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Some Results on Two Asymptotic Series of Ramanujan

by

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Contents

This thesis is dedicated to my parents Issaka Osumanu and Jamilatu Issaka. All I have and will accomplish are only possible due to their love and sacrifices.

Introduction

First, I will give a brief story about Ramanujan and his lost notebooks.

FIGURE 1: S. Ramanujan

Most famous mathematicians were educated at renowned centers of learning and were taught by inspiring teachers, if not by distinguished research mathematicians. The one exception to this rule is Srinivasa Ramanujan Aiyangar, generally regarded as India's greatest mathematician. He was born on 22 December 1887 in Erode, located in the southern Indian State of Tamil Nadu. He started to focus on mathematics at the age of 15 years and borrowed a copy of G. S. Carr's *Synopsis of Pure and Applied Mathematics* which served as his primary source of learning mathematics. At the age of 16 years, Ramanujan went to a College where he solely devoted himself to mathematics without paying attention to other subjects and consequently failed his examinations at the end of the first year. He therefore lost his scholarship and was forced to halt his formal education because his parents were poor.

During his time at the college; he began to record his mathematical discoveries in notebooks. Living in poverty with no means of financial support devoted all of his effort to mathematics and continued to record his discoveries without proofs in notebooks for the next 6 years.

He later visited G. H. Hardy, who was then a Fellow of Trinity College, Cambridge where he worked for some time and by the end of his third year in England, became ill and returned home after the World War I ended and died on 26 April 1920 the age of 32. In a letter to one of his friends, Ramanjuan admitted, “I am doing my work very slowly. My notebook is sleeping in corner for these four or five months. I am publishing only my present researches as I have not yet proved the results in my notebooks rigorously”.

He is known for his asymptotic formulas in number theory but has not received much recognition for his asymptotic methods and theorems in analysis because his beautiful asymptotic formulas, both general and specific, laid hidden in his notebooks for many years.

After his death, G. H. Hardy strongly urged that Ramanujan’s notebooks be edited and published. By “editing”, Hardy meant that each claim made by Ramanujan in his notebooks should be examined. For more information on the story of Ramanujan story, see Andrews [1], Berndt [2, 3], Hardy [9].

We will present some new results about two problems of Ramanujan from the lost notebook 1 [6] and 5 [7] in this thesis. The first problem is about an approximation of the inverse of the Digamma function, which was proposed by Ramanujan with no proof and without a formula for the general term. B. C. Berndt [6, pp. 194–195, Entry 15] also did not give any proof for the problem but only showed that, the first few terms of the proposed problem agree. The second problem is about an asymptotic formula of the Harmonic numbers in terms of the triangular numbers, which again was proposed by Ramanujan, see B. C. Berndt [7, pp. 531–532, Entry 9]. In 2012, M. D. Hirschhorn [10] gave a proof, but there is a gap in his proof which he admitted and our aim is to fill this gap.

Chapter 1

Preliminaries

1.1 Fundamental Concepts and Results

In this section, we present some fundamental concepts of asymptotic expansion which will be very useful hereinafter. For additional material, see, e.g., Olver [13].

1.1.1 Asymptotic Expansion

Let H be a domain in the complex plane. Let $f(z)$ and $g(z)$ be two continuous functions defined on H. Let z_0 be an accumulation point of H.

Definition 1.1.1. *We say "$f(z)$ is* ***big-oh*** *$g(z)$" or we write $f(z) = \mathcal{O}(g(z))$, as $z \to z_0$ to mean that there is a constant $C > 0$ and a neighbourhood U of z_0 such that $|f(z)| \leq C|g(z)|$ for all $z \in U \cap H$.*

Definition 1.1.2. *We say "$f(z)$ is* ***little-oh*** *$g(z)$" or we write $f = o(g)$, as $z \to z_0$ to mean that for every $\epsilon > 0$ and a neighbourhood U_ϵ of z_0 such that $|f(z)| \leq \epsilon|g(z)|$ for all $z \in U_\epsilon \cap H$.*

If $g(z) \neq 0$ in a neighbourhood U of z_0, then $f(z) = o(g(z))$ as $z \to z_0$ is equivalent to

$$\lim_{\substack{z \to z_0 \\ z \in U \cap H}} \frac{f(z)}{g(z)} = 0.$$

Definition 1.1.3. *We say that $f(z)$ is* ***asymptotically equal*** *to $g(z)$ as $z \to z_0$ and write $f(z) \sim g(z)$, if $f(z) = g(z) + o(g(z))$ as $z \to z_0$.*

If $g(z) \neq 0$ in a neighbourhood U of z_0, then $f(z) \sim g(z)$ as $z \to z_0$ is equivalent to

$$\lim_{\substack{z \to z_0 \\ z \in U \cap H}} \frac{f(z)}{g(z)} = 1.$$

Definition 1.1.4. *Let H be a domain in the complex plane and let z_0 be an accumulation point of H. We say that the sequence $\{\phi_k\}_{k=0}^{\infty}$ defined on H is an **asymptotic sequence** as $z \to z_0$ if*

(1) there is a neighbourhood U of z_0 such that $\phi_k(z) \neq 0$ if $z \in U$;

(2) $\phi_{k+1}(z) = o(\phi_k(z))$, as $z \to z_0$.

Definition 1.1.5 (Poincaré). *Let $f : H \to \mathbb{C}$ be continuous. We say that the (not necessarily convergent) series*

$$\sum_{k=1}^{\infty} a_k \phi_k(z)$$

*is an **asymptotic series** of $f(z)$ corresponding to the asymptotic sequence $\{\phi_k\}_{k=0}^{\infty}$, if for any $N \geq 0$ we have*

$$f(z) - \sum_{k=1}^{N} a_k \phi_k(z) = o(\phi_N(z)) \text{ as } z \to z_0. \tag{1.1}$$

In this case we write (1.1) as

$$f(z) \sim \sum_{k=1}^{\infty} a_k \phi_k(z) \text{ as } z \to z_0 \text{ in } H.$$

Definition 1.1.6. *Let $f(z)$ be defined on a domain H. If $f(z)$ has an asymptotic series corresponding to the asymptotic sequence $\phi_k(z) = z^{-k}$ as $z \to \infty$, then we call it an **asymptotic power series** of $f(z)$.*

Theorem 1.1.1. *Assume*

$$f(z) \sim \sum_{k=0}^{\infty} \frac{a_k}{z^k} \text{ and } g(z) \sim \sum_{k=0}^{\infty} \frac{b_k}{z^k} \text{ as } z \to \infty \text{ in a domain } H.$$

Then for any complex numbers α and β, we have

$$\alpha f(z) + \beta g(z) \sim \sum_{k=0}^{\infty} \frac{\alpha a_k + \beta b_k}{z^k} \text{ as } z \to \infty \text{ in the domain } H.$$

Moreover,

$$f(z)g(z) \sim \sum_{k=0}^{\infty} \frac{c_k}{z^k} \text{ as } z \to \infty \text{ in the domain } H, \text{ with } c_k = \sum_{j=0}^{k} a_{k-j} b_j.$$

If $a_0 \neq 0$*, then*

$$\frac{1}{f(z)} \sim \sum_{k=0}^{\infty} \frac{d_k}{z^k}, \text{ as } z \to \infty \text{ in the domain } H, \text{with } d_0 = \frac{1}{a_0} \text{ and } d_k = -d_0 \sum_{j=0}^{k-1} a_{k-j} d_j \text{ for } k \geq 1.$$

Theorem 1.1.2 (Uniqueness)**.** *Let* $\{\phi_k\}_{k=0}^{\infty}$ *be an asymptotic sequence. If*

$$f(z) \sim \sum_{k=0}^{\infty} a_k \phi_k(z) \text{ and } f(z) \sim \sum_{k=0}^{\infty} b_k \phi_k(z) \text{ as } z \to z_0 \text{ in } H,$$

then $a_k = b_k$ *for all* $k \geq 0$*. Moreover*

$$\begin{aligned} a_0 &= \lim_{\substack{z \to z_0 \\ z \in U \cap H}} \frac{f(z)}{\phi_0(z)}, \\ a_n &= \lim_{\substack{z \to z_0 \\ z \in U \cap H}} \frac{f(z) - \sum_{k=0}^{n-1} a_k \phi_k(z)}{\phi_n(z)}. \end{aligned}$$

1.2 The Gamma and the Digamma (or Psi) Function

In this section, we present some basic notions and concepts of the Gamma function, the Digamma function, and a bridge connecting Chapter 1 and Chapter 2 which will be useful hereinafter.

Definition 1.2.1. *The* ***Gamma function*** *also known as the Eulerian integral of the second kind can be defined in the right-half plane as*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt.$$

When $\Re(z) \leq 0$, $\Gamma(z)$ is defined by analytic continuation. It is a meromorphic function with no zeros, and with simple poles of residue $\frac{(-1)^n}{n!}$ at $z = -n$, $n \in \mathbb{N}$. The $\frac{1}{\Gamma(z)}$ is entire with simple zeros at $z = -n$, $n \in \mathbb{N}$.

Definition 1.2.2. *The* ***Digamma function*** *or the* ***Psi function*** *is given by the following expression*

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \text{ with } z \neq 0, -1, -2, \ldots.$$

The function $\psi(z)$ is meromorphic with simple poles of residue -1 at $z = -n$, $n \in \mathbb{N}$.

An integral representation of the Digamma function [14, p. 140, 5.9.12] is given by

$$\psi(z) = \int_0^{+\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right), \text{ for } \Re(z) \geq 0.$$

The functions $\psi^{(n)}(z)$, $n = 1, 2, \ldots$, are called the *polygamma* functions. In particular, $\psi'(z)$ is called the *trigamma* function, $\psi^{(2)}$, $\psi^{(3)}$, $\psi^{(4)}$ are respectively called, the *tetra-*, *penta-*, and *hexagamma* functions. The properties of these functions are obtained simply by differentiating the properties of the Digama function.

Definition 1.2.3. *The* ***Euler constant*** *also called* ***Euler–Mascheroni constant****, usually denoted by γ, is defined as*

$$\gamma = \lim_{n \to +\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n} - \log n \right).$$

The numerical value of the Euler constant is $0.57721566490153286060\ldots$.

Here we state some basic properties of the Digamma function, for additional information see [14, pp. 137–139]

1.2.1 Some Properties of the Digamma Function

(1) $\psi(z) = -\gamma - \dfrac{1}{z} + \displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right) = \lim_{n \to +\infty} \left(\log n - \sum_{k=0}^{n} \frac{1}{k+z} \right).$

(2) $\psi(1) = -\gamma$, and $\psi(2) = 1 - \gamma$.

(3) $\psi(z+1) = \psi(z) + \dfrac{1}{z} = -\gamma + \displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right).$

(4) $\psi(z+n) = \psi(z) + \displaystyle\sum_{n=1}^{n-1} \frac{1}{z+k}$, $\psi(z-n) = \psi(z) - \displaystyle\sum_{n=1}^{n-1} \frac{1}{z+k}.$

(5) $\psi(1+n) = -\gamma + \displaystyle\sum_{k=1}^{n} \frac{1}{k}.$

(6) $\psi\left(\dfrac{1}{2}\right) = -\gamma - 2\log 2.$

(7) $\psi\left(\dfrac{1}{2} \pm n\right) = -\gamma - 2\log 2 + 2\displaystyle\sum_{k=0}^{n-1} \frac{1}{2k+1}.$

(8) For $x > 0$, the Diagmma function satisfies the following functional equation

$$\psi\left(x+\frac{1}{2}\right) = 2\psi(2x) - \psi(x) - 2\log 2. \tag{1.2}$$

Definition 1.2.4. *The* ***Bernoulli polynomials*** $B_k(x)$ *are sequence of polynomials with rational coefficients and play an important role in several areas of mathematics. These polynomials can be defined by the exponential generating function*

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!},\ |x| < 2\pi.$$

The special values $B_k(0) = B_k$ *are called the* ***Bernoulli numbers****. For odd index* $k > 1$*, all the* B_k*'s are zero, and the even-indexed* B_k *alternate in sign. The first few values of the Bernoulli numbers are* $B_0 = 1$*,* $B_1 = -\frac{1}{2}$*,* $B_2 = \frac{1}{6}$*,* $B_4 = -\frac{1}{30}$*,* $B_5 = \frac{1}{42}$*,* $B_8 = -\frac{1}{30}$*.*

The following proposition gives the asymptotic expansion of the Digamma function, for more information see [14, pp. 140–141].

Proposition 1.2.1. *We have*

$$\psi(z) \sim \log z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n}\frac{1}{z^{2n}}, \tag{1.3}$$

as $z \to +\infty$*.*

From (1.2) and (1.3) we obtain the following asymptotic series

$$\psi\left(z+\frac{1}{2}\right) \sim \log z + \sum_{n=1}^{\infty} \frac{(1-2^{1-2n})B_{2n}}{2nz^{2n}}, \tag{1.4}$$

as $z \to +\infty$.

1.3 Harmonic Numbers and the Hurwitz Zeta Function

In this section, we present the fundamental concepts of the Hurwitz zeta function, Harmonic numbers, and a short history relating to some approximations of the Harmonic numbers which will be be useful hereinafter, see the paper of Villarino [15] for more details.

1.3.1 Harmonic Numbers

Definition 1.3.1. *For every natural number $n \geq 1$, the nth* ***Harmonic Number****, H_n is the nth partial sum of the harmonic series*

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

The asymptotics of H_n was determined by Euler in his famous formula, which follows from (5) in Subsection 1.2.1 and Proposition 1.2.1.

$$\begin{aligned} H_n &\sim \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \cdots \\ &= \log n + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{n^{2k}}. \end{aligned}$$

There are several approximations of H_n among which are

$$\begin{aligned} &H_n \approx \log n + \gamma + \frac{1}{2n} \quad \text{(Euler's approximation)}, \\ &H_n \approx \log n + \gamma + \frac{1}{2n + \frac{1}{3}} \quad \text{(Toth–Mare approximation)}, \\ &H_n \approx \log \sqrt{n(n+1)} + \gamma + \frac{1}{6n(n+1) + \frac{6}{5}} \quad \text{(Lodge–Ramanujan approximation)}, \\ &H_n \approx \log \left(n + \frac{1}{2}\right) + \gamma + \frac{1}{24\left(n + \frac{1}{2}\right)^2 + \frac{21}{5}} \quad \text{(Detemple–Wang approximation)}. \end{aligned}$$

1.3.2 Hurwitz Zeta Function

Here we state some definitions and representations of the Hurwitz zeta function [14, p. 607–609, Section 25.11] that we shall use in Chapter 2.

Definition 1.3.2. *The* ***Hurwitz zeta function*** *$\zeta(s, a)$ was introduced by Hurwitz (1882) and is defined by the series expansion*

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad \text{for } \Re(s) > 1 \text{ and } a \neq 0, -1, -2, \ldots. \tag{1.5}$$

The $\zeta(s, a)$ has a meromorphic continuation to the s-plane, with a simple pole at $s = 1$ with residue 1. As a function of a, with $s \neq 1$ fixed, $\zeta(s, a)$ is analytic in the half-plane $\Re(a) > 0$. Moreover, the Riemann zeta function is a special case

$$\zeta(s, 1) = \zeta(s). \tag{1.6}$$

Proposition 1.3.1 (Series representation). *We have*

$$\zeta(s, \frac{1}{2}a) = \zeta(s, \frac{1}{2}a + \frac{1}{2}) + 2^s \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s}$$

for $\Re(s) > 0$, $s \neq 1$ *and* $\Re(a) > 0$.

Proposition 1.3.2 (Integral representation). *We have*

$$\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-at} dt,$$

for $\Re(s) > -1$, $s \neq 1$ *and* $\Re(a) > 0$.

1.3.3 Triangular Numbers

Definition 1.3.3. *A **Triangular Number** enumerates the objects that can form an equilateral triangular. The nth triangular number is equal to the sum of the natural numbers from* 1 *to* n.

The sequence of Triangular numbers is given by

$$0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, \ldots,$$

The nth Triangular number can be given explicitly by the following formulas

$$m := m(n) = \sum_{k=1}^{n} k = 1 + 2 + 3 + 4 \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

1.4 Formal Power Series

In this section, we present some basic notions of formal power series and some tools that will be useful hereinafter.

Definition 1.4.1. *A **formal power series** can be defined formally as infinite sequence* $(a_0, a_1, a_3, \ldots)$ *of complex numbers; and can also be represented in the form*

$$a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{k=0}^{\infty} a_n t^n.$$

In simple terms, a formal power series is a power series (or Taylor series) in which the question of convergence is ignored.

Definition 1.4.2 (Addition). *Addition of formal power series is term by term.*

$$\sum_{k=0}^{\infty} a_k t^k + \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (a_k + b_k) t^k.$$

Definition 1.4.3 (Multiplication). *The product of two formal power series is again a formal power series:*

$$\left(\sum_{k=0}^{\infty} a_k t^k\right)\left(\sum_{k=0}^{\infty} b_k t^k\right) = \sum_{k=0}^{\infty} c_k t^k,$$

where $c_k = \sum_{n=0}^{k} a_n b_{k-n}$.

Definition 1.4.4 (Substitution). *Let* $B(t)$ *be a formal power series with constant term zero and if* $A(t) = \sum_{k=0}^{\infty} a_k t^k$ *is another formal power series, then* $A(B(t)) = \sum_{k=0}^{\infty} a_k B(t)^k$ *and* $B(t)^n$ *has no terms in* t^k *for* $k < n$.

Here we state some two important theorems about formal power series that we shall use in the next chapter (see, e.g., Comtet [8, p. 148 and pp. 133–134]).

Proposition 1.4.1 (Lagrange Inversion Formula). *Let* $f(t) = y = \sum_{k=1}^{\infty} a_k t^k$ *be a formal power series and* $a_1 \neq 0$. *Then*

$$t = \sum_{k=1}^{\infty} \frac{y^k}{k!} \left\{ \frac{d^{k-1}}{dt^{k-1}} \left[\frac{t}{f(t)}\right]^k \right\}_{t=0}.$$

Proposition 1.4.2. *The exponentiation of a formal power series is give by*

$$\exp\left(\sum_{k=1}^{\infty} a_k t^k\right) = \sum_{m=0}^{\infty} \left(\sum_{\substack{k_1, k_2, \ldots, k_m \geq 0 \\ k_1 + 2k_2 + \cdots + m k_m = m}} \prod_{j=1}^{m} \frac{1}{k_j!} (a_j)^{k_j} \right) t^m.$$

Lemma 1.4.3. *Let* $\alpha \in \mathbb{C}$ *be fixed and let* $F(t) = \sum_{k=0}^{\infty} f_k t^k$, $f_0 = 1$ *be a formal power series. Then*

$$F(t)^{\alpha} = \sum_{k=0}^{\infty} p_k t^k, \tag{1.7}$$

with $p_0 = 1$ *and*

$$p_m = \frac{1}{m} \sum_{k=1}^{m} (\alpha k - m + k) p_{m-k}(\alpha) f_k,$$

for $m \geq 1$.

Proof. Firstly, we differentiate (1.7) on both sides to obtain

$$\alpha\left(\sum_{k=0}^{\infty} f_k t^k\right)^{\alpha-1}\left(\sum_{k=1}^{\infty} k f_k t^{k-1}\right)=\sum_{k=1}^{\infty} k p_k t^{k-1},$$

and after shifting the index we arrive at

$$\alpha\left(\sum_{k=0}^{\infty} f_k t^k\right)^{\alpha-1}\left(\sum_{k=0}^{\infty} (k+1) f_{k+1} t^{k}\right)=\sum_{k=0}^{\infty} (k+1) p_{k+1} t^{k}. \tag{1.8}$$

By multiplying both sides of (1.8) by a copy of 1.7 we obtain

$$\alpha\left(\sum_{k=0}^{\infty} f_k t^k\right)^{\alpha}\left(\sum_{k=0}^{\infty} (k+1) f_{k+1} t^{k}\right)=\left(\sum_{k=0}^{\infty} (k+1) p_{k+1} t^{k}\right)\left(\sum_{k=0}^{\infty} f_k t^k\right),$$

$$\alpha\left(\sum_{k=0}^{\infty} p_k t^k\right)\left(\sum_{k=0}^{\infty} (k+1) f_{k+1} t^{k}\right)=\left(\sum_{k=0}^{\infty} (k+1) p_{k+1} t^{k}\right)\left(\sum_{k=0}^{\infty} f_k t^k\right),$$

$$\sum_{j=0}^{\infty}\left(\alpha\sum_{j=0}^{k} p_j (k-j+1) f_{k-j+1}\right) t^k=\sum_{j=0}^{\infty}\left(\sum_{j=0}^{k} (j+1) p_{j+1} f_{k-j}\right) t^k.$$

Applying the Cauchy product of two infinite series and equating coefficients, we obtain

$$\sum_{j=0}^{k} \alpha(k-j+1) p_j f_{k-j+1}=\sum_{j=0}^{k} (j+1) p_{j+1} f_{k-j},$$

where

$$\sum_{j=0}^{k} (j+1) p_{j+1} f_{k-j}=(k+1) p_{k+1} f_0+\sum_{j=0}^{k-1} (j+1) p_{j+1} f_{k-j},$$

and

$$\sum_{j=0}^{k-1} (j+1) p_{j+1} f_{k-j}=\sum_{j=0}^{k-1} (j+1) p_{j+1} f_{k+1-(j+1)}=\sum_{j=0}^{k} j p_j f_{k+1-j}.$$

Therefore

$$\sum_{j=0}^{k} \alpha(k-j+1) p_j f_{k-j+1}=(k+1) p_{k+1} f_0+\sum_{j=0}^{k} j p_j f_{k-j+1}.$$

Since $f_0=1$, we obtain

$$(k+1) p_{k+1}=\sum_{j=0}^{k} \alpha(k-j+1) p_j f_{k-j+1}-\sum_{j=0}^{k} j p_j f_{k+1-j},$$

$$(k+1)p_{k+1} = \sum_{j=0}^{k} \left(\alpha(k-j+1) - j\right) p_j f_{k-j+1},$$

$$p_{k+1} = \frac{1}{k+1} \sum_{j=0}^{k} \left(\alpha(k-j+1) - j\right) p_j f_{k-j+1}.$$

Replacing every $k+1$ by m and every j by k, we obtain

$$p_m = -\frac{1}{m} \sum_{k=1}^{m} \left(\alpha(k-m) + k\right) p_k f_{m-k},$$

which completes the proof. □

Chapter 2

Ramanujan's Inverse Digamma Approximation

In this chapter we present some new results of an unsolved problem of Ramanujan, found in the book of B. C. Berndt [6, pp. 194–195, Entry 15] titled "Ramanujan's Notebooks Part I". B. C. Berndt only showed that, the first few coefficients of an asymptotic series we are about to discuss in the subsequent section agree with the ones given by Ramanujan. But in general neither Ramanujan nor B. C. Berndt was able to come up with a formula for the general term and since then nothing has been done as of this date, that is why we deem it interesting to come up with possible solutions to the problem. This chapter is divided into two sections, the first section deals with the description of the problem. In the second section, we will give the main results that is all the theorems we have propounded.

2.1 Elaboration of the Problem

Ramanujan [6, pp. 194–195, Entry 15] first defines a by the equality

$$\log a = \psi(x+1), \tag{2.1}$$

and regards a as a function of x which is assumed to be real and positive. We know from (1.4) that $\psi(x+1) = \log(x+1/2) + o(1)$ for large x, which further implies that $a \sim x + 1/2$ for large x. This actually means that Ramanujan's notation is reasonable. Ramanujan [6, pp. 194–195, Entry 15] proposed the following asymptotic expansion without a proof and a formula for the general term

$$\left(\frac{x+\frac{1}{2}}{a}\right)^{4n} \sim 1 - \frac{n}{6a^2} + \frac{10n^2+11n}{720a^4} - \frac{70n^3+231n^2+891n}{80720a^6} + \cdots \tag{2.2}$$

as $a \to +\infty$. Let us denote the mth coefficient in Ramanujan's expansion (2.2) by $\varrho_m(n)$, so that

$$\left(\frac{x+\frac{1}{2}}{a}\right)^{4n} \sim \sum_{m=0}^{\infty} \frac{\varrho_m(n)}{a^{2m}} \quad \text{as} \quad a \to +\infty. \tag{2.3}$$

Remark 2.1.1. *If we replace n by $1/4$ in (2.2) we obtain*

$$x \sim -\frac{1}{2} + a - \frac{1}{24a} + \frac{3}{640a^3} - \frac{555}{129152a^6} + \cdots,$$

and by applying (2.1) we find

$$x \sim -\frac{1}{2} + e^{\psi(x+1)} - \frac{1}{24}e^{-\psi(x+1)} + \frac{3}{640}e^{-3\psi(x+1)} - \frac{555}{129152}e^{-5\psi(x+1)} + \cdots.$$

This means that Ramanujan's approximation really gives the inverse of the Digamma function asymptotically.

2.2 Main Results

The first aim of this section is to provide formulae for the computation of the coefficients $\varrho_m(n)$. We find that these coefficients can be derived explicitly when $n = -\frac{1}{4}$, whereas for the general complex n, we have a recurrence relation for the general term $\varrho_m(n)$ involving the $\varrho_m(-\frac{1}{4})$. The second aim is to provide an asymptotic formula for $\varrho_m(n)$ when n is fixed and $m \to +\infty$.

Here we give an explicit formula for computing $\varrho_m(n)$ when $n = -\frac{1}{4}$, which we shall use to determine a recurrence relation for the coefficients $\varrho_m(n)$.

Theorem 2.2.1. *For any $m \geq 0$ we have*

$$\varrho_m\left(-\frac{1}{4}\right) = \frac{1}{2m+1} \sum_{\substack{k_1,k_2,\ldots,k_m \geq 0 \\ k_1+2k_2+\cdots+mk_m=m}} \prod_{j=1}^{m} \frac{1}{k_j!}\left(\frac{(2m+1)(1-2^{1-2j})B_{2j}}{2j}\right)^{k_j},$$

where B_j denotes the jth Bernoulli number.

Proof. Substituting $x = z - \frac{1}{2}$ into (1.4), we obtain the following expression

$$\log\left(\frac{a}{z}\right) \sim \sum_{k=1}^{\infty} \frac{(1-2^{1-2k})B_{2k}}{2kz^{2k}},$$

as $z \to +\infty$. Then

$$\frac{1}{a} \sim \frac{1}{z}\exp\left(-\sum_{k=1}^{\infty} \frac{(1-2^{1-2k})B_{2k}}{2kz^{2k}}\right) \sim \sum_{m=0}^{\infty} \frac{\mu_m}{z^m},$$

with some real coefficients μ_m. By Olver's theorem [13, pp. 22–22], we can invert this series to arrive at the following

$$\frac{1}{z} \sim \sum_{m=0}^{\infty} \frac{\lambda_m}{a^m},$$

as $a \to +\infty$, with some real coefficients λ_m. To determine the λ_m's we can use Proposition 1.4.1 with the choice of $y = \frac{1}{a}$, $t = \frac{1}{z}$ and

$$f(t) = \sum_{m=0}^{\infty} \mu_m t^m = t\exp\left(-\sum_{m=1}^{\infty} \frac{(1-2^{1-2m})B_{2m}}{2m} t^{2m}\right),$$

to arrive at

$$t = \sum_{m=1}^{\infty} \lambda_m y^m,$$

where

$$\lambda_m = \frac{1}{m!}\left[\frac{d^{m-1}}{dt^{m-1}} \exp\left(\sum_{k=1}^{\infty} \frac{(1-2^{1-2k})B_{2k}}{2k} t^{2k}\right)^m\right]_{t=0}.$$

Returning to the original variables, we have

$$\begin{aligned}
\frac{1}{z} &\sim \sum_{m=1}^{\infty} \frac{1}{m!a^m}\left[\frac{d^{m-1}}{dt^{m-1}} \exp\left(\sum_{k=1}^{\infty} \frac{(1-2^{1-2k})B_{2k}}{2k} t^{2k}\right)^m\right]_{t=0}, \\
\frac{1}{z} &\sim \sum_{m=0}^{\infty} \frac{1}{(m+1)!a^{m+1}}\left[\frac{d^{m}}{dt^{m}} \exp\left(\sum_{k=1}^{\infty} \frac{(m+1)B_{2k}(1-2^{1-2k})}{2k} t^{2k}\right)\right]_{t=0}, \\
\frac{a}{z} &\sim \sum_{m=0}^{\infty} \frac{1}{(m+1)!a^{m}}\left[\frac{d^{m}}{dt^{m}} \exp\left(\sum_{k=1}^{\infty} \frac{(m+1)B_{2k}(1-2^{1-2k})}{2k} t^{2k}\right)\right]_{t=0}.
\end{aligned}$$

Because the coeficients of odd powers of $\frac{1}{a}$ vanish, we consider only the even powers and by applying Proposition 1.4.2, we find that

$$\frac{a}{z} \sim \sum_{m=0}^{\infty} \frac{1}{(2m+1)a^{2m}} \frac{1}{(2m)!} \left[\frac{d^{2m}}{dt^{2m}} \sum_{m=0}^{\infty} \sum_{\substack{k_1,k_2,\ldots,k_m \geq 0 \\ k_1+2k_2+\cdots+mk_m=m}} \prod_{j=1}^{m} \frac{1}{k_j!} \left(\frac{(2m+1)(1-2^{1-2j})B_{2j}}{2j} \right)^{k_j} t^{2m} \right]_{t=0}.$$

It follows from Taylor's Theorem that

$$\frac{a}{z} \sim \sum_{m=0}^{\infty} \frac{1}{(2m+1)} \sum_{\substack{k_1,k_2,\ldots,k_m \geq 0 \\ k_1+2k_2+\cdots+mk_m=m}} \prod_{j=1}^{m} \frac{1}{k_j!} \left(\frac{(2m+1)(1-2^{1-2j})B_{2j}}{2j} \right)^{k_j} \frac{1}{a^{2m}}. \tag{2.4}$$

Substituting back $z = x + \frac{1}{2}$ in (2.4) we find that

$$\left(\frac{x+\frac{1}{2}}{a} \right)^{-1} \sim \sum_{m=0}^{\infty} \frac{1}{(2m+1)} \sum_{\substack{k_1,k_2,\ldots,k_m \geq 0 \\ k_1+2k_2+\cdots+mk_m=m}} \prod_{j=1}^{m} \frac{1}{k_j!} \left(\frac{(2m+1)(1-2^{1-2j})B_{2j}}{2j} \right)^{k_j} \frac{1}{a^{2m}}.$$

With $n = -\frac{1}{4}$, we have by (2.3) that

$$\left(\frac{x+\frac{1}{2}}{a} \right)^{-1} \sim \sum_{m=0}^{\infty} \frac{\varrho_m(-\frac{1}{4})}{a^{2m}} \quad \text{as} \quad a \to +\infty.$$

It follows from the uniqueness on the coefficient of asymptotic power series that

$$\varrho_m\left(-\frac{1}{4}\right) = \frac{1}{2m+1} \sum_{\substack{k_1,k_2,\ldots,k_m \geq 0 \\ k_1+2k_2+\cdots+mk_m=m}} \prod_{j=1}^{m} \frac{1}{k_j!} \left(\frac{(2m+1)(1-2^{1-2j})B_{2j}}{2j} \right)^{k_j},$$

which completes the proof. □

Here we present a recurrence relation for the general term $\varrho_m(n)$ with the help of Lemma 1.4.3 and Theorem 2.2.1.

Theorem 2.2.2. *For $m \geq 1$, the coefficients $\varrho_m(n)$ satisfies the recurrence relation*

$$\varrho_m(n) = -\frac{1}{m} \sum_{k=1}^{m} ((4n-1)k + m) \varrho_{m-k}(n) \varrho_k(-\frac{1}{4})$$

with $\varrho_0(n) = 1$.

Proof. From (2.3), we have

$$\left(\sum_{m=0}^{\infty} \varrho_m(-1/4)t^m\right)^{-4n} = \sum_{m=0}^{\infty} \varrho_m(n)t^m.$$

Thus by Lemma 1.4.3, we obtain

$$\varrho_m(n) = -\frac{1}{m}\sum_{k=1}^{m}((4n-1)k+m)\varrho_{m-k}(n)\varrho_k(-\frac{1}{4}),$$

which completes the proof. □

Here we state some Lemmata and Theorems which we shall use to determine the asymptotic form of the coefficient $\varrho_m(n)$ as $m \to +\infty$.

Lemma 2.2.3. *For any $k \geq 0$ we have that*

$$\frac{2(2k)!}{(2\pi)^{2k}} < |B_{2k}| < \frac{4(2k)!}{(2\pi)^{2k}}.$$

Proof. It is know that

$$B_{2k} = (-1)^{k+1}\frac{2(2k)!}{(2\pi)^{2k}}\left(1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \cdots\right),$$

which means that

$$\frac{2(2k)!}{(2\pi)^{2k}} < |B_{2k}|.$$

To prove the upper bound, note that

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} \leq \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} < 2.$$

□

Lemma 2.2.4. *We have*

$$(1-2^{1-2m})B_{2m} \sim B_{2m} \sim (-1)^{m+1}\frac{2(2m)!}{(2\pi)^{2m}},$$

as $m \to +\infty$.

Proof. Clearly,

$$\lim_{m\to+\infty}\frac{(1-2^{1-2m})B_{2m}}{B_{2m}} = 1.$$

The second statement follows from the fact that

$$1 \leq \lim_{m\to+\infty} \frac{B_{2m}}{(-1)^{m+1}\frac{2(2m)!}{(2\pi)^{2m}}} = \lim_{m\to+\infty} \sum_{k=0}^{\infty} \frac{1}{k^{2m}} \leq \lim_{m\to+\infty} \sum_{j=0}^{\infty} \frac{1}{2^{2jm}} = 1,$$

where we made use of Lemma 2.2.3. □

Theorem 2.2.5 (E. A. Bender, [4]). *Let*

$$\alpha(x) = \sum_{k=1}^{\infty} \alpha_k x^k, \quad F(x) = \sum_{k=1}^{\infty} f_k x^k,$$

be two formal power series, and let

$$\beta(x) := F(\alpha(x)) = \sum_{k=1}^{\infty} \beta_k x^k.$$

Assume that $F(x)$ is analytic in x in a neighborhood of 0, $\alpha_k \neq 0$ and

(1) $\alpha_{k-1} = o(\alpha_k)$ *as* $k \to +\infty$,

(2) $\sum_{j=1}^{k-1} |\alpha_j \alpha_{k-j}| = \mathcal{O}(\alpha_{k-1})$ *as* $k \to +\infty$.

Then $\beta_k \sim f_1 \alpha_k$ *as* $k \to +\infty$.

Lemma 2.2.6. *Define the formal power series $A(x)$ with coefficients $a_1, a_2, \ldots$ via the expression*

$$A(x) = \exp\left(\sum_{k=1}^{\infty} \frac{(1-2^{1-2k})B_{2k}}{2k} x^k\right) - 1 = \sum_{k=1}^{\infty} a_k x^k.$$

Then we have

$$a_k \sim \frac{(-1)^{k+1}(2k)!}{k(2\pi)^{2k}},$$

as $k \to +\infty$.

To prove Lemma 2.2.6, we apply Theorem 2.2.5 together with Lemma 2.2.3 and Lemma 2.2.4.

Proof. We apply Theorem 2.2.5 to the formal power series

$$\alpha(x) := \sum_{k=1}^{\infty} \frac{B_{2k}(1-2^{1-2k})}{2k} x^k, \quad F(x) := e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

It follows that

$$A(x) = F(\alpha(x)) =: \beta(x) = \sum_{k=1}^{\infty} a_k x^k.$$

Applying Lemma 2.2.3 to the sequence

$$\alpha_k := \frac{(1-2^{1-2k})B_{2k}}{2k},$$

we obtain

$$\frac{(1-2^{1-2k})(2k)!}{k(2\pi)^{2k}} < |\alpha_k| < \frac{2(1-2^{1-2k})(2k)!}{k(2\pi)^{2k}},$$

for every $k \geq 1$. Then we have

$$\frac{(2k)!}{2k(2\pi)^{2k}} \leq |\alpha_k| < \frac{2(2k)!}{k(2\pi)^{2k}} \tag{2.5}$$

for every $k \geq 1$. Since $\alpha_k \neq 0$, and

$$0 \leq \lim_{k\to+\infty} \left|\frac{\alpha_{k-1}}{\alpha_k}\right| < \lim_{k\to+\infty} \frac{4(2\pi)^2}{(k-1)(2k-1)} = 0,$$

we have

$$\alpha_{k-1} = o(\alpha_k) \text{ as } k \to +\infty.$$

Hence condition (1) of Theorem 2.2.5 holds. From (2.5), we find

$$\sum_{j=1}^{k-1} |\alpha_j \alpha_{k-j}| < \sum_{j=1}^{k-1} \frac{4(2j)!(2k-2j)!}{j(k-j)(2\pi)^{2k}} < \frac{16(2k-2)!}{(k-1)(2\pi)^{2k-2}} \sum_{j=1}^{k-1} \frac{(2j-1)!(2k-2j-1)!(k-1)}{(2k-2)!}$$
$$= \frac{4(2k-2)!}{2(k-1)(2\pi)^{2k-2}} \sum_{j=0}^{k-2} \frac{(2j+1)!(2k-2j-3)!}{(2k-3)!} < 16|\alpha_{k-1}| \sum_{j=0}^{k-2} \frac{(2j+1)!(2k-2j-3)!}{(2k-3)!}.$$

We have

$$\sum_{j=0}^{k} \frac{(2j+1)!(2k-2j+1)!}{(2k+1)!} = \sum_{j=0}^{k} \frac{(2j+1)}{\binom{2k+1}{2j}} = 2 + \frac{6}{k(2k+1)} + \sum_{j=0}^{k-2} \frac{(2j+1)}{\binom{2k+1}{2j}},$$

and $\binom{2k+1}{2j} \geq Ck^3$, therefore

$$\sum_{j=1}^{k-2} \frac{(2j+1)!(2k-2j-3)!}{(2k-3)!} \leq 3 + K \sum_{k=1}^{\infty} \frac{1}{k^2} < 3 + 2K.$$

Finally, we obtain

$$\sum_{j=1}^{k-1} |\alpha_j \alpha_{k-j}| < 16|\alpha_{k-1}|(3+2k) \leq \widetilde{C}|\alpha_{k-1}|,$$

which implies that

$$\sum_{j=1}^{k-1} |\alpha_j \alpha_{k-j}| = \mathcal{O}(\alpha_{k-1}) \text{ as } k \to +\infty.$$

Since condition (1) and (2) of Theorem 2.2.6 are satisfied, we can conclude that

$$a_k \sim f_1 \alpha_k \sim \frac{(1-2^{1-2k})B_{2k}}{2k} \sim \frac{(-1)^{k+1}(2k)!}{k(2\pi)^{2k}},$$

as $k \to +\infty$. □

Theorem 2.2.7 (E. A. Bender and L. B. Richmond, [5])**.** *Let $A(x)$ be a formal power series with coefficients a_k where $a_0 = 0$ and $a_1 \neq 0$. Let p_m be the coefficient of x^m in $(1 + A(x))^{\alpha m+\beta}$ where $\alpha \neq 0$ and β are fixed complex numbers. If*

(1) $k a_{k-1} = o(a_k)$ as $k \to +\infty$,

then $p_m \sim (\alpha m + \beta) a_m$, as $m \to +\infty$.

Here we present our result concerning the asymptotic behaviour of $\varrho_m(n)$ when $n = -1/4$ by applying Lemma 2.2.6 and Theorem 2.2.7, which will be useful in our next result about the asymptotic formula of the general term $\varrho_m(n)$.

Theorem 2.2.8. *We have*

$$\varrho_m\left(-\frac{1}{4}\right) \sim (-1)^{m+1} \frac{2(2m-1)!}{(2\pi)^{2m}},$$

as $m \to +\infty$.

Proof. Let $A(x)$ be the formal power series given in Lemma 2.2.6, and define the sequence $p_0, p_1, p_2, \ldots$ by

$$(1 + A(x))^{2m+1} = \sum_{k=0}^{\infty} p_k x^k.$$

From Lemma 2.2.6 we have that

$$a_k \sim \frac{(-1)^{k+1} 2(2k-1)!}{(2\pi)^{2m}},$$

which implies that

$$\lim_{k\to+\infty} \frac{ka_{k-1}}{a_k} = \lim_{k\to+\infty} \frac{-k}{(2k-1)(2k-2)} = 0.$$

Hence the condition (1) of Theorem 2.2.7 holds. Thus, we can conclude that

$$p_m \sim (2m+1)a_m \sim \frac{(-1)^{m+1}2(2m+1)(2m-1)!}{(2\pi)^{2m}}, \tag{2.6}$$

as $m \to +\infty$. Applying Proposition 1.4.2, we can write

$$\begin{aligned}(1+A(x))^{2m+1} &= \exp\left(\sum_{k=1}^{\infty} \frac{(2m+1)(1-2^{1-2k})B_{2k}}{2k} x^k\right) \\ &= \sum_{m=0}^{\infty} \sum_{\substack{k_1,k_2,\dots,k_m\geq 0 \\ k_1+2k_2+\cdots+mk_m=m}} \prod_{j=1}^{m} \frac{1}{k_j!}\left(\frac{(2m+1)(1-2^{1-2j})B_{2j}}{2j}\right)^{k_j} x^m,\end{aligned}$$

and by comparing coefficients we obtain

$$p_m = \sum_{\substack{k_1,k_2,\dots,k_m\geq 0 \\ k_1+2k_2+\cdots+mk_m=m}} \prod_{j=1}^{m} \frac{1}{k_j!}\left(\frac{(2m+1)(1-2^{1-2j})B_{2j}}{2j}\right)^{k_j}. \tag{2.7}$$

From Theorem 2.2.1, (2.6) and (2.7) we find

$$(2m+1)\varrho_m\left(-\frac{1}{4}\right) = p_m \sim (-1)^{m+1}\frac{2(2m+1)(2m-1)!}{(2\pi)^{2m}},$$

and finally,

$$\varrho_m\left(-\frac{1}{4}\right) \sim (-1)^{m+1}\frac{2(2m-1)!}{(2\pi)^{2m}},$$

as $m \to +\infty$. □

Here we present our final result on the asymptotic formula for the general term $\varrho_m(n)$ for every complex number n. This can be achieved with the help of Theorem 2.2.5 and Theorem 2.2.8.

Theorem 2.2.9. *For every fixed complex number n, we have*

$$\varrho_m(n) \sim (-1)^m \frac{8n(2m-1)!}{(2\pi)^{2m}},$$

as $m \to +\infty$.

Proof. We apply Theorem 2.2.5 to the formal power series

$$F(x) = (1+x)^{-4n} = \sum_{k=0}^{\infty} \binom{-4n}{k} x^k, \quad \alpha(x) = \sum_{m=1}^{\infty} \varrho_m(-\frac{1}{4})x^m =: \sum_{m=1}^{\infty} \alpha_m x^m.$$

From Theorem 2.2.8 we know that

$$\alpha_m \sim (-1)^{m+1} \frac{2(2m-1)!}{(2\pi)^{2m}},$$

which implies

$$\frac{2c_1(2m-1)!}{(2\pi)^{2m}} < |\alpha_m| < \frac{2c_2(2m-1)!}{(2\pi)^{2m}}, \tag{2.8}$$

for some $c_2 > c_1 > 0$. Since $\alpha_m \neq 0$, and

$$\lim_{m\to+\infty} \left| \frac{\alpha_{m-1}}{\alpha_m} \right| = \lim_{m\to+\infty} \frac{c_2(2\pi)^2}{c_1(2m-1)} = 0,$$

the condition (1) of Theorem 2.2.5 holds. From (2.8), we have

$$\sum_{j=1}^{m-1} |\alpha_j \alpha_{m-j}| < \sum_{j=1}^{m-1} \frac{4c_2^2(2j-1)!(2m-2j-1)!}{(2\pi)^{2m}} = \frac{4c_2^2(2m-2)!}{(2\pi)^{2m}} \sum_{j=1}^{m-1} \frac{(2j-1)!(2m-2j-1)!}{(2m-2)!}$$

$$= \frac{4c_2^2(2m-2)!}{(2\pi)^{2m}} \sum_{j=1}^{m-1} \frac{1}{\binom{2m-2}{2j-1}} = \mathcal{O}\left(\frac{(2m-2)!}{(2\pi)^{2m}}\right) = \mathcal{O}(\alpha_{m-1}).$$

Hence the condition (2) of Theorem 2.2.5 also holds and it follows that

$$\varrho_m(n) \sim -4n\alpha_m = -4n\varrho_m\left(-\frac{1}{4}\right),$$

and finally by Theorem 2.2.8, we have

$$\varrho_m(n) \sim (-1)^m \frac{8n(2m-1)!}{(2\pi)^{2m}},$$

as $m \to +\infty$. □

Remark: Although the general terms are given by a complicated formula, their asymptotic behaviour are simple and it also shows the divergent character of Ramanujan's series.

Chapter 3

Ramanujan on the nth Harmonic number

Ramanujan [7, pp. 531–532, Entry 9] provided an asymptotic series for the nth Harmonic number, but he expresses the asymptotic expansion in powers of $1/m$ instead of $1/n$, where m is the nth triangular number. B. C. Berndt [7, pp. 531–532, Entry 9] says "We cannot find a natural method to produce such an asymptotic series". In 2012, M. D. Hirschhorn [10] provided a natural derivation of the asymptotic series by computing only the first 11 odd powers for the tail of the Riemann zeta function with no rigorous proof, which he admitted by saying "the reader might note that I am avoiding the word "prove" ". Our aim is to give a general expansion which works for any odd power and as a special case, our result will contain the expansions given in the result of M. D. Hirschhorn , not only that but with formula for the general coefficient and a precise error term. This means that the special case of our result together with M. D. Hirschhorn's result completely solves the problem. This chapter is divided into two sections. The first section deals with the description of the problem and a brief history of other related problems. In the second section, we present our main results.

3.1 Elaboration of the Problem

Ramanujan [7, pp. 531–532, Entry 9] proposed the following asymptotic expansion for the partial sum of the harmonic series without proof and also without a formula for the general term

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \sim \frac{1}{2}\log(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \cdots . \qquad (3.1)$$

In 2008, M. Villarino [15] gave a proof and a formula for the general term with a nice error term. M. Villarino remarked that it would be interesting to find a similar representation for $\log n!$ in terms of $\frac{1}{m}$. In 2010, G. Nemes [11] proved such an asymptotic series for $\log n!$ with a very nice error term. B. C. Bernt , M. D. Hirschhorn described (3.1) as "somewhat enigmatic" and mentioned that "we cannot find a "natural" method to produce such an asymptotic series". In 2012, M. D. Hirschhorn gave a natural derivation of (3.1) also called natural method. In proving (3.1), M. D. Hirschhorn mentions that "I hesitate to use the word "prove"" and derived the following results for the odd powers, that is $2j+1$ for $1 \leq j \leq 11$ of the tail of the Riemann zeta function.

$$\sum_{k=n+1}^{\infty} \frac{1}{k^3} \sim \frac{1}{4m} - \frac{1}{16m^2} + \frac{1}{48m^3} - \frac{1}{96m^4} + \frac{1}{120m^5} - \frac{1}{96m^6} + \cdots,$$
$$\sum_{k=n+1}^{\infty} \frac{1}{k^5} \sim \frac{1}{16m^2} - \frac{1}{24m^3} + \frac{11}{384m^3} - \frac{5}{192m^5} + \frac{13}{384m^6} - \frac{1}{16m^7} + \cdots,$$
$$\vdots$$
$$\sum_{k=n+1}^{\infty} \frac{1}{k^{23}} \sim \frac{1}{45056m^{11}} - \cdots,$$

and showed that the following expression results to (3.1)

$$\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\log 2m - \gamma = \frac{1}{3}\sum_{n+1}^{\infty} \frac{1}{k^3} + \frac{1}{5}\sum_{n+1}^{\infty} \frac{1}{k^5} + \frac{1}{7}\sum_{n+1}^{\infty} \frac{1}{k^7} + \cdots.$$

3.2 Main Results

The first aim of this section is to provide an expression for the $\sum_{k=n+1}^{\infty} 1/k^j$ where $j > 1$ with a formula for the general term and a precise error term. Secondly we will as a special case provide an expression for the odd powers, that is $j = 2i+1$ for $i \geq 1$ with a formula for the general term and nice error term. M. D. Hirschhorn just computed the first few terms of those series for the first 11 odd numbers without error term and a general term. So our result is not only going fill in this gap but will add beauty to his result base on formulas we are going to provide. We start by first proving the following lemmata which will serve as a springboard for our result.

Lemma 3.2.1. *For $\Re(s) > 0$, $s \neq 1$ and $\Re(a) > 0$, we have*

$$\zeta(s, a+1) = 2^s \zeta(s, 2a+1) - \zeta(s, a+1/2).$$

Proof. By Proposition 1.3.1 we can write

$$2^s\zeta(s,2a+1)-\zeta(s,a+\frac{1}{2})=\sum_{n=0}^{\infty}\frac{2^s}{(n+2a+1)^s}-\sum_{n=0}^{\infty}\frac{1}{(n+a+\frac{1}{2})^s}$$
$$=\sum_{n=0}^{\infty}\frac{1}{(\frac{n}{2}+a+\frac{1}{2})^s}-\sum_{n=0}^{\infty}\frac{1}{(n+a+\frac{1}{2})^s}$$

$$=\left[\frac{1}{(a+\frac{1}{2})^s}+\frac{1}{(a+1)^s}+\frac{1}{(a+\frac{3}{2})^s}+\frac{1}{(a+2)^s}+\frac{1}{(a+\frac{5}{2})^s}+\frac{1}{(a+\frac{5}{2})^s}+\cdots\right]-$$
$$\left[\frac{1}{(a+\frac{1}{2})^s}-\frac{1}{(a+\frac{3}{2})^s}-\frac{1}{(a+\frac{5}{2})^s}-\frac{1}{(a+\frac{7}{2})^s}-\cdots\right].$$

Since it is telescoping, then

$$=\left[\frac{1}{(a+1)^s}+\frac{1}{(a+2)^s}+\frac{1}{(a+3)^s}+\frac{1}{(a+4)^s}+\cdots\right]=\sum_{n=1}^{\infty}\frac{1}{(n+a)^s}$$
$$=\sum_{n=0}^{\infty}\frac{1}{(n+a+1)^s}=\zeta(s,a+1),$$

which completes the proof. □

Lemma 3.2.2. *We have*

$$\zeta(s,a+1)=\frac{(a+\frac{1}{2})^{1-s}}{s-1}+\frac{1}{2\Gamma(s)}\int_0^{+\infty}t^{s-1}\left(\frac{1}{\sinh(t/2)}-\frac{1}{t/2}\right)e^{-(a+\frac{1}{2})t}dt,$$

for $\Re(s)>-1$, $s\neq 1$ *and* $\Re(a)>-\frac{1}{2}$.

Proof. Applying Proposition 1.3.2 to Lemma 3.2.1, we obtain

$$\zeta(s,a+1)=\frac{(a+\frac{1}{2})^{1-s}}{s-1}+\left[\frac{2^s}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)e^{-(2a+1)t}dt\right]-$$
$$\left[\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)e^{-(a+\frac{1}{2})t}dt\right].$$

After simplification we deduce

$$\zeta(s,a+1)=\frac{(a+\frac{1}{2})^{1-s}}{s-1}+\frac{1}{2\Gamma(s)}\int_0^{+\infty}t^{s-1}\left(\frac{2}{e^{t/2}-1}-\frac{2}{e^t-1}-\frac{1}{t/2}\right)e^{-(a+\frac{1}{2})t}dt,$$

which is equivalent to the statement. □

Here we present a series representation of the reciprocal of the hyperbolic sine with an error term. This will enable us to evaluate the integral in Lemma 3.2.2 and obtain a series representation of the Hurwitz zeta function with a precise error term.

Lemma 3.2.3. *For any* $x > 0,\ N \geq 0,$ *we have*

$$\frac{1}{\sinh(x)} = \frac{1}{x} + \sum_{n=0}^{N-1} \frac{(2-2^{2n+2})B_{2n+2}}{(2n+2)!} x^{2n+1} + \Theta(x,N)\frac{(2-2^{2N+2})B_{2N+2}}{(2N+2)!} x^{2N+1}$$

with a suitable $0 < \Theta(x,N) < 1$.

Proof. We shall use the following well know expressions, which can be found in [14, p. 126, 4.36.5], [14, p. 592, 24.8.1 and p. 590, 24.4.27].

$$\frac{x}{\sinh(x)} = 1 + 2x^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2+\pi^2k^2}, \tag{3.2}$$

$$B_{2n+2}(x) = (-1)^n \frac{2(2n+2)!}{(2\pi)^{2n+2}} \sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{k^{2n+2}}, \tag{3.3}$$

$$B_n\left(\frac{1}{2}\right) = -(1-2^{1-n})B_n. \tag{3.4}$$

Using (3.3) and (3.4), we obtain

$$(-1)^n \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^{2n+2}} = \frac{(2\pi)^{2n+2}B_{2n+2}(\frac{1}{2})}{2(2n+2)!} = \frac{(2\pi)^{2n+2}(2^{-2n-1}-1)B_{2n+2}}{2(2n+2)!}. \tag{3.5}$$

Applying the well-know identity

$$\frac{1}{1+z} = \sum_{k=0}^{N-1} (-1)^k z^k + (-1)^N \frac{z^N}{1+z},\ z \neq 1,$$

and (3.5) we can write (3.2) as

$$\frac{x}{\sinh(x)} = 1 + 2x^2 \left[\sum_{n=0}^{N-1} \frac{(2-2^{2n+2})B_{2n+2}}{2(2n+2)!} x^{2n} + \sum_{k=1}^{\infty} (-1)^{N+k} \frac{x^{2N}}{(\pi k)^{2N}(x^2+\pi^2k^2)} \right],$$

or

$$\frac{x}{\sinh(x)} = 1 + \sum_{n=0}^{N-1} \frac{(2-2^{2n+2})B_{2n+2}}{(2n+2)!} x^{2n+2} + 2\sum_{k=1}^{\infty} (-1)^{N+k} \frac{x^{2N+2}}{(\pi k)^{2N}(x^2+\pi^2k^2)} \tag{3.6}$$

To work on the error term in (3.6), we first define

$$R_N := \frac{\sum_{k=1}^{\infty}(-1)^k \frac{1}{(\pi k)^{2N}(x^2+\pi^2k^2)}}{\sum_{k=1}^{\infty}(-1)^k \frac{1}{(\pi k)^{2N}(\pi^2k^2)}} \frac{(2-2^{2N+2})B_{2N+2}}{(2N+2)!} x^{2N+2},$$

then

$$\frac{x}{\sinh(x)} = 1 + \sum_{n=0}^{N-1} \frac{(2-2^{2n+2})B_{2n+2}}{(2n+2)!} x^{2n+2} + R_N.$$

Let

$$\Theta(x,N) := \frac{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^{2N}(x^2+\pi^2k^2)}}{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^{2N}(\pi^2k^2)}} := \frac{f(x)}{f(0)},$$

where

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^{2N}(x^2+\pi^2k^2)} \\ &= \sum_{m=0}^{\infty}\left[\frac{1}{(2m+1)^{2N}(x^2+\pi^2(2m+1)^2)} - \frac{1}{(2m+2)^{2N}(x^2+\pi^2(2m+2)^2)}\right]. \end{aligned}$$

Here we try to find the range of $\Theta(x,N)$. For the general term, we have

$$f_m(x) := \frac{1}{(2m+1)^{2N}(x^2+\pi^2(2m+1)^2)} - \frac{1}{(2m+2)^{2N}(x^2+\pi^2(2m+2)^2)} \geq 0$$

and

$$f'_m(x) = \frac{2x(2m+2)^{-2N}}{(x^2+\pi^2(2m+2)^2)} - \frac{2x(2m+1)^{-2N}}{(x^2+\pi^2(2m+1)^2)} < \frac{2x(2m+2)^{-2N}-2x(2m+1)^{-2N}}{(x^2+\pi^2(2m+2)^2)} \leq 0.$$

Because $f_m(x) > 0$ and $f'_m(x) \leq 0$ it implies that for all $x > 0$, $f(x) < f(0)$ and thus $0 < \Theta(x,N) < 1$ which completes the proof. □

Here we present our first result concerning the nth Harmonic number which will be very useful for our next result. This result can be achieved with the help of Lemma 3.2.3.

Theorem 3.2.4. *For $j > 1$, $n \geq 0$, $r \geq 0$, we have*

$$\begin{aligned} \sum_{k=1}^{n}\frac{1}{k^j} = \zeta(j) + \frac{(n+\frac{1}{2})^{1-j}}{j-1} + \sum_{p=1}^{r}\frac{(1-2^{1-2p})B_{2p}}{(2p)!}\frac{(j)_{2p-1}}{(n+\frac{1}{2})^{2p+j-1}} \\ +\theta_r(j,n)\frac{(1-2^{-2r-1})B_{2r+2}}{(2r+2)!}\frac{(j)_{2r+1}}{(n+\frac{1}{2})^{2r+j+1}}, \end{aligned}$$

where $(a)_m = \Gamma(a+m)/\Gamma(a)$ and with a suitable $0 < \theta_r(j,n) < 1$.

Proof. Using Lemma 3.2.3, we can write

$$\frac{1}{\sinh(t/2)} = \frac{1}{t/2} + \sum_{n=0}^{N-1}\frac{(2-2^{2n+2})B_{2n+2}}{(2n+2)!}\left(\frac{t}{2}\right)^{2n+1} + \Theta(t/2,N)\frac{(2-2^{2N+2})B_{2N+2}}{(2N+2)!}\left(\frac{t}{2}\right)^{2N+1},$$

and by substituting the result into Lemma 3.2.2, we obtain

$$\zeta(s,a+1) = \frac{(a+\frac{1}{2})^{1-s}}{s-1} + \frac{1}{2\Gamma(s)}\sum_{n=0}^{N-1}\frac{(2^{-2n}-2)B_{2n+2}}{(2n+2)!}\int_0^{+\infty} t^{2n+s}e^{-(a+\frac{1}{2})t}dt$$
$$+\frac{1}{2\Gamma(s)}\frac{(2^{-2N}-2)B_{2N+2}}{(2N+2)!}\int_0^{+\infty}\Theta(t/2,N)t^{2N+s}e^{-(a+\frac{1}{2})t}dt$$

Note that

$$\int_0^{+\infty} t^{2n+s}e^{-(a+\frac{1}{2})t}dt = \frac{1}{(a+\frac{1}{2})^{2n+s+1}}\int_0^{+\infty} x^{2n+s}e^{-x}dx = \frac{\Gamma(2n+s+1)}{(a+\frac{1}{2})^{2n+s+1}},$$

and by the mean value theorem for integration we have that

$$\int_0^{+\infty}\Theta(t/2,N)t^{2N+s}e^{-(a+\frac{1}{2})t}dt = \theta_N(s,a)\frac{\Gamma(2N+s+1)}{(a+\frac{1}{2})^{2N+s+1}},$$

with a suitable $0 < \theta_N(s,a) < 1$. We arrive at the following expression after combining all the partial results

$$\zeta(s,a+1) = \frac{(a+\frac{1}{2})^{1-s}}{s-1} + \sum_{n=0}^{N-1}\frac{(2^{-2n-1}-1)B_{2n+2}}{(2n+2)!}\frac{\Gamma(2n+s+1)}{\Gamma(s)(a+\frac{1}{2})^{2n+s+1}}$$
$$+\theta_N(s,a)\frac{(2^{-2N-1}-1)B_{2N+2}}{\Gamma(s)(2N+2)!}\frac{\Gamma(2N+s+1)}{(a+\frac{1}{2})^{2N+s+1}}. \tag{3.7}$$

From (1.5) and (1.6) we have that

$$\sum_{k=1}^{n}\frac{1}{k^j} = \zeta(j) - \zeta(j,n+1). \tag{3.8}$$

Substituting (3.7) into (3.8) with $s = j$ and $a = n$, we obtain obtain the statement of the theorem. □

To prove our main result, we need the following lemma, which is an easy consequence of Taylor's Theorem with Lagrange remainder.

Lemma 3.2.5. *For* $|x| < 1$ *and* $\alpha < 0$*, we have*

$$(1+x)^\alpha = \sum_{n=0}^{N-1}\binom{\alpha}{m}x^n + \Theta_N^{(\alpha)}\binom{\alpha}{N}x^N,$$

with a suitable $0 < \Theta_N^{(\alpha)} < 1$.

Now, we are ready to present our main result. Here we present an expansion of the odd powers of the Riemann zeta function in terms of the reciprocal of the nth triangular number m, with a nice error term and a formula for the general coefficient. This can be achieved with the help of Theorem 3.2.4.

Theorem 3.2.6. *For $i \geq 1$, $n \geq 0$, $r \geq 0$, we have*

$$\sum_{k=1}^{n} \frac{1}{k^{2i+1}} = \zeta(2i+1) + \frac{1}{(2m)^i} \left\{ \sum_{p=0}^{r} \frac{A_p^i}{m^p} + \kappa_r(n,i) \frac{A_{r+1}^i}{m^{r+1}} \right\},$$

with a suitable $0 < \kappa_r(n,i) < 1$, and with

$$A_p^i = \frac{(-1)^p}{8^p} \sum_{q=0}^{p} (-1)^q \binom{p+i-1}{p-q} \binom{2i+2q}{2q} \frac{(2^{2q}-2)B_{2q}}{2i+2q}.$$

Proof. Replacing j by $2i+1$ for $i \geq 1$ in Theorem 3.2.4, we obtain

$$\sum_{k=1}^{n} \frac{1}{k^{2i+1}} = \zeta(2i+1) + \frac{(n+\frac{1}{2})^{-2i}}{-2i} + \sum_{p=1}^{r} \frac{S_p^i}{(n+\frac{1}{2})^{2p+2i}} + \Theta_r(n,i) \frac{S_{r+1}^i}{(n+\frac{1}{2})^{2r+2i+2}}.$$

where $\Theta_r(n,i) = \theta_r(2i+1,n)$ and

$$S_p^i := \frac{(1-2^{1-2p})B_{2p}}{(2p)!} \frac{\Gamma(2i+2p)}{\Gamma(2i+1)}. \tag{3.9}$$

First, let us modify the term $\left(n+\frac{1}{2}\right)^{-2i}$ using the binomial series

$$\left(n+\frac{1}{2}\right)^{-2i} = \left(2m+\frac{1}{4}\right)^{-i} = (2m)^{-i}\left(1+\frac{1}{8m}\right)^{-i} = \sum_{l=0}^{\infty} \binom{-i}{l} \frac{1}{2^{3l+i} m^{l+i}}.$$

To obtain the series expansion in terms of $\frac{1}{m}$, we apply the binomial theorem again

$$\begin{aligned}\sum_{p=1}^{r} \frac{S_p^i}{(n+\frac{1}{2})^{2p+2i}} &= \sum_{p=1}^{r} \frac{S_p^i}{(2m+\frac{1}{4})^{p+i}} = \sum_{p=1}^{r} \frac{S_p^i}{(2m)^{p+i}(1+\frac{1}{8m})^{p+i}} \\ &= \sum_{p=1}^{r} \frac{S_p^i}{(2m)^{p+i}} \sum_{l=0}^{\infty} \binom{-p-i}{l} \frac{1}{8^l m^l} = \sum_{p=1}^{r} \sum_{l=0}^{\infty} \binom{-p-i}{l} \frac{S_p^i}{2^{3l+p+i} m^{l+p+i}}.\end{aligned}$$

By collecting all the partial results, we arrive at the following

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k^{2i+1}} = \zeta(2i+1) + \sum_{l=0}^{\infty} \frac{1}{-i} \binom{-i}{l} \frac{1}{2^{3l+i+1} m^{l+i}} + \sum_{p=1}^{r} \sum_{l=0}^{\infty} \binom{-p-i}{l} \frac{S_p^i}{2^{3l+p+i} m^{l+p+i}} \\ + \Theta_r(n,i) \frac{S_{r+1}^i}{(n+\frac{1}{2})^{2r+2i+2}}.\end{aligned}$$

$$= \zeta(2i+1) + \frac{1}{(2m)^i}\left[\sum_{l=0}^{\infty}\frac{1}{-i}\binom{-i}{l}\frac{1}{2^{3l+1}m^l} + \sum_{p=1}^{r}\sum_{l=0}^{\infty}\binom{-p-i}{l}\frac{S_p^i}{2^{3l+p}m^{l+p}} + E_r\right],$$

where

$$E_r := \Theta_r(n,i)\frac{S_{r+1}^i}{2^{r+1}}\frac{1}{m^{r+1}}\left(1+\frac{1}{8m}\right)^{-r-i-1}.$$

Here we try to simplify the above expression while concentrating much on the error terms

$$\sum_{l=0}^{\infty}\frac{1}{-i}\binom{-i}{l}\frac{1}{2^{3l+1}m^l} = -\frac{1}{2i} + \sum_{p=1}^{r}\frac{1}{-i}\binom{-i}{p}\frac{1}{2^{3p+1}m^p} + R_r,$$

where

$$R_r := \sum_{l=r+1}^{\infty}\frac{1}{-i}\binom{-i}{p}\frac{1}{2^{3l+1}m^l},$$

and

$$\sum_{p=1}^{r}\sum_{l=0}^{\infty}\binom{-p-i}{l}\frac{S_p^i}{2^{3l+p}m^{l+p}} = \sum_{p=1}^{r}\sum_{l=0}^{r-p}\binom{-p-i}{l}\frac{S_p^i}{2^{3l+p}m^{l+p}} + \sum_{p=1}^{r}\sum_{l=r-p+1}^{\infty}\binom{-p-i}{l}\frac{S_p^i}{2^{3l+p}m^{l+p}}.$$

Replacing every p by $p-l$ and every l by $p-q$, we obtain

$$\sum_{p=1}^{r}\sum_{l=0}^{\infty}\binom{-p-i}{l}\frac{S_p^i}{2^{3l+p}m^{l+p}} = \sum_{p=1}^{r}\sum_{q=1}^{p}\binom{-q-i}{p-q}\frac{S_q^i}{2^{3p-2q}m^p} + \epsilon_r,$$

where

$$\epsilon_r := \sum_{p=1}^{r}\sum_{l=r-p+1}^{\infty}\binom{-p-i}{l}\frac{S_p^i}{2^{3l+p}m^{l+p}}.$$

Therefore

$$\sum_{k=1}^{n}\frac{1}{k^{2i+1}}$$

$$= \zeta(2i+1) + \frac{1}{(2m)^i}\left[-\frac{1}{2i} + \sum_{p=1}^{r}\frac{1}{-i}\binom{-i}{p}\frac{1}{2^{3p+1}m^p} + \sum_{p=1}^{r}\sum_{q=1}^{p}\binom{-q-i}{p-q}\frac{S_q^i}{2^{3p-2q}m^p} + \epsilon_r + R_r + E_r\right].$$

Here the second term is the value of the third term when $q = 0$, which means that the second term can be absorbed by the third term and this simplifies to the following expression

$$\sum_{k=1}^{n}\frac{1}{k^{2i+1}}=\zeta(2i+1)+\frac{1}{(2m)^i}\left[-\frac{1}{2i}+\sum_{p=1}^{r}\left\{\sum_{q=0}^{p}\binom{-q-i}{p-q}\frac{S_q^i}{2^{3p-2q}}\right\}\frac{1}{m^p}+\epsilon_r+R_r+E_r\right].$$

Again, here the first term in the bracket is the value of the second term when $p=0$, which means that the first term can be absorbed by the second term, which results to

$$\sum_{k=1}^{n}\frac{1}{k^{2i+1}}=\zeta(2i+1)+\frac{1}{(2m)^i}\left[\sum_{p=0}^{r}\left\{\sum_{q=0}^{p}\binom{-q-i}{p-q}\frac{S_q^i}{2^{3p-2q}}\right\}\frac{1}{m^p}+\epsilon_r+R_r+E_r\right]. \tag{3.10}$$

Here we insert (3.9) into (3.10) for further simplification of the general term and as result we obtain

$$\begin{aligned}\sum_{q=0}^{p}\binom{-q-i}{p-q}\frac{S_q^i}{2^{3p-2q}} &= \sum_{q=0}^{p}(-1)^{p-q}\binom{p+i-1}{p-q}\frac{S_q^i}{2^{3p-2q}}\\ &= \frac{(-1)^p}{8^p}\sum_{q=0}^{p}(-1)^q\binom{p+i-1}{p-q}\binom{2i+2q}{2q}\frac{(2^{2q}-2)B_{2q}}{2i+2q}=:A_p^i.\end{aligned}$$

Now what is left, is to show that for $r\geq 1$ the error term has the properties specified in the theorem. We have that

$$E_r=\Theta_r(n,i)\frac{S_{r+1}^i}{2^{r+1}}\frac{1}{m^{r+1}}\left(1+\frac{1}{8m}\right)^{-r-i-1}=\sigma_r(n,i)\frac{S_{r+1}^i}{2^{r+1}}\frac{1}{m^{r+1}},$$

because $0<\left(1+\frac{1}{8m}\right)^{-r-i-1}<1$ and $0<\Theta_r(n,i)<1$ it implies that $0<\sigma_r(n,i)<1$. Using Lemma 3.2.5 for the error terms R_r and ϵ_r, we obtain

$$\begin{aligned}R_r=\sum_{l=r+1}^{\infty}\frac{1}{-i}\binom{-i}{l}\frac{1}{2^{3l+1}m^l} &= \sum_{l=r+1}^{\infty}\frac{(-1)^l}{-i}\binom{l+i-1}{l}\frac{1}{2^{3l+1}m^l}\\ &= \tau_r(n,i)\frac{(-1)^r}{i}\binom{r+i}{r+1}\frac{1}{2^{3(r+1)+1}m^{r+1}},\end{aligned}$$

and

$$\epsilon_r=\sum_{p=1}^{r}\frac{S_p^i}{2^pm^p}\sum_{l=r-p+1}^{\infty}\binom{-p-i}{l}\frac{1}{2^{3l}m^l}=\sum_{p=1}^{r}\frac{S_p^i}{2^pm^p}\sum_{l=r-p+1}^{\infty}(-1)^l\binom{l+p+i-1}{l}\frac{1}{2^{3l}m^l}$$

$$=\sum_{p=1}^{r}\frac{S_p^i}{2^p}\omega_p(n,i)\binom{r+i}{r-p+1}\frac{(-1)^{r-p+1}}{2^{3(r-p+1)}m^{r+1}}=\Omega_r(n,i)\sum_{p=1}^{r}\frac{S_p^i}{2^p}\binom{r+i}{r-p+1}\frac{(-1)^{r-p+1}}{2^{3(r-p+1)}m^{r+1}},$$

where $0<\sigma_r(n,i),\tau_r(n,i),\omega_p(n,i),\Omega_r(n,i)<1$ for $r\geq 0$ and $1\leq p\leq r$. Moreover $0<\Omega_r(n,i)<1$, because for a fixed r, all the terms with the weights $\omega_p(n,i)$ have the same sign. Since ϵ_r, E_r and R_r always have the same sign, we deduce

$$\epsilon_r + E_r + R_r = \Omega_r(n,i)\sum_{p=1}^{r}\frac{S_p^i}{2^p}\binom{r+i}{r-p+1}\frac{(-1)^{r-p+1}}{2^{3(r-p+1)}m^{r+1}} + \sigma_r(n,i)\frac{S_{r+1}^i}{2^{r+1}}\frac{1}{m^{r+1}}$$
$$+\tau_r(n,i)\frac{(-1)^r}{i}\binom{r+i}{r+1}\frac{1}{2^{3(r+1)+1}m^{r+1}}$$
$$= \kappa_r(n,i)\left\{\sum_{p=1}^{r}\frac{S_p^i}{2^p}\binom{r+i}{r-p+1}\frac{(-1)^{r-p+1}}{2^{3(r-p+1)}m^{r+1}} + \frac{S_{r+1}^i}{2^{r+1}}\frac{1}{m^{r+1}} + \frac{(-1)^r}{i}\binom{r+i}{r+1}\frac{1}{2^{3(r+1)+1}m^{r+1}}\right\},$$

where $0 < \kappa_r(n,i) < 1$. Finally, we have that

$$\epsilon_r + E_r + R_r = \kappa_r(n,i)\sum_{p=0}^{r+1}\frac{S_p^i}{2^p}\binom{r+i}{r-p+1}\frac{(-1)^{r-p+1}}{2^{3(r-p+1)}m^{r+1}} = \kappa_r(n,i)\frac{A_{r+1}^i}{m^{r+1}},$$

which completes the proof. □

Conclusion

This thesis has provided some new results of two problems which were proposed by Ramanujan.

For the first problem, which is Ramanujan's inverse Digamma approximation (2.2). Firstly we gave an explicit formula for computing the coefficients when $n = 1/4$. Secondly, we provided a recurrence relation for the general term of the asymptotic series for every complex number n. Finally, we provided an asymptotic formula for the general term. From our final result, we realized that even though the general terms looks very complicated, their asymptotic behaviour are simple which also shows the divergent character of Ramanujan's series.

For the second problem, which is about Ramanujan's asymptotic formula for the nth Harmonic number. Firstly, we provided an expression for the $\sum_{k=n+1}^{\infty} 1/k^j$ where $j > 1$ with a formula for the general term and a precise error term. Secondly, we provided an expression for the odd powers, which is in terms of the reciprocal of the nth triangular number, with a formula for the general term and a nice error term. This as a result, improved the paper of M. D. Hirschhorn entitled "Ramanujan's enigmatic formula for the harmonic series".

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