### KNOT CONCORDANCE GROUPS AND THEIR RELATIONS

A Thesis

Presented to

Department of Mathematics and Its Applications Central European University, Budapest

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

Hakan Doga

June 2015

© 2015

Hakan Doga

## ALL RIGHTS RESERVED

#### ACKNOWLEDGMENTS

I would like begin with expressing my sincere thanks to my supervisor Prof. András Stipsicz for his guidance and support during the whole thesis process. His expertise, his constructive comments and corrections were invaluable elements of my graduate study. I am also thankful to my professors and staff in the department for helping and supporting me through the whole master's degree.

I know that no matter how many times I thank my parents it will never be enough, but their support is the only thing that gives me strength to continue and I am genuinely happy that they trust me in every decision I make. Given this chance, I would like to thank them here again.

Finally, I am really grateful to have this opportunity to complete my graduate degree in Central European University surrounded by nice friends and colleagues.

## TABLE OF CONTENTS

Acknow	vledgm	ents	iii				
List of	Figures	s	v				
Chapte	er						
1.	. Basic Definitions, Theorems and Notions About Knot Concordance						
	1.1.	Introduction	1				
		1.1.1. A short outline of the work	2				
	1.2.	Three Different Concordance Relations	4				
2.	Algebraic Concordance, Isometric Structures and the Witt Group $\ .$ .						
	2.1.	Basic Constructions and Their Relation	26				
	2.2.	Invariants from Witt Groups	34				
3.	Dona	ldson's Theorem Applied to Sliceness	40				
	3.1.	Sliceness Obstruction from Donaldson	40				
		3.3.1. Some computations	47				
4.	Grid	Homology, Topological Concordance vs. Smooth Concordance .	57				
	4.1.	Grids	57				
	4.3.	Invariance of the Grid Homology; stabilization case	80				
	4.4.	au and Slice Genus	90				
Referen	nces		98				

## LIST OF FIGURES

# Figure

1.1. Right Handed Trefoil	5
1.2. A twist knot $W_k^+$	5
1.3. Right Handed Trefoil	6
1.4. Mirror image of RHT	6
1.5. Connected sum of two manifolds	7
1.6. Connected sum of RHT and LHT	8
1.7. A smooth disk in $D^4$	9
1.8. The spinning argument explained	2
1.9. A simple visualization of concordance	4
1.10. A small $D^4$ removed from the slice disk	4
1.11. How to combine two concordances by removing an arc	6
1.12. Assigning signs to the crossings	7
1.13. Stabilization of a Seifert surface and the new generators of the homology $1$	8
2.1. Stabilized Seifert surface and computation of linking numbers 2	7
2.2. A general diagram for $K(a, b, c)$	5
2.3. The knot $K(1, -5, 1)$ and the generators $\ldots \ldots \ldots \ldots \ldots \ldots 3$	8
3.1. A twist knot $W_k$ with a positive clasp	1
3.2. Seifert surface, generators and linking numbers	2
3.3. The case for genus 1 with 4 branch points	5

3.4.	2-bridge knots in general	47
3.5.	Isotoping $W_6$ to K(25,23)	48
3.6.	K(p,q) divided into two pieces	49
3.7.	Untwisting the lower part of $K(p,q)$	50
3.8.	Slam-dunk operation	52
3.9.	Lens space $L(p,q)$	52
3.10.	Using slam-dunk operation to get $\mathbb{RP}^3$	52
3.11.	Plumbing graph corresponding L(25,23)	53
4.1.	A grid diagram for two unknots in $S^3$	59
4.2.	Fixing the violation of the rule "vertical goes over horizontal" $\ldots$	60
4.3.	Drawing $W_2$ with piecewise linear line segments $\ldots \ldots \ldots \ldots$	60
4.4.	Constructing a grid diagram for $W_2$	61
4.5.	Two different grids representing the unknot	61
4.6.	Nested interval	62
4.7.	Separate intervals	62
4.8.	Intervals touching at one point	62
4.9.	Stabilization at an $X$ -marking $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$	63
4.10.	$\mathbb{X}_{SW}$ and commutations giving $\mathbb{X}_{NW}$	64
4.11.	$\mathbb{X}_{SW}$ and commutations giving $\mathbb{X}_{NE}$	65
4.12.	$\mathbb{X}_{SW}$ and commutations giving $\mathbb{X}_{SE}$	65
4.13.	$\mathbb{X}_{SW}$ and commutations giving $\mathbb{O}_{NW}$	66
4.14.	$\mathbb{X}_{SW}$ and commutations giving $\mathbb{O}_{NE}$	66
4.15.	$\mathbb{X}_{SW}$ and commutations giving $\mathbb{O}_{SW}$	67

4.16. $\mathbb{X}_{SW}$ and commutations giving $\mathbb{O}_{SE}$	67
4.17. Unknot on the toroidal grid	68
4.18. An example of a grid state	69
4.19. An example of a rectangle	70
4.20. Two different pair of rectangles from $\mathbf{x}$ to $\mathbf{z}$	72
4.21. Two different pair of rectangles from $\mathbf{x}$ to $\mathbf{z}$	73
4.22. The annuli containing an $X$ and an $O$ marking	74
4.23. A $2 \times 2$ grid diagram for the unknot	75
4.24. Rectangles contributing to the boundary of $\mathbf{x}$	75
4.25. Rectangles contributing to the boundary of $\mathbf{y}$	76
4.26. How two grids differ after stabilization	81
4.27. Rectangles containing $X_1$ or $X_2$	83
4.28. Other cases of rectangles containing $X_1$ or $X_2$	83
4.29. Matching a grid state of $\mathcal{G}$ by adding $\{c\}$	84
4.30. After adding $\{c\}$ , the shift for each coordinate of <b>x</b>	85
4.31. The rectangles $\mathbf{r_1}$ and $\mathbf{r_2}$	88
4.32. Taking the push-off of $K$ with opposite orientation $\ldots \ldots \ldots \ldots$	92
4.33. Signs coming from the crossings	92
4.34. $W_0^-(LHT)$	93
4.35. The crossings where $K_+$ , $K$ and $K_0$ differ	94
4.36. Skein rule to determine the Alexander polynomial	94
4.37. Grid diagram of $W_0^-(LHT)$	95

#### CHAPTER 1

Basic Definitions, Theorems and Notions About Knot Concordance

#### 1.1 Introduction

In the 1960s, Fox and Milnor [7] gave the definition of knot concordance and after that there has been quite a progress in this area. Main objective of the research has been to understand the structure of these concordance groups, since they are complicated and big groups. Levine [12] proved that the algebraic concordance group  $G^{\mathbb{Z}} \cong$  $\mathbb{Z}^{\infty} \oplus \mathbb{Z}_2^{\infty} \oplus \mathbb{Z}_4^{\infty}$ . Then in the 1980s, both Levine's and Casson-Gordon's [2] work led to understanding of topologically slice category of knots. A more recent approach comes from Donaldson's diagonalization theorem [6] which is originally about 4-manifold theory, but can be applied to knot theory to show that there are knots which are algebraically slice, but not smoothly slice. Freedman's [8] theorem which states that the knots with Alexander polynomial one is topologically slice also helped us construct knots which are topologically slice, but not smoothly. This construction is quite important in 4-dimensional manifold theory. With the help of these knots, it is possible to provide 4-manifolds which are homeomorphic, but not diffeomorphic to the standard Euclidean space  $\mathbb{R}^4$  (exotic  $\mathbb{R}^4$ ). Despite all these advances, there remains many open problems and conjectures. The most recent progress in the area is due to Cochran-Orr-Teichner [4]. They provide a finer filtration of the concordance group which gives us a deeper insight and understanding of the group.

#### **1.1.1** A short outline of the work

This thesis will consist of four chapters. The main purpose of this work is to combine all these theorems and concepts to provide a nice picture about knot concordance. I believe that it will be useful especially for graduate students who are planning to pursue a research in this area. I will try to put all these previously done works together and also try to fill in some gaps providing examples and computations. Throughout the text, basic knowledge of algebraic topology is assumed.

The first chapter will be the introductory chapter to provide some basic knowledge about the topic to the reader. After defining the isotopy relation, the definition of sliceness in three different categories will be given. An equivalence relation  $K_1 \sim K_2$ will be defined on the isotopy classes of knots and this requires some properties of connected sum of  $K_1$  with the mirror image of  $K_2$ . This will lead us to the notion of "concordance". We will show that this relation enables us to have a group structure by providing an inverse element. We will also define the slice genus in the topological and smooth case which will be used later in other chapters. The main theorems about the concordance groups will be mentioned and the homomorphisms between these groups will be introduced. The whole picture will look like;

$$C \stackrel{onto}{\to} C_{top} \stackrel{onto}{\to} G^{\mathbb{Z}} \cong \mathbb{Z}^{\infty} \oplus \mathbb{Z}_{2}^{\infty} \oplus \mathbb{Z}_{4}^{\infty}$$

where C denotes the smooth concordance group,  $C_{top}$  the topological concordance group and  $G^{\mathbb{Z}}$  is the algebraic concordance group. In the following chapters, if we call the first map  $\phi_1$  and the second one  $\phi_2$ , the main object will be to show that these homomorphisms have non-trivial kernel; we also introduce some invariants which can be combined to prove the isomorphism between the algebraic concordance group and  $\mathbb{Z}^{\infty} \oplus \mathbb{Z}_{2}^{\infty} \oplus \mathbb{Z}_{4}^{\infty}$ .

The following, second chapter is devoted to the isomorphism between  $G^{\mathbb{Z}}$  and  $\mathbb{Z}^{\infty} \oplus \mathbb{Z}_2^{\infty} \oplus \mathbb{Z}_4^{\infty}$ . To prove this isomorphism, Levine [12] introduced many homomorphisms from  $G^{\mathbb{Z}}$  to these summands. The algebraic concordance definition comes from the Seifert form or matrix of a given knot, therefore the techniques used in this chapter will be rather algebraic, sometimes referring to linear algebra. We will define the isometric structures and the Witt group of a ring or a finite field on which the invariants are easier to define. With this approach, we will discuss the invariants coming from these structures, which help us to detect the order of a knot in the algebraic concordance group. We will refer to Livingston's [17] example to compute the orders and show that the algebraic concordance group contains an infinite, 2-torsion and 4-torsion summand. The computation of these invariants will be algebraic and number theoretic mostly.

Knowing that a knot K is algebraically slice, Donaldson's diagonalization theorem [6] can be used to provide a further obstruction for a knot being smoothly slice. In this chapter, we will work with the knots which are algebraically slice but not smoothly slice. The important notions will be cyclic cover of a knot exterior, branched covers of  $S^3$  branched over a knot K, and also branched covers of  $D^4$  branched over the slice surface of a knot. These will be useful for the construction which will closely follow [13]. Basically, we will focus on twist knots which will be denoted with  $W_k^{\pm}$  and do the computations for the value k = 6. Throughout the chapter, we will see that by the help of intersection forms, and constructing a special 4-manifold, it is possible to extract some information about smooth sliceness of a knot. In the end, presenting these examples, we will show that the map  $\phi_1 \circ \phi_2 : C \to G^{\mathbb{Z}}$  has non-trivial kernel.

The final chapter will be dedicated to the investigation of the first homomorphism,  $\phi_1 : C \xrightarrow{onto} C_{top}$ . The kernel of this map consists of the knots which are topologically slice, but not smoothly slice. The existence of such knots will depend on Freedman's theorem [8] in which he states that the knots with Alexander polynomial one are topologically slice and the  $\tau$  invariant constructed using grid homology. We will follow the work in progress [20]. At the beginning of this last chapter, we will introduce the grid representation of knots and how to construct a chain complex using this combinatorial approach. Then the homology with two gradings, namely Maslov and Alexander gradings, will provide us the necessary tools to detect these knots. Mainly, the  $\tau$  invariant will provide a lower bound for the slice genus which helps us to distinguish topological and smooth categories. As an example, we will present the Whitehead double of the left handed trefoil with appropriate framing and show that it lies in the kernel of  $\phi_1$  and it is nontrivial.

#### **1.2** Three Different Concordance Relations

Let us start by defining our main objects of interest; knots in  $S^3$ . For the rest of this introductory chapter, we will be talking about knot concordance in three different categories which are somehow related but at the same time far from being equal.

**Definition 1.1.** A knot is an (smooth) embedding of  $S^1$  into  $S^3$ , i.e.  $S^1 \hookrightarrow S^3$ .

In general an embedding of  $S^1$  is not necessarily smooth, but for our purposes

it will be always a  $C^{\infty}$  embedding. Here are some examples of my favourite knots;





Figure 1.1: Right Handed Trefoil

Figure 1.2: A twist knot  $W_k^+$ 

Now, we would like to introduce an equivalence relation in the set of all knots which helps us to understand and somehow categorise the knots we know.

**Definition 1.2.** The knots  $K_1$  and  $K_2$  are isotopic if there is a map  $f: S^1 \times [0,1] \rightarrow S^3$  such that  $f(S^1 \times \{1\}) = K_1$  and  $f(S^1 \times \{0\}) = K_2$ , and also for every  $x \in [0,1]$  $f(S^1 \times \{x\}) = K_x$  is a knot.

Assuming that  $S^1$  is oriented, this actually gives an orientation on the knot. For the rest of the work, we will be mostly working with oriented knots and links unless stated otherwise. This definition tells us that if we have two knots say  $K_1$  and  $K_2$  in  $S^3$  and we can turn one of them into the other while having a knot in each slice, these two knots are isotopic. We can see that isotopy defines an equivalence relation in the set of knots. Reflexivity and symmetry are immediate, for transitivity one can consider two different maps, say  $f: S^1 \times [0,1] \to S^3$  and  $g: S^1 \times [0,1] \to S^3$  such that  $f(S^1 \times \{0\}) = K_1$ ,  $f(S^1 \times \{1\}) = K_2$  and also  $g(S^1 \times \{0\}) = K_2$ ,  $g(S^1 \times \{1\}) = K_3$ . We can glue these isotopies in a way that it gives an isotopy from  $K_1$  to  $K_3$ . Quite similar to the isotopy we have already defined, now we will define ambient isotopy. **Definition 1.3.** Two oriented knots  $K_1$  and  $K_2$  (or links) are ambiently isotopic if there is a smooth map  $F: S^3 \times [0,1] \to S^3$  such that  $F_x = F|_{S^3 \times \{x\}}$  is a diffeomorphism for each  $x \in [0,1]$ ,  $F_0 = id_{S^3}$ ,  $F_1(K_1) = K_2$ .

It can be seen that ambient isotopy is also an equivalence relation and we will call an equivalence class of a knot (or a link) under this equivalence class knot (or link) type.

We will weaken this condition and introduce the "concordance" relation. Before defining the concordance relation, we will recall some basic definitions and notions which will be used in the definiton of the concordance group as well.

**Definition 1.4.** Given an oriented knot  $(K, S^3)$  we can obtain three different forms of the knot K; the mirror image of K is  $mK = (K, -S^3)$ , the reverse of K is  $rK = (-K, S^3)$  and the inverse of K is defined to be  $mr(K) = (-K, -S^3)$  where the minus sign refers to the opposite orientation.

Later on mr(K) will be the inverse of K in the concordance group. To obtain the mirror image of K, one can simply consider the projection of the knot onto the plane and change all the crossings of the knot. As an example, the mirror image of the right handed trefoil will be the left handed trefoil.



Figure 1.3: Right Handed Trefoil



Figure 1.4: Mirror image of RHT

**Definition 1.5.** Given two oriented, connected n-manifolds  $M_1$  and  $M_2$ , we can define the connected sum of these manifolds  $M_1 \# M_2$  as follows; let  $N_1$  and  $N_2$  be two open n-balls embedded in  $M_1$  and  $M_2$  respectively. To construct the connected sum, we simply consider  $M_1 - N_1$  and  $M_2 - N_2$ , and take the quotient of the disjoint union of these punctured manifolds and glue them along the boundary with an orientation reversing map.

To visualize it;



Figure 1.5: Connected sum of two manifolds

When we consider the connected sum of two knots, we utilize the same idea that we used for the manifolds. Instead of *n*-balls, we delete two arcs from the knots, and glue them with arcs again. Important part is that these connecting arcs should intersect with the seperating plane only once, so the connected sum operation is welldefined. Moreover, while constructing the connected sum of two knots  $K_1$  and  $K_2$ , we do it in a way that the orientation of two knots match. Here in Figure 1.6, we can see an example of the connected sum of RHT and LHT.



Figure 1.6: Connected sum of RHT and LHT

When we define the connected sum of knots in this way, we have a well-defined binary operation in the isotopy classes of knots. It is obivous that the connected sum is commutative and associative. In this picture, the unknot serves as the identity element. This should bring the idea that we can have an abelian group, but the problem is that we do not have the inverse element. For any nontrivial knot, we cannot find a knot such that the connected sum with the new knot is the unknot. This will lead us to the concordance relation which will provide us sufficient conditions to have an abelian group, with an inverse element.

One of the main concepts of concordance will be the sliceness. First, let us define this term.

**Definition 1.6.** A knot K is called smoothly slice if there exists a smooth embedding  $(D^2, K) \hookrightarrow (D^4, S^3)$  such that  $\partial D^2 = K$ .

One can imagine this as our knot K living in  $S^3$  and bounding a smooth  $D^2$ in  $D^4$ . In a schematic picture;



Figure 1.7: A smooth disk in  $D^4$ 

It is important that the disk bounded by K is smooth. Otherwise, it will give rise to another definition called topological sliceness.

**Definition 1.7.** A knot K is called topologically slice if there exists a continuous embedding  $(D^2 \times D^2, K \times D^2) \hookrightarrow (D^4, S^3)$  such that  $\partial D^2 = K$ .

In topological sliceness, we embed K and our topological  $D^2$  with their tubular neighborhoods. It is important to realize that these two concepts are actually far from being the same. There are knots which are topologically slice, but not smoothly slice as we shall see in the last chapter. Using the smooth embedding of  $D^2$  in  $D^4$ , it can be shown that a smoothly slice knot is also topologically slice, but the converse statement is not true.

We have defined sliceness depending on whether or not our knot bounds a smooth or topological  $D^2$  in  $D^4$ . A knot does not necessarily bound a  $D^2$  in  $D^4$ , instead it can bound another surface. We will refer to these surfaces smoothly slice surface or topological slice surface. We will also define the smooth slice genus and topological slice genus based on this surfaces.

**Definition 1.8.** A surface F is called a smoothly slice surface for the knot K if there exists a smooth embedding  $(K, F) \hookrightarrow (S^3, D^4)$  where  $\partial F = K$ . In a similar fashion, F is called a topological slice surface for K if there exists a continuous embedding  $(K \times D^2, F \times D^2) \hookrightarrow (S^3, D^4)$  such that  $\partial F = K$ .

Associated to the smoothly slice surface or the topological slice surface of a knot, now we will define the smoothly slice genus and topological slice genus number. **Definition 1.9.** The smooth slice genus, denoted as  $g_s(K)$ , is defined to be  $min\{g(F)|$ F is a smoothly slice surface for K. As expected, the topological slice genus  $g_{top}(K) = min\{g(F)|\ F$  is a topological slice surface for K.

It can be seen that  $g_s(K) = 0$  and  $g_{top}(K) = 0$  implies smooth and topological sliceness respectively. Now we will mention a key theorem for the construction of concordance group.

**Theorem 1.10.** For any knot  $K \subset S^3$ , K # mr(K) is slice, where mr(K) is the inverse of K as defined previously.

Proof. It is convenient to think of a knot K in  $\mathbb{R}^3$ . Since we know that  $\mathbb{R}^3 = S^3 - \{point\}$ , we can imagine that K is in  $\mathbb{R}^3$  and all the previous constructions work. The main idea underlying this is that we can consider this extra point as a point at infinity and eventually adding this point to  $\mathbb{R}^3$ , it wraps up to  $S^3$  and this  $S^3$  is the boundary of  $D^4$ . In the schematic picture, Figure 1.8, the planar rectangle represents  $\mathbb{R}^3$  and the rectangular box it bounds represents  $\mathbb{R}^4_+$ . Now let  $\mathbb{R}^3_+ = \{(x_1, x_2, x_3, 0) \mid x_3 \ge 0\} \subset \mathbb{R}^4$  to be the positive half space and assume that the knot K is in  $\mathbb{R}^3_+$ . We will use the

spinning argument to construct the slice surface. Let A be an arc connecting K and  $\partial \mathbb{R}^3_+$  in a way that when adjoined with an arc on  $\partial \mathbb{R}^3_+$ , it gives a knot isotopic to K. Now delete a small arc from K and connect it to  $\partial \mathbb{R}^3_+$  with the arc A. Call the resulted, knotted arc A' in  $\mathbb{R}^3_+$ . For a point  $x = (x_1, x_2, x_3, 0) \in \mathbb{R}^3_+$ , we can spin it by using the formula  $x_{\theta} = (x_1, x_2, x_3 \cos \theta, x_3 \sin \theta)$  for  $0 \le \theta \le \pi$ . As shown in Figure 1.8, this spins the point in a circular trajectory landing on the other half space  $\mathbb{R}^3_-$ . We spin all the points for any  $X \subset \mathbb{R}^3_+$ , i.e.  $X_s = \{x_{\theta} \mid x \in X, 0 \le \theta \le \pi\}$ .

Hence, we can see that by spinning K, we obtain mr(K) in  $\mathbb{R}^3_-$  and furthermore  $\partial((A')_s) = K \# mr(K)$ . Of course, we do not just spin the knot, but the whole half space  $\mathbb{R}^3_+$  and we obtain  $\mathbb{R}^4_+$ . Finally, we need to see that in fact  $(A')_s$  gives a smooth  $D^2$ . It is possible to see this fact if we consider the embedding map of the knot K and the map describing the spinning. First of all, recall that we are always considering smoothly embedded knots. Consider the embedding map as a map from  $[0, 2\pi]$  to  $\mathbb{R}^3 \subset \mathbb{R}^4$ . Therefore the embedding map which gives the coordinates of the knot, say  $t \mapsto (x(t), y(t), z(t), 0)$ , is smooth. After that, we compose it with the spinning map which describes the motion in  $\mathbb{R}^4$  and can be given as  $(x(t), y(t), z(t), 0) \mapsto (x(t), y(t), z(t) \cos\theta, z(t) \cos\theta)$ . It is not difficult to see that this map is smooth. As a result, we obtain a smooth  $D^2$ .



Figure 1.8: The spinning argument explained

We have proved that K # mr(K) is slice, equivalently  $g_s(K \# mr(K)) = 0$  for any knot K. By showing this, we have found a candidate for the inverse element. As one can guess, mr(K) will be the inverse element for any knot K in the concordance group. Here comes the definition of concordance.

**Definition 1.11.** Two knots  $K_1$  and  $K_2$  are said to be concordant if  $K_1 \# mr(K_2)$  is slice, where  $mr(K_2)$  denotes the inverse of  $K_2$ . We denote it as  $K_1 \sim K_2$ .

At first sight, this definition seems a little artificial. One can say that we started with the isotopy classes of knots and somehow we need a similar relation to classify them. The following theorem shows that an equivalent definition of concordance exhibits some similarities with the notion of isotopy. **Theorem 1.12.** Two knots  $K_1$  and  $K_2$  are concordant if and only if they bound a smooth 2-manifold C diffeomorphic to a cylinder, i.e.  $S^1 \times I$ , in  $S^3 \times I$  such that  $C \cap (S^3 \times \{1\}) = K_1$  and  $C \cap (S^3 \times \{0\}) = K_2$  where I denotes the unit interval.

*Proof.* Assume that we have such smooth, proper embedding C of a cylinder,  $S^1 \times I$  such that  $C \cap (S^3 \times \{1\}) = K_1$  and  $C \cap (S^3 \times \{0\}) = K_2$ . We should construct the smooth disk bounded by  $K_1 \# mr(K_2)$ . Due to lack of dimensions we live in, it will not be easy to visualize this.

Rigorously, first we will consider the tubular neighborhood of C. So we have  $(C \times D^2) \times I$ . One can think of this as the cylinder thickened a little bit inside  $S^3 \times I$ . To obtain the connected sum of the knots, we need to cut this cylinder with a proper arc. After a suitable self-diffeomorphism of  $S^3 \times I$ , we can assume there exists an embedding of C with its tubular neighborhood;  $\phi : (S^1 \times D^2) \times I \hookrightarrow S^3 \times I$ . Let's choose an arc  $A \subset S^1$  which will cut our surface C and also  $S^3 \times I$  resulting in the smooth disk in  $D^4$  that we wanted to construct. Notice that  $\phi(A \times \{0\} \times I) \subset K_1$  and  $\phi|_{(A \times D^2 \times I)}$  is the product of inclusion  $A \times D^2 \hookrightarrow S^3$  with  $id_I$ . Now we can remove some parts to obtain the smooth disk. So remove  $((A \times D^2) \times I)$  from  $S^3 \times I$ . Moreover, remove  $A \times \{0\} \times I$  from C. Therefore we obtain the slice disk in  $D^4$  bounded by  $K_1 \# mr(K_2) \in \partial D^4$ .

For a better visualization see Figure 1.9 where the arc A and the tubular neighborhoods we remove are shown;



Figure 1.9: A simple visualization of concordance

The other part of equivalence is quite easy to see. If we are given a slice disk bounded by  $K_1 \# mr(K_2)$ , we can remove a sufficiently small open 4-ball from the center of the slice disk, then we obtain the cylinder C embedded in  $S^1 \times I$ .



Figure 1.10: A small  $D^4$  removed from the slice disk

Here we can see that  $K_1 \# mr(K_2)$  is concordant to the unknot. To obtain the precise construction, we have two copies of  $S^3$ , and at 0 level, we consider  $K_2 \# mr(K_2)$ bounding the C. We know that  $K_2$  and  $mr(K_2)$  bound a smooth disk in  $D^4$ . Closing the cylinder with that smooth disk we obtaing the embedded cylinder such that  $C \cap (S^3 \times \{1\}) = K_1$  and  $C \cap (S^3 \times \{0\}) = K_2$ .

It is not difficult to see that knot concordance is an equivalence relation on the set of knots. If two knots are isotopic, they are also concordant to each other. We know that the connected sum operation is well-defined, associative and commutative. Hence we can state the theorem we have been waiting for:

**Theorem 1.13.** Under the connected sum operation defined for the knot concordance, the knot concordance classes form an abelian group denoted by C.

Proof. Here we will just mention the obvious facts about C being an abelian group. First of all, knot concordance gives an equivalence relation. For reflexivity, given any knot K we can consider  $K \times I$ , where I is the unit interval, inside  $S^3 \times I$ , hence  $K \sim K$ is clear. For symmetry, we can just turn "upside down" the concordance between  $K_1$ and  $K_2$ . For transitivity, we have the concordances between  $K_1$ ,  $K_2$  and  $K_2$ ,  $K_3$ . By stacking them properly, it is easy to see that we obtain the concordance between  $K_1$ and  $K_3$ . To prove that connected sum is well-defined, we will just try to show you the picture. Basically, we repeat a similar procedure that we did in the previous theorem. Assume that  $K_1 \sim K_2$  and  $J_1 \sim J_2$  where  $C_1$  and  $C_2$  are corresponding concordances. We need to show that  $K_1 \# J_1 \sim K_2 \# J_2$ .



Figure 1.11: How to combine two concordances by removing an arc

As you can see in the picture, we choose two arcs again, and remove the tubular neighborhood of arcs from  $S^3 \times I$  and cut the cylinders along these arcs say  $A_1$  and  $A_2$ . More precisely, we have  $C_1 - (A_1 \times I)$  and  $C_2 - (A_2 \times I)$ . After that we glue these two concordances properly through an orientation reversing diffeomorphism identifying boundaries of these arcs and tubular neighborhoods with each other.

The associativity and the commutativity follow from the properties of the connected sum operation itself, and the identity element is the concordance class of the unknot and for any knot K, inverse element is mr(K) which is the mirror image of K with the reversed orientation. Throughout the text, we will reserve C for the smooth concordance group.

We can consider the same setting for topological concordance. We will denote the topological concordance group with  $C_{top}$ . It is already mentioned that any smoothly slice knot is also topologically slice, so we can consider the map  $\phi_1 : C \to C_{top}$  sending the smooth concordance class of K to the topological concordance class of the same knot. This map is a group homomorphism which is onto. The kernel of this homomorphism consists of the knots which are topologically slice, but not smoothly slice. In the last chapter, we will show that actually this homomorphism is not an isomorphism. Before defining the algebraic concordance group  $G^{\mathbb{Z}}$ , we will give some preliminary definitions.

First of all, we will adopt the following convention to assign a sign for the crossings of a given oriented knot  $K \subset S^3$ .



Figure 1.12: Assigning signs to the crossings

**Definition 1.14.** Let  $K_1$  and  $K_2$  be two disjoint, oriented knots and their union is represented by a diagram D. The linking number  $lk(K_1, K_2)$  of  $K_1$  with  $K_2$  is the half of the sum of the signs of those crossings where both strands are not from the same component.

Here, even though we will not argue for it, we should note that the linking number is independent of the knot diagram we choose.

**Definition 1.15.** A compact, connected, oriented and smoothly embedded surface

F with boundary in  $\mathbb{R}^3$  is called a Seifert surface of K if it is the boundary of the oriented knot (or link) K, i.e.  $\partial F = K$  and orientation coming from  $\partial F$  agrees with the orientation of K. The Seifert genus of K is defined as  $g(K) = \min\{g(F) \mid F \text{ is a Seifert surface for } K\}$ .

It is natural to ask when we are given such a knot K whether this surface is unique or not. At first sight, considering the simplest case, the unknot, we can see that there can be many different surfaces bounded by the unknot. However, the following theorem will tell us that it is possible to make these surfaces equal by using an operation called "stabilization". We shall briefly explain how this "stabilization" works.

Given the surface F, pick two points  $p, q \in Int(F)$  and connect these points with an arc  $\alpha$  in  $S^3 - F$  and assume that it approaches F from the same side. We then just delete a small disk neighborhoods of p and q and add a "tube" around  $\alpha$ . We obtain a new surface F' which inherits an orientation from F and has genus g(F') = g(F) + 1. The main idea is that, by using this operation sufficiently many times we can make two Seifert surfaces equal. Here is a picture of stabilization;



Figure 1.13: Stabilization of a Seifert surface and the new generators of the homology

**Theorem 1.16** (Reidemeister-Singer Theorem, [24]). Let  $F_1$  and  $F_2$  be two Seifert surfaces for an oriented knot K (or a link), then  $F_1$  and  $F_2$  can be stabilized sufficiently many times to obtain ambiently isotopic surfaces  $F'_1$  and  $F'_2$ .

Given a knot K, its Seifert surface F provides a bilinear form on  $H_1(F;\mathbb{Z})$ . This form is called the Seifert form and we will define it as follow. Let  $x, y \in H_1(F;\mathbb{Z})$ , then these elements can be represented by pairwise disjoint, oriented, simple closed curves;  $\gamma_x, \gamma_y$  respectively. Furthermore, let  $\gamma_x^+$  denote the push-off of  $\gamma_x$  in the positive normal direction of F.

**Definition 1.17.** The Seifert form  $\theta$  for the Seifert surface F of the knot (or link) K is defined by

$$\theta(x,y) = lk(\gamma_x,\gamma_y^+)$$

where  $x, y \in H_1(F; \mathbb{Z})$ .

It can be seen that this definition is independent of the chosen representatives. Now we can define the Seifert matrix. After chossing a basis  $\{u_1, u_2, \ldots, u_n\}$ for  $H_1(F;\mathbb{Z})$  we can represent these basis elements with embedded curves  $\alpha_1, \ldots, \alpha_n$ . The Seifert matrix is  $(S_{i,j}) = (lk(\alpha_i, \alpha_j^+))$ . We shall make use of the Seifert matrices quite frequently. Later on, we will define some invariants coming from Seifert matrices. We would like to give two remarks before proceeding to the notion of algebraic concordance.

• Parallel to our discussion about the Seifert surfaces, the Seifert matrices can be different for the same knot (or link). A similar method can be applied to the matrices and two Seifert matrices can be "stabilized" to give equivalent matrices. The main reasoning comes from that when we stabilize a Seifert surface and increase the genus by 1, we actually add two new generators to  $H_1(F;\mathbb{Z})$ . Then we can modify the Seifert matrix by extending the basis for  $H_1(F;\mathbb{Z})$ . Two Seifert matrices will be called S-equivalent if they differ by a sequence of stabilizations and basis changes.

Even though we have defined the Seifert matrices with integer entries, later in the text we will mention Seifert matrices over a field F. Here is how we can define it. An even-dimensional matrix A with entries in a field F is called a Seifert matrix or an F-Seifert matrix if it satisfies det((A − A<sup>T</sup>)(A + A<sup>T</sup>)) ≠ 0. The corresponding Seifert form is a bilinear form on a vector space V of dimension equal to the dimension of A, given by θ(x, y) = x<sup>T</sup>Ay for x, y ∈ V written as column vectors.

Now we have the necessary preparation to define algebraic concordance.

**Definition 1.18.** A  $(2n \times 2n)$  dimensional Seifert matrix A is algebraically slice if it is congruent to a matrix with a half dimensional block of zeros on the top left part, i.e. there exist a nonsingular matrix P such that;

$$PAP^T = \left(\begin{array}{cc} 0 & B \\ C & D \end{array}\right).$$

A knot K is called algebraically slice if it has a Seifert matrix which is algebraically slice.

Equivalently, we can think of algebraic sliceness in terms of the Seifert form corresponding to the Seifert matrix given in the definition. We say that this Seifert form is algebraically slice or metabolic if there is an n-dimensional summand of the underlying free  $\mathbb{Z}$ -module on which the form vanishes. This form is called the metabolizer of the form.

If a Seifert matrix of a knot is algebraically slice, then by S-equivalence, then all of them are, independent of the choice of Seifert surface.

It is natural to think about the difference between algebraically sliceness and sliceness. Before stating the relation between these two, we will just mention one theorem.

**Theorem 1.19** ([15], Theorem 3.1.2). Assume K is a slice knot, F is a Seifert surface for K and  $\theta$  the Seifert form, then there exists a summand H of  $H_1(F)$  such that;

- (1)  $rk(H) = \frac{1}{2}rk(H_1(F))$
- (2)  $\theta_{H \times H} = 0$

**Corollary 1.20.** A slice knot is algebraically slice.

Converse of the corollary above is not true as we shall see in Chapter 3. The simplest examples are the twist knots  $W_k^{\pm}$  which can be seen in Figure 1.2. In the next chapters, we will see that a twist knot  $W_k^{\pm}$  is algebraically slice if and only if  $4k + 1 = l^2$ . These examples are also the first known examples of Casson and Gordon which are algebraically slice but not slice.

We showed that under the connected sum operation, we can form an abelian group called concordance group. Now, we would like to discuss another concordance relation which will provide an abelian group again. The key elements will be the Seifert matrices, and we will utilize the orthogonal sums of matrices.

**Definition 1.21.** A  $2n \times 2n$  dimensional Seifert matrix over a field  $\mathbb{F}$  is called al-

gebraically slice or metabolic if there exists and n-dimensional subspace of  $V \cong \mathbb{F}^{2n}$ on which the corresponding form vanishes. In the language of matrices, there exists a nonsingular matrix P such that:

$$PAP^T = \left(\begin{array}{cc} 0 & B \\ C & D \end{array}\right)$$

where the top left 0 is an  $n \times n$  block of zeros.

**Definition 1.22.** Given two Seifert matrices  $A_1$  and  $A_2$ , we will call them Witt equivalent or algebraically concordant if;

$$A_1 \oplus A_2 = \left(\begin{array}{cc} A_1 & 0\\ 0 & -A_2 \end{array}\right)$$

is algebraically slice. We denote this by  $A_1 \sim A_2$ .

We have another relation on the isotopy classes of knots. We would like to show that this actually gives an equivalence relation. To prove this, besides some algebraic manipulation of matrices, we need one lemma known as Witt Cancellation Lemma.

**Lemma 1.23** ([15], Lemma 3.4.5). Let N and A be matrices of dimensions 2k and 2m respectively. Suppose that N and  $A \oplus N$  are algebraically slice and that  $N - N^T$  has non-zero determinant. Then A is algebraically slice.

**Theorem 1.24.** Algebraic concordance is an equivalence relation.

*Proof.* Let A, B and C be Seifert matrices with dim(A) = 2m and dim(B) = 2n.

(1) Reflexivity: We need to show that  $A \oplus -A$  is algebraically slice. Clearly,

$$A \oplus -A = \left(\begin{array}{cc} A & 0\\ 0 & -A \end{array}\right)$$

Let P be the  $4m \times 4m$  matrix adding (2m+i)th row to the *i*th row for  $1 \le i \le 2m$ . Clearly, P is a matrix corresponding to elementary row operations and it is nonsingular and if we consider  $P(A \oplus -A)P^T$ ;

$$P(A \oplus -A)P^{T} = \begin{pmatrix} 0 & -A \\ -A & -A \end{pmatrix}$$

which definitely contains a (2m)-dimensional diagonal block of zeros.

(2) Symmetry: Assume A ~ B. Then we know there exists a nonsingular matrix P such that P(A⊕−B)P<sup>T</sup> is algebraically slice. Hence it has (m+n) dimensional block of zeros.

$$P(A \oplus -B)P^{T} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & D \\ E & F \end{pmatrix}$$

To show that  $B \oplus -A$  is also algebraically slice, consider a new matrix Q which has rows of P with the following change; let  $R_1, R_2, ..., R_{2n+2m}$  be the rows of P. Q is obtained by switching the first 2m rows of P with the remaining 2nrows respectively. So Q has the rows  $R_{2m+1}, R_{2m+2}, ..., R_{2m+2n}, R_1, R_2, ..., R_{2m}$ . Hence if we write entries of P and Q in blocks we have,

$$P = \begin{pmatrix} P_{1_1} & P_{1_2} \\ P_{2_1} & P_{2_2} \end{pmatrix} and Q = \begin{pmatrix} P_{2_1} & P_{2_2} \\ P_{1_1} & P_{1_2} \end{pmatrix}$$

where the first row in P with entries  $P_{1_1}$  and  $P_{1_2}$  is a block of size  $2m \times (2m+2n)$ , and similarly second row is a block of size  $2n \times (2m+2n)$ . To obtain Q, we just switch these two rows. If we write down these huge matrices and check the corresponding elementary row and column operations, we will see that with this change, new matrix Q will perform the necessary changes in the first  $2n \times 2n$  block of B, and also  $2m \times 2m$  block of -A. So we have,

$$Q(B \oplus -A)Q^{T} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & -A \end{pmatrix} \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & D' \\ E' & F' \end{pmatrix}$$

Hence,  $B \sim A$ .

(3) Transitivity: Assume that A ~ B and B ~ C, i.e. A ⊕ -B and B ⊕ -C are algebraically slice. It is easy to see that we have the following relation between the orthogonal sum of algebraically slice matrices;

$$(A \oplus -B) \oplus (B \oplus -C) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & -C \end{pmatrix} = A \oplus (-B \oplus B) \oplus -C \sim (A \oplus -C) \oplus (B \oplus -B)$$

We know that  $(B \oplus -B)$  is algebraically slice, and by Witt cancellation lemma,

 $A \oplus -C$  is also algebraically slice.

Therefore, we have seen that algebraic concordance is an equivalence relation.  $\Box$ **Definition 1.25.** Equivalence classes of  $\mathbb{F}$ -Seifert matrices under the algebraic concordance relation form a group under orthogonal sum. We call this group the algebraic concordance group and denote it simply by  $G^{\mathbb{F}}$ . Generally we will focus on two cases; the integral algebraic concordance group  $G^{\mathbb{Z}}$  and the rational algebraic concordance group  $G^{\mathbb{Q}}$ . The main difference is the choice for the entries of the Seifert matrices we have defined previously.

**Theorem 1.26.** If we consider a map sending the concordance class of a knot to its algebraic concordance class of its Seifert matrix, we obtain a well-defined group homomorphism from C to  $G^{\mathbb{Z}}$ . Note that this homomorphism is also onto. Throughout the first chapter, we have given some basic definitions and theorems which helped us to have a better understanding of the subject. Keep in mind that our main objects are knots which are nothing but smooth embeddings of  $S^1$ into  $S^3$ . Starting with this simple-looking objects, we obtained abelian groups, and after that we saw that by sending one equivalence class to another, we can obtain onto homomorphisms. In the second chapter, we will analyze the structure of the algebraic concordance group.

#### CHAPTER 2

Algebraic Concordance, Isometric Structures and the Witt Group

#### 2.1 Basic Constructions and Their Relation

The algebraic concordance group was defined in the previous chapter. Here we will try to outline the ideas and invariants which are used to understand the structure of the algebraic concordance group. Levine [12] proved that;

$$G^{\mathbb{Z}} \cong \mathbb{Z}^{\infty} \oplus \mathbb{Z}_2^{\infty} \oplus \mathbb{Z}_4^{\infty}.$$

He achieved this by defining a full set of invariants. In this chapter, we will not prove this isomorphism. Following Livingston's [17] work closely, we will define the necessary constructions to show that the algebraic concordance group actually contains a sum isomorphic  $\mathbb{Z}^{\infty}$ ,  $\mathbb{Z}_{2}^{\infty}$  and  $\mathbb{Z}_{4}^{\infty}$ . We will define some algebraic invariants of knots useful not only for this purpose, but in general applicable everywhere. After that some examples will be provided to show that  $G^{\mathbb{Z}}$  has elements of order infinity, of 2 and of 4.

**Definition 2.1.** Let  $K \subset S^3$  and let A be a Seifert matrix for K. Then the determinant of K is defined as  $|det(A + A^T)|$ .

The next invariant is maybe one of the most commonly used one;

**Definition 2.2.** The Alexander polynomial of K is defined to be  $\Delta_K(t) = det(A - tA^T)$  where again A is a Seifert matrix for K.

**Definition 2.3.** Let F be a Seifert surface for K and A is a Seifert matrix of F.

The signature  $\sigma(K)$  is defined to be the signature of the symmetrized Seifert matrix  $A + A^T$ , i.e. the number of positive entries minus the number of negative entries on the diagonal form.

Clearly, we need to know that all these definitions make sense, i.e. they do not depend on the choice of a Seifert surface, or a Seifert matrix. We will give a brief argument why all these definitions only depend on the given knot.

**Lemma 2.4** ([20], Lemma 2.3.4). If F is a Seifert surface for K and F' is a stabilization of F, then there is a basis for  $H_1(F';\mathbb{Z})$  whose Seifert matrix has the form

A	$\psi$	$0 \rangle$		$\bigwedge A$	0	0
0	0	1	or	$\psi^T$	0	0
0	0	0 /		0	1	0 /

where A is a Seifert matrix for F and  $\psi$  is some vector.



Figure 2.1: Stabilized Seifert surface and computation of linking numbers

To see why this lemma is true, let  $\{u_1, \dots, u_n\}$  be a basis for  $H_1(F; \mathbb{Z})$  giving

the Seifert matrix A. After stabilizing the surface F, we add two new homology classes y and x. This means we add two rows and columns to the Seifert matrix A. Basically, we need to understand how these two new homology classes link with the others, themselves and each other. First, observe that  $lk(u_i, x^+) = lk(x, u_i^+) = 0$  for all  $1 \le i \le n$  and also  $lk(x, x^+) = 0$ . Moreover, depending on which side the stabilizing curve approaches F, we have either  $lk(x, y^+) = 0$  and  $lk(x^+, y) = 1$  or  $lk(x, y^+) = 1$  and  $lk(x^+, y) = 0$ . In Figure 2.1, the computation of the linking numbers are pictured. Now changing the basis by adding the multiples of x if necessary to the  $u_i$ 's and y, we obtain a Seifert form as expected. By Theorem 1.16, we know that given two different Seifert surfaces for K, we can stabilize them until they are ambiently isotopic. This lemma tells us how the Seifert matrices will change after a stabilization. It can be shown that with a change of basis, the invariants do not depend on the chosen Seifert matrix.

It is possible to see that the signature does not change after one stabilization. Let F' be obtained from F with one stabilization. Denote the Seifert matrix of F with A and the Seifert matrix of F' with A'. The symmetrized Seifert matrix will be;

$$A' + (A')^{T} = \begin{pmatrix} A + A^{T} & * & 0 \\ \vdots & \vdots \\ & * & 0 \\ \hline & & & & 0 \\ \hline & & & & & 0 \\ \hline & & & & & 0 \\ 0 \cdots 0 & & 1 & 0 \end{pmatrix}$$

where \* denotes some non-zero entries. Now using elementary row and column operations, first switch last two rows, then switch last two columns. We obtain;

$$\begin{pmatrix} A + A^T & 0 & * \\ \vdots & \vdots \\ & 0 & * \\ \hline 0 \cdots 0 & 0 & 1 \\ * \cdots * & 1 & 0 \end{pmatrix}$$

Now using 1's in the last row and column, we can make the entries denoted with \* 0, and then it is not difficult to see that the contribution of the small  $2 \times 2$  matrix on the bottom right corner to the signature is 0. Hence we can conclude that the signature does not change after a stabilization.

The determinant, the Alexander polynomial and the signature are all invariants which prove to be quite useful in some cases and we should be using them whenever needed. As mentioned in the previous chapter, we can consider both  $G^{\mathbb{Z}}$  and  $G^{\mathbb{Q}}$ . It is also known that there is a group homomorphism between these group sending the integral algebraic concordance class of a knot to its rational algebraic concordance class, which is injective. After mentioning this fact, we can continue with the definitions of isometric structure, Witt group and the invariants defined on the rational Witt group.

**Definition 2.5.** Let  $\mathbb{F}$  be a field of characteristic not equal to 2. An isometric structure over a field  $\mathbb{F}$  is a triple (V, Q, B) where V is a 2n-dimensional vector space, Q is a non-singular, symmetric, bilinear form on V and B is an isometry of V with respect to Q, which means Q(x, y) = Q(Bx, By) for all  $x, y \in V$ . This isometric structure will be called admissible if the characteristic polynomial of B,  $\Delta_B(t) = det(B - tI)$ satisfies  $\Delta_B(1)\Delta_B(-1) \neq 0$ .

Following Levine and Livingston, fixing a basis of  $V \cong \mathbb{F}^{2n}$ , we can represent *B* by a non-singular matrix, and B(x) is the matrix product Bx; *Q* is represented by a symmetric matrix Q(x, y) equal to the matrix product  $x^T Q y$ . Now we will define the binary operation on the isometric structures which will provide an equivalence relation on the set. Consider two isometric structures  $(V, Q_V, B_V)$  and  $(W, Q_W, B_W)$
where V and W are 2n and 2m dimensional  $\mathbb{F}$  vector spaces. Now define;

$$(V, Q_V, B_V) \oplus (W, Q_W, B_W) = (V \oplus W, Q_V \perp Q_W, B_V \oplus B_W)$$

where the direct sum of isometries  $B_V \oplus B_W : V \oplus W \to V \oplus W$  is defined to be;

$$(B_V \oplus B_W)(v \oplus w) = B_V(v) \oplus B_W(w)$$

and the orthogonal sum of forms  $Q_V \perp Q_W : (V \oplus W) \times (V \oplus W) \rightarrow \mathbb{F}$  is given by;

$$(Q_V \perp Q_W)(v_1 \oplus w_1, v_2 \oplus w_2) = Q_V(v_1, v_2) + Q_W(w_1, w_2).$$

It is not difficult to check that this is a binary operation. Now here is the definition of the equivalence relation:

**Definition 2.6.** An isometric structure (V, Q, B) of dimension 2n is called metabolic or Witt trivial, if there is an n-dimensional B-invariant subspace of V on which Q vanishes. Two isometric structures  $(V_1, Q_1, B_1), (V_2, Q_2, B_2)$  are called Witt equivalent if  $(V_1, Q_1, B_1) \oplus (V_2, -Q_2, B_2)$  is Witt trivial.

Using the definition of isometric structures, the binary operation defined on isometric structures and Witt cancellation lemma for isometric structures, one can see that this actually defines an equivalence relation on the set of isometric structures. Moreover, it induces an abelian group;

**Definition 2.7.** The abelian group of Witt classes of isometric structures with the given binary operation is denoted as  $G_{\mathbb{F}}$ .

We will not outline the details of proving that this structure is actually an abelian group. Instead, we will mention a special isometric structure which will be crucial while describing the connection between Seifert matrices and isometric structures.

Let A be a  $2n \times 2n$ , nonsingular F-Seifert matrix. Then  $(\mathbb{F}^{2n}, A + A^T, A^{-1}A^T)$ , where  $B = A + A^T$  and  $Q = A^{-1}A^T$  is an admissible isometric structure. First of all, we should see that this is an isometric structure. The nonsingularity of Q comes from the fact that A is a Seifert matrix, and we know that  $A + A^T$  has non-zero determinant. It is also clearly symmetric and bilinear from the definition. Moreover, B is a linear transformation of  $\mathbb{F}^{2n}$ , and by using some matrix multiplication manipulations it can be shown that it is an isometry with respect to Q. To see that it is also admissible, we can do the following trick;

$$\Delta_B(t) = det(A^{-1}A^T - tI) = det(A^{-1}A^T - tA^{-1}A) = det(A^{-1}(A^T - tA))$$

Determinant is multiplicative so we can write it as;

$$det(A^{-1})det(A^T - tA)$$

We know that A is non-singular so det(A) and  $det(A^{-1})$  are not equal to zero. Also for t = 1 and t = -1,  $det(A^T - A)$  and  $det(A^T + A)$  are also non-zero, because A is a Seifert matrix. Therefore, the isometric structure we have defined is admissible. Now we can define the map  $\psi_1 : G^{\mathbb{F}} \to G_{\mathbb{F}}$  sending a Seifert matrix to an isometric structure.

**Theorem 2.8.**  $G^{\mathbb{F}} \cong G_{\mathbb{F}}$  via following maps which are inverses of each other;

$$\psi_1: G^{\mathbb{F}} \to G_{\mathbb{F}}$$
$$A \mapsto (\mathbb{F}^{2n}, A + A^T, A^{-1}A^T)$$

and also  $\psi_1^{-1} = \psi_2$ 

$$\psi_2: G_{\mathbb{F}} \to G^{\mathbb{F}}$$
$$(V, Q, B) \mapsto Q(I+B)^{-1}$$

The main idea to prove this theorem relies on manipulation of matrices. We have already seen that  $(\mathbb{F}^{2n}, A + A^T, A^{-1}A^T)$  gives an admissible isometric structure. Similarly, one can check that  $Q(I + B)^{-1}$  defines a Seifert matrix. Again this will require some computation with matrices, transposes and inverses. It is sufficient to confirm that  $det(A - A^T) = 1$  for  $A = Q(I + B)^{-1}$ . We know that Q is non-singular and admissibility of the isometric structure implies that I + B is also non-singular. Check what the compositions  $\psi_1\psi_2$  and  $\psi_2\psi_1$  are:

$$\psi_2\psi_1(A) = (A + A^T)(I + A^{-1}A^T)^{-1} = A(I + A^{-1}A^T)(I + A^{-1}A^T)^{-1} = A$$

where first  $\psi_1(A) = (\mathbb{F}^{2n}, A + A^T, A^{-1}A^T))$  and then  $\psi_2$  maps this isometric structure to  $Q(I+B)^{-1}$  where  $Q = A + A^T$  and  $B = A^{-1}A^T$ .

For the other direction, we have:

$$\psi_2(V, Q, B) = Q(I+B)^{-1}.$$

Then this Seifert matrix under  $\psi_1$  maps to the isometric structure;

$$(\mathbb{F}^{2n}, Q(I+B)^{-1} + (Q(I+B)^{-1})^T, (Q(I+B)^{-1})^{-1}(Q(I+B)^{-1})^T).$$

It can not been seen directly that this actually equals to (V, Q, B). Two facts which can be found in [15] will give us the result. The first one is:

$$Q(I+B)^{-1} + (Q(I+B)^{-1})^T = Q((I+B)^{-1} + (I+B^{-1})^{-1}).$$

And the second one is:

$$(I+B)^{-1} + (I+B^{-1})^{-1} = I$$

These results primarily depend on the fact that we have an admissible isometric structure, and combining these two;

$$Q(I+B)^{-1} + (Q(I+B)^{-1})^T = Q((I+B)^{-1} + (I+B^{-1})^{-1}) = QI = QI$$

A similar argument shows that  $(Q(I+B)^{-1})^{-1}(Q(I+B)^{-1})^T = B.$ 

Hence, we have the necessary isomorphism between the algebraic concordance group  $G^{\mathbb{F}}$  and group of isometric structures  $G_{\mathbb{F}}$ .

Now we will mention some invariants defined on the Witt group. As Livingston says in [17], these invariants can be combined to give an isomorphism from the algebraic concordance group to the sum  $\mathbb{Z}^{\infty} \oplus \mathbb{Z}_2^{\infty} \oplus \mathbb{Z}_4^{\infty}$ . To this end, now we will present the definition of the Witt group. It will be useful to define homomorphisms from  $G_{\mathbb{Q}}$  to the Witt groups and use the invariants arising from here. We will consider the Witt groups over  $\mathbb{Q}$ , finite fields  $\mathbb{F}_p$  and also field of rational functions  $\mathbb{Q}(t)$  where the signature function is defined.

Let R be a commutative ring with a homomorphism  $\tau : R \to R$  such that  $\tau^2 = 1$ . These maps are called involutions. Some examples that we might utilize are rational numbers  $\mathbb{Q}$  with trivial involution, field of rational functions  $\mathbb{Q}(t)$  with the involution  $t \to t^{-1}$  and finite fields  $\mathbb{F}_p$  with trivial involution.

**Definition 2.9.** Let R be such a ring and consider a nonsingular, symmetric, bilinear form B on a free rank R-module H where symmetry is with respect to the involution  $\tau$ , i.e.  $B(x,y) = \tau(B(y,x))$ , nonsingularity will tell us the map  $H \to Hom(H,R)$  given by  $x \to \phi_x(y) = B(x, y)$  is an isomorphism, and finally bilinearity means linearity in the first variable and antilinearity in the second one, i.e.  $B(x, \alpha y) = \tau(\alpha)B(x, y)$ . Parallel to the definitions of algebraic concordance previously given, such a form B is called metabolic if there exists a submodule of M, with half rank; rank(H) = 2rank(M), and on this submodule B is trivial. Two forms B<sub>1</sub> and B<sub>2</sub> are called metabolic if the direct sum  $B_1 \oplus -B_2$  is metabolic. The set of equivalence classes is defined to be the Witt group of R and denoted as W(R).

The verification of the fact that this relation actually gives and equivalence relation and under the direct sum operation W(R) forms an Abelian group is quite similar to the proof of algebraic concordance. For further details, the reader may refer to [15].

### 2.2 Invariants from Witt Groups

Now, we can define the invariants arising from the Witt groups. We will also refer to some examples which can be computed relatively easily. First of all, observe that if we are given a rational Seifert form A, we can construct the nonsingular, symmetric form  $A + A^T$ . So the map  $A \to (A + A^T)$  gives a well-defined group homomorphism  $G_{\mathbb{Q}} \to W(\mathbb{Q})$ .

Before giving the definitions of the invariants, we will focus on a certain family of knots denoted as K(a, b, c). We will be considering these knots for computational purposes. Here a, b denotes the number of full twists and c is an odd number denoting the number of half twists between two bands. You can see how this knot looks like in this picture;



Figure 2.2: A general diagram for K(a, b, c)

For K(a, b, c) with basis elements  $\alpha$  and  $\beta$ , shown in the figure above, for the first homology, we will get the following Seifert matrix;

$$A_{K(a,b,c)} = \begin{pmatrix} a & (c+1)/2 \\ (c-1)/2 & b \end{pmatrix}$$

1. The signature function : This will be the simplest invariant on  $W(\mathbb{Q})$  and it is defined in a similar way with the previously defined signature. We consider the matrix respresenting the corresponding form and diagonalize it over  $\mathbb{Q}$ . Then, the number of positive entries minus the number of negative entries is called the signature, and denoted as  $\sigma$ . This gives a homomorphism from  $W(\mathbb{Q})$  to  $\mathbb{Z}$ . Given a rational Seifert form A, we compose two maps, i.e.  $A \to A + A^T \to \sigma(A + A^T)$  which gives the signature homomorphism. We will consider the knots K(1, k, 1) for k > 0. Look at the Seifert matrix of K(1, k, 1),

$$A_{K(1,k,1)} = \left(\begin{array}{cc} 1 & 1 \\ 0 & k \end{array}\right)$$

where k > 0. First symmetrize it,

$$A_{K(1,k,1)} + A_{K(1,k,1)}^T = \begin{pmatrix} 2 & 1 \\ 1 & 2k \end{pmatrix}$$

Now we can diagonalize it to see the signature;

$$A_{K(1,k,1)} + A_{K(1,k,1)}^T = \begin{pmatrix} 2 & 1 \\ 1 & 2k \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 \\ 0 & (4k-1)/2 \end{pmatrix}$$

It is obvious that when k > 0 this matrix has signature 2. So  $\sigma(A_{K(1,k,1)}) = 2$ . As previously discussed, the signature map gives a homomorphism in  $Hom(G_{\mathbb{Q}}, \mathbb{Z})$ . The signature itself is not sufficient to show that the algebraic concordance group contains a summand isomorphic to  $\mathbb{Z}^{\infty}$ , but sufficient to conclude that it contains  $\mathbb{Z}$ . This is due to the additivity of the signature;

$$\sigma(A \oplus B) = \sigma(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) = \sigma(A) + \sigma(B)$$

and also the fact that if A is algebraically slice, the signature is 0. We will not argue for the general case, but if a  $2 \times 2$  Seifert matrix is algebraically slice, we can see that the signature vanishes. Let A be a  $2 \times 2$  Seifert matrix which is algebraically slice. Then we know that there exists a nonsingular matrix P such that;

$$PAP^T = \left(\begin{array}{cc} 0 & b \\ c & d \end{array}\right).$$

Symmetrizing this matrix over the rationals and computing its eigenvalues we can see that the signature is 0. For a larger Seifert matrix, a similar reasoning can be used to prove the statement.

From the statements about the additivity property of the signature and the signature of an algebraically slice knot, we can conclude that the Seifert matrix  $A_{K(1,k,1)}$  is of infinite order in the algebraic concordance group. Yet, we still do not have the necessary machinery to prove that the algebraic concordance has a summand isomorphic to  $\mathbb{Z}^{\infty}$ . We will not argue for this, but this family of knots are linearly independent. Using this fact, it is possible to show the existence of  $\mathbb{Z}^{\infty}$  summand in the algebraic concordance group.

2. Invariants from  $W(\mathbb{F}_p)$ : We can consider the Witt group of finite fields and define some invariants here. For each prime p, we can define the homomorphism  $\phi_p: W(\mathbb{Q}) \to W(\mathbb{F}_p)$ . We will briefly explain how this map works, but we will not present any proofs for the arguments used in the explanation. First of all, any form over a field is Witt equivalent to a diagonal form, in matrix notation say  $\oplus_i(\alpha_i)$  where  $\alpha_i$ 's are elements from the field. The function  $\phi_p$  is defined on the generators  $(\alpha_i)$  as follows; write  $\alpha = ap^k$  where a and p are relatively prime, then  $\phi_p((\alpha)) = (a) \in W(\mathbb{F}_p)$ if k is odd and it is equal to  $(0) \in W(\mathbb{F}_p)$  if k is even. This actually gives the welldefined homomorphism. Another fact that we quote is that for prime  $p, W(\mathbb{F}_p)$  is isomorphic to  $\mathbb{Z}_2$  if  $p \equiv 1 \mod 4$ , and isomorphic to  $\mathbb{Z}_4$  if  $p \equiv 3 \mod 4$ . Using this construction, it is possible to show that there are elements in algebraic concordance group which have order 2 or 4.

Here we have exhibited two constructions which can be extracted from the Witt groups. Clearly, these are useful to understand the order of some Seifert forms in the algebraic concordance group. We will finish this chapter by quoting another theorem of Levine which identifies the order of particular forms. We will use this theorem to show the order of a particular Seifert matrix. Here is the statement; **Theorem 2.10.** Let A be a Seifert form of a knot and  $\Delta_A(t)$  denote its Alexander polynomial which is an irreducible quadratic polynomial. Then A is of finite order in the algebraic concordance group if and only if  $\Delta_A(1)\Delta_A(-1) < 0$ . In this case A is of order 4 if  $|\Delta_A(-1)| = p^{\alpha}q$  for some prime p, p and q are relatively prime, where  $\alpha$ is odd and  $p \equiv 3 \pmod{4}$ , otherwise it is of order 2.

Going back to the family of knots K(a, b, c), this time we set the values a = 1, b = -5 and c = 1. Implementing these, we get the following Seifert form;



Figure 2.3: The knot K(1, -5, 1) and the generators

For this example, if we compute its Alexander polynomial;

$$\Delta_A(t) = det(A - tA^T) = det \begin{pmatrix} 1 - t & 1 \\ -t & -5 + 5t \end{pmatrix} = -5t^2 + 11t - 5$$

which is an irreducible quadratic polynomial over  $\mathbb{Q}$ . By computing  $\Delta_A(1)\Delta_A(-1) = -21$  we can apply the previous theorem. Since  $|\Delta_A(-1)| = 21 = 3 \cdot 7$ , this form has order 4 in the algebraic concordance group.

So far, we have seen that with the help of isometric structures and certain Witt groups of rings and finite fields, it is possible to detect the order of an element in the algebraic concordance group. Clearly, these invariants are not sufficient to prove the isomorphism  $G^{\mathbb{Z}} \cong \mathbb{Z}^{\infty} \oplus \mathbb{Z}_{2}^{\infty} \oplus \mathbb{Z}_{4}^{\infty}$ , one needs to go further and define some polynomial invariants coming from the Witt groups again. Since it is beyond the scope of this work, we wanted to show that the structure of the algebraic concordance group is understood by means of this algebraic and number theoretic invariants. In the next chapter, we will discuss the homomorphism  $\phi_1 \circ \phi_2 : C \to G^{\mathbb{Z}}$  and explain that it has nontrivial kernel.

### CHAPTER 3

Donaldson's Theorem Applied to Sliceness

## 3.1 Sliceness Obstruction from Donaldson

In this chapter, the main objective is to distinguish between the algebraically slice and the smoothly slice category. To this end, we will follow a technique using the sliceness obstruction arising from Donaldson's diagonalization theorem [6] about the definite intersection forms. The main examples are the twist knots, previously denoted as  $W_k^{\pm}$  for k = 0, 1, 2, ... To simplify the notation, in this chapter we will omit the plus/minus sign in  $W_k^{\pm}$  and  $W_k$  will denote the twist knot with a positive clasp. We still should note that this sign is related to the introduction of the clasp. In Figure 3.1, both the positive and negative clasps are presented. It is known that for this set of knots,  $W_k$  is algebraically slice if and only if  $4k + 1 = l^2$  [3]. We will show that when k = 6,  $W_6$  is algebraically slice, but not smoothly slice. First of all, we would like to show that  $W_k$  is algebraically slice if and only if  $4k + 1 = l^2$ . We will utilize this fact in our construction.

**Theorem 3.1.** The twist knot  $W_k$  for k = 0, 1, 2, ... is algebraically slice if and only if  $4k + 1 = l^2$ .

*Proof.* We can directly compute the Seifert matrix of  $W_k$  by constructing one of its Seifert surfaces and computing linking numbers for the generators of the first homology of the Seifert surface. See Figure 3.2 where a Seifert surface for  $W_k$  is presented.



Figure 3.1: A twist knot  $W_k$  with a positive clasp

There are two generators of the first homology of this Seifert surface and they are represented by curves a and b in Figure 3.2. Recall the definition of linking number and how we used it in the Seifert form. We consider the push-off of one of the curves in the normal direction of the Seifert surface and compute the linking number.

In Figure 3.2, below the Seifert surface of  $W_k$ , we can see how the curves look like after we push-off one of them. We compute all the entries of this  $2 \times 2$  matrix and obtain the following matrix:

$$S_{W_k} = \left(\begin{array}{cc} -1 & 1\\ 0 & k \end{array}\right)$$



Figure 3.2: Seifert surface, generators and linking numbers

Recall that a knot  $K \subset S^3$  is algebraically slice if for a Seifert matrix of K, say S, there exists a nonsingular matrix P such that,

$$PSP^T = \left(\begin{array}{cc} 0 & . \\ . & . \end{array}\right)$$

where both P and S are  $2n\times 2n$  matrices and 0 denotes an  $n\times n$  block of 0's. So let

$$P = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$$

where  $ad - bc \neq 0$ , and  $a, b, c, d \in \mathbb{Z}$ .

Let us do the obvious computation;

$$P(S_{W_k})P^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & k \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

After computations we obtain;

$$P(S_{W_k})P^T = \begin{pmatrix} -a^2 + ab + kb^2 & . \\ . & . \end{pmatrix} = \begin{pmatrix} 0 & . \\ . & . \end{pmatrix}$$

So if we assume that  $W_k$  is algebraically slice, then this expression  $a^2 - ab - kb^2 = 0$ . Consider this quadratic as a polynomial in a or b, and compute the discriminant. In either case we obtain  $\Delta = b^2(4k+1)$  or  $\Delta = a^2(4k+1)$ . We know that in order to have rational solutions for the quadratic, the discriminant should be a square. Therefore,  $4k + 1 = l^2$  for some l.

The other direction is also similar. If we assume that  $4k + 1 = l^2$ , then we see that this quadratic has rational solutions. This implies the existence of a matrix P which will turn the Seifert matrix of K into the form we desire.

As a result, we have the necessary condition for the algebraic sliceness of  $W_k$ . For the rest of the chapter, we will be considering  $W_k$  for a particular value unless specifically pointed out in the text. We will use the value k = 6, namely the knot  $W_6$ of this family where  $4k + 1 = 4 \times 6 + 1 = 25 = l^2$ , hence l = 5. We know that  $W_6$ is algebraically slice, now using Donaldson's diagonalization theorem, we will prove that  $W_6$  is not smoothly slice. We will follow the description and methods in [13]. Before proceeding with the theory, we would like to talk about branched covers which will be used throughout the chapter. Here is the formal definition of a branched cover and some examples. For our purposes, we will only consider double branched covers later.

**Definition 3.2.** Let M and N be two n-dimensional manifolds, and A and B be n-2dimensional submanifolds respectively. Then a continuous function  $f : M \to N$  is called a branched covering branched over B if f(A) = B, f(M - A) = N - B and for any open  $U \in N - B$  f is a local homeomorphism.

This definition tells us that f behaves like a regular cover away from the branch set in N. So  $f: M - A \to N - B$  is a covering space. Each branch point  $a \in A$  has a branch index k, which means that around a, the function f is k-to-one. For this k-fold branch covers, our prototype will be the functions of the form  $f: \mathbb{C} \to \mathbb{C}$  and  $z \mapsto z^k$  where z is complex and  $k \geq 1$ .

**Example 3.2.** Let  $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$  and consider the map from this  $D^2$  to itself which is defined as  $p : z \to z^k$ . Then p is a k-fold branched covering with a unique branch point z = 0.

This is an example of a branched covering of  $D^2$  with only one branch point. One can construct another branched covering with more than one branch point. For more examples the reader can refer to [22] or [21]. But before going any further a more geometric example can provide a better understanding of the branched covers. **Example 3.3.** Let  $\Sigma_g$  be a closed, oriented genus g surface, then we can construct a 2-fold branched covering as follows:

Consider  $p: \Sigma_g \to S^2$  defined by means of a symmetry through an axis passing through  $\Sigma_g$ . From the picture, it is clear that after rotating the genus g surface around this axis, the quotient space will be homeomorphic to  $S^2$ . We obtain a 2-fold branched covering  $p: \Sigma_g \to S^2$  with 2g + 2 branch points.



Figure 3.3: The case for genus 1 with 4 branch points

For our purposes, we will focus on branched covers of  $S^3$ . Since the branch set is always codimension 2, our branch sets will be knots or links. So let  $\phi: M \to X(K)$ be a finite covering of a knot complement,  $X(K) = S^3 - K$ . For each boundary component of M (each of which is homeomorphic to  $T^2$ ) attach a solid torus  $V = S^1 \times D^2$ , by identifying each meridian of  $S^1 \times D^2$  with a preimage of a meridian of K. As a result, each meridian of V which is a curve of the form  $\{pt.\} \times \partial D^2$  is mapped to a meridian of K via a covering map. Sending each such disk  $\{pt.\} \times D^2$  of V to a normal disk of K, we can extend this map to the whole V via our prototype map  $z \mapsto z^k$  defined previously.

The most important example arises from the case of the N-fold cyclic cover of M of X(K). The boundary of M is a single torus and similarly, same map gives a cyclic N-fold cover of the meridian. Hence, attaching a single solid torus to M, we

built a new 3-manifold B. The core of the solid torus (the branched set of the map  $p: B \to S^3$ ) is mapped homeomorphically onto K. All this should give us a clearer picture of a branched cover. Before stating the Donaldson's theorem, we should define the intersection form of a 4-manifold.

**Definition 3.3.** Let X be a compact, oriented, topological 4-manifold. The symmetric bilinear form

$$Q_X: H^2(X, \partial X; \mathbb{Z}) \times H^2(X, \partial X; \mathbb{Z}) \to \mathbb{Z}$$

defined as  $Q_X(a, b) = \langle a \cup b, [X, \partial X] \rangle = a \cdot b \in \mathbb{Z}$  is called the intersection form of X. Here  $[X, \partial X]$  denotes the fundamental class in  $H_4(X, \partial X; \mathbb{Z})$  and  $\cup$  denotes the cup product. Since by Poincaré duality we have  $H_2(X; \mathbb{Z}) \cong H^2(X, \partial X; \mathbb{Z})$ ,  $Q_X$  is defined on  $H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z})$  as well.

We know that  $Q_X(a, b) = 0$  if a or b is a torsion element, which implies that  $Q_X$  can be considered on homology divided by the torsion part. This means that if we choose a basis for  $H_2(X;\mathbb{Z})/Torsion$ , we can represent  $Q_X$  by a matrix. For the applications in this chapter, we will usually use the matrix representation of the intersection form. Now we are ready to state Donaldson's theorem.

**Theorem 3.4.** If X is a smooth, closed 4-manifold with the intersection form  $Q_X$  definite, then  $Q_X$  is diagonalizable over  $\mathbb{Z}$ .

Before using this theorem to understand something about the sliceness of a knot, we will need the following lemma which will not be proved.

**Lemma 3.5.** Let  $K \subset S^3$  be a smoothly slice knot, and D be its smooth disk. Then the double branched cover of  $D^4$  branched over D denoted with  $\Sigma_2(D)$  is a rational homology ball, i.e.  $H_*(\Sigma_2(D); \mathbb{Q}) = H_*(D^4; \mathbb{Q}).$ 

Now we will proceed with some computations using the theory explained so far.

# 3.3.1 Some computations

A link in  $S^3$  is called a 2-bridge link if it has a projection with two local maxima. We will use two notations for these knots, K(p,q) or equivalently  $L(a_1, a_2, \ldots, a_n)$  where  $a_i \in \mathbb{Z}$  for  $i = 1, 2, 3, \ldots, n$ . In Figure 3.4, you can see the diagrams of 2-bridge knots depending on n being odd or even. The numbers  $a_i$ 's denote the number of crossings.



Figure 3.4: 2-bridge knots in general

The relation between these two different notations, namely K(p,q) and  $L(a_1, a_2, \ldots, a_n)$ , comes from the fact that if p and q are coprime integers such that 0 < q < p then we can write p/q as a continued fraction;

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_n}}}$$

where  $a_i < 0$  for i = 1, 2, 3, ..., n.

It is also known that if p is even, then K(p,q) is a two-component link. If p is odd, then it is a knot. Another fact we will use is that for the 2-bridge knots, K(p, p - q)is isotopic to the mirror image of K(p,q).

Now let us draw  $W_6$  and try to understand which 2-bridge knot it is isotopic to.



Figure 3.5: Isotoping  $W_6$  to K(25,23)

With a small manipulation, one can see that  $W_6$  looks pretty similar to K(25, 2), but it is not exactly the same. We need to take its mirror. This means

 $W_6$  is isotopic to K(25, 23).

Now we will use that fact that 2-fold branched cover of  $S^3$  branched over a 2-bridge knot K(p,q) is the lens space L(p,q). Here is how we can see this fact. First of all, let K(p,q) be a 2-bridge knot in  $S^3$ . We can cut the knot into two pieces by an  $S^2$  which is represented by a plane in Figure 3.6, leaving two unknotted arcs in the upper part and all the twists  $a_1, a_2, \ldots, a_n$  remaining in the lower part. Recall from Example 3.3 that the double branched cover of  $S^2$  branched over four points is a torus. It can be seen that if we utilize the same spinning argument in Example 3.3 for a solid torus, we obtain a  $D^3$  branched over two arcs. Therefore, the double branched cover of  $D^3$ branched over two arcs is a solid torus. When we cut the knot K(p,q) into these pieces, we also cut  $S^3$  into two pieces and both pieces are  $D^3$ 's. Call the upper one  $D^3_+$  and the lower one  $D^3_-$ .



Figure 3.6: K(p,q) divided into two pieces



Figure 3.7: Untwisting the lower part of K(p,q)

Now as shown in Figure 3.7, we can untwist the lower part of the knot K(p,q)until we obtain two unknotted arcs. In order to do this, we can start with  $a_n$  and introduce  $a_n$ -many twists in the opposite direction. This process will untwist  $a_n$ . Then, we are left with  $a_{n-1}, \ldots, a_1$ . We can repeat the same procedure for  $a_{n-1}$  and it will be untwisted as well. Continuing in this way, eventually we obtain two unknotted arcs in the lower part. Hence, we have two  $D^3$ 's and both of them are branched over two unknotted arcs. So the double branched covers are solid tori for each  $D^3$ . Therefore, when we glue these two  $D^3$ 's back, we also glue the corresponding double branched covers which are solid tori. While gluing the solid tori, the information about how to glue them comes from the coded information we obtained during the untwisting process and it is coded in the double branched cover of  $S^2$  branched over these four points, i.e. from the boundary of the double branched cover of  $D_{-}^{3}$  and using the information coming from the continued fraction of p/q where the  $a_i$ 's from the untwisting procedure appear. Hence, we obtain the lens space L(p,q) as the double branched cover of  $S^{3}$  branched over the knot K(p,q). For further information related to the branched covers, the reader can refer to [22].

Following [13], if a lens space L(p,q) smoothly bounds a rational homology ball W(p,q), it is possible to form a smooth negative definite 4-manifold X(p,q) by taking the union of -W(p,q) with a canonical 4-dimensional plumbing P(p,q) bounded by L(p,q). Since X(p,q) is negative definite, we can use Donaldson's theorem [6] to conclude that the intersection lattice  $Q_{X(p,q)}$  is isomorphic to the standard diagonal intersection lattice  $D^n$ , where  $n = b_2(X(p,q))$ . This means that there is an embedding  $Q_{P(p,q)} \hookrightarrow D^n$ , and since -L(p,q) = L(p,p-q) smoothly bounds the rational homology ball -W(p,q), there exists an embedding  $Q_{P(p,p-q)} \hookrightarrow D^{n'}$  where n' is the rank of  $Q_{P(p,p-q)}$ .

Now let us do the necessary computations for the intersection form. We had K(25, 23), so the double branched cover is L(25, 23). The lens space L(p, q) is obtained performing  $-\frac{p}{q}$  surgery on the unknot in  $S^3$  in general, and in our case we have  $-\frac{25}{23}$  surgery. By writing  $-\frac{25}{23}$  as a continued fraction of integers, we can think of this 3-manifold as the boundary of a 4-dimensional 2-handlebody. We represent it with the standard Kirby diagram used for 2-handles, using an operation called slam-dunk. Here is the picture for L(p,q) and a description for slam-dunk operation.



Figure 3.8: Slam-dunk operation



Figure 3.9: Lens space L(p,q)

Here is another example of a slam-dunk operation and the resulting 3-manifold;



Figure 3.10: Using slam-dunk operation to get  $\mathbb{RP}^3$ 

For more detailed information, the reader can refer to [9] and also [19]. Now by computing the continued fraction, we will obtain the plumbing graph and then work on the intersection form. For L(25, 23):



At the end, we obtain 11 - 2's and one -3. The plumbing graph will look like;



Figure 3.11: Plumbing graph corresponding L(25,23)

We would like to get a contradiction. Assume that  $W_6$  is smoothly slice, then there exists a rational homology ball W(25, 23) such that the intersection lattice of this rational homology ball, call it  $Q_{P(25,23)}$ , can be embedded into the standard diagonal lattice  $D^{12}$ . We choose generators for  $H_2(D^{12}; \mathbb{Z})$ . Assume they are  $e_1, e_2, ..., e_{12}$  such that for every i, j we have;

$$e_i e_j = -\delta_{i_j}$$

This embedding gives rise to a subset  $S = \{u_1, u_2, ..., u_{12}\} \subset D^{12}$  such that it satisfies the following conditions;

•  $u_i \cdot u_j = -2$  if i = j = 1, 2, ..., 11 and  $u_i \cdot u_j = -3$  if i = j = 12.

- $u_i \cdot u_j = 1$  if |i j| = 1 for  $i, j \in \{1, 2, ..., 12\}$ .
- $u_i \cdot u_j = 0$  if |i j| > 1 for  $i, j \in \{1, 2, ..., 12\}$ .

Let  $\phi : Q_{P(25,23)} \hookrightarrow D^{12}$  be the embedding map such that for each  $i = 1, \ldots 12$ ,  $u_i \mapsto \sum_{j=1}^{12} x_j \cdot e_j$  where each  $x_j \in \mathbb{Z}$ . We can see that this implies  $u_i^2$  will be mapped to  $(\sum_{j=1}^{12} x_j \cdot e_j)^2$  for each  $j = 1, \ldots, 12$ . Now let us start with seeing where  $u_1$  can be mapped. The map  $\phi$  tells us that  $u_1$  will mapped to a linear combination of  $x_j \cdot e_j$ 's, and we also know that  $u_1 \cdot u_1 = u_1^2 = -2$ . Hence  $u_1^2$  will be mapped to  $(\sum_{j=1}^{12} x_j \cdot e_j)^2$ . This gives the relation;

$$\left(\sum_{j=1}^{12} x_j \cdot e_j\right)^2 = x_1^2 \cdot e_1^2 + \dots + x_{12}^2 \cdot e_{12}^2 = -x_1^2 - \dots - x_{12}^2 = -2$$

So we need to have  $x_1^2 + \cdots + x_{12}^2 = 2$ . Recall that all  $x_j$ 's are integers. Therefore, this is only possible if  $x_k = \pm 1$  and  $x_l = \pm 1$  for some  $k, l \in \{1, 2, \ldots, 12\}$  and all other coefficients are 0. So  $u_1 = \pm e_k \pm e_l$  for some  $k, l \in \{1, 2, \ldots, 12\}$ . We can choose k = 1 and l = 2. Also we can choose the signs such that they satisfy the relations described above, so let  $u_1 = e_1 - e_2$ .

Similar reasoning tells us that  $u_2 = \pm e_k \pm e_l$  for some  $k, l \in \{1, 2, ..., 12\}$ . We see that  $u_2$  is connected to  $u_1$  by an edge and it is also connected to  $u_3$  by another edge. This implies that  $u_2 \cdot u_1 = 1$  and  $u_2 \cdot u_3 = 1$ . Moreover,  $u_2 \cdot u_i = 0$  for i = 4, ..., 12. Again these conditions force one of k or l to be equal to 2 and the other one can chosen to be 3. Following the previous convention, let  $u_2 = e_2 - e_3$ .

If we continue in this way, we will see that up to renumbering, this is the only choice:

$$u_1 = e_1 - e_2, u_2 = e_2 - e_3, \dots, u_{11} = e_{11} - e_{12}$$

Now we also have  $u_{12} = \sum_{i=1}^{12} x_i e_i$  where  $x_i \in \mathbb{Z}$ . Now we will try to solve the

equations for  $u_{12}$ .

First of all, for any i = 1, 2, ..., 10 we have  $u_{12} \cdot u_i = 0$ . Moreover,  $u_{11} \cdot u_{12} = 1$ . Finally if we consider  $-3 = u_{12}^2 = u_{12} \cdot u_{12} = (\sum_{i=1}^{12} x_i e_i)^2 = x_1^2 e_1^2 + \dots + x_{12}^2 e_{12}^2 = -x_1^2 - \dots - x_{12}^2 = -\sum_{i=1}^{12} x_i^2$ , we obtain  $\sum_{i=1}^{12} x_i^2 = 3$ . Recall that by the definition of the generators we know that  $e_i \cdot e_j = -1$  if i = j and  $e_i \cdot e_j = 0$  if  $i \neq j$ . If we check the equations more closely, we get equalities about  $x_i$ 's:

From the first set of equations,

 $u_{12} \cdot u_1 = u_{12} \cdot (e_1 - e_2) = -x_1 + x_2 = 0$ . Hence  $x_1 = x_2$ . If we proceed in a similar fashion;

 $u_{12} \cdot u_2 = u_{12} \cdot (e_2 - e_3) = -x_2 + x_3 = 0$ , which again implies  $x_2 = x_3$ . Then  $x_3 = x_4$ ,  $x_4 = x_5$  and so on.

What we obtain at the end is  $x_1 = x_2 = x_3 = \cdots = x_{11}$ , so all the coefficients except for the last one must be equal.

Also we had  $u_{12} \cdot u_{11} = 1$ . This gives us  $u_{11} \cdot (e_{11} - e_{12}) = -x_{11} + x_{12} = 1$ .

Now just write  $x_{12} - 1 = x_{11} = x_{10} = \dots = x_1$  in  $\sum_{i=1}^{12} x_i^2 = 3$ . The equation becomes;

$$(x_{12}-1)^2 + (x_{12}-1)^2 + \dots + (x_{12}-1)^2 + x_{12}^2 = 3$$

We are looking for integer solutions of this equation. If  $|x_{12} - 1| \ge 1$ , then it is obvious that there are no solutions satisfying the equation. In the other case where  $|x_{12} - 1| = 1$ , the equation will be  $11 + x_{12}^2 = 3$  which has no integer solutions as well. So this leaves the case  $x_{12} - 1 = 0$ . This implies that  $x_{12} = 1$ , and if we implement this value in the equation above, we have  $x_{12}^2 = 1$  whereas we should have obtained  $x_{12}^2 = 3$ . Therefore, we can conclude that there are no integer solutions for this equation. This gives us a contradiction.

The contradiction we obtained immediately implies that the intersection lattice  $Q_{P(25,23)}$  cannot be embedded into the standard diagonal intersection lattice  $D^{12}$ . Therefore, Donaldson's theorem tells us that we cannot form this smooth 4-manifold X(25,23) and L(25,23) does not bound a smooth rational homology ball. Hence, using the Lemma 3.5 we can conclude that the knot K(25,23) is not smoothly slice. We have already discussed that K(25,23) is isotopic to the twist knot  $W_6$ . This means that  $W_6$  is algebraically slice, but not smoothly slice.

This gives us what we have been looking for. The first result about this was obtained in [3]. They defined an invariant called Casson-Gordon Invariant which gives a slightly different result. Using the cyclic *n*-fold branched covers and linking forms they showed that this invariant should remain bounded as *n* goes to infinity for a particular family of knots. In fact, they used the same family of knots, namely the twist knots. They proved that  $W_k$  is not slice if and only if  $k \neq 0, 2$  which are the unknot and Stevedore's knot.

So far, using Donaldson's theorem, we have been able to show that there are knots which are algebraically slice, but not smoothly slice. In order to do so, we have borrowed many concepts from other branches of topology and combined them together. In the following chapter, we will move on to analyze the first homomorphism between the topological and the smooth concordance group.

### CHAPTER 4

Grid Homology, Topological Concordance vs. Smooth Concordance

## 4.1 Grids

We will use grid representation of knots and utilize the homology arising from this structure. My work will closely follow [20]. The invariant  $\tau$  which is extracted from the grid homology bounds the smooth slice genus from below. Constructing a knot K with trivial Alexander polynomial, but  $\tau \neq 0$  will be what we need for a knot which is topologically slice, but not smoothly. Therefore, the main aim is to exhibit this construction in order to prove that the first homomorphism we defined between smooth concordance and topological concordance group,  $\phi_1$ , has nontrivial kernel. We will start with basic definitions and preliminaries of the subject.

Recall the definition of the signature of a knot K denoted as  $\sigma(K)$ . The signature is quite useful in many cases and it can be computed using some tools from linear algebra. The problem is that, for our purposes it will not be sufficient. We have the following lower bound;

**Theorem 4.1** ([18]). Let  $K \subset S^3$  then  $|\sigma(K)| \le 2g_s(K)$ .

It implies that smoothly slice knots have signature equal to zero, but we have the same bound for topological sliceness as well; i.e.  $|\sigma(K)| \leq 2g_{top}(K)$ . Therefore, the signature itself would not be sufficient to distinguish topological and smooth category all the time. Consider the knot  $W_6$  from the Chapter 3. It has a Seifert matrix,

$$S_{W_6} = \left(\begin{array}{cc} -1 & 1\\ 0 & 6 \end{array}\right)$$

If we symmetrize it, we obtain

$$S_{W_6} + S_{W_6}^T = \left(\begin{array}{cc} -2 & 1\\ 1 & 12 \end{array}\right)$$

It is not difficult to see that  $\sigma(W_6) = 0$ , since one of the eigenvalues is positive and the other one is negative. As a result, it does not tell us anything about sliceness. For the rest of the chapter, we will see that this invariant  $\tau$  provides a better bound for slice genus.

Grid diagrams serve as another way to represent a knot K in  $S^3$ . Historically speaking, this method was first intoduced by Brunn in the late 19th century. Later, due to Cromwell's theorem which will be mentioned in the chapter, the connection between grid representations and knot types became more clear.

**Definition 4.2.** A planar grid diagram  $\mathcal{G}$  is an  $n \times n$  square on a plane with n Xand n O markings satisfying the following rules;

- Each row has a single small square marked with an X and a single square with an O marking,
- Each column has a single small square marked with an X and a single square with an O marking,
- No small square has both X and O markings at the same time.

The number n is called the grid index of  $\mathcal{G}$ . We will denote the set of O markings with  $\mathbb{O}$  and X markings with  $\mathbb{X}$ .

To present a knot or a link in  $S^3$  by a grid diagram, we do the following; given the grid  $\mathcal{G}$  with the markings, draw oriented line segments connecting the X markings to the O markings in each column. Then do the same for each row. The orientation is vertically  $X \to O$  and horizontally  $O \to X$ . The main rule is that the vertical line segments always go over the horizontal ones. Here is an example of a grid diagram;



Figure 4.1: A grid diagram for two unknots in  $S^3$ 

Lemma 4.3. Every oriented link or knot in  $S^3$  can be represented by a grid diagram. Proof. Given a knot K, to obtain a grid diagram, first we can draw our knot by piecewise linear, horizontal and vertical line segments. After that, we need to modify it to turn it into a grid diagram. There are two problems that might occur. The first one is, we can have two vertical or horizontal linear line segments corresponding to the same column or row, but we can perturb one of the line segments a little bit, so one of the line segments will correspond to another column or row. The second problem is that the rule "vertical goes over the horizontal" might be violated. Then we can modify this crossing as shown in the figure below;



Figure 4.2: Fixing the violation of the rule "vertical goes over horizontal"

Therefore, given any knot  $K \subset S^3$ , we can obtain a grid diagram representing K.

In the figures below, we exhibit the procedure of obtaining a grid diagram for Stevedore's knot which is isotopic to the twist knot  $W_2$ .



Figure 4.3: Drawing  $W_2$  with piecewise linear line segments



Figure 4.4: Constructing a grid diagram for  $W_2$ 

Clearly, it is possible to construct different grid diagrams for the same knot. Consider the simplest example, namely the unknot;



Figure 4.5: Two different grids representing the unknot

Hence, this brings the question that can we define a kind of equivalence relation for the same knot with different grid diagrams or how can we relate these diagrams, knowing that they actually represent the same knot? There are two different types of grid moves which can be best described by pictures;

### i) <u>Commutation</u>:





Figure 4.6: Nested interval

Figure 4.7: Separate intervals



Figure 4.8: Intervals touching at one

point

Commutation operation allows us to change two consecutive columns, if it satisfies certain conditions we describe here; first of all, we can see that there are intervals in each column between the X and the O marking, and we allow commutations, if one of the intervals is nested in the other (Figure 4.6), if two intervals do not intersect at all (Figure 4.7) and if one of the intervals begins where the other one ends (Figure 4.8). In the figures above, these three cases can be seen. If two intervals cross each other, the commutation is not allowed. It should be noted that same commutation moves are allowed for rows. Without distinction, we will call both row commutation and column commutation just commutation.

ii) <u>Stabilization</u>: This is a grid move which changes the size of the grid. We can define a stabilization at an X-marking or at an O-marking. For each type of marking, there are four different type of moves. In total, we have eight different moves. Also to decrease the grid size, we can destabilize the grid by performing the inverse move. Considering the notation; as an example  $X_{NW}$  denotes the northwest stabilization at an X-marking. A similar notation will be used for other types of stabilizations depending on the marking and the direction. Here we can see the 4 different stabilizations at an X-marking. Note that it is essentially the same for an O-marking.



Figure 4.9: Stabilization at an X-marking

**Theorem 4.4.** (Cromwell) Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two grid diagrams representing isotopic links, then there is a finite sequence of commutations, stabilizations and destabilizations transforming  $\mathcal{G}_1$  into  $\mathcal{G}_2$ .

This theorem tells us that a grid diagram of a knot or a link is unique up to grid moves. The following lemma describes the relation between the two types of grid moves.

**Lemma 4.5.** Any stabilization move can be obtained from a finite sequence of one  $\mathbb{X}_{SW}$  stabilization move and some commutation moves.

*Proof.* Except for  $X_{SW}$  stabilization itself which requires no commutation moves, we will explain how each stabilization can be obtained.

i)  $X_{NW}$ : Looking at the figure below, first we perform an  $X_{SW}$  stabilization. Then commuting the two rows, we obtain  $X_{NW}$ .



Figure 4.10:  $X_{SW}$  and commutations giving  $X_{NW}$ 

ii)  $\mathbb{X}_{NE}$ : After performing an  $\mathbb{X}_{SW}$  stabilization, we commute two rows again

as shown in the figure below, then commute the two consecutive columns.



Figure 4.11:  $X_{SW}$  and commutations giving  $X_{NE}$ 

**iii**)  $\mathbb{X}_{SE}$ : Starting with an  $\mathbb{X}_{SW}$  stabilization, we commute the two consecutive columns.



Figure 4.12:  $X_{SW}$  and commutations giving  $X_{SE}$ 

iv)  $\mathbb{O}_{NW}$ : As shown in the figure, stabilize the X-marking in the column
labeled  $C_1$ . Then, commute the newly created column next to  $C_1$  until it is adjacent to the column  $C_2$ . One more commutation will result in an  $\mathbb{O}_{NW}$  stabilization.



Figure 4.13:  $X_{SW}$  and commutations giving  $\mathbb{O}_{NW}$ 

**v**)  $\mathbb{O}_{NE}$ : We begin with the same stabilization  $\mathbb{X}_{SW}$ , the we keep commuting the newly created column next to  $C_1$  until it is adjacent to  $C_2$ . This gives an  $\mathbb{O}_{NE}$  stabilization.



Figure 4.14:  $X_{SW}$  and commutations giving  $\mathbb{O}_{NE}$ 

**vi**)  $\mathbb{O}_{SW}$ : After  $\mathbb{X}_{SW}$  stabilization, the commutations applied are shown in the figure below.



Figure 4.15:  $X_{SW}$  and commutations giving  $\mathbb{O}_{SW}$ 

**vii**)  $\mathbb{O}_{SE}$ : After the stabilization  $\mathbb{X}_{SW}$ , first we commute two rows  $R_1$  and  $R_2$ , then again column commutations shown in the figure give the result.



Figure 4.16:  $\mathbb{X}_{SW}$  and commutations giving  $\mathbb{O}_{SE}$ 

Therefore, we have seen that all types of stabilization moves can be obtained by starting with an  $X_{SW}$  stabilization and then performing some commutations.  $\Box$ 

We have defined the planar grid diagram, in addition, it is also possible to think of grid diagrams on a torus. We need to identify the edges of the rectangle, same as constructing the torus from a rectangle by quotienting it. The grid will lie on the surface of the torus. This toroidal identification of a grid diagram will be useful in certain cases.



Figure 4.17: Unknot on the toroidal grid

**Definition 4.6.** A grid state on a toroidal grid diagram  $\mathcal{G}$  is a 1-1 correspondence between the horizontal and vertical circles. In other words, a grid state is an n-tuple  $\mathbf{x} = (x_1, x_1, \dots, x_n)$  in the torus, such that each horizontal and vertical circle contains exactly one element of  $\mathbf{x}$ . The set of grid states on  $\mathcal{G}$  is denoted by  $S(\mathcal{G})$ .



Figure 4.18: An example of a grid state

Notice that on the 5th column and 4th row we have two dots from the grid state. This is because they are the same on the torus after the identification of the edges. From now on, we will omit the dots on the top and the right edge.

We will define two different versions of the grid chain complex with their boundary maps. In order to describe the boundary map for both versions, we need to define the rectangles on a grid diagram.

Let  $\mathbf{x}, \mathbf{y} \in S(\mathcal{G})$  be two grid states such that they differ by exactly 2 points on the diagram. This will give us a rectangle with two points of  $\mathbf{x}$  and two points of  $\mathbf{y}$  as vertices. The rectangle  $\mathbf{r}$  also inherits an orientation from the torus. Convention is that the horizontal edges of  $\mathbf{r}$  are oriented from  $\mathbf{x}$  to  $\mathbf{y}$  and the vertical ones from  $\mathbf{y}$  to  $\mathbf{x}$ . In this case, we say that  $\mathbf{r}$  is a rectangle from  $\mathbf{x}$  to  $\mathbf{y}$ . Here is an example of a rectangle;



Figure 4.19: An example of a rectangle

If we look at Figure 4.19, two grid states are denoted with black dots and little empty rectangles, call them  $\mathbf{x}$  and  $\mathbf{y}$  respectively. In two separate pictures, we can see the rectangles from  $\mathbf{x}$  to  $\mathbf{y}$  and the rectangles from  $\mathbf{y}$  to  $\mathbf{x}$ . We will denote the set of rectangles from  $\mathbf{x}$  to  $\mathbf{y}$  as  $Rect(\mathbf{x}, \mathbf{y})$ . We will define the boundary map in a way that it counts certain rectangles and gives a map from  $GC^{-}(\mathcal{G})$  to itself. Denote also  $Rect^{0}(\mathbf{x}, \mathbf{y}) = {\mathbf{r} \in Rect(\mathbf{x}, \mathbf{y}) : Int(\mathbf{r}) \cap \mathbf{x} = Int(\mathbf{r}) \cap \mathbf{y} = \emptyset}$ . These will be called empty rectangles.

**Definition 4.7** (Fully Blocked Grid Chain Complex). Assume that we are given a grid diagram  $\mathcal{G}$ . Define the fully blocked grid chain complex  $\widetilde{GC}(\mathcal{G})$  to be the complex whose underlying vector space structure is generated by all the grid states,  $S(\mathcal{G})$ .

The boundary map  $\widetilde{\partial}: \widetilde{GC}(\mathcal{G}) \to \widetilde{GC}(\mathcal{G})$  is defined as;

$$\widetilde{\partial}(\mathbf{x}) = \sum_{\mathbf{y} \in S(\mathcal{G})} \#\{\mathbf{r} \in Rect^0(\mathbf{x}, \mathbf{y}) \mid \mathbf{r} \cap \mathbb{X} = \mathbf{r} \cap \mathbb{O} = \emptyset\} \cdot \mathbf{y}$$

We can see that the boundary map counts the empty rectangles from  $\mathbf{x}$  to  $\mathbf{y}$  which

do not contain any X and O markings and # denotes counting modulo 2.

**Definition 4.8** (Unblocked Grid Chain Complex). Assume that we are given a grid diagram  $\mathcal{G}$ . Enumerate the O-markings in  $\mathcal{G}$  as  $\{O_1, O_2, \ldots, O_n\}$  and consider the polynomial ring  $\mathcal{R} = \mathbb{F}[U_1, U_2, \ldots, U_n]$ , where each  $U_i$  corresponds to the enumerated O-marking and  $\mathbb{F} = \mathbb{Z}_2$ . Define the unblocked grid chain complex  $GC^-(\mathcal{G})$  to be the free module over  $\mathcal{R}$  generated by all the grid states,  $S(\mathcal{G})$ .

Now the boundary map  $\partial: GC^{-}(\mathcal{G}) \to GC^{-}(\mathcal{G})$  is defined as follows;

$$\partial(\mathbf{x}) = \sum_{\mathbf{y} \in S(\mathcal{G})} \sum_{\mathbf{r} \in Rect^0(\mathbf{x}, \mathbf{y})} U_1^{O_1(\mathbf{r})} \cdot U_2^{O_2(\mathbf{r})} \dots U_n^{O_n(\mathbf{r})} \cdot \mathbf{y}$$

where the empty rectangle  $\mathbf{r}$  does not contain any X-markings either, i.e.  $\mathbf{r} \cap \mathbf{X} = \emptyset$ , and  $O_i(\mathbf{r})$  is 1 if  $O_i \in \mathbf{r}$  and 0 otherwise. We can see that the fully blocked grid chain complex has a simpler structure, but it provides less information compared to the unblocked theory. The following lemma tells us both of these structures are indeed chain complexes.

**Lemma 4.9** ([20], Lemma 4.6.7). The boundary maps defined in the fully blocked grid chain complex  $\widetilde{\partial}$  :  $\widetilde{GC}(\mathcal{G}) \to \widetilde{GC}(\mathcal{G})$  and in the unblocked grid chain complex  $\partial : GC^{-}(\mathcal{G}) \to GC^{-}(\mathcal{G})$  satisfy  $\widetilde{\partial}^{2} = 0$  and  $\partial^{2} = 0$ .

*Proof.* We will prove the lemma only for the fully blocked theory. To prove this, we will count rectangles. Assume that we are given a rectangle  $\mathbf{r_1} \in Rect^0(\mathbf{x}, \mathbf{y})$  and another rectangle  $\mathbf{r_2} \in Rect^0(\mathbf{y}, \mathbf{z})$  where  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  are grid states in  $\mathcal{G}$ . Recall that the boundary map  $\tilde{\partial}$  counts modulo 2, therefore for a given pair of rectangles  $(\mathbf{r_1}, \mathbf{r_2})$ , if we can show that there is another pair of rectangles  $(\mathbf{r_1}', \mathbf{r_2}')$  such that  $\mathbf{r_1'} \in Rect^0(\mathbf{x}, \mathbf{y'})$  and  $\mathbf{r_2'} \in Rect^0(\mathbf{y'}, \mathbf{z})$ , we will be done. This is equivalent to showing that there are

two ways of going from  $\mathbf{x}$  to  $\mathbf{z}$  using two separate pairs of rectangles. Notice that in the figures below, the grid states  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  and  $\mathbf{y}'$  are represented by black dots, empty triangles, empty rectangles and empty dots respectively. There are three separate cases we need to consider.

1) The first case is the one where all the vertices of two rectangles  $\mathbf{r_1}$  and  $\mathbf{r_2}$  are distinct. In this case, we have four moving coordinates, which means moving four coordinates, we obtain another grid state  $\mathbf{y}'$  that provides another pair of rectangles going from  $\mathbf{x}$  to  $\mathbf{z}$ . We can see how it works in the figure below:



Figure 4.20: Two different pair of rectangles from  $\mathbf{x}$  to  $\mathbf{z}$ 

2) In the second case, we have three moving coordinates, so we will move three coordinates of  $\mathbf{y}$ , equivalently move three empty triangles in the figure below, and obtain the new grid state  $\mathbf{y}'$  providing another pair of rectangles. Similarly, we can see how to go from  $\mathbf{x}$  to  $\mathbf{z}$  using two different pair of rectangles in the figure below:



Figure 4.21: Two different pair of rectangles from  $\mathbf{x}$  to  $\mathbf{z}$ 

Notice that it does not matter how two rectangles are positioned. In other cases where the smaller rectangle at the top is under the big rectangle or on the right or left side, we can see that the same argument works.

3) In the last case, we would like to consider two moving coordinates. But this means  $\mathbf{z} = \mathbf{x}$ , since we defined rectangles on a grid state in a way that two grid states differ only at two coordinates, and moving two coordinates of  $\mathbf{y}$  twice will bring us back to the initial picture. In this case, to go from  $\mathbf{x}$  to  $\mathbf{y}$  and then go back to  $\mathbf{x}$ , we have to go through the vertical or horizontal annulus shown in the picture below. This annulus should have width or height 1, otherwise we can see that one of the rectangles  $\mathbf{r_1}$  or  $\mathbf{r_2}$  will contain a coordinate of  $\mathbf{x}$  or  $\mathbf{y}$ . We also know that this annulus (vertical or horizontal) will contain an X and an O marking. Recall that the boundary map counts empty rectangles which do not contain any type of markings. Therefore, the boundary map will be equal to 0. See Figure 4.22 for a pictorial description:



Figure 4.22: The annuli containing an X and an O marking

In the first two cases  $\tilde{\partial}^2 = 0$ , since we are counting modulo 2 and the rectangles we count come in pairs and in the third case, because one of the rectangles we count will contain a marking, the boundary map will vanish. Therefore, we can conclude that  $\tilde{\partial}^2 = 0$ . Even though we will not present the proof in the unblocked theory, the proof is quite similar.

**Definition 4.10.** The grid homology is defined to be  $GH^{-}(\mathcal{G}) = Ker\partial/Im\partial$  as an  $\mathcal{R}$ -module where  $\mathcal{R} = \mathbb{F}[U_1, U_2, \dots, U_n]$  and  $\mathbb{F} = \mathbb{Z}_2$ . Moreover, multiplication by  $U_i$  is chain homotopic to multiplication by  $U_j$  for any  $i, j \in \{1, 2, \dots, n\}$ .

Before defining the gradings, let us consider the simplest example, the unknot represented by a  $2 \times 2$  grid, and compute the grid homology.

**Example 4.2.** Here is a  $2 \times 2$  grid for the unknot denoted as  $\mathcal{G}_{unknot}$ .



Figure 4.23: A  $2 \times 2$  grid diagram for the unknot

Since the size of the grid is 2, we have 2! = 2 grid states shown on Figure 4.23. We have also enumerated the O-markings. Our chain complex  $GC^{-}(\mathcal{G}_{unknot})$  is generated over  $\mathbb{F}[U_1, U_2]$  by the grid states  $\mathbf{x}$  and  $\mathbf{y}$ . Now we need to compute the boundaries. First, consider  $\mathbf{x}$ .



Figure 4.24: Rectangles contributing to the boundary of  $\mathbf{x}$ 

If we call the rectangle in the left  $\mathbf{r_1}$  in Figure 4.24 and the one in the right  $\mathbf{r_2}$ , then both  $\mathbf{r_1}$  and  $\mathbf{r_2}$  are in  $Rect^0(\mathbf{x}, \mathbf{y})$  and  $\mathbf{r_1} \cap \mathbb{X} = \mathbf{r_2} \cap \mathbb{X} = \emptyset$ . Therefore,  $\partial(\mathbf{x}) = U_1 \mathbf{y} + U_2 \mathbf{y}$ . For the boundary of  $\mathbf{y}$ ,



Figure 4.25: Rectangles contributing to the boundary of  $\mathbf{y}$ 

the two shaded rectangles in Figure 4.25 will contribute. For both of these rectangles, we can see that intersection with X-markings is not empty. Hence  $\partial(\mathbf{y}) =$ 0. So  $Ker(\partial) = \{p(U_1, U_2) \cdot \mathbf{y}\}$  where  $p(U_1, U_2)$  denotes a polynomial in  $U_1$  and  $U_2$ with coefficients from  $\mathbb{F} = \mathbb{Z}_2$ . Moreover,  $Im(\partial) = \{(U_1 + U_2) \cdot p(U_1, U_2) \cdot \mathbf{y}\}$ . We can see that multiplication by  $U_1$  is chain homotopic to multiplication by  $U_2$ . Let  $U_1 : GC^-(\mathcal{G}_{unknot}) \to GC^-(\mathcal{G}_{unknot})$  be the map  $U_1 \mapsto U_1 \cdot \mathbf{x}$  and  $U_2 : GC^-(\mathcal{G}_{unknot}) \to$  $GC^-(\mathcal{G}_{unknot})$  be the map  $U_2 \mapsto U_2 \cdot \mathbf{x}$ . To show that they are chain homotopic, we need to find a homomorphism of the module  $H : GC^-(\mathcal{G}_{unknot}) \to GC^-(\mathcal{G}_{unknot})$  such that  $U_1 + U_2 = \partial \cdot H + H \cdot \partial$ . We have two grid states, so let  $H(\mathbf{x}) = 0$  and  $H(\mathbf{y}) = \mathbf{x}$ . Now we can check for  $\mathbf{x}$  and  $\mathbf{y}$  if the map H satisfies the equation given above;

$$\partial \cdot H(\mathbf{x}) + H(\partial(\mathbf{x})) = H(U_1 \cdot \mathbf{y} + U_2 \cdot \mathbf{y}) = U_1 \cdot H(\mathbf{y}) + U_2 \cdot H(\mathbf{y}) = U_1 \cdot \mathbf{x} + U_2 \cdot \mathbf{x} = (U_1 + U_2) \cdot \mathbf{x}.$$

So for  $\mathbf{x}$  we have the equality. Now let us check for  $\mathbf{y}$ ;

$$\partial \cdot H(\mathbf{y}) + H(\partial(\mathbf{y})) = \partial(\mathbf{x}) = U_1 \cdot \mathbf{y} + U_2 \cdot \mathbf{y} = (U_1 + U_2) \cdot \mathbf{y}$$

Therefore multiplication by  $U_1$  is chain homotopic to multiplication by  $U_2$ . This suggests that they induce the same map on the homology. So we can conclude that the

homology  $GH^{-}(\mathcal{G}_{unknot}) = Ker(\partial)/Im(\partial) = \langle p(U_1, U_2) \cdot \mathbf{y} \rangle / \langle (U_1 + U_2) \cdot p(U_1, U_2) \cdot \mathbf{y} \rangle$  $\mathbf{y} \geq \mathbb{F}[U]$  generated by the cycle  $\mathbf{y}$ .

Grid homology will be really useful, because from this combinatorial structure, there are two gradings one can obtain, namely the Alexander and Maslov gradings. We will define these two gradings and their relation to the homology. Suppose that  $P, Q \subset \mathbb{R}^2$  are finite sets in the plane. Define

$$\begin{split} I(P,Q) &= \# \ of \ N.E. \ (northeast) \ pointing \ intervals \ from \ P \ to \ Q \\ &= |\{\{p_1,p_2\},\{q_1,q_2\} \mid p_1 < q_1,p_2 < q_2\}| \end{split}$$

Let  $J(P,Q) = \frac{1}{2}(I(P,Q) + I(Q,P))$  for symmetrizing the sum.

**Definition 4.11.** The Maslov grading is defined to be;

$$M_{\mathbb{O}}(\mathbf{x}) = J(\mathbf{x}, \mathbf{x}) - 2J(\mathbf{x}, \mathbb{O}) + J(\mathbb{O}, \mathbb{O}) + 1$$

and  $M_{\mathbb{O}}(\mathbf{x}) \in \mathbb{Z}$ .

**Definition 4.12.** The Alexander grading is;

$$A(\mathbf{x}) = \frac{1}{2}(M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{X}}(\mathbf{x})) - (\frac{n-1}{2})$$

and  $A(\mathbf{x}) \in \frac{1}{2}\mathbb{Z}$ .

The following lemma gives an important property of the Alexander grading. **Lemma 4.13.** Given a grid diagram  $\mathcal{G}$  for a knot K and a grid state  $\mathbf{x} \in S(\mathcal{G})$ , the Alexander grading  $A(\mathbf{x})$  is an integer.

We also have the following description of the Alexander and Maslov grading.

**Definition 4.14.** Let  $GC^{-}(\mathcal{G})$  be the grid chain complex generated over  $\mathbb{F}[U_1, U_2, \ldots, U_n]$ where  $\mathbb{F} = \mathbb{Z}_2$ . Then the Alexander and the Maslov grading have the following relations;

$$M(U_1^{k_1} \dots U_n^{k_n} \cdot \mathbf{x}) = M(\mathbf{x}) - 2k_1 - \dots - 2k_r$$
$$A(U_1^{k_1} \dots U_n^{k_n} \cdot \mathbf{x}) = M(\mathbf{x}) - k_1 - \dots - k_n$$

where  $\mathbf{x} \in S(\mathcal{G})$  and  $k_i$ 's are all non-negative integers.

This actually tells us that multiplication by any  $U_i$  drops the Maslov grading by 2 and the Alexander grading by 1.

Now let us compute the Alexander and the Maslov grading for  $\mathcal{G}_{unknot}$  in Figure 4.23. First, we will compute it for the grid state **x**:

 $I(\mathbf{x}, \mathbf{x}) = 1, I(\mathbf{x}, \mathbb{O}) = 3, I(\mathbb{O}, \mathbf{x}) = 1, I(\mathbb{O}, \mathbb{O}) = 1.$  Symmetrizing these we obtain  $J(\mathbf{x}, \mathbf{x}) = 1, J(\mathbf{x}, \mathbb{O}) = 2, J(\mathbb{O}, \mathbb{O}) = 1.$  Therefore, the Maslov grading  $M_{\mathbb{O}}(\mathbf{x}) = -1.$ Similarly, we compute that  $I(\mathbf{x}, \mathbb{X}) = 2, I(\mathbb{X}, \mathbf{x}) = 0, I(\mathbb{X}, \mathbb{X}) = 0.$  Again symmetrizing these we will get  $J(\mathbf{x}, \mathbb{X}) = 1, J(\mathbb{X}, \mathbb{X}) = 0.$  Hence  $M_{\mathbb{X}}(\mathbf{x}) = 0.$  Implementing these values in the formula of the Alexander grading, we see that  $A(\mathbf{x}) = -1.$ For the other grid state  $\mathbf{y}$ , similar computations show that  $M_{\mathbb{O}}(\mathbf{y}) = 0$  and  $M_{\mathbb{X}}(\mathbf{y}) =$ 

-1. Therefore  $A(\mathbf{y}) = 0$ .

Now let  $GC_d^-(\mathcal{G}, s)$  denote the vector space over  $\mathbb{F} = \mathbb{Z}_2$  generated by monomials  $U_1^{k_1}, U_2^{k_2} \dots, U_n^{k_n} \cdot \mathbf{x}$  with Maslov grading d and Alexander grading s. The following theorem reveals the properties of this bigraded structure;

**Theorem 4.15.** The construction  $GC^{-}(\mathcal{G}, \partial)$  has the structure;

• It is bigraded as a vector space;

$$GC^{-}(\mathcal{G}) = \bigoplus_{d,s \in \mathbb{Z}} GC^{-}_{d}(\mathcal{G},s)$$

- It is a chain complex, i.e.  $\partial^2 = 0$ .
- The boundary map decreases the Maslov grading by one, and it preserves the Alexander grading;

$$\partial: GC_d^-(\mathcal{G}, s) \to GC_{d-1}^-(\mathcal{G}, s)$$

• The complex comes with an endomorphism U which decreases Maslov grading by two and Alexander grading by one;

$$U: GC_d^-(\mathcal{G}, s) \to GC_{d-2}^-(\mathcal{G}, s-1)$$

The proof is a little long and technical, so we will not be presenting it here. Basically, we should understand that this structure gives a bigraded chain complex over  $\mathbb{F}[U]$  and the endomorphism is given by multiplication by any  $U_i$ . Homology arising from this bigraded chain complex inherits the same module structure.

In the following section, we will only outline the proof that grid homology is invariant under the stabilization move. In the proof of invariance under the commutation move, a similar reasoning that we will use in the next section is utilized. Given a grid diagram  $\mathcal{G}$ , if we perform a commutation move and obtain another grid diagram  $\mathcal{G}'$ , then we can define two maps  $P: GC^{-}(\mathcal{G}) \to GC^{-}(\mathcal{G}')$  and  $P': GC^{-}(\mathcal{G}) \to GC^{-}(\mathcal{G}')$ which count certain geometric regions on the grid called "pentagons". With a similar geometric interpretation used in the proof of  $\partial^2 = 0$ , it can be shown that both maps are chain maps. On the other hand, these two maps P and P' are homotopy inverses of each other. The homotopy map is provided by two maps  $H: GC^{-}(\mathcal{G}) \to GC^{-}(\mathcal{G})$ and  $H': GC^{-}(\mathcal{G}') \to GC^{-}(\mathcal{G}')$  where the maps count certain geometric regions on the grid called "hexagons". From these maps, the isomorphism on the homology is obtained. The details of invariance under the commutation move can be found in [20].

## 4.3 Invariance of the Grid Homology; stabilization case

We will begin with stating a theorem which is crucial.

**Theorem 4.16.** If  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are two different grid diagrams representing isotopic knots, then  $GH^-(\mathcal{G}_1) \cong GH^-(\mathcal{G}_2)$  as bigraded  $\mathbb{F}[U]$ -modules.

The theorem above states that the grid homology is an invariant of the knot. For that reason, the grid homology is invariant under the moves we have defined; stabilization and commutation. We will describe how homology is affected under the stabilization move. First, we shall discuss the proof for the fully blocked grid homology  $\widetilde{GH}(\mathcal{G})$ , and then we will describe what might go wrong in the unblocked grid homology  $GH^{-}(\mathcal{G})$ , and how we can fix it.

Now we have the following proposition;

**Proposition 4.17.** Let  $\mathcal{G}'$  be a stabilization of  $\mathcal{G}$ . Then, there is an isomorphism of bigraded vector spaces;

 $\widetilde{GH}_{d}(\mathcal{G}',s)\cong \widetilde{GH}_{d}(\mathcal{G},s)\oplus \widetilde{GH}_{d+1}(\mathcal{G},s+1)$ 

We will need some preparation to prove this statement. First of all, assume that the stabilization is of  $X_{SW}$  type, which means we apply a south-west stabilization at an X-marking. Recall the definition of the different types of stabilization in Figure 4.9. Just for notational purposes, we will enumerate some of the markings. Call the newly created O-marking  $O_1$ , and two splitted X-markings  $X_1$  and  $X_2$ . Also call the O-marking which is in the row under the newly created  $O_1$ , as  $O_2$ . Finally, consider the intersection of the two new lines appearing in  $\mathcal{G}'$  and call the intersection point c. Here, you can see how two grids differ.



Figure 4.26: How two grids differ after stabilization

Now we are going to use a standard construction in homological algebra called the mapping cone construction. Usually, if we are given two chain complexes  $C_1$  and  $C_2$  with boundary maps  $\partial_{C_1}$  and  $\partial_{C_2}$  respectively, and a chain map between the chain complexes  $f: C_1 \to C_2$ , i.e.  $\partial_{C_2} f = f \partial_{C_1}$ , we can form the mapping cone Cone(f)from these two complexes. It will be a chain complex of the direct sums  $C_1 \oplus C_2$ , where the new boundary map can be written in a matrix notation and acts in the following way:

$$\partial_{Cone(f)}(c_1, c_2) = \begin{pmatrix} \partial_{C_1} & 0\\ f & -\partial_{C_2} \end{pmatrix} \cdot \begin{pmatrix} c_1\\ c_2 \end{pmatrix} = (\partial_{C_1}c_1, f(c_1) - \partial_{C_2}c_2)$$

where  $(c_1, c_2)^T \in C_1 \oplus C_2$ . We can also see that  $\partial^2_{Cone(f)}(c_1, c_2) = 0$ ;

$$\partial_{Cone(f)}^{2} = \begin{pmatrix} \partial_{C_{1}} & 0\\ f & -\partial_{C_{2}} \end{pmatrix} \cdot \begin{pmatrix} \partial_{C_{1}} & 0\\ f & -\partial_{C_{2}} \end{pmatrix} = \begin{pmatrix} \partial_{C_{1}}^{2} & 0\\ f\partial_{C_{1}} - \partial_{C_{2}}f & \partial_{C_{2}}^{2} \end{pmatrix} = \begin{pmatrix} 0 & 0\\ 0 & 0 \end{pmatrix}$$

since we know that  $\partial_{C_i}^2 = 0$  for i = 1, 2. Also, f is a chain map, hence by definition  $f\partial_{C_1} - \partial_{C_2}f = 0$ . Here  $C_2$  is a subcomplex, since if we pick a cycle  $c_2$  from the complex  $C_2$  corresponding to the element  $(0, c_2) \in (0 \oplus C_2) \leq (C_1 \oplus C_2)$  and apply the boundary map, we see that it remains in  $C_2$ :

$$\partial_{Cone(f)}(0,c_2) = \begin{pmatrix} \partial_{C_1} & 0\\ f & -\partial_{C_2} \end{pmatrix} \cdot \begin{pmatrix} 0\\ c_2 \end{pmatrix} = (0,-\partial_{C_2}c_2)$$

We can decompose  $\widetilde{GC}(\mathcal{G}')$  as a mapping cone structure. Consider the grid states  $S(\mathcal{G}')$ , split it as  $I(\mathcal{G}') \cup N(\mathcal{G}')$  where  $I(\mathcal{G}')$  denotes the set of grid states which contains c as one of the grid state points, i.e. if  $\mathbf{x} \in I(\mathcal{G}')$ , then  $c \in \mathbf{x}$ . Then, it splits as a vector space;  $\widetilde{GC}(\mathcal{G}') \cong \widetilde{I} \oplus \widetilde{N}$  and these components denote the spans corresponding to the grid states. Notice that  $\widetilde{N}$  is a subcomplex of the mapping cone, since for any rectangle  $\mathbf{r} \in Rect(\mathbf{x}, \mathbf{y})$  with  $\mathbf{x} \in \widetilde{N}$  and  $\mathbf{y} \in \widetilde{I}$ , we know that this rectangle contains one of  $X_1$  or  $X_2$ . In Figure 4.27 and 4.28, we show the different cases for this, where the black dot represents the grid state  $\mathbf{x} \in \widetilde{N}$  and the small empty rectangle represents  $\mathbf{y} \in \widetilde{I}$ . The shaded and marked rectangles are all in  $Rect(\mathbf{x}, \mathbf{y})$ .



Figure 4.27: Rectangles containing  $X_1$  or  $X_2$ 



Figure 4.28: Other cases of rectangles containing  $X_1$  or  $X_2$ 

In this picture, we have  $\widetilde{\partial}_{I}^{I}: \widetilde{I} \to \widetilde{I}$  counting the rectangles from grid states of  $I(\mathcal{G}')$  to itself, and similarly we have  $\widetilde{\partial}_{N}^{N}: \widetilde{N} \to \widetilde{N}$  counting the rectangles from grid states of  $N(\mathcal{G}')$  to itself. We can represent the mapping cone with a matrix, because we have the map  $\widetilde{\partial}_{I}^{N}: \widetilde{I} \to \widetilde{N}$  which counts the rectangles  $\mathbf{r} \in Rect^{0}(\mathbf{x}, \mathbf{y})$ where  $\mathbf{x} \in I(\mathcal{G}')$  and  $\mathbf{y} \in N(\mathcal{G}')$ . Moreover, notice that  $\widetilde{\partial}_{N}^{I}$  which is the top right entry of the matrix below is 0, since any rectangle  $\mathbf{r} \in Rect^{0}(\mathbf{x}, \mathbf{y})$  with  $\mathbf{x} \in N(\mathcal{G}')$  and  $\mathbf{y} \in I(\mathcal{G}')$  must contain one of  $X_1$  or  $X_2$ . Therefore;

$$\widetilde{\partial}_{Cone(\widetilde{\partial}_{I}^{N})} = \left(\begin{array}{cc} \widetilde{\partial}_{I}^{I} & 0\\ \widetilde{\partial}_{I}^{N} & \widetilde{\partial}_{N}^{N} \end{array}\right)$$

Here we consider the  $Cone(\widetilde{\partial}_I^N) = \widetilde{GC}(\mathcal{G}')$ . Now what we would like to prove is actually;

$$\widetilde{GH}(\mathcal{G}') \cong \widetilde{GH}(\mathcal{G}) \oplus \widetilde{GH}(\mathcal{G})$$

with some grading shift. So we have three claims;

- The homology of the complex (*I*, *∂*<sup>I</sup><sub>I</sub>) is isomorphic to *GH*(*G*) (with some grading shift).
- The homology of the complex  $(\widetilde{N}, \widetilde{\partial}_N^N)$  is isomorphic to  $\widetilde{GH}(\mathcal{G})$ .
- The map induced by  $\widetilde{\partial}^I_N$  is zero in the homology.

First of all, realize that there is a 1-1 correspondence between  $S(\mathcal{G})$  and  $\widetilde{I}(\mathcal{G}')$  by matching  $\mathbf{x} \leftrightarrow \mathbf{x} \cup \{c\}$ . In the figure below, you can see how we can match two grid states.



Figure 4.29: Matching a grid state of  $\mathcal{G}$  by adding  $\{c\}$ 

We can relate the gradings of two grid states, which will also provide the necessary grading shift. Hence, we have the following lemma;

**Lemma 4.18.** If  $\mathbf{x} \in S(\mathcal{G})$  and  $\mathbf{x}' = \mathbf{x} \cup \{c\}$  is the corresponding element in  $\widetilde{I}(\mathcal{G}')$  then we have the following relations;

$$M(\mathbf{x}') = M(\mathbf{x}) - 1$$
$$A(\mathbf{x}') = A(\mathbf{x}) - 1$$

We will briefly explain the reasoning of this grading change. First, assume that we stabilized at the top, right-most corner. When we add this new coordinate  $\{c\}$ , then it increases some of the components we defined previously, such as  $I(\mathbf{x}, \mathbf{x})$ ,  $I(\mathbf{x}, \mathbb{O})$ ,  $I(\mathbb{O}, \mathbf{x})$  and finally  $I(\mathbb{O}, \mathbb{O})$ . It is not difficult to see that if the initial grid  $\mathcal{G}$ we have is an  $n \times n$  grid, then  $I(\mathbf{x}, \mathbf{x})$ ,  $I(\mathbb{O}, \mathbf{x})$  and  $I(\mathbb{O}, \mathbb{O})$  will increase by n, while  $I(\mathbf{x}, \mathbb{O})$  increases by n + 1. In Figure 4.30 below, we can see the pictorial explanation of how  $I(\mathbf{x}, \mathbf{x})$  increases by n. The black dots represent the grid state  $\mathbf{x} \in S(\mathcal{G})$ .



Figure 4.30: After adding  $\{c\}$ , the shift for each coordinate of **x** 

Therefore, if we say  $I(\mathbf{x}, \mathbf{x}) = k$ ,  $I(\mathbf{x}, \mathbb{O}) = l$ ,  $I(\mathbb{O}, \mathbf{x}) = m$  and finally  $I(\mathbb{O}, \mathbb{O}) = r$ , then it follows that  $I(\mathbf{x}', \mathbf{x}') = k + n$ ,  $I(\mathbf{x}', \mathbb{O}') = l + n + 1$ ,  $I(\mathbb{O}', \mathbf{x}') = m + n$  and finally  $I(\mathbb{O}', \mathbb{O}') = r + n$ , where  $\mathbb{O}'$  denotes the set of O-markings after the stabilization. Then,

$$M(\mathbf{x}') - M(\mathbf{x}) = k + n - 2\left(\frac{m+l+2n+1}{2}\right) + r + n + 1 - \left(k - 2\left(\frac{m+l}{2}\right) + r + 1\right)$$
$$= k - (m+l) + r - k + (m+l) - r - 1 = -1$$

Similar computations show that  $A(\mathbf{x}') = A(\mathbf{x}) - 1$ . As the next step, we should understand how this map affects the chain complexes. Here is the lemma;

**Lemma 4.19.** Let  $\tilde{e} : \tilde{I} \to \widetilde{GC}(\mathcal{G})$  be the map induced by the correspondence  $\mathbf{x} \leftrightarrow \mathbf{x} \cup \{c\}$ , then  $\tilde{e}$  induces an isomorphism between chain complexes  $(\tilde{I}, \tilde{\partial}_{I}^{I})$  and  $\widetilde{GC}_{d+1}(\mathcal{G}, s+1)$  for all (d, s) pairs where d and s denote the corresponding Maslov and Alexander gradings of  $\mathbf{x}$  in  $\mathcal{G}$ .

The map  $\tilde{e}$  is a bijection of grid states, therefore it induces an isomorphism between vector spaces  $\tilde{I}$  and  $\widetilde{GC}(\mathcal{G})$ . Our boundary map counts empty rectangles, and empty rectangles disjoint from  $\mathbb{X} \cup \mathbb{O}$  in  $\mathcal{G}$  correspond to empty rectangles disjoint from  $\mathbb{X}' \cup \mathbb{O}'$  in  $\mathcal{G}'$ . This shows that  $\tilde{e}$  is an isomorphism of the chain complexes, and in the previous lemma, we already confirmed the grading change. Now we will give the definitions of two maps which count certain rectangles, and will help us relate the homologies of two complexes;

**Definition 4.20.** Define  $\widetilde{\mathcal{H}}_{X_2}^I: \widetilde{N} \to \widetilde{I}$  for  $\mathbf{x} \in \widetilde{N}(\mathcal{G}')$  as follows;

$$\widetilde{\mathcal{H}}_{X_2}^{I}(\mathbf{x}) = \sum_{\mathbf{y} \in I(\mathcal{G}')} \#\{\mathbf{r} \in Rect^0(\mathbf{x}, \mathbf{y}) \mid Int(\mathbf{r}) \cap \mathbb{O} = \emptyset \text{ and } Int(\mathbf{r}) \cap \mathbb{X} = X_2\} \cdot \mathbf{y}$$

Similarly, the other map is defined as,  $\widetilde{\mathcal{H}}_{O_1}: \widetilde{I} \to \widetilde{N}$  for  $\mathbf{x} \in \widetilde{I}(\mathcal{G}')$ ;

$$\widetilde{\mathcal{H}}_{O_1}(\mathbf{x}) = \sum_{\mathbf{y} \in N(\mathcal{G}')} \#\{\mathbf{r} \in Rect^0(\mathbf{x}, \mathbf{y}) \mid Int(\mathbf{r}) \cap \mathbb{X} = \emptyset \text{ and } Int(\mathbf{r}) \cap \mathbb{O} = O_1\} \cdot \mathbf{y}$$

**Lemma 4.21.** The map  $\widetilde{\mathcal{H}}_{X_2}^I$  drops the Maslov and the Alexander gradings by one, and the map  $\widetilde{\mathcal{H}}_{O_1}$  increases both gradings by one. Furthermore, these maps are chain maps and they induce isomorphims on the homology.

The fact that these are chain maps can be proved using the same technique of the proof that  $\partial^2 = 0$  (Recall the proof of Lemma 4.9). We will concentrate on the claim that;

$$\mathcal{H}_{X_2}^I \circ \mathcal{H}_{O_1} = Id_{\widetilde{I}}$$

To understand what this composition map does, we can look at it on the grid. To begin with, if  $\mathbf{x} \in \widetilde{I}(\mathcal{G}')$  is a generator, the composition map counts the rectangles  $\mathbf{r_1}$ from  $\mathbf{x}$  to  $\mathbf{y}$  which contains the  $O_1$  marking and another rectangle  $\mathbf{r_2}$  touching the first one and also containing the  $X_2$  marking, moreover ending in a state in  $\widetilde{I}(\mathcal{G}')$ . If we look at the grid in Figure 4.31, there is a unique choice doing this job. This is unique, because considering the other choices of rectangles, we can see that some of them will just contain  $O_1$  and not  $X_2$  or the other way around, or they will contain some other markings different from  $O_1$  or  $X_2$ .



Figure 4.31: The rectangles  $\mathbf{r_1}$  and  $\mathbf{r_2}$ 

Therefore, we can conclude that  $\widetilde{\mathcal{H}}_{X_2}^I \circ \widetilde{\mathcal{H}}_{O_1}(\mathbf{x}) = \mathbf{x}$ . Next claim is that, the map  $\widetilde{\mathcal{H}}_{O_1} \circ \widetilde{\mathcal{H}}_{X_2}^I$  is chain homotopic to the identity map. So consider the map  $\widetilde{\mathcal{H}}_{O_1,X_2} : \widetilde{N} \to \widetilde{N}$  defined for  $\mathbf{x} \in \widetilde{N}(\mathcal{G}')$ ;

$$\widetilde{\mathcal{H}}_{O_1,X_2}(\mathbf{x}) = \sum_{\mathbf{y}\in N(\mathcal{G}')} \#\{\mathbf{r}\in Rect^0(\mathbf{x},\mathbf{y}) \mid Int(\mathbf{r})\cap \mathbb{X} = X_2 \text{ and } Int(\mathbf{r})\cap \mathbb{O} = O_1\} \cdot \mathbf{y}$$

To understand what this map does, we can follow a similar reasoning. Consider this;

$$\widetilde{\mathcal{H}}_{O_1} \circ \widetilde{\mathcal{H}}_{X_2}^I + \widetilde{\mathcal{H}}_{O_1, X_2} \circ \widetilde{\partial}_N^N + \widetilde{\partial}_N^N \circ \widetilde{\mathcal{H}}_{O_1, X_2} = Id_{\widetilde{N}}.$$

Left-hand side of the equation counts the number of composition of rectangles starting and ending at the states in  $\widetilde{N}(\mathcal{G})$  which can be given as the composition of two rectangles, say  $\mathbf{r_1}$  and  $\mathbf{r_2}$  such that  $(\mathbf{r_1} \cup \mathbf{r_2}) \cap (\mathbb{X} \cup \mathbb{O}) = \{X_2, O_1\}$ . For most of the composition of rectangles like this, we know that they can be decomposed as two such rectangles in two ways. For exceptional cases, we have the annuli containing  $X_2$  and  $O_1$  and these identify the  $Id_{\widetilde{N}}$ . Putting all these together, we conclude that  $\widetilde{\mathcal{H}}_{X_2}^I$  and  $\widetilde{\mathcal{H}}_{O_1}$  are chain homotopy equivalences.

The last claim before moving to the homological interpretation is that the chain map  $\widetilde{\partial}_{I}^{N}: (\widetilde{I}, \widetilde{\partial}_{I}^{I}) \to (\widetilde{N}, \widetilde{\partial}_{N}^{N})$  induces the trivial map on homology. Seeing that  $\widetilde{\mathcal{H}}_{X_{2}}^{I} \circ \widetilde{\partial}_{I}^{N} = 0$ will suffice. For this composition map  $\widetilde{\mathcal{H}}_{X_{2}}^{I} \circ \widetilde{\partial}_{I}^{N}$ , the only rectangles that count will be the horizontal, height-1 annulus containing  $O_{2}$  and  $X_{2}$ , but we consider rectangles which do not cross any O-markings. Therefore, the composition vanishes, and we have already seen that  $\widetilde{\mathcal{H}}_{X_{2}}^{I}$  induces an isomorphism on the homology.

Proof of Proposition 4.17. After this sequence of lemmas and explanations, we can consider the long exact sequence coming from this mapping cone and pass to the homologies. First of all, assume that we have the stabilization type  $\mathbb{X}_{SW}$ . We know that  $\widetilde{N}$  is a subcomplex of  $\widetilde{GC}(\mathcal{G}')$  with quotient  $\widetilde{I}$ . We have the following short exact sequence;

$$0\longrightarrow \widetilde{N}\longrightarrow \widetilde{GC}(\mathcal{G}')\longrightarrow \widetilde{I}\longrightarrow 0$$

The connecting homomorphism is induced by  $\widetilde{\partial}_I^N$  and we have seen that this map induces the zero map on the homology. Therefore, the long exact sequence reduces to;

$$0 \longrightarrow H(\widetilde{N}) \longrightarrow \widetilde{GH}(\mathcal{G}') \longrightarrow H(\widetilde{I}) \longrightarrow 0$$

Now if we replace the homologies above with our computations from Lemma 4.21, we obtain the result;

$$0 \longrightarrow \widetilde{GH}(\mathcal{G}) \longrightarrow \widetilde{GH}(\mathcal{G}') \longrightarrow \widetilde{GH}(\mathcal{G}) \longrightarrow 0$$

where the  $\widetilde{GH}(\mathcal{G})$  in the chain coming after  $\widetilde{GH}(\mathcal{G}')$  is the homology with the grading

shift. By Lemma 4.5 and invariance of the grid homology under the commutation move, the proof is complete.  $\hfill \Box$ 

Now we will briefly explain how this whole construction can be modified in such a way that it works in the unblocked theory as well. The main problem that arises is that we will be working over different polynomial rings before and after the stabilization, in this case, technically we cannot expect to have an isomorphism after the stabilization. Following the previous notation,  $GC^-(\mathcal{G}')$  is defined over  $\mathbb{F}[U_1, U_2, \dots, U_n]$  whereas  $GC^-(\mathcal{G})$  is defined over  $\mathbb{F}[U_2, \dots, U_n]$ . To equalize the situation, we first promote our polynomial ring by adding an extra generator and then collapse it. This means that given  $GC^-(\mathcal{G})$ , we consider  $GC^-(\mathcal{G})[U_1]$ . So far it works out perfectly, but the problem will occur when we consider the homology of this new complex. It will be bigger than the homology of  $GC^-(\mathcal{G})$ . To solve this problem, we divide it by  $U_1 - U_2$  which means considering the map;

$$U_1 - U_2 : GC^-(\mathcal{G})[U_1] \to GC^-(\mathcal{G})[U_1]$$

defined to be the multiplication by  $U_1 - U_2 \in \mathbb{F}[U_1, U_2, \cdots, U_n]$  and take its mapping cone. After this modification, we follow the previous construction and relate two mapping cones.

## 4.4 $\tau$ and Slice Genus

To summarize what we have seen so far, given a knot  $K \subset S^3$ , we represent it by a grid diagram  $\mathcal{G}$ . Then using the grid states and rectangles, we defined the boundary map on the complex generated by all grid states. We have denoted it by  $GC^{-}(\mathcal{G})$ which is an  $\mathbb{F}[U_1, \ldots, U_n]$ -module equipped with Maslov and Alexander grading. After that, from the boundary map  $\partial : GC^{-}(\mathcal{G}) \to GC^{-}(\mathcal{G})$  we extracted the homology associated to the knot K. From now on, we will denote the grid homology of K by  $GH^{-}(K)$  to emphasize the knot.

Now let us look at the structure of  $GH^-(K)$  closer. It is a finitely generated  $\mathbb{F}[U]$ module, and we know from algebra that  $\mathbb{F}[U]$  is a principal ideal domain. Hence, a finitely generated module over  $\mathbb{F}[U]$  can be written as a sum of cyclic modules. As a result, the cyclic modules will be of the form  $\mathbb{F}[U] \not< f(U) >$ . Moreover, the existence of Maslov grading forces f(U) to be a monomial, i.e.  $f(U) = U^n$ . Hence;

$$GH^{-}(K) = \mathbb{F}[U]_{(d_1,a_1)} \oplus \cdots \oplus \mathbb{F}[U]_{(d_m,a_m)} \oplus \mathbb{F}[U]_{U^{n_1}(d'_1,a'_1)} \oplus \cdots \mathbb{F}[U]_{U^{n_k}(d'_k,a'_k)}$$

where  $d_i$  and  $d'_i$  denote the Maslov grading for the corresponding generator, and  $a_i$ and  $a'_i$  denote the Alexander grading. We can say more if we have a knot K: **Proposition 4.22** ([20], Proposition 7.3.4). Given a knot  $K \subset S^3$  we have;

$$GH^{-}(K) = \mathbb{F}[U]_{(d_1,a_1)} \oplus \mathbb{F}[U]_{U^{n_1}(d'_1,a'_1)} \oplus \cdots \oplus \mathbb{F}[U]_{U^{n_k}(d'_k,a'_k)}$$

**Definition 4.23.** Define  $\tau(K) = -a_1$  where  $\tau(K)$  is the negative of the maximal Alexander grading of the elements in  $GH^-(K)$  which is not U-torsion.

The following theorem will provide the lower bound for slice genus;

**Theorem 4.24.** For a knot  $K \subset S^3$ , we have  $|\tau(K)| \leq g_s(K)$ .

Now we can consider the example of the 0-framed, negative, Whitehead double of left-handed trefoil.

To obtain the Whitehead double of a given knot K, we do the following; we consider a push-off of K and orient it with the opposite orientation of K. Call this push-off knot K'. Then the framing will be computed from the linking number lk(K, K'). After this, we also introduce a clasp which can be + or - (See Figure 3.1). The resulting knot is called the Whitehead double of K and will be denoted as  $W_k^{\pm}(K)$  where k denotes the framing and  $\pm$  denotes the chosen clasp. For our example, we will have  $W_0^{-}(LHT)$ . In the figure below, you can see how we construct this knot.



Figure 4.32: Taking the push-off of K with opposite orientation

We need to make sure that the framing is 0, so we will compute the linking number lk(K, K') as shown in Figure 4.33:



Figure 4.33: Signs coming from the crossings

Finally, we will introduce 6 crossings with minus signs to make lk(K, K') = 0and then add the minus clasp. We obtain  $W_0^-(LHT)$ :



Figure 4.34: 
$$W_0^-(LHT)$$

Now we need to show that the 0-framed Whitehead double of a knot (with positive or negative clasp) has trivial Alexander polynomial. To prove this, we need the skein relation.

**Definition 4.25.** Let  $K_+$ ,  $K_-$  and  $K_0$  be three oriented knots with projections which are the same except for one crossing. How they differ from each other at this exceptional crossing is shown in the Figure 4.35. Then we have the following relation of Alexander polynomials;

$$\Delta_{K_+}(t) - \Delta_{K_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{K_0}(t).$$

Note that in this definition  $\Delta_K(t) = |t^{-1/2}S - t^{1/2}S^T| \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$  where S is a Seifert matrix for K.



Figure 4.35: The crossings where  $K_+$ ,  $K_-$  and  $K_0$  differ

Now to see that the Alexander polynomial is trivial for  $W_0^-(K)$ , it suffices to use the skein rule at the clasp. Considering the crossing changes given in Figure 4.35 at a positive clasp, we have the following picture.



Figure 4.36: Skein rule to determine the Alexander polynomial

Notice that if we consider the top crossing in the clasp and assume that this is the crossing they differ,  $K_{-}$  will be just the unknot and we know that  $\Delta_{unkot}(t) = 1$ . Similarly,  $K_0$  will have no crossings, so we have the oriented resolution. In this case,  $K_0$  has a Seifert surface which is an annulus and there is only one generator in the homology of the Seifert surface. So the Seifert matrix is a 1 × 1 matrix. When we take the push-off of this curve which represents the generator and compute the linking number, it is not difficult to see that it equals to 0 because it is the framing. Therefore  $\Delta_{K_0} = 0$ . Implementing these in the Skein rule;

$$\Delta_{K_{+}}(t) - \Delta_{K_{-}}(t) = (t^{1/2} - t^{-1/2})\Delta_{K_{0}}(t) = \Delta_{K_{+}}(t) - 1 = (t^{1/2} - t^{-1/2}) \cdot 0$$

Therefore,  $\Delta_{K_+}(t) = 1$  which shows that the 0-framed, Whitehead double of a knot has trivial Alexander polynomial. Here we have done the computation for the + clasp. Similarly, it can be shown that the Alexander polynomial is trivial for 0framed, negative Whitehead double of any knot K.

Now we know that  $\Delta_{W_0^-(LHT)}(t) = 1$ . By Freedman's theorem [8], we can conclude that  $W_0^-(LHT)$  is topologically slice. We will proceed to computing  $\tau$  for  $W_0^-(LHT)$ . In Figure 4.37, we see a grid diagram for  $W_0^-(LHT)$  and a grid state represented by the black dots.



Figure 4.37: Grid diagram of  $W_0^-(LHT)$ 

Let us denote this specific grid state in Figure 4.37 by  $\mathbf{x}_{SW}$  where the bottom left corner of each X-marking is marked with a black dot. First of all, we have  $I(\mathbf{x}_{SW}, \mathbf{x}_{SW}) = 36$ ,  $I(\mathbf{x}_{SW}, \mathbb{O}) = 41$ ,  $I(\mathbb{O}, \mathbf{x}_{SW}) = 29$  and finally  $I(\mathbb{O}, \mathbb{O}) = 35$ . Symmetrizing these, we obtain  $J(\mathbf{x}_{SW}, \mathbf{x}_{SW}) = 36$ ,  $J(\mathbf{x}_{SW}, \mathbb{O}) = 35$ ,  $J(\mathbb{O}, \mathbb{O}) = 35$ . Hence;

$$M_{\mathbb{O}}(\mathbf{x}_{SW}) = 36 - 2 \cdot 35 + 35 + 1 = 2.$$

Continuing the computation, we obtain  $I(\mathbf{x}_{SW}, \mathbb{X}) = 49$ ,  $I(\mathbb{X}, \mathbf{x}_{SW}) = 36$  and finally  $I(\mathbb{X}, \mathbb{X}) = 36$ . After symmetrizing, the results are  $J(\mathbf{x}_{SW}, \mathbb{X}) = \frac{85}{2}$ ,  $J(\mathbb{X}, \mathbb{X}) = 36$ . Hence;

$$M_{\mathbb{X}}(\mathbf{x}_{SW}) = 36 - 2 \cdot \frac{85}{2} + 36 + 1 = -12.$$

Recall the definition of the Alexander grading and implement these values in the formula.

$$A(\mathbf{x}_{SW}) = \frac{1}{2}(M_{\mathbb{O}}(\mathbf{x}_{SW}) - M_{\mathbb{X}}(\mathbf{x}_{SW})) - (\frac{n-1}{2}) = \frac{1}{2}(2+12) - \frac{13-1}{2} = 1$$

For  $\mathbf{x}_{SW}$ , it can be shown that this element is non-torsion, and it gives the maximum Alexander grading. We will not argue for the non-torsion part, but recall from Chapter 3 that we have constructed a Seifert surface for the twist knots and computed the 2 × 2 Seifert matrix, and a twist knot is the Whitehead double of the unkot. As a result, our knot also has a genus 1 Seifert surface. Moreover, we have the inequality  $g_s(K) \leq g(K)$  where g(K) denotes the Seifert genus. This inequality can be seen from pushing the interior of a Seifert surface of a given knot K inside the interior of  $D^4$ . This suggests that we have the following inequality for a given knot K;  $|\tau(K)| \leq g_s(K) \leq g(K)$ . In our case  $g(W_0^-(LHT)) = 1$ . Therefore,  $\tau$  cannot be smaller than -1, equivalently there cannot be another grid state  $\mathbf{y}$  with  $A(\mathbf{y}) > 1$ . It follows that  $\mathbf{x}_{SW}$  gives the maximum Alexander grading and  $\tau(W_0^-(LHT)) = -1$ . We have already seen that  $W_0^-(LHT)$  is topologically slice by Freedman's theorem [8]. Now by Theorem 4.24,  $|\tau(W_0^-(LHT))| \leq g_s(W_0^-(LHT))$ . Since  $\tau(W_0^-(LHT)) = -1$ , we can conclude that  $g_s(W_0^-(LHT)) \neq 0$ . This immediately implies that  $W_0^-(LHT)$ is not smoothly slice.

To sum up, we have utilized the grid homology construction to obtain a lower bound for  $g_s(K)$ . Combining the  $\tau$  invariant we extracted from the grid homology with Freedman's theorem [8] which states that any knot in  $S^3$  with trivial Alexander polynomial is topologically slice, we have exhibited the example  $W_0^-(LHT)$  which is topologically slice, but not smoothly slice. Therefore, the map  $\phi_1 : C \to C_{top}$  has non-trivial kernel.

## REFERENCES

- Auckly, D. Two-fold Branched Covers, Journal of Knot Theory and Its Ramifications (2014)
- [2] Casson, A.J. and Gordon, C. Cobordism of Classical Knots, A la Recherche de la Topologie Perdue Progress in Mathematics, Volume 62, pp. 181-197 (1986)
- [3] Casson, A.J. and Gordon, C On slice knots in dimension three, Proc. Symp. Pure Math. pp. 39-53 (1978)
- [4] Cochran, T., Orr, K. and Teichner, P. Knot concordance, Whitney towers and L2 signatures, Ann. of Math. (2), Vol. 157, no. 2, 433-519 (2003)
- [5] Collins, J. On the concordance orders of knots, University of Edinburgh, Department of Mathematics (2011)
- [6] Donaldson, S. K. An application of gauge theory to four-dimensional topology, Journal of Differential Geometry 18, no. 2, 279-315. (1983)
- [7] Fox, R. and Milnor, J. Singularities of 2-spheres in 4-space and cobordism of knots, Osaka Journal of Mathematics 3, 257-267. (1966)
- [8] Freedman, Michael H.; Quinn, Frank Topology of 4-manifolds, Princeton Mathematical Series 39. Princeton, NJ: Princeton University Press., pp. 357-453 (1990)
- [9] Gompf, R. and Stipsicz A. 4-Manifolds and Kirby Calculus, American Mathematican Society (1999)

- [10] Jabuka, S. Rational Witt Classes of Pretzel Knots, Osaka Journal of Mathematics, Volume 47, Number 4, 977-1027 (2010)
- [11] Lesieutre, J. Slice Knots and the Concordane Group, Harvard University, Department of Mathematics (2009)
- [12] Levine, J. Invariants of Knot Cobordism, Inventiones mathematicae, volume 8;
   pp. 98 110 (1969)
- [13] Lisca, P. Lens spaces, rational balls and the ribbon conjecture, Geom. Topol., 11:429-472, (2007)
- [14] Lisca, P. Sums of lens spaces bounding rational balls, Algebr. Geom. Topol.,
  7:2141-2164, (2007)
- [15] Livingston, C. and Naik, S. Introduction to Knot Concordance, Work in Progress
- [16] Livingston, C. The Algebraic Concordance Order of a Knot, Journal of Knot Theory and Its Ramifications Vol. 19, No. 12, 1693-1711 (2010)
- [17] Livingston, C. A Survey of Classical Knot Concordance, The Handbook of Knot Theory, Elsevier B.V., pp. 321-347 (2005)
- [18] Murasugi, K. On a Certain Numerical Invariant of Link Types, Transactions of the American Mathematical Society, Vol. 117, pp 387-422, (1965)
- [19] Ozbagci, B. and Stipsicz, A. I. Surgery on Contact 3-Manifolds and Stein Surfaces, Verlag Berlin Heidelberg. Springer (2004)
- [20] Ozsváth, P.S. and Stipsicz, A.I. and Szabó, Z. Grid Homology for knots and links, Work in progress

- [21] Prasolov, V. V. and Sossinsky A. B. Knots, Links, Braids and 3-manifolds, American Mathematical Society (1996)
- [22] Rolfsen, D. Knots and Links, Providence, Rhode Island. AMS Chelsea Publishing (2003)
- [23] Saveliev, N. Lectures on the topology of 3-manifolds: an introduction to the Casson invariant, Berlin. de Gruyter (1999)
- [24] Singer, J. Three-dimensional manifolds and their Heegaard diagrams., Trans. Amer. Math. Soc., 35(1):88-111, (1993)
- [25] Teichner, P. Slice Knots:Knot Theory in the 4th Dimension Lecture Notes in San Diego (2001)