Characterizing of digraphs with every edge in a fixed number of cycles

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Submitted to Central European University Department of Mathematics And Its Applications

Mathematics and its Applications
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Contents

Abstract

About 40 years ago A. Ádám described the structure of the digraphs with every vertex being in at most two cycles. He provided constructive theorem for it and raised the problem: How to describe the digraphs with every edge at most in two cycles? Later Zelinka and Győri proved that Necklace is the only directed graph with the property that every edge is contained in exactly 2 directed cycles. Győri also provided both constructive and direct structure theorems of more general problem with every edge being in at most 2 directed cycles. He also described that if we know solution for every edge being in at most k cycles then we can easily characterize digraphs with every vertex in at most $k + 1$ cycles. In the present note, we give a characterization of directed graphs with the property that every edge is contained in exactly 3 directed cycles. We will also provide interesting examples for the digraphs with every edge in k of cycles for some higher k. This is a joint paper with my coursemate Abhishek Methuku and supervisor Ervin Győri.

Acknowledgements

First of all, I would like to thank my supervisor Ervin Győri firstly for making me interested in graph theory and then for being an excellent supervisor and for teaching basically everything I know now in graph theory.

I would also like to thank the co-author of the solution of the main theorem Abhishek Methuku, who helped me to gain confidence for thinking about open problems and without whom I wouldn't even start thinking about this (and many other) problem.

I want to thank my friends Nika Salia and Francisco Gonzalez De Cossio (and basically all my course mates) for being around and for reminding me every important mail that I forgot to check.

I would also like to thank to all CEU professors, especially Karoly Boroczky and Pal Hegedus for being patient about my slight irresponsible behaviours sometimes.

Chapter 1

Introduction

1.1 Basic definitions

Definition 1. *A directed graph (Digraph) is a pair $D = (V, E)$ where $V = V(D)$ is a set of vertices and $E = E(D)$ is a set of ordered pairs (directed edges) of elements of V.*

Definition 2. *A set $P = \bigcup_{i=1}^{n}(x_1, y_1)$ is called a walk in D if $x_i, y_i \in E(D)$ for all i and $y_i = x_{i+1}$ for $1 \leq i \leq n-1$.*

Definition 3. *A set $P = \bigcup_{i=1}^{n}(x_1, y_1)$ is called a path in D if $x_i = y_j$ if and only if $i - j = 1$.*

Definition 4. *A set $C = \bigcup_{i=1}^{n}(x_1, y_1)$ is called a directed cycle in D if $x_i = y_j$ if and only if $i - j = 1$ or $j - 1 = n - 1$.*

Definition 5. *A digraph D is called strongly connected if for every $x, y \in V(D)$ there is a path from x to y and there is a path from y to x.*

Definition 6. *A vertex is called a cut-vertex of a connected digraph if there are $v, x, y \in V(D)$ such that every path that contains x and y, also contains v. (In other words, after deleting v graph becomes not connected).*

1.2 Statements of theorems

Theorem 7 (A. Ádám [2], [4])**.** *The members of the digraphs with every vertex in at most 2 cycles is family of digraphs produced by means of the following construction:*

Construction 0.

step 1.

Let T be an undirected tree with at least one edge. For each vertex $v \in V(T)$ we denote with $e_1^v, e_2^v, ..., e_{d(v)}^v$ the edges incident to v (obviously, every edge gets two notations).

step 2.

Let us form a digraph D_1 in the following way: the vertices of D_1 correspond one-to-one with edges of T; if the vertex $w \in V(D_1)$ corresponds to the edge $e_p^p = e_q^Q \in E(T)$ then edges go from w to the vertices corresponding to e_{p+1}^p and e_{q+1}^Q and only to these vertices (in case $p = d(P)$ e_1^p plays the role of e_{p+1}^p, similarly in case of $q = d(Q)$).(D_1 may contain loops)

step 3.

Choose a subset V' of the vertex-set $V(D_1)$ arbitrarily. For any vertex $v \in V'$ perform the following procedure:

Replace v with two vertices v' and v".

if an edge was entering to v then let it enter to v'.

if an edge was going out of v then let it go out from v".

finally add a new edge leading from v' to v" to the graph.

Let D_2 denote the produced graph (it may contain loops).

step 4.

Instead of any edge e of D_2 we take a directed path of arbitrary length (Directions of the path and the edge are the same). If e was a loop then instead of it we take a cycle of arbitrary length > 1.

Ádám raised a problem how to describe digraphs with every edge at most in 2 cycles, which was solved by Győri. Before stating that theorem let's discuss slightly different class of digraphs. Main question of our paper is following.

Question 8. *Let $k \geq 1$ be an integer. Which directed graphs satisfy the property that every edge is contained in exactly k directed cycles?*

First, let us make two simple observations which will allow us to assume that the directed graphs under study are strongly connected and have no cut vertices.

Let D be a directed graph such that every edge of D is in exactly k directed cycles. If D is disconnected, then each connected component of D satisfies the property that every edge is in exactly k cycles which are contained inside the component. Therefore, each component satisfies our desired property. So we can assume that D is connected. That is, for any two vertices $u, v \in V(D)$, there is a directed path from u to v. Since every edge of this path is contained in a cycle, it follows that there is also a directed path from v to u. Thus D is in fact, strongly connected.

If $D = D_1 \cup D_2$ such that $V(D_1) \cap V(D_2) = \{v\}$ then every cycle of D is completely contained in either D_1 or D_2. This means that D_1 and D_2 also satisfy our desired property and so we can assume that D has no such cut vertices.

We will need a definition before we can state our results.

Definition 9. *A Necklace is a system of directed cycles* $\mathcal{C} = \{C_i\}_{i=1}^m$ *(*$m > 1$*) such that* $V(C_i) \cap V(C_{i+1}) = \{r_i\}$ *for every* $1 \leq i \leq m-1$*,* $V(C_m) \cap V(C_1) = \{r_m\}$ *and* $V(C_i) \cap V(C_j) = \emptyset$ *when* $|i-j| \neq 0, 1 \pmod{}$*).*

A necklace-path between vertices u *and* v *is as a system of directed cycles* $\mathcal{C} = \{C_i\}_{i=1}^m$ *such that* $u \in C_1$*,* $v \in C_m$ *and* $C_i \cap C_{i+1} = \{r_i\}$ *(*$r_1 \neq u, r_{m-1} \neq v$*) for every* $1 \leq i \leq m-1$ *and* $C_i \cap C_j = \emptyset$ *when* $|i-j| > 1$*.*

We will refer to the vertices r_i *and* u, v *as* key-vertices*.*

Now, if $k = 1$, Question 8 can be easily answered as the only directed graph with this property is a directed cycle. For $k = 2$, Zelinka [3] Győri [1] showed that up to isomorphism, *Necklace* is the only directed graph with this property.

Theorem 10 (B.Zelinka [3])**.** *The following two sentences are equivalent:*

1. *D is a strongly connected digraph without cut vertices.*

2. *D is a necklace.*

Later Győri [1] generalized the problem (answered the question raised by Ádám) with every edge being in at most two cycles.

Theorem 11 (Győri [1])**.** *Let* ϵ_2 *be a class of strongly connected digraphs with no cut vertex, Then* ϵ_2 *is a union of the set of graphs constructed by the following two constructions.*

Construction 1. Take $k > 1$ vertex-disjoint cycles $C_1, C_2, ..., C_k$ and chose $2k$ distinct vertices $x_i y_i$ such that $x_i, y_i \in V(C_i)$ for $i = 1, 2, ..., k$. Then unite vertex pairs (y_i, x_{i+1}) ($i = 1, 2, ..., k-1$) and the pair (y_k, x_1). (note that this is nothing but construction of necklace)

Construction 2. At every step of construction of a digraph D we define a certain subset of the edge-set $E(D)$ and denote it with E_D.

Step 1.

Take a cycle C, Let's $E_c = E(C)$ and C be considered as an additional path.

step 2.

Take a digraph D constructed by means of **Construction 2** and consider a directed path P in D such that $E(P) \subset E_D$ and for every additional path P_0, $E(P_0) \cap (E_D \setminus E(P)) \neq \emptyset$. Let x and y denote the initial and the terminal vertex of the path P, respectively. Then add directed path R to the digraph D which leads from the vertex y to x and for which $V(R) \cap V(D) = x, y$, $E(R) \cap E(D) = \emptyset$. Let's $D' = D \cup R$, then let $E_{D'} = E_D \cup E(R) \setminus E(P)$, and R be an additional path.

step 3.

Let $D_1, D_2, ..., D_k$ be vertex-disjoint digraphs constructed by means of **Construction 2**. For every $i = 1, 2, ..., k$ choose a directed path P_i in D_i such that $E(P_i) \subseteq E_{D_i}$ and for every additional path R_i in D_i, $E(R_i) \cap (E_{Di} \setminus E(P_i)) \neq \emptyset$ (as in *step* 2). Let x_i and

y_i denote the initial and the terminal vertex of the path P_i for every $i = 1, 2, ..., k$. Then unite vertex pairs (y_i, x_{i+1}) ($i = 1, 2, ..., k-1$) and add a directed path P leading from y_k to x_1 to the arising graph D such that $V(P) \cap V(D) = y_k, x_1$ and $E(P) \cap E(D) = \emptyset$. let D' be the obtained graph, then let $E_{D'} = E(P) \cup (\bigcup_{i=1}^{k} E_{D_i}) \setminus (\bigcup_{i=1}^{k} E(P_i))$. and let P be additional path.

Let's now consider different construction which tells us more about the structure.

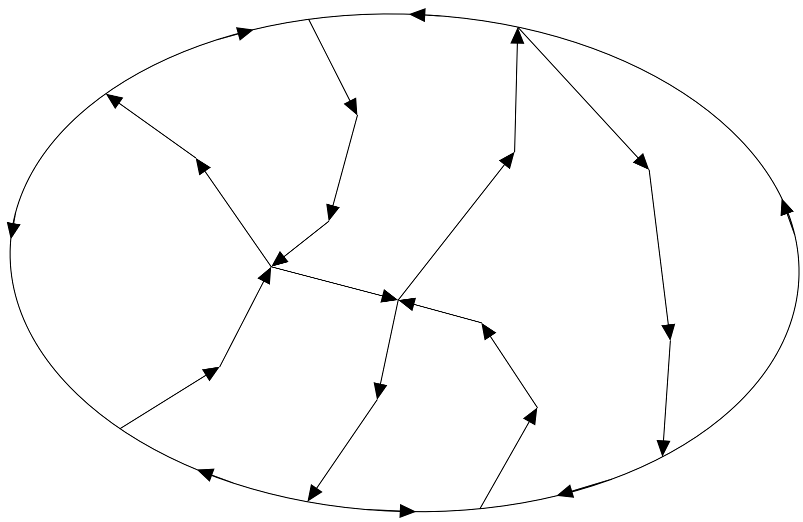


Figure 1.1: A digraph constructed by **Construction 3.**

Construction 3. Consider an undirected 2-connected (cut-vertex free) planar graph such that the unbounded face is adjacent to every bounded face and the degree of every vertex x which doesn't lie on boundary of the unbounded face is even. Direct the edges so that boundary of every unbounded face is a cycle (Degree condition gives opportunity to do it). (see figure 1.2)

Theorem 12 (Győri [1])**.** *Construction 2. is equivalent of the Construction 3..*

The main result of the paper is characterisation of the graphs with every edge in exactly 3 cycles. The following is the main theorem of this paper.

Theorem 13. *If D is finite, connected digraph with no cut-vertices where every edge is contained in exactly 3 directed cycles then D is a graph consisting of three disjoint necklace-paths between two vertices. 1.2*

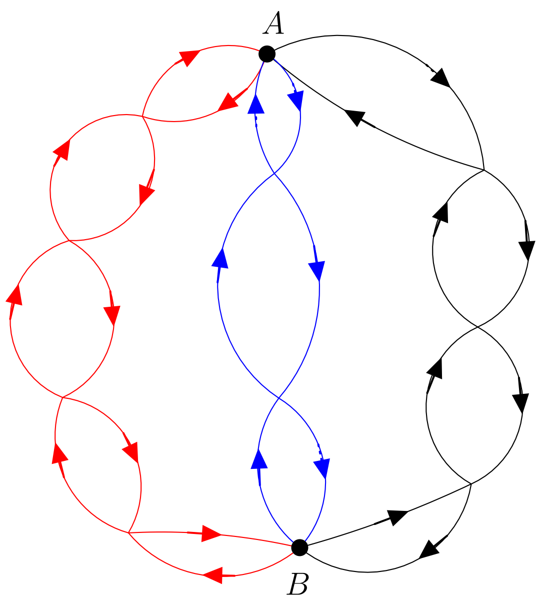


Figure 1.2: Answer of the main theorem

It is interesting to note that, while for $k = 2$ and $k = 3$ we have only one type of directed graphs, with the property that every edge is in exactly k directed cycles, the situation is quite different for $k = 4$. It can be easily checked that the directed graph consisting of 4 disjoint necklace-paths between two vertices satisfies the desired property for $k = 4$. But there exists another different graph with this property as well, which is shown in the following figure. Later we will explain the example in a more detailed way and construct new examples using it.

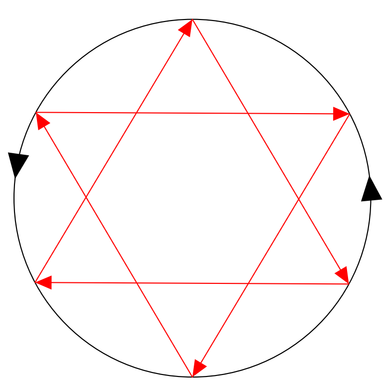


Figure 1.3: Example for $k = 4$

Chapter 2

Strongly connected digraphs with every edge in 3 cycles

2.1 Definitions, lemmas

To prove the main theorem, we need some definitions and lemmas. Suppose that D such that every edge is contained in exactly 3 directed cycles.

Definition 14. *Let C be a directed cycle in our graph $D = (V, E)$. We call a directed path in D, a* Path-chord *(P-chord, for short), of C if it connects two different vertices $P, Q \in V(C)$ and there are no other vertices of C on this path except P and Q. We denote a path chord from P to Q by $\overrightarrow{PQ}$.*

A segment *of C created by a P-chord $\overrightarrow{PQ}$ is defined as a directed cycle which consists of the P-chord $\overrightarrow{PQ}$ and the arc QP of C (the subpath of C from Q to P). Typically, we refer to such a directed cycle as a segment.*

Lemma 15. *If v is a vertex of the directed graph G, then $d_{in}(v) = d_{out}(v)$.*

Proof. Let $C_1, C_2, \ldots, C_m$ be the set of cycles going through v. If there are k_1 ingoing edges at v, then since each edge is in exactly 3 cycles, there are $3k_1$ cycles going through v. If there are k_2 outgoing edges at v, by the same argument, this implies that there are $3k_2$ cycles going through v. Therefore, $3k_1 = 3k_2$, and so $k_1 = k_2$, as desired. $\square$

Lemma 16. *If there is an ingoing (outgoing resp.) path-chord $\mathcal{P}_I$ into a vertex V of the cycle C, then there is at least one outgoing (ingoing resp.) path-chord $\mathcal{P}_O$ from V.*

Proof. Let e_1 and e_2 be the two edges of the cycle C with a common point V. Let's say e_1 is in a segment created by $\mathcal{P}_I$. By contradiction let's assume that there is no outgoing pathchord from V which means that every cycle which contains e_2, also contains e_1 but then e_1 is in a segment which doesn't contain e_2 which means e_1 is in more cycles than e_2, a contradiction. $\square$

Lemma 17. *If $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are P-chords of a cycle C, then R must belong to the directed arc $\widehat{PQ}$ of C.*

Proof. Assume by contradiction that R is an inner point of the directed arc $\widehat{QP}$. The edges of the arc $\widehat{RP}$ are contained in segments created by P-chords $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ and C itself. So they are contained in three cycles already. Also, they are contained in the fourth cycle created by the subpath of the union of path-chords $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ and the directed arc $\widehat{RP}$, a contradiction. □

Lemma 18. *If the end vertices of two P-chords are different then they cannot have a common vertex.*

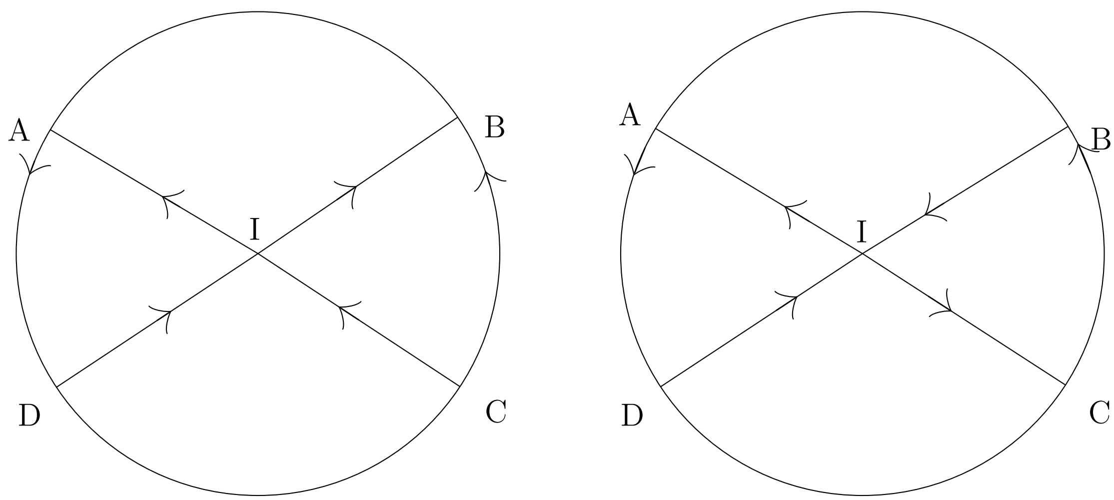


Figure 2.1: Intersecting P-chords

Proof. Let the end vertices of these two P-chords be D, C, B and A in this order on the directed cycle C. Suppose by contradiction that I is one of the common vertices of the two P-chords. Without loss of generality, we can consider only two cases for the orientation of the two P-chords with respect to I:

Case 1. *IA and IB are outgoing paths and CI and DI are ingoing paths.*

Directed paths IA, IB, CI and DI have no common vertices with C except for A, B, C, D as they are parts of P-chords. This means that we can use the walks $DI \cup IA$, $CI \cup IA$, $DI \cup IB$ to obtain 3 P-chords DA, CA and DB. Now consider any edge of C that belongs to the directed arc $\widehat{AD}$. This edge is in 3 segments formed by the P-chords DA, CA, DB and it is also in C, a contradiction.

Case 2. *IA and IC are outgoing paths and BI and DI are ingoing paths.*

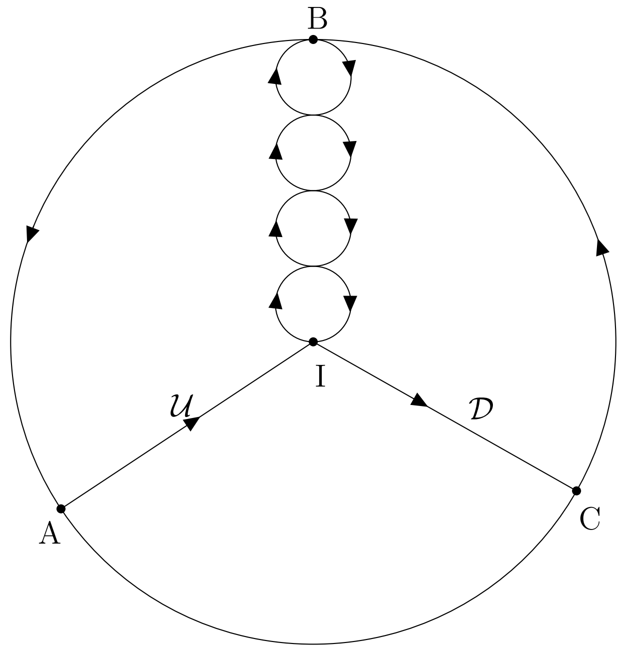


Figure 2.2: Intersecting P-chords

Similarly to Case 1, we again have P-chords DA, DC and BA creating 3 segments such that any edge of the arc $\widehat{AD}$ lies in all of the them which implies that this edge is in 4 cycles, a contradiction.

$\square$

Lemma 19. *If there is a path chord $\mathcal{U}$ from vertex A to vertex B and there is a path chord $\mathcal{D}$ from vertex B to vertex C and there are no other path chords from A to B or from B to C contained in $\mathcal{U} \cup \mathcal{D}$, then there is a vertex I such that $\mathcal{U} \cup \mathcal{D}$ is a union of three components, necklace-path between I and B and paths AI and IC such that intersection of every two components is the vertex I. 2.2*

Remark 20. *Note that if $A \neq C$, then in the statement of the above lemma, we don't need the condition that there are no other path chords from A to B or from B to C (except $\mathcal{U}$ and $\mathcal{D}$) contained in $\mathcal{U} \cup \mathcal{D}$, since it follows from the fact that there is a path-chord from A to C (unless $\mathcal{U} \cap \mathcal{D}$ is just B) and edges on the arcs BA and CB are already in three cycles.*

Proof of Lemma 19. Let the vertices on $\mathcal{U}$ from A to B be $A_1, A_2, A_3 \dots A_r$ (in this order) where $A_1 = A$ and $A_r = B$. Let the intersection vertices of $\mathcal{D}$ with $\mathcal{U}$ as we travel from B to C be $A_{i_1}, A_{i_2} \dots A_{i_s}$ in this order. Now notice that if $i_j < i_{j+1}$, and if $\mathcal{U}$ and $\mathcal{D}$ do not coincide between A_{i_j} and $A_{i_{j+1}}$, we have a new path from A to B, a contradiction. So assume that $\mathcal{U}$ and $\mathcal{D}$ coincide between A_{i_j} and $A_{i_{j+1}}$. Let the longest common subpath $\mathcal{P}_C$ of $\mathcal{U}$ and $\mathcal{D}$ containing A_{i_j} and $A_{i_{j+1}}$ be from A_{i_p} to A_{i_q} ($i_p < i_q$). Note that the edges in $\mathcal{P}_C$

are already in segments created by $\mathcal{U}$ and $\mathcal{D}$ and so it is contained in two cycles. The path from $A_{i_{p-1}}$ to A_{i_p} (notice $i_{p-1} > i_p$) in $\mathcal{D}$ and the path from A_{i_p} to A_{i_q} in $\mathcal{U}$ form a cycle which contains this common path $\mathcal{P}_C$ between A_{i_j} and $A_{i_{j+1}}$.

If $A_{i_{q+1}}$ exists (If A and C are the same, it surely exists), then the path from $A_{i_{q+1}}$ to A_{i_q} (notice $i_{q+1} < i_q$) on $\mathcal{U}$ and the path from A_{i_q} to $A_{i_{q+1}}$ on $\mathcal{D}$ create a fourth cycle which contains $\mathcal{P}_C$, a contradiction.

If $A_{i_{q+1}}$ does not exist (note that this implies that A and C are different), then the subpath of $\mathcal{U}$ from A to A_{i_p} and the subpath of $\mathcal{D}$ from A_{i_p} to C are disjoint. Hence they create a path chord from A to C which clearly contains $\mathcal{P}_C$. That is, we found a fourth cycle for the edges in $\mathcal{P}_C$, a contradiction.

This means that the first intersection vertex on $\mathcal{U}$ is the last intersection on $\mathcal{D}$. So if we call this intersection vertex, I then the lemma follows. □

If $A = C$, then we have the following corollary of Lemma 19.

Corollary 21. *If there is a path $\mathcal{U}$ from vertex A to vertex B and there is a path $\mathcal{D}$ from vertex B to vertex A and there are no other paths from A to B or from B to A contained in $\mathcal{U} \cup \mathcal{D}$, then $\mathcal{U} \cup \mathcal{D}$ is a necklace-path between A and B.*

Lemma 22. *Let $k \geq 1$ be an integer. If we have a subgraph G_k and a necklace-path $\mathcal{N}_k$ with endvertices S_k and T_k such that they satisfy conditons:*

1. *$\mathcal{N}_k \setminus \{S_k, T_k\}$ and $G_k \setminus \{S_k, T_k\}$ belong to different components of $G \setminus \{S_k, T_k\}$.*
2. *Every edge of $\mathcal{N}_k$ is contained in exactly two cycles of the subgraph $G_{k+1} := G_k \cup \mathcal{N}_k$.*
3. *Every path between S_k and T_k is contained in $G_k \cup \mathcal{N}_k$. (Using the first property these paths are contained either in G_k or in $\mathcal{N}_k$)*

then, there exists a necklace-path $\mathcal{N}_{k+1}$ with its end vertices S_{k+1} and T_{k+1} in $\mathcal{N}_k$ such that $\mathcal{N}_{k+1}$ and G_{k+1} satisfies the same conditions as above.

Proof of lemma 22. Let the path from T_k to S_k be $\mathcal{U}$ and the path from S_k to T_k be $\mathcal{D}$. Since each edge of $\mathcal{U}$ must be in another cycle, there is a system of cycles $\mathcal{C} = \{C_i\}$ covering the walk $\mathcal{U}$. By property 1 of the Lemma, it's clear that, every path between T_k and S_k contained in $\mathcal{C} \cup \mathcal{N}_k$ is either $\mathcal{U}$ or $\mathcal{D}$.

Claim 1. *For any two points A and B on $\mathcal{U}$ where A is closer to T_k, there is no path from A to B which isn't contained in $\mathcal{U}$ (which also means that this path doesn't contain every edge of $\mathcal{U}$ between A and B).*

Proof of Claim. If there is such a path, then the subpath of the union of paths from T_k to A in $\mathcal{U}$, this path and the path from B to S_k is different from $\mathcal{U}$, a contradiction. □

Let the intersection of a $C_i \in \mathcal{C}$ with $\mathcal{U}$ be called the base of C_i. Clearly each base is a connected path because otherwise there will be two points A and B again as in the previous paragraph, leading to a contradiction. Similarly, the following claim is true.

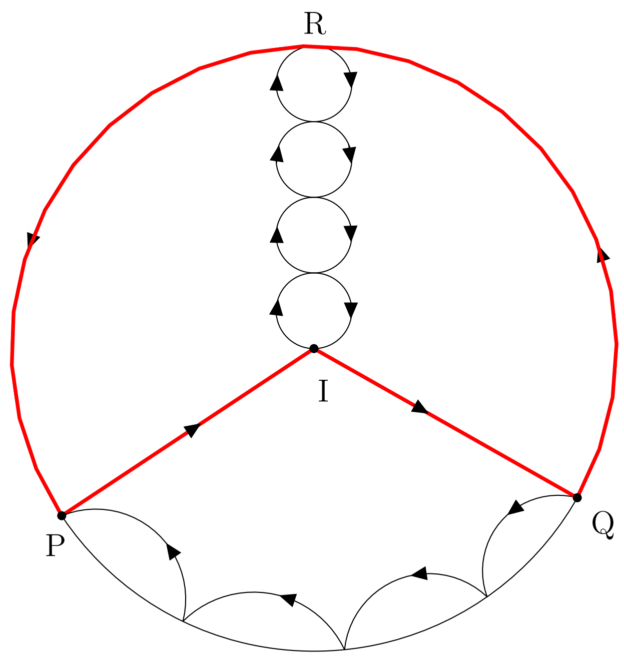


Figure 2.3: Figure shows edges which are already in 3 cycles

Claim 2. *If the bases of two cycles are disjoint, then cycles are disjoint.*

Proof of Claim. If they intersect, it clearly contradicts Claim 1. □

Claim 3. *No cycle $C_i \in \mathcal{C}$ contains two or more key vertices of $\mathcal{N}_k$.*

Proof. Assume by contradiction that C_i contains key-vertices R_i and R_{i+1}. Then, there will be path from R_{i+1} to R_i which isn't contained in $\mathcal{D}$, creating a path from S_k to R_k different from $\mathcal{D}$. a contradiction. □

Claim 4. *For every cycle $C_i \in \mathcal{C}$, there is a cycle C_j such that $C_i \cap C_j$ contains a vertex V which is not in $\mathcal{U}$.*

Proof. Assume by contradiction that there is no such C_j. By Claim 3 one of the end-vertices V of base of C_i is not a key vertex. Then clearly there is a subpath of $\bigcup_{C_i \in \mathcal{C}} C_i$ from S_k to T_k which contains V, and so is different from $\mathcal{D}$, a contradiction. □

By the above chaim, it is clear that there is a cycle $C_K \in C_i \cup C_{i+1}$ which contains the bases of both cycles C_i and C_{i+1}. But then the edges in the bases of these cycles are contained in four cycles, unless $C_K = \mathcal{N}_k^j$ where $\mathcal{N}_k$ is defined by the system of cycles $\{\mathcal{N}_k^j\}$.

By symmetry, there is a cycle system for the path $\mathcal{D}$ and clearly C_i and C_{i+1} are members of it. Therefore, their bases in $\mathcal{D}$ are two paths having only one common vertex V'. Now if we look at the cycle $\mathcal{N}_k^j$, the directed arc of the cycle C_i from V' to V and the directed arc of C_{i+1} from V to V' are path chords of $\mathcal{N}_k^j$. Let union of these two path-chords be $\mathcal{N}_{k+1}$. By

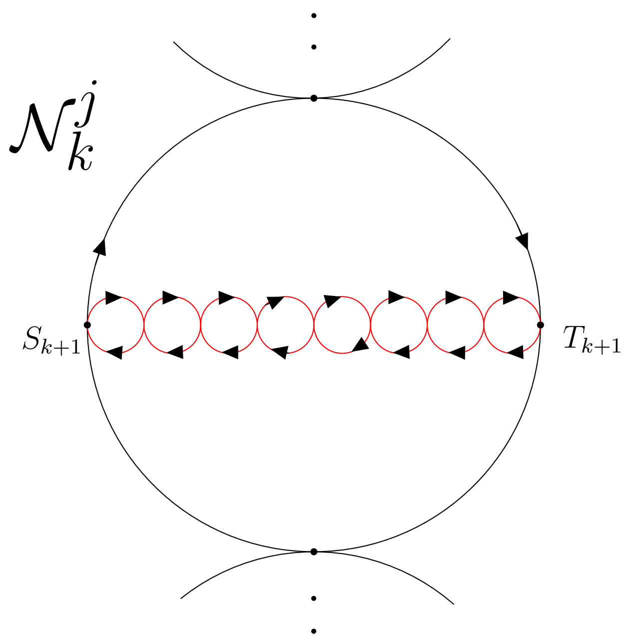


Figure 2.4: New necklace path

Corollary 21, $\mathcal{N}_{k+1}$ is a necklace-path between V and V' (there can be no other path chord between V and V' as they would create a fourth cycle for some edges in $\mathcal{N}_k^j$).

Let $V := S_{k+1}$ and $V' := T_{k+1}$. We claim that $\mathcal{N}_{k+1}$ and G_{k+1} satisfy all three properties of our Lemma.

Since all the edges of the cycle $\mathcal{N}_k^j$ are already in 3 cycles in the graph $G_{k+1} \cup \mathcal{N}_{k+1}$, every pathchord of this cycle is contained in the graph $G_{k+1} \cup \mathcal{N}_{k+1}$. The same is true for any path between S_{k+1} and T_{k+1} because if there is a path which isn't contained in $G_{k+1} \cup \mathcal{N}_{k+1}$, there will also be a path-chord which is not contained in $G_{k+1} \cup \mathcal{N}_{k+1}$ as every path between two points of a cycle is the union of the pathchords of this cycle. Therfore, the third property of the lemma holds for $k+1$ also.

We claim that $\mathcal{N}_{k+1} \setminus \{S_{k+1}, T_{k+1}\}$ and $G_{k+1} \setminus \{S_{k+1}, T_{k+1}\}$ belong to different components of $G \setminus \{S_{k+1}, T_{k+1}\}$. First we show that $\mathcal{N}_{k+1} \setminus \{S_{k+1}, T_{k+1}\}$ and $\mathcal{N}_k \setminus \{S_{k+1}, T_{k+1}\}$ belong to different components of $G \setminus \{S_{k+1}, T_{k+1}\}$. Suppose by contradiction that there are vertices $v_1 \in \mathcal{N}_k \setminus \{S_{k+1}, T_{k+1}\}$ and $v_2 \in \mathcal{N}_{k+1} \setminus \{S_{k+1}, T_{k+1}\}$ such that either $v_1 = v_2$ or there is a path between them in $G \setminus \{S_{k+1}, T_{k+1}\}$ which has no common vertex with $\mathcal{N}_k \cup \mathcal{N}_{k+1}$ except v_1 and v_2. In the second case, w.l.o.g we may assume that this path is from v_1 to v_2. It is easy to see that from v_2 to v_1 there is a path completely contained in $\mathcal{N}_k \cup \mathcal{N}_{k+1}$ since $\mathcal{N}_k \cup \mathcal{N}_{k+1}$ is strongly connected. This path contains either the vertex S_{k+1} or T_{k+1} and so it must contain an edge e of N_k^j. Since these two paths do not intersect anywhere except v_1 and v_2, we have a fourth cycle for e in N_k^j, a contradiction. Now, it is easy to see that $\mathcal{N}_{k+1}$ and G_k are in different components (otherwise the first property of our lemma fails for k). Therefore, the first property of our lemma holds for $k+1$.

It is easy to check that each edge e of $\mathcal{N}_{k+1}$ is contained only in two cycles. One of them is a segment of $\mathcal{N}_k^j$ created by a path-chord containing e and another is the cycle of the cycle system defining $\mathcal{N}_{k+1}$. Therefore, the second property of our Lemma holds for $k+1$. $\square$

Lemma 23. *If $\overrightarrow{PR}$ and $\overrightarrow{RQ}$ are two P-chords with $P \neq Q \neq R$, then they cannot have a common vertex other than R.*

Proof. By Lemma 17, Q is not in the directed arc RP. So we assume that Q is in the directed arc PR for the rest of the proof. Assume by contradiction that the P-chords $\overrightarrow{PR}$ and $\overrightarrow{RQ}$ have a common vertex and let I be the closest such vertex to P (i.e., there are no common vertices on the directed path PI).

Clearly, there is a P-chord from P to Q because there is a walk PIQ. Every edge in the directed arc RP is contained in the segments created by PR, PIQ and is also on the cycle C. So these edges cannot be on any other cycle. Similarly, the same is true for edges on QR. This clearly implies that there is no path chord starting or ending in the inner points of QR and RP.

By Lemma 19, it follows that between I and R there is a necklace-path $\mathcal{N}_1$ and PI and IQ don't have any other common vertex except I. Since edges on the arc QRP of the cycle C are already in 3 cycles, any other path chord should start and end on the PQ arc of C.

It is easy to see that direction of these path chords must be opposite to the direction of the arc PQ of C. Therefore, it follows from Lemma 16 that there are vertices Q_i $1 \leq i \leq m$ where $Q_1 = Q$ and $Q_m = P$ such that there is a path chord from Q_i to Q_{i+1} for each $1 \leq i < m$.

Notice that if the path chord Q_iQ_{i+1} has a common vertex with the path chord RQ different from Q, then we would have a path chord from R to Q_{i+1}, a contradiction. Similarly, Q_iQ_{i+1} doesn't have a common vertex with the path chord PR other than P.

Let's consider the cycle C' consisting of directed arc QRP of the cycle C and paths PI and IQ. Now it's easy to see that between P and Q, there is a necklace-path, say $\mathcal{N}$ and between I and R, there is a necklace-path $\mathcal{N}_1$. It is easy to see that all the edges of C' are in 3 cycles already. So C' can't have any pathcords which are not contained in $\mathcal{N} \cup \mathcal{N}_1$.

Let $G_1 := C' \cup \mathcal{N}$. We claim that $\{I, R\}$ is a vertex cut of G and that vertices of $\mathcal{N}_1$ and $G_1 \setminus \{I, R\}$ are in different components of $G \setminus \{I, R\}$. Assume by contradiction that $\{I, R\}$ is not a vertex cut of G. Then, if there exists a path between $U \in C \setminus \{I, R\}$ and $V \in \mathcal{N}_1$, we will find a path-chord between U and R or between U and I, a contradiction. So there should be a path between $\mathcal{N}_1$ and $\mathcal{N}$ disjoint from C' or they have a common vertex. In both cases, since necklace-paths are strongly connected, there is a path-chord between I and Q or R and Q, contradiction.

By the above arguments and the fact that every pathchord of C' is either in $\mathcal{N}_1$ or in $\mathcal{N}$, the three conditions of Lemma 22 can be easily checked for $k = 1$. (That is, G_1 and $\mathcal{N}_1$ satisfy the properties of Lemma 22). By induction on k, and using the fact that Lemma 22 holds, we have infinitely many such $\mathcal{N}_k$ and G_k, and since $\mathcal{N}_{k+1}$ is edge-disjoint (even vertex-disjoint) from G_k, we will need infinitely many edges in G, a contradiction. $\square$

So, by Lemma 18 and Lemma 23, it follows that two any two P-chords $\overrightarrow{AB}$ and $\overrightarrow{CD}$ don't intersect unless $B = D$ or $A = C$, or $A = D$ and $B = C$.

Lemma 24. *Every edge on the cycle C is in exactly two segments.*

Proof. Suppose by contradition that there exists an edge on the cycle C which belongs to a cycle C_u which is neither a segment nor C. Let AB be a longest arc which contains this edge and is in cycle C_u. Since C_u isn't a segment $(C_u \setminus AB) \cap C \neq \emptyset$. So we can take closest point M and farthest point N from B on $(C_u \setminus AB) \cap C$. It's easy to see that there exist p-chords $\overrightarrow{BM}$ and $\overrightarrow{NA}$ (note that M and N can be the same) and that the segments created by them cointain the arc AB of the cycle C, a contradiction. □

Lemma 25. *If there are exactly two ingoing path chords at a vertex, then there are exactly two outgoing path chords (notice that we can't have three ingoing path chords at a vertex).*

Proof. Let's assume by contradiction that there are two ingoing and one outgoing patchords at V. Then it is easy to see that one of the two edges incident on V in C is in exactly two segments and the other one in only 1. a contradiction of lemma 24. □

Lemma 26. *The set of all path-chords is a cycle or it can be partitioned into two components such that each component is a cycle not crossing itself or a necklace-path.*

Proof. Notice that by Lemma 17 if a cycle consisting only of p-chords doesn't cross itself, then every edge of the cycle C is contained in exactly one of the segments created by these p-chords. If a cycle crosses itself, every edge of the cycle C is exactly in a fixed number (more than one) of segments and in our case, this number cannot be more than 2. Consider a path-chord A_1A_2. By Lemma 16, since A_1A_2 is an ingoing path-chord at A_2, there is an outgoing path-chord at A_2, say A_2A_3. Continuing this way, it is easy to see that we will eventually reach a point A_i such that $A_i = A_j$ for some $j < i$. Notice that by lemma18 and lemma 23 $C_1 = \bigcup_{r=j}^{i-1} A_rA_{r+1}$ is a cycle or a necklace-path (here we also used Lemma 19). If C_1 crosses itself then every edge of C will be covered by two segments so we can't have any other p-chord. If C_1 doesn't cross itself, segments created by the set of p-chords not contained in C_1 cover every edge of C once. So from the remaining p-chords we can chose one and proceed with the same procedure as above (here we used Lemma 25) to create a C_2 which is either a cycle not crossing itself or a necklace-path. Since every edge of C is contained in exactly two s egments created by p-chords of C_1 and C_2 there can't be any other p-chord which means that C_1 and C_2 partition the set of all p-chords, as desired.

□

Now we are ready to prove our main theorem.

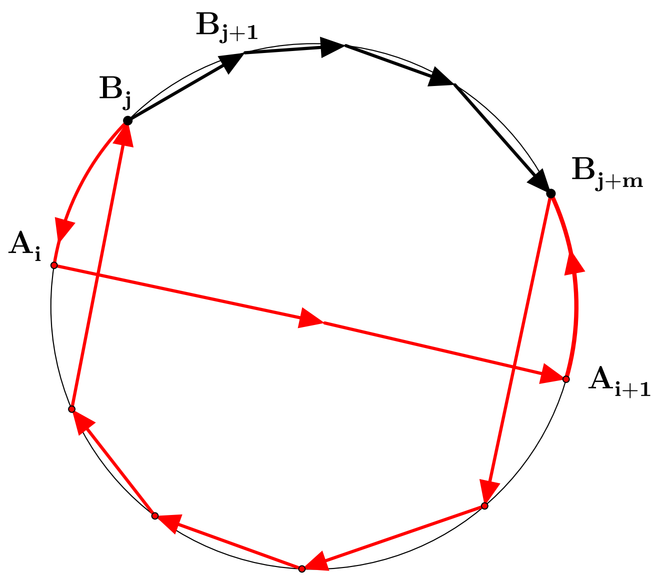
2.2 Proof of Theorem 13

We consider two cases.

Case 1. *If we have two* crossing *p-chords:*

Let $\overrightarrow{A_1B_1}$ and $\overrightarrow{A_2B_2}$ be two crossing P-chords s.t. A_2 is on the directed arc B_1A_1 and B_2 on A_1B_1. We know that there are cycles or necklace-paths C_1 and C_2 (C_1 might be ths same as C_2), consisting only of path chords (by Lemma 26) containing $\overrightarrow{A_1B_1}$ and $\overrightarrow{A_2B_2}$ respectively. We claim that C_1 and C_2 are disjoint because if they are not, there is a path $\mathcal{P}$ between A_2 and B_1 consisting only of p-chords. Since edges on the arc B_1A_2 are already in two segments, there is no ingoing or outgoing p-chord at any vertex inside the arc, so P is disjoint from the arc B_1A_2. This means we found a fourth cycle $\mathcal{P} \cup B_1A_2$ for edges of the arc B_1A_2, a contradiction.

Let C_1 be the union of the path-chords $\overrightarrow{A_iA_{i+1}}$ for $0 \leq i \leq k$ where the addition in the subscript is modulo k and let C_2 be the union of the path chords B_jB_{j+1} for $0 \leq j \leq l$ where the addition in the subscript is modulo l. We claim that for every i, there is at most one vertex of C_2 on the directed arc $A_{i+1}A_i$. Suppose by contradiction that there are vertices $B_j, B_{j+1}, \ldots, B_{j+m}$ on the directed arc $A_{i+1}A_i$ where $m > 0$. Now let's consider the cycle by replacing the walk in C_2 between B_j and B_{j+m} with the walk $B_jA_iA_{i+1}B_{j+m}$. It's clear that this cycle is a fourth cycle for the edges of the arc B_jA_i, a contradiction.



By symmetry, we can also conclude that for every j, there is at most one vertex of C_1 on the directed arc $B_{j+1}B_j$. Thus the vertices of C_1 and C_2 alternate on the cycle C and so $k = l$. W.l.o.g assume that B_j is on the directed arc $A_{j+1}A_j$. If $k \geq 2$, then replace the walk between A_0 and A_3 in the cycle C_1 by the walk consisting of $\overrightarrow{A_0A_1}$, directed arc A_1B_0, $\overrightarrow{B_0B_1}$, $\overrightarrow{B_1B_2}$, directed arc B_2A_2, $\overrightarrow{A_2A_3}$ and we'll get a fourth cycle for the edges in the directed arc A_1B_0, a contradiction. Therefore, $k = 1$. That is, both C_1 and C_2 must be necklace-paths.

Let $G_1 = C \cup C_2$ and $\mathcal{N}_1 = C_1$ in the Lemma 22. As we have already shown for exactly similar case in proof of Lemma 23, all 3 conditions of Lemma 22 hold for $k = 1$, giving us a contradiction.

Case 2. *If there are no crossing p-chords:*

Then by lemma 26, there are C_1 and C_2 such that each of them is a cycle or a necklace-path. Let C_1 be the union of the path chords $\overrightarrow{A_iA_{i+1}}$ for $0 \leq i \leq k$ where the addition in the subscript is modulo k and let C_2 be the union of the path chords B_jB_{j+1} for $0 \leq j \leq l$ where the addition in the subscript is modulo l. Since we know that there are no crossing P-chords, it is easy to see that we have only two cases:

Case 2a. *If one of the two cycles is completely contained in a segment created by one of path-chords of the other cycle*

Assume w.l.o.g that C_2, is completely contained in a segment created by one of path-chords of C_1, say A_iA_{i+1}. If C_2 is completely contained in the segment A_iA_{i+1}, let B_j be the closest point of $\{B_j \mid 0 \leq j \leq l\}$ to A_i on the cycle C and let B_m be the closest point from A_{i+1} to $\{B_j \mid 0 \leq j \leq l\}$ on the cycle C.

If $B_j = A_i$ and $B_m = A_{i+1}$, then replace the edges of C_1 from A_i to A_{i+1} by $\overrightarrow{A_iB_{j+1}}, \overrightarrow{B_{j+1}B_{j+2}}, \overrightarrow{B_{m-1}A_{i+1}}$ to create a cycle, say C_1' and let us replace the edges of the C_2 from B_j to B_m by $\overrightarrow{A_iA_{i+1}}$, creating a new cycle say C_2'. Clearly, C_1' and C_2' are disjoint and we are in Case 2b, as $V(C) \cap V(C_2') \subset V(C) \cap V(C_1')$.

So we may assume w.l.o.g that $B_j \neq A_i$. Now, the edges in the directed arc B_jA_i are in a fourth cycle created by the directed arc B_jA_i, path chord $\overrightarrow{A_iA_{i+1}}$, directed arc $A_{i+1}B_m$ (this may just be a single vertex if $A_{i+1} = B_m$) and the path chord $\overrightarrow{B_mB_j}$, a contradiction.

Case 2b. $\{B_j \mid 0 \leq j \leq l\} \subset \{A_i \mid 0 \leq i \leq k\}$

First let's show that G is contained in $C \cup C_1 \cup C_2$ (i.e., there are no more edges in G) by showing that every edge in $C \cup C_1 \cup C_2$ is in at least 3 cycles. It is easy to see that each edge of the cycle C is in two segments, which means it is in 3 cycles. If an edge e is in C_1 and C_2, then it is obvious that it is in at least three cycles (namely C_1, C_2 and some segment of C). Now, let's say $e \in C_i$ and is not in C_j (where $\{i, j\} = \{1, 2\}$). We know that C_i and C_j have at least two common vertices. Therefore there exist two vertices I_1 and I_2 on C_i, such that e belongs to the directed arc I_1I_2 of the cycle C_i and there are no vertices of C_j on this arc. Then the path in C_j from I_2 to I_1 and this directed arc I_1I_2 of C_i create a cycle. Clearly e is in this cycle and it is easy to see that this cycle is different from C_i and so e is in 3 cycles. So every edge is in at least three cycle. We proved that if G is our desired answer, then it contains $C \cup C_1 \cup C_2$ and since every edge in $C \cup C_1 \cup C_2$ is already in 3 cycles there can't be any other edges in G (Using the fact that G has no cut vertices).

We claim that $l = 1$. Suppose by contradiction that $l \geq 2$. Then, between B_0 and B_1 there are two different (edge-disjoint) paths $\mathcal{P}_1$ and $\mathcal{P}_2$ contained in C_1 and C_2 respectively. Similarly between B_1 and B_2 there are two different paths $\mathcal{Q}_1$ and $\mathcal{Q}_2$ contained in C_1 and

C_2 respectively. This implies that between B_0 and B_2 there are four different paths. It is easy to see that all these four paths are disjoint (except vertices B_0 and B_2) from the path B_2B_0 in C_2. This means, that the edges in the path B_2B_0 are contained in four cycles, a contradiction. Therefore, $l = 1$ and C_2 consists of only two pathchords between B_0 and B_1.

Assume that C_1 and C intersect in vertices other than B_0 and B_1 and w.l.o.g we can assume that they intersect on the directed arc B_1B_0. Let A_0 be first intersection of C_1 with C after B_0. Every inner vertex on the directed arc A_0B_0 of C has degree 2 as $C \cap C_2$ is only $\{B_0, B_1\}$. Now, if there is an inner vertex u on the directed arc B_0A_0 of C_1 with degree more than 2 (i.e., degree at least 4 by Lemma 15), it means there is another ingoing edge e at u and this edge must belong to C_2 (since C_1 and C do not have a common point with any inner point of the directed arc B_0A_0). Clearly u must be either on the path-chord B_0B_1 or B_1B_0. In the first case, then there is a path-chord from B_0 to A_0 containing e. Since this path-chord contains A, it is not in C_2 and since it contains e, it is not in C_1, and we found a path-chord which is neither in C_1 nor C_2, a contradiction. Similarly, in the second case, we can find a new path-chord from B_1 to A_0 containing u, a contradiction. Therefore, no such u exists and so every inner vertex on the directed arc B_0A_0 of the cycle C_1 has degree 2.

Now consider the cycle C' formed by the union of the directed arc B_0A_0 of the cycle C_1 and the directed arc A_0B_0 of the cycle C. Every vertex of C' has degree 2 except B_0 and A_0. If C' was the chosen as the starting cycle instead of C, then all the path chords of C' can be partitioned into C'_1 and C'_2 such that $G = C' \cup C'_1 \cup C'_2$ where C'_2 is a necklace-path and C'_1 is either a cycle or a necklace-path. But since, all the vertices of C' have degree 2 except B_0 and A_0, C'_1 can only be a necklace-path (between B_0 and A_0, of course).

Let $C'_i := \mathcal{U}_i \cup \mathcal{D}_i$ for each $i = 1, 2$ where $\mathcal{U}_i$ is a path from B_0 to A_0 and $\mathcal{D}_i$ is a path from B_0 to A_0. In $C'_1 \cup C'_2$, it is easy to see that there are exactly two paths from B_0 to A_0 and exactly two paths from A_0 to B_0. We claim that there exist two vertices A_u and B_u such that between B_0 and B_u, and between A_u and A_0 the paths $\mathcal{U}_1$ and $\mathcal{U}_2$ are identical and they are vertex disjoint between B_u and A_u (A_u might be the same as A_0 and B_u might be the same as B_0). Otherwise, there exists a common vertex x such that $\mathcal{U}_1$ and $\mathcal{U}_2$ are not identical between B_0 and x and they are not identical between x and A_0 giving us four different paths from B_0 to A_0, a contradiction. For the same reason, there exist two vertices A_d and B_d such that between B_0 and B_d, and between A_d and A_0 the paths $\mathcal{D}_1$ and $\mathcal{D}_2$ are identical and they are vertex disjoint between B_d and A_d (A_d might be the same as A_0 and B_d might be the same as B_0).

We claim that $B_u = B_d$ and $A_u = A_d$. By Corollary 21, $\mathcal{U}_1 \cup \mathcal{U}_2$ is edge-disjoint from $\mathcal{D}_1 \cup \mathcal{D}_2$. Without loss of generality, it's enough to show that $B_u = B_d$. If $B_u = B_d = B_0$, then we are done. So w.l.o.g, let's say $B_u \neq B_0$. Since the number of ingoing edges is less than the number of outgoing edges at B_u in $\mathcal{U}_1 \cup \mathcal{U}_2$, by Lemma 15, there should be some edges from $\mathcal{D}_1 \cup \mathcal{D}_2$ incident on B_u (because there can't be any more edges incident on B_u from $\mathcal{U}_1 \cup \mathcal{U}_2$ or C'). The only way this can happen is if B_u is a vertex in $\mathcal{D}_1 \cup \mathcal{D}_2$ which has more ingoing edges than outgoing edges but such a vertex can only be B_d. So $B_u = B_d$. Moreoever, it is easy to see that the sub-graph which consists of sub-paths of $\mathcal{U}_1, \mathcal{U}_2, \mathcal{D}_1, \mathcal{D}_2$ between B_0 and B_u is a necklace path as it is a sub-graph of C'_1 (and C'_2). Similarly, the sub-graph between A_u and A_0 is a necklace path.

Let's call the subpath of $\mathcal{D}_i$ and $\mathcal{U}_i$ between $A_u = A_d$ and $B_u = B_d$ be $\mathcal{D}_i^s$ and $\mathcal{U}_i^s$ respectively for each $i = 1, 2$. If e is an edge in $\mathcal{U}_i^s$, it is easy to see that the cycle consisting of $\mathcal{U}_i$ from B_0 to A_0 and the directed arc of the cycle C' from A_0 to B_0 contains e. Therefore, e is in at most two cycles which are completely contained in $\mathcal{D}_i^s \cup \mathcal{U}_i^s$. But it is easy to see that e is in two cycles which are contained in $\mathcal{U}_i^s \cup \mathcal{D}_1^s$ and $\mathcal{U}_i^s \cup \mathcal{D}_2^s$ respectively. Therefore, each e is contained in exactly two cycles and so by Theorem 10, $\mathcal{D}_1^s \cup \mathcal{U}_1^s \cup \mathcal{D}_2^s \cup \mathcal{U}_2^s$ is isomorphic to two disjoint necklace-paths between $A_u = A_d$ and $B_u = B_d$ which implies that G is isomorphic to three disjoint necklace-paths between $A_u = A_d$ and $B_u = B_d$, as desired.

Chapter 3

Concluding Remarks

3.1 Infinite example

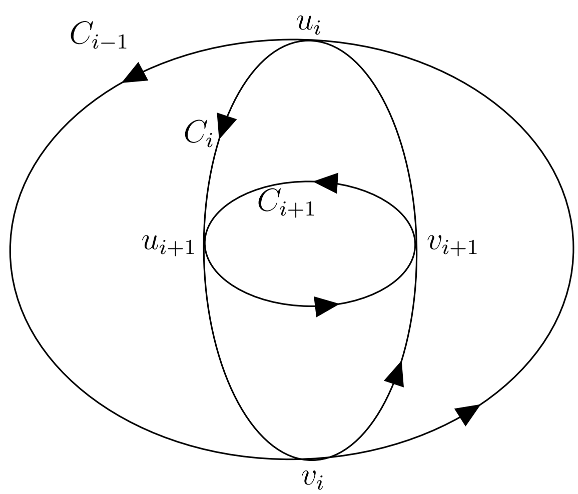


Figure 3.1: Infinite example

It is worth noticing that finiteness of the digraph is necessary condition. If we allow the graph to be infinite than we have a different solution for the problem. Below we will construct this example.

Let's D be a digraph consisting of directed cycles c_i $i \in \mathbb{Z}$ such that $V(c_i) \cap V(c_{i+1}) = v_i + 1, u_i + 1$ and $V(c_i) \cap V(c_j) = \emptyset$ when $|i - j| > 1$. Then it's easy to see that every edge is in exactly 3 cycles. Let's consider the edges a directed arc $u_i v_i$ of c_i (notice that $\overrightarrow{u_i v_i}$ is a path-chord in a cycle c_{i-1}) then 3 cycles containing them are: a segment of c_{i-1} created by the path-chord $\overrightarrow{u_i v_i}$, a cycle c_i itself, and the segment of the cycle c_i created by either a path-chord $\overrightarrow{u_{i+1} v_{i+1}}$ or by $\overrightarrow{v_{i+1} u_{i+1}}$. Notice that it is possible to constract similar example with necklaces instead of the cycles.

3.2 special example for $k = 4$

It is quite obvious that for any $k > 1$, if we take a digraph consisting of k disjoint necklace paths between two vertices then this graph will satisfy the condition that every edge is in exactly k cycles. It is interesting to note that, while for $k = 2$ and $k = 3$ we have only this type of directed graphs the situation is quite different for $k = 4$. Apart from the directed graph consisting of 4 disjoint necklace-paths between two vertices there exists another different graph with this property as well. Bellow we will construct such directed graph D.

Let's take a directed cycle C with six different A_i $i = 0, 1, 2, 3, 4, 5$ vertices on it such that A_{i+1} is on a directed arc $A_{i+2}A_i$ (addition in the subscript is modulo 6). Let D be a digraph wich consists of a directed cycle C and the directed edges $\overrightarrow{A_iA_{i+2}}$ be in $E(D)$. Then we claim that every edge in D is exactly in 4 directed cycles. Let's first consider edges of the directed arc A_1A_0. It is obvious that they are in two segments created by directed path-chords A_0A_2 and A_5A_1, one cycle c and a cycle consisting of path-chords (edges) $\overrightarrow{A_0A_2}$, $\overrightarrow{A_2A_4}$, $\overrightarrow{A_3A_5}$, $\overrightarrow{A_5A_1}$ directed arcs A_1A_0 and A_4A_3. So similarly we can say that all the directed edges of a cycle c are in 4 cycles. Now let's consider edges of a path-chord $\overrightarrow{A_0A_2}$. The 4 cycles containing them are: a segment created by $\overrightarrow{A_0A_2}$, a cycle consisting of the path-chords $\overrightarrow{A_0A_2}$ $\overrightarrow{A_2A_4}$ $\overrightarrow{A_4A_0}$, a cycle consisting of $\overrightarrow{A_0A_2}$, $\overrightarrow{A_2A_4}$, $\overrightarrow{A_3A_5}$, $\overrightarrow{A_5A_1}$ directed arcs A_1A_0 and A_4A_3, and a cycle consisting of $\overrightarrow{A_1A_3}$, $\overrightarrow{A_3A_5}$, $\overrightarrow{A_4A_0}$, $\overrightarrow{A_0A_2}$ directed arcs A_2A_1 and A_5A_4. So similarly we can conclude that every edge of all of the path-chords are in 4 cycles.

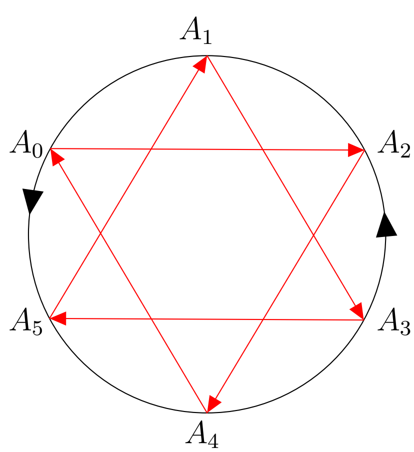


Figure 3.2: Example for $k = 4$

3.3 Constructing different examples for bigger k

Take two digraphs D_1 and D_2 constructed in a similar way as in the previous section (see figure 3.2). Let's say $A_i \in D_1$ $i = 0,1,2,3,4,5$ and $B_i \in D_2$ $i = 0,1,2,3,4,5$ are the similar vertices (figure 3.2) on the main cycle of digraphs. Let's unite vertices A_0 and B_0, and also A_3 and B_3 and call the produced digraph D. (so D=$D_1 \cap D_2$ where $A_0 = B_0$ and $A_3 = B_3$).

Claim 5. *Every edge of the digraph D is contained exactly in 12 cycles.*

Proof. let's at first consider D_1 only and let's count all the possible paths from vertex A_0 to A_3. These paths are:

1. A path consisting of $\overrightarrow{A_0A_2}$, $\overrightarrow{A_2A_4}$ and directed arc A_4A_3.
2. A path consisting of $\overrightarrow{A_0A_2}$, directed arc A_2A_1 and $\overrightarrow{A_1A_3}$.
3. A path consisting of a directed arc A_0A_5, $\overrightarrow{A_5A_1}$ and $\overrightarrow{A_1A_3}$
4. A directed arc A_0A_3

Let's observe that edges of path-chords $\overrightarrow{A_0A_2}$, $\overrightarrow{A_1A_3}$ and directed arcs A_0A_5 and A_4A_3 are in 2 different paths from A_0 to A_3. Notice that they are not in any path from A_3 to A_0. Also edges of path-chords $\overrightarrow{A_2A_4}$, $\overrightarrow{A_5A_1}$ and directed arcs A_2A_1 and A_5A_4 are exactly in one path from A_0 to A_3. It's easy to see that by symmetry they will be exactly in one path from A_3 to A_0 too. And edges of all remaining parts of the D_1 are in 0 paths from A_0 to A_3 and symmetry suggests that they are in exactly 2 paths from A_3 to A_0. So overall every edge of D_1 is exactly in two paths between A_0 to A_3.

Now let's consider cycles of D which aren't contained in either in D_1 or D_2. It is easy to see that these cycles consist of one path between $A_0 = B_0$ and $A_3 = B_3$ in $D1$ and one path (opposite direction) between $A_0 = B_0$ and $A_3 = B_3$ in $D2$. So each path between $A_0 = B_0$ and $A_3 = B_3$ is in 4 this kind of cycles, and since each edge of D_1(and similarly pf D_2) is in 2 paths between $A_0 = B_0$ and $A_3 = B_3$ we can conclude that every edge is in $2 * 4 = 8$ different cycles of D which aren't cycles of D_1 or D_2. So overall every edge of D will be in exactly 12(8+4 internal cycles of D_1 or D_2) directed cycles. □

Corollary 27. *Let's say we have n different D_i digraphs constructed in a similar way as in 5. And let's say vertices $A_i^j \in D_j$ $i = 0,1,2,3,4,5$ $(1 \leq j \leq n)$. Let construct a graph D from D_i $1 \leq j \leq n$ by uniting pair of vertices A_3^j A_0^{j+1} for $1 \leq j \leq n-1$ and the pair A_3^n A_0^1. Then every edge of D is exactly $4^{n-1} * 2 + 4$ edges. Pfroof is same as proof of claim 5.*

3.4 Open problems

The first open problem that naturally arises from the previous example is that for $k = 4$ if these two graphs (4 necklace paths and the example above) are the only answers.

Also it is interesting to find algorithms to construct different general solutions for higher k.

Also problem solved by E. Győri[1] suggests that there might be a nice constructive solution for every edge being in at most 3 cycles.

Bibliography

[1] E. Győri, Strongly connected digraphs with few cycles, Algebraic Methods in Graph Theory, Proc. Coll. Math. Soc. János Bolyai, Szeged, 1978, 251-265.

[2] A. Ádám, On some generalizations of cyclic networks, $ActaCybernetica, 1(1971), 105-119$.

[3] B. Zelinka, Strongly connected digraphs in which each edge is contained exactly two cycles, Acta Cybernetica, 4(1979), 203-205.

[4] A. Ádám, On some open problemsof applied automaton theory and graph theory, $ActaCybernetica, 1(1976), 187-214$.