# Computations and comparison of generalized Montréal functors

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Submitted to
Central European University
Department of Mathematics and its Applications

In partial fulfilment of requirements
for the degree of
Doctor of Philosophy in Mathematics

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Budapest, Hungary
2015

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**Acknowledgments**

First and foremost I would like to thank my supervisor, Gergely Zábrádi, for introducing me to this beautiful topic and proposing this intuitive project. Your ideas, continuous help, valuable comments and all our useful discussions fundamentally contributed to this dissertation itself as well as were critically important to my development as a scientist.

I am grateful to the anonymous referee of paper [9] for his very careful reading and his helpful remarks, and also to Levente Nagy for reading through parts of earlier versions of my work.

The Central European University and the Rényi Institute of the Hungarian Academy of Sciences are acknowledged for supporting me the past few years. I thank the staff of both institutions for their assistance in various issues.

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## Abstract

In this thesis we examine the functors $D_{SV}$ of Schneider and Vigneras ([17]) and $D_\xi^\vee$ of Breuil ([3]) generalizing the so called Montréal functor $D$ of Colmez ([4]).

Let $G = \mathbf{G}(F)$ be the $F$-points of a $F$-split reductive group $\mathbf{G}$ defined over $\mathbb{Z}_p$ for a finite extension $F|\mathbb{Q}_p$ with connected centre and split Borel $\mathbf{B} = \mathbf{TN}$. Let $o$ be the ring of integers in a finite extension $K|\mathbb{Q}_p$, and $\varpi \in o$ be an uniformizer.

In chapter 2 we compute $D_{SV}$ attaching a module over the Iwasawa algebra $\Lambda(N_0)$ of certain compact subgroup $N_0 \leq N$ to a $B$-representation for irreducible modulo $\varpi$ principal series of the group $G = \mathbf{GL_n}(F)$.

Chapter 3 and some parts of chapter 4 are joint work with Gergely Zábrádi. We show that Breuil's [3] pseudocompact $(\varphi, \Gamma)$-module $D_\xi^\vee(\pi)$ attached to a smooth $o$-torsion representation $\pi$ of $B = \mathbf{B}(\mathbb{Q}_p)$ is isomorphic to the pseudocompact completion of the basechange $\mathcal{O}_\mathcal{E} \otimes_{\Lambda(N_0),\ell} \widetilde{D_{SV}}(\pi)$ to Fontaine's ring (via a Whittaker functional $\ell\colon N_0 = \mathbf{N}(\mathbb{Z}_p) \to \mathbb{Z}_p$) of the étale hull $\widetilde{D_{SV}}(\pi)$ of $D_{SV}$.

Both in [17] and [3] the functional $\ell$ was generic. In the last chapter we examine the case when $\ell$ is chosen to be $\ell = \ell_\alpha$, the projection of $N_0$ onto a root subgroup of a simple root $\alpha$ of $\mathbf{G}$, which is nongeneric. We extend the results of Breuil to this situation, moreover we define an étale action of the submonoid $T_+ \leq T$ on the noncommutative multivariable version $D_{\xi,\ell,\infty}^\vee(\pi)$ of $D_\xi^\vee(\pi)$ enabling us to go backwards to the representations of $G$. We also show some disadvantages of this choice of $\ell$.

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# Contents



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# Chapter 1

# Introduction

## 1.1 Local Langlands correspondence

At first, we catch a glimpse of local class field theory (see for example [19]) as an antecedent of the local Langlands conjectures.

Let $p$ be a prime number and $\mathbb{Q}_p$ be the $p$-adic field. Let $F|\mathbb{Q}_p$ be a field extension—in general it can be any local field—, $F^*$ be the multiplicative group of $F$, and $E$ be an algebraically closed field.

The main theorem of local class field theory gives the Artin homomorphism $\theta : \mathbf{GL_1}(F) \simeq F^* \to \mathrm{Gal}(\overline{F}|F)^{ab}$, which induces an isomorphism on the profinite completion $\widehat{F^*}$ of $F^*$.

Since $\mathbf{GL_1}(F)$ is abelian, the irreducible $E$-representations of $\mathbf{GL_1}(F)$ are the homomorphisms $\mathbf{GL_1}(F) \to E^*$, which are this way related to the homomorphisms $\mathrm{Gal}(\overline{F}|F)^{ab} \to E^*$ corresponding to one dimensional $E$-representations of the absolute Galois group of $F$.

The precise statements depend on the field $E$, and we do not explain them in details here.

The local Langlands conjectures are generalizations of this, namely for $\mathbf{GL_n}$ the aim is to relate certain irreducible $E$-representations of $\mathbf{GL_n}(F)$ with certain continuous $n$ dimensional $E$-representations of $\mathrm{Gal}(\overline{F}|F)$. This correspondence shall be compatible with different structures (such as $\varepsilon$- and $L$-factors) on these representations.

In the situation $E = \overline{\mathbb{Q}_\ell}$ ($\ell \neq p$ is a prime number) and hence also if $E = \mathbb{C}$

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Harris and Taylor ([11]), and independently Henniart ([12]) estabilished the correspondence.

However, the $p$-adic version $E = \overline{\mathbb{Q}_p}$ of the conjectures (which are closely related to the $p$-characteristic version) seems to be much more involved. A satisfactory explanation comes from the representation theory of $\mathbf{GL_n}(F)$: there are much more $p$-adic representation than $\ell$-adic. By now the correspondence for $\mathbf{GL_2}(\mathbb{Q}_p)$ is very well understood through the work of Colmez [4], [5] and others (see [1] for an overview). In other cases the conjecturial picture is not clear yet.

One can see the problem even for $\mathbf{GL_2}(F)$ with $F \neq \mathbb{Q}_p$ as follows: On the Galois side nothing really different happens as we change from $\mathbb{Q}_p$ to $F$. On the other hand, the dimension of $\mathbf{GL_2}(F)$ as a $p$-adic analytic group is bigger than that of $\mathbf{GL_2}(\mathbb{Q}_p)$, consequently the representation theory of $\mathbf{GL_2}(\mathbf{F})$ is much more complicated than that of $\mathbf{GL_2}(\mathbb{Q}_p)$. In particular there is no possible naive 1-1 correspondence (see [2]).

Since that many efforts have been done to generalize parts of Colmez's results. The aim of this thesis is to examine and compare the functors of Schneider-Vigneras ([17]) and Breuil ([3]) going towards the Galois side (we call these "generalized Montréal" functors).

## 1.2 The correspondence for $\mathbf{GL_2}(\mathbb{Q}_p)$

To review Colmez's work let $K|\mathbb{Q}_p$ be a finite extension with ring of integers $o$, uniformizer $\varpi$ and residue field $k$.

The starting point is Fontaine's [13] theorem that the category of $o$-torsion Galois representations of $\mathbb{Q}_p$ is equivalent to the category of torsion $(\varphi, \Gamma)$-modules over $\mathcal{O}_{\mathcal{E}} = \varprojlim_h o/\varpi^h((X))$.

Recall that a $(\varphi, \Gamma)$-module $D$ is an $\mathcal{O}_{\mathcal{E}}$-module with additional actions of the Frobenius $\varphi$ and the group $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ which are commutative, satisfying the étale property: the map $\mathcal{O}_{\mathcal{E}} \otimes_\varphi D \to D$, $\lambda \otimes d \mapsto \lambda\varphi(d)$ is an isomorphism or equivalently

$$D \simeq \bigoplus_{\lambda \in \mathcal{O}_{\mathcal{E}}/\varphi(\mathcal{O}_{\mathcal{E}})} \lambda\varphi(D) = \bigoplus_{i=0}^{p-1} (1+X)^i \varphi(D).$$

Let $A_{\mathbb{Q}_p}$ be those elements $f \in \mathcal{O}_{\mathcal{E}}$ which have coefficients in $\mathbb{Z}_p$ (the ring of $p$-adic integers) and $A$ be the $p$-adic completion of the maximal unramified



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extension $A^{nr}_{\mathbb{Q}_p}$ of $A_{\mathbb{Q}_p}$. We have actions of $\varphi$ and $\Gamma$ on $A$. Let $\Gamma$ and $\chi : \Gamma \to \mathbb{Z}_p^*$ be the cyclotomic character with kernel $\mathcal{H}$.

The category equivalence of Fontaine is realized by these exact functors: For an étale $(\varphi, \Gamma)$-module $D$, $V(D) = (o \cdot A \otimes_{\mathcal{O}_\mathcal{E}} D)^{\varphi=1}$ is a Galois representation of $\mathbb{Q}_p$. For a Galois representation $V$, $D(V) = (A \otimes_{\mathbb{Z}_p} V)^{\mathcal{H}}$ is an étale $(\varphi, \Gamma)$-module.

One of Colmez's breakthroughs was that he managed to relate $p$-adic (and mod $p$) representations of $G^{(2)} = \mathbf{GL_2}(\mathbb{Q}_p)$ to $(\varphi, \Gamma)$-modules, too.

The so-called "Montréal-functor" $D$ associates to a smooth $o$-torsion representation $\pi$ of the standard Borel subgroup $B^{(2)}$ of $G^{(2)}$ a torsion $(\varphi, \Gamma)$-module over $\mathcal{O}_\mathcal{E}$. We can construct it in the following way:

Let $T^{(2)} \leq B^{(2)}$ be the maximal torus and $N_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ be a compact open subgroup of the unipotent radical of $B^{(2)}$, $T_+$ be the submonoid $\{t \in T | tN_0t^{-1} \subseteq N_0\}$ in $T$, and $B_+ = N_0T_+$.

Let $\Pi$ be a smooth (the action of $G^{(2)}$ is locally constant) $o$-representation of $G^{(2)}$ of finite length. For a certain (sufficiently small) generating $B_+$-subrepresentation $M$ of $\Pi$ (which is denoted by $I^{\Pi}_{\mathbb{Z}_p}(W)$ in [4]) $D(\Pi)$ is defined as the localization $M^\vee[1/X]$ of the Pontryagin dual of $M$. The functor $\Pi \mapsto D(\Pi)$ is contravariant and exact.

The way Colmez goes back to representations of $G^{(2)}$ requires the following construction.

Let $D$ be an étale $(\varphi, \Gamma)$-module over $\mathcal{E} = \mathcal{O}_\mathcal{E}[1/p]$. For all $d \in D$ there are unique $d_i \in D$ such that $d = \sum_{i=0}^{p-1}(1+X)^i\varphi(d_i)$. Set $\psi(d) = d_0$, thus $\psi$ is a left inverse of $\varphi$. With the help of that we can define a $\begin{pmatrix} \mathbb{Q}_p \setminus \{0\} & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}$-equivariant sheaf of $K$-vectorspaces over $\mathbb{Q}_p$, with global sections

$$D \boxtimes \mathbb{Q}_p = \{(d^{(n)})_{n\in\mathbb{N}} | \forall n : d^{(n)} \in D, \psi(d^{(n)}) = d^{(n-1)}\}$$

This can be done for the smallest compact $\psi$-invariant generating ${\mathcal{O}_\mathcal{E}}^+ = o[[X]]$-submodule $D^\natural \leq D$ as well.

After choosing a character $\delta \colon \mathbb{Q}_p^* \to o^*$ we can extend this sheaf to a $G^{(2)}$-equivariant sheaf $\mathfrak{Y} \colon U \mapsto D \boxtimes_\delta U$ ($U \subseteq \mathbb{P}^1$ open) of $K$-vectorspaces on the



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projective space $\mathbb{P}^1(\mathbb{Q}_p) \cong G^{(2)}/B^{(2)}$. This sheaf has the following properties: $(i)$ the centre of $G^{(2)}$ acts via $\delta$ on $D \boxtimes_\delta \mathbb{P}^1$; $(ii)$ we have $D \boxtimes_\delta \mathbb{Z}_p \cong D$ as a module over the monoid $\begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ (where we regard $\mathbb{Z}_p$ as an open subspace in $\mathbb{P}^1 = \mathbb{Q}_p \cup \{\infty\}$).

Whenever $D$ is 2-dimensional and $\delta$ is the character corresponding to the Galois representation of $\bigwedge^2 D$ via local class field theory, we set $\Pi(D) = D \boxtimes_\delta \mathbb{P}^1 / D^\natural \boxtimes_\delta \mathbb{P}^1$, where

$$D^\natural \boxtimes_\delta \mathbb{P}^1 = \{x \in D \boxtimes_\delta \mathbb{P}^1 | \mathrm{Res}_{\mathbb{Q}_p}(x) \in D^\natural \boxtimes_\delta \mathbb{Q}_p\}$$

is a $G$-invariant submodule of $D \boxtimes_\delta \mathbb{P}^1$. $\Pi(D)$ is an irreducible smooth representation of $G^{(2)}$.

We have $D(\Pi(D)) = \check{D}$, where $\check{D} = \mathrm{Hom}(D, \mathcal{E})$ is the dual $(\varphi, \Gamma)$-module. Moreover the $G$-representation of global sections $D \boxtimes_\delta \mathbb{P}^1$ admits a short exact sequence

$$0 \to \Pi(\check{D})^\vee \to D \boxtimes_\delta \mathbb{P}^1 \to \Pi(D) \to 0.$$

It also turns out, that this relation has the other required properties as well.

## 1.3 Generalized Montréal functors

By now there are more different approaches to generalize Colmez's functor $D$ to reductive groups $G$ other than $\mathbf{GL_2}(\mathbb{Q}_p)$. We briefly recall these generalized Montréal functors here.

The approach by Schneider and Vigneras [17] starts with the set $\mathcal{B}_+(\pi)$ of generating $B_+$-subrepresentations $W \leq \pi$. The Pontryagin dual $W^\vee = \mathrm{Hom}_o(W, K/o)$ of each $W$ admits a natural action of the inverse monoid $B_+^{-1}$. Moreover, the action of $N_0 \leq B_+^{-1}$ on $W^\vee$ extends to an action of the Iwasawa algebra $\Lambda(N_0) = o[[N_0]]$. For $W_1, W_2 \in \mathcal{B}_+(\pi)$ we also have $W_1 \cap W_2 \in \mathcal{B}_+(\pi)$ (Lemma 2.2 in [17]) therefore we may take the inductive limit $D_{SV}(\pi) = \varinjlim_{W \in \mathcal{B}_+(\pi)} W^\vee$. In [17] it is denoted by $D(\pi)$, however, in order to avoid confusion we denote it by $D_{SV}(\pi)$ (also note that the notation $V$ is used for the $o$-torsion representation that we denote by $\pi$). In general, $D_{SV}(\pi)$ does not have good properties: for instance it may not admit a



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canonical right inverse of the $T_+$-action making $D_{SV}(\pi)$ an étale $T_+$-module over $\Lambda(N_0)$. However, by taking a resolution of $\pi$ by compactly induced representations of $B$, one may consider the derived functors $D^i_{SV}$ of $D_{SV}$ for $i \geq 0$ producing étale $T_+$-modules $D^i_{SV}(\pi)$ over $\Lambda(N_0)$. Note that the functor $D_{SV}$ is neither left- nor right exact, but takes injective (resp. surjective) maps to surjective (resp. injective) maps. The fundamental open question of [17] whether the topological localizations $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D^i_{SV}(\pi)$ are finitely generated over $\Lambda_\ell(N_0)$ in case when $\pi$ comes as a restriction of a smooth admissible representation of $G$ of finite length. One can pass to usual 1-variable étale $(\varphi, \Gamma)$-modules—still not necessarily finitely generated—over $\mathcal{O}_{\mathcal{E}}$ via the map $\ell\colon \Lambda_\ell(N_0) \to \mathcal{O}_{\mathcal{E}}$ which step is an equivalence of categories for finitely generated étale $(\varphi, \Gamma)$-modules (Thm. 8.20 in [18]).

More recently, Breuil [3] managed to find a different approach, producing a pseudocompact (ie. projective limit of finitely generated) $(\varphi, \Gamma)$-module $D^\vee_\xi(\pi)$ over $\mathcal{O}_{\mathcal{E}}$ when $\pi$ is killed by a power $\varpi^h$ of the uniformizer $\varpi$. In [3] (and also in [17]) $\ell$ is a *generic* Whittaker functional, namely $\ell$ is chosen to be the composite map

$$\ell\colon N_0 \to N_0/(N_0 \cap [N, N]) \cong \prod_{\alpha \in \Delta} N_{\alpha,0} \xrightarrow{\sum_{\alpha\in\Delta} u_\alpha^{-1}} \mathbb{Z}_p \ .$$

Breuil passes right away to the space of $H_0$-invariants $\pi^{H_0}$ of $\pi$ where $H_0$ is the kernel of the group homomorphism $\ell\colon N_0 \to \mathbb{Z}_p$. By the assumption that $\pi$ is smooth, the invariant subspace $\pi^{H_0}$ has the structure of a module over the Iwasawa algebra $\Lambda(N_0/H_0)/\varpi^h \cong o/\varpi^h[[X]]$. Moreover, it admits a semilinear action of $F$ which is the Hecke action of $s = \xi(p)$: For any $m \in \pi^{H_0}$ we define

$$F(m) = \mathrm{Tr}_{H_0/sH_0s^{-1}}(sm) = \sum_{u \in J(H_0/sH_0s^{-1})} usm \ .$$

So $\pi^{H_0}$ is a module over the skew polynomial ring $\Lambda(N_0/H_0)/\varpi^h[F]$ (defined by the identity $FX = (sXs^{-1})F = ((X+1)^p - 1)F)$. We consider those $(i)$ finitely generated $\Lambda(N_0/H_0)/\varpi^h[F]$-submodules $M \subset \pi^{H_0}$ that are $(ii)$ invariant under the action of $\Gamma$ and are $(iii)$ *admissible* as a $\Lambda(N_0/H_0)/\varpi^h$-module, ie. the Pontryagin dual $M^\vee = \mathrm{Hom}_o(M, o/\varpi^h)$ is finitely generated over $\Lambda(N_0/H_0)/\varpi^h$. Note that this admissibility condition $(iii)$ is equivalent to the usual admissibility condition in smooth representation theory, ie. that



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for any (or equivalently for a single) open subgroup $N' \leq N_0/H_0$ the fixed points $M^{N'}$ form a finitely generated module over $o$. We denote by $\mathcal{M}(\pi^{H_0})$ the—via inclusion partially ordered—set of those submodules $M \leq \pi^{H_0}$ satisfying $(i), (ii), (iii)$. Note that whenever $M_1, M_2$ are in $\mathcal{M}(\pi^{H_0})$ then so is $M_1 + M_2$. It is shown in [4] (see also [6] and Lemma 2.6 in [3]) that for $M \in \mathcal{M}(\pi^{H_0})$ the localized Pontryagin dual $M^\vee[1/X]$ naturally admits a structure of an étale $(\varphi, \Gamma)$-module over $o/\varpi^h((X))$. Therefore Breuil [3] defines

$$D^\vee_\xi(\pi) = \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} M^\vee[1/X] \ .$$

By construction this is a projective limit of usual $(\varphi, \Gamma)$-modules. Moreover, $D^\vee_\xi$ is right exact and compatible with parabolic induction [3]. It can be characterized by the following universal property: For any (finitely generated) étale $(\varphi, \Gamma)$-module over $o/\varpi^h((X)) \cong o/\varpi^h[[\mathbb{Z}_p]][([1]-1)^{-1}]$ (here $[1]$ is the image of the topological generator of $\mathbb{Z}_p$ in the Iwasawa algebra $o/\varpi^h[[\mathbb{Z}_p]]$) we may consider continuous $\Lambda(N_0)$-homomorphisms $\pi^\vee \to D$ via the map $\ell\colon N_0 \to \mathbb{Z}_p$ (in the weak topology of $D$ and the compact topology of $\pi^\vee$). These all factor through $(\pi^\vee)_{H_0} \cong (\pi^{H_0})^\vee$. So we may require these maps be $\psi_s$- and $\Gamma$-equivariant where $\Gamma = \xi(\mathbb{Z}_p \setminus \{0\})$ acts naturally on $(\pi^{H_0})^\vee$ and $\psi_s\colon (\pi^{H_0})^\vee \to (\pi^{H_0})^\vee$ is the dual of the Hecke-action $F\colon \pi^{H_0} \to \pi^{H_0}$ of $s$ on $\pi^{H_0}$. Any such continuous $\psi_s$- and $\Gamma$-equivariant map $f$ factors uniquely through $D^\vee_\xi(\pi)$. However, it is not known in general whether $D^\vee_\xi(\pi)$ is nonzero for smooth irreducible representations $\pi$ of $G$ (restricted to $B$).

Even more recently Scholze and Grosse-Klönne proposed different methods, which are just mentioned here. For $G = \mathbf{GL_n}(F)$ Scholze ([20]) uses a finiteness result of the $p$-adic cohomology of the Lubin-Tate tower to get a representation of the Galois group $\mathrm{Gal}_F$, he also gets an additional action of a central division algebra $D/F$. Grosse-Klönne ([14]) uses the $G$-equivariant coefficient system on the Bruhat Tits building attached to $\pi$ with some additional information to construct a functor of this type, which is also exact and for $\mathbf{GL_2}(\mathbb{Q}_p)$ is the same as the classical functor $D$.

## 1.4 Summary of results

The thesis is mostly based on the papers [9] and [10].

In chapter 2 we compute $D_{SV}$ for principal series representations of $G = \mathbf{GL_n}(F)$.



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In order to that, we need to understand the $B_+$-module structure of the principal series. In section 2.2 we decompose $G$ into open $N_0$-invariant subsets $U_w$, indexed by the elements $w$ of Weyl group. The action of $B_+$ respects this structure in the following sense: if $w, w' \in W$, $y \in U_w$ and $b \in B_+$ such that $b^{-1}y \in U_{w'}$, then $w' \preceq w$ for certain ordering on $W$.

With the help of this we prove in section 2.3 that there exists a minimal element $M_0$ in the set of generating $B_+$-subrepresentations of $\pi$: namely the $B_+$-submodules generated by the "characteristic functions" of the sets $U_w w$ for $w$ in $W$.

Now we have $D_{SV}(\pi) = M_0^\vee$ - the dual of this minimal $B_+$-subrepresentation. We do not know whether it is finitely generated or it has rank 1 as a module over the modulo $p$ Iwasawa algebra $\Omega(N_0)$. However, we show that in some sense only a rank 1 quotient of $D_{SV}(\pi)$ is relevant if we want to get an étale $(\varphi, \Gamma)$-module.

In the last section we point out some properties of $M_0$, which sheds some light on why the picture for principal series is more difficult compared to the case of subquotients defined by the Bruhat filtration.

In chapter 3 we relate the functors $D_{SV}$ and $D_\xi^\vee$.

Our first result is the construction of a noncommutative multivariable version of $D_\xi^\vee(\pi)$. Let $\pi$ be a smooth $o$-torsion representation of $B$ such that $\varpi^h\pi = 0$. The idea here is to take the invariants $\pi^{H_k}$ for a family of open normal subgroups $H_k \leq H_0$ with $\bigcap_{k\geq 0} H_k = \{1\}$. Now $\Gamma$ and the quotient group $N_0/H_k$ act on $\pi^{H_k}$ (we choose $H_k$ so that it is normalized by both $\Gamma$ and $N_0$). Further, we have a Hecke-action of $s$ given by $F_k = \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot)$. As in [3] we consider the set $\mathcal{M}_k(\pi^{H_k})$ of finitely generated $\Lambda(N_0/H_k)[F_k]$-submodules of $\pi^{H_k}$ that are stable under the action of $\Gamma$ and admissible as a representation of $N_0/H_k$. In section 3.1 we show that for any $M_k \in \mathcal{M}_k(\pi^{H_k})$ there is an étale $(\varphi, \Gamma)$-module structure on $M_k^\vee[1/X]$ over the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$. So the projective limit

$$D_{\xi,\ell,\infty}^\vee(\pi) = \varprojlim_{k\geq 0} \varprojlim_{M_k \in \mathcal{M}_k(\pi^{H_k})} M_k^\vee[1/X]$$

is a pseudocompact étale $(\varphi, \Gamma)$-module over $\Lambda_\ell(N_0)/\varpi^h = \varprojlim_k \Lambda(N_0/H_k)/\varpi^h[1/X]$. Moreover, we also give a natural isomorphism $D_{\xi,\ell,\infty}^\vee(\pi)_{H_0} \cong D_\xi^\vee(\pi)$ showing that $D_{\xi,\ell,\infty}^\vee(\pi)$ corresponds to $D_\xi^\vee(\pi)$ via (the



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projective limit of) the equivalence of categories in Thm. 8.20 in [18]. Moreover, the natural map $\pi^\vee \to D_\xi^\vee(\pi)$ factors through the projection map $D^\vee_{\xi,\ell,\infty}(\pi) \twoheadrightarrow D^\vee_\xi(\pi) = D^\vee_{\xi,\ell,\infty}(\pi)_{H_0}$. Note that this shows that $D^\vee_{\xi,\ell,\infty}(\pi)$ is naturally attached to $\pi$—not just simply via the equivalence of categories (loc. cit.)—in the sense that any $\psi$- and $\Gamma$-equivariant map from $\pi^\vee$ to an étale $(\varphi,\Gamma)$-module over $o/\varpi^h((X))$ factors uniquely through the corresponding multivariable $(\varphi,\Gamma)$-module.

In section 3.2 we develop these ideas further and show that the natural map $\pi^\vee \to D^\vee_{\xi,\ell,\infty}(\pi)$ factors through the map $\pi^\vee \to D_{SV}(\pi)$. In fact, we show (Prop. 3.2.4) that $D^\vee_{\xi,\ell,\infty}(\pi)$ has the following universal property: Any continuous $\psi_s$- and $\Gamma$-equivariant map $f\colon D_{SV} \to D$ into a finitely generated étale $(\varphi,\Gamma)$-module $D$ over $\Lambda_\ell(N_0)$ factors uniquely through $\mathrm{pr} = \mathrm{pr}_\pi\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$. The association $\pi \mapsto \mathrm{pr}_\pi$ is a natural transformation between the functors $D_{SV}$ and $D^\vee_{\xi,\ell,\infty}$. One application is that Breuil's functor $D^\vee_\xi$ vanishes on compactly induced representations of $B$ (see Corollary 3.2.3).

In order to be able to compute $D^\vee_{\xi,\ell,\infty}(\pi)$ (hence also $D^\vee_\xi(\pi)$) from $D_{SV}(\pi)$ we introduce the notion of the *étale hull* of a $\Lambda(N_0)$-module with a $\psi$-action of $T_+$ (or of a submonoid $T_* \leq T_+$). Here a $\Lambda(N_0)$-module $D$ with a $\psi$-action of $T_+$ is the analogue of a $(\psi,\Gamma)$-module over $o[[X]]$ in this multivariable noncommutative setting. The étale hull $\widetilde{D}$ of $D$ (together with a canonical map $\iota\colon D \to \widetilde{D}$) is characterized by the universal property that any $\psi$-equivariant map $f\colon D \to D'$ into an étale $T_+$-module $D'$ over $\Lambda(N_0)$ factors uniquely through $\iota$. It can be constructed as a direct limit $\varinjlim_{t\in T_+} \varphi_t^* D$ where $\varphi_t^* D = \Lambda(N_0) \otimes_{\varphi_t,\Lambda(N_0)} D$ (Prop. 3.3.4). We show (Thm. 3.3.9 and the remark thereafter) that the pseudocompact completion of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ is canonically isomorphic to $D^\vee_{\xi,\ell,\infty}(\pi)$ as they have the same universal property.

In order to go back to representations of $G$ we need an étale action of $T_+$ on $D^\vee_{\xi,\ell,\infty}(\pi)$, not just of $\xi(\mathbb{Z}_p \setminus \{0\})$. This is only possible if $tH_0t^{-1} \leq H_0$ for all $t \in T_+$ which is not the case for generic $\ell$. So in the last chapter we equip $D^\vee_{\xi,\ell,\infty}(\pi)$ with an étale action of $T_+$ (extending that of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$) in case $\ell = \ell_\alpha$ is the projection of $N_0$ onto a root subgroup $N_{\alpha,0} \cong \mathbb{Z}_p$ for some simple root $\alpha$ in $\Delta$. Moreover, we show (Prop. 4.1.5) that the map $\mathrm{pr}\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is $\psi$-equivariant for this extended action, too. Note that $D^\vee_{\xi,\ell,\infty}(\pi)$ may not be the projective limit of finitely generated étale

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$T_+$-modules over $\Lambda_\ell(N_0)$ as we do not necessarily have an action of $T_+$ on $M^\vee_\infty[1/X]$ for $M \in \mathcal{M}(\pi^{H_0})$, only on the projective limit.

Let $P \leq G$ be a parabolic subgroup with Levi decomposition $P = L_P N_P$. We show in section 4.2 that the compatibility with parabolic induction [3] Theorem 6.1 goes through in this situation:

$$D^\vee_\xi\left(\mathrm{Ind}^G_{P^-}\pi_P\right) \cong \begin{cases} D^\vee_\xi(\pi_P) & \text{if } N_\alpha \subseteq L_P \\ o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee & \text{if } N_\alpha \subseteq N_P \end{cases},$$

where Ord is the ordinary part similar to the definition of Emerton (cf Definition 3.1.9 in [7]).

We present the results of section 4 in [10], where a $G$-equivariant sheaf $\mathfrak{Y}$ on $G/B$ is attached to $D^\vee_{\xi,\ell,\infty}(\pi)$ and a natural transformation $\beta_{G/B}$ from $(\cdot)^\vee$ to $\pi \to \mathfrak{Y}$ is constructed, which is compatible with a reverse functor.

In section 4.4 we show some disadvantages of the choice $\ell = \ell_\alpha$: $D^\vee_\xi$ vanishes for the twist of a modulo $p$ supercuspidal representation $\pi^{(2)}$ of $\mathbf{GL_2}(\mathbb{Q}_p)$ by a character $\chi$. Moreover $D^\vee_\xi$ is not exact even for extensions of principal series $\pi_P = \pi^{(2)} \otimes \chi$.

The mostly folklore computation with $(\varphi, \Gamma)$-modules which is needed for the latter result is carried out in section 4.5.

## 1.5 Notations

Let $F, K \leq \overline{\mathbb{Q}}_p$ finite extensions of $\mathbb{Q}_p$. Let $o_F$, respectively $o_K$ be the rings of integers in $F$, respectively in $K$, $\varpi_F \in o_F$ and $\varpi_K \in o_K$ be the uniformizers, $\nu_F$ and $\nu_K$ be the standard valuations and $k_F = o_F/\varpi_F o_F$, $k_K = o_K/\varpi_K o_K$ be the residue fields.

Let $G = \mathbf{G}(F)$ be the $F$-points of a $F$-split connected reductive group $\mathbf{G}$ defined over $\mathbb{Z}_p$ with connected centre and a fixed split Borel subgroup $\mathbf{B} = \mathbf{TN}$. Put $B = \mathbf{B}(F)$, $T = \mathbf{T}(F)$, and $N = \mathbf{N}(F)$. We denote by $\Phi_+$ the set of roots of $T$ in $N$, by $\Delta \subset \Phi_+$ the set of simple roots, and by $u_\alpha : \mathbb{G}_a \to N_\alpha$, for $\alpha \in \Phi_+$, a $F$-homomorphism onto the root subgroup $N_\alpha$ of $N$ such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $x \in F$ and $t \in \mathbf{T}(\mathbb{Q}_p)$, and $N_0 = \prod_{\alpha\in\Phi_+} u_\alpha(o_F)$ is a subgroup of $N$. We put $N_{\alpha,0} = u_\alpha(o_F)$ for the image of $u_\alpha$ on $o_F$.



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Let $W = N_G(T)/Z_G(T)$ denote the Weyl group of $G$ and $\prec$ denote the strong Bruhat ordering of $W$ (see [15] II. 13.7): we say $w' \prec w$ for $w \neq w' \in W$ if there exist transpositions $w_1, w_2, \dots, w_i \in W$ such that $w' = ww_1w_2 \dots w_i$ and $l(w) > l(ww_1) > l(ww_1w_2) > \dots > l(ww_1w_2 \dots w_i)$.

We denote by $T_+$ the monoid of dominant elements $t$ in $\mathbf{T}(\mathbb{Q}_p)$ such that $\nu_F(\alpha(t)) \geq 0$ for all $\alpha \in \Phi_+$, by $T_0 \subset T_+$ the maximal subgroup, by $T_{++}$ the subset of strictly dominant elements, i.e. $\nu_F(\alpha(t)) > 0$ for all $\alpha \in \Phi_+$, and we put $B_+ = N_0T_+, B_0 = N_0T_0$. The natural conjugation action of $T_+$ on $N_0$ extends to an action on the Iwasawa $o_K$-algebra $\Lambda(N_0) = o_K[[N_0]]$. For $t \in T_+$ we denote this action of $t$ on $\Lambda(N_0)$ by $\varphi_t$. The map $\varphi_t\colon \Lambda(N_0) \to \Lambda(N_0)$ is an injective ring homomorphism with a distinguished left inverse $\psi_t\colon \Lambda(N_0) \to \Lambda(N_0)$ satisfying $\psi_t \circ \varphi_t = \mathrm{id}_{\Lambda(N_0)}$ and $\psi_t(u\varphi_t(\lambda)) = \psi_t(\varphi_t(\lambda)u) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$ and $\lambda \in \Lambda(N_0)$.

Each simple root $\alpha$ gives a $F$-homomorphism $x_\alpha : N \to \mathbb{G}_a$ with section $u_\alpha$. We denote by $\ell_\alpha : N_0 \to F \overset{\mathrm{Tr}_{F/\mathbb{Q}_p}}{\to} \mathbb{Z}_p$, resp. $\iota_\alpha\colon o_F \to N_0$, the restriction of $x_\alpha$, resp. $u_\alpha$, to $N_0$, resp. $o_F$.

Since the centre of $G$ is assumed to be connected, there exists a cocharacter $\xi\colon F^* \to T$ such that $\alpha \circ \xi$ is the identity on $F^*$ for each $\alpha \in \Delta$. If $F = \mathbb{Q}_p$ we put $\Gamma = \xi(\mathbb{Z}_p^*) \leq T$ and often denote the action of $s = \xi(p)$ by $\varphi = \varphi_s$.

For an $o_K$-representation $\pi$ let $\pi^\vee = \mathrm{Hom}_{o_K}(\pi, K/o_K)$ be the Pontryagin dual of $\pi$. Pontryagin duality sets up an anti-equivalence between the category of torsion $o_K$-modules and the category of all compact linear-topological $o_K$-modules.

By a smooth $o_K$-torsion representation of $G$ (resp. of $B = \mathbf{B}(F)$) we mean a torsion $o_K$-module $\pi$ together with a smooth (ie. stabilizers are open) and linear action of the group $G$ (resp. of $B$). $\pi$ is admissible if for any $U \leq G$ open subgroup, the vector space $k_K \otimes_{o_K} \pi^U$ is finite dimensional.

For example, if $\mathbf{G} = \mathbf{GL_n}$ and $F = \mathbb{Q}_p$, $B$ is the subgroup of upper triangular matrices, $N$ consists of the strictly upper triangular matrices (1 on the diagonal), $T$ is the diagonal subgroup, $N_0 = \mathbf{N}(\mathbb{Z}_p)$, the simple roots are $\alpha_1, \dots, \alpha_{n-1}$ where $\alpha_i(\mathrm{diag}(t_1, \dots, t_n)) = t_it_{i+1}^{-1}$, $x_{\alpha_i}$ sends a matrix to its $(i, i+1)$-coefficient, $u_{\alpha_i}(\cdot)$ is the strictly upper triangular matrix, with $(i, i+1)$-coefficient $\cdot$ and 0 everywhere else.



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Let $C^\infty(G)$ (respectively $C_c^\infty(G)$) denote the set of locally constant $G \to k_K$ functions (respectively locally constant functions with compact support), with the group $G$ acting by left multiplication ($gf : x \mapsto f(g^{-1}x)$ for $f \in C^\infty(G)$ and $g, x \in G$).

Let $G_0 \leq G$ be a compact open subgroup and $\Lambda(G_0)$ denote the completed group ring of the profinite group $G_0$ over $o_K$. Any smooth $o_K$-representation $\pi$ is the union of its finite $G_0$-subrepresentations, therefore $\pi^\vee$ is a left $\Lambda(G_0)$-module (through the inversion map on $G_0$).

Let $\Omega(G_0) = \Lambda(G_0)/\varpi_K\Lambda(G_0)$. $\Omega(N_0)$ is noetherian and has no zero divisors, so it has a fraction (skew) field. If $M$ is a $\Omega(N_0)$-module, by the rank of $M$ we mean $\dim_{k_K}(\mathrm{Frac}(\Omega(N_0)) \otimes_{\Omega(N_0)} M)$.

Let $\ell\colon N_0 \to \mathbb{Z}_p$ (for now) any surjective group homomorphism and denote by $H_0 \lhd N_0$ the kernel of $\ell$. The ring $\Lambda_\ell(N_0)$, denoted by $\Lambda_{H_0}(N_0)$ in [17], is a generalisation of the ring $\mathcal{O}_\mathcal{E}$, which corresponds to $\Lambda_{\mathrm{id}}(N_0^{(2)})$ where $N_0^{(2)}$ is the $\mathbb{Z}_p$-points of the unipotent radical of a split Borel subgroup in $\mathbf{GL_2}$. We refer the reader to [17] for the proofs of some of the following claims.

The maximal ideal $\mathcal{M}(H_0)$ of the completed group $o_K$-algebra $\Lambda(H_0) = o_K[[H_0]]$ is generated by $\varpi_k$ and by the kernel of the augmentation map $o_K[[H_0]] \to o_K$.

The ring $\Lambda_\ell(N_0)$ is the $\mathcal{M}(H_0)$-adic completion of the localization of $\Lambda(N_0)$ with respect to the Ore subset $S_\ell(N_0)$ of elements which are not in the ideal $\mathcal{M}(H_0)\Lambda(N_0)$. The ring $\Lambda(N_0)$ can be viewed as the ring $\Lambda(H_0)[[X]]$ of skew Taylor series over $\Lambda(H_0)$ in the variable $X = [u] - 1$ where $u \in N_0$ and $\ell(u)$ is a topological generator of $\ell(N_0) = \mathbb{Z}_p$. Then $\Lambda_\ell(N_0)$ is viewed as the ring of infinite skew Laurent series $\sum_{n\in\mathbb{Z}} a_n X^n$ over $\Lambda(H_0)$ in the variable $X$ with $\lim_{n\to-\infty} a_n = 0$ for the compact topology of $\Lambda(H_0)$. For a different characterization of this ring in terms of a projective limit $\Lambda_\ell(N_0) \cong \varprojlim_{n,k} \Lambda(N_0/H_k)[1/X]/\varpi_K^n$ for $H_k \lhd N_0$ normal subgroups contained and open in $H_0$ satisfying $\bigcap_{k\geq 0} H_k = \{1\}$ see also [23].

For a finite index subgroup $\mathcal{G}_2$ in a group $\mathcal{G}_1$ we denote by $J(\mathcal{G}_1/\mathcal{G}_2) \subset \mathcal{G}_1$ a (fixed) set of representatives of the left cosets in $\mathcal{G}_1/\mathcal{G}_2$.

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# Chapter 2

# The Schneider-Vigneras functor for principal series

## 2.1 Principal series

In this chapter fix $n \in \mathbb{N}$, and let $G = \mathbf{GL_n}(\mathbf{F})$, and $G_0 = \mathbf{GL_n}(o_F)$.

Let $B$ be the set of upper triangular matrices in $G$, $T$ the set of diagonal matrices, $N$ the set of upper triangular unipotent matrices. Let $N^-$ be the lower unipotent matrices - the opposite of $N$ - and $N_0 = N \cap G_0$ - a totally decomposed compact open subgroup of $N$ - those matrices wich has coefficients in $o_F$.

By the abuse of notation let $w \in W$ denote also the permutation matrices - representatives of $W$ in $G$ (with $w_{ij} = 1$ if $w(j) = i$, and $w_{ij} = 0$ otherwise), and also the corresponding permutation of the set $\{1, 2, \ldots, n\}$. For $w \in W$ denote length of $w$—the length of the shortest word representing $w$ in the terms of the standard generators of $W$—by $l(w)$.

Let the kernel of the projection $pr : G_0 \to \mathbf{GL_n}(k_F)$ be $U^{(1)}$. This is a compact open pro-$p$ normal subgroup of $G_0$. We have $G = G_0 B$ and $U^{(1)} \subset (N^- \cap U^{(1)})B$.

Let

$$\chi = \chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n : T \to k_K^*$$

be a locally constant character of $T$ with $\chi_i : F^* \to k_K^*$ multiplicative. Note that for all $i$ we have $\chi_i(1 + \pi_F o_F) = 1$ and $\chi_i(o_F^*) \subset k_F^* \cap k_K^* \leq \overline{\mathbb{F}_p}^*$. Since $T \simeq B/[B, B]$, also denote the correspondig $B \to k_K^*$ character by $\chi$. Let

$$\pi = \mathrm{Ind}_B^G(\chi) = \{f \in C^\infty(G) | \forall g \in G, b \in B : f(gb) = \chi^{-1}(b) f(g)\}$$



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$\pi$ is called a principal series representation of $G$. $\pi$ is irreducible exactly when for all $i$ we have $\chi_i \neq \chi_{i+1}$ ([16], theorem 4). For any open right $B$-invariant subset $X \subset G$ we write $\mathrm{Ind}_B^X = \{f \in \mathrm{Ind}_B^G(\chi) | f|_{G\setminus X} \equiv 0\}$.

We can understand the stucture of $\pi$ better (see [21], section 4.), by the Bruhat decomposition $G = \bigcup_{w\in W} BwB$. Fix a total ordering $\prec_T$ refining the Bruhat ordering $\prec$ of $W$, and let

$$w_1 = \mathrm{id}_W \prec_T w_2 \prec_T w_3 \prec_T \cdots \prec_T w_{n!} = w_0.$$

Let us denote by $G_m = \bigcup_{1\leq l\leq m} Bw_lB$ - a closed subset of $G$. We obtain a descending $B$-invariant filtration of $\pi$ by

$$\pi_m = \mathrm{Ind}_B^{G\setminus G_m}(\chi) = \{F \in \mathrm{Ind}_B^G(\chi) | F|_{G_m} \equiv 0\} \qquad (0 < m \leq n!),$$

with quotients $\pi_{m-1}/\pi_m$ via $f \mapsto f(\cdot w_m)$ isomorphic to $\pi(w_m, \chi) = C_c^\infty(N/N'_{w_m})$ (see [17], section 12), where $N'_{w_m} = N \cap w_m N w_m^{-1}$, with $N$ acting by left translations and $T$ acting via

$$(t\phi)(n) = \chi(w_m^{-1} t w_m)\phi(t^{-1}nt).$$

For any $w \in W$ put

$$N_w = \{n \in N | \forall i < j, w^{-1}(i) < w^{-1}(j) : n_{ij} = 0\} = N \cap wN^-w^{-1} \leq N,$$

and $N_{0,w} = N_0 \cap N_w$. Then we have the following form of the Bruhat decomposition $G = \coprod_{w\in W} N_w wB$.

## 2.2 The action of $B_+$ on $G$

The first goal is to partition $G$ to $N_0$-invariant open subsets $\{U_w | w \in W\}$ indexed by the Weyl-group, which are respected by the $B_+$-action in the sense that if $x \in U_w$ $b \in B_+$ then there exists $w' \preceq w$ in $W$ such that $b^{-1}x \in U_{w'}$.

**Definition** Let for any $w \in W$ $r_w : N^- \cap G_0 \to \mathbf{G}(k_F), n^- \mapsto pr(wn^-w^{-1})$, $R_w = wr_w^{-1}(\mathbf{N}(k_F))$, $R = \cup_{w\in W} R_w$.

We have that

$$R_w = \left\{ (a_{ij}) \in G | \forall i,j : a_{ij} \begin{cases} = 1, & \text{if } w^{-1}(i) = j \\ = 0, & \text{if } w^{-1}(i) < j \\ \in o_F, & \text{if } w^{-1}(i) > j \text{ and } w(j) > i \\ \in \varpi_F o_F, & \text{if } w^{-1}(i) > j \text{ and } w(j) < i \end{cases} \right\}$$



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For $n = 3$ in details (with $o = o_F$ and $\varpi = \varpi_F$):

| $w$ | $R_w$ | $w$ | $R_w$ |
|---|---|---|---|
| $\mathrm{id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 \\ \varpi o & 1 & 0 \\ \varpi o & \varpi o & 1 \end{pmatrix}$ | $(23) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 \\ \varpi o & o & 1 \\ \varpi o & 1 & 0 \end{pmatrix}$ |
| $(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} o & 1 & 0 \\ 1 & 0 & 0 \\ \varpi o & \varpi o & 1 \end{pmatrix}$ | $(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} o & o & 1 \\ 1 & 0 & 0 \\ \varpi o & 1 & 0 \end{pmatrix}$ |
| $(132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} o & 1 & 0 \\ o & \varpi o & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | $(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} o & o & 1 \\ o & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ |

Let $\mathbf{N}(k_F)$ be the $k_F$-points of $\mathbf{N}$ (the upper triangular unipotent matrices with coefficients in $k_F$). $k_F$ has canonical (multiplicative) injection to $o_F \subset F$, hence any subgroup $\mathbf{H}(k_F) \leq \mathbf{N}(k_F)$ is mapped injectively to $N_0$ (however this is not a group homomorphism). We denote this subset of $N_0$ by $\widetilde{\mathbf{H}(k_F)}$.

**Proposition 2.2.1** *A set of double coset representatives of $U^{(1)} \setminus G/B$ is $\cup_{w \in W} \widetilde{\mathbf{N_w}(k_F)} w$. Every element of $G$ can be written uniquely in the form $rb$ with $r \in R$ and $b \in B$.*

**Proof** By the Bruhat decomposition of $\mathbf{G}(k_F)$ a set of double coset representatives of $U^{(1)} \setminus G_0/(B \cap G_0)$ is the set as above. Since $G = G_0 B$, we have the first part of proposition.

Let $g = unwb \in G$ with $u \in U^{(1)}$, $w \in W$, $n \in \widetilde{\mathbf{N_w}(k_F)}$ and $b \in B$. Then $g = w(w^{-1}nw)u'b$ with $u' = w^{-1}n^{-1}unw \in U^{(1)}$. But then there exist $n' \in N^- \cap U^{(1)}$ and $b' \in B$ such that $u' = n'b'$. Then $g = w(w^{-1}nwn')(b'b)$, where $w^{-1}nwn' \in r_w^{-1}(\mathbf{N}(k_F))$ because of the definition of $N_w$.

For any $w \in W$ we clearly have $U^{(1)}\widetilde{\mathbf{N_w}(k_F)}wB = R_w B$. Hence the uniqueness follows: if $rb = r'b'$ then there exists $w \in W$ such that $r, r' \in R_w$ and $b'b^{-1} = (r'^{-1}w^{-1})(wr) \in B \cap N^- = \{\mathrm{id}\}$. □

**Definition** For any $w \in W$ let $U_w = U^{(1)}\widetilde{\mathbf{N_w}(k_F)}wB$. This way we partitioned $G$ into open subsets indexed by the Weyl group. We obviously have $U_w = R_w B$.

**Corollary 2.2.2** *For any $w \in W$ we have that $U_w$ is (left) $N_0$-invariant.*



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**Proof** Let $n' \in N_0$ and $x = unwb \in U^{(1)}\widetilde{\mathbf{N_w}(k_F)}wB$. We have $N_0 = N_{0,w}(N'_w \cap N_0)$, thus $n'n = mm'$ for some $m \in N_{0,w}$ and $m' \in N'_w \cap N_0$, moreover we can write $m = m_1 m_0 \in (N_w \cap U^{(1)})\widetilde{\mathbf{N_w}(k_F)}$. By the definition of $N'_w$

$$n'x = (n'un'^{-1}m_1)m_0 w(w^{-1}m'wb) \in U^{(1)}\widetilde{\mathbf{N_w}(k_F)}wB,$$

meaning that $U_w$ is $N_0$-invariant. $\square$

**Proposition 2.2.3** *Let $y \in U_w = R_w B$, $nt \in B_+ = N_0 T_+$, and $x = t^{-1}n^{-1}y \in U_{w'} = R_{w'}B$. Then $w' \preceq w$.*

**Proof** Let $y = rb$ with $r \in R_w$ and $b \in B$. By the previous proposition we may assume that $n = \mathrm{id}$. If $t = \mathrm{diag}(t_1, t_2, \ldots, t_n) \in G_0$, then

$$x = w(w^{-1}t^{-1}w(w^{-1}r)w^{-1}tw)(w^{-1}t^{-1}wb),$$

where $w^{-1}t^{-1}w(w^{-1}r)w^{-1}tw \in r_w^{-1}(\mathbf{N}(k_F))$, because it is in $N^-$ and the coefficients under the diagonal have the same valuation as those in $w^{-1}r$. $T_+$ as a monoid is generated by $T \cap G_0$, the center $Z(G)$ and the elements with the form $(\varpi_F, \varpi_F, \ldots, \varpi_F, 1, 1, \ldots, 1)$, hence it is enough to prove the proposition for such $t$-s.

So fix $t = (t_1 = \varpi_F, t_2 = \varpi_F, \ldots, t_l = \varpi_F, t_{l+1} = 1, t_{l+2} = 1, \ldots, t_n = 1)$, $r = (r_{ij})$ and try to write $x$ in the form as in Proposition 2.2.1. For all $j = 0, 1, 2, \ldots, n$ we construct inductively a decomposition $x = (t^{(j)})^{-1}r^{(j)}b^{(j)}$ together with $w^{(j)} \in W$, where

- $w^{(j+1)} \preceq w^{(j)}$ for $j < n$ and such that the first $j$ columns of $w^{(j)}$ are the same as the first $j$ columns of $w^{(j+1)}$,

- $t^{(j)} = \mathrm{diag}(t_i^{(j)}) \in T$ with

$$t_i^{(j)} = \begin{cases} 1, & \text{if } (w^{(j)})^{-1}(i) \leq j \\ t_i, & \text{if } (w^{(j)})^{-1}(i) > j \end{cases},$$

- $r^{(j)} \in R_{w^{(j)}}$, and if we change the first $j$ columns of $r^{(j)}$ to the first $j$ columns of $(t^{(j)})^{-1}r^{(j)}$ it is still in $R_{w^{(j)}}$ (by de definition of $t^{(j)}$ it is enough to verify the condition for $(t^{(j)})^{-1}r^{(j)}$),

- $b^{(j)} \in B$.



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Then $w^{(n)} \preceq w^{(n-1)} \preceq w^{(n-2)} \preceq \cdots \preceq w^{(1)} = w$. However for $j = n$ we have $t^{(n)} = \mathrm{id}$, hence $w^{(n)} = w'$ by disjointness of the sets $R_v B$ for $v \in W$, so we have the proposition.

For $j = 0$ we have $t^{(0)} = t, r^{(0)} = r, b^{(0)} = b$ and $w^{(0)} = w$. From $j$ to $j+1$:

- If $w^{(j)}(j+1) \leq l$, then let $w^{(j+1)} = w^{(j)}$, so $t^{(j+1)} = e^{-1}_{w^{(j)}(j+1)} t^{(j)}$, where for $1 \leq k \leq n$ we denote $e_k = e_k(\varpi_F)$ the diagonal matrix with $\varpi_F$ in the $k$-th row and 1 everywhere else. We can choose $r^{(j+1)} = e^{-1}_{w^{(j)}(j+1)} r^{(j)} e_{j+1}$, and $b^{(j+1)} = e^{-1}_{j+1} b^{(j)}$.

  Then the first $j$ columns of $(t^{(j+1)})^{-1} r^{(j+1)}$ are equal of those of $(t^{(j)})^{-1} r^{(j)}$, and the entries at place $(i, j+1)$ with $i \neq w^{(j+1)}(j+1)$ are multiplied by $\varpi_F$. Because of the conditions for $r^{(j)}$, this is in $R_{w^{(j+1)}}$. The other conditions for $w^{(j+1)}, t^{(j+1)}, r^{(j+1)}$ and $b^{(j+1)}$ obviously hold.

- If $w^{(j)}(j+1) > l$ and if $\nu_F(r^{(j)}_{i,j+1}) \geq 1$ for all $i \leq l$, then it suffices to choose $w^{(j+1)} = w^{(j)}, t^{(j+1)} = t^{(j)}, r^{(j+1)} = r^{(j)}$ and $b^{(j+1)} = b^{(j)}$.

- Assume that $w^{(j)}(j+1) > l$ and that there exists $i \leq l$ such that $\nu_F(r^{(j)}_{i,j+1}) = 0$. Let $i_0$ be the maximal such $i$. Then choose $w^{(j+1)}(j+1) = i_0$, and $t^{(j+1)} = e^{-1}_{i_0} t^{(j)}$.

  Let $r' = e^{-1}_{i_0} r^{(j)} e_{j+1}((r^{(j)}_{i_0,j+1})^{-1} \cdot \varpi)$, where $e_j(\alpha)$ is the diagonal matrix with $\alpha \in F$ in the $j$-th row and 1 everywhere else. Note that $r'_{i_0,j+1} = 1$ and $r'$ differs from $r^{(j)}$ only in the $i_0$-th row and the $j+1$-st column. But $(t^{(j+1)})^{-1} r'$ is not in $\mathbf{GL_n}(o_F)$ - for example $\nu_F(r'_{i_0,(w^{(j)})^{-1}(i_0)}) = -1$, and there might be some other elements of $r'$ in the $i_0$-th row and columns between the $j+2$-nd and $j' = (w^{(j)})^{-1}(i_0)$-th.

  To see this note first that $w^{(j)}(j+1) > l \geq i_0$, so $(w^{(j)})^{-1}(i_0) \neq j+1$. In particular the right multiplication with $e_{j+1}$ does not change the entry at place $(i_0, (w^{(j)})^{-1}(i_0))$. Since $r^{(j)} \in R_{w^{(j)}}$, the defining conditions of $R_{w^{(j)}}$ and that $(w^{(j)})^{-1}(i_0) \neq j+1$ imply $(w^{(j)})^{-1}(i_0) > j+1$. Thus $(t^{(j)}_{i_0})^{-1} = (t_{i_0})^{-1} = \varpi_F^{-1}$, since $i_0 \leq l$. By the definition of $R_{w^{(j)}}$ we have $r^{(j)}_{i_0,(w^{(j)})^{-1}(i_0)} = 1$ . Therefore $r'_{i_0,(w^{(j)})^{-1}(i_0)} = \varpi_F^{-1}$ which has valuation -1.

  But note, that in the $j+1$-st column of $r'$ the $i_0$-th element is 1, all the other has valuation at least 1. Thus the first $j+1$ columns of $(t^{(j+1)})^{-1} r'$

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satisfy the condition for the first $j+1$ columns of $(t^{(j+1)})^{-1}r^{(j+1)}$ - this is meaningful, because we already fixed the first $j+1$ columns of $w^{(j+1)}$.

So we want to find $r^{(j+1)} = r'b'$ with $b' \in B$ such that the first $j+1$ columns of $b'$ is those of the identity matrix, and $(t^{(j+1)})^{-1}r^{(j+1)} \in R_{w^{(j+1)}}$ for some $w^{(j)} \preceq w^{(j+1)}$.

Let $j_0 = j+1$, and if $j_i < j'$ then

$$j_{i+1} = \min\{h | j+1 < h, r'_{i_0,h} \notin o_F, w^{(j)}(j_i) > w^{(j)}(h)\}.$$

We claim that the set on the right hand side contains $j'$ if $j_i < j'$. We prove it by induction on $i$. For $i = 0$ we already verified it. Assume by contradiction that $w^{(j)}(j_i) < i_0 = w^{(j)}(j')$. Since $j' > j_i$ we get $r^{(j)}_{i_0,j_i} \in \varpi_F o_F$, because $r^{(j)} \in R_{w^{(j)}}$. But then $r'_{i_0,j_i} \in o_F$, because $r' \in e_{i_0}^{-1} r^{(j)} \cdot \mathrm{Mat}(o_F)$, contradicting the defining conditions of $j_i$. Thus we have $w^{(j)}(j_i) \geq i_0 = w^{(j)}(j')$.

Let $s$ be minimal such that $j_s = j'$ and set $j_{s+1} = n+1$. We claim that $r^{(j+1)}$ will be in $R_{w^{(j+1)}}$ with $w^{(j+1)} = w^{(j)}(j_{s-1}, j_s)(j_{s-2}, j_{s-1}) \dots (j_0, j_1)$. Then the condition $w^{(j+1)} \prec w^{(j)}$ holds, because the multiplication from right with each transposition $(j_i, j_{i+1})$ decreases the inversion number and the length respectively, by the definition of $j_{i+1}$.

For the existence of a $b' \in B$ such that $r'b' \in R_{w^{(j+1)}}$ we prove the following statements inductively:

**Lemma 2.2.4** *For all $j+1 \leq k \leq n$ there exist*

- *$b'^{(k)} \in B$ such that the first $k$ column of $r'^{(k)} = r'b'^{(k)}$ satisfy the defining condition for the first $k$ column in $R_{w^{(j+1)}}$, and if we have $k < n$ then $r'^{(k)}$ and $r'^{(k+1)}$ differ only in the $k+1$-st column.*
- *a linear combination $s^{(k)}$ of the columns $j+1, j+2, \dots, k$ in $r'^{(k)}$ for which we have*

$$s_i^{(k)} = \begin{cases} 1, & \text{if } i = i_0 \\ 0, & \text{if } (w^{(j+1)})^{-1}(i) \leq k, \text{ and } i \neq i_0 \\ \varpi_F x, & \text{for some } x \in o_F \text{ otherwise} \end{cases}$$

*and the maximal $i$ such that $\nu_F(s_i^{(k)}) = 1$ is $w^{(j)}(j_{i'})$, where $i'$ is so, that $j_{i'} \leq k < j_{i'+1}$.*



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**Proof** This holds for $k = j+1$ with $b'^{(j+1)} = \mathrm{id}$, $r'^{(j+1)} = r'$ and $s^{(j+1)}$ the $j+1$-st column of $r'$. To verify the condition for $s^{(j+1)}$ note that $r'_{(w^{(j)}(j+1),j+1)} = \varpi_F$ and if $i > j+1$, then by the definition of $R_{w^{(j)}}$ we have that $r^{(j)}_{i,j+1}$ has valuation at least 1 and $r'_{(i,j+1)} = \varpi_F (r^{(j)}_{i_0,j+1})^{-1} r^{(j)}_{i,j+1}$ has valuation at least 2.

Assume that we have $r'^{(k)}$, $b'^{(k)}$ and $s^{(k)}$. Let $i'$ be so that $j_{i'} \leq k < j_{i'+1}$ and $s'$ be the $k+1$-st column of $r'^{(k)}$ (which is equal with the $k+1$-st column of $r'$, thus for $i \neq i_0$ we have $s'_i = r^{(j)}_{i,k+1}$) and $s'' = s' - r'^{(k)}_{(i_0,k+1)} s^{(k)}$. Then by the conditions on $s'$ we can change the $k+1$-st column of $r'^{(k)}$ to $s''$ with multiplication from right by an element $b'' \in B$. Moreover $s''_{i_0} = 0$, and the element in $s''$ with minimal valuation and biggest row index is the $w^{(j+1)}(k+1)$-st:

- If $\nu_F(r'^{(k)}_{(i_0,k+1)}) \geq 0$ then for $i \neq i_0$ we have $s'_i \equiv s''_i = s'_i - r'^{(k)}_{(i_0,k+1)} s^{(k)}_i$ mod $\varpi_F$, hence the element with minimal valuation is in the row $w^{(j+1)}(k+1) = w^{(j)}(k+1)$ (because $r^{(j)} \in R_{w^{(j)}}$ and $j_{i'+1} \neq k+1$).
- If $\nu_F(r'^{(k)}_{(i_0,k+1)}) < 0$ then it is -1 and for $i \neq i_0$ we have $s''_i = r^{(j)}_{(i,k+1)} - r'^{(k)}_{(i_0,k+1)} \cdot s^{(k)}_i$. Where on the right hand side the first term has positive valuation for $i > w^{(j)}(k+1)$ and 0 valuation for $i = w^{(j)}(k+1)$ (because $r^{(j)} \in R_{w^{(j)}}$), and the second has valuation 0=-1+1 for $i = w^{(j)}(j_{i'})$ and at least 1 for $i > w^{(j)}(j_{i'})$ (by the induction hypothesis on $s^{(k)}$). Moreover $j_{i'} \neq k+1$, because $j_{i'} \leq k$, hence $w^{(j)}(j_{i'}) \neq w^{(j)}(k+1)$.

  If $w^{(j)}(j_{i'}) < w^{(j)}(k+1)$ then $j_{i'+1} \neq k+1$ and $w^{(j)}(k+1) = w^{(j+1)}(k+1)$. If $w^{(j)}(j_{i'}) > w^{(j)}(k+1)$ then $j_{i'+1} = k+1$ and $w^{(j+1)}(k+1) = w^{(j+1)}(j_{i'+1}) = w^{(j)}(j_{i'})$.

By multiplying this column with $(s''_{w^{(j+1)}(k+1)})^{-1}$ we get the element $r'^{(k+1)}$ (we also have to multiply the $k+1$-st row of $b''$ with $s''_{w^{(j+1)}(k+1)}$, this is $b'^{(k+1)}$). This satisfies the condition for the $k+1$-st row of $R_{w^{(j+1)}}$ because the defining conditions for $r^{(j)} \in R_{w^{(j)}}$, $s^{(k)}$ and the equality

$$\{i|(w^{(j+1)})^{-1}(i) < k+1\} = \{i|(w^{(j)})^{-1}(i) < k+1\} \setminus \{w^{(j)}(j_{i'})\} \cup \{i_0\}.$$

The last thing to verify is the existence of an appropriate linear combination $s^{(k+1)}$. Let $s^{(k+1)} = s^{(k)} - s^{(k)}_{w^{(j+1)}(k+1)} (s''_{w^{(j+1)}(k+1)})^{-1} \cdot s''$. Since



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$\nu_F(s^{(k)}_{w^{(j+1)}(k+1)}) > 0$, we have $\nu_F(s_i^{(k+1)}) > 0$ if $i \neq i_0$, and by the previous argument also $s^{(k+1)}_{w^{(j+1)}(j')} = 0$ for $j' \leq k+1$ and $j' \neq j+1$.

If $w^{(j+1)}(k+1) > w^{(j)}(j_{i'})$, then $s^{(k)}_{w^{(j+1)}(k+1)} > 1$ and $s^{(k+1)} \equiv s^{(k)}$ mod $\varpi_F^2$. If $w^{(j+1)}(k+1) < w^{(j)}(j_{i'})$ then by the definition of $R_{w^{(j+1)}}$ for all $i > w^{(j+1)}(k+1)$ we have $\nu(s_i'') > 1$ and again $s_i^{(k+1)} \equiv s_i^{(k)}$ mod $\varpi_F^2$. If $w^{(j+1)}(k+1) = w^{(j)}(j_{i'})$, then by the definition of $R_{w^{(j)}}$ we have $s'_{w^{(j)}(j_{i'})} = r'_{(w^{(j)}(j_{i'}),k+1)} = 0$, $s''_{w^{(j+1)}(k+1)} = 0 - r'^{(k)}_{(i_0,k+1)} s^{(k)}_{w^{(j)}(j_{i'})}$ and $s^{(k+1)} =$

$$= s^{(k)} - s^{(k)}_{w^{(j)}(j_{i'})}(-r'^{(k)}_{(i_0,k+1)} s^{(k)}_{w^{(j)}(j_{i'})})^{-1} \cdot \left(s' - r'^{(k)}_{(i_0,k+1)} s^{(k)}\right) = (r'^{(k)}_{(i_0,k+1)})^{-1} s',$$

which satisfies the condition because $s'$ is the $j_{i'+1} = k+1$-st column of $r'^{(k)}$ and because of the definition of $R_{w^{(j)}}$. □

To finish the proof we set $b' = b'^{(n)}$, $r^{(j+1)} = r'b'^{(n)} \in R_{w^{(j+1)}}$ and $b^{(j+1)} = (b'^{(n)})^{-1}(r^{(j)}_{i_0,j+1} \cdot e^{-1}_{j+1}) b^{(j)} \in B$.

□

**Corollary 2.2.5** *For any $w \in W$ we have $BwB = N_w wB \subset \cup_{w' \preceq w} U_{w'}$. In particular for any $0 < m_0 \leq n!$ we have that*

$$\bigcup_{m \geq m_0} U_{w_m} \subset G \setminus G_{m_0 - 1} = \bigcup_{m \geq m_0} Bw_m B.$$

**Proof** Let $x = n_w wb \in N_w wB$. Then there exists $t \in T_+$ such that $n' = tn_w t^{-1} \in N_0$. Thus $x = t^{-1}n'w(w^{-1}tw)b = t^{-1}n'wb''$ with $b'' \in B$. By the previous proposition for $w = w \cdot \mathrm{id} \in R_w B$ and $(n')^{-1}t \in B_+$, there exist $w' \prec w$, $r_{w'} \in R_{w'}$ and $b' \in B$ such that $t^{-1}n'w = r_{w'}b'$, hence $x = r_{w'}(b'b'') \in U_{w'}$. The second assertion follows from that:

$$\bigcup_{m \geq m_0} U_{w_m} = G \setminus \bigcup_{1 \leq m < m_0} U_{w_m} \subset G \setminus \bigcup_{1 \leq m < m_0} Bw_m B = G \setminus G_{m_0-1}.$$

□

**Remark** We can achieve the results of this section not only for $\mathbf{GL_n}$, but different groups: let $G' = \mathbf{G}'(F)$ be such that



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- $G'$ is isomorphic to a closed subgroup in $G$ which we also denote by $G'$,
- In $G'$ a maximal torus is $T' = T \cap G'$, a Borel subgroup $B' = B \cap G'$ with unipotent radical $N' = N \cap G'$, such that $N_{G'}(T') = N_G(T) \cap G'$ and hence $W' \leq W$ with $w_0 \in W'$, with representatives $w'$ of $W'$ in $G'_0 \leq G_0$ such that the representatives $w$ of $W$ in $G$ can be written in the form $w = w't$ such that $t \in T \cap G_0$.
- $G'_0 = G_0 \cap G'$ with $G' = G'_0 B'$ and
- $U'^{(1)} = U^{(1)} \cap G'$ such that $U'^{(1)} \subset (N'^{-} \cap U'^{(1)})B'$ for $N'^{-} = w_0 N' w_0$.

For example these condititons are satisfied for the group $\mathbf{SL_n}$.

The proof of the first proposition works for such $G'$, and from a decomposition $x = r'b' \in R'_w B' \subset G'$ we get some $r \in R_w$ and $b \in B$ such that $x = rb \in G$. Hence the $B'_+$-action on $G'$ respects the restriction of $\prec$ to $W'$ in the sense that if $x \in R_{w'}B'$ and $b' \in B'$ then there exists $w'' \preceq w'$ in $W'$ such that $b'^{-1}x \in R'_{w''}B'$.

## 2.3 Generating $B_+$-subrepresentations

For any torsion $o_K$-module $X$ with $o_K$-linear $B$-action denote the (partially ordered) set of generating $B_+$-subrepresentations of $X$ (those $B_+$-submodules $M$ of $X$ for which $BM = X$) by $\mathcal{B}_+(X)$.

For example $\mathrm{Ind}_B^{U_{w_0}}(\chi) \simeq C^\infty(N_0)$ is the minimal generating $B_+$-subrepresentation of the Steinberg representation $\pi_{n!-1} = \mathrm{Ind}_B^{Bw_0B}(\chi) \simeq C_c^\infty(N)$. (cf [17], Lemma 2.6)

**Proposition 2.3.1** *Let $X$ be a smooth admissible and irreducible torsion $o_K$-representation of $G$. Then $M_0 = B_+ X^{U^{(1)}}$ is a generating $B_+$-subrepresentation of $X$. For any $M \in \mathcal{B}_+(X)$ there exists a $t_+ \in T_+$ such that $t_+ M_0 \subset M$.*

**Proof** $X$ is a $\varpi_K$ vectorspace as well, because $\varpi_K X \leq X$, hence by the irreducibility it is either $0$ or $X$, and since $X$ is torsion $\varpi_K X = X$ gives $X = 0$.

$BM_0$ is a $B$-subrepresentation, and also a $G_0$-subrepresentation (because $U^{(1)} \lhd G_0$). $G_0 B = BG_0 = G$, so $BM_0$ is a $G$-subrepresentation of $X$. $M_0$ is



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not $\{0\}$, since $U^{(1)}$ is pro-$p$ and since $X$ is irreducible $BM_0 = X$, hence $M_0$ is generating. And $M_0$ is clearly a $B_+$-submodule of $X$.

$X$ is admissible, hence $X^{U^{(1)}}$ has a finite generating set, say $R$. Let $M$ be as in the proposition. For any $r \in R$ there exists an element $t_r \in T_+$ such that $t_r r \in M$ ([17], Lemma 2.1). The cardinality of $R$ is finite, hence for $t_+ = \prod_{r\in R} t_r$ we have $t_r^{-1}t_+ \in T_+$ for all $r \in R$, and then $t_+ M_0 \subset M$. $\square$

From now on let $\pi = \mathrm{Ind}_B^G(\chi)$ as before and $M_0 = B_+\pi^{U^{(1)}}$. Then $\pi^{U^{(1)}}$ (as a vector space) is generated by

$$f_r : \begin{cases} urb & \mapsto \quad \chi^{-1}(b) \\ y \neq urb & \mapsto \quad 0 \end{cases} \qquad \left( r \in U^{(1)} \setminus G/B = \bigcup_{w\in W} \widetilde{\mathbf{N_w}(k_F)} w \right).$$

If we denote the coset $U^{(1)}wB$ also with $w$, then $\pi^{U^{(1)}}$ is generated by $\{f_w | w \in W\}$ as an $N_0$-module. Hence any $f \in M_0$ can be written in the form $\sum_{i=1}^s \lambda_i n_i t_i f_{w_i}$ for some $\lambda_i \in k_K, n_i \in N_0, t_i \in T_+$ and $w_i \in W$.

**Proposition 2.3.2** *$M_0$ is minimal in $\mathcal{B}_+(\pi)$.*

**Remark** In [17] section 12 Schneider and Vigneras treated the case of the subquotients $\pi_{m-1}/\pi_m$. Unfortunately $M_0$ does not generally give the minimal generating $B_+$-subrepresentation of $\pi_{m-1}/\pi_m$ on this subqoutient, since that their method does not work on the whole $\pi$. It is not true even for $\mathbf{GL_3}(\mathbb{Q}_p)$: an explicit example is shown in Corollary 2.5.2.

**Proof** By the previous proposition, it is enough to show, that for any $t' \in T_+$ we have $M_0 \subset B_+t'M_0$.

If $t' \in G_0$, then $t'^{-1} \in T_+$ thus we have $B_+t' = B_+$, and $B_+t'M_0 = B_+M_0 = M_0$. The same is true for central elements $t' \in Z(G)$. So it is enough to prove for $t' = (\varpi_F, \varpi_F, \dots, \varpi_F, 1, 1, \dots, 1)$ that $M_0 \subset B_+t'M_0$.

Let $j_0 \in \mathbb{N}$ be such that $t'_{j_0} = \varpi_F$ and $t'_{j_0+1} = 1$. We need to show, that for all $w \in W$ we have $f_w \in B_+t'M_0$. We prove it by descending induction on $w$ with respect to $\prec$.

Let us denote $N_{j_0}^{(1)} = \{n \in N \cap U^{(1)} | \forall i < j, (j_0 - i)(j - j_0) < 0 : n_{ij} = 0\}$, $N_{w,j_0} = N_w \cap N_{j_0}^{(1)}$ and

$$J_{w,j_0} = J(N_{w,j_0}/t'N_{w,j_0}t'^{-1}) \subset N_0 \cap U^{(1)}.$$

It is enough to prove the following:



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**Lemma 2.3.3** *Let* $g = \sum_{m \in J_{w,j_0}} mt'f_w$. *Then* $\chi(w^{-1}t'w)f_w - g$ *is in* $\sum_{w' : w \prec w'} N_0 f_{w'}$.

We claim that for $r \in R_w$ we have

$$t'f_w(r) = \begin{cases} \chi(w^{-1}t'w), & \text{if } \forall i \leq j_0 < j, w^{-1}(i) > w^{-1}(j) : r_{ij} \in \varpi_F^2 o_F, \\ 0, & \text{otherwise.} \end{cases}$$

$t'f_w(r) = f(t'^{-1}r)$ is nonzero if and only if $t'^{-1}r \in U^{(1)}wB$. Following the proof of Proposition 2.2.3, it is equivalent to that for all $1 \leq j \leq n$ we have $w = w^{(j)}$ and that the first $j$ column of $(t^{(j)})^{-1}r^{(j)}$ is as the first $j$ column of $U^{(1)}w$. This holds if and only if $r_{ij} \in \varpi_F^2 o_F$ for all $i$ and $j$ as above. Then we have $r^{(n)} = t'^{-1}rw^{-1}t'w$ and $b^{(n)} = w^{-1}(t')^{-1}w$, hence our claim.

Therefore $\chi(w^{-1}t'w)f_w|_{U_w} = \sum_{m \in J_{w,j_0}} mt'f_w|_{U_w}$. Hence by the induction hypothesis and Proposition 2.2.3 it suffices to prove that $g$ is $U^{(1)}$-invariant.

To do that, first notice that since $f_w$ is $U^{(1)}$-invariant, we have that $t'f_w$ is $t'U^{(1)}t'^{-1}$-invariant. Moreover, since for all $m \in J_{w,j_0}$ we have $m \in N_0 \cap U^{(1)} \subseteq t'N_0t'^{-1}$, $m$ normalizes $t'U^{(1)}t'^{-1}$, $mt'f_w$ is also $t'U^{(1)}t'^{-1}$-invariant, and so is $g$.

On the other hand, we can write

$$g = \sum_{m \in J_{w,j_0}} mt'f_w = \sum_{m \in J_{w,j_0}} t'(t'^{-1}mt')f_w = t'\left( \sum_{n \in t'^{-1}N_{w,j_0}t'/N_{w,j_0}} nf_w \right),$$

where the sum in the bracket on the right hand side is obviously $t'^{-1}N_{w,j_0}t'$-invariant, hence $g$ is $N_{w,j_0}$-invariant.

Denote $N'_{w,j_0} = N'_w \cap N_{j_0}^{(1)}$. Then $N_{w,j_0}$ centralizes $t'^{-1}N'_{w,j_0}t'$: let $n_0 = \mathrm{id} + m_0 \in t'^{-1}N'_{w,j_0}t'$, $n \in N_{w,j_0}$,

$$(n^{-1}n_0n - n_0)_{xy} = (n^{-1}m_0n - m_0)_{xy} = \sum_{x \leq s \leq t \leq y} (n^{-1})_{xs}(m_0)_{st}n_{ty} - (m_0)_{xy},$$

and by the definition $N_{j_0}^{(1)}$, $(m_0)_{st}$ is 0, unless $s \leq j_0 \leq t$ and hence $(n^{-1})_{xs}m_{st}n_{ty} = 0$, unless $x = s$ and $y = t$.

By the definiton of $N'_w$ we have $w^{-1}N'_{w,j_0}w \subset B$, so for any $u \in U^{(1)}$ and $n_0 \in t'^{-1}N'_{w,j_0}t' \subset G_0$ we have $n_0uw = (n_0un_0^{-1})w(w^{-1}n_0w) \in U^{(1)}wB$, and hence $f_w$ is $t'^{-1}N'_{w,j_0}t'$-invariant.



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Altogether for any representative $n \in J_{w,j_0}$

$$nf_w(n_0x) = f_w(n^{-1}n_0x) = f_w(n_0n^{-1}x) = f_w(n^{-1}x) = nf_w(x),$$

meaning that $nf_w$ is $t'^{-1}N'_{w,j_0}t'$-invariant, and $t'nf_w$ is $N'_{w,j_0}$-invariant. So $g$ is also $N'_{w,j_0}$-invariant.

$U^{(1)}$ is contained in $\langle t'U^{(1)}t'^{-1}, N_{w,j_0}, N'_{w,j_0}\rangle$, so $g$ is $U^{(1)}$-invariant, and we are done. $\square$

**Corollary 2.3.4** *For any $f \in M_0$ there exists $t \in T_+$ such that $f$ can be written in form $\sum_{i=1}^{s} \lambda_i n_i t f_{w_i}$ for some $\lambda_i \in k_K, n_i \in N_0$ and $w_i \in W$.*

Define the $k_K[B_+]$-submodules $M_{0,m} = \sum_{m'>m} B_+ f_{w_{m'}} \leq \mathrm{Ind}_B^{G_m}(\chi)$. We obtain a descending filtration $M_0 = M_{0,0} \geq M_{0,1} \geq \cdots \geq M_{0,n!} = 0$. Then $M_{0,n!-1} = \mathrm{Ind}_B^{U_{w_0}}(\chi)$ is the minimal generating subrepresentation of $\pi_{n!-1}$.

**Proposition 2.3.5** *Let $1 < m \leq n!$, $w = w_{m-1}$ and $n' \in N'_{0,w} = N'_w \cap N_0$ and $t \in T_+$. Then $g = n'tf_w - tf_w \in M_{0,m}$.*

**Proof** For $w' \prec w$ we have $tf_w|_{U_{w'}} = n'tf_w|_{U_{w'}} = 0$ and following the proof of Proposition 2.2.3 we get $n'tf_w|_{U_w} = tf_w|_{U_w}$. Moreover $g$ is $tU^{(1)}t^{-1}$-invariant, thus it is contained in $\sum_{m'>m-1} tf_{w_{m'}} \subset M_{0,m}$. $\square$

**Corollary 2.3.6** *For any $f \in M_0$ there exists $t \in T_+$ such that $f$ can be written in form $\sum_{i=1}^{s} \lambda_i n_i t f_{w_i}$ for some $\lambda_i \in k_K$, $w_i \in W$ and $n_i \in N_{0,w_i}$.*

**Remarks**

1. $\pi$ is the modulo $\varpi_K$ reduction of the $p$-adic principal series representation. This can be done with any $l \in \mathbb{N}$ for the modulo $\varpi_K^l$ reduction. Then the $\varpi_K$-torsion part of the minimal generating $B_+$-representation is exactly $M_0$.

2. This can be carried out in the same way for groups $G' = \mathbf{G}'(F)$ as in the previous section satisfying moreover $N_0 \subset G'$. For example $\mathbf{G}' = \mathbf{SL_n}$ has this property (but its center is not connected), or $G' = P$ for arbitary $P \leq G$ parabolic subgroup has also (but these are not reduvtive).



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## 2.4 The Schneider-Vigneras functor

Following Schneider and Vigneras ([17], section 2) we introduce the functor $D$ from torsion $o_K$-modules to modules over the Iwasawa algebra of $N_0$.

Let us denote the completed group ring of $N_0$ over $o_K$ by $\Lambda(N_0)$, and define

$$D_{SV}(\rho) = \varinjlim_{M \in \mathcal{B}_+(\rho)} M^\vee,$$

as an $\Lambda(N_0)$-module, equipped with a natural $T_+^{-1}$-action $\psi$.

On $D_{SV}(\pi)$ the action of $\varpi_K$ is 0, hence we can view it as a $\Omega(N_0) = \Lambda(N_0)/\varpi_K\Lambda(N_0)$-module.

By Proposition 2.3.2 we have

**Proposition 2.4.1** *The $\Omega(N_0)$-module $D_{SV}(\pi)$ is equal to $M_0^\vee$.*

**Remarks**

1. We do not now whether $D_{SV}(\pi)$ is finitely generated or it has rank 1 as an $\Omega(N_0)$-module.

2. On $M_0$ we have an action of $U^{(1)}$: if $x \in U^{(1)}$, $n \in N_0, t \in T_+$ and $w \in W$ then we can write $n^{-1}xn = n_1 n_2 \in U^{(1)}$ with $n_1 \in N_0$ and $n_2 \in B^- T \cap U^{(1)}$ (with $B^- = N^- T$), thus

$$xntf_w = n(n^{-1}xn)tf_w = (nn_1)t(t^{-1}n_2 t)f_w = (nn_1)tf_w \in M_0,$$

since $t^{-1}n_2 t \in U^{(1)}$ and $f_w$ is $U^{(1)}$-invariant. Thus on $D_{SV}(\pi)$ there is an action of $\Lambda(U^{(1)})$, therefore an action of $\Lambda(I)$ (with $I$ denoting the Iwahori subgroup).

Till this point we considered only the $\Lambda(N_0)$-module structure of $D_{SV}(\pi)$. Now we shall examine the $\psi$-action as well. We need to get an étale module from $D_{SV}(\pi)$, thus we examine the $\psi$-invariant images of $D_{SV}(\pi)$ in an étale module.

Let $D$ be a topologically étale (see [18] the first lines of Section 4) $(\varphi, \Gamma)$-module over $\Omega(N_0)$, with the following properties:

- $D$ is torsion-free as an $\Omega(N_0)$-module,
- on $D$ the topology is Hausdorff,
- $D$ has a basis of neighborhoods of 0, containing $\varphi$-invariant $\Omega(N_0)$-submodules ($O \leq D$ open such that $\varphi_t(O) \subseteq O$ for all $t \in T_+$).



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**Theorem 2.4.2** *If $D$ is as above and $F : D_{SV}(\pi) \to D$ is a continuous $\psi$-invariant map (where $\psi$ is the canonical left inverse of $\varphi$ on $D$), then $F$ factors through the natural map $F_0 : D_{SV}(\pi) \to D_{SV}(\pi_{n!-1})$: there exists a continuous $\psi$-invariant map $G : D_{SV}(\pi_{n!-1}) \to D$ such that $F = F_0 \circ G$.*

**Proof** $\overline{D_{SV}(\pi) - tors}$ is in the kernel of $F$ (the torsion submodules exist, because the rings are Ore rings).

In $M_0/(M_0 \cap \pi_{n!-1})$ there are no nontrivial $k_K[N_0]$-divisible elements, because if $f \in M_0$ the image of it in $M_0/(M_0 \cap \pi_{n!-1})$ is $f' = f|_{G \backslash Bw_0B}$. Assume by contradiction that $f'$ is $k_K[N_0]$-divisible. If it is nontrivial, then there exists $bw_mb \in G$ such that $f(bw_mb) \neq 0$ with some $m < n!$ Let $n' \in N'_{0,w_m} = N_0 \cap w_m N_0 w_m^{-1}$ with $n' \neq \mathrm{id}$, and $[n'] - [\mathrm{id}] \in k_K[N_0]$. Then for any $g \in M_0$ we have

$$([n'] - [\mathrm{id}])g(w_m) = g(n'^{-1}w_m) - g(w_m) = g(w_m(w_m^{-1}n'^{-1}w_m)) - g(w_m) = 0,$$

because $w_m^{-1}n'^{-1}w_m \in N$. Thus $f'$ is not divisible by $[n'] - [\mathrm{id}]$.

It follows that $F$ factors through $(M_0 \cap \pi_{n!-1})^\vee$: The fact that there are no nontrivial divisible submodules in $M_0/(M_0 \cap \pi_{n!-1})$ implies that for any (closed) submodule the maps $f \mapsto \lambda f$ are not surjective for all $\lambda \in k_K[N_0]^\vee$. Hence dual maps are not injective for all $\lambda$ - it has no torsionfree quotient arising as a dual of a submodule of $M_0/(M_0 \cap \pi_{n!-1})$, thus $(M_0/(M_0 \cap \pi_{n!-1}))^\vee \leq \overline{D_{SV}(\pi) - tors}$. Now consider the exact sequence

$$0 \to M_0 \cap \pi_{n!-1} \to M_0 \to M_0/(M_0 \cap \pi_{n!-1}) \to 0.$$

We claim that $F$ factors through $M_{0,n!-1}^\vee$ as well. If $f \in (M_0 \cap \pi_{n!-1})^\vee$ such that $f|_{M_{0,n!-1}} \equiv 0$, then $\psi_t(u^{-1}f)|_{t^{-1}M_{0,n!-1}} \equiv 0$ for all $u \in N_0$:

The $\psi$-action on $D_{SV}(\pi)$ comes from the $T_+$-action on $\pi$, hence $\psi_t(u^{-1}f)(t^{-1}x) = (u^{-1}f)(tt^{-1}x) = f(ux) = 0$ if $x \in M_{0,n!-1}$.

For all $O \subseteq D$ open subset there exists $t \in T_+$ such that $\mathrm{Ker}(f \mapsto f|_{t^{-1}M_{0,n!-1}}) \subset F^{-1}(O)$, since $F$ is continuous and $\bigcup_{t \in T_+} t^{-1}M_{0,n!-1} = \pi_{0,n!-1}$. If $O$ is $\varphi$ and $N_0$-invariant as well, then

$$F(f) = \sum_{u \in N_0/tN_0t^{-1}} u\varphi_t(F(\psi_t(u^{-1}f)) \subseteq O.$$

Then $F(f) = 0$ by the Hausdorff property.

By [17], Proposition 12.1, we have $D_{SV}(\pi_{n!-1}) = M_{0,n!-1}^\vee$, which completes the proof. □



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**Remarks** 1. For this we do not need the $\Gamma$-action of $D$, the statement is true for $D$ étale $\varphi$-modules with continuous $N_0$ and $\varphi$-action.

2. Let $D'$ be the maximal quotient of $D_{SV}(\pi)$, which is torsionfree, Haussdorff and on which the action of $\psi$ is nondegenerate in the following sense: for all $d \in D' \setminus \{0\}$ and $t \in T_+$ there exists $u \in N_0$ such that $\psi_t(ud) \neq 0$. Then the natural map from $D'$ to $D_{SV}(\pi_{n!-1})$ is bijective.

3. By [22] section 4 if $F = \mathbb{Q}_p$, we have that $D^0(\pi_{n!-1}) = D_{SV}(\pi_{n!-1})$ and $D^i(\pi_{n!-1}) = 0$ for $i > 0$.

Following [17] we choose a surjective homomorphism $\ell : N_0 \to \mathbb{Q}_p$. Then we can get $(\varphi, \Gamma)$-modules from $D_{SV}(\pi)$: Let $\Lambda_\ell(N_0)$ denote the ring $\Lambda_{N_1}(N_0)$ of [17] with $N_1 = \mathrm{Ker}(\ell)$, with maximal ideal $\mathcal{M}_\ell(N_0)$, $\Omega_\ell(N_0) = \Lambda_\ell(N_0)/\varpi_K \Lambda_\ell(N_0)$ and $D_\ell(\pi) = \Omega_\ell(N_0) \otimes_{\Omega(N_0)} D_{SV}(\pi)$.

**Corollary 2.4.3** *Let $D$ be a finitely generated topologically étale $(\varphi, \Gamma)$-module over $\Omega_\ell(N_0)$, and $F' : D_\ell(\pi) \to D$ a continuous map. Then $F'$ factors through the natural map $F'_0 : D_\ell(\pi) \to D_\ell(\pi_{n!-1})$.*

**Proof** If $D$ is a finitely generated topologically étale $(\varphi, \Gamma)$-module over $\Omega_\ell(N_0)$, then it automatically satisfies the conditions above:

$D$ is étale, hence $\Omega_\ell(N_0)$-torsion free (Theorem 8.20 in [18]), thus $\Omega(N_0)$-torsion free as well. It is Hausdorff, since finitely generated and the weak topology is Haussdorff on $\Omega_\ell(N_0)$ (Lemma 8.2.iii in [17]).

We only need to verify the condition for the neighborhoods. The sets $\mathcal{M}_\ell(N_0)^k D + \Omega(N_0) \otimes_{k[[X]]} X^n \ell(D)^{++}$ (where $\ell(D)$ is the étale $(\varphi, \Gamma)$-module attached to $D$ at the category equivalence [18] Theorem 8.20) are open $\varphi$-invariant $\Omega(N_0)$ submodules and form a basis of neighborhoods of 0 in the weak topology of $D$.

Thus $D_{SV}(\pi) \to D_\ell(\pi) \to D$ factors through $D_{SV}(\pi) \to D_{SV}(\pi_{n!-1})$, hence the corollary. □

## 2.5 Some properties of $M_0$

In this section we point out some properties of $M_0$, which make the picture more difficult than the known case of subqoutients $\pi_{m-1}/\pi_m$. Recall ([17]

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section 12) that $\pi_{m-1}/\pi_m \simeq \pi(w_m, \chi)$, which has a minimal generating $B_+$-subrepresentation

$$M(w_m, \chi) = C^\infty(N_0/N'_{w_m} \cap N_0) \in \mathcal{B}_+(\pi(w_m, \chi)).$$

**Proposition 2.5.1** *Let $n = 3$, $F = \mathbb{Q}_p$, then $M_0 \cap \pi_{n!-1} \supsetneq M_{0,n!-1}$.*

**Corollary 2.5.2** *Thus $M_0 \cap \pi_{n!-1}$ is not equal to the minimal generating $B_+$-subrepresentation of $\pi_{n!-1}$, which is $C^\infty(N_0) = M_{0,n!-1}$ ([17] section 12).*

**Proof** Assume that $\chi = \chi_1 \otimes \chi_2 \otimes \chi_3 : T \to k_K^*$ is a character, such that neither $\chi_1/\chi_2$, nor $\chi_2/\chi_3$ is trivial on $o_K^*$. Similar construction can be carried out in the other cases.

Let $\prec_T$ be the following total ordering of the Weyl group of $G$ refining the Bruhat ordering:

$$w_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \prec_T w_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \prec_T w_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \prec_T$$

$$\prec_T w_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \prec_T w_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \prec_T w_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = w_0.$$

And let

$$h = \sum_{a=0}^{p^2-1} \sum_{b=0}^{p^2-1} \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f_{w_2} \in M_0,$$

$$f = h - \frac{1}{\chi_3(p^2)} \sum_{a=0}^{p^3-1} \sum_{b=0}^{p^3-1} h\left( \begin{pmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f_{w_5}.$$

Then it is easy to verify that $f \in M_0 \cap \pi_5$, and that $f(z) \neq 0$ for

$$z = \begin{pmatrix} p^2 & 0 & 1 \\ 1 & 0 & 0 \\ p & 1 & 0 \end{pmatrix} \in Bw_0B \setminus N_0w_0B.$$

Thus $f \notin M_{0,5} = B_+f_6 \subseteq \{f \in \pi | \mathrm{supp}(f) \leq N_0w_0B\}$. □



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However, if $f \in M_0 \cap \pi_5$ then $\mathrm{supp}(f)$ is contained in $Bw_0B \cap \bigcup_{i>3} R_iB$: A straightforward computation shows that for any $n \in N_0$, $t \in T_+$, $w \in W$ and

- for any $r \in R_{w_1}$ we have $ntf_w(r) = ntf_w(w_1)$. Let $r' = w_1 \in G_5$,
- for any $r \in R_{w_2}$ we have $ntf_w(r) = ntf_w(r')$ for

$$r' = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & 0 & 0 \\ \beta' & 0 & 1 \end{pmatrix} \in G_5, \text{ where } r = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & 0 & 0 \\ \beta' & \gamma' & 1 \end{pmatrix},$$

- for any $r \in R_{w_3}$ we have $ntf_w(r) = ntf_w(r')$ for

$$r' = \begin{pmatrix} 1 & 0 & 0 \\ \alpha' - \beta\gamma' & \gamma & 1 \\ 0 & 1 & 0 \end{pmatrix} \in G_5, \text{ where } r = \begin{pmatrix} 1 & 0 & 0 \\ \alpha' & \gamma & 1 \\ \beta' & 1 & 0 \end{pmatrix}.$$

Thus if $i < 4$ and $r \in R_{w_i}$, then since $r' \notin Bw_0B$ we have $f(r) = f(r') = 0$.

**Proposition 2.5.3** *The quotients $M_{0,m-1}/M_{0,m-1} \cap \pi_m$ via $f \mapsto f(\cdot w_m)$ are isomorphic to $M(w_m, \chi)$.*

**Proof** It is obvious, that $f(\cdot w_m) \equiv 0$ implies $f|_{G_m \setminus G_{m-1}} \equiv 0$ and $f \in M_{0,m-1} \cap \pi_m$. Hence the map $M_{0,m-1}/M_{0,m-1} \cap \pi_m \to M(w_m, \chi)$, $f \mapsto f(\cdot w_m)$ is injective.

Let $t_0 = \mathrm{diag}(\varpi_F^{n-1}, \varpi_F^{n-2}, \dots, \varpi_F, 1) \in T_+$, and for any $l \in \mathbb{N}$ let $U^{(l)} = \mathrm{Ker}(G_0 \to \mathbf{G}(o_F/\varpi_F^l o_F))$. For $x = rb \in R_{w_m}B$ we have

$$\sum_{n \in (N_0 \cap U^{(l)})/t_0^l N_0 t_0^{-l}} nt_0^l f_{w_m}(rb) = \begin{cases} \chi^{-1}(b), & \text{if } r \in U^{(l)} w_m, \\ 0, & \text{if not.} \end{cases}$$

The image of these generate $M(w_m, \chi)$ as an $N_0$-module, so $f \mapsto f(\cdot w_m)$ is surjective. $\square$

Since $M_{0,m} \leq \pi_m$, $M(w_m, \chi)$ is naturally a quotient of $M_{0,m-1}/M_{0,m}$, we have $D_{SV}(\pi_{m-1}/\pi_m) \leq (M_{0,m-1}/M_{0,m})^\vee$.

**Proposition 2.5.4** *For $m = 1$ and $m = n! - n + 1, n! - n + 2, \dots, n!$ $(M_{0,m-1}/M_{0,m})^\vee = D_{SV}(\pi_{m-1}/\pi_m)$. For other $m$-s it is not true, for example if $n = 3$, $F = \mathbb{Q}_p$ and $m = 2, 3$.*



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**Proof** By the previous proposition it is enough to show that $M_{0,m} = M_{0,m-1} \cap \pi_m$ for $m = 1$ and $m > n! - n$.

For $m = 1$ the quotient is obviously $k_K$, for $m > n! - n$ we have $w \prec w_m$ implies $w = w_{n!}$, so if $f \in B_+ f_{w_m} \cap \pi_{m-1} = B_+ f_{w_m} \cap \pi_{n!-1}$, then $\mathrm{supp}(f) \subset U^{(1)} R^{(1)}_{w_{n!-1}} B$. But

$$M_{0,n!-1} \simeq C^\infty(N_0) \simeq \{f \in \pi_{n!-1} | \mathrm{supp}(f) \subset U^{(1)} R_{w_{n!-1}} B\}.$$

The fuction $f$ constructed in the beginning of this section is in $M_{0,1} \cap \pi_2 \setminus M_{0,2}$. The same can be done for $m = 3$. $\square$

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# Chapter 3

# Comparison of functors

## 3.1 A $\Lambda_\ell(N_0)$-variant of Breuil's functor

Our first goal is to associate a $(\varphi,\Gamma)$-module over $\Lambda_\ell(N_0)$ (not just over $\mathcal{O}_{\mathcal{E}}$) to a smooth $o$-torsion representation $\pi$ of $G$ in the spirit of [3] that corresponds to $D^\vee_\xi(\pi)$ via the equivalence of categories of [18] between $(\varphi,\Gamma)$-modules over $\mathcal{O}_{\mathcal{E}}$ and over $\Lambda_\ell(N_0)$.

From now on let $o = o_K, \varpi = \varpi_K$. Let $H_k$ be the normal subgroup of $N_0$ generated by $s^k H_0 s^{-k}$, ie. we put

$$H_k = \langle n_0 s^k H_0 s^{-k} n_0^{-1} \mid n_0 \in N_0 \rangle \ .$$

$H_k$ is an open subgroup of $H_0$ normal in $N_0$ and we have $\bigcap_{k\geq 0} H_k = \{1\}$. Denote by $F_k$ the operator $\mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot)$ on $\pi$ and consider the skew polynomial ring $\Lambda(N_0/H_k)/\varpi^h[F_k]$ where $F_k\lambda = (s\lambda s^{-1})F_k$ for any $\lambda \in \Lambda(N_0/H_k)/\varpi^h$. The set of finitely generated $\Lambda(N_0/H_k)[F_k]$-submodules of $\pi^{H_k}$ that are stable under the action of $\Gamma$ and admissible as a representation of $N_0/H_k$ is denoted by $\mathcal{M}_k(\pi^{H_k})$.

**Lemma 3.1.1** *We have $F = F_0$ and $F_k \circ \mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k\cdot) = \mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k\cdot) \circ F_0$ as maps on $\pi^{H_0}$.*

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**Proof** We compute

$$\begin{aligned}F_k \circ \mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k\cdot) = \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot) \circ \mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k\cdot) = \\ \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ \mathrm{Tr}_{sH_ks^{-1}/s^{k+1}H_0s^{-k-1}} \circ (s^{k+1}\cdot) = \\ \mathrm{Tr}_{H_k/s^{k+1}H_0s^{-k-1}} \circ (s^{k+1}\cdot) = \\ \mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ \mathrm{Tr}_{s^kH_0s^{-k}/s^{k+1}H_0s^{-k-1}} \circ (s^{k+1}\cdot) = \\ \mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k\cdot) \circ \mathrm{Tr}_{H_0/sH_0s^{-1}} \circ (s\cdot) = \\ \mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k\cdot) \circ F_0 \ .\end{aligned}$$

□

Note that if $M \in \mathcal{M}(\pi^{H_0})$ then $\mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^kM)$ is a $s^kN_0s^{-k}H_k$-subrepresentation of $\pi^{H_k}$. So in view of the above Lemma we define $M_k$ to be the $N_0$-subrepresentation of $\pi^{H_k}$ generated by $\mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^kM)$, ie. $M_k = N_0\mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^kM)$. By Lemma 3.1.1 $M_k$ is a $\Lambda(N_0/H_k)/\varpi^h[F_k]$-submodule of $\pi^{H_k}$.

**Lemma 3.1.2** *For any $M \in \mathcal{M}(\pi^{H_0})$ the $N_0$-subrepresentation $M_k$ lies in $\mathcal{M}_k(\pi^{H_k})$.*

**Proof** Let $\{m_1, \dots, m_r\}$ be a set of generators of $M$ as a $\Lambda(N_0/H_0)/\varpi^h[F]$-module. We claim that the elements $\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^km_i)$ $(i = 1, \dots, r)$ generate $M_k$ as a module over $\Lambda(N_0/H_k)/\varpi^h[F_k]$. Since both $H_k$ and $s^kH_0s^{-k}$ are normalized by $s^kN_0s^{-k}$, for any $u \in N_0$ we have

$$\mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^kus^{-k}\cdot) = (s^kus^{-k}\cdot) \circ \mathrm{Tr}_{H_k/s^kH_0s^{-k}} \ . \tag{3.1}$$

Therefore by continuity we also have

$$\mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k\lambda s^{-k}\cdot) = (s^k\lambda s^{-k}\cdot) \circ \mathrm{Tr}_{H_k/s^kH_0s^{-k}}$$

for any $\lambda \in \Lambda(N_0/H_0)/\varpi^h$. Now writing any $m \in M$ as $m = \sum_{j=1}^r \lambda_j F^{i_j} m_j$ we compute

$$\begin{aligned}\mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \sum_{j=1}^r \lambda_j F^{i_j} m_j) = \sum_{j=1}^r (s^k\lambda s^{-k}) F_k^{i_j} \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^km_j) \in \\ \in \sum_{j=1}^r \Lambda(N_0/H_k)/\varpi^h[F_k]\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^km_j) \ .\end{aligned}$$



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For the stability under the action of $\Gamma$ note that $\Gamma$ normalizes both $H_k$ and $s^kH_0s^{-k}$ and the elements in $\Gamma$ commute with $s$.

Since $M$ is admissible as an $N_0$-representation, $s^kM$ is admissible as a representation of $s^kN_0s^{-k}$. Further by (3.1) the map $\mathrm{Tr}_{H_k/s^kH_0s^{-k}}$ is $s^kN_0s^{-k}$-equivariant therefore its image is also admissible. Finally, $M_k$ can be written as a finite sum

$$\sum_{u\in J(N_0/s^kN_0s^{-k}H_k)} u\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)$$

of admissible representations of $s^kN_0s^{-k}$ therefore the statement. $\square$

**Lemma 3.1.3** *Fix a simple root $\alpha \in \Delta$ such that $\ell(N_{\alpha,0}) = \mathbb{Z}_p$. Then for any $M \in \mathcal{M}(\pi^{H_0})$ the kernel of the trace map*

$$\mathrm{Tr}_{H_0/H_k}\colon Y_k = \sum_{u\in J(N_{\alpha,0}/s^kN_{\alpha,0}s^{-k})} u\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM) \to N_0F^k(M) \qquad (3.2)$$

*is finitely generated over $o$. In particular, the length of $Y_k^\vee[1/X]$ as a module over $o/\varpi^h((X))$ equals the length of $M^\vee[1/X]$.*

**Proof** Since any $u \in N_{\alpha,0} \leq N_0$ normalizes both $H_0$ and $H_k$ and we have $N_{\alpha,0}H_0 = N_0$ by the assumption that $\ell(N_{\alpha,0}) = \mathbb{Z}_p$, the image of the map (3.2) is indeed $N_0F^k(M)$. Moreover, by the proof of Lemma 2.6 in [3] the quotient $M/N_0F^k(M)$ is finitely generated over $o$. Therefore we have $M^\vee[1/X] \cong (N_0F^k(M))^\vee[1/X]$ as a module over $o/\varpi^h((X))$. In particular, their length are equal:

$$l = \mathrm{length}_{o/\varpi^h((X))}M^\vee[1/X] = \mathrm{length}_{o/\varpi^h((X))}(N_0F^k(M))^\vee[1/X] \ .$$

We compute

$$\begin{aligned} l = \mathrm{length}_{o/\varpi^h((X))}M^\vee[1/X] &= \mathrm{length}_{o/\varpi^h((\varphi^k(X)))}(s^kM)^\vee[1/X] \geq \\ &\geq \mathrm{length}_{o/\varpi^h((\varphi^k(X)))}(\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM))^\vee[1/X] = \\ = \mathrm{length}_{o/\varpi^h((X))}(o/\varpi^h[[X]] &\otimes_{o/\varpi^h[[\varphi^k(X)]]} \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM))^\vee[1/X] \geq \\ &\geq \mathrm{length}_{o/\varpi^h((X))}Y_k^\vee[1/X] \ . \end{aligned}$$

By the existence of a surjective map (3.2) we must have equality in the above inequality everywhere. Therefore we have $\mathrm{Ker}(\mathrm{Tr}_{H_0/H_k})^\vee[1/X] = 0$, which shows that $\mathrm{Ker}(\mathrm{Tr}_{H_0/H_k})$ is finitely generated over $o$, because $M$ is admissible, and so is $\mathrm{Ker}(\mathrm{Tr}_{H_0/H_k}) \leq M$. $\square$



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The kernel of the natural homomorphism

$$\Lambda(N_0/H_k)/\varpi^h \to \Lambda(N_0/H_0)/\varpi \cong k[[X]]$$

is a nilpotent prime ideal in the ring $\Lambda(N_0/H_k)/\varpi^h$. We denote the localization at this ideal by $\Lambda(N_0/H_k)/\varpi^h[1/X]$. For the justification of this notation note that any element in $\Lambda(N_0/H_k)/\varpi^h[1/X]$ can uniquely be written as a formal Laurent-series $\sum_{n\gg-\infty} a_n X^n$ with coefficients $a_n$ in the finite group ring $o/\varpi^h[H_0/H_k]$. Here $X$—by an abuse of notation—denotes the element $[u_0]-1$ for an element $u_0 \in N_{\alpha,0} \leq N_0$ with $\ell(u_0) = 1 \in \mathbb{Z}_p$. The ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ admits a conjugation action of the group $\Gamma$ that commutes with the operator $\varphi$ defined by $\varphi(\lambda) = s\lambda s^{-1}$ (for $\lambda \in \Lambda(N_0/H_k)/\varpi^h[1/X]$). A $(\varphi,\Gamma)$-module over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ is a finitely generated module over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ together with a semilinear commuting action of $\varphi$ and $\Gamma$. Note that $\varphi$ is no longer injective on the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ for $k \geq 1$, in particular it is not flat either. However, we still call a $(\varphi,\Gamma)$-module $D_k$ over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ étale if the natural map

$$1 \otimes \varphi \colon \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} D_k \to D_k$$

is an isomorphism of $\Lambda(N_0/H_k)/\varpi^h[1/X]$-modules. For any $M \in \mathcal{M}(\pi^{H_0})$ we put

$$M_k^\vee[1/X] = \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\Lambda(N_0/H_k)/\varpi^h} M_k^\vee$$

where $(\cdot)^\vee$ denotes the Pontryagin dual $\mathrm{Hom}_o(\cdot, K/o)$.

The group $N_0/H_k$ acts by conjugation on the finite $H_0/H_k \lhd N_0/H_k$. Therefore the kernel of this action has finite index. In particular, there exists a positive integer $r$ such that $s^r N_{\alpha,0} s^{-r} \leq N_0/H_k$ commutes with $H_0/H_k$. Therefore the group ring $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$ is contained as a subring in $\Lambda(N_0/H_k)/\varpi^h[1/X]$.

**Lemma 3.1.4** *As modules over the group ring $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$ we have an isomorphism*

$$M_k^\vee[1/X] \to o/\varpi^h((\varphi^r(X)))[H_0/H_k] \otimes_{o/\varpi^h((\varphi^r(X)))} Y_k^\vee[1/X] \ .$$

*In particular, $M_k^\vee[1/X]$ is induced as a representation of the finite group $H_0/H_k$, so the reduced (Tate-) cohomology groups $\tilde{H}^i(H', M_k^\vee[1/X])$ vanish for all subgroups $H' \leq H_0/H_k$ and $i \in \mathbb{Z}$.*



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**Proof** By the definition of $M_k$ we have a surjective $o/\varpi^h[[\varphi^r(X)]][H_0/H_k]$-linear map

$$f\colon o/\varpi^h[[\varphi^r(X)]][H_0/H_k]\otimes_{o/\varpi^h[[\varphi^r(X)]]}Y_k\to M_k$$

sending $\lambda\otimes y$ to $\lambda y$ for $\lambda\in o/\varpi^h[[\varphi^r(X)]][H_0/H_k]$ and $y\in Y_k$. Further, by Lemma 3.1.3 the kernel of the restriction of $f$ to the $H_0/H_k$-invariants

$$(o/\varpi^h[[\varphi^r(X)]][H_0/H_k]\otimes_{o/\varpi^h[[\varphi^r(X)]]}Y_k)^{H_0/H_k}=(\sum_{h\in H_0/H_k}h)\otimes Y_k$$

is finitely generated over $o$. By taking the Pontryagin dual of $f$ and inverting $X$ we obtain an injective $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$-homomorphism

$$\begin{aligned}f^\vee[1/X]\colon M_k^\vee[1/X]\to (o/\varpi^h[[\varphi^r(X)]][H_0/H_k]\otimes_{o/\varpi^h[[\varphi^r(X)]]}Y_k)^\vee[1/X]\cong\\ \cong o/\varpi^h((\varphi^r(X)))[H_0/H_k]\otimes_{o/\varpi^h((\varphi^r(X)))}(Y_k^\vee[1/X])\end{aligned}$$

that becomes surjective after taking $H_0/H_k$-coinvariants. Since $M_k^\vee[1/X]$ is a finite dimensional representation of the finite $p$-group $H_0/H_k$ over the local artinian ring $o/\varpi^h((X))$ with residual characteristic $p$, the map $f^\vee[1/X]$ is in fact an isomorphism as its cokernel has trivial $H_0/H_k$-coinvariants. $\square$

Denote by $H_{k,-}/H_k$ the kernel of the group homomorphism

$$s(\cdot)s^{-1}\colon N_0/H_k\to N_0/H_k\ .$$

It is a finite normal subgroup contained in $H_0/H_k\leq N_0/H_k$. If $k$ is big enough so that $H_k$ is contained in $sH_0s^{-1}$ then we have $H_{k,-}=s^{-1}H_ks$, otherwise we always have $H_{k,-}=H_0\cap s^{-1}H_ks$. The ring homomorphism

$$\varphi\colon \Lambda(N_0/H_k)/\varpi^h\to\Lambda(N_0/H_k)/\varpi^h$$

factors through the quotient map $\Lambda(N_0/H_k)/\varpi^h\twoheadrightarrow\Lambda(N_0/H_{k,-})/\varpi^h$. We denote by $\tilde\varphi$ the induced ring homomorphism

$$\tilde\varphi\colon\Lambda(N_0/H_{k,-})/\varpi^h\to\Lambda(N_0/H_k)/\varpi^h\ .$$

Note that $\tilde\varphi$ is injective and makes $\Lambda(N_0/H_k)/\varpi^h$ a free module of rank

$$\begin{aligned}\nu=|\mathrm{Coker}(s(\cdot)s^{-1}\colon N_0/H_k\to N_0/H_k)|=\\ =p|\mathrm{Coker}(s(\cdot)s^{-1}\colon H_0/H_k\to H_0/H_k)|=\\ =p|\mathrm{Ker}(s(\cdot)s^{-1}\colon H_0/H_k\to H_0/H_k)|=p|H_{k,-}/H_k|\end{aligned}$$

over $\Lambda(N_0/H_{k,-})/\varpi^h$.



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**Lemma 3.1.5** *We have a series of isomorphisms of $\Lambda(N_0/H_k)/\varpi^h[1/X]$-modules*

$$\begin{aligned}\mathrm{Tr}^{-1} = \mathrm{Tr}^{-1}_{H_{k,-}/H_k}\colon (\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h} M_k)^\vee[1/X] &\overset{(1)}{\to} \\ \overset{(1)}{\to} \mathrm{Hom}_{\Lambda(N_0/H_k),\varphi}(\Lambda(N_0/H_k), M_k^\vee[1/X]) &\overset{(2)}{\to} \\ \overset{(2)}{\to} \mathrm{Hom}_{\Lambda(N_0/H_{k,-}),\tilde{\varphi}}(\Lambda(N_0/H_k), (M_k^\vee[1/X])^{H_{k,-}}) &\overset{(3)}{\to} \\ \overset{(3)}{\to} \Lambda(N_0/H_k) \otimes_{\Lambda(N_0/H_{k,-}),\tilde{\varphi}} M_k^\vee[1/X]^{H_{k,-}} &\overset{(4)}{\to} \\ \overset{(4)}{\to} \Lambda(N_0/H_k) \otimes_{\Lambda(N_0/H_{k,-}),\tilde{\varphi}} (M_k^\vee[1/X])_{H_{k,-}} &\overset{(5)}{\to} \\ \overset{(5)}{\to} \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h,\varphi} M_k^\vee[1/X] &\ .\end{aligned}$$

**Proof** (1) follows from the adjoint property of $\otimes$ and Hom. The second isomorphism follows from noting that the action of the ring $\Lambda(N_0/H_k)$ over itself via $\varphi$ factors through the quotient $\Lambda(N_0/H_{k,-})$ therefore $H_{k,-}$ acts trivially on $\Lambda(N_0/H_k)$ via this map. So any module-homomorphism $\Lambda(N_0/H_k) \to M_k^\vee[1/X]$ lands in the $H_{k,-}$-invariant part $M_k^\vee[1/X]^{H_{k,-}}$ of $M_k^\vee[1/X]$. The third isomorphism follows from the fact that $\Lambda(N_0/H_k)$ is a free module over $\Lambda(N_0/H_{k,-})$ via $\tilde{\varphi}$. The fourth isomorphism is given by (the inverse of) the trace map $\mathrm{Tr}_{H_{k,-}/H_k}\colon (M_k^\vee[1/X])_{H_{k,-}} \to M_k^\vee[1/X]^{H_{k,-}}$ which is an isomorphism by Lemma 3.1.4. The last isomorphism follows from the isomorphism $(M_k^\vee[1/X])_{H_{k,-}} \cong \Lambda(N_0/H_{k,-}) \otimes_{\Lambda(N_0/H_k)} M_k^\vee[1/X]$. $\square$

**Remark** Here $\varphi$ always acted only on the ring $\Lambda(N_0/H_k)$, hence denoting $\varphi_t$ the action $n \mapsto tnt^{-1}$ for a fixed $t \in T_+$ and choosing $k$ large enough such that $tH_0t^{-1} \geq H_k$ we get analogously an isomorphism

$$\begin{aligned}\mathrm{Tr}^{-1}_{t^{-1}H_kt/H_k}\colon (\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi_t,\Lambda(N_0/H_k)/\varpi^h} M_k)^\vee[1/X] &\to \\ \to \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h,\varphi_t} M_k^\vee[1/X] &\ .\end{aligned}$$

We denote the composite of the five isomorphisms in Lemma 3.1.5 by $\mathrm{Tr}^{-1}$ emphasising that all but (4) are tautologies. Our main result in this section is the following generalization of Lemma 2.6 in [3].

**Proposition 3.1.6** *The map*

$$\mathrm{Tr}^{-1} \circ (1 \otimes F_k)^\vee[1/X]\colon \quad M_k^\vee[1/X] \to \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] \tag{3.3}$$



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*is an isomorphism of $\Lambda(N_0/H_k)/\varpi^h[1/X]$-modules. Therefore the natural action of $\Gamma$ and the operator*

$$\begin{array}{rcl}\varphi\colon M_k^\vee[1/X] & \to & M_k^\vee[1/X]\\ f & \mapsto & (\mathrm{Tr}^{-1}\circ(1\otimes F_k)^\vee[1/X])^{-1}(1\otimes f)\end{array}$$

*make $M_k^\vee[1/X]$ into an étale $(\varphi,\Gamma)$-module over the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$.*

**Proof** Since $M_k$ is finitely generated over $\Lambda(N_0/H_k)/\varpi^h[F_k]$ by Lemma 3.1.2, the cokernel $C$ of the map

$$1\otimes F_k\colon \Lambda(N_0/H_k)/\varpi^h\otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h} M_k\to M_k \tag{3.4}$$

is finitely generated as a module over $\Lambda(N_0/H_k)/\varpi^h$. Further, it is admissible as a representation of $N_0$ (again by Lemma 3.1.2), therefore $C$ is finitely generated over $o$. In particular, we have $C^\vee[1/X]=0$ showing that (3.3) is injective.

For the surjectivity put $Y_k=\sum_{u\in J(N_{\alpha,0}/s^kN_{\alpha,0}s^{-k})}u\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)$. This is an $o/\varpi^h[[X]]$-submodule of $M_k$. By Lemma 3.1.3 we have

$$\begin{aligned}&\mathrm{length}_{o/\varpi^h((\varphi^r(X)))}(Y_k^\vee[1/X])=\\ &=|N_{\alpha,0}:s^rN_{\alpha,0}s^{-r}|\mathrm{length}_{o/\varpi^h((X))}(Y_k^\vee[1/X])=p^rl\ .\end{aligned}$$

By Lemma 3.1.4 we obtain

$$\begin{aligned}&\mathrm{length}_{o/\varpi^h((\varphi^r(X)))}M_k^\vee[1/X]=\\ &=|H_0:H_k|\cdot\mathrm{length}_{o/\varpi^h((\varphi^r(X)))}Y_k^\vee[1/X]=|H_0:H_k|p^rl\ .\end{aligned}$$

Consider the ring homomorphism

$$\varphi\colon \Lambda(N_0/H_k)/\varpi^h[1/X]\to\Lambda(N_0/H_k)/\varpi^h[1/X]\ . \tag{3.5}$$

Its image is the subring $\Lambda(sN_0s^{-1}H_k/H_k)/\varpi^h[1/\varphi(X)]$ over which the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ is a free module of rank $\nu=|N_0:sN_0s^{-1}H_k|=p|H_{k,-}:H_k|$. So we obtain

$$\begin{aligned}&p\mathrm{length}_{o((\varphi^r(X)))}\Lambda(N_0/H_k)/\varpi^h[1/X]\otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]}M_k^\vee[1/X]=\\ &=\mathrm{length}_{o((\varphi^{r+1}(X)))}\Lambda(N_0/H_k)/\varpi^h[1/X]\otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]}M_k^\vee[1/X]=\\ &=\nu\mathrm{length}_{o((\varphi^{r+1}(X)))}\Lambda(sN_0s^{-1}H_k/H_k)/\varpi^h[1/\varphi(X)]\otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]}\\ &\otimes M_k^\vee[1/X]\overset{(*)}{=}\nu\mathrm{length}_{o((\varphi^r(X)))}M_k^\vee[1/X]_{H_{k,-}}=\\ &=\nu\mathrm{length}_{o((\varphi^r(X)))}(o/\varpi^h[H_0/H_{k,-}]\otimes_{o/\varpi^h}Y_k^\vee[1/X])=\\ &=\nu|H_0:H_{k,-}|p^rl=p|H_0:H_k|p^rl=p\mathrm{length}_{o/\varpi^h((\varphi^r(X)))}M_k^\vee[1/X]\ .\end{aligned}$$



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Here the equality $(*)$ follows from the fact that the map $\varphi$ induces an isomorphism between $\Lambda(N_0/H_{k,-})/\varpi^h[1/X]$ and $\Lambda(sN_0s^{-1}H_k/H_k)/\varpi^h[1/\varphi(X)]$ sending the subring $o((\varphi^r(X)))$ isomorphically onto $o((\varphi^{r+1}(X)))$.

This shows that (3.3) is an isomorphism as it is injective and the two sides have equal length as modules over the artinian ring $o/\varpi^h((X))$. $\square$

**Remark** We also obtain in particular that the map (3.4) has finite kernel and cokernel. Hence there exists a finite $\Lambda(N_0/H_k)/\varpi^h$-submodule $M_{k,*}$ of $M_k$ such that the kernel of $1 \otimes F_k$ is contained in the image of $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_{k,*}$ in $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_k$. We denote by $M_k^*$ the image of $1 \otimes F_k$.

Note that for $k = 0$ we have $M_0 = M$. Let now $0 \leq j \leq k$ be two integers. By Lemma 3.1.4 the space of $H_j$-invariants of $M_k$ is equal to $\mathrm{Tr}_{H_j/H_k}(M_k)$ upto finitely generated modules over $o$. On the other hand, we compute

$$\begin{aligned} N_0F_j^{k-j}(M_j) = N_0\mathrm{Tr}_{H_j/s^{k-j}H_js^{j-k}} \circ (s^{k-j}\cdot) \circ \mathrm{Tr}_{H_j/s^jH_0s^{-j}}(s^jM) = \\ = N_0\mathrm{Tr}_{H_j/s^kH_0s^{-k}}(s^kM) = N_0\mathrm{Tr}_{H_j/H_k} \circ \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM) = \\ = \mathrm{Tr}_{H_j/H_k}(N_0\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)) = \mathrm{Tr}_{H_j/H_k}(M_k) \end{aligned}$$

since both $H_k$ and $H_j$ are normal in $N_0$ whence we have $(u\cdot) \circ \mathrm{Tr}_{H_j/H_k} = \mathrm{Tr}_{H_j/H_k} \circ (u\cdot)$ for all $u \in N_0$. So taking $H_j/H_k$-coinvariants of $M_k^\vee[1/X]$, we have a natural identification

$$\begin{aligned} M_k^\vee[1/X]_{H_j/H_k} \cong (M_k^{H_j/H_k})^\vee[1/X] \cong \\ \cong (\mathrm{Tr}_{H_j/H_k}(M_k))^\vee[1/X] = (N_0F_j^{k-j}(M_j))^\vee[1/X] \cong M_j^\vee[1/X] \end{aligned} \tag{3.6}$$

induced by the inclusion $N_0F_j^{k-j}(M_j) \subseteq M_k^{H_j} \subseteq M_k$.

**Lemma 3.1.7** *We have* $\mathrm{Tr}_{H_j/H_k} \circ F_k = F_j \circ \mathrm{Tr}_{H_j/H_k}$.

**Proof** We compute

$$\begin{aligned} \mathrm{Tr}_{H_j/H_k} \circ F_k = \mathrm{Tr}_{H_j/H_k} \circ \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot) = \\ \mathrm{Tr}_{H_j/sH_ks^{-1}} \circ (s\cdot) = \mathrm{Tr}_{H_j/sH_js^{-1}} \circ \mathrm{Tr}_{sH_js^{-1}/sH_ks^{-1}}(s\cdot) = \\ \mathrm{Tr}_{H_j/sH_js^{-1}} \circ (s\cdot)\mathrm{Tr}_{H_j/H_k} = F_j \circ \mathrm{Tr}_{H_j/H_k} \ . \end{aligned}$$

$\square$



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**Proposition 3.1.8** *The identification* (3.6) *is* $\varphi$ *and* $\Gamma$*-equivariant.*

**Proof** It suffices to treat the case when $k$ is large enough so that we have $H_{k,-} = s^{-1}H_k s$. So from now on we assume $H_k \leq sH_0s^{-1} \leq sN_0s^{-1}$. As $\Gamma$ acts both on $M_k$ and $M_j$ by multiplication coming from the action of $\Gamma$ on $\pi$, the map (3.6) is clearly $\Gamma$-equivariant. In order to avoid confusion we are going to denote the map $\varphi$ on $M_k^\vee[1/X]$ (resp. on $M_j^\vee[1/X]$) temporarily by $\varphi_k$ (resp. by $\varphi_j$). Let $f$ be in $M_k^\vee$ such that its restriction to $M_{k,*}$ is zero (see the Remark after Prop. 3.1.6).

We regard $f$ as an element in $(M_k^*/M_{k,*})^\vee \leq (M_k^*)^\vee$. We are going to compute $\varphi_k(f)$ and $\varphi_j(f_{|\mathrm{Tr}_{H_j/H_k}(M_k^*)})$ explicitly and find that the restriction of $\varphi_k(f)$ to $\mathrm{Tr}_{H_j/H_k}(M_k^*)$ is equal to $\varphi_j(f_{|\mathrm{Tr}_{H_j/H_k}(M_k^*)})$. Note that we have an isomorphism $M_k^\vee[1/X] \cong M_k^{*\vee}[1/X] \cong (M_k^*/M_{k,*})^\vee[1/X]$ (resp. $M_j^\vee[1/X] \cong \mathrm{Tr}_{H_j/H_k}(M_k^*)^\vee[1/X]$).

Let $m \in M_k^* \leq M_k$ be in the form

$$m = \sum_{u \in J((N_0/H_k)/s(N_0/H_k)s^{-1})} uF_k(m_u)$$

with elements $m_u \in M_k$ for $u \in J((N_0/H_k)/s(N_0/H_k)s^{-1})$. By the remark after Proposition 3.1.6 $M_k^*$ is a finite index submodule of $M_k$. Note that the elements $m_u$ are unique upto $M_{k,*} + \mathrm{Ker}(F_k)$. Therefore $\varphi_k(f) \in (M_k^*)^\vee$ is well-defined by our assumption that $f_{|M_{k,*}} = 0$ noting that the kernel of $F_k$ equals the kernel of $\mathrm{Tr}_{H_{k,-}/H_k}$ since the multiplication by $s$ is injective and we have $F_k = s \circ \mathrm{Tr}_{H_{k,-}/H_k}$. So we compute

$$\begin{aligned}\varphi_k(f)(m) = ((1\otimes F_k)^\vee)^{-1}(\mathrm{Tr}_{H_{k,-}/H_k}(1\otimes f))(m) = \\ = ((1\otimes F_k)^\vee)^{-1}(1\otimes \mathrm{Tr}_{H_{k,-}/H_k}(f))(\sum_{u \in J((N_0/H_k)/s(N_0/H_k)s^{-1})} uF_k(m_u)) = \\ = \mathrm{Tr}_{H_{k,-}/H_k}(f)(F_k^{-1}(u_0F_k(m_{u_0}))) = f(\mathrm{Tr}_{H_{k,-}/H_k}((s^{-1}u_0s)m_{u_0}))\end{aligned} \tag{3.7}$$

where $u_0$ is the single element in $J(N_0/sN_0s^{-1})$ corresponding to the coset of 1. In order to simplify notation put $f_*$ for the restriction of $f$ to $\mathrm{Tr}_{H_j/H_k}(M_k)$ and

$$U = J(N_0/sN_0s^{-1}) \cap H_jsN_0s^{-1} \ .$$

Note that we have $0 = \varphi_j(f_*)(uF_j(m'))$ for all $m' \in M_j$ and

$$u \in J(N_0/sN_0s^{-1}) \setminus U \ .$$



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Therefore using Lemma 3.1.7 we obtain

$$\begin{aligned}
\varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} m) &= \varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} \sum_{u\in J(N_0/sN_0s^{-1})} uF_k(m_u)) = \\
&= \varphi_j(f_*)(\sum_{u\in J(N_0/sN_0s^{-1})} uF_j \circ \mathrm{Tr}_{H_j/H_k}(m_u)) = \\
&= \sum_{u\in U} f(\mathrm{Tr}_{H_{j,-}/H_j}(s^{-1}\overline{u}s\mathrm{Tr}_{H_j/H_k}(m_u))) = \\
&= \sum_{u\in U} f(s^{-1}\overline{u}s\mathrm{Tr}_{H_{j,-}/H_k}(m_u))
\end{aligned} \tag{3.8}$$

where for each $u \in U$ we choose a fixed $\overline{u}$ in $sN_0s^{-1} \cap H_ju$. Note that $f(s^{-1}\overline{u}s\mathrm{Tr}_{H_{j,-}/H_k}(m_u))$ does not depend on this choice: If $\overline{u_1} \in sN_0s^{-1} \cap H_ju$ is another choice then we have $(\overline{u_1})^{-1}\overline{u} \in sN_0s^{-1} \cap H_j$ whence $s^{-1}(\overline{u_1})^{-1}\overline{u}s$ lies in $H_{j,-} = N_0 \cap s^{-1}H_js$ so we have

$$\begin{aligned}
s^{-1}\overline{u}s\mathrm{Tr}_{H_{j,-}/H_k}(m_u) &= s^{-1}\overline{u_1}ss^{-1}(\overline{u_1})^{-1}\overline{u}s\mathrm{Tr}_{H_{j,-}/H_k}(m_u) = \\
&= s^{-1}\overline{u_1}s\mathrm{Tr}_{H_{j,-}/H_k}(m_u) \ .
\end{aligned}$$

Moreover, the equation (3.8) also shows that $\varphi_j(f_*)$ is a well-defined element in $(\mathrm{Tr}_{H_j/H_k}(M_k^*))^\vee$. On the other hand, for the restriction of $\varphi_k(f)$ to $\mathrm{Tr}_{H_j/H_k}(M_k)$ we compute

$$\begin{aligned}
\varphi_k(f)(\mathrm{Tr}_{H_j/H_k} m) &= \varphi_k(f)(\sum_{w\in J(H_j/H_k)} w \sum_{u\in J(N_0/sN_0s^{-1})} uF_k(m_u)) = \\
&= \sum_{w\in J(H_j/H_k)} \sum_{u\in J(N_0/sN_0s^{-1})} \varphi_k(f)(wuF_k(m_u)) = \\
&= \sum_{\substack{u\in U \\ w\in J(H_j/H_k)\cap(sN_0s^{-1}u^{-1})}} f(\mathrm{Tr}_{H_{k,-}/H_k}((s^{-1}wus)m_u)) = \\
&= f(\sum_{v=s^{-1}wu\overline{u}^{-1}s\in J(H_{j,-}/H_{k,-})} \mathrm{Tr}_{H_{k,-}/H_k} \sum_{u\in U} vs^{-1}\overline{u}sm_u) = \\
&= \sum_{u\in U} f(s^{-1}\overline{u}s\mathrm{Tr}_{H_{j,-}/H_k}(m_u))
\end{aligned}$$

that equals $\varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} m)$ by (3.8). Finally, let now $f \in M_k^\vee$ be arbitrary. Since $M_{k,*}$ is finite, there exists an integer $r \geq 0$ such that $X^r f$



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vanishes on $M_{k,*}$. By the above discussion we have $\varphi_k(X^r f)(\mathrm{Tr}_{H_j/H_k} m) = \varphi_j(X^r f_*)(\mathrm{Tr}_{H_j/H_k} m)$. The statement follows noting that $\varphi(X^r)$ is invertible in the ring $\Lambda(N_0/H_j)/\varpi^h[1/X]$. $\square$

So we may take the projective limit $M_\infty^\vee[1/X] = \varprojlim_k M_k^\vee[1/X]$ with respect to these quotient maps. The resulting object is an étale $(\varphi,\Gamma)$-module over the ring

$$\varprojlim_k \Lambda(N_0/H_k)/\varpi^h[1/X] \cong \Lambda_\ell(N_0)/\varpi^h \ .$$

Moreover, by taking the projective limit of (3.6) with respect to $k$ we obtain a $\varphi$- and $\Gamma$-equivariant isomorphism $(M_\infty^\vee[1/X])_{H_j} \cong M_j^\vee[1/X]$. So we just proved

**Corollary 3.1.9** *For any object $M \in \mathcal{M}(\pi^{H_0})$ the $(\varphi,\Gamma)$-module $M^\vee[1/X]$ over $o/\varpi^h((X))$ corresponds to $M_\infty^\vee[1/X]$ via the equivalence of categories in Theorem 8.20 in [18].*

Note that whenever $M \subset M'$ are two objects in $\mathcal{M}(\pi^{H_0})$ then we have a natural surjective map $M'^\vee_\infty[1/X] \twoheadrightarrow M_\infty^\vee[1/X]$. So in view of the above corollary we define

$$D^\vee_{\xi,\ell,\infty}(\pi) = \varprojlim_{k\geq 0, M\in\mathcal{M}(\pi^{H_0})} M_k^\vee[1/X] = \varprojlim_{M\in\mathcal{M}(\pi^{H_0})} M_\infty^\vee[1/X] \ .$$

We call two elements $M, M' \in \mathcal{M}(\pi^{H_0})$ equivalent ($M \sim M'$) if the inclusions $M \subseteq M + M'$ and $M' \subseteq M + M'$ induce isomorphisms $M^\vee[1/X] \cong (M+M')^\vee[1/X] \cong M'^\vee[1/X]$. This is equivalent to the condition that $M$ equals $M'$ upto finitely generated $o$-modules. In particular, this is an equivalence relation on the set $\mathcal{M}(\pi^{H_0})$. Similarly, we say that $M_k, M_k' \in \mathcal{M}_k(\pi^{H_k})$ are equivalent if the inclusions $M_k \subseteq M_k + M_k'$ and $M_k' \subseteq M_k + M_k'$ induce isomorphisms

$$M_k^\vee[1/X] \cong (M_k + M_k')^\vee[1/X] \cong M_k'^\vee[1/X].$$

**Proposition 3.1.10** *The maps*

$$\begin{array}{rcl} M & \mapsto & N_0 \mathrm{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M) \\ \mathrm{Tr}_{H_0/H_k}(M_k) & \leftarrow\!\shortmid & M_k \end{array}$$



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*induce a bijection between the sets $\mathcal{M}(\pi^{H_0})/\sim$ and $\mathcal{M}_k(\pi^{H_k})/\sim$. In particular, we have*

$$D^\vee_{\xi,\ell,\infty}(\pi) = \varprojlim_{k\geq 0} \varprojlim_{M_k\in\mathcal{M}_k(\pi^{H_k})} M_k^\vee[1/X] \ .$$

**Proof** We have $\mathrm{Tr}_{H_0/H_k}(N_0\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)) = N_0\mathrm{Tr}_{H_0/s^kH_0s^{-k}}(s^kM) = N_0F^k(M)$ which is equivalent to $M$. Conversely,

$$N_0\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^k\mathrm{Tr}_{H_0/H_k}(M_k)) = N_0\mathrm{Tr}_{H_k/s^kH_ks^{-k}}(s^kM_k) = N_0F_k^k(M_k)$$

is equivalent to $M_k$ as it is the image of the map

$$1\otimes F_k^k\colon \Lambda(N_0/H_k)/\varpi^h\otimes_{\varphi^k,\Lambda(N_0/H_k)/\varpi^h}\to M_k$$

having finite cokernel. □

We equip the pseudocompact $\Lambda_\ell(N_0)$-module $D^\vee_{\xi,\ell,\infty}(\pi)$ with the weak topology, ie. with the projective limit topology of the weak topologies of $M^\vee_\infty[1/X]$. (The weak topology on $\Lambda_\ell(N_0)$ is defined in section 8 of [17].) Recall that the sets

$$O(M,l,l') = f^{-1}_{M,l}(\Lambda(N_0/H_l)\otimes_{u_\alpha} X^{l'}M^\vee[1/X]^{++}) \qquad (3.9)$$

for $l,l'\geq 0$ and $M\in\mathcal{M}(\pi^{H_0})$ form a system of neighbourhoods of 0 in the weak topology of $D^\vee_{\xi,\ell,\infty}(\pi)$. Here $f_{M,l}$ is the natural projection map $f_{M,l}\colon D^\vee_{\xi,\ell,\infty}(\pi)\twoheadrightarrow M_l^\vee[1/X]$ and $M^\vee[1/X]^{++}$ denotes the set of elements $d\in M^\vee[1/X]$ with $\varphi^n(d)\to 0$ in the weak topology of $M^\vee[1/X]$ as $n\to\infty$.

## 3.2 A natural transformation from $D_{SV}$ to $D^\vee_{\xi,\ell,\infty}$

**Lemma 3.2.1** *Let $W$ be in $\mathcal{B}_+(\pi)$ and $M\in\mathcal{M}(\pi^{H_0})$. There exists a positive integer $k_0>0$ such that for all $k\geq k_0$ we have $s^kM\subseteq W$. In particular, both $M_k = N_0\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)$ and $N_0F^k(M)$ are contained in $W$ for all $k\geq k_0$.*

**Proof** By the assumption that $M$ is finitely generated over $\Lambda(N_0/H_0)/\varpi^h[F]$ and $W$ is a $B_+$-subrepresentation it suffices to find an integer $s^{k_0}$ such that we have $s^{k_0}m_i$ lies in $W$ for all the generators $m_1,\dots,m_r$ of $M$. This, however, follows from Lemma 2.1 in [17] noting that the powers of $s$ are cofinal in $T_+$. □



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In particular, we have a homomorphism $W^\vee \twoheadrightarrow M_k^\vee$ of $\Lambda(N_0)$-modules induced by this inclusion. We compose this with the localization map $M_k^\vee \to M_k^\vee[1/X]$ and take projective limits with respect to $k$ in order to obtain a $\Lambda(N_0)$-homomorphism

$$\mathrm{pr}_{W,M}\colon W^\vee \to M_\infty^\vee[1/X] \ .$$

**Lemma 3.2.2** *The map* $\mathrm{pr}_{W,M}$ *is* $\psi_s$*- and* $\Gamma$*-equivariant.*

**Proof** The $\Gamma$-equivariance is clear as it is given by the multiplication by elements of $\Gamma$ on both sides. For the $\psi_s$-equivariance let $k>0$ be large enough so that $H_k$ is contained in $sH_0s^{-1} \leq sN_0s^{-1}$ (ie. $H_{k,-} = s^{-1}H_ks$) and $M_k$ is contained in $W$. Let $f$ be in $W^\vee = \mathrm{Hom}_o(W, o/\varpi^h)$ such that $f_{|N_0sM_{k,*}} = 0$. By definition we have $\psi_s(f)(w) = f(sw)$ for any $w \in W$. Denote the restriction of $f$ to $M_k$ by $f_{|M_k}$ and choose an element $m \in M_k^* \leq M_k$ written in the form

$$m = \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u) = \sum_{u \in J(N_0/sN_0s^{-1})} us\mathrm{Tr}_{H_{k,-}/H_k}(m_u) \ .$$

Then we compute

$$\begin{aligned}
f_{|M_k}(m) &= \sum_{u \in J(N_0/sN_0s^{-1})} f(us\mathrm{Tr}_{H_{k,-}/H_k}(m_u)) = \\
&= \sum_{u \in J(N_0/sN_0s^{-1})} (u^{-1}f)(s\mathrm{Tr}_{H_{k,-}/H_k}(m_u)) = \\
&= \sum_{u \in J(N_0/sN_0s^{-1})} \psi_s(u^{-1}f)(\mathrm{Tr}_{H_{k,-}/H_k}(m_u)) = \\
&\overset{(3.7)}{=} \sum_{u \in J(N_0/sN_0s^{-1})} \varphi(\psi_s(u^{-1}f)_{|M_k})(F_k(m_u)) = \\
&= \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi_s(u^{-1}f)_{|M_k})(uF_k(m_u)) = \\
&= \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi_s(u^{-1}f)_{|M_k})(m)
\end{aligned}$$

as for distinct $u, v \in J(N_0/sN_0s^{-1})$ we have $u\varphi(f_0)(vF_k(m_v)) = 0$ for any $f_0 \in (M_k^*)^\vee$. So by inverting $X$ and taking projective limits with respect to



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$k$ we obtain

$$\mathrm{pr}_{W,M}(f) = \sum_{u\in J(N_0/sN_0s^{-1})} u\varphi(\mathrm{pr}_{W,M}(\psi_s(u^{-1}f)))$$

as we have $(M_k^*)^\vee[1/X] \cong M_k^\vee[1/X]$. However, since $M_\infty^\vee[1/X]$ is an étale $(\varphi,\Gamma)$-module over $\Lambda_\ell(N_0)/\varpi^h$ we have a unique decomposition of $\mathrm{pr}_{W,M}(f)$ as

$$\mathrm{pr}_{W,M}(f) = \sum_{u\in J(N_0/sN_0s^{-1})} u\varphi(\psi(u^{-1}\mathrm{pr}_{W,M}(f)))$$

so we must have $\psi(\mathrm{pr}_{W,M}(f)) = \mathrm{pr}_{W,M}(\psi_s(f))$. For general $f\in W^\vee$ note that $N_0sM_{k,*}$ is killed by $\varphi(X^r)$ for $r\geq 0$ big enough, so we have

$$\begin{aligned} X^r\psi(\mathrm{pr}_{W,M}(f)) = \psi(\mathrm{pr}_{W,M}(\varphi(X^r)f)) = \\ = \mathrm{pr}_{W,M}(\psi_s(\varphi(X^r)f)) = X^r\mathrm{pr}_{W,M}(\psi_s(f)). \end{aligned}$$

The statement follows since $X^r$ is invertible in $\Lambda_\ell(N_0)$. $\square$

By taking the projective limit with respect to $M\in\mathcal{M}(\pi^{H_0})$ and the injective limit with respect to $W\in\mathcal{B}_+(\pi)$ we obtain a $\psi_s$- and $\Gamma$-equivariant $\Lambda(N_0)$-homomorphism

$$\mathrm{pr} = \varinjlim_W \varprojlim_M \mathrm{pr}_{W,M}\colon D_{SV}(\pi)\to D^\vee_{\xi,\ell,\infty}(\pi)\ .$$

**Remarks** 1. The natural maps $\pi^\vee\to D^\vee_\xi(\pi)$ and $\pi^\vee\to D^\vee_{\xi,\ell,\infty}(\pi)$ both factor through the map $\pi^\vee\twoheadrightarrow D_{SV}(\pi)$.

2. The natural topology on $D_{SV}$ obtained as the quotient topology from the compact topology on $\pi^\vee$ via the surjective map $\pi^\vee\twoheadrightarrow D_{SV}(\pi)$ is compact, but may not be Hausdorff in general. However, if $\mathcal{B}_+(\pi)$ contains a minimal element (as in the case of the principal series see Proposition 2.3.2) then it is also Hausdorff. However, the map pr factors through the maximal Hausdorff quotient of $D_{SV}(\pi)$, namely $\overline{D}_{SV}(\pi) = (\bigcap_{W\in\mathcal{B}_+(\pi)} W)^\vee$. Indeed, pr is continuous and $D^\vee_{\xi,\ell,\infty}(\pi)$ is Hausdorff, so the kernel of pr is closed in $D_{SV}(\pi)$ (and contains 0).

3. Assume that $h=1$, ie. $\pi$ is a smooth representation in characteristic $p$. Then $D^\vee_{\xi,\ell,\infty}(\pi)$ has no nonzero $\Lambda(N_0)/\varpi$-torsion. Hence the $\Lambda(N_0)/\varpi$-torsion part of $D_{SV}(\pi)$ is contained in the kernel of pr.



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4. If $D_{SV}(\pi)$ has finite rank and its torsion free part is étale over $\Lambda(N_0)$ then $\Lambda_\ell(N_0)\otimes_{\Lambda(N_0)} D_{SV}(\pi)$ is also étale and of finite rank $r$ over $\Lambda_\ell(N_0)$. Moreover, the map $\Lambda_\ell(N_0)\otimes_{\Lambda(N_0)}\mathrm{pr} : \Lambda_\ell(N_0)\otimes_{\Lambda(N_0)}D_{SV}(\pi) \to D_{\xi,\ell,\infty}(\pi)$ has dense image by Lemma 3.2.1. Thus $D^\vee_{\xi,\ell,\infty}(\pi)$ has rank at most $r$ over $\Lambda_\ell(N_0)$.

One can show the above Remark 2 algebraically, too. Let $M \in \mathcal{M}(\pi^{H_0})$ be arbitrary. Then the map $1 \otimes \mathrm{id}_{M^\vee} \colon M^\vee \to M^\vee[1/X]$ has finite kernel, so the image $(1 \otimes \mathrm{id}_{M^\vee})(M^\vee)$ is isomorphic to $M_0^\vee$ for some finite index submodule $M_0 \leq M$. Moreover, $M_0^\vee$ is a $\psi$- and $\Gamma$-invariant treillis in $D = M^\vee[1/X] = M_0^\vee[1/X]$. Therefore the map $(1 \otimes F)^\vee$ is injective on $M_0^\vee$ since it is injective after inverting $X$ and $M_0^\vee$ has no $X$-torsion. This means that $1 \otimes F \colon o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]],\varphi} M_0 \to M_0$ is surjective, ie. we have $M_0 = N_0F^k(M_0)$ for all $k \geq 0$. However, for any $W \in \mathcal{B}_+(\pi)$ and $k$ large enough (depending a priori on $W$) we have $N_0F^k(M_0) \subseteq W$, so we deduce $M_0 \subset \cap_{W\in\mathcal{B}_+} W$.

**Corollary 3.2.3** *If $\pi = \mathrm{Ind}_{B_0}^B \pi_0$ is a compactly induced representation of $B$ for some smooth $o/\varpi^h$-representation $\pi_0$ of $B_0$ then we have $D^\vee_\xi(\pi) = 0$. In particular, $D^\vee_\xi$ is not exact on the category of smooth $o/\varpi^h$-representations of $B$. (However, it may still be exact on a smaller subcategory with additional finiteness conditions.)*

**Proof** By the 2nd remark above the map $\pi^\vee \to D^\vee_\xi(\pi)$ factors through the maximal Hausdorff quotient $\overline{D}_{SV}(\pi)$ of $D_{SV}(\pi)$. By Lemma 3.2 in [17], we have $\overline{D}_{SV}(\pi) = (\bigcap_\sigma W_\sigma)^\vee$ where the $B_+$-subrepresentations $W_\sigma$ are indexed by order-preserving maps $\sigma\colon T_+/T_0 \to \mathrm{Sub}(\pi_0)$ where $\mathrm{Sub}(\pi_0)$ is the partially order set of $B_0$-subrepresentations of $\pi_0$. The explicit description of the $B_+$-subrepresentations $W_\sigma$ (there denoted by $M_\sigma$) before Lemma 3.2 in [17] shows that we have in fact $\bigcap_\sigma W_\sigma = \{0\}$ whence the natural map $\pi^\vee \to D^\vee_\xi(\pi)$ is zero. However, by the construction of this map this can only be zero if $D^\vee_\xi(\pi) = 0$.

Since the principal series arises as a quotient of a compactly induced representation, the exactness of $D^\vee_\xi$ would imply the vanishing of $D^\vee_\xi$ on the principal series, too—which is not the case by Ex. 7.6 in [3]. $\square$

**Proposition 3.2.4** *Let $D$ be an étale $(\varphi,\Gamma)$-module over $\Lambda_\ell(N_0)/\varpi^h$, and $f : D_{SV}(\pi) \to D$ be a continuous $\psi_s$ and $\Gamma$-equivariant $\Lambda(N_0)$-homomorphism.*



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*Then $f$ factors uniquely through* pr, *ie. there exists a unique $\psi$- and $\Gamma$-equivariant $\Lambda(N_0)$-homomorphism $\hat{f} : D^\vee_{\xi,\ell,\infty}(\pi) \to D$ such that $f = \hat{f} \circ \mathrm{pr}$.*

**Proof** Note that the uniqueness of $\hat{f}$ follows from Lemma 3.2.1 since any continuous $\Lambda_\ell(N_0)$-homomorphism of $D^\vee_{\xi,\ell,\infty}(\pi)$ factors through $M^\vee_\infty[1/X]$ for some $M \in \mathcal{M}(\pi^{H_0})$. Indeed, if $\hat{f}'$ is another lift then the image of pr is contained in the kernel of $\hat{f} - \hat{f}'$.

At first we construct a homomorphism $\hat{f}_{H_0} : D^\vee_\xi = (D^\vee_{\xi,\ell,\infty})_{H_0} \to D_{H_0}$ such that the following diagram commutes:

$$\begin{tikzcd}
D_{SV}(\pi) \arrow[r, "\mathrm{pr}"] \arrow[dr, "f"'] & D^\vee_{\xi,\ell,\infty}(\pi) \arrow[r, "(\cdot)_{H_0}"] & D^\vee_\xi(\pi) \arrow[d, "\hat{f}_{H_0}"] \\
 & D \arrow[r, "(\cdot)_{H_0}"'] & D_{H_0}
\end{tikzcd}$$

Consider the composite map $f' : \pi^\vee \to D_{SV}(\pi) \xrightarrow{f} D \to D_{H_0}$. Note that $f'$ is continuous and $D_{H_0}$ is Hausdorff, so $\mathrm{Ker}(f')$ is closed in $\pi^\vee$. Therefore $M_0 = (\pi^\vee/\mathrm{Ker}(f'))^\vee$ is naturally a subspace in $\pi$. We claim that $M_0$ lies in $\mathcal{M}(\pi^{H_0})$. Indeed, $M_0^\vee$ is a quotient of $\pi^\vee_{H_0}$, hence $M_0 \leq \pi^{H_0}$ and it is $\Gamma$-invariant since $f'$ is $\Gamma$-equivariant. $M_0$ is admissible because it is discrete, hence $M_0^\vee$ is compact, equivalently finitely generated over $o/\varpi^h[[X]]$, because $M_0^\vee$ can be identified with a $o/\varpi^h[[X]]$-submodule of $D_{H_0}$ which is finitely generated over $o/\varpi^h((X))$. The last thing to verify is that $M$ is finitely generated over $o/\varpi^h[[X]][F]$, which follows from the following

**Lemma 3.2.5** *Let $D$ be an étale $(\varphi,\Gamma)$-module over $o/\varpi^h((X))$ and $D_0 \subset D$ be a $\psi$ and $\Gamma$-invariant compact (or, equivalently, finitely generated) $o/\varpi^h[[X]]$ submodule. Then $D_0^\vee$ is finitely generated as a module over $o/\varpi^h[[X]][F]$ where for any $m \in D_0^\vee = \mathrm{Hom}_o(D_0, o/\varpi^h)$ we put $F(m)(f) = m(\psi(f))$ (for all $f \in D_0$).*

**Proof** As the extension of finitely generated modules over a ring is again finitely generated, we may assume without loss of generality that $h = 1$ and $D$ is irreducible, ie. $D$ has no nontrivial étale $(\varphi,\Gamma)$-submodule over $o/\varpi((X))$.

If $D_0 = \{0\}$ then there is nothing to prove. Otherwise $D_0$ contains the smallest $\psi$ and $\Gamma$ stable $o[[X]]$-submodule $D^\natural$ of $D$. So let $0 \neq m \in D_0^\vee$



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be arbitrary such that the restriction of $m$ to $D^\natural$ is nonzero and consider the $o/\varpi[[X]][F]$-submodule $M = o/\varpi[[X]][F]m$ of $D_0^\vee$ generated by $m$. We claim that $M$ is not finitely generated over $o$. Suppose for contradiction that the elements $F^r m$ are not linearly independent over $o/\varpi$. Then we have a polynomial $P(x) = \sum_{i=0}^n a_i x^i \in o/\varpi[x]$ such that $0 = P(F)m(f) = m(\sum a_i \psi^i(f)) = m(P(\psi)f)$ for any $f \in D^\natural \subset D_0$. However, $P(\psi)\colon D^\natural \to D^\natural$ is surjective by Prop. II.5.15. in [5], so we obtain $m_{|D^\natural} = 0$ which is a contradiction. In particular, we obtain that $M^\vee[1/X] \neq 0$. However, note that $M^\vee[1/X]$ has the structure of an étale $(\varphi,\Gamma)$-module over $o/\varpi((X))$ by Lemma 2.6 in [3]. Indeed, $M$ is admissible, $\Gamma$-invariant, and finitely generated over $o/\varpi[[X]][F]$ by construction. Moreover, we have a natural surjective homomorphism $D = D_0[1/X] = (D_0^\vee)^\vee[1/X] \to M^\vee[1/X]$ which is an isomorphism as $D$ is assumed to be irreducible. Therefore we have $(D_0^\vee/M)^\vee[1/X] = 0$ showing that $D_0^\vee/M$ is finitely generated over $o$. In particular, both $M$ and $D_0^\vee/M$ are finitely generated over $o/\varpi[[X]][F]$ therefore so is $D_0^\vee$. $\square$

Now $D_0 = M_0^\vee$ is a $\psi$- and $\Gamma$-invariant $o/\varpi^h[[X]]$-submodule of $D$ therefore we have an injection $f_0\colon M_0^\vee[1/X] \hookrightarrow D$ of étale $(\varphi,\Gamma)$-modules. The map $\hat{f}_{H_0}\colon D_\xi^\vee \to D_{H_0}$ is the composite map $D_\xi^\vee \twoheadrightarrow M_0^\vee[1/X] \hookrightarrow D$. It is well defined and makes the above diagram commutative, because the map

$$\pi^\vee \to D_{SV}(\pi) \xrightarrow{\mathrm{pr}} D_{\xi,\ell,\infty}^\vee(\pi) \xrightarrow{(\cdot)_{H_0}} D_\xi^\vee(\pi) \to M_0^\vee[1/X]$$

is the same as $\pi^\vee \to M_0^\vee \to M_0^\vee[1/X]$.

Finally, by Corollary 3.1.9 $M^\vee[1/X]$ (resp. $D_{H_0}$) corresponds to $M_\infty^\vee[1/X]$ (resp. to $D$) via the equivalence of categories in Theorem 8.20 in [18] therefore $f_0$ can uniquely be lifted to a $\varphi$- and $\Gamma$-equivariant $\Lambda_\ell(N_0)$-homomorphism $f_\infty\colon M_\infty^\vee[1/X] \hookrightarrow D$. The map $\hat{f}$ is defined as the composite $D_{\xi,\ell,\infty}^\vee \twoheadrightarrow M_\infty^\vee[1/X] \hookrightarrow D$. Now the image of $f - \hat{f}\circ\mathrm{pr}$ is a $\psi_s$-invariant $\Lambda(N_0)$-submodule in $(H_0-1)D$ therefore it is zero by Lemma 8.17 and the proof of Lemma 8.18 in [18]. Indeed, for any $x \in D_{SV}(\pi)$ and $k \geq 0$ we may write $(f-\hat{f}\circ\mathrm{pr})(x)$ in the form $\sum_{u\in J(N_0/s^kN_0s^{-k})} u\varphi^k((f-\hat{f}\circ\mathrm{pr})(\psi^k(u^{-1}x)))$ that lies in $(H_k-1)D$. $\square$

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## 3.3 Étale hull

In this section we construct the étale hull of $D_{SV}(\pi)$: an étale $T_+$-module $\widetilde{D_{SV}}(\pi)$ over $\Lambda(N_0)$ with an injection $\iota : D_{SV}(\pi) \to \widetilde{D_{SV}}(\pi)$ with the following universal property: For any étale $(\varphi, \Gamma)$-module $D'$ over $\Lambda(N_0)$, and $\psi_s$- and $\Gamma$-equivariant map $f : D_{SV}(\pi) \to D'$, $f$ factors through $\widetilde{D_{SV}}(\pi)$, ie. there exists a unique $\psi$- and $\Gamma$-equivariant $\Lambda(N_0)$-homomorphism $\widetilde{f} : \widetilde{D_{SV}}(\pi) \to D'$ making the diagram

$$\begin{tikzcd} D_{SV}(\pi) \arrow[r, "\iota"] \arrow[d, "f"'] & \widetilde{D_{SV}}(\pi) \arrow[dl, "\widetilde{f}"] \\ D' & \end{tikzcd}$$

commutative. Moreover, if we assume further that $D'$ is an étale $T_+$-module over $\Lambda(N_0)$ and the map $f$ is $\psi_t$-equivariant for all $t \in T_+$ then the map $\widetilde{f}$ is $T_+$-equivariant.

**Definition** Let $D$ be a $\Lambda(N_0)$-module and $T_* \leq T_+$ be a submonoid. Assume moreover that the monoid $T_*$ (or in the case of $\psi$-actions the inverse monoid $T_*^{-1}$) acts $o$-linearly on $D$, as well.

We call the action of $T_*$ a $\varphi$-action (relative to the $\Lambda(N_0)$-action) and denote the action of $t$ by $d \mapsto \varphi_t(d)$, if for any $\lambda \in \Lambda(N_0)$, $t \in T_*$ and $d \in D$ we have $\varphi_t(\lambda d) = \varphi_t(\lambda)\varphi_t(d)$. Moreover, we say that the $\varphi$-action is *injective* if for all $t \in T_*$ the map $\varphi_t$ is injective. The $\varphi$-action of $T_*$ is *nondegenerate* if for all $t \in T_*$ we have

$$D = \sum_{u \in J(N_0/tN_0t^{-1})} \mathrm{Im}(u \circ \varphi_t) = \sum_{u \in J(N_0/tN_0t^{-1})} u(\varphi_t(D)) \ .$$

We call the action of $T_*^{-1}$ a $\psi$-action of $T_*$ (relative to the $\Lambda(N_0)$-action) and denote the action of $t^{-1} \in T_*^{-1}$ by $d \mapsto \psi_t(d)$, if for any $\lambda \in \Lambda(N_0)$, $t \in T_*$ and $d \in D$ we have $\psi_t(\varphi_t(\lambda)d) = \lambda\psi_t(d)$. Moreover, we say that the $\psi$-action of $T_*$ is *surjective* if for all $t \in T_*$ the map $\psi_t$ is surjective. The $\psi$-action of $T_*$ is *nondegenerate* if for all $t \in T_*$ we have

$$\{0\} = \bigcap_{u \in J(N_0/tN_0t^{-1})} \mathrm{Ker}(\psi_t \circ u^{-1}) \ .$$

The nondegeneracy is equivalent to the condition that for any $t \in T_*$ $\mathrm{Ker}(\psi_t)$ does not contain any nonzero $\Lambda(N_0)$-submodule of $D$.

We say that a $\varphi$- and a $\psi$-action of $T_*$ are *compatible* on $D$, if



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$(\varphi\psi)$ for any $t \in T_*$, $\lambda \in \Lambda(N_0)$, and $d \in D$ we have $\psi_t(\lambda\varphi_t(d)) = \psi_t(\lambda)d$.

Note that with $\lambda = 1$ we also have $\psi_t \circ \varphi_t = \mathrm{id}_D$ for any $t \in T_*$ assuming $(\varphi\psi)$.

We also consider $\varphi$- and $\psi$-actions of the monoid $\mathbb{Z}_p \setminus \{0\}$ on $\Lambda(N_0)$-modules via the embedding $\xi\colon \mathbb{Z}_p \setminus \{0\} \to T_+$. Modules with a $\varphi$-action (resp. $\psi$-action) of $\mathbb{Z}_p \setminus \{0\}$ are called $(\varphi, \Gamma)$-modules (resp. $(\psi, \Gamma)$-modules).

For example, the natural $\varphi$- and $\psi$-actions of $T_+$ on $\Lambda(N_0)$ are compatible.

**Remarks**

1. Note that the $\psi$-action of the monoid $T_*$ is in fact an action of the inverse monoid $T_*^{-1}$. However, we assume $T_+$ to be commutative so it may also be viewed as an action of $T_*$.

2. Pontryagin duality provides an equivalence of categories between compact $\Lambda(N_0)$-modules with a continuous $\psi$-action of $T_*$ and discrete $\Lambda(N_0)$-modules with a continuous $\varphi$-action of $T_*$. The surjectivity of the $\psi$-action corresponds to the injectivity of $\varphi$-action. Moreover, the $\psi$-action is nondegenerate if and only if so is the corresponding $\varphi$-action on the Pontryagin dual.

If $D$ is a $\Lambda(N_0)$-module with a $\varphi$-action of $T_*$ then there exists a homomorphism

$$\Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} D \to D, \lambda \otimes d \mapsto \lambda\varphi_t(d) \tag{3.10}$$

of $\Lambda(N_0)$-modules. We say that the $T_*$-action on $D$ is *étale* if the above map is an isomorphism. The $\varphi$-action of $T_*$ on $D$ is étale if and only if it is injective and for any $t \in T_*$ we have

$$D = \bigoplus_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(D) \ . \tag{3.11}$$

Similarly, we call a $\Lambda(N_0)$-module together with a $\varphi$-action of the monoid $\mathbb{Z}_p \setminus \{0\}$ an étale $(\varphi, \Gamma)$-module over $\Lambda(N_0)$ if the action of $\varphi = \varphi_s$ is étale.

If $D$ is an étale $T_*$-module over $\Lambda(N_0)$ then there exists a $\psi$-action of $T_*$ compatible with the étale $\varphi$-action (see [17] Section 6).

Dually, if $D$ is a $\Lambda(N_0)$-module with a $\psi$-action of $T_*$ then there exists a map

$$\begin{array}{rcl} \iota_t\colon D & \to & \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} D \\ d & \mapsto & \displaystyle\sum_{u \in J(N_0/tN_0t^{-1})} u \otimes \psi_t(u^{-1}d) \ . \end{array}$$



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**Lemma 3.3.1** *For any $t \in T_*$ the map $\iota_t$ is a homomorphism of $\Lambda(N_0)$-modules. It is injective for all $t \in T_*$ if and only if the $\psi$-action of $T_*$ on $D$ is nondegenerate.*

**Proof** Fix $t \in T_*$. For any $\lambda \in \Lambda(N_0)$ and $u, v \in N_0$ we put $\lambda_{u,v} = \psi_t(u^{-1}\lambda v)$. Note that for any fixed $v \in N_0$ we have

$$\lambda v = \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(\lambda_{u,v})$$

and for any fixed $u \in N_0$ we have

$$u^{-1}\lambda = \sum_{v \in J(N_0/tN_0t^{-1})} \varphi_t(\lambda_{u,v})v^{-1} \ .$$

So we compute

$$\begin{aligned}\iota_t(\lambda x) &= \sum_{u \in J(N_0/tN_0t^{-1})} u \otimes \psi_t(u^{-1}\lambda x) = \\ &= \sum_{u,v \in J(N_0/tN_0t^{-1})} u \otimes \psi_t(\varphi_t(\lambda_{u,v})v^{-1}x) = \\ &= \sum_{u,v \in J(N_0/tN_0t^{-1})} u \otimes \lambda_{u,v}\psi_t(v^{-1}x) = \\ &= \sum_{u,v \in J(N_0/tN_0t^{-1})} u\varphi_t(\lambda_{u,v}) \otimes \psi_t(v^{-1}x) = \\ &= \sum_{v \in J(N_0/tN_0t^{-1})} \lambda v \otimes \psi_t(v^{-1}x) = \lambda\iota_t(x) \ .\end{aligned}$$

The second statement follows from noting that $\Lambda(N_0)$ is a free right module over itself via the map $\varphi_t$ with free generators $u \in J(N_0/tN_0t^{-1})$. $\square$

**Lemma 3.3.2** *Let $D$ be a $\Lambda(N_0)$-module with a $\psi$-action of $T_*$ and $t \in T_*$. Then there exists a $\psi$-action of $T_*$ on $\varphi_t^*D = \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} D$ making the homomorphism $\iota_t$ $\psi$-equivariant. Moreover, if we assume in addition that the $\psi$-action on $D$ is nondegenerate then so is the $\psi$-action on $\varphi_t^*D$.*

**Proof** Let $t' \in T_*$ be arbitrary and define the action of $\psi_{t'}$ on $\varphi_t^*D$ by putting

$$\psi_{t'}(\lambda \otimes d) = \sum_{u' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}d) \text{ for } \lambda \in \Lambda(N_0), d \in D \ ,$$



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and extending $\psi_{t'}$ to $\varphi_t^* D$ $o$-linearly. Note that we have

$$\psi_{t'}(\varphi_{t'}(\mu)\lambda \otimes d) =$$
$$= \sum_{u' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\varphi_{t'}(\mu)\lambda\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}d) = \mu\psi_{t'}(\lambda \otimes d) \ .$$

Moreover, the map $\psi_{t'}$ is well-defined since we have

$$\begin{aligned}
\psi_{t'}(\lambda\varphi_t(\mu) \otimes d) &= \sum_{v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(\mu)\varphi_t(v')) \otimes \psi_{t'}(v'^{-1}d) = \\
&= \sum_{v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(\mu v')) \otimes \psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'\varphi_{t'}(\mu_{u',v'}))) \otimes \psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\varphi_t(\mu_{u',v'}) \otimes \psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u')) \otimes \mu_{u',v'}\psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u')) \otimes \psi_{t'}(\varphi_{t'}(\mu_{u',v'})v'^{-1}d) = \\
&= \sum_{u' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}\mu d) = \psi_{t'}(\lambda \otimes \mu d) \ ,
\end{aligned}$$

where $\mu_{u',v'} = \psi_{t'}(u'^{-1}\mu v')$. Introducing the notation $J' = J(N_0/t'N_0t'^{-1})$ and $J'' = J(N_0/t''N_0t''^{-1})$ we further compute

$$\begin{aligned}
\psi_{t''}(\psi_{t'}(\lambda \otimes d)) &= \psi_{t''}\Big(\sum_{u' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}d)\Big) = \\
&= \sum_{u'' \in J''} \sum_{u' \in J'} \psi_{t''}(\psi_{t'}(\lambda\varphi_t(u'))\varphi_t(u'')) \otimes \psi_{t''}(u''^{-1}\psi_{t'}(u'^{-1}d)) = \\
&= \sum_{u'' \in J''} \sum_{u' \in J'} \psi_{t''}(\psi_{t'}(\lambda\varphi_t(u'\varphi_{t'}(u'')))) \otimes \psi_{t''}(\psi_{t'}(\varphi_{t'}(u'')^{-1}u'^{-1}d)) = \\
&= \psi_{t''t'}(\lambda \otimes d)
\end{aligned}$$

showing that it is indeed a $\psi$-action of the monoid $T_*$.



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For the second statement of the Lemma we compute

$$\psi_{t'}(\iota_t(x)) =$$

$$= \sum_{u'\in J(N_0/t'N_0t'^{-1})} \sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}\psi_t(u^{-1}x)) =$$

$$= \sum_{u'\in J(N_0/t'N_0t'^{-1})} \sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u\varphi_t(u')) \otimes \psi_{t'}(\psi_t(\varphi_t(u')^{-1}u^{-1}x)) \ .$$

Note that in the above sum $u\varphi_t(u')$ runs through a set of representatives for the cosets $N_0/tt'N_0t'^{-1}t^{-1}$. Moreover, $v = \psi_{t'}(u\varphi_t(u'))$ is nonzero if and only if $u\varphi_t(u')$ lies in $t'N_0t'^{-1}$ and the nonzero values of $v$ run through a set $J'(N_0/tN_0t^{-1})$ of representatives of the cosets $N_0/tN_0t^{-1}$. In case $v \neq 0$ we have $\psi_{t'}(\varphi_t(u')^{-1}u^{-1}x) = \psi_{t'}(\varphi_t(u')^{-1}u^{-1})\psi_{t'}(x)$. So we obtain

$$\psi_{t'}(\iota_t(x)) = \sum_{v\in J'(N_0/tN_0t^{-1})} v \otimes \psi_t(\psi_{t'}(\varphi_{t'}(v)x)) =$$

$$= \sum_{v\in J'(N_0/tN_0t^{-1})} v \otimes \psi_t(v^{-1}\psi_{t'}(x)) = \iota_t(\psi_{t'}(x)) \ .$$

Assume now that the $\psi$-action of $T_*$ on $D$ is nondegenerate. Any element in $x \in \varphi_t^*D$ can be uniquely written in the form $\sum_{u\in J(N_0/tN_0t^{-1})} u \otimes x_u$. Assume that for a fixed $t' \in T_*$ we have $\psi_{t'}(u_0'^{-1}x) = 0$ for all $u_0' \in N_0$. Then we compute

$$0 = \psi_{t'}(u_0'^{-1}x) =$$

$$= \sum_{u'\in J(N_0/t'N_0t'^{-1})} \sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u_0'^{-1}u\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}x_u) \ .$$

Put $y = u_0'^{-1}u\varphi_t(u')$. For any fixed $u_0'$ the set

$$\{y \mid u \in J(N_0/tN_0t^{-1}), u' \in J(N_0/t'N_0t'^{-1})\}$$

forms a set of representatives of $N_0/tt'N_0(tt')^{-1}$, and we have $\psi_{t'}(y) \neq 0$ if and only if $y$ lies in $t'N_0t'^{-1}$ in which case we have $\psi_{t'}(y) = t'^{-1}yt'$. So the nonzero values of $\psi_{t'}(y)$ run through a set of representatives of $N_0/tN_0t^{-1}$. Since we have the direct sum decomposition $\varphi_t^*D = \bigoplus_{v\in J(N_0/tN_0t^{-1})} v \otimes D$ we obtain $\psi_{t'}(u'^{-1}x_u) = 0$ for all $u' \in J(N_0/t'N_0t'^{-1})$ and $u \in J(N_0/tN_0t^{-1})$ such that $y = u_0'^{-1}u\varphi_t(u')$ is in $t'N_0t'^{-1}$. However, for any choice of $u'$ and $u$ there exists such a $u_0'$, so we deduce $x = 0$. $\square$



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**Proposition 3.3.3** *Let $D$ be a $\Lambda(N_0)$-module with a $\psi$-action of $T_*$. The following are equivalent:*

1. *There exists a unique $\varphi$-action on $D$, which is compatible with $\psi$ and which makes $D$ an étale $T_*$-module.*

2. *The $\psi$-action is surjective and for any $t \in T_*$ we have*

$$D = \bigoplus_{u_0 \in J(N_0/tN_0t^{-1})} \bigcap_{\substack{u \in J(N_0/tN_0t^{-1}) \\ u \neq u_0}} \mathrm{Ker}(\psi_t \circ u^{-1}) \ . \tag{3.12}$$

   *In particular, the action of $\psi$ is nondegenerate.*

3. *The map $\iota_t$ is bijective for all $t \in T_*$.*

**Proof** $1 \implies 3$ In this case the map $\iota_t$ is the inverse of the isomorphism (3.10) so it is bijective by the étale property.

$3 \implies 2$: The injectivity of $\iota_t$ shows the nondegeneracy of the $\psi$-action. Further if $1 \otimes d = \iota_t(x)$ then we have $\psi_t(x) = d$ so the $\psi$-action is surjective. Moreover, $\iota_t^{-1}(u_0 \otimes D)$ equals $\bigcap_{u_0 \neq u \in J(N_0/tN_0t^{-1})} \mathrm{Ker}(\psi_t \circ u^{-1})$ therefore $D$ can be written as a direct sum (3.12).

$2 \implies 1$: Fix $t \in T_*$. For any $d \in D$ we have to choose $\varphi_t(d)$ such that $\psi_t(\varphi_t(d)) = d$. By the surjectivity of $\psi_t$ we can choose $x \in D$ such that $\psi_t(x) = d$. Using the assumption we can write $x = \sum_{u_0 \in J(N_0/tN_0t^{-1})} x_{u_0}$, with

$$x_{u_0} \in \bigcap_{\substack{u \in J(N_0/tN_0t^{-1}) \\ u \neq u_0}} \mathrm{Ker}(\psi_t \circ u^{-1}) \ .$$

By the compatibility ($\varphi\psi$) we should have

$$\varphi_t(d) \in \bigcap_{\substack{u \in J(N_0/tN_0t^{-1}) \\ u \neq 1}} \mathrm{Ker}(\psi_t \circ u^{-1})$$

as we have $\psi_t(u) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$.

A convenient choice is $\varphi_t(d) = x_1$, and there exists exactly one such element in $D$: if $x'$ would be an other, then

$$x_1 - x' \in \bigcap_{u \in J(N_0/tN_0t^{-1})} \mathrm{Ker}(\psi_t \circ u^{-1}) = \{0\} \ .$$

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This shows the uniqueness of the $\varphi$-action. Further, $x_1 = \varphi_t(d) = 0$ would mean that $x$ lies in $\mathrm{Ker}(\psi_t)$ whence $d = \psi_t(x) = 0$—therefore the injectivity. Similarly, by definition we also have $x_{u_0} = u_0\varphi_t \circ \psi_t(u_0^{-1}x)$ for all $u_0 \in J(N_0/tN_0t^{-1})$. By the surjectivity of the $\psi$-action any element in $D$ can be written of the form $\psi_t(u_0^{-1}x)$ for any fixed $u_0 \in J(N_0/tN_0t^{-1})$ so we obtain

$$u_0\varphi_t(D) = \bigcap_{u_0 \neq u \in J(N_0/tN_0t^{-1})} \mathrm{Ker}(\psi_t \circ u^{-1}) \ .$$

The étale property (3.11) follows from this using our assumption 2. Moreover, this also shows $\psi_t(u\varphi_t(d)) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$ which implies $(\varphi\psi)$ using that $\psi_t \circ \varphi_t = \mathrm{id}_D$ by construction. Finally, $\varphi_t(\lambda)\varphi_t(d) - \varphi_t(\lambda d)$ lies in the kernel of $\psi_t \circ u_0^{-1}$ for any $u_0 \in J(N_0/tN_0t^{-1})$, $\lambda \in \Lambda(N_0)$ and $d \in D$, so it is zero. $\square$

From now on if we have an étale $T_*$-module over $\Lambda(N_0)$ we a priori equip it with the compatible $\psi$-action, and if we have a $\Lambda(N_0)$-module with a $\psi$-action, which satisfies the above property 2, we equip it with the compatible $\varphi$-action, which makes it étale. The construction of the étale hull and its universal property is given in the following

**Proposition 3.3.4** *For any $\Lambda(N_0)$-module $D$, with a $\psi$-action of $T_*$ there exists an étale $T_*$-module $\widetilde{D}$ over $\Lambda(N_0)$ and a $\psi$-equivariant $\Lambda(N_0)$-homomorphism $\iota\colon D \to \widetilde{D}$ with the following universal property: For any $\psi$-equivariant $\Lambda(N_0)$-homomorphism $f : D \to D'$ into an étale $T_*$-module $D'$ we have a unique morphism $\widetilde{f} : \widetilde{D} \to D'$ of étale $T_*$-modules over $\Lambda(N_0)$ making the diagram*

$$\begin{tikzcd} D \arrow[r, "\iota"] \arrow[d, "f"'] & \widetilde{D} \arrow[dl, "\widetilde{f}"] \\ D' & \end{tikzcd}$$

*commutative. $\widetilde{D}$ is unique upto a unique isomorphism. If we assume the $\psi$-action on $D$ to be nondegenerate then $\iota$ is injective.*

**Proof** We will construct $\widetilde{D}$ as the injective limit of $\varphi_t^*D$ for $t \in T_*$. Consider the following partial order on the set $T_*$: we put $t_1 \leq t_2$ whenever we have $t_2t_1^{-1} \in T_*$. Note that by Lemma 3.3.2 we obtain a $\psi$-equivariant isomorphism



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$\varphi^*_{t_2t_1^{-1}}\varphi^*_{t_1}D\cong\varphi^*_{t_2}D$ for any pair $t_1\leq t_2$ in $T_*$. In particular, we obtain a $\psi$-equivariant map $\iota_{t_1,t_2}\colon\varphi^*_{t_1}D\to\varphi^*_{t_2}D$. Applying this observation to $\varphi^*_{t_1}D$ for a sequence $t_1\leq t_2\leq t_3$ we see that the $\Lambda(N_0)$-modules $\varphi^*_tD$ $(t\in T_*)$ with the $\psi$-action of $T_*$ form a direct system with respect to the connecting maps $\iota_{t_1,t_2}$. We put

$$\widetilde{D}=\varinjlim_{t\in T_*}\varphi^*_tD$$

as a $\Lambda(N_0)$-module with a $\psi$-action of $T_*$. For any fixed $t'\in T_*$ we have

$$\varphi^*_{t'}\widetilde{D}=\Lambda(N_0)\otimes_{\Lambda(N_0),\varphi_{t'}}\varinjlim_{t\in T_*}\varphi^*_tD\cong$$

$$\cong\varinjlim_{t\in T_*}\Lambda(N_0)\otimes_{\Lambda(N_0),\varphi_{t'}}\varphi^*_tD\cong\varinjlim_{t't\in T_*}\varphi^*_{t't}D\cong\widetilde{D}$$

showing that there exists a unique $\varphi$-action of $T_*$ on $\widetilde{D}$ making $\widetilde{D}$ an étale $T_*$-module over $\Lambda(N_0)$ by Proposition 3.3.3.

For the universal property, let $f:D\to D'$ be an $\psi$-equivariant map into an étale $T_*$-module $D'$ over $\Lambda(N_0)$. By construction of the map $\varphi_t$ on $\widetilde{D}$ $(t\in T_*)$ we have $\varphi_t(\iota(x))=(1\otimes x)_t$ where $(1\otimes x)_t$ denotes the image of $1\otimes x\in\varphi^*_tD$ in $\widetilde{D}$. So we put

$$\widetilde{f}((\lambda\otimes x)_t)=\lambda\varphi_t(f(x))\in D'$$

and extend it $o$-linearly to $\widetilde{D}$. Note right away that $\widetilde{f}$ is unique as it is $\varphi_t$-equivariant. The map $\widetilde{f}\colon\widetilde{D}\to D'$ is well-defined as we have

$$\widetilde{f}(\iota_{t,tt'}(1\otimes_t x))=\widetilde{f}(\sum_{u'\in N_0/t'N_0t'^{-1}}u'\otimes_{t'}\psi_{t'}(u'^{-1}\otimes_t x))=$$

$$=\sum_{u',v'\in N_0/t'N_0t'^{-1}}\widetilde{f}(u'\otimes_{t'}\psi_{t'}(u'^{-1}\varphi_t(v'))\otimes_t\psi_{t'}(v'^{-1}x))=$$

$$=\sum_{u',v'\in N_0/t'N_0t'^{-1}}\widetilde{f}(u'\varphi_{t'}\circ\psi_{t'}(u'^{-1}\varphi_t(v'))\otimes_{tt'}\psi_{t'}(v'^{-1}x))=$$

$$=\sum_{v'\in N_0/t'N_0t'^{-1}}\widetilde{f}(\varphi_t(v')\otimes_{tt'}\psi_{t'}(v'^{-1}x))=$$

$$=\sum_{v'\in N_0/t'N_0t'^{-1}}\varphi_t(v')\varphi_{tt'}(f(\psi_{t'}(v'^{-1}x)))=$$

$$=\sum_{v'\in N_0/t'N_0t'^{-1}}\varphi_t(v'\varphi_{t'}\circ\psi_{t'}(v'^{-1}f(x)))=\varphi_t(f(x))=\widetilde{f}(1\otimes_t x)$$



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noting that $\iota_{t,tt'}$ is a $\Lambda(N_0)$-homomorphism. Here the notation $\otimes_t$ indicates that the tensor product is via the map $\varphi_t$. By construction $\widetilde{f}$ is a homomorphism of étale $T_*$-modules over $\Lambda(N_0)$ satisfying $\widetilde{f} \circ \iota = f$.

The injectivity of $\iota$ in case the $\psi$-action on $D$ is nondegenerate follows from Lemmata 3.3.1 and 3.3.2. □

**Example** If $D$ itself is étale then we have $\widetilde{D} = D$.

**Corollary 3.3.5** *The functor $D \mapsto \widetilde{D}$ from the category of $\Lambda(N_0)$-modules with a $\psi$-action of $T_*$ to the category of étale $T_*$-modules over $\Lambda(N_0)$ is exact.*

**Proof** $\Lambda(N_0)$ is a free $\varphi_t(\Lambda(N_0))$-module, so $\Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} -$ is exact, and so is the direct limit functor. □

**Corollary 3.3.6** *Assume that $D$ is a $\Lambda(N_0)$-module with a nondegenerate $\psi$-action of $T_*$ and $f\colon D \to D'$ is an injective $\psi$-equivariant $\Lambda(N_0)$-homomorphism into the étale $T_*$-module $D'$ over $\Lambda(N_0)$. Then $\widetilde{f}$ is also injective.*

**Proof** Since $D$ is nondegenerate we may identify $\varphi_t^* D$ with a $\Lambda(N_0)$-submodule of $\widetilde{D}$. Assume that $x = \sum_{u \in J(N_0/tN_0t^{-1})} u \otimes_t x_u \in \varphi_t^* D$ lies in the kernel of $\widetilde{f}$. Then $x_u = \psi_t(u^{-1}x) \in D \subseteq \varphi_t^* D \subseteq \widetilde{D}$ ($u \in J(N_0/tN_0t^{-1})$) also lies in the kernel of $\widetilde{f}$. However, we have $\widetilde{f}(x_u) = f(x_u)$ showing that $x_u = 0$ for all $u \in J(N_0/tN_0t^{-1})$ as $f$ is injective. □

**Example** Let $D$ be a (classical) irreducible étale $(\varphi, \Gamma)$-module over $k((X))$ and $D_0 \subset D$ a $\psi$- and $\Gamma$-invariant treillis in $D$. Then we have $\widetilde{D_0} \cong D$ unless $D$ is 1-dimensional and $D_0 = D^\natural$ in which case we have $\widetilde{D_0} = D_0$.

**Proof** If $D$ is 1-dimensional then $D^\natural = D^+$ is an étale $(\varphi, \Gamma)$-module over $k[[X]]$ (Prop. II.5.14 in [5]) therefore it is equal to its étale hull. If $\dim D > 1$ then we have $D^\natural = D^\# \subseteq D_0$ by Cor. II.5.12 and II.5.21 in [5]. By Corollary 3.3.6 $\widetilde{D^\#} \subseteq \widetilde{D_0}$ injects into $D$ and it is $\varphi$- and $\psi$-invariant. Since $D^\#$ is not $\varphi$-invariant (Prop. II.5.14 in [5]) and it is the maximal compact $o[[X]]$-submodule of $D$ on which $\psi$ acts surjectively (Prop. II.4.2 in [5]) we obtain that $\widetilde{D_0}$ is not compact. In particular, its $X$-divisible part is nonzero therefore equals $D$ as the $X$-divisible part of $\widetilde{D_0}$ is an étale $(\varphi, \Gamma)$-submodule of the irreducible $D$. □

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**Proposition 3.3.7** *The $T_+^{-1}$ action on $D_{SV}(\pi)$ is a surjective nondegenerate $\psi$-action of $T_+$.*

**Proof** Let $d \in D_{SV}(\pi)$ and $t \in T_+$. Since the action of both $t$ and $\Lambda(N_0)$ on $D_{SV}(\pi)$ comes from that on $\pi^\vee$ we have $t^{-1}\varphi_t(\lambda)d = t^{-1}t\lambda t^{-1}d = \lambda t^{-1}d$, so this is indeed a $\psi$-action. The surjectivity of each $\psi_t$ follows from the injectivity of the multiplication by $t$ on each $W \in \mathcal{B}_+(\pi)$. Finally, if $W$ is in $\mathcal{B}_+(\pi)$ then so is $t^*W = \sum_{u \in J(N_0/tN_0t^{-1})} utW$ for any $t \in T_+$. Take an element $d \in D_{SV}(\pi)$ lying in the kernel of $\psi_t(u^{-1}\cdot)$ for all $u \in J(N_0/tN_0t^{-1})$. Then there exists a generating $B_+$-subrepresentation $W$ of $\pi$ such that the restriction of $t^{-1}u^{-1}d$ to $W$ is zero for all $u \in J(N_0/tN_0t^{-1})$. Then the restriction of $d$ to $t^*W$ is zero showing that $d$ is zero in $D_{SV}(\pi)$ therefore the nondegeneracy. Alternatively, the nondegeneracy of the $\psi$-action also follows from the existence of a $\psi$-equivariant injective map $D_{SV}(\pi) \hookrightarrow D^0_{SV}(\pi)$ into an étale $T_+$-module $D^0_{SV}(\pi)$ ([17] Proposition 3.5 and Remark 6.1). $\square$

**Question** Let $D^{(0)}_{SV}(\pi)$ as in [17]. We have that $D^{(0)}_{SV}(\pi)$ is an étale $T_*$-module over $\Lambda(N_0)$ ([17] Proposition 3.5) and $f : D_{SV}(\pi) \hookrightarrow D^{(0)}_{SV}(\pi)$ is a $\psi$-equivariant map ([17] Remark 6.1). By the universal property of the étale hull and Corollary 3.3.6 $\widetilde{D_{SV}}(\pi)$ also injects into $D^{(0)}_{SV}(\pi)$. Whether or not this injection is always an isomorphism is an open question. In case of the Steinberg representation this is true by Proposition 11 in [22].

We call the submonoid $T'_* \leq T_* \leq T_+$ cofinal in $T_*$ if for any $t \in T_*$ there exists a $t' \in T'_*$ such that $t \leq t'$. For example $\xi(\mathbb{Z}_p \setminus \{0\})$ is cofinal in $T_+$.

**Corollary 3.3.8** *Let $D$ be a $\Lambda(N_0)$-module with a $\psi$-action of $T_*$ and denote by $\widetilde{D}$ (resp. by $\widetilde{D}'$) the étale hull of $D$ for the $\psi$-action of $T_*$ (resp. of $T'_*$). Then we have a natural isomorphism $\widetilde{D}' \xrightarrow{\sim} \widetilde{D}$ of étale $T'_*$-modules over $\Lambda(N_0)$. More precisely, if $f\colon D \to D_1$ is a $\psi$-equivariant $\Lambda(N_0)$-homomorphism into an étale $T'_*$-module $D_1$ then $f$ factors uniquely through $\iota\colon D \to \widetilde{D}$.*

**Proof** Since $T'_* \leq T_*$ is cofinal in $T_*$ we have

$$\varinjlim_{t' \in T'_*} \varphi^*_{t'} D \cong \varinjlim_{t \in T_*} \varphi^*_t D = \widetilde{D}.$$

$\square$



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By Corollary 3.3.8 there exists a homomorphism $\widetilde{\mathrm{pr}} : \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ of étale $(\varphi, \Gamma)$-modules over $\Lambda(N_0)$ such that $\mathrm{pr} = \widetilde{\mathrm{pr}} \circ \iota$. Our main result in this section is the following

**Theorem 3.3.9** *$D^\vee_{\xi,\ell,\infty}(\pi)$ is the pseudocompact completion of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ in the category of étale $(\varphi, \Gamma)$-modules over $\Lambda_\ell(N_0)$, ie. we have*

$$D^\vee_{\xi,\ell,\infty}(\pi) \cong \varprojlim_D D$$

*where $D$ runs through the finitely generated étale $(\varphi, \Gamma)$-modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ by a closed submodule. This holds in* any *topology on $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ making both the maps $1 \otimes \iota \colon D_{SV}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$, $d \mapsto 1 \otimes \iota(d)$ and $1 \otimes \widetilde{\mathrm{pr}} \colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ continuous.*

**Remark** Since the map $\mathrm{pr} \colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is continuous, there exists such a topology on $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$. For instance we could take either the final topology of the map $D_{SV}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ or the initial topology of the map $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$.

**Proof** The homomorphism $\widetilde{\mathrm{pr}}$ factors through the map $1 \otimes \mathrm{id} \colon \widetilde{D_{SV}}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ since $D^\vee_{\xi,\ell,\infty}(\pi)$ is a module over $\Lambda_\ell(N_0)$, so we obtain a homomorphism

$$1 \otimes \widetilde{\mathrm{pr}} \colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$$

of étale $(\varphi, \Gamma)$-modules over $\Lambda_\ell(N_0)$. At first we claim that $1 \otimes \widetilde{\mathrm{pr}}$ has dense image. Let $M \in \mathcal{M}(\pi^{H_0})$ and $W \in \mathcal{B}_+(\pi)$ be arbitrary. Then by Lemma 3.2.1 the map $\mathrm{pr}_{W,M,k} \colon W^\vee \to M_k^\vee$ is surjective for $k \geq 0$ large enough. This shows that the natural map

$$1 \otimes \mathrm{pr}_{W,M,k} \colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} W^\vee \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} M_k^\vee \cong M_k^\vee[1/X]$$

is surjective. However, $1 \otimes \mathrm{pr}_{W,M,k}$ factors through $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi)$ by the Remarks after Lemma 3.2.2. In particular, the natural map

$$1 \otimes \mathrm{pr}_{M,k} \colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \to M_k^\vee[1/X]$$



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is surjective for all $M \in \mathcal{M}(\pi^{H_0})$ and $k \geq 0$ large enough (whence in fact for all $k \geq 0$). This shows that the image of the map

$$1 \otimes \mathrm{pr} \colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$$

is dense whence so is the image of $1 \otimes \widetilde{\mathrm{pr}}$. By the assumption that $1 \otimes \widetilde{\mathrm{pr}}$ is continuous we obtain a surjective homomorphism

$$\widehat{1 \otimes \widetilde{\mathrm{pr}}} \colon \varprojlim_D D \to D^\vee_{\xi,\ell,\infty}(\pi)$$

of pseudocompact $(\varphi, \Gamma)$-modules over $\Lambda_\ell(N_0)$ where $D$ runs through the finitely generated étale $(\varphi, \Gamma)$-modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$.

Let $0 \neq (x_D)_D$ be in the kernel of $\widehat{1 \otimes \widetilde{\mathrm{pr}}}$. Then there exists a finitely generated étale $(\varphi, \Gamma)$-module $D$ over $\Lambda_\ell(N_0)$ with a surjective continuous homomorphism $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \twoheadrightarrow D$ such that $x_D \neq 0$. By Proposition 3.2.4 this map factors through $D^\vee_{\xi,\ell,\infty}(\pi)$ contradicting to the assumption $\widehat{1 \otimes \widetilde{\mathrm{pr}}}((x_D)_D) = 0$. $\square$

**Remark** Breuil's functor $D^\vee_\xi$ can therefore be computed from $D_{SV}$ the following way: For a smooth $o/\varpi^h$-representation $\pi$ we have

$$D^\vee_\xi(\pi) \cong (\varprojlim_D D)_{H_0} \cong \varprojlim_D D_{H_0}$$

where $D$ runs through the finitely generated étale $(\varphi, \Gamma)$-modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ by a closed submodule.

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# Chapter 4

# Nongeneric $\ell$

Assume from now on that $\ell = \ell_\alpha$ is a nongeneric Whittaker functional defined by the projection of $N_0$ onto $N_{\alpha,0} \cong \mathbb{Z}_p$ for some simple root $\alpha \in \Delta$.

## 4.1 The action of $T_+$

Our goal in this section is to define a $\varphi$-action of $T_+$ on $D^\vee_{\xi,\ell,\infty}(\pi)$ or equivalently, on $D^\vee_\xi(\pi)$ extending the action of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$ and making $D^\vee_{\xi,\ell,\infty}(\pi)$ an étale $T_+$-module over $\Lambda_\ell(N_0)$. Let $t \in T_+$ be arbitrary. Note that by the choice of this $\ell$ we have $tH_0t^{-1} \subseteq H_0$. In particular, $T_+$ acts via conjugation on the ring $\Lambda(N_0/H_0) \cong o[[X]]$; we denote the action of $t \in T_+$ by $\varphi_t$. This action is via the character $\alpha$ mapping $T_+$ onto $\mathbb{Z}_p\setminus\{0\}$. In particular, $o[[X]]$ is a free module of finite rank over itself via $\varphi_t$. Moreover, we define the Hecke action of $t \in T_+$ on $\pi^{H_0}$ by the formula $F_t(m) := \mathrm{Tr}_{H_0/tH_0t^{-1}}(tm)$ for any $m \in \pi^{H_0}$. For $t, t' \in T_+$ we have

$$\begin{aligned} F_{t'} \circ F_t &= \mathrm{Tr}_{H_0/t'H_0t'^{-1}} \circ (t'\cdot) \circ \mathrm{Tr}_{H_0/tH_0t^{-1}} \circ (t\cdot) = \\ &= \mathrm{Tr}_{H_0/t'H_0t'^{-1}} \circ \mathrm{Tr}_{t'H_0t'^{-1}/t'tH_0t^{-1}t'^{-1}} \circ (t't\cdot) = F_{t't} \ . \end{aligned}$$

For any $M \in \mathcal{M}(\pi^{H_0})$ we put $F_t^* M := N_0 F_t(M)$.

**Lemma 4.1.1** *For any $M \in \mathcal{M}(\pi^{H_0})$ we have $F_t^* M \in \mathcal{M}(\pi^{H_0})$.*

**Proof** We have

$$\begin{aligned} F(F_t^* M) = F(N_0 F_t(M)) \subset N_0 F F_t(M) = \\ = N_0 F_{st}(M) = N_0 F_t(F(M)) \subseteq F_t^* M \ . \end{aligned}$$



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So $F_t^* M$ is a module over $\Lambda(N_0/H_0)/\varpi^h[F]$. Moreover, if $m_1, \dots m_r$ generates $M$, then the elements $F_t(m_i)$ $(1 \leq i \leq r)$ generate $F_t^* M$, so it is finitely generated. The admissibility is clear as $F_t^* M = \sum_{u \in J(N_0/tN_0t^{-1})} uF_t(M)$ is the sum of finitely many admissible submodules. Finally, $F_t^* M$ is stable under the action of $\Gamma$ as $F_t$ commutes with the action of $\Gamma$. $\square$

By the definition of $F_t^* M$ we have a surjective $o/\varpi^h[[X]]$-homomorphism

$$1 \otimes F_t \colon o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]],\varphi_t} M \twoheadrightarrow F_t^* M$$

which gives rise to an injective $o/\varpi^h((X))$-homomorphism

$$(1 \otimes F_t)^\vee[1/X] \colon (F_t^* M)^\vee[1/X] \hookrightarrow o/\varpi^h((X)) \otimes_{o/\varpi^h((X)),\varphi_t} M^\vee[1/X] \ . \tag{4.1}$$

Moreover, there is a structure of an $o/\varpi^h[[X]][F]$-module on

$$o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]],\varphi_t} M$$

by putting $F(\lambda \otimes m) := \varphi_t(\lambda) \otimes F(m)$. Similarly, the group $\Gamma$ also acts on $o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]],\varphi_t} M$ semilinearly. The map $1 \otimes F_t$ is $F$ and $\Gamma$-equivariant as $F_t$, $F$, and the action of $\Gamma$ all commute. We deduce that $(1 \otimes F_t)^\vee[1/X]$ is a $\varphi$- and $\Gamma$-equivariant map of étlae $(\varphi, \Gamma)$-modules.

Note that for any $t \in T_+$ there exists a positive integer $k \geq 0$ such that $t \leq s^k$, ie. $t' := t^{-1}s^k$ lies in $T_+$. So we have $F_t^*(F_{t'}^* M) = F_{s^k}^* M = N_0 F^k(M) \subseteq M$. So we obtain an isomorphism $M^\vee[1/X] \cong (F_{s^k}^* M)^\vee[1/X] = (F_t^*(F_{t'}^* M))^\vee[1/X]$ as $M/N_0F^k(M)$ is finitely generated over $o$.

**Lemma 4.1.2** *The map* (4.1) *is an isomorphism of étale* $(\varphi, \Gamma)$*-modules for any* $M \in \mathcal{M}(\pi^{H_0})$ *and* $t \in T_+$.

**Proof** The composite $(1 \otimes F_{t'})^\vee[1/X] \circ (1 \otimes F_t)^\vee[1/X] = (1 \otimes F^k)^\vee[1/X]$ is an isomorphism by Lemma 2.6 in [3]. So $(1 \otimes F_t)^\vee[1/X]$ is also an isomorphism as both $(1 \otimes F_t)^\vee[1/X]$ and $(1 \otimes F_{t'})^\vee[1/X]$ are injective. $\square$

Now taking projective limits we obtain an isomorphism of pseudocompact étale $(\varphi, \Gamma)$-modules

$$\begin{aligned} (1 \otimes F_t)^\vee[1/X] \colon D_\xi^\vee(\pi) \ &\to \ \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} (o/\varpi^h((X)) \otimes_{o/\varpi^h((X)),\varphi_t} M^\vee[1/X]) \\ (m)_{(F_t^* M)^\vee[1/X]} \ &\mapsto \ ((1 \otimes F_t)^\vee[1/X](m))_{M^\vee[1/X]} \ . \end{aligned}$$



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Moreover, since $o((X))$ is finite free over itself via $\varphi_t$, we have an identification

$$\varprojlim_{M\in\mathcal{M}(\pi^{H_0})} (o/\varpi^h((X))\otimes_{o/\varpi^h((X)),\varphi_t} M^\vee[1/X])\cong$$
$$\cong o/\varpi^h((X))\otimes_{o/\varpi^h((X)),\varphi_t} D_\xi^\vee(\pi)\ .$$

Using the maps $(1\otimes F_t)^\vee[1/X]$ we define a $\varphi$-action of $T_+$ on $D_\xi^\vee(\pi)$ by putting $\varphi_t(d):=((1\otimes F_t)^\vee[1/X])^{-1}(1\otimes d)$ for $d\in D_\xi^\vee(\pi)$.

**Proposition 4.1.3** *The above action of $T_+$ extends the action of $\xi(\mathbb{Z}_p\setminus\{0\})\leq T_+$ and makes $D_\xi^\vee(\pi)$ into an étale $T_+$-module over $o/\varpi^h[[X]]$.*

**Proof** By the definition of the $T_+$-action it is indeed an extension of the action of the monoid $\mathbb{Z}_p\setminus\{0\}$. For $t,t'\in T_+$ we compute

$$\begin{aligned}\varphi_{t'}\circ\varphi_t(d)=((1\otimes F_{t'})^\vee[1/X])^{-1}\circ((1\otimes F_t)^\vee[1/X])^{-1}(1\otimes d)=\\ =((1\otimes F_t)^\vee[1/X]\circ(1\otimes F_{t'})^\vee[1/X])^{-1}(1\otimes d)=\\ =((1\otimes F_{tt'})^\vee[1/X])^{-1}(1\otimes d)=\varphi_{tt'}(d)=\varphi_{t't}(d)\ .\end{aligned}$$

Further, we have

$$\begin{aligned}\varphi_t(\lambda d)=((1\otimes F_t)^\vee[1/X])^{-1}(1\otimes\lambda d)=((1\otimes F_t)^\vee[1/X])^{-1}(\varphi_t(\lambda)\otimes d)=\\ =\varphi_t(\lambda)((1\otimes F_t)^\vee[1/X])^{-1}(1\otimes d)=\varphi_t(\lambda)\varphi_t(d)\end{aligned}$$

showing that this is indeed a $\varphi$-action of $T_+$. The étale property follows from the fact that $(1\otimes F_t)^\vee[1/X]$ is an isomorphism for each $t\in T_+$. $\square$

The inclusion $u_\alpha\colon\mathbb{Z}_p\to N_{\alpha,0}\leq N_0$ induces an injective ring homomorphism—still denoted by $u_\alpha$ by a certain abuse of notation—$u_\alpha\colon\widehat{o((X))}^p\hookrightarrow\Lambda_\ell(N_0)$ where $\widehat{o((X))}^p$ denotes the $p$-adic completion of the Laurent-series ring $o((X))$. For each $t\in T_+$ this gives rise to a commutative diagram

$$\begin{tikzcd}\widehat{o((X))}^p \arrow[r, "u_\alpha"] \arrow[d, "\varphi_t"'] & \Lambda_\ell(N_0) \arrow[d, "\varphi_t"] \\ \widehat{o((X))}^p \arrow[r, "u_\alpha"] & \Lambda_\ell(N_0)\end{tikzcd}$$



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with injective ring homomorphisms. On the other hand, by the equivalence of categories in Thm. 8.20 in [18] we have a $\varphi$- and $\Gamma$-equivariant identification $M^\vee_\infty[1/X] \cong \Lambda_\ell(N_0) \otimes_{\widehat{o((X))}^p, u_\alpha} M^\vee[1/X]$. Therefore tensoring the isomorphism (4.1) with $\Lambda_\ell(N_0)$ via $u_\alpha$ we obtain an isomorphism

$$\begin{aligned}(1 \otimes F_t)^\vee_\infty[1/X] \colon (F_t^* M)^\vee_\infty[1/X] \cong \Lambda_\ell(N_0) \otimes_{u_\alpha} (F_t^* M)^\vee[1/X] \to \\ \to \Lambda_\ell(N_0) \otimes_{u_\alpha} o/\varpi^h((X)) \otimes_{o/\varpi^h((X)),\varphi_t} M^\vee[1/X] \cong \\ \cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\varphi_t} \Lambda_\ell(N_0) \otimes_{u_\alpha} M^\vee[1/X] \cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\varphi_t} M^\vee_\infty[1/X] \ .\end{aligned} \tag{4.2}$$

Taking projective limits again we deduce an isomorphism

$$\begin{array}{rcl}(1 \otimes F_t)^\vee_\infty[1/X] \colon D^\vee_{\xi,\ell,\infty}(\pi) & \to & \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\varphi_t} D^\vee_{\xi,\ell,\infty}(\pi) \\ (m)_{(F_t^* M)^\vee_\infty[1/X]} & \mapsto & ((1 \otimes F_t)^\vee_\infty[1/X](m))_{M^\vee_\infty[1/X]}\end{array}$$

for all $t \in T_+$ using the identification

$$\varprojlim_{M \in \mathcal{M}(\pi^{H_0})} (\Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\varphi_t} M^\vee_\infty[1/X]) \cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\varphi_t} D^\vee_{\xi,\ell,\infty}(\pi) \ .$$

Using the maps $(1 \otimes F_t)^\vee_\infty[1/X]$ we define a $\varphi$-action of $T_+$ on $D^\vee_{\xi,\ell,\infty}(\pi)$ by putting $\varphi_t(d) := ((1 \otimes F_t)^\vee_\infty[1/X])^{-1}(1 \otimes d)$ for $d \in D^\vee_{\xi,\ell,\infty}(\pi)$.

**Corollary 4.1.4** *The above action of $T_+$ extends the action of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$ and makes $D^\vee_{\xi,\ell,\infty}(\pi)$ into an étale $T_+$-module over $\Lambda_\ell(N_0)$. The reduction map $D^\vee_{\xi,\ell,\infty}(\pi) \to D^\vee_\xi(\pi)$ is $T_+$-equivariant for the $\varphi$-action.*

We can view this $\varphi$-action of $T_+$ in a different way: Let us define $F_{t,k} := \mathrm{Tr}_{H_k/tH_kt^{-1}} \circ (t\cdot)$. Then we have a map

$$1 \otimes F_{t,k} \colon \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h,\varphi_t} M_k \to F_{t,k}^* M_k := N_0 F_{t,k}(M_k) \ , \tag{4.3}$$

where we have $F_{t,k}^* M \in \mathcal{M}_k(\pi^{H_k})$. Let $k$ be large enough such that we have $tH_0t^{-1} \geq H_k$. After taking Pontryagin duals, inverting $X$, taking projective limit and using the remark after Lemma 3.1.5 we obtain a homomorphism of étale $(\varphi, \Gamma)$-modules

$$\varprojlim_k \mathrm{Tr}^{-1}_{t^{-1}H_kt} \circ (1 \otimes F_{t,k})^\vee[1/X] \colon (F_t^* M)^\vee_\infty[1/X] \to \Lambda_\ell(N_0) \otimes_{\varphi_t} M^\vee_\infty[1/X] \ . \tag{4.4}$$



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This map is indeed $\Gamma$- and $\varphi$-equivariant because we compute

$$\begin{aligned} F_k \circ F_{t,k} = \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot) \circ \mathrm{Tr}_{H_k/tH_kt^{-1}} \circ (t\cdot) = \\ = \mathrm{Tr}_{H_k/s^ktH_kt^{-1}s^{-k}} \circ (s^kt\cdot) = \\ = \mathrm{Tr}_{H_k/tH_kt^{-1}} \circ (t\cdot) \circ \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot) = F_{t,k} \circ F_k \ . \end{aligned}$$

Now we have two maps (4.2) and (4.4) between $(F_t^*M)^\vee_\infty[1/X]$ and $\Lambda_\ell(N_0) \otimes_{\varphi_t} M^\vee_\infty[1/X]$ that agree after taking $H_0$-coinvariants by definition. Hence they are equal by the equivalence of categories in Thm. 8.20 in [18].

We obtain in particular that the map (4.3) has finite kernel and cokernel as it becomes an isomorphism after taking Pontryagin duals and inverting $X$. Hence there exists a finite $\Lambda(N_0/H_k)/\varpi^h$-submodule $M_{t,k,*}$ of $M_k$ such that the kernel of $1 \otimes F_{t,k}$ is contained in the image of $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_{t,k,*}$ in $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_k$. We denote by $M^*_{t,k} \leq F^*_{t,k}M_k$ the image of $1 \otimes F_{t,k}$. We conclude that as in Proposition 3.1.6, we can describe the $\varphi_t$-action in the following way:

$$\begin{aligned} \varphi_t \colon M_k^\vee[1/X] &\to (F^*_{t,k}M_k)^\vee[1/X] \\ f &\mapsto (\mathrm{Tr}^{-1}_{t^{-1}H_kt/H_k} \circ (1 \otimes F_{t,k})^\vee[1/X])^{-1}(1 \otimes f) \end{aligned} \quad (4.5)$$

Being an étale $T_+$-module over $\Lambda_\ell(N_0)$ we equip $D^\vee_{\xi,\ell,\infty}(\pi)$ with the $\psi$-action of $T_+$: $\psi_t$ is the canonical left inverse of $\varphi_t$ for all $t \in T_+$.

**Proposition 4.1.5** *The map* $\mathrm{pr}\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ *is $\psi$-equivariant for the $\psi$-actions of $T_+$ on both sides.*

**Proof** We proceed as in the proofs of Proposition 3.1.8 and Lemma 3.2.2. We fix $t \in T_+$, $W \in \mathcal{B}_+(\pi)$ and $M \in \mathcal{M}(\pi^{H_0})$ and show that $\mathrm{pr}_{W,M}$ is $\psi_t$-equivariant. Fix $k$ such that $F^*_{t,k}M_k \leq W$ and $tH_0t^{-1} \geq H_k$.

At first we compute the formula analogous to (3.7). Let $f$ be in $M_k^\vee$ such that its restriction to $M_{t,k,*}$ is zero and $m \in M^*_{t,k} \leq F^*_{t,k}M_k$ be in the form

$$m = \sum_{u \in J(N_0/tN_0t^{-1})} uF_{t,k}(m_u)$$

with elements $m_u \in M_k$ for $u \in J(N_0/tN_0t^{-1})$. $M^*_{t,k}$ is a finite index submodule of $F^*_{t,k}M_k$. Note that the elements $m_u$ are unique upto $M_{t,k,*} + \mathrm{Ker}(F_{t,k})$. Therefore $\varphi_t(f) \in (M^*_{t,k})^\vee$ is well-defined by our assumption that $f_{|M_{t,k,*}} = 0$



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noting that the kernel of $F_{t,k}$ equals the kernel of $\mathrm{Tr}_{t^{-1}H_kt/H_k}$ since the multiplication by $t$ is injective and we have $F_{t,k} = t \circ \mathrm{Tr}_{t^{-1}H_kt/H_k}$. So we compute

$$\begin{aligned} \varphi_t(f)(m) = ((1 \otimes F_{t,k})^\vee)^{-1}(\mathrm{Tr}_{t^{-1}H_kt/H_k}(1 \otimes f))(m) = \\ = ((1 \otimes F_{t,k})^\vee)^{-1}(1 \otimes \mathrm{Tr}_{t^{-1}H_kt/H_k}(f))(\sum_{u \in J((N_0/H_k)/t(N_0/H_k)t^{-1})} uF_{t,k}(m_u)) = \\ = \mathrm{Tr}_{t^{-1}H_kt/H_k}(f)(F_{t,k}^{-1}(u_0F_{t,k}(m_{u_0}))) = f(\mathrm{Tr}_{t^{-1}H_kt/H_k}((t^{-1}u_0t)m_{u_0})) \end{aligned} \tag{4.6}$$

where $u_0$ is the single element in $J(N_0/tN_0t^{-1})$ corresponding to the coset of 1.

Now let $f$ be in $W^\vee$ such that the restriction $f_{|N_0tM_{t,k,*}} = 0$. By definition we have $\psi_t(f)(w) = f(tw)$ for any $w \in W$. Choose an element $m \in M_{t,k}^* \subset F_{t,k}^*M_k$ written in the form

$$m = \sum_{u \in J(N_0/tN_0t^{-1})} uF_{t,k}(m_u) = \sum_{u \in J(N_0/tN_0t^{-1})} ut\mathrm{Tr}_{t^{-1}H_kt/H_k}(m_u) \ .$$

Then we compute

$$\begin{aligned} f_{|F_{t,k}^*M_k}(m) &= \sum_{u \in J(N_0/tN_0t^{-1})} f(ut\mathrm{Tr}_{t^{-1}H_kt/H_k}(m_u)) = \\ &= \sum_{u \in J(N_0/tN_0t^{-1})} \psi_t(u^{-1}f)(\mathrm{Tr}_{t^{-1}H_kt/H_k}(m_u)) = \\ &\overset{(4.6)}{=} \sum_{u \in J(N_0/tN_0t^{-1})} \varphi_t(\psi_t(u^{-1}f)_{|F_{t,k}^*M_k})(F_{t,k}(m_u)) = \\ &= \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}f)_{|M_k})(uF_{t,k}(m_u)) = \\ &= \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}f)_{|M_k})(m) \end{aligned}$$

as for distinct $u, v \in J(N_0/tN_0t^{-1})$ we have $u\varphi_t(f_0)(vF_{t,k}(m_v)) = 0$ for any $f_0 \in (M_{t,k}^*)^\vee$. So by inverting $X$ and taking projective limits with respect to $k$ we obtain

$$\mathrm{pr}_{W,F_t^*M}(f) = \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(\mathrm{pr}_{W,M}(\psi_t(u^{-1}f)))$$



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as we have $(M_{t,k}^*)^\vee[1/X] \cong (F_{t,k}^* M)^\vee[1/X]$. Since the map (4.2) is an isomorphism we may decompose $\mathrm{pr}_{W,F_t^*M}(f)$ uniquely as

$$\mathrm{pr}_{W,F_t^*M}(f) = \sum_{u\in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}\mathrm{pr}_{W,F_t^*M}(f)))$$

so we must have $\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \mathrm{pr}_{W,M}(\psi_t(f))$. For general $f \in W^\vee$ note that $N_0 s M_{t,k,*}$ is killed by $\varphi_t(X^r)$ for $r \geq 0$ big enough, so we have

$$\begin{aligned} X^r\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \psi_t(\mathrm{pr}_{W,F_t^*M}(\varphi_t(X^r)f)) = \\ = \mathrm{pr}_{W,M}(\psi_t(\varphi_t(X^r)f)) = X^r \mathrm{pr}_{W,M}(\psi_t(f)) \ . \end{aligned}$$

Since $X^r$ is invertible in $\Lambda_\ell(N_0)$, we obtain

$$\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \mathrm{pr}_{W,M}(\psi_t(f))$$

for any $f \in W^\vee$. The statement follows taking the projective limit with respect to $M \in \mathcal{M}(\pi^{H_0})$ and the inductive limit with respect to $W \in \mathcal{B}_+(\pi)$.
$\square$

## 4.2 Compatibility with parabolic induction

Let $P = L_P N_P$ be a parabolic subgroup of $G$ containing $B$ with Levi component $L_P$ and unipotent radical $N_P$ and let $\pi_P$ be a smooth $o/\varpi^h$-representation of $L_P$ that we view as a representation of $P^-$ via the quotient map $P^- \twoheadrightarrow L_P$ where $P^- = L_P N_{P^-}$ is the parabolic subgroup opposite to $P$. Since $T$ is contained in $L_P$, we may consider the same cocharacter $\xi\colon \mathbb{Q}_p^* \to T$ for the group $L_P$ instead of $G$. Further, we put $N_{L_P} = N \cap L_P$ and $N_{L_P,0} = N_0 \cap L_P$.

As in [3] denote by $W = N_G(T)/T$ (resp. by $W_P = (N_G(T) \cap L_P)/T$) the Weyl group of $G$ (resp. of $L_P$) and by $w_0 \in W$ the element of maximal length. We have a canonical system

$$K_P = \{w \in W \mid w^{-1}(\Phi_P^+) \subseteq \Phi_+\}$$

of representatives (the Kostant representatives) of the right cosets $W_P\backslash W$ where $\Phi_P^+$ denotes the set of positivie roots of $L_P$ with respect to the Borel subgroup $L_P \cap B$. We have a generalized Bruhat decomposition

$$G = \coprod_{w\in K_P} P^- wB = \coprod_{w\in K_P} P^- wN \ .$$



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Now let $\pi_P$ be a smooth representation of $L_P$ over $o/\varpi^h$. We regard $\pi_P$ as a representation of $P^-$ via the quotient map $P^- \twoheadrightarrow L_P$. Then the parabolically induced representation $\mathrm{Ind}_{P^-}^G \pi_P$ admits [21] (see also [7] §4.3) a filtration by $B$-subrepresentations whose graded pieces are contained in

$$\mathcal{C}_w(\pi_P) = c - \mathrm{Ind}_{P^-}^{P^- wN} \pi_P$$

for $w \in K_P$ where $c-\mathrm{Ind}_{P^-}^{*}$ stands for the space of locally constant functions on $* \supseteq P^-$ with compact support modulo $P^-$. $B$ acts on $\mathcal{C}_w(\pi_P)$ by right translations. Moreover, the first graded piece equals $\mathcal{C}_1(\pi_P)$.

**Lemma 4.2.1** *Let $\pi' \leq \mathcal{C}_w(\pi_P)$ be any $B$-subrepresentation for some $w \in K_P \setminus \{1\}$. Then we have $D_\xi^\vee(\pi') = 0$.*

**Proof** By the right exactness of $D_\xi^\vee$ (Prop. 2.7($ii$) in [3]) it suffices to treat the case $\pi' = \mathcal{C}_w(\pi_P)$. For this the same argument works as in Prop. 6.2 [3] with the following modification:

The particular shape of $\ell$ is only used in Lemma 6.5 in [3] (note that the subgroup $H_0 = \mathrm{Ker}(\ell\colon N_0 \to \mathbb{Z}_p)$ is denoted by $N_1$ therein). For an element $w \neq 1$ in the Weyl group we have $(w^{-1}N_{P^-}w \cap N_0)\backslash N_0/H_0 = \{1\}$ if and only if $H_0$ does not contain $w^{-1}N_{P^-}w \cap N_0$. Whenever $w^{-1}N_{P^-}w \cap N_0 \not\subseteq H_0$, the statement of Lemma 6.5 in [3] is true and there is nothing to prove.

In case we have $\{1\} \neq w^{-1}N_{P^-}w \cap N_0 \subseteq H_0$, the statement of Lemma 6.5 is not true for $\ell = \ell_\alpha$. However, the argument using it in the proof of Prop. 6.2 can be replaced by the following: the operator $F$ acts on the space $\mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{H_0}$ nilpotently. Indeed, the trace map $\mathrm{Tr}_{H_0/sH_0s^{-1}}$

$$\mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{sH_0s^{-1}} \to \mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{H_0}$$

is zero as each double coset $(w^{-1}N_{P^-}w \cap H_0)\backslash H_0/sH_0s^{-1}$ has size divisible by $p$ and any function in $\mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{sH_0s^{-1}}$ is constant on these double cosets. The statement follows from Prop. 2.7($iii$) in [3]. $\square$

In order to extend Thm. 6.1 in [3] (the compatibility with parabolic induction) to our situation ($\ell = \ell_\alpha$) we need to distinguish two cases: whether the root subgroup $N_\alpha$ is contained in $L_P$ or in $N_P$. Similarly to [7] we define the $s^{\mathbb{Z}}N_{L_P}$-ordinary part $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ of a smooth representation $\pi_P$ of $L_P$ as follows. We equip $\pi_P^{N_{L_P,0}}$ with the Hecke action $F_P = \mathrm{Tr}_{N_{L_P,0}/sN_{L_P,0}s^{-1}} \circ (s\cdot)$ of $s$ making $\pi_P^{N_{L_P,0}}$ a module over the polynomial ring $o/\varpi^h[F_P]$ and put

$$\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P) = \mathrm{Hom}_{o/\varpi^h[F_P]}(o/\varpi^h[F_P, F_P^{-1}], \pi_P^{N_{L_P,0}})_{F_P-fin}$$



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where $F_P - fin$ stands for those elements in the Hom-space whose orbit under the action of $F_P$ is finite. By Lemmata 3.1.5 and 3.1.6 in [7] we may identify $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ with an $o/\varpi^h[F_P]$-submodule in $\pi_P^{N_{L_P,0}}$ by sending a map $f \in \mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ to its value $f(1) \in \pi_P^{N_{L_P,0}}$ at $1 \in o/\varpi^h[F_P, F_P^{-1}]$.

**Proposition 4.2.2** *Let $\pi_P$ be a smooth locally admissible representation of $L_P$ over $o/\varpi^h$ which we view by inflation as a representation of $P^-$. We have an isomorphism*

$$D_\xi^\vee\left(\mathrm{Ind}_{P^-}^G \pi_P\right) \cong \begin{cases} D_\xi^\vee(\pi_P) & \text{if } N_\alpha \subseteq L_P \\ o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee & \text{if } N_\alpha \subseteq N_P \end{cases}$$

*as étale $(\varphi,\Gamma)$-modules. In particular, for $P = B$ we have $D_\xi^\vee(\mathrm{Ind}_{B^-}^G\pi_B) \cong o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\pi_B^\vee$, ie. the value of $D_\xi^\vee$ at the principal series is the same $(\varphi,\Gamma)$-module of rank 1 regardless of the choice of $\ell$ (generic or not).*

**Proof** By Lemma 4.2.1 and the right exactness of $D_\xi^\vee$ (Prop. 2.7($ii$) in [3]) it suffices to show that $D_\xi^\vee(\mathcal{C}_1(\pi_P))$ is isomorphic either to $D_\xi^\vee(\pi_P)$ or $o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee$. Moreover, the proof of Prop. 6.7 in [3] goes through without modification so we have an isomorphism $D_\xi^\vee(\mathcal{C}_1(\pi_P)) \cong D^\vee((\mathrm{Ind}_{P^-\cap N_0}^{N_0}\pi_P)^{H_0})$. Hence we are reduced to computing $D^\vee((\mathrm{Ind}_{P^-\cap N_0}^{N_0}\pi_P)^{H_0})$ in terms of $\pi_P$. We further have an identification

$$\mathrm{Ind}_{P^-\cap N_0}^{N_0}\pi_P \cong \mathcal{C}(N_{P,0},\pi_P) \cong \mathcal{C}(N_{P,0}, o/\varpi^h)\otimes_{o/\varpi^h}\pi_P$$

by equation (40) in [3]. We need to distinguish two cases.

*Case 1:* $N_\alpha \subseteq L_P$. In this case we have $N_{P,0} \subseteq H_0$. Hence we deduce $(\mathcal{C}(N_{P,0}, o/\varpi^h)\otimes_{o/\varpi^h}\pi_P)^{H_0} = \pi_P^{H_0/N_{P,0}} = \pi_P^{H_{P,0}}$. So we have

$$D_\xi^\vee\left(\mathrm{Ind}_{P^-}^G\pi_P\right) \cong D^\vee((\mathrm{Ind}_{P^-\cap N_0}^{N_0}\pi_P)^{H_0}) \cong D^\vee(\pi_P^{H_{P,0}}) \cong D_\xi^\vee(\pi_P)$$

in this case as claimed.

*Case 2:* $N_\alpha \subseteq N_P$. In this case we have $N_{L_P,0} \subseteq H_0$ and $N_{P,0}/(N_{P,0}\cap H_0) \cong \mathbb{Z}_p$. So we have an identification

$$\mathcal{C}(N_{P,0},\pi_P)^{H_0} \cong \mathcal{C}(N_{P,0}/(N_{P,0}\cap H_0), \pi_P^{N_{L_P,0}}) \cong \mathcal{C}(\mathbb{Z}_p, \pi_P^{N_{L_P,0}})\ .$$



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Here the Hecke action $F = F_G = \mathrm{Tr}_{H_0/sH_0s^{-1}} \circ (s\cdot)$ of $s$ on the right hand side is given by the formula

$$F_G(f)(a) = \begin{cases} F_P(f(a/p)) & \text{if } a \in p\mathbb{Z}_p \\ 0 & \text{if } a \in \mathbb{Z}_p \setminus p\mathbb{Z}_p \end{cases},$$

where $F_P = \mathrm{Tr}_{N_{L_P,0}/sN_{L_P,0}s^{-1}} \circ (s\cdot)$ denotes the Hecke action of $s$ on $\pi_P^{N_{L_P,0}}$.

Now let $M$ be a finitely generated $o/\varpi^h[[X]][F]$ submodule of $\mathcal{C}(\mathbb{Z}_p, \pi_P^{N_{L_P,0}})$ that is stable under the action of $\Gamma$ and is admissible as a representation of $\mathbb{Z}_p$. By possibly passing to a finite index submodule of $M$ we may assume without loss of generality that the natural map $M^\vee \to M^\vee[1/X]$ is injective whence the map $\mathrm{id} \otimes F \colon o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]],F} M \to M$ is surjective.

Let $f \in M$ be arbitrary. By continuity of $f$ there exists an integer $n \geq 0$ such that $f$ is constant on the cosets of $p^n\mathbb{Z}_p$. Writing $f = \sum_{i=0}^{p^n-1} [i] \cdot F^n(f_i)$ (where $[i]\cdot$ denotes the multiplication by the group element $i \in \mathbb{Z}_p$) by the surjectivity of $\mathrm{id} \otimes F$ we find that each $f_i$ is necessarily constant as a function on $\mathbb{Z}_p$ satisfying $F_P^n(f_0(0)) = f(0)$.

Put $M_* = \{f(0) \mid f \in M\} \subseteq \pi_P^{N_{L_P,0}}$. By the previous discussion $F_P$ acts surjectively on $M_*$ and is generated by the values of elements in $M^{\mathbb{Z}_p}$ (ie. constant functions) as a module over $o/\varpi^h[F_P]$. By the admissibility of $M$ we deduce that $M^{\mathbb{Z}_p}$ hence $M_*$ is finite (or, equivalently, finitely generated over $o/\varpi^h$). We deduce that in fact we have $M = \mathcal{C}(\mathbb{Z}_p, M_*)$, ie. $M^\vee \cong o/\varpi^h[[X]] \otimes_{o/\varpi^h} M_*^\vee$.

Conversely, whenever we have a $o/\varpi^h[F_P]$-submodule $M' \leq \pi_P^{N_{L_P,0}}$ that is finitely generated over $o/\varpi^h$ and on which $F_P$ acts surjectively (hence bijectively as the cardinality of $o/\varpi^h$ is finite) then for $M = \mathcal{C}(\mathbb{Z}_p, M')$ we have $M' = M_*$, $M \in \mathcal{M}(\mathcal{C}(\mathbb{Z}_p, \pi_P^{N_{L_P,0}}))$, and $M^\vee \cong o/\varpi^h[[X]] \otimes_{o/\varpi^h} (M')^\vee$ is $X$-torsion free. In particular, we compute

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$$
\begin{aligned}
D_\xi^\vee(\mathcal{C}_1(\pi_P)) \cong & \varprojlim_{M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}}))} M^\vee[1/X] \cong \\
\cong & \varprojlim_{\substack{M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}})),\\ M^\vee\hookrightarrow M^\vee[1/X]}} o/\varpi^h((X))\otimes_{o/\varpi^h} M_*^\vee \cong \\
& o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}(\varinjlim_{\substack{M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}})),\\ M^\vee\hookrightarrow M^\vee[1/X]}} M_*)^\vee = \\
& = o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee
\end{aligned}
$$

as claimed. □

**Remark** For $N_\alpha \subseteq N_P$ we have the equivalent description $D_\xi^\vee\left(\mathrm{Ind}_{P^-}^G\pi_P\right) \cong \varprojlim_{M\in\mathcal{M}(\pi'_P)} o/\varpi^h[[X]][1/X]\otimes_{o/\varpi^h} M^\vee$, where

$$\pi'_P = (\pi_P^{H_0})_{F_P^\infty=0} = \pi_P^{H_0}/\langle x\in\pi_P^{H_0} | \exists n\in\mathbb{N} : F_P^n x = 0\rangle,$$

and the action of $\varphi$ (resp. $\Gamma$) on $o/\varpi^h[[X]][1/X]\otimes M^\vee$ is the unique $o/\varpi^h[[X]][1/X]$-semilinear action such that $\varphi(f)(m) = f(\xi(p^{-1})m)$ for $f\in M^\vee$ and $m\in M$ (resp. $x(f)(m) = f(\xi(x^{-1})m)$ for $x\in\mathbb{Z}_p^*\simeq\Gamma$, $f\in M^\vee$ and $m\in M$).

## 4.3 Compatibility with a reverse functor

In this section the results of [10], section 4 are presented without proofs.

In [18] the functor $D\mapsto\mathfrak{Y}$ is generalized to arbitrary $\mathbb{Q}_p$-split reductive groups $G$ with connected centre. Let $D$ be an étale $(\varphi,\Gamma)$-module finitely generated over $\mathcal{O}_\mathcal{E}$ and choose a character $\delta\colon \mathrm{Ker}(\alpha)\to o^*$. Then we may let the monoid $\xi(\mathbb{Z}_p\setminus\{0\})\mathrm{Ker}(\alpha)\leq T$ (containing $T_+$) act on $D$ via the character $\delta$ of $\mathrm{Ker}(\alpha)$ and via the natural action of $\mathbb{Z}_p\setminus\{0\}\cong\varphi^{\mathbb{N}_0}\times\Gamma$ on $D$. This way we also obtain a $T_+$-action on $\Lambda_\ell(N_0)\otimes_{u_\alpha} D$ making $\Lambda_\ell(N_0)\otimes_{u_\alpha} D$ an étale $T_+$-module over $\Lambda_\ell(N_0)$. In [18] a $G$-equivariant sheaf $\mathfrak{Y}$ on $G/B$ is attached to $D$ such that its sections on $\mathcal{C}_0 = N_0w_0B/B\subset G/B$ is $B_+$-equivariantly



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isomorphic to the étale $T_+$-module $(\Lambda_\ell(N_0) \otimes_{u_\alpha} D)^{bd}$ over $\Lambda(N_0)$ consisting of bounded elements in $\Lambda_\ell(N_0) \otimes_{u_\alpha} D$ (see [18] section 9).

The construction of a $G$-equivariant sheaf on $G/B$ with sections on $\mathcal{C}_0 = N_0 w_0 B/B \subset G/B$ isomorphic to a dense $B_+$-stable $\Lambda(N_0)$-submodule $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ of $D^\vee_{\xi,\ell,\infty}(\pi)$ is not immediate from the work [18] as only the case of finitely generated modules over $\Lambda_\ell(N_0)$ is treated in there. However, the most natural definition of bounded elements in $D^\vee_{\xi,\ell,\infty}(\pi)$ works: The $\Lambda(N_0)$-submodule $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ is defined as the union of $\psi$-invariant compact $\Lambda(N_0)$-submodules of $D^\vee_{\xi,\ell,\infty}(\pi)$. The image of $\widetilde{\mathrm{pr}}\colon \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is contained in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ and the constructions of [18] can be carried over to this situation. The resulting $G$-equivariant sheaf on $G/B$ is denoted by $\mathfrak{Y} = \mathfrak{Y}_{\alpha,\pi}$.

Now consider the functors $(\cdot)^\vee\colon \pi \mapsto \pi^\vee$ and the composite

$$\mathfrak{Y}_{\alpha,\cdot}(G/B)\colon \pi \mapsto D^\vee_{\xi,\ell,\infty}(\pi) \mapsto \mathfrak{Y}_{\alpha,\pi}(G/B)$$

both sending smooth, admissible $o/\varpi^h$-representations of $G$ of finite length to topological representations of $G$ over $o/\varpi^h$. There exists is a natural transformation $\beta_{G/B}$ from $(\cdot)^\vee$ to $\mathfrak{Y}_{\alpha,\cdot}$. This generalizes Thm. IV.4.7 in [4]. The proof of this relies on the observation that the maps $\mathcal{H}_g\colon D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \to D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ in fact come from the $G$-action on $\pi^\vee$. More precisely, for any $g \in G$ and $W \in \mathcal{B}_+(\pi)$ we have maps

$$(g\cdot)\colon (g^{-1}W \cap W)^\vee \to (W \cap gW)^\vee$$

where both $(g^{-1}W \cap W)^\vee$ and $(W \cap gW)^\vee$ are naturally quotients of $W^\vee$. These maps fit into a commutative diagram

$$\begin{tikzcd}
W^\vee \arrow[r] \arrow[rr, bend left] \arrow[d, "\mathrm{pr}_W"] & (g^{-1}W \cap W)^\vee \arrow[r, "g\cdot"'] \arrow[d] & (W \cap gW)^\vee \arrow[d] \\
D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \arrow[r] \arrow[rr, bend right] & \mathrm{res}^{\mathcal{C}_0}_{g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}) \arrow[r, "g\cdot"] & \mathrm{res}^{\mathcal{C}_0}_{\mathcal{C}_0 \cap g\mathcal{C}_0}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd})
\end{tikzcd}$$

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allowing us to construct the map $\beta_{G/B}$. The proof of this is similar to that of Thm. IV.4.7 in [4]. However, unlike that proof we do not need the full machinery of "standard presentations" in Ch. III.1 of [4] which is not available at the moment for groups other than $\mathbf{GL_2}(\mathbb{Q}_p)$.

## 4.4 Counterexamples

In [3] the Whittaker functional $\ell$ is assumed to be generic. However, even if $\ell$ is not generic, the functor $D^\vee_\xi$ (hence also $D^\vee_{\xi,\ell,\infty}$) is right exact. Here we show that in this case $D^\vee_\xi$ is not faithful and the restriction of $D^\vee_\xi$ to the category $SP_{o/\varpi^h}$ is not exact in general.

From now on let $h = 1$, thus we are over $k = o/\varpi$, and $G = \mathbf{GL_3}(\mathbb{Q}_p)$. Then $|\Delta| = 2$, say $\Delta = \{\alpha, \beta\}$, fix the parabolic subgroup $P$ such that $L_P \cong \mathbf{GL_2}(\mathbb{Q}_p) \times T'$ where $T'$ is a torus and $\ell = \ell_\alpha$. Let the superscript ${}^{(2)}$ denote the analogous construction of the subgroups $B, T, N, T_0$ and element $s$ of $G$ in case $G = \mathbf{GL_2}(\mathbb{Q}_p)$. Let $\pi_P \cong \pi^{(2)} \otimes_k \chi$ with $\pi^{(2)}$ a smooth admissible representation of $\mathbf{GL_2}(\mathbb{Q}_p)$.

**Proposition 4.4.1** *Let $\pi_P \cong \pi^{(2)} \otimes \chi$ be the twist of a supercuspidal modulo $p$ representation $\pi^{(2)}$ of $\mathbf{GL_2}(\mathbb{Q}_p)$ by a character $\chi$ of the torus. Then we have*

$$\dim_{k((X))} D^\vee_\xi \left(\mathrm{Ind}_{P^-}^G \pi_P\right) = \begin{cases} 0 & \text{if } N_\beta \subset L_P \\ 2 & \text{if } N_\alpha \subset L_P \end{cases}.$$

**Proof** We use the compatibility with parabolic induction (Proposition 4.2.2). Note that the torus $T^{(2)}$ is generated by $s^{(2)}$ and $T_0^{(2)}$. So in the case when $N_\beta \subset L_P$ we have an isomorphism

$$\mathrm{Ord}_{s^{\mathbb{Z}} N_{L_P}}(\pi_P) \cong (\mathrm{Ord}_{B^{(2)}}(\pi_2) \otimes \chi)_{|k[F_P]} = 0$$

by the adjunction formula of Emerton's ordinary parts (Thm. 4.4.6 in [7]). In the other case we apply Thm. 0.10 in [4]. $\square$

Now let $\chi = \mathrm{id}$ and $\pi_P = \pi^{(2)} \otimes \mathrm{id}$ be a representation of $L_P \cong \mathbf{GL_2}(\mathbb{Q}_p) \times T'$ such that $N_\beta \subset L_P$.

By definition ([3], section 3) the $k[[X]]$-module structure of $\pi_P^{H_0}$ is isomorphic to those of $\pi^{(2)}$, the $\mathbb{Z}_p^*$-actions are the same, and

$$F_P m = \sum_{i=0}^{p-1} (1+X)^i F^{(2)} m \qquad \text{for } m \in \pi_P^{H_0} = \pi_P^{N_0^{(2)}}.$$



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Let $M^{(2)} \in \mathcal{M}(\pi^{(2)})$ and consider the $k$-vectorspace $(M^{(2)})^\vee / X(M^{(2)})^\vee = (M^{H_0})^\vee$. $M^{H_0}$ is $F_P$-invariant thus we have an action of $F_P$ on the dual. We describe it with the $\psi$ coming from the étale $(\varphi, \Gamma)$-module structure of $(M^{(2)})^\vee[1/X]$ (cf. Lemma 2.6 and the part after Lemma 3.1 in [3]):

$$F_P(d + X(M^{(2)})^\vee) = \psi\left(\sum_{i=0}^{p-1}(1+X)^i d\right) + X(M^{(2)})^\vee \qquad (d \in (M^{(2)})^\vee).$$

**Proposition 4.4.2** *Let $\pi^{(2)}$ be an extension of principal series:*

$$0 \to \pi_1^{(2)} = \mathrm{Ind}_{B^{(2)-}}^{\mathbf{GL_2}(\mathbb{Q}_p)}(\chi_1 \otimes \chi_2) \xrightarrow{i} \pi^{(2)} \xrightarrow{j} \pi_2^{(2)} = \mathrm{Ind}_{B^{(2)-}}^{\mathbf{GL_2}(\mathbb{Q}_p)}(\chi_1' \otimes \chi_2') \to 0,$$

*and $D(\pi^{(2)})$ be the $(\varphi, \Gamma)$-module attached to $\pi^{(2)}$ by the classical Montréal functor $D$. Then $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee$ is a quotient of*

$$(\Lambda/X\Lambda)_{F_P^\infty=0} = (\Lambda/X\Lambda)/\langle d \in \Lambda/X\Lambda | \exists n \in \mathbb{N} : F^n d = 0\rangle$$

*for a certain lattice $\Lambda$ containing the smallest $\psi$-invariant lattice $D^\natural(\pi^{(2)}) \subset D(\pi^{(2)})$.*

**Proof** As before, we have $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P) \cong \mathrm{Ord}_{B^{(2)}}(\pi^{(2)}) \otimes \mathrm{id} \cong \mathrm{Ord}_{B^{(2)}}(\pi^{(2)})$. Let us denote it with $\mathrm{Ord}^{(2)}$.

We have $\dim_k(\mathrm{Ord}^{(2)}) \leq 2$, because the ordinary parts of the principal series are 1 dimensional over $k$ (Theorem 4.2.12 in [8]), and the functor $\pi \mapsto \mathrm{Ord}(\pi)$ is left exact (Proposition 3.2.4 in [7]).

For a principal series representation $\pi_0^{(2)}$, if $M \in \mathcal{M}(\pi_0^{(2)})$ such that $M^\vee[1/X]$ is nontrivial, then we have $\mathrm{Ord}_{B^{(2)}}(\pi_0^{(2)}) \leq M^{N_0^{(2)}}$. The minimal generating $B_+$-subrepresentation $M_0 \in \mathcal{M}(\pi_0^{(2)})$ of the Steinberg representation is of that kind. Assume indirectly that $M^{N_0^{(2)}}$ does not contain the ordinary part for some $M \in \mathcal{M}(\pi_0^{(2)})$. We have $\dim_{k((X))}(M'^\vee[1/X]) \leq 1$ for all $M' \in \mathcal{M}(\pi_0^{(2)})$. But then by Lemma 2.1 in [3] we would have $M' = M + M_0 \in \mathcal{M}(\pi_0^{(2)})$ and $\dim_{k((X))}(M'^\vee[1/X]) \geq 2$.

We show, that there exists $M' \in \mathcal{M}(\pi^{(2)})$ such that $\mathrm{Ord}^{(2)} \leq M'$.

If $\dim_k(\mathrm{Ord}^{(2)}) = 1$, then $\mathrm{Ord}^{(2)} \cong \mathrm{Ord}_{B^{(2)}}(\pi_1^{(2)})$ which is contained in the Steinberg representation $M_1 \leq \pi_1^{(2)}$. Thus $\mathrm{Ord}^{(2)} \leq M' = i(M_1) \in \mathcal{M}(\pi^{(2)})$.



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If $\dim_k(\mathrm{Ord}^{(2)}) = 2$, we use the fact that $\mathrm{Ord}_{B^{(2)}}$ is the right adjoint of $\mathrm{Ind}_{B^{(2)-}}^{\mathbf{GL_2}(\mathbb{Q}_p)}$ ([7] Theorem 4.4.6). We have

$$0 \to \chi_1 \otimes \chi_2 \to U \cong \mathrm{Ord}^{(2)} \to \chi_1' \otimes \chi_2' \to 0.$$

Thus the isomorphism $U \to \mathrm{Ord}^{(2)}$ gives an isomorphism $\mathrm{Ind}_{B^{(2)-}}^{\mathbf{GL_2}(\mathbb{Q}_p)}(U) \to \pi^{(2)}$.

Let $M'$ be the $k[[X]][F]$-representation generated by $\mathrm{Ord}^{(2)}$. $M' \in \mathcal{M}(\pi^{(2)})$, because any $f \in M$ viewed as a function $G \to U$ has support in $N_0^{(2)} B^{(2)-}$, thus $M'^{\vee}$ is admissible.

Moreover we can choose $M$ such that $M^{\vee}[1/X] \cong D(\pi^{(2)})$: let $M'' \in \mathcal{M}(\pi^{(2)})$ be such that $M'^{\vee}[1/X] \cong D(\pi^{(2)})$. Then we also have $M = M' + M'' \in \mathcal{M}(\pi^{(2)})$ (cf Lemma 2.1 in [3]).

Set $\Lambda = M^{\vee} \leq M^{\vee}[1/X]$. This is $\psi$-invariant and generates $D(\pi^{(2)})$, thus it contains $D^{\natural}(\pi^{(2)})$. We got that $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^{\vee}$ is a quotient of $\Lambda/X\Lambda$. Moreover since $F_P$ acts surjectively on $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$, the dual is a quotient of $(\Lambda/X\Lambda)_{F_P^{\infty}=0}$. □

**Corollary 4.4.3** *Let $\chi_1 \neq \chi_2$, $\chi_1' = \chi_2\overline{\varepsilon}^{-1}$ and $\chi_2' = \chi_1\overline{\varepsilon}$ with $\chi_1 \neq \chi_1'$ and $\overline{\varepsilon} : \mathbb{Q}_p^* \cong p^{\mathbb{Z}} \times \mathbb{Z}_p^* \to \mathbb{Z}_p^* \to \mathbb{F}_p^*$ denoting the modulo $p$ cyclotomic character. Then we have an exact sequence*

$$0 \to \mathrm{Ind}_{P^-}^G(\pi_1^{(2)} \otimes \mathrm{id}) \to \pi = \mathrm{Ind}_{P^-}^G(\pi^{(2)} \otimes \mathrm{id}) \to \mathrm{Ind}_{P^-}^G(\pi_2^{(2)} \otimes \mathrm{id}) \to 0,$$

*but the natural map $D_{\xi}^{\vee}(\mathrm{Ind}_{P^-}^G(\pi_2^{(2)} \otimes \mathrm{id})) \to D_{\xi}^{\vee}(\mathrm{Ind}_{P^-}^G(\pi^{(2)} \otimes \mathrm{id}))$ is not injective.*

**Proof** The above sequence is exact, because both $- \otimes \mathrm{id}$ and $\mathrm{Ind}_{P^-}^G(-)$ are exact.

By Proposition 4.2.2 we have $D_{\xi}^{\vee}(\mathrm{Ind}_{P^-}^G(\pi_2^{(2)} \otimes \mathrm{id})) \cong k((X)) \otimes \mathrm{Ord}_{B^{(2)}}(\pi_2^{(2)})$ and $D_{\xi}^{\vee}(\mathrm{Ind}_{P^-}^G(\pi^{(2)} \otimes \mathrm{id})) \cong k((X)) \otimes \mathrm{Ord}_{B^{(2)}}(\pi^{(2)})$ (here we also used that $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi) \cong \mathrm{Ord}_{B^{(2)}}(\pi^{(2)})$ as before).

For any extension $D$ of the $(\varphi, \Gamma)$-modules $D(\pi_1^{(2)})$ and $D(\pi_2^{(2)})$ there exists an extension $\pi^{(2)}$ of the two principal series with $D(\pi^{(2)}) = D$, since the functor $D$ is essentially surjective (see Thm 0.17(iii) in [4]) and we have $\dim_{\mathbb{F}_p}(\mathrm{Ext}(\pi_2^{(2)}, \pi_1^{(2)})) = 1$ (see [8] Prop. 4.3.15(2)).



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Thus it suffices to prove, that there exists a nontrivial extension $D$ and that for any lattice $\Lambda \supseteq D^\natural$ the action $F_P$ on $\Lambda/X\Lambda$ has nontrivial kernel. This is done in the following section. $\square$

## 4.5 Extensions of 1 dimensional $(\varphi, \Gamma)$-modules

The most part of the following is folklore, however I could not find it anywhere, so I wrote it down. Let $p$ be an odd prime and $\gamma \in \Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ be a topological generator. Let $\chi : \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \to \mathbb{Z}_p^*$ be the cyclotomic character.

For $f(X) = \sum_n \lambda_n X^n \in \mathbb{F}_p((X))^*$, write $\deg(f(X)) = \min\{n|\lambda_n \neq 0\}$.

**Proposition 4.5.1** *Let $D$ be a one dimensional $(\varphi, \Gamma)$-module over $\mathbb{F}_p((X))$. Then there exists a basis $\{e\}$ of $D$ and $\lambda, \mu \in \mathbb{F}_p^*$ such that the $\varphi(e) = \lambda e$ and $\gamma(e) = \mu e$.*

**Proof** Let $e_0$ be any generator of $D$. Then $\varphi(e_0) = f(X)e$ for some $f \in \mathbb{F}_p((X))$. We can write $f(X) = \lambda_0 X^n f'(X)$ with $\lambda_0 \in \mathbb{F}_p^*$, $n \in \mathbb{Z}$ and $f'(X) \in 1 + X\mathbb{F}_p[[X]]$.

If we change the basis to $e = h(X)e_0$ for any $h(X) \in \mathbb{F}_p((X))^*$, we have $\varphi(h(X)e) = h(X^p)\varphi(e) = (h(X^p)/h(X) \cdot \lambda_0 X^n f'(X))(h(X)e)$. After choosing $h(X) = X^{\lfloor n/p \rfloor} \prod_{j=0}^{\infty} f'(X^{p^j})$ (which is convergent in $\mathbb{F}_p((X))$, since $f'(0) = 1$), we have that $\varphi(e) = \lambda_0 X^m e$, where $0 \leq m < p$ and $p|n - m$.

Let $\gamma(e) = g(X)e = \mu_0 X^l g'(X)e$ with $\mu_0 \in \mathbb{F}_p^*$, $l \in \mathbb{Z}$ and $g'(X) \in 1 + X\mathbb{F}_p[[X]]$. Then we have $\varphi(\gamma(e)) = \gamma(\varphi(e))$, where on the left hand side we have:

$$\varphi(\gamma(e)) = \varphi(\mu_0 X^l g'(X)e) = \lambda_0 \mu_0 X^{pl} g'(X^p) X^m e.$$

On the right hand side

$$\gamma(\varphi(e)) = \gamma(\lambda_0 X^m e) = \lambda_0 \mu_0 ((1 + X)^{\chi(\gamma)} - 1)^m X^l g'(X)e.$$

Thus we have $X^{pl+m} g'(X^p) = ((1 + X)^{\chi(\gamma)} - 1)^m X^l g'(X)$, comparing the degrees and the leading coefficients gives $l = m = 0$, $g'(X) = 1$ and we have the proposition. $\square$

Recall the following definitions of Colmez (cf [5]): For a $(\varphi, \Gamma)$-module $D$

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- we define $D^{nr} = \cap_{n\in\mathbb{N}}\varphi^n(D) \leq D$,
- $D^\natural \leq D$ to be the smallest $\psi$-invariant lattice and
- $D^\# \leq D$ to be the biggest $\psi$-invariant lattice on which $\psi$ acts surjectively.

**Corollary 4.5.2** *If $D$ is one dimensional with a basis $e$ as above, we have $D^{nr} = \mathbb{F}_p e$, $D^\natural = k[[X]]e$ and $D^\# = X^{-1}k[[X]]e$.*

**Proof** The first two statements are clear, the last comes from the facts that $\psi(X^{-1}e) = \psi(\sum_{i=0}^{p-1}(1+X)^i\varphi(X^{-1}e)) = X^{-1}e$ and that $\psi(X^m e) \in X^{m+1}k[[X]]e$ if $m < -1$. □

**Remark** For any $\lambda_0, \mu_0 \in \mathbb{F}_p^*$ there exists a one dimensional $(\varphi,\Gamma)$-module, such that the matrix of $\varphi$ (respectively $\gamma$) is $\lambda_0$ (respectively $\mu_0$). It is easy to see that in this case the action of $\varphi$ is étale and the action of $\gamma$ extends continuously to $\Gamma$.

Altogether there are $(p-1)^2$ one dimensional $(\varphi,\Gamma)$-modules over $\mathbb{F}_p((X))$.

Now let $D_1$ and $D_2$ be one dimensional $(\varphi,\Gamma)$-modules over $\mathbb{F}_p((X))$. We determine the extensions of $D_2$ by $D_1$. By the previous proposition we might choose a basis $\{e_i'\}$ in $D_i$ such that $\varphi(e_i') = \lambda_i e_i'$ and $\gamma(e_i') = \mu_i e_i'$ for $i = 1, 2$ and $\lambda_i, \mu_i \in \mathbb{F}_p^*$.

**Proposition 4.5.3**

- *If $D$ is an extension of $D_2$ by $D_1$, then in an appropriate basis $\{e_1, e_2\} \subset D$ we have $\varphi(e_1) = \lambda_1 e_1$, $\varphi(e_2) = f(X)e_1 + \lambda_2 e_2$, $\gamma(e_1) = \mu_1 e_1$, $\gamma(e_2) = g(X)e_1 + \mu_2 e_2$, with $f(X) = \sum_i \alpha_i X^i$ and $g(X) \in \mathbb{F}_p((X))$, such that $\alpha_i = 0$ if a) $i > 0$ or b) $i < 0$ and $p|i$, and*
$$\mu_1 f((1+X)^{\chi(\gamma)} - 1) - \mu_2 f(X) = \lambda_1 g(X^p) - \lambda_2 g(X).$$
*If $\lambda_1 \neq \lambda_2$ we can also have $\alpha_0 = 0$.*

- *Let $f(X), g(X) \in \mathbb{F}_p((X))$ as above. Then there exists a 2 dimensional $(\varphi,\Gamma)$-module $D$, for which the above statements hold. If $f'(X) \neq \alpha f(X)$ for any $\alpha \in \mathbb{F}_p^*$ and $g'(X)$ are as above with a $(\varphi,\Gamma)$-module $D'$, then $D \not\cong D'$.*



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**Proof**

- We may choose a basis $\{e_1, e_2\}$ in $D$ such that $e_1$ is the image of $e_1'$ and $e_2$ is a preimage of $e_2'$. Then there exist $f(X), g(X) \in \mathbb{F}_p((X))$ such that $\varphi(e_2) = f(X)e_1 + \lambda_2 e_2$ and $\gamma(e_2) = g(X)e_1 + \mu_2 e_2$.

  We have $\varphi(\gamma(e_2)) = \varphi(g(X)e_1 + \mu_2 e_2) = (\lambda_1 g(X^p) + \mu_2 f(X))e_1 + \lambda_2\mu_2 e_2$ and $\gamma(\varphi(e_2)) = \gamma(f(X)e_1 + \lambda_2 e_2) = (\mu_1 f((1+X)^{\chi(\gamma)} - 1) + \lambda_2 g(x))e_1 + \lambda_2\mu_2 e_2$, thus

  $$\mu_1 f((1+X)^{\chi(\gamma)} - 1) - \mu_2 f(X) = \lambda_1 g(X^p) - \lambda_2 g(X).$$

  Now we look at the basis $\{e_1, e_2 + h(X)e_1\}$ for $h(X) \in \mathbb{F}_p((X))^*$. We have $\varphi(e_2 + h(X)e_1) = (f(X) + \lambda_1 h(X^p) - \lambda_2 h(X))e_1 + \lambda_2(e_2 + h(X)e_1)$ and $\gamma(e_2 + h(X)e_1) = (g(X) + \mu_1 h((1+X)^{\chi(\gamma)} - 1) - \mu_2 h(X))e_1 + \mu_2(e_2 + h(X)e_1)$.

  Let $i_0 = pj_0 < 0$ minimal such that $\alpha_{i_0} \neq 0$. Then setting $h(X) = -\lambda_1^{-1}\alpha_{i_0}X^{j_0}$ and $e_2 = e_2 + h(X)e_1$ we can change $\lambda_{i_0} = 0$. Thus we may assume, that $\alpha_{pj_0} = 0$ for $j_0 < 0$.

  If $\lambda_1 \neq \lambda_2$, then change $e_2$ to $e_2 - \alpha_0(\lambda_1 - \lambda_2)^{-1}$, then $\lambda_0 = 0$. For $i > 0$ we can set $\alpha_i = 0$ inductively.

- It is clear, that the action of $\varphi$ is étale. (the matrix of $\varphi$ is upper triangular)

  We need that the action of $\gamma$ extends continuously to $\Gamma$. We claim that it is always true if $\gamma$ has matrix $\begin{pmatrix} \mu_1 & g(X) \\ 0 & \mu_2 \end{pmatrix}$. Let $k_n \in \mathbb{N}$ such that $\gamma^{k_n}$ converges in $\Gamma$. It suffices to verify, that for all $j \in \mathbb{Z}$ there exists $N(j)$ such that for $n, m > N(j)$ in $\gamma^{k_n}(e_2) - \gamma^{k_m}(e_2)$ the coefficient of $X^{j'}$ for $j' \leq j$ is 0. We have

  $$\gamma^k(e_2) = \left(\sum_{i=0}^{k-1} \mu_1^i \mu_2^{k-1-i} g((1+X)^{\chi(\gamma)^i} - 1)\right)e_1 + \mu_2^k e_2,$$

  Let $d = \deg(g)$ and $l = \max\{j - d, j + 1\}$. The convergence of $\gamma^{k_n}$ yields that there exists $N'(j)$ such that for all $n, m > N'(j)$ we have $(p-1)p^l | k_n - k_m$. If $n, m > N'(j)$ then for any $i \in \mathbb{N}$ we have $\mu_2^{k_n - i} = \mu_2^{k_m - i}$ and

  $$X^j | g((1+X)^{\chi(\gamma)^i} - 1) - g((1+X)^{\chi(\gamma)^{k_n - k_m + i}} - 1).$$



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Suppose that $k_n \geq k_m$. Then for $q = (p-1)p^j$ and for some $h(X), h'(X) \in \mathbb{F}_p[[X]]$ we have

$$\gamma^{k_n}(e_2) - \gamma^{k_m}(e_2) =$$

$$= \left( \sum_{i=0}^{k_n - k_m - 1} \mu_1^i \mu_2^{k_n-1-i} g((1+X)^{\chi(\gamma)^i} - 1) + X^j h(X) \right) e_1 =$$

$$= \left( \frac{k_n - k_m}{q} \left( \sum_{i=0}^{q-1} \mu_1^i \mu_2^{k_n-1-i} g((1+X)^{\chi(\gamma)^i} - 1) \right) + X^j h'(X) \right) e_1 =$$

$$= X^j h'(X) e_1,$$

since $pq | k_n - k_m$. Thus $N(j) = N'(j)$ is a convenient choice.

To see that for different choices of $f(X)$ we get different modules let $\{d_1, d_2\}$ be an other basis in $D$, such that the matrix of $\varphi$ (and $\gamma$) is upper triangular. We will show, that then $d_1 = \alpha e_1$ with $\alpha \in \mathbb{F}_p^*$, unless $f(X) = 0$, which is sufficient for the proposition.

Let $d_1 = a(X)e_1 + b(X)e_2$. $\lambda d_1 = \varphi(d_1) = (\lambda_1 a(X^p) + f(X)b(X^p))e_1 + \lambda_2 b(X^p) e_2$, thus we have $\lambda_2 b(X^p) = \lambda b(X)$, meaning either $\lambda = \lambda_2$ and $b(X) = \beta \in \mathbb{F}_p^*$ or $b(X) = \beta = 0$. We also have $\lambda_1 a(X^p) + f(X)\beta = \lambda a(X)$. Then by the properties of $f(X)$ we have that the coefficients of $X^i$ in $a(X)$ with $i > 0$ is 0, and $\deg(a) = 0$, because otherwise the coefficient of $X^{p \deg(a)}$ is nonzero on the left hand side and 0 on the right. Thus $a(X) = \alpha$ and $f(X) = \delta$ with $\alpha, \delta \in \mathbb{F}_p$. If $\lambda_1 \neq \lambda_2$, then $f(X) = 0$ (see the last statement in the first part of the proposition). If $\lambda_1 = \lambda_2$, then $\lambda_1 \alpha + \delta\beta = \lambda_1 a(X^p) + f(X)\beta = \lambda a(X) = \lambda_1 \alpha$, thus either $\delta = f(X) = 0$ or $\beta = 0$ hence $d = \alpha e_1$.

$\square$

**Corollary 4.5.4** *If $\lambda_1 \neq \lambda_2$, then there exists a nontrivial extension of $D_2$ by $D_1$.*

**Proof** Let $(1+X)^{\chi(\gamma)} - 1 = X(\rho + Xh(X))$, and $n$ with $1 - p \leq n < 0$ such that $\mu_1 \rho^n = \mu_2$. We can choose $f(X) = \sum_{i=n}^{-1} \alpha_i X^i$ such that $\mu_1 f((1+X)^{\chi(\gamma)} - 1) - \mu_2 f(X) \in \mathbb{F}_p[[X]]$, because for $i > n$ we have $\mu_1 \rho^i \neq \mu_2$, and we can choose the $\alpha_i$-s inductively in increasing order. Thus there exists $g(X)$ such that the condition for $f(X)$ and $g(X)$ is satisfied. $\square$



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**Remark** By the modulo $p$ Langlands-correspondence for $\mathbf{GL_2}(\mathbb{Q}_p)$ these 2-dimensional $(\varphi,\Gamma)$-modules (which are the extension of two 1-dimensional ones) correspond to extension of principal series representations of $\mathbf{GL_2}(\mathbb{Q}_p)$.

Let $\pi = \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ and $\pi' = \mathrm{Ind}_B^G(\chi_1' \otimes \chi_2')$ (with $\chi_i, \chi_j' : \mathbb{Q}_p^* \to \mathbb{F}_p^*$ characters) be principal series of $\mathbf{GL_2}(\mathbb{Q}_p)$. By [6], Proposition 4.3.15. there exists nontrivial extension of $\pi'$ by $\pi$ if and only if either $\chi_1 = \chi_1'$ and $\chi_2' = \chi_2$, or $\chi_1 = \chi_2'\overline{\chi}^{-1}$ and $\chi_2 = \chi_1'\overline{\chi}$ (where $\overline{\chi}$ is the modulo $p$ reduction of the cyclotomic character).

The $(\varphi,\Gamma)$-module $D(\pi)$ attached to $\pi$ is not $D_1$ or $D_2$. $D_i$ contains information only of $\chi_i$. However from $D$ we can recover $\pi$ and $\pi'$ (and the other way around): $\chi_i|_{1+p\mathbb{Z}_p} = \chi_j'|_{1+p\mathbb{Z}_p} = 1$ we have $\chi_1'(p) = \lambda_1$, $\chi_1'(\gamma) = \mu_1$, $\chi_1(p) = \lambda_2$ and $\chi_1(\gamma) = \mu_2$ (cf. the part before Thérorème 0.9 in [4]). If $\chi_1' \neq \chi_1$, then $\chi_2 = \chi_1'\overline{\chi}$ and $\chi_2' = \chi_1\overline{\chi}$.

**Proposition 4.5.5** *Let $D$ be as in the previous proposition. Then*

$$\dim_{\mathbb{F}_p}(D^{nr}) = \begin{cases} 2, & \text{if } f(X) \in \mathbb{F}_p \subset \mathbb{F}_p((X)), \\ 1, & \text{otherwise.} \end{cases}$$

**Proof** We have

$$\varphi^n(a(X)e_1 + b(X)e_2) =$$
$$\left(\lambda_1^n a(X^{p^n}) + \sum_{i=0}^{n-1} \lambda_1^i \lambda_2^{n-1-i} f(X^{p^i}) b(X^{p^n})\right) e_1 + \lambda_2^n b(X^{n^p}) e_2.$$

If $d = a_0(X)e_1 + b_0(X)e_2 \in D^{nr}$, and $\mathrm{pr} : D \to D_2$, then $\mathrm{pr}(d) \in D_2^{nr} = \mathbb{F}_p e_2'$, hence if $d = \varphi^n(a(X)e_1 + b(X)e_2) \in D^{nr}$, then $b(X) = \beta \in \mathbb{F}_p$.

In $f$ the coefficients of $X^{pj}$ with $j < 0$ are 0, hence in the above sum the coefficient of $X^{p^{n-1}\deg(f)}$ is not 0. Thus if $d \in \varphi^n(D)$, then either $\deg(a_0) \leq p^{n-1}\deg(f)$ or $\deg(a_0) \geq 0$. Hence if $d \in D^{nr}$, we have $\deg(a_0) = 0$, and $a(X) = \alpha \in \mathbb{F}_p$.

If $\deg(f) < 0$, then we must have $\beta = 0$. □

**Proposition 4.5.6** *Let $D$ be as in Lemma 4.5.3 such that $-p < \deg(f) < 0$. Then $D^\natural = X^{-1}\mathbb{F}_p[[X]]e_1 + \mathbb{F}_p[[X]]e_2$.*

**Proof** Let $\Lambda = X^{-1}\mathbb{F}_p[[X]]e_1 + \mathbb{F}_p[[X]]e_2$. It is a $k((X))$-generating submodule, we show that it is $\psi$-invariant as well. Let $d \in \Lambda$. We can write it in



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the form $d = \sum_{i=0}^{p-1}(1+X)^i\varphi(\alpha_i(X)e_1 + \beta_i(X)e_2)$, and a simple computation shows that $\alpha_i(X) \in X^{-1}\mathbb{F}_p[[X]]$ and $\beta_i(X) \in \mathbb{F}_p[[X]]$ for all $i$. Then $\psi(d) = \alpha_0(X)e_1 + \beta_0(X)e_2 \in \Lambda$. Thus $D^\natural \subseteq \Lambda$.

$\mathbb{F}_p[[X]]e_1 \subset D^\natural$, because if $D' \to D$ is injective, then so is $D'^\natural \to D^\natural$ (cf [5] Prop. II.5.17($ii$).), and $\mathbb{F}_p((X))e_1 \hookrightarrow D$ as a $(\varphi,\Gamma)$-module, with $D^\natural(\mathbb{F}_p((X))e_1) = \mathbb{F}_p[[X]]e_1$.

We also have that if $D \to D'$ is surjective, then so is $D^\natural \to D'^\natural$ (cf [5] Prop. II.5.17($iii$).), thus we have an element in the form $d = \lambda X^{-1}e_1 + \lambda_2 e_2$ in $D^\natural$ with some $\lambda \in \mathbb{F}_p$ because $\mathbb{F}_p[[X]]e_1 \leq D^\natural$. Then we have

$$d = \varphi(e_2) + (\lambda X^{-1} - f(X))e_1 = \varphi(e_2) + \sum_{i=0}^{p-1}(1+X)^i\varphi(\alpha_i(X)e_1)$$

with $\alpha_i(X) \in X^{-1}\mathbb{F}_p[[X]]$. We have $\alpha_i(X) \in \mathbb{F}_p[[X]]$ for $i < p + \deg(f)$.

If $\lambda X^{-1} \neq f(X)$, then we also have $\alpha_{p+\deg(f)}(X) \notin \mathbb{F}_p[[X]]$, thus $\psi((1+X)^{-(p+\deg(f))}d) = \alpha_{p+\deg(f)}e_1$, meaning $\Lambda \subseteq D^\natural$.

If $\lambda X^{-1} = f(X)$, then $\psi(d) = e_2 \in D^\natural$ and also $\lambda^{-1}(d - \lambda_2 e_2) = X^{-1}e_1 \in D^\natural$, and we again have $\Lambda \subseteq D^\natural$. $\square$

**Corollary 4.5.7** *If $D$ is as above, then the action $F_P$ defined in the previous section has a nontrivial kernel for any $\Lambda \supseteq D^\natural$.*

**Proof** Recall that $F_P : d + X\Lambda = \psi(\sum_{i=0}^{p-1}(1+X)^i d) + X\Lambda$.

Let $d = X^m e_1 \in \Lambda \cap D_1$ such that $m = \min\{m | m \in \mathbb{Z}, X^m e_1 \in \Lambda\}$. By the Proposition 4.5.6 we have $m \leq -1$. Then $d + X\Lambda \notin X\Lambda$, hence it is enough to prove that $\psi(\sum_{i=0}^{p-1}(1+X)^i d) \in X^{m+1}\mathbb{F}_p[[X]]e_1 \subset X\Lambda$.

If $m < -1$, then it is clear, because then $\Lambda \cap D_1 \supsetneq D_1^{\#}$, hence $\psi$ is not surjective on it, meaning $\psi(d') \in X^{m+1}\mathbb{F}_p[[X]]e_1$ for any $d' \in \Lambda \cap D_1$, especially for $d' = \sum_{i=0}^{p-1}(1+X)^i d$.

If $m = -1$, then

$$\psi\left(\sum_{i=0}^{p-1}(1+X)^i\frac{1}{X}e_1\right) = \psi\left(\sum_{i=0}^{p-1}(1+X)^i\left(\sum_{j=0}^{p-1}(1+X)^j\varphi\left(\frac{1}{X}\right)\right)e_1\right) =$$
$$= \psi\left(\sum_{i,j=0}^{p-1}(1+X)^{i+j}\varphi\left(\frac{1}{X}\right)e_1\right) = \lambda_1(1+(p-1)(1+X))\frac{1}{X}e_1 =$$
$$= \lambda_1(p-1)e_1 \in \mathbb{F}_p[[X]]e_1.$$

$\square$



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