The Bieberbach Conjecture

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Introduction

The now-proven Bieberbach conjecture (1916) is one of the most famous and challenging problems of mathematics. It considers the class S of univalent functions of the form

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots, |z| < 1.$$

It asserts that for every $f \in S$ and every n $|a_n| \leq n$ holds.
This conjecture was not settled for nearly 70 years. Though there were many attempts, the best results that were achieved were not general. It inspired development of lots of complicated algorithms and tools that became useful in theory of univalent functions. In particular, it motivated the development of the Loewener parametric method in 1923, which became one of the milestones in proof of this conjecture. In 1971 I.M. Milin constructed a sequence of functionals for the coefficients of the expansion $\sum_{k=1}^{\infty} c_k z^k$ for a branch of $log\left(\frac{f(z)}{z}\right)$. Together with Lebedev they assumed that this functionals were nonpositive. They also showed that their conjecture implies the on of Bieberbach. Finally, in 1984 Louis De Branges came up with the proof of this conjecture as well as some stronger ones. He created a functional associated with Lebedev-Milin conjecture, which varied monotonically along Loewener chains. He also suggested and solved a system of differential equations to make this functional manageable. In the end, he used a positivity result for hypergeometric functions to verify it's monotonicity.
But De Branges' original argument was complicated. Many connected important tools and methods were not thoroughly presented in his research, as well as, the applications, which made it not accessible to many readers. Therefore, the main idea of the current thesis is to synthesize all the theory, methods and tools in an clear way and search for recent applications.
This thesis is structured in the following way. In the first chapter we will state the problem itself and consider the proof for the most trivial case $n = 2$. In the following chapter we will discuss additional subclasses of functions, which proved to be very useful in complex analysis. The third chapter will be fully dedicated to the De Branges theorem, which confirms the Bieberbach conjecture. Thus, in chapters four and five, we will consider a very important tool, used in proof of De Branges theorem - the Loewner chains and Loewner differential equations. In the end we arrive to some applications of the Loewner parametric method.

Chapter 1

Elaboration of the problem

First we need to give a definition of univalent functions:

Definition 1.1. *A holomorphic* $1-1$ *function on an open subset of the complex plane is called univalent.*

The well-known Riemann mapping theorem states, that:

Theorem 1.2. *If* D *be an arbitrary simply connected domain in complex plane* $\mathbb{C}$, *which is not the whole plane, then there exists a conformal mapping* $w = f(z) : U \to D$, *where* $U := \{|z| < 1\}$.

Moreover, there exists a unique such mapping that takes D into the origin and has a positive derivative there. This fundamental result allows us to formulate lots of extremal problems in the plane for normalized univalent functions. Therefore properties of such functions became of interest.
Now let us consider the class S of univalent functions of the form

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots, |z| < 1, \tag{1.1}$$

such $f(z)$ are normalized by requiring $f(0) = 0, f'(0) = 1$.

Examples

Let's consider a conformal map $w : U \to D$, where $U := \{|z| \leq 1\}, D := \{\Re w > 0\}$

$$w = \frac{1+z}{1-z} = 1 + 2z + \ldots + 2z^n + \ldots$$

Squaring gives a conformal map $w_1 : U \to \mathbb{C} \setminus (-\infty, 0]$ of the form

$$w_1 = \left(\frac{1+z}{1-z}\right)^2$$

After normalization we obtain a very important function $K_0(z) : U \to \mathbb{C} \setminus (-\infty, -\frac{1}{4}]$

$$K_0(z) = \frac{1}{4}\left[\frac{1+z^2}{1-z} - 1\right] = \frac{z^2}{1-z^2} = z + 2z^2 + \ldots + nz^n + \ldots \tag{1.2}$$

which is also called the *Koebe function*. It and its rotations

$$K_\theta(z) = e^{-i\theta} K_0(e^{i\theta} z) = \frac{z}{1 - e^{i\theta} z^2} \tag{1.3}$$

provide the solution to many extremal problems for the class S.
One of the examples of such results is Koebe's distortion theorem, which state, that:

Theorem 1.3. *For all $f \in S$ inequalities*

$$|f'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \frac{|z|}{(1+|z|)^2} \le |f'(z)| \le \frac{|z|}{(1-|z|)^2} \tag{1.4}$$

hold, with strict equality for all $z \neq 0$ unless f is a Koebe function.

But the more important fact about the Koebe function and it's rotations is that they become of great interest when dealing with coefficient estimates. In particular, in 1916 Bieberbach suggested, that:

Theorem 1.4. *For any $f(z) \in S$*

$$|a_n| \le n, n = 2, 3, \ldots \tag{1.5}$$

and equality holds for any given n only for Koebe function $K(z)$ and its rotations $K_\theta(z)$.

This conjecture became of great interest immediately, as it didn't have any strict confirmation. Bieberbach himself proved his conjecture for the case of $n = 2$.

Theorem 1.5. *For any $f(z) \in S$ $|a_2| \le 2$ and equality holds only for Koebe function $K(z)$ and its rotations $K_\theta(z)$.*

Proof. Let's consider function $F(z) = [f(z^2)]^{-\frac{1}{2}} = \frac{1}{z} - \frac{1}{2}a_2 z + \ldots$. It is univalent in $U \setminus \{0\}$. For $r \in (0,1)$ let consider a circle $U_r = \{|z| < r\}$ and let C_r be the image under F of U_r. Clearly, C_r is a simple closed curve. Switching to polar coordinates, we obtain $F(re^{i\alpha}) = Re^{i\Psi}, 0 < \alpha < 2\pi$. Then, as the area enclosed by C_r is positive, we have

$$\frac{1}{2}\int\limits_{C_r} R^2 d\Psi > 0, \tag{1.6}$$

where the integration is performed along C_r in the counterclockwise direction.

From Cauchy-Riemann equation $R\Psi_\alpha = rR_r$ in polar coordinates it follows that $d\Psi = \frac{r}{R}R_r$. After we substitute it in (1.6), we get

$$-\frac{d}{dr}\int_0^{2\pi} |F(re^{i\alpha})|^2 d\alpha = 4\pi\left(r^{-3} - \mid \frac{1}{2}a_2 \mid^2 r - \dots\right) > 0$$

As $r \to 1$ we deduce that $|a_2| \leq 2$. Equality holds only if $F(z) = z^{-1} - \lambda z$, where $|\lambda| = |\frac{1}{2}a_2| = 1$,and thus $f(z) = K_\theta(z)$. ■

Later, in 1923, using the partial equation, which was later named after him, Loewner proved, that $|a_3| \leq 3$ for every $f \in S$. Then there were developed various methods for injective holomorphic functions, which resulted in tedious proofs for other special cases $n = 4, 5$ and 6 of the Bieberbach conjecture.

Regarding the general case, from (4) and Cauchy's inequality for the coefficients of power series it was obtained, that $|a_n| < en^2$. In 1925 Littlewood settled, that $|a_n| < en$ for all $f \in S$ as $n \to \infty$. This result was improved by FitzGerald and Horowitz in 1970's, who showed, that $|a_n| < 1.07n$. There is also a nice result of Hayman's, that states, that $\lim_{n\to\infty} \frac{|a_n|}{n}$ exists for every $f \in S$ and smaller then 1 unless f is the Koebe function.

Chapter 2

Related questions and their implication to the original one

2.1 Odd functions and Robertson's conjecture

For the functions $f \in S$, mentioned above, it is useful to consider related odd functions of the form

$$f^*(z) = \sqrt{f(z^2)} = b_1 z + b_3 z^3 + \ldots + b_{2n-1} z^{2n-1} + \ldots, b_1 = 1 \tag{2.1}$$

where coefficients correspond to the equations $a_n = b_1 b_{2n-1} + \ldots + b_{2n-1} b_1$.Such functions also belong to the class S.
If f is the Koebe function, then

$$f^*(z) = \sqrt{\frac{z^2}{(1-z^2)^2}} = \frac{z}{1-z^2} = z + z^3 + z^{2n-1} + \ldots$$

Assuming, that every function $f \in S$ can be transformed into corresponding odd function f^*, the question of estimation of coefficients of f^* and connection of this estimates of one for f became of interest as well. In 1932 Littlewood and Paley settled that there exists constant $c \leq 14$, such that for every odd function in S

$$|b_{2n-1}| \leq c, n = 1, 2, \ldots.$$

They also conjectured that the true bound is given by $c = 1$. But this was disproven later, in 1933, by Fekete and Szego. They found an example of odd function from S for which $|b_5| > 1$.
Later Robertson settled a bridge between the Bieberbach conjecture for functions in S and estimations for coefficients of odd functions:

Proposition 2.1. *Bieberbach conjecture for function $f \in S$ with coefficients a_i would instantly follow from the inequality*

$$\sum_{k=1}^{n} |b_{2k-1}|^2 \leq n \tag{2.2}$$

on coefficients for a corresponding odd function.

This became known as Robertson conjecture.

2.2 An exponentiation approach and Lebedev-Milin conjecture

Another related subclass of functions considered along with the odd ones is subclass of logarithmic transforms of the injective holomorphic functions. Let's consider a function $\frac{f(z)}{z}$. It's holomorphic and zero-free on U. We take a branch of $\log\left(\frac{f(z)}{z}\right)$ and focus on the expansion

$$\log\left(\frac{f(z)}{z}\right) = \sum_{k=1}^{\infty} c_k z^k \tag{2.3}$$

for f in S. In case, if f is the Koebe function, we have that $c_k = \frac{2}{k}$.
In 1920, Nevanlinna considered the case for the image domains $f(U)$ that are star-shaped relative to the origin. Here a geometric argument at the boundary shows that

$$\Re z\frac{f'(z)}{f(z)} = \Re\left(1 + \sum_{k=1}^{\infty} k c_k z^k\right).$$

Hence, by a well-known Caratheodory inequality for functions with positive part, he obtained that $k|c_k| \leq 2$ or $|c_k| \leq \frac{2}{k}$.
As mentioned above, the inequality $|c_k| \leq \frac{2}{k}$ or $k|c|^2 - \frac{4}{k} \leq 0$ is not true for every f in S. But, in 1971, Lebedev and Milin conjectured that it might be true in the following average sense:

$$\Omega_n = \sum_{p=1}^{n-1}\sum_{k=1}^{p}\left(k|c_k|^2 - \frac{4}{k}\right) = \sum_{k=1}^{n-1}\left(k|c_k|^2 - \frac{4}{k}\right)(n-k) \leq 0 \tag{2.4}$$

for $n = 2, 3, \ldots$ and all $f \in S$.

Examples

Based on De Branges' method described in Chapter 3, it is easy to prove the Lebedev-Milin conjecture for $n = 2$ and $n = 3$. Let's see it's connection with Bieberbach's one.
For $n = 2$ from (2.4) we obtain $|c_1|^2 \leq 4$ or $|c_1| \leq 2$. Since

$$\frac{f(z)}{z} = 1 + a_2 z + a_3 z^2 + \ldots = \exp(c_1 z + c_2 z^2 + \ldots) = 1 + c_1 z + \left(\frac{1}{2}c_1^2 + c_2\right)z^2 + \ldots, \tag{2.5}$$

after comparing corresponding coefficients, we immediately get Bieberbach's inequality:

$$|a_2| = |c_1| \le 2.$$

For $n = 3$ (2.4) gives inequality $|c_1|^2 + |c_2|^2 \le 5$. Since

$$\frac{f(z)}{z} = 1+a_2z+a_3z^2+\ldots = \exp(c_1z+c_2z^2+\ldots) = 1+c_1z+\left(\frac{1}{2}c_1^2+c_2\right)z^2+\ldots,$$

we obtain that

$$|a_3| = \left|\frac{1}{2}c_1^2 + c_2\right| \le \frac{1}{2}|c_1|^2 + |c_2| \le \frac{5}{2} - \frac{1}{2}|c_2|^2 + |c_2| \le 3 - \frac{1}{2}(|c_2|-1) \le 3$$

For a general case there is another very useful inequality for the coefficients of

$$\sum_{k=0}^{\infty} \beta_k z^k = \exp\left(\sum_{k=1}^{\infty} \gamma_k z^k\right)$$

which was proved by Lebedev and Milin.
It asserts that

$$\sum_{k=0}^{n-1} |\beta_k|^2 \le n \exp\left\{\frac{1}{n}\sum_{p=1}^{n-1}\sum_{k=1} p\left(k|\gamma_k|^2 - \frac{1}{k}\right)\right\}, n = 1, 2, \ldots \tag{2.6}$$

Let's consider identity function

$$b_1 + b_3z + \ldots + b_{2n-1}z^{n-1} + \ldots = \frac{f^*(z)}{z^{\frac{1}{2}}} =$$
$$= \left(\frac{f(z)}{z}\right)^{\frac{1}{2}} = \exp\left(\frac{1}{2}\log\frac{f(z)}{z}\right) = \exp\left(\frac{1}{2}\sum_{k=1}^{\infty} c_k z^k\right)$$

Using (2.6) one concludes that

$$|b_1|^2 + \ldots + |b_{2n-1}|^2 \le n \exp\left(\frac{\Omega_n}{4n}\right) \tag{2.7}$$

Thus, we obtain

Theorem 2.2. *If the Lebedev-Milin conjecture (2.4) holds for $f \in S$ and a certain n, then the Robertson conjecture (2.2) holds for corresponding f^* and same n, so that also the Bieberbach conjecture (1.5) must be true for the same n. Moreover, if $\Omega_n < 0$ for some n, then one has a strict inequality in (2.4) and hence also in (2.2).*

Chapter 3

De Branges Theorem

In 1984 Louis De Brange has proved Lebedev-Milin conjecture (2.4) and thereby also the Robertson (2.2) and the Bieberbach (1.5) conjectures. We will present his ideas in a less complicated way synthesizing several articles of Fitzgerald and Pommerenke [3], of Grinshpan [4] and of Korevaar [5].

Theorem 3.1 (L.De Branges)**.** *Let $f : U \to D$ be from S, let the power-series coefficients a_n be defined as in (1.1) and logarithmic coefficients c_k by (2.3). Then the conjecture Lebedev - Milin inequality (2.4) and hence the conjectured Bieberbach inequality (1.5) are true for every $n \geq 1$. Equality in (1.5) and hence in (2.4) holds for $n \geq 2$ if and only if f is a Koebe function (1.2).*

Proof. To make it more accessible for the reader the proof will be split into several steps.

- **Step 1:** We make D nice.
  For proof of Lebedev - Milin conjecture (2.4), it may be assumed that f maps U onto a domain D, which is bounded by analytic Jordan curve. Indeed, for any $f \in S$ and $0 < \rho < 1$ we can define another map

$$f^1(z) = \frac{1}{\rho} f(\rho z) = z + a_2 \rho z^2 + \ldots + a_n \rho^{n-1} z^n + \ldots$$

  This function maps U onto the domain $\frac{1}{\rho} f(\rho U)$. This domain is bounded by analytic Jordan curve, which is $(\frac{1}{\rho})$ times the image of the circle $|z| = \rho$ under f. As

$$\log \frac{f^1(z)}{z} = \log \frac{f(\rho z)}{\rho z} = \sum_{k=1}^{\infty} c_k \rho^k z^k,$$

  we obtain that $c_k^1 = c_k \rho^k$ for f^1. Hence if (2.4) has been proved for c_k^1, then it follows for the coefficients c_k by letting ρ tend to 1.

- **Step 2:** Loewner chains.
Given $D = f(U)$ as in step 1, it is easy to construct a continuously increasing family of connected domains $D_t, 0 \leq t < \infty$, such that

$$D_0 = D, D_s \subsetneq D_t, \text{if } s < t \text{ and } \lim_{t\to\infty} D_t = \mathbb{C} \tag{3.1}$$

For this family of domains we can define a Loewner chain (see Chapter 4) $f_t(z) : U \to D_t, f_t(z) = f(z,t),\ o \leq t < \infty$ starting at $f(z)$, such that

$$f_t(0) = 0,\ f_t'(0) > 0$$

Assuming $f_t'(0) = e^t$, we get the corresponding family of functions

$$f_t(z) = f(z,t) = e^t(z + a_2(t)z^2 + \ldots),\ 0 \leq t < \infty;\quad f_0(z) = f(z) \tag{3.2}$$

The functions $f_t(z)$ of Loewner chain satisfy the partial differential equation of Loewner:

$$\frac{\partial f}{\partial t} = z\frac{\partial f}{\partial z}p(z,t), \tag{3.3}$$

where

$$p(z,t) \text{ is analytic in z, } \Re p(z,t) > 0, p(0,t) = 1. \tag{3.4}$$

- **Step 3:** Logarithmic coefficients for $\frac{f(z,t)}{e^t}$.
As it was mentioned in Chapter 2 it's natural to consider the expansions

$$\log\frac{f(z,t)}{e^t z} = \sum_{k=1}^{\infty} c_k(t)z^k, \tag{3.5}$$

where $\frac{f(z,t)}{e^t z} \in S$. So there exist some constants A_k, such that $|a_k(t)| \leq A_k, \forall t$. Hence by recursion from (2.5) there will be constants C_k such that

$$|c_k(t)| \leq C_k, \forall t \tag{3.6}$$

This relation can be differentiated with respect to t and with respect to z. We substitute it to Loewner equation (3.3), setting

$$p(z,t) = 1 + 2\sum_{k=1}^{\infty} d_k(t)z^k. \tag{3.7}$$

Thus, equating the coefficients of like powers of z, we obtain the system of differential equations

$$c_k'(t) = 2d_k(t) + kc_k(t) + 2\sum_{j=1}^{k-1} jc_j(t)d_{k-j}(t), k = 1, 2, \ldots. \tag{3.8}$$

- **Step 4:** The auxiliary functional Ω.
  For fixed n we introduce the auxiliary functional

$$\Omega(t) = \Omega_n = \sum_{k=1}^{n-1} \left(k|c_k(t)|^2 - \frac{4}{k} \right) \sigma_k(t), \tag{3.9}$$

  where weight functions $\sigma_k(t)$ are chosen in a suitable manner. It's desired that relation $\Omega(0) \leq 0$ will be the Lebedev-Milin conjecture (2.4). For that we need to show that $\Omega(t)$ is a non-decreasing function of t, which vanishes at $t \to +\infty$, i. e.

$$\Omega'(t) \geq 0 \quad for \quad 0 \leq t < \infty, \tag{3.10}$$

  while $\Omega(t) \to 0$ as $t \to \infty$.

  Therefore we obtain the following properties of σ_k:

  - Considering $c_k(0) = c_k$ we get

$$\sigma_k(0) = n - k, k = 1, \ldots, n-1 \tag{3.11}$$

  - As every $c_k(t)$ is bounded (3.7), it's sufficient that

$$\lim_{t \to \infty} \sigma_k(t) = 0, k = 1, \ldots, n-1, \tag{3.12}$$

    So that (3.10) is satisfied.

  - De Branges conditions:

$$\sigma_k - \sigma_{k+1} = -\left(\frac{\sigma'_k}{k} + \frac{\sigma'_{k+1}}{k+1} \right), k = 1, 2, \ldots, n-1; \;\; \sigma_n \equiv 0. \tag{3.13}$$

- **Step 5:** Differential equation conditions on the σ_k.
  Using the differential equations (3.8) for $c_k(t)$ and property (3.13) of σ_k, while calculating $\Omega'(t)$, we obtain that it can be written in the form

$$\Omega' = -\sum_{k=1}^{n-1} Q_k(c, d) \sigma'_k(t), \tag{3.14}$$

  where Q_k are nonnegative functions of the $c_k(t)$ and the $d_k(t)$. Hence we would like to indicate the precise form of $Q_k(c, d)$. Using Herglotz representation for holomorphic functions on the unit disc with positive real part, we obtain that

$$p(z, t) = \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu_t(\theta),$$

where μ_t is a positive Borel measure of total mass equal to $p(0,t) = 1$. So from (3.7) we get that

$$d_k(t) = \int_{-\pi}^{\pi} e^{-ik\theta} d\mu_t(\theta).$$

Introducing the sums

$$S_k = \sum_{j=1}^{k} jc_j(t)e^{ij\theta}, \; S_0 = 0 \tag{3.15}$$

we obtain that $kc_k(t) = (S_k - S_{k-1})e^{-ik\theta}$. Thus we can rewrite (3.8) in the following form:

$$c_k' = \int_{-\pi}^{\pi} (2 + S_{k-1} + S_k)e^{-ik\theta} d\mu_t(\theta)$$

Applying this result to (3.14) we get

$$Q_k(c,d) = \frac{1}{k} \int_{-\pi}^{\pi} |2 + S_{k-1} + S_k|^2 d\mu_t \geq 0. \tag{3.16}$$

- **Step 6:** Explicit form of the σ_k'.
  Condition (3.10) for Ω' given in the form (3.14) will hold, if we could guarantee that
  $$\sigma_k' \leq 0, k = 1, \ldots, n-1. \tag{3.17}$$
  A very important observation is that the first σ_{n-1} and the next $\sigma_{n-2}, \ldots, \sigma_1$ are completely determined by the system of differential equations (3.13) and the initial conditions (3.11).

Examples

For $n = 2$ one has $\sigma_2 \equiv 0$. From (3.13) we obtain that $\sigma_1 = e^{-t}$. Thus $\sigma_1' \leq 0$ and Lebedev-Milin inequality (2.4) holds. Therefore the Bieberbach conjecture (1.5) also holds.

For $n = 3$ one has $\sigma_3 \equiv 0$. After solving the system of differential equations

$$\sigma_2 - \sigma_3 = -\left(\frac{\sigma_2'}{2} + \frac{\sigma_3'}{3}\right)$$
$$\sigma_1 - \sigma_2 = -\left(\sigma_1' + \frac{\sigma_2'}{2}\right),$$

obtained from (3.13), we see that $\sigma_2 = e^{-2t}$, $\sigma_1 = 4e^{-t} - 2e^{-2t}$. Here again $\sigma_k \le 0$, $k = 1, 2$, which proves that the Lebedev-Milin inequality (2.4) and hence the Bieberbach inequality hold (1.5).

For general n De Branges found solution of his system of differential equations (3.13) with initial conditions (3.11) in a form

$$\sigma_k(t) = k \sum_{\nu=0}^{n-k-1} (-1)^\nu \frac{(2k+\nu+1)_\nu (2k+2\nu+2)_{n-k-1-\nu}}{(k+\nu)\nu!(n-k-1-\nu)!} e^{-\nu t - kt}, \tag{3.18}$$

$k = 1, \ldots, n-1$, where

$$(a)_\nu = a(a+1)\ldots(a+\nu-1) \text{ for } \nu \ge 1,\ (a)_0 = 1.$$

Taking derivative of (3.18) we obtain

$$-\frac{\sigma_k'}{k} e^{kt} = \sum_{\nu=0}^{n-k-1} (-1)^\nu \frac{(2k+\nu+1)_\nu (2k+2\nu+2)_{n-k-1-\nu}}{\nu!(n-k-1-\nu)!} e^{-\nu t} \tag{3.19}$$

For relatively small n it could be immediately verified that sums from(3.19) are positive on $(0, \infty)$, which proves Lebedev-Milin inequality (2.4) and hence the Bieberbach inequality hold (1.5). But what happens when we have larger values of n?

After series of researches it was discovered that those sums from(3.19) are generalized hypergeometric functions of a very special type, which are known to be positive. Therefore De Branges proof of Lebedev-Milin conjecture (2.4) was complete.

- **Step 7:** The case of equality.
  Let us consider arbitrary function $f \in S$ and an associated Loewner chain. It's obvious that equality holds for (2.4) if and only if $\Omega' \equiv 0$. Since $\sigma_k(t) < 0$ on $(0, \infty)$ for $1 \le k \le n-1$, then it's required that $Q_k(c, d) \equiv 0$ and $Q_1(c, d) \equiv 0$, in particular. From (3.16) with positive μ_t we get

$$2 + S_1 = 2 + c_1(t)e^{i\theta} = 0,\ a.\ e.$$

Thus absolutely continuous part of μ_t must be zero. It follows that $|c_1(t)| \equiv 2$ or $|c_1| = 2$ and hence $|a_2| = 2$. Then from Theorem 1.5 f is Koebe function.

■

Chapter 4

Loewner equation

As mentioned before the Bieberbach conjecture inspired development of lots of complicated algorithms and tools that became useful in theory of univalent functions. In particular, it motivated the development of the Loewner parametric method in 1923, which allowed him to prove the first non-elementary case of this conjecture. This theory was later developed by other authors in a way that it helped to solve many extremal problems on the class S. Lets briefly consider this method.
First we need to give a definition of single-slit mappings:

Definition 4.1. *A holomorphic function mapping the unit disc $U \subset \mathbb{C}$ onto the complement in $\mathbb{C}$ of the Jordan arc is called a single-slit mapping.*

In his paper Loewner proved that the class of single-slit mappings is a dense subset of the class S of all normalized univalent functions f in the unit disk. This argument was called Loewner representation theorem for single-slit mappings. The proof of this important property can be found in [2]. Changing it slightly we'll show that some stronger argument is true.

Lemma 4.2. *To each $f \in S$ there corresponds a sequence of single-slit mappings $f_n \in S, n = 1, 2, \ldots,$, such that $f_{\rightarrow} f$ uniformly on compact subsets of U as $n \to \infty$, and the boundary of each $f_n(U), n \geq 1$, contains a subray of the negative real axis.*

Proof. As in the first step of proof of De Branges theorem (see Chapter 3) it may be assumed that f maps U onto a domain D, which is bounded by analytic Jordan curve C. Thus there exists a subray L of the negative real axis that belongs to the complement of $\bar{D}$ except for its endpoint $w_L \in C$. Let J_n be a Jordan arc that runs from infinity along L to the point w_L and then along C to a point w_n. Let G_n be the complement of J_n and $g_n : U \to G_n$, such that $g_n(0) = 0,\ g_n'(0) > 0$. The endpoints w_n are chosen in the such way that $J_n \subset J_{n+1}$ and $w_n \to w_L$.
Then $D \subset \bigcap\limits_{n=1}^{\infty} G_n$, D contains the origin. According to the Caratheodory convergence theorem, $g_n \to f$ uniformly on compact subsets of U as $n \to \infty$. Therefore

$$g_n'(0) \to f'(0) = 1.$$

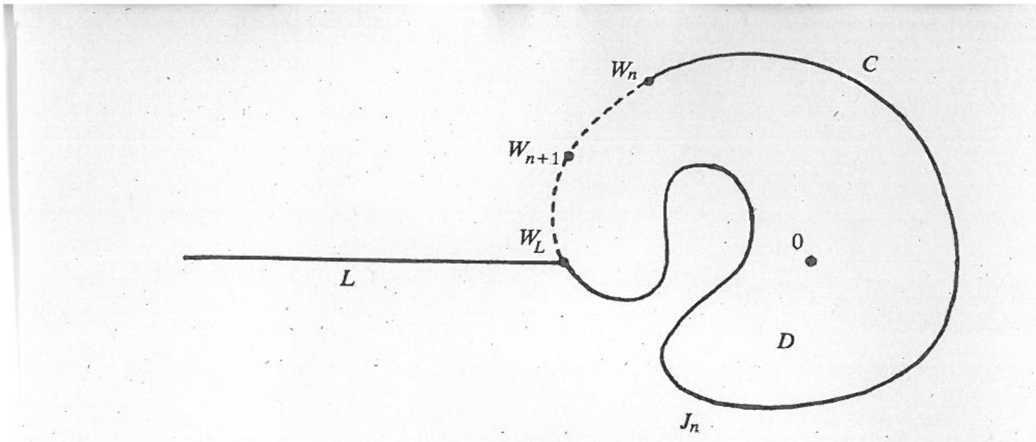

Figure 1

So we may take $f_n = \frac{g_n}{g_n'(0)},\ n \geq 1.$ ■

Thus, the single-slit mappings that does not include a subray of the negative real axis are dense in S.
Along with the representation theorem, Loewner also introduced a method to parametrize single-slit maps.
Let f be a single-slit map, whose image in $\mathbb{C}$ avoids the Jordan arc $J = \{V(t) : 0 \leq t < \infty\}$ extending from $V(0)$ to infinity. For each $t > 0$ let $f_t(z) = f(z,t)$ denote the $1-1$ unique map of U onto the plane less the portion of J from $V(t)$ such that $f_t(0) = 0$ and $\frac{\partial f_t(0)}{\partial z} > 0$, and let $f_0(z) = f(z)$. The parametrization $V(t)$ can be chosen so that $\frac{\partial f_t(0)}{\partial z} = e^t,\ t > 0$. In this case, it is easy to see that the images of f_t form an increasing family of simply connected domains. Such family of mappings (f_t) is called a *Loewner chain*. Again, to the Loewner chain (f_t) we can associate a family $(\phi_{s,t} = \phi(z,s,t) := (f_t^{-1} \circ f_s(z))$, for $0 \leq s \leq t$ of holomorphic self-maps of the unit disc U into itself, but not onto itself, such that 0 is carried to 0. Hence by Schwarz lemma

$$|\phi(z,s,t)| < |z| = |\phi(z,s,s)| \text{ for all } z \neq 0.$$

Let us assume that $\frac{\partial \phi}{\partial t}$ exists and is analytic in z. Then the angle between the vector $\frac{\partial \phi}{\partial t}$ for $t = s$ and vector $-z$ must be bounded by $\frac{1}{2}\pi$. It follows that

$$\frac{\partial \phi}{\partial t}\,|_{t=s} = -zp(z,s), \tag{4.1}$$

where $p : U \times [0,\infty) \to \mathbb{C}$ is a *normalized parametric Herglotz function*, which satisfies the following conditions:

- $p(0,z) \equiv 0$,
- $p(z,t)$ is holomorphic for all $t \geq 0$,

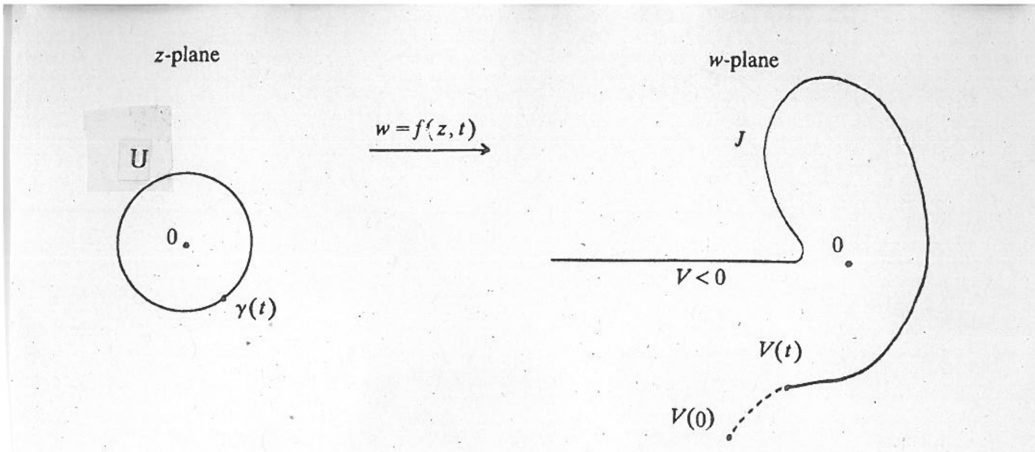

Figure 2

- $p(z, t)$ is measurable for all $z \in U$,
- $\Re p(z, t) \geq 0$ for all $t \geq 0$ and $z \in U$.

Equation (4.1) is also called *general radial Loewner ODE.*
From the definition of ϕ,

$$f_t \circ \phi_{s,t} = f_s(z)$$

Differentiating with respect to t and setting $t = s$, we obtain

$$\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial z}\frac{\partial \phi}{\partial t} = 0 \text{ for } t = s \tag{4.2}$$

Combination of (4.1) and (4.2) give an equation

$$\frac{\partial f_t}{\partial t} = z\frac{\partial f_t}{\partial z}p(z, t) \text{ for } t = s. \tag{4.3}$$

This equation is called *general radial Loewner PDE.*
General radial Loewner equations yield a one-to-one correspondence between Loewner chains and normalized parametric Herglotz functions. They are also important as they allow one to get estimates and growth bounds for f_t and $\phi_{t,s}$ starting from well-known estimates and growth bounds for maps, such as $p(z, t)$, having their image in the right half-plane. This approach let Pommerenke [1] solve the so-called "embedding problem", showing that for any $f \in S$ it is possible to find a Loewner chain (f_t), which starts in f, i. e. $f_0 = f$.

Chapter 5

Applications

Unfortunately, the Bieberbach conjecture does not have applications itself. But it is interesting to consider applications of Loewener parametric method, which was inspired by the original problem.

Loewner's crucial observation was that the family (f_t), which was introduced in Chapter 4, can be described by differential equations. In the general case of not univalent maps (f_t), one can choose a parametrization $V(t)$ in a way that there will exist a continuous function $K : [0, \infty) \to \partial D$, such that f_t satisfies

$$\frac{\partial \phi_{s,t}}{\partial t} = z\frac{K(t)+z}{K(t)-z}\frac{\partial f_t(z)}{\partial z} \tag{5.1}$$

This equation is usually called *slit-redial Loewner PDE* and function K is called the *driving term*. This PDE is the first one of several so-called *evolution equations*. Loewner also remarked that the associated family of holomorphic self-maps of U $(\phi_{s,t}) := (f_t^{-1} \circ f_s)$ for $0 \leq s \leq t$ gives solutions of characteristic equation

$$\frac{\partial f_t(z)}{\partial t} = z\frac{K(t)+z}{K(t)-z} \text{ for } t = s. \tag{5.2}$$

Equation (5.2) is known nowadays as the *slit-radial Loewner ODE.*

These two slit-radial Loewner equations can be studied on their own without any reference to parametrized families of the univalent maps. Restricting Loewner ODE (5.2) one will obtain a unique solution. However, without conditions on the driving term, the solutions of this equation are in general not of slit type. Therefore the problem of understanding exactly which driving terms produce slit solutions of (5.2) has become, and still is, a basic problem in the theory.

In 1946, Kufarev [1] proposed an evolution equation in upper-half plane analogous to the one introduced by Loewner in the unit disc. In 1968, Kufarev, Sobolev and Sporysheva [1] established a parametric method, based on this equation, for the class of univalent functions in the upper half-plane, which is known to be related to physical problems in hydrodinamics. Lets briefly introduce this evolution equation.

Let J be a Jordan arc in the upper half-plane $\mathbb{H}$ with starting point $J(0) = 0$. Then there exists a unique conformal map $f_t : \mathbb{H} \setminus J[0,t] \to \mathbb{H}$ with the normalization

$$f_t(z) = z + \frac{c(t)}{z} + O\left(\frac{1}{z^2}\right) \tag{5.3}$$

After reparametrization of the curve J, one can assume that $c(t) = 2t$. Under this normalization, one can show that f_t satisfies the following differential equation

$$\frac{\partial f_t(z)}{\partial t} = \frac{2}{f_t(z) - h(t)}, \; f_0(z) = z, \tag{5.4}$$

where $h(z)$ is a continuous real-valued function. On the contrary, given a continuous function $h : [0, \infty) \to \mathbb{R}$, one can consider the following initial value problem for each $z \in \mathbb{H}$:

$$\frac{\partial \phi_{s,t}(z)}{\partial t} = \frac{2}{\phi(z,s,t) - h(t)}, \; \phi(0,s,t) = z, \tag{5.5}$$

where $\phi(z,s,t)$ is the associated family of holomorphic self-maps of U. This equation is known nowadays as *the chordal Loewner differential equation.*
In 2000 Schramm had a very simple but very effective idea of replacing the function $h(z)$ and in (5.5) by a Brownian motion, i. e. $h(t) := \sqrt{k}B_t$. The resulting chordal Loewner equation was called *stochastic Loewner evolution with parameter $k \geq 0$* (SLE_k). The SLE_k depends on the choice of the Brownian motion and the value of parameter k:

- $k = 2$ corresponds to the loop-erased random walk, or equivalently, branches of the uniform spanning tree.
- For $k = \frac{8}{3}$ SLE_k has the restriction property and is conjectured to be the scaling limit of self-avoiding random walks. A version of it is the outer boundary of Brownian motion. This case also arises in the scaling limit of critical percolation on the triangular lattice.
- $k = 3$ is the limit of interfaces for the Ising model.
- For $0 \leq k \leq 4$ the curve $J(t)$ is simple (with probability 1).
- $k = 4$ corresponds to the path of the harmonic explorer and contour lines of the Gaussian free field.
- For $k = 6$ SLE_k has the locality property. This arises in the scaling limit of critical percolation on the triangular lattice and conjecturally on other lattices.
- For $4 < k < 8$ the curve $J(t)$ intersects itself and every point is contained in a loop but the curve is not space-filling (with probability 1).

- $k = 8$ corresponds to the path separating the uniform spanning tree from its dual tree.
- For $k = 8$ the curve $J(t)$ is space-filling (with probability 1).

The SLE_6 was used by Lawler, Schramm and Werner in 2001 to prove the conjecture of Mandelbrot (1982) that the boundary of planar Brownian motion has fractal dimension 4/3. Moreover Smirnov proves that critical percolation on the triangular lattice was related to SLE_6. Combined with earlier work of Harry Kesten, this led to the determination of many of the critical exponents for percolation. This breakthrough, in turn, allowed further analysis of many aspects of this model.
For the case of SLE_2 Lawler, Schramm and Werner showed that it is a limit for loop-erased random walk. This allowed derivation of many quantitative properties of loop-erased random walk. The related random Peano curve outlining the uniform spanning tree was shown to converge to SLE_8.
Another important application of the Loewner method is its extension to several complex variables. The first one to propose this was J.Pfaltzgraff, who in 1974 extended the basic Loewner theory to $\mathbb{C}^n$, in order to obtain bounds and growth estimates for some classes of univalent mappings defined in the unit ball of $\mathbb{C}^n$. The theory was later developed by other authors as well, but still there is no clear descriptions available. Most of the literature in higher dimensions is devoted to the radial Loewner equation on the unit ball of complex Banach spaces. But the class of univalent mappings on the unit ball of $\mathbb{C}^n$ is not compact, so special subclasses should be considered. All of this makes this theory much more complicated than in dimension one. Therefor there is not yet a satisfactory answer to the question of whether it is possible to associate to an evolution family on a ball $\mathbb{C}^n$ a Loewner chain with image on $\mathbb{C}^n$ solving the Loewner PDE.

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Biographies of Bieberbach, Loewner and de Branges

Ludwig Georg Elias Moses Bieberbach(4 December 1886 1 September 1982) was a German mathematician. He was born in Goddelau, near Darmstadt, studied at Heidelberg and under Felix Klein at Gottingen, receiving his doctorate in 1910. His dissertation was titled "On the theory of automorphic functions" (German: Theorie der automorphen Funktionen). He began working as a Privatdozent at Konigsberg in 1910 and as Professor ordinarius at the University of Basel in 1913. He taught at the University of Frankfurt in 1915 and the University of Berlin from 192145.

Bieberbach wrote a habilitation thesis in 1911 about groups of Euclidean motions identifying conditions under which the group must have a translational subgroup whose vectors span the Euclidean space that helped solve Hilbert's 18th problem. He worked on complex analysis and its applications to other areas in mathematics. He is known for his work on dynamics in several complex variables, where he obtained results similar to Fatou's. In 1916 he formulated the Bieberbach conjecture, stating a necessary condition for a holomorphic function to map the open unit disc injectively into the complex plane in terms of the function's Taylor series. In 1984 Louis de Branges proved the conjecture (for this reason, the Bieberbach conjecture is sometimes called de Branges' theorem). There is also a Bieberbach theorem on space groups. In 1928 Bieberbach wrote a book with Issai Schur titled "Uber die Minkowskische Reduktiontheorie der positiven quadratischen Formen".

Charles Loewner (29 May 1893 8 January 1968) was an American mathematician. His name was Karel Lowner in Czech and Karl Lowner in German.

Karl Loewner was born into a Jewish family in Lany, about 30 km from Prague, where his father Sigmund Lowner was a store owner.

Loewner received his PhD from the University of Prague in 1917 under supervision of Georg Pick; then he spent some years at the University of Berlin and Cologne. In 1930, he returned to Charled University of Prague as a professor. When the Nazis occupied Prague, he was imprisoned. Luckily, after paying the "emigration tax" he was allowed to leave the country with his family and move to the US. Although he had a job offer at Louisville University, he had to start his life from scratch. In the US he worked at Brown University, Syracuse University and eventually Stanford University, where he remained until his death in 1968.

Loewner's work covers wide areas of complex analysis and differential geometry. But one of his central mathematical contributions is the proof of the Bieberbach conjecture in the first highly nontrivial case of the third coefficient. The technique he introduced, the Loewner differential equation, has had far-reaching implications in geometric function theory. It was used in the final solution of the Bieberbach conjecture by Louis de Branges in 1984.

Louis de Branges de Bourcia (born August 21, 1932) is a French-American mathematician. He is the Edward C. Elliott Distinguished Professor of Mathematics at Purdue University in West Lafayette, Indiana. He is best known for proving the long-standing Bieberbach conjecture in 1984, now called de Branges's theorem.

Born to American parents who lived in Paris, de Branges moved to the US in 1941 with his mother and sisters. His native language is French. He did his undergraduate studies at the Massachusetts Institute of Technology, and received a PhD in mathematics from Cornell University. His advisors were Wolfgang Fuchs and then-future Purdue colleague Harry Pollard. He spent two years at the Institute for Advanced Study and another two at the Courant Institute of Mathematical Sciences. He was appointed to Purdue in 1962.

An analyst, de Branges has made incursions into real, functional, complex, harmonic (Fourier) and Diophantine analyses. As far as particular techniques and approaches are concerned, he is an expert in spectral and operator theories.

Early in 1984 he completed a manuscript of 385 pages, which culminated in a proof of the Bieberbach conjecture. This proof was not initially accepted by the mathematical community - many mathematicians were skeptical because de Branges had earlier announced some false results, including a claimed proof of the invariant subspace conjecture in 1964. It took verification by a team of mathematicians at Steklov Institute of Mathematics in Leningrad to validate de Branges' proof, a process that took several months and led later to significant simplification of the main argument. The original proof uses hypergeometric functions and innovative tools from the theory of Hilbert spaces of entire functions, largely developed by de Branges.

Actually, the correctness of the Bieberbach conjecture was only the most important consequence of de Branges' proof, which covers a more general problem, the Milin conjecture. The particular analysis tools he has developed, although largely successful in tackling the Bieberbach conjecture, have been mastered by only a handful of other mathematicians (many of whom have studied under de Branges). This poses another difficulty to verification of his current work, which is largely self-contained. During most of his working life, he published articles as the sole author.

Two named concepts arose out of de Branges' work. An entire function satisfying a particular inequality is called a de Branges function. Given a de Branges function, the set of all entire functions satisfying a particular relationship to that function, is called a de Branges space.