Symmetry and Structure of graphs

by

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Submitted to

Central European University

Department of Department of Mathematics and its Applications

In partial fulfillment of the requirements for the degree of Master of Science

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Budapest, Hungary

2014

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Abstract

The thesis surveys results on structure and symmetry of graphs. Structure and symmetry of graphs can be handled by graph homomorphisms and graph automorphisms - the two approaches are compatible. Two graphs are called homomorphically equivalent if there is a graph homomorphism between the two graphs back and forth. Being homomorphically equivalent is an equivalence relation, and every class has a vertex minimal element called the graph core. It turns out that transitive graphs have transitive cores. The possibility of a structural result regarding transitive graphs is investigated. We speculate that almost all transitive graphs are cores. The interplay between graph products, graph retractions and graph cores is described.

Keywords: graph homomorphism, graph core, transitive graphs, graph retraction

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Notations, elementary definitions

I will be using the following notations.

- If I am talking about both graphs and groups in the same context, then I denote graphs with capital latin letters and groups with capital greek letters.
- I generally introduce a graph as $G(V,E)$, the vertex set being $V(G)$ and the edge set being $E(G)$.
- In a graph if u and v vertices are connected by an edge, I denote this relationship by either $u \sim v$ or $[u,v]$ or one of the following: $u \in \mathcal{N}(v)$ or $v \in \mathcal{N}(u)$, meaning that one vertex belongs to the neighbourhood of the other.
- Let G be a graph, u a vertex of it. By $d(u)$ I denote the number of neighbours of u, and call this quantity either the degree or valency of u. To slightly abuse notation, if u,v are two vertices in the graph, then by $d(u,v)$ I denote their distance - the length of the shortest path joining u and v. If no such path exists then the distance is define infinity.
- α stands for the independence number, ω stands for the clique number and χ stands for the chromatic number of a graph.
- Let G be a graph. Then I denote its complement by $\bar{G}$.
- If G is a graph then a subgraph relation is denoted by $\leq$ and an induced subgraph relation is denoted by $\lhd$.
- If X is a set and Γ is a group acting on it, $x \in X$ and $\gamma \in \Gamma$, then I use the $x\gamma$ notation. I denote the fact that Γ acts on X by $X \curvearrowleft \Gamma$.
- By the girth of a graph I mean the shortest cycle in the graph. If the graph is a tree or a forest, then its girth is assumed to be infinity. The odd girth of a graph is the shortest odd cycle in the graph.

- Let $\mathscr{G}(n)$ denote the set of all the finite graphs on n vertices, $\mathscr{G}_P(n)$ denote the set of all the finite graphs on n vertices which have some property P (such as three-colourable, vertex-critical, odd-cycle free, et cetera). We say that almost all graphs have property P if $\lim_{n\to\infty} \frac{|\mathscr{G}_P(n)|}{|\mathscr{G}(n)|} = 1$.

- If G is a graph, then I denote its adjacency matrix by A_G.

- Let n be a positive integer. The symbol $[n]$ denotes the set $\{1, \ldots, n\}$.

Chapter 1

Introduction

This thesis attempts to give an overview of results concerning the structure of finite graphs. My motivation for choosing this topic evolved from studying probability on finite (or infinite but discrete) structures. Most structures used in that subject exhibit very strong symmetry properties which fact I always found intriguing. Symmetry certainly makes arguments much more elegant and simple, however I sometimes felt that imposing very strong symmetry conditions might not be necessary.

Probably the most often used criterion for strong symmetry of graphs is vertex-transitivity. Loosely, this means that the graphs looks exactly the same from every vertex - hence transitivity is a global property of a graph. This usually means that understanding a graph locally is enough, transitivity can extend local observations to hold for an entire object. My question was the following: what happens if we slightly modify a transitive graph locally? Intuitively, not much should have changed in terms of the global structure of the graph, with the exception of a few errors. As expected, this is a very difficult question even to formulate precisely.

This line of thinking turned me towards investigating what tools are available to describe structure of graphs, preferably in a way that is compatible with the group theoretic approach used to describe the symmetries of graphs. Graph homomorphisms and retracts provide such a framework with already a large body of results and many open questions. Graph homomorphisms are mappings that preserve the adjacency relation, hence the local structure of a graph. The study of the existence and counting of graph homomorphisms is closely related to the study of graph colourings and counting proper colourings: graph colouring questions can be reformulated by asking whether a homomorphism exists from a graph to a complete graph. Graph retracts are defined as pairs of homomorphisms from the graph into itself - graphically if a graph contains structural repetition, we could fold some of the repetition into a smaller structure contained in the graph. It turns out that every graph contains a vertex-minimal, unretractive structure that it can be folded back to, this is called the core of the graph.

Graphs with isomorphic cores have homomorphisms between each other back and forth - they can be considered structurally similar. Such graphs are called homomorphically equivalent, and this relation is an equivalence over all finite graphs. A simple but very important result is due to Welzl [31]: every transitive graph has transitive core. This results gives hope that a meaningful distance from transitivity concept could be established based on homomorphically equivalent classes. For that hope to survive however, we need to learn whether homomorphically equivalent classes of transitive graphs are closed enough under natural operations.

A question that occurs naturally is the following: how closed are homomorphically equivalent classes under graph operations that preserve or create structure? Such operations are for example graph products. This area of research still contains many open questions, even for the arguably simpler case of graph colourings. Generally, even seemingly tame structural operations such as the Cartesian product behave erraticly with respect to taking cores. Intuitively this is not very surprising as the Cartesian product is a commutative operation, whereas the core is in some sense the "only structurally dominant part" in the graph. A result aligning with this intuition is the fact that the lexicographic product of graphs (a non-commutative product) is much more well-behaved.

A closely related question is whether we can classify transitive graphs, which is based on the fact that the size of the core of a transitive graph divides the size of the transitive graph - consequently with some luck the graph can be partitioned into copies of the core. This is not always the case, but when it is, I speculate that it might be possible to obtain a natural generalization of every product and thus a classification for such transitive graphs in terms of their cores.

As I am concerned with structure of graphs in general, another important question to answer is how frequent graphs with structure are. I recite the classic result of Erdős and Rényi [7] that almost all graphs are asymmetric. A result in similar flavour is that of Koubek and Rödl [22], telling us that almost all graphs are unretractive. I speculate that almost all transitive graphs are unretractive as well.

The structure of the thesis is as follows. In the first chapter I give brief definitions for graph symmetry, transitivity, automorphism groups and recite some elementary results, mostly without proof. The first chapter is mainly built on Godsil and Royle's book [10]. Chapter two is concerned with ways in which one can create a graph from others in a structured way, namely graph products and similar constructions. I attempted to reduce chapter two to the minimum, more claims will follow about such constructions scattered around the later chapters. Perhaps the most comprehensive source of information on this topic is Hammack, Imrich and Klavžar's book [15]. Chapter three is devoted to symmetry and lack of symmetry of graphs. My main source of information was Babai's survey article [3]. Chapter four is concerned with graph homomorphisms with the main focus being strong graph retracts and graph cores. For this the two main sources were Hahn and Tardif's survey article [14] and Hell and Nešetřil's book [18].

Chapter 2

Graph operations

2.1 Graph join and union

Definition 2.1 (Graph join):

Let G, H be two graphs. Their join, denoted by $G + H$ is defined as follows:

1. $V(G+H) = V(G) \cup V(H)$,

2. $E(G+H) = E(G) \cup E(H) \cup \{[u, v] : u \in V(G), v \in V(H)\}$.

A similar, yet opposite operation is what I will call the disjoint union of graphs.

Definition 2.2 (Graph disjoint union):

Let G, H be two graphs. Their disjoint union, denoted by $G \cup H$ is defined as follows:

1. $V(G \cup H) = V(G) \cup V(H)$,

2. $E(G \cup H) = E(G) \cup E(H)$.

2.2 Cartesian product

Definition 2.3 (Cartesian product):

Let G, H be two graphs. Their Cartesian product, denoted by $G \square H$ is defined as follows:

1. $V(G \square H) = V(G) \times V(H)$,

2. *and* $[(x_1, y_1), (x_2, y_2)] \in E(G \square H)$ *if either*

- $x_1 = x_2$ *and* $[y_1, y_2] \in E(H)$*, or*
- $y_1 = y_2$ *and* $[x_1, x_2] \in E(G)$*.*

Note 2.3.1:

Because the product tends to generate "boxes", we may also call the product Box product for brevity.

Claim 2.4:

The Cartesian product is associative and commutative.

Claim 2.5:

Let (u, v)*,* $(x, y) \in V(G\square H)$*. Then*

$$d_{G\square H}((u, v), (x, y)) = d_G(u, x) + d_H(v, y).$$

Claim 2.6:

Let A_G *be the adjacency matrix of the graph* G *and* A_H *be the adjacency matrix of graph* H*. Suppose that* $|V(G)| = n_G$ *and* $|V(H)| = n_H$*. Then the adjacency matrix* $A_{G\square H} = A_G \otimes I_{n_H} + I_{n_G} \otimes A_H$*.*

Note 2.6.1:

Based on this, the Cartesian product is also called the Kronecker product or the tensor product of graphs.

Theorem 2.7 (Sabidussi-Vizing: Prime factorization):

Every connected graph has a unique representation as a product of prime graphs, up to isomorphism and the order of the factors.

If a graph is not connected then the factorization may not be unique. A counterexample is

$$(K_1 \cup K_2 \cup K_2^2)\square(K_1 \cup K_2^3) = (K_1 \cup K_2^2 \cup K_2^4)\square(K_1 \cup K_2),$$

where $\cup$ is the disjoint union of two graphs - no edges are created between copies.

Note 2.7.1:

The above prime factorization theorem sadly does not tell us too much about the structure of graphs as most graphs are $\square$-primes.

Corollary 2.7.1:

An important consequence of the theorem is the cancellation property: if $G\square K \cong H\square K$ *then* $G \cong H$*. This claim is true even for non-connected graphs.*

Theorem 2.8 (Imrich, Miller):

Let G be a connected graph. Suppose $\phi \in \mathrm{Aut}(G)$ and the prime factor decomposition of G is $G = G_1\Box\ldots\Box G_n$. Then there is a permutation $\pi \in S_n$ and isomorphisms $\phi_i : G_{\pi(i)} \to G_i$ for which:

$$\phi(x_1,\ldots,x_n) = (\phi_1(x_{\pi(1)}),\ldots,\phi_k(x_{\pi(k)})).$$

Corollary 2.8.1:

The automorphism group of the Cartesian product of connected prime graphs is isomorphic to the automorphism group of the disjoint union of the factors.

Claim 2.9:

Let $G = G_1\Box\ldots\Box G_n$. G is transitive if and only if G_i is transitive for all $i = 1,\ldots,n$.

Corollary 2.9.1:

Let G be a transitive graph. Then its $\Box$-factor decomposition is unique.

2.3 Direct product

Definition 2.10 (Direct product or categorial product):

Let G,H be two graphs. The direct product of the two graphs, denoted $G \times H$ is defined as follows:

- $V(G \times H) = V(G) \times V(H)$,
- $(x_1,x_2) \sim (y_1,y_2)$ *if*
  - $x_1 \sim y_1$ *and*
  - $x_2 \sim y_2$.

Claim 2.11:

All the equalities are understood up to isomorphism.

- *The direct product is commutative and associative,*
- *The direct product distributes with the disjoint union:* $G \times (H_1 \cup H_2) \cong G \times H_1 \cup G \times H_2$,
- $G \times K_1 \cong G$, *so the one point graph is a unit.*

Claim 2.12:

A direct product of graphs is connected if at most one of the factors is bipartite. If the product has k bipartite factors then there are 2^{k-1} disjoint components.

Theorem 2.13:

A direct product has a transitive automorphism group if each factor has a transitive automorphism group.

2.4 Strong product

Definition 2.14 (Strong product):

Let G, H be two graphs. The direct product of the two graphs, denoted $G \boxtimes H$ is defined as follows:

- $V(G \times H) = V(G) \times V(H)$,
- $(x_1, x_2) \sim (y_1, y_2)$ *if*
  - $[(x_1, x_2), (y_1, y_2)] \in E(G \square H)$, *or*
  - $[(x_1, x_2), (y_1, y_2)] \in E(G \times H)$.

Claim 2.15:

All the equalities are understood up to isomorphism.

- *The strong product is commutative and associative,*
- *The strong product distributes with the disjoint union:* $G \boxtimes (H_1 \cup H_2) \cong G \boxtimes H_1 \cup G \boxtimes H_2$,
- $G \boxtimes K_1^s \cong G$, *so the one point graph with a loop on it is a unit.*

Claim 2.16:

Let G, H be two graphs, then

$$d_{G \boxtimes H}((a,b),(c,d)) = \max\{d_G(a,c), d_H(b,d)\},$$

and this can be generalized for multiple factors.

Corollary 2.16.1:

A strong product of graphs is connected if and only if every factor is connected.

2.5 Lexicographic product

Definition 2.17:

Let G, H be two graphs. The lexicographic product of the two graphs, denoted $G[H]$ is defined as follows:

- $V(G[H]) = V(G) \times V(H)$,
- $(x_1, x_2) \sim (y_1, y_2)$ *if*
  - $x_1 \sim y_1$ *or*
  - $x_1 = y_1$ *and* $x_2 \sim y_2$.

A special case of the lexicographic product is the graph multiple operation.

Definition 2.18:

Let G be a graph, $n > 0$ an integer. The n-multiple of G is the graph $G[\bar{K_n}]$, the lexicographic product of G with the n-vertex empty graph.

Claim 2.19 (Properties):

All the equalities are understood up to isomorphism.

- *The lexicographic product is not commutative.*
- *The lexicographic product is associative,*
- *the lexicographic product distributes with the disjoint union: $G[H_1 \cup H_2] \cong G[H_1] \cup G[H_2]$ from one side, but not the other.*
- $\bar{G}[\bar{H}] = \overline{G[H]}$.

Claim 2.20:

A lexicographic product of graphs is connected if and only if the leftmost factor is connected.

Theorem 2.21:

A lexicographic product of two graphs has a transitive automorphism group if and only if both factors have a transitive automorphism group.

Chapter 3

Automorphism groups of graphs

Definition 3.1 (Graph automorphism):

Let G be a graph. We say that $\phi : G \to G$ is an automorphism of G if

- *ϕ is a bijection on the set of vertices,*
- *ϕ respectes neighbourhood: if $u \sim v$ then $\phi(u) \sim \phi(v)$.*

Graph automorphisms are special permutations of the vertex set of the graph. The following claim is trivial.

Claim 3.2:

Let G be a graph. The set of all of its automorphisms forms a group with composition. We denote this group by $\mathrm{Aut}(G)$, and call it the full automorphism group of G.

Note 3.2.1:

In the following text I call any group of automorphisms of a graph an automorphism group.

The following claims provide very simple tools to observe how automorphisms act on graphs.

Lemma 3.3:

Let G be a graph, $u, v \in V(G)$ and $\phi \in \mathrm{Aut}(G)$ such that $u\phi = v\phi$. Then $d(u) = d(v)$.

Lemma 3.4:

Let G be a graph, $u, v \in V(G)$ and $\phi \in \mathrm{Aut}(G)$. Then $d(u, v) = d(u\phi, v\phi)$.

Lemma 3.5:

$\mathrm{Aut}(\bar{G}) = \mathrm{Aut}(G)$.

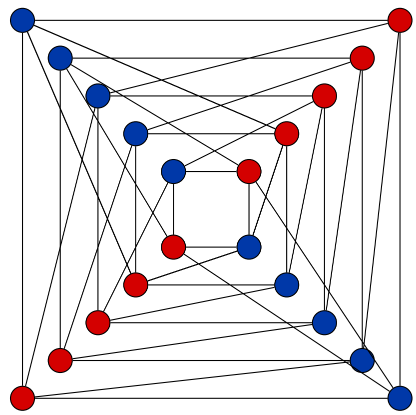

Figure 3.1: The Folkman graph

3.1 Transitive graphs

This section deals with the definitons and most elementary properties of various classes of transitive graphs, following [10].

Definition 3.6 (Vertex-transitive graph):

Let $G(V,E)$ be a graph. We say that it is vertex-transitive, if its automorphism group acts on its vertices transitively.

In other terms, for any vertices x, y there exists $\sigma \in \mathrm{Aut}(G)$ such that $x\sigma = y$.

Definition 3.7 (Edge-transitive graph):

A graph is called edge-transitive, if its automorphism group acts transitively on the edges.

This means that for any two unordered pairs of neighbouring vertices (x, y), (u, v) there is $\sigma \in \mathrm{Aut}(G)$ such that $[x, y]\sigma = [u, v]$. The notation here means that either $x\sigma = u$ and $y\sigma = v$, or $x\sigma = v$ and $y\sigma = u$.

Definition 3.8 (Arc-transitive graph):

Let $G(V,E)$ be a graph. We say that it is arc-transitive, if its automorphism group acts on ordered pairs of its neighbouring vertices transitively[1].

Note 3.8.1:

Edge-transitivity does not imply vertex-transitivity.

Proof: A counterexample is the Folkman graph (see figure 3.1), which is (the smallest) regular graph that is edge transitive, but not vertex-transitive. The Folkman graph is a bipartite graph on 20 vertices.

□

[1]Ordered pairs of vertices are sometimes called arcs, sometimes flags, sometimes directed lines in the literature.

The fact that the counterexample is bipartite is not a coincidence.

Claim 3.9:

Let G be an edge-transitive graph with no isolated vertices. If G is not vertex transitive, then $\mathrm{Aut}(G)$ *has exactly two orbits, and no edge goes inside the orbits.*

Proof:

- Since G is not vertex-transitive, the action of $\mathrm{Aut}(G)$ has at least two orbits on the set of vertices. We need to show that there are no more than two orbits.

- Fix an edge $[x,y]$ such that its endpoints lie in different orbits. Pick orbit Ω_1, a vertex $v \in \Omega_1$, and pick an edge $[u,v]$. By edge-transitivity, there is an automorphism $\phi \in \mathrm{Aut}(G)$ such that $[u,v]\phi = [x,y]$.

- Consequently $v\phi \in \{x,y\}$, thus one of the vertices is in Ω_1 together with v. Doing this for all adjacent pairs immediately shows that there are exactly two orbits.

- Suppose that an edge is contained in one vertex-orbit. By edge transitivity, this edge can be mapped to $[x,y]$. Using the previous argument it is clear that either x or y could then belong to two orbits, which is impossible. Hence the graph is bipartite, and the two colour classes are the two vertex-orbits of the group action.

□

The following claim is evident.

Claim 3.10:

Let G be an arc-transitive graph. Then it is both edge and vertex-transitive.

Claim 3.11:

Let G be an edge and vertex-transitive graph which is not arc-transitive. Then the valency of the graph must be even.

Proof:

- Let $\Gamma \doteq \mathrm{Aut}(G)$, $x \in V(G)$ an arbitrary vertex.

- Pick a neighbour y of x, let Ω be the orbit of the action $V(G) \times V(G) \curvearrowleft \Gamma$ containing (x,y).

- Since G is edge-transitive, for any arc $(u,v) \in V(G) \times V(G)$, there is a $\phi \in \Gamma$ automorphism that maps (u,v) either to (x,y) or to (y,x), say to (x,y).

- Since G is not arc-transitive, $(y,x) \notin \Omega$ as otherwise using vertex-transitivity, we could unite orbits. Consequently Ω is not symmetric.

- This all means that the edge set of G can be written as $\Omega \cup \Omega^T$. In both orbits, the out degree of x must be the same, hence the valency of x must be even.

$\square$

There are many other notions of transitivity, we concern ourselves with one more.

Definition 3.12 (Nonedge-transitivity):

We say that a graph G is nonedge-transitive if its automorphism group acts transitively on unordered pairs of non-adjacent vertices.

3.2 Cayley graphs

Definition 3.13:

Let Γ be a group, C be an inverse closed subset of Γ not containing the identity element. The Cayley-graph $\mathrm{Cay}(\Gamma, C)$ *is defined as follows:*

- *Let $V(\mathrm{Cay}(\Gamma, C)) = \Gamma$, and*

- $E(\mathrm{Cay}(\Gamma, C)) \doteq \{gh : hg^{-1} \in C\}$.

Note that if C is inverse closed, then the obtained graph is naturally undirected, otherwise directedness arises by taking the arcs (g,h) where $hg^{-1} \in C$.

Definition 3.14:

Let Γ be a group and S be an inverse-closed generating set of Γ. A normal Cayley graph is a Cayley graph for which $x^{-1}sx \in S$ for any $x \in \Gamma$, $s \in S$.

Theorem 3.15:

Cayley graphs are vertex transitive.

Proof:

- Right multiplication by any $g \in \Gamma$ is an automorphism: $(yg)(xg)^{-1} = ygg^{-1}x^{-1} = yx^{-1}$ hence $x \sim y$ if and only if $xg \sim yg$.

- The permutations given like this form a subgroup of the full automorphism group of the Cayley graph, and of course the subgroup is isomorphic to Γ.

- For any g, h consider the automorphism created by $x \mapsto xg^{-1}h$. Clearly this maps g to h. Hence the subgroup acts transitively on the Cayley graph.

□

Definition 3.16:

Let Γ be a group, $X \curvearrowleft \Gamma$. We say that the action is semi-regular if the elements of Γ have no fixed points except for the identity.

Note 3.16.1:

By the orbit-stabilizer lemma, all the orbits have size $|\Gamma|$.

Definition 3.17:

Let Γ be a group, $X \curvearrowleft \Gamma$. If the action is semi-regular and transitive then we call it regular. Clearly then $|\Gamma| = |X|$.

Note 3.17.1:

A group acts on itself regularly by right multiplication. Of course this action is transitive and semi-regular.

Lemma 3.18:

Let Γ be a group, C be an inverse-closed subset of Γ not containing the identity. Then the automorphism group of the Cayley-graph $\mathrm{Cay}(\Gamma, C)$ contains a subgroup isomorphic to Γ.

Proof: This is proved by the proof of 3.15. □

Lemma 3.19:

If a group Γ acts regularly on the vertices of a graph G, then G is a Cayley graph for Γ with some inverse closed subset.

Proof:

- Let $u \in V(G)$. Since Γ acts regularly on G, we have a unique ϕ_{uv} such that u is mapped to some other vertex v.
- A natural choice for C is the "neighbour maps": $C \doteq \{g_{uv} : v \sim u\}$.
- If $x, y \in V(G)$, then $x \sim y$ if and only if $xg_{ux}^{-1} \sim yg_{ux}^{-1}$. Hence $u \sim u(g_{uy}g_{ux}^{-1})$.
- Hence $x \sim y$ if and only if $g_{uy}g_{ux}^{-1} \in C$.
- If we identify every vertex x with the group element g_{ux}, then we get a Cayley graph of Γ with C.

□

Claim 3.20:

There are vertex transitive graphs which are not Cayley graphs.

Proof: The Petersen graph and the dodecahedron are two small counterexamples.

The Petersen graph is a graph on 10 vertices, hence the corresponding group has to have 10 elements. By the Sylow theorems, only two such groups exist, C_{10} and D_5. It is straightforward to check that the Petersen graph cannot be a Cayley graph for either of those two. □

Theorem 3.21 (Sabidussi, 1964 [29]):

If G is a vertex-transitive graph then there is some multiple[2] *of G which is a Cayley-graph.*

Proof: We construct a Cayley graph from the automorphism group of G that is the multiple of G.

- Fix a vertex v and let $\Gamma \doteq \mathrm{Aut}(G)$.
- Define $S := \{\sigma \in \Gamma : v \sim v\sigma\}$, the set of automorphisms that move v to a neighbour.
- S is inverse closed: let $[v, v\sigma]$ be an edge, then applying σ^{-1} yields $v \sim v\sigma^{-1}$. The identity is not contained in S as there are no loops. Consequently $\mathrm{Cay}(\Gamma, S)$ is a properly defined Cayley-graph.
- Let $T_x \doteq \{\sigma \in \Gamma : v\sigma^{-1} \sim x\}$, the set of automorphism whose inverse maps v to x.
- Let $\sigma_1 \in T_{x_1}$ and $\sigma_2 \in T_{x_2}$. In the Cayley-graph $\sigma_1 \sim \sigma_2$ if and only if $\sigma_2^{-1}\sigma_1 \in S$.
- $\sigma_2^{-1}\sigma_1 \in S$ if and only if $v \sim v\sigma_2^{-1}\sigma_1$, which is the case if and only if $v\sigma_1^{-1} \sim v\sigma_2^{-1}$.
- By definition of T_x, $v\sigma_1^{-1} \sim v\sigma_2^{-1}$ holds if and only if $x_1 \sim x_2$.
- By transitivity of G, all T_y sets are nonempty and have the same size, as T_y^{-1} is a left coset of the stabilizer of v for any $y \in V(G)$. Thus $V(\mathrm{Cay}(\Gamma, S))$ can be partitioned into sets of same size in such a way that two elements σ_1 and σ_2 belonging to different partitions are neighbours if and only if the corresponding vertices were neighbours in G.
- Thus $\mathrm{Cay}(\Gamma, S) \cong G[\bar{K_n}]$ with $n = |\mathrm{Stab}(v)|$.

□

[2] Multiple is meant in the sense of 2.18: a k-multiple of G is $G[\bar{K_k}]$.

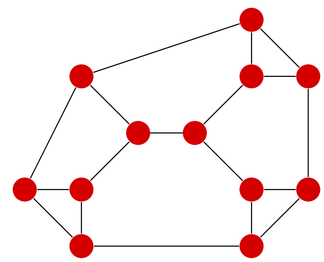

Figure 3.2: Frucht's graph

3.3 Asymmetric graphs

This section deals with graphs that have no symmetries. We follow the paper of Erdős and Rényi [7], describing its main results with some added commentary and examples from various other sources. The results describe how frequent such graphs are, and how much we have to change them to get a graph with some symmetry.

Definition 3.22:

Let G be a graph. We call it asymmetric, if $\mathrm{Aut}(G)$ *is the trivial group.*

Note 3.22.1:

Like transitivity, asymmetry is a global property, the graph has to "look completely different" from every vertex.

Example 3.22.1:

The smallest regular asymmetric graph is Frucht's graph (Fig. 3.2).

Definition 3.23:

Let $G(V, E)$ be a graph. An edit of the graph is adding one edge, or removing one edge in a way that the graph remains connected and no double edge or loop is created.

The degree of asymmetry of a graph, denoted $A(G)$ is the minimal number of edits needed to get a graph with at least one symmetry.

We need the following lemmas.

Lemma 3.24:

$A(G) = A(\bar{G})$.

Proof: A symmetry of a graph is a symmetry of the complement as well (edges go to edges and non-edges go to non-edges).

Suppose that G can be edited in $A(G) = d + a$ steps, d being the number of deletions and a the number of additions. This means that in the complement, applying d additions and a deletions in the corresponding places, we get the complement of the graph G was edited to. Since this process can be done in any direction, we get the equality. $\square$

Lemma 3.25:

Let G be a graph with components G_i, $i = 1, \dots, k$. Then $A(G) \leq \min_{i \in \{1,\dots,k\}} A(G_i)$.

Proof: This claim follows from the fact that the automorphism group of a graph with several components contains the direct product of the automorphism group of the components. $\square$

The following is Erdős and Rényi's first theorem concerning asymmetric graphs. The proof I recite is due to the original authors.

Theorem 3.26:

Let $G(V, E)$ be a simple, connected graph on n vertices. Then

$$A(G) \leq \left[\frac{n-1}{2}\right]$$

Proof:

- Suppose that $n \geq 6$. Let the vertices of the graph be $v_1, \dots, v_n$, and $d_k \doteq \deg v_k$, the number of neighbours of v_k.

- Let d_{jk} be the number of common neighbours of v_j and v_k. We define $d_{jj} = 0$.

- Double counting yields the following connection between degrees and common neighbours:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} = \sum_{k=1}^{n} d_k(d_k - 1). \tag{3.1}$$

  The right hand side counts the number of pairs of edges that share vertex k for each k, the left hand side counts the edge pairs not based on their middle, but on their endpoints.

- Let δ_{jk} be the indicator function of $[v_j, v_k] \in E(G)$.

- Let $\Delta_{jk} \doteq d_j + d_k - 2d_{jk} - 2\delta_{jk}$, the number of not-common neighbours of v_k and v_j. Again define $\Delta_{jj} = 0$.

- Suppose that for any pair v_j, v_k we delete Δ_{jk} edges from the graph. Then we create two vertices that only have common neighbours, hence a transposition is an automorphism of the new graph. This is our approach, we just have to find the minimum Δ_{jk} and hope that we get a nice upper bound on it.

- We clearly have
$$A(G) \le \min_{j\neq k\in\{1,\dots,n\}} \Delta_{jk} \le \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\Delta_{ij}}{n(n-1)}, \tag{3.2}$$
since the minimum element of a set of numbers is always at most as big as the average.

- We have that
$$\sum_{i=1}^{n}\sum_{j=1}^{n}\Delta_{ij} = 2\sum_{k=1}^{n} v_k(n-1-v_k).$$
This can be deduced from the definition of Δ_{jk} and equality 3.1.

- Since
$$v_k(n-1-v_k) = \left(\frac{n-1}{2}\right)^2 - \left(v_k - \frac{n-1}{2}\right)^2,$$
we can get the upper bound
$$2\sum_{k=1}^{n} v_k(n-1-v_k) \le \frac{n(n-1)^2}{2} \text{ if } n = 2t+1,$$
$$2\sum_{k=1}^{n} v_k(n-1-v_k) \le \frac{n\Big((n-1)^2-1\Big)}{2}, \text{ if } n = 2t.$$

- Plugging the previous inequalities into the inequality 3.2, we are done.

□

Note 3.26.1:

The theorem sadly is not as enlightening as it first seems to be. The proof shows that the editing creates the simplest possible type of automorphism, a transposition of two vertices sharing the same neighbourhood.

The proof effectively reduces the problem of creating an automorphism by editing into finding the least amount of edits needed to create two vertices sharing the same neighbourhoods.

Automorphisms that have fairly few fixed vertices tend to imply a large structure in the graph. Transpositions on the other hand imply the most local type of structure that can be rarely exploited in applications or proofs.

The following is a very important theorem: asymmetric graphs are ubiquitous.

Definition 3.27 (Random graph):

Throughout the text, by a random graph on n vertices we mean a probability space on the set of graphs of n vertices, described by mutually independent events $([i,j] \in E(G))$, each having probability $\frac{1}{2}$.

Theorem 3.28 (Erdős, Rényi (1963)):

Let Γ be a random graph on n vertices. Let $\epsilon > 0$ be arbitrary. If $P_n(\epsilon)$ is the probability that the random graph can be edge-edited in $\frac{n(1-\epsilon)}{2}$ steps to yield a nontrivial automorphism, then we have $\lim_{n\to\infty} P_n(\epsilon) = 0$.

Proof:

- Let us denote the vertices of the random graph by $v_1, \ldots, v_n$. In this random graph model, the probability of any graph appearing on n vertices is $2^{-\binom{n}{2}}$. Let $\xi_{j,k}$ be the indicating variable for each edge $[v_j, v_k]$. We define $\xi_{j,k} = \xi_{k,j}$ and $\xi_{j,j} = 0$. Hence for every $j \neq k$, $\xi_{j,k}$ is Bernoulli distributed with parameter $p = \frac{1}{2}$.

- Possible automorphisms of the random graphs are from S_n, so we consider permutations from S_n in their cycle representation with cycle lengths $c_1, \ldots, c_k$. Let $\pi \in S_n$.

- We calculate the probability that a random graph has π as an automorphism. It turns out that this probability is
$$P(G\pi = G) = \frac{2^{\sum_{1\le a<b\le r}\gcd(c_a,c_b)+\sum_{a=1}^{r}\left[\frac{c_a}{2}\right]}}{2^{\binom{n}{2}}},$$
where c_a is the cycle length of cycle a in the cycle representation of the permutation π and r is the total number of cycles in the cycle representation.

  - We need to know how many edges are still free to choose after the choice of π and $\xi_{j,k}$. Clearly if π is an automorphism then $\xi_{j\pi,k\pi} = \xi_{j,k}$ as edges map to edges and non-edges map to non-edges. Iterating, one edge yields the following constraints:
$$\xi_{j,k} = \xi_{j\pi,k\pi} = \xi_{j\pi^2,k\pi^2} = \xi_{j\pi^3,k\pi^3} = \ldots.$$

  - Suppose that j belongs to cycle a and k belongs to cycle b. All of these constraints are repeated after the $\mathrm{lcm}(c_a, c_b)$-th term. Since there are $c_a \cdot c_b$ choices altogether, we get that there remain $\gcd(c_a, c_b)$ free choices.

  - Suppose that j and k both belong to some cycle a. Then there are $\left[\frac{c_a}{2}\right]$ free choices to be made.

  - If there are r disjoint cycles in the cycle representation, then there are
$$\sum_{1\le a<b\le r}\gcd(c_a,c_b) + \sum_{a=1}^{r}\left[\frac{c_a}{2}\right]$$

free choices to be made, hence

$$2^{\sum_{1\le a<b\le r}\gcd(c_a,c_b)+\sum_{a=1}^{r}\lceil \frac{c_a}{2}\rceil}$$

random graphs have π as an automorphism.

- We have the probability for any $\pi \in S_n$ that a random graph has π as an automorphism. Suppose G has π as an automorphism. We are interested in the number of graphs that can be edited in m steps into G. The edges to be edited can be chosen in $\binom{\binom{n}{2}}{m}$ ways, hence the number of graphs that can be edited into a graph having π as an automorphism cannot be more than

$$\binom{\binom{n}{2}}{m}2^{\sum_{1\le a<b\le r}\gcd(c_a,c_b)+\sum_{a=1}^{r}\lceil \frac{c_a}{2}\rceil}.$$

- Let q be the number of points which π does not fix, and $t=n-q$ be the number of points fixed. In the cycle decomposition, every fixed point corresponds to a 1-cycle, and these commute with all the other cycles, so we can define $c_1=\cdots=c_t=1$. Then $q=\sum_{i=t+1}^{r}c_i \le 2(r-t)$, as each cycle over index t is at least of length 2. From this $r\le \frac{n+t}{2}$.

- Since $\gcd(c_a,c_b)\le \frac{c_a+c_b}{2}$,

$$\sum_{1\le a<b\le r}\gcd(c_a,c_b)+\sum_{a=1}^{r}\Big\lceil \frac{c_a}{2}\Big\rceil \le \binom{t}{2}+\frac{(r-t)(n+t)}{2}, \tag{3.3}$$

using the inequality on r again yields:

$$\sum_{1\le a<b\le r}\gcd(c_a,c_b)+\sum_{a=1}^{r}\Big\lceil \frac{c_a}{2}\Big\rceil \le \binom{t}{2}+\frac{n^2-t^2}{4}. \tag{3.4}$$

- Using standard approximations and Stirling's formula, we obtain that the number of graphs that can be edited in at most $\frac{n(1-\epsilon)}{2}$ steps cannot exceed

$$2^{\frac{t^2-n^2}{4}+\mathcal{O}(n\log n)}.$$

We now proceed in function of q, the number of elements not fixed by a selected permutation π. Denote by $P_n(\epsilon,q)$ the probability that a random graph of order n can be edited by changing at most $\frac{n(1-\epsilon)}{2}$ edges into a graph admitting an automorphism π which moves exactly q elements.

- We clearly have $P_n(\epsilon)\le \sum_{q=2}^{n}P_n(\epsilon,q)$.

- The number of permutations that fix t elements is at most $\binom{n}{t}(n-t)! = 2^{\mathcal{O}(n\log n)}$.

- For $n \geq q \geq \sqrt[4]{n}$, we get that

$$\sum_{q=\sqrt[4]{n}}^{n} P_n(\epsilon, q) \leq 2^{-\frac{n^{\frac{5}{4}}}{2}+\mathcal{O}(n\log n)},$$

as $t^2 = n^2 - 2n\sqrt[4]{n} + \sqrt{n}$.

- We now estimate the case when $\sqrt[4]{n} \geq q \geq 5$. The number of permutations leaving $n-q$ elements fixed is at most $\binom{n}{q}q! \leq n^{\sqrt[4]{n}}$ (again, by Sterling's formula). When we edit the graph, we can concentrate on edges that are moved by the automorphism: edges between non-fixed vertices and edges with one fixed and one non-fixed vertex endpoint. Thus an upper bound for the number of choices for edges to be edited:

$$\sum_{m<\frac{n(1-\epsilon)}{2}} \binom{\binom{q}{2}+q(n-q)}{m} \leq n\binom{nq}{\frac{n(1-\epsilon)}{2}} = 2^{f(\alpha)nq+\mathcal{O}(\log n)},$$

where $\alpha = \frac{1-\epsilon}{2q}$ and $f(\alpha) = \alpha\log_2\frac{1}{\alpha} + (1-\alpha)\log_2\frac{1}{1-\alpha}$. f has positive derivative in $(0, \frac{1}{2})$, so it is increasing, and for $q \geq 5$, $\alpha < \frac{1}{10}$, hence $f(\alpha) < 0.47$. Thus

$$P_n(\epsilon, q) = 2^{\mathcal{O}(\sqrt[4]{n}\log n)+\frac{n^2-(n-q)^2}{4}+\binom{n-q}{2}-\binom{n}{2}+0.47nq}.$$

Summing and taking trivial approximations yields:

$$\sum_{q=5}^{\sqrt[4]{n}} P_n(\epsilon, q) = 2^{-0.03qn+\mathcal{O}(\sqrt{n})}.$$

- We turn to transpositions ($q = 2$). There are $\binom{n}{2}$ transpositions, and the number of graphs yielding a transposition is $2^{\binom{n-1}{2}+1}$, hence the probability that a graph yields a transposition is at most $2^{-n+\mathcal{O}(\log n)}$. When we edit the graph to yield a transposition, only edges that have exactly one non-fixed vertex are interesting. Also it is enough to take editing as deleting, since a transposition is only possible if two vertices share the same neighbourhood, and this can be done in the same number of steps by only deleting, or deleting and adding at the same time. Hence, f being defined as previously, we have that the number of graphs that can be edited to yield a transposition in at most m steps is

$$\sum_{m<\frac{n(1-\epsilon)}{2}} \binom{n-2}{m} = 2^{f(\frac{1-\epsilon}{2})n+\mathcal{O}(\log n)}.$$

Since $f(x) < 1$ for every $x \neq \frac{1}{2}$, $P_n(\epsilon, 2) \leq 2^{-c(\epsilon)n}$, where $c(\epsilon)$ is some positive constant only depending on ϵ.

- If $q = 3$, we can only have 3-cycles as automorphisms. There are $2\binom{n}{3}$ such automorphisms, thus there are at most
$$2\binom{n}{3}2^{\binom{n-2}{2}+1-\binom{n}{2}} = 2^{-2n+\mathcal{O}(\log n)}$$
graphs on n vertices yielding exactly two 3-cycles as an automorphism group. Using the same line of thought as before, we can obtain
$$P_n(\epsilon, 3) = \mathcal{O}\left(\left(\frac{\sqrt{3}}{2}\right)^n\right).$$

- For $q = 4$ there are two possible configurations. Either the four non-fixed vertices are moved by a 4-cycle, or they are moved by two disjoint transpositions. We deal with the two cases separately and get that
$$P(G \text{ yields a four-cycle}) \leq 6\binom{n}{4}2^{\binom{n-3}{2}+2-\binom{n}{2}} = 2^{-3n+\mathcal{O}(\log n)}, \tag{3.5}$$
$$P(G \text{ yields two transpositions}) \leq 6\binom{n}{4}2^{\binom{n-2}{2}+3-\binom{n}{2}} = 2^{-2n+\mathcal{O}(\log n)}. \tag{3.6}$$
Using the same techniques as before, we can obtain the number of graphs that can be edited into one of these two types, and finally:
$$P_n(\epsilon, 4) \leq 2^{-0.68n+\mathcal{O}(\log n)}.$$

□

Chapter 4

Graph homomorphism and graph structure

Generally the term "homomorphism" is meant as a "structure-preserving mapping" in many areas of mathematics and it is natural to work with such mappings in case of graphs. Homomorphisms turn out to be the generalization of colourings, which is probably the most important driving force of research.

Our goals are slightly different as we approach homomorphisms with the goal of gaining better understanding of graph retracts, and through retracts a possibly better understanding of the inner structure of graphs. We will refer to some of the colouring results to show the connections and because examples can be quickly drawn from known results on colourings. The key concept of the chapter is the graph core: loosely speaking the core of a graph is the smallest graph onto which the graph can be folded back to, thus it represents an "irreducible structure" in terms of retracts.

In the ending part of the chapter, we concern ourselves with rigid graphs. These are the graphs that exhibit the least structure possible in our framework. Similarly to asymmetric graphs, it turns out that rigid graphs are very frequent: almost every graph is rigid, and almost every graph is a core.

4.1 Homomorphisms

Definition 4.1 (Graph homomorphism and related concepts):

Let G, H be graphs and $\phi : V(G) \to V(H)$ an adjacency preserving mapping. Then

- *We call ϕ a homomorphism from G to H without further restrictions.*

- *We call ϕ a bijective homomorphism if it is a homomorphism and a bijection on the vertex set.*
- *We call ϕ an isomorphism if it is a bijective homomorphism and its inverse is also a homomorphism.*
- *If $H = G$ and ϕ is a homomorphism, we call it an endomorphism.*
- *We call ϕ a faithful homomorphism if it is a homomorphism and $\phi(G)$ is an induced subgraph of H.*
- *We call ϕ a full homomorphism if it is a homomorphism and $[u, v] \in E(G)$ if and only if $[\phi(u), \phi(v)] \in E(H)$.*
- *We call ϕ a complete homomorphism if it is a homomorphism, it is faithful and it is surjective.*
- *If there exists a homomorphism from G to H, we denote this relation by $G \to H$. If there is a homomorphism back and forth between G and H, we denote this by $G \longleftrightarrow H$. If no homomorphism exists from G to H, then we write $G \not\to H$. We denote the set of homomorphisms from G to H by* $\mathrm{Hom}(G, H)$.
- *If $f : G \to H$ is a homomorphism, then the preimages $f^{-1}(y)$ are the fibres of f.*
- *The fibres of f determine a partition of G which we call the kernel of f.*
- *Given G and a kernel partition π, we can define a graph G/π as follows:*
  - *Let $V(G)$ be the sets in the partition.*
  - *Two partition elements v, w are adjacent if there is an edge going between any element of v and w.*

Note 4.1.1:

- If we enable loops in G/π whenever neighbouring vertices appear in the same set of the partition, then there is a natural homomorphism $f_\pi : G \to G/\pi$, the images being given by π. Otherwise π has to contain independent vertices for such a homomorphism to exist.
- Let G be a graph, π_1, π_2 be two partitions of the vertex set, π_1 finer than π_2. Then there is a homomorphism from $G/\pi_1 \to G/\pi_2$ if we enable loops.

I give the following small claims without proofs as most of them are straight consequences of elementary facts and the definitions.

Claim 4.2:

- *The $\to$ relation is transitive: if $G \to H$ and $H \to K$ then $G \to K$. Consequently the $\longleftrightarrow$ relation is transitive as well.*

- *A mapping f from a k-long path $P_k = x_0 e_0 x_1 \dots x_k$ to another graph G is a homomorphism if and only if $f(x_0)f(x_1)\dots f(x_k)$ is a walk in G.*

- *Consequently homomorphisms never increase the distance of two vertices in a graph.*

- *Similarly a mapping f from a k-long cycle C_k to another graph G is a homomorphism if and only if the image is a closed walk in G.*

- *Contrary to groups, there might not be a homomorphism from a graph to another. There is no homomorphism from K_n to its complement. Also, there is no homomorphism from a small odd cycle to a larger one (this is the consequence of the previous small claim).*

- *A bijective homomorphism may not be a graph isomorphism. It could turn out that the inverse is not a homomorphism. Consider K_n and one of its spanning trees. The embedding of the spanning tree into K_n is trivially a homomorphism, and it is bijective on the vertex set, but clearly no homomorphism can map K_n to a tree for $n > 2$.*

- *Endomorphisms form a monoid.*

- *Suppose ϕ is a faithful homomorphism. Pick $u \sim v \in H$. Then there must be an edge going between the sets $\phi^{-1}(u)$ and $\phi^{-1}(v)$.*

- *A graph is k-colourable if and only if there is a homomorphism from it to K_k.*

- *Consequently if $G \to H$, then $\chi(G) \leq \chi(H)$.*

Note 4.2.1:

The last claim gives us the ability to decide the possible existence of a homomorphism between two graphs based on the chromatic number. Clearly if $\chi(G) > \chi(H)$ then there is no possible homomorphism from G to H.

Let G, H be two graphs. Let $m(G, H)$ be the size of a vertex-maximal induced subgraph of G which is homomorphic to H. The proof of the following is due to Hell and Nesetril [18].

Theorem 4.3:

Let G, H, K be graphs, H is vertex-transitive. If $G \to H$, then

$$\frac{m(G,K)}{|G|} \geq \frac{m(H,K)}{|H|}.$$

Proof:

- Suppose $H_1, H_2, \ldots, H_q$ are all the induced subgraphs of G of size $m(H,K)$ who are homomorphic to K.
- Since H is vertex transitive, every vertex of H belongs exactly to the same number of induced subgraphs homomorphic to K. Denote this number p.
- Hence $qm(H,K) = p|H|$.
- Fix a homomorphism $f : G \to H$ and let $G_i \doteq f^{-1}(V(H_i))$.
- By transitivity each G_i is homomorphic to K and each vertex of G belongs to p subgraphs G_i.
- Hence $qm(G,K) \geq p|G|$. Combining this with $qm(H,K) = p|H|$, the claim follows.

□

The previous theorem is the more general version of what is known in the literature as the 'No-Homomorphism Lemma'.

Definition 4.4:

The independence ratio of a graph

$$i(G) = \frac{\alpha(G)}{|V(G)|},$$

where $\alpha(G)$ is the independence number, the size of the largest independent set in G.

Theorem 4.5 (No-homomorphism lemma, Albertson, Collins (1985) [1]):

Let G, H be graph, H vertex transitive and $G \to H$. Then

$$i(G) \geq i(H).$$

Note 4.5.1:

The name comes from the negation: if $i(H) > i(G)$ then there is no homomorphism from G to H.

Note 4.5.2:

If we take K to be the one point graph, the lemma is just a special case of Theorem 4.3. The original proof of the lemma is basically the same as the proof of the more general theorem.

Note 4.5.3:

An easy consequence of the no-homomorphism lemma is the fact that there is no homomorphism from smaller odd cycles to larger ones. Suppose $k < n$ and take

$$\frac{\alpha(C_{2k+1})}{2k+1} \geq \frac{\alpha(C_{2n+1})}{2n+1},$$

if there is a homomorphism from C_{2k+1} to C_{2n+1}. Since $\alpha(C_{2k+1}) = k$ by taking every second vertex, we would have $\frac{k}{2k+1} \geq \frac{n}{2n+1}$ from which $k \geq n$, a contradiction.

Note 4.5.4:

The no-homomorphism lemma requires the computation of the independence number of a graph, which is generally an NP-hard problem, even for transitive graphs. However, for special cases, the lemma makes it easy to verify that no homomorphism can exist from one graph to another, even more so in the special case of transitive graphs (c.f. Corollary. 4.21.1).

We now define special homomorphisms.

Definition 4.6 (Folding):

Let G be a graph. If there is a homomorphism ϕ that identifies exactly two vertices at distance two from each other while fixing all others, we call this homomorphism a simple folding.

A homomorphism is a folding if it is a composition of simple foldings.

Definition 4.7 (Retract):

Let G, H be two graphs. We say that H is a retract of G if there exist two homomorphisms:

- $\Phi : G \to H$ *is called a retraction,*
- $\Psi : H \to G$ *is called a co-retraction,*

if $\Psi\Phi : H \to H$ is the identity mapping of H.

Note 4.7.1:

Some authors define retracts with endomorphisms and fixing the co-retraction to be an identity mapping into G. Then the retraction is a homomorphism which has to act as the identity on H.

Note 4.7.2:

Using the fact that if $G \to H$, then $\chi(G) \leq \chi(H)$ we immediately get that if G can be retracted to H then $\chi(G) = \chi(H)$. The reverse is obviously false: any two odd cycles have chromatic number 3, but there is no homomorphism from the smaller to the larger.

The following claim and its proof is taken from Godsil and Royle's book [10].

Claim 4.8 (Godsil, Royle [10]):

Let G be a connected graph and $H \leq G$. Then if $f : V(G) \to V(H)$ is a retraction, it is a folding, hence a composition of simple foldings.

Proof: The proof is by induction on the number of vertices.

- Let $f : G \to H$ be a retraction, that is f is a homomorphism and $f|_H = \mathrm{id}_H$.
- Since G is connected, there is a vertex $w \in H$ adjacent to some $v \in G$.
- f fixes w while maps v to some neighbour of w. Denote this neighbour of w as $u(\in H)$.
- Let π be the partition of G be all vertices as singletons except for $\{v, u\}$. Define the graph $G_1 \doteq G/\pi$ as in Definition 4.1.
- There is a homomorphism $f_1 : G \to G_1$ with kernel π and f_1 has a finer kernel than f, so there must exist some homomorphism $f_2 : G_1 \to H$ such that $xf_1f_2 = xf$ for every $x \in V(G)$. (No loops can occur here as u and uf were not neighbours by definition.)
- Since f_1 maps each vertex of H to itself, $H \leq G_1$. Note that this is not completely precise. f_1 maps each vertex of H to a singleton set, corresponding to a vertex in G/π, except for u which vertex is mapped to $\{v, u\}$. However, viewing H as $G/\ker(f)$ we can see that $\{v, u\} \subseteq uf^{-1}$, hence there is an isomorphic image of H in G_1 as a subgraph.
- As $f = f_1f_2$ and $H \leq G_1$, we have that f_2 is a retraction from G_1 to H. Since f_1 is a simple folding and we have a retraction, by induction the claim follows.

□

Lemma 4.9:

Let $\phi : G \to G$ be an endomorphism. Then there is some positive integer n for which ϕ^n is a retraction.

4.2 Homomorphic equivalence and ordering

Definition 4.10:

Let G, H be graphs. If there is a homomorphism from G to H we denote this relation by $G \to H$.

If there is a homomorphism back and forth, we say that G and H are homomorphically equivalent and denote this relation with $G \longleftrightarrow H$.

Lemma 4.11:

$\longleftrightarrow$ is an equivalence relation.

Definition 4.12:

Let G be a graph. We denote the homomorphically equivalent class of G with $\mathscr{H}(G)$.

Claim 4.13:

Let G be a graph, H be a multiple of G. Then $G \longleftrightarrow H$.

Claim 4.14:

Theorem 3.21 with Claim 4.13 shows that every transitive graph is homomorphically equivalent to a Cayley graph. In other words, every homomorphically equivalent class of graphs contains a Cayley graph.

Note 4.14.1:

The Cayley graphs do not have a special role in this setup. A group can have Cayley graphs in many homomorphically equivalent classes, and not all transitive classes[1] have a minimal element as a Cayley graph.

4.3 Cores

The concept of core is one of the main concepts in the thesis. Cores are graphs which are "irreducible" in the sense that they cannot be further retracted (or more graphically, folded) into themselves, meaning that all endomorphisms are automorphisms. Cores are minimal elements of homomorphically equivalent classes of graphs. The most important question concerning cores is how much they characterize their homomorphically equivalent classes. It turns out that even for vertex-transitive graphs, this is a very complex problem, with many open questions and possible research directions.

Definition 4.15 (Core of a graph):

Let G be a graph. The graph $G^\star$ is called a core[2] of the graph of G if

- *$G^\star$ is a retract of G,*
- *$G^\star$ is unretractive: no retraction exists on it.*

Note 4.15.1:

Based on the definition we call graphs admitting no retractions cores as well - these are graphs that are the cores of themselves. I will use one word for the two meanings freely, as I believe it might only cause minimal confusion.

[1] See Theorem 4.21.

[2] In the early literature, cores were also called minimal graphs, as in [31] and [9].

Claim 4.16 (Hell, Nešetřil (1992), [17]):

Let G, H be graphs. The following properties hold:

1. *Every finite graph has a core.*
2. *The core of the graph G is unique up to isomorphism.*
3. *The core is an induced subgraph of G.*
4. *If $G \longleftrightarrow H$ then their cores are isomorphic.*
5. *If G is a core graph then it is the unique graph of smallest order up to isomorpism in $\mathscr{H}(G)$.*

Proof:

1. Let $\mathscr{K} \doteq \{H \leq G : \exists \phi : G \to H \text{ homomorphism}\}$ is a finite set and has a minimal element with respect to inclusion.
2. Suppose H_1, H_2 are cores and $f_1 : G \to H_1$ and $f_2 : G \to H_2$ homomorphisms. Since H_2 is a core, the restriction of f_1 to H_2 is a surjective homomorphism, and similarly for f_2 and H_1. Hence the two cores are isomorphic.
3. Any retraction has to act as an automorphism on the core of G. If the core would be a non-induced subgraph, this property would be clearly violated.
4. $G \longleftrightarrow H$ means that there is a homomorphism $f_1 : G \to H$ and $f_2 : H \to G$. Let $H^\star$ be the core of H, and $G^\star$ be the core of G. Clearly $H^\star \longleftrightarrow G^\star$. Since both graphs are cores any pair of back and forth homomorphisms cannot be a retraction. Therefore the sizes of the vertex sets are equal, the homomorphisms are surjective and thus the cores are isomorphic.
5. Suppose there are non isomorphic homomorphically equivalent cores of the same size H_1, H_2. If both back and forth homomorphisms were surjective, the graphs would be isomorphic. Then there must be a proper retraction between the two graphs, hence one of them has bigger vertex set size than the other. This is a contradiction, so all cores in a class must be isomorphic.

□

Example 4.16.1:

1. Odd cycles are cores.
2. Complete graphs are cores.
3. Every planar graph which contains an induced K_4 has K_4 as its core.

4. Let G be a vertex-critical graph, such as the Myczielski graphs, Zykov graphs and stable Kneser graphs (also known as Schrijver graphs). Then G is core.

5. The core of a bipartite graph is either the one point graph or K_2.

Proof:

1. Clearly any odd cycle has chromatic number 3 and contains no induced subgraphs of chromatic number 3.

2. Given a complete graph, all induced subgraphs are complete and of smaller chromatic number.

3. Let G denote our planar graph. By the four colour theorem, there is a homomorphism from G to K_4, and because G contains an induced K_4 we can embed K_4 into G trivially. This gives a retraction, and K_4 is core.

4. If a graph G is vertex critical, then all induced subgraphs have smaller chromatic number hence there can be no homomorphism from G.

$\square$

We have very few necessary conditions on cores, the following is due to Fellner [9].

Claim 4.17 (Fellner (1982), [9]):

Let G be a core graph on n vertices. Then $\omega(G) < \frac{n}{2}$.

Corollary 4.17.1:

Let G be a core graph that is not complete. Then $\bar{G}$ is not bipartite.

Generally, one has to work hard to show that a graph or family of graphs is core. The following claim provides a large set of transitive core graphs. The proof is interesting because it exploits the underlying properties that define the class.

Claim 4.18 ([14]):

Any Kneser graph $K(m,n)$ is a core.

First we obtain the independence numbers of the Kneser family.

Claim 4.19:

$\alpha(K(m,n)) = \binom{m-1}{n-1}$.

Proof:

- Kneser graphs are defined by taking all the $\binom{m}{n}$ n size subsets of $[m]$ as vertices and connecting those who are disjoint.

- Take the complement: vertices are connected in the complement if the corresponding sets intersect. We can use the Erdős-Ko-Rado theorem [2] to find a largest clique in the complement graph: it is at most size $\binom{m-1}{n-1}$.
- The largest clique in the complement graph is the largest independent set in the original, hence we have that $\alpha(K(m,n)) \leq \binom{m-1}{n-1}$.
- The bound is sharp, as seen by $T_k \doteq \{A \in \binom{[m]}{n} : k \in A\}$, $|T_k| = \binom{m-1}{n-1}$ and T_k describes an independent set by definition of the Kneser graph.

$\square$

Proof of Claim 4.18 : We show that if ϕ is an endomorphism on $K(m,n)$, then it has to be an automorphism, thus the graph is unretractive.

- Let ϕ be an endomorphism. If we take the T_k sets defined in the previous proof, then $|\phi^{-1}(T_k)| = \alpha(K(m,n))$ and thus $\phi^{-1}(T_k) = T_{k\psi}$ with some $[n] \to [n]$ mapping ψ.
- Using the definition, for $A \in \binom{[m]}{n}$, $k \in \phi(A)$ if and only if $\psi(k) \in A$. This establishes ψ as a bijection, and hence ϕ is the automorphism induced by ψ.

$\square$

4.3.1 Cores and symmetry

We turn our attention to cores of vertex transitive graphs. The three most important results are Theorems 4.21, 4.22 and 4.22.1, as these hint on a nice structure of vertex-transitive graphs in terms of their cores. Although Example 4.24.1 shows that we have to be cautious with our expectations, I speculate that some structural result in terms of cores can be obtained for vertex transitive graphs.

Claim 4.20:

If two vertex-transitive graphs are homomorphically equivalent then their independence ratio is the same.

Proof: This is the immediate consequence of the No-homomorphism lemma 4.5. $\square$

The following theorem shows that the most important class of graphs for us is closed under taking cores.

Theorem 4.21 (Welzl (1984), [31]):

The core of a vertex transitive graph is vertex transitive.

Proof:

- Let G be the graph, H its core.
- Let ϕ be the retraction, ψ be the co-retraction and $\alpha_{u\psi v\psi}$ an automorphism of G mapping vertices $u\psi$ to $v\psi$ (u, v are in the core).
- Then of course $u\psi\alpha_{u\psi v\psi}\phi = v$ for any pair of vertices in the core. We have constructed an automorphism.

□

Corollary 4.21.1:

A vertex transitive graph and its core have the same independence ratio.

Corollary 4.21.2:

Let G be a vertex transitive graph. If $\alpha(G)$ and $|V(G)|$ are relatively prime then G is a core.

Theorem 4.22 (Hahn, Tardif (1997) [14]):

Let G be a vertex transitive graph and $G^\star$ its core. If $\phi : G \to G^\star$ is a homomorphism then all the fibres $u\phi^{-1}$, $u \in V(G^\star)$ are of the same cardinality.

Proof: The proof is by double counting:

- Let $z \in V(G)$, $\Gamma \doteq \mathrm{Aut}(G)$.
- Let $S \doteq \{(\sigma, g) : \sigma \in \Gamma, g \in V(G^\star), g\sigma\phi = z\}$: the pair of elements in the core and automorphisms that move the element into the fibre of z.
- Obviously $\sigma\phi$ restricted on $G^\star$ is an automorphism of $G^\star$, and thus there is exactly one $g \in G^\star$ such that $z = g\sigma\phi$. Thus since for any automorphism there is exactly one such vertex of the core, $|S| = |\Gamma|$.
- The set of automorphisms mapping an element y to a fibre of z is a coset of the stabilizer of y, thus by the Orbit-Stabilizer lemma we have

$$|\Gamma| = |S| = |V(G^\star)| \cdot |\phi^{-1}(z)| \cdot \frac{|\Gamma|}{|V(G)|},$$

from which the claim follows.

□

Corollary 4.22.1:

Given a transitive graph G, the size of its core $G^\star$ has to divide the size of G.

Corollary 4.22.2:

If a graph G is vertex transitive and the size of its vertex set is a prime number, then G must be a core graph.

The theorem raises a very interesting question: in which cases can a transitive graph G be partitioned into copies of its core? If this is the case, then we could write the adjacency matrix of G in a block form:

$$A_G = \begin{bmatrix} A & B_{12} & B_{13} & \dots & B_{1n} \\ B_{12}^T & A & B_{23} & \dots & B_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ B_{1n}^T & B_{2n}^T & B_{3n}^T & \dots & A \end{bmatrix}. \tag{4.1}$$

For graphs that can be written in this form, retract properties might be possible to be characterised in terms of the blocks B_{ij} which describe how the disjoint copies of the core are connected.

The question has been recently, partially answered by Roberson [27], [26]. We first describe his results concerning cores that have half the size of the original graph.

Claim 4.23 (Roberson (2013)[27]):

Let G be a vertex transitive graph with core $G^\star$ such that $|V(G)| = 2|V(G^\star)|$. At least one copy of $G^\star$ is contained in G, let ϕ be the retraction that maps G to the core, and let G_2 be the induced subgraph of G on the vertices $V(G) \setminus V(G^\star)$. Let B be the bipartite graph consisting of edges of G that go between $G^\star$ and G_2. The following are true.

1. *$G^\star \cong G_2$,*
2. *$\phi|_{G_2}$ is an isomorphism from X_2 to X_1,*
3. *B is regular.*

Proof: First note that since both $G, G^\star$ are transitive, they are regular and thus the half of B that lies in $G^\star$ is certainly regular. Denote the valency of G by d, and the valency of $G^\star$ by d_1.

1. By 4.22, the fibres of the homomorphism ϕ are of the same size for every vertex in $G^\star$. Since $|V(G_2)| = |V(G^\star)|$ and one element of the fibre belongs to $G^\star$, the other belongs to G_1, we have that ϕ is a bijection. Since ϕ is a homomorphism, we have that G_2 is isomorphic to a spanning subgraph of $G^\star$[3]. Consequently G_2 has maximum degree at most d_1 as well, thus every vertex of

[3] As pointed out in Claim 4.2, a bijective homomorphism may not be an isomorphism.

B has to have degree at least $d - d_1$. Since in a bipartite graph the sum of degrees on one side is equal to the other, we get that B is regular and G_2 is regular as well. The claim follows.

2. Since $G_2 \cong G^\star$ and ϕ is an isomorphism that maps G_2 to $G^\star$, we already proved this claim with the previous argument.

3. We have already established regularity of B.

$\square$

For arc-transitive graphs, the following characterization theorem is possible for half-size cores.

Theorem 4.24 (Roberson (2013)[27]):

Let G be an arc-transitive graph, $G^\star$ its core and $|V(G)| = 2|V(G^\star)|$. Then either $G \cong X_1 \square \bar{K}_2$ or $G \cong X_1[\bar{K}_2]$.

These theorems might feed the false hope that it is possible to describe all possible vertex transitive graphs as either cores, or graphs made up of disjoint blocks of cores with some connectivity graph giving the relations between the core layers. This is not the case, as the following example suggests. The example is due to [27].

Example 4.24.1:

Let K_{2n} be a complete graph on an even number of vertices and $L(K_{2n})$ denote its line graph[4] Since K_{2n} is $(2n-1)$-regular, the line graph surely contains a clique of size $2n-1$. An edge colouring of K_{2n} corresponds to a vertex-colouring of $L(K_{2n})$, hence by Vizing's theorem $L(K_{2n}) \to K_{2n-1}$ and since any complete graph is a core, we have found the core of $L(K_{2n})$.

To every vertex of K_{2n} corresponds a $(2n-1)$-clique in $L(K_{2n})$, and every edge in K_{2n} connects two vertices, thus any two cliques of $L(K_{2n})$ intersect in a vertex, hence no core partition is possible.

The reason of failure in this example can be attributed to the fact that the graph is too densely populated by cores to make a partitioning possible. If the size ratio of the graph and its core is n, we need to find exactly n disjoint copies of the core in the graph to make a block structure possible.

Despite our previous counterexample, there is a class of transitive graphs for which the block structure is guaranteed to exist.

Theorem 4.25 (Roberson (2013) [27]):

Let G be a normal Cayley graph and $G^\star$ be the core of G. Then there exists a partition $\{V_1, \ldots, V_m\}$ of $V(G)$ such that each V_i induces a graph in G that is isomorphic to $G^\star$.

[4]A line graph $L(G)$ of a graph G is created by taking the edge set of G as the vertex set of $L(G)$ and two vertices of $L(G)$ are adjacent if the corresponding edges in G are incident.

We will need two lemmas to prove the theorem.

Lemma 4.26:

Let G be a graph, $G^\star$ its core. Suppose ϕ is an endomorphism which maps $x, y \in V(G)$ to the same vertex. Let u, v be two vertices of the core of G. There is no automorphism that maps $\{u, v\}$ to $\{x, y\}$.

Proof:

- Suppose by contradiction that $\alpha \in \mathrm{Aut}(G)$ is such that $u\alpha = x$ and $v\alpha = y$.
- Let ρ be a retraction to some core of G which contains u and v.
- Take two elements a, b which are mapped by ρ to u and v. Then take the automorphism α on u and v to obtain x, y. Then take ϕ to merge the two elements.
- Since the image of ρ was a core and ϕ was an endomorphism that merged two vertices, we obtained a retract of a core graph, a contradiction.

□

Lemma 4.27:

Let $G = \mathrm{Cay}(\Gamma, S)$ be a Cayley-graph. Let ϕ be an endomorphism of $\mathrm{Cay}(\Gamma, S)$ whose images are some core K. If $y \in V(K)$ and $a \in y\phi^{-1}$, then the sets $a^{-1}V(K)$ are pairwise disjoint.

Proof:

- By contradiction suppose that there is a pair $a, b \in y\phi^{-1}$, such that $a^{-1}V(K) \cap b^{-1}V(K) \neq \emptyset$. Equivalently, there exist $c, d \in V(K)$ such that $a^{-1}c = b^{-1}d$.
- Let $\alpha : V(\mathrm{Cay}(\Gamma, S)) \to \mathrm{Cay}(\Gamma, S)$ be defined as $x\alpha = xc^{-1}a$. Since this is a right shift, α is an automorphism.
- Note that $c\alpha = a$ and $d\alpha = b$, since $b = dc^{-1}a$. This is a contradiction with the previous lemma as ϕ maps a and b.

□

Proof of Theorem 4.25 : Let ϕ be a retraction from G to $G^\star$.

- All fibres of ϕ are of the same size and the size of such a fibre has to divide $|V(G)|$, in fact $|V(G^\star)| \cdot |\phi^{-1}(v)| = |V(G)|$ for any $v \in G^\star$.
- From the previous lemma, taking the sets $a^{-1}V(G^\star)$ are disjoint for different $a \in \phi^{-1}(v)$ elements, giving the required partitioning.

- Since X is a normal Cayley graph, left translations are automorphisms as well, hence for any $a \in \phi^{-1}(v)$, $a^{-1}V(G^\star)$ induces an isomorphic image of the core.

□

Roberson notes that only two kinds of vertex-transitive graphs are known yet: those which yield a disjoint block structure, and those who don't but have complete graphs as cores. This, together with a previous remark on the "core-density" property of the counterexample leads me to believe the following.

Speculation 1:

Either a graph is decomposable into a block form described by (4.1)*, or its core is complete.*

It seems that complete cores could have a special role, this is underlined by the following claim.

Theorem 4.28 (Cameron, Kazanidis (2008) [4]):

Let G be a nonedge-transitive graph. Then either the core of G is a complete graph or G itself is a core.

Cameron and Kazanidis gave the following conjecture:

Conjecture 2 (Cameron, Kazanidis (2008) [4]):

Let G be a strongly regular graph[5]. Then either the core of G is complete, or G is a core.

4.4 Product structure

This chapter relies on [15], [3], [14], [23].

There are many results to chose from related to graph homomorphisms and graph products. Our main focus however is to understand graph structure, hence we will mainly try to understand graph products in terms of core graphs.

4.4.1 The Cartesian product

In this subsection I use Cartesian and Box products interchangeably. The main point of this section is to demonstrate that the Cartesian product and retractions are not compatible notions: we can rarely gain any control over the core of a graph if we know it is a Cartesian product and the cores of the factors are known. Even in the case of vertex-transitive graphs, the core of a Cartesian product of two

[5]A strongly regular graph is a regular graph described by two more parameters: every two adjacent vertices have λ common neighbours, and every two nonadjacent vertices have μ common neighbours. Strongly regular graphs can be asymmetric and for example $C_4 \square P_2$ (the 3-dimensional cube) is transitive but not strongly regular.

transitive cores might be surprising.

The easiest case to control is when we take a graph G and investigate how the product $G\square G$ relates to G in terms of retractions.

Claim 4.29:

Let G, H be graphs. Then $G \to G\square H$ and $H \to G\square H$.

Consequently, we are interested whether $G\square G \to G$ holds.

Lemma 4.30:

Let G and H be two graphs. Then $\chi(G\square H) = \max\{\chi(G), \chi(H)\}$.

Definition 4.31:

We say that some graph G is hom-idempotent if $G\square G \to G$.

The following three claims are easy consequences of Theorem 4.40, nevertheless, investigating them separately might show some characteristics of the problem.

Claim 4.32:

Complete graphs are hom-idempotent.

Proof:

- The graph $K_n\square K_n$ is transitive by Claim 2.9, so by symmetry we only concern ourselves with n copies of the same vertex.
- Label K_n with numbers $1,\ldots,n$, and suppose that $K_n\square K_n$ is labeled with ordered pairs in a consistent manner. Note that in $K_n\square K_n$, there are exactly n copies of the vertex 1 (namely the coordinates $(1,.)$), and these form a K_n as well.
- Define the map $f_t : V(K_n\square K_n) \to V(K_n)$ as $f_t(k,l) = (t, l+(k-t))$ where addition is understood modulo n.
- The map "collapses" the graph to the t-th layer in the first coordinate and to ensure that neighbours map to neighbours the collapsing is done while "twisting" the graph.
- We claim that f_t is a homomorphism.
- First of all, for every (t,s), $s = 1,\ldots,n$, f acts as the identity.
- Clearly $(k,s) \sim (k,r)$. These get mapped to $(t, s+(k-t))$ and $(t, r+(k-t))$, obviously neighbours as the target graph is a K_n.

- Take $(l,s) \sim (k,s)$. The image of these is $(t, s+(l-t)), (t, s+(k-t))$. Again, these are neighbours.

$\square$

Note 4.32.1:

The main idea when checking whether a graph can be retracted to K_n is to see whether we can map the vertices in a way that no two neighbours get mapped to the same vertex. If we can do that, we are basically done. This is not very surprising since this is exactly how we colour a graph.
Checking other retracts is more complicated as we also have to worry about the image of edges being edges. This simple remark is underlined by the following claim.

Note 4.32.2:

The claim follows from Lemma 4.30 immediately. Also $K_n \square K_m \to K_n$ if $n > m$.

Claim 4.33:

Odd cycles are hom-idempotent.

Claim 4.34 (Albertson, Collins (1985), [1]):

The Petersen graph is not hom-idempotent, in fact $K(5,2)\square K(5,2)$ *is core.*

The three previous cases would suggest that if G is a core, then either $G\square G$ has G as its core, or $G\square G$ is itself a core. It turns out that the situation is much more complicated: it could happen that the core is a Cartesian product of two (proper) subgraphs of G. For disconnected graphs, there are cases when even this is not true.

The previous result can be generalized for any power of any Kneser graphs.

Theorem 4.35 (Albertson, Collins (1985), [1]):

Let $s > r$ *be two positive integeres.* $\square_{i=1}^{s} K(m,n) \to \square_{i=1}^{r} K(m,n)$.

We now turn to the general question of hom-idempotency. The following results hints that normal Cayley graphs might have an important role again.

Claim 4.36 (Hahn, Hell, Poljak (1995) [12]):

Let $\mathrm{Cay}(\Gamma, S)$ *be a normal Cayley graph. Then* $\phi : \mathrm{Cay}(\Gamma,S)\square\mathrm{Cay}(\Gamma,S) \to \mathrm{Cay}(\Gamma,S)$ *defined by* $\phi(x,y) = xy$ *is a homomorphism, hence* $\mathrm{Cay}(\Gamma,S)$ *is hom-idempotent.*

Proof:

- Normal Cayley graphs have the property that both the left and the right translations are automorphisms of them.

- ϕ acts as a right translation on the left factor and as a left translation on the right factor. Since the graph is a normal Cayley graph, it means that ϕ is a homomorphism. The claim follows.

$\square$

Definition 4.37 (Homomorphism graph):

Let G, H be two graphs. The homomorphism graph $\mathrm{Hom}(G,H)$ *is defined as follows:*

- $V(\mathrm{Hom}(G,H)) = \{\phi : G \to H, \phi \in \mathrm{Hom}(G,H)\}$,
- $\phi \sim \psi$ *if and only if for every* $u \in V(G)$ $\phi(u) \sim \psi(u)$.

Claim 4.38:

Let G, H, K be graphs.

1. *Let* $\phi : G \square H \to K$ *be a homomorphism. Then the map* $\psi : G \to \mathrm{Hom}(H,K)$ *defined by* $\psi(u) = \phi_u(v) \doteq \phi(u,v)$ *is a homomorphism.*
2. *Let* $\psi : G \to \mathrm{Hom}(H,K)$ *be a homomorphism. Then the map* $\phi : G \square H \to K$ *defined by* $\phi(u,v) = \phi_u(v)$ *is a homomorphism where* $\phi_u = \psi(u)$.

Corollary 4.38.1:

K is hom-idempotent if and only if $K \longleftrightarrow \mathrm{Hom}(K,K)$.

Claim 4.39:

Let K be a core. Then $\mathrm{Hom}(K,K)$ *is a normal Cayley-graph.*

Proof:

- If K is a core, then $V(\mathrm{Hom}(K,K)) = \mathrm{Aut}(K)$.
- Two automorphisms, ϕ, ψ are adjacent if and only if they map the same vertex to two a neighbouring pair: $[u\phi, u\psi] \in E(K)$. Consequently $[u, u\psi\phi^{-1}] \in E(K)$, so we can find a Cayley graph which represents $\mathrm{Hom}(K,K) \cong \mathrm{Cay}(\mathrm{Aut}(K), S)$:

  $$S \doteq \{\sigma \in \mathrm{Aut}(K) : [u, u\sigma] \in \mathrm{Aut}(K),\ \forall u \in K\}.$$

  Clearly, the elements of S are the shifts of K.
- We need to show that $\phi\psi\phi^{-1} \in S$ to prove that the Cayley graph is normal. We surely have $[u\phi, u\phi\sigma]$ for $\sigma \in S$ and $\phi \in \mathrm{Aut}(K)$. Hence for any $u \in V(K)$, $[u, u\phi\sigma\phi^{-1}] \in E(K)$, and thus $\phi\psi\phi^{-1} \in S$ as we needed.

$\square$

The previous claims together prove the main result here.

Theorem 4.40 (Hahn, Hell, Poljak (1995) [12]):

A graph is hom-idempotent if and only if it is homomorphically equivalent to a normal Cayley-graph.

4.4.2 The direct product

Conjecture 3 (Hedetniemi's conjecture):

Let G, H be two graphs. Then $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.

4.4.3 The strong product

The following two statements are interesting if contrasted to the results on lexicographic products.

Theorem 4.41 (Imrich, Klavžar):

Let G, H be connected graphs, R a retract of $G \boxtimes H$. Then there are subgraphs $G' \leq G$ and $H' \leq H$ such that $R \cong G' \boxtimes H'$.

Theorem 4.42:

Let G, H be connected, triangle free graphs, R a retract of $G \boxtimes H$. Then there exist retracts $G' \leq G$, $H' \leq H$ such that $R \cong G' \boxtimes H'$. Consequently $(G \boxtimes H)^\star = G^\star \boxtimes H^\star$.

4.4.4 The lexicographic product

The lexicographic product meshes well with the core concept.

Claim 4.43:

Let G, H be connected graphs. Then the core of $G[H]$ is $G'[H^\star]$, where $G' \leq G$ and G' is a core.

Note 4.43.1:

It is important to note that G' is a subgraph of G, not necessarily an induced subgraph. This is similar to the result for the Cartesian product.

Claim 4.44:

Suppose that G and H are both connected graphs and G does not contain any triangles. Then $(G[H])^\star = G^\star[H^\star]$.

4.4.5 Graph join

Claim 4.45 (Fellner (1982), [9]):

Let G, H be graphs. G, H are cores if and only if $G + H$ is a core.

Corollary 4.45.1:

Let G be a graph. G is core if and only if $G + K_n$ is a core as well.

Corollary 4.45.2:

Wheel graphs that are formed by an odd cycle and one middle point are cores, as they can be represented as $C_{2k+1} + K_1$.

4.5 Rigid graphs

Rigid graphs exhibit no structure or symmetry that can be described in terms of automorphisms or endomorphisms. In this section we show that almost all graphs are rigid and almost all graphs are cores - analogously to the result about asymmetric graphs. I recite two proofs, Kötters' proof [21] based on an idea hinted at by Babai [3] and a proof based on the original proof of Koubek and Rödl [22] found sketched in [18].

Definition 4.46:

A graph is called retract-rigid or simply rigid, if it has only the identity as endomorphism.

Note 4.46.1:

Rigid graphs are the asymmetric cores.

Theorem 4.47 (Kötters (2009), [21]):

Let R be the graph property that the graph is unretractive. Then almost all graphs are unretractive:

$$\frac{|\mathcal{G}_R(n)|}{|\mathcal{G}_n|} = \mathcal{O}\left(n^2\left(\frac{3}{4}\right)^n\right).$$

To prove the theorem, we do some preliminary work. In the proof of Erdős and Rényi (3.28), we counted the number of graphs that admit a certain type of automorphism. We are basically doing the same, just for endomorphisms. We first have to get rid of "fake endomorphisms": automorphisms.

Definition 4.48:

We call a $\phi : [n] \to [n]$ mapping idempotent if $\phi^2 = \phi$. Let $\ell_n^\star$ denote the nonidentity idempotent mappings of $[n]$ into itself.

Note 4.48.1:

Since n is finite, for every $\phi : [n] \to [n]$ mapping there are powers a, $a+k$ such that $\phi^a = \phi^{a+k}$. Choose a to be the minimal such number. If there is an idempotent power of ϕ, it can be represented as ϕ^{a+r}, where $0 \le r \le k$. For an idempotent power, we have that $\phi^{a+r} = \phi^{2(a+r)}$, which is only possible if $2(a+r) = a+r+mk$ for some integer m. Consequently $a+r = mk$. Since $0 \le r \le k$, there is exactly one such r thus one idempotent power for any ϕ.

Note that any automorphism has an idempotent power as well, and it is exactly the identity. Hence no automorphisms belong to $\ell_n^\star$.

We have thus establised:

$$\sum_{\phi \in \ell_n^\star} |\{G \in \mathcal{G}(n) : \phi \in \mathrm{End}(G)\}| \ge |\mathcal{G}_R(n)|.$$

Lemma 4.49:

Let ϕ be a nonidentity idempotent endomorphism of the graph G on n vertices. Let $v_1, \dots, v_k$ be the fixed points of ϕ. Denote by C_i the ϕ-preimage of c_i, we call these the ϕ-orbits of G.

We have:

$$|\{G \in \mathscr{G}(n) : \phi \in \mathrm{End}(G)\}| = \prod_{1 \le i < j \le k} (1 + 2^{|C_i||C_j|-1}).$$

Proof:

- An endomorphism cannot map neighbouring vertices to the same vertex over simple graphs, hence there must be no edge going inside any ϕ-orbits.
- Then define $C_{ij} \doteq \{\{u, v\} : u \in C_i, v \in C_j\}$[6], and we immediately get:

$$E(G) = \bigcup_{1 \le i \le j \le k} (E(G) \cap C_{ij}).$$

- Obviously $E(G) \cap C_{i,i} = \emptyset$ as noted before. Since ϕ is an endomorphism, there is a $[c_i, c_j]$ edge if there is an edge anywhere going between C_i and C_j.
- Consequently, there are $2^{|C_i||C_j|-1}$ free choices to be made if $E(G) \cap C_{ij} \neq \emptyset$, as $[c_i, c_j] \in E(G) \cap C_{ij}$.
- Thus, altogether there are really $\prod_{1 \le i < j \le k} (1 + 2^{|C_i||C_j|-1})$ possible choices for edge sets.

□

Proof of Theorem 4.47 : Let ϕ be an idempotent nonidentity endomorphism on n vertices.

- Rewrite the previous lemma as

$$|\{G \in \mathscr{G}(n) : \phi \in \mathrm{End}(G)\}| = 2^{\sum_{1 \le i < j \le k} |C_i||C_j|} \prod_{1 \le i < j \le k} \Big(2^{-|C_i||C_j|} + \frac{1}{2}\Big). \tag{4.2}$$

- Let t_ϕ be the number of ϕ-orbits consisting of only one element (only fixed points) and q_ϕ be the number of ϕ-orbits consisting of more than one elements, thus $t_\phi + q_\phi = k$.
- Suppose (without loss of generality) that when we list the C_i sets, the first q_ϕ are exactly the ϕ-orbits consisting of more than one element. We have four cases based on the size of t_ϕ and q_ϕ respectively.

[6]The formalism might not be completely clear here: C_{ij} consists of unordered pairs of elements from C_i and C_j.

- If $t_\phi \le (n - n^{0.9})$ and $q_\phi \le n^{0.6}$, we have that

$$\sum_{1\le i<j\le k} |C_i||C_j| \le \frac{1}{2}\bigg(\Big(\sum_{i=1}^{k}|C_i|\Big)^2 - \sum_{i=1}^{k}|C_i|^2\bigg) \le \frac{1}{2}\bigg(n^2 - \sum_{i=1}^{k}|C_i|^2\bigg), \tag{4.3}$$

$$\sum_{i=1}^{q_\phi}|C_i|^2 \ge \frac{1}{q_\phi}\Big(\sum_{i=1}^{q_\phi}|C_i|\Big)^2 = \frac{(n-t_\phi)^2}{q_\phi} \ge \sqrt{n}. \tag{4.4}$$

$$\sum_{1\le i<j\le k} |C_i||C_j| \le \frac{1}{2}(n^2 - \sqrt{n}). \tag{4.5}$$

Thus, since the second part of product (4.2) is bounded by one, we have that

$$|\{G \in \mathscr{G}(n) : \phi \in \mathrm{End}(G)\}| \le 2^{\binom{n}{2}} 2^{-\frac{1}{2}(\sqrt{n}-n)}. \tag{4.6}$$

- From here on, instead of the above bound for $2^{\sum_{1\le i<j\le k}|C_i||C_j|}$, we use the trivial bound:

$$2^{\sum_{1\le i<j\le k}|C_i||C_j|} \le 2^{\binom{n}{2}}. \tag{4.7}$$

- If $q_\phi \ge n^{0.6}$, the product term in (4.2) contains $\binom{q_\phi}{2}$ factors of size at least 2, hence

$$\prod_{1\le i<j\le k}\Big(2^{-|C_i||C_j|} + \frac{1}{2}\Big) \le \Big(2^{-4} + \frac{1}{2}\Big)^{\binom{q_\phi}{2}} \le \Big(\frac{9}{16}\Big)^{\frac{n^{1.2}-n^{0.6}}{2}}. \tag{4.8}$$

- If $n - 3 \ge t_\phi \ge n - n^{0.9}$, then there are $q_\phi t_\phi$ factors in (4.2) where $|C_i| \ge 2$ and $|C_j| = 1$. There are two cases. Either there is one nontrivial ϕ-orbit, or there are more. If there are at least two, we can obtain the bound

$$\prod_{1\le i<j\le k}\Big(2^{-|C_i||C_j|} + \frac{1}{2}\Big) \le \Big(\frac{9}{16}\Big)^{t_\phi} \le \Big(\frac{9}{16}\Big)^{n-n^{0.9}}, \tag{4.9}$$

and if there is only one, then $|C_1| \ge 3$, hence

$$\prod_{1\le i<j\le k}\Big(2^{-|C_i||C_j|} + \frac{1}{2}\Big) \le \Big(\frac{5}{8}\Big)^{n-n^{0.9}}. \tag{4.10}$$

Using estimation (4.7), we get

$$|\{G \in \mathscr{G}(n) : \phi \in \mathrm{End}(G)\}| \le 2^{\binom{n}{2}}\Big(\frac{5}{8}\Big)^{n-n^{0.9}}.$$

- If $t_\phi = n-2$, then using the fact that there is exactly one ϕ-orbit of size two, with a similar estimation as in the previous case, we get

$$|\{G \in \mathscr{G}(n) : \phi \in \operatorname{End}(G)\}| \leq 2^{\binom{n}{2}}\Big(\frac{3}{4}\Big)^{n-2}.$$

- It remains to estimate $\sum_{\phi \in \ell_n^\star} |\{G \in \mathscr{G}(n) : \phi \in \operatorname{End}(G)\}|$ for which we need to know how many mappings we have for each case, denote these $M_i(n)$. For the first two cases, n^n will do. For the third case, there are at least $n - n^{0.9}$ fixed points, hence

$$M_3(n) \leq \sum_{l > n-n^{0.9}} \binom{n}{l} l^{n-l} \leq \sum_{l > n-n^{0.9}} n^{2n^{0.9}} \leq n^{2n^{0.9}+1}.$$

$M_4(n) = n(n-1)$ as the number of the fixed points is $n-2$.

- The claim follows as

$$\begin{aligned}
\frac{\mathscr{G}_U(n)}{\mathscr{G}(n)} &\leq \frac{\sum_{\phi \in \ell_n^\star} |\{G \in \mathscr{G}(n) : \phi \in \operatorname{End}(G)\}|}{2^{\binom{n}{2}}} \leq \\
&\leq \frac{M_1(n) \cdot 2^{\binom{n}{2}} \cdot 2^{-\frac{1}{2}(\sqrt{n}-n)} + M_2(n) \cdot 2^{\binom{n}{2}} \cdot \left(\frac{9}{16}\right)^{\frac{n^{1.2}-n^{0.6}}{2}} + M_3(n) \cdot 2^{\binom{n}{2}} \cdot \left(\frac{5}{8}\right)^{n-n^{0.9}} + M_4(n) \cdot 2^{\binom{n}{2}} \left(\frac{3}{4}\right)^{n-2}}{2^{\binom{n}{2}}} \\
&= n^n \left(2^{-\frac{1}{2}(\sqrt{n}-n)} + \Big(\frac{9}{16}\Big)^{\frac{n^{1.2}-n^{0.6}}{2}}\right) + n^{2n^{0.9}+1}\Big(\frac{5}{8}\Big)^{n-n^{0.9}} + n(n-1)\Big(\frac{3}{4}\Big)^{n-2} \\
&= \mathscr{O}\left(n^2 \Big(\frac{3}{4}\Big)^{n-2}\right).
\end{aligned}$$

□

The proof of Koubek and Rödl depends on a combination of results.

Theorem 4.50:

The random graph on n vertices[7] *has with probabiltiy* $1 - o(c^{-\log n})$ *(*$c > 1$ *is a constant, and* $\epsilon > 0$ *is arbitrarily small) the following properties:*

1. *All degrees are at least* $\frac{n}{2}(1-\epsilon)$ *and at most* $\frac{n}{2}(1+\epsilon)$*,*
2. *the number of common neighbours of any two vertices is at least* $\frac{n}{4}(1-\epsilon)$ *and at most* $\frac{n}{4}(1+\epsilon)$*,*
3. *both the largest clique and the largest independent set have fewer than* $2\log(1+\epsilon)n$ *vertices,*
4. *each set of* $m > 30\log n$ *vertices induces a subgraph with at most* $\frac{3}{4}\binom{m}{2}$ *edges,*

[7] c.f. Definition 3.27.

5. *there is a set of $m > 60 \log n$ disjoint pairs of vertices (a_i, b_i), $i = 1, \ldots, m$, such that when each a_i is identified with its pair b_i, the resulting graph has more than $\frac{3}{4}\binom{m}{2}$ edges amongst the a_i vertices.*

Before giving the proof for this theorem, we show the main theorem.

Theorem 4.51:

Let G, H be two graphs on n vertices, satisfying the properties $(1), \ldots, (5)$. Then every homomorphism from G to H is injective.

Proof:

- Let $\phi : G \to H$ be a homomorphism, that is not injective: there exist $x \neq y \in G$, $z \in H$ such that $x\phi = y\phi = z$.
- Let $A \doteq \mathcal{N}(x) \cup \mathcal{N}(y) \subseteq G$, $B \doteq \mathcal{N}(z) \subseteq H$.
- By claims 1 and 2 of 4.50, we have that

$$|A| \geq n(1-\epsilon) - \frac{n}{4}(1+\epsilon) = \frac{3}{4}n - \frac{5}{4}n\epsilon.$$

  B can have at most $\frac{n}{2}(1+\epsilon)$ vertices.
- Since f cannot map neighbours to the same vertex, at most $2\log(1+\epsilon)n$ vertices of A can be mapped to the same vertex in B by Claim 3 of 4.50.
- Consequently there are at least $30 \log n$ vertices in B such that at least two elements are mapped to them, and in the preimage there are at least $m > 60 \log n$ vertices such that there are at least $\frac{3}{4}\binom{m}{2}$ edges among the preimages. Mapping those back with ϕ, we get a contradiction with 4.
- Thus ϕ has to be an injective homomorphism.

$\square$

Corollary 4.51.1:

Almost every graph is a core.

Proof of Theorem 4.50 :

1. Let X_{ij} be the random variable that takes value 1 if ij is an edge and takes value -1 if ij is a non-edge. Clearly, $E[X_{ij}] = 0$. Since expectation is linear, $E[\sum_i \sum_j] = 0$. A special case of Chernoff's inequality [5] tells us that $P[|\sum_j X_{ij} - \frac{n}{2}| \geq \frac{n\epsilon}{2}] \leq e^{-\frac{\epsilon^2 n}{2}}$. Since $\sum_j X_{ij}$ counts the number of neighbours of i, we get the claim.

2. Let i_1, i_2 be fixed vertices. Define $Y_{i_1,i_2,j}$ as the random variable that takes value 1 if $X_{i_1,j} = 1$ and $X_{i_2,j} = 1$, and 0 otherwise. Clearly $P[Y_{i_1,i_2,j} = 1] = \frac{1}{4}$. We use the more general form of the Chernoff inequality now to obtain that

$$P\left[\left|\frac{\sum_{j=1}^{n} Y_{i_1,i_2,j}}{n} - \frac{1}{4}\right| \geq \frac{\epsilon}{4}\right] \leq e^{-2D\left(\frac{1}{4}+\epsilon, \frac{1}{4}\right)n},$$

where $D(x,y) = x \log\frac{x}{y} + (1-x)\log\frac{1-x}{1-y}$, and $D(x,y) > 0$. This proves our claim.

3. This is just a reformulation of the classical result of Erdős on the lower bound for symmetric Ramsey numbers, see [6], [2, p. 2.]. The bound says that we need at least $(1+\epsilon)n$ vertices in the graph to guarantee cliques or co-cliques of size $2\log_2(1+\epsilon)n$.

4. Again we use the Chernoff inequality. Let X_{ij} be the random variable as in the first point. Then $\sum_{1\leq i<j\leq m} X_{ij}$ counts all the edges that can occur in the subgraph of m points, there are $\binom{m}{2}$ summands. Applying the Chernoff inequality, we have that

$$P\left[\sum_{1\leq i<j\leq m} X_{ij} > \frac{3}{4}\binom{m}{2}\right] \leq \exp\left(\frac{-\left(\frac{3}{4}\binom{m}{2}\right)^2}{2\binom{m}{2}}\right) = e^{-cm(m-1)} = \frac{n}{n^{-c_2\log n}},$$

where $c, c_2 > 0$ are constants, thus the last term tends to 0 with the desired rate.

5. We claim that choosing m disjoint pairs of vertices, there are more than $\frac{3}{4}\binom{m}{2}$ such indices that at least one of the following neighbouring relations holds: $a_i \sim a_j$, $b_i \sim b_j$, $a_i \sim b_j$, $a_j \sim b_i$. Hence,the probability we are looking for is $p = 1-\left(\frac{1}{2}\right)^4$. Define a Bernoulli variable X_{ij} for each index pair, which takes value 1 if there is no edge out of the four, and 0 if there is. We can now use the Chernoff bound:

$$P\left[\frac{\sum_{1\leq i<j\leq m} X_{ij}}{\binom{m}{2}} > \frac{1}{4}\right] \leq exp\left(-D\left(\frac{1}{4}, \frac{1}{16}\right)\binom{m}{2}\right) = e^{-cm(m-1)} = \frac{n}{n^{-c_2\log n}},$$

where $c, c_2 > 0$ are constants as $D\left(\frac{1}{4}, \frac{1}{16}\right) > 0$.

□

Chapter 5

Closing remarks

In the thesis, I have organized what I consider the most important facts to know about how graph homomorphisms, graph products and graph automorphisms relate to each other and graph structure in general. I want to stress that there is much more known, I have omitted entire fields of research which merit mention. An excellent comprehensive reference for further results is [18] and [14]. A different aspect that I completely omitted is brought forward in [8] and [20].

As a closing, here are the three questions that I currently find most intriguing, based on the thougths in [27, Chapter 3].

- Can most non-core transitive graphs be described by a block structure where disjoint copies of the core are connected in some pattern? Which are the graphs for which this is possible? My suspicion is that most non-core transitive graphs can be described by a block structure and that the exceptions come from very dense cores.

- Given a transitive core K, generating a graph from it by taking the previous block structure, what is the probability that the resulting graph is transitive? What is the probability that it is a transitive core? Without assuming a transitive core and a block structure, the first question is hopeless to answer, however here I feel there might be some result possible.

- Are almost all transitive graphs cores? My suspicion is that almost all transitive graphs are cores.

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