PHD DISSERTATION

Existence, Regularity and Perturbation Theory for Some Semilinear Differential Equations in Hilbert Spaces

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Submitted to

CENTRAL EUROPEAN UNIVERSITY
Department of Mathematics and its Applications

In partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics and its Applications

Budapest, Hungary
2013

Contents

Abstract

We consider in a real Hilbert space H the Cauchy problem $(P_0)\colon u'(t)+Au(t)+Bu(t)=f(t)$, $0<t<T$; $u(0)=u_0$, where $A\colon D(A)\subset H\to H$ is a maximal monotone linear operator, $B\colon H\to H$ is a Lipschitz monotone (nonlinear) operator, and $f\colon [0,T]\to H$ is a given function. A typical example of problem (P_0) is the semilinear heat equation when $-A$ is the Laplace operator Δ with the homogeneous Dirichlet boundary conditions. We associate with problem (P_0) the following elliptic-like regularizations: $(P_1^\varepsilon)\colon -\varepsilon u''(t)+u'(t)+Au(t)+Bu(t)=f(t)$, $0<t<T$; $u(0)=u_0$, $u(T)=u_T$, and $(P_2^\varepsilon)\colon -\varepsilon u''(t)+u'(t)+Au(t)+Bu(t)=f(t)$, $0<t<T$; $u(0)=u_0$, $u'(T)=u_T$, where $\varepsilon>0$ is a small parameter. Problems (P_1^ε) and (P_2^ε) are essentially different and require different methods of investigation. We discuss the existence, uniqueness and higher regularity for the solutions of problems (P_0), (P_1^ε) and (P_2^ε). Then we establish the asymptotic expansions of order zero for the solutions of problems (P_1^ε) and (P_2^ε), as well as an asymptotic expansion of order one for the solution of problem (P_2^ε). A boundary layer of order zero occurs in problem (P_1^ε) with respect to the norm of $C([0,T];H)$, but the boundary layer of order zero is not visible with respect to the norm of $L^2(0,T;H)$. Problem (P_2^ε) turns out to be a regular perturbation problem of order zero with respect to the norm of $C([0,T];H)$, hence, it is also a regular perturbation problem of order zero with respect to the norm of $L^2(0,T;H)$. However, when we establish the asymptotic expansion of order one for the solution of problem (P_2^ε), a boundary layer of order one occurs with respect to the norm of $C([0,T];H)$, but this boundary layer is not visible with respect to the norm of $L^2(0,T;H)$.

Acknowledgements

I want to express my deep gratitude to Professor Gheorghe Moroşanu, my supervisor, for the professional guidance and support he gave me during the research period, as well as during the first year of my PhD studies, at the Central European University. I have greatly benefited from his immense knowledge, experience and kindness. I found him a brilliant teacher, an experienced and very helpful supervisor, and a very kind person.

I am also grateful to Elvira Kadvany, the department coordinator, for her assistance in resolving various administrative issues.

My special thoughts go to my parents and my wife for their never-ending support. This thesis is dedicated to them.

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Budapest, 2013

Introduction

The main purpose of this thesis is to study, in a real Hilbert space H, a Cauchy problem, and its elliptic-like regularizations. More explicitly, we consider the following Cauchy problem:

$$u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T, \quad u(0) = u_0, \tag{P_0}$$

where $A\colon D(A) \subset H \to H$ is a maximal monotone linear operator, $B\colon H \to H$ is a Lipschitz monotone (nonlinear) operator, $T > 0$ is a given time instant, $u_0 \in H$ is a given initial state, and $f\colon [0, T] \to H$ is a given function.

An important special case of problem (P_0) is the semilinear heat equation, when $-A$ is the Laplace operator Δ, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $H = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, and $Bv = \beta \circ v$, for all $v \in L^2(\Omega)$, where $\beta\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz nondecreasing function.

We also consider two elliptic-like regularizations of (P_0), i.e., the following elliptic-like higher order differential equations, involving a small parameter $\varepsilon > 0$, with the same condition at $t = 0$ as problem (P_0), and another boundary condition at $t = T$:

$$\left| \begin{aligned} -\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T, \\ u(0) = u_0, \quad u(T) = u_T, \end{aligned} \right. \tag{P_1^ε}$$

or

$$\left| \begin{aligned} -\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T, \\ u(0) = u_0, \quad u'(T) = u_T. \end{aligned} \right. \tag{P_2^ε}$$

Problems (P_1^ε) and (P_2^ε) are essentially different and require different methods of investigation. We first investigate the existence, uniqueness, and regularity results for the solutions of problems (P_0), (P_1^ε) and (P_2^ε). While the results are important in themselves, we need them in order to validate the asymptotic expansions for the solutions of problems (P_1^ε) and (P_2^ε). It turns out that, for a given f, the solutions of problems (P_1^ε) and (P_2^ε) are more regular than the solution of problem (P_0).

Next, we investigate if the solution u of problem (P_0) can be approximated by the solutions u_ε's of problems (P_1^ε) and (P_2^ε), which are more regular than u. We show that problem (P_1^ε) is a **singular perturbation problem of the boundary layer type (of order zero)**, i.e.,

$\|u_\varepsilon - u\|_{C([0,T];H)}$ *does not tend to zero as* $\varepsilon \to 0^+$*, but for a small* $0 < \delta < T$, $\|u_\varepsilon - u\|_{C([0,T-\delta];H)}$ *tends to zero as* $\varepsilon \to 0^+$*, and there exists a function,* $i = i(\tau)$*, where* $\tau := (T-t)/\varepsilon$*, such that* $\|u_\varepsilon - u - i\|_{C([0,T];H)}$ *tends to zero as* $\varepsilon \to 0^+$.

The function i is called a **boundary layer function of order zero**, and τ is called the **fast variable**. The boundary layer function of order zero fills the gap between u_ε and u in the boundary layer $[T-\delta, T]$. We call the expansion

$$u_\varepsilon(t) = u(t) + i(\tau) + r_\varepsilon(t), \quad 0 \le t \le T,$$

an **asymptotic expansion of order zero**.

However, we show that problem (P_1^ε) is a **regular perturbation problem** of order zero with respect to the norm of $L^2(0,T;H)$, i.e., $\|u_\varepsilon - u\|_{L^2(0,T;H)}$ tends to zero as $\varepsilon \to 0^+$. So the boundary layer of order zero is not visible with respect to the norm of $L^2(0,T;H)$.

We show that problem (P_2^ε) is a regular perturbation problem of order zero with respect to the norm of $C([0,T];H)$, hence, it is also a regular perturbation problem of order zero with respect to the norm of $L^2(0,T;H)$.

Problem (P_1^ε) or (P_2^ε) is said to have an asymptotic expansion of order $N \ge 1$ if its solution u_ε has the following expansion:

$$u_\varepsilon(t) = [u(t) + i(\tau)] + \sum_{k=1}^{N} \varepsilon^k [u_k(t) + i_k(\tau)] + r_\varepsilon(t), \quad 0 \le t \le T,$$

where:

u is the solution of (P_0);

$\tau := (T-t)/\varepsilon$ is the fast variable;
$i = i(\tau)$ is a boundary layer of order zero
u, u_k, $k = 1, \cdots, N$, are the first $N+1$ regular terms;
i, i_k, $k = 1, \cdots, N$, are the corresponding boundary layer functions;

$$\|r_\varepsilon\|_X = \mathcal{O}(\varepsilon^{N+\alpha}), \quad \alpha > 0, \quad X = C([0,T];H) \text{ or } X = L^2(0,T;H).$$

When we investigate the asymptotic expansion of order one for the elliptic-like regularization (P_2^ε), a boundary layer of order one occurs with respect to the norm of $C([0,T];H)$, but the boundary layer of order one is not visible with respect to the norm of $L^2(0,T;H)$.

This thesis is divided into four chapters.

Chapter 1 (Preliminaries)**.** We quickly go through some well-known results related to the topics, which we need in the next chapters, such as Sobolev spaces, vector-valued Sobolev spaces, maximal monotone nonlinear operators, semigroups of contractions in a real Hilbert space, C_0-semigroups of contractions, and the evolution equations in a real Hilbert space. In fact, the vector-valued Sobolev spaces are what we need to prove our results, while the (scalar-valued) Sobolev spaces are used just to give some examples. Semigroups of contractions, and C_0-semigroups, have many applications in partial differential equations, as well as in other fields, but we mention only those results which we need.

Chapter 2 (Presentation of the Problems)**.** We give a presentation of all problems which we want to investigate, a historical background of singular perturbation problems of boundary layer type, a brief survey of some known results related to the problems similar to the main problems discussed in this thesis.

Chapter 3 (Existence, Uniqueness and Regularity Theorems)**.** We discuss the existence, uniqueness and regularity results for problems (P_0), (P_1^ε) and (P_2^ε). In turns out that every level of regularity for the solution of (P_0) can be reached under appropriate conditions. But, instead of going on indefinitely, we just prove the regularity results up to order two, i.e., we show that the solution u of problem (P_0) belongs to $W^{2,2}(0,T;H)$ or $C^2([0,T];H)$ under appropriate conditions, though problem (P_0) is a first order differential equation. We need the regularity results to validate the asymptotic expansions for the elliptic-like regularizations (P_1^ε) and (P_2^ε). However, it seems possible to consider the fourth order elliptic-like regularizations for which we need more regularity for the solution of problem (P_0) to validate the asymptotic expansions.

Chapter 4 (Asymptotic Expansions)**.** We validate the asymptotic expansions for the solutions of the elliptic-like regularizations (P_1^ε) and (P_2^ε). We consider the asymptotic expansions of order zero for the elliptic-like regularizations (P_1^ε) and (P_2^ε), as well as the asymptotic expansion of order one for the elliptic-like regularization (P_2^ε). At the end, we mention some open problems related to the main problems discussed in this thesis.

Chapter 1

Preliminaries

In this chapter, we will go through some well-known results which we will need in the next chapters. We will not present the proof of any statement, but we will give the appropriate references to every statement. We will always assume that all operators are single-valued, nonlinear and unbounded unless otherwise specified.

1.1 Some function spaces

Let X be a real Banach space with norm $\|\cdot\|$, and let $\Omega \subset \mathbb{R}^n$ be a nonempty Lebesgue measurable set. A function $u\colon \Omega \to X$ is called **strongly measurable** if $u^{-1}(U)$ is Lebesgue measurable for every open $U \subset X$, and there exists a set N of Lebesgue measure zero such that $f(\Omega \setminus N)$ is separable. A function $u\colon \Omega \to X$ is called **Bochner integrable** if it is strongly measurable, and $\|u\|\colon \Omega \to \mathbb{R}$ is Lebesgue integrable.

We denote by $L^p(\Omega; X)$, $1 \leq p < \infty$, the space of (equivalence classes with respect to the equality a. e. in Ω of) strongly measurable functions $u\colon \Omega \to X$ such that $\|u\|^p\colon \Omega \to \mathbb{R}$ is Lebesgue integrable over Ω. $L^p(\Omega; X)$ is a real Banach space with the norm $\|u\|_{L^p(\Omega;X)} = \left(\int_\Omega \|u(x)\|^p dx\right)^{1/p}$.

We denote by $L^\infty(\Omega; X)$ the space of (equivalence classes with respect to the equality a. e. in Ω of) strongly measurable functions $u\colon \Omega \to X$ such that $\|u\|\colon \Omega \to \mathbb{R}$ is essentially bounded in Ω. $L^\infty(\Omega; X)$ is a real Banach space with the norm $\|u\|_{L^\infty(\Omega;X)} = \operatorname{ess\,sup}_{x\in\Omega}\|u(x)\|$.

We denote by $L^p_{\text{loc}}(\Omega; X)$, $1 \le p \le \infty$, the space of (equivalence classes with respect to the equality a. e. in Ω of) strongly measurable functions $u\colon \Omega \to X$ such that the restriction of u to every compact set $K \subset \Omega$ is in $L^p(K; X)$.

Usually, we identify an equivalence class of $L^p(\Omega; X)$, $L^p_{\text{loc}}(\Omega; X)$ with one of its representatives. If $X = \mathbb{R}$, we write $L^p(\Omega)$, $L^p_{\text{loc}}(\Omega)$ instead of $L^p(\Omega; \mathbb{R})$, $L^p_{\text{loc}}(\Omega; \mathbb{R})$, respectively. If Ω is an interval in $\mathbb{R}$, say $\Omega = (a, b)$, $-\infty \le a < b \le +\infty$, we write $L^p(a, b; X)$, $L^p_{\text{loc}}(a, b; X)$ instead of $L^p((a, b); X)$, $L^p_{\text{loc}}((a, b); X)$, respectively.

Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, and let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We call an $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}_0^n$ a multi-index. We use the following symbols associated with α

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$
$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}.$$

Let Ω be a nonempty open subset of $\mathbb{R}^n$. We denote by $C_0^\infty(\Omega)$ the space of all functions $\varphi\colon \Omega \to \mathbb{R}$ such that φ has continuous partial derivatives of any order, and φ has compact support in Ω. Let $u \in L^1_{\text{loc}}(\Omega)$, we say that $v \in L^1_{\text{loc}}(\Omega)$ is the **weak or distributional partial derivative** of u of order α, denoted by $D^\alpha u$, if

$$\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|}\int_\Omega v\varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

We define $W^{k,p}(\Omega)$, $1 \le p \le \infty$, $k \in \mathbb{N}$, as

$$W^{k,p}(\Omega) = \left\{u \in L^p(\Omega)\colon D^\alpha u \in L^p(\Omega) \quad \forall 0 < |\alpha| \le k\right\}.$$

$W^{k,p}(\Omega)$, $1 \le p < \infty$, $k \in \mathbb{N}$, is a real Banach space with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{0\le|\alpha|\le k}\|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p}.$$

$W^{k,\infty}(\Omega)$ is a real Banach space with the norm

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{0\le|\alpha|\le k}\|D^\alpha u\|_{L^\infty(\Omega)}.$$

We also define, for $1 \le p \le \infty$,

$$W_0^{k,p}(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in } W^{k,p}(\Omega).$$

$W^{k,p}(\Omega), W_0^{k,p}(\Omega)$, with the above norms, are called **Sobolev spaces**.

We define $W^{k,p}_{\mathrm{loc}}(\Omega)$, $1 \le p \le \infty$, $k \in \mathbb{N}$, as

$$W^{k,p}_{\mathrm{loc}}(\Omega) = \{u \in L^p_{\mathrm{loc}}(\Omega) \colon D^\alpha u \in L^p_{\mathrm{loc}}(\Omega) \quad \forall 0 < |\alpha| \le k\}.$$

If $p = 2$, we also write $H^k(\Omega)$, $H_0^k(\Omega)$, $H^k_{\mathrm{loc}}(\Omega)$ instead of $W^{k,2}(\Omega)$, $W_0^{k,2}(\Omega)$, $W^{k,2}_{\mathrm{loc}}(\Omega)$, respectively. $H^k(\Omega)$, for $k \in \mathbb{N}$, is a Hilbert space with respect to the scalar product

$$(u,v)_{H^k(\Omega)} = \sum_{0 \le |\alpha| \le k} \int_\Omega D^\alpha u(x)\, D^\alpha v(x) dx.$$

We now define the Sobolev spaces of vector-valued functions. Let Ω be an open interval in $\mathbb{R}$, say (a,b), $-\infty \le a < b \le +\infty$. We denote by $C_0^\infty(a,b)$ the space of all functions $\varphi \colon (a,b) \to \mathbb{R}$ such that φ has continuous derivatives of any order, and φ has compact support in (a,b). Let $u \in L^1_{\mathrm{loc}}(a,b;X)$, we say that $v \in L^1_{\mathrm{loc}}(a,b;X)$ is the **jth weak or distributional derivative** of u, denoted by $u^{(j)}$, if

$$\int_a^b \frac{d^j\varphi}{dt^j}\, u = (-1)^j \int_a^b \varphi\, v \quad \forall \varphi \in C_0^\infty(a,b).$$

We define $W^{k,p}(a,b;X)$, $1 \le p \le \infty$, $k \in \mathbb{N}$, as

$$W^{k,p}(a,b;X) = \left\{u \in L^p(a,b;X) \colon u^{(j)} \in L^p(a,b;X) \quad \forall 1 \le j \le k\right\}.$$

$W^{k,p}(a,b;X)$, $1 \le p < \infty$, $k \in \mathbb{N}$, is a real Banach space with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \left(\sum_{j=0}^k \|u^{(j)}\|^p_{L^p(a,b;X)}\right)^{1/p}.$$

$W^{k,\infty}(a,b;X)$ is a real Banach space with the norm

$$\|u\|_{W^{k,\infty}(a,b;X)} = \max_{0 \le j \le k} \|u^{(j)}\|_{L^\infty(a,b;X)}.$$

$W^{k,p}(a,b;X)$, with the above norms, are called **Sobolev spaces of vector-valued functions**.

We define $W^{k,p}_{\mathrm{loc}}(a,b;X)$, $1 \le p \le \infty$, $k \in \mathbb{N}$, as

$$W^{k,p}_{\mathrm{loc}}(a,b;X) = \left\{ u \in L^p_{\mathrm{loc}}(a,b;X) \colon u^{(j)} \in L^p_{\mathrm{loc}}(a,b;X) \quad \forall 1 \le j \le k \right\}.$$

If $p = 2$, we also write $H^k(a,b;X)$, $H^k_{\mathrm{loc}}(a,b;X)$ instead of $W^{k,2}(a,b;X)$, $W^{k,2}_{\mathrm{loc}}(a,b;X)$, respectively. If X is a real Hilbert space with scalar product $(\cdot,\cdot)$, then $H^k(a,b;X)$, $k \in \mathbb{N}$, is a Hilbert space with respect to the scalar product

$$(u,v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \left(u^{(j)}(t), v^{(j)}(t) \right) dt.$$

In the rest of this section, we shall assume that $-\infty < a < b < +\infty$.

Theorem 1.1 (see, e.g., [22])**.** *Let X be a reflexive Banach space. Let $u \colon [a,b] \to X$ be an absolutely continuous function. Then u is differentiable* a. e. *on (a,b), $\frac{du}{dt} \in L^1(a,b;X)$ and*

$$u(t) = u(a) + \int_a^t \frac{du}{ds}(s)ds, \qquad a \le t \le b.$$

We denote by $A^{k,p}(a,b;X)$, where $k \in \mathbb{N}$ and $1 \le p \le \infty$, the space of all absolutely continuous functions $u \colon [a,b] \to X$ such that $d^j u/dt^j$, for $j = 0,1,\dots,k-1$, are absolutely continuous on $[a,b]$, and $d^k u/dt^k \in L^p(a,b;X)$.

Theorem 1.2 (see, e.g., [9, p. 19])**.** *Let X be a Banach space and let $u \in L^p(a,b)$, $1 \le p \le \infty$. Then the following conditions are equivalent:*

(i) $u \in W^{k,p}(a,b;X)$

(ii) *There exists $u_1 \in A^{k,p}(a,b;X)$ such that $u(t) = u_1(t)$* a. e. *on (a,b).*

So, we can identify $W^{k,p}(a,b;X)$ with $A^{k,p}(a,b;X)$. Moreover, if X is reflexive, then by Theorem 1.1 we can identify $W^{1,1}(a,b;X)$ with the space of all X-valued absolutely continuous functions on $[a,b]$, and $W^{1,\infty}(a,b;X)$ can be identified with the space of all Lipschitz X-valued functions on $[a,b]$.

For further information on Bochner integral, see, e.g., D.L. Cohn [14], and S. Lang [23]; for further information on Sobolev spaces, see, e.g., R.A. Adams [1], S. Kesavan [21], W. Rudin [31], L. Schwartz [33], and K. Yosida [37]; for further information on vector-valued Sobolev spaces, see, e.g., V. Barbu [9], and L. Schwartz [32].

1.2 Monotone operators

Let H be a real Hilbert space with the scalar product $(\cdot,\cdot)$, and the induced norm $\|\cdot\|$. As we mentioned earlier, we will always assume that all operators are single-valued, nonlinear and unbounded unless otherwise specified. However, most of the results of this section and of the next section can be extended to multivalued operators or to the general Banach spaces.

A set $S \subset H \times H$ is said to be **monotone** if

$$(x_1 - x_2, y_1 - y_2) \geq 0 \quad \forall (x_1, y_1), (x_2, y_2) \in S.$$

An operator $A\colon D(A) \subset H \to H$ is called **monotone** if its graph is a monotone subset of $H \times H$, i.e.,

$$(x_1 - x_2, Ax_1 - Ax_2) \geq 0, \quad \forall x_1, x_2 \in D(A).$$

Let $A\colon D(A) \subset H \to H$ be a linear operator. Then, obviously, A is monotone if and only if

$$(x, Ax) \geq 0, \quad \forall x \in D(A).$$

A monotone linear operator is called a **positive** linear operator.

A monotone operator $A\colon D(A) \subset H \to H$ is said to be **maximal monotone** if its graph is not properly contained in any monotone subset of $H \times H$.

Theorem 1.3 (see, e.g., [25, p. 20])**.** *Let $A\colon D(A) \subset H \to H$ be a maximal monotone operator. Then A is demiclosed, i.e.,*

$$\begin{aligned} x_n \to x \quad &\textit{strongly in} \quad H, \\ Ax_n \to y \quad &\textit{weakly in} \quad H \\ \Rightarrow x \in D(A) \quad &\textit{and} \quad y = Ax. \end{aligned}$$

Theorem 1.4 (G. Minty; see, e.g., [9, p. 39])**.** *Let $A\colon D(A) \subset H \to H$ be a monotone operator. Then A is maximal monotone if and only if $R(I + \lambda A) = H$ for some $\lambda > 0$ or, equivalently, for all $\lambda > 0$.*

Theorem 1.5 (G. Minty; see, e.g., [25, p. 25]). *Let $A\colon H \to H$ be a monotone operator. Then A is maximal monotone if it is hemicontinuous, i.e., for all $x, y \in H$*

$$A(x+ty) \to Ax \quad \textit{weakly as} \quad t \to 0.$$

Theorem 1.6 (R.T. Rockafellar; see, e.g., [11, p. 36]). *Let $A\colon D(A) \subset H \to H$ and $B\colon D(B) \subset H \to H$ be two maximal monotone operators. Then $A+B$ is maximal monotone if*

$$D(A) \cap \operatorname{Int}(D(B)) \neq \emptyset.$$

Definition 1.7. *Let $A\colon D(A) \subset H \to H$ be a maximal operator and $\lambda > 0$. We define the following well-known operators (which will be shown single-valued):*

$$J_\lambda = (I + \lambda A)^{-1},$$
$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda),$$

where J_λ is called the ***resolvent*** *of A, and A_λ is called the* ***Yosida approximation*** *of A.*

From Theorem 1.4, we have

$$D(J_\lambda) = D(A_\lambda) = H, \quad \forall \lambda > 0.$$

Theorem 1.8 (see, e.g., [25, p. 21]). *Let $A\colon D(A) \subset H \to H$ be a maximal monotone operator. Then, for every $\lambda > 0$,*

(i) *J_λ is nonexpansive, i.e., Lipschitz with constant 1. So, in particular, both J_λ and A_λ are single-valued ;*

(ii) $A_\lambda(x) = A(J_\lambda(x)), \quad \forall x \in H$;

(iii) A_λ *is monotone, and Lipschitz with constant* $1/\lambda$;

(iv) $\|A_\lambda(x)\| \leq \|A(x)\|, \quad \forall x \in D(A)$;

(v) $\lim_{\lambda\to 0} A_\lambda(x) = A(x), \quad \forall x \in D(A)$;

(vi) $\overline{D(A)}$ *is convex, and* $\lim_{\lambda\to 0} J_\lambda(x) = Proj_{\overline{D(A)}}\, x, \quad \forall x \in H$.

Remark 1.9. *If $A\colon D(A) \subset H \to H$ is a maximal monotone linear operator, then it can be shown easily that $\overline{D(A)} = H$, see, e.g., [12, p. 181]. So by (vi) of Theorem 1.8, we have that $\lim_{\lambda\to 0} J_\lambda(x) = x$, for all $x \in H$.*

Definition 1.10. *Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set, and let $A\colon D(A) \subset H \to H$ be a maximal monotone operator. We define $\bar{A}\colon D(\bar{A}) \subset L^2(\Omega; H) \to L^2(\Omega; H)$ by*

$$D(\bar{A}) = \left\{u \in L^2(\Omega; H);\ A \circ u \in L^2(\Omega; H)\right\},$$
$$\bar{A}u = A \circ u \quad \forall u \in D(\bar{A}).$$

*The operator $\bar{A}$ is called the **canonical extension** of A to $L^2(\Omega; H)$ or the **realization** of A on $L^2(\Omega; H)$.*

It is easy to see that $\bar{A}$ is monotone. Moreover, if either $m(\Omega) < \infty$, where m is the Lebesgue measure, or $A0 = 0$, then $\bar{A}$ is maximal monotone. Indeed, if $g \in L^2(\Omega; H)$, then we set

$$u = (I + A)^{-1}g = [J_1 g - J_1 0] + [J_1 0] \in L^2(\Omega; H)$$
$$\Rightarrow A \circ u = g - u \in L^2(\Omega; H),$$

so $u \in D(\bar{A})$, and $u + \bar{A}u = g$. Hence, $\bar{A}$ is maximal monotone.
Note that, for $\lambda > 0$, the realizations of $(I + \lambda A)^{-1}$ and A_λ to $L^2(\Omega; H)$ are $(I + \lambda\bar{A})^{-1}$ and $(\bar{A})_\lambda$, respectively.

Definition 1.11. *Let $C \subset H$ be a convex subset. A function $\varphi : C \to (-\infty, +\infty]$ is called **convex** if*

$$\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y), \quad \forall x, y \in C, \quad \forall 0 < t < 1.$$

Geometrically, if $H = \mathbb{R}$, it means that any line segment joining any two points of the graph of φ lies above the graph of φ.

We can always extend the domain of φ as below

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in C \\ +\infty & \text{if } x \notin C. \end{cases}$$

It is easy to see that $\widetilde{\varphi} : H \to (-\infty, +\infty]$ is convex. So, we can always assume that φ is defined everywhere. φ is called **proper** if $\varphi \not\equiv +\infty$, and $D(\varphi) = \{\varphi(x) \neq +\infty\}$ is called the **effective domain** of φ.

A function $\varphi\colon H \to (-\infty, \infty]$ is said to be **lower semicontinuous** at $x_0 \in H$ if $\liminf_{x \to x_0} \varphi(x) = \varphi(x_0)$, or equivalently, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varphi(x) \geq \varphi(x_0) - \varepsilon$ if $\|x - x_0\| < \delta$.

Let $\varphi\colon H \to (-\infty, \infty]$ be a proper convex function. Its **subdifferential** $\partial\varphi\colon D(\partial\varphi) \subset H \to H$, in general a multivalued function, is defined by

$$\partial\varphi(x) = \begin{cases} \{y \in H\colon \varphi(v) \geq \varphi(x) + (v - x, y) \quad \forall v \in H\} & \text{if } x \in D(\varphi), \\ \emptyset & \text{if } x \notin D(\varphi). \end{cases}$$

So, $D(\partial\varphi)$ is contained in $D(\varphi)$.

Geometrically, when $H = \mathbb{R}$, $\partial\varphi(x)$ is the set of slopes of all lines passing through $(x, \varphi(x))$ and lying below the graph of φ.

Theorem 1.12 (see, e.g., [25, p. 36])**.** *Let $\varphi\colon H \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function, then $\partial\varphi$ is a maximal monotone operator.*

Definition 1.13. *Let $A\colon D(A) \subset H \to H$ be a linear operator with dense domain, i.e., $\overline{D(A)} = H$. We define a linear operator $A^\star : D(A^\star) \subset H \to H$,* *called the **adjoint** of A, as follows.*

$$D(A^\star) = \{y \in H; \exists c \geq 0 \text{ such that } |(Ax, y)| \leq c\|x\| \ \ \forall x \in D(A)\}.$$

It is easy to see that $D(A^\star)$ is a subspace containing at least 0 element. Since $D(A)$ is dense in H, if $y \in D(A^\star)$, then the map $x \mapsto (Ax, y)$ has a unique continuous linear extension on H which, by Riesz-Fréchet representation theorem, is given by an element of H which we denote by $A^\star(y)$. We have

$$(Ax, y) = (x, A^\star y), \quad \forall x \in D(A), \ \forall y \in D(A^\star).$$

*We say that A is **self-adjoint** if*

$$D(A^\star) = D(A) \quad \text{and} \quad A^\star = A.$$

Theorem 1.14 (see, e.g., [10] or [37, p. 259])**.** *Let $A\colon D(A) \subset H \to H$ be a positive self-adjoint linear operator. Then there exists a unique positive self-adjoint operator $B\colon D(B) \subset H \to H$ such that $D(A) \subset D(B)$, and $B^2 = A$ on $D(A)$. We call B the square root of A, and denote it by $A^{1/2}$.*

Theorem 1.15 (see, e.g., [25, p. 43])**.** *Let A be a positive self-adjoint linear operator. Then the function $\varphi\colon H \to (-\infty, +\infty]$ defined by*

$$\varphi(x) = \begin{cases} \frac{1}{2}\,\|A^{1/2}(x)\|^2, & \text{if } x \in D(A^{1/2}), \\ +\infty, & \text{otherwise}, \end{cases}$$

is proper, convex and lower semicontinuous. Moreover, $A = \partial\varphi$.

For further information on monotone operators, see, e.g., V. Barbu [9], H. Brézis [11], G. Moroşanu [25], and R.E. Showalter [34].

1.3 Semigroups of operators

Let C be a nonempty closed subset of H, which is a real Hilbert space with the scalar product $(\cdot,\cdot)$ and the induced norm $\|\cdot\|$. A **semigroup** on C is a family of operators $\mathcal{S} = \{S(t)\colon C \to C,\ t \geq 0\}$ satisfying the following conditions:

$$S(t_1+t_2)x = S(t_1)S(t_2)x, \quad \forall x \in C, \quad t_1, t_2 \geq 0, \quad \text{and}$$

$$S(0)x = x, \quad \forall x \in C.$$

A semigroup $\mathcal{S} = \{S(t)\colon C \to C,\ t \geq 0\}$ is called a **continuous semigroup of contractions** if it satisfies the following conditions:
(a) for every $x \in C$, the mapping $t \mapsto S(t)x$ is continuous on $[0,+\infty)$;
(b) $\|S(t)x - S(t)y\| \leq \|x-y\|, \quad \forall x, y \in C, \quad t \geq 0.$

A semigroup $\mathcal{S} = \{S(t)\colon H \to H,\ t \geq 0\}$ is called a C_0**-semigroup** if it satisfies (a), and each $S(t)$ is a continuous linear operator from H into H. If a C_0- semigroup also satisfies (b), then it is called a C_0**-semigroup of contractions**. Note that if $\mathcal{S}$ is a C_0-semigroup, then (a) and (b) are equivalent to the following conditions (a)$'$ and (b)$'$, respectively:
(a)$'$ for every $x \in H$, we have $\lim_{t\to 0^+} S(t)x = x$;
(b)$'$ $\|S(t)\|_{\mathcal{L}(H)} \leq 1 \quad \forall t \geq 0.$

Let $\mathcal{S} = \{S(t)\colon C \to C,\ t \geq 0\}$ be a semigroup. Then the **infinitesimal generator** of the semigroup $\mathcal{S}$, say A, is defined by

$$A\colon D(A) \subset H \to H,$$

$$D(A) = \left\{ x \in H\colon \lim_{h\to 0^+} \frac{S(h)x - x}{h} \text{ exists in } H \right\},$$

$$A(x) = \lim_{h\to 0^+} \frac{S(h)x - x}{h}, \qquad \forall x \in D(A).$$

Theorem 1.16 (Hille-Yosida Theorem: Lumer-Phillips form in Hilbert spaces, see, e.g., [17, p. 26] or [28, p. 14])**.** *Let $A\colon D(A) \subset H \to H$ be a linear operator. Then A is maximal monotone if and only if $-A$ is the infinitesimal generator of a C_0-semigroup of contractions on H.*

Definition 1.17. *A function $u \in C([0,T];H)$ is said to be a* ***strong solution*** *of the following Cauchy problem*

$$\frac{du}{dt} + Au(t) = f(t), \quad 0 < t < T, \tag{1.1}$$
$$u(0) = u_0,$$

if u is absolutely continuous on every compact subinterval of $(0,T)$, $u(0) = u_0$, and u satisfies (1.1) a. e. on $(0,T)$.

Theorem 1.18 (see, e.g., [25, p. 48])**.** *Let $A\colon D(A) \subset H \to H$ be maximal monotone, and let $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$. Then there exists a unique strong solution $u \in W^{1,\infty}(0,T;H)$ of the following Cauchy problem*

$$\frac{du}{dt} + Au(t) = f(t), \quad 0 < t < T, \quad u(0) = u_0.$$

Moreover, u is differentiable from the right at every point in $[0,T)$ and

$$\frac{d^+u}{dt}(t) + Au(t) = f(t), \qquad \forall t \in [0,T), \tag{1.2}$$

$$\left\|\frac{d^+u}{dt}(t)\right\| \le \|f(0) - A(u_0)\| + \int_0^t \left\|\frac{df}{ds}(s)\right\| ds, \qquad \forall t \in [0,T). \tag{1.3}$$

If $u, \bar{u}$ are the strong solutions corresponding to $(u_0, f), (\bar{u}_0, \bar{f}) \in D(A) \times W^{1,1}(0,T;H)$ then

$$\|u(t) - \bar{u}(t)\| \le \|u_0 - \bar{u}_0\| + \int_0^t \|f(t) - \bar{f}(t)\| dt, \quad \forall t \in [0,T]. \tag{1.4}$$

Remark 1.19. *Theorem 1.18 still holds if A is replaced with $A+B$, where $A\colon D(A) \subset H \to H$ is maximal monotone and $B\colon H \to H$ is a Lipschitz operator. The only modifications appear in estimates (1.3) and (1.4) which become:*

$$\left\|\frac{d^+u}{dt}(t)\right\| \le e^{\omega t}\left(\|f(0) - A(u_0)\| + \int_0^t e^{-\omega s}\left\|\frac{df}{ds}\right\| ds\right), \quad \forall t \in [0,T), \tag{1.5}$$

$$\|u(t)-\bar{u}(t)\| \leq e^{\omega t}\left(\|u_0-\bar{u}_0\| + \int_0^t e^{-\omega s}\|f(t)-\bar{f}(t)\|dt\right), \quad \forall t \in [0,T], \tag{1.6}$$

where ω is the Lipschitz constant of B (see, e.g., [11, p. 105] or [34, p. 181]).

Remark 1.20. *Assuming all conditions of Theorem 1.18 are satisfied, then by (1.2), we have that $u(t) \in D(A)$, for all $t \in [0,T)$. Moreover, if u happens to be in $C^1([0,T];H)$, then by (1.2), we have that $u'(t) + Au(t) = f(t)$, for all $t \in [0,T)$. By stretching the domain of f, we can extend these results for all $t \in [0,T]$. Finally, by Remark 1.19, these results hold when A is replaced by $A+B$, where $B\colon H \to H$ is a Lipschitz operator.*

Let $A\colon D(A) \subset H \to H$ be a maximal monotone operator. Consider the Cauchy problem

$$\frac{du}{dt} + Au(t) = 0, \quad t > 0, \quad u(0) = u_0.$$

By Theorem 1.18, we have that for any $u_0 \in D(A)$ there exists a strong solution $u(t)$, $t \geq 0$ of the above Cauchy problem. We set

$$S(t)u_0 = u(t), \quad t \geq 0.$$

It easily follows from the estimate (1.4) that for any $t \geq 0$, $S(t)$ is a contraction on $D(A)$ and so $S(t)$ can be extended as a contraction on $\overline{D(A)}$. Moreover, it is obvious that the family $\{S(t)\colon \overline{D(A)} \to \overline{D(A)}; t \geq 0\}$ is a semigroup of contractions. From (1.2), we have that the infinitesimal generator of this semigroup is $-A$.
On the other hand, from a result of M.G. Crandall and A. Pazy (see, e.g., H. Brézis [11, p. 114]), we know that if C is a nonempty closed convex set in H and $\{S(t)\colon C \to C; t \geq 0\}$ is a continuous semigroup of contractions, then there exists a unique maximal monotone operator A such that $\overline{D(A)} = C$ and the given semigroup coincides to that generated by $-A$.

For further information on semigroups of operators, see, e.g., V. Barbu [9], H. Brézis [11], and R.E. Showalter [34]; for further information on C_0-semigroups of contractions and the abstract Cauchy problem, see, e.g., H. Brézis [12], J.A. Goldstein [17], E. Hille and R.S. Phillips [18], and A. Pazy [28].

1.4 Singular perturbations

Consider the problems:

$$\begin{aligned}&(P_0)\colon\, L_0u = f_0;\\ &(P_\varepsilon)\colon\, L_0u + \varepsilon L_1 u = f_0 + \varepsilon f_1,\end{aligned}$$

where $\varepsilon > 0$ is a small parameter, L_0, L_1 are given operators which do not depend on ε, f_0, f_1 are given functions, and u is the unknown function. Usually, problem (P_ε) has more boundary conditions than problem (P_0).

Problem (P_ε) is called **regularly perturbed** with respect to some norm $\|\cdot\|$ if there exists a solution u of problem (P_0) such that

$$\|u_\varepsilon - u\| \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where u_ε is the solution of problem (P_ε).

Otherwise, problem (P_ε) is said to be **singularly perturbed** with respect to the norm $\|\cdot\|$.

Problem (P_ε) can be regularly perturbed with respect to one norm, but singularly perturbed with respect to another norm.

Example (Singular perturbation problem of the boundary layer type)**.**
Consider the following problems:

$$\begin{aligned}&(P_0)\colon\, u'(t) = 2t, \quad 0 < t < 1, \quad u(1) = 0;\\ &(P_\varepsilon)\colon\, \varepsilon u''(t) + u'(t) = 2t, \quad 0 < t < 1, \quad u(0) = 0 = u(1).\end{aligned}$$

The solution u_ε of problem (P_ε) can be expressed as

$$u_\varepsilon(t) = (t^2 - 1) + e^{-t/\varepsilon} + r_\varepsilon(t), \quad 0 \le t \le 1,$$

where $r_\varepsilon(t)$ converges uniformly to zero in $[0,1]$ as $\varepsilon \to 0^+$. Therefore, u_ε converges uniformly, as $\varepsilon \to 0^+$, to the function $u(t) := t^2 - 1$ on every interval $[\delta, 1]$, $0 < \delta < 1$, but not on the whole interval $[0,1]$. Problem (P_ε) is a singularly perturbed problem with respect to the sup norm. The function $e^{-t/\varepsilon}$ is called a **boundary layer function**, and it fills the gap between u_ε and u in the boundary layer $[0, \delta]$.
Note that problem (P_ε) is a regular perturbation problem with respect to the

L^p-norm, for $1 \leq p < \infty$, since $\|u_\varepsilon - u\|_{L^p(0,1)}$ tends to zero as $\varepsilon \to 0^+$. So the boundary layer is invisible in the space $L^p(0,1)$, for $1 \leq p < \infty$.

For further information on singular perturbations, see, e.g., L. Barbu and G. Moroşanu [7], W. Eckhaus [15], E.M. de Jager and J. Furu [19], J.L. Lions [24], R.E. O'Malley [26], [27], and A.B. Vasilieva, V.F. Butuzov and L.V. Kalachev [35].

Chapter 2

Presentation of the Problems

Boundary layers are often present in physical reality. A well known example is the so-called Prandtl boundary layer. In 1904, L. Prandtl, a German physicist, was able to explain the resistance of a fluid, with low viscosity, flowing past a solid body by advancing the hypothesis that the effect of viscosity is concentrated in a narrow layer near the surface of the body. The flow velocity changes quickly across this boundary layer (called Prandtl boundary layer). There are many examples of discontinuities and quick transitions, which occur either at the boundary or inside the corresponding spacial domain (see, e.g., K.O. Friedrichs [16]). Such phenomena (called "asymptotic" by Friedrichs [16]) may result from approximate description, in particular by neglecting small parameters. This leads, in many cases, to singular perturbations.

The theory of singular perturbations is well developed in the case of linear problems associated with ordinary and partial differential equations. In 1957, M.I. Vishik and L.A. Lyusternik [36] launched their method which has proved to be very useful in studying linear partial differential equations with singular perturbations. Since then a great deal of work has been devoted to this subject, including several monographs, such as A.B. Vasilieva, V.F. Butuzov and L.V. Kalachev [35], and J.L. Lions [24]. Lions' book represented a great advance since the whole theory was done in a functional analysis framework. Specifically, Lions considered abstract linear evolution equations

$$u'(t) + Au(t) = f(t), \quad 0 < t < T, \tag{2.1}$$

and regularizations of the form

$$\pm\, \varepsilon u''(t) + u'(t) + Au(t) = f(t), \quad 0 < t < T, \tag{2.2}$$

where $\varepsilon > 0$ is small parameter. The sign in front of the equation (2.2) is very important. In order to explain this, let us consider the case when $-A$ is the Laplace operator $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$, with some boundary condition(s) (e.g., the Dirichlet boundary condition, or combined boundary conditions on different parts of the boundary).
In this case, equation (2.1) is the classical heat equation, while equation (2.2) is a hyperbolic equation if $\varepsilon = +1$, and an elliptic one if $\varepsilon = -1$. The problem is whether the solution of equation (2.1), subjected to an initial condition, is approximated by the solutions of equation (2.2) for ε small. The methods of investigation depend essentially on the (hyperbolic or elliptic) character of equation (2.2). For a general linear operator A, equation (2.2) is said to be hyperbolic-like if $\varepsilon = +1$, and elliptic-like if $\varepsilon = -1$. The general linear case has also been extensively discussed by L. Barbu and G. Moroşanu [7] by using new techniques.

In applications we mostly meet nonlinear, in particular semilinear, partial differential equations. As a typical example, consider the semilinear heat equation

$$u_t(t,x) - \Delta u(t,x) + \beta(u(t,x)) = f(t,x), \quad 0 < t < T,\ x \in \Omega,$$

with the Dirichlet boundary condition

$$u(t,x) = 0, \quad \text{for} \quad 0 < t < T,\ x \in \partial\Omega,$$

and the initial condition

$$u(0,x) = u_0(x), \quad \text{for} \quad x \in \Omega,$$

where T is a given positive number, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, Δ is the Laplace operator with respect to $x = (x_1, \ldots, x_n)$, i.e., $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$, and $\beta \colon \mathbb{R} \to \mathbb{R}$ is a nonlinear function.
Now we consider the following elliptic regularizations of the semilinear heat equation, i.e., we consider a higher order equation which is elliptic in (t,x), containing a small parameter ε,

$$-\varepsilon u_{tt}(t,x) + u_t(t,x) - \Delta u(t,x) + \beta(u(t,x)) = f(t,x), \quad 0 < t < T,\ x \in \Omega,$$

with the Dirichlet boundary condition on $\partial\Omega$, as well as two-point boundary conditions in the form

$$u(0,x) = u_0(x), \quad u(T,x) = u_T(x), \quad \text{for} \quad x \in \Omega,$$

or

$$u(0,x) = u_0(x), \quad u_t(T,x) = u_T(x), \quad \text{for} \quad x \in \Omega.$$

It is also possible to consider the following hyperbolic regularization

$$\varepsilon u_{tt}(t,x) + u_t(t,x) - \Delta u(t,x) + \beta(u(t,x)) = f(t,x), \quad 0 < t < T, \ x \in \Omega,$$

with the Dirichlet boundary condition on $\partial\Omega$ and two initial conditions in the form

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_T(x), \quad \text{for} \quad x \in \Omega.$$

The above elliptic and hyperbolic regularizations of the semilinear heat equation have been discussed by L. Barbu and G. Moroşanu [7, pp. 209-226]. An elliptic regularization of the above semilinear heat equation, when $\beta(u) = u^3$, was discussed by J.L. Lions [24, pp. 424-427]. L. Barbu and E. Cosma [8] have discussed the elliptic regularizations of the semilinear heat equation with the following initial condition and the Neumann boundary condition

$$u(0,x) = u_0(x), \quad -\frac{\partial u}{\partial \nu}(t,x) = \alpha(u(t,x)),$$

where ν is the outer unit normal to the boundary $\partial\Omega$, and $\alpha\colon \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function.

In what follows we replace some semilinear problems with singularly perturbed, higher order problems, admitting solutions which are more regular and approximate the solutions of the original problems. We will instead do everything in more general settings, i.e., we start with the following semilinear problem

$$\begin{cases} u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T, & (E) \\ u(0) = u_0, & (IC) \end{cases} \tag{P_0}$$

where $A\colon D(A) \subset H \to H$ is a maximal monotone linear operator, H is a real Hilbert space, $B\colon H \to H$ is a monotone Lipschitz nonlinear operator, $T > 0$ is given time instant, $u_0 \in H$ is a given initial state, and $f\colon [0,T] \to H$ is a given function.

So, the semilinear heat equation, with the Dirichlet boundary condition, mentioned above becomes a special case of (P_0) if we set

$$\begin{cases} H = L^2(\Omega), \\ A = -\Delta, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ Bu = \beta \circ u. \end{cases}$$

The Dirichlet boundary condition is included in the definition of the domain of $A = -\Delta$.

We will next consider the following two elliptic-like regularizations of (P_0).

$$\begin{cases} -\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T, \\ u(0) = u_0, \quad u(T) = u_T. \end{cases} \tag{P_1^ε}$$

$$\begin{cases} -\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T, \\ u(0) = u_0, \quad u'(T) = u_T. \end{cases} \tag{P_2^ε}$$

We will not consider the hyperbolic-like regularizations of (P_0) which have been discussed by A. Perjan when A is a linear, symmetric, strongly positive operator, and B is a nonlinear operator (see, e.g., [29] and [30]). As we said above, L. Barbu and G. Moroşanu [7, pp. 185-208] as well as J.L. Lions [24, pp. 491-495] also discussed the hyperbolic-like regularizations of the linear case (i.e., when $B = 0$).

It is worth mentioning that our analysis covers many applications, in particular the semilinear heat equation mentioned above. The elliptic-like regularizations (P_1^ε) and (P_2^ε) for the linear case (i.e., $B = 0$) have been studied by J.L. Lions [24, pp. 407-420], where asymptotic expansions (of order zero and one, respectively) have been established. He called them elliptic-evolution problems. Some examples were also provided there. Lions explained (see [24, p. IX]) that sometimes it might be useful to consider regularizations of problem (P_0) that provide good solutions approximating the solution of (P_0) for ε small. This regularization method was also called by Lions the method of artificial viscosity (due to the additional term involving ε). It is also possible to consider the fourth order elliptic-like regularizations of problem (P_0) whose solutions are expected to be even more regular with respect to t.

The rest of this dissertation is organized as follows: Chapter 3 is concerned with the existence, uniqueness, and regularity of the solutions of problems (P_0), (P_1^ε) and (P_2^ε). While the results of Chapter 3 are important in themselves, we need them in Chapter 4 in order to validate the asymptotic expansion for the solutions of problems (P_1^ε) and (P_2^ε).

Chapter 3

Existence, Uniqueness and Regularity Theorems

In this chapter we will discuss existence, uniqueness and regularity theorems for problems (P_0) and its elliptic regularizations (P_1^ε) and (P_2^ε) mentioned in Chapter 2. As usual H represents a real Hilbert space with scalar product $(\cdot,\cdot)$ and the induced norm $\|\cdot\|$. All operators are single-valued and nonlinear unless otherwise specified. Recall problem (P_0):

$$\begin{cases} u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T, & (E) \\ u(0) = u_0, & (IC) \end{cases} \qquad (P_0)$$

where $T > 0$ is a given time instant, $u_0 \in H$ is a given initial state, $f\colon [0,T] \to H$ is a given function, and A, B satisfy the following conditions:

(H1) $A\colon D(A) \subset H \to H$ is a maximal monotone linear operator, or equivalently, $-A$ is the infinitesimal generator of a C_0-semigroup of contractions on H, say $\{S(t)\colon H \to H;\, t \geq 0\}$;

(H2) $B\colon H \to H$ is a monotone Lipschitz operator with a Lipschitz constant C, i.e.,

$$\|Bx - By\| \leq C\|x - y\| \quad \forall x, y \in H.$$

Remark 3.1. *Since B is a monotone Lipschitz operator, it follows from Theorem 1.5 that B is maximal monotone. Moreover, it follows from Theorem 1.6* *that $A+B$ is maximal monotone with $D(A+B)=D(A)$.*

Lemma 3.2. *Let X be a Banach space, and $E\colon X\to X$ be Fréchet differentiable satisfying $\sup\{\|E'(x)\|_{\mathcal{L}(X)} : x\in X\}=L<\infty$, where $\mathcal{L}(X)$ is the space of continuous linear operators from X to X, and E' is the Fréchet derivative of E. Then*

$$\|Ex-Ey\|\le L\|x-y\| \quad \forall x,y\in X. \tag{3.1}$$

Proof. The proof is not new but we give it for the convenience of the reader. Let $f\in X^*$ be arbitrary, where X^* is the dual of X. Consider $g\colon[0,1]\to\mathbb{R}$ defined as

$$g(t)=(f\circ E)(y+t(x-y)).$$

By the mean value theorem for real valued functions, there exists a $c\in(0,1)$ such that

$$g(1)-g(0)=g'(c).$$

So,

$$\begin{aligned}&f(Ex)-f(Ey)=f[E'(y+c(x-y))(x-y)]\\ &\Rightarrow f(Ex-Ey)=f[E'(y+c(x-y))(x-y)].\end{aligned}$$

If $Ex-Ey=0$, then (3.1) is trivial. Assume that $Ex-Ey\neq 0$, then by the Hahn-Banach theorem, there exists $f\in X^*$ satisfying $\|f\|=1$, and $f(Ex-Ey)=\|Ex-Ey\|$. From this (3.1) follows easily. □

Remark 3.3. *Let X be a Banach space. By using similar steps as in Lemma 3.2, we can prove that if $E\colon X\to X$ is Fréchet differentiable, and $E' : X\to\mathcal{L}(X)$ is bounded on bounded sets, then E is Lipschitz on bounded sets.*

3.1 Existence, uniqueness and regularity theorems for problem (P_0)

In this section we will discuss some existence, uniqueness and regularity theorems for problem (P_0). Consider the Cauchy problem associated with a nonlinear operator $Q\colon D(Q) \subset H \to H$,

$$u'(t) + Qu(t) = f(t), \quad 0 < t < T, \quad u(0) = u_0, \tag{P}$$

where $T > 0$ is a given time instant, $u_0 \in \overline{D(Q)}$ is a given initial state.

In Chapter 1, we defined the strong solution of problem (P), and discussed some existence theorem (see, Theorem 1.18 and Remark 1.19). Now we define the weak solution of (P).

Definition 3.4. *Let $f \in L^1(0, T; H)$. A function $u \in C([0, T]; H)$ is said to be a **weak solution** of problem (P) if there exist sequences $\{u_n\} \subset C([0, T]; H)$ and $\{f_n\} \subset L^1(0, T; H)$ such that each u_n is absolutely continuous on every compact subinterval of $(0, T)$ and $u_n'(t) + Qu_n(t) = f_n(t)$ a. e. $t \in (0, T)$, for each n; $u_n \to u$ in $C([0, T]; H)$; $u(0) = u_0$; and $f_n \to f$ in $L^1(0, T; H)$.*

Now we are going to state and prove some existence results for problem (P_0). For the convenience of the reader, we first recall some known existence results for problem (P):

Lemma 3.5 (H. Brézis; see, e.g., [25, p. 56])**.** *If Q is the subdifferential of a proper, convex, lower semicontinuous function $\varphi\colon H \to (-\infty, +\infty]$, $u_0 \in \overline{D(Q)}$ and $f \in L^2(0, T; H)$, then problem (P) has a unique strong solution u, such that $t^{1/2}u' \in L^2(0, T; H)$, $t \to \varphi(u(t))$ is integrable on $[0, T]$ and absolutely continuous on $[\delta, T]$, $\forall \delta \in (0, T)$. If, in addition, $u_0 \in D(\varphi)$, then $u' \in L^2(0, T; H)$.*

Lemma 3.6 (see, e.g., [7, p. 30])**.** *If $Q := A + F(t, \cdot)$, where A is a maximal monotone linear operator, $F(\cdot, z) \in L^1(0, T; H)$ for all $z \in H$, and there exists* *a constant* $\omega > 0$ *such that*

$$\|F(t,z_1) - F(t,z_2)\| \le \omega \|z_1 - z_2\| \quad \forall t \in [0,T],\ z_1, z_2 \in H,$$

then, for every $u_0 \in H$*, there exists a unique* $u \in C([0,T];\, H)$ *which satisfies the following integral equation*

$$u(t) = S(t)u_0 - \int_0^t S(t-s)F(s,u(s))ds, \quad \forall t \in [0,T].$$

The solution of this integral equation is called a mild solution of the Cauchy problem

$$\begin{cases} u'(t) + Au(t) + F(t,u(t)) = 0, \quad 0 < t < T, \\ u(0) = u_0. \end{cases} \tag{3.2}$$

Lemma 3.7 (H. Brézis [11, pp. 106-107])**.** *Assume that all the assumptions of Lemma 3.6 hold. If, in addition, A is self-adjoint and* $F(\cdot,z) \in L^2(0,T;\, H)$ *for all* $z \in H$*, then there exists a unique strong solution* u *of the Cauchy problem* (3.2)*, such that* $t^{1/2}u' \in L^2(0,T;H)$*.*

Lemma 3.8 (T. Kato [20])**.** *Assume that* $Q = Q(t,\cdot)$ *(i.e.,* Q *is time dependent),* $Q(t,\cdot)$ *is single-valued, maximal monotone, with* $D(Q(t,\cdot)) = D$ *for all* $t \in [0,T]$ *(i.e.,* $D(Q(t,\cdot))$ *is independent of* t*), and the following condition is satisfied*

$$\|Q(t,z) - Q(s,z)\| \le L|t-s|(1 + \|z\| + \|Q(s,z)\|),$$

for all $z \in D$*,* $s,t \in [0,T]$*, where* L *is a positive constant. Then, for every* $u_0 \in D$*, problem* (P) *has a unique strong solution* $u \in W^{1,\infty}(0,T;\, H)$*.*

Theorem 3.9. *Assume (H1) and (H2). If* $u_0 \in H$ *and* $f \in L^1(0,T;H)$*, then* (P_0) *has a unique mild solution* $u \in C([0,T];H)$*, i.e.,*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)[f(s) - Bu(s)]\, ds, \quad 0 \le t \le T. \tag{3.3}$$

If $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$, then $u \in C^1([0,T];H)$ and it is a strong solution of (P_0), satisfying (P_0) for all $t \in [0,T]$.

Proof. The first part follows from Lemma 3.6 if we choose $F(t,z) := Bz - f(t)$. Now, let $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$. Then by Remark 1.19, (P_0) has a unique strong solution $u \in W^{1,\infty}(0,T;H)$, and it is also a mild solution. Note that

$$\begin{aligned} u(t) &= S(t)u_0 + \int_0^t S(t-s)[f(s) - Bu(s)]\, ds \\ &= S(t)u_0 + \int_0^t S(s)[f(t-s) - Bu(t-s)]\, ds, \quad 0 \le t \le T. \end{aligned}$$

This implies that

$$\begin{aligned} u'(t) = S(t)(f(0) &- Au_0 - Bu_0) \\ &+ \int_0^t S(t-s)[f'(s) - (Bu)'(s)]\, ds, \quad 0 \le t \le T. \end{aligned}$$

Hence, $u' \in C([0,T];H)$. By Remark 1.20, we have that u satisfies (P_0) for all $t \in [0,T]$. This completes the proof. □

Remark 3.10. *If $u_0 \in H$ and $f \in L^1(0,T;H)$ then the mild solution u of problem (P_0) is also a weak solution. Indeed, let $(u_0^n, f_n) \in D(A) \times W^{1,1}(0,T;H)$ approximate (u_0, f) in $H \times L^1(0,T;H)$. Denote by u_n the strong solution of (P_0) with $u_0 := u_0^n$ and $f := f_n$. Then,*

$$u_n(t) = S(t)u_0^n + \int_0^t S(t-s)[f_n(s) - Bu_n(s)]\, ds, \quad 0 \le t \le T. \tag{3.4}$$

Therefore, since $S(t)$ is a contraction for each $t \ge 0$ and B is Lipschitz, we have

$$\|u_n(t) - u_m(t)\| \le \|u_0^n - u_0^m\| + \|f_n - f_m\|_{L^1(0,T;H)} + C\int_0^t \|u_n(s) - u_m(s)\| ds,$$

for all $t \in [0,T]$.

It follows by Gronwall's Lemma that

$$\|u_n(t) - u_m(t)\| \leq \left(\|u_0^n - u_0^m\| + \|f_n - f_m\|_{L^1(0,T;H)}\right) e^{Ct}, \quad \forall t \in [0,T].$$

This shows that u_n converges in $C([0,T];H)$ and its limit $\tilde{u}$ is a weak solution of problem (P_0). By passing to the limit in (3.4) we can see that $\tilde{u}$ is also a mild solution of the same problem, so by the uniqueness of the mild solution $\tilde{u} = u$.

Theorem 3.11. *Assume that (H1) and (H2) hold and, in addition, that A is self-adjoint. If $u_0 \in H$ and $f \in L^2(0,T;H)$, then problem (P_0) has a unique strong solution u, such that $t^{1/2}u' \in L^2(0,T;H)$.*

Proof. The result follows easily by Lemma 3.7. □

Theorem 3.12 (Higher Regularity)**.** *Assume that (H1) and (H2) hold and, in addition, that A is self-adjoint. If $u_0 \in D(A)$ and $f \in W^{1,2}(0,T;H)$, B is differentiable and $B' \colon H \to \mathcal{L}(H)$ is bounded on bounded sets, then the solution u of problem (P_0) belongs to $C^1([0,T];H)$ and u' is differentiable* a. e.*, with $t^{1/2}u'' \in L^2(0,T;H)$. If in addition $f(0) - Au_0 - Bu_0 \in D(A^{1/2})$, then $u \in W^{2,2}(0,T;H)$.*

Proof. If $u_0 \in D(A)$ and $f \in W^{1,2}(0,T;H)$ it follows by Theorem 3.9 that $u \in C^1([0,T];H)$. Obviously u satisfies the equation

$$u'(t) = S(t)(f(0) - Au_0 - Bu_0) + \int_0^t S(t-s)[f'(s) - B'(u(s))u'(s)]ds \quad (3.5)$$

for all $t \in [0,T]$. Now, consider the equation (obtained from (E) of problem (P_0) by formal differentiation)

$$\begin{cases} v'(t) + Av(t) + B'(u(t))v(t) = f'(t), \quad 0 \leq t \leq T, \\ v(0) = f(0) - Au_0 - Bu_0. \end{cases} \quad (CP)$$

(CP) has a mild solution $v = v(t) \in C([0,T]; H)$,

$$v(t) = S(t)\left(f(0) - Au_0 - Bu_0\right) + \int_0^t S(t-s)[f'(s) - B'(u(s))v(s)]ds, \quad (3.6)$$

for all $t \in [0,T]$. From (3.5) and (3.6) we derive

$$\begin{aligned} \|v(t) - u'(t)\| &\leq \int_0^t \|B'(u(s))\|_{\mathcal{L}(H)} \|v(s) - u'(s)\| ds \\ &\leq K \int_0^t \|v(s) - u'(s)\| ds, \quad 0 \leq t \leq T, \end{aligned}$$

where $K > 0$ is some constant (see Remark 3.3). This implies $v(t) = u'(t)$ for all $0 \leq t \leq T$.

In fact, since A is self-adjoint, the above Cauchy problem (CP) has a strong solution v, with $\sqrt{t}\, v' \in L^2(0,T;H)$ (cf. Lemma 3.7). Therefore, $\sqrt{t}\, u'' \in L^2(0,T;H)$. Now, if in addition $f(0) - Au_0 - Bu_0 \in D(A^{1/2})$, then the solution $v = u'$ of problem (CP) belongs to $W^{1,2}(0,T;H)$. This follows by Lemma 3.5, where $\varphi(x) = (1/2)\|A^{1/2}x\|^2$ for $x \in D(A^{1/2})$, and $\varphi(x) = +\infty$ for $x \in H \setminus D(A^{1/2})$. So the proof is complete. $\square$

If A is not self-adjoint, then a higher regularity result holds under more restrictive conditions, as shown in the next theorem.

Theorem 3.13. *Assume (H1) and (H2). If $f \in W^{2,\infty}(0,T;H)$, $u_0 \in D(A)$, $f(0) - Au_0 - Bu_0 \in D(A)$, and B is twice differentiable with B', B'' bounded on bounded sets, then problem (P_0) has a unique solution $u \in C^2([0,T];H)$.*

Proof. Taking into account Theorem 3.9, it suffices to prove that the above (CP) has a solution $v \in C^1([0,T];H)$. Since $v = u'$ this would conclude the proof. We will apply Kato's theorem (Lemma 3.8). To this purpose, let us replace (CP) by an equivalent one which fits in the framework of Kato's theorem. Let $C_1 > C$. Multiply the equation (CP) by $e^{-C_1 t}$. Denoting

$w(t) = e^{-C_1 t} v(t)$, we obtain the following Cauchy problem (which is equivalent to (CP))

$$\begin{cases} w'(t) + Aw(t) + (C_1 I + B'(u(t)))\, w(t) = e^{-C_1 t} f'(t), \quad 0 \le t \le T, \\ w(0) = f(0) - Au_0 - Bu_0. \end{cases} \tag{$\widetilde{CP}$}$$

The operator $A + C_1 I + B'(u(t))$ is maximal monotone for all $t \in [0,T]$ and its domain is $D(A)$ for all $t \in [0,T]$. Thus, by Kato's existence result, $(\widetilde{CP})$ has a unique solution $w \in W^{1,\infty}(0,T;H)$. Therefore, $v = u' \in W^{1,\infty}(0,T;H)$. In fact, $v = u'$ is a mild solution of problem (CP) (see (3.6)) and satisfies

$$u''(t) = v'(t) = S(t)v'(0) + \int_0^t S(t-s)\,[f''(s) - B''(u(s))u'(s)v(s) \\ - B'(u(s))v'(s)]\, ds, \quad 0 \le t \le T,$$

which shows that $u'' \in C([0,T];H)$. This concludes the proof. □

Remark 3.14. *It is worth pointing out that by using the above ideas every level of regularity for the solution of* (P_0) *can be reached under appropriate conditions.*

3.2 Existence, uniqueness and regularity theorems for problem (P_1^ε)

In this section we will discuss the elliptic-like regularization (P_1^ε) of problem (P_0) mentioned in Chapter 2. Recall (P_1^ε)

$$\begin{cases} -\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T, \\ u(0) = u_0, \quad u(T) = u_T. \end{cases} \tag{P_1^ε}$$

We can assume $\varepsilon = 1$ without any loss of generality (otherwise, use substitution $t = \varepsilon s$). So (P_1^ε) becomes

$$\begin{cases} -u'' + u' + Au + Bu = f(t), \quad 0 < t < T, \\ u(0) = u_0, \quad u(T) = u_T. \end{cases} \tag{P_1}$$

We will first discuss the existence and uniqueness of the solution of problem (P_1). For existence, we will use a result of A.R. Aftabizadeh and N.H. Pavel [2], however, the uniqueness will be shown directly by using monotonicity of A and B. It is worth mentioning that it is also possible to prove the existence and uniqueness of the solution of problem (P_1) by using steps similar to those used in Theorem 3.20. In the rest of this section, we will prove some other existence, uniqueness and regularity theorems under appropriate conditions on f and B.

Theorem 3.15 (A.R. Aftabizadeh and N.H. Pavel [2])**.** *Let $Q\colon D(Q) \subset H \to H$, $\beta_1\colon D(\beta_1) \subset H \to H$, $\beta_2\colon D(\beta_2) \subset H \to H$ be multivalued maximal monotone operators; $0,\ a,\ b \in D(Q)$; $f \in L^2(0,T;H)$;*

$$(Q_\lambda x - Q_\lambda y, v) \geq 0\ \forall \lambda > 0\ \forall v, x, y \in H,\ x - y \in D(\beta_1),\ v \in \beta_1(x-y),$$

$$(Q_\lambda x - Q_\lambda y, v) \geq 0\ \forall \lambda > 0\ \forall v, x, y \in H,\ x - y \in D(\beta_2),\ v \in \beta_2(x-y),$$

where Q_λ denotes the Yosida approximation of Q; either $D(\beta_1)$ or $D(\beta_2)$ is bounded; $p, r \in W^{1,\infty}(0,T)$ and $p(t) \geq c > 0\ \forall t \in [0,T]$. Then there exists at least one $u \in W^{2,2}(0,T;H)$ such that

$$p(t)u''(t) + r(t)u'(t) \in Qu(t) + f(t), \quad 0 < t < T,$$

$$u(0) - a \in D(\beta_1), \quad u'(0) \in \beta_1(u(0) - a),$$

$$u(T) - b \in D(\beta_2), \quad -u'(T) \in \beta_2(u(T) - b).$$

If, in addition, at least one of the operators Q, β_1, β_2 is injective, then u satisfying above conditions is unique.

Theorem 3.16. *If (H1), (H2) hold, $u_0, u_T \in D(A)$ and $f \in L^2(0,T;H)$, then problem (P_1) has a unique solution $u \in W^{2,2}(0,T;H)$.*

Proof. $Q = A + B$ is a maximal monotone operator, with $D(Q) = D(A)$ (see Remark 3.1). Let φ be the indicator function of the set $\{0\} \subset H$, i.e.,

$$\varphi(x) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

Let $\partial\varphi$ be the subdifferential of φ, then

$$\partial\varphi(x) = \begin{cases} H & \text{if } x = 0, \\ \emptyset & \text{if } x \neq 0. \end{cases}$$

If we choose $\beta_1 = \beta_2 = \partial\varphi$, $p \equiv 1$, $r \equiv -1$, then by Theorem 3.15 there exists at least one solution $u \in W^{2,2}(0,T;H)$ for problem (P_1).

In fact, u is unique. Indeed, if $v \in W^{2,2}(0,T;H)$ is another solution of (P_1), then

$$\begin{cases} -(u-v)'' + (u-v)' + A(u-v) + Bu - Bv = 0, \quad 0 \le t \le T, \\ (u-v)(0) = 0, \quad (u-v)(T) = 0. \end{cases}$$

If we multiply the above equation by $(u-v)$, and use the monotonicity of A and B, we obtain

$$-\int_0^T ((u-v)'', u-v)\, dt + \int_0^T ((u-v)', u-v)\, dt \le 0.$$

This implies

$$\int_0^T \|u'-v'\|^2 dt + \underbrace{\frac{1}{2}\|u-v\|^2\Big|_{t=0}^{t=T}}_{=0} \le 0,$$

hence $u' - v' \equiv 0$, i.e., $u - v$ is a constant function. In fact, $u \equiv v$ since $u(0) = v(0)$. $\square$

Theorem 3.17. *Assume (H1) and (H2) hold. If $u_0, u_T \in H$ and $f \in L^2(0,T;H)$, then problem (P_1) has a unique solution $u \in C([0,T];H) \cap W^{2,2}_{\mathrm{loc}}(0,T;H)$, with $t^{1/2}(T-t)^{1/2}u'$, $t^{3/2}(T-t)^{3/2}u'' \in L^2(0,T;H)$.*

Proof. Note that $\overline{D(A)} = H$ since A is a maximal monotone linear operator. Therefore, $\overline{D(Q)} = \overline{D(A)} = H$, where $Q = A + B$. We will use a technique similar to that of R.E. Bruck [13]. We continue the proof with the following claim.

Claim 1. Let u, v be two solutions of (P_1) with boundary conditions $u(0), u(T)$ and $v(0), v(T)$, respectively, and let u, v satisfy the properties specified in the statement of Theorem 3.17, then $t \mapsto e^{-t}\|u(t) - v(t)\|^2$ is a convex function on $[0, T]$, and

$$\|u - v\|_{C([0,T];H)} \leq \max\left\{e^{T/2}\|u(0) - v(0)\|, \|u(T) - v(T)\|\right\}, \tag{3.7}$$

$$\int_0^T t(T-t)\|u' - v'\|^2 dt \leq 2T\left(e^T\|u(0) - v(0)\|^2 + \|u(T) - v(T)\|^2\right). \tag{3.8}$$

Proof of Claim 1. Let

$$g(t) = \frac{1}{2}\|u(t) - v(t)\|^2, \quad 0 \leq t \leq T.$$

Obviously, $g \in C([0, T])$ and

$$g'' = (u'' - v'', u - v) + \|u' - v'\|^2, \tag{3.9}$$

for a. e. $t \in (0, T)$. From (3.9) and (P_1) we get

$$\begin{aligned} g'' &= (u' - v', u - v) + (Qu - Qv, u - v) + \|u' - v'\|^2 \\ &\geq g' + \|u' - v'\|^2 \quad \text{a. e. } t \in (0, T). \end{aligned}$$

Therefore,

$$
\begin{aligned}
(e^{-t}g)'' &= e^{-t}(g'' - 2g' + g) \\
&\geq e^{-t}\left(\|u' - v'\|^2 - g' + g\right) \\
&= e^{-t}\left(\|u' - v'\|^2 - (u' - v', u - v) + \frac{1}{2}\|u - v\|^2\right) \\
&\geq e^{-t}\left(\|u' - v'\|^2 - \|u' - v'\|\,\|u - v\| + \frac{1}{2}\|u - v\|^2\right) \\
&= e^{-t}\left(\frac{1}{2}\|u' - v'\|^2 + \frac{1}{2}\left(\|u' - v'\| - \|u - v\|\right)^2\right) \\
&\geq \frac{1}{2}e^{-t}\|u' - v'\|^2
\end{aligned} \tag{3.10}
$$

for a. e. $t \in (0, T)$, hence $t \mapsto e^{-t}g(t)$ is a convex function. This yields

$$
e^{-T}g(t) \leq e^{-t}g(t) \leq \max\left\{g(0), e^{-T}g(T)\right\} \quad \forall t \in [0, T],
$$

i.e., (3.7) holds. In order to prove estimation (3.8), consider the function (as in [13]) $\beta_\delta(t) := \min\{t - \delta, T - \delta - t\}$ for a small $\delta > 0$. If we multiply (3.10) by β_δ and integrate over $[\delta, T - \delta]$, we obtain

$$
\begin{aligned}
\frac{1}{2}\int_\delta^{T-\delta} e^{-t}\beta_\delta(t)\|u' - v'\|^2 dt &\leq \int_\delta^{T-\delta} \beta_\delta(t)\left(e^{-t}g\right)'' dt \\
&= -\int_\delta^{T-\delta} \beta_\delta'(t)\left(e^{-t}g\right)' dt \\
&= e^{-\delta}g(\delta) + e^{-T+\delta}g(T - \delta) - 2e^{-T/2}g(T/2) \\
&\leq g(\delta) + e^{-T+\delta}g(T - \delta).
\end{aligned}
$$

Letting $\delta \to 0^+$ and applying the Fatou's lemma yields

$$
\frac{1}{2}\int_0^T \beta(t)\|u' - v'\|^2 dt \leq e^T g(0) + g(T),
$$

where $\beta(t) := \min\{t, T - t\}$. This inequality implies (3.8), so the proof of Claim 1 is complete. □

Claim 2. Let $u \in W^{2,2}(0,T;H)$ be the solution of (P_1) with $u_0, u_T \in D(A)$, and $f \in L^2(0,T;H)$. Let $u_\lambda \in W^{2,2}(0,T;H)$ be the solution of the problem

$$-u_\lambda'' + u_\lambda' + A_\lambda u_\lambda = f - Bu, \quad 0 \le t \le T, \tag{3.11a}$$

$$u_\lambda(0) = u_0, \quad u_\lambda(T) = u_T, \tag{3.11b}$$

where $\lambda > 0$ and A_λ denotes the Yosida approximation of A. The existence of u_λ follows by Theorem 3.16, where $A := 0$, $B := A_\lambda$, and $f(t) := f(t) - Bu(t)$. Then,

$$u_\lambda \to u,\ u_\lambda' \to u' \text{ in } C\left([0,T];H\right), \tag{3.12}$$

$$\text{and } u_\lambda'' \to u'' \text{ weakly in } L^2(0,T;H) \text{ as } \lambda \to 0^+. \tag{3.13}$$

Proof of Claim 2. Define for $\lambda > 0$ and $t \in [0,T]$

$$\begin{cases} u^*(t) = \frac{T-t}{T}u_0 + \frac{t}{T}u_T, \\ v_\lambda(t) = u_\lambda(t) - u^*(t). \end{cases}$$

Obviously, v_λ satisfies the problem

$$\begin{cases} -v_\lambda'' + v_\lambda' + A_\lambda v_\lambda = f - Bu - A_\lambda u^* + \frac{1}{T}(u_0 - u_T), \quad 0 \le t \le T, \\ v_\lambda(0) = 0 = v_\lambda(T). \end{cases} \tag{3.14}$$

If we multiply (3.14) by v_λ and integrate over $[0,T]$, we obtain

$$\begin{aligned} &-\int_0^T (v_\lambda'', v_\lambda)\, dt + \underbrace{\frac{1}{2}\int_0^T \frac{d}{dt}\|v_\lambda\|^2 dt}_{=0} + \underbrace{\int_0^T (A_\lambda v_\lambda, v_\lambda)\, dt}_{\ge 0} \\ &= \int_0^T \left(f - Bu + \frac{1}{T}(u_0 - u_T) - A_\lambda u^*, v_\lambda\right) dt. \end{aligned} \tag{3.15}$$

Since

$$\begin{aligned} \|A_\lambda u^*(t)\| &= \left\|\frac{T-t}{T}A_\lambda u_0 + \frac{t}{T}A_\lambda u_T\right\| \\ &\le \frac{T-t}{T}\|Au_0\| + \frac{t}{T}\|Au_T\| \le \max\left\{\|Au_0\|, \|Au_T\|\right\} < \infty, \end{aligned} \tag{3.16}$$

for all $t \in [0,T]$, we derive from (3.15)

$$\int_0^T \|v_\lambda'\|^2 dt \le K \left(\int_0^T \|v_\lambda\|^2 dt \right)^{1/2}, \tag{3.17}$$

where $K > 0$ is a constant. On the other hand, we have

$$v_\lambda(t) = \int_0^t v_\lambda'(s) ds, \quad 0 \le t \le T. \tag{3.18}$$

From (3.18), we have

$$\|v_\lambda\|_{C([0,T];H)} \le T^{1/2} \|v_\lambda'\|_{L^2}, \tag{3.19}$$

and

$$\|v_\lambda\|_{L^2} \le T \|v_\lambda'\|_{L^2}, \tag{3.20}$$

where $L^2 := L^2(0,T;H)$.

From (3.17) and (3.20), we get

$$\{v_\lambda'; \lambda > 0\} \text{ is bounded in } L^2(0,T;H). \tag{3.21}$$

From (3.19) and (3.21), we get

$$\{v_\lambda; \lambda > 0\} \text{ is bounded in } C([0,T];H). \tag{3.22}$$

Since

$$\begin{aligned} \frac{d}{dt}\left(v_\lambda', A_\lambda v_\lambda\right) &= \left(v_\lambda'', A_\lambda v_\lambda\right) + \underbrace{\left(v_\lambda', A_\lambda v_\lambda'\right)}_{\ge 0} \\ &\ge \left(v_\lambda'', A_\lambda v_\lambda\right) \quad \text{for a. e. } t \in (0,T). \end{aligned} \tag{3.23}$$

Integrating (3.23) over $[0,T]$, and using (3.14), we have

$$0 \ge \int_0^T \left(v_\lambda'', A_\lambda v_\lambda\right) dt = \int_0^T \left(v_\lambda' + A_\lambda v_\lambda - f_\lambda, A_\lambda v_\lambda\right) dt,$$

where

$$f_\lambda(t) := f(t) - Bu(t) - A_\lambda u^*(t) + \frac{1}{T}(u_0 - u_T).$$

Therefore, by (3.16) and (3.21), one gets

$$\begin{aligned}\|A_\lambda v_\lambda\|_{L^2}^2 &\le \|v_\lambda'\|_{L^2}\,\|A_\lambda v_\lambda\|_{L^2} + \|f_\lambda\|_{L^2}\|A_\lambda v_\lambda\|_{L^2}\\ &\le K_1\|A_\lambda v_\lambda\|_{L^2} \quad \text{(by (3.21))},\end{aligned}$$

so

$$\{A_\lambda v_\lambda; \lambda > 0\} \text{ is bounded in } L^2(0,T;H). \tag{3.24}$$

Finally, by using (3.21), (3.24) and the first equation of (3.14), we have

$$\{u_\lambda''; \lambda > 0\} \text{ is bounded in } L^2(0,T;H). \tag{3.25}$$

For $\lambda, \mu > 0$, we have from (3.11)

$$\begin{aligned}-\int_0^T \left(u_\lambda'' - u_\mu'', u_\lambda - u_\mu\right) dt + \int_0^T \left(u_\lambda' - u_\mu', u_\lambda - u_\mu\right) dt&\\ + \int_0^T \left(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu\right) dt &= 0,\end{aligned}$$

which implies that

$$\begin{aligned}\int_0^T \left\|u_\lambda' - u_\mu'\right\|^2 dt = &-\int_0^T \underbrace{\left(A_\lambda u_\lambda - A_\mu u_\mu, J_\lambda u_\lambda - J_\mu u_\mu\right)}_{\ge 0} dt\\ &- \int_0^T \left(A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu\right) dt\\ \le &K_2(\lambda+\mu),\end{aligned}$$

where $J_\lambda = (I + \lambda A)^{-1}$. This shows that $\{u_\lambda'; \lambda > 0\}$ is a Cauchy sequence in L^2, hence convergent in L^2 as $\lambda \to 0^+$. Since

$$\begin{aligned}\|u_\lambda(t) - u_\mu(t)\| &= \left\|\int_0^T \left(u_\lambda'(s) - u_\mu'(s)\right) ds\right\|\\ &\le T^{1/2}\left\|u_\lambda' - u_\mu'\right\|_{L^2} \quad \forall t \in [0,T],\end{aligned}$$

$\{u_\lambda; \lambda > 0\}$ converges in $C([0,T];H)$. Denote its limit by $\hat{u}$. Summarizing, we have $\hat{u} \in W^{2,2}(0,T;H)$ and

$$u_\lambda \to \hat{u} \quad \text{in} \quad C([0,T];H), \tag{3.26}$$

$$u'_\lambda \to \hat{u}' \quad \text{in} \quad L^2(0,T;H), \tag{3.27}$$

$$u''_\lambda \to \hat{u}'' \quad \text{weakly in } L^2(0,T;H), \text{ as } \lambda \to 0^+. \tag{3.28}$$

In fact, since by (3.28) the sequence $\{u'_\lambda; \lambda > 0\}$ is equicontinuous,

$$u'_\lambda \to \hat{u}' \text{ in } C([0,T];H) \text{ as } \lambda \to 0^+.$$

It is easily seen that

$$J_\lambda u_\lambda \to \hat{u} \text{ in } C([0,T];H).$$

Indeed, for all $t \in [0,T]$,

$$\begin{aligned} \|J_\lambda u_\lambda(t) - \hat{u}(t)\| &\le \|J_\lambda u_\lambda(t) - u_\lambda(t)\| + \|u_\lambda(t) - \hat{u}(t)\| \\ &= \lambda \|A_\lambda u_\lambda(t)\| + \|u_\lambda(t) - \hat{u}(t)\| \\ &\le K_3\lambda + \|u_\lambda(t) - \hat{u}(t)\| \quad (\text{by } (3.24)), \end{aligned}$$

which confirms our assertion.

Using the above pieces of information on u_λ, we can pass to the limit as $\lambda \to 0^+$ in (3.11a) regarded as an equation in L^2 to obtain

$$-\hat{u}'' + \hat{u}' + A\hat{u} = f - Bu, \quad 0 \le t \le T. \tag{3.29a}$$

We also have (see (3.11b) and (3.26))

$$\hat{u}(0) = u_0, \quad \hat{u}(T) = u_T. \tag{3.29b}$$

Now, from (P_1), (3.29a) and (3.29b) we can easily see that $\hat{u} \equiv u$. This completes the proof of Claim 2. $\square$

Claim 3. Let $u \in W^{2,2}(0,T;H)$ be the solution of (P_1) with $u_0, u_T \in D(A)$ and $f \in L^2(0,T;H)$. Then, there exist constants $C_1, C_2 > 0$ such that

$$\|u''\|_{L^2_{**}} \leq C_1 \left(\|f\|_{L^2} + \|u'\|_{L^2_*} + \|u\|_{C([0,T];H)}\right) + C_2, \tag{3.30}$$

where (as in [13]) $L^2_* := L^2(0,T;H;\beta(t)dt)$, $L^2_{**} := L^2(0,T;H;\beta^3(t)dt)$.

Proof of Claim 3. Consider again problem (3.11). From the obvious inequality

$$\begin{aligned}\frac{d}{dt}\left(u'_\lambda, A_\lambda u_\lambda\right) &= \left(u''_\lambda, A_\lambda u_\lambda\right) + \left(u'_\lambda, A_\lambda u'_\lambda\right)\\ &\geq \left(u''_\lambda, A_\lambda u_\lambda\right),\end{aligned}$$

we derive by multiplication by β^3 and integration over $[0,T]$,

$$-3\int_0^T \beta^2\beta'\left(u'_\lambda, A_\lambda u_\lambda\right)dt \geq \int_0^T \beta^3\left(u'_\lambda + A_\lambda u_\lambda - f + Bu, A_\lambda u_\lambda\right)dt.$$

It follows that

$$\begin{aligned}\|A_\lambda u_\lambda\|^2_{L^2_{**}} &\leq 3\|u'_\lambda\|_{L^2_*}\|A_\lambda u_\lambda\|_{L^2_{**}} + \|u'_\lambda\|_{L^2_{**}}\|A_\lambda u_\lambda\|_{L^2_{**}} +\\ &\quad + \|f\|_{L^2_{**}}\|A_\lambda u_\lambda\|_{L^2_{**}} + \|Bu\|_{L^2_{**}}\|A_\lambda u_\lambda\|_{L^2_{**}}\\ &\leq K_4\|A_\lambda u_\lambda\|_{L^2_{**}}\left(\|f\|_{L^2} + \|u'_\lambda\|_{L^2_*} + \|u\|_{C([0,T];H)}\right) + K_5,\end{aligned}$$

and so

$$\|A_\lambda u_\lambda\|_{L^2_{**}} \leq K_4\left(\|f\|_{L^2} + \|u'_\lambda\|_{L^2_*} + \|u\|_{C([0,T];H)}\right) + K_6.$$

From (3.11a) we then derive

$$\|u''_\lambda\|_{L^2_{**}} \leq C_1\left(\|f\|_{L^2} + \|u'_\lambda\|_{L^2_*} + \|u\|_{C([0,T];H)}\right) + C_2. \tag{3.31}$$

By (3.12), (3.13) and (3.31) we obtain (3.30). □

Proof of Theorem 3.17 *(continuation)***.** Let us approximate $u_0, u_T \in H$ by $u_{0n}, u_{Tn} \in D(A)$, i.e.,

$$\|u_{0n} - u_0\| \to 0, \quad \|u_{Tn} - u_T\| \to 0 \quad \text{as} \quad n \to \infty.$$

By Theorem 3.16, problem (P_1) with $u(0) = u_{0n}$, $u(T) = u_{Tn}$ has a unique solution $u_n \in W^{2,2}(0,T;H)$. Now, estimates (3.7), (3.8) and (3.30) come into play showing that there exists a function $u \in C([0,T];H) \cap W^{2,2}_{\text{loc}}(0,T;H)$, with $u' \in L^2_*$, $u'' \in L^2_{**}$, such that

$$u_n \to u \quad \text{in} \quad C([0,T];H), \tag{3.32}$$

$$u_n' \to u' \quad \text{in} \quad L^2_*, \tag{3.33}$$

$$u_n'' \to u'' \quad \text{weakly in} \quad L^2_{**}. \tag{3.34}$$

Regarding the equation

$$-u_n'' + u_n' + Au_n + Bu_n = f$$

as one in the space $L^2(\delta, T-\delta; H)$ for positive small δ's, we obtain by (3.32), (3.33) and (3.34) that u satisfies for a. e. $t \in (0,T)$ the equation

$$-u'' + u' + Au + Bu = f.$$

In addition, by (3.32),

$$u(0) = \lim u_n(0) = u_0, \quad u(T) = \lim u_n(T) = u_T.$$

The uniqueness of the solution follows by (3.7). □

Theorem 3.18. *Assume (H1) and (H2) hold. If $u_0, u_T \in D(A)$ and $f \in W^{1,2}(0,T;H)$, then problem (P_1) has a unique solution $u \in W^{2,2}(0,T;H) \cap W^{3,2}_{\text{loc}}(0,T;H)$, with $t^{3/2}(T-t)^{3/2}u''' \in L^2(0,T;H)$. If $u_0, u_T \in H$ and $f \in W^{1,2}(0,T;H)$, then $u \in C([0,T];H) \cap W^{3,2}_{\text{loc}}(0,T;H)$, with $t^{1/2}(T-t)^{1/2}u'$, $t^{3/2}(T-t)^{3/2}u''$, $t^{5/2}(T-t)^{5/2}u''' \in L^2(0,T;H)$.*

Proof. Assume first $u_0, u_T \in D(A)$ and $f \in W^{1,2}(0,T;H)$. By Theorem 3.16 problem (P_1) has a unique solution $u \in W^{2,2}(0,T;H)$. Consider again problem (3.11). We know that u_λ approximates u in the sense of (3.12) and (3.13). Note that $u_\lambda \in W^{3,2}(0,T;H)$ and

$$-u_\lambda''' + u_\lambda'' + A_\lambda u_\lambda' = (f - Bu)' \quad \text{for a. e. } t \in (0,T).$$

Now, if we multiply by β^3 the inequality

$$\frac{d}{dt}\left(u_\lambda'', A_\lambda u_\lambda'\right) \geq \left(u_\lambda''', A_\lambda u_\lambda'\right),$$

and then integrate over $[0,T]$, we get

$$-3\int_0^T \beta^2\beta'\left(u_\lambda'', A_\lambda u_\lambda'\right) dt \geq \int_0^T \beta^3\left(u_\lambda'' - f' + (Bu)' + A_\lambda u_\lambda', A_\lambda u_\lambda'\right) dt.$$

As in the proof of Claim 3, we find

$$\|u_\lambda'''\|_{L^2_{**}} \leq \widetilde{C_1}\left(\|f'\|_{L^2} + \|u_\lambda''\|_{L^2_*} + \|u'\|_{C([0,T];H)}\right) + \widetilde{C_2}. \tag{3.35}$$

According to (3.12), (3.13) and (3.35) $u''' \in L^2_{**}$ and

$$\beta^{3/2}u_\lambda''' \to \beta^{3/2}u''' \text{ weakly in } L^2, \text{ as } \lambda \to 0^+.$$

Now assume $u_0, u_T \in H$ and $f \in W^{1,2}(0,T;H)$. By Theorem 3.17 above, problem (P_1) has a unique solution $u \in C([0,T];H) \cap W^{2,2}_{\text{loc}}(0,T;H)$, with $u' \in L^2_*$, $u'' \in L^2_{**}$. So all we have to prove is that $u \in W^{3,2}_{\text{loc}}(0,T;H)$ and $t^{5/2}(T-t)^{5/2}u''' \in L^2(0,T;H)$. As usual, we approximate u_0, u_T by $u_{0n}, u_{Tn} \in D(A)$, and denote by u_n the solution of the problem

$$\begin{cases} -u_n'' + u_n' + Au_n + Bu_n = f, \quad 0 \leq t \leq T, \\ u_n(0) = u_{0n}, \quad u_n(T) = u_{Tn}. \end{cases}$$

From the proof of Theorem 3.17, we know that (u_n) satisfies (3.32)-(3.34). Now, for an $n \in \mathbb{N}$ (arbitrary but fixed) and $\lambda > 0$, denote by $u_{n\lambda}$ the solution of the problem

$$\begin{cases} -u''_{n\lambda} + u'_{n\lambda} + A_\lambda u_{n\lambda} = f - Bu_n, \quad 0 \le t \le T, \\ u_{n\lambda}(0) = u_{0n}, \quad u_{n\lambda}(T) = u_{Tn}. \end{cases} \tag{3.36}$$

We know from Claim 2 that

$$\begin{cases} u_{n\lambda} \to u_n,\ u'_{n\lambda} \to u'_n \text{ in } C([0,T];H) \text{ and} \\ u''_{n\lambda} \to u''_n \text{ weakly in } L^2(0,T;H) \text{ as } \lambda \to 0^+. \end{cases} \tag{3.37}$$

Obviously, $u_{n\lambda} \in W^{3,2}(0,T;H)$ and satisfies the equation

$$-u'''_{n\lambda} + u''_{n\lambda} + A_\lambda u'_{n\lambda} = (f - Bu_n)' \text{ for a. e. } t \in (0,T).$$

Let us multiply by β^5 the inequality

$$\frac{d}{dt}\left(u''_{n\lambda}, A_\lambda u'_{n\lambda}\right) \ge \left(u'''_{n\lambda}, A_\lambda u'_{n\lambda}\right),$$

and integrate over $[0,T]$. Thus we derive the estimate

$$\left(\int_0^T \beta^5 \,\|u'''_{n\lambda}\|^2\right)^{1/2} \le \widehat{C_1}\left(\|u''_{n\lambda}\|_{L^2_{**}} + \|u'_n\|_{L^2_*} + \|f'\|_{L^2}\right) + \widehat{C_2}. \tag{3.38}$$

As in the proof of Claim 3, we obtain from (3.36) an estimate similar to (3.31)

$$\|u''_{n\lambda}\|_{L^2_{**}} \le C_1\left(\|f\|_{L^2} + \|u'_{n\lambda}\|_{L^2_*} + \|u_n\|_{C([0,T];H)}\right) + C_2. \tag{3.39}$$

Using (3.39) in (3.38) we obtain

$$\begin{aligned} \left(\int_0^T \beta^5 \|u'''_{n\lambda}\|^2\right)^{1/2} \le D_1 \big(\|f\|_{L^2} &+ \|f'\|_{L^2} + \|u'_{n\lambda}\|_{L^2_*} \\ &+ \|u'_n\|_{L^2_*} + \|u_n\|_{C([0,T];H)}\big) + D_2, \end{aligned} \tag{3.40}$$

where D_1, D_2 are positive constants. Since $u'_{n\lambda} \to u'_n$ in $C([0,T];H)$ as $\lambda \to 0^+$ (see (3.37)), we deduce from (3.40) that $\{u'''_{n\lambda}; \lambda > 0\}$ is bounded in $L^2(0,T;H;\beta^5(t)dt)$, so $\beta^{5/2}u'''_{n\lambda} \to \beta^{5/2}u'''_n$ weakly in $L^2(0,T;H)$. Letting $\lambda \to 0^+$ in (3.40) we get

$$\begin{aligned}\|\beta^{5/2}u'''_n\|_{L^2} \leq D_1 \big(\|f\|_{L^2} + \|f'\|_{L^2} + 2\|u'_n\|_{L^2_*} \\ + \|u_n\|_{C([0,T];H)}\big) + D_2.\end{aligned} \tag{3.41}$$

Since (u_n) satisfies (3.32) and (3.33), it follows from (3.41) that $\beta^{5/2}u''' \in L^2(0,T;H)$. Thus the proof of the Theorem 3.18 is complete. □

Remark 3.19. *By using the above method, we can obtain higher regularity for u under appropriate regularity assumptions on f and B.*

3.3 Existence, uniqueness and regularity theorems for problem (P_2^ε)

In this section we will discuss the elliptic-like regularization (P_2^ε) of problem (P_0) mentioned in Chapter 2. Recall (P_2^ε)

$$\begin{cases}-\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T,\\ u(0) = u_0, \quad u'(T) = u_T.\end{cases} \tag{P_2^ε}$$

We can assume $\varepsilon = 1$ without any loss of generality (otherwise, use substitution $t = \varepsilon s$). So (P_2^ε) becomes

$$\begin{cases}-u'' + u' + Au + Bu = f(t), \quad 0 < t < T, & (F)\\ u(0) = u_0, \quad u'(T) = u_T. & (BC)\end{cases} \tag{P_2}$$

We will first prove an existence and uniqueness theorem for problem (P_2). The technique we use is not completely new, but is adapted to this particular problem. In the rest of this section, we will prove some other existence, uniqueness and regularity theorems under appropriate conditions on f and B.

Theorem 3.20. *Assume (H1) and (H2) hold, $u_0, u_T \in D(A)$ and $f \in L^2(0,T;H)$, then problem (P_2) has a unique solution $u \in W^{2,2}(0,T;H)$.*

Proof. **Claim 1.** The operator $Q\colon D(Q) \in X \to X$, where $X = L^2(0,T;H)$, $D(Q) = \{v \in X : v', v'' \in X, v(0) = u_0, v'(T) = u_T\}$, and $Q(v) = -v'' + v'$, is maximal monotone.

Proof of Claim 1. Monotonicity of Q follows easily: for $v_1, v_2 \in D(Q)$ we have

$$\begin{aligned}
&(Qv_1 - Qv_2, v_1 - v_2)_X \\
&= -(v_1' - v_2', v_1 - v_2)\Big|_0^T + \int_0^T \|v_1' - v_2'\|^2 dt + \frac{1}{2}\|v_1 - v_2\|^2\Big|_0^T \\
&= \int_0^T \|v_1' - v_2'\|^2 dt + \frac{1}{2}\|v_1(T) - v_2(T)\|^2 \geq 0.
\end{aligned}$$

For the maximality of Q, we need to prove that for all $g \in X$, there exists a $v \in D(Q)$ satisfying the equation

$$-v'' + v' + v = g. \tag{3.42}$$

The general solution of (3.42) is given by

$$v(t) = c_1 e^{r_1 t} + c_1 e^{r_1 t} + v_p(t), \tag{3.43}$$

with

$$v_p(t) = \frac{1}{r_1 - r_2}\int_0^t \left[e^{r_2(t-s)} - e^{r_1(t-s)}\right] g(s)ds,$$

where r_1, r_2 are the roots of the characteristic equation $-r^2 + r + 1 = 0$ and $c_1, c_2 \in H$ are arbitrary. Obviously, $v \in H^2(0,T;H)$ for all $c_1, c_2 \in H$. Imposing to v, given by (3.43), the conditions

$$v(0) = v_0, \quad v'(T) = u_T,$$

one can determine uniquely c_1, c_2. $\square$

Now, we denote by $\bar{A}, \bar{B}$ the realizations (canonical extensions) of A, B to $X = L^2(0,T;H)$, respectively. For all $\lambda > 0$, $Q + \bar{A}_\lambda + \bar{B}$ is maximal monotone in X, so there exists a unique $u_\lambda \in D(Q)$ satisfying the equation

$$Qu_\lambda + \bar{A}_\lambda u_\lambda + \bar{B}u_\lambda + \lambda u_\lambda = f,$$

(cf. Minty's theorem).

In other words, u_λ satisfies the problem

$$-u_\lambda'' + u_\lambda' + \bar{A}_\lambda u_\lambda + \bar{B}u_\lambda + \lambda u_\lambda = f, \quad 0 < t < T, \tag{3.44}$$

$$u_\lambda(0) = u_0, \quad u_\lambda'(T) = u_T.$$

Claim 2. $\{u_\lambda; 0 < \lambda \le \lambda_0\}$ is bounded in $C([0,T];H)$ and $\{u_\lambda'; 0 < \lambda \le \lambda_0\}$ is bounded in $X = L^2(0,T;H)$, for some $\lambda_0 > 0$ arbitrary but fixed.

Proof of Claim 2. Define for $\lambda > 0$ and $t \in [0,T]$

$$\begin{cases} u^*(t) := u_0 + u_T\, t, \\ v_\lambda(t) := u_\lambda(t) - u^*(t). \end{cases}$$

Obviously, $u^*(t) \in D(A)$ for all $t \in [0,T]$, and v_λ satisfies the problem

$$\begin{aligned} -v_\lambda'' + v_\lambda' + A_\lambda v_\lambda + B(v_\lambda + u^*) + \lambda v_\lambda &= f - u_T - A_\lambda u^* - \lambda u^* \\ &:= f_\lambda, \quad 0 < t < T, \end{aligned} \tag{3.45}$$

$$v_\lambda(0) = 0, \quad v_\lambda'(T) = 0.$$

Note that for all $\lambda > 0$, $0 \le t \le T$, we have

$$\|A_\lambda u^*(t)\| = \|A_\lambda u_0 + tA_\lambda u_T\| \le \|Au_0\| + T\|Au_T\|.$$

Now we multiply (3.45) by v_λ and integrate over $[0,T]$:

$$\begin{aligned} -\int_0^T (v_\lambda'', v_\lambda)\,dt + \frac{1}{2}\int_0^T \frac{d}{dt}\|v_\lambda\|^2 dt + \int_0^T (A_\lambda v_\lambda, v_\lambda)\,dt \\ + \int_0^T (B(v_\lambda + u^*) - B(u^*), v_\lambda)\,dt + \int_0^T (B(u^*), v_\lambda)\,dt \\ + \lambda \int_0^T \|v_\lambda\|^2 dt = \int_0^T (f_\lambda, v_\lambda)\,. \end{aligned}$$

Therefore, using the monotonicity of A_λ, B, we get

$$\int_0^T \|v_\lambda'\|^2 dt \le \|B(u^*)\|_X \;\|v_\lambda\|_X + \|f_\lambda\|_X \;\|v_\lambda\|_X. \tag{3.46}$$

Obviously, since B is Lipschitz,

$$\begin{aligned} \|Bu^*(t)\| &\le \|Bu^*(t) - Bu^*(0)\| + \|Bu^*(0)\| \\ &\le C\|u^*(t) - u_0\| + \|Bu_0\| \\ &\le CT\|u_T\| + \|Bu_0\|. \end{aligned} \tag{3.47}$$

By (3.46) and (3.47), it follows that

$$\int_0^T \|v_\lambda'\|^2 dt \le C_0 \|v_\lambda\|_X, \quad \text{for all } \lambda \in (0, \lambda_0], \tag{3.48}$$

where C_0 is a positive constant. Note that

$$v_\lambda(t) = \int_0^t v_\lambda'(s) ds,$$

so

$$\|v_\lambda\|_{C([0,T];H)} \le T^{1/2}\;\|v_\lambda'\|_X, \tag{3.49}$$

and

$$\|v_\lambda\|_X \le T\;\|v_\lambda'\|_X. \tag{3.50}$$

By (3.48)-(3.50), we can see that $\{v_\lambda; 0 < \lambda \le \lambda_0\}$ is bounded in $C([0,T];H)$ and $\{v_\lambda'; 0 < \lambda \le \lambda_0\}$ is bounded in X. These properties are satisfied by u_λ too, so the proof of Claim 2 is complete. □

Claim 3. $\{A_\lambda u_\lambda; 0 < \lambda \le \lambda_0\}$ and $\{u''_\lambda; 0 < \lambda \le \lambda_0\}$ are both bounded in X.

Proof of Claim 3. It is sufficient to prove these properties for v_λ defined above. Since

$$\begin{aligned}\frac{d}{dt}\left(v'_\lambda, A_\lambda v_\lambda\right) &= \left(v''_\lambda, A_\lambda v_\lambda\right) + \underbrace{\left(v'_\lambda, A_\lambda v'_\lambda\right)}_{\ge 0} \\ &\ge \left(v''_\lambda, A_\lambda v_\lambda\right) \quad \text{for a. e. } t \in (0,T),\end{aligned}$$

we obtain

$$0 \ge \int_0^T \left(v''_\lambda, A_\lambda v_\lambda\right) dt = \int_0^T \left(A_\lambda v_\lambda + v'_\lambda + B(v_\lambda + u^*) + \lambda v_\lambda - f_\lambda, A_\lambda v_\lambda\right). \tag{3.51}$$

Note that

$$\begin{aligned}\|B\left(v_\lambda(t) + u^*(t)\right)\| &\le \|B\left(v_\lambda(t) + u^*(t)\right) - B0\| + \|B0\| \\ &\le C\|v_\lambda(t) + u^*(t)\| + \|B0\| \\ &\le C\|v_\lambda(t)\| + C\|u_0\| + CT\|u_T\| + \|B0\|,\end{aligned}$$

so, by Claim 2,

$$\sup_{\substack{0\le t\le T \\ 0<\lambda\le\lambda_0}} \|B\left(v_\lambda(t) + u^*(t)\right)\| < \infty. \tag{3.52}$$

From (3.51), (3.52) and Claim 2, we can see that $\{A_\lambda v_\lambda; 0 < \lambda \le \lambda_0\}$ is bounded in X, and so is $\{v''_\lambda; 0 < \lambda \le \lambda_0\}$ (see (3.45)). $\square$

Claim 4. u_λ converges, as $\lambda \to 0^+$, in $C([0,T]; H)$.

Proof of Claim 4. For $\lambda, \mu > 0$ we obtain from (3.44)

$$
\begin{aligned}
-\int_0^T \left(u_\lambda'' - u_\mu'', u_\lambda - u_\mu\right) dt + \int_0^T \left(u_\lambda' - u_\mu', u_\lambda - u_\mu\right) dt \\
+ \int_0^T (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu)\, dt \\
= -\int_0^T (Bu_\lambda - Bu_\mu, u_\lambda - u_\mu) - \int_0^T (\lambda u_\lambda - \mu u_\mu, u_\lambda - u_\mu)\, dt \\
\le -\int_0^T (\lambda u_\lambda - \mu u_\mu, u_\lambda - u_\mu)\, dt.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_0^T \|u_\lambda' - u_\mu'\|^2 \le -\int_0^T & \underbrace{(A_\lambda u_\lambda - A_\mu u_\mu, J_\lambda u_\lambda - J_\mu u_\mu)}_{\ge 0} dt \\
& - \int_0^T (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu)\, dt \\
& - \int_0^T (\lambda u_\lambda - \mu u_\mu, u_\lambda - u_\mu)\, dt \\
& \le C_1(\lambda + \mu),
\end{aligned}
$$

where $J_\lambda = (I + \lambda A)^{-1}$ and C_1 is a positive constant. The above estimate shows that (u_λ') is a Cauchy sequence in X, hence convergent in X as $\lambda \to 0^+$. Since

$$
\begin{aligned}
\|u_\lambda(t) - u_\mu(t)\| &= \left\| \int_0^t \left(u_\lambda' - u_\mu'\right) ds \right\| \\
&\le T^{1/2} \|u_\lambda' - u_\mu'\|_X,
\end{aligned}
$$

it follows that u_λ converges to some u in $C([0,T]; H)$ as $\lambda \to 0^+$. In fact, by Claim 3, we have $u \in W^{2,2}(0,T; H)$ and, as $\lambda \to 0^+$,

$$u_\lambda \to u \quad \text{in } C([0,T]; H), \tag{3.53}$$

$$u_\lambda' \to u' \quad \text{in } X, \tag{3.54}$$

$$u_\lambda'' \to u'' \quad \text{weakly in } X. \tag{3.55}$$

Even more, since (3.55) (u'_λ) is equicontinuous, we have

$$u'_\lambda \to u' \quad \text{in} \quad C([0,T];H). \tag{3.56}$$

$\square$

Proof of Theorem 3.20 *(Continuation)***.** On the other hand,

$$\begin{aligned} \|J_\lambda u_\lambda(t) - u(t)\| &\leq \|J_\lambda u_\lambda(t) - u_\lambda(t)\| + \|u_\lambda(t) - u(t)\| \\ &= \lambda\|A_\lambda u_\lambda(t)\| + \|u_\lambda(t) - u(t)\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|J_\lambda u_\lambda - u\|_X &\leq \lambda\|A_\lambda u_\lambda\|_X + \|u_\lambda - u\|_X \\ &\leq C_2\lambda + T^{1/2}\|u_\lambda - u\|_{C([0,T];H)}, \end{aligned}$$

where C_2 is a positive constant.

By (3.53) we get

$$J_\lambda u_\lambda \to u \quad \text{in } X \text{ as } \lambda \to 0^+. \tag{3.57}$$

Note that

$$A_\lambda u_\lambda = A(J_\lambda u_\lambda). \tag{3.58}$$

By using (3.53)-(3.55), (3.57) and (3.58), we can pass to limit as $\lambda \to 0^+$ in (3.44) regarded as an equation in X, and conclude that u satisfies (G) of problem (P_2) for a. e. $t \in (0,T)$. We also have by (3.53) and (3.56) that u satisfies (BC) of problem (P_2).

For uniqueness, let $\tilde{u} \in W^{2,2}(0,T;H)$ be another solution. Then,

$$-(u-\tilde{u})'' + (u-\tilde{u})' + A(u-\tilde{u}) + Bu - B\tilde{u} = 0 \quad \text{for a. e. } t \in (0,T), \tag{3.59}$$

$$(u-\tilde{u})(0) = 0 = (u' - \tilde{u}')(T). \tag{3.60}$$

We multiply (3.59) by $u-\tilde{u}$ and integrate over $[0,T]$

$$\int_0^T \|u'-\tilde{u}'\|^2 dt + \underbrace{\frac{1}{2}\|u-\tilde{u}\|^2\Big|_0^T}_{\geq 0} = -\int_0^T (A(u-\tilde{u}), u-\tilde{u})\, dt$$
$$- \int_0^T (Bu - B\tilde{u}, u-\tilde{u})\, dt \leq 0,$$

since both A and B are monotone. So $(u-\tilde{u})' \equiv 0 \Rightarrow u \equiv \tilde{u}$ (cf. (3.60)). The proof of Theorem 3.20 is complete. □

Theorem 3.21. *Assume that (H1) and (H2) hold. If $u_0, u_T \in H$ and $f \in L^2(0,T;H)$, then problem (P_2) has a unique generalized (in the sense explained below) solution $u \in C([0,T];H) \cap W^{2,2}_{\mathrm{loc}}(0,T;H)$, with $t^{1/2}u'$, $t^{3/2}(T-t)^{3/2}u'' \in L^2(0,T;H)$.*

Proof. Note that $\overline{D(A)} = \overline{D(A+B)} = H$.

Claim 1. If u, v are two solutions of (F) of problem (P_2) with the properties specified in the above statement (of Theorem 3.21), then

$$\|u-v\|_{C([0,T];H)} \leq \max\left\{ e^{T/2}\|u(0)-v(0)\|, \|u(T)-v(T)\| \right\} \tag{3.61}$$

$$\int_0^T t\left(\|u-v\|^2 + \|u'-v'\|^2\right) dt \leq C_3\big(\, \|u(0)-v(0)\|^2 + \|u'(T)-v'(T)\|^2\big)\,, \tag{3.62}$$

where C_3 is a positive constant.

Proof of Claim 1. Define

$$g(t) = \frac{1}{2}\|u(t)-v(t)\|^2, \quad 0 \leq t \leq T.$$

By a computation similar to the proof of Claim 1 of Theorem 3.17, we get

$$\left(e^{-t}g\right)'' \geq e^{-t}\left(\|u'-v'\|^2 - (u'-v', u-v) + \frac{1}{2}\|u-v\|^2 \right), \quad 0 \leq t \leq T.$$

This implies

$$\left(e^{-t}g\right)'' \geq c\, e^{-t}\left(\|u-v\|^2 + \|u'-v'\|^2\right), \quad 0 \leq t \leq T, \tag{3.63}$$

where c is a small positive constant (e.g., $c = \frac{1}{8}$).

Estimate (3.63) shows that $t \mapsto e^{-t}g(t)$ is a convex function and consequently

$$e^{-t}g(t) \leq \max\left\{g(0), e^{-T}g(T)\right\}, \quad 0 \leq t \leq T,$$

i.e., inequality (3.61) holds.

Now we multiply (3.63) by t and integrate over $[0,T]$:

$$\begin{aligned}
&c\int_0^T te^{-t}\left(\|u-v\|^2 + \|u'-v'\|^2\right) dt \leq \int_0^T t(e^{-t}g)''dt \\
&\quad = t(e^{-t}g)'\Big|_0^T - \int_0^T (e^{-t}g)'dt \\
&\quad = te^{-t}(g'-g)\Big|_0^T - e^{-t}g\Big|_0^T \\
&\quad = Te^{-T}\left((u'(T)-v'(T), u(T)-v(T)) - \frac{1}{2}\|u(T)-v(T)\|^2\right) \\
&\qquad\quad - \frac{e^{-T}}{2}\|u(T)-v(T)\|^2 + \frac{1}{2}\|u(0)-v(0)\|^2 \\
&\quad \leq Te^{-T}\left(\frac{1}{2}\|u'(T)-v'(T)\|^2 + \frac{1}{2}\|u(T)-v(T)\|^2 - \frac{1}{2}\|u(T)-v(T)\|^2\right) \\
&\qquad\quad + \frac{1}{2}\|u(0)-v(0)\|^2 \\
&\quad \leq \frac{Te^{-T}}{2}\|u'(T)-v'(T)\|^2 + \frac{1}{2}\|u(0)-v(0)\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
c\, e^{-T}\int_0^T t\left(\|u-v\|^2 + \|u'-v'\|^2\right) dt \leq{} & \frac{Te^{-T}}{2}\|u'(T)-v'(T)\|^2 \\
& + \frac{1}{2}\|u(0)-v(0)\|^2,
\end{aligned}$$

thus (3.62) holds with

$$C_3 = \frac{1}{2c}\max\left\{T, e^T\right\}.$$

$\square$

Claim 2. Let $u \in W^{2,2}(0,T;H)$ be a solution of problem (P_2), where $u_0, u_T \in D(A)$ and $f \in L^2(0,T;H)$. Let $u_\lambda \in W^{2,2}(0,T;H)$ be the unique solution of the problem

$$-u_\lambda'' + u_\lambda' + A_\lambda u_\lambda = f - Bu, \quad 0 \le t \le T, \tag{3.64}$$

$$u_\lambda(0) = u_0, \quad u_\lambda'(T) = u_T, \tag{3.65}$$

where $\lambda > 0$ and A_λ denotes the Yosida approximation of A. The existence of u_λ follows by Theorem 3.20, where $A = 0$, $B = A_\lambda$, and $f(t) := f(t) - Bu(t)$. Then, as $\lambda \to 0^+$,

$$u_\lambda \to u, \quad u_\lambda' \to u' \quad \text{in} \quad C([0,T];H), \tag{3.66}$$

$$u_\lambda'' \to u'' \quad \text{weakly in} \quad X = L^2(0,T;H). \tag{3.67}$$

Proof of Claim 2. As in the proof of Theorem 3.20, denote

$$v_\lambda(t) = u_\lambda(t) - u^*(t), \quad \text{where}$$

$$u^*(t) = u_0 + u_T\, t, \quad 0 \le t \le T.$$

Obviously, v_λ satisfies the problem

$$-v_\lambda'' + v_\lambda' + A_\lambda v_\lambda = f_\lambda, \quad 0 < t < T, \tag{3.68}$$

$$v_\lambda(0) = 0 = v_\lambda'(T), \tag{3.69}$$

where

$$f_\lambda(t) := f(t) - Bu(t) - A_\lambda u^*(t) - u_T.$$

If we multiply (3.68) by v_λ and integrate over $[0,T]$, we obtain an estimate similar to (3.48) of Claim 2 in the proof of Theorem 3.20 (here we do not need any upper bound for $\lambda > 0$).

Continuing the proof along the lines of the proof of Theorem 3.20, we can show that there exists a $\tilde{u} \in W^{2,2}(0,T;H)$ such that

$$u_\lambda \to \tilde{u},\ u_\lambda' \to \tilde{u}' \text{ in } C([0,T];H) \text{ and } u_\lambda'' \to \tilde{u}'' \text{ weakly in } X, \text{ as } \lambda \to 0^+.$$

Moreover, $\tilde{u}$ is a solution of the problem

$$-\tilde{u}'' + \tilde{u}' + A\tilde{u} + Bu = f, \quad 0 < t < T, \tag{3.70}$$

$$\tilde{u}(0) = u_0, \quad \tilde{u}'(T) = u_T. \tag{3.71}$$

From (P_2) and (3.70)-(3.71) one can easily obtain that $\tilde{u} \equiv u$, thus completing the proof of Claim 2. □

Claim 3. If $u \in W^{2,2}(0,T;H)$ is a solution of problem (P_2), where $u_0, u_T \in D(A)$ and $f \in L^2(0,T;H)$, then there are positive constants C_4, C_5 such that

$$\|u''\|_{X_{**}} \le C_4 \left(\|f\|_X + \|u'\|_{X_*} + \|u\|_{C([0,T];H)}\right) + C_5, \tag{3.72}$$

where $X = L^2(0,T;H)$, $X_* = L^2(0,T;H;tdt)$, $X_{**} = L^2(0,T;H;\beta^3(t)dt)$, $\beta(t) = \min\{t, T-t\}$.

Proof of Claim 3. Consider again problem (3.64), (3.65). If we multiply the obvious inequality

$$\frac{d}{dt}\left(u_\lambda', A_\lambda u_\lambda\right) \ge \left(u_\lambda'', A_\lambda u_\lambda\right)$$

by $\beta^3(t)$ and integrate over $[0,T]$, we obtain

$$-3\int_0^T \beta^2\beta'\left(u_\lambda', A_\lambda u_\lambda\right)dt \ge \int_0^T \beta^3\left(u_\lambda' + A_\lambda u_\lambda + Bu - f, A_\lambda u_\lambda\right)dt.$$

This implies

$$\begin{aligned}\|A_\lambda u_\lambda\|_{X_{**}}^2 \le 3\|A_\lambda u_\lambda\|_{X_{**}} &\left(\int_0^T \beta(t)\|u_\lambda'\|^2 dt\right)^{1/2} + \|A_\lambda u_\lambda\|_{X_{**}}\, \|u_\lambda'\|_{X_{**}} \\ &+ \|A_\lambda u_\lambda\|_{X_{**}}\, \|Bu\|_{X_{**}} + \|A_\lambda u_\lambda\|_{X_{**}}\, \|f\|_{X_{**}},\end{aligned}$$

so

$$\|A_\lambda u_\lambda\|_{X_{**}} \leq 3\|u'_\lambda\|_{X_*} + \|u'_\lambda\|_{X_{**}} + \|Bu\|_{X_{**}} + \|f\|_{X_{**}}. \tag{3.73}$$

Since B is Lipschitz and

$$C([0,T];H) \subset X \subset X_* \subset X_{**},$$

with continuous injections, we obtain from (3.73) estimate (3.72) for $u := u_\lambda$, i.e.,

$$\|u''_\lambda\|_{X_{**}} \leq C_4 \left(\|f\|_X + \|u'_\lambda\|_{X_*} + \|u\|_{C([0,T];H)}\right) + C_5. \tag{3.74}$$

To conclude the proof of Claim 3, we just have to use Claim 2 (see (3.66) and (3.67)) and pass to the limit in (3.74) as $\lambda \to 0^+$. $\square$

Proof of Theorem 3.21 *(Continuation).* Let $u_{0n}, u_{Tn} \in D(A)$ such that $\|u_{0n} - u_0\| \to 0$, $\|u_{Tn} - u_T\| \to 0$, as $n \to \infty$.
Let $u_n \in W^{2,2}(0,T;H)$ be the solution of the problem (P_2) with u_{0n}, u_{Tn} instead of u_0, u_T. By estimate (3.62) of Claim 1, (u_n) is a Cauchy sequence in X_*. Consequently, there exists a $u \in X_*$ such that $u' \in X_*$ and

$$u_n \to u,\ u'_n \to u' \text{ in } X_*.$$

It follows that

$$u_n \to u \text{ in } C([\delta,T];H) \text{ for each } 0 < \delta < T,$$

so in particular $\|u_n(T) - u(T)\| \to 0$. According to the estimate (3.61) of Claim 1, we see that u_n converges in $C([0,T];H)$, hence $u \in C([0,T];H)$ and $u(0) = u_0$.
By the estimate (3.72) of Claim 3, u''_n is bounded in X_{**}. This implies that $u'' \in X_{**}$ and

$$u''_n \to u'' \text{ weakly in } X_{**}.$$

Regarding the equation

$$-u_n'' + u_n' + Au_n + Bu_n = f$$

as one in the space $L^2(\delta, T-\delta; H)$ for positive small δ's, we see that u satisfies equation (F) of problem (P_2) for a. e. $t \in (0,T)$.
If $u_T \in D(A)$, one can use $u_{Tn} = u_T$ for all $n \in \mathbb{N}$. Using the change

$$\tilde{u}_\lambda(t) = u_\lambda(t) - t\, u_T$$

in (3.64), (3.65), we obtain

$$-\tilde{u}_\lambda'' + \tilde{u}_\lambda' + A_\lambda \tilde{u}_\lambda = f - Bu - tA_\lambda u_T - u_T =: \tilde{f}_\lambda,$$

$$\tilde{u}_\lambda(0) = u_0, \quad \tilde{u}_\lambda'(0) = 0,$$

with $\{\tilde{f}_\lambda; \lambda > 0\}$ bounded in X (note that $\|A_\lambda u_T\| \leq \|Au_T\|$). If we replace $\beta(t)$ in the proof of Claim 3 by $\tilde{\beta}(t) = t$, we obtain (3.72) with $X_{**} = L^2(0,T; t^3 dt)$. Since u_n'' is bounded in $L^2(\delta, T; H)$, it follows that u_n' converges to u' in $C([\delta, T]; H)$, so $u'(T) = u_T$. Therefore, u is a solution of problem (P_2), with $t^{1/2}u', t^{3/2}u'' \in L^2(0,T;H)$.
If both $u_0, u_T \in H \setminus D(A)$, then there exists a generalized solution u, i.e., u satisfies all properties specified in the statement of Theorem 3.21, except $u'(T) = u_T$. Uniqueness follows by a standard argument. $\square$

Theorem 3.22. *Assume that (H1), (H2) hold. If $u_0, u_T \in D(A)$ and $f \in W^{1,2}(0,T;H)$, then problem (P_2) has a unique solution $u \in W^{2,2}(0,T;H) \cap W^{3,2}_{\mathrm{loc}}(0,T;H)$ with $t^{3/2}(T-t)^{3/2}u''' \in L^2(0,T;H)$. If $u_0 \in H$, $u_T \in D(A)$, $f \in W^{1,2}(0,T;H)$, then $u \in C([0,T];H)$, $t^{1/2}u'$, $t^{3/2}u''$, $t^{5/2}(T-t)^{5/2}u''' \in L^2(0,T;H)$.*

Proof. Assume first that $u_0, u_T \in D(A)$ and $f \in W^{1,2}(0,T;H)$. By Theorem 3.20 there exists a unique solution $u \in W^{2,2}(0,T;H)$. Consider again problem (3.64), (3.65) for $\lambda > 0$. We see that $u_\lambda \in W^{3,2}(0,T;H)$ and

$$-u_\lambda''' + u_\lambda'' + A_\lambda u_\lambda' = f' - (Bu)' \quad \text{for a. e. } t \in (0,T).$$

Starting from the obvious inequality

$$\frac{d}{dt}\,(u_\lambda'', A_\lambda u_\lambda') \geq (u_\lambda''', A_\lambda u_\lambda')$$

we obtain after multiplying by $\beta^3(t)$ and integrating over $[0,T]$ that

$$\|u_\lambda'''\|_{X_{**}} \leq \widetilde{C_4}\left(\|f\|_X + \|u_\lambda''\|_{X_*} + \|u'\|_{C([0,T];H)}\right) + \widetilde{C_5}. \tag{3.75}$$

By (3.66), (3.67), (3.75) it follows that $u''' \in X_{**}$ and $u_\lambda''' \to u'''$ in X_{**}.
Now assume that $u_0 \in H$, $u_T \in D(A)$ and $f \in W^{1,2}(0,T;H)$. By Theorem 3.21, there exists a unique solution $u \in C([0,T];H) \cap W^{2,2}_{\text{loc}}(0,T;H)$, with $t^{1/2}u'$, $t^{3/2}u'' \in L^2(0,T;H)$. So all we have to prove is that u'' is differentiable for a. e. $t \in (0,T)$ and $t^{5/2}(T-t)^{5/2}u''' \in L^2(0,T;H)$. We approximate u_0 by $u_{0n} \in D(A)$ and denote by u_n the solution of the problem

$$\begin{cases} -u_n'' + u_n' + Au_n + Bu_n = f, \quad 0 < t < T, \\ u_n(0) = u_{0n}, \quad u_n'(T) = u_T. \end{cases}$$

By the proof of Theorem 3.21,

$$u_n \to u \text{ in } C([0,T];H),\ t^{1/2}u_n' \to t^{1/2}u' \text{ in } L^2(0,T;H), \text{ and}$$

$$t^{3/2}u_n'' \to t^{3/2}u'' \text{ weakly in } L^2(0,T;H).$$

Now, for a fixed $n \in \mathbb{N}$ and $\lambda > 0$, denote by $u_{n\lambda}$ the solution of the problem

$$\begin{cases} -u_{n\lambda}'' + u_{n\lambda}' + A_\lambda u_{n\lambda} = f - Bu_n, \quad 0 < t < T, \\ u_{n\lambda}(0) = u_{0n}, \quad u_{n\lambda}'(T) = u_T. \end{cases}$$

We can assume that $u_T = 0$ (otherwise, we change u: $\tilde{u}(t) = u(t) - t\,u_T$). We know from the proof of Theorem 3.21 that

$$\begin{cases} u_{n\lambda} \to u_n, \quad u'_{n\lambda} \to u'_n \quad \text{in } C([0,T];H) \text{ and} \\ u''_{n\lambda} \to u''_n \quad \text{weakly in } X = L^2(0,T;H), \text{ as } \lambda \to 0^+. \end{cases}$$

Obviously, $u_{n\lambda} \in W^{3,2}(0,T;H)$ and

$$-u'''_{n\lambda} + u''_{n\lambda} + A_\lambda u'_{n\lambda} = f' - (Bu_n)' \quad \text{for a. e. } t \in (0,T).$$

We multiply the inequality

$$\frac{d}{dt}\,(u''_{n\lambda}, A_\lambda u'_{n\lambda}) \geq (u'''_{n\lambda}, A_\lambda u'_{n\lambda})$$

by $\beta^5(t)$ and integrate over $[0,T]$ to get (see (3.73), (3.74))

$$\begin{aligned}\left(\int_0^T \beta^5 \|u'''_{n\lambda}\|^2\right)^{1/2} \leq D \Bigg(\|u''_{n\lambda}\|_{X_{**}} &+ \left(\int_0^T \beta^5 \|u''_{n\lambda}\|^2\right)^{1/2} \\ &+ \left(\int_0^T \beta^5 \|(Bu_n)'\|^2\right)^{1/2} + \left(\int_0^T \beta^5 \|f'\|^2\right)^{1/2}\Bigg).\end{aligned} \tag{3.76}$$

Similarly (see (3.74))

$$\begin{gathered}\|u''_{n\lambda}\|_{X_{**}} \leq \widetilde{C_4}\left(\|f\|_X + \|u'_{n\lambda}\|_{X_*} + \|u_n\|_{C([0,T];H)}\right) + \widetilde{C_5} \\ (\text{in fact, here } X_{**} = L^2(0,T;H;t^3dt)).\end{gathered} \tag{3.77}$$

By (3.76) and (3.77), we get

$$\begin{aligned}\left(\int_0^T \beta^5 \|u'''_{n\lambda}\|^2\right)^{1/2} \leq D_1 \big(\|f\|_X + \|f'\|_X &+ \|u'_{n\lambda}\|_{X_*} \\ &+ \|u'_n\|_{X_*} + \|u_n\|_{C([0,T];H)}\big) + D_2.\end{aligned} \tag{3.78}$$

Since $u'_{n\lambda} \to u'_n$ in $C([0,T];H)$, (3.78) implies

$$\begin{aligned}\left(\int_0^T \beta^5 \|u'''_{n\lambda}\|^2\right)^{1/2} \leq D_1 \big(\|f\|_X + \|f'\|_X &+ 2\|u'_n\|_{X_*} \\ &+ \|u_n\|_{C([0,T];H)}\big) + D_2.\end{aligned}$$

which shows that $\left(\beta^{5/2}u'''_{n\lambda}\right)_{\lambda>0}$ is bounded in X and its limit $\beta^{5/2}u'''_n$ satisfies

$$\left(\int_0^T \beta^5\|u'''_n\|^2\right)^{1/2} \le D_1\left(\|f\|_X + \|f'\|_X + 2\|u'_n\|_{X_*} + \|u_n\|_{C([0,T];H)}\right) + D_2. \tag{3.79}$$

Since $u_n \to u$ in $C([0,T];H)$ and $u'_n \to u'$ in X_* it follows from (3.79) that $\beta^{5/2}u''' \in X$. □

Remark 3.23. *By using the above method, we can obtain higher regularity for u under appropriate regularity assumptions on f and B.*

Remark 3.24. *From the last two results (Theorems 3.21 and 3.22), we see that the boundary condition $u'(T) = u_T$ brings some difficulties as compared to the previous case when $u(T) = u_T$, and less regularity is obtained near $t = T$. Thus problems (P_1^ε) and (P_2^ε) are essentially different. This difference will be noticed again in Chapter 4, where different boundary layer phenomena are identified.*

Chapter 4

Asymptotic Expansions

We have seen in Chapter 3 that the solutions of problems (P_1^ε) and (P_2^ε) are more regular than those of problem (P_0). For example, if $u_0, u_T \in H$ and $f \in L^2(0,T;H)$, then the solution of problem (P_0) belongs to $C([0,T];H)$, while the solutions of problems (P_1^ε) and (P_2^ε) belong to $C([0,T];H) \cap W^{2,2}_{\text{loc}}(0,T;H)$ (cf. Theorems 3.9, 3.17 and 3.21). Now if $u_0, u_T \in D(A)$ and $f \in W^{1,2}(0,T;H)$, then the solution of problem (P_0) belongs to $C^1([0,T];H)$, while the solutions of problems (P_1^ε) and (P_2^ε) belong to $W^{2,2}(0,T;H) \cap W^{3,2}_{\text{loc}}(0,T;H)$ (cf. Theorems 3.9, 3.18 and 3.22).

We also expect that the solutions of the elliptic-like regularizations (P_1^ε) and (P_2^ε) approximate the solution of problem (P_0) as $\varepsilon \to 0^+$. We will show in what follows that this is indeed the case under suitable conditions on the data. However, a boundary layer occurs near $t = T$, and so the solutions u_ε's of the elliptic-like regularizations (P_1^ε) and (P_2^ε) must be corrected by adding boundary layer functions in order to obtain a good approximation for the solution of problem (P_0).

In this chapter we will establish asymptotic expansions of order zero for the elliptic-like regularizations (P_1^ε) and (P_2^ε), as well as asymptotic expansion of order one for the elliptic-like regularization (P_2^ε). As we mentioned in Chapter 2 that the elliptic and hyperbolic regularizations of the semilinear heat equation, which is a special case of problem (P_0), have been discussed by L. Barbu and G. Moroşanu [7, pp. 209-226]. So, in our general case, we also expect the following asymptotic expansion of order zero to hold for the elliptic-like regularizations (P_1^ε) and (P_2^ε)

$$u_\varepsilon(t) = u(t) + i(\tau) + r_\varepsilon(t), \quad 0 \le t \le T, \tag{4.1}$$

where $\tau := \frac{T-t}{\varepsilon}$ is the stretched (fast) variable, $u = u(t)$ is the solution of the reduced problem (P_0), $i = i(\tau)$ is the boundary layer function, and $r_\varepsilon = r_\varepsilon(t)$ is the remainder (of order zero). We will see that in case of problem (P_2^ε), we have $i \equiv 0$, and the problem is regularly perturbed (of order zero). For the first order asymptotic expansion for the elliptic-like regularization (P_2^ε), we expect the following

$$u_\varepsilon(t) = u(t) + \varepsilon\left[u_1(t) + i_1(\tau)\right] + r_\varepsilon(t). \tag{4.2}$$

4.1 Asymptotic expansion of order zero for problem (P_1^ε)

In this section, we will discuss asymptotic expansion of order zero for problem (P_1^ε). It turns out that problem (P_1^ε) is regularly perturbed in $L^2(0,T;H)$, while it is singularly perturbed in $C([0,T];H)$. We will also derive the estimates for remainder of order zero with respect to the norms of $L^2(0,T;H)$ and $C([0,T];H)$.

Assuming that all functions involved in (4.1) are smooth enough, we can identify these functions by heuristic arguments. We have

$$u'_\varepsilon(t) = u'(t) - \frac{1}{\varepsilon}\frac{d}{d\tau}i(\tau) + r'_\varepsilon(t),$$

$$u''_\varepsilon(t) = u''(t) + \frac{1}{\varepsilon^2}\frac{d^2}{d\tau^2}i(\tau) + r''_\varepsilon(t).$$

so

$$\begin{aligned} -\varepsilon\left[u''(t) + \frac{1}{\varepsilon^2}\frac{d^2}{d\tau^2}(\tau) + r''_\varepsilon(t)\right] + \left[u'(t) - \frac{1}{\varepsilon}\frac{di}{d\tau}(\tau) + r'_\varepsilon(t)\right] \\ + Au(t) + Ai(\tau) + Ar_\varepsilon(t) + Bu_\varepsilon = f(t). \end{aligned} \tag{4.3}$$

If we identify the coefficients of ε^{-1}, ε^0, we get

$$\frac{d^2 i}{d\tau^2} + \frac{di}{d\tau} = 0, \quad \tau > 0 \quad \text{with} \quad i(0) = u_T - u(T), \tag{4.4}$$

u satisfies (P_0), and r_ε satisfies

$$\begin{cases} -\varepsilon(u+r_\varepsilon)'' + r_\varepsilon' + Ar_\varepsilon + B(u_\varepsilon) = B(u) - Ai, \\ r_\varepsilon(0) = -i(T/\varepsilon), \quad r_\varepsilon(T) = 0. \end{cases} \tag{R_ε}$$

From (4.4) we get (note that $i(\infty) = 0$)

$$i(\tau) = (u_T - u(T))\, e^{-\tau}. \tag{4.5}$$

Condition $i(\infty) = 0$ should be read as: i is negligible away from the boundary layer. For more details on the heuristic procedure to determine asymptotic expansions, see, e.g., [7]. In what follows we validate expansion (4.1).

Theorem 4.1. *Assume that (H1) and (H2) hold, $u_0, u_T \in D(A)$, A is strongly positive, i.e., $(Ax, x) \geq c\|x\|^2\ \forall x \in D(A)$, for some $c > 0$, and $f \in W^{1,1}(0, T; H)$. Then, for every $\varepsilon > 0$, the solution u_ε of problem (P_1^ε) admits the following asymptotic expansion*

$$u_\varepsilon(t) = u(t) + i(\tau) + r_\varepsilon(t), \quad 0 \leq t \leq T,\ \tau := (T-t)/\varepsilon,$$

where u is the solution of problem (P_0), $i(\tau) = (u_T - u(T))\, e^{-\tau}$ is the boundary layer function, and the remainder $r_\varepsilon = r_\varepsilon(t)$ satisfies problem (R_ε). Moreover, for $0 < \varepsilon < 1$, we have the following estimates

$$\begin{aligned} \|r_\varepsilon\|_{C([0,T];H)} &= \mathcal{O}(\varepsilon^{1/4}), \\ \|r_\varepsilon\|_{L^2(0,T;H)} &= \mathcal{O}(\varepsilon^{1/2}), \\ \|u_\varepsilon - u\|_{L^2(0,T;H)} &= \mathcal{O}\left(\varepsilon^{1/2}\right). \end{aligned}$$

Thus, problem (P_1^ε) is regularly perturbed in $L^2(0, T; H)$, while it is singularly perturbed in $C([0, T]; H)$.

Proof. By Theorems 3.9 and 3.16, we have

$$r_\varepsilon = u_\varepsilon - u - i \in C^1([0, T]; H), \tag{4.6}$$

and

$$u + r_\varepsilon = u_\varepsilon - i \in W^{2,2}(0,T;H).$$

Note that $u(T) \in D(A)$, so $i(\tau) \in D(A)$ for all $\tau \geq 0$. It is easy to check that r_ε, defined by (4.6), satisfies problem (R_ε).

In order to homogenize the first boundary condition for r_ε, we set

$$\bar{r}_\varepsilon(t) = r_\varepsilon(t) + \alpha_\varepsilon(t), \quad 0 \leq t \leq T,$$

where

$$\alpha_\varepsilon(t) = (1 - t/T) i(T/\varepsilon).$$

Obviously, $\bar{r}_\varepsilon$ satisfies the problem

$$-\varepsilon \left(u + \bar{r}_\varepsilon\right)'' + \bar{r}_\varepsilon' + A\bar{r}_\varepsilon + Bu_\varepsilon = h_\varepsilon + Bu, \quad 0 \leq t \leq T, \tag{4.7a}$$

$$\bar{r}_\varepsilon(0) = 0, \quad \bar{r}_\varepsilon(T) = 0, \tag{4.7b}$$

where

$$h_\varepsilon(t) := -i(T/\varepsilon) + A\alpha_\varepsilon(t) - Ai(\tau). \tag{4.8}$$

Multiplying (4.7a) by $\bar{r}_\varepsilon$ and integrating over $[0,T]$, we obtain

$$\begin{aligned} \varepsilon \int_0^T \left((u + \bar{r}_\varepsilon)', \bar{r}_\varepsilon'\right) dt + \frac{1}{2}\int_0^T \frac{d}{dt}\|\bar{r}_\varepsilon\|^2 dt + \int_0^T (A\bar{r}_\varepsilon, \bar{r}_\varepsilon)\, dt \\ + \int_0^T (Bu_\varepsilon, \bar{r}_\varepsilon)\, dt = \int_0^T (h_\varepsilon, \bar{r}_\varepsilon)\, dt + \int_0^T (Bu, \bar{r}_\varepsilon)\, dt. \end{aligned} \tag{4.9}$$

Since B is monotone, we derive from (4.9)

$$\begin{aligned} \varepsilon \int_0^T \|\bar{r}_\varepsilon'\|^2 dt + \int_0^T (A\bar{r}_\varepsilon, \bar{r}_\varepsilon)\, dt \leq \int_0^T (Bu - B(u - \alpha_\varepsilon + i), \bar{r}_\varepsilon)\, dt \\ \|h_\varepsilon\|_{L^2}\, \|\bar{r}_\varepsilon\|_{L^2} + \varepsilon\|u'\|_{L^2}\|\bar{r}_\varepsilon'\|_{L^2}. \end{aligned} \tag{4.10}$$

Note that

$$\begin{cases} \|i(T/\varepsilon)\| = \mathcal{O}(\varepsilon^j) \quad \forall j \geq 1 \quad \forall 0 < \varepsilon < 1, \\ \|\alpha_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^j) \quad \forall j \geq 1 \quad \forall 0 < \varepsilon < 1, \\ \|i\|_{L^2} = \mathcal{O}(\varepsilon^{1/2}) \quad \forall \varepsilon > 0, \\ \|Ai\|_{L^2} = \|e^{-\tau} A(u_T - u(T))\|_{L^2} \\ \qquad = \|A(u_T - u(T))\| \, \|e^{-\tau}\|_{L^2(0,T)} = \mathcal{O}(\varepsilon^{1/2}) \quad \forall \varepsilon > 0, \\ \|h_\varepsilon\| = \mathcal{O}(\varepsilon^{1/2}) \quad \forall 0 < \varepsilon < 1. \end{cases} \tag{4.11}$$

We get an estimate

$$\|B(u + i - \alpha_\varepsilon) - Bu\|_{L^2} \leq C\|i - \alpha_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^{1/2}) \quad \forall 0 < \varepsilon < 1. \tag{4.12}$$

From (4.10) and (4.12) we obtain

$$\varepsilon\|\bar{r}_\varepsilon'\|_{L^2}^2 + c\|\bar{r}_\varepsilon\|_{L^2}^2 \leq M\varepsilon^{1/2}\|\bar{r}_\varepsilon\|_{L^2} + \varepsilon\|u'\|_{L^2} \, \|\bar{r}_\varepsilon'\|_{L^2} \quad \forall 0 < \varepsilon < 1, \tag{4.13}$$

where $M > 0$ is some constant.

$$\begin{aligned} M\varepsilon^{1/2}\|\bar{r}_\varepsilon\|_{L^2} &= \frac{1}{\sqrt{c}} M\varepsilon^{1/2} \sqrt{c} \, \|\bar{r}_\varepsilon\|_{L^2} \\ &\leq \frac{1}{2c} M^2\varepsilon + \frac{c}{2}\|\bar{r}_\varepsilon\|_{L^2}^2 \qquad (\text{using } ab \leq \frac{a^2}{2} + \frac{b^2}{2}). \end{aligned} \tag{4.14}$$

By (4.13) and (4.14), we have

$$\varepsilon\|\bar{r}_\varepsilon'\|_{L^2}^2 + \frac{c}{2}\|\bar{r}_\varepsilon\|_{L^2}^2 \leq \frac{1}{2c} M^2\varepsilon + \varepsilon\|u'\|_{L^2} \, \|\bar{r}_\varepsilon'\|_{L^2} \quad \forall 0 < \varepsilon < 1. \tag{4.15}$$

By (4.15), we have

$$\|\bar{r}_\varepsilon'\|_{L^2}^2 \leq \frac{1}{2c} M^2 + \|u'\|_{L^2} \, \|\bar{r}_\varepsilon'\|_{L^2} \quad \forall 0 < \varepsilon < 1$$

$$\Rightarrow \left(\|\bar{r}_\varepsilon'\|_{L^2} - \frac{1}{2}\|u'\|_{L^2}\right)^2 \leq \frac{1}{2c} M^2 + \frac{1}{4}\|u'\|_{L^2}^2 \quad \forall 0 < \varepsilon < 1,$$

which shows that

$$\{\|\bar{r}_\varepsilon'\|_{L^2}; 0 < \varepsilon < 1\} \text{ is bounded.} \tag{4.16}$$

By (4.15), we also have

$$\frac{c}{2}\|\bar{r}_\varepsilon\|_{L^2}^2 \leq \frac{1}{2c}M^2\varepsilon + \varepsilon\|u'\|_{L^2}\,\|\bar{r}_\varepsilon'\|_{L^2} \quad \forall 0 < \varepsilon < 1. \tag{4.17}$$

By (4.16) and (4.17), we have

$$\|\bar{r}_\varepsilon\|_{L^2} = \mathcal{O}\left(\varepsilon^{1/2}\right) \quad \forall 0 < \varepsilon < 1. \tag{4.18}$$

By (4.11) and (4.18), we have

$$\|r_\varepsilon\|_{L^2} = \mathcal{O}\left(\varepsilon^{1/2}\right) \quad \forall 0 < \varepsilon < 1. \tag{4.19}$$

Note that

$$\begin{aligned} \|r_\varepsilon(t)\|^2 = \int_0^t \left(\|r_\varepsilon\|^2\right)' &= 2\int_0^t (r_\varepsilon, r_\varepsilon') \\ &\leq 2\|r_\varepsilon\|_{L^2}\,\|r_\varepsilon'\|_{L^2}. \end{aligned} \tag{4.20}$$

From (4.16), (4.19) and (4.20), we obtain

$$\|r_\varepsilon\|_{C([0,T];H)} = \mathcal{O}\left(\varepsilon^{1/4}\right) \quad \forall 0 < \varepsilon < 1. \tag{4.21}$$

It is easy to see that problem (P_1^ε) is singularly perturbed in $C([0,T];H)$. Assume, on contrary, that it is regularly perturbed, i.e.,

$$\lim_{\varepsilon \to 0^+} \|u_\varepsilon - u\|_{C([0,T];H)} = 0,$$

which together with (4.21) contradicts the fact that $\|i\|_{C([0,T];H)} = \|u_T - u(T)\| \neq 0$ in general.

However, by (4.11) and (4.19), we have

$$\|u_\varepsilon - u\|_{L^2} = \mathcal{O}\left(\varepsilon^{1/2}\right) \quad \forall 0 < \varepsilon < 1,$$

which shows that problem (P_1^ε) is regularly perturbed in $L^2(0,T;H)$. □

Remark 4.2. *The results presented in this paper cover many specific problems in PDEs, in particular the semilinear heat equation mentioned in Chapter 2. For this particular example, one can obtain, in addition to (4.21) (which reads in this case $\|r_\varepsilon\|_{C([0,T];L^2(\Omega))} = \mathcal{O}(\varepsilon^{1/4})$) the following estimate*

$$\|u_\varepsilon - u\|_{L^2(0,T;H_0^1(\Omega))} = \mathcal{O}(\varepsilon^{1/2}). \tag{4.22}$$

Indeed, an inspection of the proof of Theorem 4.1, shows that

$$(A\bar{r}_\varepsilon, \bar{r}_\varepsilon)_{L^2(0,T;L^2(\Omega))} = \int_0^T \int_\Omega \nabla_x \bar{r}_\varepsilon \,.\, \nabla_x \bar{r}_\varepsilon \, dx\, dt = \mathcal{O}(\varepsilon),$$

which implies (4.22), since

$$\begin{aligned} \|i\|_{L^2(0,T;H_0^1(\Omega))} &= \|u_1 - u(T)\| \left(\int_0^T e^{-2\left(\frac{T-t}{\varepsilon}\right)} \, dt \right)^{1/2} \\ &= \mathcal{O}(\varepsilon^{1/2}). \end{aligned}$$

So in this case the boundary layer function i can be included in the remainder term r_ε, i.e., i disappears from expansion (4.1). In other words, the boundary layer is not visible in $L^2(0,T;H_0^1(\Omega))$ and problem (P_1^ε) is regularly perturbed in this space (while it is singularly perturbed in $C([0,T];L^2(\Omega))$.

4.2 Asymptotic expansion of order zero for problem (P_2^ε)

In this section, we will discuss the asymptotic expansion of order zero for problem (P_2^ε). In turns out that problem (P_2^ε) is regularly perturbed in $C([0,T];H)$ (hence, it is also regularly perturbed in $L^2(0,T;H)$). We will also derive the estimates for remainder of order zero with respect to the norms of $L^2(0,T;H)$ and $C([0,T];H)$.

We are looking for an expansion of order zero. We expect a discrepancy at $t = T$, so the following expression is expected to hold

$$u_\varepsilon(t) = u(t) + i(\tau) + r_{0\varepsilon}(t), \quad 0 \le t \le T, \tag{4.23}$$

where $\tau = (T-t)/\varepsilon$, $\varepsilon > 0$ is the stretched (fast) variable, $u = u(t)$ is the solution of problem (P_0), $i = i(\tau)$ is the boundary layer function, and $r_{0\varepsilon} = r_{0\varepsilon}(t)$ is the remainder of order zero.

Assuming that all functions involved in (4.23) are smooth enough, we can identify these functions by heuristic arguments. We have

$$u_\varepsilon'(t) = u'(t) - \frac{1}{\varepsilon}\frac{d}{d\tau}i(\tau) + r_{0\varepsilon}'(t). \tag{4.24}$$

$$u_\varepsilon''(t) = u''(t) + \frac{1}{\varepsilon^2}\frac{d^2}{d\tau^2}i(\tau) + r_{0\varepsilon}''(t). \tag{4.25}$$

so

$$\begin{aligned} -\varepsilon\left[u''(t) + \frac{1}{\varepsilon^2}\frac{d^2}{d\tau^2}(\tau) + r_{0\varepsilon}''(t)\right] + \left[u'(t) - \frac{1}{\varepsilon}\frac{di}{d\tau}(\tau) + r_{0\varepsilon}'(t)\right] \\ + Au(t) + Ai(\tau) + Ar_{0\varepsilon}(t) + B\left(u(t) + i(\tau) + r_{0\varepsilon}(t)\right) = f(t). \end{aligned} \tag{4.26}$$

By identification in (4.26), we get

$$\varepsilon^0 : \begin{cases} u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T, \\ u(0) = u_0. \end{cases}$$

$$\varepsilon^{-1} : \qquad \frac{d^2 i}{d\tau^2} + \frac{di}{d\tau} = 0 \quad \Rightarrow i(\tau) = e^{-\tau}y \quad \text{for some } y \in H.$$

$$\text{From (4.24)}: \quad \underbrace{u_\varepsilon'(T)}_{u_T} = u'(T) - \frac{1}{\varepsilon}\frac{di}{d\tau}(0) + r_{0\varepsilon}'(T)$$

$$\Rightarrow \frac{di}{d\tau}(0) = 0 \quad \Rightarrow i \equiv 0, \quad \text{and} \quad r_{0\varepsilon}'(T) = u_T - u'(T).$$

Finally,

$$\begin{cases} -\varepsilon(u + r_{0\varepsilon})'' + r_{0\varepsilon}' + Ar_{0\varepsilon} + B(u + r_{0\varepsilon}) - Bu = 0, \\ r_{0\varepsilon}(0) = 0, \quad r_{0\varepsilon}'(T) = u_T - u'(T). \end{cases} \tag{$R_{0\varepsilon}$}$$

Theorem 4.3. *Assume that (H1) and (H2) hold, $u_0, u_T \in D(A)$, A is strongly positive, i.e., $(Ax, x) \geq c\|x\|^2 \; \forall x \in D(A)$, for some $c > 0$, and $f \in W^{1,1}(0, T; H)$. Then, for every $\varepsilon > 0$, the solution u_ε of problem (P_2^ε) admits the following asymptotic expansion*

$$u_\varepsilon(t) = u(t) + r_{0\varepsilon}(t), \quad 0 \leq t \leq T,$$

where u is the solution of problem (P_0), and the remainder $r_{0\varepsilon} = r_{0\varepsilon}(t)$ satisfies problem ($R_{0\varepsilon}$). Moreover, for $0 < \varepsilon < 1$, we have the following estimates

$$\|r_{0\varepsilon}\|_{C([0,T];H)} = \mathcal{O}(\varepsilon^{1/4}),$$
$$\|r_{0\varepsilon}\|_{L^2(0,T;H)} = \mathcal{O}(\varepsilon^{1/2}).$$

Hence, problem (P_2^ε) is regularly perturbed in $C([0, T]; H)$.

Proof. By Theorems 3.9 and 3.20, we have

$$r_{0\varepsilon} = u_\varepsilon - u \in C^1([0, T]; H), \tag{4.27}$$

and

$$u + r_{0\varepsilon} = u_\varepsilon \in W^{2,2}(0, T; H).$$

It is easy to check that $r_{0\varepsilon}$, defined by (4.27), satisfies problem ($R_{0\varepsilon}$).

If we multiply the first equation of ($R_{0\varepsilon}$) by $r_{0\varepsilon}(t)$, and use the monotonicity of B and the strong positivity of A, we obtain

$$-\varepsilon\left((u + r_{0\varepsilon})'', r_{0\varepsilon}\right) + (r'_{0\varepsilon}, r_{0\varepsilon}) + c\|r_{0\varepsilon}\|^2_{L^2} \leq 0. \tag{4.28}$$

Integrating (4.28) over $[0, T]$ gives

$$-\varepsilon\left[\left((u + r_{0\varepsilon})', r_{0\varepsilon}\right)\Big|_0^T - \int_0^T \left((u + r_{0\varepsilon})', r'_{0\varepsilon}\right)\right] + \frac{1}{2}\|r_{0\varepsilon}(T)\|^2 + c\|r_{0\varepsilon}\|^2_{L^2} \leq 0$$

$$\Rightarrow -\varepsilon\left[(u_T, r_{0\varepsilon}(T)) - \int_0^T (u', r'_{0\varepsilon}) - \int_0^T (r'_{0\varepsilon}, r'_{0\varepsilon})\right] + \frac{1}{2}\|r_{0\varepsilon}(T)\|^2 + c\|r_{0\varepsilon}\|_{L^2}^2 \leq 0$$

$$\begin{aligned}\Rightarrow \varepsilon\|r'_{0\varepsilon}\|_{L^2}^2 + \frac{1}{2}\|r_{0\varepsilon}(T)\|^2 + c\|r_{0\varepsilon}\|_{L^2}^2 &\\ \leq \varepsilon\|u_T\|\,\|r_{0\varepsilon}(T)\| + \varepsilon\|u'\|_{L^2}\,\|r'_{0\varepsilon}\|_{L^2} &\\ \leq \frac{\varepsilon^2}{2}\|u_T\|^2 + \frac{1}{2}\|r_{0\varepsilon}(T)\|^2 + \varepsilon\|u'\|_{L^2}\,\|r'_{0\varepsilon}\|_{L^2} &\\ \leq \frac{\varepsilon}{2}\|u_T\|^2 + \frac{1}{2}\|r_{0\varepsilon}(T)\|^2 + \varepsilon\|u'\|_{L^2}\,\|r'_{0\varepsilon}\|_{L^2} \quad \forall 0 < \varepsilon < 1,&\end{aligned}$$

which implies that

$$\varepsilon\|r'_{0\varepsilon}\|_{L^2}^2 + c\|r_{0\varepsilon}\|_{L^2}^2 \leq \frac{\varepsilon}{2}\|u_T\|^2 + \varepsilon\|u'\|_{L^2}\,\|r'_{0\varepsilon}\|_{L^2} \quad \forall 0 < \varepsilon < 1. \tag{4.29}$$

From (4.29), we have

$$\|r'_{0\varepsilon}\|_{L^2}^2 \leq \frac{1}{2}\|u_T\|^2 + \|u'\|_{L^2}\,\|r'_{0\varepsilon}\|_{L^2} \quad \forall 0 < \varepsilon < 1$$

$$\Rightarrow \left(\|r'_{0\varepsilon}\|_{L^2} - \frac{1}{2}\|u'\|_{L^2}\right)^2 \leq \frac{1}{2}\|u_T\|^2 + \frac{1}{4}\|u'\|_{L^2}^2 \quad \forall 0 < \varepsilon < 1,$$

which implies that

$$\{\|r'_{0\varepsilon}\|_{L^2}; 0 < \varepsilon < 1\} \text{ is bounded.} \tag{4.30}$$

From (4.29), we also have

$$c\|r_{0\varepsilon}\|_{L^2}^2 \leq \frac{\varepsilon}{2}\|u_T\|^2 + \varepsilon\|u'\|_{L^2}\,\|r'_{0\varepsilon}\|_{L^2} \quad \forall 0 < \varepsilon < 1. \tag{4.31}$$

From (4.30) and (4.31), we have

$$\|r_{0\varepsilon}\|_{L^2} = \mathcal{O}\left(\varepsilon^{1/2}\right) \quad \forall 0 < \varepsilon < 1. \tag{4.32}$$

Note that

$$\begin{aligned}\|r_{0\varepsilon}(t)\|^2 = \int_0^t \left(\|r_{0\varepsilon}\|^2\right)' = 2\int_0^t (r_{0\varepsilon}, r'_{0\varepsilon}) &\\ \leq 2\|r_{0\varepsilon}\|_{L^2}\,\|r'_{0\varepsilon}\|_{L^2}.&\end{aligned} \tag{4.33}$$

From (4.30), (4.32) and (4.33), we obtain

$$\|r_{0\varepsilon}\|_{C([0,T];H)} = \mathcal{O}\left(\varepsilon^{1/4}\right) \quad \forall 0 < \varepsilon < 1.$$

This completes the proof. □

Remark 4.4. *In fact, Theorems 4.1 and 4.3 hold under weaker assumptions on B, more precisely it suffices to assume that*

$$B\colon H \to H \text{ is monotone and Lipschitz on bounded sets.} \tag{4.34}$$

To argue, let us revisit the proofs of Theorems 4.1 and 4.3. We used Theorems 3.16 and 3.20, and the second part of Theorem 3.9.
Theorem 3.16 is true if we assume the weaker condition (4.34), since we need only the fact that $Q = A + B$ is maximal monotone. Theorem 3.20 is also true if we use the weaker condition.
The second part of Theorem 3.9 is also valid if B satisfies (4.34). Indeed, for $r > 0$, let us consider the operator $B_r = B \circ \phi_r$, where ϕ_r is the radial retraction,

$$\phi_r(x) = \begin{cases} x & \text{if} \quad \|x\| \le r, \\ \frac{r}{\|x\|}x & \text{if} \quad \|x\| \ge r. \end{cases}$$

Since ϕ_r is Lipschitz (see, e.g., [6, p. 55]), it follows that B_r is Lipschitz on H (with a Lipschitz constant depending on r). If $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$, then problem (P_0) with B_r instead of B has a strong solution $u_r \in C^1([0,T];H)$ (cf. Theorem 3.9). If we multiply by $u_r(t)$ the equation

$$u_r'(t) + Au_r(t) + B_r u_r(t) - B0 = f(t) - B0,$$

take into account the monotonicity of A and B, and the fact that $u_r(t) = \phi_r u_r(t)$ times a nonnegative coefficient (A need not be strongly positive), we *obtain*

$$\frac{1}{2}\frac{d}{dt}\|u_r(t)\|^2 \leq \left(\|f(t)\| + \|B0\|\right)\|u_r(t)\|,$$

which, by integrating over $[0,t]$ *and simple computations, implies that*

$$\|u_r\|_{C([0,T];H)} \leq M,$$

where M *is a constant depending on* $\|f\|_{C([0,T];H)}$ *and* $\|B0\|$ *(and independent of* r*). Therefore, if we choose* $r > M$*, then* u_r *is a solution of* (P_0)*. The rest of the proofs of Theorems 4.1 and 4.3 work well for* B *satisfying* (4.34)*. We just point out that in* (4.12) *the arguments of* B *belong to a ball in* H *whose radius depends on* $\|u\|_{C([0,T];H)}$ *but is independent of* ε*. So Theorems 4.1 and 4.3 actually cover a larger class of* B*'s.*

4.3 Asymptotic expansion of order one for problem (P_2^ε)

We are looking for an asymptotic expansion of order one. We expect a discrepancy at $t = T$, and we have seen in the previous section that the boundary layer function (of order zero) vanishes, so we expect the following expansion to hold

$$u_\varepsilon(t) = u(t) + \varepsilon\left[u_1(t) + i_1(\tau)\right] + r_\varepsilon(t), \tag{4.35}$$

where $\tau = (T-t)/\varepsilon$, $\varepsilon > 0$ is the stretched (fast) variable, $u = u(t)$ is the solution of problem (P_0), $i_1 = i_1(\tau)$ is the boundary layer function (of order one), and $r_\varepsilon = r_\varepsilon(t)$ is the remainder of order one.

Next, we assume that all functions involved in (4.35) are smooth enough, so that we can identify these functions by heuristic arguments. If we use (4.35) in (P_2^ε), and identify the coefficients of ε^{-1}, ε^0, we get

$$\varepsilon^0, t\colon \begin{cases} u'(t) + Au(t) + Bu(t) = f(t), \quad 0 < t < T, \\ u(0) = u_0. \end{cases}$$

$$\varepsilon^0, \tau: \quad \frac{d^2 i_1}{d\tau^2} + \frac{di_1}{d\tau} = 0, \qquad i_1'(0) = u'(T) - u_T,\ i_1(\infty) = 0$$
$$\Rightarrow i_1(\tau) = [u_T - u'(T)]\, e^{-\tau}.$$
$$\begin{cases} u_1'(t) + Au_1(t) + B'(u(t))u_1(t) = u''(t), \\ u_1(0) = 0. \end{cases} \tag{4.36}$$
$$\begin{cases} -\varepsilon(r_\varepsilon + \varepsilon u_1)'' + r_\varepsilon' + Ar_\varepsilon + Bu_\varepsilon - Bu - \varepsilon B'(u(t))u_1(t) = -\varepsilon A i_1(\tau), \\ r_\varepsilon(0) = -\varepsilon i_1(T/\varepsilon), \quad r_\varepsilon'(T) = -\varepsilon u_1'(T). \end{cases} \tag{$R_{1\varepsilon}$}$$

Assume that $u_0 \in D(A)$, $f(0) - Au_0 - Bu_0 \in D(A)$, A is self-adjoint, $f \in W^{2,\infty}(0,T;H)$, and B is twice differentiable with B', B'' bounded on bounded sets.

By Theorem 3.13, problem (P_0) has a unique solution $u \in C^2([0,T];H)$.

We show that problem (4.36) has a unique strong solution $u_1 \in W^{1,2}(0.T;H)$. Let
$$F(t,z) = B'(u(t))z - u''(t), \quad \text{where } t \in [0,T],\ z \in H,$$
then by Lemma 3.6, problem (4.36) has a unique mild solution $u_1 \in C([0,T];H)$. Now consider
$$w'(t) + Aw(t) = g(t), \quad w(0) = 0, \quad 0 < t < T, \tag{4.37}$$
where
$$g(t) := -B'(u(t))u_1(t) + u''(t), \quad \text{and} \quad g \in L^2(0,T;H).$$
By Lemma 3.5, there exists a unique strong solution w of (4.37), and $w' \in L^2(0,T;H)$. It is easy to show that $w \equiv u_1$. Hence, $u_1 \in W^{1,2}(0,T;H)$ is the unique strong solution of (4.36).

Now, we show that $u'(t) \in D(A)$, for all $t \in [0,T]$. As in the proof of Theorem 3.12, we have that $v = u'$ is the unique strong solution of
$$\begin{cases} v'(t) + Av(t) + B'(u(t))v(t) = f'(t), \quad 0 \le t \le T, \\ v(0) = f(0) - Au_0 - Bu_0. \end{cases} \tag{4.38}$$
Now consider the problem
$$\begin{cases} v'(t) + Av(t) = f'(t) - B'(u(t))u'(t), \quad 0 \le t < T, \\ v(0) = f(0) - Au_0 - Bu_0. \end{cases} \tag{4.39}$$

$v = u'$ is a strong solution of problem (4.39). By Remark 1.20, we have that $u'(t) \in D(A)$, for all $0 \le t \le T$. Hence, Ai_1 is well-defined.

Theorem 4.5. *Assume that (H1) and (H2) hold, $u_0, u_T \in D(A)$, $f(0) - Au_0 - Bu_0 \in D(A)$, A is self-adjoint, A is strongly positive, i.e., $(Ax, x) \ge c\|x\|^2$ $\forall x \in D(A)$, for some $c > 0$, $f \in W^{2,\infty}(0,T;H)$, and B is twice differentiable with B', B'' bounded on bounded sets. Then, for every $\varepsilon > 0$, the solution u_ε of problem (P_2^ε) admits the following asymptotic expansion*

$$u_\varepsilon(t) = u(t) + \varepsilon[u_1(t) + i_1(\tau)] + r_\varepsilon(t), \quad 0 \le t \le T,\ \tau := (T-t)/\varepsilon,$$

where u is the solution of problem (P_0), $i_1(\tau) = (u_T - u'(T))\, e^{-\tau}$ is the boundary layer function of order one, u_1 is the unique strong solution of problem (4.36), and the remainder $r_\varepsilon = r_\varepsilon(t)$ is a strong solution of problem ($R_{1\varepsilon}$). Moreover, for $0 < \varepsilon < 1$, we have the following estimates

$$\|r_\varepsilon\|_{C([0,T];H)} = \mathcal{O}\left(\varepsilon^{5/4}\right),$$

$$\|r_\varepsilon\|_{L^2(0,T;H)} = \mathcal{O}\left(\varepsilon^{3/2}\right),$$

$$\|r'_\varepsilon\|_{L^2(0,T;H)} = \mathcal{O}(\varepsilon),$$

$$\|u_\varepsilon - u - \varepsilon u_1\|_{L^2(0,T;H)} = \mathcal{O}\left(\varepsilon^{3/2}\right),$$

$$\|u_\varepsilon - u\|_{C([0,T];H)} = \mathcal{O}(\varepsilon).$$

Proof. Note that

$$r_\varepsilon = u_\varepsilon - u - \varepsilon u_1 - \varepsilon i_1 \in W^{1,2}(0,T;H), \tag{4.40}$$

$$r_\varepsilon + \varepsilon u_1 = u_\varepsilon - u - \varepsilon i_1 \in W^{2,2}(0,T;H).$$

It is easy to check that r_ε, defined by (4.40), is a strong solution of problem ($R_{1\varepsilon}$).

Now, we derive the estimates given in the statement of Theorem 4.5.

Let

$$\bar{r}_\varepsilon(t) = r_\varepsilon(t) + \alpha_\varepsilon, \quad 0 < t < T, \tag{4.41}$$

where

$$\alpha_\varepsilon = \varepsilon i_1 \left(T/\varepsilon \right). \tag{4.42}$$

Note that

$$\|\alpha_\varepsilon\| = \mathcal{O}(\varepsilon^j) \quad \forall j \geq 1 \quad \forall 0 < \varepsilon < 1. \tag{4.43}$$

Then $\bar{r}_\varepsilon$ satisfies

$$\begin{cases} -\varepsilon(\bar{r}_\varepsilon + \varepsilon u_1)'' + \bar{r}_\varepsilon' + A\bar{r}_\varepsilon + Bu_\varepsilon - Bu - \varepsilon B'(u(t))u_1(t) = h_\varepsilon(t), \\ \text{where} \quad h_\varepsilon(t) = -\varepsilon A i_1(\tau) + A\alpha_\varepsilon, \\ \bar{r}_\varepsilon(0) = 0, \quad \bar{r}_\varepsilon'(T) = -\varepsilon u_1'(T). \end{cases} \tag{4.44}$$

By doing the same calculations as in Theorem 4.1, we have

$$\|h_\varepsilon\|_{L^2} = \mathcal{O}\left(\varepsilon^{3/2}\right), \quad \forall 0 < \varepsilon < 1.$$

Note

$$\begin{aligned} &Bu_\varepsilon - Bu - \varepsilon B'(u(t))u_1(t) \\ &= \left[Bu_\varepsilon - B(u_\varepsilon - \bar{r}_\varepsilon)\right] + \left[B(u_\varepsilon - \bar{r}_\varepsilon) - Bu - \varepsilon B'(u(t))u_1(t)\right]. \end{aligned} \tag{4.45}$$

Let

$$\beta_\varepsilon(t) = \varepsilon u_1(t) + \varepsilon i_1(\tau) - \alpha_\varepsilon.$$

Then

$$u_\varepsilon - \bar{r}_\varepsilon = u + \beta_\varepsilon,$$

$$\begin{aligned} &B(u_\varepsilon - \bar{r}_\varepsilon) - Bu - \varepsilon B'(u(t))u_1(t) \\ &= \varepsilon B'(u(t))i_1(\tau) - B'(u(t))\alpha_\varepsilon + \frac{1}{2}B''(u(t) + \theta\beta_\varepsilon(t))\,\beta_\varepsilon(t)\beta_\varepsilon(t), \end{aligned} \tag{4.46}$$

where θ varies with t, and satisfies $0 < \theta < 1$.

Note that

$$\|\beta_\varepsilon\|_{C([0,T];H)} = \mathcal{O}(\varepsilon), \quad \|i_1\|_{L^2} = \mathcal{O}\left(\varepsilon^{1/2}\right). \tag{4.47}$$

If B' and B'' are bounded on bounded set, then from (4.43), (4.46) and (4.47), we have

$$\|B(u_\varepsilon - \bar{r}_\varepsilon) - Bu - \varepsilon B'(u)u_1\|_{L^2} = \mathcal{O}\left(\varepsilon^{3/2}\right). \tag{4.48}$$

If we multiply the first equation of (4.44) by $\bar{r}_\varepsilon$ and integrate over $[0,T]$, then by using (4.45), monotonicity of B, and (4.48), we get

$$-\varepsilon\left[((\bar{r}_\varepsilon + \varepsilon u_1)', \bar{r}_\varepsilon)\Big|_0^T - \int_0^T ((\bar{r}_\varepsilon + \varepsilon u_1)', \bar{r}_\varepsilon')\right] + \frac{1}{2}\|\bar{r}_\varepsilon(T)\|^2 + c\|\bar{r}_\varepsilon\|_{L^2}^2 \leq K\varepsilon^{3/2}\|\bar{r}_\varepsilon\|_{L^2},$$

where $K > 0$ is some constant. This implies

$$\begin{aligned} \varepsilon\|\bar{r}_\varepsilon'\|_{L^2}^2 + \frac{1}{2}\|\bar{r}_\varepsilon(T)\|^2 + c\|\bar{r}_\varepsilon\|_{L^2}^2 & \\ \leq K\varepsilon^{3/2}\|\bar{r}_\varepsilon\|_{L^2} + \varepsilon^2\|u_1'\|_{L^2}\,\|\bar{r}_\varepsilon'\|_{L^2} & \\ \Rightarrow \varepsilon\|\bar{r}_\varepsilon'\|_{L^2}^2 + c\|\bar{r}_\varepsilon\|_{L^2}^2 \leq K\varepsilon^{3/2}\|\bar{r}_\varepsilon\|_{L^2} + \varepsilon^2\|u_1'\|_{L^2}\,\|\bar{r}_\varepsilon'\|_{L^2}. & \end{aligned}$$

By using the same steps as in Theorem 4.1, we get

$$\varepsilon\|\bar{r}_\varepsilon'\|_{L^2}^2 + \frac{c}{2}\|\bar{r}_\varepsilon\|_{L^2}^2 \leq \mathcal{O}(\varepsilon^3) + \varepsilon^2\|u_1'\|_{L^2}\,\|\bar{r}_\varepsilon'\|_{L^2}. \tag{4.49}$$

By (4.49), we have

$$\|\bar{r}_\varepsilon'\|_{L^2}^2 \leq \mathcal{O}(\varepsilon^2) + \varepsilon\|u_1'\|_{L^2}\,\|\bar{r}_\varepsilon'\|_{L^2}$$

$$\Rightarrow \left(\|\bar{r}_\varepsilon'\|_{L^2} - \frac{\varepsilon}{2}\|u_1'\|_{L^2}\right)^2 \leq \mathcal{O}(\varepsilon^2) + \frac{\varepsilon^2}{4}\|u_1'\|_{L^2}^2,$$

which implies that

$$\|\bar{r}_\varepsilon'\|_{L^2} = \mathcal{O}(\varepsilon). \tag{4.50}$$

From (4.49) and (4.50), we have

$$\|\bar{r}_\varepsilon\|_{L^2} = \mathcal{O}\left(\varepsilon^{3/2}\right). \tag{4.51}$$

From (4.41), (4.43), (4.50) and (4.51), we have

$$\|r'_\varepsilon\|_{L^2(0,T;H)} = \mathcal{O}(\varepsilon), \tag{4.52}$$

$$\|r_\varepsilon\|_{L^2(0,T;H)} = \mathcal{O}\left(\varepsilon^{3/2}\right). \tag{4.53}$$

But

$$\begin{aligned}\|r_\varepsilon(t)\|^2 &= \int_0^t \left(\|r_\varepsilon(s)\|^2\right)' ds \\ &= 2\int_0^t (r_\varepsilon(s), r'_\varepsilon(s))\, ds \leq 2\|r_\varepsilon\|_{L^2}\, \|r'_\varepsilon\|_{L^2}.\end{aligned} \tag{4.54}$$

From (4.52), (4.53) and (4.54), we have

$$\|r_\varepsilon\|_{C([0,T];H)} = \mathcal{O}\left(\varepsilon^{5/4}\right). \tag{4.55}$$

Note

$$\|i_1\|_{C([0,T];H)} = \|u_T - u'(T)\|, \quad \|i_1\|_{L^2(0,T;H)} = \mathcal{O}\left(\varepsilon^{1/2}\right). \tag{4.56}$$

From (4.53), (4.55) and (4.56), we get

$$\|u_\varepsilon - u - \varepsilon u_1\|_{C([0,T];H)} = \mathcal{O}(\varepsilon),$$

$$\|u_\varepsilon - u - \varepsilon u_1\|_{L^2(0,T;H)} = \mathcal{O}\left(\varepsilon^{3/2}\right),$$

$$\|u_\varepsilon - u\|_{C([0,T];H)} = \mathcal{O}(\varepsilon).$$

This completes the proof. □

Remark 4.6. *In Theorem 4.3, we derived the estimate* $\|u_\varepsilon - u\|_{C([0,T];H)} = \mathcal{O}\left(\varepsilon^{1/4}\right)$*, while in Theorem 4.5, we derived the estimate* $\|u_\varepsilon - u\|_{C([0,T];H)} = \mathcal{O}(\varepsilon)$ *without using Theorem 4.3. But, we derived the estimate in Theorem 4.3 under much weaker conditions.*

Some open problems:
There are several open problems related to the problems we have discussed in this thesis.

It is expected that a first order asymptotic expansion holds for problem (P_1^ε) under additional assumptions on u_0, u_T, f, A and B. This problem appears harder than the first order asymptotic expansion for problem (P_2^ε) we have discussed since the boundary layer function of order zero for problem (P_2^ε) is identically zero, while problem (P_1^ε) is a singular perturbation problem of boundary layer type (of order zero).

It is possible to consider higher order regularizations of problem (P_0), e.g., fourth order regularizations of problem (P_0) whose solutions are expected to be even more regular than solutions of elliptic-like regularizations (P_1^ε) and (P_2^ε) we have discussed.

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I hereby declare that the dissertation contains no material accepted for any other degree in any other institution.

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I hereby declare that Muhammad Ahsan is the primary author of the following joint papers which are based on the material contained in this thesis:

1. Ahsan, M. and Moroşanu, G., *Some regularizations of semilinear evolution equations*, Proceedings of the International Conference Nonlinear Difference and Differential Equations and their Applications, Ruse, October 3rd-October 6th, 2012, pp. 1-6.

2. Ahsan, M. and Moroşanu, G., *Elliptic-like regularization of semilinear evolution equations*, J. Math. Anal. Appl. 396 (2012) 759-771.

Gheorghe Moroşanu

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Statement of Objection

Student's name: Muhammad Ahsan

Program: PhD in Mathematics

Dissertation title: Existence, Regularity and Perturbation Theory for Some Semilinear Differential Equations in Hilbert Spaces

Dissertation supervisor(s): Gheorghe Moroşanu

I wish to name individual(s) whose presence in the Dissertation Committee I object to:

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If you marked YES, please name the individual(s):

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