Master's Thesis

First-order Differential Inclusions Governed by Maximal Monotone Operators

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2015

Acknowledgments

I am mostly grateful to Gheorghe Moroşanu for his guidance before and during the realization of this thesis; I feel honored for working with such a great mathematician. I am also thankful to him for introducing me to this fascinating theory.

I also express my sincere acknowledgments to the following persons. To Karoly Boroczky and Pál Hegedűs for their constant concern for their students and for their support; for allowing me to form part of this great program. To all the excellent teachers I had during these two years at the CEU and that now form part of my mathematical education. To Melinda Balazs and Elvira Kadvany for coordinating efficiently the Mathematics Department. To Robin Bellers for his english corrections in this thesis. To the staff at CEU in general and particularly to George Soros for his financial support.

I am specially thankful to my parents for their constant support, advice and encouragement. I dedicate this work to them, to my brother and to my closest friends.

Contents

Introduction

If you would be a real seeker after truth, it is necessary that at least once in your life you doubt, as far as possible, all things. Descartes

We exhibit an existence and uniqueness result for a differential inclusion of first order, with initial condition, governed by a maximal monotone operator on a Hilbert space. In order to do this, we first expose the main proofs in maximal monotone theory needed for the existence and uniqueness theorem. We also present the concept of subdifferential of a convex function and show the proof of its maximal monotonicity under certain conditions. This theory helps to solve equations when there is a lack of linearity and continuity of the associated operators. The importance of this is then exhibited by three main applications: The Heat (Diffusion) Equation, the Dynamics of a Model for Wheeled Vehicles (see [7]) and the Nonlinear Wave Equation. The Nonlinear Wave Equation is solved assuming very mild conditions on the nonlinear term.

The main objective is to exhibit an understandable, very detailed, well-written and self-contained survey on the subject. To present the results and concepts in a natural way and making a deep analysis. When possible, simplified proofs and generalizations will be made. Main references for this thesis are [10] and [19].

One very efficient way to describe some natural phenomena is through Partial Differential Equations. This field of mathematics is constantly explaining and solving real world problems coming from very different contexts. To mention some of them: Physics, Economics, Finance, Biology, Social sciences, etc. Moreover, the theory of Partial Differential Equations also contributes to pure mathematics. One of the main tools for developing this theory is based on Functional Analysis. Nonlinear Analysis also plays an essential role. Here the concept of maximal monotonicity results in a very rich theory. In the last fifty years, a growing number of scientists have shown strong interest in these topics since a very wide class of differential inclusions in Hilbert spaces are governed by this kind of operators. The first and most important reason that makes us consider such kind of operators is due to the existence and uniqueness theorem presented in this thesis.

A big number of partial differential equations of diverse kind can be solved by working with one specific abstract framework. The equation we want to study is:

$$(p)\begin{cases} u' + Au \ni f, \text{ on } (0,T) \\ u(0) = u_0, \end{cases}$$

where H is a real Hilbert space, T is a positive real number, $u : [0, T) \to H$ is the unknown function, $A : \mathrm{D}(A) \subseteq H \to H$ is a possibly multi-valued operator, $f : (0, T) \to H$ is a given function and u_0 is an element of H.

The usual trade-off when considering this general equation is that the order is decreased but the underlying space will no longer be finite-dimensional. When the operator A is linear and continuous then existence and uniqueness is easily shown. When we have linearity and a weaker assumption, that is the operator is closed plus some other extra hypotheses, the known Hille-Yosida theorem solves the equation. Within the frame of nonlinearity, if the operator is Lipschitz continuous we can also get this result by a simple application of the Banach fixed point theorem. The problem arises from more precise equations, as the mentioned Heat (Diffusion) Equation and the Nonlinear wave equation, where an extra nonlinear term appears. In this case continuity is also not satisfied because of the Laplacian nature. In order to solve this and many other equations that arise naturally in applications, a different existence and uniqueness theorem is needed. The aim is to define a property for an operator A such that if it is satisfied, then we can find a sequence of Lipschitz continuous operators approximating A in some sense; and hope convergence of the corresponding solutions, which we know exist, to a good candidate for solution of our equation associated to A. The property that works is called maximal monotonicity and it is defined on Hilbert spaces H for possibly multi-valued operators. Now, identify the operator with its graph. By monotonicity we mean a generalization of the concept of non-decreasing functions to Hilbert spaces using the inner product. This is, A is monotone if for all $[x_1, y_1], [x_2, y_2] \in A$ we have

$$(x_1 - x_2, y_1 - y_2) \geq 0.$$

By maximality we refer to the property of being maximal as a monotone subset of $H \times H$. One of the fundamental properties that make these maximal monotone operators satisfy their purpose is their demiclosedness. Next we state a simplified version of the main theorem. If $A : \mathrm{D}(A) \subseteq H \to H$ is a maximal monotone operator, $u_0 \in \mathrm{D}(A)$ and $f \in W^{1,1}(0, T; H)$ then there exists a unique "solution" of (p).

This paragraph is based on [9]. Monotone operators were first introduced by George Minty in 1960, in order to study electrical networks, in his famous paper [18]; although they were presented in the setting of partial differential equations by Browder in [12]. During the first ten years of maximal monotonicity, the foundations of the modern theory were settled mainly by: Minty, Browder, Asplund, Rockafellar and Zarantonello. One of the most important and celebrated results was a characterization of maximal monotonicity given by Minty (presented in this thesis). On the other hand, Rockafellar achieved considerable results around 1970, for example, concerning the maximal monotonicity of the sum of maximal monotone operators and of the subdifferentials. By 1975 the main ideas about maximal monotonicity in Hilbert spaces (and even in reflexive Banach spaces) were clear; an important and satisfactory monograph was published by Brézis around that time, namely [10]. It was in the eighties when Fitzpatrick studied his famous function and Phelps, in his monograph [21], brought together for the first time the concepts of convex function, monotone operator and differentiability. During the nineties Fitzpatrick and Phelps made some advances in the context of non-reflexive spaces. However, maximal monotone operator theory is far from being completely understood and a large number of modern researchers continue developing the field.

The useful characterization for maximal monotonicity given by Minty states that a monotone operator A is maximal monotone precisely when $I + A$ is surjective. The other main theorem that we will cover in this thesis is the perturbation result of Rockafellar: conditions for the sum of two maximal monotone operators to be maximal monotone. Once the importance of maximal monotonicity is in evidence, the machine for creating maximal monotone operators, called the subdifferential, comes into play. The subdifferential is a multi-valued generalization of the directional derivative in Hilbert spaces. It turns out that the subdifferential of a proper, convex and lower semicontinuous function is always maximal monotone. Minty's characterization is essential in showing this. When dealing with applications, using subdifferentials in this way provides a useful path. An example of this is the application we will give to the braking system of a car, where the associated operator is the subdifferential of such kind of function. The other two problems that we present are the Heat (Diffusion) Equation and the Nonlinear Wave equation; respectively a parabolic and a hyperbolic partial differential equation. The classical applications of the existence theorem include these two equations; they have been studied for more than two hundred years and are fundamental in Science.

The organization of the material in this thesis is as follows. The first chapter is divided into four sections. The first one introduces the concept of the Yosida approximation of an operator and shows its main properties. The second section is dedicated to the basic properties of maximal monotone operators; a proof of Minty's theorem is presented here. The third section addresses the conditions on the class of maximal monotone operators to be closed under addition. Some of the results previously presented have generalizations to Banach spaces, since these versions are used later, the fourth section makes some remarks about them. Chapter Two consists of three sections. First, the main function spaces used in this thesis are briefly reviewed. The second section presents the main result of the thesis, the existence and uniqueness theorem; the material is separated in different lemmas and the needed tools come from the first chapter. The last section presents the concept of subdifferential and gives a proof of its main theorem (see above). The idea is to use the existence result along with the subdifferential theory too solve partial differential equations. This is illustrated in the last chapter, which is again divided into three sections. They correspond respectively to the Heat (Diffusion) Equation, the Braking System of a Vehicle and the Nonlinear Wave Equation. Although the first two sections are relatively short, the third one is longer and studies also a generalization of the Nonlinear Wave Equation.

For reading this thesis it is recommended to be familiar with the basic results and techniques of Mathematical Analysis. For example, good background material can be found in [11]. Other than this, the thesis is practically self-contained.

Chapter 1

Preliminaries

The aim of this chapter is to present, with proofs, the basic results about Maximal Monotone Operators. All of them will be used for the main proof of existence and uniqueness. Here the theory is studied in general real Hilbert spaces. Throughout this thesis H denotes a real Hilbert space with norm $\|\cdot\|$ induced by the inner product $(\cdot,\cdot)$. If r is a positive real number and u an element of H, then the corresponding open ball is denoted by

$$B_r(u) := \{v \in H : \|u - v\| < r\}.$$

Elements of the Cartesian product of two sets will be denoted using $[\cdot,\cdot]$. Cardinality and Lebesgue measure will be denoted by $|\cdot|$. As usual, we use $\rightharpoonup$ to denote convergence in a normed space in the weak sense. We use $\overset{*}{\rightharpoonup}$ for weak star convergence in the dual of a normed space. Finally, the abbreviation a.e. stands for almost everywhere. The proof of Minty's theorem is based on [13].

Maximal Monotone operators started being considered around fifty years ago. Many differential equations that arise from all kinds of applications are modeled using this kind of operators. The usual methods concerning existence, uniqueness, regularity, etc. did not work to solve these problems since the operators are usually not linear and not continuous. Hence, the concept of Maximal Monotonicity resulted in being very useful. We remark that this theory is very rich and beautiful by itself, even without mentioning its applications.

We are concerned with multi-valued operators, not only with functions. The reason is that many problems coming from applications have to be represented by inclusions instead of equations. Moreover, with multi-valued operators we do not have to worry about the existence of an inverse and we can extend the notion of differentiability as we will do in the second chapter. The best way of thinking of these operators is just as subsets of the Cartesian product.

Let X be any set. Consider a subset A of the Cartesian product $X^2 := X \times X$. Given $u \in X$ there are two options: there is $v \in X$ such that $[u, v] \in A$ or there is no such v. Let us name

$$\mathrm{D}(A) := \{u \in X : \exists v \in H, \text{ such that } [u, v] \in A\}.$$

Of course given $u \in \mathrm{D}(A)$ there might be many such v's, let

$$A(u) = \{v \in X : [u, v] \in A\}.$$

To express this we will use the notation $A : \mathrm{D}(A) \subseteq X \to X$. We call A (multi-valued) operator on X and $\mathrm{D}(A)$ the domain of A. As probably expected, the range of A is defined as

$$\mathrm{R}(A) := \bigcup_{u \in \mathrm{D}(A)} A(u).$$

We say that A is surjective if $\mathrm{R}(A) = H$. The inverse of A is just

$$A^{-1} := \{[u, v] \in X^2 : [v, u] \in A\}.$$

Notice that $\mathrm{D}(A^{-1}) = \mathrm{R}(A)$ and $\mathrm{R}(A^{-1}) = \mathrm{D}(A)$. When for each $u \in \mathrm{D}(A)$ we have that $|A(u)| = 1$ we say that A is single-valued. We identify functions with single-valued operators in the obvious way. An operator A such that $\mathrm{D}(A) = X$ is called an everywhere defined operator. When X is a vector space we can define the sum and the scalar multiplication. Let A and B be two operators on X and $\lambda \in \mathrm{R}$. We define the operators

$$A+B := \{[x, y+z] \in X^2 : [x, y] \in A \text{ and } [x, z] \in B\}, \quad \lambda A := \{[x, \lambda y] \in X^2 : [x, y] \in A\}.$$

In order to get interesting results we assume that X is equipped with a rich algebraic and topological structure, more precisely from now on we take $X := H$ real Hilbert space. *Throughout this thesis, the simple sentence A is an operator will mean $A : \mathrm{D}(A) \subseteq H \to H$ is a (possibly) multi-valued operator on H.*

1.1 Yosida Approximation

Let A be an operator on H and $f : [0, \infty) \to H$ a function. Recall that our main goal, vaguely speaking, is to show existence of u for a problem of the type $u' + Au \ni f$; where A is not necessarily continuous nor linear. As mentioned before, this problem is relatively easy to solve when A is an everywhere defined Lipschitz continuous function (Cauchy-Lipschitz-Picard theorem). A first natural step then is to find a sequence of Lipschitz continuous functions that approximate A in some sense. The definition of this sequence is not trivial; it carries the name of the Japanese mathematician K. Yosida.

Definition 1.1. *The Yosida approximation of an operator A corresponding to $\lambda > 0$ is defined as*

$$A_\lambda := \frac{1}{\lambda}(I - (I + \lambda A)^{-1}),$$

where I denotes the identity on H.

It will be useful to put

$$J_\lambda := (I + \lambda A)^{-1},$$

this is called the Resolvent of A corresponding to $\lambda > 0$ (if there is a chance of confusion we write $J_{A,\lambda}$). Then we can simply express

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda).$$

Notice that $\mathrm{D}(J_\lambda) = \mathrm{D}(A_\lambda) = \mathrm{R}(I + \lambda A)$. Let us illustrate these definitions and the idea of approximating a non-Lipschitz continuous function with Lipschitz ones.

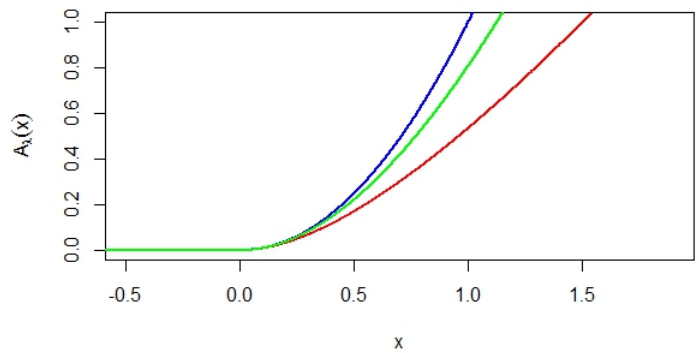

Figure 1.1: The graphs of A and A_λ, for $\lambda = \frac{1}{2}, \frac{1}{8}$, respectively in blue, red and green.

Example 1.2. *Let $A : \mathbb{R} \to \mathbb{R}$ be given by*

$$A(x) := \begin{cases} 0, \ \textit{if } x < 0 \\ x^2, \ \textit{if } x \geq 0. \end{cases}$$

A straight forward computation gives for $\lambda > 0$,

$$A_\lambda(x) = \begin{cases} 0, \ \textit{if } x < 0 \\ \frac{x}{\lambda} + \frac{1}{2\lambda^2}(1 - (1 + 4\lambda x)^{\frac{1}{2}}), \ \textit{if } x \geq 0. \end{cases}$$

The function A is not Lipschitz continuous. For each $\lambda > 0$, A_λ is an everywhere defined function, it is Lipschitz continuous and approximates A pointwise as $\lambda \to 0^+$. The Lipschitz continuity follows easily from the fact that the square root function is Lipschitz continuous on $[1, \infty)$. The pointwise convergence follows from applying L'Hôpital's Rule twice. See Figure 1.1.

Example 1.3. *Let λ be a positive real number and suppose that A is the operator from Example 1.2. Then*

$$J_\lambda(x) = \begin{cases} 0, \ \textit{if } x < 0 \\ \frac{1}{2\lambda}(-1 + (1 + 4\lambda x)^{\frac{1}{2}}), \ \textit{if } x \geq 0. \end{cases}$$

Here for each positive λ we even have that J_λ is a non-expansive function (Lipschitz continuous with constant one). See Figure 1.2.

Let A be an operator on H. Of course, in general, we would not get the approximation property and the Lipschitz continuity. Let us see which properties A must satisfy in order to go in this direction. The first obvious desirable property for A_λ is to be a function. Take any $u \in \mathrm{D}(J_\lambda)$ and let $v_1, v_2 \in J_\lambda(u)$. Then $v_1 - v_2 + \lambda(A(v_1) - A(v_2)) \ni 0$ and multiplying by $v_1 - v_2$ we get $\|v_1 - v_2\|^2 + \lambda(A(v_1) - A(v_2), v_1 - v_2) = 0$. So in order to get $v_1 = v_2$ we need $(A(v_1) - A(v_2), v_1 - v_2) \geq 0$. For example, if $H = \mathbb{R}$ and A

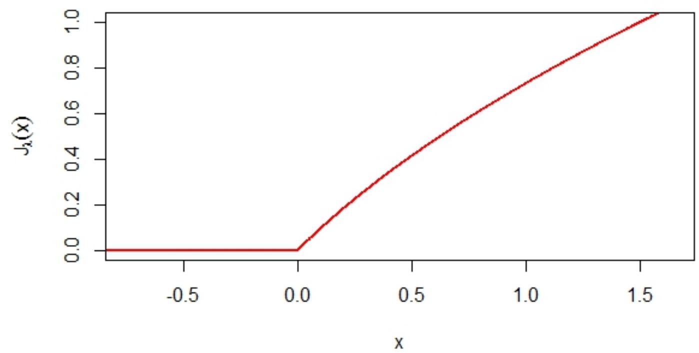

Figure 1.2: The graph of J_λ for $\lambda = \frac{1}{2}$.

is a function then we are asking A to be non-decreasing on the range of the Resolvent. So we are just looking to generalize the idea of non-decreasing function to multi-valued operators on general Hilbert spaces.

Definition 1.4. *We say an operator A is monotone if for all $[u_1, v_1], [u_2, v_2] \in A$ we have*

$$(u_1 - u_2, v_1 - v_2) \geq 0.$$

We equivalently say sometimes that A is a monotone subset of $H \times H$. A stronger notion of monotonicity that will be used later in the thesis is the following.

Definition 1.5. *An operator A is strongly monotone if there exists c positive real number such that for all $[u_1, v_1], [u_2, v_2] \in A$ we have*

$$(u_1 - u_2, v_1 - v_2) \geq c\|u_1 - u_2\|^2.$$

In the case that A is single-valued and linear, we say A is strongly positive if it is strongly monotone.

Example 1.6. *A very important first example of a monotone operator is the so-called Sign Operator A on $H := \mathbb{R}$ (see Figure 1.3). It is defined as*

$$A(x) := \begin{cases} -1, \text{ if } x < 0 \\ [-1, 1] \subseteq \mathbb{R}, \text{ if } x = 0 \\ 1, \text{ if } x > 0. \end{cases}$$

Now an example in a function space. We will refer to it later in the chapter.

Example 1.7. *Let Ω be an open subset of $\mathbb{R}^n$. The (minus) Laplacian operator $-\Delta : H_0^1(\Omega) \cap H^2(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega)$ given by*

$$-\Delta u := -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

is monotone.

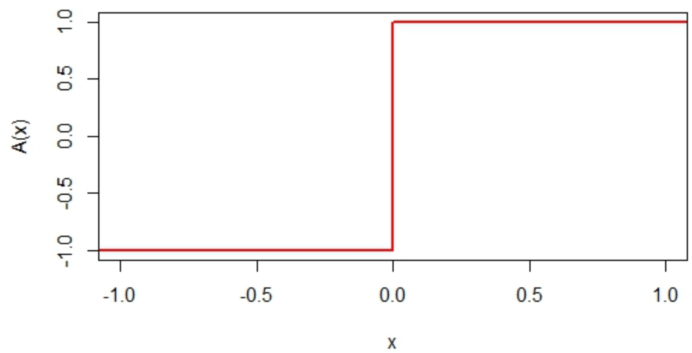

Figure 1.3: Graph of the Sign Operator.

Proof. Let $u, v \in H_0^1(\Omega) \cap H^2(\Omega)$. The formula of integration by parts gives

$$\int_\Omega (u-v)\Delta(v-u) = -\int_\Omega \nabla(u-v)\nabla(v-u).$$

Hence,

$$\begin{aligned}(u-v, -\Delta u + \Delta v)_{L^2(\Omega)} &:= \int_\Omega (u-v)(-\Delta u + \Delta v) \\ &= \int_\Omega |\nabla(u-v)|^2 \geq 0.\end{aligned}$$

■

Example 1.8. *Consider any non-expansive (this is, Lipschitz continuous with constant* 1*) function* $T : \mathrm{D}(T) \subseteq H \to H$*. Then* $I + T$ *is a monotone operator.*

Proof. We have that for all u, v in $\mathrm{D}(T)$,

$$\begin{aligned}((I+T)(u) - (I+T)(v), u-v) &= \|u-v\|^2 + (T(u) - T(v), u-v) \\ &\geq \|u-v\|^2 - \|T(u) - T(v)\|\|u-v\| \\ &\geq 0.\end{aligned}$$

■

Analogous we can get $I - T$ is also monotone.

Example 1.9. *Consider* C *a non-empty, convex and closed subset of* H*. The projection* $P : D(P) = H \to H$ *on* C *is monotone.*

Proof. By basic properties of this operator we know that for all $u \in H$ and all $c \in C$ we have $(u - P(u), P(u) - c) \geq 0$. Let $u_1, u_2 \in H$ and use the inequality to get

$$(u_1 - P(u_1), P(u_1) - P(u_2)) \geq 0,\ (u_2 - P(u_2), P(u_2) - P(u_1)) \geq 0.$$

It follows that

$$(P(u_1) - P(u_2), P(u_2) - P(u_1)) + (u_1 - u_2, P(u_1) - P(u_2)) \geq 0,$$

and hence

$$(u_1 - u_2, P(u_1) - P(u_2)) \geq \|P(u_1) - P(u_2)\|^2 \geq 0.$$

■

The next proposition states the discussion above and shows that the complicated definition of A_λ actually seems to be going in the right way. For simplicity we call

$$D := \mathrm{D}(J_\lambda) = \mathrm{D}(A_\lambda) = \mathrm{R}(I + \lambda A).$$

Proposition 1.10. *Let A be a monotone operator and $\lambda > 0$. Then A_λ is a Lipschitz continuous function with constant $\frac{1}{\lambda}$.*

Proof. We already showed that J_λ, and hence A_λ are functions. Let $u \in D$ and $v = A_\lambda(u)$. Then $u - \lambda v = J_\lambda(u)$ and $u - \lambda v + \lambda A(u - \lambda v) \ni u$. It follows that

$$A(u - \lambda v) = A(J_\lambda(u)) \ni A_\lambda(u). \tag{1.1}$$

Now let $u, v \in D$ and notice that $\lambda(A_\lambda(u) - A_\lambda(v)) + J_\lambda(u) - J_\lambda(v) = u - v$. Multiplying by $A_\lambda(u) - A_\lambda(v)$, using equation (1.1) and that A is monotone we get

$$\begin{aligned} \lambda\|A_\lambda(u) - A_\lambda(v)\|^2 &\leq (u - v, A_\lambda(u) - A_\lambda(v)) \\ &\leq \|u - v\|\|A_\lambda(u) - A_\lambda(v)\|. \end{aligned}$$

■

Observation 1.11. *From the proof above we get a very useful relation between the Yosida approximation and the resolvent that it is constantly used in this thesis. If the operator A is monotone then, for all u in H we have*

$$A_\lambda(u) \in A(J_\lambda(u)).$$

Consider an operator A. The second reason for being interested in the Yosida approximation is because it actually approximates A. This approximation will be pointwise so we need a single-valued version of A. The proper way of getting it is the following. Assign to each u in $\mathrm{D}(A)$ the projection of 0 into the set $A(u)$. In order to have a well-defined projection we need the sets $A(u)$ to be convex and closed in H. In this case we define the minimal section of A as the function $A^0 : \mathrm{D}(A^0) = \mathrm{D}(A) \subseteq H \to H$ such that $A^0(u) := v_u$; where v_u is the unique element of $A(u)$ satisfying $\|v_u\| \leq \|v\|$ for every v in $A(u)$. One last concept before the proposition. If A is a closed subset of $H \times H$ then we say A is a closed operator. We want to make this definition slightly stronger.

Definition 1.12. *Let A be an operator. We say that A is demiclosed if for every sequence $([u_n, v_n])$ in A, the fact that $u_n \to u$ strongly and $v_n \rightharpoonup v$ weakly in H implies that $[u, v]$ is in A.*

Proposition 1.13. *Let A be a monotone demiclosed operator such that for each u in* $\mathrm{D}(A)$ *the set $A(u)$ is convex and closed. Suppose that λ is a positive real number. Then, for all u in* $\mathrm{D}(A) \cap D$*:*

$$\lim_{\lambda \to 0^+} A_\lambda(u) = A^0(u).$$

Proof. First we show J_λ is non-expansive. Take $u_1, u_2 \in D$ and let $v_i := J_\lambda(u_i)$ for $i = 1, 2$. Then $u_1 - u_2 \in v_1 - v_2 + \lambda(A(v_1) - A(v_2))$. Multiplying by $v_1 - v_2$ and using the monotonicity of A it follows that

$$\begin{aligned} \|v_1 - v_2\|^2 &\leq (u_1 - u_2, v_1 - v_2) \\ &\leq \|u_1 - u_2\| \|v_1 - v_2\|. \end{aligned}$$

Now fix $u \in \mathrm{D}(A) \cap D$ and let $v := A^0(u)$. The latter gives us that for all $\lambda > 0$:

$$\|A_\lambda(u)\| = \frac{1}{\lambda} \|J_\lambda(u + \lambda v) - J_\lambda(u)\| \leq \|v\|.$$

We know then that there exists a sequence of positive real numbers (λ_n) converging to 0 as $n \to \infty$ and $w \in H$ such that $A_{\lambda_n}(u) \rightharpoonup w$ in H as $n \to \infty$. This implies that

$$\|w\| \leq \liminf_{n \to \infty} \|A_{\lambda_n}(u)\| \leq \|v\|.$$

On the other hand, $\|u - J_{\lambda_n}(u)\| \leq \lambda_n \|v\|$ and $J_{\lambda_n}(u) \to u$ in H as $n \to \infty$. Using the Observation 1.11 and the demiclosedness of A we get $[u, w] \in A$. The uniqueness of the Projection gives then $w = v$ and, since w is determined, $A_\lambda(u) \rightharpoonup v$ as $\lambda \to 0^+$. Finally, the fact that $\limsup_{\lambda \to 0^+} \|A_\lambda(u)\| \leq \|v\|$ implies

$$A_\lambda(u) \to v, \text{ strongly in } H \text{ as } \lambda \to 0^+.$$

■

Observation 1.14. *From the proof we also get that, for A satisfying the hypotheses of the last Proposition, J_λ is non-expansive. Then, by using Example 1.8 we get A_λ is monotone. It is also clear that J_λ is monotone since A is. Moreover, we got the bound $\|A_\lambda(u)\| \leq \|A^0(u)\|$ for all $\lambda > 0$ and all u in* $\mathrm{D}(A) \cap D$*.*

1.2 Maximal Monotone Operators

Everything seems to be working in the direction we want for operators that are monotone, demiclosed and have closed and convex pointwise images. But we still do not have the everywhere defined part for the Yosida approximations. This is, we want $D := \mathrm{D}(A_\lambda) = \mathrm{R}(I + \lambda A) = H$ for each positive λ. The question now is: What kind of operators satisfy these four properties? The next definition is the most important for this thesis.

Definition 1.15. *Let A be a monotone subset of $H \times H$. We say A is maximal monotone if it is maximal among the monotone subsets of $H \times H$.*

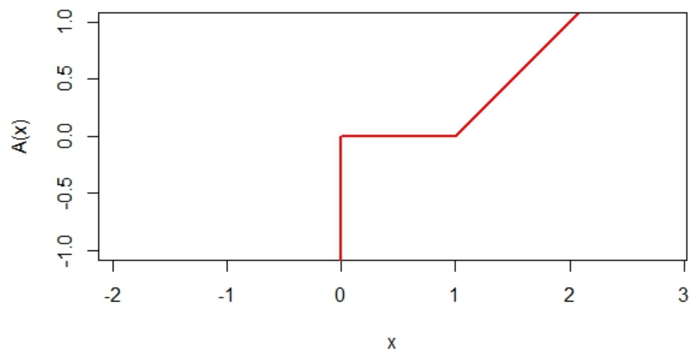

Figure 1.4: Graph of the maximal monotone operator A.

This is, and operator A on H is maximal monotone if and only if A is monotone and for every B monotone operator on H satisfying $A \subseteq B$ we have $A = B$. In a more explicit way: an operator A is maximal monotone if and only if it is monotone and for every $[w, z] \in H \times H$,

$$(u - w, v - z) \geq 0, \, \forall [u, v] \in A \text{ implies } [w, z] \in A.$$

Example 1.16. *Let $A : \mathrm{D}(A) \subseteq \mathbb{R} \to \mathbb{R}$ be given by $D(A) := [0, \infty)$ and*

$$A(x) := \begin{cases} (-\infty, 0], \; \textit{if } x = 0 \\ 0, \; \textit{if } 0 < x < 1 \\ x - 1, \; \textit{if } x \geq 1. \end{cases}$$

It is easy to see that A is in fact a maximal monotone operator; A is a monotone subset of $\mathbb{R}^2$ and you cannot add a vector in $\mathbb{R}^2$ to A such that the monotonicity is preserve (see Figure 1.4).

Saying that an operator A is maximal monotone is also equivalent to the equality $A = \{[w, z] \in H \times H : (u - w, v - z) \geq 0, \, \forall [u, v] \in A\}$. This makes maximal monotone operators useful, they can be expressed in terms of the inner product of our space H. From this expression it is a straight forward exercise to show that if A is maximal monotone then it is demiclosed and for each $u \in \mathrm{D}(A)$, $A(u)$ is a closed convex subset of H. Notice that if A is maximal monotone then A^{-1} is also. Clearly, the sign operator is an example of a maximal monotone operator. Now suppose $H := \mathbb{R}$. As intuition might suggests, everywhere defined non-decreasing continuous functions are in fact examples of maximal monotone operators. This will be shown later in this chapter for general Hilbert spaces.

The next central theorem says that the fourth property we want is also true, for this kind of operators the Yosida approximations are everywhere defined (see observation 1.19 below). Moreover, it gives a characterization of maximal monotone operators given by Minty. The definition of maximal monotone operators might be very nice and elegant but when showing theorems and even in applications we do not use this definition frequently; we use this characterization. Before showing this celebrated theorem, we state without proof the known Ky Fan Inequality. The usual result is true with much weaker assumptions but we state it as we use it. Concerning this inequality the reader is referred to [2].

Lemma 1.17 (Ky Fan). *Let K be a convex and compact subset of $\mathbb{R}^n$. Let also $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function such that $\phi(\cdot, y)$ is continuous for each $y \in K$ and $\phi(x, \cdot)$ is linear for each $x \in K$. Moreover, assume that $\phi(x, x) \leq 0$ for all $x \in K$. Then there exist $x_0 \in K$ such that $\phi(x_0, y) \leq 0$ for all $y \in K$.*

Theorem 1.18 (Minty). *Let A be a monotone operator. Then A is maximal monotone if and only if $I + A$ is surjective.*

Proof. First let us suppose A is maximal monotone, we need to show $\mathrm{R}(I + A) = H$. Assume we showed that $0 \in \mathrm{R}(I + A)$ and take any $y \in H$. Then, since $-y + A$ is maximal monotone, we can find $u \in D(A)$ such that $0 \in u - y + A(u)$ and $y \in u + A(u)$. So it suffices to show

$$0 \in \mathrm{R}(I + A). \tag{1.2}$$

For each $[w, z] \in A$ let $c(w, z) := \{u \in H : (z + u, w - u) \geq 0\}$. Then by maximal monotonicity of A, (1.2) is equivalent to

$$\bigcap_{[w,z]\in A} c(w, z) \neq \emptyset. \tag{1.3}$$

The sets $c(w, z)$ are convex, closed and bounded in H. It is straight forward to show the first two properties so we check only boundedness. Let $[w, z] \in A$ and suppose by contradiction that $c(w, z)$ is not bounded. Then we can find a sequence $(u_n) \subseteq c(w, z)$ such that $\|u_n\| \to \infty$ as $n \to \infty$. It follows that as $n \to \infty$,

$$\begin{aligned} 0 &\geq -(z + u_n, w - u_n) \\ &\geq -\|z\|\|w - u_n\| - \|u_n\|\|w\| + \|u_n\|^2 \to \infty. \end{aligned}$$

Hence the sets are in fact bounded. Being closed and convex, the sets $c(w, z)$ are weakly closed in H. The boundedness then implies weakly sequential compactness. It then follows, by the Eberlain-Smulian theorem, that $c(w, z)$ are compact in the weak topology of H. This means that (1.3) is equivalent to showing that finite intersections of the sets $c(w, z)$ are non-empty. Fix n positive integer and take $[w_1, z_1], .., [w_n, z_n] \in A$, it suffices to show that there is $u_0 \in H$ satisfying $(z_i + u_0, w_i - u_0) \geq 0$ for each $i = 1, .., n$. Consider the convex and compact subset of $\mathbb{R}^n$,

$$K := \{\lambda = [\lambda_1, .., \lambda_n] \in \mathbb{R}^n : \lambda_i \geq 0 \text{ for all } i = 1, .., n \text{ and } \sum_{i=1}^{n} \lambda_i = 1\},$$

and define $s(\lambda) := \sum_{i=1}^{n} \lambda_i w_i$. Moreover, let $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be given by

$$\phi(\lambda, \mu) = \sum_{i=1}^{n} \mu_i (s(\lambda) + z_i, s(\lambda) - w_i).$$

This function is clearly continuous on the first coordinate and linear in the second one. Let $\lambda \in K$ and express

$$\phi(\lambda, \lambda) = \sum_{i=1}^{n} \lambda_i (s(\lambda), s(\lambda) - w_i) + \sum_{i=1}^{n} \lambda_i (z_i, s(\lambda) - w_i). \tag{1.4}$$

Using that $\lambda_1 + .. + \lambda_n = 1$, we get the first term of the sum in (1.4) is zero since:

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i (s(\lambda), s(\lambda) - w_i) &= \sum_{i,j=1}^{n} \lambda_i \lambda_j (s(\lambda), w_j - w_i) \\ &= -\sum_{i=1}^{n} \lambda_i (s(\lambda), s(\lambda) - w_i). \end{aligned}$$

On the other hand,

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j (z_i - z_j, w_j - w_i) = 2 \sum_{i,j=1}^{n} \lambda_i \lambda_j (z_i, w_j - w_i).$$

Finally, the monotonicity of A implies

$$\phi(\lambda, \lambda) = \frac{1}{2} \sum_{i,j=1}^{n} \lambda_i \lambda_j (z_i - z_j, w_j - w_i) \leq 0.$$

This means that the hypotheses of the Ky Fan Inequality are satisfied and there exists λ_0 in K such that $\phi(\lambda_0, \mu) \leq 0$ for all μ in K. We take $u_0 := s(\lambda_0)$. Given $i = 1, .., n$ consider $\mu := e_i$ in K to conclude $(u_0 + z_i, w_i - u_0) \geq 0$ as wanted.

Conversely, suppose that $\mathrm{R}(I + A) = H$ for A monotone operator. Let us assume by contradiction that A is not maximal monotone. This means that there is $[u_0, v_0] \in H \times H - A$ such that

$$(u - u_0, v - v_0) \geq 0, \forall [u, v] \in A.$$

Since $u_0 + v_0 \in R(I + A)$, we can find $u_1 \in D(A)$ such that $u_0 + v_0 \in u_1 + A(u_1)$. Let $v_1 \in A(u_1)$ satisfy $u_0 + v_0 = u_1 + v_1$. It follows that $(v_0 - v_1, v_1 - v_0) = (u_1 - u_0, v_1 - v_0) \geq 0$ and hence, $[u_1, v_1] = [u_0, v_0] \notin A$. So A has to be maximal monotone. ■

Observation 1.19. *It is not hard to see from the proof that if A is a monotone operator then: A is maximal monotone if and only if $I + \lambda A$ is surjective for all $\lambda > 0$; if and only if there exists $\lambda > 0$ such that $I + \lambda A$ is surjective.*

Example 1.20. *Recall the Sign Operator given in Example 1.6. This operator is clearly maximal monotone and*

$$(I + A)(x) = \begin{cases} x - 1, \text{ if } x < 0 \\ [-1, 1], \text{ if } x = 0 \\ x + 1, \text{ if } x > 0 \end{cases}$$

is a surjective operator (see Figure 1.5).

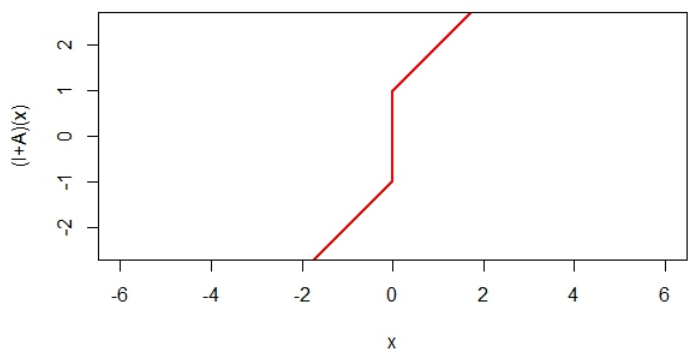

Figure 1.5: Graph of the surjective operator $I + A$.

Example 1.21. *Let Ω be an open, bounded and smooth subset of $\mathbb{R}^n$. We saw in Example 1.7 that $-\Delta$ is a monotone operator on $L^2(\Omega)$. To conclude the operator is maximal monotone we use Minty's theorem. Let $f \in L^2(\Omega)$, we need to exhibit $u \in H_0^1(\Omega) \cap H^2(\Omega)$ such that*

$$u - \Delta u = f, \text{ a.e. on } \Omega.$$

This is a known differential equation. Due to the Lax-Milgram theorem and results in Regularity Theory it has a unique solution. See respectively [3] and [17].

As said before, maximal monotone operators are demiclosed, have minimal section and now we know that their Yosida approximations are everywhere defined. We next summarize what we already showed.

Corollary 1.22. *Let A be a maximal monotone operator on H. Then, for each positive λ, J_λ and A_λ are everywhere defined, monotone and Lipschitz continuous functions with respective constants 1 and $\frac{1}{\lambda}$. Moreover, the following is satisfied:*

$$\begin{aligned}
&A_\lambda(u) \in A(J_\lambda(u)) \ \forall u \in H, \\
&\|A_\lambda(u)\| \le \|A^0(u)\| \ \forall u \in \mathrm{D}(A), \\
&\lim_{\lambda \to 0^+} A_\lambda(u) = A^0(u) \ \forall u \in \mathrm{D}(A).
\end{aligned}$$

■

The operator A given in Example 1.2 is clearly a maximal monotone operator. The properties of the last corollary are evident for this example; see also Example 1.3.

A way of getting new maximal monotone operators from old is the following. Let Ω be a Lebesgue measurable set in $\mathbb{R}^n$ and A a maximal monotone operator on H. The idea is to get an operator on $L^2(\Omega; H)$ through composition, this is, $u \mapsto A(u)$. Of course this does not make sense when A is multi-valued so we have to complicate a little bit the definition. Although studied when Ω is an interval, to review the definitions involving such a function space see the section of function spaces in the second chapter of this thesis.

Definition 1.23. *The canonical extension of A to $L^2(\Omega; H)$ is $\overline{A} : \mathrm{D}(\overline{A}) \subseteq L^2(\Omega; H) \to L^2(\Omega; H)$ given by:*

$$\overline{A}(u) := \{v \in L^2(\Omega; H) \,:\, v(x) \in A(u(x)) \text{ for a.e. } x \in \Omega\},$$

where its domain are those $u \in L^2(\Omega; H)$ such that $u(x) \in D(A)$ for a.e. $x \in \Omega$ and v as above exists.

Observation 1.24. *Notice that taking the canonical extension and taking the Yosida Approximation commutes. Namely, if A is an operator, Ω is a Lebesgue measurable subset of $\mathbb{R}^n$ and λ is positive, then*

$$(\overline{A})_\lambda = \overline{(A_\lambda)}.$$

The analogous statement holds for the resolvent of A.

Example 1.25. *If A is maximal monotone and Ω is a Lebesgue measurable subset of $\mathbb{R}^n$ such that $|\Omega| < \infty$, then $\overline{A}$ is maximal monotone.*

Proof. Monotonicity of $\overline{A}$ is clear from monotonicity of A and the integral. By Minty's theorem it suffices to show $L^2(\Omega; H) = \mathrm{R}(I + \overline{A})$. Let $g \in L^2(\Omega : H)$ and define $u : \Omega \to H$ as

$$u(x) := J_1(g(x)) = (I + A)^{-1}(g(x)).$$

Since J_1 is non-expansive, then for a.e. $x \in \Omega$

$$\begin{aligned} \|u(x)\| &\leq \|J_1(g(x)) - J_1(0)\| + \|J_1(0)\| \\ &\leq \|g(x)\| + \|J_1(0)\| \end{aligned}$$

and $u \in L^2(\Omega; H)$. Moreover, since $g(x) \in u(x) + A(u(x))$ for a.e. $x \in \Omega$, then $g \in u + \overline{A}(u)$ as wanted. ■

Observation 1.26. *With a similar proof to the above one, we have the following. If A is maximal monotone and $0 \in A(0)$ then $\overline{A}$ is maximal monotone.*

As already mentioned, if an everywhere defined function on $\mathbb{R}$ is continuous and non-decreasing, then it achieves maximal monotonicity. This generalizes nicely to arbitrary Hilbert spaces. In fact a weaker version of continuity is enough to conclude this. The theorem offers an easy way of showing maximal monotonicity.

Definition 1.27. *Let A be an everywhere defined single-valued operator on H. We say that A is hemicontinuous if for all $u, v \in H$*

$$A(u + tv) \rightharpoonup A(u) \text{ weakly in } H \text{ as } t \to 0.$$

Theorem 1.28. *Let A be an everywhere defined single-valued operator. If A is monotone and hemicontinuous, then A is maximal monotone.*

Proof. By contradiction, assume A is not maximal monotone. Then we can find $u_0, v_0 \in H$ such that $v_0 \neq A(u_0)$ and such that for all $u \in H$

$$(A(u) - v_0, u - u_0) \geq 0.$$

Taking $w \in H$, $t \in [0,1)$ and u as $tu_0 + (1-t)w$,

$$(A(tu_0 + (1-t)w) - v_0, (t-1)u_0 + (1-t)w) \geq 0.$$

Hence, we have that for all $t \in [0,1)$ and all $w \in H$

$$(A((1-t)u_0 + tw) - v_0, w - u_0) \geq 0.$$

Using hemicontinuity of A it follows that for all $w \in H$, $(A(u_0) - v_0, w - u_0) \geq 0$; so we make $w := u_0 - A(u_0) + v_0$ to get $A(u_0) = v_0$. ■

Observation 1.29. *As a result of the last theorem we now have: $I + T$ and $I - T$ are maximal monotone operators when T is an everywhere defined non-expansive function (see Example 1.8). In particular, for A maximal monotone and for every positive λ, A_λ and J_λ are maximal monotone. Another example of a maximal monotone operator that we now have is the Projection on a non-empty, convex and closed subset of H (see Example 1.9).*

From Minty's theorem we know that if A is maximal monotone then $I+A$ is surjective. But how far are maximal monotone operators from being surjective? Or, which property is necessary for a maximal monotone operator to get surjectivity? The answer is coercivity.

Definition 1.30. *An operator A on H is coercive if there exists $u \in H$ such that the following is true. For every sequence $([u_n, v_n]) \subseteq A$, satisfying $\|u_n\| \to \infty$ as $n \to \infty$ we have*

$$\lim_{n\to\infty} \frac{(u_n - u, v_n)}{\|u_n\|} = \infty.$$

Theorem 1.31. *If A is a coercive and maximal monotone operator, then A is surjective.*

Proof. Fix $v \in H$. For each $\lambda > 0$, Minty's theorem shows the existence of $u_\lambda \in D(A)$ such that $v \in \lambda u_\lambda + A(u_\lambda)$. Let $[u_\lambda, v_\lambda] \in A$ verify $v = \lambda u_\lambda + v_\lambda$. Consider $u_0 \in H$ given by the definition of coercivity and multiply by $u_\lambda - u_0$ to get for all $\lambda > 0$,

$$\lambda\|u_\lambda\|^2 - \lambda(u_\lambda, u_0) + (v_\lambda, u_\lambda - u_0) = (v, u_\lambda - u_0). \tag{1.5}$$

We claim that $\|u_\lambda\|$ is bounded for bounded λ. Suppose this is not the case, this is, there exists a bounded sequence of positive real numbers (λ_n) such that $\|u_{\lambda_n}\| \to \infty$ as $n \to \infty$. Using (1.5) we can get for each n,

$$\begin{aligned}\frac{(v_{\lambda_n}, u_{\lambda_n} - u_0)}{\|u_{\lambda_n}\|} &\leq \frac{(v, u_{\lambda_n} - u_0) + \lambda_n(u_{\lambda_n}, u_0)}{\|u_{\lambda_n}\|} \\ &\leq \lambda_n\|u_0\| + \|v\| + \frac{\|u_0\|\|v\|}{\|u_{\lambda_n}\|}.\end{aligned}$$

The right term in the last inequality is bounded, contradicting the coercivity of A. Hence, we can find a sequence $(\lambda_n) > 0$ converging to zero such that $u_{\lambda_n} \rightharpoonup u$ weakly in H as $n \to \infty$. On the other hand, notice that $\|v - v_{\lambda_n}\| = \lambda_n\|u_{\lambda_n}\|$; giving $v_{\lambda_n} \to v$ strongly in H as $n \to \infty$. Using that A^{-1} is maximal monotone, in particular demiclosed, we conclude $[u, v] \in A$ as wanted. ■

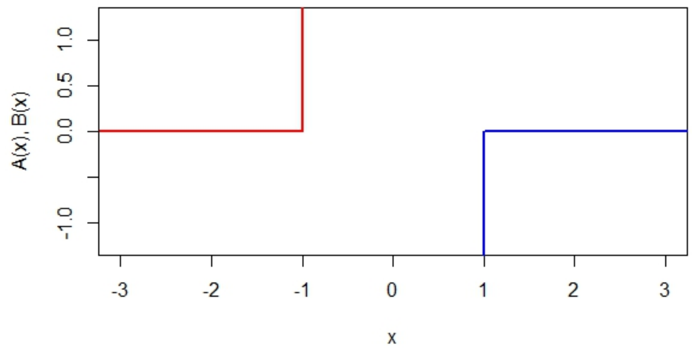

Figure 1.6: Graphs of the maximal monotone operators A and B respectively in red and blue.

1.3 Sum of Maximal Monotone Operators

In applications it occurs frequently that the associated equation is governed by a sum of maximal monotone operators. Unfortunately, although the sum of monotone operators is in fact monotone, the sum of maximal monotone operators needs not to be maximal monotone. What are the conditions needed for such conclusion? The domains of the operators should play a role in this; it is easy to construct two maximal monotone operators such that their domains do not intersect. Hence the sum will be the empty set which of course is not maximal monotone. More explicitly, consider $H = \mathbb{R}$ and A, B operators on H defined as follows:

$$A(x) := \begin{cases} 0, \text{ if } x \in (-\infty, -1) \\ [0, \infty), \text{ if } x = -1 \end{cases} \qquad \text{and} \quad B(x) := \begin{cases} (-\infty, 0], \text{ if } x = 1 \\ 0, \text{ if } x \in (1, \infty) \end{cases},$$

where $\mathrm{D}(A) := (-\infty, -1]$ and $\mathrm{D}(B) := [1, \infty)$. See Figure 1.6.

For the theorem we first need two lemmas; first we will show that the sum of a maximal monotone operator with a particular type of maximal monotone operator is again maximal monotone.

Lemma 1.32. *Let A be a maximal monotone operator and T be a single-valued, everywhere defined, monotone and Lipschitz continuous operator. Then $A + T$ is maximal monotone.*

Proof. Consider $c > 0$ Lipschitz constant for T and take $\lambda > 0$ such that $c < \frac{1}{\lambda}$. Fix any $v \in H$ and define the function $f : H \to H$ as

$$f(u) := (I + \lambda A)^{-1}(v - \lambda T(u)) = J_\lambda(v - \lambda T(u)).$$

By Corollary 1.22, we have that J_λ is a single-valued, everywhere defined, non-expansive operator. Then f is a contraction (Lipschitz continuous with constant lower than 1), this is, for all $u, w \in H$:

$$\begin{aligned}\|f(u) - f(w)\| &\leq \|\lambda T(u) - \lambda T(w)\| \\ &\leq c\lambda\|u - w\|.\end{aligned}$$

It follows from the Banach fixed point theorem the existence of $u \in H$ such that $u = f(u)$; this is

$$v \in u + \lambda A(u) + \lambda T(u)$$

and $\mathrm{R}(I + \lambda(A + T)) = H$. Since $A + T$ is monotone, using Minty's theorem we conclude that $A + T$ is maximal monotone. ∎

Let A and B be maximal monotone operators on H such that $\mathrm{D}(A) \cap \mathrm{D}(B) \neq \emptyset$. Moreover, suppose there exists $\epsilon > 0$ such that the sphere

$$S := \{z \in H : \|z\| = \epsilon\} \subseteq \mathrm{D}(B) - \mathrm{D}(A),$$

where $\mathrm{D}(B) - \mathrm{D}(A) := \{u \in H : \exists v \in \mathrm{D}(B), \exists w \in \mathrm{D}(A) \text{ such that } u = v - w\}$. Our aim is to show that $A + B$ is maximal monotone. The last lemma suggests the natural idea. By Corollary 1.22 and the previous lemma we have that $A + B_\lambda$ is maximal monotone for each $\lambda > 0$. We want to show that $\mathrm{R}(I + A + B) = H$; fix any $v \in H$. There exists $u_\lambda \in \mathrm{D}(A)$ such that $v \in u_\lambda + A(u_\lambda) + B_\lambda(u_\lambda)$. Now a first step is to have some kind of boundedness. This is what the next lemma gives.

Lemma 1.33. *In the context of the discussion above;* $\{u_\lambda : \lambda > 0\}$ *and* $\{B_\lambda(u_\lambda) : \lambda > 0\}$ *are bounded in* H.

Proof. Fix for now $\lambda > 0$ and let $v_\lambda \in A(u_\lambda)$ such that $v = u_\lambda + v_\lambda + B_\lambda(u_\lambda)$. Consider some $u_0 \in \mathrm{D}(A) \cap \mathrm{D}(B)$. Then, using the monotonicity of A:

$$\begin{aligned}(u_\lambda - u_0, v) &= (u_\lambda - u_0, v_\lambda - A^0(u_0)) + (u_\lambda - u_0, A^0(u_0) + u_\lambda + B_\lambda(u_\lambda)) \\ &\geq (u_\lambda - u_0, B_\lambda(u_\lambda) - B_\lambda(u_0)) + (u_\lambda - u_0, B_\lambda(u_0) + A^0(u_0) + u_\lambda).\end{aligned}$$

Using the monotonicity of B_λ it follows that

$$(u_\lambda - u_0, u_0 - u_\lambda) + (u_\lambda - u_0, -u_0 + v - B_\lambda(u_0) - A^0(u_0)) = (u_\lambda - u_0, v - B_\lambda(u_0) - A^0(u_0) - u_\lambda) \geq 0.$$

This implies

$$\|u_\lambda - u_0\| \leq \|v - u_0 - B_\lambda(u_0) - A^0(u_0)\|,$$

and using Corollary 1.22 we get for all $\lambda > 0$:

$$\|u_\lambda\| \leq 2\|u_0\| + \|v\| + \|B^0(u_0)\| + \|A^0(u_0)\|. \tag{1.6}$$

To have that $\{B_\lambda(u_\lambda) : \lambda > 0\}$ is a bounded subset of H, it suffices to apply the uniform boundedness principle to the family of linear and bounded functions on H, $\{(B_\lambda(u_\lambda), \cdot) : \lambda > 0\}$. In order to apply it, it is clearly enough to check this family is pointwise bounded at S.

Let $b - a = z \in S$, where $a \in \mathrm{D}(A)$ and $b \in \mathrm{D}(B)$. Using the monotonicity of B_λ:

$$(u_\lambda, -B_\lambda(u_\lambda)) + (-b, -B_\lambda(u_\lambda)) \leq (u_\lambda - b, -B_\lambda(b)),$$

and

$$(B_\lambda(u_\lambda), b) \leq (B_\lambda(u_\lambda), u_\lambda) - (B_\lambda(b), u_\lambda - b).$$

It follows that

$$\begin{aligned}(B_\lambda(u_\lambda), z) &\leq (B_\lambda(u_\lambda), u_\lambda) - (B_\lambda(b), u_\lambda - b) - (B_\lambda(u_\lambda), a) \\ &\leq (v - u_\lambda - v_\lambda, u_\lambda - a) + \|B^0(b)\|\|u_\lambda - b\|.\end{aligned}$$

On the other hand, using again that A is monotone we get

$$(v - u_\lambda - v_\lambda, u_\lambda - a) \leq (v - A^0(a) - u_\lambda, u_\lambda - a).$$

The last two inequalities, together with (1.6), imply

$$(B_\lambda(u_\lambda), z) \leq \|u_\lambda - a\|(\|v - u_\lambda\| + \|A^0(a)\|) + \|B^0(b)\|\|u_\lambda - b\| \leq c(z),$$

for some constant c depending on z and for all $\lambda > 0$. Considering $-z \in S$ we can conclude: For each $z \in S$, there exists a constant $c(z)$ such that

$$|(B_\lambda(u_\lambda), z)| \leq c(z),\ \forall \lambda > 0.$$

■

Now we apply the last lemma to the mentioned theorem. Notice that if A and B are two operators such that $0 \in \mathrm{Int}(\mathrm{D}(B) - \mathrm{D}(A))$, then $\mathrm{D}(A) \cap \mathrm{D}(B) \neq \emptyset$. Moreover, there is a sphere centered at 0 contained in $\mathrm{D}(B) - \mathrm{D}(A)$.

Theorem 1.34 (Attouch)**.** *Let A and B be maximal monotone operators. If the zero vector is in the interior of the algebraic difference of the domains, then $A+B$ is maximal monotone.*

Proof. Fix any $v \in H$. By Lemma 1.32 we have that for $\lambda > 0$, $A + B_\lambda$ is maximal monotone. Let $u_\lambda \in \mathrm{D}(A)$ be such that $v \in u_\lambda + A(u_\lambda) + B_\lambda(u_\lambda)$. From Lemma 1.33 it follows that $\{u_\lambda : \lambda > 0\}$ and $\{B_\lambda(u_\lambda) : \lambda > 0\}$ are bounded in H. We will now show that (u_λ) is a Cauchy sequence. Let $\lambda, \mu > 0$, we have that $u_\lambda - u_\mu \in A(u_\mu) - A(u_\lambda) + B_\mu(u_\mu) - B_\lambda(u_\lambda)$; multiplying this by $u_\lambda - u_\mu$ and using A is monotone

$$\|u_\lambda - u_\mu\|^2 \leq (u_\lambda - u_\mu, B_\mu(u_\mu) - B_\lambda(u_\lambda)). \tag{1.7}$$

Notice that from Corollary 1.22:

$$\begin{aligned}(B_\lambda(u_\lambda) - B_\mu(u_\mu), \lambda B_\lambda(u_\lambda) - \mu B_\mu(u_\mu)) &= (B_\lambda(u_\lambda) - B_\mu(u_\mu), -J_{B,\lambda}(u_\lambda) + J_{B,\mu}(u_\mu)) \\ &\quad + (B_\lambda(u_\lambda) - B_\mu(u_\mu), u_\lambda - u_\mu) \\ &\leq (B_\lambda(u_\lambda) - B_\mu(u_\mu), u_\lambda - u_\mu).\end{aligned}$$

Using this last inequality and (1.7), there exists some constant c such that for every $\lambda, \mu > 0$:

$$\|u_\lambda - u_\mu\|^2 \leq (\|B_\lambda(u_\lambda)\| + \|B_\mu(u_\mu)\|)(\lambda\|B_\lambda(u_\lambda)\| + \mu\|B_\mu(u_\mu)\|) \leq c(\lambda + \mu).$$

This implies there is $u \in H$ such that

$$u_\lambda \to u \text{ strongly in } H \text{ as } \lambda \to 0^+. \tag{1.8}$$

Going to a subsequence if necessary, there is also $w \in H$ such that

$$B_\lambda(u_\lambda) \rightharpoonup w \text{ weakly in } H \text{ as } \lambda \to 0^+. \tag{1.9}$$

Hence, as $\lambda \to 0^+$:

$$\|J_{B,\lambda}(u_\lambda) - u\| \leq \|u_\lambda - u\| + \lambda\|B_\lambda(u_\lambda)\| \to 0.$$

The demiclosedness of B gives

$$[u, w] \in B. \tag{1.10}$$

Let $v_\lambda \in A(u_\lambda)$ be such that $v_\lambda = v - u_\lambda - B_\lambda(u_\lambda)$; then (v_λ) is bounded and, going through a subsequence if necessary, $v_\lambda \rightharpoonup z$ weakly in H as $\lambda \to 0^+$. Using now that A is demiclosed we get

$$[u, z] \in A.$$

Finally, this together with (1.8), (1.9) and (1.10) yield $v \in u + A(u) + B(u)$. We then conclude by Minty's theorem. ■

Observation 1.35 (Rockafellar Condition). *Let A and B be maximal monotone operators. To get maximal monotonicity for $A + B$, sometimes is easier to check that* $\mathrm{Int}(\mathrm{D}(B)) \cap \mathrm{D}(A) \neq \emptyset$. *This clearly implies that* $0 \in \mathrm{Int}(\mathrm{D}(B) - \mathrm{D}(A))$.

Observation 1.36. *Other than the maximality of the operators A and B, the only hypothesis needed to conclude $A + B$ is maximal monotone is that* $0 \in \mathrm{Int}(\mathrm{D}(B) - \mathrm{D}(A))$. *In fact, as mentioned before, it suffices to have that:*

1) *The domains intersect.*
2) *There is some sphere centered at zero contained in* $\mathrm{D}(B) - \mathrm{D}(A)$.

The first of these conditions is used to show that (u_λ) is bounded. Having this, the second condition gives the boundedness of $(B_\lambda(u_\lambda))$.

In applications sometimes the hypotheses of the latter theorem are not satisfied but the following are.

Corollary 1.37. *Let A and B be maximal monotone operators. If their domains intersect and if for all $[u, v] \in A$ and all $\lambda > 0$:*

$$(B_\lambda(u), v) \geq 0,$$

then $A + B$ is maximal monotone.

Proof. With the notation of the proof of Attouch's theorem, we had that $v = u_\lambda + v_\lambda + B_\lambda(u_\lambda)$, where $v_\lambda \in A(u_\lambda)$. Multiplying the equation by v_λ we get

$$\|v_\lambda\|^2 + (v_\lambda, B_\lambda(u_\lambda)) = (v, v_\lambda) - (u_\lambda, v_\lambda).$$

It follows that

$$\|v_\lambda\| \le \|v - u_\lambda\|,$$

implying the boundedness of $(B_\lambda(u_\lambda))$. The proof is complete when considering the previous observation. ∎

1.4 Maximal Monotone Operators on Banach Spaces

Theorems 1.28, 1.31 and 1.34 hold in a different context; we will need to use them later in this sense. To define a monotone operator we used the inner product of the real Hilbert space H. The definition can be generalized naturally to real Banach spaces. Let V be a real Banach space with norm $\|\cdot\|$; we denote its topological dual space by V^* and its norm $\|\cdot\|_*$. If $v \in V^*$ and $u \in V$ then we put $(u, v)_{VV^*} := v(u)$. Let also $A : D(A) \subseteq V \to V^*$ be a multi-valued operator on V, this is, a subset of $V \times V^*$. The latter will be indicated by just saying that A is an operator from V to V^*. We say that A is a monotone operator if

$$(u_1 - u_2, v_1 - v_2)_{VV^*} \ge 0, \; \forall [u_1, v_1], [u_2, v_2] \in A.$$

As before, we say that A is a maximal monotone operator if A is monotone and it is maximal with this property among the subsets of $V \times V^*$. If A is everywhere defined and single-valued then we say it is hemicontinuous if for every $u, v \in V$ as $t \to 0$ we have:

$$A(u + tv) \to A(u),$$

in the weak star topology of V^*. The corresponding definitions of sum of operators, surjectivity, strong monotonicity and coercivity can be formulated also in the obvious way. Later in the thesis we will need the next lemma. Its proof follows easily from the definitions.

Lemma 1.38. *Let A and B be operators from V to V^*. If A is strongly monotone and B is monotone, then A is coercive and $A + B$ is strongly monotone.*

Now suppose that V is a real Hilbert space which is not identified with its dual. Consider its structure of real Banach space to apply the previous definitions. The above mentioned theorems can be restated as follows.

Theorem 1.39. *Let A be an operator from V to V^*. If A is monotone and hemicontinuous, then A is maximal monotone.*

Theorem 1.40. *Let A be an operator from V to V^*. If A is maximal monotone and coercive, then A is surjective.*

Theorem 1.41. *Let A and B be operators from V to V^*. If $\mathrm{D}(A) \cap \mathrm{Int}(\mathrm{D}(B)) \neq \emptyset$, then $A + B$ is maximal monotone.*

The original theorems along with the Riesz Representation theorem are used to show these versions.

Even when V is just a real Banach space there are a lot of similar results concerning Monotone Operator Theory. For example, define the following operator $F : D(F) = V \to V^*$ as

$$F(u) := \{v \in V^* : (u, v)_{VV^*} = \|u\|^2 = \|v\|_*^2\}.$$

Then Minty's theorem generalizes in the following way. Let V be reflexive and A a monotone operator from V to V^*. Then A is maximal monotone if and only if $A + F$ is surjective. This theorem allows us to extend, in a useful way, the concepts of resolvent and Yosida approximation in this context. The surjectivity theorem also holds, this is: If V is reflexive and A is a coercive maximal monotone operator from V to V^*, then A is surjective. Theorem 1.41 above also remains true for a reflexive real Banach space V; but the most famous open problem in monotone operator theory is the following, see [22].

Conjecture 1.42. *Suppose that V is any real Banach space and let A and B be operators from V to V^*. If $\mathrm{D}(A) \cap \mathrm{Int}(\mathrm{D}(B)) \neq \emptyset$, then $A + B$ is maximal monotone.*

When extending the theory to Banach spaces, reflexivity almost always plays a fundamental role. In this context we refer to [6].

Chapter 2

Existence of Solutions and Subdifferentials

This is the main chapter of the thesis. Recall that H denotes a real Hilbert space and let T be a positive real number. We are interested in the following problem:

$$(p)\begin{cases} u'(t) + Au(t) \ni f(t),\ t \in (0,T) \\ u(0) = u_0, \end{cases}$$

where A operator on H, $f : [0,T] \to H$ function and $u_0 \in H$ are given. The unknown function is $u : [0,T] \to H$. This is a first-order differential inclusion, with initial condition u_0, governed by the operator A. Usually the variable t is interpreted as time and the equalities in (p) hold in H; so we also say this is an evolution inclusion in H.

A huge number of differential equations/inclusions that arise from applications can be written in the form of (p). The only thing one has to do is to choose T, H, A, u_0 and f such that the problem (p) results in an equivalent formulation of our original problem.

By studying the problem in this abstract setting, we are able to treat a large amount of problems at the same time.

Numerous questions can be formulated concerning problem (p). The most important one being about the existence or non-existence of a "solution". The second, in the case of having existence, is about uniqueness. The first goal of this chapter is to show the existence and uniqueness of a "solution" for (p) when the given setting satisfies some conditions. The central condition is the maximal monotonicity of A; this will be needed to follow the strategy discussed in the previous chapter. Hence, Corollary 1.22 will play a fundamental role here. Other questions that can be asked about (p) are related to the regularity of its solutions. This will not be treated in detail in this thesis but we refer the reader to [19]. For the sake of completeness, in this chapter we will also show the existence result for the case when A is a Lipschitz continuous function (which we will need).

The second goal is, after making clear the importance of maximal monotonicity, to present the subdifferential of a function. This is a multi-valued generalization of the directional derivative in Hilbert spaces. We are interested in this, since as we will show, the subdifferential of a certain class of functions are always maximal monotone operators.

We now introduce the main function spaces that will be used in this thesis.

2.1 Function Spaces

This section recalls the definitions of function spaces used in this thesis, particularly in this chapter. The purpose is not to introduce or build on these definitions but only to have them as references for the proofs. We also omit showing that the concepts are well-defined.

As we said, depending on the conditions of the given data we may conclude the existence of a solution for (p). Obviously, we want to make these conditions as weak as possible. It turns out that it is enough for f to be in a not so restrictive function space, namely $W^{1,1}(0,T;H)$. We will define this space here among many others. First we generalize in a natural way the integral for vector-valued functions defined on an open interval. All the usual integrals here are considered to be in the Lebesgue sense.

Let V be a real Banach space with norm $\|\cdot\|$ and T be a positive real number. Let $f:(0,T)\to V$ be a function. We say f is simple if we can find n positive integer, $E_1,..,E_n$ disjoint Lebesgue measurable subsets of $(0,T)$ and $c_1,..,c_n\in V$ such that

$$f=\sum_{i=1}^{n} c_i 1_{E_i},$$

where 1_{E_i} denotes the characteristic function of E_i on $(0,T)$. The Bochner integral of such a function is defined as

$$\int_0^T f:=\sum_{i=1}^{n} c_i|E_i|\in V.$$

We say f is strongly measurable if there exists a sequence (f_n) of simple functions converging to f pointwise a.e. on $(0,T)$ as n tends to ∞. So as one might expect, we say that f is Bochner integrable if f is strongly measurable, say by (f_n), and as n tends to ∞:

$$\int_0^T \|f-f_n\|\to 0.$$

The Bochner integral for such a function f is defined as

$$\int_0^T f:=\lim_{n\to\infty}\int_0^T f_n\in V.$$

The usual results, well interpreted, hold for this kind of vector-valued integrals. One of the main tools is Bochner's theorem which states that f is Bochner integrable if and only if f is strongly measurable and $\int_0^T\|f\|<\infty$.

Now let $p\in[1,\infty)$. The Lebesgue space $L^p(0,T;V)$ is the set of strongly measurable functions $u:(0,T)\to V$ such that $\int_0^T\|u\|^p<\infty$. To be more precise, each element of this space is an equivalence class. This class is induced by the relation of being equal a.e. in $(0,T)$ defined in the set of strongly measurable functions. Of course, in practice we just consider them as functions unless needed otherwise. We can give it the norm

$$\|u\|_{L^p(0,T;V)}:=\left(\int_0^T\|u\|^p\right)^{\frac{1}{p}}.$$

Similarly, the Lebesgue space $L^\infty(0,T;V)$ is the set of strongly measurable functions (classes) $u:(0,T)\to V$ such that there exists $c\geq 0$ satisfying $\|u(x)\|\leq c$ for a.e. x in $(0,T)$. The norm for this space is

$$\|u\|_{L^\infty(0,T;V)} := \inf\{c\geq 0 : \|u(x)\|\leq c \text{ for a.e. } x\in(0,T)\}.$$

These generalized versions of the usual Lebesgue spaces are still real Banach spaces for every $p\in[1,\infty]$. Moreover, the Duality theorem extends also. Assume the usual convention when dividing by zero; let $p\in[1,\infty)$ and $q=\frac{p}{p-1}$. The next isomorphism means isometric linear bijection.

Theorem 2.1 (Duality)**.** *If V is reflexive, then $L^q(0,T;V^*)\cong L^p(0,T;V)^*$. Explicitly,*

$$L^q(0,T;V^*)\ni v\mapsto \left(u\mapsto \int_0^T (u(t),v(t))_{VV^*}dt\right)\in L^p(0,T;V)^*.$$

Observation 2.2. *In particular, if $H:=V$ is a Hilbert space we get*

$$L^q(0,T;H)\cong L^p(0,T;H)^*,$$

given by

$$L^q(0,T;H)\ni v\mapsto \left(u\mapsto \int_0^T (u,v)\right)\in L^p(0,T;H)^*.$$

Let us define now the Sobolev spaces for vector-valued functions on $(0,T)$. In an analogous way, we can define the Lebesgue spaces $L^p(A;V)$ for any Lebesgue measurable subset A of R. We say then that $u\in L^1_{loc}(0,T;V)$ if, after restricting its domain, $u\in L^1(K;V)$ for every K compact subset of $(0,T)$. Denote also by $C_c^\infty(0,T)$ the set of all infinitely differentiable functions $\phi:(0,T)\to \mathrm{R}$ with a compact support in $(0,T)$.

Definition 2.3. *We say that $u\in L^1_{loc}(0,T;V)$ is weakly differentiable if there exists $v\in L^1_{loc}(0,T;V)$ such that*

$$\int_0^T \phi' u = -\int_0^T \phi v,\ \forall\phi\in C_c^\infty(0,T).$$

In this case v is called the weak derivative of u and denoted by Du.

Definition 2.4. *The Sobolev space for k non-negative integer and $p\in[1,\infty]$ is*

$$W^{k,p}(0,T;V) := \{u\in L^p(0,T;V) : D^i u\in L^p(0,T;V),\ \forall i\in[1,k]\},$$

and we equipped it with the following norms. If $p\neq\infty$,

$$\|u\|_{W^{k,p}(0,T;V)} := \left(\sum_{i=0}^{k}\|D^i u\|^p_{L^p(0,T;V)}\right)^{\frac{1}{p}}.$$

And if $p=\infty$,

$$\|u\|_{W^{k,p}(0,T;V)} := \max_{0\leq i\leq k}\|D^i u\|_{L^p(0,T;V)}.$$

Again we have that $W^{k,p}(0,T;V)$ is a real Banach space for all such k and p. Of special interest is when $H := V$ is a real Hilbert space and $p = 2$. Let $(\cdot,\cdot)$ denote the inner product of H. In this case we put

$$H^k(0,T;H) := W^{k,2}(0,T;H),$$

which is again a Hilbert space. In fact, its norm is induced by the inner product

$$(u,v)_{H^k(0,T;H)} := \sum_{i=0}^{k} \int_0^T (D^i u, D^i v).$$

In the particular case when $V := \mathbb{R}$, with its usual norm, we omit writing V in the definition of this spaces. This is, we just write $L^p(0,T)$, $W^{k,p}(0,T)$ and $H^k(0,T)$ and we recover the known spaces. When there is no risk of confusion, we just write u' instead of Du for the weak derivative of u.

For a complete introduction to the concept of the Bochner integral see [14]. To learn more about the vector-valued Lebesgue and Sobolev spaces we refer the reader to [4]. Finally, for a first study of Lebesgue and Sobolev spaces see [11].

2.2 Existence of Solutions

The aim of this section is to show the main theorem of the thesis. Throughout this section T denotes a positive real number. We will show the existence and uniqueness of a "solution" for the problem:

$$(p) \begin{cases} u' + Au \ni f, \text{ on } (0,T) \\ u(0) = u_0, \end{cases}$$

where A is a maximal monotone operator on H, $f \in W^{1,1}(0,T;H)$ and $u_0 \in D(A)$. Moreover, the solution u will have a desirable property: $u \in W^{1,\infty}(0,T;H)$. We need several preliminary results. Among them: the existence theorem when A is a Lipschitz continuous function, which is the central argument; an important technical lemma which actually gives $u \in W^{1,\infty}(0,T;H)$; and a constantly used inequality of the Gronwall type and its consequences (uniqueness will follow easily from this).

Let V be a real Banach space with norm $\|\cdot\|$. Consider the problem

$$(q) \begin{cases} u' + Bu = f, \text{ on } [0,T] \\ u(0) = u_0, \end{cases}$$

where $B : V \to V$ is an everywhere defined Lipschitz continuous function, $f \in C([0,T];V)$ and $u_0 \in V$. Recall that $C([0,T];V)$ is a real Banach space under the usual supremum norm $\|v\|_\infty := \sup_{t\in[0,T]} \|v(t)\|$.

Lemma 2.5. *There exists a unique $u \in C^1([0,T];V)$ satisfying (q).*

Proof. Let $W := C([0,T];V)$ and define the function $F : W \to W$ given by

$$F(v)(t) := u_0 - \int_0^t B(v) + \int_0^t f, \, \forall t \in [0,T].$$

We translate the problem (q) to an equivalent integral equation using basic properties of the Bochner integral; $u \in W$ is a solution of (q) in $C^1([0,T];V)$ if and only if u is a fixed point of F. Consider a Lipschitz constant $L > 0$ for B and define in W the so-called Bielecki norm

$$\|v\|_B := \sup_{t\in[0,T]} e^{-2Lt}\|v(t)\|.$$

Clearly this is an equivalent norm to the supremum one, hence $(W, \|\cdot\|_B)$ is a real Banach space. Therefore, it suffices to show F is a contraction to conclude with the Banach fixed point theorem. In fact, for all $v, w \in W$ and all $t \in [0,T]$ the following is satisfied:

$$\begin{aligned}\|F(v)(t) - F(w)(t)\| &= \left\| \int_0^t B(v) - B(w) \right\| \\ &\le L \int_0^t \|v - w\| \\ &\le L\|v - w\|_B \int_0^t e^{2Ls} ds \\ &\le \frac{e^{2Lt}}{2} \|v - w\|_B.\end{aligned}$$

It follows that for all $v, w \in W$,

$$\|F(v) - F(w)\|_B \le \frac{1}{2}\|v - w\|_B.$$

■

Notice that at 0 the derivative of u only makes sense from the right, we will frequently avoid the extra notation since it is deduced from the context. Now let us recall a basic analysis result. Let $f, g : [0,T] \to \mathrm{R}$ be any functions and (f_k) a sequence in $C^1([0,T])$ such that f_k converges pointwise to f. Moreover, assume that (f_k') converges uniformly to g. Then $f \in C^1([0,T])$ and $f' = g$. This motivates the following lemma, which is essential for the existence theorem. Consider Observation 2.2.

Lemma 2.6. *Let (u_k) be a sequence in $C^1([0,T];H)$. Suppose*

$$\begin{aligned}&u_k \to u \text{ in } C([0,T];H) \text{ and} \\ &u_k' \overset{*}{\rightharpoonup} v \text{ weakly star in } L^\infty(0,T;H),\end{aligned}$$

then $u \in W^{1,\infty}(0,T;H)$ and $u' = v$.

Proof. By definition we have that for every $w \in L^1(0,T;H)$,

$$\int_0^T (u_k', w) \to \int_0^T (v, w).$$

Take any $\varphi \in C_c^\infty(0,T)$ and any $h \in H$. Then as k tends to ∞,

$$\left|\int_0^T (\varphi(t)u_k'(t) - \varphi(t)v(t), h)dt\right| \leq \|\varphi\|_\infty \left|\int_0^T (u_k'(t) - v(t), h)dt\right| \to 0.$$

On the other hand, it is not hard to see that

$$\int_0^T (\varphi(t)u_k'(t) - \varphi(t)v(t), h)dt = \left(\int_0^T \varphi(t)u_k'(t) - \varphi(t)v(t)dt, h\right).$$

The two previous statements imply that

$$\int_0^T \varphi u_k' \rightharpoonup \int_0^T \varphi v \text{ weakly in } H.$$

Using integration by parts and the Lebesgue Dominated Convergence theorem, as k tends to ∞:

$$\int_0^T \varphi u_k' = -\int_0^T \varphi' u_k \to -\int_0^T \varphi' u.$$

This is, u is weakly differentiable and $u' = v$. ■

An important estimation tool in analysis is the known lemma named after T. H. Gronwall. It works for estimating functions satisfying certain differential inequality and it has several different versions. The next lemma resembles this results and will be used frequently in the proof of existence and uniqueness. The proof is inspired in [10].

Lemma 2.7 (Gronwall Type)**.** *Let $a < b$ and c be real numbers. Consider $0 \leq \psi \in L^1(a,b)$ and $h \in C([a,b])$ such that for every $t \in [a,b]$:*

$$\frac{1}{2}h^2(t) \leq \frac{1}{2}c^2 + \int_a^t \psi h.$$

Then, for all $t \in [a,b]$:

$$|h(t)| \leq |c| + \int_a^t \psi.$$

Proof. Notice that in fact $\psi h \in L^1(a,b)$. The following inequalities are satisfied for a.e. t in (a,b). Take any $\epsilon > 0$ and define

$$m_\epsilon(t) := \frac{1}{2}(|c| + \epsilon)^2 + \int_a^t \psi h.$$

Then m_ϵ is differentiable for a.e. $t \in (a,b)$ and $m_\epsilon' = \psi h$. Using our assumption and the latter we have

$$m_\epsilon \geq \frac{1}{2}h^2 \text{ and } \psi\sqrt{2m_\epsilon} \geq m_\epsilon'. \tag{2.1}$$

Using again our hypothesis:

$$m_\epsilon \geq \frac{1}{2}\epsilon^2.$$

This is the reason for introducing ϵ, because now we have $\sqrt{m_\epsilon}$ is also differentiable for a.e. $t \in (a, b)$ with derivative $\frac{1}{2\sqrt{m_\epsilon}} m'_\epsilon$. It follows from (2.1) that $\frac{1}{\sqrt{2}}\psi \geq (\sqrt{m_\epsilon})'$, integrating this inequality from a to t:

$$\sqrt{m_\epsilon(t)} - \sqrt{m_\epsilon(a)} \leq \frac{1}{\sqrt{2}} \int_a^t \psi.$$

This together with (2.1) and the definition of m_ϵ imply, for every $t \in [a, b]$,

$$|h(t)| \leq (|c| + \epsilon) + \int_a^t \psi.$$

We conclude by making $\epsilon \to 0$. ■

Lemma 2.8. *Let A be a monotone operator on H, $u_1, u_2 \in W^{1,1}(0, T; H)$ and $v_1, v_2 \in L^1(0, T; H)$. Suppose that for a.e. $t \in (0, T)$ we have*

$$[u_1(t), v_1(t)], [u_2(t), v_2(t)] \in A.$$

Then for every $t \in [0, T]$,

$$\|u_1(t) - u_2(t)\| \leq \|u_1(0) - u_2(0)\| + \int_0^t \|v_1 + u'_1 - (v_2 + u'_2)\|.$$

Proof. Define $f : [0, T] \to \mathbb{R}$ by $f(t) := \frac{1}{2}\|u_1(t) - u_2(t)\|^2$. This function is differentiable a.e. in $(0, T)$ and by the chain rule

$$f'(t) = (u_1(t) - u_2(t), u'_1(t) - u'_2(t)).$$

Hence, using the monotonicity of A:

$$\begin{aligned} f'(t) &\leq (u_1(t) - u_2(t), v_1(t) + u'_1(t) - v_2(t) - u'_2(t)) \\ &\leq \|u_1(t) - u_2(t)\| \|v_1(t) + u'_1(t) - v_2(t) - u'_2(t)\|. \end{aligned}$$

Integrate the inequality to get for every $t \in [0, T]$,

$$\begin{aligned} f(t) - f(0) &= \frac{1}{2}\|u_1(t) - u_2(t)\|^2 - \frac{1}{2}\|u_1(0) - u_2(0)\|^2 \\ &\leq \int_0^t \|u_1 - u_2\| \|v_1 + u'_1 - v_2 - u'_2\|. \end{aligned}$$

We finish the proof by applying Lemma 2.7. ■

It will be useful to refer to the next evident identity in the existence result.

Observation 2.9. *If $\lambda, \mu > 0$ and A is a maximal monotone operator on H, then*

$$u - v = J_\lambda(u) - J_\mu(v) + \lambda A_\lambda(u) - \mu A_\mu(v),\ \forall u, v \in H.$$

For the rest of the section, fix an operator $A : \mathrm{D}(A) \subseteq H \to H$, $f : [0,T] \to H$ a function and $u_0 \in H$. Consider again our problem

$$(p) \begin{cases} u' + Au \ni f, \text{ on } (0,T) \\ u(0) = u_0. \end{cases}$$

What properties a solution of (p) should satisfy? First of all $u : [0,T] \to H$ should be a function such that $u(0) = u_0$. Moreover, u should satisfy $u'(t) + A(u(t)) \ni f(t)$ for a.e. $t \in (0,T)$; so u has to be differentiable a.e. on $(0,T)$ and has to verify $u(t) \in D(A)$ for a.e. t in $(0,T)$.

Definition 2.10. *We say that $u \in C([0,T]; H)$ is a solution of (p) if the conditions above are satisfied.*

We point out some details. In fact, we are going to be able to find u satisfying the inclusion for all t in $[0,T)$ but using its right derivative. As we said, further we will ask $f \in W^{1,1}(0,T;H)$ so it does not make sense to want to find u satisfying the inclusion for every t. It is a known fact that such an f will have a continuous representative. So whenever necessary, we will consider f as an element of $C([0,T];H)$. Moreover, we would not show directly that u is differentiable a.e., we will show that $u \in W^{1,\infty}(0,T;H)$ which implies it.

Lemma 2.11 (Uniqueness)**.** *Suppose f belongs to $L^1(0,T;H)$. There exists at most one solution u in $W^{1,1}(0,T;H)$ of problem (p).*

Proof. Suppose that u and v are solutions of (p) such that $u, v \in W^{1,1}(0,T;H)$. Then, we have that a.e. on $(0,T)$:

$$f - u' \in A(u) \text{ and } f - v' \in A(v).$$

Since we also have $u(0) = v(0) = u_0$, by applying Lemma 2.8 we get for all $t \in (0,T)$:

$$\|u(t) - v(t)\| = 0.$$

∎

Observation 2.12 (Continuous dependence on the data)**.** *Similarly, if the data of the problem f and u_0 change slightly, then the corresponding solutions, if any, would not differ much. More precisely, suppose u solves (p) and v solves (p) after changing f for g, both in $L^1(0,T;H)$, and u_0 for v_0. Then*

$$\|u - v\|_\infty \le \|u_0 - v_0\| + \int_0^T \|f - g\|.$$

Observation 2.13. *Suppose that $f \in W^{1,1}(0,T;H)$, A is maximal monotone and $u_0 \in \mathrm{D}(A)$. Using Corollary 1.22 and Lemma 2.5, we have that for each positive λ there exist a unique $u_\lambda \in C^1([0,T];H)$ satisfying*

$$(q_\lambda) \begin{cases} u_\lambda'(t) + A_\lambda(u_\lambda(t)) = f(t), \ \forall t \in [0,T] \\ u_\lambda(0) = u_0. \end{cases}$$

The condition $u_0 \in \mathrm{D}(A)$ is not necessary for this conclusion but will be important in the next lemmas.

A first step for proving the existence of a solution of (p) is showing that u_λ converges to u, first we show boundedness under A_λ.

Lemma 2.14 (Boundedness)**.** *In the context discussed above, $A_\lambda(u_\lambda(t))$ is uniformly bounded in H for all t in $[0,T]$ and all positive λ.*

Proof. Fix for now $\lambda > 0$ and h real number. For each $t, t+h$ in $[0,T]$ we have

$$\begin{gathered} f(t) - u_\lambda'(t) \in A_\lambda(u_\lambda(t)), \\ f(t+h) - u_\lambda'(t+h) \in A_\lambda(u_\lambda(t+h)). \end{gathered}$$

Hence, Lemma 2.8 implies that for all $t, t+h$ in $[0,T]$:

$$\|u_\lambda(t+h) - u_\lambda(t)\| \leq \|u_\lambda(h) - u_0\| + \int_0^t \|f(s+h) - f(s)\| ds.$$

Dividing by h, tending h to zero (from the right) and using the Lebesgue Dominated Convergence theorem we get:

$$\|u_\lambda'(t)\| \leq \|u_\lambda'(0)\| + \int_0^t \|f'(s)\| ds, \ \forall t \in [0,T].$$

Using now that u_λ verifies (q_λ), u_0 is in $\mathrm{D}(A)$ and Corollary 1.22, we get for all $t \in [0,T]$:

$$\begin{aligned} \|A_\lambda(u_\lambda(t))\| &= \|f(t) - u_\lambda'(t)\| \\ &\leq \|f(t)\| + \|f(0)\| + \|A^0(u_0)\| + \int_0^T \|f'(s)\| ds. \end{aligned}$$

Since $f \in C([0,T];H)$, we conclude that there exists a constant $C > 0$ such that for all t in $[0,T]$ and all positive λ:

$$\|A_\lambda(u_\lambda(t))\| \leq C.$$

■

To which u should the sequence (u_λ) converge? As is common in this field, we would not have an explicit expression of the candidate for solution u. Instead, we want to use that our space $C([0,T];H)$ has no "holes", this is, it is a complete space. So all we need to show existence is that the sequence (u_λ) is a Cauchy sequence. We will use the previous lemma for achieving this.

Lemma 2.15 (Convergence)**.** *In the context of Observation 2.13, there exists a function u such that as $\lambda \to 0^+$,*

$$u_\lambda \to u \text{ in } C([0,T];H).$$

Proof. Fix for now any λ and μ positive real numbers. The next relations hold for every t in $[0,T]$. Since u_λ, u_μ solve their respective problems:

$$u'_\lambda(t) - u'_\mu(t) + A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t)) = 0.$$

By the chain rule then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 &= (u_\lambda(t) - u_\mu(t), u'_\lambda(t) - u'_\mu(t)) \\ &= (u_\mu(t) - u_\lambda(t), A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))).\end{aligned}$$

Hence, by the identity in Observation 2.9 we have

$$\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 = -2(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t)), \lambda A_\lambda(u_\lambda(t)) - \mu A_\mu(u_\mu(t))) \tag{2.2}$$
$$-2(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t)), J_\lambda(u_\lambda(t)) - J_\mu(u_\mu(t))). \tag{2.3}$$

We express the term in (2.3) this way since we want to use the monotonicity of A. In fact, by Corollary 1.22 we have that for all $\nu > 0$:

$$A_\nu(u_\nu(t)) \in A(J_\nu(u_\nu(t))),$$

giving

$$(J_\lambda(u_\lambda(t)) - J_\mu(u_\mu(t)), A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))) \geq 0.$$

Concerning the term in (2.2), notice that

$$(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t)), \lambda A_\lambda(u_\lambda(t)) - \mu A_\mu(u_\mu(t))) \geq -(\lambda + \mu)\|A_\lambda(u_\lambda(t))\|\|A_\mu(u_\mu(t))\|.$$

Hence, by Lemma 2.14 there exists a constant $C > 0$ such that for all $\lambda, \mu > 0$:

$$\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 \leq C(\lambda + \mu).$$

Take any $t \in (0,T)$ and $\lambda, \mu > 0$. Consider the function $h : [0,t] \to \mathbb{R}$ given by $h(s) := \|u_\lambda(s) - u_\mu(s)\|^2$. Applying the Mean Value theorem to it we get

$$\|u_\lambda(t) - u_\mu(t)\| \leq Ct^{\frac{1}{2}}(\lambda + \mu)^{\frac{1}{2}}.$$

Finally, it follows that for all positive λ and μ:

$$\|u_\lambda - u_\mu\|_\infty \leq CT^{\frac{1}{2}}(\lambda + \mu)^{\frac{1}{2}}.$$

This means that (u_λ) is a Cauchy sequence, since $C([0,T];H)$ is a Banach space, we conclude that (u_λ) converges to some u in $C([0,T];H)$ as $\lambda \to 0^+$. ■

The next natural step is to see how the sequence (u'_λ) is behaving. We want it to converge to u' from the last theorem. The idea to get this is to weaken a lot the topology in the adequate space. Then the existence result will be finished by applying the demi-closedness of the canonical extension of A with respect to $(0,T)$. Let us write the theorem and see this in detail.

Theorem 2.16 (Existence). *If A is maximal monotone, f is in $W^{1,1}(0,T;H)$ and u_0 belongs to $\mathrm{D}(A)$, then problem (p) has a solution in $W^{1,\infty}(0,T;H)$.*

Proof. Consider the sequence (u_λ) like in Observation 2.13. By the lemma above, we have that

$$u_\lambda \to u \text{ in } C([0,T];H) \text{ as } \lambda \to 0^+, \tag{2.4}$$

in particular this implies $u(0) = u_0$.

On the other hand, since $L^1(0,T;H)$ is separable, the known sequential version of the Banach-Alaoglu theorem implies that the closed unit ball in $L^1(0,T;H)^*$ is sequentially compact in the weak star topology. From Lemma 2.14 it is clear that (u'_λ) is bounded in $L^\infty(0,T;H)$. Hence, there exists a function v such that as $\lambda \to 0^+$ and going through a subsequence

$$u'_\lambda \overset{*}{\rightharpoonup} v \text{ weakly star in } L^\infty(0,T;H).$$

Making use of Lemma 2.6, $u \in W^{1,\infty}(0,T;H)$ and $u' = v$. In particular this implies weak convergence in $L^2(0,T;H)$, so in fact

$$A_\lambda(u_\lambda) \rightharpoonup f - u' \text{ weakly in } L^2(0,T;H).$$

Moreover, using Lemma 2.14 again, there exists a constant $C > 0$ such that for every $t \in [0,T]$ and every $\lambda > 0$:

$$\begin{aligned} \|J_\lambda(u_\lambda(t)) - u(t)\| &\le \lambda \|A_\lambda(u_\lambda(t))\| + \|u_\lambda(t) - u(t)\| \\ &\le \lambda C + \|u_\lambda - u\|_\infty. \end{aligned}$$

This implies, by (2.4), that

$$J_\lambda(u_\lambda) \to u \text{ in } L^2(0,T;H) \text{ as } \lambda \to 0^+.$$

Finally,

$$A_\lambda(u_\lambda(t)) \in A(J_\lambda(u_\lambda(t)))$$

and we can use the demiclosedness of the canonical extension of A to $L^2(0,T;H)$. This gives for a.e. $t \in (0,T)$ that $u(t) \in \mathrm{D}(A)$, and

$$f(t) - u'(t) \in A(u(t)).$$

■

Actually, we do have some control on the inclusion, that is, the solution u is actually satisfying a differential equation on all $[0,T)$. Vaguely writing, it turns out that $u'(t)$ falls always into an extreme of the subset $f(t) - A(u(t))$. But in this case, we will have to consider the derivative from the right. The importance of this is that once we have the equation, we can get a bound for the right derivative of the solution. This is stated precisely in the next proposition and its corollary.

Recall that if an operator A on H is maximal monotone, then for $u \in \mathrm{D}(A)$ we denoted by $A^0(u)$ the unique element in H satisfying

$$\|A^0(u)\| = \min\{\|v\| : v \in A(u)\}.$$

More generally, if C is any subset of H which is non-empty, convex and closed, then we can use the notation C^0 to indicate the unique element of minimum norm in C.

We state an easy general inequality that we will use in the next proof.

Observation 2.17. *If $u, v, w \in H$, then*

$$(u - v, v - w) \leq \frac{1}{2}\|u - w\|^2 - \frac{1}{2}\|v - w\|^2.$$

Proposition 2.18. *Suppose the hypotheses of the theorem above and let u be the solution of (p). Then, u is differentiable from the right on all $[0, T)$ and*

$$\frac{d^+u}{dt}(t) = (f(t) - A(u(t)))^0,\ \forall t \in [0, T).$$

Proof. Consider the sequence (u_λ) like in Observation 2.13. Let us fix for now any $t \in [0, T]$. By the proof of the theorem above and by Lemma 2.14 we have

$$J_\lambda(u_\lambda(t)) \to u(t) \text{ in } H \text{ as } \lambda \to 0^+$$

and, going through a subsequence, $(A_\lambda(u_\lambda(t)))$ converges weakly in H. As usual, we also have for every positive λ:

$$A_\lambda(u_\lambda(t)) \in A(J_\lambda(u_\lambda(t)))$$

so the demiclosedness of A implies $u(t) \in \mathrm{D}(A)$, and this is true for every t in $[0, T]$. Now fix any $t_0 \in [0, T)$. We have for a.e. h such that $t_0 + h \in [0, T]$,

$$f(t_0 + h) - u'(t_0 + h) \in A(u(t_0 + h)),$$
$$f(t_0) - (f(t_0) - A(u(t_0)))^0 \in A(u(t_0)).$$

Hence, Lemma 2.8 gives for all $t_0 + h \in [0, T]$:

$$\begin{aligned}\|u(t_0 + h) - u(t_0)\| &\leq \int_0^h \|f(t_0 + s) - f(t_0) + (f(t_0) - A(u(t_0)))^0\| ds \\ &\leq h\|(f(t_0) - A(u(t_0)))^0\| + \int_0^h \|f(t_0 + s) - f(t_0)\| ds.\end{aligned}$$

Dividing by h, tending it to zero (from the right) and using a basic property of the integral, it follows that

$$\limsup_{h \to 0^+} \frac{\|u(t_0 + h) - u(t_0)\|}{h} \leq \|(f(t_0) - A(u(t_0)))^0\|. \tag{2.5}$$

Going through a subsequence if necessary, let $z \in H$ satisfy

$$\frac{u(t_0+h)-u(t_0)}{h} \rightharpoonup z \text{ weakly as } h \to 0^+. \tag{2.6}$$

On the other hand, let $[v,w] \in A$. By the chain rule and the monotonicity of A, for a.e. t in $[0,T]$:

$$\begin{aligned}\frac{1}{2}\frac{d}{\partial t}\|u(t)-v\|^2 &= (u(t)-v, u'(t)) \\ &\leq (u(t)-v, f(t)-w).\end{aligned}$$

If we integrate this last expression from s to t, where $s,t \in [0,T]$, we get

$$\frac{1}{2}\|u(t)-v\|^2 \leq \frac{1}{2}\|u(s)-v\|^2 + \int_s^t (u(r)-v, f(r)-w)dr.$$

Hence, using Observation 2.17 we know for all $t_0+h \in [0,T]$,

$$\begin{aligned}(u(t_0+h)-u(t_0), u(t_0)-v) &\leq \frac{1}{2}\|u(t_0+h)-v\|^2 - \frac{1}{2}\|u(t_0)-v\|^2 \\ &\leq \int_{t_0}^{t_0+h} (u(s)-v, f(s)-w)ds.\end{aligned}$$

Now divide by h and take the limit as h approaches zero from the right. Conclude from (2.6) that for all $[v,w] \in A$,

$$(z, u(t_0)-v) \leq (u(t_0)-v, f(t_0)-w).$$

But A is maximal monotone, then the latter implies

$$z \in f(t_0) - A(u(t_0)). \tag{2.7}$$

From the weak convergence in (2.6) and from (2.5) we have

$$\|z\| \leq \liminf_{h \to 0^+} \frac{\|u(t_0+h)-u(t_0)\|}{h} \leq \|(f(t_0)-A(u(t_0)))^0\|.$$

So the last two statements together imply that

$$z = (f(t_0)-A(u(t_0)))^0$$

and that as $h \to 0^+$,

$$\frac{\|(u(t_0+h)-u(t_0))\|}{h} \to \|(f(t_0)-A(u(t_0)))^0\|.$$

Notice that z does not depend on the subsequence in (2.6) and no matter how h tends to zero from the right,

$$\frac{u(t_0+h)-u(t_0)}{h} \to (f(t_0)-A(u(t_0)))^0, \text{ strongly in } H.$$

This is, u is differentiable from the right at t_0 and its derivative at it is the latter limit. ■

Corollary 2.19. *Under the hypotheses of Theorem 2.16, if u is the solution of (p) then*

$$\left\|\frac{d^+u}{dt}(t)\right\| \leq \|(f(0)-A(u_0))^0\| + \int_0^t \|f'\|,\ \forall t \in [0,T).$$

Proof. Let h be any real number, for a.e. $t, t+h \in [0,T]$ we have

$$\begin{aligned}&f(t+h)-u'(t+h) \in A(u(t+h)),\\ &f(t)-u'(t) \in A(u(t)).\end{aligned}$$

Then, by Lemma 2.8 it follows that for every $t \in [0,T]$:

$$\|u(t+h)-u(t)\| \leq \|u(h)-u_0\| + \int_0^t \|f(s+h)-f(s)\|ds. \tag{2.8}$$

On the other hand, for a.e. $h \in [0,T]$ it is satisfied

$$\begin{aligned}&f(h)-u'(h) \in A(u(h)),\\ &f(h)-(f(h)-A(u_0))^0 \in A(u_0).\end{aligned}$$

So again by Lemma 2.8, for all $h \in [0,T]$:

$$\|u(h)-u_0\| \leq \int_0^h \|(f(s)-A(u_0))^0\|ds.$$

This together with (2.8) implies that for all t in $[0,T)$:

$$\begin{aligned}\left\|\frac{d^+u}{dt}(t)\right\| &= \lim_{h\to 0^+} \frac{\|u(t+h)-u(t)\|}{h}\\ &\leq \lim_{h\to 0^+} \frac{1}{h}\int_0^h \|(f(s)-A(u_0))^0\|ds + \lim_{h\to 0^+} \frac{1}{h}\int_0^t \|f(s+h)-f(s)\|ds.\end{aligned}$$

Clearly, the right side of this inequality is

$$\|(f(0)-A(u_0))^0\| + \int_0^t \|f'\|.$$

■

For the sake of clarity, we now state the complete theorem we showed in this section. We remark that this theorem is by Y. Komura and has several extensions.

Theorem 2.20 (Existence and Uniqueness)**.** *Suppose that A is maximal monotone, f is in $W^{1,1}(0,T;H)$ and u_0 belongs to $\mathrm{D}(A)$. Then, there exists a unique solution u of (p) in $W^{1,\infty}(0,T;H)$. Moreover, u is differentiable from the right at every t in $[0,T)$ and*

$$\begin{aligned}&\frac{d^+u}{dt}(t) = (f(t)-A(u(t)))^0,\\ &\left\|\frac{d^+u}{dt}(t)\right\| \leq \|(f(0)-A(u_0))^0\| + \int_0^t \|f'\|.\end{aligned}$$

■

The maximal monotonicity of A was essential for the proof. But a straight forward modification of it gives the same result for an operator A with the following property: there exists $\omega > 0$ such that $A+\omega I$ is a maximal monotone operator. The only difference is in the inequality stated above.

On the other hand, the conditions $f \in W^{1,1}(0,T;H)$ and $u_0 \in \mathrm{D}(A)$ were also important for this result but more can be said. As usual, here there is also a notion of "weak solution" which, to conclude existence and uniqueness, allows to have only $f \in L^1(0,T;H)$ and u_0 in the closure of $\mathrm{D}(A)$. The proof follows easily from the existence theorem presented here. In this context we refer the reader to [19]. In the case that H is of finite dimension, we still have a unique (strong) solution under these limiting conditions on the data, see for example [10]. Finally, now that we have solutions, the next step would be concerned with regularity and their stability. Although this is not included in this thesis, for a complete study in this direction see [19].

The existence theorem shown in this section can be very versatile. It is common in applications that the setting to which we would like to apply the existence theorem does not seem to match the exact needed form. For example, consider the section regarding wheeled vehicles of the last chapter. There, a simplified version of the model has (p) as the associated problem. If we like to use a more precise model, like the one discussed in [7], we would have to consider a problem of the form:

$$(p_0)\begin{cases} u' + MAu \ni f, \text{ on } (0,T) \\ u(0) = u_0, \end{cases}$$

where A is a maximal monotone operator on $\mathbb{R}^n$, M is a symmetric and strongly positive n-square matrix with real coefficients, f is in $L^1(0,T;\mathbb{R}^n)$ and u_0 belongs to the closure of $\mathrm{D}(A)$. The difference is that MA is not necessarily maximal monotone and we cannot apply the existence theorem directly. The next beautiful chain of reasoning was suggested by my advisor, G. Moroşanu. Notice that M is in particular positive definite which implies it has an inverse. Using the continuity of M and its symmetry, it follows easily that M^{-1} remains strongly positive. But A is monotone, then clearly $M^{-1}+A$ is strongly monotone; in particular it is coercive. On the other hand, Theorem 1.28 implies that M^{-1} is a maximal monotone operator. Since Rockafellar's condition is trivially satisfied, $M^{-1}+A$ is also maximal monotone and Theorem 1.31 gives that $M^{-1}+A$ is surjective. This provides the equation

$$\mathrm{R}(I+MA) = \mathbb{R}^n.$$

Unfortunately, monotonicity of MA might fail. Let us define a new inner product in $\mathbb{R}^n$ given by

$$(x,y)_0 := (M^{-1}x, y).$$

This is in fact an inner product by the hypotheses on M and it is straight forward to check that MA is a monotone operator on $(\mathbb{R}^n, (\cdot,\cdot)_0)$. It follows from Minty's theorem that MA is in fact maximal monotone on that space. Since all norms are equivalent in $\mathbb{R}^n$, we showed the following corollary of the existence theorem.

Corollary 2.21. (p_0) *has a unique solution.*

■

This paragraph is based on [5]. A Hilbert space might have more structure than needed to conclude an existence result as the one in this section. One of the reasons of studying a more general case comes from the fact that, although a Banach space, L^p is not a Hilbert space for $p \neq 2$. Consider any real Banach space V. At the end of the first chapter we already made some remarks about a more general theory of maximal monotone operators from V to its dual; recall from there the definition of the duality operator F. A way to proceed towards a more general framework for the existence theorem of this section is the following. Let A be a subset of $V \times V$. The next notion plays the role of monotonicity here. We say A is accretive if for all $[u_1, v_1], [u_2, v_2]$ in A there exists w in $F(u_1 - u_2)$ such that

$$(v_1 - v_2, w)_{VV^*} \geq 0.$$

More than a notion of maximality among the accretive operators, what we need is the next definition. The subset A is m-accretive if it is accretive and

$$\mathrm{R}(I + A) = V.$$

It turns out that if A is m-accretive, u_0 is in the closure of $\mathrm{D}(A)$ and f belongs to $L^1(0, T; V)$, then problem (p) has a unique "mild solution". In the case that V is reflexive, u_0 belongs to $\mathrm{D}(A)$ and f is an element of $W^{1,1}(0, T; V)$ we have this is a "strong solution" and belongs to $W^{1,\infty}(0, T; V)$.

2.3 Subdifferential

Now that the importance of maximal monotone operators is evident, we need a practical way of showing that a specific operator is of this kind. We do have Minty's characterization, but unfortunately the conditions are not always easy to check. In this section we will exhibit a machine, called the subdifferential, for creating maximal monotone operators. There is also a stronger version of the existence and uniqueness theorem for maximal monotone operators coming from subdifferentials which we will state later.

The main results of this section are two. The first states that the mentioned machine in fact gives maximal monotone operators. To give a proof of this, we will need two lemmas. The second exhibits a relation between the concepts of subdifferential and canonical extension of an operator. These results will be necessary for the last chapter, dedicated to applications.

Throughout this section φ will denote a function defined on H and taking values in $(-\infty, \infty]$. The subset

$$\mathrm{D}(\varphi) := \{u \in H : \varphi(u) < \infty\}$$

is called the effective domain of φ. We say the function is proper if this subset is non-empty. We will also assume from now on that φ is proper.

Consider the following way to get maximal monotone operators. Take any real-valued convex function defined on $\mathbb{R}$ and suppose it is continuously differentiable. It follows that f' is non-decreasing and continuous, hence Theorem 1.28 implies it is maximal monotone. In order to get the largest amount of maximal monotone operators in this way, we would like to relax the conditions on f. It turns out that, since we are not looking specifically for single-valued maximal monotone operators, differentiability is not really necessary. Instead, we would prefer a more general concept of derivative that allows it to be multi-valued.

Let f be a real-valued convex function defined on the real line and suppose it is differentiable at $t \in \mathbb{R}$. Then, for all λ in $(0,1)$ and all s in $\mathbb{R}$:

$$f(t) - f(s) \leq \frac{f(t) - f(t + \lambda(s-t))}{\lambda}.$$

Taking the limit when λ tends to zero from the right,

$$f(t) - f(s) \leq f'(t)(t-s). \tag{2.9}$$

Let us now recall the definitions of convexity and directional derivative in an arbitrary Hilbert space. We say that φ is convex if for all u and v in H and for all λ in $(0,1)$ we have

$$\varphi(\lambda u + (1-\lambda)v) \leq \lambda\varphi(u) + (1-\lambda)\varphi(v),$$

where the usual conventions with respect to ∞ are considered. The function φ is Gâteaux differentiable at $u \in H$ if there exists $\nabla\varphi(u) \in H$ such that for every $v \in H$,

$$\lim_{\tau \to 0} \frac{f(u + \tau v) - f(u)}{\tau} = (\nabla\varphi(u), v).$$

Inspired by the inequality in (2.9), the next definition gives a multi-valued generalization of the Gâteaux derivative for convex functions.

Definition 2.22. *The subdifferential of φ is the operator $\partial\varphi : \mathrm{D}(\partial\varphi) \subseteq H \to H$ given by*

$$\partial\varphi(u) := \{w \in H : \varphi(u) - \varphi(v) \leq (w, u-v),\ \forall v \in H\},$$

where its domain consist of all vectors for which this set is non-empty.

Observation 2.23. *Notice that $\partial\varphi$ is always a monotone operator on H and that clearly*

$$\mathrm{D}(\partial\varphi) \subseteq \mathrm{D}(\varphi).$$

In fact, when φ is convex and continuous it can be shown that these domains have the same closure and the same interior (see [19]).

As expected, it is simple to verify that this definition is in fact a generalization. We state this in the next proposition.

Proposition 2.24. *If φ is convex and Gâteaux differentiable at $u \in H$, then*

$$\partial\varphi(u) = \{\nabla\varphi(u)\}.$$

Figure 2.1: In red the boundary of a convex subset. In blue and green respectively, the tangent line and outer normal ray at its smooth point 0.

Our first example of a subdifferential is the following. Take H as $\mathbb{R}$ and consider the absolute value function given by $\varphi(x) := |x|$. The last proposition allows us to restrict our attention to its subdifferential at $x = 0$. The function φ is not differentiable there since instead of one tangent line to the graph at 0, we have a family of such lines. These lines have all slopes going from -1 to 1 which suggests that $\partial\varphi$ is the sign operator. This can be shown by the definition in a trivial way. Notice also that φ is convex and continuous and recall that the sign operator is maximal monotone.

This example generalizes in a natural way to a general Hilbert space H; if φ is the norm function then

$$\partial\varphi(u) = \begin{cases} \frac{u}{\|u\|}, \text{ if } u \neq 0 \\ \\ \overline{B_1(0)}, \text{ if } u = 0. \end{cases}$$

Consider a convex subset C of $\mathbb{R}^n$ and suppose x is in the boundary of C. For simplicity, think of $x = 0$. If x is smooth, that is, if there is a unique tangent line to C at x, then there is a unique outward normal ray at x. If x is not smooth, then we will find a family of such rays. In any case, the set of vectors in these rays is called the normal cone to C at x and it is denoted by $N_C(x)$ (see Figure 2.1). Explicitly,

$$N_C(x) := \{y \in \mathbb{R}^n : (x - z, y) \geq 0, \, \forall z \in C\}.$$

Written in this way, this notion can be considered for H instead of $\mathbb{R}^n$. We are interested in the normal cones because of their relation to the subdifferential of indicator functions. In fact, notice that the following is true.

Example 2.25. *Let C be a non-empty, convex and closed subset of H. Let I_C be its indicator function, that is:*

$$I_C(u) := \begin{cases} 0, \text{ if } u \in C \\ \infty, \text{ if } u \notin C. \end{cases}$$

Then, $\mathrm{D}(\partial I_C) = C$ *and*

$$\partial I_C(u) = \begin{cases} 0, \text{ if } u \in \operatorname{Int} C \\ N_C(u), \text{ if } u \in \partial C, \end{cases}$$

where ∂C denotes the boundary of C.

In order to get maximal monotone operators from φ we use its convexity but not even continuity is needed; a slightly weaker notion will suffice. We next recall the definition of lower semicontinuity for real-valued functions defined on a topological space (X, τ). The function $\psi : X \to (-\infty, \infty]$ is said to be lower semicontinuous (lsc) if for every real number λ the subset

$$\{x \in X : \psi(x) \leq \lambda\}$$

is closed in X. It can be easily checked that lsc is equivalent to having the next identity for every x in X:

$$\psi(x) = \liminf_{y \to x} \psi(y) := \sup_{U \in \tau_x} \inf\{\psi(y) : y \in U - \{x\}\},$$

where we denoted

$$\tau_x := \{U \in \tau : x \in U \text{ and } U - \{x\} \neq \emptyset\}.$$

Similarly, we can define the corresponding

$$\limsup_{y \to x} \psi(y) := \inf_{U \in \tau_x} \sup\{\psi(y) : y \in U - \{x\}\}.$$

It is in fact not hard to show that a proper, convex and lsc function is actually continuous on the interior of its effective domain.

Let us start now with the two necessary lemmas for showing that the subdifferential of a proper, convex and lsc function is maximal monotone. Since φ is a proper function the following lemma states that, under certain conditions, φ achieves its minimum in an element of its effective domain.

Lemma 2.26. *Let φ be convex and lsc. If*

$$\lim_{\|u\| \to \infty} \varphi(u) = \infty,$$

then φ has a minimum on H.

Proof. Denote

$$d := \inf_{u \in H} \varphi(u).$$

Since φ is proper, we can find a sequence (u_n) in $\mathrm{D}(\varphi)$ such that

$$\lim_{n \to \infty} \varphi(u_n) = d < \infty. \tag{2.10}$$

It follows by the last hypothesis that (u_n) is bounded and, going through a subsequence, there exists $u_0 \in H$ such that

$$u_n \rightharpoonup u_0 \text{ weakly in } H. \tag{2.11}$$

On the other hand, the first two hypotheses imply the subsets $(\lambda \in \mathbb{R})$:

$$\{u \in H : \varphi(u) \leq \lambda\}$$

are convex and closed and hence, by Mazur's theorem, they are weakly closed in H. Hence, φ is weakly lsc. This together with (2.10) and (2.11) give that $\varphi(u_0) \leq d$. ■

The second lemma bounds a proper, convex and lsc function with an affine function. Although the result is also intuitively clear, a proof is needed.

Lemma 2.27. *If φ is convex and lsc, then there exists $v_0 \in H$ and $\lambda_0 \in \mathbb{R}$ such that for every $u \in H$:*

$$\varphi(u) > (v_0, u) + \lambda_0.$$

Proof. Consider the epigraph of φ given by

$$E := \{[u, \lambda] \in H \times \mathbb{R} : \varphi(u) \leq \lambda\}.$$

This is a convex and closed subset of the Cartesian product. Fix any $u_0 \in \mathrm{D}(\varphi)$ and $\epsilon > 0$ and notice that $[u_0, \varphi(u_0) - \epsilon] \notin E$. By a version of the Hahn-Banach theorem and by the Riesz Representation theorem, there exists $[w, \mu] \in H \times \mathbb{R}$ such that for every $[u, \lambda]$ in E:

$$(w, u) + \lambda\mu < (w, u_0) + (\varphi(u_0) - \epsilon)\mu.$$

Since for all $u \in \mathrm{D}(\varphi)$,

$$[u, \varphi(u)] \in E$$

then we get $\mu < 0$ when taking u_0 as u. Therefore, a straight forward computation shows that for every $u \in H$:

$$\varphi(u) > (v_0, u) + \lambda_0,$$

where we denoted

$$v_0 := -\frac{1}{\mu} w,$$

$$\lambda_0 := \frac{1}{\mu}(w, u_0) + \varphi(u_0) - \epsilon.$$

■

Theorem 2.28 (Rockafellar)**.** *If φ is convex and lsc, then $\partial\varphi$ is maximal monotone.*

Proof. As mentioned before, $\partial\varphi$ is clearly a monotone operator. By Minty's theorem, it suffices to check $\mathrm{R}(I + \partial\varphi) = H$. Fix any $w \in H$; we will show that there exists u_0 in H such that for every v in H:

$$\varphi(u_0) - \varphi(v) \leq (u_0 - v, w - u_0).$$

Define $\psi : H \to (-\infty, \infty]$ such that

$$\psi(u) := \frac{1}{2}\|u\|^2 + \varphi(u) - (u, w).$$

It is clear that ψ is also a proper, convex and lsc function. Moreover, since by Lemma 2.27 we have that φ is bounded from below by an affine function:

$$\lim_{\|u\| \to \infty} \psi(u) = \infty.$$

Hence, Lemma 2.26 gives $u_0 \in \mathrm{D}(\psi)$ such that

$$\psi(u_0) = \inf_{u \in H} \psi(u).$$

On the other hand, since for all u in H:

$$\frac{1}{2}\|u_0\|^2 + (u - u_0, u) - \frac{1}{2}\|u\|^2 = \frac{1}{2}\|u_0 - u\|^2 \geq 0,$$

then we get

$$\begin{aligned}\varphi(u_0) - \varphi(u) &\leq \psi(u_0) - \psi(u) + (u, u - u_0) + (u_0 - u, w) \\ &\leq (w - u, u_0 - u).\end{aligned}$$

If we set $u := tu_0 + (1-t)v$ for t in $(0,1)$ and v in H, it follows from the convexity of φ and the latter that

$$(1-t)(\varphi(u_0) - \varphi(v)) \leq (w - u, u_0 - u).$$

A direct computation shows that also

$$(1-t)(w - u_0, u_0 - v) + (1-t)^2\|u_0 - v\|^2 = (w - u, u_0 - u),$$

and we conclude then

$$\varphi(u_0) - \varphi(v) \leq (w - u_0, u_0 - v) + (1-t)\|u_0 - v\|^2.$$

Finally, tend t to one to get the result. ■

One might expect the subdifferential to be linear, unfortunately this is not always the case. Although it is trivial to check that for positive real λ:

$$\partial(\lambda\varphi) = \lambda\partial\varphi,$$

the additivity requires more attention. Consider any proper functions $\varphi_1, \varphi_2 : H \to (-\infty, \infty]$. It is straight forward to verify

$$\partial\varphi_1 + \partial\varphi_2 \subseteq \partial(\varphi_1 + \varphi_2).$$

The theorem above together with Rockafellar's condition (Observation 1.35) imply the equality in the following context.

Corollary 2.29. *Let $\varphi_1, \varphi_2 : H \to (-\infty, \infty]$ be proper, convex and lsc. Assume that*

$$\operatorname{Int}\mathrm{D}(\partial\varphi_1) \cap \mathrm{D}(\partial\varphi_2) \neq \emptyset.$$

Then,

$$\partial\varphi_1 + \partial\varphi_2 = \partial(\varphi_1 + \varphi_2).$$

The natural question now is if all maximal monotone operators on H arise as subdifferentials of proper, convex and lsc functions from H to $(-\infty, \infty]$. This is not the case, even for $H := \mathbb{R}^2$ we can find a counterexample of this (see [15]). Nevertheless, the answer is affirmative when $H := \mathbb{R}$. The proof of this fact is elementary but a bit long and technical so we will omit the proof (see [19]).

Proposition 2.30. *Let A be a maximal monotone operator on $\mathbb{R}$. Then, there exists $\psi : \mathbb{R} \to (-\infty, \infty]$ proper, convex and lsc function such that $A = \partial\psi$. Moreover, we can find $a \leq b$ elements of $[-\infty, \infty]$ such that*

$$(a, b) \subseteq \mathrm{D}(A) \subseteq [a, b]$$

and ψ given by

$$\psi(x) := \begin{cases} \displaystyle\int_{x_0}^{x} A^0(s)ds, \ \textit{if } x \in [a, b] \\ \\ \infty, \ \textit{if } x \in \mathbb{R} - [a, b] \end{cases}$$

works, where $x_0 \in \mathrm{D}(A)$ is arbitrary and fixed.

The idea in the proof is to take $a := \inf \mathrm{D}(A)$, $b := \sup \mathrm{D}(A)$ and to use the maximal monotonicity of A. Then showing that ψ has the desired properties and that actually has A as subdifferential is done directly.

To end this section, suppose that we have an operator A on H and that φ is convex and lsc. Furthermore, assume that $A = \partial\varphi$ and let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$. By definition, the canonical extension of A using Ω is an operator on $L^2(\Omega; H)$. Is it true that $\overline{A}$ remains being a subdifferential of such a function? This is the case when Ω is not too "big" and we have an explicit expression for this function.

Corollary 2.31. *Suppose Ω is a Lebesgue measurable subset of $\mathbb{R}^n$ with finite measure. Assume also that φ is convex and lsc. Let us define $\phi : L^2(\Omega; H) \to (-\infty, \infty]$ as*

$$\phi(f) := \begin{cases} \displaystyle\int_{\Omega} \varphi(f), \ \textit{if } \varphi(f) \in L^1(\Omega) \\ \\ \infty, \ \textit{otherwise.} \end{cases}$$

Then, ϕ is proper, convex and lsc and $\partial\phi = \overline{\partial\varphi}$.

Proof. We start by noticing that Lemma 2.27 implies ϕ cannot take the value $-\infty$. Using that the measure of Ω is finite and that φ is proper, we have that ϕ is also a proper function. It is also convex since φ is. Fix any t real number and define

$$M := \{f \in L^2(\Omega; H) : \phi(f) \leq t\}.$$

We claim that M is a closed subset of $L^2(\Omega; H)$. In fact, take any sequence (f_k) in M and suppose it converges to some f_0 in that space; we may suppose that the convergence is also a.e. on Ω. Let us fix any $[u_0, v_0]$ in $\partial\varphi$ (so $\varphi(u_0)$ is a real number) and define $\psi : H \to (-\infty, \infty]$ as

$$\psi(u) := \varphi(u) - \varphi(u_0) - (v_0, u - u_0).$$

It is clear that ψ is a proper, convex and lsc function; by the definition of subdifferential we also have that this function is non-negative. Hence, as a consequence of Fatou's Lemma, it follows that

$$\int_\Omega \liminf_{k\to\infty} \psi(f_k) \leq \liminf_{k\to\infty} \int_\Omega \psi(f_k).$$

On the other hand, notice that for a.e. x in Ω:

$$\psi(f_0(x)) = \liminf_{v\to f_0(x)} \psi(v) \leq \liminf_{k\to\infty} \psi(f_k(x)),$$

where the first inferior limit is the topological version defined in this section and the second is the usual one. By using the definition of ψ we conclude

$$\int_\Omega \varphi(f_0) \leq \liminf_{k\to\infty} \phi(f_k) \leq t$$

and $f_0 \in M$. That is, ϕ is in fact a lsc function. By Theorem 2.28 and by example 1.25, it suffices to show that $\overline{\partial\varphi} \subseteq \partial\phi$. But this inclusion follows from the definition of subdifferential and the monotonicity of the integral. ■

Consider our usual problem (p) from last section. The existence and uniqueness theorem shown there is stated for general maximal monotone operators. The next result was shown by Brézis. When the operator arises as a subdifferential of a proper, convex and lsc function, we can relax the conditions on the data of (p) and still have existence and uniqueness. Namely, u_0 can be in the closure of the domain of A and f in $L^2(0, T; H)$. Moreover, if this is the case, the solution u satisfies $(\sqrt{t})u' \in L^2(0, T; H)$.

Chapter 3

Applications

Let us write a brief summary of what we done so far. In the first chapter of this thesis some of the classical results about maximal monotone operator theory were presented. These tools were enough to show, in the second chapter, the main result of the thesis: the existence theorem. After presenting this, that chapter also introduced the concept of subdifferential; we provided a way of checking if an operator is maximal monotone, namely, if it is the subdifferential of a function satisfying some properties that are relatively easy to check. Evidently, our goal now is to use the existence theorem along with the just mentioned Rockafellar's result to solve specific differential equations that model real-world phenomena. Some results from the first chapter will also be helpful.

The present chapter is divided into three sections, going from simple to more complex. The first deals with the well-known Heat (Diffusion) Equation, which is a partial differential equation of parabolic type. The second with a publication from 2014 about the braking system of a car, see [7]. In the third section the Nonlinear Wave Equation is studied, this is a hyperbolic partial differential equation. It is firstly exposed in a restrictive way, then in a more general and useful presentation. Getting a solution in both cases is not trivial; many results from earlier chapters will arise naturally.

3.1 Heat (Diffusion) Equation

The Heat Equation describes the conduction of heat in solids, see Figure (3.1). It was introduced in 1807 by Fourier and has been widely studied since then. Applications of it arise not only in physics but in many other fields of science such as Probability Theory and Financial Mathematics. For a complete survey on the history involving this equation, refer to [20]. Derivation of the actual equation can be found easily through the appropriate literature.

This equation is involved in the following problem:

$$(he)\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + \beta(u) = f, \text{ on } (0,T)\times(0,1) \\ u = 0, \text{ on } (0,T)\times\{0,1\} \\ u(0,\cdot) = u_0, \text{ on } (0,1), \end{cases}$$

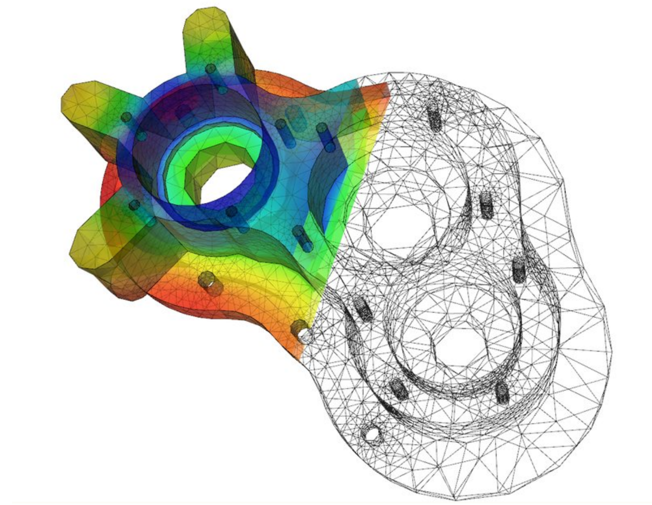

Figure 3.1: Heat transfer in a pump casing; internal heating and cooling at the boundary. This image was created by solving the Heat Equation; see “Differential Equation” in Wikipedia.

where T is a positive real number, $u : [0,T) \times [0,1] \to \mathbb{R}$ is the unknown function, $\beta \in C^1(\mathbb{R})$ is non-decreasing, f belongs to $W^{1,1}(0,T;L^2(0,1))$ and

$$u_0 \in H_0^1(0,1) \cap H^2(0,1).$$

Additionally, we will suppose that

$$\beta(0) = 0.$$

To make a simple first example of an application, we will limit ourselves to show the existence of a function u satisfying (he) without discussing its properties. The first step is to write this problem in the form of our known (p) from the previous chapter. Let $H := L^2(0,1)$, $g : (0,T) \to H$ as $g(t) := f(t,\cdot)$, $A : \mathrm{D}(A) \subseteq H \to H$ given by

$$A(u) := -u'' + \overline{\beta}(u)$$

and

$$\mathrm{D}(A) := H_0^1(0,1) \cap H^2(0,1),$$

where $\overline{\beta}$ is the canonical extension of β to $L^2(0,1)$. Notice that by our assumptions, $\mathrm{D}(\overline{\beta}) = L^2(0,1)$. Consider the modified problem

$$(p) \begin{cases} v' + A(v) = g, \text{ on } (0,T) \\ v(0) = u_0. \end{cases}$$

It is easy to see that if $v \in W^{1,\infty}(0,1;H)$ is a solution of (p), then $u(t,x) := v(t)(x)$ satisfies problem (he). The only delicate point is the boundary condition of (he), notice that this has been incorporated into the domain of A. Since g is in $W^{1,1}(0,T;H)$ and $u_0 \in \mathrm{D}(A)$, it suffices to show that A is a maximal monotone operator and to apply the existence theorem. Let $B : \mathrm{D}(B) := H_0^1(0,1) \cap H^2(0,1) \subseteq L^2(0,1) \to L^2(0,1)$ be given by

$$B(u) := -u''.$$

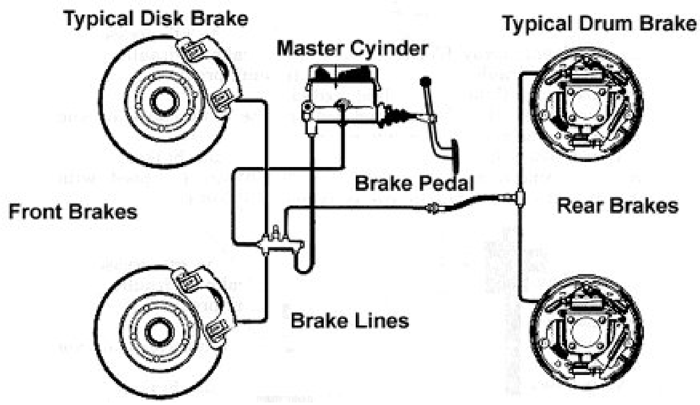

Figure 3.2: Typical Automotive Braking System

Recall from Example 1.21 and Theorem 1.28 that B and $\overline{\beta}$ are maximal monotone operators. Moreover, their domains clearly intersect. Hence, by Corollary 1.37 and Observation 1.24, we only need to show that

$$(B(u), \overline{\beta}_\lambda(u))_H \geq 0,$$

for every $u \in \mathrm{D}(B)$ and every positive λ. In fact, integrating by parts provides

$$\int_0^1 B(u)\beta_\lambda(u) = -u'(1)\beta_\lambda(u(1)) + u'(0)\beta_\lambda(u(0)) + \int_0^1 (u')^2\beta'_\lambda(u) = \int_0^1 (u')^2\beta'_\lambda(u),$$

where we used that u belongs to $H_0^1(0,1)$ and that $\beta_\lambda(0) = 0$. This expression is non-negative since β_λ is a non-decreasing function.

We treated the simplest case which can be very restrictive. For a much wider study of the applications of the existence theorem to the Heat Equation we refer the reader to [5].

3.2 Braking System of Wheeled Vehicles

This section is based on a publication of J. Bastien; [7]. He studies the dynamics of a wheel under two friction forces. These forces are induced by: the ground and the break pad. The wheel is also subjected to a motor torque. See Figure (1.9).

Deriving the equation modeling this phenomena relies strongly on Coulomb's law. In this context, the sign operator σ arises naturally. A simplified version of the problem has the form:

$$(br)\begin{cases} u' + a = 0,\ v' + a + b = f, \text{ on } (0,T) \\ a \in c_1\sigma(u+v),\ b \in c_2\sigma(v), \text{ on } (0,T) \\ u(0) = u_0,\ v(0) = v_0, \end{cases}$$

where T is a positive real number, $u, v : [0,T) \to \mathbb{R}$ and $a, b : (0,T) \to \mathbb{R}$ are the unknown functions, $f : (0,T) \to \mathbb{R}$ is a given function, c_1, c_2 are positive real numbers and u_0, v_0 are given real constants.

We now present the appropriate setting for using the existence and uniqueness theorem; as previously said, the more accurate model would make use of Corollary 2.21. For α and β positive real numbers, we consider the everywhere defined operator $A_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$A_{\alpha,\beta}(x_1, x_2) := \{[y_1, y_2] \in \mathbb{R}^2 : y_1 \in \alpha\sigma(x_1 + x_2) \text{ and } y_2 \in y_1 + \beta\sigma(x_2)\}, \tag{3.1}$$

where σ is the sign operator. Define also $F : (0, T) \to \mathbb{R}^2$ as

$$F(t) := [0, f(t)].$$

Suppose u, v, a, b satisfy (br) and set $U : [0, T) \to \mathbb{R}^2$ as

$$U(t) := (u(t), v(t)).$$

It follows that on $(0, T)$:

$$U' = (-a, -a - b + f),$$
$$(a, a + b) \in A_{c_1,c_2}(u, v).$$

Therefore, we have that

$$U' + A_{c_1,c_2}(U) \ni F,$$

and $U(0) = [u_0, v_0]$. Conversely, suppose $U := (u, v)$ satisfies the last inclusion with its initial condition. Then $u(0) = u_0$, $v(0) = v_0$. Moreover, for each t in $(0, T)$ we can find $[y_1(t), y_2(t)], [z_1(t), z_2(t)]$ in $\mathbb{R}^2$ such that $[y_1(t), y_2(t)] \in A_{c_1,c_2}(U(t))$ and

$$\begin{aligned}
&u'(t) + y_1(t) = 0,\\
&v'(t) + y_2(t) = f(t),\\
&y_1(t) = c_1 z_1(t),\\
&y_2(t) = y_1 + c_2 z_2(t),\\
&z_1(t) \in \sigma(u(t) + v(t)),\\
&z_2(t) \in \sigma(v(t)).
\end{aligned}$$

Then we make

$$a(t) := y_1(t),\ b(t) := c_2 z_2(t)$$

to conclude that u, v, a, b satisfy (br). Hence, without analyzing further details, we will limit ourselves to show that $A_{\alpha,\beta}$ is a maximal monotone operator on $\mathbb{R}^2$. But we do mention the following. The conditions on f and $[u_0, v_0]$ in order to have existence and uniqueness for the modified problem are not so restrictive; $H := \mathbb{R}^2$ is finite dimensional (see the end of Section 2.2).

Proposition 3.1. *Let α and β be positive real numbers. Then, the operator $A_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{R}^2$ given by (3.1) is maximal monotone.*

Proof. Define $\phi_1, \phi_2 : \mathbb{R}^2 \to \mathbb{R}$ as

$$\phi_1(x_1, x_2) := |x_1 + x_2|,$$

$$\phi_2(x_1, x_2) := |x_2|.$$

We next compute their subdifferentials. Let $x := [x_1, x_2]$ be in $\mathbb{R}^2$ and suppose that x_1+x_2 is not zero. Then, for every $y := (y_1, y_2)$ in $\mathbb{R}^2$:

$$\begin{aligned}
\lim_{\tau\to 0} \frac{\phi_1(x+\tau y) - \phi_1(x)}{\tau} &= \lim_{\tau\to 0} \frac{|x_1 + x_2 + \tau(y_1+y_2)|^2 - |x_1+x_2|^2}{\tau(|x_1 + x_2 + \tau(y_1+y_2)| + |x_1+x_2|)} \\
&= \frac{(x_1+x_2)(y_1+y_2)}{|x_1+x_2|} \\
&= (y, [\sigma(x_1+x_2), \sigma(x_1+x_2)])_{\mathbb{R}^2}.
\end{aligned}$$

Since ϕ_1 is convex, it follows from Proposition 2.24 that

$$\partial\phi_1(x) = \{[\sigma(x_1+x_2), \sigma(x_1+x_2)]\}.$$

If $x_1 + x_2 = 0$ one can show directly from the definition of subdifferential that

$$\partial\phi_1(x) = \{[y_1, y_2] \in \mathbb{R}^2 : y_1 = y_2 \in [-1, 1]\}.$$

With a similar procedure one gets

$$\partial\phi_2(x_1, x_2) = \begin{cases} (0, -1), \text{ if } x_2 < 0 \\ \{0\} \times [-1, 1], \text{ if } x_2 = 0 \\ (0, 1), \text{ if } x_2 > 0. \end{cases}$$

Now let $\phi := \alpha\phi_1 + \beta\phi_2$. Notice that ϕ_1 and ϕ_2 are convex and continuous functions with everywhere defined subdifferentials. By Corollary 2.29 it follows that

$$\partial\phi = \alpha\partial\phi_1 + \beta\partial\phi_2.$$

On the other hand, by Rockafellar's theorem, it suffices to show that

$$\partial\phi = A_{\alpha,\beta}$$

to conclude the proof; but this is in fact the case and it can be checked by a direct computation. ■

3.3 Nonlinear Wave Equation

The Wave Equation describes different kinds of waves that occur in nature. For example: water, sound and light waves (see Figure 3.3). This equation arises in many different fields of Physics, to mention some of them: Fluid Dynamics, Acoustics and Electromagnetics. In its simplest version, the Wave Equation was introduced by J.-B. d'Alembert in 1746;

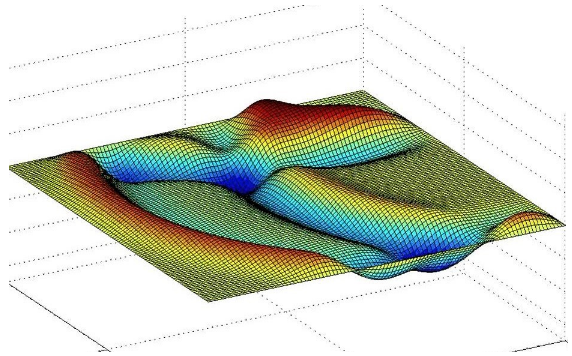

Figure 3.3: Solution to the Wave Equation in two dimensions at a particular time.

its derivation is based on Hooke's law. The study of nonlinear waves started with the works of Stokes, 1847 and Riemann, 1858. For a complete overview of the Wave Equation see [23].

The Wave Equation that we will study is the following:

$$(we)\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \beta\left(\dfrac{\partial u}{\partial t}\right) \ni f, \text{ on } (0,T)\times\Omega \\ u = 0, \text{ on } (0,T)\times\partial\Omega \\ u(0,\cdot) = u_0, \text{ on } \Omega \\ \dfrac{\partial^+ u}{\partial t}(0,\cdot) = v_0, \text{ on } \Omega, \end{cases}$$

where T is a positive real number, Ω is an open (non-empty), bounded and smooth subset of $\mathbb{R}^n$, $u : [0,T)\times\overline{\Omega} \to \mathbb{R}$ is the unknown function, β is an operator on $\mathbb{R}$ and $f : (0,T)\times\Omega \to \mathbb{R}$, $u_0, v_0 : \Omega \to \mathbb{R}$ are given functions. *Moreover, we assume that β is maximal monotone.* The assumptions in this paragraph will be considered throughout this section. The so-called String Equation arises as a particular case when taking Ω as $(0,1)$ and β as the sign operator. This is discussed at the end of this section.

As we did with previously, we next construct the corresponding setting for applying the existence theorem. The natural way is the following. Let f be in $W^{1,1}(0,T;L^2(\Omega))$ and suppose $[u_0, v_0]$ belongs to $(H_0^1(\Omega)\cap H^2(\Omega))\times H_0^1(\Omega)$. Consider the Laplacian operator

$$-\Delta : H_0^1(\Omega)\cap H^2(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega)$$

and the canonical extension of β to $L^2(\Omega)$. Put $H := H_0^1(\Omega)\times L^2(\Omega)$; as usual, the inner product for this Hilbert space is

$$([u_1, v_1], [u_2, v_2]) := \int_\Omega (\nabla u_1, \nabla u_2)_{\mathbb{R}^n} + \int_\Omega v_1 v_2.$$

Define $A_1 : \mathrm{D}(-\Delta)\times(H_0^1(\Omega)\cap \mathrm{D}(\overline{\beta})) \subseteq H \to H$ as

$$A_1(u,v) := [-v, -\Delta u + \overline{\beta}(v)].$$

Moreover, let $h : (0,T) \to H$ be given by

$$h(t) := [0, f(t,\cdot)]$$

and set $w_0 := (u_0, v_0)$, which belongs to H. The associated problem is then:

$$\begin{cases} w' + A_1(w) \ni h, \text{ on } (0,T) \\ w(0) = w_0. \end{cases} \tag{3.2}$$

Unfortunately, the maximal monotonicity of A_1 would not come without a cost.

Lemma 3.2. *Suppose that β is everywhere defined and that for every v in $L^2(\Omega)$,*

$$\beta^0(v) \in L^2(\Omega). \tag{3.3}$$

Then, A_1 is a maximal monotone operator on H.

Proof. By our hypotheses, it is clear that

$$\mathrm{D}(\overline{\beta}) = L^2(\Omega).$$

Showing that A_1 is monotone follows from the integration by parts formula and the monotonicity of $\overline{\beta}$. In fact, let $u := [u_1, u_2], v := [v_1, v_2]$ be in $\mathrm{D}(A_1)$ and $x := [x_1, x_2], y := [y_1, y_2]$ be respectively in $A_1(u)$ and $A_1(v)$. Then, there exists $z := [z_1, z_2] \in L^2(\Omega)^2$ such that $z_1 \in \overline{\beta}(u_2)$, $z_2 \in \overline{\beta}(v_2)$ and

$$x = [-u_2, -\Delta u_1 + z_1],\ y = [-v_2, -\Delta v_1 + z_2].$$

Hence,

$$\begin{aligned} (u - v, x - y) &= ([u_1 - v_1, u_2 - v_2], [-u_2 + v_2, -\Delta u_1 + z_1 + \Delta v_1 - z_2]) \\ &= \int_\Omega \nabla(u_1 - v_1)\nabla(v_2 - u_2) + \int_\Omega (u_2 - v_2)(\Delta(-u_1 + v_1) + z_1 - z_2) \\ &= \int_\Omega (u_2 - v_2)(z_1 - z_2) = (u_2 - v_2, z_1 - z_2)_{L^2(\Omega)} \\ &\geq 0. \end{aligned}$$

Now let us prove its maximality; by Minty's theorem, it suffices to show

$$\mathrm{R}(I + A_1) = H.$$

Fix any $[g, h]$ in H. Consider the everywhere defined operator $B : L^2(\Omega) \to L^2(\Omega)$ given by

$$B(u) := \overline{\beta}(u - g).$$

The maximal monotonicity of $\overline{\beta}$ makes B monotone and, by Minty's theorem, gives also

$$\mathrm{R}(I + \overline{\beta}) = L^2(\Omega).$$

So if v is in $L^2(\Omega)$, we can find u in that space such that

$$u + \overline{\beta}(u) \ni v - g.$$

It follows that

$$(u + g) + B(u + g) \ni v$$

and B is maximal monotone. By Example 1.21 the Laplacian operator $-\Delta$ is also maximal monotone, then the Rockafellar's condition implies that $-\Delta + B$ is maximal monotone and therefore,

$$\mathrm{R}(I - \Delta + B) = L^2(\Omega).$$

In particular, we can find p in $H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$p - \Delta p + B(p) \ni h + g.$$

Notice that $[p, p - g]$ is in $\mathrm{D}(A_1)$ and

$$[p, p - g] + A_1(p, p - g) \ni [g, h].$$

■

Before continuing with an observation of this lemma, we make precise the next notion.

Definition 3.3. *We say that a function $g : \mathbb{R} \to \mathbb{R}$ has sublinear growth if we can find c_1, c_2 real numbers such that c_1 is positive and for every t in $\mathbb{R}$,*

$$|g(t)| \leq c_1|t| + c_2.$$

Observation 3.4. *More can be said about condition (3.3). This condition states that the so-called superposition operator associated with β^0 is well-defined on $L^2(\Omega)$. Since a non-decreasing function on $\mathbb{R}$ has at most countable discontinuities (which are of the "jump type"), the following can be deduced from Krasnoselskii's theorem (see [16]): condition (3.3) is equivalent to the sublinear growth of β^0. In applications, sublinear growth can be easily checked. For a different approach to the proof of the mentioned equivalence, see also [1].*

On the other hand, we say a function $g : \mathbb{R} \to \mathbb{R}$ is a version of β if for every real number t, $g(t)$ belongs to $\beta(t)$. Notice that the latter lemma only required the canonical extension of β to be everywhere defined. Therefore, the minimal section of β does not play an essential role; instead of it, any version of β can be used in condition (3.3).

It is worth noticing that in the proof of the latter lemma we used both implications in Minty's theorem; this does not occur frequently in this thesis. This lemma implies that, for such a β, there exists a unique solution $w := [w_1, w_2] \in W^{1,\infty}(0, T; H)$ for the problem in (3.2). Then just take $u : [0, T] \times \Omega \to \mathbb{R}$ as $u(t, x) := w_1(t)(x)$ to "solve" (we). We summarize the results in the next precise theorem.

Theorem 3.5. *Suppose T is a positive real number and Ω an open, bounded and smooth subset of $\mathbb{R}^n$. Let β be a maximal monotone operator on $\mathbb{R}$ and assume that f belongs to $W^{1,1}(0,T;L^2(\Omega))$ and*

$$[u_0, v_0] \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega).$$

Additionally, suppose that β is everywhere defined and that it has a version with sublinear growth. Then, there exists a function $u : [0,T] \times \Omega \to \mathbb{R}$ such that

$$u \in W^{1,\infty}(0,T;H_0^1(\Omega)) \cap W^{2,\infty}(0,T;L^2(\Omega))$$

and, for a.e. (t,x) in $(0,T) \times \Omega$,

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) + \beta\left(\frac{\partial u}{\partial t}(t,x)\right) \ni f(t,x).$$

Moreover, for a.e. x in Ω,

$$u(0,x) = u_0(x),\ \frac{\partial^+ u}{\partial t}(0,x) = v_0(x).$$

■

But what about the case when β does not satisfy the additional conditions stated in the previous lemma? It turns out that problem (we) is still satisfied but in a more general sense; the inclusion will hold now in the dual space of $H_0^1(\Omega)$. We will follow a similar strategy to show this. Going in this direction, consider again problem (we). For the following, we do need to suppose that

$$0 \in \beta(0).$$

By Proposition 2.30, β should be a subdifferential. Namely, if $\psi : \mathbb{R} \to (-\infty, \infty]$ is given by

$$\psi(x) := \begin{cases} \displaystyle\int_0^x \beta^0(s)ds, \text{ if } x \in [a,b] \\ \\ \infty, \text{ if } x \in \mathbb{R} - [a,b], \end{cases}$$

where $a \leq b$ are in the extended real line and verify

$$(a,b) \subseteq D(\beta) \subseteq [a,b],$$

then ψ is proper, convex and lsc and

$$\beta = \partial\psi.$$

Now, by Corollary 2.31, the canonical extension of β is also a subdifferential of the following proper, convex and lsc function: $\phi : L^2(\Omega) \to (-\infty, \infty]$ defined as

$$\phi(v) := \begin{cases} \displaystyle\int_\Omega \psi(v), \text{ if } \psi(v) \in L^1(\Omega) \\ \\ \infty, \text{ otherwise.} \end{cases}$$

Let us restrict the function domain of ϕ to $H_0^1(\Omega)$ and call this new function φ. It is clear that $\varphi : H_0^1(\Omega) \to (-\infty, \infty]$ remains proper, convex and even lsc; Rockafellar's result gives then that $\partial\varphi$ is a maximal monotone operator on $H_0^1(\Omega)$. For convenience, let the dual of $H_0^1(\Omega)$ be denoted by $H^{-1}(\Omega)$ and consider the mapping $F : H_0^1(\Omega) \to H^{-1}(\Omega)$ given by the Riesz Representation theorem, that is, F has the rule

$$F(u)v = \int_\Omega (\nabla u, \nabla v)_{\mathbb{R}^n};$$

conclude that $F(\partial\varphi)$ is maximal monotone (see Section 1.4). Moreover, set $\widetilde{\beta} := F(\partial\varphi)$ and notice that $\widetilde{\beta} : \mathrm{D}(\widetilde{\beta}) \subseteq H_0^1(\Omega) \to H^{-1}(\Omega)$ is given by

$$\widetilde{\beta}(v) := \{z \in H^{-1}(\Omega) : \varphi(v) - \varphi(w) \leq (v - w, z)_{H_0^1(\Omega), H^{-1}(\Omega)},\ \forall w \in H_0^1(\Omega)\},$$

its domain are those $v \in H_0^1(\Omega)$ such that the latter set is non-empty. In the next diagram the arrow means "used in the definition of":

$$\beta \to \psi \to \phi \to \varphi \to \widetilde{\beta}.$$

Consider the inclusion

$$i : H_0^1(\Omega) \hookrightarrow L^2(\Omega),$$

which is linear and bounded, and its adjoint operator

$$i^* : L^2(\Omega)^* \to H^{-1}(\Omega).$$

Since we can identify $L^2(\Omega)$ with $L^2(\Omega)^*$ under the duality mapping R, given $w \in L^2(\Omega)$ we will sometimes denote also by w the element $i^*(R(w)) \in H^{-1}(\Omega)$. By the known Rellich-Kondrachov's theorem, the inclusion i is compact. Therefore, Schauder's theorem implies that i^* is also compact and so it is $i^*(R)$.

The generalization of the inclusion in (we) can be written as follows:

$$\frac{\partial^2 u}{\partial t^2}(t, \cdot) + F(u(t, \cdot)) + \widetilde{\beta}\left(\frac{\partial u}{\partial t}(t, \cdot)\right) \ni f(t, \cdot), \tag{3.4}$$

for a.e. t in $(0, T)$ and for $f : (0, T) \to L^2(\Omega)$. Notice that, using the formula of integration by parts, F is in fact directly related with the Laplacian operator $-\Delta$. We now give an appropriate setting to deal with this redefined problem. Recall that $H := H_0^1(\Omega) \times L^2(\Omega)$ and define the operator $A : \mathrm{D}(A) \subseteq H \to H$ as

$$\begin{aligned}\mathrm{D}(A) &:= \{[p, q] \in H_0^1(\Omega)^2 : (F(p) + \widetilde{\beta}(q)) \cap i^* R[L^2(\Omega)] \neq \emptyset\},\\ A(p, q) &:= \{-q\} \times (i^* R)^{-1}[F(p) + \widetilde{\beta}(q)].\end{aligned}$$

Lemma 3.6 (Brézis). *A is a maximal monotone operator on H.*

Proof. First we show that A is monotone. Take $u_1 := [p_1, q_1]$ in $\mathrm{D}(A)$ and v_1 in $A(u_1)$. By the definition of A, we can find $r_1 \in L^2(\Omega)$ and $z_1 \in \widetilde{\beta}(q_1)$ such that $v_1 = [-q_1, r_1]$ and

$$i^* R(r_1) = F(p_1) + z_1.$$

Do the same for $u_2 := [p_2, q_2]$ in $\mathrm{D}(A)$ and v_2 in $A(u_2)$ to get the corresponding r_2 and z_2. The monotonicity of $\widetilde{\beta}$, the expressions of z_1, z_2 and the definition of adjoint operator imply:

$$\begin{aligned}
0 &\le (q_1 - q_2, z_1 - z_2)_{H_0^1, H^{-1}} \\
&= (q_1 - q_2, i^*(\cdot, r_1 - r_2)_{L^2})_{H_0^1, H^{-1}} + (q_1 - q_2, p_2 - p_1)_{H_0^1} \\
&= (q_1 - q_2, r_1 - r_2)_{L^2} + (q_2 - q_1, p_1 - p_2)_{H_0^1} \\
&= (u_1 - u_2, v_1 - v_2).
\end{aligned}$$

Now we proceed by showing the maximality of A. By Minty's theorem, it suffices to verify the equality

$$\mathrm{R}(I + A) = H.$$

Going in this direction, we need to consider the function

$$G := i^* R i + F : H_0^1(\Omega) \to H^{-1}(\Omega),$$

as mentioned in the previous discussion, we can think of G simply as $I - \Delta$. It is clear that G is continuous, moreover, a straight forward computation shows that G is strongly monotone. In fact, for u_1, u_2 in $H_0^1(\Omega)$,

$$(u_1 - u_2, G(u_1) - G(u_2))_{H_0^1, H^{-1}} = \|u_1 - u_2\|^2_{L^2(\Omega)} + \|u_1 - u_2\|^2_{H_0^1}.$$

It follows from Theorem 1.39, Theorem 1.41 and Lemma 1.38 that G is maximal monotone, $G + \widetilde{\beta}$ is maximal monotone and that $G + \widetilde{\beta}$ is coercive. Finally, by Theorem 1.40, we conclude that $G + \widetilde{\beta}$ is surjective. Therefore, take any $[g, h]$ in H and consider $q \in \mathrm{D}(\widetilde{\beta})$ such that

$$i^* R(h) - F(g) \in (G + \widetilde{\beta})(q).$$

Then $[q + g, q]$ belongs to $H_0^1(\Omega)^2$ and

$$i^* R(h - q) \in (F(q + g) + \widetilde{\beta}(q)) \cap i^* R[L^2(\Omega)],$$

that is, $[q + g, q]$ is in $\mathrm{D}(A)$ and

$$[-q, h - q] \in A(q + g, q).$$

This implies that

$$[g, h] \in [q + g, q] + A(q + g, q).$$

■

Assume that $f \in W^{1,1}(0, T; L^2(\Omega))$ and $w_0 := [u_0, v_0] \in \mathrm{D}(A)$; recall that $h(t) := [0, f(t, \cdot)]$. By Theorem 2.20, there exists a unique solution $w \in W^{1,\infty}(0, T; H)$ of problem

$$\begin{cases} w' + A(w) \ni h, \text{ on } (0, T) \\ w(0) = w_0. \end{cases}$$

Moreover, w is differentiable from the right in $[0, T)$ and there

$$\frac{\partial^+ w}{\partial t} = (h - A(w))^0.$$

Put $w := [w_1, w_2]$ and define $u : [0, T] \times \Omega \to \mathbb{R}$ as

$$u(t, x) := w_1(t)(x).$$

Then u satisfies the following two properties. Clearly, for a.e. x in Ω,

$$u(0, x) = u_0(x).$$

And if $[z_1, z_2]$ in $A(w(0))$ is such that

$$\frac{\partial^+ w}{\partial t}(0) = h(0) - [z_1, z_2],$$

then $z_1 = -v_0$. It follows that for a.e. x in Ω,

$$\frac{\partial^+ u}{\partial t}(0, x) = v_0(x).$$

On the other hand, for a.e. t in $(0, T)$,

$$[w_1'(t), w_2'(t)] + [-w_2(t), z(t)] = [0, f(t, \cdot)],$$

where $z(t)$ belongs to $L^2(\Omega)$ and, for some τ in $\widetilde{\beta}(w_2(t))$,

$$i^* R(z(t)) = F(w_1(t)) + \tau.$$

Therefore, $w_1' = w_2$ is in $W^{1,\infty}(0, T; L^2(\Omega))$ and, for a.e. t in $(0, T)$,

$$i^* R\left(\frac{\partial^2 u}{\partial t^2}(t, \cdot)\right) + F(u(t, \cdot)) + \widetilde{\beta}\left(\frac{\partial u}{\partial t}(t, \cdot)\right) \ni i^* R(f(t, \cdot)).$$

As a consequence of this, we have the next theorem.

Theorem 3.7. *Suppose T is a positive real number and Ω an open, bounded and smooth subset of $\mathbb{R}^n$. Let β be a maximal monotone operator on $\mathbb{R}$ satisfying $0 \in \beta(0)$ and consider $\mathrm{D}(A)$ as previously constructed. Assume also that f is in $W^{1,1}(0, T; L^2(\Omega))$ and $[u_0, v_0]$ in $\mathrm{D}(A)$. Then, there exists a function $u : [0, T] \times \Omega \to \mathbb{R}$ such that*

$$u \in W^{1,\infty}(0, T; H_0^1(\Omega)) \cap W^{2,\infty}(0, T; L^2(\Omega))$$

and, for a.e. t in $(0, T)$, the following holds in $H^{-1}(\Omega)$:

$$\frac{\partial^2 u}{\partial t^2}(t, \cdot) + F(u(t, \cdot)) + \widetilde{\beta}\left(\frac{\partial u}{\partial t}(t, \cdot)\right) \ni f(t, \cdot).$$

Moreover, for a.e. x in Ω,

$$u(0, x) = u_0(x), \ \frac{\partial^+ u}{\partial t}(0, x) = v_0(x).$$

■

Observation 3.8. *In the lemmas of this section we used the operators A_1 and A, both on H. It is interesting to see that, under the hypotheses of Lemma 3.2, these two operators coincide. In fact, it is not hard to show that $A_1 \subseteq A$ and conclude by maximality of A_1.*

We finish the thesis by presenting a concrete application of Theorem 3.5. As a particular case of the Wave Equation we get the String Equation:

$$(se)\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial^2 v}{\partial x^2} + \sigma(\dfrac{\partial v}{\partial t}) \ni g, \text{ on } (0,1)^2 \\ v(\cdot,0) = v(\cdot,1) = 0, \text{ on } (0,1), \end{cases}$$

where $v : (0,1) \times [0,1] \to \mathbb{R}$ is the unknown function, σ is the sign operator and $g : (0,1)^2 \to \mathbb{R}$ is a given function. Under certain assumptions, it describes the vibration of an elastic string with fixed ends; for example, the string of a guitar. Since the sign operator clearly verifies the hypotheses of Theorem 3.5, problem (se) can be "solved" if g is an element of $W^{1,1}(0,1;L^2(0,1))$.

Although the applications shown in this chapter are diverse and require different methods, they all reflect the importance of the existence theorem. A much larger number of problems can be solve using this procedure; as shown by the study of dynamics of wheeled vehicles, this theorem keeps being extremely useful in modern science.

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