1, Q_2, Q_3 \in \mathfrak{S}(I)$

**Lemma 3.4.5.** *For* $\mathbf{z} \in (\mathfrak{S})_\omega$ *we have:*

$$\mathbf{z} \in \cap_{j\in\omega}((d^{\{1,2\}})^{-1}(\{j\}))_\omega \text{ or } \mathbf{z} \notin (\mathfrak{S}^{\{1,2\}+})_\omega.$$

*Proof.* For $\mathbf{z} \in (\mathfrak{S}^{\{1,2\}+})_\omega$ we can choose $\mathbf{z}^* \in \partial$ satisfying Lemma 3.4.4. Then we are done by the assumption on $c^{\{1,2\}}$.

□

Let $I \in \mathcal{I}(\{1,2\})$ be such that $\mathfrak{S}(I) = \{S \in (\mathfrak{S} \setminus \mathfrak{S}^{\{1,2\}+}) : S \cap \partial = I\}$ is infinite. For each such $I$ we color the squares in $\mathfrak{S}(I)$ the following way:

For $R \in (\mathfrak{S}(I))_\omega$ a side of $R$ is a subset of $[\mathbf{x}, \mathbf{y}]$ by the fact above. Choose an enumeration $\{Q_n : n \in \omega\} = \mathfrak{S}(I)$ with the property that the $y$ coordinate of $\mathbf{v_4}(Q_n)$ is not increasing. Then we can easily find

$$d^{\{1,2\}} : \{Q_n : n \in \omega\} \to \omega$$

satisfying $\cap_{j\in\omega}((d^{\{1,2\}})^{-1}(\{j\}))_\omega = (\{Q_n : n \in \omega\})_\omega$. Let $d^{\{1,2\}}$ be 0 for the still uncolored squares in $\mathfrak{S}^{\{1,2\}}$.

**Lemma 3.4.6.** $(\mathfrak{S}^{\{1,2\}})_\omega = \cap_{j\in\omega}((d^{\{1,2\}})^{-1}(\{j\}))_\omega$.

*Proof.* For $\mathbf{z} \in (\mathfrak{S}^{\{1,2\}})_\omega$ choose $\mathbf{z}^*$ by Lemma 3.4.4. Then either there is $I \in \mathcal{I}(\{1,2\})$ with $\mathbf{z}^* \in (\mathfrak{S}(I))_\omega$ or $\mathbf{z}^* \in (\mathcal{I}(\{1,2\}))_\omega$. In both cases we are easily done by the above observations. □

By Lemma 3.4.6 we have that $d^{\{1,2\}}$ fulfills $(i)$ and $(ii)$ of Theorem 3.4.2.

### 3.4.4 Choosing a subset of $\mathfrak{S}^a$ for $a \in \mathcal{E}_{2,o}$ (Step 3) of the strategy)

In this section we choose

$$\mathfrak{M}^a \subseteq \mathfrak{S}^a$$

for $a \in \mathcal{E}_{2,o}$ which is:

$\bullet_1$ regular enough to be able to apply Corollary 2.2.1 (see Lemma 3.4.7 $\bullet_1$);

$\bullet_2$ big enough that the intersections of the squares with the boundary cover $\omega$-fold exactly the points which are covered $\omega$-fold by squares in $\mathfrak{S}^a$ (see Lemma 3.4.7 $\bullet_2$).

Now we start the construction of $\mathfrak{M}^a$:

**Lemma 3.4.7.**

*Suppose* $|\mathfrak{S}^a| = \omega$ *for* $a \in \mathcal{E}_{2,o}$. *Then there is* $\{S_n : n \in \omega\} = \mathfrak{M}^a \in [\mathfrak{S}^a]^\omega$ *with*

$$I(S_n) = \partial(\mathbf{A}_n, \mathbf{B}_n),\ J(S_n) = \partial(\mathbf{b}_n, \mathbf{a}_n) \text{ such that}$$

$\bullet_1$ $\{\mathbf{A}_n, \mathbf{B}_n\} \subseteq (\partial_{\{1\}}, <_{\{1,3\}})$ *and* $\{\mathbf{a}_n, \mathbf{b}_n\} \subseteq (\partial_{\{3\}}, <_{\{1,3\}})$ *fulfill conditions* $(i) - (iv)$ *of Corollary 2.2.1;*

$\bullet_2$ $(\mathfrak{M}^a)_\omega \cap \partial = (\mathfrak{S}^a)_\omega \cap \partial$.

*Proof.* By symmetry it is enough to prove our statement for $a = \{1, 3\}$.

**Claim 3.4.8.** There are $\subseteq$-maximal elements in every subset of $\{I(S) : S \in \mathfrak{S}^{\{1,3\}}\}$.

*Proof of the claim.*

Observe first that since $a = \{1, 3\}$, if $I(S_1) \subseteq I(S_2)$ for $S_1, S_2 \in \mathfrak{S}^{\{1,3\}}$ then $\mathbf{v_1}(S_1) \in \overline{S_2}$. Then we argue by contradiction. If there would be $I(S_1) \subsetneq I(S_2) \subsetneq ...$, then $\mathbf{v_1}(S_1) \in \overline{S_n}$ for all $n \in \omega$, therefore $\mathbf{v_1}(S_1) \in \overline{(\mathfrak{S})_\omega}$. Since $S_1 \in \mathfrak{S}^{\{1,3\}}$ it is a contradiction, hence we are done.

$\square$

Now let

$$\mathfrak{M}^{\{1,3\}} = \{S \in \mathfrak{S}^{\{1,3\}} : I(S) \ \textit{is} \ \subseteq \textit{-maximal in} \ \{I(S) : S \in \mathfrak{S}^{\{1,3\}}\}\}.$$

**Claim 3.4.9.** $(\mathfrak{M}^{\{1,3\}})_\omega \cap \partial = (\mathfrak{S}^{\{1,3\}})_\omega \cap \partial$.

*Proof of the claim.*

Assume on the contrary that there is $\mathbf{x} \in ((\mathfrak{S}^{\{1,3\}})_\omega \cap \partial) \setminus ((\mathfrak{M}^{\{1,3\}})_\omega \cap \partial)$, i.e. $|\{S \in \mathfrak{S}^{\{1,3\}} : \mathbf{x} \in S\}| = \omega$ and $|\{S \in \mathfrak{M}^{\{1,3\}} : \mathbf{x} \in S\}| < \omega$. By Claim 3.4.8 there is $S \in \mathfrak{M}^{\{1,3\}}$ with $|\{Q \in \mathfrak{S}^{\{1,3\}} : I(Q) \subseteq I(S)\}| = \omega$. But $I(Q) \subseteq I(S)$ implies $\mathbf{v_3}(S) \in \overline{Q}$ meaning $\mathbf{v_3}(S) \in \overline{(\mathfrak{S})_\omega}$, which is a contradiction by $S \in \mathfrak{S}^{\{1,3\}}$.

$\square$

So we verified $\bullet_2$ of Lemma 3.4.7.

To continue our proof, let $S_1, S_2 \in \mathfrak{M}^{\{1,3\}}$ be different. Note that the boundary of $S_1$ and $S_2$ intersect in 2 points. Let us denote them by $\mathbf{q_1}$ and $\mathbf{q_2}$. Let $(x_1, y_1) = \mathbf{v_1}(S_1)$ and $(x_2, y_2) = \mathbf{v_1}(S_2)$. We know that $y_1 = y_2$ can not stand by the definition of $\mathfrak{M}^{\{1,3\}}$ so by symmetry we can assume $y_1 > y_2$. From this $x_1 > x_2$ holds again by the definition of $\mathfrak{M}^{\{1,3\}}$,

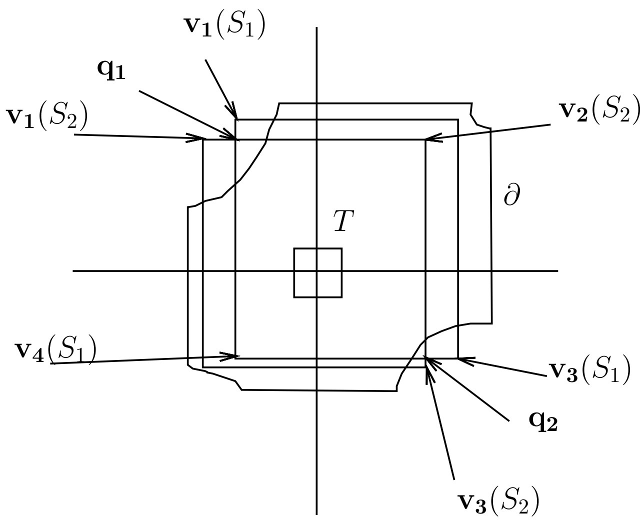


Figure 3.5: The definition of $\mathbf{q_1}$ and $\mathbf{q_2}$

so

$$\mathbf{q_1} = [\mathbf{v_1}(S_1), \mathbf{v_4}(S_1)] \cap [\mathbf{v_1}(S_2), \mathbf{v_2}(S_2)] \text{ and } \mathbf{q_2} = [\mathbf{v_3}(S_1), \mathbf{v_4}(S_1)] \cap [\mathbf{v_3}(S_2), \mathbf{v_2}(S_2)].$$

**Claim 3.4.10.** The following statements are true:

$(a)$ $\mathbf{q_1} \in \overline{(\mathfrak{S})_\omega}$ iff $\mathbf{q_2} \in \overline{(\mathfrak{S})_\omega}$,

$(b)$ $\mathbf{q_1}, \mathbf{q_2} \notin \partial$.

*Proof.*

$(a)$: If $\mathbf{q_1} \in \overline{(\mathfrak{S})_\omega}$ then by Theorem 3.2.3 (2) there is $R \in \mathfrak{S}_{lim}$ with $\mathbf{q_1} \in \overline{R}$. Since $\mathbf{v_1}(S_1), \mathbf{v_1}(S_2) \notin \overline{R}$ (as they are $\notin \overline{(\mathfrak{S})_\omega}$), $\mathbf{q_2} \in \overline{R}$ so by Corollary 3.2.5 $\mathbf{q_2} \in \overline{(\mathfrak{S})_\omega}$. And vice versa.

$(b)$: By contradiction. If e.g. $\mathbf{q_1} \in \partial$ then there is $R \in \mathfrak{S}_{lim}$ with $\mathbf{q_1} \in \partial R$. But then either $\mathbf{v_3}(S_1) \in \overline{R}$ or $\mathbf{v_3}(S_2) \in \overline{R}$. Contradiction.

We are done with Claim 3.4.10.

$\square$

**Claim 3.4.11.** For $S_1, S_2 \in \mathfrak{M}^{\{1,3\}}$ the following statements are equivalent:

(1) $I(S_1) \cap I(S_2) = \emptyset$,

(2) $J(S_1) \cap J(S_2) = \emptyset$.

*Proof of the claim.*

(1) $\Leftrightarrow \mathbf{q_1} \in \overline{(\mathfrak{S})_\omega} \Leftrightarrow \mathbf{q_2} \in \overline{(\mathfrak{S})_\omega} \Leftrightarrow$ (2).

□

- $(i)$ and $(ii)$ of Corollary 2.2.1 is true by the definition of $\mathfrak{M}^{\{1,3\}}$.
- $(iii)$ of Corollary 2.2.1 is true by Claim 3.4.11.
- $(iv)$ of Corollary 2.2.1 is true by Lemma 3.4.1 (2) using that $y_1 > y_2$ and $x_1 > x_2$ for $(x_1, y_1) = \mathbf{v_1}(S_1)$ and $(x_2, y_2) = \mathbf{v_1}(S_2)$.

We finished the proof of Lemma 3.4.7.

□

After this lemma we give the definition of $\mathcal{I}(a)$ for $a \in \mathcal{E}_{2,o}$:

**Definition 3.4.12.**

For $a \in \mathcal{E}_{2,o}$ let

$$\mathcal{I}(a) = \{I(S) : S \in \mathfrak{M}^a\}.$$

### 3.4.5 Step 4) of the strategy

We will use the following trivial facts:

**Fact 3.4.13.** *Suppose* $1 \le i \le 4$, $(x, y) = \mathbf{x} \in \mathbf{R}_i^2$ *and* $(v, w) = \mathbf{v_i}(S)$ *with some* $S \in \mathfrak{S}$. *Then the following statements are equivalent:*

$(i)$ $\mathbf{x} \in S$;

$(ii)$ $|x| < |v|$ *and* $|y| < |w|$.

**Fact 3.4.14.**

(1) *Let* $(x,y) = \mathbf{x} \in \mathbf{R}^2_1$ *and* $(v,w) = \mathbf{v_1}(S)$ *for some* $S \in \mathfrak{S}$. *Then the following statements are equivalent:*

*(i)* $|x| < |v|$ *and* $|y| < |w|$;

*(ii)* $v < x$ *and* $y < w$.

(2) *Let* $(x,y) = \mathbf{x} \in \mathbf{R}^2_2$, $(z,w) = \mathbf{v_1}(S)$ *and* $(v,w) = \mathbf{v_2}(S)$ *for some* $S \in \mathfrak{S}$. *Then the following statements are equivalent:*

*(i)* $|x| < |v|$ *and* $|y| < |w|$;

*(ii)* $x < z+1$ *and* $y < w$.

(3) *Let* $(x,y) = \mathbf{x} \in \mathbf{R}^2_3$, $(z,u) = \mathbf{v_1}(S)$ *and* $(v,w) = \mathbf{v_3}(S)$ *for some* $S \in \mathfrak{S}$. *Then the following statements are equivalent:*

*(i)* $|x| < |v|$ *and* $|y| < |w|$;

*(ii)* $x < z+1$ *and* $u-1 < y$.

(4) *Let* $(x,y) = \mathbf{x} \in \mathbf{R}^2_4$, $(v,u) = \mathbf{v_1}(S)$ *and* $(v,w) = \mathbf{v_4}(S)$ *for some* $S \in \mathfrak{S}$. *Then the following statements are equivalent:*

*(i)* $|x| < |v|$ *and* $|y| < |w|$;

*(ii)* $v < x$ *and* $u-1 < y$.

For $0 \le r$ let us define the horizontal line $y = r$ by $h(r)$ and for $r \le 0$ let us define the vertical line $x = r$ by $u(y)$.

$h(r) \cap \overline{(\mathfrak{S})_\omega}$ is a horizontal segment if it is not empty and let us define its endpoints by $\mathbf{h_1}(r) = (h_1, r) \in \partial_{\{1\}}$ and $\mathbf{h_2}(r) = (h_2, r) \in \partial_{\{2\}}$. If $h(r) \cap \overline{(\mathfrak{S})_\omega}$ is empty then let $\mathbf{h_1}(r) = \mathbf{h_2}(r) = (0, Y_M)$.

Similarly $u(r) \cap \overline{(\mathfrak{S})_\omega}$ is a vertical segment if it is not empty and let us define its endpoints by $\mathbf{u_1}(r) = (r, u_1) \in \partial_{\{1\}}$ and $\mathbf{u_2}(r) = (r, u_2) \in \partial_{\{4\}}$. If $u(r) \cap \overline{(\mathfrak{S})_\omega}$ is empty then let $\mathbf{u_1}(r) = \mathbf{u_2}(r) = (X_m, 0)$.

With these notation using Lemma 3.4.1 (2) we have the following fact:

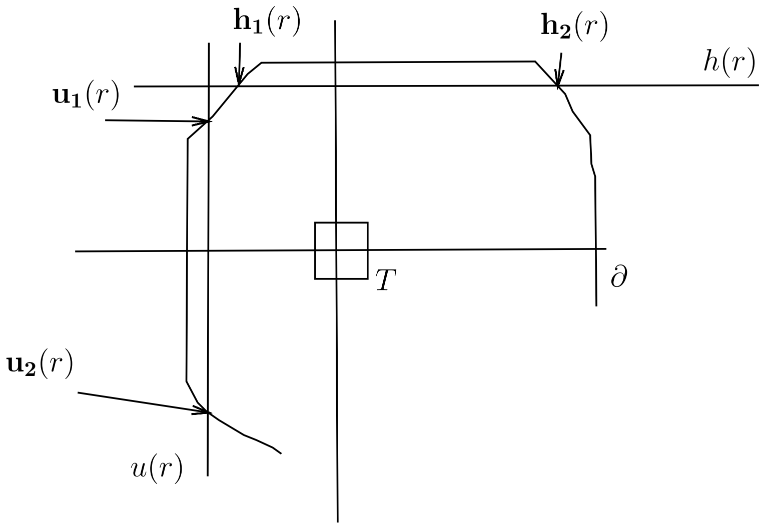


Figure 3.6: The definition of $\mathbf{h_1}(r), \mathbf{h_2}(r)$ $(0 \leq r \leq Y_M)$ and $\mathbf{u_1}(r), \mathbf{u_2}(r)$ $(X_m \leq r \leq 0)$

**Fact 3.4.15.**

(1) $r_1 < r_2$ *with* $0 \leq r_1 \leq Y_M \Leftrightarrow \mathbf{h_1}(r_1) <_{\{1\}} \mathbf{h_1}(r_2) \Leftrightarrow \mathbf{h_2}(r_2) <_{\{2\}} \mathbf{h_2}(r_1)$*;*

(2) $r_1 < r_2$ *with* $X_m \leq r_2 \leq 0 \Leftrightarrow \mathbf{u_1}(r_1) <_{\{1\}} \mathbf{u_1}(r_2) \Leftrightarrow \mathbf{u_2}(r_2) <_{\{4\}} \mathbf{u_2}(r_1)$.

**Lemma 3.4.16.** *Let* $D \in [\partial]^\omega$ *satisfying* $(\circ 1) - (\circ 3)$ *and* $a \in \mathcal{E}_1 \cup \mathcal{E}_{2,o} \cup \mathcal{E}_3$*. Let*

$$\mathcal{X}(a) := \{\mathbf{x} \in int(\mathfrak{S})_\omega : \exists \mathbf{p}, \mathbf{q} \in D \text{ and } \varepsilon \in 4 \text{ such that } \mathbf{x} \in S \Leftrightarrow I(S) \in (\partial_a)_{\mathbf{p},\mathbf{q},\varepsilon}\}.$$

*Then* $\mathcal{X}(a)$ *is dense in* $int(\mathfrak{S})_\omega$*.*

*Proof.* By rotation it is enough to prove the lemma for $a = \{1\}, \{1,3\}$ and $\{2,3,4\}$.

Choose $V \subseteq int(\mathfrak{S})_\omega$ open, nonempty.

• $a = \{1\}$:

1) Suppose we have a point, $(x, y) = \mathbf{x} \in \mathbf{R}^2_1 \cap int(\mathfrak{S})_\omega$. By Fact 3.4.13 and Fact 3.4.14 we know that the following is true for all $S \in \mathfrak{S}^{\{1\}}$ and $(v, w) = \mathbf{v_1}(S)$:

$$\mathbf{x} \in S \Leftrightarrow v < x \text{ and } y < w.$$

$X_m \leq v \leq 0$ so by Fact 3.4.15 (2) we have $v < x \Leftrightarrow \mathbf{u_1}(v) <_{\{1\}} \mathbf{u_1}(x)$. $0 \leq w \leq Y_M$ so by Fact 3.4.15 (1) we have $\mathbf{h_1}(y) <_{\{1\}} \mathbf{h_1}(w)$. Note that $\mathbf{u_1}(v)$ and $\mathbf{h_1}(w)$ are the endpoints of $I(S)$.

If $\mathbf{R}_1^2 \cap V \neq \emptyset$, then by the assumptions on $D$ we can find $(x, y) = \mathbf{x} \in \mathbf{R}_1^2 \cap V$ such that $\mathbf{u_1}(x), \mathbf{h_1}(y) \in D$. So

$$\mathbf{p} = \mathbf{u_1}(x), \mathbf{q} = \mathbf{h_1}(y) \text{ and } \varepsilon = 2.$$

proves the statement in this case.

The proof of the other cases are similar to 1) using Fact 3.4.13, Fact 3.4.14 and Fact 3.4.15, so we just define $\mathbf{p}, \mathbf{q}$ and $\varepsilon$ in these cases:

2) If $\mathbf{R}_2^2 \cap V \neq \emptyset$ then we can find $(x, y) = \mathbf{x} \in \mathbf{R}_2^2 \cap V$ such that $\mathbf{u_1}(x-1), \mathbf{h_1}(y) \in D$, so

$$\mathbf{p} = \mathbf{u_1}(x-1), \mathbf{q} = \mathbf{h_1}(y) \text{ and } \varepsilon = 3$$

proves the statement in this case.

3) If $\mathbf{R}_3^2 \cap V \neq \emptyset$ then we can find $(x, y) = \mathbf{x} \in \mathbf{R}_3^2 \cap V$ such that $\mathbf{u_1}(x-1), \mathbf{h_1}(y+1) \in D$, so

$$\mathbf{p} = \mathbf{u_1}(x-1), \mathbf{q} = \mathbf{h_1}(y+1) \text{ and } \varepsilon = 1$$

proves the statement in this case.

4) If $\mathbf{R}_4^2 \cap V \neq \emptyset$ then we can find $(x, y) = \mathbf{x} \in \mathbf{R}_4^2 \cap V$ such that $\mathbf{u_1}(x), \mathbf{h_1}(y+1) \in D$, so

$$\mathbf{p} = \mathbf{u_1}(x), \mathbf{q} = \mathbf{h_1}(y+1) \text{ and } \varepsilon = 0$$

proves the statement in this case.

The definition of the points and the proof of the statement for $a = \{1, 3\}$ is exactly the same.

Using that $(\partial_{\{2\}}, <_{\{2\}})$ and $(\partial_{\{4\}}, <_{\{4\}})$ are the restrictions of $(\partial_{\{2,3,4\}}, <_{\{2,3,4\}})$ to $\partial_{\{2\}}$ and $\partial_{\{4\}}$ respectively we can define the points for $a = \{2,3,4\}$ similarly to the $a = \{1\}$ case. The proofs are also similar.

- $a = \{2,3,4\}$:

1) If $\mathbf{R}^2_1 \cap V \neq \emptyset$ then we can find $(x,y) = \mathbf{x} \in \mathbf{R}^2_1 \cap V$ such that $\mathbf{u_2}(x), \mathbf{h_2}(y) \in D$, so

$$\mathbf{p} = \mathbf{u_2}(x), \mathbf{q} = \mathbf{h_2}(y) \text{ and } \varepsilon = 1$$

proves the statement in this case.

2) If $\mathbf{R}^2_2 \cap V \neq \emptyset$ then we can find $(x,y) = \mathbf{x} \in \mathbf{R}^2_2 \cap V$ such that $\mathbf{u_2}(x-1), \mathbf{h_2}(y) \in D$, so

$$\mathbf{p} = \mathbf{u_2}(x-1), \mathbf{q} = \mathbf{h_2}(y) \text{ and } \varepsilon = 0$$

proves the statement in this case.

3) If $\mathbf{R}^2_3 \cap V \neq \emptyset$ then we can find $(x,y) = \mathbf{x} \in \mathbf{R}^2_3 \cap V$ such that $\mathbf{u_2}(x-1), \mathbf{h_2}(y+1) \in D$, so

$$\mathbf{p} = \mathbf{u_2}(x-1), \mathbf{q} = \mathbf{h_2}(y+1) \text{ and } \varepsilon = 2$$

proves the statement in this case.

4) If $\mathbf{R}^2_4 \cap V \neq \emptyset$ then we can find $(x,y) = \mathbf{x} \in \mathbf{R}^2_4 \cap V$ such that $\mathbf{u_2}(x), \mathbf{h_2}(y+1) \in D$, so

$$\mathbf{p} = \mathbf{u_2}(x), \mathbf{q} = \mathbf{h_2}(y+1) \text{ and } \varepsilon = 3$$

proves the statement in this case.

We are done with the proof of Lemma 3.4.16.

$\square$

Now finish Step 4) and the proof of Theorem 3.4.2.

For $a \in \mathcal{E}_1 \cup \mathcal{E}_3$ apply Corollary 2.1.22 with $X = \partial_a$, $\mathfrak{Q} = \mathcal{I}(a)$ and $E = D \cap \partial_a$.

For $a = \{1,3\}$ apply Corollary 2.1.22 with $X = \partial_{\{1\}}$, $\mathfrak{Q} = \mathcal{I}(\{1,3\})$ and $E = D \cap \partial_{\{1\}}$.

For $a = \{2,4\}$ apply Corollary 2.1.22 with $X = \partial_{\{4\}}$, $\mathfrak{Q} = \mathcal{I}(\{2,4\})$ and $E = D \cap \partial_{\{2\}}$.

So for $a \in \mathcal{E}_1 \cup \mathcal{E}_3 \cup \mathcal{E}_{2,o}$ we have $e^a : \mathcal{I}(a) \to \omega$ satisfying $(i)$ and $(ii)$ of Corollary 2.1.22 and let

$$d^a(S) = e^a(I(S))$$

for $S \in \cup_{a \in \mathcal{E}_1 \cup \mathcal{E}_3} \mathfrak{S}^a \bigcup \cup_{a \in \mathcal{E}_{2,o}} \mathfrak{M}^a$, and let $d(S) = 0$ for $S \in \cup_{a \in \mathcal{E}_{2,o}} (\mathfrak{S}^a \setminus \mathfrak{M}^a)$.

Now we want to prove that $d^a$ satisfies $(i)$ and $(ii)$ of Theorem 3.4.2:

• Proof of $(ii)$ of Theorem 3.4.2:

∘ for $a \in \mathcal{E}_1 \cup \mathcal{E}_3$ choose $\mathbf{p}$ and $\mathbf{q}$ as the endpoints of $\partial_a$ and $\varepsilon = 1$, and apply $(ii)$ of Corollary 2.1.22 for $e^a$. This proves the statement.

∘ For $a = \{1,3\}$ we know that $\partial \cap (\mathfrak{M}^{\{1,3\}})_\omega = \partial \cap (\mathfrak{S}^{\{1,3\}})_\omega$ by $\bullet_2$ of Lemma 3.4.7 and choosing $\mathbf{p}$ and $\mathbf{q}$ as the endpoints of $\partial_{\{1\}}$ and $\varepsilon = 1$ and applying $(ii)$ of Corollary 2.1.22 for $e^{\{1,3\}}$ proves the statement on $\partial_{\{1\}}$.

∘ On $\partial_{\{3\}}$ the statement is true by $\bullet_1$ and $\bullet_2$ of Lemma 3.4.7 and Corollary 2.1.22, since for $\mathbf{x} \in (\{J(S) : S \in \mathfrak{M}^{\{1,3\}}\})_\omega$ we can find $\mathbf{y}(\mathbf{x}) \in \partial_{\{1\}} \cap \{I(S) : S \in \mathfrak{M}^{\{1,3\}}\})_\omega$ such that with finitely many exceptions the same squares in $\mathfrak{M}^{\{1,3\}}$ contains $\mathbf{x}$ and $\mathbf{y}(\mathbf{x})$ by Corollary 2.2.1. Then we are done by $(ii)$ of Corollary 2.1.22 as above.

• Proof of $(i)$ of Theorem 3.4.2:

if $\mathbf{x} \in (\mathfrak{S}^a)_\omega \cap \partial$, then we are done as above. If $\mathbf{x} \in (\mathfrak{S}^a)_\omega \cap int(\mathfrak{S})_\omega$ then we can find $R^{\mathbf{x}} \in (\mathfrak{S}^a)_{lim}$ with $\mathbf{x} \in R^{\mathbf{x}}$ by (1) of Theorem 3.2.3. If $\mathbf{x} \in \mathbf{R}^2_i$, then by Fact 3.4.13 and Lemma 3.4.16 we can find $\mathbf{y} \in R^{\mathbf{x}}$ and $\mathbf{p}, \mathbf{q} \in D \cap \partial_a$ and $\varepsilon \in 4$ such that for all $S \in \mathfrak{S}^a$ we have

$$I(S) \in (\partial_a)_{\mathbf{p},\mathbf{q},\varepsilon} \Leftrightarrow \mathbf{y} \in S \Rightarrow \mathbf{x} \in S.$$

As $\mathbf{y} \in R^{\mathbf{x}}$ we have $|(\mathcal{I}(a))_{\mathbf{p},\mathbf{q},\varepsilon}| = \omega$, then by $(i)$ and $(ii)$ of Corollary 2.1.22 we are done.

We are done with Theorem 3.4.2.

□

**Theorem 3.4.17.** *There exists* $c : \mathfrak{S} \to \omega$ *such that* $(\mathfrak{S})_\omega = \cap_{j\in\omega}(c^{-1}(\{j\}))_\omega$.

*Proof.* There exists $d : \mathfrak{S} \to \omega$ satisfying $(i)$ and $(ii)$ of Theorem 3.4.2.

So

$$\partial \cap (\mathfrak{S})_\omega = \cap_{j>1}(d^{-1}(\{j\}))_\omega \text{ and } int(\mathfrak{S})_\omega = int(d^{-1}(\{0\}) \cup d^{-1}(\{1\}))_\omega$$

by $(ii)$ of Theorem 3.4.2.

Let

$$\mathfrak{S}_1 = \{int(\mathfrak{S})_\omega \cap R : R \in \mathfrak{S} \setminus \cup_{j>1} d^{-1}(\{j\})\}.$$

By the fact that an open subset of the plane is a $\sigma$-compact space and by Lemma 1.2.2 there exists $d_1 : \mathfrak{S}_1 \to \omega$ with $int(\mathfrak{S})_\omega = \cap_{j\in\omega}(d_1^{-1}(\{j\}))_\omega$.

Let

$$c(R) = \begin{cases} d_1(R \cap int(\mathfrak{S})_\omega) & \text{if } R \in (d^{-1}(\{0\}) \cup d^{-1}(\{1\})); \\ d(R) - 2 & \text{if } R \in \cup_{j>1} d^{-1}(\{j\}). \end{cases}$$

which fulfills the requirements of the theorem.

□

**Theorem 3.4.18.** *Suppose* $\mathfrak{T} \in [\mathcal{T}_C]^{\leq\omega}$. *Then* $\mathfrak{T}$ *is* $\omega$*-decomposable over* $(\mathfrak{T})_\omega$.

*Proof.* Consider a grid with distance 1/3 and put an open square with length of side 1/3 onto each point of the grid. Let $\{Q_i : i \in \omega\}$ be the set of these squares of side length 1/3. Let $\mathfrak{T}_i' = \{S \in \mathfrak{S} : Q_i \subseteq S\}$, and consider $\mathfrak{T}_i \subseteq \mathfrak{T}_i'$ which are disjoint, and $\cup_{i\in\omega}\mathfrak{T}_i = \cup_{i\in\omega}\mathfrak{T}_i'$. By elementary geometry the following statements are true:

(1) for $S \in \mathfrak{T}$ there is $i \in \omega$ such that $S \in \mathfrak{T}_i$;

(2) for $x \in (\mathfrak{T})_\omega$ there is $i \in \omega$ with $x \in (\mathfrak{T}_i)_\omega$.

By Theorem 3.4.17 there is $c_i : \mathfrak{T}_i \to \omega$ with $(\mathfrak{T}_i)_\omega = \cap_{j\in\omega}(c_i^{-1}(\{j\}))_\omega$ for all $i \in \omega$.

By (1) and (2) $c = \cup_{i\in\omega} c_i$ proves the statement of the theorem.

□

## 3.5 Closed square case

## 3.6 Back to the open case

**Theorem 3.6.1.** *Suppose $\mathfrak{C} \in [\mathcal{T}_{\overline{E}}]^{\leq\omega}$. Then $\mathfrak{C}$ is $\omega$-decomposable over $(\mathfrak{C})_\omega$.*

*Proof.* We will denote the elements of $\mathfrak{C}$ by $U$. For $U \in \mathfrak{C}$ let us denote the set of vertices of $U$ by $v(U)$ and let $U^{-v} = U \setminus v(U)$. For $\mathfrak{D} \in [\mathcal{T}_{\overline{E}}]^{\leq\omega}$ let $\mathfrak{D}^{-v} = \{U^{-v} : U \in \mathfrak{D}\}$ and $int(\mathfrak{D}) = \{int(U) : U \in \mathfrak{D}\}$.

Note that $(\mathfrak{D}^{-v})_\omega = (\mathfrak{D})_\omega$ for $\mathfrak{D} \in [\mathcal{T}_{\overline{E}}]^{\leq\omega}$, since the multiplicity of each translate is 1.

- Let

$$X = (\mathfrak{C}^{-v})_\omega \setminus (int(\mathfrak{C}))_\omega.$$

For $x \in X$ pick $l(x)$, an axis-parallel line with $x \in l(x)$, let $L = \{l(x) : x \in X\}$ and write $L = \{l_u : u \in |L|\}$ (we know that $|L| \leq \omega$ as $|\mathfrak{C}| = \omega$).

- First we choose disjoint subsets $\{\mathfrak{C}_i : i \in \omega\}$ of $\mathfrak{C}$ in the following way: for $i \in \omega$ let

$$\mathfrak{C}_i = \{S \in \mathfrak{C} \setminus \cup_{k<i}\mathfrak{C}_k : \text{ a side of } S \text{ is a subset of } l_i\}.$$

- Then let

$$tr_i = \{S^{-v} \cap l_i : S \in \mathfrak{C}_i\}.$$

By applying Corollary 2.1.11 on each components of $\cup tr_i$, we know that there exists a partition

$$\cup^*_{s\in\omega} tr_{i,s} = tr_i$$

with:

(1) $\cup tr_{i,0} = \cup tr_i$;

(2) $(tr_{i,u})_\omega = (tr_i)_\omega$ for $u \geq 1$.

For $i, s \in \omega$ let

$$\mathfrak{C}_{i,s} = \{V \in \mathfrak{C}_i : V^{-v} \cap l_i \in tr_{i,s}\}.$$

Note that (1) and (2) above and elementary geometry imply that

$(a)$ $\cup \mathfrak{C}_{i,0} = \cup \mathfrak{C}_i$;

$(b)$ $(\mathfrak{C}_{i,u})_\omega = (\mathfrak{C}_i)_\omega$ for $u \geq 1$.

Also note that $\cup_{i\in\omega}(\mathfrak{C}_i)_\omega \supseteq X$.

• Now we provide $c : \mathfrak{C} \to \omega$ proving Theorem 3.6.1.

By Theorem 3.4.18 we know that there exists

$$d : int(\mathfrak{C} \setminus \cup_{i\in\omega, s\geq 1}\mathfrak{C}_{i,s}) \to \omega$$

with

$$(int(\mathfrak{C} \setminus \cup_{i\in\omega, s\geq 1}\mathfrak{C}_{i,s}))_\omega = \cap_{j\in\omega}(d^{-1}(\{j\}))_\omega.$$

Let us define $c : \mathfrak{C} \to \omega$ the following way:

$$c(U) = \begin{cases} s-1 & \text{if } U \in \mathfrak{C}_{i,s} \text{ for some } i \in \omega, s \geq 1; \\ d(int(U)) & \text{if } U \in \mathfrak{C} \setminus \cup_{i\in\omega, s\geq 1}\mathfrak{C}_{i,s}. \end{cases}$$

We prove that this coloring satisfies the requirement of Theorem 3.6.1.



**Claim 3.6.2.** The following statement holds:

$$(\mathfrak{C})_\omega = \cup_{i\in\omega}(\mathfrak{C}_i)_\omega \cup (int(\mathfrak{C} \setminus \cup_{i\in\omega, s\geq 1}\mathfrak{C}_{i,s}))_\omega.$$

*Proof of the claim.* $\subseteq$ is trivial, so we want to prove $\supseteq$.

Let $\mathbf{x} \in (\mathfrak{C})_\omega$. By the definition of $\mathfrak{C}_i$ either there is $i \in \omega$ with $\mathbf{x} \in (\mathfrak{C}_i)_\omega$ or $x \in (\mathfrak{C} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s})_\omega$.

Suppose $\mathbf{x} \in (\mathfrak{C} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s})_\omega \setminus \cup_{i\in\omega}(\mathfrak{C}_i)_\omega$. We prove that $\mathbf{x} \in (int(\mathfrak{C} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s}))_\omega$ holds. Suppose by contradiction that $\mathbf{x} \in ((\mathfrak{C}\setminus\cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s})_\omega\setminus\cup_{i\in\omega}(\mathfrak{C}_i)_\omega)\setminus(int(\mathfrak{C}\setminus\cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s}))_\omega = ((\mathfrak{C} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s})_\omega \setminus (int(\mathfrak{C} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s}))_\omega) \setminus \cup_{i\in\omega}(\mathfrak{C}_i)_\omega$. Using this we have that $\mathbf{x} \in X$, which is a contradiction, since $\cup_{i\in\omega}(\mathfrak{C}_i)_\omega$ covers $X$. Hence we are done with the statement.

So we proved the claim.

□

By Claim 3.6.2 and by the definition of $c$ we are done with Theorem 3.6.1.

□

## 3.7 The proof of Theorem 1.3.7

*Proof.*

Proof of (1): by hereditarily Lindelöfness of the plane choose $\mathcal{G}_j \in [\mathcal{F} \setminus \cup_{i<j}\mathcal{G}_i]^{\leq\omega}$ with $\cup\mathcal{G}_j = \cup\mathcal{F} \setminus \cup_{i<j}\mathcal{G}_i$ for $j \in \omega$. Note that $(\cup\mathcal{G}_j)_\omega = (\mathcal{F})_\omega$. So by applying Theorem 3.4.18 for $\cup\mathcal{G}_j$ we are done.

Proof of (2): Note that $(\mathcal{H}^{-v})_\omega = (\mathcal{H})_\omega$ for any $\mathcal{H} \subseteq \mathcal{T}_{\overline{E}}$. The following fact is well-known:

**Fact 3.7.1.** *Let $\mathcal{C}$ be a set of closed polygons without vertices. Then there is $\mathcal{C}' \in [\mathcal{C}]^{\leq\omega}$ with*

$$\cup\mathcal{C} = \cup\mathcal{C}'.$$

Apply this fact to choose $\mathcal{G}_j \in [\mathcal{F} \setminus \cup_{i<j}\mathcal{G}_i]^{\leq\omega}$ with $\cup\mathcal{G}_j^{-v} = \cup(\mathcal{F} \setminus \cup_{i<j}\mathcal{G}_i)^{-v}$ for $j \in \omega$. Note that $(\cup\mathcal{G}_j)_\omega = ((\cup\mathcal{G}_j)^{-v})_\omega = (\mathcal{F}^{-v})_\omega = (\mathcal{F})_\omega$. So by applying Theorem 3.6.1 for $\cup\mathcal{G}_j$ we are done.

We are done with Theorem 1.3.7.

□

# 4 Constructions

In this section we describe constructions, showing the sharpness of Theorem 1.3.7.

Each construction works similarly:

- first we describe an elementary statement, then
- using the elementary statement we construct $A \subseteq \mathbf{R}^2$ in $\omega$ steps and a covering of $A$, which can not be decomposed.

## 4.1 Axis-parallel rectangles with side length between $1-\varepsilon$ and 1

*Proof of Theorem 1.3.8.* We prove (1), the proof of (2) is similar.

Let $l$ be the $x=-y$ line and $\vec{\mathbf{v}}$ be the vector from the origin to $(1,1)$, and for $A \subseteq \mathbf{R}^2$, $\lambda \in \mathbf{R}$ let $A+\lambda\vec{\mathbf{v}}$ be the translation of $A$ with $\lambda\vec{\mathbf{v}}$.

We will use the following elementary geometric statement repeatedly:

**Lemma 4.1.1.** *For all $\varepsilon > 0$ there are $\varepsilon_1$ and $\varepsilon_2$ with the following property:*

*For all $I \subseteq l$ interval with $|I| < \varepsilon_1$ and for all $I_1 \subseteq I$, $I_2 \subseteq I + (1-\varepsilon_2)\vec{\mathbf{v}}$ closed (open) intervals there is $R \in \mathcal{R}_\varepsilon$ ($R \in \mathcal{Q}_\varepsilon$) with*

$$R \cap (l \cup (l + (1-\varepsilon_2)\vec{\mathbf{v}})) = I_1 \cup I_2.$$

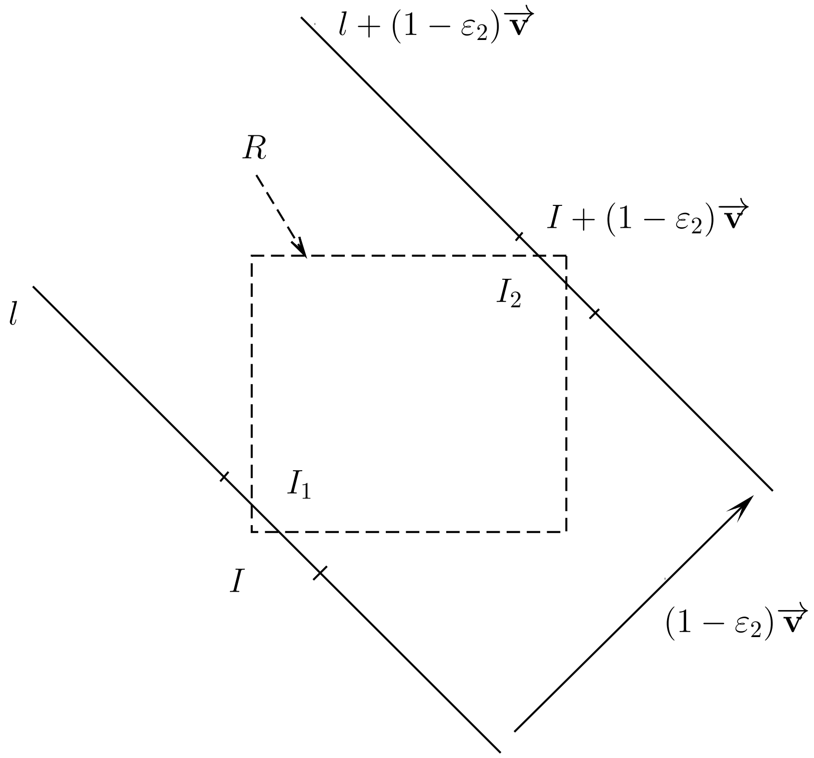


Figure 4.1: Lemma 4.1.1

*Proof.* The proof is immediate by Figure 4.1 and left to the reader.

□

We construct $A$ and $\mathcal{R}$ such that $A \subseteq (\mathcal{R})_\omega$ and $\mathcal{R}$ is not 2 decomposable over $A$:

1.) Fix $\varepsilon_1, \varepsilon_2$ for $\varepsilon$ satisfying Lemma 4.1.1.

2.) Let $I \subseteq l$ with $|I| < \varepsilon_1$ arbitrary and let $I_0 = I$, $J_0 = I + (1-\varepsilon_2)\overrightarrow{\mathbf{v}}$.

3.) Let $\omega_0^{<\omega} = \{\langle s_0, s_1, ..., s_j\rangle \in \omega^{<\omega} : s_0 = 0\}$.

4.) For $j \geq 1$ and $s = \langle s_0, s_1, ..., s_j\rangle \in \omega_0^{<\omega}$ let $s^- = \langle s_0, s_1, ..., s_{j-1}\rangle$ and $|s| = j$.

In the *jth step* ($j \geq 1$) for all $s \in \omega_0^{<\omega}$ with $|s| = j$ choose (see Figure 4.2) :

$\bullet_1$ $I_s \subseteq (I_0 \setminus \cup_{1\leq|s'|\leq j, s'\neq s} I_{s'})$ closed intervals with with $\sum_{1\leq|s|\leq j} |I_s| < \frac{|I_0|}{2}$, and

$\bullet_2$ $J_s \subseteq J_{s^-}$ closed intervals with $J_{\langle s^-, i\rangle} \cap J_{\langle s^-, j\rangle} = \emptyset$ for all $i \neq j \in \omega$.

In the *jth step* ($j \geq 2$) for all $s \in \omega_0^{<\omega}$ with $|s| = j$ choose:

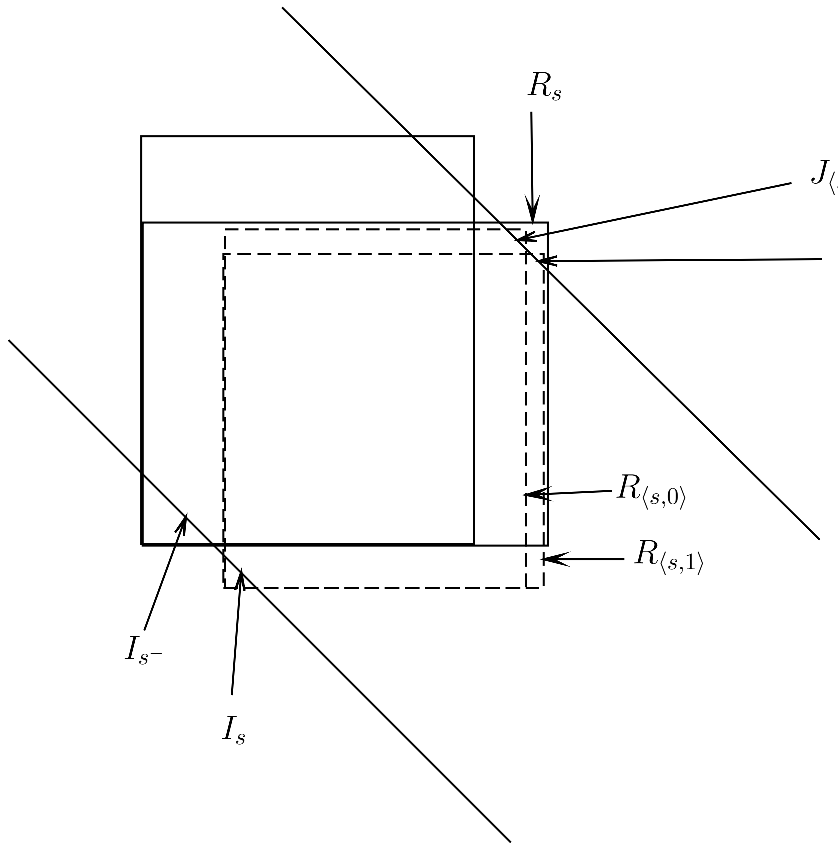


Figure 4.2: The jth step

$\bullet_3$ choose $R_s \in \mathcal{R}_\varepsilon$ with $R_s \cap (l \cup (l + (1-\varepsilon_2)\overrightarrow{\mathbf{v}})) = I_{s^-} \cup J_s$ by Lemma 4.1.1.

And finally let

$\bullet_4$ $A = \cap_{j\geq 2} \cup_{|s|=j} J_s \bigcup \cup_{|s|\geq 1} I_s$ and $\mathcal{R} = \{R_s : s \in \omega_0^{<\omega}, |s| \geq 2\}$.



First we prove that $A$ and $\mathcal{R}$ satisfies (1.1):

**Claim 4.1.2.** $A \subseteq (\mathcal{R})_\omega$.

*Proof.*

- $I_s \subseteq R_t$ if $t = \langle s, i\rangle$ for $i \in \omega$.
- for $\mathbf{x} \in \cap_{j\geq 2} \cup_{|s|=j} J_s$ one can choose $\langle t_0(\mathbf{x}), t_1(\mathbf{x}), ...\rangle \in \omega^\omega$ with $\mathbf{x} \in J_{\langle t_0(\mathbf{x}), t_1(\mathbf{x}), ..., t_j(\mathbf{x})\rangle}$

for all $j \geq 1$ meaning $\mathbf{x} \in R_{\langle t_0(\mathbf{x}), t_1(\mathbf{x}), ..., t_j(\mathbf{x})\rangle}$ by $\bullet_3$.

□

Now we want to prove that $A$ and $\mathcal{R}$ satisfies (1.2):

**Claim 4.1.3.** For all partition $\mathcal{R}_1 \cup^* \mathcal{R}_2 = \mathcal{R}$ either $\cup\mathcal{R}_1 \not\supseteq A$ or $\cup\mathcal{R}_2 \not\supseteq A$.

*Proof.* By $\bullet_1$ and $\bullet_{3.1}$ for $s, t \in \omega_0^{<\omega}$ with $|s|, |t| \geq 1$ the following is true:

a) $I_s \subseteq R_t$ if $t = \langle s, i \rangle$ for $i \in \omega$ and $I_s \cap R_t = \emptyset$ if $t \neq \langle s, i \rangle$ for $i \in \omega$.

So by the fact that $(\cap_{j\geq 2} \cup_{|s|=j} J_s) \cap (\cup_{|s|\geq 1} I_s) = \emptyset$ for all $s \in \omega_0^{<\omega}, |s| \geq 1$ and for any $c : \mathcal{R} \to 2$ with $\cup c^{-1}(\{k\}) \supseteq \cup_{|s|\geq 1} I_s$ $(k \in 2)$ one can find $n(s) \in \omega$ with $I_s \subseteq R_{\langle s, n(s)\rangle}$ and $c(R_{\langle s,n(s)\rangle}) = 0$. Let $\{t_i \in \omega^i : i \geq 1 \textit{ with } t_{i+1} = \langle t_i, n(t_i)\rangle\}$.

By $\bullet_2$ for $\mathbf{x} \in \cap_{i\geq 1} J_{t_i}$ if $\mathbf{x} \in J_s$ for some $s \in \omega^v$ then $s = t_v$, meaning, that $A \ni \mathbf{x} \notin \cup c^{-1}(\{1\})$.

□

We are done with Theorem 1.3.8.

□

Note that in the construction of $A$ in the proof of Theorem 1.3.8 we can choose $I_s$ $(|s| \geq 1)$ and $J_s$ $(|s| \geq 1)$ with:

$\circ_1$ $\cap_{j\geq 2} \cup_{|s|=j} J_s \subseteq (l + (1 - \varepsilon_2)\overrightarrow{\mathbf{v}})$ is a closed set minus countably many points.

$\circ_2$ $\cup_{|s|\geq 1} I_s \subseteq l$ is also a closed set minus one point.

Choosing $\varepsilon_2$ small enough we can choose $\mathcal{R}_1 \in [\mathcal{R}_\varepsilon]^\omega$ with $\cup\mathcal{R}_1 \subseteq \mathbf{R}^2 \setminus A \subseteq (\mathcal{R}_1)_\omega$, resulting in a bit strengthening of Theorem 4.1.5:

**Theorem 4.1.4.** *For all $\varepsilon > 0$ there is $\mathcal{R} \in [\mathcal{R}_\varepsilon]^\omega$ with:*

*(i) $(\mathcal{R})_\omega \supseteq \mathbf{R}^2$;*

*(ii) $\mathcal{R}$ is not 2-decomposable.*

**Remark.** Note that Theorem 4.1.4 is a strengthening of the following:

**Theorem 4.1.5.** *([2], Theorem 7.2)*

*There exists $\mathcal{R}$, a countable family of axis-parallel closed rectangles with:*

*(i)* $(\mathcal{R})_\omega \supseteq \mathbf{R}^2$*;*

*(ii)* $\mathcal{R}$ *is not 2-decomposable.*

## 4.2 Closed unit squares with small rotation

The proof of Theorem 1.3.9 is similar to Theorem 1.3.8, we only need to use points instead of the intervals $I_s$ ($s \in \omega_0^{<\omega}$) and to use the following elementary statement instead of Lemma 4.1.1:

Let $l$ be the $x = -y$ line, $\overrightarrow{\mathbf{v}}$ be the vector from the origin to $(1, 1)$.

**Lemma 4.2.1.** *For any* $\varepsilon > 0$ *we can choose* $\varepsilon_1 > 0$ *such that for any* $I \subseteq l + (1 - \varepsilon_1)\overrightarrow{\mathbf{v}}$ *and* $A \subseteq l$ *finite set, we can find* $\{I_i : i \in \omega\} \subseteq I$ *disjoint intervals,* $\{S_i : i \in \omega\} \subseteq \mathcal{S}_\varepsilon$ *and* $\mathbf{p} \in (l \setminus A)$ *such that for all* $i \in \omega$*:*

$\bullet_1$ $\mathbf{v_4}(S_i) = \mathbf{p}$*;*

$\bullet_2$ $S_i \cap (l + (1 - \varepsilon_1)\overrightarrow{\mathbf{v}}) = I_i$*.*

## 4.3 Axis-parallel closed squares with side length between $1-\varepsilon$ and $1$

*Proof of Theorem 1.3.10.* Let $C$ be the closed unit square, $\frac{1}{4} > \varepsilon > 0$ and let

$$\mathcal{T}(C, \varepsilon) = \{C_i : i \in \omega\}$$

a set of axis-parallel closed squares with:

$\bullet_1$ $C_0 = C$,

$\bullet_2$ the side length of $C_i$ is less than the side length of $C_{i-1}$ minus $\varepsilon^{i+1}$ $(i \geq 1)$,

$\bullet_3$ the $\overrightarrow{\mathbf{v_4}(C_{i-1})\mathbf{v_4}(C_i)}$ vector is the $\overrightarrow{(0,0)(0,\varepsilon^i)}$ vector.

Using the construction $\mathcal{T}(C,\varepsilon)$, let us introduce some notation:

$\circ_1$ Let $\mathbf{p}(\mathcal{T}(C,\varepsilon)) = (0, \frac{\varepsilon}{1-\varepsilon})$. Note that $\mathbf{p}(\mathcal{T}(C,\varepsilon)) \in C_i$ for all $i \in \omega$, since $\varepsilon < \frac{1}{4}$.

$\circ_2$ Let $\mathcal{A}(\mathcal{T}(C,\varepsilon)) = \{A_i : i \in \omega\}$, the following set of open rectangles:

$\circ_{2.1}$ let $A_0$ be the open square with $(1,1)$ and $(1-\varepsilon^2, 1-\varepsilon^2)$ as opposite vertices. Note that $A_0 \subseteq C_0 \setminus (\cup_{j \in (\omega \setminus \{0\})} C_j)$,

$\circ_{2.2}$ let $A_i = int(C_i \setminus (\cup_{j \in (\omega \setminus \{i\})} C_j))$ for $i \geq 1$.

$\circ_3$ For any transformation $t$ of $\mathbf{R}^2$ let

$\circ_{3.1}$ $\mathcal{T}(t(C),\varepsilon) = \{t(C_i) : i \in \omega\}$,

$\circ_{3.2}$ $\mathbf{p}(\mathcal{T}(t(C),\varepsilon)) = t(\mathbf{p}(\mathcal{T}(C,\varepsilon)))$,

$\circ_{3.3}$ $\mathcal{A}(\mathcal{T}(t(C),\varepsilon)) = \{t(A_i) : i \in \omega\}$.

Let us denote by $B(\mathbf{x}, r)$ the 2 dimensional ball around $\mathbf{x}$ with radius $r$. Let us mention the following easy fact witout proof:

**Fact 4.3.1.** *For any $t$, a transformation of $\mathbf{R}^2$ with $t(C)$ axis-parallel and $\varepsilon > 0$ there is $t_1$, a transformation of $\mathbf{R}^2$ and $\varepsilon_1 > 0$ with:*

*(i)* $t_1(C)$ *is axis-parallel,*

*(ii)* $int(t(C)) \supseteq \mathcal{T}(t_1(C), \varepsilon_1)$,

*(iii)* $B(\mathbf{v_4}(t(C_i)), \varepsilon) \supseteq \mathbf{p}(\mathcal{T}(t_1(C), \varepsilon_1))$,

*(iv)* $B(\mathbf{v_2}(t(C_i)), \varepsilon) \supseteq \mathcal{A}(\mathcal{T}(t_1(C), \varepsilon_1))$.

We start to describe our construction:

- Let $\omega_0^{<\omega} = \{\langle s_0, s_1, ..., s_j\rangle \in \omega^{<\omega} : s_0 = 0\}$, and
- For $j \geq 1$ and $s = \langle s_0, s_1, ..., s_j\rangle \in \omega_0^{<\omega}$ let $s^- = \langle s_0, s_1, ..., s_{j-1}\rangle$, $|s| = j$ and

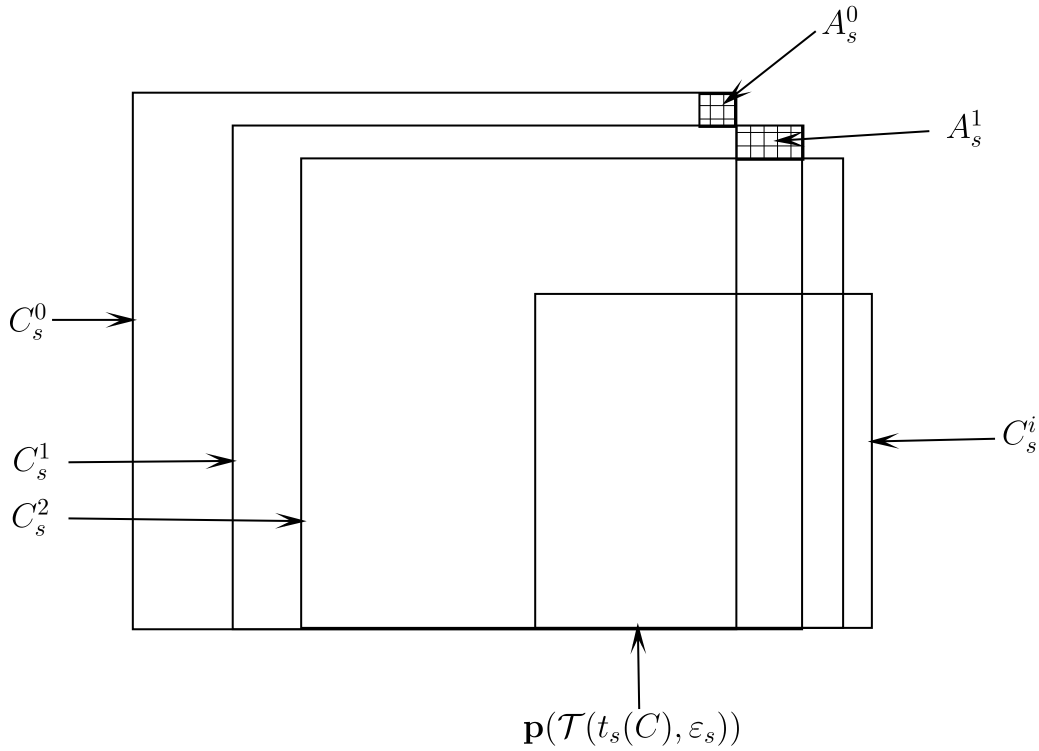


Figure 4.3: The construction of $\mathcal{T}(t_s(C), \varepsilon_s)$

let $s|i = \langle s_0, s_1, ..., s_{i-1}\rangle$ for $i < |s|$.

In the *jth* step ($j \geq 1$) for $s \in \omega_0^{<\omega}$, $|s| = j$ we define

$$\mathcal{T}(t_s(C), \varepsilon_s),$$

where $t_s$ is a transformation of $\mathbf{R}^2$. To do this for $s \in \omega_0^{<\omega}$, $|s| = j$, ($j \geq 1$) we define:

1.) $x_s$ and $y_s$, for

$$\mathbf{v_4}(t_s(C)) = (\sum_{j \leq |s|} x_{s|j}, \sum_{j \leq |s|} y_{s|j}),$$

2.) $|t_s(C)|$, and

3.) $\varepsilon_s$.

Now we write down what assumptions do we need on 1.)- 3.):

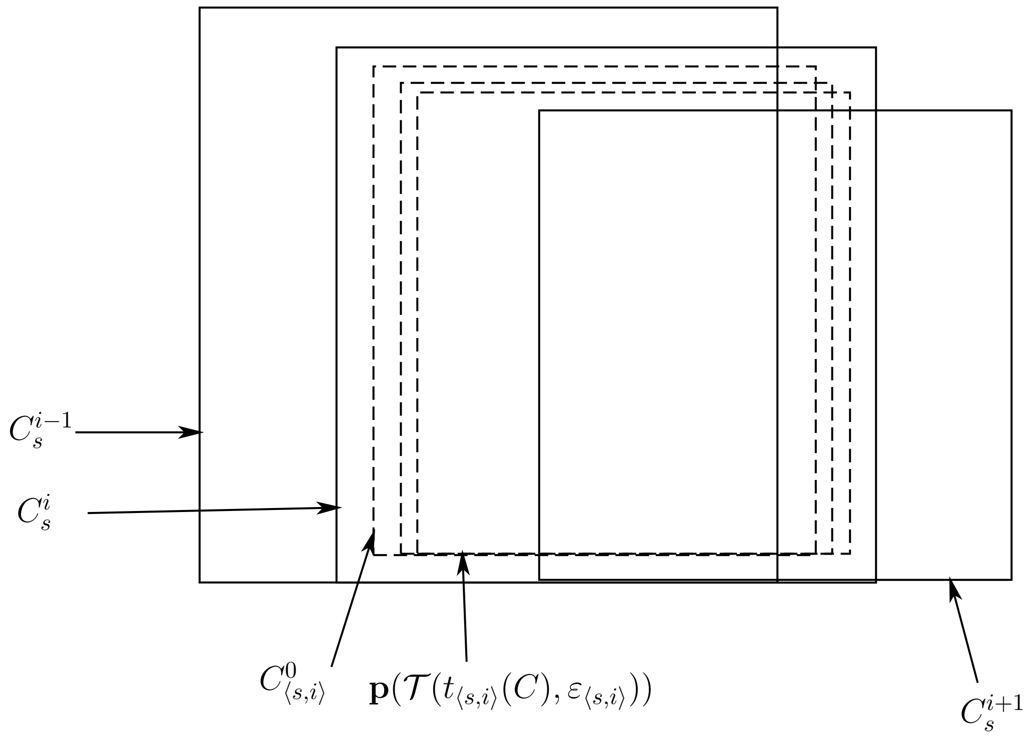


Figure 4.4: A step

For $s \in \omega_0^{<\omega}$ and $i \in \omega$ let $A_{i,s} = t_s(A_i)$ and $C_{i,s} = t_s(C_i)$.

If $s = \langle s^-, i\rangle = \langle s_0, s_1, ..., s_w, 0, 0, ..., 0, i\rangle$ with $s_w \neq 0$ let:

a0) $\mathcal{T}(t_s(C), \varepsilon_s) \subseteq \mathcal{U}_\varepsilon$

a1) $C_{i,s^-} \supseteq t_s(C, \varepsilon_s)$,

a2) $A_{i,s^-} \supseteq \mathcal{A}(\mathcal{T}(t_s(C), \varepsilon_s))$,

b) $y_s + \frac{\varepsilon_s}{1-\varepsilon_s} < \varepsilon^{i+1}_{s^-}$, and

c) $x_{\langle s^-,0\rangle} > x_{\langle s^-,1\rangle} > ...$ with

$$\sum_{j\leq |s|} x_{\langle s^-,0\rangle|j} < \sum_{j\leq w-1} x_{\langle s_0,s_1,...,s_w-1,0...0\rangle|j}.$$

Using Fact 4.3.1 we can easily choose $x_s, y_s, |t_s(C)|$ and $\varepsilon_s$ satisfying a1)-c).

Note that

**Claim 4.3.2.** $\mathbf{p}(\mathcal{T}(t_s(C),\varepsilon_s)) \in C_{j,r} \Leftrightarrow$

1.) $r = s|k$ for some $k < |s|$ and $j \leq s_k$, or

2.) $r = s$ and $i \in \omega$.

*Proof.* It can be easily seen by induction using b) and c).

□

Let

$$A = \cup_{s\in\omega_0^{<\omega}} \mathbf{p}(\mathcal{T}(t_s(C),\varepsilon_s)) \bigcup \cap_{j\geq 1} \cup_{|s|=j, s\in\omega_0^{<\omega}} \mathcal{A}(\mathcal{T}(t_s(C),\varepsilon_s)), and$$

$$\mathcal{U} = \cup_{s\in\omega_0^{<\omega}} \mathcal{T}(t_s(C),\varepsilon_s).$$

**Lemma 4.3.3.** *The following statements are true:*

*a)* $A \subseteq (\cup_{s\in\omega_0^{<\omega}} \mathcal{T}(t_s(C),\varepsilon_s))_\omega$,

*b)* $A \not\subseteq \cap_{j\in 2} c^{-1}(\{j\})_\omega$ *for all* $c : \cup_{s\in\omega_0^{<\omega}} \mathcal{T}(t_s(C),\varepsilon_s) \to 2$.

*Proof.* a) is immediate by the construction.

b) by Claim 4.3.2 for all $s \in \omega_0^{<\omega}$ there are infinitely many $i \in \omega$ with $c(C_{i,s}) = 0$, so we can choose $t \in \omega^\omega$ with $c(C_{t|i}) = 0$ for all $i \in \omega$. But then $\cap_{i\in\omega} A_{t|i} \not\subseteq c^{-1}(\{1\})_\omega$.

□

We are done with Theorem 1.3.10.

□

# 5 Open questions

However we think that with much more work we could prove our main result for (open or closed) convex symmetric polygons instead of the (open or closed) unit square we do not know the following:

**Question 5.0.4.** *Let $P$ be an open convex polygon and $\mathcal{F} \subseteq \mathcal{T}_P$ (resp.$\mathcal{T}_{\overline{P}}$). Is $\mathcal{F}$ $\omega$-decomposable over $(\mathcal{F})_\omega$?*

Or can we prove something for disks?

**Question 5.0.5.** *Let $D$ be the open unit disk and $\mathcal{F} \subseteq \mathcal{T}_D$ (resp.$\mathcal{T}_{\overline{D}}$). Is $\mathcal{F}$ $\omega$-decomposable over $(\mathcal{F})_\omega$?*

Finally it worth to pose as a question the $\omega$-fold and generalized version of Pach's conjecture:

**Question 5.0.6.** *Let $C$ be a convex planar set and $\mathcal{F} \subseteq \mathcal{T}_C$. Is $\mathcal{F}$ $\omega$-decomposable over $(\mathcal{F})_\omega$?*

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