THE ROLE OF RESURGENCE IN THE THEORY OF ASYMPTOTIC EXPANSIONS

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"... why are asymptotic theorems so much simpler than finite approximations? Infinity does not correspond to the popular image. It is a guiding light, a star that draws us to finite ways of thinking. God knows why." – Stanisław Ulam

Abstract

In the present dissertation, we summarize our systematic investigation of asymptotic expansions of special functions. We shall first give a detailed account of the general theory of the resurgence properties of special functions given by certain integral representations. This general theory is then employed to obtain a number of properties of the asymptotic expansions of various special functions, including explicit and realistic error bounds, asymptotics for the late coefficients and exponentially improved asymptotic expansions.

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Prologue

It was already observed by mathematicians of the 18th century that certain divergent series may be used for approximating functions that in some sense can be regarded as the "sum" of these series. Some early examples of such series are Abraham De Moivre's and James Stirling's expansions for the logarithm of the factorial [4, pp. 482–483] or the approximation of the harmonic numbers by Leonhard Euler [121, pp. 38–39]. Usually, the error committed by truncating these series at any term is of the order of the first term omitted. Adrien-Marie Legendre [52] and Thomas J. Stieltjes [105] called such series "semi-convergent". In 1886, J. Henri Poincaré [101] published the definition of an *asymptotic power series*, according to which $\sum_{n=0}^{\infty} a_n/z^n$ is said to be the asymptotic power series of a function $f(z)$ if

$$\lim_{z\to\infty} z^N \left(f(z) - \sum_{n=0}^{N} \frac{a_n}{z^n} \right) = 0,$$

for all non-negative integers N in an appropriate sector of the complex z-plane. It is easy to see that if a function possesses an asymptotic power series expansion, then the coefficients of this series are uniquely determined; nevertheless, the series does not, in general, determine the function itself. In most cases, the asymptotic form of a function is a linear combination of asymptotic power series with some functions as coefficients. Such a development is called the *asymptotic expansion* of the function.

Poincaré's definition of an asymptotic power series implies the property of the error term we have already mentioned: the error committed by truncating an asymptotic power series at any term is of the order of the first term omitted. Consequently, the error decreases algebraically to zero as the variable tends to infinity. Nearly four decades before Poincaré, Sir George G. Stokes [107, 108] investigated the function

$$\mathrm{Ai}(z) \stackrel{\mathrm{def}}{=} \frac{1}{2\pi\mathrm{i}} \int_{\infty e^{-\frac{\pi}{3}\mathrm{i}}}^{\infty e^{\frac{\pi}{3}\mathrm{i}}} \exp\left(\frac{t^3}{3} - zt \right) \mathrm{d}t.$$

Now known as Airy's function, it was introduced by George B. Airy [3] to describe certain physical phenomena. Stokes showed that Airy's function has a so-called "factorially divergent" asymptotic power series, meaning that the terms initially get smaller but then increase factorially (the ratio of consecutive coefficients is proportional to their index). More importantly, he showed that if this series is truncated not at a fixed order N but at its numerically smallest term (whose index depends on z), the error term becomes exponentially small, which is far beyond the algebraically small error guaranteed by Poincaré's definition. Such truncation of an asymptotic power series is now called an *optimal truncation*, and the resulting approximation is called a *superasymptotic* approximation. Another important observation of Stokes was that the asymptotic expansion of a function can be different in different sectors of the complex plane, and the change between these forms is apparently discontinuous. The boundaries of these (Stokes) sectors are called *Stokes lines*, and the discontinuous change in the form of the asymptotic expansion is called *Stokes phenomenon*.

Stieltjes [105] also investigated asymptotic power series truncated at their numerically smallest term. He demonstrated through several examples that if the coefficients of an asymptotic power series of real variable are alternate in sign, then the error after optimal truncation is approximately half of the first omitted term. In 1937, John R. Airey [2] defined the "converging factor" of an asymptotic power series to be the ratio of the remainder and the first omitted term. He showed, using formal methods, that if an alternating asymptotic power series is truncated at or near its numerically least term, the converging factor can be expanded into a new asymptotic power series in inverse powers of the truncation index. According to the work of Stieltjes, the leading order of these re-expansions is $1/2$. Using his theory, Airey was able to compute, in particular, the Bessel and allied functions to very high accuracy. In 1952, Jeffrey C. P. Miller [66] made similar investigations of functions defined by differential equations and gave recurrence formulae for the computation of the coefficients in the asymptotic power series of their converging factors.

In a series of papers [29–34] and in a research monograph published in 1973 [35], the theoretical physicist Robert B. Dingle incorporated earlier and new, original ideas into a comprehensive theory which had a substantial impact on later developments in modern asymptotics. Dingle's intuition was that asymptotic expansions are exact coded representations of functions, and the main task of asymptotics is to decode them. Dingle made the important observation that, although the early terms of an asymptotic power series can rapidly get extremely complicated, the high-order (late) coefficients of a wide class of asymptotic power series diverge in a universal and simple way. More precisely, the approximate form of the late coefficients is always a "factorial divided by a power".[1] An application of Émile Borel's summation method for the approxi-

[1]Such a behaviour of the late coefficients in some special cases had been already known

mation of the late coefficients then enabled him to transform the divergent tail of an asymptotic power series into a new series in terms of certain integrals that he called "basic terminants". The resulting series, Dingle's "interpretation" of the asymptotic power series, can yield extremely accurate approximations assuming optimal truncation, far beyond the accuracy of Stokes' superasymptotics. Though very innovative, Dingle's results relied on formal rather than rigorous methods, which was perhaps the main reason that his ideas had been largely unappreciated for a long time.

In 1988, exploiting Dingle's late coefficient approximations, Sir Michael V. Berry [5,6] provided a new interpretation of the Stokes phenomenon. He found that, assuming optimal truncation, the transition between two different asymptotic expansions in adjacent Stokes sectors is effected smoothly and not discontinuously as in previous explanations of the Stokes phenomenon; moreover, the form of this transition is universal for all factorially divergent asymptotic power series. Nevertheless, his analysis was based on Dingle's formal theory, whence it was not rigorously justified. Motivated by Berry's breakthrough in the subject, Frank W. J. Olver [90, 92] re-expanded the remainder term in the asymptotic expansion of the confluent hypergeometric function in terms of new functions he called *terminant functions*, indicating their close relation to Dingle's basic terminants. Olver's new powerful expansion was valid in a much larger region than the original asymptotic expansion of the confluent hypergeometric function, covering three different Stokes sectors. Assuming optimal truncation, Olver gave complete asymptotic expansions for the terminant functions whose leading order behaviours near a Stokes line were in agreement with Berry's universal smoothing law, thereby providing a rigorous mathematical basis for Berry's results. Later in 1994, Olver and Adri B. Olde Daalhuis [85] generalized these new *exponentially improved asymptotic expansions* to solutions of second-order linear differential equations and justified Dingle's late term approximations for this case (see also [93] and [94]).

In 1990, Berry and Christopher J. Howls [7] considered the asymptotic behaviour of certain solutions of the one-dimensional Helmholtz equation. Based on a "resurgence formula" of Dingle that relates late terms to early terms, they found that the remainder of the asymptotic expansion can be repeatedly re-expanded using optimal truncation at each stage, leading to ultimately accurate approximations, termed *hyperasymptotics*. These re-expansions were given in terms of certain multiple integrals which are now called *hyperterminants*, the generalizations of Olver's terminant functions. The derivation of their re-

by the time of Dingle. E. Meissel [64] proved that the coefficients in the asymptotic power series of the Bessel function $J_\nu(\nu)$ behave in this way. Another example is G. N. Watson's [116] approximation for the high-order coefficients in the asymptotic expansion of the incomplete gamma function $\Gamma(z, z)$.

sults were, however, completely formal based on Dingle's non-rigorous resurgence formula and the Borel summation method. In 1995, in the important papers [84, 86], Olde Daalhuis and Olver developed a mathematically rigorous hyperasymptotic theory for the asymptotic solutions of second-order linear differential equations, a vast extension of their previous results on exponentially improved asymptotic expansions. With a clever choice of a series of truncations, different from the optimal truncation, they showed that each step of the re-expansion process reduces the order of the error term by the same exponentially-small factor, while at the same time increases the region of validity. In some special cases, their theory also justifies the similar results of Berry and Howls.

In 1991, in their groundbreaking paper [8], Berry and Howls studied the asymptotic expansions of integrals of the form

$$\int_{\mathscr{C}} \mathrm{e}^{-zf(t)} g\left(t\right) \mathrm{d}t,$$

where $f(t)$ has several first-order saddle points in the complex t-plane, i.e., points t^* such that $f'(t^*) = 0$ but $f''(t^*) \neq 0$, and the path of integration $\mathscr{C}$ passes through one of these points. Asymptotic expansions for such integrals can be constructed by *the method of steepest descents*, originally introduced by P. Debye [25] in 1909 to study the large-ν behaviour of the Bessel function $J_\nu(\nu z)$. Berry and Howls reformulated the method of steepest descents by showing that the remainder term in the asymptotic power series of such integrals can be written explicitly in terms of closely related integrals or sometimes in terms of the original integral itself. This surprising property of the remainder term of an asymptotic power series is called *resurgence*. It has to be mentioned that this resurgence property of integrals with saddles, in some special cases, was already discovered by Dingle [35, pp. 480, 482 and 484] under the name "dispersion relation". The main difference between Debye's method and the reformulation by Berry and Howls is that the former one uses only the local properties of $f(t)$ at the saddle point through which the path $\mathscr{C}$ passes, whereas in the latter one, global properties of the integrand play role. It is these global properties that make the resurgence formula of Berry and Howls so powerful. This resurgence formula can be used to provide a rigorous proof of Dingle's approximations for the late coefficients, implying that the asymptotic expansions of integrals with saddles always diverge factorially. It also incorporates the Stokes phenomenon of the asymptotic power series in a simple way and shows the change in the form when a Stokes line is crossed. The original aim of Berry and Howls in developing the resurgence formula was to extend their hyperasymptotic theory to integrals with saddles. The integrals appearing in the remainder can themselves be expanded into truncated asymptotic expansions with explicit remainder terms, and the repetition of this process leads to the hyperasymptotic expansion of the original integral. William G. C. Boyd [13] demonstrated how the theory of Berry and Howls can be used to provide computable bounds for asymptotic expansions arising from an application of the method of steepest descents. The resurgence formula was later extended by Howls to integrals with finite endpoints [45] and to multidimensional integrals [46] (see also [27]).

It is not possible to review here all the developments that have taken place in the past few decades in the field of asymptotics. Instead, we refer the reader to the expository papers by Berry and Howls [9], John P. Boyd [11] and the references therein.

During my early investigations in the field of asymptotic analysis, I gradually came to realize that, using the exciting new tools developed in the past few decades, a much deeper understanding could be obtained of the well-known asymptotic expansions of special functions. Apart from a few examples, this had not been attempted for many of the important special functions in mathematics. This observation led me to begin a systematic investigation of asymptotic expansions of special functions; the present dissertation is a summary of my pertaining research thus far.[2] The theory of Berry and Howls and the work of Boyd [12–16] form the basis of these studies, which were further influenced by the ideas of Cornelis S. Meijer [60–62].

Our investigation for a given function consists of the following four main steps:

(i) Starting with a suitable integral representation of the function, we derive its asymptotic expansion, but in an exact form: we truncate it after a finite number of terms and give its remainder term as an explicit integral expression. The truncated asymptotic expansion and the integral expression for the remainder together form the exact resurgence relation for the function. We also address the problem of computing the coefficients of the asymptotic expansion.

(ii) Based on the resurgence relation and by utilizing further properties of the function and its integral representation, we provide explicit, numerically computable bounds for the remainder term. The regions of validity of these bounds cover the whole domain where the original asymptotic expansion holds true.

(iii) We derive efficient approximations for the late coefficients of the asymptotic expansion with precise bounds on their error terms, thereby justifying many of the formal results of Dingle.

[2]To restrict the dissertation to readable proportions, some of our related results are omitted, such as those on the incomplete gamma function [77,78].

(iv) Using the resurgence relation, we obtain an exponentially improved asymptotic expansion for our function, which is similar to those found by Olver and Olde Daalhuis but is supplied with an explicit, numerically computable bound for its remainder term.

The real difficulty in asymptotics is not in writing down the asymptotic expansion but in deriving explicit, computable bounds on its remainder term (step (ii)) which are indispensable for numerical applications. In this connection, Johannes G. van der Corput [35, p. 405] [114, pp. 3–4] remarks: "It is often very difficult to find such an upper bound, or the upper bound obtained may be so weak that it is quite useless[...] In complicated problems it is advisable to restrict oneself first to pure asymptotic expansions without trying to find a numerical upper bound for the absolute value of the error term[...]". Differential equation methods were developed by Olver [95] to obtain asymptotic expansions with explicit error bounds of varying degrees of complexity. The functions studied usually contain additional parameters besides their arguments. When the large-argument asymptotics is considered, Olver's error bounds, up to a factor of 2, are as sharp as one can reasonably expect. However, if we are interested in asymptotic expansions where both the argument and one or more of the parameters are large, these error bounds can become quite cumbersome. An advantage of using resurgence relations, is that we are able to obtain explicit and sharp bounds even in those cases where Olver's methods yield complicated expressions or do not apply at all. It has to be emphasized that we provide estimates not only for the remainders of the asymptotic expansions of the functions but also for the remainders of their exponentially improved asymptotic expansions (step (iv)), thereby giving a powerful numerical tool for their computation.

Our program of systematically investigating asymptotic expansions of special functions is far from being complete; there is still a large variety of functions that has to be considered. Besides the classical asymptotic expansions studied in the present dissertation, other, more complex topics should be treated in the near future such as transitional region expansions, uniform asymptotic expansions, asymptotics of difference equations.

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CHAPTER 1

GENERAL THEORY

The aim of this chapter is to present a detailed account of the resurgence properties of integrals of the form

$$\int_{\mathscr{P}^{(X)}(\theta)} \mathrm{e}^{-zf(t)} g(t)\,\mathrm{d}t \quad \text{and} \quad \int_{\mathscr{C}^{(X)}(\theta)} \mathrm{e}^{-zf(t)} g(t)\,\mathrm{d}t, \tag{1.1}$$

where z is a (large) complex variable, $\theta = \arg z$, $f(t)$ and $g(t)$ are complex functions satisfying certain analyticity requirements. The contours of integration $\mathscr{P}^{(X)}(\theta)$ and $\mathscr{C}^{(X)}(\theta)$ are paths of steepest descent in the complex t-plane. The path $\mathscr{P}^{(X)}(\theta)$ emanates from a finite point X and ends at infinity, whereas the path $\mathscr{C}^{(X)}(\theta)$ is a doubly infinite contour passing through X. In the latter case, X is required to be a saddle point of $f(t)$, i.e., $f'(X) = 0$. (The reader who is not familiar with the method of steepest descents is referred to the recent book of Paris [98] for a detailed introduction to the subject.) The material covered in this chapter forms the basis of our investigations on the asymptotic expansions of special functions, discussed in later parts of this work.

The chapter consists of four main sections. In the first two sections, we derive explicit representations for the remainder terms of the asymptotic expansions of the integrals (1.1). In these representations, closely related integrals make their appearance explaining the term "resurgence". Our presentation is based on the papers by Berry and Howls [8], Boyd [13], and Howls [45], but in many aspects it is more detailed and rigorous. In addition, we also treat the case of second-order saddle points which was not considered by these authors and is not discussed anywhere else in the literature. Extensions to higher-order saddles are also possible, but because of their rare occurrence in direct applications, we do not consider these cases. Section 1.3 is devoted to the asymptotic analysis of the high-order (late) coefficients appearing in the asymptotic expansions of the integrals (1.1). Its results provide a secure basis for the formal investigations of Dingle [35, Ch. VII] which he originally based on a theorem of Darboux [21]. Some of the questions we address in this section were also considered by Boyd [15]. In Section 1.4, we discuss the Stokes phenomenon, the continuation rules for the resurgence formulae and the re-expansion of the remainder terms in the asymptotic expansions of the integrals (1.1). Repeated re-expansions would lead to the hyperasymptotic theory of Berry and Howls [8], but we shall consider only the first stage of this process (the exponential improvement).

1.1 The resurgence properties of integrals with finite endpoints

This section is about the resurgence properties of integrals with finite endpoints. We distinguish three different cases, according to the local behaviour of the phase function (the function $f(t)$ in (1.1)) at the finite endpoint of the contour of integration.

1.1.1 Linear dependence at the endpoint

In this subsection, we study the resurgence properties of integrals of the type

$$I^{(e)}(z) = \int_{\mathscr{P}^{(e)}(\theta)} \mathrm{e}^{-zf(t)} g(t)\,\mathrm{d}t, \tag{1.2}$$

where $\mathscr{P}^{(e)}(\theta)$ is the steepest descent path emanating from a finite endpoint e and passing to infinity down the valley of $\mathfrak{Re}[-\mathrm{e}^{\mathrm{i}\theta}(f(t)-f(e))]$. The functions $f(t)$ and $g(t)$ are assumed to be analytic in a domain $\Delta^{(e)}$ which will be specified below. We suppose further that $f'(e) \neq 0$, $|f(t)| \to +\infty$ in $\Delta^{(e)}$, and $f(t)$ has first-order saddle points in the complex t-plane at $t = t^{(p)}$ labelled by $p \in \mathbb{N}$.[1] We denote by $\mathscr{C}^{(p)}(\theta)$ the steepest descent path through the saddle $t^{(p)}$.

The domain $\Delta^{(e)}$ is defined as follows. Consider all the steepest descent paths for different values of θ, which emerge from the endpoint e. In general these paths can end either at infinity or at a singularity of $f(t)$. We assume that all of them end at infinity. Since there are no branch points of $f(t)$ along these paths, any point in the t-plane either cannot be reached by any path of steepest

[1] In general, $f(t)$ can have higher-order saddle points as well. This assumption is made for the sake of simplicity and because it almost always holds in applications. All the subsequent arguments can be modified in a straightforward way for the case of higher-order saddles.

descent issuing from e, or else by only one. A continuity argument shows that the set of all the points which can be reached by a steepest descent path from e forms the closure of a domain in the t-plane. It is this domain which we denote by $\Delta^{(e)}$.

It is convenient to consider instead of the integral (1.2), its slowly varying part, defined by

$$T^{(e)}(z) \stackrel{\text{def}}{=} z\mathrm{e}^{zf(e)} I^{(e)}(z) = z\int_{\mathscr{P}^{(e)}(\theta)} \mathrm{e}^{-z(f(t)-f(e))} g(t)\,\mathrm{d}t. \tag{1.3}$$

First, we make the transformation

$$s = z\left(f(t) - f(e)\right). \tag{1.4}$$

Since $f'(e) \neq 0$, according to the Taylor formula, we have

$$s/z = f'(e)(t-e) + \mathcal{O}(|t-e|^2)$$

when t is close to e, showing the linear dependence at the endpoint e. The right-hand side of (1.4) is a monotonic function of t, unless perhaps when $\mathscr{P}^{(e)}(\theta)$ passes through a saddle point of $f(t)$. As θ varies, the contour $\mathscr{P}^{(e)}(\theta)$ varies smoothly until it encounters a saddle point $t^{(m)}$ when $\theta = -\sigma_{em}$ with

$$\sigma_{em} \stackrel{\text{def}}{=} \arg(f(t^{(m)}) - f(e)).$$

Such saddle points are called *adjacent* to the endpoint e. From now on, we shall assume that the steepest descent path $\mathscr{P}^{(e)}(\theta)$ in (1.2) does not passes through any of the saddle points of $f(t)$, and that θ is restricted to an interval

$$-\sigma_{em_1} < \theta < -\sigma_{em_2}, \tag{1.5}$$

where $t^{(m_1)}$ and $t^{(m_2)}$ are adjacent to e. We shall suppose that $f(t)$ and $g(t)$ grow sufficiently rapidly at infinity so that the integral (1.2) converges for all values of θ in the interval (1.5). We remark that since $\mathscr{P}^{(e)}(\theta) = \mathscr{P}^{(e)}(\theta + 2\pi)$, we have

$$0 < \sigma_{em_1} - \sigma_{em_2} \leq 2\pi,$$

and $\sigma_{em_1} - \sigma_{em_2} = 2\pi$ occurs, for example, when there is only one saddle point that is adjacent to e.

With these assumptions, as t travels along $\mathscr{P}^{(e)}(\theta)$, s increases monotonically from 0 to $+\infty$. Therefore, corresponding to each positive value of s, there is a value of t, say $t(s/z)$, satisfying (1.4) with $t(s/z) \in \mathscr{P}^{(e)}(\theta)$. In terms of s, we have

$$T^{(e)}(z) = z\int_0^{+\infty} e^{-s} g(t)\frac{dt}{ds}ds = \int_0^{+\infty} e^{-s}\frac{g(t(s/z))}{f'(t(s/z))}ds. \tag{1.6}$$

Since there is no saddle point of $f(t)$ on the contour $\mathscr{P}^{(e)}(\theta)$, the quantity $g(t(s/z))/f'(t(s/z))$ is an analytic function of t in a neighbourhood of $\mathscr{P}^{(e)}(\theta)$, and by the residue theorem[2]

$$\frac{g(t(s/z))}{f'(t(s/z))} = \operatorname*{Res}_{t=t(s/z)} \frac{g(t)}{f(t)-f(e)-s/z} = \frac{1}{2\pi i}\oint_{(t(s/z)+)} \frac{g(t)}{f(t)-f(e)-s/z}dt.$$

Substituting this expression into (1.6) gives us an alternative representation of the integral $T^{(e)}(z)$,

$$T^{(e)}(z) = \int_0^{+\infty} e^{-s}\frac{1}{2\pi i}\oint_{\Gamma^{(e)}(\theta)} \frac{g(t)}{f(t)-f(e)-s/z}dt ds,$$

where the infinite contour $\Gamma^{(e)}(\theta)$ encircles the path $\mathscr{P}^{(e)}(\theta)$ in the positive direction within $\Delta^{(e)}$ (see Figure 1.1(a)). This integral will exist provided that $g(t)/f(t)$ decays sufficiently rapidly at infinity in $\Delta^{(e)}$. Otherwise, we can define $\Gamma^{(e)}(\theta)$ as a finite loop contour surrounding $t(s/z)$ and consider the limit

$$\lim_{S\to+\infty}\int_0^{S} e^{-s}\frac{1}{2\pi i}\oint_{\Gamma^{(e)}(\theta)} \frac{g(t)}{f(t)-f(e)-s/z}dt ds.$$

Now, we employ the well-known expression for non-negative integer N

$$\frac{1}{1-x} = \sum_{n=0}^{N-1} x^n + \frac{x^N}{1-x},\ x \neq 1, \tag{1.7}$$

to expand the denominator in powers of $s/z\,(f(t)-f(e))$ and find that

$$T^{(e)}(z) = \sum_{n=0}^{N-1}\frac{1}{z^n}\int_0^{+\infty} s^n e^{-s}\frac{1}{2\pi i}\oint_{\Gamma^{(e)}(\theta)} \frac{g(t)}{(f(t)-f(e))^{n+1}}dt ds + R_N^{(e)}(z), \tag{1.8}$$

where

$$R_N^{(e)}(z) = \frac{1}{2\pi i z^N}\int_0^{+\infty} s^N e^{-s}\oint_{\Gamma^{(e)}(\theta)} \frac{g(t)}{(f(t)-f(e))^{N+1}}\frac{dt}{1-s/z\,(f(t)-f(e))}ds. \tag{1.9}$$

[2]If $P(t)$ and $Q(t)$ are analytic in a neighbourhood of t_0 with $P(t_0)=0$ and $P'(t_0)\neq 0$, then $Q(t_0)/P'(t_0) = \operatorname*{Res}_{t=t_0} Q(t)/P(t)$.

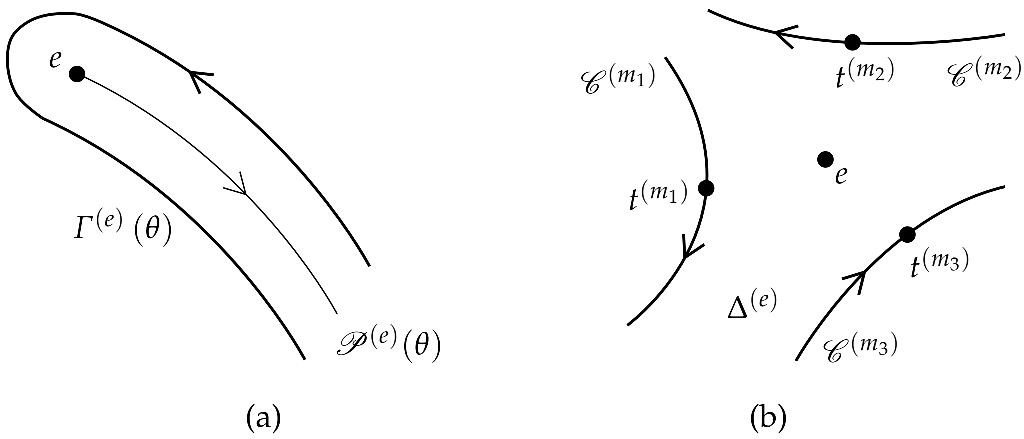

Figure 1.1. *(a) The contour $\Gamma^{(e)}(\theta)$ surrounding the path $\mathscr{P}^{(e)}(\theta)$. (b) Three saddle points $t^{(m)}$ adjacent to the endpoint e together with the corresponding adjacent contours $\mathscr{C}^{(m)}$, forming the boundary of the domain $\Delta^{(e)}$.*

Again, a limiting process is used in (1.9) if necessary. Throughout this work, if not stated otherwise, empty sums are taken to be zero. The path $\Gamma^{(e)}(\theta)$ in the sum can be shrunk into a small circle around e, and we arrive at

$$T^{(e)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(e)}}{z^n} + R_N^{(e)}(z), \tag{1.10}$$

with

$$a_n^{(e)} = \frac{\Gamma(n+1)}{2\pi \mathrm{i}} \oint_{(e+)} \frac{g(t)}{(f(t)-f(e))^{n+1}} \mathrm{d}t \tag{1.11}$$

$$= \left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(g(t) \left(\frac{t-e}{f(t)-f(e)} \right)^{n+1} \right) \right]_{t=e}. \tag{1.12}$$

If we neglect the remainder term $R_N^{(e)}(z)$ in (1.10) and formally extend the sum to infinity, the result becomes the well-known asymptotic expansion of an integral with linear endpoint (cf. [35, eq. (12), p. 114] or [98, eq. (1.2.21), p. 14]). A representation equivalent to (1.11) was derived by Dingle [35, eq. (9), p. 113]. The expression (1.12) is a special case of Perron's formula for the coefficients in steepest descent expansions (see, for instance, [70]).

In the next step, we deform $\Gamma^{(e)}(\theta)$ in (1.9) to the boundary of the domain $\Delta^{(e)}$. But before we do so, we wish to take a closer look at this boundary. We claim that it is the union $\bigcup_m \mathscr{C}^{(m)}(-\sigma_{em})$ of steepest descent paths through the adjacent saddles $t^{(m)}$ (see Figure 1.1(b)). The paths $\mathscr{C}^{(m)}(-\sigma_{em})$ are called the *adjacent contours*. We assume that each adjacent contour contains only one saddle point. To prove our claim, we first choose $\theta = \theta'$ to be an angle such that $\mathscr{P}^{(e)}(\theta')$ does not pass through any of the saddle points of $f(t)$. As we vary θ, the path $\mathscr{P}^{(e)}(\theta)$ varies smoothly until it encounters an adjacent saddle point $t^{(m)}$ when $\theta = -\sigma_{em}$.[3] Let us define $\mathscr{P}_{\pm}^{(e)}(-\sigma_{em})$ via

$$\lim_{\delta \to 0+} \mathscr{P}^{(e)}(-\sigma_{em} \pm \delta) = \mathscr{P}_{\pm}^{(e)}(-\sigma_{em}). \tag{1.13}$$

We note that $\mathscr{P}_{-}^{(e)}(-\sigma_{em}) \neq \mathscr{P}_{+}^{(e)}(-\sigma_{em})$. To see that this discontinuity must occur, we can argue as follows. If the path $\mathscr{P}^{(e)}(\theta)$ were change continuously across the adjacent saddle $t^{(m)}$ then, for a suitable $\delta > 0$, the contours $\mathscr{P}^{(e)}(-\sigma_{em} + \delta)$ and $\mathscr{C}^{(m)}(-\sigma_{em} + \delta)$ would intersect each other giving two different paths of steepest descent from the point of intersection. But if $\delta > 0$ is chosen suitably, then this point of intersection is not a saddle point of $f(t)$, and hence it can not be the endpoint of two different steepest descent contours. If $t \in \mathscr{P}_{-}^{(e)}(-\sigma_{em})$, then $\mathfrak{Im}[\mathrm{e}^{-\mathrm{i}\sigma_{em}}(f(t) - f(e))] = 0$ and therefore

$$\begin{aligned} &\mathfrak{Im}[\mathrm{e}^{-\mathrm{i}\sigma_{em}}(f(t) - f(t^{(m)}))] \\ &\qquad = \mathfrak{Im}[\mathrm{e}^{-\mathrm{i}\sigma_{em}}(f(t) - f(e))] - \mathfrak{Im}[\mathrm{e}^{-\mathrm{i}\sigma_{em}}(f(t^{(m)}) - f(e))] = 0. \end{aligned}$$

In other words, any point on $\mathscr{P}_{-}^{(e)}(-\sigma_{em})$ lies on either a steepest ascent or a steepest descent path issuing from the saddle point $t^{(m)}$. Since the quantity $\mathfrak{Re}[\mathrm{e}^{-\mathrm{i}\sigma_{em}}(f(t) - f(e))]$ is monotonically increasing along the segment of $\mathscr{P}_{-}^{(e)}(-\sigma_{em})$ joining the points e and $t^{(m)}$, this segment must be part of a steepest ascent path running into $t^{(m)}$. Afterwards, $\mathscr{P}_{-}^{(e)}(-\sigma_{em})$ turns sharply through a right angle at $t^{(m)}$ and continues its descent to infinity down the valley of $\mathfrak{Re}[-\mathrm{e}^{-\mathrm{i}\sigma_{em}}(f(t) - f(t^{(m)}))]$. All these observations on $\mathscr{P}_{-}^{(e)}(-\sigma_{em})$ are certainly true for the contour $\mathscr{P}_{+}^{(e)}(-\sigma_{em})$ as well. Along the segment which joins e and $t^{(m)}$, $\mathscr{P}_{-}^{(e)}(-\sigma_{em})$ and $\mathscr{P}_{+}^{(e)}(-\sigma_{em})$ must coincide, but because they are not identical, at $t^{(m)}$ they will split and continue along the two opposite steepest descent directions from $t^{(m)}$ (cf. Figure 1.2). Taking into account that any

[3] A discontinuity in the change of the path $\mathscr{P}^{(e)}(\theta)$ may also occur when, for a critical value θ^* of θ, the contour $\mathscr{P}^{(e)}(\theta^*)$ splits into two parts: a path linking e to infinity and a doubly infinite contour. This situation happens, for example, in the resurgence analysis of the generalized Bessel function [122]. Our assumption that $|f(t)| \to +\infty$ in $\Delta^{(e)}$ excludes this case, because $\mathrm{e}^{\mathrm{i}\theta^*}(f(t) - f(e))$ must have a finite limit as $t \to \infty$ along the part of $\mathscr{P}^{(e)}(\theta^*)$ which emerges from e.

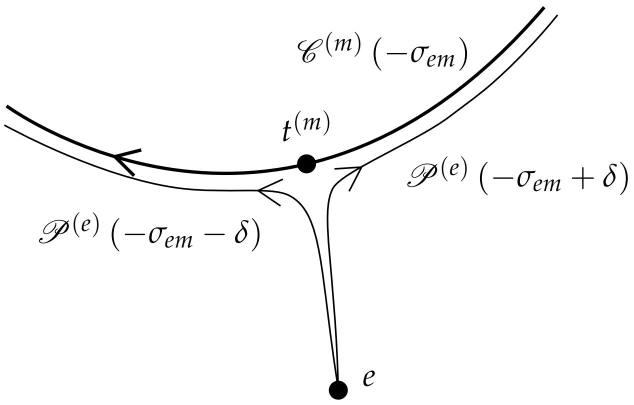

Figure 1.2. *The discontinuous change of the steepest descent path $\mathscr{P}^{(e)}(\theta)$ from e as the adjacent saddle $t^{(m)}$ is passed ($\delta > 0$).*

point of the complex t-plane can be reached by at most one steepest descent path from e, it follows that $\mathscr{C}^{(m)}(-\sigma_{em})$ indeed contributes to the boundary of the domain $\Delta^{(e)}$. We rotate θ further and visit the adjacent saddles one-by-one until $\theta = \theta' + 2\pi$, in which case $\mathscr{P}^{(e)}(\theta)$ returns to its original state. This argument works only if the set of saddle points that are adjacent to e is non-empty and finite, which we shall assume to be the case. Since $\mathscr{P}^{(e)}(\theta)$ passes to infinity, it changes continuously whenever $\theta \neq -\sigma_{em}$ and any point in the complex t-plane can be reached by at most one such path, there are no further points on the boundary of $\Delta^{(e)}$ other than those situated on an adjacent contour.

By expanding $\Gamma^{(e)}(\theta)$ to the boundary of $\Delta^{(e)}$, we obtain

$$\begin{aligned} R_N^{(e)}(z) = {} & \frac{1}{2\pi \mathrm{i} z^N} \int_0^{+\infty} s^N \mathrm{e}^{-s} \\ & \times \sum_m (-1)^{\gamma_{em}} \int_{\mathscr{C}^{(m)}} \frac{g(t)}{(f(t)-f(e))^{N+1}} \frac{\mathrm{d}t}{1 - s/z\,(f(t)-f(e))} \mathrm{d}s, \end{aligned} \tag{1.14}$$

where $\mathscr{C}^{(m)} = \mathscr{C}^{(m)}(-\sigma_{em})$ and the summation is over each of the adjacent contours. The exponents γ_{em} are the *orientation anomalies*; they take the value 0 if the sense of the deformed $\Gamma^{(e)}(\theta)$ and the chosen orientation of the relevant $\mathscr{C}^{(m)}(-\sigma_{em})$ are identical, 1 otherwise. This expansion process is justified provided that (i) $f(t)$ and $g(t)$ are analytic in the domain $\Delta^{(e)}$, (ii) the quantity $g(t)/f^{N+1}(t)$ decays sufficiently rapidly at infinity in $\Delta^{(e)}$, and (iii) there are no zeros of the denominator $1 - s/z\,(f(t)-f(e))$ within the region R through which the loop $\Gamma^{(e)}(\theta)$ is deformed. The first condition simply repeats one of the assumptions from the very beginning of our analysis. To meet the second condition, we require that $g(t)/f^{N+1}(t) = o(|t|^{-1})$ as $t \to \infty$ in $\Delta^{(e)}$. The third condition is satisfied according to the following argument. The zeros of the denominator are those points of the t-plane for which $\mathrm{e}^{\mathrm{i}\theta}(f(t) - f(e))$ is real and positive, in particular the points of the path $\mathscr{P}^{(e)}(\theta)$. Furthermore, no components of the set defined by the equation $\arg[\mathrm{e}^{\mathrm{i}\theta}(f(t) - f(e))] = 0$ other than $\mathscr{P}^{(e)}(\theta)$ can lie within $\Delta^{(e)}$, otherwise $f(t)$ would have branch points along those components. By observing that $\mathscr{P}^{(e)}(\theta)$ is different for different values of θ mod 2π, we see that the locus of the zeros of the denominator $1 - s/z(f(t) - f(e))$ inside $\Delta^{(e)}$ is precisely the contour $\mathscr{P}^{(e)}(\theta)$, which is wholly contained within $\Gamma^{(e)}(\theta)$ and so these zeros are external to R.

At this point, it is convenient to introduce the so-called *singulants* $\mathcal{F}_{em}$ (originally defined by Dingle [35, p. 147]) via

$$\mathcal{F}_{em} \stackrel{\text{def}}{=} f(t^{(m)}) - f(e), \quad \arg \mathcal{F}_{em} = \sigma_{em}.$$

Before we proceed to the last step of our analysis, we wish to consider the convergence of the double integrals in (1.14) further. It is convenient to make the change of variable from t to v by

$$f(t) - f(e) = v\mathrm{e}^{\mathrm{i}\sigma_{em}}, \tag{1.15}$$

where $v \geq |\mathcal{F}_{em}|$. Note that $\mathrm{e}^{-\mathrm{i}\sigma_{em}}(f(t) - f(e))$ is a monotonic function of t on each half of the contour $\mathscr{C}^{(m)}(-\sigma_{em})$ before and after the saddle point $t^{(m)}$. Hence, corresponding to each value of v, there are two values of t, say $t_\pm(v)$, that satisfy (1.15). The convergence of the double integrals in (1.14) will be assured provided that the real double integrals

$$\int_0^{+\infty} \int_{|\mathcal{F}_{em}|}^{+\infty} \frac{s^N \mathrm{e}^{-s}}{v^{N+1}} \left| \frac{g(t_\pm(v))}{f'(t_\pm(v))} \right| \mathrm{d}v \mathrm{d}s$$

exist. (The assumption (1.5) implies that the factor $[1 - s/z(f(t) - f(e))]^{-1}$ in (1.14) is bounded above by a constant.) These real double integrals will exist if and only if the single integrals

$$\int_{|\mathcal{F}_{em}|}^{+\infty} \frac{1}{v^{N+1}} \left| \frac{g(t_\pm(v))}{f'(t_\pm(v))} \right| \mathrm{d}v \tag{1.16}$$

exist. From now on, we suppose that the integrals in (1.16) exist for each of the adjacent contours.

Along each of the contours $\mathscr{C}^{(m)}(-\sigma_{em})$ in (1.14), we perform the change of variable from s and t to u and t by

$$s = \mathrm{e}^{-\mathrm{i}\sigma_{em}}(f(t) - f(e))u = |\mathcal{F}_{em}|u + \mathrm{e}^{-\mathrm{i}\sigma_{em}}(f(t) - f(t^{(m)}))u \tag{1.17}$$

to find

$$R_N^{(e)}(z) = \frac{1}{2\pi i z^N} \sum_m \frac{(-1)^{\gamma_{em}}}{e^{i(N+1)\sigma_{em}}} \int_0^{+\infty} \frac{u^N e^{-|\mathcal{F}_{em}|u}}{1 - u/ze^{i\sigma_{em}}} \times \int_{\mathscr{C}^{(m)}} e^{-e^{-i\sigma_{em}}(f(t)-f(t^{(m)}))u} g(t)\, dt du. \tag{1.18}$$

The change of variable (1.17) is justified because the infinite double integrals in (1.14) are absolutely convergent, which is justified because of the requirement that the integrals (1.16) exist. By analogy with (1.3), we define

$$T^{(m)}(w) \stackrel{\text{def}}{=} w^{\frac{1}{2}} \int_{\mathscr{C}^{(m)}(\arg w)} e^{-w(f(t)-f(t^{(m)}))} g(t)\, dt \tag{1.19}$$

with $\arg(w^{\frac{1}{2}}) \stackrel{\text{def}}{=} \frac{1}{2}\arg w$. This is the slowly varying part of an integral over a doubly infinite contour through a first-order saddle point. With this notation, (1.18) becomes

$$R_N^{(e)}(z) = \frac{1}{2\pi i z^N} \sum_m \frac{(-1)^{\gamma_{em}}}{e^{i(N+\frac{1}{2})\sigma_{em}}} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}} e^{-|\mathcal{F}_{em}|u}}{1 - u/ze^{i\sigma_{em}}} T^{(m)}\left(ue^{-i\sigma_{em}}\right) du. \tag{1.20}$$

This result provides an exact form for the remainder in the asymptotic expansion of an integral with linear endpoint, expressed as a sum involving integrals through the adjacent saddles. Equations (1.10) and (1.20) together yield the exact resurgence formula for $T^{(e)}(z)$.

1.1.2 Quadratic dependence at the endpoint

We consider the resurgence properties of integrals of the form

$$I^{(k/2)}(z) = \int_{\mathscr{P}^{(k)}(\theta)} e^{-zf(t)} g(t)\, dt, \tag{1.21}$$

where $\mathscr{P}^{(k)}(\theta)$ is one of the two paths of steepest descent emerging from the first-order saddle point $t^{(k)}$ of $f(t)$ and passing to infinity down the valley of $\mathfrak{Re}[-e^{i\theta}(f(t) - f(t^{(k)}))]$ (the other path can be expressed as $\mathscr{P}^{(k)}(\theta + 2\pi)$). The functions $f(t)$ and $g(t)$ are assumed to be analytic in a domain $\Delta^{(k)}$, whose closure is the set of all the points that can be reached by a steepest descent path emanating from $t^{(k)}$. We suppose further that $|f(t)| \to +\infty$ in $\Delta^{(k)}$, and $f(t)$ has several other first-order saddle points in the complex t-plane at $t = t^{(p)}$ indexed by $p \in \mathbb{N}$. As in the linear endpoint case, $\mathscr{C}^{(p)}(\theta)$ denotes the steepest descent path through the saddle $t^{(p)}$, in particular $\mathscr{C}^{(k)}(\theta) = \mathscr{P}^{(k)}(\theta) \cup$

$\mathscr{P}^{(k)}(\theta + 2\pi)$. The notation "$k/2$" is used for consistency with later sections on integrals having doubly infinite steepest paths.

It is again convenient to consider instead of the integral (1.21), its slowly varying part, given by

$$T^{(k/2)}(z) \stackrel{\text{def}}{=} 2z^{\frac{1}{2}} e^{zf(t^{(k)})} I^{(k/2)}(z) = 2z^{\frac{1}{2}} \int_{\mathscr{P}^{(k)}(\theta)} e^{-z(f(t)-f(t^{(k)}))} g(t)\, dt. \tag{1.22}$$

The square root is defined to be positive on the positive real line and is defined by analytic continuation elsewhere. Similarly to the linear endpoint case, the path $\mathscr{P}^{(k)}(\theta)$ may pass through other saddle points $t^{(m)}$ when $\theta = -\sigma_{km}$ with

$$\sigma_{km} \stackrel{\text{def}}{=} \arg(f(t^{(m)}) - f(t^{(k)})).$$

We call such saddle points adjacent to $t^{(k)}$. We shall assume that the steepest descent path $\mathscr{P}^{(k)}(\theta)$ in (1.21) does not encounter any of the saddle points of $f(t)$ other than $t^{(k)}$, and that θ is restricted to an interval

$$-\sigma_{km_1} < \theta < -\sigma_{km_2}, \tag{1.23}$$

where $t^{(m_1)}$ and $t^{(m_2)}$ are adjacent to $t^{(k)}$. We shall suppose that $f(t)$ and $g(t)$ grow sufficiently rapidly at infinity so that the integral (1.21) converges for all values of θ in the interval (1.23). We note that since $\mathscr{P}^{(k)}(\theta) = \mathscr{P}^{(k)}(\theta + 4\pi)$, we have

$$0 < \sigma_{km_1} - \sigma_{km_2} \leq 4\pi,$$

and $\sigma_{km_1} - \sigma_{km_2} = 4\pi$ arises, for example, when there is only one saddle point that is adjacent to $t^{(k)}$.

The next step is to give a parametrization of the integrand in (1.22) along $\mathscr{P}^{(k)}(\theta)$ similar to that of (1.4) in the linear endpoint case. Since $t^{(k)}$ is a first-order saddle point of $f(t)$, according to the Taylor formula, we have

$$f(t) - f(t^{(k)}) = \frac{1}{2} f''(t^{(k)})(t - t^{(k)})^2 + \mathcal{O}(|t - t^{(k)}|^3)$$

when t is close to $t^{(k)}$. This locally quadratic behaviour at the endpoint $t^{(k)}$ suggests that instead of a linear transformation (like in (1.4)), one may use the quadratic parametrization

$$s^2 = z(f(t) - f(t^{(k)})). \tag{1.24}$$

The advantage of this parametrization is that the inverse mapping is, at least locally, a single-valued analytic function of s (cf. [69, eq. (3.23), p. 49]). The right-hand side of (1.24) is real and positive if either $t \in \mathscr{P}^{(k)}(\theta)$ or $t \in \mathscr{P}^{(k)}(\theta + 2\pi)$, the other steepest descent path from $t^{(k)}$. Furthermore, if the path $\mathscr{P}^{(k)}(\theta+2\pi)$ does not pass through any of the saddle points of $f(t)$ other than $t^{(k)}$, then (1.24) gives a one-to-one correspondence between the real s-axis and the steepest descent path $\mathscr{C}^{(k)}(\theta)$. To parametrize $\mathscr{P}^{(k)}(\theta)$, we choose the positive reals and thus

$$T^{(k/2)}(z) = 2z^{\frac{1}{2}}\int_0^{+\infty} \mathrm{e}^{-s^2} g(t)\frac{\mathrm{d}t}{\mathrm{d}s}\mathrm{d}s = 2\int_0^{+\infty}\mathrm{e}^{-s^2}\frac{2s}{z^{\frac{1}{2}}}\frac{g(t(s/z^{\frac{1}{2}}))}{f'(t(s/z^{\frac{1}{2}}))}\mathrm{d}s,$$

where $t = t(s/z^{\frac{1}{2}})$ is the unique solution of the equation (1.24) with $t(s/z^{\frac{1}{2}}) \in \mathscr{P}^{(k)}(\theta)$. Since there are no saddle points of $f(t)$ on the contour $\mathscr{P}^{(k)}(\theta)$ other than $t^{(k)}$, the quantity

$$\frac{2s}{z^{\frac{1}{2}}}\frac{g(t(s/z^{\frac{1}{2}}))}{f'(t(s/z^{\frac{1}{2}}))} = \frac{2(f(t(s/z^{\frac{1}{2}})) - f(t^{(k)}))^{\frac{1}{2}}}{f'(t(s/z^{\frac{1}{2}}))}g(t(s/z^{\frac{1}{2}}))$$

is an analytic function of t in a neighbourhood of $\mathscr{P}^{(k)}(\theta)$.[4] (For the analyticity of the factor $(f(t) - f(t^{(k)}))^{\frac{1}{2}}$ in $\Delta^{(k)}$, consider the paragraph after equation (1.26) below.) Therefore, by the residue theorem

$$\begin{aligned}\frac{2(f(t(s/z^{\frac{1}{2}}))-f(t^{(k)}))^{\frac{1}{2}}}{f'(t(s/z^{\frac{1}{2}}))}g(t(s/z^{\frac{1}{2}})) &= \operatorname*{Res}_{t=t(s/z^{\frac{1}{2}})}\frac{g(t)}{(f(t)-f(t^{(k)}))^{\frac{1}{2}}-s/z^{\frac{1}{2}}}\\ &= \frac{1}{2\pi\mathrm{i}}\oint_{(t(s/z^{\frac{1}{2}})+)}\frac{g(t)}{(f(t)-f(t^{(k)}))^{\frac{1}{2}}-s/z^{\frac{1}{2}}}\mathrm{d}t.\end{aligned}$$

Consequently, we have

$$T^{(k/2)}(z) = \int_0^{+\infty}\mathrm{e}^{-s^2}\frac{1}{\pi\mathrm{i}}\oint_{\Gamma^{(k)}(\theta)}\frac{g(t)}{(f(t)-f(t^{(k)}))^{\frac{1}{2}}-s/z^{\frac{1}{2}}}\mathrm{d}t\mathrm{d}s, \tag{1.25}$$

where $\Gamma^{(k)}(\theta)$ is an infinite loop contour which encloses $\mathscr{P}^{(k)}(\theta)$ in the anticlockwise sense within $\Delta^{(k)}$. This integral will exist provided that $g(t)/f^{\frac{1}{2}}(t)$ decreases in magnitude sufficiently rapidly at infinity in $\Delta^{(k)}$. Otherwise, we can define $\Gamma^{(k)}(\theta)$ as a finite loop contour encircling $t(s/z^{\frac{1}{2}})$ and consider the limit

$$\lim_{S\to+\infty}\int_0^{S}\mathrm{e}^{-s^2}\frac{1}{\pi\mathrm{i}}\oint_{\Gamma^{(k)}(\theta)}\frac{g(t)}{(f(t)-f(t^{(k)}))^{\frac{1}{2}}-s/z^{\frac{1}{2}}}\mathrm{d}t\mathrm{d}s. \tag{1.26}$$

[4]The apparent singularity at $t = t^{(k)}$ is removable.

The factor $(f(t) - f(t^{(k)}))^{\frac{1}{2}}$ in (1.25) is defined in the domain $\Delta^{(k)}$ as follows. First, we observe that $f(t) - f(t^{(k)})$ has a double zero at $t = t^{(k)}$ and is non-zero elsewhere in $\Delta^{(k)}$ (because any point in $\Delta^{(k)}$, different from $t^{(k)}$, can be reached from $t^{(k)}$ by a path of descent). Second, $\mathscr{P}^{(k)}(\theta)$ is a periodic function of θ with (least) period 4π. Hence, we may define the square root so that $(f(t) - f(t^{(k)}))^{\frac{1}{2}}$ is a single-valued analytic function of t in $\Delta^{(k)}$. The correct choice of the branch of $(f(t) - f(t^{(k)}))^{\frac{1}{2}}$ is determined by the requirement that s is real and positive on $\mathscr{P}^{(k)}(\theta)$, which can be fulfilled by setting $\arg[(f(t) - f(t^{(k)}))^{\frac{1}{2}}] = -\frac{\theta}{2}$ for $t \in \mathscr{P}^{(k)}(\theta)$. With any other definition of $(f(t) - f(t^{(k)}))^{\frac{1}{2}}$, the representation (1.25) would be invalid.

Now, we apply the expression (1.7) to expand the denominator in (1.25) in powers of $s/[z(f(t) - f(t^{(k)}))]^{\frac{1}{2}}$. One thus finds

$$T^{(k/2)}(z) = \sum_{n=0}^{N-1} \frac{1}{z^{\frac{n}{2}}} \int_0^{+\infty} s^n \mathrm{e}^{-s^2} \frac{1}{\pi \mathrm{i}} \oint_{\Gamma^{(k)}(\theta)} \frac{g(t)}{(f(t) - f(t^{(k)}))^{\frac{n+1}{2}}} \mathrm{d}t \mathrm{d}s + R_N^{(k/2)}(z)$$

with

$$\begin{aligned} R_N^{(k/2)}(z) = {} & \frac{1}{\pi \mathrm{i} z^{\frac{N}{2}}} \int_0^{+\infty} s^N \mathrm{e}^{-s^2} \\ & \times \oint_{\Gamma^{(k)}(\theta)} \frac{g(t)}{(f(t) - f(t^{(k)}))^{\frac{N+1}{2}}} \frac{\mathrm{d}t}{1 - s/[z(f(t) - f(t^{(k)}))]^{\frac{1}{2}}} \mathrm{d}s. \end{aligned} \tag{1.27}$$

Again, a limiting process is used in (1.27) if necessary. The contour $\Gamma^{(k)}(\theta)$ in the sum can be shrunk into a small loop around $t^{(k)}$, and we arrive at

$$T^{(k/2)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(k/2)}}{z^{\frac{n}{2}}} + R_N^{(k/2)}(z), \tag{1.28}$$

where the coefficients are given by

$$a_n^{(k/2)} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi \mathrm{i}} \oint_{(t^{(k)}+)} \frac{g(t)}{(f(t) - f(t^{(k)}))^{\frac{n+1}{2}}} \mathrm{d}t \tag{1.29}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n+1)} \left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(g(t) \left(\frac{(t - t^{(k)})^2}{f(t) - f(t^{(k)})} \right)^{\frac{n+1}{2}} \right) \right]_{t = t^{(k)}}. \tag{1.30}$$

If we omit the remainder term $R_N^{(k/2)}(z)$ in (1.28) and formally extend the sum to infinity, the result becomes the well-known asymptotic expansion of an integral with quadratic endpoint (cf. [35, eq. (20), p. 119] or [98, eq. (1.2.18), p. 13]). A representation equivalent to (1.29) was obtained by Copson [20, p. 69] and Dingle [35, eq. (19), p. 119]. The expression (1.30) is again a special case of Perron's formula [70].

By analogy with the linear endpoint case, we now deform $\Gamma^{(k)}(\theta)$ in (1.27) by expanding it to the boundary of the domain $\Delta^{(k)}$. Note that in identifying the boundary of the corresponding domain $\Delta^{(e)}$ in the linear endpoint case, we did not use any specific properties of the finite endpoint of the steepest descent path. Therefore, without any further considerations, the boundary is again the union $\bigcup_m \mathscr{C}^{(m)}(-\sigma_{km})$ of adjacent contours (at least in the case that the set of adjacent saddle points is non-empty and finite). We suppose that each of these contours contains only one saddle point. The expansion process is possible provided that (i) $f(t)$ and $g(t)$ have no singularities in the domain $\Delta^{(k)}$, (ii) the quantity $g(t)/f^{\frac{N+1}{2}}(t)$ decays sufficiently rapidly at infinity in $\Delta^{(k)}$, and (iii) there are no zeros of the denominator $1-s/[z(f(t)-f(t^{(k)}))]^{\frac{1}{2}}$ within the region through which the loop $\Gamma^{(k)}(\theta)$ is deformed. The first condition is fulfilled because of our assumption that $f(t)$ and $g(t)$ are analytic in $\Delta^{(k)}$. To meet the second condition, we require that $g(t)/f^{\frac{N+1}{2}}(t)=o(|t|^{-1})$ as $t\to\infty$ in $\Delta^{(k)}$. The third condition is satisfied by an argument akin to that in the linear endpoint case. With these assumptions, we can write

$$R_N^{(k/2)}(z)=\frac{1}{\pi \mathrm{i} z^{\frac{N}{2}}}\int_0^{+\infty} s^N \mathrm{e}^{-s^2} \times\sum_m (-1)^{\gamma_{km}}\int_{\mathscr{C}^{(m)}}\frac{g(t)}{(f(t)-f(t^{(k)}))^{\frac{N+1}{2}}}\frac{\mathrm{d}t}{1-s/[z(f(t)-f(t^{(k)}))]^{\frac{1}{2}}}\mathrm{d}s, \tag{1.31}$$

where $\mathscr{C}^{(m)}=\mathscr{C}^{(m)}(-\sigma_{km})$ and the summation is over each of the adjacent contours. The orientation anomalies γ_{km} are defined as above and take the value 0 if the deformed $\Gamma^{(k)}(\theta)$ and the chosen orientation of the relevant $\mathscr{C}^{(m)}(-\sigma_{km})$ have the same orientation, and 1 otherwise.

Following the linear case, we define the singulants by

$$\mathcal{F}_{km}\overset{\text{def}}{=}f(t^{(m)})-f(t^{(k)}),\quad \arg\mathcal{F}_{km}=\sigma_{km}$$

and shall make a change of variable in (1.31) to bring the expression for $R_N^{(k/2)}(z)$ into its final form. But before we do so, we make simple requirements to ensure the absolute convergence of the double integrals in (1.31). For convenience, we introduce the change of variable in (1.31) from t to v by

$$f(t)-f(t^{(k)})=v\mathrm{e}^{\mathrm{i}\sigma_{km}}, \tag{1.32}$$

with $v \geq |\mathcal{F}_{km}|$. Note that $\mathrm{e}^{-\mathrm{i}\sigma_{km}}(f(t) - f(t^{(k)}))$ is a monotonic function of t on both halves of the contour $\mathscr{C}^{(m)}(-\sigma_{km})$ before and after the saddle point $t^{(m)}$. Thus, corresponding to each value of v, there are two values of t, say $t_{\pm}(v)$, that satisfy (1.32). The convergence of the double integrals in (1.31) will be guaranteed provided that the real single integrals

$$\int_{|\mathcal{F}_{km}|}^{+\infty} \frac{1}{v^{\frac{N+1}{2}}} \left| \frac{g(t_{\pm}(v))}{f'(t_{\pm}(v))} \right| \mathrm{d}v \tag{1.33}$$

exist (cf. equation (1.16)). Hereafter, we assume that the integrals in (1.33) exist for each of the adjacent contours.

Along each of the contours $\mathscr{C}^{(m)}(-\sigma_{km})$ in (1.31), we make the change of variable from s and t to u and t by

$$s^2 = \mathrm{e}^{-\mathrm{i}\sigma_{km}}(f(t) - f(t^{(k)}))u = |\mathcal{F}_{km}|\, u + \mathrm{e}^{-\mathrm{i}\sigma_{km}}(f(t) - f(t^{(m)}))u \tag{1.34}$$

to find

$$\begin{aligned} R_N^{(k/2)}(z) = \frac{1}{2\pi \mathrm{i} z^{\frac{N}{2}}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{2}\sigma_{km}}} \int_0^{+\infty} \frac{u^{\frac{N-1}{2}} \mathrm{e}^{-|\mathcal{F}_{km}|u}}{1 - (u/z\mathrm{e}^{\mathrm{i}\sigma_{km}})^{\frac{1}{2}}} \\ \times \int_{\mathscr{C}^{(m)}} \mathrm{e}^{-\mathrm{e}^{-\mathrm{i}\sigma_{km}}(f(t)-f(t^{(m)}))u} g(t)\, \mathrm{d}t\mathrm{d}u. \end{aligned} \tag{1.35}$$

The change of variable (1.34) is justified because the infinite double integrals in (1.31) are absolutely convergent, which is justified because of the requirement that the integrals (1.33) exist. With the notation (1.19), the representation (1.35) becomes

$$R_N^{(k/2)}(z) = \frac{1}{2\pi \mathrm{i} z^{\frac{N}{2}}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathrm{e}^{\mathrm{i}\frac{N}{2}\sigma_{km}}} \int_0^{+\infty} \frac{u^{\frac{N}{2}-1} \mathrm{e}^{-|\mathcal{F}_{km}|u}}{1 - (u/z\mathrm{e}^{\mathrm{i}\sigma_{km}})^{\frac{1}{2}}} T^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right) \mathrm{d}u. \tag{1.36}$$

This result provides an exact form for the remainder in the asymptotic expansion of an integral with quadratic endpoint, expressed as a sum involving integrals over doubly infinite contours passing through the adjacent saddles. Equations (1.28) and (1.36) together give the exact resurgence formula for $T^{(k/2)}(z)$.

1.1.3 Cubic dependence at the endpoint

In this last subsection, we study the resurgence properties of integrals of the type

$$I^{(k/3)}(z) = \int_{\mathscr{P}^{(k)}(\theta)} \mathrm{e}^{-zf(t)} g(t)\, \mathrm{d}t, \tag{1.37}$$

where $\mathscr{P}^{(k)}(\theta)$ is now one of the three paths of steepest descent issuing from the second-order saddle point $t^{(k)}$ of $f(t)$ and passing to infinity down the valley of $\mathfrak{Re}[-\mathrm{e}^{\mathrm{i}\theta}(f(t)-f(t^{(k)}))]$ (the other two paths may be expressed as $\mathscr{P}^{(k)}(\theta+2\pi)$ and $\mathscr{P}^{(k)}(\theta+4\pi)$). The assumptions on $f(t)$ and $g(t)$ are the same as in the quadratic case, except we suppose here that all the saddles of $f(t)$ are of second order. The domain $\Delta^{(k)}$ is defined analogously as for the case of quadratic endpoint. We denote by $\mathscr{C}^{(p)}(\theta)$ one of the three doubly infinite steepest descent paths through the saddle $t^{(p)}$ labelled by $p \in \mathbb{N}$. The notation "$k/3$" is again used for consistency with later sections on integrals having doubly infinite steepest paths.

Instead of the integral (1.37), we consider its slowly varying part, given by

$$T^{(k/3)}(z) \stackrel{\text{def}}{=} 3z^{\frac{1}{3}}\mathrm{e}^{zf(t^{(k)})}I^{(k/3)}(z) = 3z^{\frac{1}{3}}\int_{\mathscr{P}^{(k)}(\theta)} \mathrm{e}^{-z(f(t)-f(t^{(k)}))}g(t)\,\mathrm{d}t. \tag{1.38}$$

The cube root is defined to be positive on the positive real line and is defined by analytic continuation elsewhere. The notation of adjacency is identical to the quadratic case, except that θ here needs to change by 6π before $\mathscr{P}^{(k)}(\theta)$ returns to its original state. We restrict θ to an interval of the form (1.23) and shall assume that $f(t)$ and $g(t)$ grow sufficiently rapidly at infinity so that the integral (1.37) converges for all values of θ in that interval.

The local behaviour of $f(t)$ at the saddle point $t^{(k)}$ is given by

$$f(t)-f(t^{(k)}) = \frac{1}{6}f'''(t^{(k)})(t-t^{(k)})^3 + \mathcal{O}(|t-t^{(k)}|^4),$$

which suggests the cubic parametrization

$$s^3 = z(f(t)-f(t^{(k)})) \tag{1.39}$$

of the integrand in (1.38) along $\mathscr{P}^{(k)}(\theta)$. The advantage of this parametrization is that the inverse mapping is, at least locally, single-valued. The right-hand side of (1.39) is real and positive if t lies on any of the three steepest descent paths from $t^{(k)}$. Therefore, we may parametrize the steepest descent paths by the lines $\arg s = 0$, $\arg s = \frac{2\pi}{3}$ and $\arg s = \frac{4\pi}{3}$ via (1.39), as long as these contours do not encounter saddles of $f(t)$ different from $t^{(k)}$. However, the association is not purely a matter of choice; it has to preserve the cyclic order of the lines in the s-plane and the phases of the steepest path mod 6π in the t-plane. To parametrize $\mathscr{P}^{(k)}(\theta)$, we choose the positive real line and hence

$$T^{(k/3)}(z) = 3z^{\frac{1}{3}}\int_0^{+\infty} \mathrm{e}^{-s^3}g(t)\frac{\mathrm{d}t}{\mathrm{d}s}\mathrm{d}s = 3\int_0^{+\infty} \mathrm{e}^{-s^3}\frac{3s^2}{z^{\frac{2}{3}}}\frac{g(t(s/z^{\frac{1}{3}}))}{f'(t(s/z^{\frac{1}{3}}))}\mathrm{d}s, \tag{1.40}$$

where $t = t(s/z^{\frac{1}{3}})$ is the unique solution of the equation (1.39) with $t(s/z^{\frac{1}{3}}) \in \mathscr{P}^{(k)}(\theta)$. Since the contour $\mathscr{P}^{(k)}(\theta)$ does not pass through any of the saddle points of $f(t)$ other than $t^{(k)}$, the quantity

$$\frac{3s^2}{z^{\frac{2}{3}}}\frac{g(t(s/z^{\frac{1}{3}}))}{f'(t(s/z^{\frac{1}{3}}))} = \frac{3(f(t(s/z^{\frac{1}{3}})) - f(t^{(k)}))^{\frac{2}{3}}}{f'(t(s/z^{\frac{1}{3}}))} g(t(s/z^{\frac{1}{3}}))$$

is an analytic function of t in a neighbourhood of $\mathscr{P}^{(k)}(\theta)$. (For the analyticity of the factor $(f(t) - f(t^{(k)}))^{\frac{1}{3}}$ in $\Delta^{(k)}$, consider the paragraph after equation (1.42) below.) Whence, according to the residue theorem

$$\begin{aligned}\frac{3(f(t(s/z^{\frac{1}{3}})) - f(t^{(k)}))^{\frac{2}{3}}}{f'(t(s/z^{\frac{1}{3}}))} g(t(s/z^{\frac{1}{3}})) &= \operatorname*{Res}_{t=t(s/z^{\frac{1}{3}})} \frac{g(t)}{(f(t) - f(t^{(k)}))^{\frac{1}{3}} - s/z^{\frac{1}{3}}} \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{(t(s/z^{\frac{1}{3}})+)} \frac{g(t)}{(f(t) - f(t^{(k)}))^{\frac{1}{3}} - s/z^{\frac{1}{3}}} \mathrm{d}t.\end{aligned}$$

Substituting this expression into (1.40) leads to an alternative representation for the integral $T^{(k/3)}(z)$ in the form

$$T^{(k/3)}(z) = \int_0^{+\infty} \mathrm{e}^{-s^3} \frac{3}{2\pi \mathrm{i}} \oint_{\Gamma^{(k)}(\theta)} \frac{g(t)}{(f(t) - f(t^{(k)}))^{\frac{1}{3}} - s/z^{\frac{1}{3}}} \mathrm{d}t\mathrm{d}s, \tag{1.41}$$

where the infinite contour $\Gamma^{(k)}(\theta)$ encircles the path $\mathscr{P}^{(k)}(\theta)$ in the positive direction within $\Delta^{(k)}$. This integral will exist provided that $g(t)/f^{\frac{1}{3}}(t)$ decays sufficiently rapidly at infinity in $\Delta^{(k)}$. Otherwise, we can take $\Gamma^{(k)}(\theta)$ to be a finite loop contour surrounding $t(s/z^{\frac{1}{3}})$ and consider the limit

$$\lim_{S \to +\infty} \int_0^{S} \mathrm{e}^{-s^3} \frac{3}{2\pi \mathrm{i}} \oint_{\Gamma^{(k)}(\theta)} \frac{g(t)}{(f(t) - f(t^{(k)}))^{\frac{1}{3}} - s/z^{\frac{1}{3}}} \mathrm{d}t\mathrm{d}s. \tag{1.42}$$

The function $f(t) - f(t^{(k)})$ has a triple zero at $t = t^{(k)}$ and is non-zero elsewhere in $\Delta^{(k)}$. Also, the least period of $\mathscr{P}^{(k)}(\theta)$ is 6π, and therefore we can define the cube root so that $(f(t) - f(t^{(k)}))^{\frac{1}{3}}$ in (1.41) is a single-valued analytic function of t in $\Delta^{(k)}$. The correct choice of the branch of $(f(t) - f(t^{(k)}))^{\frac{1}{3}}$ is determined by the requirement that s is real and positive on $\mathscr{P}^{(k)}(\theta)$, which can be satisfied by setting $\arg[(f(t) - f(t^{(k)}))^{\frac{1}{3}}] = -\frac{\theta}{3}$ for $t \in \mathscr{P}^{(k)}(\theta)$. With this definition of the cube root, s has angle $\frac{2\pi}{3}$ on $\mathscr{P}^{(k)}(\theta + 2\pi)$ and has angle $\frac{4\pi}{3}$ on $\mathscr{P}^{(k)}(\theta + 4\pi)$, the other two steepest descent paths from $t^{(k)}$.

Next, we use the expression (1.7) to expand the denominator in (1.41) in powers of $s/[z(f(t)-f(t^{(k)}))]^{\frac{1}{3}}$, giving us

$$T^{(k/3)}(z)=\sum_{n=0}^{N-1}\frac{1}{z^{\frac{n}{3}}}\int_0^{+\infty}s^n\mathrm{e}^{-s^3}\frac{3}{2\pi\mathrm{i}}\oint_{\Gamma^{(k)}(\theta)}\frac{g(t)}{\left(f(t)-f(t^{(k)})\right)^{\frac{n+1}{3}}}\mathrm{d}t\mathrm{d}s+R_N^{(k/3)}(z)$$

with the remainder

$$\begin{aligned}R_N^{(k/3)}(z)&=\frac{3}{2\pi\mathrm{i}z^{\frac{N}{3}}}\int_0^{+\infty}s^N\mathrm{e}^{-s^3}\\&\quad\times\oint_{\Gamma^{(k)}(\theta)}\frac{g(t)}{\left(f(t)-f(t^{(k)})\right)^{\frac{N+1}{3}}}\frac{\mathrm{d}t}{1-s/[z(f(t)-f(t^{(k)}))]^{\frac{1}{3}}}\mathrm{d}s.\end{aligned}\tag{1.43}$$

Again, a limiting process is used in (1.43) if necessary. The contour $\Gamma^{(k)}(\theta)$ in the sum is shrunk to a small closed contour encircling the saddle point $t^{(k)}$, and we arrive at

$$T^{(k/3)}(z)=\sum_{n=0}^{N-1}\frac{a_n^{(k/3)}}{z^{\frac{n}{3}}}+R_N^{(k/3)}(z),\tag{1.44}$$

where the coefficients are given by

$$a_n^{(k/3)}=\frac{\Gamma\left(\frac{n+1}{3}\right)}{2\pi\mathrm{i}}\oint_{(t^{(k)}+)}\frac{g(t)}{\left(f(t)-f(t^{(k)})\right)^{\frac{n+1}{3}}}\mathrm{d}t\tag{1.45}$$

$$=\frac{\Gamma\left(\frac{n+1}{3}\right)}{\Gamma(n+1)}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(g(t)\left(\frac{(t-t^{(k)})^3}{f(t)-f(t^{(k)})}\right)^{\frac{n+1}{3}}\right)\right]_{t=t^{(k)}}.\tag{1.46}$$

If we neglect the remainder term $R_N^{(k/3)}(z)$ in (1.44) and formally extend the sum to infinity, the result becomes the known asymptotic expansion of an integral with cubic endpoint (cf. [35, eq. (29), p. 125]). A representation equivalent to (1.45) was derived by Copson [20, p. 69] and Dingle [35, eq. (28), p. 125], while the expression (1.46) is again a special case of Perron's formula [70].

The contour $\Gamma^{(k)}(\theta)$ in (1.43) is now deformed by expanding it onto the boundary of $\Delta^{(k)}$. We assume that the set of saddle points which are adjacent to $t^{(k)}$ is non-empty and finite. Under this assumption, the boundary of $\Delta^{(k)}$ can be written as a union $\bigcup_m \mathscr{C}^{(m)}(-\sigma_{km})$, where $\mathscr{C}^{(m)}(-\sigma_{km})$ is one of the three doubly infinite steepest descent paths through the adjacent saddle $t^{(m)}$. It is supposed that each of these contours contains only one saddle point. The third steepest descent path from $t^{(m)}$ is always external to the domain $\Delta^{(k)}$ (for otherwise there were two different paths of steepest descent emanating from regular points). The expansion process is legitimate as long as (i) $f(t)$ and $g(t)$ have no singularities in $\Delta^{(k)}$, (ii) the quantity $g(t)/f^{\frac{N+1}{3}}(t)$ decreases in magnitude sufficiently rapidly at infinity in the domain $\Delta^{(k)}$, and (iii) the denominator $1 - s/[z(f(t) - f(t^{(k)}))]^{\frac{1}{3}}$ has no zeros in the region through which the contour $\Gamma^{(k)}(\theta)$ is deformed. The first condition is satisfied because of the assumption that $f(t)$ and $g(t)$ are analytic in $\Delta^{(k)}$. To meet the second condition, we require that $g(t)/f^{\frac{N+1}{3}}(t) = o(|t|^{-1})$ as $t \to \infty$ in $\Delta^{(k)}$. The third condition is fulfilled by an argument similar to that in the linear endpoint case. With these provisos, we have

$$R_N^{(k/3)}(z) = \frac{3}{2\pi i z^{\frac{N}{3}}} \int_0^{+\infty} s^N e^{-s^3} \times \sum_m (-1)^{\gamma_{km}} \int_{\mathscr{C}^{(m)}} \frac{g(t)}{(f(t) - f(t^{(k)}))^{\frac{N+1}{3}}} \frac{dt}{1 - s/[z(f(t) - f(t^{(k)}))]^{\frac{1}{3}}} ds, \tag{1.47}$$

where $\mathscr{C}^{(m)} = \mathscr{C}^{(m)}(-\sigma_{km})$ and the summation is over each of the adjacent contours. The orientation anomalies γ_{km} are defined analogously to the linear and quadratic endpoint cases.

Following the linear and quadratic cases, we now impose simple conditions to guarantee the absolute convergence of the double integrals in (1.47). We define the singulants $\mathcal{F}_{km}$ analogously to the case of quadratic endpoint and introduce the change of integration variable from t to v by

$$f(t) - f(t^{(k)}) = v e^{i\sigma_{km}}, \tag{1.48}$$

where $v \geq |\mathcal{F}_{km}|$. Since $e^{-i\sigma_{km}}(f(t) - f(t^{(k)}))$ is a monotonic function of t on each half of the contour $\mathscr{C}^{(m)}(-\sigma_{km})$ before and after the saddle point $t^{(m)}$, corresponding to each value of v, there are two values of t, say $t_\pm(v)$, that satisfy (1.48). The convergence of the double integrals in (1.47) will be ensured provided that the real single integrals

$$\int_{|\mathcal{F}_{km}|}^{+\infty} \frac{1}{v^{\frac{N+1}{3}}} \left| \frac{g(t_\pm(v))}{f'(t_\pm(v))} \right| dv \tag{1.49}$$

exist (cf. equations (1.16) and (1.33)). Henceforth, we assume that the integrals in (1.49) exist for each of the adjacent contours.

On each of the contours $\mathscr{C}^{(m)}(-\sigma_{km})$ in (1.47), we perform the change of variable from s and t to u and t via

$$s^3 = e^{-i\sigma_{km}}(f(t) - f(t^{(k)}))u = |\mathcal{F}_{km}| u + e^{-i\sigma_{km}}(f(t) - f(t^{(m)}))u$$

to obtain

$$R_N^{(k/3)}(z) = \frac{1}{2\pi \mathrm{i} z^{\frac{N}{3}}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{3}\sigma_{km}}} \int_0^{+\infty} \frac{u^{\frac{N-2}{3}} \mathrm{e}^{-|\mathcal{F}_{km}|u}}{1-(u/z\mathrm{e}^{\mathrm{i}\sigma_{km}})^{\frac{1}{3}}} \times \int_{\mathscr{C}^{(m)}} \mathrm{e}^{-\mathrm{e}^{-\mathrm{i}\sigma_{km}}(f(t)-f(t^{(m)}))u} g(t)\,\mathrm{d}t\mathrm{d}u. \tag{1.50}$$

This change of variable is permitted because the infinite double integrals in (1.47) are absolutely convergent, which is a consequence of the requirement that the integrals (1.49) exist. By analogy with (1.38), we define

$$T^{(2m/3)}(w) \stackrel{\text{def}}{=} w^{\frac{1}{3}} \int_{\mathscr{C}^{(m)}(\arg w)} \mathrm{e}^{-w(f(t)-f(t^{(m)}))} g(t)\,\mathrm{d}t \tag{1.51}$$

with $\arg(w^{\frac{1}{3}}) \stackrel{\text{def}}{=} \frac{1}{3}\arg w$. This is the slowly varying part of an integral over a doubly infinite contour through a second-order saddle point. With this notation, (1.50) becomes

$$R_N^{(k/3)}(z) = \frac{1}{2\pi \mathrm{i} z^{\frac{N}{3}}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathrm{e}^{\mathrm{i}\frac{N}{3}\sigma_{km}}} \int_0^{+\infty} \frac{u^{\frac{N}{3}-1} \mathrm{e}^{-|\mathcal{F}_{km}|u}}{1-(u/z\mathrm{e}^{\mathrm{i}\sigma_{km}})^{\frac{1}{3}}} T^{(2m/3)}(u\mathrm{e}^{-\mathrm{i}\sigma_{km}})\mathrm{d}u. \tag{1.52}$$

This is an exact form of the remainder in the asymptotic expansion of an integral with cubic endpoint, expressed as a sum involving integrals through the adjacent saddles. Equations (1.44) and (1.52) together provide the exact resurgence relation for $T^{(k/3)}(z)$.

1.2 The resurgence properties of integrals with saddles

In this section, we investigate the resurgence properties of integrals over doubly infinite contours passing through saddles of the phase function. We consider the cases of first- and second-order saddle points separately.

1.2.1 Quadratic dependence at the saddle point

We study the resurgence properties of integrals of the form

$$I^{(k)}(z) = \int_{\mathscr{C}^{(k)}(\theta)} \mathrm{e}^{-zf(t)} g(t)\,\mathrm{d}t, \tag{1.53}$$

where $\mathscr{C}^{(k)}(\theta)$ is the doubly infinite path of steepest descent passing through the first-order saddle point $t^{(k)}$ of $f(t)$ along the two valleys of $\mathfrak{Re}[-\mathrm{e}^{\mathrm{i}\theta}(f(t) -$

$f(t^{(k)}))]$. The functions $f(t)$ and $g(t)$ are assumed to be analytic in a domain $\Delta^{(k)}$, whose closure is the set of all the points that can be reached by a path of steepest descent which emanates from $t^{(k)}$. We suppose further that $|f(t)| \to +\infty$ in $\Delta^{(k)}$, and $f(t)$ has several other first-order saddle points in the complex t-plane situated at $t = t^{(p)}$ and indexed by $p \in \mathbb{N}$.

For convenience, we define and consider the slowly varying part of the integral (1.53),

$$T^{(k)}(z) \stackrel{\text{def}}{=} z^{\frac{1}{2}} \mathrm{e}^{zf(t^{(k)})} I^{(k)}(z) = z^{\frac{1}{2}} \int_{\mathscr{C}^{(k)}(\theta)} \mathrm{e}^{-z(f(t)-f(t^{(k)}))} g(t)\,\mathrm{d}t, \tag{1.54}$$

where the square root is defined to be positive on the positive real line and is defined by analytic continuation elsewhere (cf. equation (1.19)). After a phase change of 2π, the orientation of $\mathscr{C}^{(k)}(\theta)$ reverses, and $I^{(k)}(z)$ changes sign. Thus, the integral $I^{(k)}(z)$ is double-valued and the function $T^{(k)}(z)$ is single-valued. With the notation of Subsection 1.1.2, the contour of integration can be written as $\mathscr{C}^{(k)}(\theta) = \mathscr{P}^{(k)}(\theta) \cup \mathscr{P}^{(k)}(\theta + 2\pi)$. We choose the orientation of these paths so that both $\mathscr{P}^{(k)}(\theta)$ and $\mathscr{P}^{(k)}(\theta + 2\pi)$ lead away from $t^{(k)}$, and the orientation of $\mathscr{C}^{(k)}(\theta)$ is the same as that of $\mathscr{P}^{(k)}(\theta)$. With this convention and the definition (1.22), we may write

$$\begin{aligned} T^{(k)}(z) &= z^{\frac{1}{2}} \int_{\mathscr{P}^{(k)}(\theta)} \mathrm{e}^{-z(f(t)-f(t^{(k)}))} g(t)\,\mathrm{d}t - z^{\frac{1}{2}} \int_{\mathscr{P}^{(k)}(\theta+2\pi)} \mathrm{e}^{-z(f(t)-f(t^{(k)}))} g(t)\,\mathrm{d}t \\ &= \frac{1}{2}\left(T^{(k/2)}(z) + T^{(k/2)}\left(z\mathrm{e}^{2\pi \mathrm{i}}\right)\right). \end{aligned} \tag{1.55}$$

Observe that the sets of adjacent saddles are identical for the integrals $T^{(k/2)}(z)$ and $T^{(k/2)}\left(z\mathrm{e}^{2\pi \mathrm{i}}\right)$. We assume that neither $\mathscr{P}^{(k)}(\theta)$ nor $\mathscr{P}^{(k)}(\theta + 2\pi)$ encounter any of the saddle points of $f(t)$ different from $t^{(k)}$, and that θ is restricted to an interval

$$-\sigma_{km_1} < \theta < -\sigma_{km_2}, \tag{1.56}$$

where $t^{(m_1)}$ and $t^{(m_2)}$ are adjacent to $t^{(k)}$. We suppose that $f(t)$ and $g(t)$ increase in magnitude sufficiently rapidly at infinity so that the integral (1.53) converges for all values of θ in the interval (1.56). Since the orientation of $\mathscr{C}^{(k)}(\theta)$ reverses after a phase change of 2π, we have

$$0 < \sigma_{km_1} - \sigma_{km_2} \leq 2\pi,$$

and $\sigma_{km_1} - \sigma_{km_2} = 2\pi$ occurs, for example, when there is only one saddle point that is adjacent to $t^{(k)}$.

The resurgence relation for $T^{(k)}(z)$ can be written down directly from the known results for $T^{(k/2)}(z)$. We use (1.28) with $2N$ in place of N to expand the functions $T^{(k/2)}(z)$ and $T^{(k/2)}(ze^{2\pi i})$ in (1.55), which, after some simplification, yields

$$T^{(k)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(k)}}{z^n} + R_N^{(k)}(z), \tag{1.57}$$

where the coefficients $a_n^{(k)}$ are given by

$$\begin{aligned} a_n^{(k)} = a_{2n}^{(k/2)} &= \frac{\Gamma\left(n+\frac{1}{2}\right)}{2\pi i} \oint_{(t^{(k)}+)} \frac{g(t)}{(f(t)-f(t^{(k)}))^{n+\frac{1}{2}}} \mathrm{d}t \\ &= \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)} \left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} \left(g(t) \left(\frac{(t-t^{(k)})^2}{f(t)-f(t^{(k)})} \right)^{n+\frac{1}{2}} \right) \right]_{t=t^{(k)}} . \end{aligned} \tag{1.58}$$

The ratio of the two gamma functions in front of the second expression may be simplified to $\sqrt{\pi}2^{-2n}/\Gamma(n+1)$. The remainder term $R_N^{(k)}(z)$ is given in terms of the remainders of the expansions for $T^{(k/2)}(z)$ and $T^{(k/2)}(ze^{2\pi i})$ as

$$R_N^{(k)}(z) = \frac{1}{2}\left(R_{2N}^{(k/2)}(z) + R_{2N}^{(k/2)}(ze^{2\pi i})\right).$$

The final result follows by employing the representation (1.36) for the right-hand side and combining the terms corresponding to the same adjacent saddles. This step is justified provided that (i) the set of saddle points which are adjacent to $t^{(k)}$ is non-empty and finite, (ii) each of the adjacent contours contains only one saddle point, (iii) $g(t)/f^{N+\frac{1}{2}}(t) = o(|t|^{-1})$ as $t \to \infty$ in $\Delta^{(k)}$, and (iv) the integrals in (1.33), with $2N$ in place of N, exist for each of the adjacent contours. Under these conditions, we have

$$R_N^{(k)}(z) = \frac{1}{2\pi i z^N} \sum_m \frac{(-1)^{\gamma_{km}}}{e^{iN\sigma_{km}}} \int_0^{+\infty} \frac{u^{N-1} e^{-|\mathcal{F}_{km}|u}}{1-u/ze^{i\sigma_{km}}} T^{(m)}\left(ue^{-i\sigma_{km}}\right) \mathrm{d}u, \tag{1.59}$$

which is the required representation of the remainder, expressed as a sum involving integrals through the adjacent saddles. Equations (1.57) and (1.59) together yield the exact resurgence formula for $T^{(k)}(z)$.

1.2.2 Cubic dependence at the saddle point

Here we consider the resurgence properties of integrals of the type

$$I^{(2k/3)}(z) = \int_{\mathscr{C}^{(k)}(\theta)} e^{-zf(t)} g(t)\, \mathrm{d}t, \tag{1.60}$$

where $\mathscr{C}^{(k)}(\theta)$ is one of the three doubly infinite paths of steepest descent passing through the second-order saddle point $t^{(k)}$ of $f(t)$ along the two valleys of $\mathfrak{Re}[-\mathrm{e}^{\mathrm{i}\theta}(f(t)-f(t^{(k)}))]$. The functions $f(t)$ and $g(t)$ are supposed to be analytic in a domain $\Delta^{(k)}$, whose closure is the set of all the points that can be reached by a path of steepest descent issuing from $t^{(k)}$. Furthermore, we assume that $|f(t)| \to +\infty$ in $\Delta^{(k)}$, and $f(t)$ has several other second-order saddle points in the complex t-plane at $t = t^{(p)}$ labelled by $p \in \mathbb{N}$.

Instead of the integral (1.60), we consider its slowly varying part, given by

$$T^{(2k/3)}(z) \stackrel{\text{def}}{=} z^{\frac{1}{3}} \mathrm{e}^{zf(t^{(k)})} I^{(2k/3)}(z) = z^{\frac{1}{3}} \int_{\mathscr{C}^{(k)}(\theta)} \mathrm{e}^{-z(f(t)-f(t^{(k)}))} g(t)\,\mathrm{d}t, \tag{1.61}$$

where the cube root is defined to be positive when $\theta = 0$, and it is defined elsewhere by analytic continuation (cf. equation (1.51)). With the notation of Subsection 1.1.3, the contour of integration can be written as either $\mathscr{C}^{(k)}(\theta) = \mathscr{P}^{(k)}(\theta) \cup \mathscr{P}^{(k)}(\theta + 2\pi)$, $\mathscr{C}^{(k)}(\theta) = \mathscr{P}^{(k)}(\theta) \cup \mathscr{P}^{(k)}(\theta + 4\pi)$ or $\mathscr{C}^{(k)}(\theta) = \mathscr{P}^{(k)}(\theta + 2\pi) \cup \mathscr{P}^{(k)}(\theta + 4\pi)$, depending on the choice of the doubly infinite steepest descent path in (1.60). We consider only the first case, the other two can be treated in a similar manner. The orientations of the contours are chosen so that both $\mathscr{P}^{(k)}(\theta)$ and $\mathscr{P}^{(k)}(\theta + 2\pi)$ lead away from $t^{(k)}$, and the orientation of $\mathscr{C}^{(k)}(\theta)$ is the same as that of $\mathscr{P}^{(k)}(\theta)$. With this convention and the definition (1.38), we can write

$$\begin{aligned} T^{(2k/3)}(z) &= z^{\frac{1}{3}} \int_{\mathscr{P}^{(k)}(\theta)} \mathrm{e}^{-z(f(t)-f(t^{(k)}))} g(t)\,\mathrm{d}t - z^{\frac{1}{3}} \int_{\mathscr{P}^{(k)}(\theta+2\pi)} \mathrm{e}^{-z(f(t)-f(t^{(k)}))} g(t)\,\mathrm{d}t \\ &= \frac{1}{3}\left(T^{(k/3)}(z) - \mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}} T^{(k/3)}(z\mathrm{e}^{2\pi\mathrm{i}})\right). \end{aligned} \tag{1.62}$$

We assume that the integration contours $\mathscr{P}^{(k)}(\theta)$ and $\mathscr{P}^{(k)}(\theta + 2\pi)$ do not encounter any of the saddle points of $f(t)$ other than $t^{(k)}$, and that θ is restricted to an interval of the form (1.56). Furthermore, we suppose that $f(t)$ and $g(t)$ increase in magnitude sufficiently rapidly at infinity so that the integral (1.60) converges for all values of θ in that interval.

We now apply (1.44) to expand the integrals $T^{(k/3)}(z)$ and $T^{(k/3)}(z\mathrm{e}^{2\pi\mathrm{i}})$ in (1.62), which, after some simplification, gives

$$T^{(2k/3)}(z) = -\frac{2}{3\mathrm{i}} \sum_{n=0}^{N-1} \mathrm{e}^{-\frac{\pi(n+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(n+1)}{3}\right) \frac{a_n^{(k/3)}}{z^{\frac{n}{3}}} + R_N^{(2k/3)}(z), \tag{1.63}$$

with the remainder

$$R_N^{(2k/3)}(z) = \frac{1}{3}\left(R_N^{(k/3)}(z) - \mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}} R_N^{(k/3)}(z\mathrm{e}^{2\pi\mathrm{i}})\right). \tag{1.64}$$

The resurgence relation for $T^{(2k/3)}(z)$ follows by applying the representation (1.52) for the right-hand side of (1.64) and combining the terms which correspond to the same adjacent saddles. This step is justified provided that (i) the set of saddle points that are adjacent to $t^{(k)}$ is non-empty and finite, (ii) each of the adjacent contours contains only one saddle point, (iii) $g(t)/f^{\frac{N+1}{3}}(t) = o(|t|^{-1})$ as $t \to \infty$ in $\Delta^{(k)}$, and (iv) the integrals in (1.49) exist for each of the adjacent contours. With these assumptions, we find

$$R_N^{(2k/3)}(z) = \frac{1}{6\pi i z^{\frac{N}{3}}} \sum_m \frac{(-1)^{\gamma_{km}}}{e^{i\frac{N}{3}\sigma_{km}}} \int_0^{+\infty} u^{\frac{N}{3}-1} e^{-|\mathcal{F}_{km}|u} \times \left(\frac{1}{1-(u/ze^{i\sigma_{km}})^{\frac{1}{3}}} - \frac{e^{-\frac{2\pi(N+1)}{3}i}}{1-(u/ze^{i\sigma_{km}})^{\frac{1}{3}} e^{-\frac{2\pi}{3}i}} \right) T^{(2m/3)}(ue^{-i\sigma_{km}}) du, \tag{1.65}$$

bearing in mind that the sets of adjacent saddles are identical for the functions $T^{(k/3)}(z)$ and $T^{(k/3)}(ze^{2\pi i})$. Equations (1.63) and (1.65) together form the exact resurgence relation for $T^{(2k/3)}(z)$.

1.3 Asymptotic expansions for the late coefficients

The aim of this section is to investigate the asymptotic behaviour of the coefficients $a_n^{(e)}$, $a_n^{(k/2)}$ and $a_n^{(k/3)}$ introduced in Section 1.1. It is well known that the early coefficients are determined by the local properties of the integrand at the finite endpoint of the integration contour. However, as we shall see very soon, the form of the coefficients when n gets large is dictated by the global properties of the integrand, more precisely by the behaviour of the integrand at all the adjacent saddle points.

There has been a recent interest in finding explicit formulae for these coefficients (see [56], [70], [119] and [120]). In general, the form of the early coefficients gets extremely complicated rapidly (see, e.g., [98, eq. (1.2.15), pp. 11–12]). Conversely, the asymptotics of the late coefficients admits a surprisingly simple and universal form and hence provides an efficient way of computing $a_n^{(e)}$, $a_n^{(k/2)}$ and $a_n^{(k/3)}$ for large n. Furthermore, if the set of saddle points of the phase function is finite, these asymptotic approximations for the late coefficients can be used to determine, in a purely algebraic manner, which saddles are adjacent to the endpoint and which are not. For a detailed discussion of this method, together with examples, the reader is referred to the paper of Howls [46].

1.3.1 Asymptotic evaluation of the coefficients $a_n^{(e)}$

Throughout this subsection, we suppose that all the conditions under which formula (1.20) holds true are satisfied. Suppose we put $N+1$ in place of N in (1.10), and consider the difference between the two right-hand sides. Using (1.20), one immediately infers that

$$a_n^{(e)} = \frac{1}{2\pi \mathrm{i}} \sum_m \frac{(-1)^{\gamma_{em}}}{\mathrm{e}^{\mathrm{i}\left(n+\frac{1}{2}\right)\sigma_{em}}} \int_0^{+\infty} u^{n-\frac{1}{2}} \mathrm{e}^{-|\mathcal{F}_{em}|u} T^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{em}}\right) \mathrm{d}u, \tag{1.66}$$

where we have written n in place of N. This is a third representation for the coefficients $a_n^{(e)}$, different from (1.11) or (1.12). Now, under appropriate conditions, the functions $T^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{em}}\right)$ can be expanded into truncated asymptotic power series in inverse powers of u with remainders

$$T^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{em}}\right) = \sum_{r=0}^{N_m-1} \frac{a_r^{(m)}}{u^r} \mathrm{e}^{\mathrm{i}r\sigma_{em}} + R_{N_m}^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{em}}\right) \tag{1.67}$$

(cf. equation (1.57)). Substituting these expansions into (1.66) and integrating term-by-term gives us

$$a_n^{(e)} = \frac{1}{2\pi \mathrm{i}} \sum_m \frac{(-1)^{\gamma_{em}}}{\mathcal{F}_{em}^{n+\frac{1}{2}}} \left(\sum_{r=0}^{N_m-1} a_r^{(m)} \mathcal{F}_{em}^r \Gamma\left(n-r+\frac{1}{2}\right) + A_{N_m}^{(e)}(n) \right), \tag{1.68}$$

with the remainder terms

$$A_{N_m}^{(e)}(n) = |\mathcal{F}_{em}|^{n+\frac{1}{2}} \int_0^{+\infty} u^{n-\frac{1}{2}} \mathrm{e}^{-|\mathcal{F}_{em}|u} R_{N_m}^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{em}}\right) \mathrm{d}u.$$

To ensure the convergence of these integrals, we require $N_m \leq n$. Formula (1.68) relates the late coefficients of the asymptotic power series of an integral with linear endpoint to the early coefficients of the asymptotic power series of integrals over doubly infinite contours passing through the adjacent saddles. Expansions of type (1.68) are called inverse factorial series in the literature. Numerically, their character is similar to the character of asymptotic power series, because the consecutive gamma functions decrease asymptotically by a factor n. In most applications, it is possible to establish explicit bounds for the remainders $A_{N_m}^{(e)}(n)$. With these bounds in hand, we can use (1.68) for the numerical computation of the coefficients $a_n^{(e)}$ for large n.

If there is a value of m, say $\tilde{m}$, for which $|\mathcal{F}_{e\tilde{m}}|$ is less than $|\mathcal{F}_{em}|$ for all the other adjacent saddles, then at leading order

$$a_n^{(e)} \sim (-1)^{\gamma_{e\tilde{m}}} \frac{a_0^{(\tilde{m})}}{2\pi \mathrm{i}} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\mathcal{F}_{e\tilde{m}}^{n+\frac{1}{2}}}$$

as $n \to +\infty$.[5] This reveals the characteristic "factorial divided by a power" growth of the late coefficients of the asymptotic expansion of an integral with linear endpoint discovered originally by Dingle using the Darboux theorem. Dingle called $\mathcal{F}_{e\tilde{m}}$ the "chief singulant". In addition, he gave a complete expansion which might also be derived from (1.68); one has to take the contribution only from the adjacent saddle which corresponds to the chief singulant, neglect the remainder term and formally extend the sum over r to infinity (see [35, eq. (13), p. 145]).

Dingle also suggested another way of deriving approximations for the late coefficients by using the method of steepest descents itself. In the remaining part of this subsection, we provide a rigorous treatment of this approach. We begin by deforming the small loop around e in (1.11) out to the boundary of $\Delta^{(e)}$ and hence obtain

$$\begin{aligned} a_n^{(e)} &= \frac{\Gamma(n+1)}{2\pi \mathrm{i}} \sum_m (-1)^{\gamma_{em}} \int_{\mathscr{C}^{(m)}} \frac{g(t)}{(f(t)-f(e))^{n+1}} \mathrm{d}t \\ &= \frac{\Gamma(n+1)}{2\pi \mathrm{i}} \sum_m \frac{(-1)^{\gamma_{em}}}{\mathcal{F}_{em}^{n+1}} \int_{\mathscr{C}^{(m)}} \mathrm{e}^{-(n+1)(h(t)-h(t^{(m)}))} g(t)\, \mathrm{d}t, \end{aligned} \tag{1.69}$$

where we have taken $h(t) = \log(f(t) - f(e))$. We would like to apply the method of steepest descents for the integrals in (1.69). The saddle points of $h(t)$ occur at the zeros of $f'(t)$ that is, they coincide with the saddle points of $f(t)$. Since

$$h(t) - h(t^{(m)}) = \log\left(1 + \frac{f(t) - f(t^{(m)})}{\mathcal{F}_{em}}\right),$$

for each value of m in (1.69), the quantity $h(t) - h(t^{(m)})$ is real and positive for values of t on each half of the contour $\mathscr{C}^{(m)}(-\sigma_{em})$ before and after the saddle point $t^{(m)}$. Thus, the adjacent contours are paths of steepest descent for the integrands in (1.69). Furthermore, each of these contours contains exactly one saddle point of $h(t)$. Therefore, we can apply the method of steepest descents directly to the integrals in (1.69) to obtain the complete asymptotic expansion

$$a_n^{(e)} \sim \frac{1}{2\pi \mathrm{i}} \frac{\Gamma(n+1)}{(n+1)^{\frac{1}{2}}} \sum_m \frac{(-1)^{\gamma_{em}}}{\mathcal{F}_{em}^{n+1}} \sum_{r=0}^{\infty} \frac{b_r^{(m)}}{(n+1)^r} \tag{1.70}$$

[5]Here we assume that $a_0^{(\tilde{m})} \neq 0$, or equivalently that $g(t^{(\tilde{m})}) \neq 0$.

as $n \to +\infty$. The leading coefficients are as follows

$$b_0^{(m)} = \left(\frac{2\pi\mathcal{F}_{em}}{f''}\right)^{\frac{1}{2}} g,$$
$$b_1^{(m)} = \left(\frac{2\pi\mathcal{F}_{em}}{f''}\right)^{\frac{1}{2}} \frac{9f''^3 g + \mathcal{F}_{em}\left(5f'''^2 g - 3f''f''''g - 12f''f'''g' + 12f''^2 g''\right)}{24f''^3},$$

where the various derivatives of $f(t)$ and $g(t)$ are evaluated at the adjacent saddle points $t^{(m)}$. The expressions for the higher coefficients are successively more complicated. The correct branch of the square root in forming these coefficients is specified as follows. As we remarked above, the quantity $h(t) - h(t^{(m)})$ in (1.69) is real. Moreover, as $n \to +\infty$ the value of the integral in (1.69) becomes gradually dominated by the local behaviour of the integrand at the saddle point $t^{(m)}$. Hence, the correct branch of the square root must be chosen so that $\arg((\mathcal{F}_{em})^{\frac{1}{2}}) - \arg(f''^{\frac{1}{2}}(t^{(m)}))$ equals the angle which the tangential direction vector at $t = t^{(m)}$ forms with the positive real axis, taking into consideration the direction in which the contour $\mathscr{C}^{(m)}(-\sigma_{em})$ is traversed. Our result (1.70) is in agreement with that of Dingle [35, exer. 11, pp. 153–154], provided that only the contribution from the saddle corresponding to the chief singulant (which is supposed to be unique) is taken into account. Alternatively, (1.70) can be derived directly from (1.68) by introducing the large-n expansions

$$\frac{\Gamma\left(n - r + \frac{1}{2}\right)}{\Gamma(n+1)} \sim \frac{1}{(n+1)^{r+\frac{1}{2}}}\left(1 + \frac{(2r+1)(2r+3)}{8(n+1)} + \frac{(2r+1)(2r+3)(2r+5)(6r+5)}{384(n+1)^2} + \cdots\right)$$

and rearranging the inner sums in descending powers of $n+1$. This alternative approach also provides us with expressions for the coefficients $b_r^{(m)}$ involving the singulant $\mathcal{F}_{em}$ and the coefficients $a_r^{(m)}$.

Although the expansion (1.70) might initially appear to be simpler in form, using the inverse factorial series for approximating late coefficients is a more natural choice, due to their having explicit remainders and the direct relation of their terms to the coefficients of the asymptotic power series of integrals through the adjacent saddles.

1.3.2 Asymptotic evaluation of the coefficients $a_n^{(k/2)}$

Throughout this subsection, we assume that all the conditions under which the representation (1.36) holds are satisfied. Similarly to the formula (1.66) for $a_n^{(e)}$, we can express the coefficients $a_n^{(k/2)}$ as

$$a_n^{(k/2)} = \frac{1}{2\pi \mathrm{i}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathrm{e}^{\mathrm{i}\frac{n}{2}\sigma_{km}}} \int_0^{+\infty} u^{\frac{n}{2}-1} \mathrm{e}^{-|\mathcal{F}_{km}|u} T^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right) \mathrm{d}u, \tag{1.71}$$

by making use of (1.36). This is a third representation for the coefficients $a_n^{(k/2)}$, different from (1.29) or (1.30). Now, under suitable conditions, the integrals $T^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right)$ can be expanded into truncated asymptotic power series in descending powers of u with remainder terms,

$$T^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right) = \sum_{r=0}^{N_m-1} \frac{a_r^{(m)}}{u^r} \mathrm{e}^{\mathrm{i}r\sigma_{km}} + R_{N_m}^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right). \tag{1.72}$$

We substitute these expansions into (1.71) and integrate term-by-term to obtain the inverse factorial expansion

$$a_n^{(k/2)} = \frac{1}{2\pi \mathrm{i}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathcal{F}_{km}^{\frac{n}{2}}} \left(\sum_{r=0}^{N_m-1} a_r^{(m)} \mathcal{F}_{km}^r \Gamma\left(\frac{n}{2} - r\right) + A_{N_m}^{(k/2)}(n) \right), \tag{1.73}$$

with the remainders

$$A_{N_m}^{(k/2)}(n) = |\mathcal{F}_{km}|^{\frac{n}{2}} \int_0^{+\infty} u^{\frac{n}{2}-1} \mathrm{e}^{-|\mathcal{F}_{km}|u} R_{N_m}^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right) \mathrm{d}u.$$

To insure the convergence of these integrals, we require $2N_m < n$. Formula (1.73) links the late coefficients of the asymptotic power series of an integral with quadratic endpoint to the early coefficients of the asymptotic power series of integrals over doubly infinite contours passing through the adjacent saddles. With suitable bounds on the remainders $A_{N_m}^{(k/2)}(n)$, the result (1.73) provides an effective way of calculating numerically the coefficients $a_n^{(k/2)}$ for large values of n. Equation (1.73) is the full and rigorous form of Dingle's formal expansion for the late coefficients in the asymptotic power series of integrals with quadratic endpoints (see [35, eq. (12), p. 145]).

If there is a value $\widetilde{m}$ of m, for which $|\mathcal{F}_{k\widetilde{m}}|$ is less than $|\mathcal{F}_{km}|$ for all the other adjacent saddles, then at leading order

$$a_n^{(k/2)} \sim (-1)^{\gamma_{k\widetilde{m}}} \frac{a_0^{(\widetilde{m})}}{2\pi \mathrm{i}} \frac{\Gamma\left(\frac{n}{2}\right)}{\mathcal{F}_{k\widetilde{m}}^{\frac{n}{2}}}$$

as $n \to +\infty$, implying that the late coefficients in quadratic endpoint expansions also behave like a "factorial divided by a power".

Boyd [15] also addressed the problem of finding asymptotic approximations for the coefficients $a_n^{(k/2)}$. He considered the particular case when n is even, that is, he studied the coefficients $a_n^{(k)}$ in the asymptotic power series of an integral through a first-order saddle point (cf. equation (1.57)). In what follows, we derive complete expansions in inverse powers of $n+1$ (for n not necessarily even) of which Boyd's approximations are special cases. The derivation is analogous to that of (1.70). We start by expanding the small loop around $t^{(k)}$ in (1.29) to the boundary of the domain $\Delta^{(k)}$ and thus obtain

$$\begin{aligned} a_n^{(k/2)} &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi \mathrm{i}} \sum_m (-1)^{\gamma_{km}} \int_{\mathscr{C}^{(m)}} \frac{g(t)}{\left(f(t) - f(t^{(k)})\right)^{\frac{n+1}{2}}} \mathrm{d}t \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi \mathrm{i}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathcal{F}_{km}^{\frac{n+1}{2}}} \int_{\mathscr{C}^{(m)}} \mathrm{e}^{-\frac{n+1}{2}\left(h(t) - h(t^{(m)})\right)} g(t)\, \mathrm{d}t, \end{aligned} \tag{1.74}$$

where we have denoted $h(t) = \log(f(t) - f(t^{(k)}))$. We observe that the saddle points of $h(t)$ coincide with the saddles of $f(t)$, and that the adjacent contours $\mathscr{C}^{(m)}(-\sigma_{km})$ are paths of steepest descent for the integrands in (1.74). Since each adjacent contour contains exactly one saddle point of $h(t)$, we can apply the method of steepest descents directly to the integrals in (1.74) to obtain the asymptotic expansion

$$a_n^{(k/2)} \sim \frac{1}{2\pi \mathrm{i}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{(n+1)^{\frac{1}{2}}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathcal{F}_{km}^{\frac{n+1}{2}}} \sum_{r=0}^{\infty} \frac{2^{r+\frac{1}{2}} b_r^{(m)}}{(n+1)^r} \tag{1.75}$$

as $n \to +\infty$. The initial two coefficients are given by

$$\begin{aligned} b_0^{(m)} &= \left(\frac{2\pi \mathcal{F}_{km}}{f''}\right)^{\frac{1}{2}} g, \\ b_1^{(m)} &= \left(\frac{2\pi \mathcal{F}_{km}}{f''}\right)^{\frac{1}{2}} \frac{9f''^3 g + \mathcal{F}_{km}\left(5f'''^2 g - 3f'' f'''' g - 12 f'' f''' g' + 12 f''^2 g''\right)}{24 f''^3}, \end{aligned}$$

where $f(t)$, $g(t)$ and their derivatives are evaluated at the adjacent saddle points $t^{(m)}$. The correct branch of the square root in these expressions is specified analogously as for the expansion (1.70). In particular, the leading order approximation for $a_n^{(k)} = a_{2n}^{(k/2)}$ is then found to be

$$a_n^{(k)} \sim -\mathrm{i} \frac{\Gamma\left(n + \frac{1}{2}\right)}{\left(2\pi\left(n + \frac{1}{2}\right)\right)^{\frac{1}{2}}} \sum_m \frac{(-1)^{\gamma_{km}} g(t^{(m)})}{\mathcal{F}_{km}^n f''^{\frac{1}{2}}(t^{(m)})} \sim -\mathrm{i} \frac{\Gamma(n)}{(2\pi)^{\frac{1}{2}}} \sum_m \frac{(-1)^{\gamma_{km}} g(t^{(m)})}{\mathcal{F}_{km}^n f''^{\frac{1}{2}}(t^{(m)})}$$

as $n \to +\infty$, in agreement with the results obtained by Boyd (cf. [15, eqs. (13) and (17)]). Another way of deriving (1.75) is by substituting into (1.73) asymptotic expansions for the gamma function ratios $\Gamma\left(\frac{n}{2}-r\right)/\Gamma\left(\frac{n+1}{2}\right)$ in inverse powers of $n+1$.

1.3.3 Asymptotic evaluation of the coefficients $a_n^{(k/3)}$

Throughout this subsection, we suppose that all the conditions under which formula (1.52) holds are satisfied. We proceed as before and express the coefficients $a_n^{(k/3)}$ in terms of integrals through the adjacent saddles as

$$a_n^{(k/3)} = \frac{1}{2\pi \mathrm{i}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathrm{e}^{\mathrm{i}\frac{n}{3}\sigma_{km}}} \int_0^{+\infty} u^{\frac{n}{3}-1} \mathrm{e}^{-|\mathcal{F}_{km}|u} T^{(2m/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right) \mathrm{d}u, \tag{1.76}$$

by making use of the formula (1.52). This is a third representation for the coefficients $a_n^{(k/3)}$, different from (1.45) or (1.46). Following the previous subsections, we substitute into (1.76) truncated asymptotic power series expansions for the functions $T^{(2m/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right)$. Changing the orientation of the contours $\mathscr{C}^{(m)}\left(-\sigma_{km}\right)$ (and hence the values of the orientation anomalies γ_{km}) if necessary, we may assume without loss of generality that each of these expansions has the form

$$\begin{aligned} T^{(2m/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right) = &-\frac{2}{3\mathrm{i}} \sum_{r=0}^{N_m-1} \mathrm{e}^{-\frac{\pi(r+1)}{3}\mathrm{i}} \sin\left(\frac{\pi\left(r+1\right)}{3}\right) \frac{a_r^{(m/3)}}{u^{\frac{r}{3}}} \mathrm{e}^{\mathrm{i}\frac{r}{3}\sigma_{km}} \\ &+ R_{N_m}^{(2m/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right), \end{aligned} \tag{1.77}$$

(cf. equation (1.63)) with an appropriate branch of $\left(f\left(t\right)-f\left(t^{(m)}\right)\right)^{\frac{1}{3}}$ in the expression defining the coefficients $a_r^{(m/3)}$. Substitution of the expansions (1.77) into (1.76) and term-by-term integration yields

$$\begin{aligned} a_n^{(k/3)} = \frac{1}{3\pi} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathcal{F}_{km}^{\frac{n}{3}}} &\Bigg(\sum_{r=0}^{N_m-1} \mathrm{e}^{-\frac{\pi(r+1)}{3}\mathrm{i}} \sin\left(\frac{\pi\left(r+1\right)}{3}\right) \\ &\times a_r^{(m/3)} \mathcal{F}_{km}^{\frac{r}{3}} \Gamma\left(\frac{n-r}{3}\right) + A_{N_m}^{(k/3)}\left(n\right) \Bigg), \end{aligned} \tag{1.78}$$

with the remainder terms

$$A_{N_m}^{(k/3)}\left(n\right) = \frac{3\left|\mathcal{F}_{km}\right|^{\frac{n}{3}}}{2\mathrm{i}} \int_0^{+\infty} u^{\frac{n}{3}-1} \mathrm{e}^{-|\mathcal{F}_{km}|u} R_{N_m}^{(2m/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km}}\right) \mathrm{d}u.$$

To guarantee the convergence of these integrals, we require $N_m < n$. Formula (1.78) relates the late coefficients of the asymptotic power series of an integral with cubic endpoint to the early coefficients of the asymptotic power series of integrals through the adjacent saddles. For in the case that computable bounds are available for the remainders $A_{N_m}^{(k/3)}(n)$, the result (1.78) provides an efficient way of calculating numerically the coefficients $a_n^{(k/3)}$ for large values of n. Equation (1.78) is the complete and rigorous form of Dingle's formal expansion for the late coefficients in the asymptotic power series of integrals with cubic endpoints (see [35, eq. (20), p. 147]).

If there is a value $\widetilde{m}$ of m, for which $|\mathcal{F}_{k\widetilde{m}}|$ is less than $|\mathcal{F}_{km}|$ for all the other adjacent saddle points, then at leading order

$$a_n^{(k/3)} \sim (-1)^{\gamma_{k\widetilde{m}}} \frac{a_0^{(\widetilde{m}/3)} \mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{2\sqrt{3}\pi} \frac{\Gamma\left(\frac{n}{3}\right)}{\mathcal{F}_{k\widetilde{m}}^{\frac{n}{3}}}$$

as $n \to +\infty$, revealing that the late coefficients in cubic endpoint expansions also have the "factorial divided by a power" form.

For the sake of completeness, we also discuss the corresponding asymptotic expansion in descending powers of $n+1$. The derivation is akin to those of (1.70) and (1.75). We begin by deforming the small loop around $t^{(k)}$ in (1.45) over the adjacent contours and hence obtain

$$\begin{aligned} a_n^{(k/3)} &= \frac{\Gamma\left(\frac{n+1}{3}\right)}{2\pi \mathrm{i}} \sum_m (-1)^{\gamma_{km}} \int_{\mathscr{C}^{(m)}} \frac{g(t)}{\left(f(t) - f(t^{(k)})\right)^{\frac{n+1}{3}}} \mathrm{d}t \\ &= \frac{\Gamma\left(\frac{n+1}{3}\right)}{2\pi \mathrm{i}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathcal{F}_{km}^{\frac{n+1}{3}}} \int_{\mathscr{C}^{(m)}} \mathrm{e}^{-\frac{n+1}{3}\left(h(t) - h(t^{(m)})\right)} g(t)\, \mathrm{d}t, \end{aligned} \tag{1.79}$$

where we have taken $h(t) = \log(f(t) - f(t^{(k)}))$. We note that the set of saddle points of $h(t)$ is identical to the set of saddles of $f(t)$, and that the adjacent contours $\mathscr{C}^{(m)}(-\sigma_{km})$ are paths of steepest descent for the integrands in (1.79). Since each adjacent contour passes through only one saddle point of $h(t)$, we can apply the method of steepest descents directly to the integrals in (1.79). By changing the orientation of the contours $\mathscr{C}^{(m)}(-\sigma_{km})$ if necessary, the complete asymptotic expansion can be written as

$$a_n^{(k/3)} \sim \frac{1}{3\pi} \frac{\Gamma\left(\frac{n+1}{3}\right)}{(n+1)^{\frac{1}{3}}} \sum_m \frac{(-1)^{\gamma_{km}}}{\mathcal{F}_{km}^{\frac{n+1}{3}}} \sum_{r=0}^{\infty} \mathrm{e}^{-\frac{\pi(r+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(r+1)}{3}\right) \frac{3^{\frac{r+1}{3}} b_r^{(m/3)}}{(n+1)^{\frac{r}{3}}} \tag{1.80}$$

when $n \to +\infty$. The leading coefficients are given as follows

$$b_0^{(m/3)} = \Gamma\left(\frac{1}{3}\right)\left(\frac{6\mathcal{F}_{km}}{f'''(t^{(m)})}\right)^{\frac{1}{3}} g(t^{(m)}),$$
$$b_1^{(m/3)} = \Gamma\left(\frac{2}{3}\right)\left(\frac{6\mathcal{F}_{km}}{f'''(t^{(m)})}\right)^{\frac{2}{3}} \frac{6f'''(t^{(m)})g'(t^{(m)}) - f''''(t^{(m)})g(t^{(m)})}{6f'''(t^{(m)})},$$

with a suitable branch of the cube roots. An alternative way of deriving the result (1.80) is by substituting into (1.78) asymptotic expansions for the gamma function ratios $\Gamma\left(\frac{n-r}{3}\right)/\Gamma\left(\frac{n+1}{3}\right)$ in negative powers of $n+1$.

1.4 Exponentially improved asymptotic expansions

The main emphasis of this section is on the Stokes phenomenon and the exponentially improved asymptotic expansions. In Subsection 1.4.1, we discuss how the resurgence formulae can be continued analytically across the Stokes lines. In Subsection 1.4.2, the concept of the terminant function is introduced, which will be utilized in Subsections 1.4.3–1.4.5 to derive exponentially improved asymptotic expansions for the functions $T^{(e)}(z)$, $T^{(k/2)}(z)$ and $T^{(k/3)}(z)$ (the corresponding results for $T^{(k)}(z)$ and $T^{(2k/3)}(z)$ may be derived using the connection of these functions with $T^{(k/2)}(z)$ and $T^{(k/3)}(z)$, respectively).

1.4.1 Stokes phenomenon and continuation rules

Consider the integral $T^{(e)}(z)$ defined in (1.3), with $\theta = \arg z$ being restricted to an interval

$$-\sigma_{em_1} < \theta < -\sigma_{em_2}, \tag{1.81}$$

where $t^{(m_1)}$ and $t^{(m_2)}$ are adjacent to e. Suppose that $f(t)$ and $g(t)$ grow rapidly enough at infinity for the integral (1.3) to converge for all values of θ in the interval (1.81). As we vary θ in the range (1.81), the steepest descent contour $\mathscr{P}^{(e)}(\theta)$ varies in a continuous manner, and (1.3) defines an analytic function of the complex variable z. When θ passes through the value $-\sigma_{em}$ ($m = m_1$ or m_2) however, the path $\mathscr{P}^{(e)}(\theta)$ hits the adjacent saddle $t^{(m)}$ and jumps, resulting in a discontinuity of $T^{(e)}(z)$. Nevertheless, the function defined by $T^{(e)}(z)$ in the sector (1.81) can still be continued analytically beyond the ray $\theta = -\sigma_{em}$ by adding to the original integral (1.3) a further contribution, which is exponentially small in magnitude compared to $T^{(e)}(z)$. To understand the origin of this contribution, consider the discontinuous change of $\mathscr{P}^{(e)}(\theta)$ as θ passes through $-\sigma_{em}$, illustrated schematically in Figure 1.3. For simplicity, we assume that θ increases through $-\sigma_{em}$, that is $m = m_2$. The case of $m = m_1$ can be treated in

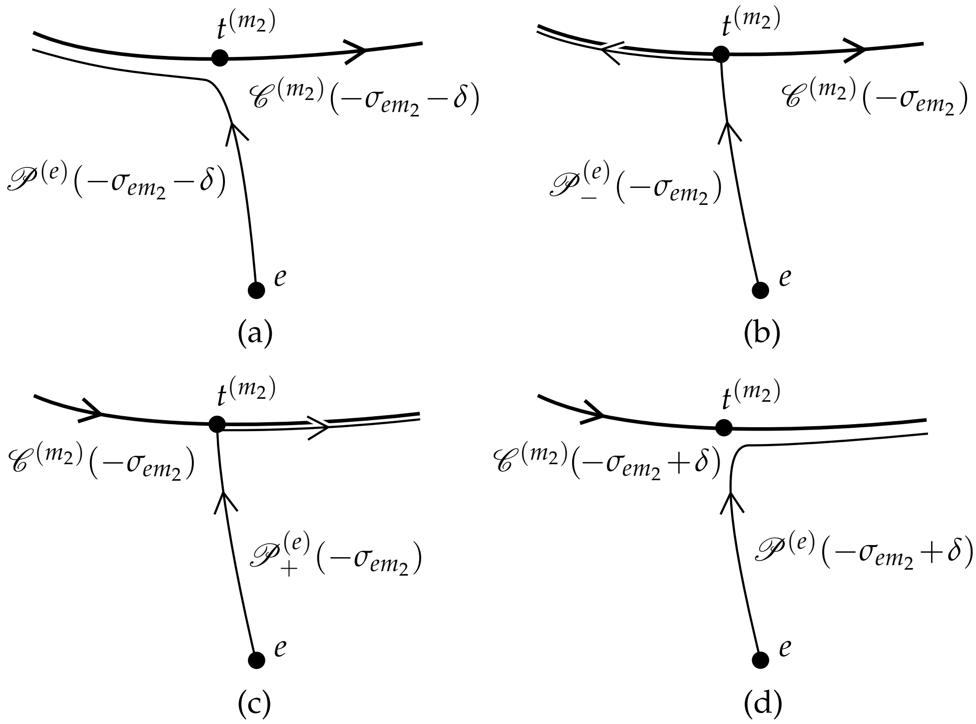

Figure 1.3. *An example of a Stokes phenomenon ($\delta > 0$). (a) The steepest descent path from e before it encounters the adjacent saddle $t^{(m_2)}$. (b) The steepest descent path from e connects to the adjacent saddle $t^{(m_2)}$. (c) The discontinuous change of the steepest descent path from e gives rise to a discontinuity of the function $T^{(e)}(z)$. (d) The steepest descent path from e has passed through the adjacent saddle $t^{(m_2)}$. The analytic continuation of $T^{(e)}(z)$ now includes the integral along the adjacent contour $\mathscr{C}^{(m_2)}$.*

a similar way. It is seen from Figure 1.3 (b) and (c) that if $\theta = -\sigma_{em_2}$, then the following equality holds

$$
\begin{aligned}
& z\int_{\mathscr{P}_{-}^{(e)}(-\sigma_{em_2})} \mathrm{e}^{-z(f(t)-f(e))} g(t)\,\mathrm{d}t = \\
& = z\int_{\mathscr{P}_{+}^{(e)}(-\sigma_{em_2})} \mathrm{e}^{-z(f(t)-f(e))} g(t)\,\mathrm{d}t + (-1)^{\gamma_{em_2}} z\int_{\mathscr{C}^{(m_2)}(-\sigma_{em_2})} \mathrm{e}^{-z(f(t)-f(e))} g(t)\,\mathrm{d}t \\
& = z\int_{\mathscr{P}_{+}^{(e)}(-\sigma_{em_2})} \mathrm{e}^{-z(f(t)-f(e))} g(t)\,\mathrm{d}t + (-1)^{\gamma_{em_2}} z^{\frac{1}{2}} \mathrm{e}^{-\mathcal{F}_{em_2} z} T^{(m_2)}(z)\,,
\end{aligned}
$$

where the contours $\mathscr{P}_{\pm}^{(e)}(-\sigma_{em_2})$ are those defined in (1.13). It is now clear that if we wish to continue our function analytically through the line $\theta = -\sigma_{em_2}$, the quantity

$$(-1)^{\gamma_{em_2}} z^{\frac{1}{2}} \mathrm{e}^{-\mathcal{F}_{em_2} z} T^{(m_2)}(z) \tag{1.82}$$

has to be added to the integral $T^{(e)}(z)$ when $\theta > -\sigma_{em_2}$. This result is in agreement with the observation made by Dingle based on his formal theory of terminants [35, eq. (7), p. 454]. Now, $T^{(m_2)}(z) = \mathcal{O}(1)$ for large z (cf. equation (1.57)) and $\mathfrak{Re}(\mathcal{F}_{em_2} z) > 0$ if θ is close to $-\sigma_{em_2}$, implying that (1.82) is indeed exponentially small compared to $T^{(e)}(z)$. With the above argument, the function remains analytic until θ reaches another critical value. At this point, the contour $\mathscr{P}^{(e)}(\theta)$ (or $\mathscr{C}^{(m_2)}(\theta)$) runs into an adjacent saddle and changes discontinuously again; therefore the introduction of an additional term is required in order to keep our function continuous (and analytic).

Let us now consider how the introduction of (1.82) affects the asymptotic expansion of the function. In the sector (1.81), we have the asymptotic power series

$$T^{(e)}(z) \sim \sum_{n=0}^{\infty} \frac{a_n^{(e)}}{z^n} \tag{1.83}$$

as $z \to \infty$. This can be verified by estimating the remainder term $R_N^{(e)}(z)$ in (1.10), or by a direct application of the method of steepest descents to the integral (1.3). Suppose now that $-\sigma_{em_2} < \theta < -\sigma_{em_3}$, where $t^{(m_3)}$ is adjacent to e (or to $t^{(m_2)}$). Furthermore, assume that the integrals $T^{(e)}(z)$ and $T^{(m_2)}(z)$ exist and are analytic in this sector. The asymptotic expansion of the integral $T^{(e)}(z)$ in this sector is still given by (1.83). However, the (analytic) function that was originally defined by (1.3) is now expressed as the sum of $T^{(e)}(z)$ and (1.82), and therefore it has the asymptotic expansion

$$\begin{aligned} T^{(e)}(z) + (-1)^{\gamma_{em_2}} z^{\frac{1}{2}} \mathrm{e}^{-\mathcal{F}_{em_2} z} T^{(m_2)}(z) \sim \\ \sim \sum_{n=0}^{\infty} \frac{a_n^{(e)}}{z^n} + (-1)^{\gamma_{em_2}} z^{\frac{1}{2}} \mathrm{e}^{-\mathcal{F}_{em_2} z} \sum_{r=0}^{\infty} \frac{a_r^{(m_2)}}{z^r} \end{aligned} \tag{1.84}$$

as $z \to \infty$ in the sector $-\sigma_{em_2} < \theta < -\sigma_{em_3}$. Thus, the introduction of the quantity (1.82) results in a discontinuous change in the form of the asymptotic expansion of the function across the ray $\theta = -\sigma_{em_2}$. The sudden appearance of the second (exponentially small) expansion in (1.84) is an example of the Stokes phenomenon (see, e.g., [98, Sec. 1.7.1]). The line $\theta = -\sigma_{em_2}$, that gives rise to this extra contribution, is a Stokes line. Hence, in a sense the Stokes phenomenon is a consequence of the presence of adjacent saddles, as is the divergence of the asymptotic expansions in (1.83) and (1.84) (for the latter, see [13]). Now, let δ be a small fixed positive real number and consider the sector

$$-\sigma_{em_2} < \theta < -\sigma_{em_2} + \frac{\pi}{2} - \delta. \tag{1.85}$$

For such values of θ, $\mathfrak{Re}\left(\mathcal{F}_{em_2} z\right) = \left|\mathcal{F}_{em_2} z\right| \sin\delta > 0$, and consequently the second expansion in (1.84) is exponentially small compared to any of the terms in the first one for large z. Even if $-\sigma_{em_3} < -\sigma_{em_2} + \frac{\pi}{2} - \delta$, the additional expansions arising from further Stokes phenomena do not contribute asymptotically in the sector (1.85). A similar analysis of the case $m = m_1$ then shows that the analytic continuation of $T^{(e)}(z)$ through the rays $\theta = -\sigma_{em_1}, -\sigma_{em_2}$ has the asymptotic power series given on the right-hand side of (1.83) as $z \to \infty$ in the larger sector

$$-\sigma_{em_1} - \frac{\pi}{2} + \delta < \theta < -\sigma_{em_2} + \frac{\pi}{2} - \delta.$$

We would like to emphasize, however, that in the sector $-\sigma_{em_2} < \theta < \min\left(-\sigma_{em_3}, -\sigma_{em_2} + \frac{\pi}{2} - \delta\right)$ for example, the asymptotic expansion (1.84) is numerically a better approximation to the function than the asymptotic power series (1.83). On the line $\theta = -\sigma_{em_2} + \frac{\pi}{2}$, the two asymptotic expansions in (1.84) become comparable in magnitude to each other, which means that it is an *anti-Stokes line*. We remark that in some references, the notions of Stokes and anti-Stokes lines are interchanged.

We shall now show that the resurgence formula, which we discussed in Subsection 1.1.1, automatically incorporates the Stokes phenomenon. The resurgence relation given by (1.10) and (1.20) was proved under the condition that θ is restricted to an interval of the form (1.81). This is the largest possible domain of validity, because the denominators of the integrands corresponding to $m = m_1$ and $m = m_2$ have zeros at $u = z\mathrm{e}^{\mathrm{i}\sigma_{em_1}}$ and $u = z\mathrm{e}^{\mathrm{i}\sigma_{em_2}}$, respectively. Consider the problem of analytic continuation of $T^{(e)}(z)$ into the sector $-\sigma_{em_2} < \theta < -\sigma_{em_3}$, where $t^{(m_3)}$ is adjacent to e (or to $t^{(m_2)}$). It is evident from (1.10), that we can restrict ourselves to the investigation of $R_N^{(e)}(z)$. Each term in (1.20) is analytic when $\theta = -\sigma_{em_2}$, except that corresponding to $m = m_2$. As θ increases beyond $-\sigma_{em_2}$, the pole at $u = z\mathrm{e}^{\mathrm{i}\sigma_{em_2}}$ arising from the denominator is entrapped: consequently, the analytic continuation of $R_N^{(e)}(z)$ to the sector $-\sigma_{em_2} < \theta < -\sigma_{em_3}$ is still given by the formula in (1.20), but with the term

$$\frac{(-1)^{\gamma_{em_2}}}{z^{N-1}} z^{N-\frac{1}{2}} \mathrm{e}^{-\mathcal{F}_{em_2} z} T^{(m_2)}(z) = (-1)^{\gamma_{em_2}} z^{\frac{1}{2}} \mathrm{e}^{-\mathcal{F}_{em_2} z} T^{(m_2)}(z)$$

added to the right-hand side. This is precisely the extra contribution (1.82) that we have found earlier by examining the discontinuous change of the steepest descent path $\mathscr{P}^{(e)}(\theta)$. Note that this extra contribution is independent of the truncation index N of the asymptotic power series, a known aspect of the Stokes phenomenon.

It can be shown in an analogous manner that the resurgence formulae for integrals with quadratic and cubic endpoints also incorporate the Stokes phenomenon. First, we remove the roots in the denominators of the integrands in (1.36) and (1.52) by the rationalizations

$$\frac{1}{1-(u/z\mathrm{e}^{\mathrm{i}\sigma_{km}})^{\frac{1}{2}}}=\frac{1}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km}}}+\frac{1}{z^{\frac{1}{2}}}\frac{1}{\mathrm{e}^{\mathrm{i}\frac{1}{2}\sigma_{km}}}\frac{u^{\frac{1}{2}}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km}}}\tag{1.86}$$

and

$$\frac{1}{1-(u/z\mathrm{e}^{\mathrm{i}\sigma_{km}})^{\frac{1}{3}}}=\frac{1}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km}}}+\frac{1}{z^{\frac{1}{3}}}\frac{1}{\mathrm{e}^{\mathrm{i}\frac{1}{3}\sigma_{km}}}\frac{u^{\frac{1}{3}}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km}}}+\frac{1}{z^{\frac{2}{3}}}\frac{1}{\mathrm{e}^{\mathrm{i}\frac{2}{3}\sigma_{km}}}\frac{u^{\frac{2}{3}}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km}}},\tag{1.87}$$

respectively. A Stokes phenomenon occurs when $\mathscr{P}^{(k)}(\theta)$ encounters an adjacent saddle $t^{(m)}$, which introduces an exponentially small contribution. This extra contribution may be written in both cases as a sum of residues which arise from the poles at $u = z\mathrm{e}^{\mathrm{i}\sigma_{km}}$. They are found to be

$$\pm(-1)^{\gamma_{km}}2\mathrm{e}^{-\mathcal{F}_{km}z}T^{(m)}(z)\quad\text{and}\quad\pm(-1)^{\gamma_{km}}3\mathrm{e}^{-\mathcal{F}_{km}z}T^{(2m/3)}(z)\tag{1.88}$$

in the cases of integrals with quadratic and cubic endpoints, respectively (cf. [35, eqs. (13) and (18), pp. 455–456]). The upper or lower signs are taken in (1.88) according as θ increases or decreases through $-\sigma_{em}$.

1.4.2 The terminant function

The idea of re-expanding the remainder of a truncated asymptotic expansion into another asymptotic expansion, in order to improve its numerical efficacy, dates back to the 1886 paper of Stieltjes [105]. In Stieltjes' work, the object of re-expansion was not the remainder but the so-called converging factor, which is the ratio of the remainder and the first omitted term. The converging factors of many of the known asymptotic expansions were later studied extensively by Airey [2] in 1937 and by Miller [66] in 1952. Numerical computations confirmed the soundness of their results, nevertheless, the methods used by these authors were entirely formal and non-rigorous. In his 1974 book [95, pp. 522–536], Olver revisited the problem and proved rigorous results for the converging factors of the asymptotic expansions of the exponential integral and the confluent hypergeometric function. Olver's ideas were widely extended later by Olde Daalhuis [82, 83] in the early 1990's. The general disadvantage of these type of re-expansions is that they are valid only in relatively small sectors of the complex plane.

A different theory of converging factors was developed by Dingle in a series of papers [29–34] written in the 1950's, and in a research monograph published in 1973 [35]. Dingle showed that the converging factors of a wide class of asymptotic expansions can be expressed in terms of certain functions he called "basic terminants" and demonstrated the power of his method through various examples. Nonetheless, Dingle's investigations were based on interpretive, rather than rigorous, methods.

In the late 1980's, based on the ideas of Dingle, Berry [5,6] gave an interesting formal argument which shows that, assuming optimal truncation, the transition between the two different asymptotic expansions in adjacent Stokes sectors is effected smoothly and not discontinuously as in previous explanations of the Stokes phenomenon. Berry's work was put in a rigorous mathematical framework by Olver [90,92], Boyd [12], Jones [49], Berry and Howls [8] and Paris [97] using integral methods, and by Olver and Olde Daalhuis [85] using differential equation methods.

In this work, we follow the approach of [8] and establish re-expansions of the remainder terms of (optimally) truncated asymptotic expansions of integrals of the form (1.1). The truncation error of these re-expansions is then found to be exponentially small in comparison with the original asymptotic expansion, hence the term "exponential improvement". These exponentially improved asymptotic expansions, in contrast with those given by Olde Daalhuis [82,83], are not expressed in terms of elementary functions, on the other hand they are valid in large sectors of the complex plane.

Following Olver [90], we define the terminant function $T_p(w)$, for $\mathfrak{Re}(p) > 0$, by

$$T_p(w) \stackrel{\text{def}}{=} \frac{\mathrm{e}^{\pi \mathrm{i} p} w^{1-p} \mathrm{e}^{-w}}{2\pi \mathrm{i}} \int_0^{+\infty} \frac{t^{p-1}\mathrm{e}^{-t}}{w+t} \mathrm{d}t \tag{1.89}$$

when $|\varphi| < \pi$, and elsewhere by analytic continuation (here and subsequently, we write $\varphi = \arg w$). This function will be the building block of our exponentially improved asymptotic expansions. The terminant function is directly related to the incomplete gamma function via (cf. [96, eq. 8.6.4, p. 177])

$$T_p(w) = \frac{\mathrm{e}^{\pi \mathrm{i} p}\Gamma(p)}{2\pi \mathrm{i}}\Gamma(1-p,w).$$

Dingle's original terminants $\Lambda_p(w)$, $\Pi_p(w)$ may be expressed as

$$\Lambda_p(w) = -\frac{2\pi \mathrm{i}\mathrm{e}^{-\pi \mathrm{i} p}}{\Gamma(p+1)} w^{p+1}\mathrm{e}^{w} T_{p+1}(w)$$

and

$$\Pi_p(w) = \frac{\pi e^{-\pi i p}}{\Gamma(p+1)} w^{p+1} \left(e^{\left(w+\frac{\pi p}{2}\right)i} T_{p+1}\left(w e^{\frac{\pi}{2}i}\right) - e^{-\left(w+\frac{\pi p}{2}\right)i} T_{p+1}\left(w e^{-\frac{\pi}{2}i}\right) \right),$$

although we shall not use this notation in our work. Dingle also gave alternative re-expansions in terms of "reduced" and "compound" derivatives of these terminants [35, eqs. (10) and (12), p. 433], which have not been investigated yet and might be the subjects of future research.

Various properties of the terminant function can be deduced from those of the incomplete gamma function. For the derivation of exponentially improved asymptotic expansions, the asymptotic behaviour of $T_p(w)$ when w is large and $p \approx |w|$ has to be known. In the important paper [93, eqs. (2.9) and (2.11)], Olver gave the following uniform estimates, valid when w is large and $|p - |w||$ is bounded:[6]

$$T_p(w) = \begin{cases} e^{-w}\mathcal{O}(e^{-|w|}) & \text{if} \quad |\varphi| \le \pi, \\ \mathcal{O}(1) & \text{if} \quad \pi \le |\varphi| \le 3\pi - \delta, \end{cases} \tag{1.90}$$

with an arbitrary small positive δ. Concerning the smooth interpretation of the Stokes phenomenon, the following more precise asymptotic formulae can be used:

$$T_p(w) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(c(\varphi) \left(\frac{1}{2}|w| \right)^{\frac{1}{2}} \right) + \mathcal{O}\left(\frac{e^{-\frac{1}{2}|w| c^2(\varphi)}}{|w|^{\frac{1}{2}}} \right) \tag{1.91}$$

provided $-\pi + \delta \le \varphi \le 3\pi - \delta$, and

$$e^{-2\pi i p} T_p(w) = -\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(-\overline{c(-\varphi)} \left(\frac{1}{2}|w| \right)^{\frac{1}{2}} \right) + \mathcal{O}\left(\frac{e^{-\frac{1}{2}|w| \overline{c^2(-\varphi)}}}{|w|^{\frac{1}{2}}} \right) \tag{1.92}$$

provided $-3\pi + \delta \le \varphi \le \pi - \delta$. Here erf denotes the error function [96, eq. 7.2.1, p. 160], and the quantity $c(\varphi)$ is defined implicitly by the equation

$$\frac{1}{2}c^2(\varphi) = 1 + i(\varphi - \pi) - e^{i(\varphi - \pi)}$$

and corresponds to the branch of $c(\varphi)$ which has the following Taylor series expansion in the neighborhood of $\varphi = \pi$:

$$c(\varphi) = (\varphi - \pi) + \frac{i}{6}(\varphi - \pi)^2 - \frac{1}{36}(\varphi - \pi)^3 - \frac{i}{270}(\varphi - \pi)^4 + \cdots. \tag{1.93}$$

The asymptotic approximations (1.91) and (1.92) are, in fact, the leading terms of two asymptotic expansions due to Olver [90, 91]. Slightly different asymptotic expansions were given by Boyd [12].

[6]In the paper [93], Olver uses the alternative notation $F_p(w) = i e^{-\pi i p} T_p(w)$ for the terminant function.

1.4.3 Exponentially improved expansion for $T^{(e)}(z)$

Consider the integral $T^{(e)}(z)$ given in (1.3), with $\theta = \arg z$ being restricted to an interval

$$-\sigma_{em_1} < \theta < -\sigma_{em_2}, \tag{1.94}$$

where $t^{(m_1)}$ and $t^{(m_2)}$ are adjacent to e. Suppose that $f(t)$ and $g(t)$ grow sufficiently rapidly at infinity so that the integral (1.3) converges for all values of θ in the interval (1.94). In order to make the presentation simpler, we also assume that there are no further saddles adjacent to e other than $t^{(m_1)}$ and $t^{(m_2)}$ and that $|\mathcal{F}_{em_1}| = |\mathcal{F}_{em_2}|$.[7] By (1.10) and (1.20) we have, under appropriate conditions, that

$$T^{(e)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(e)}}{z^n} + R_N^{(e)}(z), \tag{1.95}$$

with

$$\begin{aligned} R_N^{(e)}(z) = {} & \frac{1}{2\pi i z^N} \frac{(-1)^{\gamma_{em_1}}}{e^{i\left(N+\frac{1}{2}\right)\sigma_{em_1}}} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}} e^{-|\mathcal{F}_{em_1}|u}}{1 - u/z e^{i\sigma_{em_1}}} T^{(m_1)}\left(u e^{-i\sigma_{em_1}}\right) du \\ & + \frac{1}{2\pi i z^N} \frac{(-1)^{\gamma_{em_2}}}{e^{i\left(N+\frac{1}{2}\right)\sigma_{em_2}}} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}} e^{-|\mathcal{F}_{em_2}|u}}{1 - u/z e^{i\sigma_{em_2}}} T^{(m_2)}\left(u e^{-i\sigma_{em_2}}\right) du \end{aligned}$$

for all non-zero values of z in the sector (1.94) and with a sufficiently large N. We now proceed by replacing the factors $T^{(m_1)}\left(u e^{-i\sigma_{em_1}}\right)$ and $T^{(m_2)}\left(u e^{-i\sigma_{em_2}}\right)$ in the integrands by their truncated asymptotic expansions (1.67). This is reasonable because the integrands are dominated by their behaviour near $u = N/|\mathcal{F}_{em_1}| = N/|\mathcal{F}_{em_2}|$, whose value will be chosen later to be $\approx |z|$ with z being large. We thus find that

$$\begin{aligned} R_N^{(e)}(z) = {} & \frac{1}{2\pi i z^N} \frac{(-1)^{\gamma_{em_1}}}{e^{i\left(N+\frac{1}{2}\right)\sigma_{em}}} \sum_{r=0}^{N_{m_1}-1} a_r^{(m_1)} e^{ir\sigma_{em_1}} \int_0^{+\infty} \frac{u^{N-r-\frac{1}{2}} e^{-|\mathcal{F}_{em_1}|u}}{1 - u/z e^{i\sigma_{em_1}}} du \\ & + \frac{1}{2\pi i z^N} \frac{(-1)^{\gamma_{em_2}}}{e^{i\left(N+\frac{1}{2}\right)\sigma_{em_2}}} \sum_{r=0}^{N_{m_2}-1} a_r^{(m_2)} e^{ir\sigma_{em_2}} \int_0^{+\infty} \frac{u^{N-r-\frac{1}{2}} e^{-|\mathcal{F}_{em_2}|u}}{1 - u/z e^{i\sigma_{em_2}}} du \\ & + R_{N_{m_1},N_{m_2}}^{(e)}(z), \end{aligned} \tag{1.96}$$

[7] It appears that this is the most common case in applications. If $|\mathcal{F}_{em_1}| \neq |\mathcal{F}_{em_2}|$, the exponentially improved expansion becomes more elaborate; an example of this case is provided by the Hankel and Bessel functions of large order and argument discussed in Section 3.1.

with

$$
\begin{aligned}
R_{N_{m_1},N_{m_2}}^{(e)}(z) = {} & \frac{1}{2\pi \mathrm{i} z^N} \frac{(-1)^{\gamma_{em_1}}}{\mathrm{e}^{\mathrm{i}\left(N+\frac{1}{2}\right)\sigma_{em_1}}} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}}\mathrm{e}^{-|\mathcal{F}_{em_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{em_1}}} R_{N_{m_1}}^{(m_1)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{em_1}}\right)\mathrm{d}u \\
& + \frac{1}{2\pi \mathrm{i} z^N} \frac{(-1)^{\gamma_{em_2}}}{\mathrm{e}^{\mathrm{i}\left(N+\frac{1}{2}\right)\sigma_{em_2}}} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}}\mathrm{e}^{-|\mathcal{F}_{em_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{em_2}}} R_{N_{m_2}}^{(m_2)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{em_2}}\right)\mathrm{d}u
\end{aligned}
\tag{1.97}
$$

for $-\sigma_{em_1} < \theta < -\sigma_{em_2}$. To ensure the convergence of the integrals in (1.97), we require $N_{m_1}, N_{m_2} \leq N$. Because $0 < \sigma_{em_1} - \sigma_{em_2} \leq 2\pi$ and therefore

$$\left|\arg\left(\mathcal{F}_{em_1} z \mathrm{e}^{-\pi \mathrm{i}}\right)\right| = |\theta + \sigma_{em_1} - \pi| < \pi$$

and

$$\left|\arg\left(\mathcal{F}_{em_2} z \mathrm{e}^{\pi \mathrm{i}}\right)\right| = |\theta + \sigma_{em_2} + \pi| < \pi,$$

the definition (1.89) can be used to express the integrals in (1.96) in terms of terminant functions as follows:

$$
\begin{aligned}
\int_0^{+\infty} \frac{u^{N-r-\frac{1}{2}}\mathrm{e}^{-|\mathcal{F}_{em_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{em_1}}}\mathrm{d}u &= -\frac{z\mathrm{e}^{\mathrm{i}\sigma_{em_1}}}{|\mathcal{F}_{em_1}|^{N-r-\frac{1}{2}}} \int_0^{+\infty} \frac{t^{N-r-\frac{1}{2}}\mathrm{e}^{-t}}{\mathcal{F}_{em_1} z\mathrm{e}^{-\pi \mathrm{i}} + t}\mathrm{d}t \\
&= -2\pi \mathrm{i} \mathrm{e}^{\mathrm{i}\left(N-r+\frac{1}{2}\right)\sigma_{em_1}} z^{N-r+\frac{1}{2}} \mathrm{e}^{-\mathcal{F}_{em_1} z} T_{N-r+\frac{1}{2}}\left(\mathcal{F}_{em_1} z\mathrm{e}^{-\pi \mathrm{i}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^{+\infty} \frac{u^{N-r-\frac{1}{2}}\mathrm{e}^{-|\mathcal{F}_{em_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{em_2}}}\mathrm{d}u &= -\frac{z\mathrm{e}^{\mathrm{i}\sigma_{em_2}}}{|\mathcal{F}_{em_2}|^{N-r-\frac{1}{2}}} \int_0^{+\infty} \frac{t^{N-r-\frac{1}{2}}\mathrm{e}^{-t}}{\mathcal{F}_{em_2} z\mathrm{e}^{\pi \mathrm{i}} + t}\mathrm{d}t \\
&= 2\pi \mathrm{i} \mathrm{e}^{\mathrm{i}\left(N-r+\frac{1}{2}\right)\sigma_{em_2}} z^{N-r+\frac{1}{2}} \mathrm{e}^{-\mathcal{F}_{em_2} z} T_{N-r+\frac{1}{2}}\left(\mathcal{F}_{em_2} z\mathrm{e}^{\pi \mathrm{i}}\right).
\end{aligned}
$$

Thus from (1.95) and (1.96)

$$
\begin{aligned}
T^{(e)}(z) = {} & \sum_{n=0}^{N-1} \frac{a_n^{(e)}}{z^n} + (-1)^{\gamma_{em_1}+1} z^{\frac{1}{2}} \mathrm{e}^{-\mathcal{F}_{em_1} z} \sum_{r=0}^{N_{m_1}-1} \frac{a_r^{(m_1)}}{z^r} T_{N-r+\frac{1}{2}}\left(\mathcal{F}_{em_1} z\mathrm{e}^{-\pi \mathrm{i}}\right) \\
& + (-1)^{\gamma_{em_2}} z^{\frac{1}{2}} \mathrm{e}^{-\mathcal{F}_{em_2} z} \sum_{r=0}^{N_{m_2}-1} \frac{a_r^{(m_2)}}{z^r} T_{N-r+\frac{1}{2}}\left(\mathcal{F}_{em_2} z\mathrm{e}^{\pi \mathrm{i}}\right) \\
& + R_{N_{m_1},N_{m_2}}^{(e)}(z).
\end{aligned}
\tag{1.98}
$$

This is the full and rigorous form of Dingle's formal re-expansion of the remainder term of an asymptotic power series of an integral with linear endpoint (see [35, eqs. (3) and (6), pp. 452–453]).

We now derive from (1.98) the exponentially improved asymptotic expansion for $T^{(e)}(z)$ by choosing N suitably as a function of z. For large values of z, the way to obtain the best approximation from the asymptotic power series (1.83) is by truncating it just before its numerically least term. Using the asymptotic behaviour of the coefficients $a_n^{(e)}$ (see Subsection 1.3.1), it is seen that the index of this least term is $n \approx |\mathcal{F}_{em_1}||z| = |\mathcal{F}_{em_2}||z|$. Therefore, $N \approx |\mathcal{F}_{em_1}||z| = |\mathcal{F}_{em_2}||z|$ seems to be a natural choice in (1.98). In the case of most applications, it can be shown that there exist positive constants $C_{N_{m_1}}$ and $C_{N_{m_2}}$ such that $\left|R_{N_{m_1}}^{(m_1)}(u\mathrm{e}^{-\mathrm{i}\sigma_{em_1}})\right| \leq C_{N_{m_1}}|u|^{-N_{m_1}}$ and $\left|R_{N_{m_2}}^{(m_2)}(u\mathrm{e}^{-\mathrm{i}\sigma_{em_2}})\right| \leq C_{N_{m_2}}|u|^{-N_{m_2}}$ hold for any u with $\arg u$ lying in a small neighbourhood of the origin. Assuming that this holds, it is not hard to prove that, with the above special choice of N, we have

$$R_{N_{m_1},N_{m_2}}^{(e)}(z) = \mathcal{O}_{N_{m_1}}\left(\frac{\mathrm{e}^{-|\mathcal{F}_{em_1}||z|}}{|z|^{N_{m_1}-\frac{1}{2}}}\right) + \mathcal{O}_{N_{m_2}}\left(\frac{\mathrm{e}^{-|\mathcal{F}_{em_2}||z|}}{|z|^{N_{m_2}-\frac{1}{2}}}\right) \tag{1.99}$$

as $z \to \infty$ in the closed sector $-\sigma_{em_1} \leq \theta \leq -\sigma_{em_2}$[8], provided that N_{m_1} and N_{m_2} are fixed or small in comparison with $N = \mathcal{O}(|z|)$. From (1.99) and Olver's estimation (1.90), we infer that $R_{N_{m_1},N_{m_2}}^{(e)}(z)$ has the order of magnitude of the first omitted terms of the second and third series in (1.98). The expression (1.98), with the specific choice of $N \approx |\mathcal{F}_{em_1}||z| = |\mathcal{F}_{em_2}||z|$, is the exponentially improved asymptotic expansion for $T^{(e)}(z)$.

We close this subsection by discussing briefly the smooth transition of the Stokes discontinuities discovered originally by Berry. We shall show that if the asymptotic power series (1.83) is truncated near its numerically least term, the first few terms of the second asymptotic expansion in (1.84) are "switched on" in a rapid and smooth way as θ increases through $-\sigma_{em_2}$ (the discussion of the case when θ decreases though $-\sigma_{em_1}$ is similar). To this end, we make the following assumptions: $N \approx |\mathcal{F}_{em_1}||z| = |\mathcal{F}_{em_2}||z|$ (i.e., the asymptotic power series (1.83) is truncated near its numerically least term), $\delta \leq \sigma_{em_1} - \sigma_{em_2} \leq 2\pi - \delta$ (i.e., the adjacent saddles are bounded away from each other) and $-\sigma_{em_2} - \delta \leq \theta \leq -\sigma_{em_2} + \delta$ with some small positive δ (i.e., z varies in a small neighbourhood of the Stokes line). We may use analytic continuation to extend the functions on right-hand side of the equality (1.98) to the sector $-\sigma_{em_2} \leq \theta \leq -\sigma_{em_2} + \delta$. In most applications, even when $-\sigma_{em_2} \leq \theta \leq -\sigma_{em_2} + \delta$, the remainder term $R_{N_{m_1},N_{m_2}}^{(e)}(z)$ has the order of magnitude of the first omitted terms of the second

[8]For $\theta = -\sigma_{em_1}, -\sigma_{em_2}$, we define $R_{N_{m_1},N_{m_2}}^{(e)}(z)$ using analytic continuation.

and third series in (1.98). Assuming that this holds, we may write

$$\begin{aligned}T^{(e)}(z) \approx \sum_{n=0}^{N-1}\frac{a_n^{(e)}}{z^n} + (-1)^{\gamma_{em_1}+1} z^{\frac{1}{2}}\mathrm{e}^{-\mathcal{F}_{em_1}z}\sum_{r=0}\frac{a_r^{(m_1)}}{z^r}T_{N-r+\frac{1}{2}}\left(\mathcal{F}_{em_1}z\mathrm{e}^{-\pi\mathrm{i}}\right)\\ + (-1)^{\gamma_{em_2}} z^{\frac{1}{2}}\mathrm{e}^{-\mathcal{F}_{em_2}z}\sum_{r=0}\frac{a_r^{(m_2)}}{z^r}T_{N-r+\frac{1}{2}}\left(\mathcal{F}_{em_2}z\mathrm{e}^{\pi\mathrm{i}}\right),\end{aligned} \tag{1.100}$$

where $\sum_{r=0}$ means that the sum is restricted to the first few terms of the series. Because $\delta \leq \sigma_{em_1} - \sigma_{em_2} \leq 2\pi - \delta$, $-\sigma_{em_2} - \delta \leq \theta \leq -\sigma_{em_2} + \delta$ and therefore

$$\left|\arg(\mathcal{F}_{em_1}z\mathrm{e}^{-\pi\mathrm{i}})\right| = \left|\theta + \sigma_{em_1} - \pi\right| \leq \pi$$

and

$$\pi - \delta \leq \arg(\mathcal{F}_{em_2}z\mathrm{e}^{\pi\mathrm{i}}) = \left|\theta + \sigma_{em_2} + \pi\right| \leq \pi + \delta,$$

the formulae (1.90), (1.91) and (1.93) may be applied to express the asymptotic behaviour of the terminant functions:

$$T_{N-r+\frac{1}{2}}\left(\mathcal{F}_{em_1}z\mathrm{e}^{-\pi\mathrm{i}}\right) = \mathrm{e}^{\mathcal{F}_{em_1}z}\mathcal{O}(\mathrm{e}^{-|\mathcal{F}_{em_1}||z|}) \tag{1.101}$$

and

$$T_{N-r+\frac{1}{2}}\left(\mathcal{F}_{em_2}z\mathrm{e}^{\pi\mathrm{i}}\right) \sim \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\left(\sigma_{em_2} + \theta\right)\left(\frac{1}{2}\left|\mathcal{F}_{em_2}\right||z|\right)^{\frac{1}{2}}\right), \tag{1.102}$$

provided that $|z|$ is large and r is small in comparison with $N \approx |\mathcal{F}_{em_1}||z| = |\mathcal{F}_{em_2}||z|$. Hence, from (1.100), (1.101) and (1.102), we may assert that

$$\begin{aligned}T^{(e)}(z) \approx \sum_{n=0}^{N-1}\frac{a_n^{(e)}}{z^n}\\ + (-1)^{\gamma_{em_2}} z^{\frac{1}{2}}\mathrm{e}^{-\mathcal{F}_{em_2}z}\sum_{r=0}\frac{a_r^{(m_2)}}{z^r}\left(\frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\left(\sigma_{em_2} + \theta\right)\left(\frac{1}{2}\left|\mathcal{F}_{em_2}\right||z|\right)^{\frac{1}{2}}\right)\right).\end{aligned} \tag{1.103}$$

The effect of the error function in (1.103) is to "switch on" the first few terms of the second asymptotic expansion in (1.84): the values of the error function are -1 for $\theta < -\sigma_{em_2}$ up to an exponentially small error, are almost 1 for $\theta > -\sigma_{em_2}$, and change rapidly but smoothly near $\theta = -\sigma_{em_2}$.

1.4.4 Exponentially improved expansion for $T^{(k/2)}(z)$

Consider the integral $T^{(k/2)}(z)$ given in (1.22), with $\theta = \arg z$ being restricted to an interval of the form

$$-\sigma_{km_1} < \theta < -\sigma_{km_2}, \tag{1.104}$$

where $t^{(m_1)}$ and $t^{(m_2)}$ are adjacent to $t^{(k)}$. Assume that $f(t)$ and $g(t)$ grow rapidly enough at infinity for the integral (1.22) to converge for all values of θ in the interval (1.104). To make the presentation simpler, we also assume that there are no further saddles adjacent to $t^{(k)}$ other than $t^{(m_1)}$ and $t^{(m_2)}$ and that $|\mathcal{F}_{km_1}| = |\mathcal{F}_{km_2}|$, $0 < \sigma_{km_1} - \sigma_{km_2} \le 2\pi$.[9] By (1.28) we have, under suitable conditions, that

$$T^{(k/2)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(k/2)}}{z^{\frac{n}{2}}} + R_N^{(k/2)}(z), \tag{1.105}$$

where, by (1.36) and (1.86),

$$\begin{aligned} R_N^{(k/2)}(z) = {} & \frac{1}{2\pi \mathrm{i} z^{\frac{N}{2}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N}{2}\sigma_{km_1}}} \int_0^{+\infty} \frac{u^{\frac{N}{2}-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}} T^{(m_1)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}}\right)\mathrm{d}u \\ & + \frac{1}{2\pi \mathrm{i} z^{\frac{N+1}{2}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{2}\sigma_{km_1}}} \int_0^{+\infty} \frac{u^{\frac{N-1}{2}}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}} T^{(m_1)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}}\right)\mathrm{d}u \\ & + \frac{1}{2\pi \mathrm{i} z^{\frac{N}{2}}} \frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N}{2}\sigma_{km_2}}} \int_0^{+\infty} \frac{u^{\frac{N}{2}-1}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}} T^{(m_2)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}}\right)\mathrm{d}u \\ & + \frac{1}{2\pi \mathrm{i} z^{\frac{N+1}{2}}} \frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{2}\sigma_{km_2}}} \int_0^{+\infty} \frac{u^{\frac{N-1}{2}}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}} T^{(m_2)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}}\right)\mathrm{d}u \end{aligned}$$

for all non-zero values of z in the sector (1.104) and with a sufficiently large N. Following the steps in Subsection 1.4.3, we replace the factors $T^{(m_1)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}}\right)$ and $T^{(m_2)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}}\right)$ in the integrands by their truncated asymptotic expansions (1.72), and hence find that

$$\begin{aligned} R_N^{(k/2)}(z) = {} & \frac{1}{2\pi \mathrm{i} z^{\frac{N}{2}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N}{2}\sigma_{km_1}}} \sum_{r=0}^{N_{m_1}-1} a_r^{(m_1)} \mathrm{e}^{\mathrm{i}r\sigma_{km_1}} \int_0^{+\infty} \frac{u^{\frac{N}{2}-r-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}}\mathrm{d}u \\ & + \frac{1}{2\pi \mathrm{i} z^{\frac{N+1}{2}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{2}\sigma_{km_1}}} \sum_{r=0}^{N_{m_1}-1} a_r^{(m_1)} \mathrm{e}^{\mathrm{i}r\sigma_{km_1}} \int_0^{+\infty} \frac{u^{\frac{N-1}{2}-r}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}}\mathrm{d}u \\ & + \frac{1}{2\pi \mathrm{i} z^{\frac{N}{2}}} \frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N}{2}\sigma_{km_2}}} \sum_{r=0}^{N_{m_2}-1} a_r^{(m_2)} \mathrm{e}^{\mathrm{i}r\sigma_{km_2}} \int_0^{+\infty} \frac{u^{\frac{N}{2}-r-1}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}}\mathrm{d}u \\ & + \frac{1}{2\pi \mathrm{i} z^{\frac{N+1}{2}}} \frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{2}\sigma_{km_2}}} \sum_{r=0}^{N_{m_2}-1} a_r^{(m_2)} \mathrm{e}^{\mathrm{i}r\sigma_{km_2}} \int_0^{+\infty} \frac{u^{\frac{N-1}{2}-r}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}}\mathrm{d}u \\ & + R_{N_{m_1},N_{m_2}}^{(k/2)}(z), \end{aligned} \tag{1.106}$$

[9] If $2\pi < \sigma_{km_1} - \sigma_{km_2} \le 4\pi$, the exponentially improved expansion becomes more elaborate; an example of this case is provided by the the incomplete gamma function $\Gamma(z,z)$ discussed in the paper [77] of the present author.

with

$$\begin{aligned}
R_{N_{m_1},N_{m_2}}^{(k/2)}(z) &= \frac{1}{2\pi \mathrm{i} z^{\frac{N}{2}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N}{2}\sigma_{km_1}}} \int_0^{+\infty} \frac{u^{\frac{N}{2}-1} \mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}} R_{N_{m_1}}^{(m_1)}(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}})\mathrm{d}u \\
&+ \frac{1}{2\pi \mathrm{i} z^{\frac{N+1}{2}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{2}\sigma_{km_1}}} \int_0^{+\infty} \frac{u^{\frac{N-1}{2}} \mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}} R_{N_{m_1}}^{(m_1)}(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}})\mathrm{d}u \\
&+ \frac{1}{2\pi \mathrm{i} z^{\frac{N}{2}}} \frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N}{2}\sigma_{km_2}}} \int_0^{+\infty} \frac{u^{\frac{N}{2}-1} \mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}} R_{N_{m_2}}^{(m_2)}(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}})\mathrm{d}u \\
&+ \frac{1}{2\pi \mathrm{i} z^{\frac{N+1}{2}}} \frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{2}\sigma_{km_2}}} \int_0^{+\infty} \frac{u^{\frac{N-1}{2}} \mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}} R_{N_{m_2}}^{(m_2)}(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}})\mathrm{d}u
\end{aligned} \tag{1.107}$$

for $-\sigma_{km_1} < \theta < -\sigma_{km_2}$. To insure the convergence of the integrals in (1.107), we make the requirement $2N_{m_1}, 2N_{m_2} < N$. The integrals in (1.106) can be expressed in terms of terminant functions by making use of the assumption that $0 < \sigma_{km_1} - \sigma_{km_2} \leq 2\pi$. Thus from (1.105) and (1.106), we obtain

$$\begin{aligned}
T^{(k/2)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(k/2)}}{z^{\frac{n}{2}}} &+ (-1)^{\gamma_{km_1}} 2\mathrm{e}^{-\mathcal{F}_{km_1} z} \sum_{r=0}^{N_{m_1}-1} \frac{a_r^{(m_1)}}{z^r} \\
&\times \mathrm{e}^{-2\pi\left(\frac{N}{2}-r\right)\mathrm{i}} \frac{T_{\frac{N}{2}-r}\left(\mathcal{F}_{km_1} z\mathrm{e}^{-\pi\mathrm{i}}\right) - T_{\frac{N+1}{2}-r}\left(\mathcal{F}_{km_1} z\mathrm{e}^{-\pi\mathrm{i}}\right)}{2} \\
&+ (-1)^{\gamma_{km_2}} 2\mathrm{e}^{-\mathcal{F}_{km_2} z} \sum_{r=0}^{N_{m_2}-1} \frac{a_r^{(m_2)}}{z^r} \\
&\times \frac{T_{\frac{N}{2}-r}\left(\mathcal{F}_{km_2} z\mathrm{e}^{\pi\mathrm{i}}\right) + T_{\frac{N+1}{2}-r}\left(\mathcal{F}_{km_2} z\mathrm{e}^{\pi\mathrm{i}}\right)}{2} \\
&+ R_{N_{m_1},N_{m_2}}^{(k/2)}(z) .
\end{aligned} \tag{1.108}$$

This is the complete and rigorous form of Dingle's formal re-expansion of the remainder term of an asymptotic power series of an integral with quadratic endpoint (cf. [35, eqs. (10) and (12), p. 454]).

To deduce from (1.108) the exponentially improved asymptotic expansion of $T^{(k/2)}(z)$, we may argue as in the previous subsection. From the asymptotic properties of the coefficients $a_n^{(k/2)}$ (see Subsection 1.3.2), we infer that the optimal choice for N is $N \approx 2\left|\mathcal{F}_{km_1}\right||z| = 2\left|\mathcal{F}_{km_2}\right||z|$. Assume that there exist positive constants $C_{N_{m_1}}$ and $C_{N_{m_2}}$ such that $\left|R_{N_{m_1}}^{(m_1)}(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}})\right| \leq C_{N_{m_1}}|u|^{-N_{m_1}}$ and $\left|R_{N_{m_2}}^{(m_2)}(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}})\right| \leq C_{N_{m_2}}|u|^{-N_{m_2}}$ hold for any u with $\arg u$ lying in a small neighbourhood of the origin (this is the case for most applications). Then it is not hard to prove that, with the above special choice of N, we have

$$R^{(k/2)}_{N_{m_1},N_{m_2}}(z) = \mathcal{O}_{N_{m_1}}\left(\frac{\mathrm{e}^{-|\mathcal{F}_{km_1}||z|}}{|z|^{N_{m_1}}}\right) + \mathcal{O}_{N_{m_2}}\left(\frac{\mathrm{e}^{-|\mathcal{F}_{km_2}||z|}}{|z|^{N_{m_2}}}\right) \tag{1.109}$$

as $z \to \infty$ in the closed sector $-\sigma_{km_1} \le \theta \le -\sigma_{km_2}$, as long as N_{m_1} and N_{m_2} are fixed or small compared to $N = \mathcal{O}(|z|)$. From (1.109) and Olver's estimation (1.90), we see that $R^{(k/2)}_{N_{m_1},N_{m_2}}(z)$ has the order of magnitude of the first omitted terms of the second and third series in (1.108). The expression (1.108), with the specific choice of $N \approx 2|\mathcal{F}_{km_1}||z| = 2|\mathcal{F}_{km_2}||z|$, is the exponentially improved asymptotic expansion for $T^{(k/2)}(z)$.

The smooth transition of the Stokes discontinuities may be discussed analogously to the case of $T^{(e)}(z)$, the details are left to the reader.

1.4.5 Exponentially improved expansion for $T^{(k/3)}(z)$

Consider the integral $T^{(k/3)}(z)$ given in (1.38), with $\theta = \arg z$ being restricted to an interval of the form (1.104) where $t^{(m_1)}$ and $t^{(m_2)}$ are adjacent to $t^{(k)}$. Suppose that $f(t)$ and $g(t)$ grow sufficiently rapidly at infinity so that the integral (1.38) converges for all values of θ in the interval (1.104). For simplicity, we also assume that there are no further saddles adjacent to $t^{(k)}$ other than $t^{(m_1)}$ and $t^{(m_2)}$ and that $|\mathcal{F}_{km_1}| = |\mathcal{F}_{km_2}|$ and $0 < \sigma_{km_1} - \sigma_{km_2} \le 2\pi$. By (1.28) we have, under suitable conditions, that

$$T^{(k/3)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(k/3)}}{z^{\frac{n}{3}}} = R_N^{(k/3)}(z) \tag{1.110}$$

where, by (1.52) and (1.87),

$$\begin{aligned}
R_N^{(k/3)}(z) = {} & \frac{1}{2\pi \mathrm{i} z^{\frac{N}{3}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N}{3}\sigma_{km_1}}} \int_0^{+\infty} \frac{u^{\frac{N}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}} T^{(2m_1/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}}\right)\mathrm{d}u \\
& + \frac{1}{2\pi \mathrm{i} z^{\frac{N+1}{3}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{3}\sigma_{km_1}}} \int_0^{+\infty} \frac{u^{\frac{N+1}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}} T^{(2m_1/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}}\right)\mathrm{d}u \\
& + \frac{1}{2\pi \mathrm{i} z^{\frac{N+2}{3}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N+2}{3}\sigma_{km_1}}} \int_0^{+\infty} \frac{u^{\frac{N+2}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}} T^{(2m_1/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}}\right)\mathrm{d}u \\
& + \frac{1}{2\pi \mathrm{i} z^{\frac{N}{3}}} \frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N}{3}\sigma_{km_2}}} \int_0^{+\infty} \frac{u^{\frac{N}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}} T^{(2m_2/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}}\right)\mathrm{d}u
\end{aligned}$$

$$+\frac{1}{2\pi\mathrm{i}z^{\frac{N+1}{3}}}\frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{3}\sigma_{km_2}}}\int_0^{+\infty}\frac{u^{\frac{N+1}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}}T^{(2m_2/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}}\right)\mathrm{d}u$$
$$+\frac{1}{2\pi\mathrm{i}z^{\frac{N+2}{3}}}\frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N+2}{3}\sigma_{km_2}}}\int_0^{+\infty}\frac{u^{\frac{N+2}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}}T^{(2m_2/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}}\right)\mathrm{d}u$$

for all non-zero values of z in the sector (1.104) and with a sufficiently large N. Following the steps in Subsections 1.4.3 and 1.4.4, we replace the factors $T^{(2m_1/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}}\right)$ and $T^{(2m_2/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}}\right)$ in the integrands by their truncated asymptotic expansions (1.77), and thus find that

$$\begin{aligned}
R_N^{(k/3)}(z) = {} & \frac{1}{3\pi z^{\frac{N}{3}}}\frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N}{3}\sigma_{km_1}}}\sum_{r=0}^{N_{m_1}-1}\mathrm{e}^{-\frac{\pi(r+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(r+1)}{3}\right)a_r^{(m_1/3)}\mathrm{e}^{\mathrm{i}\frac{r}{3}\sigma_{km_1}} \\
& \times\int_0^{+\infty}\frac{u^{\frac{N-r}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}}\mathrm{d}u \\
& +\frac{1}{3\pi z^{\frac{N+1}{3}}}\frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{3}\sigma_{km_1}}}\sum_{r=0}^{N_{m_1}-1}\mathrm{e}^{-\frac{\pi(r+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(r+1)}{3}\right)a_r^{(m_1/3)}\mathrm{e}^{\mathrm{i}\frac{r}{3}\sigma_{km_1}} \\
& \times\int_0^{+\infty}\frac{u^{\frac{N-r+1}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}}\mathrm{d}u \\
& +\frac{1}{3\pi z^{\frac{N+2}{3}}}\frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N+2}{3}\sigma_{km_1}}}\sum_{r=0}^{N_{m_1}-1}\mathrm{e}^{-\frac{\pi(r+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(r+1)}{3}\right)a_r^{(m_1/3)}\mathrm{e}^{\mathrm{i}\frac{r}{3}\sigma_{km_1}} \\
& \times\int_0^{+\infty}\frac{u^{\frac{N-r+2}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}}\mathrm{d}u \\
& +\frac{1}{3\pi z^{\frac{N}{3}}}\frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N}{3}\sigma_{km_2}}}\sum_{r=0}^{N_{m_2}-1}\mathrm{e}^{-\frac{\pi(r+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(r+1)}{3}\right)a_r^{(m_2/3)}\mathrm{e}^{\mathrm{i}\frac{r}{3}\sigma_{km_2}} \\
& \times\int_0^{+\infty}\frac{u^{\frac{N-r}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}}\mathrm{d}u \\
& +\frac{1}{3\pi z^{\frac{N+1}{3}}}\frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{3}\sigma_{km_2}}}\sum_{r=0}^{N_{m_2}-1}\mathrm{e}^{-\frac{\pi(r+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(r+1)}{3}\right)a_r^{(m_2/3)}\mathrm{e}^{\mathrm{i}\frac{r}{3}\sigma_{km_2}} \\
& \times\int_0^{+\infty}\frac{u^{\frac{N-r+1}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}}\mathrm{d}u \\
& +R_{N_{m_1},N_{m_2}}^{(k/3)}(z),
\end{aligned} \tag{1.111}$$

with

$$
\begin{aligned}
R_{N_{m_1},N_{m_2}}^{(k/3)}(z) = {} & \frac{1}{2\pi \mathrm{i} z^{\frac{N}{3}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N}{3}\sigma_{km_1}}} \int_0^{+\infty} \frac{u^{\frac{N}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}} R_{N_{m_1}}^{(2m_1/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}}\right)\mathrm{d}u \\
& + \frac{1}{2\pi \mathrm{i} z^{\frac{N+1}{3}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{3}\sigma_{km_1}}} \int_0^{+\infty} \frac{u^{\frac{N+1}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}} R_{N_{m_1}}^{(2m_1/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}}\right)\mathrm{d}u \\
& + \frac{1}{2\pi \mathrm{i} z^{\frac{N+2}{3}}} \frac{(-1)^{\gamma_{km_1}}}{\mathrm{e}^{\mathrm{i}\frac{N+2}{3}\sigma_{km_1}}} \int_0^{+\infty} \frac{u^{\frac{N+2}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_1}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_1}}} R_{N_{m_1}}^{(2m_1/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}}\right)\mathrm{d}u \\
& + \frac{1}{2\pi \mathrm{i} z^{\frac{N}{3}}} \frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N}{3}\sigma_{km_2}}} \int_0^{+\infty} \frac{u^{\frac{N}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}} R_{N_{m_2}}^{(2m_2/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}}\right)\mathrm{d}u \\
& + \frac{1}{2\pi \mathrm{i} z^{\frac{N+1}{3}}} \frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N+1}{3}\sigma_{km_2}}} \int_0^{+\infty} \frac{u^{\frac{N+1}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}} R_{N_{m_2}}^{(2m_2/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}}\right)\mathrm{d}u \\
& + \frac{1}{2\pi \mathrm{i} z^{\frac{N+2}{3}}} \frac{(-1)^{\gamma_{km_2}}}{\mathrm{e}^{\mathrm{i}\frac{N+2}{3}\sigma_{km_2}}} \int_0^{+\infty} \frac{u^{\frac{N+2}{3}-1}\mathrm{e}^{-|\mathcal{F}_{km_2}|u}}{1-u/z\mathrm{e}^{\mathrm{i}\sigma_{km_2}}} R_{N_{m_2}}^{(2m_2/3)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}}\right)\mathrm{d}u
\end{aligned}
\tag{1.112}
$$

for $-\sigma_{km_1} < \theta < -\sigma_{km_2}$. To ensure the convergence of the integrals in (1.112), we require $N_{m_1}, N_{m_2} < N$. We can express the integrals in (1.111) in terms of terminant functions by using the assumption that $0 < \sigma_{km_1} - \sigma_{km_2} \le 2\pi$. Thus from (1.110) and (1.111), we obtain

$$
\begin{aligned}
& T^{(k/3)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(k/3)}}{z^{\frac{n}{3}}} \\
& + (-1)^{\gamma_{km_1}+1} 3\mathrm{e}^{-\mathcal{F}_{km_1}z} \frac{2}{3\mathrm{i}} \sum_{r=0}^{N_{m_1}-1} \mathrm{e}^{-\frac{\pi(r+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(r+1)}{3}\right) \frac{a_r^{(m_1/3)}}{z^{\frac{r}{3}}} \mathrm{e}^{-2\pi\frac{N-r}{3}\mathrm{i}} \\
& \times \frac{T_{\frac{N-r}{3}}\left(\mathcal{F}_{km_1} z\mathrm{e}^{-\pi\mathrm{i}}\right) + \mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}} T_{\frac{N-r+1}{3}}\left(\mathcal{F}_{km_1} z\mathrm{e}^{-\pi\mathrm{i}}\right) + \mathrm{e}^{-\frac{4\pi}{3}\mathrm{i}} T_{\frac{N-r+2}{3}}\left(\mathcal{F}_{km_1} z\mathrm{e}^{-\pi\mathrm{i}}\right)}{3} \\
& + (-1)^{\gamma_{km_2}+1} 3\mathrm{e}^{-\mathcal{F}_{km_2}z} \frac{2}{3\mathrm{i}} \sum_{r=0}^{N_{m_2}-1} \mathrm{e}^{-\frac{\pi(r+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(r+1)}{3}\right) \frac{a_r^{(m_2/3)}}{z^{\frac{r}{3}}} \\
& \times \frac{T_{\frac{N-r}{3}}\left(\mathcal{F}_{km_2} z\mathrm{e}^{\pi\mathrm{i}}\right) + T_{\frac{N-r+1}{3}}\left(\mathcal{F}_{km_2} z\mathrm{e}^{\pi\mathrm{i}}\right) + T_{\frac{N-r+2}{3}}\left(\mathcal{F}_{km_2} z\mathrm{e}^{\pi\mathrm{i}}\right)}{3} \\
& + R_{N_{m_1},N_{m_2}}^{(k/3)}(z).
\end{aligned}
\tag{1.113}
$$

This is the full and rigorous form of Dingle's formal re-expansion of the remainder term of an asymptotic power series of an integral with cubic endpoint (see [35, eqs. (16) and (17), p. 456]).

To deduce from (1.113) the exponentially improved asymptotic expansion of $T^{(k/3)}(z)$, we may argue as in the previous subsections. From the asymptotic behaviour of the coefficients $a_n^{(k/3)}$ (see Subsection 1.3.3), we infer that the optimal choice for N is $N \approx 3|\mathcal{F}_{km_1}||z| = 3|\mathcal{F}_{km_2}||z|$. Suppose that there exist positive constants $C_{N_{m_1}}$ and $C_{N_{m_2}}$ such that

$$\left|R_{N_{m_1}}^{(2m_1/3)}(u\mathrm{e}^{-\mathrm{i}\sigma_{km_1}})\right| \leq C_{N_{m_1}}|u|^{-N_{m_1}/3} \text{ and } \left|R_{N_{m_2}}^{(2m_2/3)}(u\mathrm{e}^{-\mathrm{i}\sigma_{km_2}})\right| \leq C_{N_{m_2}}|u|^{-N_{m_2}/3}$$

hold for any u with $\arg u$ lying in a small neighbourhood of the origin (this is the case for most applications). Then it is not hard to show that, with the above special choice of N, we have

$$R_{N_{m_1},N_{m_2}}^{(k/3)}(z) = \mathcal{O}_{N_{m_1}}\left(\frac{\mathrm{e}^{-|\mathcal{F}_{km_1}||z|}}{|z|^{\frac{N_{m_1}}{3}}}\right) + \mathcal{O}_{N_{m_2}}\left(\frac{\mathrm{e}^{-|\mathcal{F}_{km_2}||z|}}{|z|^{\frac{N_{m_2}}{3}}}\right) \tag{1.114}$$

as $z \to \infty$ in the closed sector $-\sigma_{km_1} \leq \theta \leq -\sigma_{km_2}$, as long as N_{m_1} and N_{m_2} are fixed or small in comparison with $N = \mathcal{O}(|z|)$. From (1.114) and Olver's result (1.90), we infer that $R_{N_{m_1},N_{m_2}}^{(k/3)}(z)$ has the order of magnitude of the first omitted terms of the second and third series in (1.113). The expression (1.113), with the specific choice of $N \approx 3|\mathcal{F}_{km_1}||z| = 3|\mathcal{F}_{km_2}||z|$, is the exponentially improved asymptotic expansion for $T^{(k/3)}(z)$.

The smooth transition of the Stokes discontinuities can be discussed analogously to the case of $T^{(e)}(z)$, we leave the details to the reader.

CHAPTER 2

ASYMPTOTIC EXPANSIONS FOR LARGE ARGUMENT

In this chapter, we shall consider applications to various special functions of the theory developed in Chapter 1. We derive the exact forms of the well-known large-argument asymptotic expansions of these functions; more precisely, instead of giving the classical formal asymptotic expansion of such a function, we provide its truncated asymptotic expansion and an explicit resurgence-type formula for its remainder term. The resurgence formulae are then exploited in several different ways: they are utilized to provide error bounds, asymptotic expansions for the high-order coefficients and exponentially improved asymptotic expansions complete with error bounds. Most of the functions considered here contain an additional parameter which is fixed or small compared to the argument. Asymptotic expansions where both the argument and the parameter are large will be studied in the next chapter.

The chapter consists of four main sections. In Section 2.1, the large-argument asymptotic expansions of the Hankel functions, the Bessel functions, the modified Bessel functions and those of their derivatives are treated. In Sections 2.2 and 2.3, similar results are given for the closely related Anger–Weber-type, Struve and modified Struve functions, and their derivatives. Finally, Section 2.4 deals with the classical asymptotic expansions of the gamma function and its reciprocal.

2.1 Hankel, Bessel and modified Bessel functions

The large-z asymptotic expansions of the Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$, and the modified Bessel functions $K_\nu(z)$

and $I_\nu(z)$ have a long and rich history. The earliest known attempt to obtain an asymptotic expansion is that of Poisson [102] in 1823 where the special case of $J_0(x)$ with x being positive was considered. He derived a complete asymptotic expansion based on the differential equation satisfied by $J_0(x)$, but gave no investigation of the remainder term. A similar (formal) analysis of $J_1(x)$ is due to Hansen [43, pp. 119–123] from 1843. The asymptotic expansion of $J_n(x)$ for arbitrary integer order was first given by Jacobi [48] in 1849. A rigorous treatment of Poisson's expansion was provided by Lipschitz [54] in 1859 with the aid of contour integration; here Jacobi's result was also studied briefly. The general asymptotic expansions of $J_\nu(z)$ and $Y_\nu(z)$, with a fixed complex ν and large complex z, were obtained (rigorously) by Hankel [42] in his memoir written in 1868. He also gave the corresponding expansions of the Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$. The asymptotic expansion of the modified Bessel function $K_\nu(z)$ was established by Kummer [51] in 1837; this result was reproduced, with the addition of the corresponding formula for $I_\nu(z)$, by Kirchhoff [50] in 1854. For a more detailed historical account, the reader is referred to Watson's monumental treatise on the theory of Bessel functions [117, pp. 194–196].

In modern notation, Hankel's expansions may be written

$$H_\nu^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\omega} \sum_{n=0}^{\infty} \mathrm{i}^n \frac{a_n(\nu)}{z^n}, \tag{2.1}$$

as $z \to \infty$ in the sector $-\pi + \delta \le \theta \le 2\pi - \delta$;

$$H_\nu^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\omega} \sum_{n=0}^{\infty} (-\mathrm{i})^n \frac{a_n(\nu)}{z^n}, \tag{2.2}$$

as $z \to \infty$ in the sector $-2\pi + \delta \le \theta \le \pi - \delta$;

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\cos\omega \sum_{n=0}^{\infty} (-1)^n \frac{a_{2n}(\nu)}{z^{2n}} - \sin\omega \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m+1}(\nu)}{z^{2m+1}}\right) \tag{2.3}$$

and

$$Y_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\sin\omega \sum_{n=0}^{\infty} (-1)^n \frac{a_{2n}(\nu)}{z^{2n}} + \cos\omega \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m+1}(\nu)}{z^{2m+1}}\right), \tag{2.4}$$

as $z \to \infty$ in the sector $|\theta| \le \pi - \delta$, with δ being an arbitrary small positive constant, $\theta = \arg z$ and $\omega = z - \frac{\pi}{2}\nu - \frac{\pi}{4}$ (see, e.g., [96, Sec. 10.17]). The square root in these expansions is defined to be positive on the positive real line and is defined by analytic continuation elsewhere. The coefficients $a_n(\nu)$ are polynomials in ν^2 of degree n; their explicit form will be given in Subsection 2.1.1 below. The results established by Poisson, Hansen and Jacobi are all special cases of the asymptotic expansion (2.3). If 2ν equals an odd integer, then the right-hand sides of (2.1)–(2.4) terminate and represent the corresponding function exactly.

The analogous expansions for the modified Bessel functions are

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{-z} \sum_{n=0}^{\infty} \frac{a_n(\nu)}{z^n}, \tag{2.5}$$

as $z \to \infty$ in the sector $|\theta| \le \frac{3\pi}{2} - \delta$; and

$$I_\nu(z) \sim \frac{\mathrm{e}^{z}}{(2\pi z)^{\frac{1}{2}}} \sum_{n=0}^{\infty} (-1)^n \frac{a_n(\nu)}{z^n} \pm \mathrm{i}\mathrm{e}^{\pm\pi\mathrm{i}\nu} \frac{\mathrm{e}^{-z}}{(2\pi z)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{a_m(\nu)}{z^m}, \tag{2.6}$$

as $z \to \infty$ in the sectors $-\frac{\pi}{2} + \delta \le \pm\theta \le \frac{3\pi}{2} - \delta$, with δ being an arbitrary small positive constant (see, for instance, [96, Sec. 10.40]). The square root in these expansions is defined to be positive when $\theta = 0$, and it is defined elsewhere by analytic continuation. The original result of Kirchhoff omits the second component of the asymptotic expansion (2.6), which is permitted if we restrict z to the smaller sector $|\theta| \le \frac{\pi}{2} - \delta$. (In this sector, the second component is exponentially small compared to any of the terms in the first component for large z and is therefore negligible.) The expansion (2.5) terminates and is exact when 2ν equals an odd integer.

It is important to note that these asymptotic expansions are not uniform with respect to ν; we have to require $\nu^2 = o(|z|)$ in order to satisfy Poincaré's definition. There exist other types of large-z expansions which are valid under the weaker condition $\nu = o(|z|)$ (see, for instance, [10] or [44]); however these expansions do not lend themselves to treatment with our methods.

This section is organized as follows. In Subsection 2.1.1, we obtain resurgence formulae for the Hankel, Bessel and modified Bessel functions, and their derivatives, for large argument. Error bounds for the asymptotic expansions of these functions are established in Subsection 2.1.2. Subsection 2.1.3 deals with the asymptotic behaviour of the corresponding late coefficients. Finally, in Subsection 2.1.4, we derive exponentially improved asymptotic expansions for the above mentioned functions.

2.1.1 The resurgence formulae

In this subsection, we investigate the resurgence properties of the Hankel, Bessel and modified Bessel functions, and that of their derivatives, for large argument.

Perhaps the most convenient way is to start with the study of the modified Bessel function $K_\nu(z)$, as the analogous results for the other functions can be deduced in a simple way through their various relations with $K_\nu(z)$.

For our purposes, the most appropriate integral representation of $K_\nu(z)$ is

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{-z\cosh t}\mathrm{e}^{\nu t}\mathrm{d}t, \tag{2.7}$$

which is valid for $|\theta| < \frac{\pi}{2}$ and every complex ν [96, eq. 10.32.9, p. 252]. The function $\cosh t$ has infinitely many first-order saddle points in the complex t-plane situated at $t = t^{(k)} = \pi i k$ with $k \in \mathbb{Z}$. The path of steepest descent $\mathscr{C}^{(0)}(0)$ through $t^{(0)} = 0$ coincides with the real axis, and its orientation is chosen so that it runs from left to right. Hence we may write

$$K_\nu(z) = \frac{\mathrm{e}^{-z}}{2z^{\frac{1}{2}}}T^{(0)}(z), \tag{2.8}$$

where $T^{(0)}(z)$ is given in (1.54) with the specific choices of $f(t) = \cosh t$ and $g(t) = \mathrm{e}^{\nu t}$. The problem is therefore one of quadratic dependence at the saddle point, which we discussed in Subsection 1.2.1. To determine the domain $\Delta^{(0)}$ corresponding to this problem, the adjacent saddles and contours have to be identified. When $\theta = \pm\pi$, the path $\mathscr{C}^{(0)}(\theta)$ connects to the saddle points $t^{(1)} = \pi\mathrm{i}$ and $t^{(-1)} = -\pi\mathrm{i}$, and these are therefore adjacent to $t^{(0)} = 0$. The corresponding adjacent contours $\mathscr{C}^{(1)}(-\pi)$ and $\mathscr{C}^{(-1)}(\pi)$ are horizontal lines parallel to the real axis (see Figure 2.1); this in turn shows that there cannot be further saddles adjacent to $t^{(0)}$ other than $t^{(1)}$ and $t^{(-1)}$. The domain $\Delta^{(0)}$ is formed by the set of all points between the adjacent contours.

By analytic continuation, the representation (2.8) is valid in a wider range than (2.7), namely in $|\theta| < \pi$. Following the analysis in Subsection 1.2.1, we expand $T^{(0)}(z)$ into a truncated asymptotic power series with remainder,

$$T^{(0)}(z) = \sum_{n=0}^{N-1}\frac{a_n^{(0)}}{z^n} + R_N^{(0)}(z).$$

The conditions posed in Subsection 1.2.1 hold good for the domain $\Delta^{(0)}$ and the functions $f(t) = \cosh t$ and $g(t) = \mathrm{e}^{\nu t}$; closer attention is needed only in the case of the requirement that $g(t)/f^{N+\frac{1}{2}}(t) = o(|t|^{-1})$ as $t \to \infty$ in $\Delta^{(0)}$. It is readily verified that this requirement is satisfied precisely when $|\mathfrak{Re}(\nu)| < N + \frac{1}{2}$. The orientations of the adjacent contours are chosen to be identical to that of $\mathscr{C}^{(0)}(0)$, consequently the orientation anomalies are $\gamma_{01} = 1$ and $\gamma_{0-1} = 0$, respectively. The relevant singulant pair is given by

$$\mathcal{F}_{0\pm1} = \cosh(\pm\pi\mathrm{i}) - \cosh 0 = -2, \quad \arg\mathcal{F}_{0\pm1} = \sigma_{0\pm1} = \pm\pi.$$

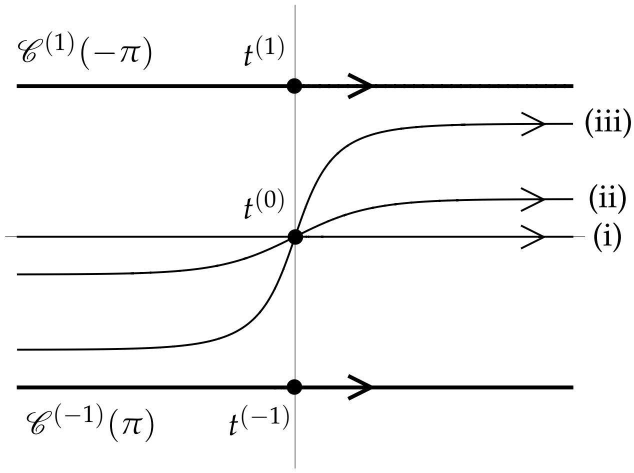

Figure 2.1. *The steepest descent contour $\mathscr{C}^{(0)}(\theta)$ associated with the modified Bessel function of large argument through the saddle point $t^{(0)} = 0$ when (i) $\theta = 0$, (ii) $\theta = -\frac{\pi}{4}$ and (iii) $\theta = -\frac{3\pi}{4}$. The paths $\mathscr{C}^{(1)}(-\pi)$ and $\mathscr{C}^{(-1)}(\pi)$ are the adjacent contours for $t^{(0)}$. The domain $\Delta^{(0)}$ comprises all points between $\mathscr{C}^{(1)}(-\pi)$ and $\mathscr{C}^{(-1)}(\pi)$.*

We thus find that for $|\theta| < \pi$, $N \geq 0$ and $|\mathfrak{Re}(\nu)| < N + \frac{1}{2}$, the remainder $R_N^{(0)}(z)$ may be written

$$\begin{aligned} R_N^{(0)}(z) = &-\frac{(-1)^N}{2\pi i z^N} \int_0^{+\infty} \frac{u^{N-1} e^{-2u}}{1+u/z} T^{(1)}\left(u e^{-\pi i}\right) du \\ &+ \frac{(-1)^N}{2\pi i z^N} \int_0^{+\infty} \frac{u^{N-1} e^{-2u}}{1+u/z} T^{(-1)}\left(u e^{\pi i}\right) du. \end{aligned} \tag{2.9}$$

It is possible to arrive at a simpler result, by observing that we can express the functions $T^{(1)}\left(u e^{-\pi i}\right)$ and $T^{(-1)}\left(u e^{\pi i}\right)$ in terms of $T^{(0)}(u)$. Indeed, by shifting the contour $\mathscr{C}^{(1)}(-\pi)$ downwards by πi, we establish that

$$\begin{aligned} T^{(1)}\left(u e^{-\pi i}\right) &= u^{\frac{1}{2}} e^{-\frac{\pi}{2} i} \int_{\pi i - \infty}^{\pi i + \infty} e^{u(\cosh t - \cosh(\pi i))} e^{\nu t} dt \\ &= -i e^{\pi i \nu} u^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-u(\cosh t - 1)} e^{\nu t} dt = -i e^{\pi i \nu} T^{(0)}(u). \end{aligned}$$

Likewise, one can show that $T^{(-1)}\left(u e^{\pi i}\right) = i e^{-\pi i \nu} T^{(0)}(u)$. Therefore, the representation (2.9) simplifies to

$$R_N^{(0)}(z) = (-1)^N \frac{\cos(\pi \nu)}{\pi} \frac{1}{z^N} \int_0^{+\infty} \frac{u^{N-1} e^{-2u}}{1+u/z} T^{(0)}(u)\, du \tag{2.10}$$

for all non-zero values of z in the sector $|\theta| < \pi$, provided that $N \geq 0$ and $|\mathfrak{Re}(\nu)| < N + \frac{1}{2}$.

We may now connect the above results with the asymptotic expansion (2.5) of $K_\nu(z)$ by writing

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{-z} \left(\sum_{n=0}^{N-1} \frac{a_n(\nu)}{z^n} + R_N^{(K)}(z,\nu)\right), \tag{2.11}$$

with the notation $a_n(\nu) = (2\pi)^{-\frac{1}{2}} a_n^{(0)}$ and $R_N^{(K)}(z,\nu) = (2\pi)^{-\frac{1}{2}} R_N^{(0)}(z)$. Formulae (2.8) and (2.10) then imply

$$R_N^{(K)}(z,\nu) = (-1)^N \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\cos(\pi\nu)}{\pi} \frac{1}{z^N} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}}\mathrm{e}^{-u}}{1+u/z} K_\nu(u)\,\mathrm{d}u \tag{2.12}$$

under the same conditions which were required for (2.10) to hold. Equations (2.11) and (2.12) together yield the exact resurgence formula for $K_\nu(z)$. We remark that the special cases of (2.12) when $-\frac{1}{2} < \nu < \frac{1}{2}$ and when $N = 0$ were also given by Boyd [12] and Erdélyi et al. [39, ent. (39), p. 230], respectively.

Taking $a_n(\nu) = (2\pi)^{-\frac{1}{2}} a_n^{(0)}$ and (1.58) into account, we obtain the following representation for the coefficients $a_n(\nu)$:

$$a_n(\nu) = \frac{1}{2^n \Gamma(n+1)} \left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} \left(\mathrm{e}^{\nu t} \left(\frac{1}{2} \frac{t^2}{\cosh t - 1}\right)^{n+\frac{1}{2}}\right)\right]_{t=0}. \tag{2.13}$$

The expansion of the higher derivatives using Leibniz's formula confirms that $a_n(\nu)$ is indeed a polynomial in ν^2 of degree n.[1] Although (2.13) expresses the coefficients $a_n(\nu)$ in a closed form, it does not provide an efficient method to compute them. A more useful expression can be obtained as follows. Since $a_n(\nu) = z^n(R_n^{(K)}(z,\nu) - R_{n+1}^{(K)}(z,\nu))$, we immediately infer from (2.12) that

$$a_n(\nu) = (-1)^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\cos(\pi\nu)}{\pi} \int_0^{+\infty} u^{n-\frac{1}{2}} \mathrm{e}^{-u} K_\nu(u)\,\mathrm{d}u, \tag{2.14}$$

as long as $|\mathfrak{Re}(\nu)| < n + \frac{1}{2}$. The right-hand side can be evaluated explicitly using the known expression for the Mellin transform of $\mathrm{e}^{-u}K_\nu(u)$ (see, e.g., [38,

[1] Since $\left(\frac{1}{2}\frac{t^2}{\cosh t - 1}\right)^{n+\frac{1}{2}}$ is an even function of t, its even derivatives vanish at $t = 0$. Alternatively, we can appeal to the symmetry relation $K_\nu(z) = K_{-\nu}(z)$ and the uniqueness theorem on the coefficients of asymptotic power series.

ent. (28), p. 331]), giving

$$a_n(\nu) = (-1)^n \frac{\cos(\pi\nu)}{\pi} \frac{\Gamma\left(n+\frac{1}{2}+\nu\right)\Gamma\left(n+\frac{1}{2}-\nu\right)}{2^n \Gamma(n+1)} \tag{2.15}$$

$$= \frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots(4\nu^2-(2n-1)^2)}{8^n \Gamma(n+1)}. \tag{2.16}$$

The condition $|\mathfrak{Re}(\nu)| < n + \frac{1}{2}$ may now be removed by appealing to analytic continuation. This is the representation that was originally given by Hankel.

To obtain the analogous result for the asymptotic expansion (2.6) of the modified Bessel function $I_\nu(z)$, we may proceed as follows. We start with the functional relation

$$I_\nu(z) = \mp \frac{\mathrm{i}}{\pi} K_\nu\left(z\mathrm{e}^{\mp\pi\mathrm{i}}\right) \pm \frac{\mathrm{i}}{\pi} \mathrm{e}^{\pm\pi\mathrm{i}\nu} K_\nu(z)$$

(see, for instance, [96, eq. 10.34.3, p. 253]) and substitute by means of (2.11) to arrive at

$$\begin{aligned} I_\nu(z) = \frac{\mathrm{e}^z}{(2\pi z)^{\frac{1}{2}}} &\left(\sum_{n=0}^{N-1} (-1)^n \frac{a_n(\nu)}{z^n} + R_N^{(K)}\left(z\mathrm{e}^{\mp\pi\mathrm{i}},\nu\right) \right) \\ &\pm \mathrm{i}\mathrm{e}^{\pm\pi\mathrm{i}\nu} \frac{\mathrm{e}^{-z}}{(2\pi z)^{\frac{1}{2}}} \left(\sum_{m=0}^{M-1} \frac{a_m(\nu)}{z^m} + R_M^{(K)}(z,\nu) \right). \end{aligned} \tag{2.17}$$

Assuming that $0 < \pm\theta < \pi$, $N, M \geq 0$ and $|\mathfrak{Re}(\nu)| < \min\left(N+\frac{1}{2}, M+\frac{1}{2}\right)$, equations (2.17) and (2.12) then yield the exact resurgence formula for $I_\nu(z)$.

We may derive the corresponding expressions for the z-derivatives by substituting the results (2.11) and (2.17) into the right-hand sides of the functional relations $-2K'_\nu(z) = K_{\nu-1}(z) + K_{\nu+1}(z)$ and $2I'_\nu(z) = I_{\nu-1}(z) + I_{\nu+1}(z)$ (for these, see [96, eq. 10.29.1, p. 251]). Upon matching our notation with that of [96, Sec. 10.40], we find

$$K'_\nu(z) = -\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{-z} \left(\sum_{n=0}^{N-1} \frac{b_n(\nu)}{z^n} + R_N^{(K')}(z,\nu) \right) \tag{2.18}$$

and

$$\begin{aligned} I'_\nu(z) = \frac{\mathrm{e}^z}{(2\pi z)^{\frac{1}{2}}} &\left(\sum_{n=0}^{N-1} (-1)^n \frac{b_n(\nu)}{z^n} + R_N^{(K')}\left(z\mathrm{e}^{\mp\pi\mathrm{i}},\nu\right) \right) \\ &\mp \mathrm{i}\mathrm{e}^{\pm\pi\mathrm{i}\nu} \frac{\mathrm{e}^{-z}}{(2\pi z)^{\frac{1}{2}}} \left(\sum_{m=0}^{M-1} \frac{b_m(\nu)}{z^m} + R_M^{(K')}(z,\nu) \right), \end{aligned} \tag{2.19}$$

where $2b_n(\nu) = a_n(\nu-1) + a_n(\nu+1)$ and $2R_N^{(K')}(z,\nu) = R_N^{(K)}(z,\nu-1) + R_N^{(K)}(z,\nu+1)$. The complete resurgence formulae can now be written down by employing (2.12). For this, the following assumptions are made: in (2.18), we suppose that $|\theta| < \pi$, $N \geq 1$ and $|\mathfrak{Re}(\nu)| < N - \frac{1}{2}$; whereas in (2.19), we suppose that $0 < \pm\theta < \pi$, $N, M \geq 1$ and $|\mathfrak{Re}(\nu)| < \min\left(N-\frac{1}{2}, M-\frac{1}{2}\right)$. With these provisos, we have

$$R_N^{(K')}(z,\nu) = (-1)^N \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\cos(\pi\nu)}{\pi} \frac{1}{z^N} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}}\mathrm{e}^{-u}}{1+u/z} K'_\nu(u)\,\mathrm{d}u \tag{2.20}$$

(and the same for M). We can derive explicit representations for the coefficients $b_n(\nu)$ by substituting into $2b_n(\nu) = a_n(\nu-1) + a_n(\nu+1)$ those various expressions for the $a_n(\nu)$'s which we have already given earlier. In particular, we find that $b_0(\nu) = 1$ and

$$\begin{aligned} b_n(\nu) &= \frac{1}{2^n \Gamma(n+1)} \left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} \left(\mathrm{e}^{\nu t}\cosh t \left(\frac{1}{2}\frac{t^2}{\cosh t - 1}\right)^{n+\frac{1}{2}}\right)\right]_{t=0} \\ &= (-1)^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\cos(\pi\nu)}{\pi} \int_0^{+\infty} u^{n-\frac{1}{2}}\mathrm{e}^{-u} K'_\nu(u)\,\mathrm{d}u && (2.21) \\ &= (-1)^{n+1} \frac{\cos(\pi\nu)}{\pi} \frac{\Gamma\left(n-\frac{1}{2}+\nu\right)\Gamma\left(n-\frac{1}{2}-\nu\right)(4\nu^2+4n^2-1)}{2^{n+2}\Gamma(n+1)} && (2.22) \\ &= \frac{((4\nu^2-1^2)(4\nu^2-3^2)\cdots(4\nu^2-(2n-3)^2))(4\nu^2+4n^2-1)}{8^n\Gamma(n+1)} \end{aligned}$$

for $n \geq 1$. For the formula (2.21) to hold, the restriction $|\mathfrak{Re}(\nu)| < n - \frac{1}{2}$ is necessary.

Let us now turn our attention to the resurgence properties of the Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$. These functions are directly related to the modified Bessel function $K_\nu(z)$ through the connection formulae

$$H_\nu^{(1)}(z) = \frac{2}{\pi\mathrm{i}}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\nu} K_\nu\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right), \quad -\frac{\pi}{2} \leq \theta \leq \pi \tag{2.23}$$

and

$$H_\nu^{(2)}(z) = -\frac{2}{\pi\mathrm{i}}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\nu} K_\nu\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right), \quad -\pi \leq \theta \leq \frac{\pi}{2} \tag{2.24}$$

(see, e.g., [96, eq. 10.27.8, p. 251]). We substitute (2.12) into the right-hand sides and match the notation with those of (2.1) and (2.2) in order to obtain

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\omega} \left(\sum_{n=0}^{N-1} \mathrm{i}^n \frac{a_n(\nu)}{z^n} + R_N^{(K)}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}, \nu\right)\right) \tag{2.25}$$

and

$$H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\omega}\left(\sum_{n=0}^{N-1}(-\mathrm{i})^n\frac{a_n(\nu)}{z^n} + R_N^{(K)}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\nu\right)\right). \tag{2.26}$$

We now impose the following conditions: in (2.25), we assume that $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, $N \geq 0$ and $|\mathfrak{Re}(\nu)| < N + \frac{1}{2}$; while in (2.26), we assume that $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$, $N \geq 0$ and $|\mathfrak{Re}(\nu)| < N + \frac{1}{2}$. Under these conditions, the expression (2.12) is applicable, and so the resurgence formulae follow at once. Note that here we used implicitly an analytic continuation in the variable z; the connection relations (2.23) and (2.24) imply the expansions (2.25) and (2.26) only in more restricted domains.

The resurgence relations for the z-derivatives can be most readily obtained by substituting (2.25) and (2.26) into the right-hand sides of the connection formulae $2H_\nu^{(1)\prime}(z) = H_{\nu-1}^{(1)}(z) - H_{\nu+1}^{(1)}(z)$ and $2H_\nu^{(2)\prime}(z) = H_{\nu-1}^{(2)}(z) - H_{\nu+1}^{(2)}(z)$ (cf. [96, eq. 10.6.1, p. 222]). Considering the notation of [96, Sec. 10.17], we can write

$$H_\nu^{(1)\prime}(z) = \mathrm{i}\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\omega}\left(\sum_{n=0}^{N-1}\mathrm{i}^n\frac{b_n(\nu)}{z^n} + R_N^{(K')}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}},\nu\right)\right) \tag{2.27}$$

and

$$H_\nu^{(2)\prime}(z) = -\mathrm{i}\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\omega}\left(\sum_{n=0}^{N-1}(-\mathrm{i})^n\frac{b_n(\nu)}{z^n} + R_N^{(K')}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\nu\right)\right). \tag{2.28}$$

We now make the following assumptions: in (2.27), we suppose that $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, $N \geq 1$ and $|\mathfrak{Re}(\nu)| < N - \frac{1}{2}$; whereas in (2.28), we suppose that $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$, $N \geq 1$ and $|\mathfrak{Re}(\nu)| < N - \frac{1}{2}$. With these requirements, formula (2.20) applies and together with (2.27) and (2.28) yields the exact resurgence relations for $H_\nu^{(1)\prime}(z)$ and $H_\nu^{(2)\prime}(z)$.

From the expressions (2.25) and (2.26) for the Hankel functions, we can obtain the corresponding resurgence formulae for the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$. By substituting into the functional relation $2J_\nu(z) = H_\nu^{(1)}(z) + H_\nu^{(2)}(z)$ and employing Euler's formula $\mathrm{e}^{\pm\mathrm{i}\omega} = \cos\omega \pm \mathrm{i}\sin\omega$, we readily establish that

$$\begin{aligned} J_\nu(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \Bigg(\cos\omega\left(\sum_{n=0}^{N-1}(-1)^n\frac{a_{2n}(\nu)}{z^{2n}} + R_{2N}^{(J)}(z,\nu)\right) \\ -\sin\omega\left(\sum_{n=0}^{N-1}(-1)^n\frac{a_{2n+1}(\nu)}{z^{2n+1}} - R_{2N+1}^{(J)}(z,\nu)\right)\Bigg). \end{aligned} \tag{2.29}$$

The remainder terms $R_{2N}^{(J)}(z,\nu)$ and $R_{2N+1}^{(J)}(z,\nu)$ can be expressed by the single formula

$$R_L^{(J)}(z,\nu) = (-1)^{\lfloor L/2\rfloor}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\cos(\pi\nu)}{\pi}\frac{1}{z^L}\int_0^{+\infty}\frac{t^{L-\frac{1}{2}}\mathrm{e}^{-t}}{1+(t/z)^2}K_\nu(t)\,\mathrm{d}t, \tag{2.30}$$

provided that $|\theta| < \frac{\pi}{2}$, $L \geq 0$ and $|\mathfrak{Re}(\nu)| < L + \frac{1}{2}$. (Note that the last condition is equivalent to the requirement $|\mathfrak{Re}(\nu)| < 2N + \frac{1}{2}$ for (2.29).) It is possible to arrive at a slightly more general result in which the truncation indices of the series in (2.29) can be different. For this purpose, let M be a non-negative integer such that $M \geq N$. We expand the denominator in the integrand of (2.30) by means of (1.7) and use (2.14) and (2.30) to deduce

$$-R_{2N+1}^{(J)}(z,\nu) = \sum_{m=N}^{M-1}(-1)^m\frac{a_{2m+1}(\nu)}{z^{2m+1}} - R_{2M+1}^{(J)}(z,\nu). \tag{2.31}$$

Note that the use of the formulae (2.14) and (2.30) is permitted because $|\mathfrak{Re}(\nu)| < 2N + \frac{1}{2} \leq 2m + \frac{1}{2} < 2M + \frac{3}{2}$. Combining equality (2.31) with (2.29) yields

$$\begin{aligned} J_\nu(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\Bigg(&\cos\omega\left(\sum_{n=0}^{N-1}(-1)^n\frac{a_{2n}(\nu)}{z^{2n}} + R_{2N}^{(J)}(z,\nu)\right) \\ &- \sin\omega\left(\sum_{m=0}^{M-1}(-1)^m\frac{a_{2m+1}(\nu)}{z^{2m+1}} - R_{2M+1}^{(J)}(z,\nu)\right)\Bigg) \end{aligned} \tag{2.32}$$

(cf. equation (2.3)). The case $M < N$ can be handled similarly; we replace n and N by m and M in (2.29) and expand the remainder $R_{2M}^{(J)}(z,\nu)$ into a sum of $R_{2N}^{(J)}(z,\nu)$ and $N - M$ other terms. In summary, if $|\theta| < \frac{\pi}{2}$, $N, M \geq 0$ and $|\mathfrak{Re}(\nu)| < \min\left(2N + \frac{1}{2}, 2M + \frac{3}{2}\right)$, equations (2.32) and (2.30) together constitute an exact resurgence formula for $J_\nu(z)$.

In a similar way, starting with the connection formula $2\mathrm{i}Y_\nu(z) = H_\nu^{(1)}(z) - H_\nu^{(2)}(z)$, the analogous expression for the Bessel function $Y_\nu(z)$ is found to be

$$\begin{aligned} Y_\nu(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\Bigg(&\sin\omega\left(\sum_{n=0}^{N-1}(-1)^n\frac{a_{2n}(\nu)}{z^{2n}} + R_{2N}^{(J)}(z,\nu)\right) \\ &+ \cos\omega\left(\sum_{m=0}^{M-1}(-1)^m\frac{a_{2m+1}(\nu)}{z^{2m+1}} - R_{2M+1}^{(J)}(z,\nu)\right)\Bigg) \end{aligned} \tag{2.33}$$

(cf. equation (2.4)). Under the assumptions $|\theta| < \frac{\pi}{2}$, $N, M \geq 0$ and $|\mathfrak{Re}(\nu)| < \min\left(2N + \frac{1}{2}, 2M + \frac{3}{2}\right)$, equations (2.33) and (2.30) yield the required resurgence formula for $Y_\nu(z)$.

We close this subsection by discussing the corresponding resurgence relations for the z-derivatives $J_\nu'(z)$ and $Y_\nu'(z)$. The simplest way to derive these relations is by substituting the expressions (2.32) and (2.33) into the connection formulae $2J_\nu'(z) = J_{\nu-1}(z) - J_{\nu+1}(z)$ and $2Y_\nu'(z) = Y_{\nu-1}(z) - Y_{\nu+1}(z)$ (cf. [96, eq. 10.6.1, p. 222]). Carrying out the necessary calculations and matching our notation with that of [96, Sec. 10.17], we find

$$\begin{aligned} J_\nu'(z) = -\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} &\left(\sin\omega\left(\sum_{n=0}^{N-1}(-1)^n\frac{b_{2n}(\nu)}{z^{2n}} + R_{2N}^{(J')}(z,\nu)\right)\right. \\ &\left.+\cos\omega\left(\sum_{m=0}^{M-1}(-1)^m\frac{b_{2m+1}(\nu)}{z^{2m+1}} - R_{2M+1}^{(J')}(z,\nu)\right)\right) \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} Y_\nu'(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} &\left(\cos\omega\left(\sum_{n=0}^{N-1}(-1)^n\frac{b_{2n}(\nu)}{z^{2n}} + R_{2N}^{(J')}(z,\nu)\right)\right. \\ &\left.-\sin\omega\left(\sum_{m=0}^{M-1}(-1)^m\frac{b_{2m+1}(\nu)}{z^{2m+1}} - R_{2M+1}^{(J')}(z,\nu)\right)\right), \end{aligned} \tag{2.35}$$

where $2R_L^{(J')}(z,\nu) = R_L^{(J)}(z,\nu-1) + R_L^{(J)}(z,\nu+1)$. The complete resurgence formulae now follow by applying (2.30). To this end, assume that $|\theta| < \frac{\pi}{2}$, $L \geq 1$ and $|\mathfrak{Re}(\nu)| < L - \frac{1}{2}$. With these conditions, the remainder $R_L^{(J')}(z,\nu)$ has the following integral representation:

$$R_L^{(J')}(z,\nu) = (-1)^{\lfloor L/2\rfloor}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\cos(\pi\nu)}{\pi}\frac{1}{z^L}\int_0^{+\infty}\frac{t^{L-\frac{1}{2}}e^{-t}}{1+(t/z)^2}K_\nu'(t)\,dt.$$

The corresponding requirements for the expressions (2.34) and (2.35) are $|\theta| < \frac{\pi}{2}$, $N \geq 1$, $M \geq 0$ and $|\mathfrak{Re}(\nu)| < \min\left(2N - \frac{1}{2}, 2M + \frac{1}{2}\right)$.

2.1.2 Error bounds

In this subsection, we derive computable bounds for the remainders $R_N^{(K)}(z,\nu)$, $R_N^{(K')}(z,\nu)$, $R_N^{(J)}(z,\nu)$ and $R_N^{(J')}(z,\nu)$. Unless otherwise stated, we assume that $N \geq 0$ and $|\mathfrak{Re}(\nu)| < N + \frac{1}{2}$ when dealing with $R_N^{(K)}(z,\nu)$ and $R_N^{(J)}(z,\nu)$, and $N \geq 1$ and $|\Re\mathfrak{e}(\nu)| < N - \frac{1}{2}$ are assumed in the cases of $R_N^{(K')}(z,\nu)$ and $R_N^{(J')}(z,\nu)$. We would like to emphasize that the requirement $|\Re\mathfrak{e}(\nu)| < N + \frac{1}{2}$ (respectively, $|\Re\mathfrak{e}(\nu)| < N - \frac{1}{2}$) is not a serious restriction. Indeed, the index of the numerically least term of the asymptotic expansion (2.5), for example, is $n \approx 2|z|$. Therefore, it is reasonable to choose the optimal $N \approx 2|z|$, whereas the condition $\nu^2 = o(|z|)$ has to be fulfilled in order to obtain proper approximations from (2.11).

Bounds for $R_N^{(K)}(z,\nu)$ differing from those we shall derive here were given by Olver using differential equation methods (see [95, exer. 13.2, p. 269] or [96, Subsec. 10.40(iii)]). Boyd [12] obtained bounds for $R_N^{(K)}(z,\nu)$ when $|\theta| \leq \pi$ and $-\frac{1}{2} < \nu < \frac{1}{2}$ starting with the integral representation (2.12). A thorough analysis of $R_N^{(J)}(z,\nu)$ was given in a series of papers by Meijer [60] (see also Döring's paper [36]). Further results concerning the estimation of $R_N^{(K)}(z,\nu)$ and $R_N^{(J)}(z,\nu)$ can be found in Watson's book [117, Ch. VII].

Throughout this work, we shall frequently use the following inequality in constructing error bounds:

$$\frac{1}{|1+re^{i\alpha}|} \leq \begin{cases} |\csc\alpha| & \text{if } \frac{\pi}{2} < |\alpha \bmod 2\pi| < \pi, \\ 1 & \text{if } |\alpha \bmod 2\pi| \leq \frac{\pi}{2}, \end{cases} \tag{2.36}$$

where $r > 0$. The proof is elementary and is left to the reader.

First, we consider the estimation of the remainder terms $R_N^{(K)}(z,\nu)$ and $R_N^{(K')}(z,\nu)$. We begin by replacing in (2.12) the function $K_\nu(u)$ by its integral representation (2.7) and performing the change of variable from u and t to s and t via $s = u\cosh t$. We therefore find

$$\begin{aligned} R_N^{(K)}(z,\nu) = (-1)^N \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\cos(\pi\nu)}{2\pi} \frac{1}{z^N} \int_0^{+\infty} s^{N-\frac{1}{2}} e^{-s} \\ \times \int_{-\infty}^{+\infty} \frac{e^{-\frac{s}{\cosh t}} e^{\nu t} \cosh^{-N-\frac{1}{2}} t}{1 + s/(z\cosh t)} dt ds. \end{aligned} \tag{2.37}$$

We will need the analogous formula for the coefficients $a_N(\nu)$ in deriving our error bounds, which can be most readily obtained by substituting (2.37) into the relation $a_N(\nu) = z^N (R_N^{(K)}(z,\nu) - R_{N+1}^{(K)}(z,\nu))$. Thus we have

$$a_N(\nu) = (-1)^N \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\cos(\pi\nu)}{2\pi} \int_0^{+\infty} s^{N-\frac{1}{2}} e^{-s} \int_{-\infty}^{+\infty} e^{-\frac{s}{\cosh t}} e^{\nu t} \cosh^{-N-\frac{1}{2}} t dt ds. \tag{2.38}$$

Now, from (2.37), one immediately establishes the inequality

$$\left|R_N^{(K)}(z,\nu)\right| \le \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{|\cos(\pi\nu)|}{2\pi} \frac{1}{|z|^N} \int_0^{+\infty} s^{N-\frac{1}{2}} e^{-s} \times \int_{-\infty}^{+\infty} \frac{e^{-\frac{s}{\cosh t}} e^{\mathfrak{Re}(\nu)t} \cosh^{-N-\frac{1}{2}} t}{|1+s/(z\cosh t)|} dt ds.$$

We estimate $1/|1+s/(z\cosh t)|$ using (2.36) and then compare the result with (2.38) in order to obtain the error bound[2]

$$\left|R_N^{(K)}(z,\nu)\right| \le \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|} \frac{|a_N(\mathfrak{Re}(\nu))|}{|z|^N} \begin{cases} |\csc\theta| & \text{if } \frac{\pi}{2} < |\theta| < \pi, \\ 1 & \text{if } |\theta| \le \frac{\pi}{2}. \end{cases} \tag{2.39}$$

If $2\mathfrak{Re}(\nu)$ is an odd integer, then the limiting value has to be taken in this bound. The existence of the limit can be seen from the representation (2.16) of the coefficients $a_N(\mathfrak{Re}(\nu))$ by taking into account the assumption $|\mathfrak{Re}(\nu)| < N + \frac{1}{2}$. The bound (2.39) in the case that ν is real was also given by Watson [117, p. 219]. Likewise, one can show that

$$\left|R_N^{(K')}(z,\nu)\right| \le \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|} \frac{|b_N(\mathfrak{Re}(\nu))|}{|z|^N} \begin{cases} |\csc\theta| & \text{if } \frac{\pi}{2} < |\theta| < \pi, \\ 1 & \text{if } |\theta| \le \frac{\pi}{2}. \end{cases} \tag{2.40}$$

Again, if $2\mathfrak{Re}(\nu)$ is an odd integer, the limiting value is taken in this estimate.

Consider now the special case when z is positive and ν is real. With these assumptions, we have $0 < 1/(1+s/(z\cosh t)) < 1$ in (2.37) and together with (2.38), the mean value theorem of integration shows that

$$R_N^{(K)}(z,\nu) = \frac{a_N(\nu)}{z^N} \Theta_N(z,\nu), \tag{2.41}$$

where $0 < \Theta_N(z,\nu) < 1$ is an appropriate number that depends on z, ν and N. In other words, the remainder term $R_N^{(K)}(z,\nu)$ does not exceed the first neglected term in absolute value and has the same sign provided that $z > 0$ and $-N - \frac{1}{2} < \nu < N + \frac{1}{2}$. This is a well-known property of $R_N^{(K)}(z,\nu)$ (see, for instance, [96, Subsec. 10.40(ii)] or [117, p. 207]). We can prove in a similar manner that

$$R_N^{(K')}(z,\nu) = \frac{b_N(\nu)}{z^N} \Xi_N(z,\nu), \tag{2.42}$$

[2] We note that this bound could have been established directly from (2.12). The alternative representation (2.37) is essential only for the derivation of the estimate (2.43) below. A similar remark applies throughout this work when considering error bounds.

where $0 < \Xi_N(z,\nu) < 1$ is a suitable number which depends on z, ν and N.

In the case that ν is real (or $\mathfrak{Im}(\nu)$ is relatively small) and z lies in the right half-plane, the estimates (2.39) and (2.40) are as sharp as it is reasonable to expect. However, although acceptable in much of the second and fourth quadrants, the bounds (2.39) and (2.40) become inappropriate near the Stokes lines $\theta = \pm\pi$. We now show how alternative estimates can be established that are suitable for the sectors $\frac{\pi}{2} < |\theta| < \frac{3\pi}{2}$ (which include the Stokes lines $\theta = \pm\pi$). We may use (2.11) and (2.20) to define the remainder terms $R_N^{(K)}(z,\nu)$ and $R_N^{(K')}(z,\nu)$ in the sectors $\pi \le |\theta| < \frac{3\pi}{2}$. We choose, for all θ in the range $\frac{\pi}{2} < |\theta| < \frac{3\pi}{2}$ any angle $\varphi = \varphi(\theta)$ which has the following properties: $\frac{\pi}{2} < |\theta - \varphi| < \pi$, and $0 < \varphi < \frac{\pi}{2}$ when $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ while $-\frac{\pi}{2} < \varphi < 0$ when $-\frac{3\pi}{2} < \theta < -\frac{\pi}{2}$. Consider the estimation of $R_N^{(K)}(z,\nu)$. We deform the contour of integration of the s-integral in (2.37) by rotating it through the angle φ. One therefore finds, using analytic continuation, that

$$\begin{aligned}
R_N^{(K)}(z,\nu) = (-1)^N \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\cos(\pi\nu)}{2\pi} \frac{1}{z^N} \int_0^{+\infty e^{i\varphi}} s^{N-\frac{1}{2}} e^{-s} \\
\times \int_{-\infty}^{+\infty} \frac{e^{-\frac{s}{\cosh t}} e^{\nu t} \cosh^{-N-\frac{1}{2}} t}{1 + s/(z\cosh t)} dt ds \\
= (-1)^N \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\cos(\pi\nu)}{2\pi} \frac{1}{z^N} \left(\frac{e^{i\varphi}}{\cos\varphi}\right)^{N+\frac{1}{2}} \int_0^{+\infty} u^{N-\frac{1}{2}} e^{-\frac{ue^{i\varphi}}{\cos\varphi}} \\
\times \int_{-\infty}^{+\infty} \frac{e^{-\frac{ue^{i\varphi}}{\cosh t\cos\varphi}} e^{\nu t} \cosh^{-N-\frac{1}{2}} t}{1 + ue^{i\varphi}/(z\cosh t\cos\varphi)} dt du,
\end{aligned}$$

for $\frac{\pi}{2} < |\theta - \varphi| < \pi$. In passing to the second equality, we have made the change of integration variable from s to u by $u = se^{-i\varphi}\cos\varphi$. Employing the inequality (2.36) and then taking into consideration the expression (2.38), we obtain the error bound

$$\left|R_N^{(K)}(z,\nu)\right| \le \frac{|\csc(\theta-\varphi)|}{\cos^{N+\frac{1}{2}}\varphi} \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|} \frac{|a_N(\mathfrak{Re}(\nu))|}{|z|^N}, \tag{2.43}$$

provided that $\frac{\pi}{2} < |\theta - \varphi| < \pi$. We would like to choose φ to minimize the right-hand side of (2.43). Conveniently, for any $\frac{\pi}{2} < |\theta| < \frac{3\pi}{2}$ there is only one such choice for φ, as can be seen from the following lemma of Meijer [60, pp. 953–954].

Lemma 2.1.1. *Let χ be a fixed positive real number, and let θ be a fixed angle such that $\frac{\pi}{2} < |\theta| < \frac{3\pi}{2}$. Consider the problem of minimizing the quantity $|\csc(\theta-\varphi)|/\cos^\chi\varphi$*

in φ with respect to the following conditions: $\frac{\pi}{2} < |\theta - \varphi| < \pi$, and $0 < \varphi < \frac{\pi}{2}$ when $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ while $-\frac{\pi}{2} < \varphi < 0$ when $-\frac{3\pi}{2} < \theta < -\frac{\pi}{2}$. Under these conditions, the minimization problem has a unique solution φ^* that satisfies the implicit equation

$$(\chi + 1)\cos(\theta - 2\varphi^*) = (\chi - 1)\cos\theta,$$

and has the property that $0 < \varphi^* < -\frac{\pi}{2} + \theta$ if $\frac{\pi}{2} < \theta < \pi$, $-\pi + \theta < \varphi^* < \frac{\pi}{2}$ if $\pi \le \theta < \frac{3\pi}{2}$, $0 < \varphi^* < \frac{\pi}{2} + \theta$ if $-\pi < \theta < -\frac{\pi}{2}$ and $-\frac{\pi}{2} < \varphi^* < \pi + \theta$ if $-\frac{3\pi}{2} < \theta \le -\pi$.

Combining inequality (2.43) and Lemma 2.1.1 (when $\chi = N + \frac{1}{2}$), we obtain the desired error bounds for the sectors $\frac{\pi}{2} < |\theta| < \frac{3\pi}{2}$. Note that the ranges of validity of the bounds (2.39) and (2.43) together cover that of the asymptotic expansion (2.5) for $K_\nu(z)$. One may likewise show that for the remainder term $R_N^{(K')}(z,\nu)$,

$$\left|R_N^{(K')}(z,\nu)\right| \le \frac{|\csc(\theta - \varphi^*)|}{\cos^{N+\frac{1}{2}}\varphi^*} \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|} \frac{|b_N(\mathfrak{Re}(\nu))|}{|z|^N} \tag{2.44}$$

if $\frac{\pi}{2} < |\theta| < \frac{3\pi}{2}$, where φ^* is the minimizing value given by Lemma 2.1.1 with the specific choice of $\chi = N + \frac{1}{2}$.

We may simplify these bounds if θ is close to the Stokes lines as follows. When $\theta = \pi$, the minimizing value φ^* provided by Lemma 2.1.1 (when $\chi = N + \frac{1}{2}$) can be written explicitly as

$$\varphi^* = \operatorname{arccot}\left(\left(N + \tfrac{1}{2}\right)^{\frac{1}{2}}\right).$$

With this specific value of φ^* we may assert that for $\frac{\pi}{2} + \varphi^* < \theta \le \pi$,

$$\frac{\csc(\theta - \varphi^*)}{\cos^{N+\frac{1}{2}}\varphi^*} \le \frac{\csc(\pi - \varphi^*)}{\cos^{N+\frac{1}{2}}\varphi^*} = \left(\frac{2N+3}{2N+1}\right)^{\frac{2N+3}{4}} \left(N + \frac{1}{2}\right)^{\frac{1}{2}} \le \sqrt{\mathrm{e}(N+1)}.$$ [3]

On the other hand, if $N \ge 1$, we have

$$\sqrt{\mathrm{e}(N+1)} \ge \sqrt{2\mathrm{e}} \ge \csc\theta$$

for $\frac{\pi}{2} < \theta \le \frac{\pi}{2} + \varphi^* \le \frac{\pi}{2} + \operatorname{arccot}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right)$. Combining these inequalities with the error bounds (2.39) and (2.43), we can infer that

$$\left|R_N^{(K)}(z,\nu)\right| \le \sqrt{\mathrm{e}(N+1)} \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|} \frac{|a_N(\mathfrak{Re}(\nu))|}{|z|^N} \tag{2.45}$$

[3]Note that the sequence $\left(\frac{2N+3}{2N+1}\right)^{\frac{2N+3}{4}}\left(N+\frac{1}{2}\right)^{\frac{1}{2}}(N+a)^{-\frac{1}{2}}$ is monotonically increasing if and only if $a \ge 1$, in which case it has limit $\sqrt{\mathrm{e}}$.

as long as $\frac{\pi}{2} < \theta \le \pi$ and $N \ge 1$. An application of the Schwarz reflection principle shows that this bound is also valid when $-\pi \le \theta < -\frac{\pi}{2}$. The estimate (2.45) is stronger and more general than the analogous bound derived by Boyd [12, eq. (14)]. In the case that ν is real or $\mathfrak{Im}(\nu)$ is relatively small in magnitude, the bound (2.45) is also sharper than that arising from differential equation methods. One may similarly show that

$$\left|R_N^{(K')}(z,\nu)\right| \le \sqrt{\mathrm{e}(N+1)}\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_N(\mathfrak{Re}(\nu))|}{|z|^N}, \tag{2.46}$$

provided $\frac{\pi}{2} < |\theta| \le \pi$. The estimates (2.45) and (2.46) may be used in conjunction with our earlier results (2.39) and (2.40), respectively.

The appearance of the factor $\sqrt{\mathrm{e}(N+1)}$ in these bounds may give the impression that these estimates are unrealistic for large N, but this is not the case. Suppose, for simplicity, that ν is real. By applying Stirling's formula to the representation (2.16) of the coefficients $a_N(\nu)$, it can readily be shown that inequality (2.45) implies

$$R_N^{(K)}(z,\nu) = \mathcal{O}_\nu\left(\left(\frac{N}{2|z|}\right)^N \mathrm{e}^{-N}\right)$$

for large N. When the asymptotic expansion (2.5) is truncated near its numerically least term, i.e., when $N \approx 2|z|$, the above estimate is reduced to

$$R_N^{(K)}(z,\nu) = \mathcal{O}_\nu(\mathrm{e}^{-2|z|}).$$

This is exactly the magnitude of the exponentially small contribution that appears in the asymptotic expansion of $K_\nu(z)$ as the Stokes lines $\theta = \pm\pi$ are crossed (cf. Proposition 2.1.3 below). Therefore the presence of the factor $\sqrt{\mathrm{e}(N+1)}$ is entirely reasonable and our bounds are realistic near the Stokes lines.

The estimation of the remainder terms $R_N^{(J)}(z,\nu)$ and $R_N^{(J')}(z,\nu)$ can be done in essentially the same way as the estimations of $R_N^{(K)}(z,\nu)$ and $R_N^{(K')}(z,\nu)$ above, and therefore we omit the details. In this way, we may first obtain the analogues of the bounds (2.39) and (2.40),

$$\left|R_N^{(J)}(z,\nu)\right| \le \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_N(\mathfrak{Re}(\nu))|}{|z|^N}\begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \le \frac{\pi}{4} \end{cases}$$

and

$$\left|R_N^{(J')}(z,\nu)\right| \le \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_N(\mathfrak{Re}(\nu))|}{|z|^N}\begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \le \frac{\pi}{4}, \end{cases}$$

respectively. If $2\mathfrak{Re}(\nu)$ is an odd integer, then the limiting values are taken in these bounds.

For the special case when z is positive and ν is real, one finds

$$R_N^{(J)}(z,\nu) = (-1)^{\lceil N/2 \rceil} \frac{a_N(\nu)}{z^N} \Theta_N(z,\nu), \; R_N^{(J')}(z,\nu) = (-1)^{\lceil N/2 \rceil} \frac{b_N(\nu)}{z^N} \Xi_N(z,\nu).$$

Here $0 < \Theta_N(z,\nu) < 1$ and $0 < \Xi_N(z,\nu) < 1$ are appropriate numbers that depend on z, ν and N (cf. equations (2.41) and (2.42)). In particular, the remainders are of the same sign as, and numerically less than, the first terms neglected. In the case of $R_N^{(J)}(z,\nu)$, this is a well-known result (see, e.g., [96, Subsec. 10.17(iii)] or [117, p. 209]).

Let us now consider estimates which are suitable for the sectors $\frac{\pi}{4} < |\theta| < \pi$. For $\frac{\pi}{2} \le |\theta| < \pi$, the remainder terms $R_N^{(J)}(z,\nu)$ and $R_N^{(J')}(z,\nu)$ may be defined via (2.32) and (2.34). We find in a manner analogous to the derivation of (2.43) (and (2.44)) the error bounds

$$\left|R_N^{(J)}(z,\nu)\right| \le \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{N+\frac{1}{2}}\varphi^*} \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|} \frac{|a_N(\mathfrak{Re}(\nu))|}{|z|^N} \tag{2.47}$$

and

$$\left|R_N^{(J')}(z,\nu)\right| \le \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{N+\frac{1}{2}}\varphi^*} \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|} \frac{|b_N(\mathfrak{Re}(\nu))|}{|z|^N}. \tag{2.48}$$

The angle φ^* is chosen so as to minimize $|\csc(2(\theta-\varphi))| / \cos^{N+\frac{1}{2}}\varphi$ as a function of φ. The following lemma of Meijer [60, p. 956] guarantees the existence and uniqueness of such a minimizer.

Lemma 2.1.2. *Let χ be a fixed positive real number, and let θ be a fixed angle such that $\frac{\pi}{4} < |\theta| < \pi$. Consider the problem of minimizing $|\csc(2(\theta-\varphi))| / \cos^{\chi}\varphi$ in φ with respect to the following conditions: $\frac{\pi}{4} < |\theta-\varphi| < \frac{\pi}{2}$, and $0 < \varphi < \frac{\pi}{2}$ when $\frac{\pi}{4} < \theta < \pi$ while $-\frac{\pi}{2} < \varphi < 0$ when $-\pi < \theta < -\frac{\pi}{4}$. Under these conditions, the minimization problem has a unique solution φ^* that satisfies the implicit equation*

$$(\chi+2)\cos(2\theta-3\varphi^*) = (\chi-2)\cos(2\theta-\varphi^*),$$

and has the property that $0 < \varphi^* < -\frac{\pi}{4}+\theta$ if $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, $-\frac{\pi}{2}+\theta < \varphi^* < -\frac{\pi}{4}+\theta$ if $\frac{\pi}{2} \le \theta < \frac{3\pi}{4}$, $-\frac{\pi}{2}+\theta < \varphi^* < \frac{\pi}{2}$ if $\frac{3\pi}{4} \le \theta < \pi$, $\frac{\pi}{4}+\theta < \varphi^* < 0$ if $-\frac{\pi}{2} < \theta < -\frac{\pi}{4}$, $\frac{\pi}{4}+\theta < \varphi^* < \frac{\pi}{2}+\theta$ if $-\frac{3\pi}{4} < \theta \le -\frac{\pi}{2}$ and $-\frac{\pi}{2} < \varphi^* < \frac{\pi}{2}+\theta$ if $-\pi < \theta \le -\frac{3\pi}{4}$.

We remark that a bound equivalent to (2.47) was proved by Meijer [60].

The two simple estimates below are appropriate near the Stokes lines $\theta = \pm\frac{\pi}{2}$ and can be obtained from (2.47) and (2.48) using an argument akin to the derivation of (2.45) (and (2.46)):

$$\left|R_N^{(J)}(z,\nu)\right| \le \frac{1}{2}\sqrt{\mathrm{e}(N+2)}\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_N(\mathfrak{Re}(\nu))|}{|z|^N} \tag{2.49}$$

and

$$\left|R_N^{(J')}(z,\nu)\right| \le \frac{1}{2}\sqrt{\mathrm{e}(N+2)}\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_N(\mathfrak{Re}(\nu))|}{|z|^N},$$

where $\frac{\pi}{4} < |\theta| \le \frac{\pi}{2}$ and $N \ge 3$. The bound (2.49) is sharper and more general than the analogous estimate given by Döring [36, eq. (3.9a)].

2.1.3 Asymptotics for the late coefficients

In this subsection, we investigate the asymptotic nature of the coefficients $a_n(\nu)$ and $b_n(\nu)$ as $n \to +\infty$. One may assume that 2ν is not an odd integer because if $n \ge |\nu| + \frac{1}{2}$, the coefficients are identically zero for such values of ν. We begin by considering the $a_n(\nu)$'s. We replace the function $K_\nu(u)$ in (2.14) by the truncated asymptotic expansion

$$K_\nu(u) = \left(\frac{\pi}{2u}\right)^{\frac{1}{2}}\mathrm{e}^{-u}\left(\sum_{m=0}^{M-1}\frac{a_m(\nu)}{u^m} + R_M^{(K)}(u,\nu)\right), \tag{2.50}$$

where $M \ge 0$, and from (2.39),

$$\left|R_M^{(K)}(u,\nu)\right| \le \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|}{u^M} \tag{2.51}$$

provided $|\mathfrak{Re}(\nu)| < M + \frac{1}{2}$. Thus, from (2.14), (2.50) and (2.51), and provided $n \ge 1$,

$$a_n(\nu) = (-1)^n\frac{\cos(\pi\nu)}{2^n\pi}\left(\sum_{m=0}^{M-1}2^m a_m(\nu)\Gamma(n-m) + A_M(n,\nu)\right) \tag{2.52}$$

where

$$|A_M(n,\nu)| \le \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}2^M|a_M(\mathfrak{Re}(\nu))|\Gamma(n-M), \tag{2.53}$$

as long as $0 \le M \le n-1$. For given large n, the least value of the bound (2.53) occurs when $M \approx \frac{n}{2}$. With this choice of M, the ratio of the error bound to the

values of n, ν and M	$n = 50$, $\nu = 3$, $M = 25$
exact numerical value of $\lvert a_n(\nu)\rvert$	$0.2048807425250108601102333780421908 \times 10^{48}$
approximation (2.52) to $\lvert a_n(\nu)\rvert$	$0.2048807425250108353400340489533 7929 \times 10^{48}$
error	$0.2477019928885083979 \times 10^{32}$
error bound using (2.53)	$0.4920071790575388977 \times 10^{32}$
values of n, ν and M	$n = 50$, $\nu = 5 + 2\mathrm{i}$, $M = 25$
exact numerical value of $\lvert a_n(\nu)\rvert$	$0.6973131453257225518777511152832201 0 \times 10^{50}$
approximation (2.52) to $\lvert a_n(\nu)\rvert$	$0.6973131453256961625578597752658592 5 \times 10^{50}$
error	$0.2638931989134001736085 \times 10^{37}$
error bound using (2.53)	$0.6728131140705554142647 \times 10^{37}$
values of n, ν and M	$n = 100$, $\nu = 5$, $M = 50$
exact numerical value of $\lvert a_n(\nu)\rvert$	$0.3001832682706904003594769730605823 8 \times 10^{126}$
approximation (2.52) to $\lvert a_n(\nu)\rvert$	$0.3001832682706904003594769730606067 7 \times 10^{126}$
error	-0.2439×10^{95}
error bound using (2.53)	0.4856×10^{95}
values of n, ν and M	$n = 100$, $\nu = 7 + 9\mathrm{i}$, $M = 50$
exact numerical value of $\lvert a_n(\nu)\rvert$	$0.1610949654164792993448279373235780 2 \times 10^{138}$
approximation (2.52) to $\lvert a_n(\nu)\rvert$	$0.1610949654164792993468222600495913 8 \times 10^{138}$
error	$-0.199432272601336 \times 10^{118}$
error bound using (2.53)	$0.7119489063948507 \times 10^{119}$

Table 2.1. *Approximations for $\lvert a_n(\nu)\rvert$ with various n and ν, using* (2.52).

leading term in (2.52) is $\mathcal{O}_\nu(n^{-\frac{1}{2}}2^{-n})$. This is the best accuracy we can achieve using the truncated version of the expansion (2.52). Numerical examples illustrating the efficacy of (2.52), truncated optimally, are given in Table 2.1.

One may likewise show that for the coefficients $b_n(\nu)$,

$$b_n(\nu) = (-1)^{n+1}\frac{\cos(\pi\nu)}{2^n\pi}\left(\sum_{m=0}^{M-1} 2^m b_m(\nu)\,\Gamma(n-m) + B_M(n,\nu)\right)$$

where

$$|B_M(n,\nu)| \le \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}2^M\,|b_M(\mathfrak{Re}(\nu))|\,\Gamma(n-M)\,, \tag{2.54}$$

provided that $n \geq 2$, $1 \leq M \leq n-1$ and $|\Re\mathfrak{e}(\nu)| < M - \frac{1}{2}$. One readily establishes that the least value of the bound (2.54) occurs when $M \approx \frac{n}{2}$.

An alternative set of approximations might be derived using the representations (2.15) and (2.22) in conjunction with the known asymptotic expansion of the gamma function $\Gamma(w+a)$ with large w and fixed a (see, e.g., [112, Sec. 6.4]). This approach, however, does not provide bounds for the error terms.

2.1.4 Exponentially improved asymptotic expansions

In this subsection, we give exponentially improved asymptotic expansions for the Hankel, Bessel and modified Bessel functions as well as for their derivatives. Re-expansions for the remainder terms of the asymptotic expansions of the Bessel functions $J_0(z)$ and $Y_0(z)$ were first derived by Stieltjes [105]. Following his ideas, Watson [117, pp. 213–214] gave an analogous re-expansion of the remainder term $R_N^{(K)}(z,0)$. Although both Stieltjes and Watson assumed that z is positive, their results can be proved to hold in the sector $|\theta| \leq \frac{\pi}{2} - \delta$ with an arbitrary fixed small positive δ. More general expansions were established later, using formal methods, by Airey [2] and Dingle [32] [35, pp. 441 and 450]. Their work was placed on rigorous mathematical foundations by Olver [90] and Boyd [12], who derived an exponentially improved asymptotic expansion for the modified Bessel function $K_\nu(z)$ valid when $\frac{\pi}{2} \leq |\theta| \leq \frac{3\pi}{2}$.

Here, we shall re-consider the result of Olver and Boyd. We derive a re-expansion for the remainder term $R_N^{(K)}(z,\nu)$ with the largest possible domain of validity and with explicit error bound. Analogous re-expansions for the remainders $R_N^{(K')}(z,\nu)$, $R_N^{(J)}(z,\nu)$ and $R_N^{(J')}(z,\nu)$ are also provided.

Proposition 2.1.3. *Let M be an arbitrary fixed non-negative integer, and let ν be a fixed complex number. Suppose that $|\theta| \leq \frac{5\pi}{2} - \delta$ with an arbitrary fixed small positive δ, $|z|$ is large and $N = 2|z| + \rho$ with ρ being bounded. Then*

$$R_N^{(K)}(z,\nu) = 2\mathrm{i}\mathrm{e}^{2z}\cos(\pi\nu)\sum_{m=0}^{M-1}(-1)^m\frac{a_m(\nu)}{z^m}T_{N-m}(2z) + R_{N,M}^{(K)}(z,\nu), \tag{2.55}$$

$$R_N^{(K')}(z,\nu) = -2\mathrm{i}\mathrm{e}^{2z}\cos(\pi\nu)\sum_{m=0}^{M-1}(-1)^m\frac{b_m(\nu)}{z^m}T_{N-m}(2z) + R_{N,M}^{(K')}(z,\nu), \tag{2.56}$$

where

$$R_{N,M}^{(K)}(z,\nu),\ R_{N,M}^{(K')}(z,\nu) = \mathcal{O}_{M,\nu,\rho}\left(\frac{\mathrm{e}^{-2|z|}}{|z|^M}\right) \tag{2.57}$$

for $|\theta| \leq \pi$, *and*

$$R_{N,M}^{(K)}(z,\nu), \ R_{N,M}^{(K')}(z,\nu) = \mathcal{O}_{M,\nu,\rho,\delta}\left(\frac{e^{2\mathfrak{Re}(z)}}{|z|^M}\right) \tag{2.58}$$

for $\pi \leq |\theta| \leq \frac{5\pi}{2} - \delta$.

Watson's and Airey's expansions may be deduced from Proposition 2.1.3 by inserting into (2.55) the known asymptotic expansions of the terminant functions $T_{N-m}(2z)$ (see, e.g., [91] or [99, p. 261]).

Proposition 2.1.3 in conjunction with (2.11), (2.17)–(2.19) and (2.25)–(2.28) yields the exponentially improved asymptotic expansions for the Hankel and modified Bessel functions and their derivatives. In particular, formula (2.55) together with (2.11) embraces the three asymptotic expansions (2.5) and

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left(\sum_{n=0}^{\infty}\frac{a_n(\nu)}{z^n} \pm 2ie^{2z}\cos(\pi\nu)\sum_{m=0}^{\infty}(-1)^m\frac{a_m(\nu)}{z^m}\right)$$

which holds when $z \to \infty$ in the sectors $\frac{\pi}{2} + \delta \leq \pm\theta \leq \frac{5\pi}{2} - \delta$ (see, e.g., [89]); furthermore, they give the smooth transition across the Stokes lines $\theta = \pm\pi$.

The main results of this subsection are the explicit bounds on $R_{N,M}^{(K)}(z,\nu)$ and $R_{N,M}^{(K')}(z,\nu)$ presented in Theorem 2.1.4 below. Note that in these results, N does not necessarily depend on z. (Evidently, $R_{N,M}^{(K)}(z,\nu)$ and $R_{N,M}^{(K')}(z,\nu)$ can be defined for arbitrary positive integer N via (2.55) and (2.56), respectively.) We remark that the special case of the estimate (2.59) when $-\frac{1}{2} < \nu < \frac{1}{2}$ was also proved by Boyd [12].

Theorem 2.1.4. *Let N and M be arbitrary fixed non-negative integers such that $M < N$, and let ν be a fixed complex number. Then we have*

$$\begin{aligned}\left|R_{N,M}^{(K)}(z,\nu)\right| \leq 2\left|e^{2z}\cos(\pi\nu)\right|&\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|}{|z|^M}\left|T_{N-M}(2z)\right| \\ &+\frac{|\cos(\pi\nu)|}{\pi}\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|\,\Gamma(N-M)}{2^{N-M}|z|^N}\end{aligned} \tag{2.59}$$

provided that $|\theta| \leq \pi$ *and* $|\mathfrak{Re}(\nu)| < M + \frac{1}{2}$, *and*

$$\begin{aligned}\left|R_{N,M}^{(K')}(z,\nu)\right| \leq 2\left|e^{2z}\cos(\pi\nu)\right|&\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_M(\mathfrak{Re}(\nu))|}{|z|^M}\left|T_{N-M}(2z)\right| \\ &+\frac{|\cos(\pi\nu)|}{\pi}\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_M(\mathfrak{Re}(\nu))|\,\Gamma(N-M)}{2^{N-M}|z|^N}\end{aligned} \tag{2.60}$$

for $|\theta| \leq \pi$, $|\mathfrak{Re}(\nu)| < M - \frac{1}{2}$ *and* $M \geq 1$. *If* $2\mathfrak{Re}(\nu)$ *is an odd integer, then limiting values are taken in these bounds.*

When ν is real or $\mathfrak{Im}(\nu)$ is small, the first terms on the right-hand sides of the inequalities (2.59) and (2.60) are of the same order of magnitude as the first neglected terms in the expansions (2.55) and (2.56), respectively. It can be shown that for large $N - M$, the second terms are comparable with, or less than, the corresponding first terms (except near the zeros of $T_{N-M}(2z)$). The proof is identical to that given by Boyd [12] for the case of $R_{N,M}^{(K)}(z,\nu)$ with $-\frac{1}{2} < \nu < \frac{1}{2}$, and it is therefore not pursued here.

The analogous results for $R_N^{(J)}(z,\nu)$ and $R_N^{(J')}(z,\nu)$ can be deduced immediately from Proposition 2.1.3 using the connection formulae

$$\begin{aligned} 2R_{2N}^{(J)}(z,\nu) &= R_{2N}^{(K)}(ze^{\frac{\pi}{2}i},\nu) + R_{2N}^{(K)}(ze^{-\frac{\pi}{2}i},\nu), \\ 2iR_{2N+1}^{(J)}(z,\nu) &= R_{2N}^{(K)}(ze^{\frac{\pi}{2}i},\nu) - R_{2N}^{(K)}(ze^{-\frac{\pi}{2}i},\nu), \\ 2R_N^{(J')}(z,\nu) &= R_N^{(J)}(z,\nu-1) + R_N^{(J)}(z,\nu+1), \end{aligned} \tag{2.61}$$

and they are as follows.

Proposition 2.1.5. *Let* M *be an arbitrary fixed non-negative integer, and let* ν *be a fixed complex number. Suppose that* $|\theta| \leq 2\pi - \delta$ *with an arbitrary fixed small positive* δ, $|z|$ *is large and* $N = |z| + \rho$ *with* ρ *being bounded. Then*

$$\begin{aligned} R_{2N}^{(J)}(z,\nu) = {}& ie^{2iz}\cos(\pi\nu)\sum_{m=0}^{M-1} i^m \frac{a_m(\nu)}{z^m} T_{2N-m}(2ze^{\frac{\pi}{2}i}) \\ &+ ie^{-2iz}\cos(\pi\nu)\sum_{m=0}^{M-1}(-i)^m \frac{a_m(\nu)}{z^m} T_{2N-m}(2ze^{-\frac{\pi}{2}i}) + R_{2N,M}^{(J)}(z,\nu), \end{aligned}$$

$$\begin{aligned} R_{2N+1}^{(J)}(z,\nu) = {}& e^{2iz}\cos(\pi\nu)\sum_{m=0}^{M-1} i^m \frac{a_m(\nu)}{z^m} T_{2N-m}(2ze^{\frac{\pi}{2}i}) \\ &- e^{-2iz}\cos(\pi\nu)\sum_{m=0}^{M-1}(-i)^m \frac{a_m(\nu)}{z^m} T_{2N-m}(2ze^{-\frac{\pi}{2}i}) + R_{2N+1,M}^{(J)}(z,\nu), \end{aligned}$$

$$\begin{aligned} R_{2N}^{(J')}(z,\nu) = {}& -ie^{2iz}\cos(\pi\nu)\sum_{m=0}^{M-1} i^m \frac{b_m(\nu)}{z^m} T_{2N-m}(2ze^{\frac{\pi}{2}i}) \\ &- ie^{-2iz}\cos(\pi\nu)\sum_{m=0}^{M-1}(-i)^m \frac{b_m(\nu)}{z^m} T_{2N-m}(2ze^{-\frac{\pi}{2}i}) + R_{2N,M}^{(J')}(z,\nu) \end{aligned}$$

and

$$R_{2N+1}^{(J')}(z,\nu) = -\mathrm{e}^{2\mathrm{i}z}\cos(\pi\nu)\sum_{m=0}^{M-1}\mathrm{i}^m\frac{b_m(\nu)}{z^m}T_{2N-m}\left(2z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$$
$$+\mathrm{e}^{-2\mathrm{i}z}\cos(\pi\nu)\sum_{m=0}^{M-1}(-\mathrm{i})^m\frac{b_m(\nu)}{z^m}T_{2N-m}\left(2z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)+R_{2N+1,M}^{(J')}(z,\nu),$$

where

$$R_{2N,M}^{(J)}(z,\nu), R_{2N+1,M}^{(J)}(z,\nu), R_{2N,M}^{(J')}(z,\nu), R_{2N+1,M}^{(J')}(z,\nu) = \mathcal{O}_{M,\nu,\rho}\left(\frac{\mathrm{e}^{-2|z|}}{|z|^M}\right)$$

for $|\theta| \le \frac{\pi}{2}$, *and*

$$R_{2N,M}^{(J)}(z,\nu), R_{2N+1,M}^{(J)}(z,\nu), R_{2N,M}^{(J')}(z,\nu), R_{2N+1,M}^{(J')}(z,\nu) = \mathcal{O}_{M,\nu,\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\mathfrak{Im}(z)}}{|z|^M}\right)$$

for $\frac{\pi}{2} \le \pm\theta \le 2\pi - \delta$.

Explicit bounds for the error terms $R_{2N,M}^{(J)}(z,\nu)$, $R_{2N+1,M}^{(J)}(z,\nu)$, $R_{2N,M}^{(J')}(z,\nu)$ and $R_{2N+1,M}^{(J')}(z,\nu)$ may be derived using Theorem 2.1.4 together with the inequalities

$$2\left|R_{2N+1,M}^{(J)}(z,\nu)\right|, 2\left|R_{2N,M}^{(J)}(z,\nu)\right| \le \left|R_{2N,M}^{(K)}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\nu\right)\right| + \left|R_{2N,M}^{(K)}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}},\nu\right)\right|$$

and

$$2\left|R_{2N+1,M}^{(J')}(z,\nu)\right|, 2\left|R_{2N,M}^{(J')}(z,\nu)\right| \le \left|R_{2N,M}^{(K')}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\nu\right)\right| + \left|R_{2N,M}^{(K')}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}},\nu\right)\right|,$$

which can be established readily from the connection formulae (2.61).

Proof of Proposition 2.1.3 and Theorem 2.1.4. We only prove the statements for $R_N^{(K)}(z,\nu)$ and $R_{N,M}^{(K)}(z,\nu)$; the remainders $R_N^{(K')}(z,\nu)$ and $R_{N,M}^{(K')}(z,\nu)$ can be handled similarly. Let N and M be arbitrary fixed non-negative integers such that $M < N$. Suppose that $|\mathfrak{Re}(\nu)| < M + \frac{1}{2}$ and $|\theta| < \pi$. We begin by replacing the function $K_\nu(u)$ in (2.12) by its truncated asymptotic expansion (2.50) and using the definition of the terminant function, in order to obtain

$$R_N^{(K)}(z,\nu) = 2\mathrm{i}\mathrm{e}^{2z}\cos(\pi\nu)\sum_{m=0}^{M-1}(-1)^m\frac{a_m(\nu)}{z^m}T_{N-m}(2z) + R_{N,M}^{(K)}(z,\nu), \quad (2.62)$$

with

$$\begin{aligned}R_{N,M}^{(K)}(z,\nu) &= (-1)^N \frac{\cos(\pi\nu)}{\pi}\frac{1}{z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2u}}{1+u/z}R_M^{(K)}(u,\nu)\,\mathrm{d}u \\ &= (-1)^N \frac{\cos(\pi\nu)}{\pi}\mathrm{e}^{-\mathrm{i}\theta N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2r\tau}}{1+\tau\mathrm{e}^{-\mathrm{i}\theta}}R_M^{(K)}(r\tau,\nu)\,\mathrm{d}\tau.\end{aligned} \tag{2.63}$$

In passing to the second equality, we have taken $z = r\mathrm{e}^{\mathrm{i}\theta}$ and have made the change of integration variable from u to τ by $u = r\tau$. The remainder $R_M^{(K)}(r\tau,\nu)$ is given by the integral formula (2.12), which can be re-expressed in the form

$$\begin{aligned}R_M^{(K)}(r\tau,\nu) = (-1)^M\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\cos(\pi\nu)}{\pi}\frac{1}{(r\tau)^M}\int_0^{+\infty}\frac{t^{M-\frac{1}{2}}\mathrm{e}^{-t}}{1+t/r}K_\nu(t)\,\mathrm{d}t \\ + (-1)^M\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\cos(\pi\nu)}{\pi}\frac{\tau-1}{(r\tau)^M}\int_0^{+\infty}\frac{t^{M-\frac{1}{2}}\mathrm{e}^{-t}}{(1+r\tau/t)(1+t/r)}K_\nu(t)\,\mathrm{d}t.\end{aligned}$$

Noting that

$$0 < \frac{1}{1+t/r}, \frac{1}{(1+r\tau/t)(1+t/r)} < 1 \tag{2.64}$$

for positive r, τ and t, substitution into (2.63) yields the upper bound

$$\begin{aligned}\left|R_{N,M}^{(K)}(z,\nu)\right| \le \frac{|\cos(\pi\nu)|}{\pi}\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|}{|z|^M}\left|\int_0^{+\infty}\frac{\tau^{N-M-1}\mathrm{e}^{-2r\tau}}{1+\tau\mathrm{e}^{-\mathrm{i}\theta}}\mathrm{d}\tau\right| \\ + \frac{|\cos(\pi\nu)|}{\pi}\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|}{|z|^M}\int_0^{+\infty}\tau^{N-M-1}\mathrm{e}^{-2r\tau}\left|\frac{\tau-1}{\tau+\mathrm{e}^{\mathrm{i}\theta}}\right|\mathrm{d}\tau.\end{aligned}$$

In arriving at this inequality, we have made use of the representation (2.14) of the coefficients $a_M(\nu)$ and the fact that $|K_\nu(t)| \le K_{\mathfrak{Re}(\nu)}(t)$ for any $t > 0$. Since $|(\tau-1)/(\tau+\mathrm{e}^{\mathrm{i}\theta})| \le 1$ for positive τ, after simplification we find that

$$\begin{aligned}\left|R_{N,M}^{(K)}(z,\nu)\right| \le 2\left|\mathrm{e}^{2z}\cos(\pi\nu)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|}{|z|^M}\left|T_{N-M}(2z)\right| \\ + \frac{|\cos(\pi\nu)|}{\pi}\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|\,\Gamma(N-M)}{2^{N-M}|z|^N}.\end{aligned} \tag{2.65}$$

By continuity, this bound holds in the closed sector $|\theta| \le \pi$.[4] This proves the inequality (2.59).

[4]The continuity of $R_{N,M}^{(K)}(z,\nu)$ along the rays $\theta = \pm\pi$ can be seen from (2.62).

From now on, we suppose that $|z|$ is large and that $N = 2|z| + \rho$ with ρ being bounded. Using this assumption and Olver's estimate (1.90), the first term on the right-hand side of the inequality (2.65) is $\mathcal{O}_{M,\nu,\rho}(|z|^{-M} \mathrm{e}^{-2|z|})$. By employing Stirling's formula, the second term is found to be $\mathcal{O}_{M,\nu,\rho}(|z|^{-M-\frac{1}{2}} \mathrm{e}^{-2|z|})$. This establishes the estimate (2.57) for $R_{N,M}^{(K)}(z,\nu)$.

Consider now the sector $\pi \leq \theta \leq \frac{5\pi}{2} - \delta$. For such values of θ, $R_{N,M}^{(K)}(z,\nu)$ can be defined via (2.62). When z enters this sector, the pole of the integrand in (2.63) crosses the integration path. According to the residue theorem, we obtain

$$\begin{aligned} R_{N,M}^{(K)}(z,\nu) = {} & (-1)^N \frac{\cos(\pi\nu)}{\pi} \frac{1}{z^N} \int_0^{+\infty} \frac{t^{N-1}\mathrm{e}^{-2t}}{1+t/z} R_M^{(K)}(t,\nu)\,\mathrm{d}t \\ & + 2\mathrm{i}\mathrm{e}^{2z} \cos(\pi\nu)\, R_M^{(K)}(z\mathrm{e}^{-\pi\mathrm{i}},\nu) \\ = {} & R_{N,M}^{(K)}(z\mathrm{e}^{-2\pi\mathrm{i}},\nu) + 2\mathrm{i}\mathrm{e}^{2z}\cos(\pi\nu)\, R_M^{(K)}(z\mathrm{e}^{-\pi\mathrm{i}},\nu) \end{aligned}$$

for $\pi < \theta < 3\pi$. Now, by analytic continuation,

$$R_{N,M}^{(K)}(z,\nu) = R_{N,M}^{(K)}(z\mathrm{e}^{-2\pi\mathrm{i}},\nu) + 2\mathrm{i}\mathrm{e}^{2z}\cos(\pi\nu)\, R_M^{(K)}(z\mathrm{e}^{-\pi\mathrm{i}},\nu)$$

holds for any complex z, in particular for those lying in the sector $\pi \leq \theta \leq \frac{5\pi}{2} - \delta$. The asymptotic expansion (2.5) implies that $R_M^{(K)}(z\mathrm{e}^{-\pi\mathrm{i}},\nu) = \mathcal{O}_{M,\nu,\delta}(|z|^{-M})$ as $z \to \infty$ in $\pi \leq \theta \leq \frac{5\pi}{2} - \delta$. From (2.57), we infer that $R_{N,M}^{(K)}(z\mathrm{e}^{-2\pi\mathrm{i}},\nu) = \mathcal{O}_{M,\nu,\rho}(|z|^{-M}\mathrm{e}^{-2|z|})$ for large z in the sector $\pi \leq \theta \leq \frac{5\pi}{2} - \delta$. This shows that the estimate (2.58) holds for $R_{N,M}^{(K)}(z,\nu)$ when $\pi \leq \theta \leq \frac{5\pi}{2} - \delta$. The proof for the conjugate sector $-\frac{5\pi}{2} + \delta \leq \theta \leq -\pi$ is completely analogous.

Finally, consider the case when $0 \leq M \leq |\mathfrak{Re}(\nu)| - \frac{1}{2}$. We choose an integer M' such that $|\mathfrak{Re}(\nu)| < M' + \frac{1}{2}$. Then for any complex number z, we have

$$R_{N,M}^{(K)}(z,\nu) = 2\mathrm{i}\mathrm{e}^{2z}\cos(\pi\nu) \sum_{m=M}^{M'-1} (-1)^m \frac{a_m(\nu)}{z^m} T_{N-m}(2z) + R_{N,M'}^{(K)}(z,\nu).$$

The summands on the right-hand side can be estimated by Olver's result (1.90). To estimate $R_{N,M'}^{(K)}(z,\nu)$, we can use (2.57) and (2.58), which have been already proved in the case that $|\mathfrak{Re}(\nu)| < M' + \frac{1}{2}$. We thus establish

$$R_{N,M}^{(K)}(z,\nu) = \sum_{m=M}^{M'-1} \mathcal{O}_{m,\nu,\rho}\left(\frac{\mathrm{e}^{-2|z|}}{z^m}\right) + \mathcal{O}_{M',\nu,\rho}\left(\frac{\mathrm{e}^{-2|z|}}{z^{M'}}\right) = \mathcal{O}_{M,\nu,\rho}\left(\frac{\mathrm{e}^{-2|z|}}{z^M}\right)$$

as $z \to \infty$ in the sector $|\theta| \leq \pi$, and

$$R_{N,M}^{(K)}(z,\nu) = \sum_{m=M}^{M'-1} \mathcal{O}_{m,\nu,\rho,\delta}\left(\frac{\mathrm{e}^{2\mathfrak{Re}(z)}}{z^m}\right) + \mathcal{O}_{M',\nu,\rho,\delta}\left(\frac{\mathrm{e}^{2\mathfrak{Re}(z)}}{z^{M'}}\right) = \mathcal{O}_{M,\nu,\rho,\delta}\left(\frac{\mathrm{e}^{2\mathfrak{Re}(z)}}{z^M}\right)$$

as $z \to \infty$ in the sectors $\pi \le |\theta| \le \frac{5\pi}{2} - \delta$. ■

Alternatively, Proposition 2.1.3 can be proved by using the connection

$$K_\nu(z) = \pi^{\frac{1}{2}} \mathrm{e}^{-z} (2z)^\nu U\left(\nu + \tfrac{1}{2}, 2\nu + 1, 2z\right)$$

of $K_\nu(z)$ and the confluent hypergeometric function, as well as the known exponentially improved expansion for $U(a, a-b+1, z)$ (see, e.g., [92]). The statement for $R_N^{(K')}(z,\nu)$ may then be proved by employing the functional relation $K_{\nu-1}(z) + K_{\nu+1}(z) = -2K'_\nu(z)$. This approach, however, does not provide us with any explicit bounds on the error terms $R_{N,M}^{(K)}(z,\nu)$ and $R_{N,M}^{(K')}(z,\nu)$.

2.2 Anger, Weber and Anger–Weber functions

In this section, we investigate the large-z asymptotic expansions of the Anger function $\mathbf{J}_\nu(z)$, the Weber function $\mathbf{E}_\nu(z)$ and the associated Anger–Weber function $\mathbf{A}_\nu(z)$, and that of their derivatives (for definitions and basic properties, see, e.g., [96, Sec. 11.10]). The asymptotic expansions of the functions $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$ were stated without proof in 1879 by Weber [118] and in the subsequent year by Lommel [55]. They were proved as special cases of much more general formulae by Nielsen [80, p. 228] in 1904 (see also [117, Sec. 10.14]).

In modern notation, the asymptotic expansions may be written

$$\mathbf{J}_\nu(z) \sim J_\nu(z) + \frac{\sin(\pi\nu)}{\pi z}\left(\sum_{n=0}^{\infty} \frac{F_n(\nu)}{z^{2n}} - \frac{\nu}{z}\sum_{m=0}^{\infty}\frac{G_m(\nu)}{z^{2m}}\right), \tag{2.66}$$

$$\mathbf{E}_\nu(z) \sim -Y_\nu(z) - \frac{1+\cos(\pi\nu)}{\pi z}\sum_{n=0}^{\infty}\frac{F_n(\nu)}{z^{2n}} - \frac{\nu(1-\cos(\pi\nu))}{\pi z^2}\sum_{m=0}^{\infty}\frac{G_m(\nu)}{z^{2m}} \tag{2.67}$$

and

$$\mathbf{A}_\nu(z) \sim \frac{1}{\pi z}\sum_{n=0}^{\infty}\frac{F_n(\nu)}{z^{2n}} - \frac{\nu}{\pi z^2}\sum_{m=0}^{\infty}\frac{G_m(\nu)}{z^{2m}}, \tag{2.68}$$

as $z \to \infty$ in the sector $|\theta| \le \pi - \delta$, where δ denotes an arbitrary small positive constant and $\theta = \arg z$ (see, for instance, [96, Subsec. 11.11(i)]). The coefficients $F_n(\nu)$ and $G_n(\nu)$ are polynomials in ν^2 of degree n; their explicit forms will be given in Subsection 2.2.1 below.

We would like to emphasize that, just as in the case of the Bessel functions, these asymptotic expansions are not uniform with respect to ν; we have to require $\nu = o(|z|)$ in order to satisfy Poincaré's definition.

The structure of this section is similar to that of Section 2.1. In Subsection 2.2.1, we derive resurgence formulae for the Anger, Weber and Anger–Weber functions, and their derivatives, for large argument. We give computable error bounds for the asymptotic expansions of these functions in Subsection 2.2.2. In Subsection 2.2.3, the asymptotic nature of the corresponding late coefficients is considered. Finally, in Subsection 2.2.4, we obtain exponentially improved asymptotic expansions for the above mentioned functions.

2.2.1 The resurgence formulae

In this subsection, we investigate the resurgence properties of the Anger, Weber and Anger–Weber functions, and their derivatives, for large argument. We will begin with the study of the Anger–Weber function $\mathbf{A}_\nu(z)$, and will obtain the corresponding results for the other functions using their functional relations with $\mathbf{A}_\nu(z)$. The Anger–Weber function $\mathbf{A}_\nu(z)$ may be defined by the integral

$$\mathbf{A}_\nu(z) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_0^{+\infty} \mathrm{e}^{-z\sinh t}\mathrm{e}^{-\nu t}\mathrm{d}t, \tag{2.69}$$

which converges when $|\theta| < \frac{\pi}{2}$ and for every complex ν [96, eq. 11.10.4, p. 295]. The function $\sinh t$ has infinitely many first-order saddle points in the complex t-plane located at $t = t^{(k)} = \pi\left(k+\frac{1}{2}\right)\mathrm{i}$ with $k \in \mathbb{Z}$. The path of steepest descent $\mathscr{P}^{(o)}(0)$ issuing from the origin coincides with the positive real axis, and its orientation is chosen so that it leads away from 0. Hence we may write

$$\mathbf{A}_\nu(z) = \frac{1}{\pi z}T^{(o)}(z),$$

where $T^{(o)}(z)$ is given in (1.3) with the specific choices of $f(t) = \sinh t$ and $g(t) = \mathrm{e}^{-\nu t}$. The problem is therefore one of linear dependence at the endpoint, which we considered in Subsection 1.1.1. To identify the domain $\Delta^{(o)}$ corresponding to this problem, we have to determine the adjacent saddles and contours. When $\theta = -\frac{\pi}{2}$, the path $\mathscr{P}^{(o)}(\theta)$ connects to the saddle point $t^{(0)} = \frac{\pi}{2}\mathrm{i}$, whereas when $\theta = \frac{\pi}{2}$, it connects to the saddle point $t^{(-1)} = -\frac{\pi}{2}\mathrm{i}$. These are therefore adjacent to 0. The corresponding adjacent contours $\mathscr{C}^{(0)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(-1)}\left(\frac{\pi}{2}\right)$ are horizontal lines parallel to the real axis (see Figure 2.2); this in turn shows that there cannot be further saddles adjacent to the origin other than $t^{(0)}$ and $t^{(-1)}$. The domain $\Delta^{(o)}$ is formed by the set of all points between the adjacent contours.

Following the analysis in Subsection 1.1.1, we expand $T^{(o)}(z)$ into a truncated asymptotic power series with remainder,

$$T^{(o)}(z) = \sum_{n=0}^{N-1}\frac{a_n^{(o)}}{z^n} + R_N^{(o)}(z).$$

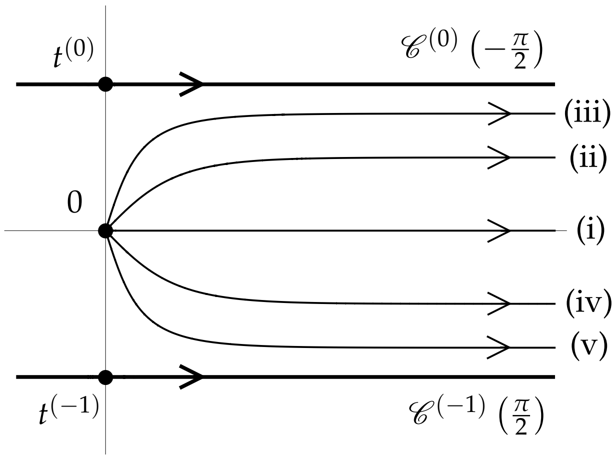

Figure 2.2. *The steepest descent contour $\mathscr{P}^{(o)}(\theta)$ associated with the Anger–Weber function of large argument emanating from the origin when (i) $\theta = 0$, (ii) $\theta = -\frac{\pi}{4}$, (iii) $\theta = -\frac{2\pi}{5}$, (iv) $\theta = \frac{\pi}{4}$ and (v) $\theta = \frac{2\pi}{5}$. The paths $\mathscr{C}^{(0)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(-1)}\left(\frac{\pi}{2}\right)$ are the adjacent contours for 0. The domain $\Delta^{(o)}$ comprises all points between $\mathscr{C}^{(0)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(-1)}\left(\frac{\pi}{2}\right)$.*

The conditions posed in Subsection 1.1.1 hold true for the domain $\Delta^{(o)}$ and the functions $f(t) = \sinh t$, $g(t) = \mathrm{e}^{-\nu t}$; only the requirement that $g(t)/f^{N+1}(t) = o(|t|^{-1})$ as $t \to \infty$ in $\Delta^{(o)}$ needs closer attention. It is readily verified that this requirement is fulfilled precisely when $|\mathfrak{Re}(\nu)| < N+1$. The orientations of the adjacent contours are chosen to be identical to that of $\mathscr{P}^{(o)}(0)$, therefore the orientation anomalies are $\gamma_{o0} = 1$ and $\gamma_{o-1} = 0$, respectively. The relevant singulant pair is given by

$$\mathcal{F}_{o0} = \sinh\left(\tfrac{\pi}{2}\mathrm{i}\right) - \sinh 0 = \mathrm{i},\ \arg \mathcal{F}_{o0} = \sigma_{o0} = \tfrac{\pi}{2}\ \text{ and}$$
$$\mathcal{F}_{o-1} = \sinh\left(-\tfrac{\pi}{2}\mathrm{i}\right) - \sinh 0 = -\mathrm{i},\ \arg \mathcal{F}_{o-1} = \sigma_{o-1} = -\tfrac{\pi}{2}.$$

We thus find that

$$\begin{aligned} R_N^{(o)}(z) = {} & \frac{(-\mathrm{i})^N \mathrm{e}^{\frac{\pi}{4}\mathrm{i}}}{2\pi z^N} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}}\mathrm{e}^{-u}}{1+\mathrm{i}u/z} T^{(0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \\ & + \frac{\mathrm{i}^N \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}}{2\pi z^N} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}}\mathrm{e}^{-u}}{1-\mathrm{i}u/z} T^{(-1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u, \end{aligned} \tag{2.70}$$

with $|\theta| < \frac{\pi}{2}$, $N \geq 0$ and $|\mathfrak{Re}(\nu)| < N+1$. A representation simpler than (2.70) is available. To derive it, we note that the functions $T^{(0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$ and $T^{(-1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$

can be written in terms of the modified Bessel function $K_\nu(u)$. Indeed, by shifting the contour $\mathscr{C}^{(0)}\left(-\frac{\pi}{2}\right)$ downwards by $\frac{\pi}{2}\mathrm{i}$, we deduce that

$$\begin{aligned} T^{(0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) &= u^{\frac{1}{2}}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}\int_{\frac{\pi}{2}\mathrm{i}-\infty}^{\frac{\pi}{2}\mathrm{i}+\infty}\mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\left(\sinh t-\sinh\left(\frac{\pi}{2}\mathrm{i}\right)\right)}\mathrm{e}^{-\nu t}\mathrm{d}t \\ &= \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\nu}u^{\frac{1}{2}}\mathrm{e}^{u}\int_{-\infty}^{+\infty}\mathrm{e}^{-u\cosh t}\mathrm{e}^{-\nu t}\mathrm{d}t = 2\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\nu}u^{\frac{1}{2}}\mathrm{e}^{u}K_\nu(u), \end{aligned}$$

using the integral formula (2.7) and the symmetry relation $K_{-\nu}(u) = K_\nu(u)$. Similarly, one can show that $T^{(-1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) = 2\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\nu}u^{\frac{1}{2}}\mathrm{e}^{u}K_\nu(u)$. Thus, the representation (2.70) can be simplified to

$$\begin{aligned} R_N^{(o)}(z) = {}& \frac{(-\mathrm{i})^N\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\nu}}{\pi z^N}\int_0^{+\infty}\frac{u^N}{1+\mathrm{i}u/z}K_\nu(u)\,\mathrm{d}u \\ &+\frac{\mathrm{i}^N\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\nu}}{\pi z^N}\int_0^{+\infty}\frac{u^N}{1-\mathrm{i}u/z}K_\nu(u)\,\mathrm{d}u \end{aligned}$$

for all z in the sector $|\theta| < \frac{\pi}{2}$, as long as $N \geq 0$ and $|\mathfrak{Re}(\nu)| < N+1$.

We may now connect the above results with the asymptotic expansion (2.68) of the Anger–Weber function $\mathbf{A}_\nu(z)$. We write $2N$ in place of N and match the notation with that of (2.68) in order to obtain

$$\mathbf{A}_\nu(z) = \frac{1}{\pi z}\left(\sum_{n=0}^{N-1}\frac{F_n(\nu)}{z^{2n}} + R_N^{(\mathbf{A})}(z,\nu)\right) - \frac{\nu}{\pi z^2}\left(\sum_{n=0}^{N-1}\frac{G_n(\nu)}{z^{2n}} + \widetilde{R}_N^{(\mathbf{A})}(z,\nu)\right), \tag{2.71}$$

where $F_n(\nu) = a_{2n}^{(o)}$ and $G_n(\nu) = -a_{2n+1}^{(o)}/\nu$. The remainder terms are related to $R_{2N}^{(o)}(z)$ via $R_N^{(\mathbf{A})}(z,\nu) - (\nu/z)\widetilde{R}_N^{(\mathbf{A})}(z,\nu) = R_{2N}^{(o)}(z)$ and can be expressed as

$$R_N^{(\mathbf{A})}(z,\nu) = (-1)^N\frac{2}{\pi}\cos\left(\frac{\pi\nu}{2}\right)\frac{1}{z^{2N}}\int_0^{+\infty}\frac{u^{2N}}{1+(u/z)^2}K_\nu(u)\,\mathrm{d}u \tag{2.72}$$

and

$$\widetilde{R}_N^{(\mathbf{A})}(z,\nu) = (-1)^N\frac{2}{\pi\nu}\sin\left(\frac{\pi\nu}{2}\right)\frac{1}{z^{2N}}\int_0^{+\infty}\frac{u^{2N+1}}{1+(u/z)^2}K_\nu(u)\,\mathrm{d}u, \tag{2.73}$$

provided $|\mathfrak{Re}(\nu)| < 2N+1$ and $|\mathfrak{Re}(\nu)| < 2N+2$, respectively. We have just proved (2.73) under the condition $|\mathfrak{Re}(\nu)| < 2N+1$, but analytic continuation in ν shows that it is actually valid in the slightly larger domain $|\mathfrak{Re}(\nu)| < 2N+2$.[5]

[5]For this, note that if $u \to 0+$, then $K_\nu(u) = \mathcal{O}(u^{-|\mathfrak{Re}(\nu)|})$ for $\nu \neq 0$ and $K_\nu(u) = \mathcal{O}(\log u)$ for $\nu = 0$ (cf. [96, eqs. 10.30.2 and 10.30.3, p. 252]).

We can arrive at a more general result than (2.71), as we did for the Bessel function $J_\nu(z)$. We find

$$\mathbf{A}_\nu(z) = \frac{1}{\pi z}\left(\sum_{n=0}^{N-1} \frac{F_n(\nu)}{z^{2n}} + R_N^{(\mathbf{A})}(z,\nu)\right) - \frac{\nu}{\pi z^2}\left(\sum_{m=0}^{M-1} \frac{G_m(\nu)}{z^{2m}} + \widetilde{R}_M^{(\mathbf{A})}(z,\nu)\right) \quad (2.74)$$

in a manner similar to the derivation of (2.32), using formulae (2.75) and (2.76) below. If $|\theta| < \frac{\pi}{2}$, $N, M \geq 0$ and $|\mathfrak{Re}(\nu)| < \min(2N+1, 2M+2)$, equations (2.74), (2.72), and (2.73) with M in place of N together yield the desired resurgence formula for the function $\mathbf{A}_\nu(z)$.

Taking $F_n(\nu) = a_{2n}^{(o)}$, $G_n(\nu) = -a_{2n+1}^{(o)}/\nu$ and (1.12) into consideration, we obtain the following explicit formulae for the coefficients:

$$F_n(\nu) = \left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\left(\mathrm{e}^{-\nu t}\left(\frac{t}{\sinh t}\right)^{2n+1}\right)\right]_{t=0}$$

and

$$G_n(\nu) = -\frac{1}{\nu}\left[\frac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}}\left(\mathrm{e}^{-\nu t}\left(\frac{t}{\sinh t}\right)^{2n+2}\right)\right]_{t=0}.$$

Expanding the higher derivatives by Leibniz's formula shows that $F_n(\nu)$ and $G_n(\nu)$ are indeed polynomials in ν^2 of degree n. However, these expressions are not very effective for the practical computation of $F_n(\nu)$ and $G_n(\nu)$. We can obtain more useful expressions as follows. Since $F_n(\nu) = z^{2n}(R_n^{(\mathbf{A})}(z,\nu) - R_{n+1}^{(\mathbf{A})}(z,\nu))$ and $G_n(\nu) = z^{2n}(\widetilde{R}_n^{(\mathbf{A})}(z,\nu) - \widetilde{R}_{n+1}^{(\mathbf{A})}(z,\nu))$, we immediately deduce from (2.72) and (2.73) that

$$F_n(\nu) = (-1)^n \frac{2}{\pi}\cos\left(\frac{\pi\nu}{2}\right)\int_0^{+\infty} u^{2n} K_\nu(u)\,\mathrm{d}u \quad (2.75)$$

for $|\mathfrak{Re}(\nu)| < 2n+1$, and

$$G_n(\nu) = (-1)^n \frac{2}{\pi\nu}\sin\left(\frac{\pi\nu}{2}\right)\int_0^{+\infty} u^{2n+1} K_\nu(u)\,\mathrm{d}u \quad (2.76)$$

for $|\mathfrak{Re}(\nu)| < 2n+2$. The right-hand sides can be evaluated explicitly using the known formula for the Mellin transform of the modified Bessel function $K_\nu(u)$ (see, e.g., [38, ent. (26), p. 331]), giving

$$F_n(\nu) = (-1)^n \frac{2^{2n}}{\pi}\cos\left(\frac{\pi\nu}{2}\right)\Gamma\left(n+\frac{1}{2}+\frac{\nu}{2}\right)\Gamma\left(n+\frac{1}{2}-\frac{\nu}{2}\right) \quad (2.77)$$

$$= (\nu^2-1^2)(\nu^2-3^2)\cdots(\nu^2-(2n-1)^2) \quad (2.78)$$

and

$$G_n(\nu) = (-1)^n \frac{2^{2n+1}}{\pi\nu} \sin\left(\frac{\pi\nu}{2}\right) \Gamma\left(n+1+\frac{\nu}{2}\right) \Gamma\left(n+1-\frac{\nu}{2}\right) \tag{2.79}$$
$$= (\nu^2 - 2^2)(\nu^2 - 4^2)\cdots(\nu^2 - (2n)^2). \tag{2.80}$$

The restrictions on ν may now be dropped by appealing to analytic continuation. These are the representations that were originally given by Weber.

We may derive the corresponding expression for the z-derivative by substituting (2.74) into the right-hand side of the functional relation $2\mathbf{A}'_\nu(z) = \mathbf{A}_{\nu+1}(z) - \mathbf{A}_{\nu-1}(z)$ (this can be proved straightforwardly from the definition (2.69)). One thus finds

$$\mathbf{A}'_\nu(z) = \frac{\nu}{\pi z}\left(\sum_{n=1}^{N-1} \frac{2nG_{n-1}(\nu)}{z^{2n}} + R_N^{(\mathbf{A}')}(z,\nu)\right) - \frac{1}{\pi z^2}\left(\sum_{m=0}^{M-1} \frac{(2m+1)F_m(\nu)}{z^{2m}} + \widetilde{R}_M^{(\mathbf{A}')}(z,\nu)\right), \tag{2.81}$$

with the notation $2\nu R_N^{(\mathbf{A}')}(z,\nu) = R_N^{(\mathbf{A})}(z,\nu+1) - R_N^{(\mathbf{A})}(z,\nu-1)$, $2\widetilde{R}_M^{(\mathbf{A}')}(z,\nu) = (\nu+1)\widetilde{R}_M^{(\mathbf{A})}(z,\nu+1) - (\nu-1)\widetilde{R}_M^{(\mathbf{A})}(z,\nu-1)$. When identifying the coefficients, we have made use of the relations $2(2m+1)F_m(\nu) = (\nu+1)G_m(\nu+1) - (\nu-1)G_m(\nu-1)$ and $4n\nu G_{n-1}(\nu) = F_n(\nu+1) - F_n(\nu-1)$, which can be verified directly from (2.77) and (2.79). The complete resurgence formula can now be written down by applying (2.72) and (2.73). For this, we make the following assumptions: $|\theta| < \frac{\pi}{2}$, $N \geq 1$, $M \geq 0$ and $|\mathfrak{Re}(\nu)| < \min(2N, 2M+1)$. With these provisos, the remainders can be written

$$R_N^{(\mathbf{A}')}(z,\nu) = (-1)^N \frac{2}{\pi\nu} \sin\left(\frac{\pi\nu}{2}\right) \frac{1}{z^{2N}} \int_0^{+\infty} \frac{u^{2N}}{1+(u/z)^2} K'_\nu(u)\,\mathrm{d}u \tag{2.82}$$

and

$$\widetilde{R}_M^{(\mathbf{A}')}(z,\nu) = (-1)^{M+1} \frac{2}{\pi} \cos\left(\frac{\pi\nu}{2}\right) \frac{1}{z^{2M}} \int_0^{+\infty} \frac{u^{2M+1}}{1+(u/z)^2} K'_\nu(u)\,\mathrm{d}u. \tag{2.83}$$

From (2.82) and (2.83) we immediately infer that

$$F_n(\nu) = (-1)^{n+1} \frac{2}{\pi(2n+1)} \cos\left(\frac{\pi\nu}{2}\right) \int_0^{+\infty} u^{2n+1} K'_\nu(u)\,\mathrm{d}u \tag{2.84}$$

for $|\mathfrak{Re}(\nu)| < 2n+1$, and

$$G_n(\nu) = (-1)^{n+1} \frac{2}{\pi\nu(2n+2)} \sin\left(\frac{\pi\nu}{2}\right) \int_0^{+\infty} u^{2n+2} K'_\nu(u)\,\mathrm{d}u \tag{2.85}$$

for $|\mathfrak{Re}(\nu)| < 2n+2$.

Let us now consider the resurgence properties of the Anger function $\mathbf{J}_\nu(z)$ and the Weber function $\mathbf{E}_\nu(z)$. These functions are related to the Anger–Weber function $\mathbf{A}_\nu(z)$ through the connection formulae $\mathbf{J}_\nu(z) = J_\nu(z) + \sin(\pi\nu)\,\mathbf{A}_\nu(z)$ and $\mathbf{E}_\nu(z) = -Y_\nu(z) - \cos(\pi\nu)\,\mathbf{A}_\nu(z) - \mathbf{A}_{-\nu}(z)$ (see, e.g., [96, eqs. 11.10.15 and 11.10.16, p. 296]). We substitute (2.74) into the right-hand sides and match the notation with those of (2.66) and (2.67) in order to obtain

$$\begin{aligned}\mathbf{J}_\nu(z) = J_\nu(z) + \frac{\sin(\pi\nu)}{\pi z}\Bigg(&\sum_{n=0}^{N-1}\frac{F_n(\nu)}{z^{2n}} - \frac{\nu}{z}\sum_{m=0}^{M-1}\frac{G_m(\nu)}{z^{2m}}\\ &+ R_N^{(\mathbf{A})}(z,\nu) - \frac{\nu}{z}\widetilde{R}_M^{(\mathbf{A})}(z,\nu)\Bigg)\end{aligned}\tag{2.86}$$

and

$$\begin{aligned}\mathbf{E}_\nu(z) = -Y_\nu(z) &- \frac{1+\cos(\pi\nu)}{\pi z}\left(\sum_{n=0}^{N-1}\frac{F_n(\nu)}{z^{2n}} + R_N^{(\mathbf{A})}(z,\nu)\right)\\ &- \frac{\nu(1-\cos(\pi\nu))}{\pi z^2}\left(\sum_{m=0}^{M-1}\frac{G_m(\nu)}{z^{2m}} + \widetilde{R}_M^{(\mathbf{A})}(z,\nu)\right).\end{aligned}\tag{2.87}$$

In arriving at these expressions, we have made use of the fact that the coefficients $F_n(\nu)$, $G_m(\nu)$ and the remainders $R_M^{(\mathbf{A})}(z,\nu)$, $\widetilde{R}_M^{(\mathbf{A})}(z,\nu)$ are all even functions of ν. If $|\theta| < \frac{\pi}{2}$, $N, M \geq 0$ and $|\mathfrak{Re}(\nu)| < \min(2N+1, 2M+2)$, equations (2.86), (2.87), (2.72), and (2.73) with M in place of N together give the exact resurgence formulae for $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$.

We finish this subsection by discussing the corresponding resurgence relations for the z-derivatives $\mathbf{J}'_\nu(z)$ and $\mathbf{E}'_\nu(z)$. The simplest way to derive these relations is by substituting the expressions (2.86) and (2.87) into the connection formulae $2\mathbf{J}'_\nu(z) = \mathbf{J}_{\nu-1}(z) - \mathbf{J}_{\nu+1}(z)$ and $2\mathbf{E}'_\nu(z) = \mathbf{E}_{\nu-1}(z) - \mathbf{E}_{\nu+1}(z)$ (cf. [96, eqs. 11.10.34 and 11.10.35, p. 297]). We thus establish

$$\begin{aligned}\mathbf{J}'_\nu(z) = J'_\nu(z) + \frac{\sin(\pi\nu)}{\pi z}\Bigg(&\nu\sum_{n=1}^{N-1}\frac{2nG_{n-1}(\nu)}{z^{2n}} - \frac{1}{z}\sum_{m=0}^{M-1}\frac{(2m+1)F_m(\nu)}{z^{2m}}\\ &+ \nu R_N^{(\mathbf{A}')}(z,\nu) - \frac{1}{z}\widetilde{R}_N^{(\mathbf{A}')}(z,\nu)\Bigg)\end{aligned}\tag{2.88}$$

and

$$\mathbf{E}'_\nu(z) = -Y'_\nu(z) + \frac{\nu(1-\cos(\pi\nu))}{\pi z}\left(\sum_{n=1}^{N-1}\frac{2nG_{n-1}(\nu)}{z^{2n}} + R_N^{(\mathbf{A}')}(z,\nu)\right) + \frac{1+\cos(\pi\nu)}{\pi z^2}\left(\sum_{m=0}^{M-1}\frac{(2m+1)F_m(\nu)}{z^{2m}} + \widetilde{R}_M^{(\mathbf{A}')}(z,\nu)\right). \tag{2.89}$$

Under the assumptions $|\theta| < \frac{\pi}{2}$, $N \geq 1$, $M \geq 0$ and $|\mathfrak{Re}(\nu)| < \min(2N, 2M+1)$, equations (2.82) and (2.83) apply and, together with (2.88) and (2.89), yield the required resurgence formulae for $\mathbf{J}'_\nu(z)$ and $\mathbf{E}'_\nu(z)$.

By neglecting the remainder terms in (2.81), (2.88) and (2.89) and formally extending the sums to infinity, we obtain asymptotic expansions for the functions $\mathbf{A}'_\nu(z)$, $\mathbf{J}'_\nu(z)$ and $\mathbf{E}'_\nu(z)$. Alternatively, one can deduce these expansions directly from (2.66), (2.67) and (2.68) by term-wise differentiation. This latter approach in turn shows that these asymptotic expansions are valid in the sector $|\theta| < \pi - \delta$, with δ being an arbitrary small positive constant.

2.2.2 Error bounds

This subsection is devoted to obtaining computable bounds for the remainders $R_N^{(\mathbf{A})}(z,\nu)$, $\widetilde{R}_N^{(\mathbf{A})}(z,\nu)$, $R_N^{(\mathbf{A}')}(z,\nu)$ and $\widetilde{R}_N^{(\mathbf{A}')}(z,\nu)$. Throughout it, unless otherwise stated, we assume the following: that $N \geq 0$ and $|\mathfrak{Re}(\nu)| < 2N+1$ when considering $R_N^{(\mathbf{A})}(z,\nu)$ and $\widetilde{R}_N^{(\mathbf{A}')}(z,\nu)$; that $N \geq 0$ and $|\mathfrak{Re}(\nu)| < 2N+2$ when considering $\widetilde{R}_N^{(\mathbf{A})}(z,\nu)$; that $N \geq 1$ and $|\mathfrak{Re}(\nu)| < 2N$ when considering $R_N^{(\mathbf{A}')}(z,\nu)$. We should emphasize that, just as in the case of the Bessel functions, the above restrictions on $|\mathfrak{Re}(\nu)|$ are not serious. Indeed, the indices of the numerically smallest terms of the two asymptotic power series in (2.68), for example, are $n, m \approx \frac{1}{2}|z|$. Therefore, it is reasonable to choose the optimal $N, M \approx \frac{1}{2}|z|$, whereas the condition $\nu = o(|z|)$ has to be fulfilled in order to obtain useful approximations from (2.74).

Watson [117, Sec. 10.14] gave bounds, based on (2.69), for the remainders of the asymptotic expansions of $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$, when z is positive and ν is real. A detailed analysis of $R_N^{(\mathbf{A})}(z,\nu)$ and $\widetilde{R}_N^{(\mathbf{A})}(z,\nu)$ for complex z and ν, including estimates, was provided by Meijer [60].

The procedure of obtaining error bounds is essentially the same as in the case of the Bessel and modified Bessel functions discussed in Subsection 2.1.2, and therefore we omit the details. The following estimates are valid in the right half-plane and are useful when z is bounded away from the Stokes lines $\theta = \pm\frac{\pi}{2}$:

$$\left|R_N^{(\mathbf{A})}(z,\nu)\right| \le \frac{\left|\cos\left(\frac{\pi\nu}{2}\right)\right|}{\left|\cos\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\frac{\left|F_N\left(\mathfrak{Re}(\nu)\right)\right|}{|z|^{2N}}\begin{cases}|\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2},\\ 1 & \text{if } |\theta| \le \frac{\pi}{4},\end{cases} \tag{2.90}$$

$$\begin{aligned}\left|\widetilde{R}_N^{(\mathbf{A})}(z,\nu)\right| \le{}& \frac{\left|\mathfrak{Re}(\nu)\right|}{|\nu|}\frac{\left|\sin\left(\frac{\pi\nu}{2}\right)\right|}{\left|\sin\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\\ &\times\frac{\left|G_N\left(\mathfrak{Re}(\nu)\right)\right|}{|z|^{2N}}\begin{cases}|\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2},\\ 1 & \text{if } |\theta| \le \frac{\pi}{4},\end{cases}\end{aligned} \tag{2.91}$$

$$\begin{aligned}\left|R_N^{(\mathbf{A}')}(z,\nu)\right| \le{}& \frac{\left|\mathfrak{Re}(\nu)\right|}{|\nu|}\frac{\left|\sin\left(\frac{\pi\nu}{2}\right)\right|}{\left|\sin\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\\ &\times\frac{\left|2NG_{N-1}\left(\mathfrak{Re}(\nu)\right)\right|}{|z|^{2N}}\begin{cases}|\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2},\\ 1 & \text{if } |\theta| \le \frac{\pi}{4}\end{cases}\end{aligned} \tag{2.92}$$

and

$$\begin{aligned}\left|\widetilde{R}_N^{(\mathbf{A}')}(z,\nu)\right| \le{}& \frac{\left|\cos\left(\frac{\pi\nu}{2}\right)\right|}{\left|\cos\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\\ &\times\frac{\left|(2N+1)F_N\left(\mathfrak{Re}(\nu)\right)\right|}{|z|^{2N}}\begin{cases}|\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2},\\ 1 & \text{if } |\theta| \le \frac{\pi}{4}.\end{cases}\end{aligned} \tag{2.93}$$

If $\mathfrak{Re}(\nu)$ is an odd integer, then the limiting values have to be taken in (2.90) and (2.93). The existence of these limits can be seen from expression (2.78) for the coefficients $F_N(\mathfrak{Re}(\nu))$ by taking into account the assumption $|\mathfrak{Re}(\nu)| < 2N+1$. Similarly, if $\mathfrak{Re}(\nu)$ is an even integer, the limiting values are taken in (2.91) and (2.92). If $\mathfrak{Re}(\nu) \neq 0$, the existence of these limits follows from (2.80) and the restrictions on $|\mathfrak{Re}(\nu)|$; otherwise we use the well-known fact $\lim_{w\to 0}\sin w/w = 1$.

For the special case when z is positive and ν is real, one finds that

$$R_N^{(\mathbf{A})}(z,\nu) = \frac{F_N(\nu)}{z^{2N}}\Theta_N(z,\nu),\ \widetilde{R}_N^{(\mathbf{A})}(z,\nu) = \frac{G_N(\nu)}{z^{2N}}\widetilde{\Theta}_N(z,\nu),$$

$$R_N^{(\mathbf{A}')}(z,\nu) = \frac{2NG_{N-1}(\nu)}{z^{2N}}\Xi_N(z,\nu),\ \widetilde{R}_N^{(\mathbf{A}')}(z,\nu) = \frac{(2N+1)F_N(\nu)}{z^{2N}}\widetilde{\Xi}_N(z,\nu).$$

Here $0 < \Theta_N(z,\nu), \widetilde{\Theta}_N(z,\nu), \Xi_N(z,\nu), \widetilde{\Xi}_N(z,\nu) < 1$ are suitable numbers that depend on z, ν and N. In particular, the remainder terms do not exceed the corresponding first neglected terms in absolute value, and they have the same sign provided that $z > 0$ and that $|\nu|$ satisfies the restrictions given at the beginning of this subsection. This property of $R_N^{(\mathbf{A})}(z,\nu)$ and $\widetilde{R}_N^{(\mathbf{A})}(z,\nu)$ was also proved by Watson [117, p. 315] using methods different from ours.

The estimates (2.90)–(2.93) become singular as θ approaches $\pm\frac{\pi}{2}$ and are therefore not suitable near the Stokes lines $\theta = \pm\frac{\pi}{2}$. We now give alternative bounds that are appropriate for the sectors $\frac{\pi}{4} < |\theta| < \pi$ (which include the Stokes lines $\theta = \pm\frac{\pi}{2}$). We may use (2.74) and (2.81) to define the remainder terms in the sectors $\frac{\pi}{2} \leq |\theta| < \pi$. The bounds are as follows

$$\left|R_N^{(\mathbf{A})}(z,\nu)\right| \leq \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{2N+1}\varphi^*}\frac{\left|\cos\left(\frac{\pi\nu}{2}\right)\right|}{\left|\cos\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\frac{|F_N(\mathfrak{Re}(\nu))|}{|z|^{2N}}, \tag{2.94}$$

$$\left|\widetilde{R}_N^{(\mathbf{A})}(z,\nu)\right| \leq \frac{|\csc(2(\theta-\varphi^{**}))|}{\cos^{2N+2}\varphi^{**}}\frac{|\mathfrak{Re}(\nu)|}{|\nu|}\frac{\left|\sin\left(\frac{\pi\nu}{2}\right)\right|}{\left|\sin\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\frac{|G_N(\mathfrak{Re}(\nu))|}{|z|^{2N}}, \tag{2.95}$$

$$\left|R_N^{(\mathbf{A}')}(z,\nu)\right| \leq \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{2N+1}\varphi^*}\frac{|\mathfrak{Re}(\nu)|}{|\nu|}\frac{\left|\sin\left(\frac{\pi\nu}{2}\right)\right|}{\left|\sin\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\frac{|2NG_{N-1}(\mathfrak{Re}(\nu))|}{|z|^{2N}} \tag{2.96}$$

and

$$\left|\widetilde{R}_N^{(\mathbf{A}')}(z,\nu)\right| \leq \frac{|\csc(2(\theta-\varphi^{**}))|}{\cos^{2N+2}\varphi^{**}}\frac{\left|\cos\left(\frac{\pi\nu}{2}\right)\right|}{\left|\cos\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\frac{|(2N+1)F_N(\mathfrak{Re}(\nu))|}{|z|^{2N}} \tag{2.97}$$

for $\frac{\pi}{4} < |\theta| < \pi$. Here φ^* and φ^{**} are the minimizing values given by Lemma 2.1.2 with the specific choices of $\chi = 2N+1$ and $\chi = 2N+2$, respectively. Note that the ranges of validity of the bounds (2.90)–(2.93) and (2.94)–(2.97) together cover that of the asymptotic expansions of the Anger, Weber and Anger–Weber functions and their derivatives. We remark that bounds equivalent to (2.94) and (2.95) were proved by Meijer [60].

The following simple estimates are suitable near the Stokes lines $\theta = \pm\frac{\pi}{2}$ and can be obtained from (2.94)–(2.97) using an argument similar to that given in Subsection 2.1.2:

$$\left|R_N^{(\mathbf{A})}(z,\nu)\right| \leq \frac{1}{2}\sqrt{e\left(2N+\tfrac{5}{2}\right)}\frac{\left|\cos\left(\frac{\pi\nu}{2}\right)\right|}{\left|\cos\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\frac{|F_N(\mathfrak{Re}(\nu))|}{|z|^{2N}},$$

$$\left|\widetilde{R}_N^{(\mathbf{A})}(z,\nu)\right| \leq \frac{1}{2}\sqrt{e\left(2N+\tfrac{7}{2}\right)}\frac{|\mathfrak{Re}(\nu)|}{|\nu|}\frac{\left|\sin\left(\frac{\pi\nu}{2}\right)\right|}{\left|\sin\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\frac{|G_N(\mathfrak{Re}(\nu))|}{|z|^{2N}},$$

$$\left|R_N^{(\mathbf{A}')}(z,\nu)\right| \leq \frac{1}{2}\sqrt{e\left(2N+\tfrac{5}{2}\right)}\frac{|\mathfrak{Re}(\nu)|}{|\nu|}\frac{\left|\sin\left(\frac{\pi\nu}{2}\right)\right|}{\left|\sin\left(\frac{\pi\mathfrak{Re}(\nu)}{2}\right)\right|}\frac{|2NG_{N-1}(\mathfrak{Re}(\nu))|}{|z|^{2N}}$$

and

$$\left|\widetilde{R}_N^{(\mathbf{A}')}(z,\nu)\right| \le \tfrac{1}{2}\sqrt{\mathrm{e}\left(2N+\tfrac{7}{2}\right)}\frac{\left|\cos\left(\frac{\pi\nu}{2}\right)\right|}{\left|\cos\left(\frac{\pi\Re\mathfrak{e}(\nu)}{2}\right)\right|}\frac{\left|(2N+1)\,F_N\left(\Re\mathfrak{e}\left(\nu\right)\right)\right|}{|z|^{2N}},$$

where $\frac{\pi}{4} < |\theta| \le \frac{\pi}{2}$ and $N \ge 1$. These bounds might be used in conjunction with with our earlier results (2.90)–(2.93).

2.2.3 Asymptotics for the late coefficients

In this subsection, we consider the asymptotic behaviour of the coefficients $F_n(\nu)$ and $G_n(\nu)$ for large n. One may assume that 2ν is not an odd integer when considering $F_n(\nu)$ and that it is not a non-zero even integer when dealing with $G_n(\nu)$, because otherwise, these coefficients are identically zero for such values of ν if n is sufficiently large. We substitute into the right-hand sides of (2.75) and (2.76) the truncated asymptotic expansion (2.50) of $K_\nu(u)$ and use the error bound (2.51) to arrive at following expansions. Firstly,

$$F_n(\nu) = (-1)^n\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos\left(\frac{\pi\nu}{2}\right)\left(\sum_{m=0}^{M-1} a_m(\nu)\,\Gamma\left(2n-m+\frac{1}{2}\right)+A_M(n,\nu)\right) \tag{2.98}$$

where

$$\left|A_M(n,\nu)\right| \le \frac{\left|\cos\left(\pi\nu\right)\right|}{\left|\cos\left(\pi\Re\mathfrak{e}\left(\nu\right)\right)\right|}\left|a_M\left(\Re\mathfrak{e}\left(\nu\right)\right)\right|\Gamma\left(2n-M+\frac{1}{2}\right), \tag{2.99}$$

provided $n \ge 0$, $0 \le M < 2n+\frac{1}{2}$ and $\left|\Re\mathfrak{e}\left(\nu\right)\right| < M+\frac{1}{2}$. Secondly,

$$G_n(\nu) = (-1)^n\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{\nu}\sin\left(\frac{\pi\nu}{2}\right)\left(\sum_{m=0}^{M-1} a_m(\nu)\,\Gamma\left(2n-m+\frac{3}{2}\right)+B_M(n,\nu)\right) \tag{2.100}$$

where

$$\left|B_M(n,\nu)\right| \le \frac{\left|\cos\left(\pi\nu\right)\right|}{\left|\cos\left(\pi\Re\mathfrak{e}\left(\nu\right)\right)\right|}\left|a_M\left(\Re\mathfrak{e}\left(\nu\right)\right)\right|\Gamma\left(2n-M+\frac{3}{2}\right), \tag{2.101}$$

as long as $n \ge 0$, $0 \le M < 2n+\frac{3}{2}$ and $\left|\Re\mathfrak{e}\left(\nu\right)\right| < M+\frac{1}{2}$. For given large n, the least values of the bounds (2.99) and (2.101) occur when $M \approx \frac{4n}{3}$. With this choice of M, the ratios of the error bounds to the leading terms in (2.98) and (2.100) are $\mathcal{O}_\nu(n^{-\frac{1}{2}}9^{-n})$. This is the best accuracy available from truncating the expansions (2.98) and (2.100).

values of n, ν and M	$n = 20$, $\nu = 2$, $M = 27$
exact numerical value of $\|F_n(\nu)\|$	$0.10753759651550311126362302989411847 \times 10^{48}$
approximation (2.98) to $\|F_n(\nu)\|$	$0.10753759651550311126436817593883359 \times 10^{48}$
error	$-0.74514604471512 \times 10^{27}$
error bound using (2.99)	$0.150004254243613 \times 10^{28}$
approximation (2.102) to $\|F_n(\nu)\|$	$0.10753759651550311126335593901006385 \times 10^{48}$
error	$0.26709088405462 \times 10^{27}$
error bound using (2.103)	$0.51832048187004 \times 10^{27}$
values of n, ν and M	$n = 20$, $\nu = 4 + 4\mathrm{i}$, $M = 27$
exact numerical value of $\|F_n(\nu)\|$	$0.27351869444288624491826250578549711 \times 10^{50}$
approximation (2.98) to $\|F_n(\nu)\|$	$0.27351869444288641362495417331163060 \times 10^{50}$
error	$-0.16870669166752613350 \times 10^{35}$
error bound using (2.99)	$0.89989353365282590234 \times 10^{35}$
approximation (2.102) to $\|F_n(\nu)\|$	$0.27351869444288619110422177885737773 \times 10^{50}$
error	$0.53814040726928111938 \times 10^{34}$
error bound using (2.103)	$0.32156531428449840421 \times 10^{35}$
values of n, ν and M	$n = 30$, $\nu = 4$, $M = 40$
exact numerical value of $\|F_n(\nu)\|$	$0.97537883219564525333191584258450035 \times 10^{81}$
approximation (2.98) to $\|F_n(\nu)\|$	$0.97537883219564525333191584258258989 \times 10^{81}$
error	0.191045×10^{52}
error bound using (2.99)	0.377882×10^{52}
approximation (2.102) to $\|F_n(\nu)\|$	$0.97537883219564525333191584258518855 \times 10^{81}$
error	-0.68821×10^{51}
error bound using (2.103)	0.132873×10^{52}
values of n, ν and M	$n = 30$, $\nu = 6 + 3\mathrm{i}$, $M = 40$
exact numerical value of $\|F_n(\nu)\|$	$0.59485355287298605088656685883863293 \times 10^{83}$
approximation (2.98) to $\|F_n(\nu)\|$	$0.59485355287298605088656685174774289 \times 10^{83}$
error	$0.709089004 \times 10^{57}$
error bound using (2.99)	$0.2154510868 \times 10^{58}$
approximation (2.102) to $\|F_n(\nu)\|$	$0.59485355287298605088656686134168941 \times 10^{83}$
error	$-0.250305648 \times 10^{57}$
error bound using (2.103)	$0.777022224 \times 10^{57}$

Table 2.2. *Approximations for $|F_n(\nu)|$ with various n and ν, using* (2.98) *and* (2.102).

A different set of approximations can be derived starting from the integral formulae (2.84) and (2.85). In this way, we obtain the following expansions. Firstly,

$$F_n(\nu) = \frac{(-1)^n}{2n+1}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos\left(\frac{\pi\nu}{2}\right)\left(\sum_{m=0}^{M-1} b_m(\nu)\,\Gamma\left(2n-m+\frac{3}{2}\right)+C_M(n,\nu)\right) \tag{2.102}$$

where

$$|C_M(n,\nu)| \le \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\,|b_M(\mathfrak{Re}(\nu))|\,\Gamma\left(2n-M+\frac{3}{2}\right), \tag{2.103}$$

provided that $n \ge 0$, $1 \le M < 2n+\frac{3}{2}$ and $|\mathfrak{Re}(\nu)| < M-\frac{1}{2}$. Secondly,

$$G_n(\nu) = \frac{(-1)^n}{2n+2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin\left(\frac{\pi\nu}{2}\right)\left(\sum_{m=0}^{M-1} b_m(\nu)\,\Gamma\left(2n-m+\frac{5}{2}\right)+D_M(n,\nu)\right)$$

where

$$|D_M(n,\nu)| \le \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\,|b_M(\mathfrak{Re}(\nu))|\,\Gamma\left(2n-M+\frac{5}{2}\right), \tag{2.104}$$

provided that $n \ge 0$, $1 \le M < 2n+\frac{5}{2}$ and $|\mathfrak{Re}(\nu)| < M-\frac{1}{2}$. One readily establishes that the least values of the bounds (2.103) and (2.104) occur when $M \approx \frac{4n}{3}$. Numerical examples illustrating the efficacy of (2.98) and (2.102), truncated optimally, are given in Table 2.2.

2.2.4 Exponentially improved asymptotic expansions

The aim of this subsection is to give exponentially improved asymptotic expansions for the Anger, Weber and Anger–Weber functions as well as for their derivatives. In the case of the Anger and Weber functions, expansions somewhat similar to ours were derived, using non-rigorous methods, by Dingle [33]. The proof of our results in Proposition 2.2.1 below is essentially the same as that of Proposition 2.1.3 on the analogous expansion for the modified Bessel function, and therefore the proof is omitted.

Proposition 2.2.1. *Let M be an arbitrary fixed non-negative integer, and let ν be a fixed complex number. Suppose that $|\theta| \le 2\pi-\delta$ with an arbitrary fixed small positive δ,*

$|z|$ *is large and* $N = \frac{1}{2}|z| + \rho$ *with* ρ *being bounded. Then*

$$\begin{aligned}
R_N^{(\mathbf{A})}(z,\nu) = {} & (2\pi)^{\frac{1}{2}} \mathrm{e}^{\frac{\pi}{4}\mathrm{i}} \mathrm{e}^{\mathrm{i}z} \cos\left(\frac{\pi\nu}{2}\right) \sum_{m=0}^{M-1} \mathrm{i}^m \frac{a_m(\nu)}{z^{m-\frac{1}{2}}} T_{2N-m+\frac{1}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\
& + (2\pi)^{\frac{1}{2}} \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}} \mathrm{e}^{-\mathrm{i}z} \cos\left(\frac{\pi\nu}{2}\right) \sum_{m=0}^{M-1} (-\mathrm{i})^m \frac{a_m(\nu)}{z^{m-\frac{1}{2}}} T_{2N-m+\frac{1}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
& + R_{N,M}^{(\mathbf{A})}(z,\nu),
\end{aligned}$$

$$\begin{aligned}
\frac{\nu}{z}\widetilde{R}_N^{(\mathbf{A})}(z,\nu) = {} & (2\pi)^{\frac{1}{2}} \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}} \mathrm{e}^{\mathrm{i}z} \sin\left(\frac{\pi\nu}{2}\right) \sum_{m=0}^{M-1} \mathrm{i}^m \frac{a_m(\nu)}{z^{m-\frac{1}{2}}} T_{2N-m+\frac{3}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\
& + (2\pi)^{\frac{1}{2}} \mathrm{e}^{\frac{\pi}{4}\mathrm{i}} \mathrm{e}^{-\mathrm{i}z} \sin\left(\frac{\pi\nu}{2}\right) \sum_{m=0}^{M-1} (-\mathrm{i})^m \frac{a_m(\nu)}{z^{m-\frac{1}{2}}} T_{2N-m+\frac{3}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
& + \widetilde{R}_{N,M}^{(\mathbf{A})}(z,\nu),
\end{aligned}$$

$$\begin{aligned}
\nu R_N^{(\mathbf{A}')}(z,\nu) = {} & -(2\pi)^{\frac{1}{2}} \mathrm{e}^{\frac{\pi}{4}\mathrm{i}} \mathrm{e}^{\mathrm{i}z} \sin\left(\frac{\pi\nu}{2}\right) \sum_{m=0}^{M-1} \mathrm{i}^m \frac{b_m(\nu)}{z^{m-\frac{1}{2}}} T_{2N-m+\frac{1}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\
& - (2\pi)^{\frac{1}{2}} \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}} \mathrm{e}^{-\mathrm{i}z} \sin\left(\frac{\pi\nu}{2}\right) \sum_{m=0}^{M-1} (-\mathrm{i})^m \frac{b_m(\nu)}{z^{m-\frac{1}{2}}} T_{2N-m+\frac{1}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
& + R_{N,M}^{(\mathbf{A}')}(z,\nu)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{z}\widetilde{R}_N^{(\mathbf{A}')}(z,\nu) = {} & (2\pi)^{\frac{1}{2}} \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}} \mathrm{e}^{\mathrm{i}z} \cos\left(\frac{\pi\nu}{2}\right) \sum_{m=0}^{M-1} \mathrm{i}^m \frac{b_m(\nu)}{z^{m-\frac{1}{2}}} T_{2N-m+\frac{3}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\
& + (2\pi)^{\frac{1}{2}} \mathrm{e}^{\frac{\pi}{4}\mathrm{i}} \mathrm{e}^{-\mathrm{i}z} \cos\left(\frac{\pi\nu}{2}\right) \sum_{m=0}^{M-1} (-\mathrm{i})^m \frac{b_m(\nu)}{z^{m-\frac{1}{2}}} T_{2N-m+\frac{3}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
& + \widetilde{R}_{N,M}^{(\mathbf{A}')}(z,\nu),
\end{aligned}$$

where

$$R_{N,M}^{(\mathbf{A})}(z,\nu), \widetilde{R}_{N,M}^{(\mathbf{A})}(z,\nu), R_{N,M}^{(\mathbf{A}')}(z,\nu), \widetilde{R}_{N,M}^{(\mathbf{A}')}(z,\nu) = \mathcal{O}_{M,\nu,\rho}\left(\frac{\mathrm{e}^{-|z|}}{|z|^{M-\frac{1}{2}}}\right)$$

for $|\theta| \le \frac{\pi}{2}$, *and*

$$R_{N,M}^{(\mathbf{A})}(z,\nu), \widetilde{R}_{N,M}^{(\mathbf{A})}(z,\nu), R_{N,M}^{(\mathbf{A}')}(z,\nu), \widetilde{R}_{N,M}^{(\mathbf{A}')}(z,\nu) = \mathcal{O}_{M,\nu,\rho,\delta}\left(\frac{\mathrm{e}^{\mp\Im\mathfrak{m}(z)}}{|z|^{M-\frac{1}{2}}}\right)$$

for $\frac{\pi}{2} \le \pm\theta \le 2\pi - \delta$.

Proposition 2.2.1 together with (2.74), (2.81) and (2.86)–(2.89) gives the exponentially improved asymptotic expansions for the Anger, Weber and Anger–Weber functions and their derivatives.

In their paper [47], Howls and Olde Daalhuis investigated the hyperasymptotic properties of solutions of inhomogeneous linear differential equations with a singularity of rank one. The results in Proposition 2.2.1 can be regarded as special cases of their theory. However, our approach provides not only order estimates, but also explicit, numerically computable bounds for $R_{N,M}^{(\mathbf{A})}(z,\nu)$, $\widetilde{R}_{N,M}^{(\mathbf{A})}(z,\nu)$, $R_{N,M}^{(\mathbf{A}')}(z,\nu)$ and $\widetilde{R}_{N,M}^{(\mathbf{A}')}(z,\nu)$ which are given in the following theorem. Note that in this theorem, N may not necessarily depend on z.

Theorem 2.2.2. *Let N and M be arbitrary fixed non-negative integers such that $M \le 2N$, and let ν be a fixed complex number. Then we have*

$$\begin{aligned}\left|R_{N,M}^{(\mathbf{A})}(z,\nu)\right| \le{} & (2\pi)^{\frac{1}{2}}\left|\mathrm{e}^{\mathrm{i}z}\cos\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|}{|z|^{M-\frac{1}{2}}}\left|T_{2N-M+\frac{1}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\ & + (2\pi)^{\frac{1}{2}}\left|\mathrm{e}^{-\mathrm{i}z}\cos\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|}{|z|^{M-\frac{1}{2}}}\left|T_{2N-M+\frac{1}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\ & + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left|\cos\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|\,\Gamma\left(2N-M+\frac{1}{2}\right)}{|z|^{2N}},\end{aligned}$$

$$\begin{aligned}\left|\widetilde{R}_{N,M}^{(\mathbf{A})}(z,\nu)\right| \le{} & (2\pi)^{\frac{1}{2}}\left|\mathrm{e}^{\mathrm{i}z}\sin\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|}{|z|^{M-\frac{1}{2}}}\left|T_{2N-M+\frac{3}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\ & + (2\pi)^{\frac{1}{2}}\left|\mathrm{e}^{-\mathrm{i}z}\sin\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|}{|z|^{M-\frac{1}{2}}}\left|T_{2N-M+\frac{3}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\ & + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left|\sin\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|a_M(\mathfrak{Re}(\nu))|\,\Gamma\left(2N-M+\frac{3}{2}\right)}{|z|^{2N+1}}\end{aligned}$$

provided that $|\theta| \le \frac{\pi}{2}$ and $|\mathfrak{Re}(\nu)| < M + \frac{1}{2}$, and

$$\begin{aligned}\left|R_{N,M}^{(\mathbf{A}')}(z,\nu)\right| \le{} & (2\pi)^{\frac{1}{2}}\left|\mathrm{e}^{\mathrm{i}z}\sin\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_M(\mathfrak{Re}(\nu))|}{|z|^{M-\frac{1}{2}}}\left|T_{2N-M+\frac{1}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\ & + (2\pi)^{\frac{1}{2}}\left|\mathrm{e}^{-\mathrm{i}z}\sin\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_M(\mathfrak{Re}(\nu))|}{|z|^{M-\frac{1}{2}}}\left|T_{2N-M+\frac{1}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\ & + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left|\sin\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_M(\mathfrak{Re}(\nu))|\,\Gamma\left(2N-M+\frac{1}{2}\right)}{|z|^{2N}},\end{aligned}$$

$$\left|\widetilde{R}_{N,M}^{(\mathbf{A}')}(z,\nu)\right| \le (2\pi)^{\frac{1}{2}}\left|\mathrm{e}^{\mathrm{i}z}\cos\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_M(\mathfrak{Re}(\nu))|}{|z|^{M-\frac{1}{2}}}\left|T_{2N-M+\frac{3}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|$$
$$+ (2\pi)^{\frac{1}{2}}\left|\mathrm{e}^{-\mathrm{i}z}\cos\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_M(\mathfrak{Re}(\nu))|}{|z|^{M-\frac{1}{2}}}\left|T_{2N-M+\frac{3}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|$$
$$+ \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left|\cos\left(\frac{\pi\nu}{2}\right)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|}\frac{|b_M(\mathfrak{Re}(\nu))|\,\Gamma\left(2N-M+\frac{3}{2}\right)}{|z|^{2N+1}}$$

for $|\theta| \le \frac{\pi}{2}$, $|\mathfrak{Re}(\nu)| < M - \frac{1}{2}$ *and* $M \ge 1$. *If* $2\mathfrak{Re}(\nu)$ *is an odd integer, then limiting values are taken in these bounds.*

The proof of Theorem 2.2.2 is essentially the same as the proof of Theorem 2.1.4 and is therefore omitted.

2.3 Struve function and modified Struve function

This section concerns the large-z asymptotic expansions of the Struve function $\mathbf{H}_\nu(z)$, the modified Struve function $\mathbf{L}_\nu(z)$ and their derivatives (for definitions and basic properties, see, e.g., [96, Secs. 11.2 and 11.4]). The asymptotic expansion of $\mathbf{H}_\nu(z)$ was given in 1887 by Rayleigh [109] for the case $\nu = 0$ and in 1882 by Struve [111] for the case $\nu = 1$. The result for arbitrary values of ν was proved by Walker [115, pp. 394–395] in 1904 (see also [117, Sec. 10.42]).

In modern notation, the asymptotic expansions may be written

$$\mathbf{H}_\nu(z) \sim Y_\nu(z) + \frac{1}{\pi}\left(\frac{1}{2}z\right)^{\nu-1}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)\left(\frac{1}{2}z\right)^{2n}} \tag{2.105}$$

as $z \to \infty$ in the sector $|\theta| < \pi - \delta$, and

$$\mathbf{L}_\nu(z) \sim I_\nu(z) \pm \frac{2}{\pi\mathrm{i}}\mathrm{e}^{\pm\pi\mathrm{i}\nu}K_\nu(z) + \frac{1}{\pi}\left(\frac{1}{2}z\right)^{\nu-1}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)\left(\frac{1}{2}z\right)^{2n}} \tag{2.106}$$

as $z \to \infty$ in the sectors $-\frac{\pi}{2} + \delta < \pm\theta < \frac{3\pi}{2} - \delta$, with δ being an arbitrary small positive constant and $\theta = \arg z$ (see, for instance, [35, pp. 375–376] and [96, Subsec. 11.6(i)]). For non-integer ν, z^ν means $\mathrm{e}^{\nu\log z}$ where $\log z$ is defined to be real when $\theta = 0$, and it is defined elsewhere by analytic continuation. The asymptotic expansion (2.106) is usually stated without the term involving $K_\nu(z)$, which is permitted if we restrict z to the smaller sector $|\theta| \le \frac{\pi}{2} - \delta$. The expansions (2.105) and (2.106) terminate and are exact when 2ν equals an odd integer.

An interesting fact about the expansion (2.106), which was claimed by Dingle [35, pp. 389–391] and was proved by the present author [74], is that this expansion also has an asymptotic property when ν is large, $|\arg\nu| \leq \frac{\pi}{2} - \delta$ and z/ν is an arbitrary fixed positive number.

This section is organized as follows. In Subsection 2.3.1, we prove resurgence formulae for the Struve and modified Struve functions, and their derivatives, for large argument. In Subsection 2.3.2, we derive error bounds for the asymptotic expansions of these functions. Subsection 2.3.3 deals with the asymptotic behaviour of the corresponding late coefficients. Finally, in Subsection 2.3.4, we derive exponentially improved asymptotic expansions for the above mentioned functions.

2.3.1 The resurgence formulae

In this subsection, we study the resurgence properties of the Struve and modified Struve functions, and their derivatives, for large argument. We will begin with the study of the Struve function $\mathbf{H}_\nu(z)$, and shall obtain the corresponding results for the other functions using their functional relations with $\mathbf{H}_\nu(z)$. We start with the integral representation

$$\mathbf{H}_\nu(z) = Y_\nu(z) + \frac{2}{\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{1}{2}z\right)^\nu \int_0^{+\infty} \mathrm{e}^{-zu}(1+u^2)^{\nu-\frac{1}{2}}\mathrm{d}u, \tag{2.107}$$

which is appropriate for $|\theta| < \frac{\pi}{2}$ and every complex ν (see, e.g., [96, eq. 11.5.2, p. 292]). The change of integration variable $u = \sinh t$ transforms (2.107) to a form more suitable for our purposes:

$$\mathbf{H}_\nu(z) = Y_\nu(z) + \frac{2}{\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{1}{2}z\right)^\nu \int_0^{+\infty} \mathrm{e}^{-z\sinh t}\cosh^{2\nu} t\,\mathrm{d}t, \tag{2.108}$$

again valid for $|\theta| < \frac{\pi}{2}$ and every complex ν (cf. [58, eqs. (7) and (5), pp. 78 and 80]). We can simplify the derivation by observing that the saddle point structure of the integrand in (2.108) is identical to that of (2.69). In particular, the problem is one of linear dependence at the endpoint, and the domain $\Delta^{(o)}$ corresponding to this problem is the same as that in the case of the Anger–Weber function $\mathbf{A}_\nu(z)$ (cf. Figure 2.2). Let us write

$$\mathbf{H}_\nu(z) = Y_\nu(z) + \frac{1}{\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{1}{2}z\right)^{\nu-1} T^{(o)}(z),$$

where $T^{(o)}(z)$ is given in (1.3) with the specific choices of $f(t) = \sinh t$ and $g(t) = \cosh^{2\nu} t$. Following the analysis in Subsection 1.1.1, we expand $T^{(o)}(z)$

into a truncated asymptotic power series with remainder,

$$T^{(o)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(o)}}{z^n} + R_N^{(o)}(z).$$

The conditions posed in Subsection 1.1.1 hold good for the domain $\Delta^{(o)}$ and the functions $f(t) = \sinh t$ and $g(t) = \cosh^{2\nu} t$; only the requirement that $g(t)/f^{N+1}(t) = o(|t|^{-1})$ as $t \to \infty$ in $\Delta^{(o)}$ needs closer attention. It is not difficult to show that this requirement is satisfied precisely when $\mathfrak{Re}(\nu) < \frac{N+1}{2}$. We thus find that

$$\begin{aligned} R_N^{(o)}(z) = {} & \frac{(-\mathrm{i})^N \mathrm{e}^{\frac{\pi}{4}\mathrm{i}}}{2\pi z^N} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}} \mathrm{e}^{-u}}{1 + \mathrm{i}u/z} T^{(0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \\ & + \frac{\mathrm{i}^N \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}}{2\pi z^N} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}} \mathrm{e}^{-u}}{1 - \mathrm{i}u/z} T^{(-1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u, \end{aligned} \tag{2.109}$$

provided $|\theta| < \frac{\pi}{2}$, $N \geq 0$ and $\mathfrak{Re}(\nu) < \frac{N+1}{2}$. We may simplify the representation (2.109), as we did in the case of $\mathbf{A}_\nu(z)$. We shall show that both $T^{(0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$ and $T^{(-1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ can be expressed in terms of the modified Bessel function $K_\nu(u)$. To this end, assume temporarily that $\mathfrak{Re}(\nu) > -\frac{1}{2}$. Under this assumption, $K_\nu(u)$ admits the following integral representation originally due to Hobson:

$$K_\nu(u) = \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{1}{2}u\right)^\nu \int_0^{+\infty} \mathrm{e}^{-u\cosh t} \sinh^{2\nu} t \,\mathrm{d}t, \tag{2.110}$$

for $\mathfrak{Re}(u) > 0$ (see, e.g., [35, eq. (33), p. 64] or [96, eq. 10.32.8, p. 252]). Consider the function $T^{(0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$. First, we divide the integration contour $\mathscr{C}^{(0)}\left(-\frac{\pi}{2}\right)$ in the t-plane into two parts at the saddle point $t^{(0)}$. Second, we make the change of integration variable from t to s by $s = -t$ in the integral along the contour which lies in the left half-plane. Thus we have

$$\begin{aligned} T^{(0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) &= u^{\frac{1}{2}} \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}} \int_{\frac{\pi}{2}\mathrm{i}-\infty}^{\frac{\pi}{2}\mathrm{i}+\infty} \mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\left(\sinh t - \sinh\left(\frac{\pi}{2}\mathrm{i}\right)\right)} \cosh^{2\nu} t \,\mathrm{d}t \\ &= \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}} u^{\frac{1}{2}} \mathrm{e}^u \int_{\frac{\pi}{2}\mathrm{i}}^{\frac{\pi}{2}\mathrm{i}+\infty} \mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}} \sinh t} \cosh^{2\nu} t \,\mathrm{d}t \\ &\quad + \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}} u^{\frac{1}{2}} \mathrm{e}^u \int_{-\frac{\pi}{2}\mathrm{i}}^{-\frac{\pi}{2}\mathrm{i}+\infty} \mathrm{e}^{u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}} \sinh s} \cosh^{2\nu} s \,\mathrm{d}s. \end{aligned}$$

Now, we shift the path of integration downwards by $\frac{\pi}{2}\mathrm{i}$ in the first integral and upwards by $\frac{\pi}{2}\mathrm{i}$ in the second integral. Hence, using (2.110), we obtain

$$\begin{aligned} T^{(0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) &= 2\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}\cos\left(\pi\nu\right)u^{\frac{1}{2}}\mathrm{e}^{u}\int_0^{+\infty}\mathrm{e}^{-u\cosh t}\sinh^{2\nu}t\,\mathrm{d}t \\ &= \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}\frac{2^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}{\pi^{\frac{1}{2}}}\cos\left(\pi\nu\right)u^{-\nu+\frac{1}{2}}\mathrm{e}^{u}K_\nu\left(u\right). \end{aligned}$$

Likewise, one can prove that

$$T^{(-1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) = \mathrm{e}^{\frac{\pi}{4}\mathrm{i}}\frac{2^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}{\pi^{\frac{1}{2}}}\cos\left(\pi\nu\right)u^{-\nu+\frac{1}{2}}\mathrm{e}^{u}K_\nu\left(u\right).$$

Therefore, the representation (2.109) simplifies to

$$\begin{aligned} R_N^{(o)}\left(z\right) &= \frac{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\pi^{\frac{1}{2}}}\cos\left(\pi\nu\right)\frac{\left(-\mathrm{i}\right)^N}{\pi z^N}\int_0^{+\infty}\frac{u^{N-\nu}}{1+\mathrm{i}u/z}K_\nu\left(u\right)\mathrm{d}u \\ &\quad + \frac{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\pi^{\frac{1}{2}}}\cos\left(\pi\nu\right)\frac{\mathrm{i}^N}{\pi z^N}\int_0^{+\infty}\frac{u^{N-\nu}}{1-\mathrm{i}u/z}K_\nu\left(u\right)\mathrm{d}u. \end{aligned} \tag{2.111}$$

The restriction $\mathfrak{Re}\left(\nu\right) > -\frac{1}{2}$ is now removed by analytic continuation in ν.

We may now connect the above results with the asymptotic expansion (2.105) of $\mathbf{H}_\nu\left(z\right)$ by writing

$$\mathbf{H}_\nu\left(z\right) = Y_\nu\left(z\right) + \frac{1}{\pi}\left(\frac{1}{2}z\right)^{\nu-1}\left(\sum_{n=0}^{N-1}\frac{\pi^{\frac{1}{2}}2^{-2n}a_{2n}^{(o)}}{\Gamma\left(\nu+\frac{1}{2}\right)\left(\frac{1}{2}z\right)^{2n}} + R_N^{(\mathbf{H})}\left(z,\nu\right)\right), \tag{2.112}$$

with the notation $R_N^{(\mathbf{H})}\left(z,\nu\right) = \pi^{\frac{1}{2}}R_{2N}^{(o)}\left(z\right)/\Gamma\left(\nu+\frac{1}{2}\right)$. Thus, from (2.111),

$$R_N^{(\mathbf{H})}\left(z,\nu\right) = \left(-1\right)^N\frac{2^{\nu+1}\cos\left(\pi\nu\right)}{\pi}\frac{1}{z^{2N}}\int_0^{+\infty}\frac{u^{2N-\nu}}{1+\left(u/z\right)^2}K_\nu\left(u\right)\mathrm{d}u \tag{2.113}$$

provided that $|\theta| < \frac{\pi}{2}$, $N \geq 0$ and $\mathfrak{Re}\left(\nu\right) < N+\frac{1}{2}$. When deriving (2.112), we used implicitly the fact that $a_n^{(o)}$ vanishes for odd n. To prove this, first note that, by (1.12),

$$a_n^{(o)} = \left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\cosh^{2\nu}t\left(\frac{t}{\sinh t}\right)^{n+1}\right)\right]_{t=0}.$$

Because the quantity under the differentiation sign is an even function of t and therefore its odd-order derivatives at $t = 0$ are zero, the claim follows. It remains to show that the coefficients in (2.112) are equal to those in (2.105). Since

$\pi^{\frac{1}{2}} a_{2n}^{(o)} / \Gamma\left(\nu + \frac{1}{2}\right) = z^{2n} (R_n^{(\mathbf{H})}(z,\nu) - R_{n+1}^{(\mathbf{H})}(z,\nu))$, we infer from (2.113) that

$$\frac{\pi^{\frac{1}{2}} 2^{-2n} a_{2n}^{(o)}}{\Gamma\left(\nu + \frac{1}{2}\right)} = (-1)^n \frac{2^{\nu-2n+1}\cos(\pi\nu)}{\pi} \int_0^{+\infty} u^{2n-\nu} K_\nu(u)\,\mathrm{d}u \tag{2.114}$$

for $\Re\mathfrak{e}(\nu) < n + \frac{1}{2}$. We can evaluate the integral on the right-hand side using the known formula for the Mellin transform of the modified Bessel function $K_\nu(u)$ (see, e.g., [38, ent. (26), p. 331]), giving

$$\frac{\pi^{\frac{1}{2}} 2^{-2n} a_{2n}^{(o)}}{\Gamma\left(\nu + \frac{1}{2}\right)} = (-1)^n \frac{\cos(\pi\nu)}{\pi} \Gamma\left(n + \frac{1}{2} - \nu\right) \Gamma\left(n + \frac{1}{2}\right) = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2} - n\right)}.$$

The restriction $\Re\mathfrak{e}(\nu) < n + \frac{1}{2}$ may now be removed by analytic continuation. This is the desired form of the coefficients. Substitution into (2.112) yields

$$\mathbf{H}_\nu(z) = Y_\nu(z) + \frac{1}{\pi}\left(\frac{1}{2}z\right)^{\nu-1}\left(\sum_{n=0}^{N-1} \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2} - n\right)\left(\frac{1}{2}z\right)^{2n}} + R_N^{(\mathbf{H})}(z,\nu)\right). \tag{2.115}$$

Equations (2.115) and (2.113) together then give the exact resurgence formula for the Struve function $\mathbf{H}_\nu(z)$. We remark that the special case of (2.113) when $N = 0$ was also given by Erdélyi et al. [39, ent. (41), p. 230].

To obtain the corresponding result for the asymptotic expansion (2.106) of the modified Struve function $\mathbf{L}_\nu(z)$, we may proceed as follows. We start with the functional relation

$$\mathbf{L}_\nu(z) = I_\nu(z) \pm \frac{2}{\pi \mathrm{i}} \mathrm{e}^{\pm\pi \mathrm{i}\nu} K_\nu(z) \pm \mathrm{i}\mathrm{e}^{\pm\frac{\pi}{2}\mathrm{i}\nu}\left(\mathbf{H}_\nu\left(z\mathrm{e}^{\mp\frac{\pi}{2}\mathrm{i}}\right) - Y_\nu\left(z\mathrm{e}^{\mp\frac{\pi}{2}\mathrm{i}}\right)\right),$$

which is valid for $-\frac{\pi}{2} \leq \pm\theta \leq \pi$ (cf. [96, eqs. 11.2.2, 11.2.5 and 11.2.6, p. 288]), and substitute by means of (2.115) to arrive at

$$\begin{aligned}\mathbf{L}_\nu(z) = {} & I_\nu(z) \pm \frac{2}{\pi \mathrm{i}} \mathrm{e}^{\pm\pi \mathrm{i}\nu} K_\nu(z) \\ & + \frac{1}{\pi}\left(\frac{1}{2}z\right)^{\nu-1}\left(\sum_{n=0}^{N-1} \frac{(-1)^{n+1}\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2} - n\right)\left(\frac{1}{2}z\right)^{2n}} - R_N^{(\mathbf{H})}\left(z\mathrm{e}^{\mp\frac{\pi}{2}\mathrm{i}},\nu\right)\right).\end{aligned} \tag{2.116}$$

Assuming that $0 < \pm\theta < \pi$, $N \geq 0$ and $\Re\mathfrak{e}(\nu) < N + \frac{1}{2}$, equations (2.116) and (2.113) then yield the desired resurgence formula for $\mathbf{L}_\nu(z)$.

We may derive the corresponding expressions for the z-derivatives by substituting the results (2.115) and (2.116) into the right-hand sides of the functional relations

$$\mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = 2\mathbf{H}_\nu'(z) - \frac{1}{\pi^{\frac{1}{2}}\Gamma\left(\nu + \frac{3}{2}\right)}\left(\frac{1}{2}z\right)^\nu$$

and

$$\mathbf{L}_{\nu-1}(z) + \mathbf{L}_{\nu+1}(z) = 2\mathbf{L}'_{\nu}(z) - \frac{1}{\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{1}{2}z\right)^{\nu}$$

(for these, see [96, eqs. 11.4.24 and 11.4.26, p. 292]). One thus finds

$$\mathbf{H}'_{\nu}(z) = Y'_{\nu}(z) + \frac{1}{\pi}\left(\frac{1}{2}z\right)^{\nu-2}\left(\sum_{n=0}^{N-1}\frac{\Gamma\left(n+\frac{1}{2}\right)\left(\frac{\nu}{2}-\frac{1}{2}-n\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)\left(\frac{1}{2}z\right)^{2n}} + R_N^{(\mathbf{H}')}(z,\nu)\right) \quad (2.117)$$

and

$$\mathbf{L}'_{\nu}(z) = I'_{\nu}(z) \pm \frac{2}{\pi \mathrm{i}}\mathrm{e}^{\pm\pi\mathrm{i}\nu}K'_{\nu}(z)$$
$$+\frac{1}{\pi}\left(\frac{1}{2}z\right)^{\nu-2}\left(\sum_{n=0}^{N-1}\frac{(-1)^{n+1}\Gamma\left(n+\frac{1}{2}\right)\left(\frac{\nu}{2}-\frac{1}{2}-n\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)\left(\frac{1}{2}z\right)^{2n}} - R_N^{(\mathbf{H}')}\left(z\mathrm{e}^{\mp\frac{\pi}{2}\mathrm{i}},\nu\right)\right), \quad (2.118)$$

where $2R_N^{(\mathbf{H}')}(z,\nu) = R_N^{(\mathbf{H})}(z,\nu-1) - (z/2)^2 R_{N+1}^{(\mathbf{H})}(z,\nu+1)$. The complete resurgence formulae can now be written down by applying (2.113). For this, the following assumptions are made: in (2.117), we suppose that $|\theta| < \frac{\pi}{2}$, $N \geq 0$ and $\mathfrak{Re}(\nu) < N + \frac{1}{2}$; whereas in (2.118), we assume that $0 < \pm\theta < \pi$, $N \geq 0$ and $\mathfrak{Re}(\nu) < N + \frac{1}{2}$. With these provisos, we have

$$R_N^{(\mathbf{H}')}(z,\nu) = (-1)^N \frac{2^{\nu}\cos(\pi\nu)}{\pi}\frac{1}{z^{2N}}\int_0^{+\infty}\frac{u^{2N+1-\nu}}{1+(u/z)^2}K'_{\nu}(u)\,\mathrm{d}u.$$

Neglecting the remainder terms in (2.117) and (2.118) and formally extending the sums to infinity, we obtain asymptotic expansions for the functions $\mathbf{H}'_{\nu}(z)$ and $\mathbf{L}'_{\nu}(z)$. Alternatively, we can derive these expansions directly from (2.105) and (2.106) by term-wise differentiation. This latter approach in turn shows that these asymptotic expansions are valid in the sectors $|\theta| < \pi - \delta$ and $-\frac{\pi}{2} + \delta < \pm\theta < \frac{3\pi}{2} - \delta$, respectively, where δ is an arbitrary small positive number.

2.3.2 Error bounds

In this subsection, we provide computable bounds for the remainders $R_N^{(\mathbf{H})}(z,\nu)$ and $R_N^{(\mathbf{H}')}(z,\nu)$. Unless otherwise stated, we suppose that $N \geq 0$ and $\mathfrak{Re}(\nu) < N + \frac{1}{2}$. The index of the numerically least term of the asymptotic expansion (2.105), for example, is $n \approx \frac{1}{2}|z|$. Thus, the restriction on $\mathfrak{Re}(\nu)$ (when it is positive) involves little loss of generality, unless $|z| < 2\mathfrak{Re}(\nu)$, but in that case different approximations are appropriate anyway (cf. [117, Sec. 10.43]).

To our knowledge, the only known result concerning the estimation of the remainder $R_N^{(\mathbf{H})}(z,\nu)$ is that of Watson [117, Sec. 10.42]. Here, we shall derive bounds which are simpler than the ones given by Watson.

The procedure of deriving error estimates is akin to those discussed in Subsection 2.1.2 for the Bessel and modified Bessel functions and therefore, we omit the details. The following bounds are valid in the right half-plane and are appropriate when z does not lie close to the Stokes lines $\theta = \pm\frac{\pi}{2}$:

$$\begin{aligned}\left|R_N^{(\mathbf{H})}(z,\nu)\right| \le{}& \frac{\left|\cos(\pi\nu)\right|}{\left|\cos(\pi\mathfrak{Re}(\nu))\right|}\\ &\times\frac{\Gamma\left(N+\frac{1}{2}\right)}{\left|\Gamma\left(\mathfrak{Re}(\nu)+\frac{1}{2}-N\right)\right|\left(\frac{1}{2}\left|z\right|\right)^{2N}}\begin{cases}\left|\csc(2\theta)\right| & \text{if } \frac{\pi}{4}<\left|\theta\right|<\frac{\pi}{2},\\ 1 & \text{if } \left|\theta\right|\le\frac{\pi}{4}\end{cases}\end{aligned} \tag{2.119}$$

and

$$\begin{aligned}\left|R_N^{(\mathbf{H}')}(z,\nu)\right| \le{}& \frac{\left|\cos(\pi\nu)\right|}{\left|\cos(\pi\mathfrak{Re}(\nu))\right|}\\ &\times\frac{\Gamma\left(N+\frac{1}{2}\right)\left|\frac{\mathfrak{Re}(\nu)}{2}-\frac{1}{2}-N\right|}{\left|\Gamma\left(\mathfrak{Re}(\nu)+\frac{1}{2}-N\right)\right|\left(\frac{1}{2}\left|z\right|\right)^{2N}}\begin{cases}\left|\csc(2\theta)\right| & \text{if } \frac{\pi}{4}<\left|\theta\right|<\frac{\pi}{2},\\ 1 & \text{if } \left|\theta\right|\le\frac{\pi}{4}.\end{cases}\end{aligned} \tag{2.120}$$

If $2\mathfrak{Re}(\nu)$ is an odd integer, the limiting values are taken in these bounds. The existence of these limits is guaranteed by the zeros of the reciprocal gamma function at the non-positive integers.

In the special case when z is positive and ν is real, one can show that

$$R_N^{(\mathbf{H})}(z,\nu) = \frac{\Gamma\left(N+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-N\right)\left(\frac{1}{2}z\right)^{2N}}\Theta_N(z,\nu)$$

and

$$R_N^{(\mathbf{H}')}(z,\nu) = \frac{\Gamma\left(N+\frac{1}{2}\right)\left(\frac{\nu}{2}-\frac{1}{2}-N\right)}{\Gamma\left(\nu+\frac{1}{2}-N\right)\left(\frac{1}{2}z\right)^{2N}}\Xi_N(z,\nu),$$

with some suitable numbers $0 < \Theta_N(z,\nu), \Xi_N(z,\nu) < 1$ that depend on z, ν and N. Said differently, the remainder terms do not exceed the corresponding first neglected terms in absolute value and have the same sign provided that $z > 0$ and $\nu < N + \frac{1}{2}$. This property of $R_N^{(\mathbf{H})}(z,\nu)$ was also proved in a different way by Watson [117, p. 333].

Let us now consider estimates which are appropriate for the sectors $\frac{\pi}{4} < |\theta| < \pi$. When $\frac{\pi}{2} \le |\theta| < \pi$, the remainder terms $R_N^{(\mathbf{H})}(z,\nu)$ and $R_N^{(\mathbf{H}')}(z,\nu)$

may be defined via (2.115) and (2.117). The bounds are as follows

$$\begin{aligned}\left|R_N^{(\mathbf{H})}(z,\nu)\right| \le \frac{e^{\mathfrak{Im}(\nu)\varphi^*}\left|\csc\left(2\left(\theta-\varphi^*\right)\right)\right|}{\cos^{2N+1-\mathfrak{Re}(\nu)}\varphi^*}\frac{\left|\cos\left(\pi\nu\right)\right|}{\left|\cos\left(\pi\mathfrak{Re}\left(\nu\right)\right)\right|}\\ \times\frac{\Gamma\left(N+\frac{1}{2}\right)}{\left|\Gamma\left(\mathfrak{Re}\left(\nu\right)+\frac{1}{2}-N\right)\right|\left(\frac{1}{2}\left|z\right|\right)^{2N}},\end{aligned} \tag{2.121}$$

and

$$\begin{aligned}\left|R_N^{(\mathbf{H}')}(z,\nu)\right| \le \frac{e^{\mathfrak{Im}(\nu)\varphi^{**}}\left|\csc\left(2\left(\theta-\varphi^{**}\right)\right)\right|}{\cos^{2N+2-\mathfrak{Re}(\nu)}\varphi^{**}}\frac{\left|\cos\left(\pi\nu\right)\right|}{\left|\cos\left(\pi\mathfrak{Re}\left(\nu\right)\right)\right|}\\ \times\frac{\Gamma\left(N+\frac{1}{2}\right)\left|\frac{\mathfrak{Re}(\nu)}{2}-\frac{1}{2}-N\right|}{\left|\Gamma\left(\mathfrak{Re}\left(\nu\right)+\frac{1}{2}-N\right)\right|\left(\frac{1}{2}\left|z\right|\right)^{2N}}\end{aligned} \tag{2.122}$$

for $\frac{\pi}{4} < |\theta| < \pi$, where φ^* and φ^{**} are the minimizing values provided by Lemma 2.1.2 with the specific choices of $\chi = 2N+1-\mathfrak{Re}\left(\nu\right)$ and $\chi = 2N+2-\mathfrak{Re}\left(\nu\right)$, respectively. (It seems that, in general, we cannot minimize the quantity $e^{\mathfrak{Im}(\nu)\varphi}\left|\csc\left(2\left(\theta-\varphi\right)\right)\right|\cos^{-2N-1+\mathfrak{Re}(\nu)}\varphi$ as a function of φ in simple terms.)

We can simplify these bounds if θ is close to the Stokes lines $\theta = \pm\frac{\pi}{2}$ using an argument similar to that given in Subsection 2.1.2:

$$\begin{aligned}\left|R_N^{(\mathbf{H})}(z,\nu)\right| \le \frac{e^{\mathfrak{Im}(\nu)\varphi^*}}{2}\sqrt{e\left(2N+\frac{5}{2}-\mathfrak{Re}\left(\nu\right)\right)}\frac{\left|\cos\left(\pi\nu\right)\right|}{\left|\cos\left(\pi\mathfrak{Re}\left(\nu\right)\right)\right|}\\ \times\frac{\Gamma\left(N+\frac{1}{2}\right)}{\left|\Gamma\left(\mathfrak{Re}\left(\nu\right)+\frac{1}{2}-N\right)\right|\left(\frac{1}{2}\left|z\right|\right)^{2N}}\end{aligned} \tag{2.123}$$

with $\varphi^* = \pm\,\mathrm{arccot}((2N+2-\mathfrak{Re}\left(\nu\right))^{\frac{1}{2}})$ and $\frac{\pi}{4} < \pm\left(\theta-\varphi^*\right) \le \frac{\pi}{2}$, and

$$\begin{aligned}\left|R_N^{(\mathbf{H}')}(z,\nu)\right| \le \frac{e^{\mathfrak{Im}(\nu)\varphi^{**}}}{2}\sqrt{e\left(2N+\frac{7}{2}-\mathfrak{Re}\left(\nu\right)\right)}\frac{\left|\cos\left(\pi\nu\right)\right|}{\left|\cos\left(\pi\mathfrak{Re}\left(\nu\right)\right)\right|}\\ \times\frac{\Gamma\left(N+\frac{1}{2}\right)\left|\frac{\mathfrak{Re}(\nu)}{2}-\frac{1}{2}-N\right|}{\left|\Gamma\left(\mathfrak{Re}\left(\nu\right)+\frac{1}{2}-N\right)\right|\left(\frac{1}{2}\left|z\right|\right)^{2N}}\end{aligned} \tag{2.124}$$

with $\varphi^{**} = \pm\,\mathrm{arccot}((2N+3-\mathfrak{Re}\left(\nu\right))^{\frac{1}{2}})$ and $\frac{\pi}{4} < \pm\left(\theta-\varphi^{**}\right) \le \frac{\pi}{2}$. These bounds might be used in conjunction with with our earlier results (2.119) and (2.120).

2.3.3 Asymptotics for the late coefficients

In this subsection, we consider the asymptotic behaviour of the coefficients $\Gamma\left(n+\frac{1}{2}\right)/\Gamma\left(\nu+\frac{1}{2}-n\right)$ of the asymptotic expansion (2.105) for large n. One may assume that 2ν is not an odd integer, because otherwise, these coefficients are identically zero for such values of ν if n is sufficiently large. We substitute into the right-hand side of (2.114) the truncated asymptotic expansion (2.50) of $K_\nu\left(u\right)$ and use the error bound (2.51) to arrive at following expansion:

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)}=\left(-1\right)^n\frac{\cos\left(\pi\nu\right)}{2^{2n-\nu-\frac{1}{2}}\pi^{\frac{1}{2}}}\left(\sum_{m=0}^{M-1}a_m\left(\nu\right)\Gamma\left(2n-m-\nu+\frac{1}{2}\right)+A_M\left(n,\nu\right)\right) \tag{2.125}$$

where

$$\left|A_M\left(n,\nu\right)\right|\le\frac{\left|\cos\left(\pi\nu\right)\right|}{\left|\cos\left(\pi\mathfrak{Re}\left(\nu\right)\right)\right|}\left|a_M\left(\mathfrak{Re}\left(\nu\right)\right)\right|\Gamma\left(2n-M-\mathfrak{Re}\left(\nu\right)+\frac{1}{2}\right), \tag{2.126}$$

provided $n\ge 0, 0\le M<2n-\mathfrak{Re}\left(\nu\right)+\frac{1}{2}$ and $\left|\mathfrak{Re}\left(\nu\right)\right|<M+\frac{1}{2}$. For given large n, the least value of the bound (2.126) occurs when $M\approx\frac{4n}{3}$. With this choice of M, the ratio of the error bound to the leading term in (2.125) is $\mathcal{O}_\nu\left(n^{-\frac{1}{2}}\left(\frac{9}{4}\right)^{-n}\right)$. This is the best accuracy we can achieve using the truncated version of the expansion (2.125). Numerical examples illustrating the efficacy of (2.125), truncated optimally, are given in Table 2.3.

2.3.4 Exponentially improved asymptotic expansions

The purpose of this subsection is to give exponentially improved asymptotic expansions for the Struve and modified Struve functions and their derivatives. In the case of the Struve and modified Struve functions, expansions somewhat similar to ours were derived, using non-rigorous methods, by Dingle [33] [35, pp. 444–446]. The proof of our results in Proposition 2.3.1 below is essentially the same as that of Proposition 2.1.3 on the analogous expansion for the modified Bessel function, and therefore we omit the proof.

Proposition 2.3.1. *Let M be an arbitrary fixed non-negative integer, and let ν be a fixed complex number. Suppose that $|\theta|\le 2\pi-\delta$ with an arbitrary fixed small positive δ, $|z|$ is large and $N=\frac{1}{2}|z|+\rho$ with ρ being bounded. Then*

$$\left(\frac{1}{2}z\right)^\nu R_N^{(\mathbf{H})}\left(z,\nu\right)=\left(2\pi\right)^{\frac{1}{2}}\mathrm{i}\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{\pi\mathrm{i}\nu}\cos\left(\pi\nu\right)\sum_{m=0}^{M-1}\mathrm{i}^m\frac{a_m\left(\nu\right)}{z^{m-\frac{1}{2}}}T_{2N-m-\nu+\frac{1}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$$
$$-\left(2\pi\right)^{\frac{1}{2}}\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega}\mathrm{e}^{\pi\mathrm{i}\nu}\cos\left(\pi\nu\right)\sum_{m=0}^{M-1}\left(-\mathrm{i}\right)^m\frac{a_m\left(\nu\right)}{z^{m-\frac{1}{2}}}T_{2N-m-\nu+\frac{1}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)+R_{N,M}^{(\mathbf{H})}\left(z,\nu\right)$$

values of n, ν and M	$n = 20$, $\nu = -2$, $M = 27$
exact numerical value of $\left\lvert\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)}\right\rvert$	$0.41004693538638711649831707548500315 \times 10^{38}$
approximation (2.125) to $\left\lvert\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)}\right\rvert$	$0.41004693538638711649867195782459930 \times 10^{38}$
error	$-0.35488233959615 \times 10^{17}$
error bound using (2.126)	$0.66764488947654 \times 10^{17}$
values of n, ν and M	$n = 20$, $\nu = 4 + 4\mathrm{i}$, $M = 27$
exact numerical value of $\left\lvert\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)}\right\rvert$	$0.78075498349526202667042265408467387 \times 10^{35}$
approximation (2.125) to $\left\lvert\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)}\right\rvert$	$0.78075498349530874415993588182914557 \times 10^{35}$
error	$-0.46717489513227744471700 \times 10^{22}$
error bound using (2.126)	$0.48903700456267642929297 \times 10^{23}$
values of n, ν and M	$n = 30$, $\nu = 4$, $M = 40$
exact numerical value of $\left\lvert\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)}\right\rvert$	$0.12083274051499957030461894739579006 \times 10^{58}$
approximation (2.125) to $\left\lvert\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)}\right\rvert$	$0.12083274051499957030461894737307791 \times 10^{58}$
error	0.2271215×10^{29}
error bound using (2.126)	$0.50344031 \times 10^{30}$
values of n, ν and M	$n = 30$, $\nu = -6 + 3\mathrm{i}$, $M = 40$
exact numerical value of $\left\lvert\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)}\right\rvert$	$0.51867397900066680521091679927133678 \times 10^{76}$
approximation (2.125) to $\left\lvert\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)}\right\rvert$	$0.51867397900066680521091679928034071 \times 10^{76}$
error	-0.900393×10^{47}
error bound using (2.126)	0.4732029×10^{48}

Table 2.3. *Approximations for* $\left\lvert\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-n\right)}\right\rvert$ *with various n and ν, using* (2.125).

and

$$\left(\frac{1}{2}z\right)^{\nu-1} R_N^{(\mathbf{H}')}(z,\nu) = -(2\pi)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{\pi\mathrm{i}\nu}\cos(\pi\nu)\sum_{m=0}^{M-1}\mathrm{i}^m\frac{b_m(\nu)}{z^{m-\frac{1}{2}}}T_{2N-m-\nu+\frac{3}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$$

$$-(2\pi)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\omega}\mathrm{e}^{\pi\mathrm{i}\nu}\cos(\pi\nu)\sum_{m=0}^{M-1}(-\mathrm{i})^m\frac{b_m(\nu)}{z^{m-\frac{1}{2}}}T_{2N-m-\nu+\frac{3}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)+R_{N,M}^{(\mathbf{H}')}(z,\nu)$$

where $\omega = z - \frac{\pi}{2}\nu - \frac{\pi}{4}$ *and*

$$R_{N,M}^{(\mathbf{H})}(z,\nu), R_{N,M}^{(\mathbf{H}')}(z,\nu) = \mathcal{O}_{M,\nu,\rho}\left(\frac{\mathrm{e}^{-|z|}}{|z|^{M-\frac{1}{2}}}\right)$$

for $|\theta| \le \frac{\pi}{2}$, *and*

$$R_{N,M}^{(\mathbf{H})}(z,\nu), R_{N,M}^{(\mathbf{H}')}(z,\nu) = \mathcal{O}_{M,\nu,\rho,\delta}\left(\frac{\mathrm{e}^{\mp\Im\mathfrak{m}(z)}}{|z|^{M-\frac{1}{2}}}\right)$$

for $\frac{\pi}{2} \le \pm\theta \le 2\pi - \delta$.

Proposition 2.3.1 together with (2.115) and (2.116)–(2.118) yields the exponentially improved asymptotic expansions for the Struve function, the modified Struve function and their derivatives.

Proposition 2.3.1 might also be deduced as a special case of the hyperasymptotic theory of inhomogeneous linear differential equations with a singularity of rank one [47] (indeed, the authors of the paper [47] consider the hyperasymptotic properties of $\mathbf{H}_\nu(z)$). However, our approach provides not only order estimates, but also explicit, numerically computable bounds for $R_{N,M}^{(\mathbf{H})}(z,\nu)$ and $R_{N,M}^{(\mathbf{H}')}(z,\nu)$ which are given in the following theorem. Note that in this theorem, N may not necessarily depend on z.

Theorem 2.3.2. *Let N and M be arbitrary fixed non-negative integers, and let ν be a fixed complex number. Assume further that $M < 2N - \Re\mathfrak{e}(\nu) + \frac{1}{2}$. Then we have*

$$\left|R_{N,M}^{(\mathbf{H})}(z,\nu)\right| \le$$

$$\le (2\pi)^{\frac{1}{2}}\left|\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{\pi\mathrm{i}\nu}\cos(\pi\nu)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\Re\mathfrak{e}(\nu))|}\frac{|a_M(\Re\mathfrak{e}(\nu))|}{|z|^{M-\frac{1}{2}}}\left|T_{2N-M-\nu+\frac{1}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|$$

$$+(2\pi)^{\frac{1}{2}}\left|\mathrm{e}^{-\mathrm{i}\omega}\mathrm{e}^{\pi\mathrm{i}\nu}\cos(\pi\nu)\right|\frac{|\cos(\pi\nu)|}{|\cos(\pi\Re\mathfrak{e}(\nu))|}\frac{|a_M(\Re\mathfrak{e}(\nu))|}{|z|^{M-\frac{1}{2}}}\left|T_{2N-M-\nu+\frac{1}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|$$

$$+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}|\cos(\pi\nu)|\frac{|\cos(\pi\nu)|}{|\cos(\pi\Re\mathfrak{e}(\nu))|}\frac{|a_M(\Re\mathfrak{e}(\nu))|\,\Gamma\left(2N-M-\Re\mathfrak{e}(\nu)+\frac{1}{2}\right)}{|z|^{2N}}$$

provided that $|\theta| \le \frac{\pi}{2}$ *and* $|\mathfrak{Re}(\nu)| < M + \frac{1}{2}$, *and*

$$\begin{aligned}
&\left|R_{N,M}^{(\mathbf{H}')}(z,\nu)\right| \le \\
&\le (2\pi)^{\frac{1}{2}} \left|\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{\pi\mathrm{i}\nu}\cos(\pi\nu)\right| \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|} \frac{|b_M(\mathfrak{Re}(\nu))|}{|z|^{M-\frac{1}{2}}} \left|T_{2N-M-\nu+\frac{3}{2}}\left(z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&\quad + (2\pi)^{\frac{1}{2}} \left|\mathrm{e}^{-\mathrm{i}\omega}\mathrm{e}^{\pi\mathrm{i}\nu}\cos(\pi\nu)\right| \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|} \frac{|b_M(\mathfrak{Re}(\nu))|}{|z|^{M-\frac{1}{2}}} \left|T_{2N-M-\nu+\frac{3}{2}}\left(z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&\quad + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} |\cos(\pi\nu)| \frac{|\cos(\pi\nu)|}{|\cos(\pi\mathfrak{Re}(\nu))|} \frac{|b_M(\mathfrak{Re}(\nu))|\,\Gamma\left(2N-M-\mathfrak{Re}(\nu)+\frac{3}{2}\right)}{|z|^{2N}}
\end{aligned}$$

for $|\theta| \le \frac{\pi}{2}$, $|\mathfrak{Re}(\nu)| < M - \frac{1}{2}$ *and* $M \ge 1$. *If* $2\mathfrak{Re}(\nu)$ *is an odd integer, then limiting values are taken in these bounds.*

The proof of Theorem 2.3.2 is essentially the same as that of Theorem 2.1.4 and so, we omit it.

2.4 Gamma function and its reciprocal

The asymptotic expansion of $\log\Gamma(z)$ is one of the oldest results in the history of asymptotic analysis. It was discovered in 1730 by De Moivre and independently, in a slightly different form, by Stirling (for a detailed historical account, see [4, Subsec. 24.4]). The corresponding asymptotic expansion for $\Gamma(z)$ is due to Laplace [24, pp. 88–109] from 1812. Since the 20th century, these expansions have become standard textbook examples to illustrate various techniques, such as Watson's lemma, the method of Laplace or the method of steepest descents.

In this section, we shall study Laplace's asymptotic expansion for $\Gamma(z)$ together with the analogous result for the reciprocal gamma function. These asymptotic expansions can be written, in modern notation, as

$$\Gamma(z) \sim (2\pi)^{\frac{1}{2}} z^{z-\frac{1}{2}} \mathrm{e}^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{z^n} \tag{2.127}$$

and

$$\frac{1}{\Gamma(z)} \sim (2\pi)^{-\frac{1}{2}} z^{-z+\frac{1}{2}} \mathrm{e}^{z} \sum_{n=0}^{\infty} \frac{\gamma_n}{z^n}, \tag{2.128}$$

as $z \to \infty$ in the sector $|\theta| \le \pi - \delta$, with δ being an arbitrary small positive constant and $\theta = \arg z$ (see, e.g., [112, eq. 6.2.31, p. 71]). The square root in these expansions is defined to be positive when z is positive, and it is defined elsewhere by analytic continuation. The coefficients γ_n are rational numbers and are traditionally called the *Stirling coefficients*. Some expressions for these numbers will be given in Subsection 2.4.1 below. It is worth noting that both expansions involve the same coefficients. The simplest explanation of this interesting phenomenon is that the corresponding asymptotic expansion for $\log \Gamma(z)$ contains only negative odd powers of z (for alternative arguments, the reader is referred to [20, pp. 70–72] and [23]).

It is convenient to introduce and use throughout this work the concept of the *scaled gamma function*:

$$\Gamma^*(z) \stackrel{\text{def}}{=} \frac{\Gamma(z)}{(2\pi)^{\frac{1}{2}} z^{z-\frac{1}{2}} e^{-z}} \tag{2.129}$$

for $|\theta| < \pi$ and by analytic continuation elsewhere. Note that, by (2.127), $\Gamma^*(z) \sim \gamma_0 = 1$ for large z with $|\theta| < \pi$.

The structure of this section is as follows. In Subsection 2.4.1, we obtain resurgence formulae for the gamma function and its reciprocal. Error bounds for the asymptotic expansions of these functions are established in Subsection 2.4.2. Subsection 2.4.3 deals with the asymptotic behaviour of the Stirling coefficients. Finally, in Subsection 2.4.4, we derive exponentially improved asymptotic expansions for the gamma and reciprocal gamma functions.

2.4.1 The resurgence formulae

In this subsection, we investigate the resurgence properties of the gamma function and its reciprocal. We will begin with the study of the gamma function, and shall obtain the corresponding result for the reciprocal using its functional relation with $\Gamma(z)$. The resurgence properties of $\Gamma(z)$ were first studied in the important paper [14] by Boyd; our derivation here is based on his paper, but in many aspects it is more detailed. We start with Euler's integral representation

$$\Gamma(z) = \int_0^{+\infty} u^{z-1} e^{-u} du$$

which is valid for $|\theta| < \frac{\pi}{2}$ (see, e.g., [96, eq. 5.2.1, p. 136]). Suppose for a moment that $z > 0$. The change of integration variable $u = ze^t$ transforms Euler's integral to a form more appropriate for our purposes:

$$\Gamma(z) = z^z e^{-z} \int_{-\infty}^{+\infty} e^{-z(e^t - t - 1)} dt.$$

This representation is again valid for $|\theta| < \frac{\pi}{2}$, by analytic continuation. The function $e^t - t - 1$ has infinitely many first-order saddle points in the complex

t-plane situated at $t = t^{(k)} = 2\pi \mathrm{i} k$ with $k \in \mathbb{Z}$. The path of steepest descent $\mathscr{C}^{(0)}(0)$ through the saddle point $t^{(0)} = 0$ coincides with the real axis, and its orientation is chosen so that it runs from right to left. Thus we may write

$$\Gamma(z) = -z^{z-\frac{1}{2}} \mathrm{e}^{-z} T^{(0)}(z), \tag{2.130}$$

where $T^{(0)}(z)$ is given in (1.54) with the specific choices of $f(t) = \mathrm{e}^t - t - 1$ and $g(t) = 1$. The problem is therefore one of quadratic dependence at the saddle point, which we discussed in Subsection 1.2.1. To determine the domain $\Delta^{(0)}$ corresponding to this problem, the adjacent saddles and contours have to be identified. When $\theta = \frac{\pi}{2}$, the path $\mathscr{C}^{(0)}(\theta)$ connects to the saddle points $t^{(1)}, t^{(2)}, t^{(3)}, \ldots$, whereas when $\theta = -\frac{\pi}{2}$, it connects to the saddle points $t^{(-1)}, t^{(-2)}, t^{(-3)}, \ldots$. The corresponding adjacent contours are

$$\mathscr{C}_-^{(1)}\left(\tfrac{\pi}{2}\right) = \lim_{\delta \to 0+} \mathscr{C}^{(1)}\left(\tfrac{\pi}{2} - \delta\right) \text{ and } \mathscr{C}_+^{(-1)}\left(-\tfrac{\pi}{2}\right) = \lim_{\delta \to 0+} \mathscr{C}^{(-1)}\left(-\tfrac{\pi}{2} + \delta\right) \tag{2.131}$$

(see Figure 2.3). The domain $\Delta^{(0)}$ is formed by the set of all points between the adjacent contours.

It would seem that the method described in Subsection 1.2.1 cannot be used, as the requirements posed there are violated in two different ways. First, there are infinitely many saddle points that are adjacent to $t^{(0)}$, and second, the adjacent contours contain more than one saddle point. Nevertheless, in this specific example, the analysis can still be carried out, due to the following argument. The assumption that there are only finitely many saddles which are adjacent to $t^{(0)}$ was made in order to guarantee that the steepest descent path $\mathscr{C}^{(0)}(\theta)$ goes through only finitely many discontinuous changes before it returns to its original state. (This latter property of $\mathscr{C}^{(0)}(\theta)$ was needed in order to prove that the boundary of $\Delta^{(0)}$ is the union of the adjacent contours.) This requirement is satisfied in our case due to the fact that $\mathscr{C}^{(0)}(\theta)$ changes discontinuously only when $\theta = \pm\frac{\pi}{2} \bmod 2\pi$. We required that the adjacent contour $\mathscr{C}^{(m)}(-\sigma_{0m})$ contains exactly one saddle (namely $t^{(m)}$) to be able to identify the integral over it in terms of $T^{(m)}\left(u\mathrm{e}^{-\mathrm{i}\sigma_{0m}}\right)$. However, in our case the integrals over the adjacent contours (2.131) can also be identified, as $T^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ and $T^{(-1)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$, because $\mathscr{C}^{(1)}(\theta)$ and $\mathscr{C}^{(-1)}(\theta)$ change continuously for $|\theta| \leq \frac{\pi}{2}$.

Following the analysis in Subsection 1.2.1, we expand $T^{(0)}(z)$ into a truncated asymptotic power series with remainder,

$$T^{(0)}(z) = \sum_{n=0}^{N-1} \frac{a_n^{(0)}}{z^n} + R_N^{(0)}(z).$$

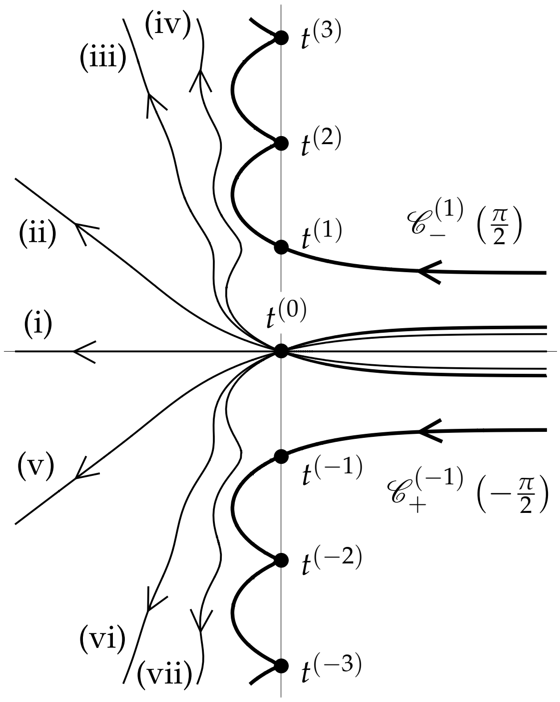

Figure 2.3. *The steepest descent contour $\mathscr{C}^{(0)}(\theta)$ associated with the gamma function through the saddle point $t^{(0)} = 0$ when (i) $\theta = 0$, (ii) $\theta = \frac{\pi}{3}$, (iii) $\theta = \frac{9\pi}{20}$, (iv) $\theta = \frac{12\pi}{25}$, (v) $\theta = -\frac{\pi}{3}$, (vi) $\theta = -\frac{9\pi}{20}$ and (vii) $\theta = -\frac{12\pi}{25}$. The paths $\mathscr{C}_-^{(1)}\left(\frac{\pi}{2}\right)$ and $\mathscr{C}_+^{(-1)}\left(-\frac{\pi}{2}\right)$ are the adjacent contours for $t^{(0)}$. The domain $\Delta^{(0)}$ comprises all points between $\mathscr{C}_-^{(1)}\left(\frac{\pi}{2}\right)$ and $\mathscr{C}_+^{(-1)}\left(-\frac{\pi}{2}\right)$.*

The conditions posed in Subsection 1.2.1 hold true for the domain $\Delta^{(0)}$ and the functions $f(t) = \mathrm{e}^t - t - 1$ and $g(t) = 1$, provided that $N \geq 1$. We choose the orientation of the adjacent contours so that $\mathscr{C}_-^{(1)}\left(\frac{\pi}{2}\right)$ is traversed in the positive direction and $\mathscr{C}_+^{(-1)}\left(-\frac{\pi}{2}\right)$ is traversed in the negative direction with respect to the domain $\Delta^{(0)}$. Consequently the orientation anomalies are $\gamma_{01} = 0$ and $\gamma_{0-1} = 1$. The relevant singulant pair is given by

$$\mathcal{F}_{0\pm 1} = \mathrm{e}^{\pm 2\pi\mathrm{i}} \mp 2\pi\mathrm{i} - 1 - \mathrm{e}^0 + 0 + 1 = \mp 2\pi\mathrm{i}, \quad \arg \mathcal{F}_{0\pm 1} = \sigma_{0\pm 1} = \mp\frac{\pi}{2}.$$

We thus find that for $|\theta| < \frac{\pi}{2}$ and $N \geq 1$, the remainder $R_N^{(0)}(z)$ can be written

$$\begin{aligned} R_N^{(0)}(z) = {} & \frac{\mathrm{i}^N}{2\pi\mathrm{i}z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1-\mathrm{i}u/z}T^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ & - \frac{(-\mathrm{i})^N}{2\pi\mathrm{i}z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1+\mathrm{i}u/z}T^{(-1)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u. \end{aligned} \tag{2.132}$$

It is possible to arrive at a simpler result, by noting that the integrals $T^{(1)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})$ and $T^{(-1)}(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}})$ are equal to the integrals $T^{(0)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})$ and $T^{(0)}(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}})$, respectively. Indeed, the contour $\mathscr{C}_-^{(1)}\left(\frac{\pi}{2}\right)$ is congruent to $\mathscr{C}_-^{(0)}\left(\frac{\pi}{2}\right)$ but is shifted upwards in the complex plane by $2\pi\mathrm{i}$, whence

$$\begin{aligned} T^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) &= u^{\frac{1}{2}}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}\int_{\mathscr{C}_-^{(1)}} \mathrm{e}^{-u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}(\mathrm{e}^t-t-1+2\pi\mathrm{i})}\mathrm{d}t \\ &= u^{\frac{1}{2}}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}\int_{\mathscr{C}_-^{(0)}} \mathrm{e}^{-u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}(\mathrm{e}^t-t-1)}\mathrm{d}t = T^{(0)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right). \end{aligned}$$

Likewise, one can show that $T^{(-1)}(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}) = T^{(0)}(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}})$. Therefore, the representation (2.132) simplifies to

$$\begin{aligned} R_N^{(0)}(z) = \frac{\mathrm{i}^N}{2\pi\mathrm{i}z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1-\mathrm{i}u/z}T^{(0)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ - \frac{(-\mathrm{i})^N}{2\pi\mathrm{i}z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1+\mathrm{i}u/z}T^{(0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \end{aligned} \tag{2.133}$$

for all non-zero values of z in the sector $|\theta| < \frac{\pi}{2}$, provided that $N \geq 1$.

We may now connect the above results with the asymptotic expansion (2.127) of $\Gamma(z)$ by writing

$$\Gamma(z) = (2\pi)^{\frac{1}{2}} z^{z-\frac{1}{2}}\mathrm{e}^{-z}\left(\sum_{n=0}^{N-1}(-1)^n\frac{\gamma_n}{z^n} + R_N(z)\right) \tag{2.134}$$

with the notation $\gamma_n = (-1)^{n+1}(2\pi)^{-\frac{1}{2}}a_n^{(0)}$ and $R_N(z) = -(2\pi)^{-\frac{1}{2}}R_N^{(0)}(z)$. Formulae (2.129), (2.130) and (2.133) then imply

$$\begin{aligned} R_N(z) = \frac{\mathrm{i}^N}{2\pi\mathrm{i}}\frac{1}{z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1-\mathrm{i}u/z}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ - \frac{(-\mathrm{i})^N}{2\pi\mathrm{i}}\frac{1}{z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1+\mathrm{i}u/z}\Gamma^*\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \end{aligned} \tag{2.135}$$

under the same conditions which were required for (2.133) to hold. Equations (2.134) and (2.135) together give the exact resurgence formula for $\Gamma(z)$. This is the result that was originally given by Boyd.

Taking $\gamma_n = (-1)^{n+1}(2\pi)^{-\frac{1}{2}}a_n^{(0)}$ and (1.58) into account, we obtain the following representation for the Stirling coefficients:

$$\gamma_n = \frac{(-1)^n}{2^n\Gamma(n+1)}\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\left(\frac{1}{2}\frac{t^2}{\mathrm{e}^t-t-1}\right)^{n+\frac{1}{2}}\right]_{t=0},$$

where the square root has its principal value. This is a well-known expression for the coefficients γ_n (see, for instance, [17] or [22]). Apparently, there is no known simple explicit representation for the γ_n's (the Mellin transform method, used in previous subsections, does not work in this case). The author [70] found the following formula involving the Stirling numbers of the second kind:

$$\gamma_n = \sum_{k=0}^{2n} \frac{(-1)^n 2^{n+k+1} \Gamma\left(3n+\frac{3}{2}\right)}{\pi^{\frac{1}{2}} (2n+2k+1) \Gamma(2n-k+1)} \sum_{j=0}^{k} \frac{(-1)^j S(2n+k+j,j)}{\Gamma(2n+k+j+1)\Gamma(k-j+1)}.$$

For another expressions, including recurrence relations, the reader is referred to the papers [70] and [75].

To obtain the analogous result for the asymptotic expansion (2.128) of the reciprocal gamma function, we may proceed as follows. First, we derive an integral representation for $R_N(z)$ which is appropriate in the sector $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. For such values of θ, $R_N(z)$ can be defined via (2.134). When z enters the sector $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, the pole of the integrand in the first integral in (2.135) crosses the integration path. According to the residue theorem, we obtain

$$\begin{aligned} R_N(z) = \mathrm{e}^{2\pi \mathrm{i} z} \Gamma^*(z) &+ \frac{\mathrm{i}^N}{2\pi \mathrm{i}} \frac{1}{z^N} \int_0^{+\infty} \frac{u^{N-1} \mathrm{e}^{-2\pi u}}{1 - \mathrm{i}u/z} \Gamma^*\left(u \mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \\ &- \frac{(-\mathrm{i})^N}{2\pi \mathrm{i}} \frac{1}{z^N} \int_0^{+\infty} \frac{u^{N-1} \mathrm{e}^{-2\pi u}}{1 + \mathrm{i}u/z} \Gamma^*\left(u \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \end{aligned} \tag{2.136}$$

for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Now, from the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we may assert that

$$\frac{1}{\Gamma^*(z)} = \left(1 - \mathrm{e}^{-2\pi \mathrm{i} z}\right) \Gamma^*\left(z \mathrm{e}^{\pi \mathrm{i}}\right) \tag{2.137}$$

holds for any z, using analytic continuation. Combining (2.134), (2.136) and (2.137), we obtain

$$\frac{1}{\Gamma(z)} = (2\pi)^{-\frac{1}{2}} z^{-z+\frac{1}{2}} \mathrm{e}^z \left(\sum_{n=0}^{N-1} \frac{\gamma_n}{z^n} + \widetilde{R}_N(z) \right) \tag{2.138}$$

with

$$\begin{aligned} \widetilde{R}_N(z) = \frac{(-\mathrm{i})^N}{2\pi \mathrm{i}} \frac{1}{z^N} &\int_0^{+\infty} \frac{u^{N-1} \mathrm{e}^{-2\pi u}}{1 + \mathrm{i}u/z} \Gamma^*\left(u \mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \\ - \frac{\mathrm{i}^N}{2\pi \mathrm{i}} \frac{1}{z^N} &\int_0^{+\infty} \frac{u^{N-1} \mathrm{e}^{-2\pi u}}{1 - \mathrm{i}u/z} \Gamma^*\left(u \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u, \end{aligned} \tag{2.139}$$

provided that $|\theta| < \frac{\pi}{2}$ and $N \geq 1$. Equations (2.138) and (2.139) together yield the desired resurgence formula for the reciprocal gamma function. We remark that the special case of (2.139) when $N = 1$ was also established by Boyd [14, eq. (4.2)].

2.4.2 Error bounds

In this subsection, we derive computable bounds for the remainders $R_N(z)$ and $\widetilde{R}_N(z)$. We assume that $N \geq 1$ unless otherwise stated. The problem of deriving error bounds for the asymptotic expansion of $\log \Gamma(z)$ has attracted many authors over the past 130 years. Some important bounds are those given by Stieltjes [106], by Lindelöf [53, pp. 93–97] and later by F. W. Schäfke and A. Sattler [104] (see also [41]). Surprising though, in the case of $\Gamma(z)$, it was not until the end of the 20th century that realistic error bounds were given. Olver [87, 88] derived bounds for $R_N(z)$, but the application of his results requires the computation of extreme values of certain implicitly defined functions. To our knowledge, the only explicit and realistic bound for $R_N(z)$ that exists in the literature is that of Boyd [14] [96, eq. 5.11.11, p. 141]. We shall derive here alternative estimates which are sharper than those given by Boyd.

We begin by re-expressing the integral representation (2.135) of the remainder term $R_N(z)$ in a form more suitable for our purposes. By the Schwarz reflection principle $\Gamma^*\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) = \overline{\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)}$ holds for $u > 0$, and therefore $\Gamma^*\left(u\mathrm{e}^{\pm\frac{\pi}{2}\mathrm{i}}\right) = \mathfrak{Re}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \pm \mathrm{i}\mathfrak{Im}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ for $u > 0$. Substituting this identity into (2.135) yields the alternative representations

$$\begin{aligned} R_{2N-1}(z) = \frac{(-1)^{N+1}}{\pi z^{2N-1}} \int_0^{+\infty} \frac{u^{2N-2}\mathrm{e}^{-2\pi u}}{1+(u/z)^2} \mathfrak{Re}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ + \frac{(-1)^N}{\pi z^{2N}} \int_0^{+\infty} \frac{u^{2N-1}\mathrm{e}^{-2\pi u}}{1+(u/z)^2} \mathfrak{Im}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \end{aligned} \tag{2.140}$$

and

$$\begin{aligned} R_{2N}(z) = \frac{(-1)^N}{\pi z^{2N}} \int_0^{+\infty} \frac{u^{2N-1}\mathrm{e}^{-2\pi u}}{1+(u/z)^2} \mathfrak{Im}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ + \frac{(-1)^N}{\pi z^{2N+1}} \int_0^{+\infty} \frac{u^{2N}\mathrm{e}^{-2\pi u}}{1+(u/z)^2} \mathfrak{Re}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u. \end{aligned} \tag{2.141}$$

Similarly to the previous subsections on error bounds, one might try to substitute by means of the formulae

$$\mathfrak{Re}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) = -\frac{u^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}} \int_{\mathscr{C}_-^{(0)}} \mathrm{e}^{-u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}(\mathrm{e}^t - t - 1)}\mathrm{d}x + \frac{u^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}} \int_{\mathscr{C}_-^{(0)}} \mathrm{e}^{-u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}(\mathrm{e}^t - t - 1)}\mathrm{d}y$$

and

$$\mathfrak{Im}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) = -\frac{u^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}} \int_{\mathscr{C}_-^{(0)}} \mathrm{e}^{-u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}(\mathrm{e}^t - t - 1)}\mathrm{d}x - \frac{u^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}} \int_{\mathscr{C}_-^{(0)}} \mathrm{e}^{-u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}(\mathrm{e}^t - t - 1)}\mathrm{d}y, \tag{2.142}$$

with $t = x + \mathrm{i}y$ and make the change of integration variable from u and t to s and t via $s = u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}(\mathrm{e}^t - t - 1)$. Proceeding in this way, however, one faces two difficulties: x is not monotonic along the contour $\mathscr{C}_{-}^{(0)}\left(\frac{\pi}{2}\right)$ and furthermore, the two integrals in (2.142) have opposite signs. Due to these difficulties, it is not clear how simple error bounds could be obtained with this approach. It is even less clear how one would derive error bounds which are expressed via the first few neglected terms of the asymptotic expansion (2.127). Nevertheless, it is possible to deduce such bounds via an alternative method in the case that $|\theta| < \frac{\pi}{2}$. First of all we will need the following lemma.

Lemma 2.4.1. *For any $u > 0$, the quantities $\mathfrak{Re}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ and $-\mathfrak{Im}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ are non-negative.*

Proof. The proof is based on the following integral representation of $\Gamma^*(z)$ due to Stieltjes:

$$\Gamma^*(z) = \exp\left(\int_0^{+\infty} \frac{Q(t)}{(t+z)^2}\mathrm{d}t\right)$$

for $|\theta| < \pi$ with $2Q(t) = t - \lfloor t \rfloor - (t - \lfloor t \rfloor)^2$ (see, e.g., [103, pp. 56–58]). By substituting $z = u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$ with $u > 0$, we find

$$\mathfrak{Re}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) = \left|\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \cos\left(\int_0^{+\infty} \frac{2tu}{(t^2+u^2)^2} Q(t)\,\mathrm{d}t\right)$$

and

$$-\mathfrak{Im}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) = \left|\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \sin\left(\int_0^{+\infty} \frac{2tu}{(t^2+u^2)^2} Q(t)\,\mathrm{d}t\right).$$

We show that the argument of the trigonometric functions is non-negative and is at most $\frac{\pi}{4}$ for any $u > 0$. This implies the statement of the lemma. The non-negativity follows from the non-negativity of $Q(t)$. On the other hand,

$$\int_0^1 \frac{2tu}{(t^2+u^2)^2} Q(t)\,\mathrm{d}t = -\frac{u}{2}\log\left(1 + \frac{1}{u^2}\right) + \frac{1}{2}\operatorname{arccot} u$$

and since $Q(t) \le \frac{1}{8}$,

$$\int_1^{+\infty} \frac{2tu}{(t^2+u^2)^2} Q(t)\,\mathrm{d}t \le \int_1^{+\infty} \frac{2tu}{(t^2+u^2)^2}\frac{1}{8}\mathrm{d}t = \frac{1}{8}\frac{u}{u^2+1}.$$

Thus, for any $u > 0$,

$$\int_0^{+\infty} \frac{2tu}{(t^2+u^2)^2} Q(t)\,\mathrm{d}t \le -\frac{u}{2}\log\left(1 + \frac{1}{u^2}\right) + \frac{1}{2}\operatorname{arccot} u + \frac{1}{8}\frac{u}{u^2+1}.$$

Then it is elementary to verify that the function on the right-hand side of this inequality is at most $\frac{\pi}{4}$. ■

We will also need formulae for the coefficients γ_N which are analogous to (2.140) and (2.141) when deriving our error bounds. These can be most readily obtained by substituting the representations (2.140) and (2.141) into the relation $\gamma_N = (-1)^N z^N (R_N(z) - R_{N+1}(z))$. Thus we have

$$\gamma_{2N-1} = \frac{(-1)^N}{\pi} \int_0^{+\infty} u^{2N-2} \mathrm{e}^{-2\pi u} \mathfrak{Re}\,\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \tag{2.143}$$

and

$$\gamma_{2N} = \frac{(-1)^N}{\pi} \int_0^{+\infty} u^{2N-1} \mathrm{e}^{-2\pi u} \mathfrak{Im}\,\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u. \tag{2.144}$$

Now, from (2.140), (2.141) and Lemma 2.4.1, one immediately establishes the inequalities

$$\begin{aligned} \left|R_{2N-1}(z)\right| \le & \frac{1}{\pi\left|z\right|^{2N-1}} \int_0^{+\infty} \frac{u^{2N-2}\mathrm{e}^{-2\pi u}}{\left|1+(u/z)^2\right|} \mathfrak{Re}\,\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \\ & - \frac{1}{\pi\left|z\right|^{2N}} \int_0^{+\infty} \frac{u^{2N-1}\mathrm{e}^{-2\pi u}}{\left|1+(u/z)^2\right|} \mathfrak{Im}\,\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \end{aligned}$$

and

$$\begin{aligned} \left|R_{2N}(z)\right| \le & -\frac{1}{\pi\left|z\right|^{2N}} \int_0^{+\infty} \frac{u^{2N-1}\mathrm{e}^{-2\pi u}}{\left|1+(u/z)^2\right|} \mathfrak{Im}\,\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \\ & + \frac{1}{\pi\left|z\right|^{2N+1}} \int_0^{+\infty} \frac{u^{2N}\mathrm{e}^{-2\pi u}}{\left|1+(u/z)^2\right|} \mathfrak{Re}\,\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u. \end{aligned}$$

We estimate $1/\left|1+(u/z)^2\right|$ using (2.36) and then compare the results with (2.143) and (2.144) in order to obtain the required error bound

$$\left|R_N(z)\right| \le \left(\frac{\left|\gamma_N\right|}{\left|z\right|^N} + \frac{\left|\gamma_{N+1}\right|}{\left|z\right|^{N+1}}\right) \begin{cases} \left|\csc(2\theta)\right| & \text{if } \frac{\pi}{4} < \left|\theta\right| < \frac{\pi}{2}, \\ 1 & \text{if } \left|\theta\right| \le \frac{\pi}{4}. \end{cases} \tag{2.145}$$

The estimate (2.145) is sharper than the analogous bound given by Boyd [14, eq. (3.11)]. Likewise, one can show that

$$\left|\widetilde{R}_N(z)\right| \le \left(\frac{\left|\gamma_N\right|}{\left|z\right|^N} + \frac{\left|\gamma_{N+1}\right|}{\left|z\right|^{N+1}}\right) \begin{cases} \left|\csc(2\theta)\right| & \text{if } \frac{\pi}{4} < \left|\theta\right| < \frac{\pi}{2}, \\ 1 & \text{if } \left|\theta\right| \le \frac{\pi}{4}. \end{cases} \tag{2.146}$$

Consider now the special case when z is positive. Under this circumstance, we have $0 < 1/(1+(u/z)^2) < 1$ in (2.140) and (2.141). Thus, together with (2.143) and (2.144), the mean value theorem of integration shows that

$$R_N(z) = (-1)^N \frac{\gamma_N}{z^N}\Theta_N(z) + (-1)^{N+1}\frac{\gamma_{N+1}}{z^{N+1}}\Xi_N(z).$$

Here $0 < \Theta_N(z), \Xi_N(z) < 1$ are suitable numbers which depend on z and N. In particular,

$$|R_{2N-1}(z)| \le \frac{|\gamma_{2N-1}|}{z^{2N-1}} + \frac{|\gamma_{2N}|}{z^{2N}} \quad \text{and} \quad |R_{2N}(z)| \le \max\left(\frac{|\gamma_{2N}|}{z^{2N}}, \frac{|\gamma_{2N+1}|}{z^{2N+1}}\right). \tag{2.147}$$

When arriving at these inequalities, we used that $|\gamma_{2N-1}| = (-1)^N \gamma_{2N-1}$ and $|\gamma_{2N}| = (-1)^{N+1}\gamma_{2N}$ hold, which is a direct consequence of (2.143), (2.144) and Lemma 2.4.1. One can prove in a similar manner that

$$\widetilde{R}_N(z) = \frac{\gamma_N}{z^N}\widetilde{\Theta}_N(z) + \frac{\gamma_{N+1}}{z^{N+1}}\widetilde{\Xi}_N(z),$$

where $0 < \widetilde{\Theta}_N(z), \widetilde{\Xi}_N(z) < 1$ are appropriate numbers which depend on z and N. In particular,

$$\left|\widetilde{R}_{2N-1}(z)\right| \le \max\left(\frac{|\gamma_{2N-1}|}{z^{2N-1}}, \frac{|\gamma_{2N}|}{z^{2N}}\right) \quad \text{and} \quad \left|\widetilde{R}_{2N}(z)\right| \le \frac{|\gamma_{2N}|}{z^{2N}} + \frac{|\gamma_{2N+1}|}{z^{2N+1}}. \tag{2.148}$$

By examining the sign patterns of the coefficients of the asymptotic expansions (2.127) and (2.128), it can be verified that the estimates (2.147) and (2.148) are the best one can hope for.

Our bounds (2.145) and (2.146) are unrealistic near the Stokes lines $\theta = \pm\frac{\pi}{2}$ due to the presence of the factor $|\csc(2\theta)|$. We now show how alternative estimates can be established that are suitable for the sectors $0 < |\theta| < \pi$ (which include the Stokes lines $\theta = \pm\frac{\pi}{2}$) and $N \ge 2$. We may use (2.134) and (2.138) to define the remainder terms $R_N(z)$ and $\widetilde{R}_N(z)$ in the sectors $\frac{\pi}{2} \le |\theta| < \pi$. To proceed further, we need the following lemma.

Lemma 2.4.2. *For any $s > 0$ and $0 < \varphi < \frac{\pi}{2}$, we have*

$$\left|\Gamma^*\left(\frac{se^{i\left(\frac{\pi}{2}+\varphi\right)}}{\cos\varphi}\right)\right| \le \frac{1}{1-e^{-2\pi s}}. \tag{2.149}$$

Proof. Assume that $s > 0$ and $0 < \varphi < \frac{\pi}{2}$. An application of the connection formula (2.137) and the relation $\overline{\Gamma^*(z)} = \Gamma^*(\bar{z})$ shows that

$$\left|\Gamma^*\left(\frac{se^{i\left(\frac{\pi}{2}+\varphi\right)}}{\cos\varphi}\right)\right| = \frac{1}{\left(1-2e^{-2\pi s}\cos(2\pi s\tan\varphi)+e^{-4\pi s}\right)^{\frac{1}{2}}}\left|\Gamma^*\left(\frac{se^{i\left(\frac{\pi}{2}-\varphi\right)}}{\cos\varphi}\right)\right|^{-1},$$

whence

$$\left|\Gamma^*\left(\frac{s\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\varphi\right)}}{\cos\varphi}\right)\right| \le \frac{1}{1-\mathrm{e}^{-2\pi s}}\left|\Gamma^*\left(\frac{s\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\varphi\right)}}{\cos\varphi}\right)\right|^{-1}. \tag{2.150}$$

Let $z = x + \mathrm{i}y$ such that $x > 0$ and $y \neq 0$ (i.e., $|\theta| < \frac{\pi}{2}$). We show that $1/\left|\Gamma^*(z)\right|$ is bounded in the right half-plane. Indeed, if z does not lie close to the origin, this is a consequence of the asymptotics $\Gamma^*(z) \sim 1$. To see the boundedness near the origin, we note that

$$\begin{aligned}\frac{1}{\left|\Gamma^*(z)\right|} = \left|\frac{(2\pi)^{\frac{1}{2}} z^{z+\frac{1}{2}}}{\mathrm{e}^z}\right|\frac{1}{\left|\Gamma(z+1)\right|} &= (2\pi)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{\pi}{2}|y|+y\arctan\left(\frac{x}{y}\right)-x}\,|z|^{x+\frac{1}{2}}\frac{1}{\left|\Gamma(z+1)\right|}\\ &\le (2\pi)^{\frac{1}{2}}\,\mathrm{e}^{-x}\,|z|^{x+\frac{1}{2}}\frac{1}{\left|\Gamma(z+1)\right|},\end{aligned}$$

and that the reciprocal gamma function is an entire function. Since $1/\Gamma^*(z)$ is holomorphic in the sector $|\theta| < \frac{\pi}{2}$, continuous on its boundary and

$$\left|\Gamma^*\left(y\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|^{-1} = \left(1-\mathrm{e}^{-2\pi|y|}\right)^{\frac{1}{2}} \le 1,$$

by the Phragmén–Lindelöf principle (see, for instance, [113, p. 177]),

$$\frac{1}{\left|\Gamma^*(z)\right|} \le 1$$

holds for any z in the sector $|\theta| \le \frac{\pi}{2}$. Employing this inequality with $z = s\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\varphi\right)}/\cos\varphi$ in (2.150) gives (2.149). ■

Now we choose, for all θ in the range $0 < |\theta| < \pi$ any angle $\varphi = \varphi(\theta)$ which has the following properties: $0 < |\theta-\varphi| < \frac{\pi}{2}$, and $0 < \varphi < \frac{\pi}{2}$ when $0 < \theta < \pi$ while $-\frac{\pi}{2} < \varphi < 0$ when $-\pi < \theta < 0$. Consider the estimation of $R_N(z)$. Suppose, temporarily, that $0 < \theta < \pi$. We deform the contour of integration of the first integral in (2.135) by rotating it through the angle φ. One therefore finds, using analytic continuation, that

$$\begin{aligned}R_N(z) = {}& \frac{\mathrm{i}^N}{2\pi\mathrm{i}}\frac{1}{z^N}\int_0^{+\infty\mathrm{e}^{\mathrm{i}\varphi}}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1-\mathrm{i}u/z}\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u\\ &-\frac{(-\mathrm{i})^N}{2\pi\mathrm{i}}\frac{1}{z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1+\mathrm{i}u/z}\Gamma^*\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u\\ ={}& \frac{\mathrm{i}^N}{2\pi\mathrm{i}}\frac{1}{z^N}\left(\frac{\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)^N\int_0^{+\infty}\frac{s^{N-1}\mathrm{e}^{-2\pi\frac{s\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}}}{1-\mathrm{i}s\mathrm{e}^{\mathrm{i}\varphi}/(z\cos\varphi)}\Gamma^*\left(\frac{s\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\varphi\right)}}{\cos\varphi}\right)\mathrm{d}s\\ &-\frac{(-\mathrm{i})^N}{2\pi\mathrm{i}}\frac{1}{z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1+\mathrm{i}u/z}\Gamma^*\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u\end{aligned}$$

for $0 < \theta - \varphi < \frac{\pi}{2}$. In passing to the second equality, we have made the change of integration variable from u to s by $s = u\mathrm{e}^{-\mathrm{i}\varphi}\cos\varphi$. By employing the inequality (2.36), we obtain the bound

$$\begin{aligned}|R_N(z)| \le \frac{|\sec(\theta-\varphi)|}{\cos^N\varphi}\frac{1}{2\pi}\frac{1}{|z|^N}\int_0^{+\infty} s^{N-1}\mathrm{e}^{-2\pi s}\left|\Gamma^*\left(\frac{s\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\varphi\right)}}{\cos\varphi}\right)\right|\mathrm{d}s\\ + \frac{1}{2\pi}\frac{1}{|z|^N}\int_0^{+\infty} u^{N-1}\mathrm{e}^{-2\pi u}\left|\Gamma^*\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|\mathrm{d}u.\end{aligned}$$

To estimate the first integral, we apply Lemma 2.4.2, and to estimate the second integral, we use

$$\left|\Gamma^*\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| = \frac{1}{\left(1-\mathrm{e}^{-2\pi u}\right)^{\frac{1}{2}}} \le \frac{1}{1-\mathrm{e}^{-2\pi u}}. \tag{2.151}$$

The resulting integrals can then be evaluated explicitly in terms of the Riemann zeta function (cf. [96, eq. 25.5.1, p. 604]), and we thus establish

$$|R_N(z)| \le \left(\frac{|\sec(\theta-\varphi)|}{\cos^N\varphi}+1\right)\frac{\zeta(N)\,\Gamma(N)}{(2\pi)^{N+1}|z|^N} \tag{2.152}$$

for $0 < \theta - \varphi < \frac{\pi}{2}$ and $N \ge 2$. It is seen, by appealing to the Schwarz reflection principle, that this bound also holds in the conjugate sector $-\frac{\pi}{2} < \theta - \varphi < 0$ (but now with φ a negative acute angle). We would like to choose φ so that the right-hand side of (2.152) is minimized. Since $|\sec(\theta-\varphi)| = \left|\csc\left(\theta \pm \frac{\pi}{2} - \varphi\right)\right|$, the minimizing value φ^* exists and is unique; it is given by Lemma 2.1.1 of Meijer's with $\theta \pm \frac{\pi}{2}$ in place of θ and with $\chi = N$. Taking (2.152) with $\varphi = \varphi^*$, we obtain the desired error bounds for the sectors $0 < |\theta| < \pi$. Note that the ranges of validity of the bounds (2.145) and (2.152) together cover that of the asymptotic expansion (2.127) for $\Gamma(z)$. One may likewise show that for the remainder term $\widetilde{R}_N(z)$, we have

$$\left|\widetilde{R}_N(z)\right| \le \left(\frac{|\sec(\theta-\varphi^*)|}{\cos^N\varphi^*}+1\right)\frac{\zeta(N)\,\Gamma(N)}{(2\pi)^{N+1}|z|^N} \tag{2.153}$$

if $0 < |\theta| < \pi$ and $N \ge 2$.

We would like to ensure that our error bounds (2.152) and (2.153) are realistic, i.e., that they do not seriously overestimate the actual error. By appealing to Subsection 2.4.3 on late coefficients, we can see that for large N, the right-hand sides of (2.152) (with $\varphi = \varphi^*$) and (2.153) are asymptotically

$$\frac{1}{2}\left(\frac{|\sec(\theta-\varphi^*)|}{\cos^N\varphi^*}+1\right)\frac{|\gamma_N|}{|z|^N} \quad \text{and} \quad \frac{1}{2}\left(\frac{|\sec(\theta-\varphi^*)|}{\cos^N\varphi^*}+1\right)\frac{6N}{\pi}\frac{|\gamma_N|}{|z|^N}$$

for N odd and even, respectively. Therefore, if z is bounded away from the negative real axis (i.e., $|\varphi^*|$ is not very close to $\frac{\pi}{2}$) and N is large, the estimates (2.152) and (2.153) are indeed realistic.

The bounds (2.152) and (2.153) can be simplified if θ is close to the Stokes lines, in just the same way as described in Subsection 2.1.2 for the case of the modified Bessel function. One readily finds that

$$|R_N(z)|, |\widetilde{R}_N(z)| \le \left(\sqrt{\mathrm{e}\left(N+\tfrac{1}{2}\right)}+1\right)\frac{\zeta(N)\,\Gamma(N)}{(2\pi)^{N+1}|z|^N} \tag{2.154}$$

for $\frac{\pi}{4} < |\theta| \le \frac{\pi}{2}$ and $N \ge 2$. If $N \ge 3$, the bound (2.154) for $R_N(z)$ is sharper than the analogous estimate given by Boyd [14, eq. (3.14)]. The bounds (2.154) may be used in conjunction with our earlier results (2.145) and (2.146).

2.4.3 Asymptotics for the late coefficients

In this subsection, we study the asymptotic nature of the Stirling coefficients γ_n as $n \to +\infty$. Their leading order behaviour was investigated by Watson [116] using the method of Darboux, and by Diekmann [28] using the method of steepest descents. Murnaghan and Wrench [68, pp. 55–56] gave higher approximations by employing Darboux's method. Complete asymptotic expansions were derived by Dingle [35, eq. (18), p. 159], though his results were obtained by methods that were formal and interpretive, rather than rigorous. His expansions may be written, in our notation, as

$$\gamma_{2n-1} \approx \frac{(-1)^n 2}{(2\pi)^{2n}}\sum_{m=0}^{\infty}(-1)^m(2\pi)^{2m}\gamma_{2m}\Gamma(2n-2m-1)\,\zeta(2n-2m) \tag{2.155}$$

and

$$\gamma_{2n} \approx \frac{(-1)^n 2}{(2\pi)^{2n}}\sum_{m=0}^{\infty}(-1)^m(2\pi)^{2m}\gamma_{2m+1}\Gamma(2n-2m-1)\,\zeta(2n-2m). \tag{2.156}$$

Dingle's results were put in a rigorous mathematical framework by Boyd [14], who gave two different pairs of expansions for the Stirling coefficients, complete with error bounds. Boyd observed that, although his expansions are very similar to (2.155) and (2.156), Dingle's series, assuming optimal truncation, are numerically more efficient.

Here, we shall re-derive Boyd's asymptotic expansions with sharper error bounds. We also give a new pair of (formal) expansions and use it to provide a possible explanation for the remarkable accuracy of Dingle's series (2.155) and (2.156).

We begin by replacing the functions $\mathfrak{Re}\Gamma^*\left(ue^{\frac{\pi}{2}i}\right)$ and $\mathfrak{Im}\Gamma^*\left(ue^{\frac{\pi}{2}i}\right)$ in (2.143) and (2.144) by their truncated asymptotic power series

$$\mathfrak{Re}\Gamma^*\left(ue^{\frac{\pi}{2}i}\right) = \sum_{m=0}^{M-1} (-1)^m \frac{\gamma_{2m}}{u^{2m}} + \mathfrak{Re}R_{2M}\left(ue^{\frac{\pi}{2}i}\right) \tag{2.157}$$

and

$$\mathfrak{Im}\Gamma^*\left(ue^{\frac{\pi}{2}i}\right) = \sum_{m=0}^{M-1} (-1)^m \frac{\gamma_{2m+1}}{u^{2m+1}} + \mathfrak{Im}R_{2M+1}\left(ue^{\frac{\pi}{2}i}\right), \tag{2.158}$$

with $M \geq 1$, and from (2.154),

$$\left|\mathfrak{Re}R_{2M}\left(ue^{\frac{\pi}{2}i}\right)\right| \leq \left(\sqrt{e\left(2M+\tfrac{1}{2}\right)}+1\right)\frac{\zeta(2M)\,\Gamma(2M)}{(2\pi)^{2M+1}u^{2M}} \tag{2.159}$$

and

$$\left|\mathfrak{Im}R_{2M+1}\left(ue^{\frac{\pi}{2}i}\right)\right| \leq \left(\sqrt{e\left(2M+\tfrac{3}{2}\right)}+1\right)\frac{\zeta(2M+1)\,\Gamma(2M+1)}{(2\pi)^{2M+2}u^{2M+1}}. \tag{2.160}$$

Thus from (2.143), (2.144) and (2.157)–(2.160), and provided $n \geq 2$,

$$\gamma_{2n-1} = \frac{(-1)^n 2}{(2\pi)^{2n}}\left(\sum_{m=0}^{M-1}(-1)^m (2\pi)^{2m}\gamma_{2m}\Gamma(2n-2m-1) + A_M(2n-1)\right) \tag{2.161}$$

and

$$\gamma_{2n} = \frac{(-1)^n 2}{(2\pi)^{2n}}\left(\sum_{m=0}^{M-1}(-1)^m (2\pi)^{2m}\gamma_{2m+1}\Gamma(2n-2m-1) + A_M(2n)\right), \tag{2.162}$$

where

$$|A_M(2n-1)| \leq \left(\sqrt{e\left(2M+\tfrac{1}{2}\right)}+1\right)\frac{\zeta(2M)\,\Gamma(2M)}{2\pi}\Gamma(2n-2M-1) \tag{2.163}$$

and

$$|A_M(2n)| \leq \left(\sqrt{e\left(2M+\tfrac{3}{2}\right)}+1\right)\frac{\zeta(2M+1)\,\Gamma(2M+1)}{(2\pi)^2}\Gamma(2n-2M-1), \tag{2.164}$$

as long as $1 \leq M \leq n-1$. For given large n, the least values of the bounds (2.163) and (2.164) occur when $M \approx \frac{n}{2}$. With this choice of M, the ratios of the error bounds to the leading terms in (2.161) and (2.162) are $\mathcal{O}\left(4^{-n}\right)$ and

$\mathcal{O}\left(n4^{-n}\right)$, respectively. This is the best accuracy we can achieve using the truncated versions of the expansions (2.161) and (2.162).

A different set of approximations can be derived starting from the truncated asymptotic expansions[6]

$$\mathfrak{Re}\Gamma^*\left(ue^{\frac{\pi}{2}\mathrm{i}}\right) = \frac{1}{1-\mathrm{e}^{-2\pi u}}\left(\sum_{m=0}^{M-1}(-1)^m\frac{\gamma_{2m}}{u^{2m}} + \mathfrak{Re}\widetilde{R}_{2M}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right), \tag{2.165}$$

and

$$\mathfrak{Im}\Gamma^*\left(ue^{\frac{\pi}{2}\mathrm{i}}\right) = \frac{1}{1-\mathrm{e}^{-2\pi u}}\left(\sum_{m=0}^{M-1}(-1)^m\frac{\gamma_{2m+1}}{u^{2m+1}} + \mathfrak{Im}\widetilde{R}_{2M+1}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right), \tag{2.166}$$

with $M \geq 1$, and from (2.154),

$$\left|\mathfrak{Re}\widetilde{R}_{2M}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \leq \left(\sqrt{\mathrm{e}\left(2M+\tfrac{1}{2}\right)}+1\right)\frac{\zeta\left(2M\right)\Gamma\left(2M\right)}{\left(2\pi\right)^{2M+1}u^{2M}} \tag{2.167}$$

and

$$\left|\mathfrak{Im}\widetilde{R}_{2M+1}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \leq \left(\sqrt{\mathrm{e}\left(2M+\tfrac{3}{2}\right)}+1\right)\frac{\zeta\left(2M+1\right)\Gamma\left(2M+1\right)}{\left(2\pi\right)^{2M+2}u^{2M+1}}. \tag{2.168}$$

Whence from (2.143), (2.144) and (2.165)–(2.168), and provided $n \geq 3$,

$$\begin{aligned}\gamma_{2n-1} = \frac{(-1)^n 2}{(2\pi)^{2n}}\Bigg(&\sum_{m=0}^{M-1}(-1)^m(2\pi)^{2m}\gamma_{2m}\Gamma\left(2n-2m-1\right)\\ &\times\zeta\left(2n-2m-1\right)+B_M\left(2n-1\right)\Bigg)\end{aligned} \tag{2.169}$$

and

$$\begin{aligned}\gamma_{2n} = \frac{(-1)^n 2}{(2\pi)^{2n}}\Bigg(&\sum_{m=0}^{M-1}(-1)^m(2\pi)^{2m}\gamma_{2m+1}\Gamma\left(2n-2m-1\right)\\ &\times\zeta\left(2n-2m-1\right)+B_M\left(2n\right)\Bigg),\end{aligned} \tag{2.170}$$

[6]These results follow from the functional relation (2.137) and the expansion (2.138) of the reciprocal gamma function.

where

$$\left|B_M(2n-1)\right| \le \left(\sqrt{\mathrm{e}\left(2M+\tfrac{1}{2}\right)}+1\right)\frac{\zeta(2M)\,\Gamma(2M)}{2\pi}\Gamma(2n-2M-1) \times \zeta(2n-2M-1) \tag{2.171}$$

and

$$\left|B_M(2n)\right| \le \left(\sqrt{\mathrm{e}\left(2M+\tfrac{3}{2}\right)}+1\right)\frac{\zeta(2M+1)\,\Gamma(2M+1)}{(2\pi)^2}\Gamma(2n-2M-1) \times \zeta(2n-2M-1)\,, \tag{2.172}$$

as long as $1 \le M \le n-2$. One readily establishes that for large n, the least values of the bounds (2.171) and (2.172) occur when $M \approx \frac{n}{2}$. With this choice of M, the ratios of the error bounds to the leading terms in (2.169) and (2.170) are $\mathcal{O}(4^{-n})$ and $\mathcal{O}(n4^{-n})$, respectively. This is the best accuracy available from truncating the expansions (2.169) and (2.170).

The expansions (2.161), (2.162), (2.169) and (2.170) agree with those derived by Boyd; however, the error bounds we have provided are sharper.

In what follows, we shall derive a pair of enhanced approximations for the Stirling coefficients. By taking into consideration the exponentially small contributions arising from the Stokes phenomenon, Paris and Wood [100, eq. (3.4)] derived an improved asymptotic expansion for the scaled gamma function along the positive imaginary axis:

$$\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \sim \frac{1}{\left(1-\mathrm{e}^{-2\pi u}\right)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\mathrm{i}^m\frac{\gamma_m}{u^m}$$

as $u \to +\infty$. Consequently, we have

$$\mathfrak{Re}\,\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \sim \frac{1}{\left(1-\mathrm{e}^{-2\pi u}\right)^{\frac{1}{2}}}\sum_{m=0}^{\infty}(-1)^m\frac{\gamma_{2m}}{u^{2m}} \tag{2.173}$$

and

$$\mathfrak{Im}\,\Gamma^*\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \sim \frac{1}{\left(1-\mathrm{e}^{-2\pi u}\right)^{\frac{1}{2}}}\sum_{m=0}^{\infty}(-1)^m\frac{\gamma_{2m+1}}{u^{2m+1}} \tag{2.174}$$

as $u \to +\infty$. Substitution of these asymptotic expansions into the formulae (2.143) and (2.144) for the Stirling coefficients followed by term-wise integration yields

$$\gamma_{2n-1} \approx \frac{(-1)^n\,2}{(2\pi)^{2n}}\sum_{m=0}^{\infty}(-1)^m(2\pi)^{2m}\gamma_{2m}\Gamma(2n-2m-1)\,\xi(2n-2m) \tag{2.175}$$

and

$$\gamma_{2n} \approx \frac{(-1)^n 2}{(2\pi)^{2n}} \sum_{m=0}^{\infty} (-1)^m (2\pi)^{2m} \gamma_{2m+1} \Gamma(2n-2m-1)\, \xi(2n-2m). \qquad (2.176)$$

Here, the function $\xi(r)$ is given by the Dirichlet series

$$\xi(r) = \frac{(2\pi)^{r-1}}{\Gamma(r-1)} \int_0^{+\infty} \frac{u^{r-2} e^{-2\pi u}}{\left(1-e^{-2\pi u}\right)^{\frac{1}{2}}} du = \sum_{m=1}^{\infty} \frac{m}{4^{m-1}} \binom{2m-2}{m-1} \frac{1}{m^r}$$
$$= 1 + \frac{1}{2^r} + \frac{9}{8}\frac{1}{3^r} + \frac{5}{4}\frac{1}{4^r} + \frac{175}{128}\frac{1}{5^r} + \cdots,$$

provided $r > \frac{3}{2}$. The expansions (2.175) and (2.176) are formal only. One can prove their validity rigorously by constructing error bounds for the asymptotic expansions (2.173) and (2.174), but we do not pursue the details here. For large u, (2.173) and (2.174) are better approximation to $\mathfrak{Re}\Gamma^*\left(ue^{\frac{\pi}{2}i}\right)$ and $\mathfrak{Im}\Gamma^*\left(ue^{\frac{\pi}{2}i}\right)$ than either (2.157) and (2.158) or than (2.165) and (2.166). It is to be expected therefore that, assuming optimal truncation, (2.175) and (2.176) are numerically superior to (2.161) and (2.162) or to (2.169) and (2.170). Numerical computations confirm that this is indeed the case; some examples are presented in Tables 2.4 and 2.5. The fact that $\zeta(r) - \xi(r) = \mathcal{O}(3^{-r})$ for r large positive provides a possible explanation for the remarkable efficiency of Dingle's series (2.155) and (2.156).

values of n and M	$n = 51$, $M = 26$
exact numerical value of $\lvert\gamma_{2n-1}\rvert$	$0.718920823005286472090671337669485196245 \times 10^{77}$
Dingle's approximation (2.155) to $\lvert\gamma_{2n-1}\rvert$	$0.718920823005286472090671337669485196372 \times 10^{77}$
error	-0.127×10^{41}
approximation (2.161) to $\lvert\gamma_{2n-1}\rvert$	$0.718920823005286472090671337669343420137 \times 10^{77}$
error	$0.141776108 \times 10^{47}$
error bound using (2.163)	$0.305630743 \times 10^{47}$
approximation (2.169) to $\lvert\gamma_{2n-1}\rvert$	$0.718920823005286472090671337669626972607 \times 10^{77}$
error	$-0.141776362 \times 10^{47}$
error bound using (2.171)	$0.305630743 \times 10^{47}$
approximation (2.175) to $\lvert\gamma_{2n-1}\rvert$	$0.718920823005286472090671337669485196372 \times 10^{77}$
error	-0.127×10^{41}

Table 2.4. *Approximations for $\lvert\gamma_{101}\rvert$, using (2.155), (2.161), (2.169) and (2.175).*

values of n and M	$n = 50$, $M = 25$
exact numerical value of $\lvert\gamma_{2n}\rvert$	$0.238939789661593595677447537129753012 \times 10^{74}$
Dingle's approximation (2.156) to $\lvert\gamma_{2n}\rvert$	$0.238939789661593595677447537129753175 \times 10^{74}$
error	-0.163×10^{41}
approximation (2.162) to $\lvert\gamma_{2n}\rvert$	$0.238939789661593595677447537129564608 \times 10^{74}$
error	0.188403×10^{44}
error bound using (2.164)	$0.37321123 \times 10^{46}$
approximation (2.170) to $\lvert\gamma_{2n}\rvert$	$0.238939789661593595677447537129941741 \times 10^{74}$
error	-0.188729×10^{44}
error bound using (2.172)	$0.37321123 \times 10^{46}$
approximation (2.176) to $\lvert\gamma_{2n}\rvert$	$0.238939789661593595677447537129753175 \times 10^{74}$
error	-0.163×10^{41}

Table 2.5. *Approximations for* $\lvert\gamma_{100}\rvert$, *using* (2.156), (2.162), (2.170) *and* (2.176).

2.4.4 Exponentially improved asymptotic expansions

In this subsection, we derive exponentially improved asymptotic expansions for the gamma function and its reciprocal. The problem of improving the numerical performance of (2.127) by re-expanding its remainder term, was first considered by Dingle [35, pp. 461–462], using formal methods. He divided the asymptotic power series in (2.127) into two parts according to the parity of the summation index n and considered the two remainders of these series separately. An alternative re-expansion, for the remainder $R_N(z)$ of the whole expansion, was given by Boyd [14, eq. (3.25)], though he gave no investigation of the error term of this re-expansion.

Here, we shall re-derive the result of Boyd with the largest possible domain of validity and with an explicit error bound. The analogous re-expansion of the remainder $\widetilde{R}_N(z)$ is also provided.

Proposition 2.4.3. *Let M be an arbitrary fixed non-negative integer. Suppose that $|\theta| \le \pi - \delta$ with an arbitrary fixed small positive δ, $|z|$ is large and $N = 2\pi|z| + \rho$ with ρ being bounded. Then*

$$\begin{aligned} R_N(z) = {} & \mathrm{e}^{2\pi \mathrm{i} z}\sum_{m=0}^{M-1} (-1)^m \frac{\gamma_m}{z^m} T_{N-m}\left(2\pi z \mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\ & - \mathrm{e}^{-2\pi \mathrm{i} z}\sum_{m=0}^{M-1} (-1)^m \frac{\gamma_m}{z^m} T_{N-m}\left(2\pi z \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) + R_{N,M}(z), \end{aligned} \tag{2.177}$$

where

$$R_{N,M}(z) = \mathcal{O}_{M,\rho}\left(\frac{e^{-2\pi|z|}}{|z|^M}\right) \tag{2.178}$$

for $|\theta| \le \frac{\pi}{2}$, *and*

$$R_{N,M}(z) = \mathcal{O}_{M,\rho,\delta}\left(\frac{e^{\mp 2\pi \Im\mathfrak{m}(z)}}{|z|^M}\right) \tag{2.179}$$

for $\frac{\pi}{2} \le \pm\theta \le \pi - \delta$.

We remark that Boyd mistakenly gave the sign of the factor $e^{-2\pi i z}$ in (2.177) as positive. The corresponding result for $\widetilde{R}_N(z)$ is very similar, however, its range of validity is twice as large as that of the expansion for $R_N(z)$.

Proposition 2.4.4. *Let M be an arbitrary fixed non-negative integer. Suppose that $|\theta| \le 2\pi - \delta$ with an arbitrary fixed small positive δ, $|z|$ is large and $N = 2\pi|z| + \rho$ with ρ being bounded. Then*

$$\begin{aligned}\widetilde{R}_N(z) = -e^{2\pi i z}\sum_{m=0}^{M-1}\frac{\gamma_m}{z^m}T_{N-m}\left(2\pi z e^{\frac{\pi}{2}i}\right)& \\ + e^{-2\pi i z}\sum_{m=0}^{M-1}\frac{\gamma_m}{z^m}T_{N-m}\left(2\pi z e^{-\frac{\pi}{2}i}\right) + \widetilde{R}_{N,M}(z)&,\end{aligned} \tag{2.180}$$

where

$$\widetilde{R}_{N,M}(z) = \mathcal{O}_{M,\rho}\left(\frac{e^{-2\pi|z|}}{|z|^M}\right) \tag{2.181}$$

for $|\theta| \le \frac{\pi}{2}$, *and*

$$\widetilde{R}_{N,M}(z) = \mathcal{O}_{M,\rho,\delta}\left(\frac{e^{\mp 2\pi \Im\mathfrak{m}(z)}}{|z|^M}\right)$$

for $\frac{\pi}{2} \le \pm\theta \le 2\pi - \delta$.

Propositions 2.4.3 and 2.4.4 in conjunction with (2.134) and (2.138) give the exponentially improved asymptotic expansions for the gamma function and its reciprocal. In particular, formula (2.180) together with (2.138) embraces the three asymptotic expansions (2.128) and

$$\frac{1}{\Gamma(z)} \sim (2\pi)^{-\frac{1}{2}} z^{-z+\frac{1}{2}} e^{z}\left(\sum_{n=0}^{\infty}\frac{\gamma_n}{z^n} - e^{\pm 2\pi i z}\sum_{n=0}^{\infty}\frac{\gamma_n}{z^n}\right)$$

which holds when $z \to \infty$ in the sectors $\delta \le \pm\theta \le 2\pi - \delta$ (see, e.g., [75]); furthermore, they give the smooth transition across the Stokes lines $\theta = \pm\frac{\pi}{2}$.

In the following theorem, we give explicit bounds on the error terms $R_{N,M}(z)$ and $\widetilde{R}_{N,M}(z)$. Note that in these results, N may not necessarily depend on z. (The functions $R_{N,M}(z)$ and $\widetilde{R}_{N,M}(z)$ can be defined for arbitrary positive integer N via (2.177) and (2.180), respectively.)

Theorem 2.4.5. *Let N and M be arbitrary fixed positive integers such that $2 \le M < N$. Then we have*

$$\begin{aligned}\left|R_{N,M}(z)\right|, \left|\widetilde{R}_{N,M}(z)\right| \le{}& \left(\sqrt{\mathrm{e}\left(M+\tfrac{1}{2}\right)}+1\right)\frac{\zeta(M)\Gamma(M)}{(2\pi)^{M+1}|z|^M}\left|\mathrm{e}^{2\pi\mathrm{i}z}T_{N-M}\left(2\pi z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|\\ &+\left(\sqrt{\mathrm{e}\left(M+\tfrac{1}{2}\right)}+1\right)\frac{\zeta(M)\Gamma(M)}{(2\pi)^{M+1}|z|^M}\left|\mathrm{e}^{-2\pi\mathrm{i}z}T_{N-M}\left(2\pi z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|\\ &+\left(\mathrm{e}(M+1)+2\right)\frac{\zeta(M)\Gamma(M)\Gamma(N-M)}{(2\pi)^{N+2}|z|^N},\end{aligned}$$

provided that $|\theta| \le \frac{\pi}{2}$.

Proof of Proposition 2.4.3 and Theorem 2.4.5. We only prove Proposition 2.4.3 and the explicit bound for $R_{N,M}(z)$. Proposition 2.4.4 and the explicit bound for $\widetilde{R}_{N,M}(z)$ can be deduced in an analogous manner. Let N and M be arbitrary fixed positive integers such that $2 \le M < N$. Suppose that $|\theta| < \frac{\pi}{2}$. We begin by replacing the functions $\Gamma^*\left(u\mathrm{e}^{\pm\frac{\pi}{2}\mathrm{i}}\right)$ in (2.135) by their truncated asymptotic expansions

$$\Gamma^*\left(u\mathrm{e}^{\pm\frac{\pi}{2}\mathrm{i}}\right)=\sum_{m=0}^{M-1}(\pm\mathrm{i})^m\frac{\gamma_m}{u^m}+R_M\left(u\mathrm{e}^{\pm\frac{\pi}{2}\mathrm{i}}\right)$$

and using the definition of the terminant function, in order to find

$$\begin{aligned}R_N(z)={}&\mathrm{e}^{2\pi\mathrm{i}z}\sum_{m=0}^{M-1}(-1)^m\frac{\gamma_m}{z^m}T_{N-m}\left(2\pi z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\\ &-\mathrm{e}^{-2\pi\mathrm{i}z}\sum_{m=0}^{M-1}(-1)^m\frac{\gamma_m}{z^m}T_{N-m}\left(2\pi z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)+R_{N,M}(z),\end{aligned}\tag{2.182}$$

with

$$\begin{aligned}R_{N,M}(z)={}&\frac{\mathrm{i}^N}{2\pi\mathrm{i}}\frac{1}{z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1-\mathrm{i}u/z}R_M\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u\\ &-\frac{(-\mathrm{i})^N}{2\pi\mathrm{i}}\frac{1}{z^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1+\mathrm{i}u/z}R_M\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u\\ ={}&\frac{\mathrm{i}^N}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}\theta N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_M\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}\tau\\ &-\frac{(-\mathrm{i})^N}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}\theta N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}}{1+\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_M\left(r\tau\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}\tau.\end{aligned}\tag{2.183}$$

In passing to the second equality, we have taken $z = r\mathrm{e}^{\mathrm{i}\theta}$ and have made the change of integration variable from u to τ by $u = r\tau$. Let us consider the estimation of the integral in (2.183) which involves $R_M(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})$. Suppose, for a moment, that $-\pi < \arg\tau < 0$. Under this assumption, the remainder $R_M(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})$ is given by the integral representation (2.135), which can be re-expressed in the form

$$\begin{aligned}
R_M\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) = &\frac{1}{2\pi\mathrm{i}}\frac{1}{(r\tau)^M}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}}{1-t/r}\Gamma^*\left(t\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}t \\
&+\frac{1}{2\pi\mathrm{i}}\frac{\tau-1}{(r\tau)^M}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}}{(1-r\tau/t)(1-t/r)}\Gamma^*\left(t\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}t \\
&-\frac{(-1)^M}{2\pi\mathrm{i}}\frac{1}{(r\tau)^M}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}}{1+t/r}\Gamma^*\left(t\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}t \\
&-\frac{(-1)^M}{2\pi\mathrm{i}}\frac{\tau-1}{(r\tau)^M}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}}{(1+r\tau/t)(1+t/r)}\Gamma^*\left(t\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}t.
\end{aligned}$$

Let φ and φ' be arbitrary acute angles. First, we rotate the path of integration by φ in the first integral and by φ' in the second integral, after which we perform the change of integration variable from t to s via $s = t\mathrm{e}^{-\mathrm{i}\varphi}\cos\varphi$ and $s = t\mathrm{e}^{-\mathrm{i}\varphi'}\cos\varphi'$, respectively. Hence, using analytic continuation in τ, we obtain

$$\begin{aligned}
R_M\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) = &\frac{1}{2\pi\mathrm{i}}\frac{1}{(r\tau)^M}\left(\frac{\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)^M\int_0^{+\infty}\frac{s^{M-1}\mathrm{e}^{-2\pi\frac{s\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}}}{1-s\mathrm{e}^{\mathrm{i}\varphi}/(r\cos\varphi)}\Gamma^*\left(\frac{s\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\varphi\right)}}{\cos\varphi}\right)\mathrm{d}s \\
&+\frac{1}{2\pi\mathrm{i}}\frac{\tau-1}{(r\tau)^M}\left(\frac{\mathrm{e}^{\mathrm{i}\varphi'}}{\cos\varphi'}\right)^M\int_0^{+\infty}\frac{s^{M-1}\mathrm{e}^{-2\pi\frac{s\mathrm{e}^{\mathrm{i}\varphi'}}{\cos\varphi'}}}{1-r\tau\cos\varphi'/(s\mathrm{e}^{\mathrm{i}\varphi'})} \\
&\qquad\times\frac{1}{1-s\mathrm{e}^{\mathrm{i}\varphi'}/(r\cos\varphi')}\Gamma^*\left(\frac{s\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\varphi'\right)}}{\cos\varphi'}\right)\mathrm{d}s \\
&-\frac{(-1)^M}{2\pi\mathrm{i}}\frac{1}{(r\tau)^M}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}}{1+t/r}\Gamma^*\left(t\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}t \\
&-\frac{(-1)^M}{2\pi\mathrm{i}}\frac{\tau-1}{(r\tau)^M}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}}{(1+r\tau/t)(1+t/r)}\Gamma^*\left(t\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}t
\end{aligned}$$

for any $\tau > 0$. Noting that

$$0 < \frac{1}{1+t/r}, \frac{1}{(1+r\tau/t)(1+t/r)} < 1, \quad \frac{1}{\left|1-s\mathrm{e}^{\mathrm{i}\varphi}/(r\cos\varphi)\right|} \le \csc\varphi$$

and

$$\frac{1}{\left|1 - r\tau\cos\varphi' / \left(s\mathrm{e}^{\mathrm{i}\varphi'}\right)\right| \left|1 - s\mathrm{e}^{\mathrm{i}\varphi'} / \left(r\cos\varphi'\right)\right|} \le \csc^2\varphi'$$

for positive r, τ, t and s, we establish the upper bound

$$\begin{aligned}
&\left|\frac{\mathrm{i}^N}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}\theta N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_M\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}\tau\right| \le \\
&\le \frac{\csc\varphi}{\cos^M\varphi}\frac{1}{2\pi}\frac{1}{|z|^M}\int_0^{+\infty}s^{M-1}\mathrm{e}^{-2\pi s}\left|\Gamma^*\left(\frac{s\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\varphi\right)}}{\cos\varphi}\right)\right|\mathrm{d}s\frac{1}{2\pi}\left|\int_0^{+\infty}\frac{\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\mathrm{d}\tau\right| \\
&\quad + \frac{\csc^2\varphi'}{\cos^M\varphi'}\frac{1}{2\pi}\frac{1}{|z|^M}\int_0^{+\infty}s^{M-1}\mathrm{e}^{-2\pi s}\left|\Gamma^*\left(\frac{s\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\varphi'\right)}}{\cos\varphi'}\right)\right|\mathrm{d}s \\
&\qquad\qquad \times\frac{1}{2\pi}\int_0^{+\infty}\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}\left|\frac{\tau-1}{\tau+\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}\right|\mathrm{d}\tau \\
&\quad + \frac{1}{2\pi}\frac{1}{|z|^M}\int_0^{+\infty}t^{M-1}\mathrm{e}^{-2\pi t}\left|\Gamma^*\left(t\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|\mathrm{d}t\frac{1}{2\pi}\left|\int_0^{+\infty}\frac{\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\mathrm{d}\tau\right| \\
&\quad + \frac{1}{2\pi}\frac{1}{|z|^M}\int_0^{+\infty}t^{M-1}\mathrm{e}^{-2\pi t}\left|\Gamma^*\left(t\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|\mathrm{d}t\frac{1}{2\pi}\int_0^{+\infty}\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}\left|\frac{\tau-1}{\tau+\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}\right|\mathrm{d}\tau.
\end{aligned}$$

Further simplification of this bound is possible by using the inequalities (2.149) and (2.151) for the scaled gamma function, the definition of the terminant function and that $|(\tau-1)/(\tau+\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})| \le 1$ for any positive τ. We thus find

$$\begin{aligned}
&\left|\frac{\mathrm{i}^N}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}\theta N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_M\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}\tau\right| \le \\
&\qquad \le \left(\frac{\csc\varphi}{\cos^M\varphi}+1\right)\frac{\zeta(M)\,\Gamma(M)}{(2\pi)^{M+1}|z|^M}\left|\mathrm{e}^{2\pi\mathrm{i}z}T_{N-M}\left(2\pi z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&\qquad\qquad + \left(\frac{\csc^2\varphi'}{\cos^M\varphi'}+1\right)\frac{\zeta(M)\,\Gamma(M)\,\Gamma(N-M)}{(2\pi)^{N+2}|z|^N}.
\end{aligned}$$

The quantities $\csc\varphi/\cos^{-M}\varphi$ and $\csc^2\varphi'/\cos^{-M}\varphi'$, as functions of φ and φ', reach their minimum at $\varphi = \operatorname{arccot}\left(M^{\frac{1}{2}}\right)$ and $\varphi' = \operatorname{arccot}\left((M/2)^{\frac{1}{2}}\right)$, respectively. With these specific choices of φ and φ', we have

$$\frac{\csc\varphi}{\cos^M\varphi} = \left(\frac{M+1}{M}\right)^{\frac{M+1}{2}}M^{\frac{1}{2}} \le \sqrt{\mathrm{e}\left(M+\tfrac{1}{2}\right)}$$

and

$$\frac{\csc^2\varphi'}{\cos^M\varphi'} = \frac{1}{2}\left(\frac{M+2}{M}\right)^{\frac{M}{2}}(M+2) \le \frac{\mathrm{e}}{2}(M+1).$$

Consequently,

$$\begin{aligned}
&\left|\frac{\mathrm{i}^N}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}\theta N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_M\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}\tau\right| \leq \\
&\qquad \leq \left(\sqrt{\mathrm{e}\left(M+\tfrac{1}{2}\right)}+1\right)\frac{\zeta\left(M\right)\Gamma\left(M\right)}{\left(2\pi\right)^{M+1}\left|z\right|^M}\left|\mathrm{e}^{2\pi\mathrm{i}z}T_{N-M}\left(2\pi z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&\qquad\quad + \left(\frac{\mathrm{e}}{2}\left(M+1\right)+1\right)\frac{\zeta\left(M\right)\Gamma\left(M\right)\Gamma\left(N-M\right)}{\left(2\pi\right)^{N+2}\left|z\right|^N}.
\end{aligned} \tag{2.184}$$

One can prove in a similar way that

$$\begin{aligned}
&\left|-\frac{\left(-\mathrm{i}\right)^N}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}\theta N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}}{1+\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_M\left(r\tau\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}\tau\right| \leq \\
&\qquad \leq \left(\sqrt{\mathrm{e}\left(M+\tfrac{1}{2}\right)}+1\right)\frac{\zeta\left(M\right)\Gamma\left(M\right)}{\left(2\pi\right)^{M+1}\left|z\right|^M}\left|\mathrm{e}^{-2\pi\mathrm{i}z}T_{N-M}\left(2\pi z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&\qquad\quad + \left(\frac{\mathrm{e}}{2}\left(M+1\right)+1\right)\frac{\zeta\left(M\right)\Gamma\left(M\right)\Gamma\left(N-M\right)}{\left(2\pi\right)^{N+2}\left|z\right|^N}.
\end{aligned} \tag{2.185}$$

Thus, from (2.183), (2.184) and (2.185), we obtain the error bound

$$\begin{aligned}
\left|R_{N,M}\left(z\right)\right| &\leq \left(\sqrt{\mathrm{e}\left(M+\tfrac{1}{2}\right)}+1\right)\frac{\zeta\left(M\right)\Gamma\left(M\right)}{\left(2\pi\right)^{M+1}\left|z\right|^M}\left|\mathrm{e}^{2\pi\mathrm{i}z}T_{N-M}\left(2\pi z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&\quad + \left(\sqrt{\mathrm{e}\left(M+\tfrac{1}{2}\right)}+1\right)\frac{\zeta\left(M\right)\Gamma\left(M\right)}{\left(2\pi\right)^{M+1}\left|z\right|^M}\left|\mathrm{e}^{-2\pi\mathrm{i}z}T_{N-M}\left(2\pi z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&\quad + \left(\mathrm{e}\left(M+1\right)+2\right)\frac{\zeta\left(M\right)\Gamma\left(M\right)\Gamma\left(N-M\right)}{\left(2\pi\right)^{N+2}\left|z\right|^N}.
\end{aligned} \tag{2.186}$$

By continuity, this bound holds in the closed sector $\left|\theta\right| \leq \frac{\pi}{2}$. This proves Theorem 2.4.5 for $R_{N,M}\left(z\right)$.

From now on, we suppose that $\left|z\right|$ is large and that $N = 2\pi\left|z\right| + \rho$ with ρ being bounded. Using this assumption and Olver's estimate (1.90), the first two terms on the right-hand side of the inequality (2.186) are $\mathcal{O}_{M,\rho}(\left|z\right|^{-M}\mathrm{e}^{-2\pi\left|z\right|})$. An application of Stirling's formula shows that the third term satisfies the order estimate $\mathcal{O}_{M,\rho}(\left|z\right|^{-M-\frac{1}{2}}\mathrm{e}^{-2\pi\left|z\right|})$. This establishes the bound (2.178).

Consider now the sector $\frac{\pi}{2} \leq \theta \leq \pi - \delta$. For such values of θ, $R_{N,M}\left(z\right)$ can be defined via (2.182). When z enters this sector, the pole of the integrand in the first integral in (2.183) crosses the integration path. According to the residue theorem, we obtain

$$
\begin{aligned}
R_{N,M}(z) &= \mathrm{e}^{2\pi \mathrm{i} z} R_M(z) + \frac{\mathrm{i}^N}{2\pi \mathrm{i}} \frac{1}{z^N} \int_0^{+\infty} \frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1 - \mathrm{i}u/z} R_M\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \\
&\quad - \frac{(-\mathrm{i})^N}{2\pi \mathrm{i}} \frac{1}{z^N} \int_0^{+\infty} \frac{u^{N-1}\mathrm{e}^{-2\pi u}}{1 + \mathrm{i}u/z} R_M\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \\
&= \mathrm{e}^{2\pi \mathrm{i} z} R_M(z) + \widetilde{R}_{N,M}\left(z\mathrm{e}^{-\pi \mathrm{i}}\right)
\end{aligned}
$$

for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Now, by analytic continuation,

$$
R_{N,M}(z) = \mathrm{e}^{2\pi \mathrm{i} z} R_M(z) + \widetilde{R}_{N,M}\left(z\mathrm{e}^{-\pi \mathrm{i}}\right)
$$

holds for an complex z, in particular for those lying in the sector $\frac{\pi}{2} \le \theta \le \pi - \delta$. The asymptotic expansion (2.127) implies that $R_M(z) = \mathcal{O}_{M,\rho,\delta}(|z|^{-M})$ as $z \to \infty$ in $\frac{\pi}{2} \le \theta \le \pi - \delta$. From (2.181), we infer that $\widetilde{R}_{N,M}\left(z\mathrm{e}^{-\pi \mathrm{i}}\right) = \mathcal{O}_{M,\rho}(|z|^{-M}\mathrm{e}^{-2\pi|z|})$ for large z in the sector $\frac{\pi}{2} \le \theta \le \pi - \delta$. This shows that the estimate (2.179) holds true when $\frac{\pi}{2} \le \theta \le \pi - \delta$. The proof for the conjugate sector $-\pi + \delta \le \theta \le -\frac{\pi}{2}$ is completely analogous.

Finally, it remains to prove the estimates (2.178) and (2.179) when $M = 0$ or 1. Clearly, for any complex number z, we have

$$
\begin{aligned}
R_{N,0}(z) &= \mathrm{e}^{2\pi \mathrm{i} z}\gamma_0 T_N\left(2\pi z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) - \mathrm{e}^{-2\pi \mathrm{i} z}\gamma_0 T_N\left(2\pi z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
&\quad - \mathrm{e}^{2\pi \mathrm{i} z}\frac{\gamma_1}{z} T_{N-1}\left(2\pi z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) + \mathrm{e}^{-2\pi \mathrm{i} z}\frac{\gamma_1}{z} T_{N-1}\left(2\pi z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) + R_{N,2}(z)
\end{aligned}
$$

and

$$
R_{N,1}(z) = -\mathrm{e}^{2\pi \mathrm{i} z}\frac{\gamma_1}{z} T_{N-1}\left(2\pi z\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) + \mathrm{e}^{-2\pi \mathrm{i} z}\frac{\gamma_1}{z} T_{N-1}\left(2\pi z\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) + R_{N,2}(z).
$$

The terms involving the terminant functions can be estimated by Olver's result (1.90). To estimate $R_{N,2}(z,\nu)$, we can use (2.178) and (2.179), which we have already proved. We thus establish

$$
R_{N,0}(z) = \mathcal{O}_\rho(\mathrm{e}^{-2\pi|z|}) + \mathcal{O}_\rho\left(\frac{\mathrm{e}^{-2\pi|z|}}{|z|}\right) + \mathcal{O}_\rho\left(\frac{\mathrm{e}^{-2\pi|z|}}{|z|^2}\right) = \mathcal{O}_\rho(\mathrm{e}^{-2\pi|z|})
$$

and

$$
R_{N,1}(z) = \mathcal{O}_\rho\left(\frac{\mathrm{e}^{-2\pi|z|}}{|z|}\right) + \mathcal{O}_\rho\left(\frac{\mathrm{e}^{-2\pi|z|}}{|z|^2}\right) = \mathcal{O}_\rho\left(\frac{\mathrm{e}^{-2\pi|z|}}{|z|}\right)
$$

as $z \to \infty$ in the sector $|\theta| \le \frac{\pi}{2}$, and

$$R_{N,0}(z) = \mathcal{O}_{\rho,\delta}(\mathrm{e}^{\mp 2\pi \mathfrak{Im}(z)}) + \mathcal{O}_{\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi \mathfrak{Im}(z)}}{|z|}\right) + \mathcal{O}_{\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi \mathfrak{Im}(z)}}{|z|^2}\right) = \mathcal{O}_{\rho,\delta}(\mathrm{e}^{\mp 2\pi \mathfrak{Im}(z)})$$

and

$$R_{N,1}(z) = \mathcal{O}_{\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi \mathfrak{Im}(z)}}{|z|}\right) + \mathcal{O}_{\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi \mathfrak{Im}(z)}}{|z|^2}\right) = \mathcal{O}_{\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi \mathfrak{Im}(z)}}{|z|}\right)$$

as $z \to \infty$ in the sectors $\frac{\pi}{2} \le \pm\theta \le \pi - \delta$. ■

CHAPTER 3

Asymptotic Expansions for Large Parameter

In the preceding chapter, the theory developed in Chapter 1 was applied for asymptotic expansions of various special functions with large argument. Many of those functions contained an additional parameter which was assumed to be fixed or small compared to the argument. In this chapter, we treat asymptotic expansions where both the argument and the additional parameter (the order of the function) are large.

We start by considering Debye's classical asymptotic expansions for the Hankel and Bessel functions, and their derivatives, for large order and argument in Section 3.1. In Section 3.2, similar results are given for these functions when their order and argument are equal. Section 3.3 deals with several asymptotic expansions of Anger–Weber-type functions and their derivatives, for large order and argument. Finally, Section 2.4 deals with the asymptotic expansions of the Anger–Weber function of equally large order and argument.

3.1 Hankel and Bessel functions of large order and argument

We discussed in Section 2.1 asymptotic expansions for the Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ and the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ which hold when $z \to \infty$ and $\nu^2 = o(|z|)$. In this section, we consider the case when $\nu \to \infty$ and $z/\nu > 1$ is fixed, i.e., both the order and the argument are large. For this purpose, it is convenient to study the functions $H_\nu^{(1)}(\nu \sec \beta)$, $H_\nu^{(2)}(\nu \sec \beta)$, $J_\nu(\nu \sec \beta)$ and $Y_\nu(\nu \sec \beta)$ with β an arbitrary (fixed) acute angle. Approximations for $J_\nu(\nu \sec \beta)$ and $Y_\nu(\nu \sec \beta)$ were first given, using formal methods, by Lorenz [57] in 1890 and subsequently by Meissel [65] in 1892. Later in 1910, Rayleigh [110] applied the principle of stationary phase to derive a rigorous asymptotic approximation for $J_\nu(\nu \sec \beta)$. Debye [25] introduced the method of steepest descents in 1909 to tackle the problem of obtaining complete asymptotic expansions for the Hankel and Bessel functions of large order and argument. For a more detailed historical account, the reader is referred to Watson's book [117, Ch. VIII].

In modern notation, Debye's expansions may be written

$$H_\nu^{(1)}(\nu \sec \beta) \sim \frac{e^{i\xi}}{\left(\frac{1}{2}\pi\nu \tan \beta\right)^{\frac{1}{2}}} \sum_{n=0}^{\infty} (-1)^n \frac{U_n(i\cot \beta)}{\nu^n}, \tag{3.1}$$

as $\nu \to \infty$ in the sector $-\pi + \delta \le \theta \le 2\pi - \delta$;

$$H_\nu^{(2)}(\nu \sec \beta) \sim \frac{e^{-i\xi}}{\left(\frac{1}{2}\pi\nu \tan \beta\right)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{U_n(i\cot \beta)}{\nu^n}, \tag{3.2}$$

as $\nu \to \infty$ in the sector $-2\pi + \delta \le \theta \le \pi - \delta$;

$$J_\nu(\nu \sec \beta) \sim \left(\frac{2}{\pi\nu \tan \beta}\right)^{\frac{1}{2}} \left(\cos \xi \sum_{n=0}^{\infty} \frac{U_{2n}(i\cot \beta)}{\nu^{2n}} - i\sin \xi \sum_{m=0}^{\infty} \frac{U_{2m+1}(i\cot \beta)}{\nu^{2m+1}}\right) \tag{3.3}$$

and

$$Y_\nu(\nu \sec \beta) \sim \left(\frac{2}{\pi\nu \tan \beta}\right)^{\frac{1}{2}} \left(\sin \xi \sum_{n=0}^{\infty} \frac{U_{2n}(i\cot \beta)}{\nu^{2n}} + i\cos \xi \sum_{m=0}^{\infty} \frac{U_{2m+1}(i\cot \beta)}{\nu^{2m+1}}\right), \tag{3.4}$$

as $\nu \to \infty$ in the sector $|\theta| \le \pi - \delta$, where δ denotes an arbitrary small positive constant, $\theta = \arg \nu$ and $\xi = (\tan \beta - \beta)\nu - \frac{\pi}{4}$ (see, e.g., [96, Subsec. 10.19(ii)] and [121, pp. 98–100]). The square root in these expansions is defined to be positive on the positive real line and is defined by analytic continuation elsewhere. The coefficients $U_n(x)$ are polynomials in x of degree $3n$; some expressions for them will be given in Subsection 3.1.1 below.

It is important to note that the requirement $\beta^{-1} = o(|\nu|^{\frac{1}{3}})$ is necessary in order to satisfy Poincaré's definition. And so, these asymptotic expansions fail to hold as $\beta \to 0+$, i.e., when the argument approaches the order. The case when the order and argument are equal will be discussed in the next section. There exist other types of asymptotic expansions which are uniformly valid for all $\beta \geq 0$ (see, for instance, [96, Sec. 10.20]); however these expansions involve non-elementary functions and therefore our methods are not suitable for their investigation.

This section is organized as follows. In Subsection 3.1.1, we obtain resurgence formulae for the Hankel and Bessel functions, and their derivatives, for large order and argument. Error bounds for the asymptotic expansions of these functions are established in Subsection 3.1.2. Subsection 3.1.3 deals with the asymptotic behaviour of the corresponding late coefficients. Finally, in Subsection 3.1.4, we derive exponentially improved asymptotic expansions for the above mentioned functions.

3.1.1 The resurgence formulae

In this subsection, we investigate the resurgence properties of the Hankel and Bessel functions, and their derivatives, for large order and argument. It is enough to study the functions $H_\nu^{(1)}(\nu \sec \beta)$ and $H_\nu^{(1)\prime}(\nu \sec \beta)$, as the analogous results for the other functions can be deduced in a simple way through their relations with $H_\nu^{(1)}(\nu \sec \beta)$ and $H_\nu^{(1)\prime}(\nu \sec \beta)$.

We begin by considering the function $H_\nu^{(1)}(\nu \sec \beta)$. Our starting point is the Schläfli–Sommerfeld integral representation

$$H_\nu^{(1)}(z) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\pi \mathrm{i}+\infty} \mathrm{e}^{z \sinh t - \nu t} \mathrm{d}t, \tag{3.5}$$

which is valid for $|\arg z| < \frac{\pi}{2}$ and every complex ν [96, eq. 10.9.18, p. 224]. Let β be a fixed acute angle, and substitute $z = \nu \sec \beta$ to obtain

$$H_\nu^{(1)}(\nu \sec \beta) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\pi \mathrm{i}+\infty} \mathrm{e}^{-\nu(t - \sec \beta \sinh t)} \mathrm{d}t, \tag{3.6}$$

where $|\theta| < \frac{\pi}{2}$ (here and subsequently, we write $\theta = \arg \nu$). The function $t - \sec \beta \sinh t$ has infinitely many first-order saddle points in the complex t-plane situated at $t^{(r,k)} = (-1)^r \mathrm{i}\beta + 2\pi \mathrm{i}k$ with $r = 0, 1$ and $k \in \mathbb{Z}$.[1] We choose the orientation of the steepest descent path $\mathscr{C}^{(0,0)}(0)$ through $t^{(0,0)} = \mathrm{i}\beta$ so that it runs from left to right (cf. Figure 3.1). It is readily verified that the contour of

[1]The notation in this subsection ($t^{(r,k)}, \mathscr{C}^{(r,k)}(\theta), T^{(r,k)}(z)$, etc.) is a natural modification of the one-parameter notation ($t^{(k)}, \mathscr{C}^{(k)}(\theta), T^{(k)}(z)$, etc.) used in Chapter 1.

integration in (3.6) can be deformed into $\mathscr{C}^{(0,0)}(0)$, and hence we may write

$$H_\nu^{(1)}(\nu \sec\beta) = \frac{\mathrm{e}^{\mathrm{i}\left(\xi+\frac{\pi}{4}\right)}}{\pi \mathrm{i} \nu^{\frac{1}{2}}} T^{(0,0)}(\nu), \tag{3.7}$$

where $T^{(0,0)}(\nu)$ is given in (1.54) with the specific choices of $f(t) = t - \sec\beta \sinh t$ and $g(t) = 1$. The problem is therefore one of quadratic dependence at the saddle point, which we discussed in Subsection 1.2.1. To determine the domain $\Delta^{(0,0)}$ corresponding to this problem, we have to identify the adjacent saddles and contours. When $\theta = -\frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, the path $\mathscr{C}^{(0,0)}(\theta)$ connects to the saddle points $t^{(1,1)} = -\mathrm{i}\beta + 2\pi\mathrm{i}$ and $t^{(1,0)} = -\mathrm{i}\beta$, and these are therefore adjacent to $t^{(0,0)} = \mathrm{i}\beta$. Because the horizontal lines through the points $\frac{3\pi}{2}\mathrm{i}$ and $-\frac{\pi}{2}\mathrm{i}$ are asymptotes of the corresponding adjacent contours $\mathscr{C}^{(1,1)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(1,0)}\left(\frac{3\pi}{2}\right)$, respectively (see Figure 3.1), there cannot be further saddles adjacent to $t^{(0,0)}$ other than $t^{(1,1)}$ and $t^{(1,0)}$. The domain $\Delta^{(0,0)}$ is formed by the set of all points between these adjacent contours.

By analytic continuation, the representation (3.7) is valid in a wider range than (3.6), namely in $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Following the analysis in Subsection 1.2.1, we expand $T^{(0,0)}(\nu)$ into a truncated asymptotic power series with remainder,

$$T^{(0,0)}(\nu) = \sum_{n=0}^{N-1} \frac{a_n^{(0,0)}}{\nu^n} + R_N^{(0,0)}(\nu).$$

It is not difficult to verify that the conditions posed in Subsection 1.2.1 hold good for the domain $\Delta^{(0,0)}$ and the functions $f(t) = t - \sec\beta \sinh t$ and $g(t) = 1$ with any $N \geq 0$. We choose the orientation of the adjacent contours so that they are traversed in the negative sense with respect to the domain $\Delta^{(0,0)}$. Thus, the orientation anomalies are $\gamma_{0,01,1} = 1$ and $\gamma_{0,01,0} = 1$. The relevant singulant pair is given by

$$\begin{aligned} \mathcal{F}_{0,01,1} &= -\mathrm{i}\beta + 2\pi\mathrm{i} - \sec\beta \sinh(-\mathrm{i}\beta + 2\pi\mathrm{i}) - \mathrm{i}\beta + \sec\beta \sinh(\mathrm{i}\beta) \\ &= 2\mathrm{i}(\tan\beta - \beta + \pi), \quad \arg \mathcal{F}_{0,01,1} = \sigma_{0,01,1} = \tfrac{\pi}{2} \end{aligned}$$

and

$$\begin{gathered} \mathcal{F}_{0,01,0} = -\mathrm{i}\beta - \sec\beta \sinh(-\mathrm{i}\beta) - \mathrm{i}\beta + \sec\beta \sinh(\mathrm{i}\beta) = 2\mathrm{i}(\tan\beta - \beta), \\ \arg \mathcal{F}_{0,01,0} = \sigma_{0,01,0} = -\tfrac{3\pi}{2}. \end{gathered}$$

We thus find that for $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $N \geq 0$, the remainder term $R_N^{(0,0)}(\nu)$

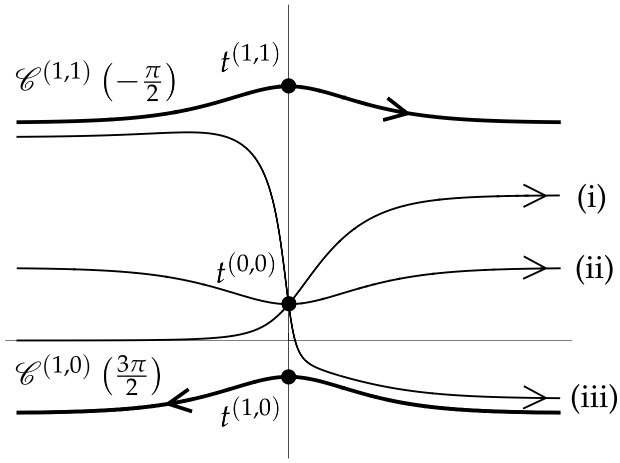

Figure 3.1. *The steepest descent contour* $\mathscr{C}^{(0,0)}(\theta)$ *associated with the Hankel function of large order and argument through the saddle point* $t^{(0,0)} = \mathrm{i}\beta$ *when (i)* $\theta = 0$, *(ii)* $\theta = \frac{\pi}{2}$ *and (iii)* $\theta = \frac{7\pi}{5}$. *The paths* $\mathscr{C}^{(1,1)}\left(-\frac{\pi}{2}\right)$ *and* $\mathscr{C}^{(1,0)}\left(\frac{3\pi}{2}\right)$ *are the adjacent contours for* $t^{(0,0)}$. *The domain* $\Delta^{(0,0)}$ *comprises all points between* $\mathscr{C}^{(1,1)}\left(-\frac{\pi}{2}\right)$ *and* $\mathscr{C}^{(1,0)}\left(\frac{3\pi}{2}\right)$.

may be written

$$\begin{aligned} R_N^{(0,0)}(\nu) = &-\frac{(-\mathrm{i})^N}{2\pi\mathrm{i}\nu^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2(\tan\beta-\beta+\pi)u}}{1+\mathrm{i}u/\nu}T^{(1,1)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ &-\frac{(-\mathrm{i})^N}{2\pi\mathrm{i}\nu^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2(\tan\beta-\beta)u}}{1+\mathrm{i}u/\nu}T^{(1,0)}\left(u\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}\right)\mathrm{d}u. \end{aligned} \tag{3.8}$$

A representation simpler than (3.8) is available. To derive it, we note that the integrals $T^{(1,1)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$ and $T^{(1,0)}\left(u\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}\right)$ are both equal to $T^{(1,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$. Indeed, the contour $\mathscr{C}^{(1,1)}\left(-\frac{\pi}{2}\right)$ is congruent to $\mathscr{C}^{(1,0)}\left(-\frac{\pi}{2}\right)$ but is shifted upwards in the complex plane by $2\pi\mathrm{i}$, whence

$$\begin{aligned} T^{(1,1)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) &= u^{\frac{1}{2}}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}\int_{\mathscr{C}^{(1,1)}}\mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}(t-\sec\beta\sinh t-\mathrm{i}(\tan\beta-\beta+2\pi))}\mathrm{d}t \\ &= u^{\frac{1}{2}}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}\int_{\mathscr{C}^{(1,0)}}\mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}(t-\sec\beta\sinh t-\mathrm{i}(\tan\beta-\beta))}\mathrm{d}t = T^{(1,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right). \end{aligned}$$

(Observe that $\mathscr{C}^{(1,0)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(1,0)}\left(\frac{3\pi}{2}\right)$ are congruent, but have opposite orientations.) The other equality $T^{(1,0)}\left(u\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}\right) = T^{(1,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$ holds because this function is single-valued. Therefore, the representation (3.8) simplifies to

$$R_N^{(0,0)}(\nu) = -\frac{(-\mathrm{i})^N}{2\pi\mathrm{i}\nu^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2(\tan\beta-\beta)u}}{1+\mathrm{i}u/\nu}\left(1+\mathrm{e}^{-2\pi u}\right)T^{(1,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \tag{3.9}$$

for all non-zero values of ν in the sector $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ with $N \geq 0$.

We may now connect the above results with the asymptotic expansion (3.1) of the Hankel function $H_\nu^{(1)}(\nu \sec\beta)$ by writing

$$H_\nu^{(1)}(\nu \sec\beta) = \frac{\mathrm{e}^{\mathrm{i}\xi}}{\left(\frac{1}{2}\pi\nu\tan\beta\right)^{\frac{1}{2}}}\left(\sum_{n=0}^{N-1}(-1)^n\frac{U_n(\mathrm{i}\cot\beta)}{\nu^n} + R_N^{(H)}(\nu,\beta)\right) \tag{3.10}$$

with the notation $U_n(\mathrm{i}\cot\beta) = (-1)^n\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}(2\pi\cot\beta)^{-\frac{1}{2}}a_n^{(0,0)}$ and $R_N^{(H)}(\nu,\beta) = \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}(2\pi\cot\beta)^{-\frac{1}{2}}R_N^{(0,0)}(\nu)$. To obtain a simple expression for the remainder term $R_N^{(H)}(\nu,\beta)$, we first note that

$$\begin{aligned} T^{(1,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) &= u^{\frac{1}{2}}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}\int_{\mathscr{C}(1,0)}\mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}(t-\sec\beta\sinh t+\mathrm{i}\beta+\sec\beta\sinh(-\mathrm{i}\beta))}\mathrm{d}t \\ &= \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}u^{\frac{1}{2}}\mathrm{e}^{(\tan\beta-\beta)u}\int_{\mathscr{C}(1,0)}\mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}(t-\sec\beta\sinh t)}\mathrm{d}t \\ &= -\pi\mathrm{i}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}u^{\frac{1}{2}}\mathrm{e}^{(\tan\beta-\beta)u}H_{-\mathrm{i}u}^{(2)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\sec\beta\right) \\ &= \pi\mathrm{i}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}u^{\frac{1}{2}}\mathrm{e}^{(\tan\beta-\beta)u}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right). \end{aligned} \tag{3.11}$$

In passing to the third equality, we have used the known Schläfli–Sommerfeld integral representation for $H_\nu^{(2)}(z)$ [96, eq. 10.9.18, p. 224], while in passing to the fourth equality, we have used the functional relation $H_\nu^{(2)}\left(z\mathrm{e}^{-\pi\mathrm{i}}\right) = -H_{-\nu}^{(1)}(z)$ (see, for instance, [96, eqs. 10.4.6 and 10.11.5, pp. 222 and 226]). The desired expression for the remainder term $R_N^{(H)}(\nu,\beta)$ now follows from (3.9), (3.11) and $R_N^{(H)}(\nu,\beta) = \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}(2\pi\cot\beta)^{-\frac{1}{2}}R_N^{(0,0)}(\nu)$:

$$\begin{aligned} R_N^{(H)}(\nu,\beta) = \frac{(-\mathrm{i})^N}{2(2\pi\cot\beta)^{\frac{1}{2}}}\frac{1}{\nu^N}\int_0^{+\infty}&\frac{u^{N-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta)u}}{1+\mathrm{i}u/\nu} \\ &\times\left(1+\mathrm{e}^{-2\pi u}\right)\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u \end{aligned} \tag{3.12}$$

for $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $N \geq 0$. Equations (3.10) and (3.12) together give the exact resurgence formula for $H_\nu^{(1)}(\nu\sec\beta)$.

Taking $U_n(\mathrm{i}\cot\beta) = (-1)^n\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}(2\pi\cot\beta)^{-\frac{1}{2}}a_n^{(0,0)}$ and (1.58) into account, we obtain the following representation for the coefficients $U_n(\mathrm{i}\cot\beta)$:

$$U_n(\mathrm{i}\cot\beta) = \frac{(-\mathrm{i}\cot\beta)^n}{2^n\Gamma(n+1)}\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\left(\frac{1}{2}\frac{t^2}{\mathrm{i}\cot\beta(t-\sinh t)+\cosh t-1}\right)^{n+\frac{1}{2}}\right]_{t=0}.$$

Although this formula expresses the coefficients $U_n(\mathrm{i}\cot\beta)$ in a closed form, it is clearly not very effective for their practical computation. A more useful expression for the polynomials $U_n(x)$, in the form of a recurrence, follows from a method based on differential equations:

$$U_n(x) = \frac{1}{2}x^2(1-x^2)U'_{n-1}(x) - \frac{1}{8}\int_0^x (5t^2-1)U_{n-1}(t)\,\mathrm{d}t$$

for $n \geq 1$ with $U_0(x) = 1$ (see, e.g., [95, eq. (7.10), p. 376] or [96, eq. 10.41.9, p. 256]). The reader may find further representations in the paper [73].

To obtain the analogous result for the asymptotic expansion (3.2) of the second Hankel function $H_\nu^{(2)}(\nu \sec\beta)$, we start with the functional relation $H_\nu^{(2)}(\nu\sec\beta) = -H_{\nu e^{\pi i}}^{(1)}(\nu e^{\pi i}\sec\beta)$ and substitute by means of (3.10) to arrive at

$$H_\nu^{(2)}(\nu\sec\beta) = \frac{\mathrm{e}^{-\mathrm{i}\xi}}{\left(\frac{1}{2}\pi\nu\tan\beta\right)^{\frac{1}{2}}}\left(\sum_{n=0}^{N-1}\frac{U_n(\mathrm{i}\cot\beta)}{\nu^n} + R_N^{(H)}(\nu e^{\pi i},\beta)\right). \qquad (3.13)$$

Assuming that $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$ and $N \geq 0$, equations (3.13) and (3.12) then yield the required resurgence formula for $H_\nu^{(2)}(\nu\sec\beta)$.

Let us now turn our attention to the resurgence properties of the derivatives $H_\nu^{(1)\prime}(\nu\sec\beta)$ and $H_\nu^{(2)\prime}(\nu\sec\beta)$.[2] From (3.5), we infer that

$$H_\nu^{(1)\prime}(\nu\sec\beta) = \frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\pi\mathrm{i}+\infty} \mathrm{e}^{-\nu(t-\sec\beta\sinh t)}\sinh t\,\mathrm{d}t \qquad (3.14)$$

with $|\theta| < \frac{\pi}{2}$. Observe that the saddle point structure of the integrand in (3.14) is identical to that of (3.6). In particular, the problem is one of quadratic dependence at the saddle point, and the domain $\Delta^{(0,0)}$ corresponding to this problem is the same as that in the case of $H_\nu^{(1)}(\nu\sec\beta)$. Since the derivation is completely analogous to that of the resurgence formula for $H_\nu^{(1)}(\nu\sec\beta)$, we omit the details and provide only the final results. With the notation of [96, Subsec. 10.19(ii)], we have

$$H_\nu^{(1)\prime}(\nu\sec\beta) = \mathrm{i}\left(\frac{\sin(2\beta)}{\pi\nu}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\xi}\left(\sum_{n=0}^{N-1}(-1)^n\frac{V_n(\mathrm{i}\cot\beta)}{\nu^n} + R_N^{(H')}(\nu,\beta)\right) \qquad (3.15)$$

and

$$H_\nu^{(2)\prime}(\nu\sec\beta) = -\mathrm{i}\left(\frac{\sin(2\beta)}{\pi\nu}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\xi}\left(\sum_{n=0}^{N-1}\frac{V_n(\mathrm{i}\cot\beta)}{\nu^n} + R_N^{(H')}(\nu e^{\pi i},\beta)\right). \qquad (3.16)$$

[2] By these derivatives, we mean $[H_\nu^{(1)\prime}(z)]_{z=\nu\sec\beta}$ and $[H_\nu^{(2)\prime}(z)]_{z=\nu\sec\beta}$, respectively.

The remainder term $R_N^{(H')}(\nu,\beta)$ has the integral representation

$$R_N^{(H')}(\nu,\beta) = -\frac{(-\mathrm{i})^N}{2(\pi\sin(2\beta))^{\frac{1}{2}}}\frac{1}{\nu^N}\int_0^{+\infty}\frac{u^{N-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta)u}}{1+\mathrm{i}u/\nu}$$
$$\times\left(1+\mathrm{e}^{-2\pi u}\right)H_{\mathrm{i}u}^{(1)\prime}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u,$$

provided $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $N \geq 1$.

The coefficients $V_n(\mathrm{i}\cot\beta)$ may be expressed in the form

$$V_n(\mathrm{i}\cot\beta) = \frac{(-\mathrm{i}\cot\beta)^n}{2^n\Gamma(n+1)}$$
$$\times\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\left((\cosh t - \mathrm{i}\cot\beta\sinh t)\left(\frac{1}{2}\frac{t^2}{\mathrm{i}\cot\beta(t-\sinh t)+\cosh t - 1}\right)^{n+\frac{1}{2}}\right)\right]_{t=0}.$$

It is known that $V_n(x)$ is a polynomial in x of degree $3n$, and that these polynomials can be represented in terms of the polynomials $U_n(x)$ since

$$V_n(x) = U_n(x) - \frac{1}{2}x(1-x^2)U_{n-1}(x) - x^2(1-x^2)U'_{n-1}(x),$$

for $n \geq 1$ and $V_0(x) = 1$ (see, for instance, [95, exer. 7.2, p. 378] or [96, eq. 10.41.9, p. 256]).

From the expressions (3.10) and (3.13) for the Hankel functions, we can obtain the corresponding resurgence formulae for the Bessel functions $J_\nu(\nu\sec\beta)$ and $Y_\nu(\nu\sec\beta)$. To this end, we substitute (3.10) and (3.13) into the functional relation $2J_\nu(\nu\sec\beta) = H_\nu^{(1)}(\nu\sec\beta) + H_\nu^{(2)}(\nu\sec\beta)$ and employ Euler's formula $\mathrm{e}^{\pm\mathrm{i}\xi} = \cos\xi \pm \mathrm{i}\sin\xi$, to establish that

$$J_\nu(\nu\sec\beta) = \left(\frac{2}{\pi\nu\tan\beta}\right)^{\frac{1}{2}}\left(\cos\xi\left(\sum_{n=0}^{N-1}\frac{U_{2n}(\mathrm{i}\cot\beta)}{\nu^{2n}} + R_{2N}^{(J)}(\nu,\beta)\right)\right.$$
$$\left.-\mathrm{i}\sin\xi\left(\sum_{n=0}^{N-1}\frac{U_{2n+1}(\mathrm{i}\cot\beta)}{\nu^{2n+1}} + \mathrm{i}R_{2N+1}^{(J)}(\nu,\beta)\right)\right). \tag{3.17}$$

The remainder terms $R_{2N}^{(J)}(\nu,\beta)$ and $R_{2N+1}^{(J)}(\nu,\beta)$ can be expressed by the single formula

$$R_L^{(J)}(\nu,\beta) = \frac{(-1)^{\lfloor L/2\rfloor}}{2(2\pi\cot\beta)^{\frac{1}{2}}}\frac{1}{\nu^L}\int_0^{+\infty}\frac{u^{L-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta)u}}{1+(u/\nu)^2} \tag{3.18}$$
$$\times\left(1+\mathrm{e}^{-2\pi u}\right)\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u,$$

provided that $|\theta| < \frac{\pi}{2}$ and $L \geq 0$. It is possible to arrive at a slightly more general result in which the truncation indices of the series in (3.17) can be different. In order to do so, we will need a formula for the coefficients $U_n\left(\mathrm{i}\cot\beta\right)$ which is analogous to (3.12). This can be obtained by substituting the representation (3.12) into the relation $U_N\left(\mathrm{i}\cot\beta\right) = (-1)^N \nu^N (R_N^{(H)}\left(\nu,\beta\right) - R_{N+1}^{(H)}\left(\nu,\beta\right))$ and replacing N by n. Thus we have

$$\begin{aligned} U_n\left(\mathrm{i}\cot\beta\right) = \frac{\mathrm{i}^n}{2\left(2\pi\cot\beta\right)^{\frac{1}{2}}} \int_0^{+\infty} & u^{n-\frac{1}{2}} \mathrm{e}^{-(\tan\beta-\beta)u} \\ & \times \left(1+\mathrm{e}^{-2\pi u}\right)\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u. \end{aligned} \tag{3.19}$$

Now, let M be an arbitrary non-negative integer such that $M \geq N$. We expand the denominator in the integrand of (3.18) by means of (1.7) and use (3.18) and (3.19) to deduce

$$\mathrm{i}R_{2N+1}^{(J)}\left(\nu,\beta\right) = \sum_{m=N}^{M-1} \frac{U_{2m+1}\left(\mathrm{i}\cot\beta\right)}{\nu^{2m+1}} + \mathrm{i}R_{2M+1}^{(J)}\left(\nu,\beta\right).$$

Combining this equality with (3.17) yields

$$\begin{aligned} J_\nu\left(\nu\sec\beta\right) = \left(\frac{2}{\pi\nu\tan\beta}\right)^{\frac{1}{2}} \Bigg(& \cos\xi \left(\sum_{n=0}^{N-1} \frac{U_{2n}\left(\mathrm{i}\cot\beta\right)}{\nu^{2n}} + R_{2N}^{(J)}\left(\nu,\beta\right)\right) \\ & -\mathrm{i}\sin\xi \left(\sum_{m=0}^{M-1} \frac{U_{2m+1}\left(\mathrm{i}\cot\beta\right)}{\nu^{2m+1}} + \mathrm{i}R_{2M+1}^{(J)}\left(\nu,\beta\right)\right)\Bigg) \end{aligned} \tag{3.20}$$

(cf. equation (3.3)). The case $M < N$ can be handled similarly; we replace n and N by m and M in (3.17) and expand the remainder $R_{2M}^{(J)}\left(\nu,\beta\right)$ into a sum of $R_{2N}^{(J)}\left(\nu,\beta\right)$ and $N-M$ other terms. In summary, if $|\theta| < \frac{\pi}{2}$ and $N, M \geq 0$, equations (3.20) and (3.18) together constitute an exact resurgence formula for the Bessel function $J_\nu\left(\nu\sec\beta\right)$.

In a similar manner, starting with the connection formula $2\mathrm{i}Y_\nu\left(\nu\sec\beta\right) = H_\nu^{(1)}\left(\nu\sec\beta\right) - H_\nu^{(2)}\left(\nu\sec\beta\right)$, the analogous expression for the Bessel function $Y_\nu\left(\nu\sec\beta\right)$ is found to be

$$\begin{aligned} Y_\nu\left(\nu\sec\beta\right) = \left(\frac{2}{\pi\nu\tan\beta}\right)^{\frac{1}{2}} \Bigg(& \sin\xi \left(\sum_{n=0}^{N-1} \frac{U_{2n}\left(\mathrm{i}\cot\beta\right)}{\nu^{2n}} + R_{2N}^{(J)}\left(\nu,\beta\right)\right) \\ & +\mathrm{i}\cos\xi \left(\sum_{m=0}^{M-1} \frac{U_{2m+1}\left(\mathrm{i}\cot\beta\right)}{\nu^{2m+1}} + \mathrm{i}R_{2M+1}^{(J)}\left(\nu,\beta\right)\right)\Bigg) \end{aligned} \tag{3.21}$$

(cf. equation (3.4)). Under the assumptions $|\theta| < \frac{\pi}{2}$ and $N, M \geq 0$, equations (3.21) and (3.18) yield the required resurgence formula for $Y_\nu(\nu\sec\beta)$.

We close this subsection by discussing the corresponding resurgence relations for the derivatives $J'_\nu(\nu\sec\beta)$ and $Y'_\nu(\nu\sec\beta)$. Perhaps the most convenient way to derive these relations is by inserting the expressions (3.15) and (3.16) into the functional equations $2J'_\nu(\nu\sec\beta) = H_\nu^{(1)\prime}(\nu\sec\beta) + H_\nu^{(2)\prime}(\nu\sec\beta)$ and $2\mathrm{i}Y'_\nu(\nu\sec\beta) = H_\nu^{(1)\prime}(\nu\sec\beta) - H_\nu^{(2)\prime}(\nu\sec\beta)$. Hence, using the technique which led from (3.17) to (3.20), we obtain

$$J'_\nu(\nu\sec\beta) = \left(\frac{\sin(2\beta)}{\pi\nu}\right)^{\frac{1}{2}}\left(-\sin\xi\left(\sum_{n=0}^{N-1}\frac{V_{2n}(\mathrm{i}\cot\beta)}{\nu^{2n}} - R_{2N}^{(J')}(\nu,\beta)\right)\right.$$
$$\left. -\mathrm{i}\cos\xi\left(\sum_{m=0}^{M-1}\frac{V_{2m+1}(\mathrm{i}\cot\beta)}{\nu^{2m+1}} + \mathrm{i}R_{2M+1}^{(J')}(\nu,\beta)\right)\right) \tag{3.22}$$

and

$$Y'_\nu(\nu\sec\beta) = \left(\frac{\sin(2\beta)}{\pi\nu}\right)^{\frac{1}{2}}\left(\cos\xi\left(\sum_{n=0}^{N-1}\frac{V_{2n}(\mathrm{i}\cot\beta)}{\nu^{2n}} - R_{2N}^{(J')}(\nu,\beta)\right)\right.$$
$$\left. -\mathrm{i}\sin\xi\left(\sum_{m=0}^{M-1}\frac{V_{2m+1}(\mathrm{i}\cot\beta)}{\nu^{2m+1}} + \mathrm{i}R_{2M+1}^{(J')}(\nu,\beta)\right)\right) \tag{3.23}$$

(cf. [96, eq. 10.19.7, p. 231]). The remainder terms $R_{2N}^{(J')}(\nu,\beta)$ and $R_{2M+1}^{(J')}(\nu,\beta)$ can be expressed by the single formula

$$R_L^{(J')}(\nu,\beta) = \frac{(-1)^{\lfloor L/2\rfloor}}{2(\pi\sin(2\beta))^{\frac{1}{2}}}\frac{1}{\nu^L}\int_0^{+\infty}\frac{u^{L-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta)u}}{1+(u/\nu)^2}$$
$$\times\left(1+\mathrm{e}^{-2\pi u}\right)H_{\mathrm{i}u}^{(1)\prime}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u,$$

where $|\theta| < \frac{\pi}{2}$ and $L \geq 1$. The corresponding requirements for the expressions (3.22) and (3.23) are $|\theta| < \frac{\pi}{2}$, $N \geq 1$ and $M \geq 0$.

3.1.2 Error bounds

In this subsection, we derive computable bounds for the remainders $R_N^{(H)}(\nu,\beta)$, $R_N^{(H')}(\nu,\beta)$, $R_N^{(J)}(\nu,\beta)$ and $R_N^{(J')}(\nu,\beta)$. Unless otherwise stated, we assume that $N \geq 0$ when dealing with $R_N^{(H)}(\nu,\beta)$ and $R_N^{(J)}(\nu,\beta)$, and $N \geq 1$ is assumed in the cases of $R_N^{(H')}(\nu,\beta)$ and $R_N^{(J')}(\nu,\beta)$.

To our knowledge, the only existing results in the literature concerning the estimation of these remainders are those of Meijer [62]. Some of the bounds we shall derive here coincide with the ones obtained by Meijer; however, our proofs are slightly simpler.

First, we consider the estimation of the remainder terms $R_N^{(H)}(\nu,\beta)$ and $R_N^{(H')}(\nu,\beta)$. For convenience, we denote $f(t) = t - \sec\beta\sinh t$. We begin by replacing in (3.12) the function $H_{\mathrm{i}u}^{(1)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta)$ by its integral representation

$$H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right) = \frac{1}{\pi\mathrm{i}}\int_{\mathscr{C}^{(0,0)}\left(\frac{\pi}{2}\right)} \mathrm{e}^{-u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}f(t)}\mathrm{d}t$$

(cf. equation (3.6)) and performing the change of variable from u and t to s and t via $s = u\mathrm{i}f(t)$ (here, and subsequently, i stands for $\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$). We therefore find

$$\begin{aligned}R_N^{(H)}(\nu,\beta) = \frac{(-\mathrm{i})^N}{2(2\pi\cot\beta)^{\frac{1}{2}}}\frac{1}{\nu^N}\int_0^{+\infty} s^{N-\frac{1}{2}}\mathrm{e}^{-s}\frac{1}{\pi}\int_{\mathscr{C}^{(0,0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-N-\frac{1}{2}}}{1+s/(\nu f(t))} \\ \times\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\right)\mathrm{e}^{-(\tan\beta-\beta)\frac{s}{\mathrm{i}f(t)}}\mathrm{d}t\mathrm{d}s.\end{aligned} \tag{3.24}$$

Since for any $t \in \mathscr{C}^{(0,0)}\left(\frac{\pi}{2}\right)$, it holds that $0 < \mathrm{i}(f(t) - f(-\mathrm{i}\beta)) = \mathrm{i}f(t) - (\tan\beta - \beta) < \mathrm{i}f(t)$, the new variable s is indeed positive on the adjacent contour $\mathscr{C}^{(0,0)}\left(\frac{\pi}{2}\right)$.

At this stage, we cannot derive simple bounds for $R_N^{(H)}(\nu,\beta)$ directly from (3.24), as t is not real on the path $\mathscr{C}^{(0,0)}\left(\frac{\pi}{2}\right)$. Therefore a further transformation of (3.24) is necessary. For any $t \in \mathscr{C}^{(0,0)}\left(\frac{\pi}{2}\right)$, we have, denoting $t = x + \mathrm{i}y$, that

$$\mathrm{i}f(t) = \mathfrak{Re}(\mathrm{i}f(t)) = \mathfrak{Re}(\mathrm{i}(x+\mathrm{i}y - \sec\beta\sinh(x+\mathrm{i}y))) = \sec\beta\cosh x\sin y - y.$$

In particular,

$$f(x+\mathrm{i}y) = f(-x+\mathrm{i}y). \tag{3.25}$$

Denote by $\mathscr{C}_1^{(0,0)}\left(\frac{\pi}{2}\right)$ and $\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)$ the parts of the steepest descent contour $\mathscr{C}^{(0,0)}\left(\frac{\pi}{2}\right)$ which lie in the left and in the right half-plane, respectively. Using (3.25) and the fact that $\mathscr{C}^{(0,0)}\left(\frac{\pi}{2}\right)$ is symmetric with respect to the imaginary axis (see Figure 3.1), we deduce

$$\begin{aligned}
&\int_{\mathscr{C}^{(0,0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-N-\frac{1}{2}}}{1+s/(\nu f(t))}\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\right)\mathrm{e}^{-(\tan\beta-\beta)\frac{s}{\mathrm{i}f(t)}}\mathrm{d}t = \\
&= \int_{\mathscr{C}_1^{(0,0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(x+\mathrm{i}y))^{-N-\frac{1}{2}}}{1+s/(\nu f(x+\mathrm{i}y))}\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(x+\mathrm{i}y)}}\right)\mathrm{e}^{-(\tan\beta-\beta)\frac{s}{\mathrm{i}f(x+\mathrm{i}y)}}\mathrm{d}(x+\mathrm{i}y) \\
&\quad + \int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(x+\mathrm{i}y))^{-N-\frac{1}{2}}}{1+s/(\nu f(x+\mathrm{i}y))}\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(x+\mathrm{i}y)}}\right)\mathrm{e}^{-(\tan\beta-\beta)\frac{s}{\mathrm{i}f(x+\mathrm{i}y)}}\mathrm{d}(x+\mathrm{i}y) \\
&= \int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(-x+\mathrm{i}y))^{-N-\frac{1}{2}}}{1+s/(\nu f(-x+\mathrm{i}y))}\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(-x+\mathrm{i}y)}}\right)\mathrm{e}^{-(\tan\beta-\beta)\frac{s}{\mathrm{i}f(-x+\mathrm{i}y)}}\mathrm{d}(x-\mathrm{i}y) \\
&\quad + \int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(x+\mathrm{i}y))^{-N-\frac{1}{2}}}{1+s/(\nu f(x+\mathrm{i}y))}\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(x+\mathrm{i}y)}}\right)\mathrm{e}^{-(\tan\beta-\beta)\frac{s}{\mathrm{i}f(x+\mathrm{i}y)}}\mathrm{d}(x+\mathrm{i}y) \\
&= 2\int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-N-\frac{1}{2}}}{1+s/(\nu f(t))}\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\right)\mathrm{e}^{-(\tan\beta-\beta)\frac{s}{\mathrm{i}f(t)}}\mathrm{d}x.
\end{aligned} \tag{3.26}$$

We thus find that the result (3.24) may be written as

$$\begin{aligned}
R_N^{(H)}(\nu,\beta) = \frac{(-\mathrm{i})^N}{2(2\pi\cot\beta)^{\frac{1}{2}}}\frac{1}{\nu^N}\int_0^{+\infty} s^{N-\frac{1}{2}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-N-\frac{1}{2}}}{1+s/(\nu f(t))} \\
\times\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\right)\mathrm{e}^{-(\tan\beta-\beta)\frac{s}{\mathrm{i}f(t)}}\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{3.27}$$

This is the form of $R_N^{(H)}(\nu,\beta)$ on which it is the most advantageous to base the derivation of our error bounds. A formula for the coefficients $U_N(\mathrm{i}\cot\beta)$ analogous to (3.27) will be needed when deriving our error bounds; it can be obtained by inserting (3.27) into the relation $U_N(\mathrm{i}\cot\beta) = (-1)^N\nu^N(R_N^{(H)}(\nu,\beta) - R_{N+1}^{(H)}(\nu,\beta))$. Hence we have

$$\begin{aligned}
U_N(\mathrm{i}\cot\beta) = \frac{\mathrm{i}^N}{2(2\pi\cot\beta)^{\frac{1}{2}}}\int_0^{+\infty} s^{N-\frac{1}{2}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)} (\mathrm{i}f(t))^{-N-\frac{1}{2}} \\
\times\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\right)\mathrm{e}^{-(\tan\beta-\beta)\frac{s}{\mathrm{i}f(t)}}\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{3.28}$$

Now, from (3.27), one immediately establishes the inequality

$$\begin{aligned}
\left|R_N^{(H)}(\nu,\beta)\right| \le \frac{1}{2(2\pi\cot\beta)^{\frac{1}{2}}}\frac{1}{|\nu|^N}\int_0^{+\infty} s^{N-\frac{1}{2}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-N-\frac{1}{2}}}{|1+s/(\nu f(t))|} \\
\times\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\right)\mathrm{e}^{-(\tan\beta-\beta)\frac{s}{\mathrm{i}f(t)}}\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

In arriving at this inequality, one uses the positivity of $\mathrm{i}f(t)$ and the monotonicity of x on the contour $\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)$. We estimate $1/\left|1+s/\left(\nu f\left(t\right)\right)\right|$ via (2.36) and then compare the result with (3.28) in order to obtain the error bound

$$\left|R_N^{(H)}\left(\nu,\beta\right)\right| \le \frac{\left|U_N\left(\mathrm{i}\cot\beta\right)\right|}{\left|\nu\right|^N}\begin{cases}\left|\sec\theta\right| & \text{if } -\frac{\pi}{2}<\theta<0 \text{ or } \pi<\theta<\frac{3\pi}{2},\\ 1 & \text{if } 0\le\theta\le\pi.\end{cases} \tag{3.29}$$

Let us now turn our attention to the estimation of the remainder $R_N^{(H')}\left(\nu,\beta\right)$. In this case, one finds that the expressions corresponding to (3.27) and (3.28) are

$$\begin{aligned}R_N^{(H')}(\nu,\beta) = &-\frac{\left(-\mathrm{i}\right)^N}{2\left(\pi\sin\left(2\beta\right)\right)^{\frac{1}{2}}}\frac{1}{\nu^N}\int_0^{+\infty}s^{N-\frac{1}{2}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)}\frac{\left(\mathrm{i}f\left(t\right)\right)^{-N-\frac{1}{2}}}{1+s/\left(\nu f\left(t\right)\right)}\\ &\qquad\times\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\right)\mathrm{e}^{-\left(\tan\beta-\beta\right)\frac{s}{\mathrm{i}f(t)}}\sinh x\cos y\,\mathrm{d}y\mathrm{d}s\\ &-\frac{\left(-\mathrm{i}\right)^N}{2\left(\pi\sin\left(2\beta\right)\right)^{\frac{1}{2}}}\frac{1}{\nu^N}\int_0^{+\infty}s^{N-\frac{1}{2}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)}\frac{\left(\mathrm{i}f\left(t\right)\right)^{-N-\frac{1}{2}}}{1+s/\left(\nu f\left(t\right)\right)}\\ &\qquad\times\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\right)\mathrm{e}^{-\left(\tan\beta-\beta\right)\frac{s}{\mathrm{i}f(t)}}\cosh x\sin y\,\mathrm{d}x\mathrm{d}s\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}V_N\left(\mathrm{i}\cot\beta\right) = &-\frac{\mathrm{i}^N}{2\left(\pi\sin\left(2\beta\right)\right)^{\frac{1}{2}}}\int_0^{+\infty}s^{N-\frac{1}{2}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)}\left(\mathrm{i}f\left(t\right)\right)^{-N-\frac{1}{2}}\\ &\qquad\times\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\right)\mathrm{e}^{-\left(\tan\beta-\beta\right)\frac{s}{\mathrm{i}f(t)}}\sinh x\cos y\,\mathrm{d}y\mathrm{d}s\\ &-\frac{\mathrm{i}^N}{2\left(\pi\sin\left(2\beta\right)\right)^{\frac{1}{2}}}\int_0^{+\infty}s^{N-\frac{1}{2}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)}\left(\mathrm{i}f\left(t\right)\right)^{-N-\frac{1}{2}}\\ &\qquad\times\left(1+\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\right)\mathrm{e}^{-\left(\tan\beta-\beta\right)\frac{s}{\mathrm{i}f(t)}}\cosh x\sin y\,\mathrm{d}x\mathrm{d}s,\end{aligned}$$

for any $N\ge 1$. From these expressions and the inequality (2.36), we establish

$$\left|R_N^{(H')}\left(\nu,\beta\right)\right| \le \frac{\left|V_N\left(\mathrm{i}\cot\beta\right)\right|}{\left|\nu\right|^N}\begin{cases}\left|\sec\theta\right| & \text{if } -\frac{\pi}{2}<\theta<0 \text{ or } \pi<\theta<\frac{3\pi}{2},\\ 1 & \text{if } 0\le\theta\le\pi,\end{cases} \tag{3.31}$$

making use of the additional facts that on the contour $\mathscr{C}_2^{(0,0)}\left(\frac{\pi}{2}\right)$ the quantity $\mathrm{i}f(t)$ is positive, x is monotonic and y increases monotonically from β to $\frac{\pi}{2}$.

In the special case when $\theta=\frac{\pi}{2}$, we have $0<1/\left(1+s/\left(\nu f\left(t\right)\right)\right)<1$ in (3.27) and together with (3.28), the mean value theorem of integration shows that

$$R_N^{(H)}\left(\nu,\beta\right) = \left(-1\right)^N\frac{U_N\left(\mathrm{i}\cot\beta\right)}{\nu^N}\Theta_N\left(\nu,\beta\right), \tag{3.32}$$

where $0 < \Theta_N(\nu, \beta) < 1$ is an appropriate number that depends on ν, β and N. We can prove in a similar manner that

$$R_N^{(H')}(\nu, \beta) = (-1)^N \frac{V_N(\mathrm{i}\cot\beta)}{\nu^N} \Xi_N(\nu, \beta), \tag{3.33}$$

where $0 < \Xi_N(\nu, \beta) < 1$ is a suitable number which depends on ν, β and N.

In the case that ν lies in the closed upper half-plane, the estimates (3.29) and (3.31) are as sharp as one can reasonably expect. However, although acceptable in much of the sectors $-\frac{\pi}{2} < \theta < 0$ and $\pi < \theta < \frac{3\pi}{2}$, the bounds (3.29) and (3.31) become inappropriate near the Stokes lines $\theta = -\frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. We now give alternative estimates that are suitable for the sectors $-\pi < \theta < 0$ and $\pi < \theta < 2\pi$ (which include the Stokes lines $\theta = -\frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$). We may use (3.10) and (3.15) to define the remainder terms in the sectors $\frac{3\pi}{2} \leq \theta < 2\pi$ and $-\pi < \theta \leq -\frac{\pi}{2}$. These alternative bounds can be derived based on the representations (3.27) and (3.30). Their derivation is similar to that of (2.43) discussed in Subsection 2.1.2, and the details are therefore omitted. The final results are as follows:

$$\left|R_N^{(H)}(\nu, \beta)\right| \leq \frac{|\sec(\theta - \varphi^*)|}{\cos^{N+\frac{1}{2}}\varphi^*} \frac{|U_N(\mathrm{i}\cot\beta)|}{|\nu|^N} \tag{3.34}$$

and

$$\left|R_N^{(H')}(\nu, \beta)\right| \leq \frac{|\sec(\theta - \varphi^*)|}{\cos^{N+\frac{1}{2}}\varphi^*} \frac{|V_N(\mathrm{i}\cot\beta)|}{|\nu|^N} \tag{3.35}$$

for $-\pi < \theta < 0$ and $\pi < \theta < 2\pi$, where φ^* is the minimizing value given by Lemma 2.1.1 with $\theta - \frac{\pi}{2}$ in place of θ and with $\chi = N + \frac{1}{2}$. Note that the ranges of validity of the bounds (3.29), (3.31), (3.34) and (3.35) together cover that of the asymptotic expansions of the functions $H_\nu^{(1)}(\nu \sec\beta)$ and $H_\nu^{(1)\prime}(\nu \sec\beta)$. We remark that the bounds (3.34) and (3.35) are equivalent to those proved by Meijer [62].

The following simple estimates are suitable near the Stokes lines $\theta = -\frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, and they can be obtained from (3.34) and (3.35) using an argument similar to that given in Subsection 2.1.2:

$$\left|R_N^{(H)}(\nu, \beta)\right| \leq \sqrt{\mathrm{e}(N+1)} \frac{|U_N(\mathrm{i}\cot\beta)|}{|\nu|^N}$$

and

$$\left|R_N^{(H')}(\nu, \beta)\right| \leq \sqrt{\mathrm{e}(N+1)} \frac{|V_N(\mathrm{i}\cot\beta)|}{|\nu|^N},$$

provided that $-\frac{\pi}{2} \le \theta < 0$ or $\pi < \theta \le \frac{3\pi}{2}$ and $N \ge 1$. These bounds may be used in conjunction with our earlier results (3.29) and (3.31), respectively.

The estimation of the remainder terms $R_N^{(J)}(\nu,\beta)$ and $R_N^{(J')}(\nu,\beta)$ can be done in essentially the same way as the estimations of $R_N^{(H)}(\nu,\beta)$ and $R_N^{(H')}(\nu,\beta)$ above, and therefore we omit the proofs. In this way, we may first obtain the analogues of the bounds (3.29) and (3.31),

$$\left|R_N^{(J)}(\nu,\beta)\right| \le \frac{\left|U_N(\mathrm{i}\cot\beta)\right|}{|\nu|^N}\begin{cases}|\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2},\\ 1 & \text{if } |\theta| \le \frac{\pi}{4}\end{cases}$$

and

$$\left|R_N^{(J')}(\nu,\beta)\right| \le \frac{\left|V_N(\mathrm{i}\cot\beta)\right|}{|\nu|^N}\begin{cases}|\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2},\\ 1 & \text{if } |\theta| \le \frac{\pi}{4},\end{cases}$$

respectively.

For the special case when ν is positive, one finds that

$$R_N^{(J)}(\nu,\beta) = (-1)^{\lceil N/2\rceil}\,\mathrm{i}^N\frac{U_N(\mathrm{i}\cot\beta)}{\nu^N}\Theta_N(\nu,\beta)$$

and

$$R_N^{(J')}(\nu,\beta) = (-1)^{\lfloor N/2\rfloor+1}\,\mathrm{i}^N\frac{V_N(\mathrm{i}\cot\beta)}{\nu^N}\Xi_N(\nu,\beta).$$

Here $0 < \Theta_N(\nu,\beta) < 1$ and $0 < \Xi_N(\nu,\beta) < 1$ are appropriate numbers that depend on ν, β and N (cf. equations (3.32) and (3.33)).

Let us now consider estimates which are suitable for the sectors $\frac{\pi}{4} < |\theta| < \pi$. For $\frac{\pi}{2} \le |\theta| < \pi$, the remainder terms $R_N^{(J)}(\nu,\beta)$ and $R_N^{(J')}(\nu,\beta)$ may be defined via (3.20) and (3.22). The bounds are as follows:

$$\left|R_N^{(J)}(\nu,\beta)\right| \le \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{N+\frac{1}{2}}\varphi^*}\frac{\left|U_N(\mathrm{i}\cot\beta)\right|}{|\nu|^N} \tag{3.36}$$

and

$$\left|R_N^{(J')}(\nu,\beta)\right| \le \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{N+\frac{1}{2}}\varphi^*}\frac{\left|V_N(\mathrm{i}\cot\beta)\right|}{|\nu|^N} \tag{3.37}$$

for $\frac{\pi}{4} < |\theta| < \pi$, where φ^* is the minimizing value given by Lemma 2.1.2 with the specific choice of $\chi = N + \frac{1}{2}$. We remark that the estimates (3.36) and (3.37) are both equivalent to those given by Meijer [62].

The two simple bounds below are appropriate near the Stokes lines $\theta = \pm\frac{\pi}{2}$ and can be obtained from (3.36) and (3.37) using an argument akin to those given in Subsection 2.1.2:

$$\left|R_N^{(J)}(\nu,\beta)\right| \le \frac{1}{2}\sqrt{\mathrm{e}(N+2)}\frac{\left|U_N(\mathrm{i}\cot\beta)\right|}{|\nu|^N}$$

and

$$\left|R_N^{(J')}(\nu,\beta)\right| \le \frac{1}{2}\sqrt{\mathrm{e}(N+2)}\frac{\left|V_N(\mathrm{i}\cot\beta)\right|}{|\nu|^N},$$

where $\frac{\pi}{4} < |\theta| \le \frac{\pi}{2}$ and $N \ge 3$.

3.1.3 Asymptotics for the late coefficients

In this subsection, we investigate the asymptotic behaviour of the coefficients $U_n(\mathrm{i}\cot\beta)$ and $V_n(\mathrm{i}\cot\beta)$ as $n \to +\infty$. A formal expansion for $U_n(\mathrm{i}\cot\beta)$ was given by Dingle [35, eq. (54), p. 170]; his result may be written, in our notation, as

$$U_n(\mathrm{i}\cot\beta) \approx \frac{(-1)^n}{2\pi\left(2\mathrm{i}(\tan\beta-\beta)\right)^n}\sum_{m=0}^{\infty}\left(2\mathrm{i}(\tan\beta-\beta)\right)^m U_m(\mathrm{i}\cot\beta)\,\Gamma(n-m). \tag{3.38}$$

We shall derive here the full and rigorous form of Dingle's expansion by truncating it after a finite number of terms and constructing its error bound. The corresponding result for the coefficients $V_n(\mathrm{i}\cot\beta)$ will also be provided.

We begin by considering the $U_n(\mathrm{i}\cot\beta)$'s. First, we split the integral representation (3.19) into two parts as follows:

$$\begin{aligned}U_n(\mathrm{i}\cot\beta) = {} & \frac{\mathrm{i}^n}{2(2\pi\cot\beta)^{\frac{1}{2}}}\int_0^{+\infty} u^{n-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta)u}\,\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u \\ & + \frac{\mathrm{i}^n}{2(2\pi\cot\beta)^{\frac{1}{2}}}\int_0^{+\infty} u^{n-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta+2\pi)u}\,\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u.\end{aligned} \tag{3.39}$$

Then we express the function $\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)$ as a truncated asymptotic expansion

$$\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right) = \frac{\mathrm{e}^{-(\tan\beta-\beta)u}}{\left(\frac{1}{2}\pi u\tan\beta\right)^{\frac{1}{2}}}\left(\sum_{m=0}^{M-1}\mathrm{i}^m\frac{U_m(\mathrm{i}\cot\beta)}{u^m} + R_M^{(H)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)\right), \tag{3.40}$$

for any $M \ge 0$, where we have, by (3.29),

$$\left|R_M^{(H)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)\right| \le \frac{\left|U_M(\mathrm{i}\cot\beta)\right|}{u^M}. \tag{3.41}$$

We substitute (3.40) into (3.39) (with $M' \ge 0$ in place of M in the second integral) and use the estimate (3.41), to establish

$$\begin{aligned}U_n(\mathrm{i}\cot\beta) = {} & \frac{(-1)^n}{2\pi(2\mathrm{i}(\tan\beta-\beta))^n} \\ & \times\left(\sum_{m=0}^{M-1}(2\mathrm{i}(\tan\beta-\beta))^m U_m(\mathrm{i}\cot\beta)\,\Gamma(n-m)+A_M(n,\beta)\right) \\ & + \frac{(-1)^n}{2\pi(2\mathrm{i}(\tan\beta-\beta+\pi))^n} \\ & \times\left(\sum_{m=0}^{M'-1}(2\mathrm{i}(\tan\beta-\beta+\pi))^m U_m(\mathrm{i}\cot\beta)\,\Gamma(n-m)+B_{M'}(n,\beta)\right)\end{aligned} \tag{3.42}$$

where

$$|A_M(n,\beta)| \leq (2(\tan\beta-\beta))^M\,|U_M(\mathrm{i}\cot\beta)|\,\Gamma(n-M) \tag{3.43}$$

and

$$|B_{M'}(n,\beta)| \leq (2(\tan\beta-\beta+\pi))^{M'}\,|U_{M'}(\mathrm{i}\cot\beta)|\,\Gamma\left(n-M'\right), \tag{3.44}$$

provided that $n \geq 1$, $0 \leq M, M' \leq n-1$. When n is large and for fixed M, M', the contribution from the second series in (3.42) is exponentially small compared to the one from the first series. If we neglect this second component and formally extend the first sum to infinity, formula (3.42) reproduces Dingle's expansion (3.38).

If n is large and β is bounded away from zero, the least values of the bounds (3.43) and (3.44) occur when $M \approx \frac{n}{2}$ and $M' \approx \frac{n}{2+\pi/(\tan\beta-\beta)}$. With these choices of M and M', the ratios of the error bounds to the corresponding leading terms in (3.42) are $\mathcal{O}_\beta\left(n^{-\frac{1}{2}}2^{-n}\right)$ and

$$\mathcal{O}_\beta\left(\left(n\frac{\tan\beta-\beta}{\tan\beta-\beta+\pi}\right)^{-\frac{1}{2}}\left(1+\frac{\tan\beta-\beta}{\tan\beta-\beta+\pi}\right)^{-n}\right),$$

respectively. Numerical examples illustrating the efficacy of Dingle's expansion (3.38) and our (3.42), both truncated optimally, are given in Table 3.1. It is seen from the computations that near $\beta = \frac{\pi}{2}$, the contribution from the second series in (3.42) becomes essential. Indeed, it can be shown that for large n and for β satisfying $\tan\beta-\beta > \pi$, the order of the main term in the second series is comparable with the last retained term in the first series, assuming optimal truncation.

values of n, β, M and M'	$n = 50$, $\beta = \frac{\pi}{6}$, $M = 25$, $M' = 1$
exact numerical value of $\|U_n(\mathrm{i}\cot\beta)\|$	$0.25922998993906050847874120769 2 \times 10^{111}$
Dingle's approximation (3.38) to $\|U_n(\mathrm{i}\cot\beta)\|$	$0.259229989939060521496039329899 \times 10^{111}$
error	$-0.13017298122207 \times 10^{95}$
approximation (3.42) to $\|U_n(\mathrm{i}\cot\beta)\|$	$0.259229989939060521496039329899 \times 10^{111}$
error	$-0.13017298122207 \times 10^{95}$
error bound using (3.43) and (3.44)	$0.26037447360811 \times 10^{95}$
values of n, β, M and M'	$n = 50$, $\beta = \frac{6\pi}{13}$, $M = 25$, $M' = 20$
exact numerical value of $\|U_n(\mathrm{i}\cot\beta)\|$	$0.225223901290126273370812162195 \times 10^{6}$
Dingle's approximation (3.38) to $\|U_n(\mathrm{i}\cot\beta)\|$	$0.225223900059708959962876639653 \times 10^{6}$
error	$0.123041731340793552254 2 \times 10^{-2}$
approximation (3.42) to $\|U_n(\mathrm{i}\cot\beta)\|$	$0.225223901290126284664000934035 \times 10^{6}$
error	$-0.11293188771840 \times 10^{-10}$
error bound using (3.43) and (3.44)	$0.22591245013109 \times 10^{-10}$
values of n, β, M and M'	$n = 75$, $\beta = \frac{\pi}{3}$, $M = 37$, $M' = 11$
exact numerical value of $\|U_n(\mathrm{i}\cot\beta)\|$	$0.297450692857018527862002612809 \times 10^{97}$
Dingle's approximation (3.38) to $\|U_n(\mathrm{i}\cot\beta)\|$	$0.297450692857018527862002983265 \times 10^{97}$
error	-0.370456×10^{73}
approximation (3.42) to $\|U_n(\mathrm{i}\cot\beta)\|$	$0.297450692857018527862002983265 \times 10^{97}$
error	-0.370456×10^{73}
error bound using (3.43) and (3.44)	0.731204×10^{73}
values of n, β, M and M'	$n = 75$, $\beta = \frac{7\pi}{15}$, $M = 37$, $M' = 31$
exact numerical value of $\|U_n(\mathrm{i}\cot\beta)\|$	$0.164100247602030019388982625583 \times 10^{17}$
Dingle's approximation (3.38) to $\|U_n(\mathrm{i}\cot\beta)\|$	$0.164100247599006703787878129388 \times 10^{17}$
error	$0.302331560110449619 5 \times 10^{6}$
approximation (3.42) to $\|U_n(\mathrm{i}\cot\beta)\|$	$0.164100247602030019388982829783 \times 10^{17}$
error	-0.204200×10^{-7}
error bound using (3.43) and (3.44)	0.403058×10^{-7}

Table 3.1. *Approximations for $|U_n(\mathrm{i}\cot\beta)|$ with various n and β, using (3.38) and (3.42).*

One may likewise show that for the coefficients $V_n(\mathrm{i}\cot\beta)$,

$$V_n(\mathrm{i}\cot\beta) = \frac{(-1)^{n+1}}{2\pi(2\mathrm{i}(\tan\beta-\beta))^n} \times \left(\sum_{m=0}^{M-1}(2\mathrm{i}(\tan\beta-\beta))^m V_m(\mathrm{i}\cot\beta)\,\Gamma(n-m) + C_M(n,\beta)\right) + \frac{(-1)^{n+1}}{2\pi(2\mathrm{i}(\tan\beta-\beta+\pi))^n} \times \left(\sum_{m=0}^{M'-1}(2\mathrm{i}(\tan\beta-\beta+\pi))^m V_m(\mathrm{i}\cot\beta)\,\Gamma(n-m) + D_{M'}(n,\beta)\right)$$

where

$$|C_M(n,\beta)| \le (2(\tan\beta-\beta))^M |V_M(\mathrm{i}\cot\beta)|\,\Gamma(n-M) \tag{3.45}$$

and

$$|D_{M'}(n,\beta)| \le (2(\tan\beta-\beta+\pi))^{M'} |V_{M'}(\mathrm{i}\cot\beta)|\,\Gamma\left(n-M'\right), \tag{3.46}$$

as long as $n \ge 2$ and $1 \le M, M' \le n-1$. One readily establishes that the least values of the bounds (3.45) and (3.46) occur again when $M \approx \frac{n}{2}$ and $M' \approx \frac{n}{2+\pi/(\tan\beta-\beta)}$, provided that n is large and β is not too close to the origin.

3.1.4 Exponentially improved asymptotic expansions

In this subsection, we give exponentially improved asymptotic expansions for the Hankel and Bessel functions, and their derivatives, for large order and argument. In the case of Bessel functions, expansions similar to ours were derived, using non-rigorous methods, by Dingle [35, eqs. (46)–(49), p. 469]. Dingle considered the contribution only from the adjacent saddle $t^{(1,0)} = -\mathrm{i}\beta$ and gave a re-expansion for $R_N^{(J)}(\nu,\beta)$ accordingly. We utilize the contributions from both adjacent saddles when deriving our exponentially improved asymptotic expansions. This allows us to capture all the exponentially small terms arising from the Stokes phenomena on the rays $\theta = -\frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. Since the moduli of the singulant pair $2\mathrm{i}(\tan\beta-\beta)$, $2\mathrm{i}(\tan\beta-\beta+\pi)$ can be quite different in magnitude (especially when β is close to 0), an extra re-expansion will be necessary before we obtain the usual expressions in terms of terminant functions.

Let us first consider this extra re-expansion in the case of the remainder term $R_N^{(H)}(\nu,\beta)$. We begin by splitting the integral representation (3.12) into two parts as follows:

$$R_N^{(H)}(\nu,\beta) = \frac{(-\mathrm{i})^N}{2(2\pi\cot\beta)^{\frac{1}{2}}}\frac{1}{\nu^N}\int_0^{+\infty}\frac{u^{N-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta)u}}{1+\mathrm{i}u/\nu}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u$$
$$+\frac{(-\mathrm{i})^N}{2(2\pi\cot\beta)^{\frac{1}{2}}}\frac{1}{\nu^N}\int_0^{+\infty}\frac{u^{N-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta+2\pi)u}}{1+\mathrm{i}u/\nu}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u.$$

Next, we expand the denominator of the integrand in the second integral by means of (1.7) to deduce

$$R_N^{(H)}(\nu,\beta) = \sum_{m=N}^{M-1}(-1)^m\frac{\widetilde{U}_m(\mathrm{i}\cot\beta)}{\nu^m} + R_{N,M}^{(H)}(\nu,\beta), \tag{3.47}$$

with[3]

$$\widetilde{U}_m(\mathrm{i}\cot\beta) = \frac{\mathrm{i}^m}{2(2\pi\cot\beta)^{\frac{1}{2}}}\int_0^{+\infty}u^{m-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta+2\pi)u}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u$$

and

$$R_{N,M}^{(H)}(\nu,\beta) = \frac{(-\mathrm{i})^N}{2(2\pi\cot\beta)^{\frac{1}{2}}}\frac{1}{\nu^N}\int_0^{+\infty}\frac{u^{N-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta)u}}{1+\mathrm{i}u/\nu}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u$$
$$+\frac{(-\mathrm{i})^M}{2(2\pi\cot\beta)^{\frac{1}{2}}}\frac{1}{\nu^M}\int_0^{+\infty}\frac{u^{M-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta+2\pi)u}}{1+\mathrm{i}u/\nu}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u, \tag{3.48}$$

for $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $M \geq N \geq 0$. Equation (3.47) gives the required re-expansion of $R_N^{(H)}(\nu,\beta)$ and $R_{N,M}^{(H)}(\nu,\beta)$ is the remainder that we shall express in terms of terminant functions.

In a similar manner, we write

$$R_N^{(H')}(\nu,\beta) = \sum_{m=N}^{M-1}(-1)^m\frac{\widetilde{V}_m(\mathrm{i}\cot\beta)}{\nu^m} + R_{N,M}^{(H')}(\nu,\beta), \tag{3.49}$$

with

$$\widetilde{V}_m(\mathrm{i}\cot\beta) = -\frac{\mathrm{i}^m}{2(\pi\sin(2\beta))^{\frac{1}{2}}}\int_0^{+\infty}u^{m-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta+2\pi)u}H_{\mathrm{i}u}^{(1)\prime}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u$$

[3]Note that the right-hand side is indeed a function of $\mathrm{i}\cot\beta$ as can be seen by writing $\tan\beta - \beta = \frac{\mathrm{i}}{\mathrm{i}\cot\beta} + \mathrm{i}\,\mathrm{arccoth}(\mathrm{i}\cot\beta)$ and $\sec\beta = \frac{\mathrm{i}}{\mathrm{i}\cot\beta}(1-(\mathrm{i}\cot\beta)^2)^{\frac{1}{2}}$.

and

$$R_{N,M}^{(H')}(\nu,\beta) = -\frac{(-\mathrm{i})^N}{2\left(\pi\sin(2\beta)\right)^{\frac{1}{2}}}\frac{1}{\nu^N}\int_0^{+\infty}\frac{u^{N-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta)u}}{1+\mathrm{i}u/\nu}H_{\mathrm{i}u}^{(1)\prime}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta)\mathrm{d}u$$
$$-\frac{(-\mathrm{i})^M}{2\left(\pi\sin(2\beta)\right)^{\frac{1}{2}}}\frac{1}{\nu^M}\int_0^{+\infty}\frac{u^{M-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta+2\pi)u}}{1+\mathrm{i}u/\nu}H_{\mathrm{i}u}^{(1)\prime}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta)\mathrm{d}u,$$

for $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $M \geq N \geq 1$. From now on, we assume that $M \geq N \geq 0$ whenever we write $R_{N,M}^{(H)}(\nu,\beta)$ and $M \geq N \geq 1$ whenever we write $R_{N,M}^{(H')}(\nu,\beta)$.

Now, we are in the position to formulate our re-expansions for the remainder terms $R_{N,M}^{(H)}(\nu,\beta)$ and $R_{N,M}^{(H')}(\nu,\beta)$, in Proposition 3.1.1 below. In this proposition, the functions $R_{N,M}^{(H)}(\nu,\beta)$ and $R_{N,M}^{(H')}(\nu,\beta)$ are extended to a sector larger than $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ via (3.47) and (3.49) using analytic continuation.

Proposition 3.1.1. *Let K and L be arbitrary fixed non-negative integers, and let β be a fixed acute angle. Suppose that $-2\pi+\delta \leq \theta \leq 3\pi-\delta$ with an arbitrary fixed small positive δ, $|\nu|$ is large and $N = 2(\tan\beta-\beta)|\nu|+\rho$, $M = 2(\tan\beta-\beta+\pi)|\nu|+\sigma$ with ρ and σ being bounded. Then*

$$\begin{aligned} R_{N,M}^{(H)}(\nu,\beta) = {} & \mathrm{e}^{-2\mathrm{i}\xi}\sum_{k=0}^{K-1}\frac{U_k(\mathrm{i}\cot\beta)}{\nu^k}T_{N-k}\left(2(\tan\beta-\beta)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\ & + \mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}\sum_{\ell=0}^{L-1}\frac{U_\ell(\mathrm{i}\cot\beta)}{\nu^\ell}T_{M-\ell}\left(2(\tan\beta-\beta+\pi)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\ & + R_{N,M,K,L}^{(H)}(\nu,\beta), \end{aligned} \tag{3.50}$$

$$\begin{aligned} R_{N,M}^{(H')}(\nu,\beta) = {} & -\mathrm{e}^{-2\mathrm{i}\xi}\sum_{k=0}^{K-1}\frac{V_k(\mathrm{i}\cot\beta)}{\nu^k}T_{N-k}\left(2(\tan\beta-\beta)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\ & - \mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}\sum_{\ell=0}^{L-1}\frac{V_\ell(\mathrm{i}\cot\beta)}{\nu^\ell}T_{M-\ell}\left(2(\tan\beta-\beta+\pi)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\ & + R_{N,M,K,L}^{(H')}(\nu,\beta), \end{aligned} \tag{3.51}$$

where

$$R_{N,M,K,L}^{(H)}(\nu,\beta), R_{N,M,K,L}^{(H')}(\nu,\beta) = \mathcal{O}_{K,\beta,\rho}\left(\frac{e^{-2|\xi|}}{|\nu|^K}\right) + \mathcal{O}_{L,\beta,\sigma}\left(\frac{e^{-2|\xi|}e^{-2\pi|\nu|}}{|\nu|^L}\right) \tag{3.52}$$

for $-\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, *and*

$$R_{N,M,K,L}^{(H)}(\nu,\beta), R_{N,M,K,L}^{(H')}(\nu,\beta) = \mathcal{O}_{K,\beta,\rho,\delta}\left(\frac{e^{2\mathfrak{Im}(\xi)}}{|\nu|^K}\right) + \mathcal{O}_{L,\beta,\sigma,\delta}\left(\frac{e^{2\mathfrak{Im}(\xi)}e^{2\pi\mathfrak{Im}(\nu)}}{|\nu|^L}\right) \tag{3.53}$$

for $-2\pi + \delta \le \theta \le -\frac{\pi}{2}$ *and* $\frac{3\pi}{2} \le \theta \le 3\pi - \delta$.

Proposition 3.1.1 in conjunction with (3.10), (3.13), (3.15), (3.16), (3.47) and (3.49) yields the exponentially improved asymptotic expansions for the Hankel functions, and their derivatives, for large order and argument. In particular, formula (3.50) together with (3.10) and (3.47) embraces the three asymptotic expansions (3.1) and

$$H_\nu^{(1)}(\nu\sec\beta) \sim \frac{e^{i\xi}}{\left(\frac{1}{2}\pi\nu\tan\beta\right)^{\frac{1}{2}}}\left(\sum_{n=0}^{\infty}(-1)^n\frac{U_n(i\cot\beta)}{\nu^n} \pm e^{-2i\xi}\sum_{k=0}^{\infty}\frac{U_k(i\cot\beta)}{\nu^k}\right.$$
$$\left.\pm e^{-2i\xi}e^{-2\pi i\nu}\sum_{\ell=0}^{\infty}\frac{U_\ell(i\cot\beta)}{\nu^\ell}\right)$$

which holds when $\nu \to \infty$ in the sectors $-\frac{\pi}{2} + \delta \le \theta \mp \frac{3\pi}{2} \le \frac{3\pi}{2} - \delta$ (see, e.g., [73]); furthermore, they give the smooth transition across the Stokes lines $\theta = -\frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.

In the following theorem, we give explicit bounds on $R_{N,M,L,K}^{(H)}(\nu,\beta)$ and $R_{N,M,L,K}^{(H')}(\nu,\beta)$. Note that in these results, N and M do not necessarily depend on ν and β. (Evidently, $R_{N,M,L,K}^{(H)}(\nu,\beta)$ and $R_{N,M,L,K}^{(H')}(\nu,\beta)$ can be defined for arbitrary positive integers N and M via (3.50) and (3.51), respectively.)

Theorem 3.1.2. *Let* N, M, K *and* L *be arbitrary fixed non-negative integers such that* $N \le M$, $K < N$, $L < M$, *and let* β *be a fixed acute angle. Then we have*

$$\begin{aligned}\left|R_{N,M,L,K}^{(H)}(\nu,\beta)\right| \le{}& \left|e^{-2i\xi}\right|\frac{|U_K(i\cot\beta)|}{|\nu|^K}\left|T_{N-K}\left(2(\tan\beta-\beta)\nu e^{-\frac{\pi}{2}i}\right)\right| \\ &+ \frac{|U_K(i\cot\beta)|\,\Gamma(N-K)}{2\pi(2(\tan\beta-\beta))^{N-K}|\nu|^N} \\ &+ \left|e^{-2i\xi}e^{-2\pi i\nu}\right|\frac{|U_L(i\cot\beta)|}{|\nu|^L}\left|T_{M-L}\left(2(\tan\beta-\beta+\pi)\nu e^{-\frac{\pi}{2}i}\right)\right| \\ &+ \frac{|U_L(i\cot\beta)|\,\Gamma(M-L)}{2\pi(2(\tan\beta-\beta+\pi))^{M-L}|\nu|^M}\end{aligned} \tag{3.54}$$

provided that $-\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, *and*

$$\begin{aligned} \left|R_{N,M,L,K}^{(H')}(\nu,\beta)\right| \le {} & \left|e^{-2i\xi}\right| \frac{|V_K(i\cot\beta)|}{|\nu|^K} \left|T_{N-K}\left(2(\tan\beta-\beta)\nu e^{-\frac{\pi}{2}i}\right)\right| \\ & + \frac{|V_K(i\cot\beta)|\,\Gamma(N-K)}{2\pi(2(\tan\beta-\beta))^{N-K}|\nu|^N} \\ & + \left|e^{-2i\xi}e^{-2\pi i\nu}\right| \frac{|V_L(i\cot\beta)|}{|\nu|^L} \left|T_{M-L}\left(2(\tan\beta-\beta+\pi)\nu e^{-\frac{\pi}{2}i}\right)\right| \\ & + \frac{|V_L(i\cot\beta)|\,\Gamma(M-L)}{2\pi(2(\tan\beta-\beta+\pi))^{M-L}|\nu|^M} \end{aligned} \tag{3.55}$$

for $-\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ *and* $K, L \ge 1$.

To see that these bounds are sharp, note that the first and third terms on the right-hand sides of the inequalities (3.54) and (3.55) are the magnitudes of the first neglected terms in the expansions (3.50) and (3.51), respectively. It can be shown that for large $N-K$ and $M-L$, the second and fourth terms are comparable with, or less than, the corresponding first and third terms (except near the zeros of the terminant functions). The proof is similar to that given by Boyd [12] in the case of the modified Bessel function $K_\nu(z)$ and is therefore not pursued here.

To derive the analogous results for $R_N^{(J)}(\nu,\beta)$ and $R_N^{(J')}(\nu,\beta)$, one can proceed as follows. Starting from $2J_\nu(\nu\sec\beta) = H_\nu^{(1)}(\nu\sec\beta) + H_\nu^{(2)}(\nu\sec\beta)$ and $2J'_\nu(\nu\sec\beta) = H_\nu^{(1)\prime}(\nu\sec\beta) + H_\nu^{(2)\prime}(\nu\sec\beta)$, it is not difficult to show that the following relations hold:

$$\begin{aligned} 2R_{2N}^{(J)}(\nu,\beta) &= R_{2N}^{(H)}(\nu e^{\pi i},\beta) + R_{2N}^{(H)}(\nu,\beta), \\ 2iR_{2N+1}^{(J)}(\nu,\beta) &= R_{2N}^{(H)}(\nu e^{\pi i},\beta) - R_{2N}^{(H)}(\nu,\beta), \\ -2R_{2N}^{(J')}(\nu,\beta) &= R_{2N}^{(H')}(\nu e^{\pi i},\beta) + R_{2N}^{(H')}(\nu,\beta), \\ 2iR_{2N+1}^{(J')}(\nu,\beta) &= R_{2N}^{(H')}(\nu e^{\pi i},\beta) - R_{2N}^{(H')}(\nu,\beta). \end{aligned}$$

We substitute (3.47) and (3.49) into these relations to obtain

$$R_{2N}^{(J)}(\nu,\beta) = \sum_{m=N}^{M-1} \frac{\widetilde{U}_{2m}(i\cot\beta)}{\nu^{2m}} + R_{2N,M}^{(J)}(\nu,\beta),$$

$$iR_{2N+1}^{(J)}(\nu,\beta) = \sum_{m=N}^{M-1} \frac{\widetilde{U}_{2m+1}(i\cot\beta)}{\nu^{2m+1}} + R_{2N+1,M}^{(J)}(\nu,\beta),$$

$$-R_{2N}^{(J')}(\nu,\beta) = \sum_{m=N}^{M-1} \frac{\widetilde{V}_{2m}(i\cot\beta)}{\nu^{2m}} + R_{2N,M}^{(J')}(\nu,\beta)$$

and

$$\mathrm{i}R_{2N+1}^{(J')}(\nu,\beta) = \sum_{m=N}^{M-1} \frac{\widetilde{V}_{2m+1}(\mathrm{i}\cot\beta)}{\nu^{2m+1}} + R_{2N+1,M}^{(J')}(\nu,\beta),$$

where remainder terms are given by

$$\begin{aligned}
2R_{2N,M}^{(J)}(\nu,\beta) &= R_{2N,2M}^{(H)}(\nu e^{\pi \mathrm{i}},\beta) + R_{2N,2M}^{(H)}(\nu,\beta),\\
2R_{2N+1,M}^{(J)}(\nu,\beta) &= R_{2N,2M}^{(H)}(\nu e^{\pi \mathrm{i}},\beta) - R_{2N,2M}^{(H)}(\nu,\beta),\\
2R_{2N,M}^{(J')}(\nu,\beta) &= R_{2N,2M}^{(H')}(\nu e^{\pi \mathrm{i}},\beta) + R_{2N,2M}^{(H')}(\nu,\beta),\\
2R_{2N+1,M}^{(J')}(\nu,\beta) &= R_{2N,2M}^{(H')}(\nu e^{\pi \mathrm{i}},\beta) - R_{2N,2M}^{(H')}(\nu,\beta).
\end{aligned} \tag{3.56}$$

Now, a direct application of Proposition 3.1.1 to the right-hand sides yields the desired re-expansions which are summarized in the following proposition.

Proposition 3.1.3. *Let K and L be arbitrary fixed non-negative integers, and let β be a fixed acute angle. Suppose that $|\theta| \le 2\pi - \delta$ with an arbitrary fixed small positive δ, $|\nu|$ is large and $N = (\tan\beta - \beta)|\nu| + \rho$, $M = (\tan\beta - \beta + \pi)|\nu| + \sigma$ with ρ and σ being bounded. Then*

$$\begin{aligned}
R_{2N,M}^{(J)}(\nu,\beta) = &-\frac{e^{2\mathrm{i}\xi}}{2}\sum_{k=0}^{K-1}(-1)^k\frac{U_k(\mathrm{i}\cot\beta)}{\nu^k}T_{2N-k}\big(2(\tan\beta-\beta)\nu e^{\frac{\pi}{2}\mathrm{i}}\big)\\
&+\frac{e^{-2\mathrm{i}\xi}}{2}\sum_{k=0}^{K-1}\frac{U_k(\mathrm{i}\cot\beta)}{\nu^k}T_{2N-k}\big(2(\tan\beta-\beta)\nu e^{-\frac{\pi}{2}\mathrm{i}}\big)\\
&-\frac{e^{2\mathrm{i}\xi}e^{2\pi\mathrm{i}\nu}}{2}\sum_{\ell=0}^{L-1}(-1)^\ell\frac{U_\ell(\mathrm{i}\cot\beta)}{\nu^\ell}T_{2M-\ell}\big(2(\tan\beta-\beta+\pi)\nu e^{\frac{\pi}{2}\mathrm{i}}\big)\\
&+\frac{e^{-2\mathrm{i}\xi}e^{-2\pi\mathrm{i}\nu}}{2}\sum_{\ell=0}^{L-1}\frac{U_\ell(\mathrm{i}\cot\beta)}{\nu^\ell}T_{2M-\ell}\big(2(\tan\beta-\beta+\pi)\nu e^{-\frac{\pi}{2}\mathrm{i}}\big)\\
&+R_{2N,M,K,L}^{(J)}(\nu,\beta),
\end{aligned}$$

$$\begin{aligned}
R_{2N+1,M}^{(J)}(\nu,\beta) = &-\frac{e^{2\mathrm{i}\xi}}{2}\sum_{k=0}^{K-1}(-1)^k\frac{U_k(\mathrm{i}\cot\beta)}{\nu^k}T_{2N-k}\big(2(\tan\beta-\beta)\nu e^{\frac{\pi}{2}\mathrm{i}}\big)\\
&-\frac{e^{-2\mathrm{i}\xi}}{2}\sum_{k=0}^{K-1}\frac{U_k(\mathrm{i}\cot\beta)}{\nu^k}T_{2N-k}\big(2(\tan\beta-\beta)\nu e^{-\frac{\pi}{2}\mathrm{i}}\big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\mathrm{e}^{2\mathrm{i}\xi}\mathrm{e}^{2\pi\mathrm{i}\nu}}{2} \sum_{\ell=0}^{L-1} (-1)^{\ell} \frac{U_{\ell}(\mathrm{i}\cot\beta)}{\nu^{\ell}} T_{2M-\ell}\big(2(\tan\beta-\beta+\pi)\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big) \\
&- \frac{\mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}}{2} \sum_{\ell=0}^{L-1} \frac{U_{\ell}(\mathrm{i}\cot\beta)}{\nu^{\ell}} T_{2M-\ell}\big(2(\tan\beta-\beta+\pi)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\big) \\
&+ R^{(J)}_{2N+1,M,K,L}(\nu,\beta),
\end{aligned}
$$

$$
\begin{aligned}
R^{(J')}_{2N,M}(\nu,\beta) = &\frac{\mathrm{e}^{2\mathrm{i}\xi}}{2} \sum_{k=0}^{K-1} (-1)^{k} \frac{V_{k}(\mathrm{i}\cot\beta)}{\nu^{k}} T_{2N-k}\big(2(\tan\beta-\beta)\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big) \\
&- \frac{\mathrm{e}^{-2\mathrm{i}\xi}}{2} \sum_{k=0}^{K-1} \frac{V_{k}(\mathrm{i}\cot\beta)}{\nu^{k}} T_{2N-k}\big(2(\tan\beta-\beta)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\big) \\
&+ \frac{\mathrm{e}^{2\mathrm{i}\xi}\mathrm{e}^{2\pi\mathrm{i}\nu}}{2} \sum_{\ell=0}^{L-1} (-1)^{\ell} \frac{V_{\ell}(\mathrm{i}\cot\beta)}{\nu^{\ell}} T_{2M-\ell}\big(2(\tan\beta-\beta+\pi)\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big) \\
&- \frac{\mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}}{2} \sum_{\ell=0}^{L-1} \frac{V_{\ell}(\mathrm{i}\cot\beta)}{\nu^{\ell}} T_{2M-\ell}\big(2(\tan\beta-\beta+\pi)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\big) \\
&+ R^{(J')}_{2N,M,K,L}(\nu,\beta)
\end{aligned}
$$

and

$$
\begin{aligned}
R^{(J')}_{2N+1,M}(\nu,\beta) = &\frac{\mathrm{e}^{2\mathrm{i}\xi}}{2} \sum_{k=0}^{K-1} (-1)^{k} \frac{V_{k}(\mathrm{i}\cot\beta)}{\nu^{k}} T_{2N-k}\big(2(\tan\beta-\beta)\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big) \\
&+ \frac{\mathrm{e}^{-2\mathrm{i}\xi}}{2} \sum_{k=0}^{K-1} \frac{V_{k}(\mathrm{i}\cot\beta)}{\nu^{k}} T_{2N-k}\big(2(\tan\beta-\beta)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\big) \\
&+ \frac{\mathrm{e}^{2\mathrm{i}\xi}\mathrm{e}^{2\pi\mathrm{i}\nu}}{2} \sum_{\ell=0}^{L-1} (-1)^{\ell} \frac{V_{\ell}(\mathrm{i}\cot\beta)}{\nu^{\ell}} T_{2M-\ell}\big(2(\tan\beta-\beta+\pi)\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big) \\
&+ \frac{\mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}}{2} \sum_{\ell=0}^{L-1} \frac{V_{\ell}(\mathrm{i}\cot\beta)}{\nu^{\ell}} T_{2M-\ell}\big(2(\tan\beta-\beta+\pi)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\big) \\
&+ R^{(J)}_{2N+1,M,K,L}(\nu,\beta),
\end{aligned}
$$

where

$$
\begin{aligned}
R^{(J)}_{2N,M,K,L}(\nu,\beta), R^{(J)}_{2N+1,M,K,L}(\nu,\beta), R^{(J')}_{2N,M,K,L}(\nu,\beta), R^{(J')}_{2N+1,M,K,L}(\nu,\beta) = \\
= \mathcal{O}_{K,\beta,\rho}\left(\frac{\mathrm{e}^{-2|\xi|}}{|\nu|^{K}}\right) + \mathcal{O}_{L,\beta,\sigma}\left(\frac{\mathrm{e}^{-2|\xi|}\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{L}}\right)
\end{aligned}
$$

for $|\theta| \le \frac{\pi}{2}$, *and*

$$R_{2N,M,K,L}^{(J)}(\nu,\beta), R_{2N+1,M,K,L}^{(J)}(\nu,\beta), R_{2N,M,K,L}^{(J')}(\nu,\beta), R_{2N+1,M,K,L}^{(J')}(\nu,\beta) = \\ = \mathcal{O}_{K,\beta,\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\mathfrak{Im}(\xi)}}{|\nu|^K}\right) + \mathcal{O}_{L,\beta,\sigma,\delta}\left(\frac{\mathrm{e}^{\mp 2\mathfrak{Im}(\xi)}\mathrm{e}^{\mp 2\pi\mathfrak{Im}(\nu)}}{|\nu|^L}\right)$$

for $\frac{\pi}{2} \le \pm\theta \le 2\pi - \delta$.

Explicit bounds for $R_{2N,M,K,L}^{(J)}(\nu,\beta)$, $R_{2N+1,M,K,L}^{(J)}(\nu,\beta)$, $R_{2N,M,K,L}^{(J')}(\nu,\beta)$ and $R_{2N+1,M,K,L}^{(J')}(\nu,\beta)$ may be derived using Theorem 3.1.2 together with the inequalities

$$2\left|R_{2N,M,K,L}^{(J)}(\nu,\beta)\right|, 2\left|R_{2N+1,M,K,L}^{(J)}(\nu,\beta)\right| \le \\ \le \left|R_{2N,2M,K,L}^{(H)}(\nu \mathrm{e}^{\pi \mathrm{i}},\beta)\right| + \left|R_{2N,2M,K,L}^{(H)}(\nu,\beta)\right|$$

and

$$2\left|R_{2N,M,K,L}^{(J')}(\nu,\beta)\right|, 2\left|R_{2N+1,M,K,L}^{(J')}(\nu,\beta)\right| \le \\ \le \left|R_{2N,2M,K,L}^{(H')}(\nu \mathrm{e}^{\pi \mathrm{i}},\beta)\right| + \left|R_{2N,2M,K,L}^{(H')}(\nu,\beta)\right|,$$

which can be established readily from the expressions (3.56).

Proof of Proposition 3.1.1 and Theorem 3.1.2. We only prove the statements for $R_{N,M}^{(H)}(\nu,\beta)$ and $R_{N,M,L,K}^{(H)}(\nu,\beta)$; the remainders $R_{N,M}^{(H')}(\nu,\beta)$ and $R_{N,M,L,K}^{(H')}(\nu,\beta)$ can be handled similarly. Let N, M, K and L be arbitrary fixed non-negative integers such that $K < N$ and $L < M$. Suppose that $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. We begin by replacing the function $\mathrm{i}H_{\mathrm{i}u}^{(1)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta)$ in (3.48) by its truncated asymptotic expansion (3.40) (with k and K in place of m and M in the first integral, and with ℓ and L in place of m and M in the second integral) and using the definition of the terminant function, in order to obtain

$$\begin{aligned} R_{N,M}^{(H)}(\nu,\beta) = {} & \mathrm{e}^{-2\mathrm{i}\xi}\sum_{k=0}^{K-1}\frac{U_k(\mathrm{i}\cot\beta)}{\nu^k}T_{N-k}\left(2(\tan\beta-\beta)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\ & + \mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}\sum_{\ell=0}^{L-1}\frac{U_\ell(\mathrm{i}\cot\beta)}{\nu^\ell}T_{M-\ell}\left(2(\tan\beta-\beta+\pi)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\ & + R_{N,M,K,L}^{(H)}(\nu,\beta), \end{aligned} \tag{3.57}$$

with

$$\begin{aligned}
R_{N,M,K,L}^{(H)}(\nu,\beta) &= \frac{(-\mathrm{i})^N}{2\pi}\frac{1}{\nu^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2(\tan\beta-\beta)u}}{1+\mathrm{i}u/\nu}R_K^{(H)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)\mathrm{d}u \\
&\quad + \frac{(-\mathrm{i})^M}{2\pi}\frac{1}{\nu^M}\int_0^{+\infty}\frac{u^{M-1}\mathrm{e}^{-2(\tan\beta-\beta+\pi)u}}{1+\mathrm{i}u/\nu}R_L^{(H)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)\mathrm{d}u \\
&= \frac{(-\mathrm{i})^N}{2\pi}\mathrm{e}^{-\mathrm{i}\theta N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2(\tan\beta-\beta)r\tau}}{1+\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_K^{(H)}\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)\mathrm{d}\tau \\
&\quad + \frac{(-\mathrm{i})^M}{2\pi}\mathrm{e}^{-\mathrm{i}\theta M}\int_0^{+\infty}\frac{\tau^{M-1}\mathrm{e}^{-2(\tan\beta-\beta+\pi)r\tau}}{1+\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_L^{(H)}\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)\mathrm{d}\tau.
\end{aligned} \tag{3.58}$$

In passing to the second equality, we have taken $\nu = r\mathrm{e}^{\mathrm{i}\theta}$ and have made the change of integration variable from u to τ by $u = r\tau$. Let us consider the estimation of the integral in (3.58) which involves $R_K^{(H)}\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)$. The remainder $R_K^{(H)}\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)$ is given by the integral representation (3.12), which can be re-expressed in the form

$$\begin{aligned}
R_K^{(H)}\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right) &= \frac{(-1)^K}{2\left(2\pi\cot\beta\right)^{\frac{1}{2}}}\frac{1}{(r\tau)^K}\int_0^{+\infty}\frac{t^{K-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta)t}}{1+t/r} \\
&\qquad\qquad \times\left(1+\mathrm{e}^{-2\pi t}\right)\mathrm{i}H_{\mathrm{i}t}^{(1)}\left(t\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}t \\
&\quad + \frac{(-1)^K}{2\left(2\pi\cot\beta\right)^{\frac{1}{2}}}\frac{\tau-1}{(r\tau)^K}\int_0^{+\infty}\frac{t^{K-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta)t}}{\left(1+r\tau/t\right)\left(1+t/r\right)} \\
&\qquad\qquad \times\left(1+\mathrm{e}^{-2\pi t}\right)\mathrm{i}H_{\mathrm{i}t}^{(1)}\left(t\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}t.
\end{aligned}$$

Taking into account the inequalities (2.64), we establish the upper bound

$$\begin{aligned}
&\left|\frac{(-\mathrm{i})^N}{2\pi}\mathrm{e}^{-\mathrm{i}\theta N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2(\tan\beta-\beta)r\tau}}{1+\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_K^{(H)}\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)\mathrm{d}\tau\right| \le \\
&\qquad \le \frac{\left|U_K\left(\mathrm{i}\cot\beta\right)\right|}{|\nu|^K}\frac{1}{2\pi}\left|\int_0^{+\infty}\frac{\tau^{N-K-1}\mathrm{e}^{-2(\tan\beta-\beta)r\tau}}{1+\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\mathrm{d}\tau\right| \\
&\qquad\quad + \frac{\left|U_K\left(\mathrm{i}\cot\beta\right)\right|}{|\nu|^K}\frac{1}{2\pi}\int_0^{+\infty}\tau^{N-K-1}\mathrm{e}^{-2(\tan\beta-\beta)r\tau}\left|\frac{\tau-1}{\tau-\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}\right|\mathrm{d}\tau.
\end{aligned}$$

In arriving at this inequality, we have made use of the representation (3.19) of the coefficients $U_K\left(\mathrm{i}\cot\beta\right)$ and the fact that $\mathrm{i}H_{\mathrm{i}t}^{(1)}\left(t\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right) > 0$ for any $t > 0$

(see, e.g., [73]). Since $|\left(\tau-1\right)/\left(\tau-\mathrm{i}e^{\mathrm{i}\theta}\right)|\leq 1$ for positive τ, after simplification we find that

$$\begin{aligned}\left|\frac{(-\mathrm{i})^N}{2\pi}\mathrm{e}^{-\mathrm{i}\theta N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2(\tan\beta-\beta)r\tau}}{1+\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_K^{(H)}\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)\mathrm{d}\tau\right| &\leq \\ \leq \left|\mathrm{e}^{-2\mathrm{i}\xi}\right|\frac{\left|U_K\left(\mathrm{i}\cot\beta\right)\right|}{|\nu|^K}\left|T_{N-K}\left(2\left(\tan\beta-\beta\right)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\ +\frac{\left|U_K\left(\mathrm{i}\cot\beta\right)\right|\Gamma\left(N-K\right)}{2\pi\left(2\left(\tan\beta-\beta\right)\right)^{N-K}|\nu|^N}.\end{aligned}\tag{3.59}$$

One can prove in a similar way that

$$\begin{aligned}\left|\frac{(-\mathrm{i})^M}{2\pi}\mathrm{e}^{-\mathrm{i}\theta M}\int_0^{+\infty}\frac{\tau^{M-1}\mathrm{e}^{-2(\tan\beta-\beta+\pi)r\tau}}{1+\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}R_L^{(H)}\left(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta\right)\mathrm{d}\tau\right| &\leq \\ \leq \left|\mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}\right|\frac{\left|U_L\left(\mathrm{i}\cot\beta\right)\right|}{|\nu|^L}\left|T_{M-L}\left(2\left(\tan\beta-\beta+\pi\right)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\ +\frac{\left|U_L\left(\mathrm{i}\cot\beta\right)\right|\Gamma\left(M-L\right)}{2\pi\left(2\left(\tan\beta-\beta+\pi\right)\right)^{M-L}|\nu|^M}.\end{aligned}\tag{3.60}$$

Thus, from (3.58), (3.59) and (3.60), we obtain the error bound

$$\begin{aligned}\left|R_{N,M,K,L}^{(H)}\left(\nu,\beta\right)\right| \leq{}& \left|\mathrm{e}^{-2\mathrm{i}\xi}\right|\frac{\left|U_K\left(\mathrm{i}\cot\beta\right)\right|}{|\nu|^K}\left|T_{N-K}\left(2\left(\tan\beta-\beta\right)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\ &+\frac{\left|U_K\left(\mathrm{i}\cot\beta\right)\right|\Gamma\left(N-K\right)}{2\pi\left(2\left(\tan\beta-\beta\right)\right)^{N-K}|\nu|^N} \\ &+\left|\mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}\right|\frac{\left|U_L\left(\mathrm{i}\cot\beta\right)\right|}{|\nu|^L}\left|T_{M-L}\left(2\left(\tan\beta-\beta+\pi\right)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\ &+\frac{\left|U_L\left(\mathrm{i}\cot\beta\right)\right|\Gamma\left(M-L\right)}{2\pi\left(2\left(\tan\beta-\beta+\pi\right)\right)^{M-L}|\nu|^M}.\end{aligned}\tag{3.61}$$

By continuity, this bound holds in the closed sector $-\frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}$. This proves Theorem 3.1.2 for $R_{N,M,K,L}^{(H)}\left(\nu,\beta\right)$.

From now on, we suppose that $|\nu|$ is large and that $N=2\left|\nu\right|\left(\tan\beta-\beta\right)+\rho$, $M=2\left|\nu\right|\left(\tan\beta-\beta+\pi\right)+\sigma$ with ρ and σ being bounded. Using these assumptions and Olver's estimate (1.90), the first and third terms on the right-hand side of the inequality (3.61) are found to be $\mathcal{O}_{K,\beta,\rho}(|\nu|^{-K}\mathrm{e}^{-2|\xi|})$ and $\mathcal{O}_{L,\beta,\sigma}(|\nu|^{-L}\mathrm{e}^{-2|\xi|}\mathrm{e}^{-2\pi|\nu|})$. By employing Stirling's formula, the second and fourth terms are $\mathcal{O}_{K,\beta,\rho}(|\nu|^{-K-\frac{1}{2}}\mathrm{e}^{-2|\xi|})$ and $\mathcal{O}_{L,\beta,\sigma}(|\nu|^{-L-\frac{1}{2}}\mathrm{e}^{-2|\xi|}\mathrm{e}^{-2\pi|\nu|})$, respectively. This establishes the estimate (3.52) for $R_{N,M,K,L}^{(H)}(\nu,\beta)$.

Consider now the sector $\frac{3\pi}{2} \leq \theta \leq 3\pi - \delta$. For such values of θ, the function $R_{N,M,K,L}^{(H)}(\nu,\beta)$ can be defined via (3.57). When ν enters this sector, the poles of the integrands in (3.58) cross the integration path. According to the residue theorem, we obtain

$$\begin{aligned}
R_{N,M,K,L}^{(H)}(\nu,\beta) &= \frac{(-\mathrm{i})^N}{2\pi}\frac{1}{\nu^N}\int_0^{+\infty}\frac{u^{N-1}\mathrm{e}^{-2(\tan\beta-\beta)u}}{1+\mathrm{i}u/\nu}R_K^{(H)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta)\mathrm{d}u \\
&\quad + \frac{(-\mathrm{i})^M}{2\pi}\frac{1}{\nu^M}\int_0^{+\infty}\frac{u^{M-1}\mathrm{e}^{-2(\tan\beta-\beta+\pi)u}}{1+\mathrm{i}u/\nu}R_L^{(H)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},\beta)\mathrm{d}u \\
&\quad + \mathrm{e}^{-2\mathrm{i}\xi}R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}},\beta) + \mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}R_L^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}},\beta) \\
&= R_{N,M,K,L}^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}},\beta) + \mathrm{e}^{-2\mathrm{i}\xi}R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}},\beta) \\
&\qquad + \mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}R_L^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}},\beta)
\end{aligned}$$

for $\frac{3\pi}{2} < \theta < \frac{7\pi}{2}$. Now, by analytic continuation, the equality

$$\begin{aligned}
R_{N,M,K,L}^{(H)}(\nu,\beta) = R_{N,M,K,L}^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}},\beta) &+ \mathrm{e}^{-2\mathrm{i}\xi}R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}},\beta) \\
&+ \mathrm{e}^{-2\mathrm{i}\xi}\mathrm{e}^{-2\pi\mathrm{i}\nu}R_L^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}},\beta)
\end{aligned}$$

holds for any complex ν, in particular for those lying in the sector $\frac{3\pi}{2} \leq \theta \leq 3\pi - \delta$. The asymptotic expansion (3.1) implies $R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}},\beta) = \mathcal{O}_{K,\beta,\delta}(|\nu|^{-K})$ and $R_L^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}},\beta) = \mathcal{O}_{L,\beta,\delta}(|\nu|^{-L})$ for large ν in $\frac{3\pi}{2} \leq \theta \leq 3\pi - \delta$. From the estimate (3.52), we infer that $R_{N,M,K,L}^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}},\beta) = \mathcal{O}_{K,\beta,\rho}(|\nu|^{-K}\mathrm{e}^{-2|\xi|}) + \mathcal{O}_{L,\beta,\sigma}(|\nu|^{-L}\mathrm{e}^{-2|\xi|}\mathrm{e}^{-2\pi|\nu|})$ as $\nu \to \infty$ in the sector $\frac{3\pi}{2} \leq \theta \leq 3\pi - \delta$. This shows that the estimate (3.53) holds for $R_{N,M,K,L}^{(H)}(\nu,\beta)$ when $\frac{3\pi}{2} \leq \theta \leq 3\pi - \delta$. The proof for the sector $-2\pi + \delta \leq \theta \leq -\frac{\pi}{2}$ is completely analogous. ■

3.2 Hankel and Bessel functions of equal order and argument

This section concerns the large-ν asymptotic expansions of the Hankel functions $H_\nu^{(1)}(\nu)$, $H_\nu^{(2)}(\nu)$, the Bessel functions $J_\nu(\nu)$, $Y_\nu(\nu)$ and their derivatives of equal order and argument. Using formal methods, a complete asymptotic expansion for $J_\nu(\nu)$, ν an integer, was given by Cauchy [18,19] in 1854 and later by Meissel [63,64] in 1891. A rigorous derivation of their result was provided by Nicholson [79] in 1908, who also considered the corresponding expansion of the

Bessel function $Y_\nu(\nu)$. In 1909, Debye [25] introduced the method of steepest descents and used it to derive the asymptotic expansions of the Hankel functions $H_\nu^{(1)}(\nu)$, $H_\nu^{(2)}(\nu)$ and the Bessel function $J_\nu(\nu)$ for large positive real ν. In a subsequent paper [26], he extended these expansions to complex values of ν. We remark that, in fact, Nicholson and Debye dealt with the more general cases of $H_\nu^{(1)}(\nu+\kappa)$, $H_\nu^{(2)}(\nu+\kappa)$, $J_\nu(\nu+\kappa)$ and $Y_\nu(\nu+\kappa)$ when $\kappa = o(|\nu|^{\frac{1}{3}})$, but, for the sake of simplicity, we restrict ourselves to the special case of $\kappa = 0$. We note that the more general case is investigated in the paper [76] of the present author.

In modern notation, Nicholson's and Debye's expansions may be written

$$H_\nu^{(1)}(\nu) \sim -\frac{2}{3\pi}\sum_{n=0}^{\infty} d_{2n} \mathrm{e}^{\frac{2\pi(2n+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(2n+1)}{3}\right)\frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}}, \tag{3.62}$$

as $\nu \to \infty$ in the sector $-\pi+\delta \le \theta \le 2\pi-\delta$;

$$H_\nu^{(2)}(\nu) \sim -\frac{2}{3\pi}\sum_{n=0}^{\infty} d_{2n} \mathrm{e}^{-\frac{2\pi(2n+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(2n+1)}{3}\right)\frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}}, \tag{3.63}$$

as $\nu \to \infty$ in the sector $-2\pi+\delta \le \theta \le \pi-\delta$;

$$J_\nu(\nu) \sim \frac{1}{3\pi}\sum_{n=0}^{\infty} d_{2n} \sin\left(\frac{\pi(2n+1)}{3}\right)\frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}} \tag{3.64}$$

and

$$Y_\nu(\nu) \sim -\frac{2}{3\pi}\sum_{n=0}^{\infty} d_{2n} \sin^2\left(\frac{\pi(2n+1)}{3}\right)\frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}}, \tag{3.65}$$

as $\nu \to \infty$ in the sector $|\theta| \le \pi-\delta$, with δ being an arbitrary small positive constant and $\theta = \arg\nu$ (see, e.g., [121, pp. 100–103]). The cube root in these expansions is defined to be positive when ν is positive, and it is defined elsewhere by analytic continuation. The coefficients d_{2n} are real numbers; some expressions for these numbers will be given in Subsection 3.2.1 below.

The structure of this section is as follows. In Subsection 3.2.1, we derive resurgence formulae for Hankel and Bessel functions, and their derivatives, for equal order and argument. Error bounds for the asymptotic expansions of these functions are established in Subsection 3.2.2. Subsection 3.2.3 deals with the asymptotic behaviour of the corresponding late coefficients. Finally, in Subsection 3.2.4, we give exponentially improved asymptotic expansions for the above mentioned functions.

3.2.1 The resurgence formulae

In this subsection, we study the resurgence properties of the Hankel and Bessel functions, and their derivatives, for equal order and argument. It is enough to study the functions $H_\nu^{(1)}(\nu)$ and $H_\nu^{(1)\prime}(\nu)$, as the analogous results for the other functions can be deduced in a simple way through their relations with $H_\nu^{(1)}(\nu)$ and $H_\nu^{(1)\prime}(\nu)$.

We begin by considering the function $H_\nu^{(1)}(\nu)$. We substitute $z = \nu$ into the Schläfli–Sommerfeld integral representation (3.5) to obtain

$$H_\nu^{(1)}(\nu) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\pi\mathrm{i}+\infty} \mathrm{e}^{-\nu(t-\sinh t)} \mathrm{d}t, \tag{3.66}$$

for $|\theta| < \frac{\pi}{2}$. The function $t - \sinh t$ has infinitely many second-order saddle points in the complex t-plane situated at $t^{(k)} = 2\pi \mathrm{i} k$ with $k \in \mathbb{Z}$. Let $\mathscr{P}^{(0)}(\theta)$ be the steepest descent path emerging from $t^{(0)} = 0$ which coincides with the negative real axis when $\theta = 0$. We set the orientation of $\mathscr{P}^{(0)}(0)$ so that it leads away from the origin. We choose $\mathscr{C}^{(0)}(\theta)$ to be the steepest descent contour through $t^{(0)} = 0$ which is the union $\mathscr{P}^{(0)}(\theta) \cup \mathscr{P}^{(0)}(\theta + 2\pi)$, and we set the orientation of $\mathscr{C}^{(0)}(0)$ to be the same as that of $\mathscr{P}^{(0)}(0)$ (see Figure 3.2). It is readily verified that the contour of integration in (3.66) can be deformed into $\mathscr{C}^{(0)}(0)$, and hence we may write

$$H_\nu^{(1)}(\nu) = -\frac{1}{\pi \mathrm{i} \nu^{\frac{1}{3}}} T^{(2,0/3)}(\nu), \tag{3.67}$$

where $T^{(2,0/3)}(\nu)$ is given in (1.61) with the specific choices of $f(t) = t - \sinh t$ and $g(t) = 1$. The problem is therefore one of cubic dependence at the saddle point, which we considered in Subsection 1.2.2. To identify the domain $\Delta^{(0)}$ corresponding to this problem, we have to determine the adjacent saddles and contours. When $\theta = \frac{3\pi}{2}$, the path $\mathscr{P}^{(0)}(\theta)$ connects to the saddle point $t^{(1)} = 2\pi\mathrm{i}$, whereas when $\theta = -\frac{3\pi}{2}$, it connects to the saddle point $t^{(-1)} = -2\pi\mathrm{i}$. These are therefore adjacent to $t^{(0)} = 0$. Because the horizontal lines through the points $\frac{3\pi}{2}\mathrm{i}$ and $-\frac{3\pi}{2}\mathrm{i}$ are asymptotes of the corresponding adjacent contours $\mathscr{C}^{(1)}\left(\frac{3\pi}{2}\right)$ and $\mathscr{C}^{(-1)}\left(-\frac{3\pi}{2}\right)$, respectively (see Figure 3.2), there cannot be other saddles adjacent to $t^{(0)}$ besides $t^{(1)}$ and $t^{(-1)}$. The domain $\Delta^{(0)}$ is formed by the set of all points between these adjacent contours.

By analytic continuation, the representation (3.67) is valid in a wider range than (3.66), namely in $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ (note that the contour $\mathscr{C}^{(0)}(\theta)$ itself encounters the saddle $t^{(1)}$ when $\theta = -\frac{\pi}{2}$). Following the analysis in Subsection

1.2.2, we expand $T^{(2,0/3)}(\nu)$ into a truncated asymptotic power series with remainder,

$$T^{(2,0/3)}(\nu) = -\frac{2}{3\mathrm{i}}\sum_{n=0}^{N-1} \mathrm{e}^{-\frac{\pi(n+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(n+1)}{3}\right)\frac{a_n^{(0/3)}}{\nu^{\frac{n}{3}}} + R_N^{(2,0/3)}(\nu).$$

It can be verified that the conditions posed in Subsection 1.2.2 hold true for the domain $\Delta^{(0)}$ and the functions $f(t) = t - \sinh t$ and $g(t) = 1$ for any $N \geq 0$. We choose the orientation of the adjacent contours so that $\mathscr{C}^{(1)}\left(\frac{3\pi}{2}\right)$ is traversed in the positive direction and $\mathscr{C}^{(-1)}\left(-\frac{3\pi}{2}\right)$ is traversed in the negative direction with respect to the domain $\Delta^{(0)}$. Thus, the orientation anomalies are $\gamma_{01} = 0$ $\gamma_{0-1} = 1$. The relevant singulant pair is given by

$$\mathcal{F}_{0\pm1} = \pm 2\pi\mathrm{i} - \sinh(\pm 2\pi\mathrm{i}) - 0 + \sinh 0 = \pm 2\pi\mathrm{i}, \quad \arg \mathcal{F}_{0\pm1} = \sigma_{0\pm1} = \mp\frac{3\pi}{2}.$$

We thus find that for $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $N \geq 0$, the remainder $R_N^{(2,0/3)}(\nu)$ can be written as

$$\begin{aligned} R_N^{(2,0/3)}(\nu) = {} & \frac{\mathrm{i}^N}{6\pi\mathrm{i}\nu^{\frac{N}{3}}}\int_0^{+\infty} u^{\frac{N}{3}-1}\mathrm{e}^{-2\pi u} \\ & \times\left(\frac{1}{1-\mathrm{i}(u/\nu)^{\frac{1}{3}}} - \frac{\mathrm{e}^{-\frac{2\pi(N+1)}{3}\mathrm{i}}}{1-\mathrm{i}(u/\nu)^{\frac{1}{3}}\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}}}\right)T^{(2,1/3)}\left(u\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ & - \frac{(-\mathrm{i})^N}{6\pi\mathrm{i}\nu^{\frac{N}{3}}}\int_0^{+\infty} u^{\frac{N}{3}-1}\mathrm{e}^{-2\pi u} \\ & \times\left(\frac{1}{1+\mathrm{i}(u/\nu)^{\frac{1}{3}}} - \frac{\mathrm{e}^{-\frac{2\pi(N+1)}{3}\mathrm{i}}}{1+\mathrm{i}(u/\nu)^{\frac{1}{3}}\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}}}\right)T^{(2,-1/3)}\left(u\mathrm{e}^{-\frac{3\pi}{2}\mathrm{i}}\right)\mathrm{d}u. \end{aligned} \tag{3.68}$$

We may now connect the above results with the asymptotic expansion (3.62) of $H_\nu^{(1)}(\nu)$ by writing

$$H_\nu^{(1)}(\nu) = -\frac{2}{3\pi}\sum_{n=0}^{N-1} d_{2n}\mathrm{e}^{\frac{2\pi(2n+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2n+1)}{3}\right)\frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}} + R_N^{(H)}(\nu), \tag{3.69}$$

with the notation $d_{2n} = -a_{2n}^{(0/3)}/\Gamma\left(\frac{2n+1}{3}\right)$ and $R_N^{(H)}(\nu) = \mathrm{i}(\pi\nu^{\frac{1}{3}})^{-1}R_{2N}^{(2,0/3)}(\nu)$. When deriving (3.69), we used implicitly the fact that $a_n^{(0/3)}$ vanishes for odd n. To prove this, first note that, by (1.46),

$$a_n^{(0/3)} = \frac{\Gamma\left(\frac{n+1}{3}\right)}{\Gamma(n+1)}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{t^3}{t-\sinh t}\right)^{\frac{n+1}{3}}\right]_{t=0}. \tag{3.70}$$

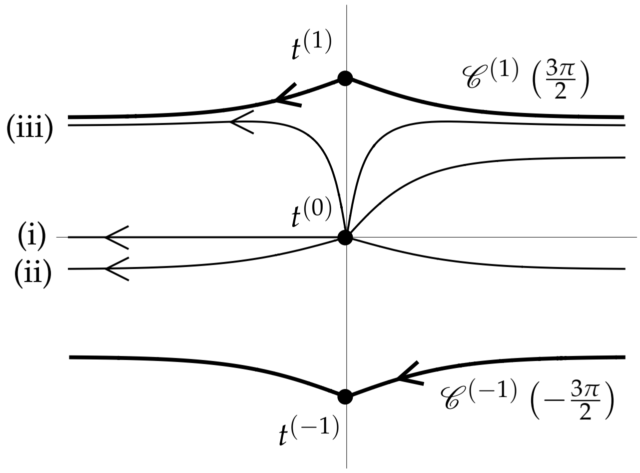

Figure 3.2. *The steepest descent contour $\mathscr{C}^{(0)}(\theta)$ associated with the Hankel function of equal order and argument through the saddle point $t^{(0)} = 0$ when (i) $\theta = 0$, (ii) $\theta = -\frac{2\pi}{5}$ and (iii) $\theta = \frac{7\pi}{5}$. The paths $\mathscr{C}^{(1)}\left(\frac{3\pi}{2}\right)$ and $\mathscr{C}^{(-1)}\left(-\frac{3\pi}{2}\right)$ are the adjacent contours for $t^{(0)}$. The domain $\Delta^{(0)}$ comprises all points between $\mathscr{C}^{(1)}\left(\frac{3\pi}{2}\right)$ and $\mathscr{C}^{(-1)}\left(-\frac{3\pi}{2}\right)$.*

Because the quantity under the differentiation sign is an even function of t and therefore its odd-order derivatives at $t = 0$ are zero, the claim follows.

It is possible to obtain a representation for the remainder $R_N^{(H)}(\nu)$ which is simpler than (3.68) by observing that we can express the functions $T^{(2,1/3)}\left(u\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}\right)$ and $T^{(2,-1/3)}\left(u\mathrm{e}^{-\frac{3\pi}{2}\mathrm{i}}\right)$ in terms of $H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$. Indeed, the contour $\mathscr{C}^{(1)}\left(\frac{3\pi}{2}\right)$ is congruent to $\mathscr{P}^{(0)}\left(-\frac{\pi}{2}\right) \cup \mathscr{P}^{(0)}\left(\frac{7\pi}{2}\right)$ but is shifted upwards in the complex plane by $2\pi\mathrm{i}$, whence[4]

$$\begin{aligned}
T^{(2,1/3)}\left(u\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}\right) &= u^{\frac{1}{3}}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}} \int_{\mathscr{C}^{(1)}\left(\frac{3\pi}{2}\right)} \mathrm{e}^{-u\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}(t-\sinh t-2\pi\mathrm{i}-\sinh(2\pi\mathrm{i}))}\mathrm{d}t \\
&= u^{\frac{1}{3}}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}} \int_{\mathscr{P}^{(0)}\left(-\frac{\pi}{2}\right)\cup\mathscr{P}^{(0)}\left(\frac{7\pi}{2}\right)} \mathrm{e}^{-u\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}(t-\sinh t)}\mathrm{d}t \\
&= u^{\frac{1}{3}}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}} \int_{\mathscr{P}^{(0)}\left(-\frac{\pi}{2}\right)\cup\mathscr{P}^{(0)}\left(\frac{7\pi}{2}\right)} \mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}(t-\sinh t)}\mathrm{d}t \\
&= -\pi u^{\frac{1}{3}} H_{-\mathrm{i}u}^{(2)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) = \pi u^{\frac{1}{3}} H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right),
\end{aligned} \tag{3.71}$$

using an argument similar to (3.11). One can prove in an analogous manner that

$$T^{(2,-1/3)}\left(u\mathrm{e}^{-\frac{3\pi}{2}\mathrm{i}}\right) = -\pi u^{\frac{1}{3}} H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right). \tag{3.72}$$

[4]We specify the orientation of $\mathscr{P}^{(0)}\left(\frac{7\pi}{2}\right)$ so that it leads into the saddle point $t^{(0)}$.

The desired expression for the remainder term $R_N^{(H)}(\nu)$ now follows from (3.71), (3.72) and $R_N^{(H)}(\nu) = \mathrm{i}(\pi\nu^{\frac{1}{3}})^{-1}R_{2N}^{(2,0/3)}(\nu)$:

$$R_N^{(H)}(\nu) = \frac{(-1)^N}{3\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty} u^{\frac{2N-2}{3}}\mathrm{e}^{-2\pi u} \times\left(\frac{1}{1+(u/\nu)^{\frac{2}{3}}} + \frac{\mathrm{e}^{\frac{\pi(2N+1)}{3}\mathrm{i}}}{1+(u/\nu)^{\frac{2}{3}}\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}}\right)H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \tag{3.73}$$

for $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $N \geq 0$. Equations (3.69) and (3.73) together yield the exact resurgence formula for $H_\nu^{(1)}(\nu)$.

By taking $d_{2n} = -a_{2n}^{(0/3)}/\Gamma\left(\frac{2n+1}{3}\right)$ and (3.70) into account, we obtain the following representation for the coefficients d_{2n}:

$$d_{2n} = \frac{1}{\Gamma(2n+1)}\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\left(\frac{t^3}{\sinh t - t}\right)^{\frac{2n+1}{3}}\right]_{t=0}, \tag{3.74}$$

where the cube root assumes its principal value. This is a known expression for the coefficients d_{2n} (see, for instance, [121, eq. (4.73), p. 102]). It seems that there is no very simple explicit representation for the d_{2n}'s. The author [73] proved the following formula involving the generalized Bernoulli polynomials:

$$6^{-\frac{2n+1}{3}}\Gamma\left(\frac{2n+1}{3}\right)d_{2n} = \sum_{k=0}^{2n}\frac{2^{2n+3k}3^{k+1}\Gamma\left(4\frac{2n+1}{3}\right)}{(2n+3k+1)\,\Gamma(2n-k+1)} \times\sum_{j=0}^{k}\frac{(-1)^j B_{2n+2k}^{(-j)}\left(-\frac{j}{2}\right)}{\Gamma(2n+2k+1)\,\Gamma(k-j+1)\,\Gamma(j+1)}.$$

For the definition and basic properties of the generalized Bernoulli polynomials, see, e.g., [59, Sec. 2.8], [67, Ch. VI] or [81, pp. 119–162]. The interested reader may find another expressions, including recurrence relations, for the coefficients d_{2n} in the paper [73].

To obtain the analogous result for the asymptotic expansion (3.63) of the second Hankel function $H_\nu^{(2)}(\nu)$, we start with the functional relation $H_\nu^{(2)}(\nu) = -H_{\nu\mathrm{e}^{\pi\mathrm{i}}}^{(1)}(\nu\mathrm{e}^{\pi\mathrm{i}})$ and substitute by means of (3.69) to arrive at

$$H_\nu^{(2)}(\nu) = -\frac{2}{3\pi}\sum_{n=0}^{N-1} d_{2n}\mathrm{e}^{-\frac{2\pi(2n+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2n+1)}{3}\right)\frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}} - R_N^{(H)}(\nu\mathrm{e}^{\pi\mathrm{i}}). \tag{3.75}$$

Assuming that $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$ and $N \geq 0$, equations (3.75) and (3.73) then yield the required resurgence formula for $H_\nu^{(2)}(\nu)$.

Let us now turn our attention to the resurgence properties of the derivatives $H_\nu^{(1)\prime}(\nu)$ and $H_\nu^{(2)\prime}(\nu)$. From (3.5), we infer that

$$H_\nu^{(1)\prime}(\nu) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\pi\mathrm{i}+\infty} \mathrm{e}^{-\nu(t-\sinh t)} \sinh t \mathrm{d}t \tag{3.76}$$

with $|\theta| < \frac{\pi}{2}$. Observe that the saddle point structure of the integrand in (3.76) is identical to that of (3.66). In particular, the problem is one of cubic dependence at the saddle point, and the domain $\Delta^{(0)}$ corresponding to this problem is the same as that in the case of $H_\nu^{(1)}(\nu)$. Since the derivation is very similar to that of the resurgence formula for the function $H_\nu^{(1)}(\nu)$, we omit the details and provide only the final results. We have

$$H_\nu^{(1)\prime}(\nu) = -\frac{2}{3\pi} \sum_{n=0}^{N-1} g_{2n} \mathrm{e}^{\frac{2\pi(2n+2)}{3}\mathrm{i}} \sin\left(\frac{\pi(2n+2)}{3}\right) \frac{\Gamma\left(\frac{2n+2}{3}\right)}{\nu^{\frac{2n+2}{3}}} + R_N^{(H')}(\nu) \tag{3.77}$$

and

$$H_\nu^{(2)\prime}(\nu) = -\frac{2}{3\pi} \sum_{n=0}^{N-1} g_{2n} \mathrm{e}^{-\frac{2\pi(2n+2)}{3}\mathrm{i}} \sin\left(\frac{\pi(2n+2)}{3}\right) \frac{\Gamma\left(\frac{2n+2}{3}\right)}{\nu^{\frac{2n+2}{3}}} + R_N^{(H')}(\nu \mathrm{e}^{\pi\mathrm{i}}), \tag{3.78}$$

where the remainder term $R_N^{(H')}(\nu)$ is given by the integral formula

$$\begin{aligned} R_N^{(H')}(\nu) = \frac{(-1)^N}{3\pi\mathrm{i}} \frac{1}{\nu^{\frac{2N+2}{3}}} \int_0^{+\infty} u^{\frac{2N-1}{3}} \mathrm{e}^{-2\pi u} \\ \times \left(\frac{1}{1+(u/\nu)^{\frac{2}{3}}} - \frac{\mathrm{e}^{\frac{\pi(2N+2)}{3}\mathrm{i}}}{1+(u/\nu)^{\frac{2}{3}}\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}}\right) H_{\mathrm{i}u}^{(1)\prime}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}u, \end{aligned} \tag{3.79}$$

provided $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $N \geq 1$.

The coefficients g_{2n} may be expressed in the form

$$g_{2n} = \frac{1}{\Gamma(2n+2)} \left[\frac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} \left(\sinh t \left(\frac{t^3}{\sinh t - t}\right)^{\frac{2n+2}{3}}\right)\right]_{t=0}. \tag{3.80}$$

The reader can find further expressions, including recurrence relations, for the coefficients g_{2n} in the paper [76]; note that in this paper g_{2n} is denoted by $6^{\frac{2n+2}{3}} D_{2n+1}(0)$.

From the expressions (3.69) and (3.75) for the Hankel functions, we can obtain the corresponding resurgence formulae for the Bessel functions $J_\nu(\nu)$ and

$Y_\nu(\nu)$. To this end, we substitute (3.69) and (3.75) into the functional relations $2J_\nu(\nu) = H_\nu^{(1)}(\nu) + H_\nu^{(2)}(\nu)$ and $2\mathrm{i}Y_\nu(\nu) = H_\nu^{(1)}(\nu) - H_\nu^{(2)}(\nu)$ and employ the identities

$$\begin{aligned} \cos\left(\frac{2\pi(2n+1)}{3}\right)\sin\left(\frac{\pi(2n+1)}{3}\right) &= -\frac{1}{2}\sin\left(\frac{\pi(2n+1)}{3}\right), \\ \sin\left(\frac{2\pi(2n+1)}{3}\right)\sin\left(\frac{\pi(2n+1)}{3}\right) &= \sin^2\left(\frac{\pi(2n+1)}{3}\right) \end{aligned} \tag{3.81}$$

for $n \geq 0$. Thus we establish that

$$J_\nu(\nu) = \frac{1}{3\pi}\sum_{n=0}^{N-1} d_{2n}\sin\left(\frac{\pi(2n+1)}{3}\right)\frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}} + R_N^{(J)}(\nu) \tag{3.82}$$

and

$$Y_\nu(\nu) = -\frac{2}{3\pi}\sum_{n=0}^{N-1} d_{2n}\sin^2\left(\frac{\pi(2n+1)}{3}\right)\frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}} + R_N^{(Y)}(\nu), \tag{3.83}$$

where $2R_N^{(J)}(\nu) = R_N^{(H)}(\nu) + R_N^{(H)}(\nu e^{\pi i})$ and $2\mathrm{i}R_N^{(Y)}(\nu) = R_N^{(H)}(\nu) - R_N^{(H)}(\nu e^{\pi i})$. The complete resurgence formulae can now be written down by employing (3.73). For this, we assume that $|\theta| < \frac{\pi}{2}$ and $N \geq 0$. With these provisos, we have

$$\begin{aligned} R_N^{(J)}(\nu) = \frac{(-1)^N}{6\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty} & u^{\frac{2N-2}{3}} e^{-2\pi u} \\ & \times\left(\frac{e^{\frac{\pi(2N+1)}{3}i}}{1+(u/\nu)^{\frac{2}{3}}e^{\frac{2\pi}{3}i}} - \frac{e^{-\frac{\pi(2N+1)}{3}i}}{1+(u/\nu)^{\frac{2}{3}}e^{-\frac{2\pi}{3}i}}\right)H_{iu}^{(1)}(ue^{\frac{\pi}{2}i})\,du \end{aligned} \tag{3.84}$$

and

$$\begin{aligned} R_N^{(Y)}(\nu) = \frac{(-1)^N}{6\pi i}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty} & u^{\frac{2N-2}{3}} e^{-2\pi u} \\ & \times\left(\frac{e^{\frac{\pi(2N+1)}{3}i}}{1+(u/\nu)^{\frac{2}{3}}e^{\frac{2\pi}{3}i}} + \frac{e^{-\frac{\pi(2N+1)}{3}i}}{1+(u/\nu)^{\frac{2}{3}}e^{-\frac{2\pi}{3}i}} + \frac{2}{1+(u/\nu)^{\frac{2}{3}}}\right)H_{iu}^{(1)}(ue^{\frac{\pi}{2}i})\,du. \end{aligned} \tag{3.85}$$

We end this subsection by discussing the corresponding resurgence relations for the derivatives $J'_\nu(\nu)$ and $Y'_\nu(\nu)$. The simplest way to derive these relations is by substituting the expressions (3.77) and (3.78) into the connection formulae $2J'_\nu(\nu) = H_\nu^{(1)\prime}(\nu) + H_\nu^{(2)\prime}(\nu)$ and $2\mathrm{i}Y'_\nu(\nu) = H_\nu^{(1)\prime}(\nu) - H_\nu^{(2)\prime}(\nu)$ and using trigonometric identities akin to those in (3.81). Hence we obtain that

$$J'_\nu(\nu) = \frac{1}{3\pi}\sum_{n=0}^{N-1} g_{2n}\sin\left(\frac{\pi(2n+2)}{3}\right)\frac{\Gamma\left(\frac{2n+2}{3}\right)}{\nu^{\frac{2n+2}{3}}} + R_N^{(J')}(\nu) \tag{3.86}$$

and

$$Y'_\nu(\nu) = \frac{2}{3\pi}\sum_{n=0}^{N-1} g_{2n}\sin^2\left(\frac{\pi(2n+2)}{3}\right)\frac{\Gamma\left(\frac{2n+2}{3}\right)}{\nu^{\frac{2n+2}{3}}} + R_N^{(Y')}(\nu), \tag{3.87}$$

with the notation $2R_N^{(J')}(\nu) = R_N^{(H')}(\nu) + R_N^{(H')}(\nu e^{\pi\mathrm{i}})$ and $2\mathrm{i}R_N^{(Y')}(\nu) = R_N^{(H')}(\nu) - R_N^{(H')}(\nu e^{\pi\mathrm{i}})$. The complete resurgence formulae can now be written down by applying (3.79). For this, we assume that $|\theta| < \frac{\pi}{2}$ and $N \geq 1$. With these assumptions, we have

$$\begin{aligned} R_N^{(J')}(\nu) = \frac{(-1)^N}{6\pi\mathrm{i}}\frac{1}{\nu^{\frac{2N+2}{3}}}&\int_0^{+\infty} u^{\frac{2N-1}{3}} e^{-2\pi u} \\ &\times\left(\frac{e^{-\frac{\pi(2N+2)}{3}\mathrm{i}}}{1+(u/\nu)^{\frac{2}{3}}e^{-\frac{2\pi}{3}\mathrm{i}}} - \frac{e^{\frac{\pi(2N+2)}{3}\mathrm{i}}}{1+(u/\nu)^{\frac{2}{3}}e^{\frac{2\pi}{3}\mathrm{i}}}\right) H_{\mathrm{i}u}^{(1)\prime}(ue^{\frac{\pi}{2}\mathrm{i}})\,\mathrm{d}u \end{aligned}$$

and

$$\begin{aligned} R_N^{(Y')}(\nu) = \frac{(-1)^N}{6\pi}\frac{1}{\nu^{\frac{2N+2}{3}}}&\int_0^{+\infty} u^{\frac{2N-1}{3}} e^{-2\pi u} \\ \times\left(\frac{e^{\frac{\pi(2N+2)}{3}\mathrm{i}}}{1+(u/\nu)^{\frac{2}{3}}e^{\frac{2\pi}{3}\mathrm{i}}} + \frac{e^{-\frac{\pi(2N+2)}{3}\mathrm{i}}}{1+(u/\nu)^{\frac{2}{3}}e^{-\frac{2\pi}{3}\mathrm{i}}} - \frac{2}{1+(u/\nu)^{\frac{2}{3}}}\right)&H_{\mathrm{i}u}^{(1)\prime}(ue^{\frac{\pi}{2}\mathrm{i}})\,\mathrm{d}u. \end{aligned}$$

By neglecting the remainder terms in (3.77), (3.78), (3.86) and (3.87) and by formally extending the sums to infinity, we obtain asymptotic expansions for the functions $H_\nu^{(1)\prime}(\nu)$, $H_\nu^{(2)\prime}(\nu)$, $J'_\nu(\nu)$ and $Y'_\nu(\nu)$. These asymptotic expansions have the same sectors of validity as the corresponding expansions (3.62)–(3.65).

3.2.2 Error bounds

In this subsection, we derive computable bounds for the remainders $R_N^{(H)}(\nu)$, $R_N^{(H')}(\nu)$, $R_N^{(J)}(\nu)$, $R_N^{(J')}(\nu)$, $R_N^{(Y)}(\nu)$ and $R_N^{(Y')}(\nu)$. Unless otherwise stated, we assume that $N \geq 0$ when dealing with $R_N^{(H)}(\nu)$, $R_N^{(J)}(\nu)$ and $R_N^{(Y)}(\nu)$, and $N \geq 1$ is assumed in the cases of $R_N^{(H')}(\nu)$, $R_N^{(J')}(\nu)$ and $R_N^{(Y')}(\nu)$.

To our best knowledge, the only known results concerning the estimation of the remainders $R_N^{(J)}(\nu)$ and $R_N^{(Y)}(\nu)$ are those of Gatteschi [40]. Here, we shall derive bounds which are simpler, more general and, presumably, much sharper than the ones given by Gatteschi.

Besides the usual inequality (2.36), we will use the following inequalities in constructing the error bounds:

$$\frac{1}{\left|1+re^{-\mathrm{i}\frac{2}{3}\alpha}e^{\frac{2\pi}{3}\mathrm{i}}\right|\left|1+re^{-\mathrm{i}\frac{2}{3}\alpha}\right|} \leq \begin{cases} |\sec\alpha| & \text{if } -\frac{\pi}{2}<\alpha<0 \text{ or } \pi<\alpha<\frac{3\pi}{2}, \\ 1 & \text{if } 0\leq\alpha\leq\pi, \end{cases} \tag{3.88}$$

$$\left|\frac{1-r^{\frac{2}{3}}e^{-\mathrm{i}\frac{4}{3}\alpha}}{1+re^{-2i\alpha}}\right| \leq 1 \text{ if } |\alpha|\leq\frac{\pi}{4} \tag{3.89}$$

and

$$\left|\frac{1-r^{\frac{1}{3}}e^{-\mathrm{i}\frac{2}{3}\alpha}}{1+re^{-2i\alpha}}\right| \leq \begin{cases} |\csc(2\alpha)| & \text{if } \frac{\pi}{4}<|\alpha|<\frac{\pi}{2}, \\ 1 & \text{if } |\alpha|\leq\frac{\pi}{4}, \end{cases} \tag{3.90}$$

where $r>0$. The proof of (3.89) is elementary while those of (3.88) and (3.90) are non-trivial and can be found in the paper [73] of the present author.

First, we consider the estimation of the remainders $R_N^{(H)}(\nu)$ and $R_N^{(H')}(\nu)$. By simple algebraic manipulation of (3.73), we deduce that

$$\begin{aligned} R_N^{(H)}(\nu) = {} & \frac{(-1)^{N+1}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty}\frac{u^{\frac{2N-2}{3}}e^{-2\pi u}e^{\frac{2\pi(2N+1)}{3}\mathrm{i}}}{\left(1+(u/\nu)^{\frac{2}{3}}e^{\frac{2\pi}{3}\mathrm{i}}\right)\left(1+(u/\nu)^{\frac{2}{3}}\right)}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(ue^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ & + \frac{(-1)^{N+1}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{\frac{2N+3}{3}}}\int_0^{+\infty}\frac{u^{\frac{2N}{3}}e^{-2\pi u}e^{\frac{2\pi(2N+1)}{3}\mathrm{i}}e^{\frac{\pi}{3}\mathrm{i}}}{\left(1+(u/\nu)^{\frac{2}{3}}e^{\frac{2\pi}{3}\mathrm{i}}\right)\left(1+(u/\nu)^{\frac{2}{3}}\right)}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(ue^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u, \end{aligned}$$

$$R_N^{(H)}(\nu) = \frac{(-1)^{N}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{\frac{2N+3}{3}}}\int_0^{+\infty}\frac{u^{\frac{2N}{3}}e^{-2\pi u}e^{\frac{2\pi(2N+1)}{3}\mathrm{i}}e^{\frac{\pi}{3}\mathrm{i}}}{\left(1+(u/\nu)^{\frac{2}{3}}e^{\frac{2\pi}{3}\mathrm{i}}\right)\left(1+(u/\nu)^{\frac{2}{3}}\right)}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(ue^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u$$

and

$$R_N^{(H)}(\nu) = \frac{(-1)^{N}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty}\frac{u^{\frac{2N-2}{3}}e^{-2\pi u}e^{\frac{2\pi(2N+1)}{3}\mathrm{i}}}{\left(1+(u/\nu)^{\frac{2}{3}}e^{\frac{2\pi}{3}\mathrm{i}}\right)\left(1+(u/\nu)^{\frac{2}{3}}\right)}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(ue^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u, \tag{3.91}$$

according to whether $N\equiv 0 \mod 3$, $N\equiv 1 \mod 3$ or $N\equiv 2 \mod 3$, respectively. Consider the case that $N\equiv 2 \mod 3$. For convenience, we denote $f(t)=t-\sinh t$. We replace in (3.91) the function $H_{\mathrm{i}u}^{(1)}\left(ue^{\frac{\pi}{2}\mathrm{i}}\right)$ by its integral representation

$$H_{\mathrm{i}u}^{(1)}\left(ue^{\frac{\pi}{2}\mathrm{i}}\right) = -\frac{1}{\pi\mathrm{i}}\int_{\mathscr{C}^{(0)}\left(\frac{\pi}{2}\right)}e^{-ue^{\frac{\pi}{2}\mathrm{i}}f(t)}\mathrm{d}t \tag{3.92}$$

(cf. equations (3.66) and (3.67)) and perform the change of integration variable from u and t to s and t via $s = u\mathrm{i}f(t)$. We thus find

$$R_N^{(H)}(\nu) = \frac{(-1)^{N+1}}{3^{\frac{1}{2}}\pi} \frac{1}{\nu^{\frac{2N+1}{3}}} \int_0^{+\infty} s^{\frac{2N-2}{3}} \mathrm{e}^{-s} \mathrm{e}^{\frac{2\pi(2N+1)}{3}\mathrm{i}} \times \frac{1}{\pi} \int_{\mathscr{C}^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+1}{3}} \mathrm{e}^{-2\pi \frac{s}{\mathrm{i}f(t)}}}{\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}} \mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}\right)\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}}\right)} \mathrm{d}t\mathrm{d}s. \tag{3.93}$$

Denote by $\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)$ the part of the steepest descent contour $\mathscr{C}^{(0)}\left(\frac{\pi}{2}\right)$ which lies in the right half-plane. (The contour $\mathscr{C}^{(0)}\left(\frac{\pi}{2}\right)$ is congruent to and has the same orientation as $\mathscr{C}^{(-1)}\left(-\frac{3\pi}{2}\right)$ but is shifted upwards in the complex plane by $2\pi\mathrm{i}$, cf. Figure 3.2.) An argument similar to (3.26) yields

$$\begin{aligned} &\int_{\mathscr{C}^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+1}{3}} \mathrm{e}^{-2\pi \frac{s}{\mathrm{i}f(t)}}}{\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}} \mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}\right)\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}}\right)} \mathrm{d}t \\ &\quad = 2\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+1}{3}} \mathrm{e}^{-2\pi \frac{s}{\mathrm{i}f(t)}}}{\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}} \mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}\right)\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}}\right)} \mathrm{d}x, \end{aligned} \tag{3.94}$$

where we have taken $x = \mathfrak{Re}(t)$. We thus find that the result (3.93) may be written as

$$R_N^{(H)}(\nu) = \frac{(-1)^{N+1}}{3^{\frac{1}{2}}\pi} \frac{1}{\nu^{\frac{2N+1}{3}}} \int_0^{+\infty} s^{\frac{2N-2}{3}} \mathrm{e}^{-s} \mathrm{e}^{\frac{2\pi(2N+1)}{3}\mathrm{i}} \times \frac{2}{\pi} \int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+1}{3}} \mathrm{e}^{-2\pi \frac{s}{\mathrm{i}f(t)}}}{\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}} \mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}\right)\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}}\right)} \mathrm{d}x\mathrm{d}s. \tag{3.95}$$

A formula for the coefficients d_{2N} analogous to (3.95) will be needed when deriving our error bounds. To avoid complications caused by the zeros of the sine function, we proceed in a different way than, for example, in the case of $U_N(\mathrm{i}\cot\beta)$ (cf. equation (3.28)). First, we use $d_{2n} = -a_{2n}^{(0/3)}/\Gamma\left(\frac{2n+1}{3}\right)$ and (1.76), together with (3.71) and (3.72), to establish

$$\Gamma\left(\frac{2N+1}{3}\right) d_{2N} = (-1)^N \int_0^{+\infty} u^{\frac{2N-2}{3}} \mathrm{e}^{-2\pi u} \mathrm{i} H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u, \tag{3.96}$$

where we have written N in place of n. Next, we replace the function $H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ by its integral representation (3.92), make the change of integration variable from u and t to s and t by $s = u\mathrm{i}f(t)$ and use a simplification analogous to (3.94). Hence we have

$$\Gamma\left(\frac{2N+1}{3}\right) d_{2N} = (-1)^{N+1} \int_0^{+\infty} s^{\frac{2N-2}{3}} \mathrm{e}^{-s} \frac{2}{\pi} \int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} (\mathrm{i}f(t))^{-\frac{2N+1}{3}} \mathrm{e}^{-2\pi \frac{s}{\mathrm{i}f(t)}} \mathrm{d}x \mathrm{d}s \tag{3.97}$$

for any $N \geq 0$. From (3.95), we infer that

$$\begin{aligned}\left|R_N^{(H)}(\nu)\right| \leq & -\frac{1}{3^{\frac{1}{2}}\pi} \frac{1}{|\nu|^{\frac{2N+1}{3}}} \int_0^{+\infty} s^{\frac{2N-2}{3}} \mathrm{e}^{-s} \\ & \times \frac{2}{\pi} \int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+1}{3}} \mathrm{e}^{-2\pi \frac{s}{\mathrm{i}f(t)}}}{\left|1 + (s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}} \mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}\right| \left|1 + (s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}}\right|} \mathrm{d}x \mathrm{d}s.\end{aligned}$$

In arriving at this inequality, one uses the positivity of $\mathrm{i}f(t)$ and the monotonicity of x on the path $\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)$. Now we apply the inequality (3.88) and then compare the result with (3.97) in order to obtain the error bound

$$\left|R_N^{(H)}(\nu)\right| \leq \frac{2}{3\pi} |d_{2N}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}} \begin{cases} |\sec\theta| & \text{if } -\frac{\pi}{2} < \theta < 0 \text{ or } \pi < \theta < \frac{3\pi}{2}, \\ 1 & \text{if } 0 \leq \theta \leq \pi. \end{cases} \tag{3.98}$$

We can prove in a similar manner that

$$\begin{aligned}\left|R_N^{(H)}(\nu)\right| \leq & \left(\frac{2}{3\pi} |d_{2N}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}} + \frac{2}{3\pi} |d_{2N+2}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}}\right) \\ & \times \begin{cases} |\sec\theta| & \text{if } -\frac{\pi}{2} < \theta < 0 \text{ or } \pi < \theta < \frac{3\pi}{2}, \\ 1 & \text{if } 0 \leq \theta \leq \pi \end{cases}\end{aligned} \tag{3.99}$$

when $N \equiv 0 \mod 3$, and

$$\left|R_N^{(H)}(\nu)\right| \leq \frac{2}{3\pi} |d_{2N+2}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}} \begin{cases} |\sec\theta| & \text{if } -\frac{\pi}{2} < \theta < 0 \text{ or } \pi < \theta < \frac{3\pi}{2}, \\ 1 & \text{if } 0 \leq \theta \leq \pi \end{cases} \tag{3.100}$$

when $N \equiv 1 \mod 3$.

Let us now turn our attention to the estimation of the remainder $R_N^{(H')}(\nu)$.

In this case, one finds that the expressions corresponding to (3.95) and (3.97) are

$$
\begin{aligned}
R_N^{(H')}(\nu) &= \frac{(-1)^{N+1}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{\frac{2N+4}{3}}}\int_0^{+\infty} s^{\frac{2N+1}{3}}\mathrm{e}^{-s}\mathrm{e}^{\frac{2\pi(2N+2)}{3}\mathrm{i}}\mathrm{e}^{\frac{\pi}{3}i} \\
&\times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+4}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}}\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}\right)\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}}\right)}\sinh x\cos y\,dy\,ds \\
&\quad + \frac{(-1)^{N+1}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{\frac{2N+4}{3}}}\int_0^{+\infty} s^{\frac{2N+1}{3}}\mathrm{e}^{-s}\mathrm{e}^{\frac{2\pi(2N+2)}{3}\mathrm{i}}\mathrm{e}^{\frac{\pi}{3}i} \\
&\times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+4}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}}\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}\right)\left(1+(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}}\right)}\cosh x\sin y\,dx\,ds
\end{aligned}
\tag{3.101}
$$

when $N \equiv 2 \mod 3$, and

$$
\begin{aligned}
\Gamma\left(\frac{2N+2}{3}\right)g_{2N} = (-1)^N &\int_0^{+\infty} s^{\frac{2N-1}{3}}\mathrm{e}^{-s} \\
&\times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} (\mathrm{i}f(t))^{-\frac{2N+2}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\sinh x\cos y\,dy\,ds \\
&+ (-1)^N\int_0^{+\infty} s^{\frac{2N-1}{3}}\mathrm{e}^{-s} \\
&\times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} (\mathrm{i}f(t))^{-\frac{2N+2}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\cosh x\sin y\,dx\,ds
\end{aligned}
\tag{3.102}
$$

for any $N \geq 1$ with $y = \mathfrak{Im}(t)$. From these expressions and the inequality (3.88), we establish

$$
\left|R_N^{(H')}(\nu)\right| \leq \frac{2}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}}\begin{cases}|\sec\theta| & \text{if } -\frac{\pi}{2}<\theta<0 \text{ or } \pi<\theta<\frac{3\pi}{2}, \\ 1 & \text{if } 0\leq\theta\leq\pi,\end{cases}
\tag{3.103}
$$

making use of the additional facts that on the contour $\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)$ the quantity $if(t)$ is positive, x is monotonic and y decreases monotonically from $\frac{\pi}{2}$ to 0. One may likewise show that

$$
\left|R_N^{(H')}(\nu)\right| \leq \frac{2}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}}\begin{cases}|\sec\theta| & \text{if } -\frac{\pi}{2}<\theta<0 \text{ or } \pi<\theta<\frac{3\pi}{2}, \\ 1 & \text{if } 0\leq\theta\leq\pi\end{cases}
\tag{3.104}
$$

when $N \equiv 0 \mod 3$, and

$$\left|R_N^{(H')}(\nu)\right| \le \left(\frac{2}{3\pi} |g_{2N}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}} + \frac{2}{3\pi} |g_{2N+2}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}} \right) \times \begin{cases} |\sec\theta| & \text{if } -\frac{\pi}{2} < \theta < 0 \text{ or } \pi < \theta < \frac{3\pi}{2}, \\ 1 & \text{if } 0 \le \theta \le \pi \end{cases} \tag{3.105}$$

when $N \equiv 1 \mod 3$, respectively.

In the special case that $\theta = \frac{\pi}{2}$, the formula (3.95) can be re-expressed in the form

$$R_N^{(H)}(\nu) = -\frac{1}{3^{\frac{1}{2}}\pi \mathrm{i}} \frac{1}{|\nu|^{\frac{2N+1}{3}}} \int_0^{+\infty} s^{\frac{2N-2}{3}} \mathrm{e}^{-s} \times \frac{2}{\pi} \int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{1-\left(s/\left(|\nu|\,\mathrm{i}f(t)\right)\right)^{\frac{2}{3}}}{1-\left(s/\left(|\nu|\,\mathrm{i}f(t)\right)\right)^{2}} \left(\mathrm{i}f(t)\right)^{-\frac{2N+1}{3}} \mathrm{e}^{-2\pi \frac{s}{\mathrm{i}f(t)}} \mathrm{d}x\mathrm{d}s.$$

For any $s > 0$, we have

$$0 < \frac{1-\left(s/\left(|\nu|\,\mathrm{i}f(t)\right)\right)^{\frac{2}{3}}}{1-\left(s/\left(|\nu|\,\mathrm{i}f(t)\right)\right)^{2}} < 1 \tag{3.106}$$

on the contour $\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)$ and so, using (3.97) and the mean value theorem of integration,

$$R_N^{(H)}(\nu) = -\frac{2}{3\pi} d_{2N} \mathrm{e}^{\frac{2\pi(2N+1)}{3}\mathrm{i}} \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+1}{3}\right)}{\nu^{\frac{2N+1}{3}}} \Theta_N(\nu),$$

when $N \equiv 2 \mod 3$ and with $0 < \Theta_N(\nu) < 1$ being an appropriate number that depends on ν and N. We find in an analogous manner that

$$R_N^{(H)}(\nu) = -\frac{2}{3\pi} d_{2N} \mathrm{e}^{\frac{2\pi(2N+1)}{3}\mathrm{i}} \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+1}{3}\right)}{\nu^{\frac{2N+1}{3}}} \widetilde{\Theta}_N(\nu)$$

when $N \equiv 0 \mod 3$, and

$$R_N^{(H)}(\nu) = \frac{2}{3\pi} d_{2N+2} \mathrm{e}^{\frac{2\pi(2N+3)}{3}\mathrm{i}} \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+3}{3}\right)}{\nu^{\frac{2N+3}{3}}} \widehat{\Theta}_N(\nu)$$

when $N \equiv 1 \mod 3$. Here $0 < \widetilde{\Theta}_N(\nu), \widehat{\Theta}_N(\nu) < 1$ are suitable numbers that depend on ν and N. Similarly, the formula (3.101) can be re-expressed in the form

$$\begin{aligned}
R_N^{(H')}(\nu) = {} & \frac{1}{3^{\frac{1}{2}}\pi}\frac{1}{|\nu|^{\frac{2N+4}{3}}}\int_0^{+\infty} s^{\frac{2N+1}{3}}\mathrm{e}^{-s} \\
& \times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{1-\left(s/\left(|\nu|\,\mathrm{i}f(t)\right)\right)^{\frac{2}{3}}}{1-\left(s/\left(|\nu|\,\mathrm{i}f(t)\right)\right)^{2}} \left(\mathrm{i}f(t)\right)^{-\frac{2N+4}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\sinh x\cos y\mathrm{d}y\mathrm{d}s \\
& + \frac{1}{3^{\frac{1}{2}}\pi}\frac{1}{|\nu|^{\frac{2N+4}{3}}}\int_0^{+\infty} s^{\frac{2N+1}{3}}\mathrm{e}^{-s} \\
& \times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{1-\left(s/\left(|\nu|\,\mathrm{i}f(t)\right)\right)^{\frac{2}{3}}}{1-\left(s/\left(|\nu|\,\mathrm{i}f(t)\right)\right)^{2}} \left(\mathrm{i}f(t)\right)^{-\frac{2N+4}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\cosh x\sin y\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

By taking into account the inequality (3.106) and the representation (3.102), the mean value theorem of integration shows that

$$R_N^{(H')}(\nu) = -\frac{2}{3\pi}g_{2N+2}\mathrm{e}^{\frac{2\pi(2N+4)}{3}\mathrm{i}}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{\nu^{\frac{2N+4}{3}}}\Xi_N(\nu)$$

when $N \equiv 2 \mod 3$ and with $0 < \Xi_N(\nu) < 1$ being an appropriate number that depends on ν and N. We find in a similar way that

$$R_N^{(H')}(\nu) = -\frac{2}{3\pi}g_{2N}\mathrm{e}^{\frac{2\pi(2N+2)}{3}\mathrm{i}}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{\nu^{\frac{2N+2}{3}}}\widetilde{\Xi}_N(\nu)$$

when $N \equiv 0 \mod 3$, and

$$R_N^{(H')}(\nu) = \frac{2}{3\pi}g_{2N}\mathrm{e}^{\frac{2\pi(2N+2)}{3}\mathrm{i}}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{\nu^{\frac{2N+2}{3}}}\widehat{\Xi}_N(\nu)$$

when $N \equiv 1 \mod 3$. Here $0 < \widetilde{\Xi}_N(\nu), \widehat{\Xi}_N(\nu) < 1$ are suitable numbers that depend on ν and N.

In the case that ν lies in the closed upper half-plane, the estimates (3.98)–(3.100) and (3.103)–(3.105) are as sharp as it is reasonable to expect. However, although acceptable in much of the sectors $-\frac{\pi}{2} < \theta < 0$ and $\pi < \theta < \frac{3\pi}{2}$, the bounds (3.98)–(3.100) and (3.103)–(3.105) become inappropriate near the Stokes lines $\theta = -\frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. We now provide alternative estimates that are suitable for the sectors $-\pi < \theta < 0$ and $\pi < \theta < 2\pi$ (which include the Stokes lines $\theta = -\frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$). We may use (3.69) and (3.77) to define the remainder terms $R_N^{(H)}(\nu)$ and $R_N^{(H')}(\nu)$ in the sectors $\frac{3\pi}{2} \le \theta < 2\pi$ and $-\pi < \theta \le -\frac{\pi}{2}$. These alternative bounds can be derived based on the inequality (3.88) and the representations (3.95) and (3.101) (or on the analogous formulae for the other two cases). Their derivation is similar to that of (2.43) discussed in Subsection 2.1.2, and the details are therefore omitted. One finds that in the sectors $-\pi < \theta < 0$ and $\pi < \theta < 2\pi$, the remainder $R_N^{(H)}(\nu)$ can be estimated as follows:

$$\begin{aligned}\left|R_N^{(H)}(\nu)\right| \le\; & \frac{\left|\sec\left(\theta-\varphi^*\right)\right|}{\cos^{\frac{2N+1}{3}}\varphi^*}\frac{2}{3\pi}\left|d_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{\left|\nu\right|^{\frac{2N+1}{3}}}\\ &+\frac{\left|\sec\left(\theta-\varphi^{**}\right)\right|}{\cos^{\frac{2N+3}{3}}\varphi^{**}}\frac{2}{3\pi}\left|d_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{\left|\nu\right|^{\frac{2N+3}{3}}}\end{aligned} \tag{3.107}$$

when $N \equiv 0 \mod 3$,

$$\left|R_N^{(H)}(\nu)\right| \le \frac{\left|\sec\left(\theta-\varphi^{**}\right)\right|}{\cos^{\frac{2N+3}{3}}\varphi^{**}}\frac{2}{3\pi}\left|d_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{\left|\nu\right|^{\frac{2N+3}{3}}} \tag{3.108}$$

when $N \equiv 1 \mod 3$, and

$$\left|R_N^{(H)}(\nu)\right| \le \frac{\left|\sec\left(\theta-\varphi^{*}\right)\right|}{\cos^{\frac{2N+1}{3}}\varphi^{*}}\frac{2}{3\pi}\left|d_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{\left|\nu\right|^{\frac{2N+1}{3}}} \tag{3.109}$$

when $N \equiv 2 \mod 3$. Here φ^* and φ^{**} are the minimizing values given by Lemma 2.1.1 with $\theta - \frac{\pi}{2}$ in place of θ and with $\chi = \frac{2N+1}{3}$ and $\chi = \frac{2N+3}{3}$, respectively. Similarly, for $-\pi < \theta < 0$ and $\pi < \theta < 2\pi$, the remainder term $R_N^{(H')}(\nu)$ satisfies the following bounds:

$$\left|R_N^{(H')}(\nu)\right| \le \frac{\left|\sec\left(\theta-\varphi^{*}\right)\right|}{\cos^{\frac{2N+2}{3}}\varphi^{*}}\frac{2}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{\left|\nu\right|^{\frac{2N+2}{3}}} \tag{3.110}$$

if $N \equiv 0 \mod 3$,

$$\begin{aligned}\left|R_N^{(H')}(\nu)\right| \le\; & \frac{\left|\sec\left(\theta-\varphi^*\right)\right|}{\cos^{\frac{2N+2}{3}}\varphi^*}\frac{2}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{\left|\nu\right|^{\frac{2N+2}{3}}}\\ &+\frac{\left|\sec\left(\theta-\varphi^{**}\right)\right|}{\cos^{\frac{2N+4}{3}}\varphi^{**}}\frac{2}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{\left|\nu\right|^{\frac{2N+4}{3}}}\end{aligned} \tag{3.111}$$

if $N \equiv 1 \mod 3$, and

$$\left|R_N^{(H')}(\nu)\right| \le \frac{\left|\sec\left(\theta-\varphi^{**}\right)\right|}{\cos^{\frac{2N+4}{3}}\varphi^{**}}\frac{2}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{\left|\nu\right|^{\frac{2N+4}{3}}} \tag{3.112}$$

if $N \equiv 2 \mod 3$. Here φ^* and φ^{**} are the minimizing values given by Lemma 2.1.1 with $\theta - \frac{\pi}{2}$ in place of θ and with $\chi = \frac{2N+2}{3}$ and $\chi = \frac{2N+4}{3}$, respectively.

Note that the ranges of validity of the bounds (3.98)–(3.100), (3.103)–(3.105) and (3.107)–(3.112) together cover that of the asymptotic expansions of the functions $H_\nu^{(1)}(\nu)$ and $H_\nu^{(1)\prime}(\nu)$.

The following simple estimates are suitable for the sectors $-\frac{\pi}{2} \leq \theta < 0$ and $\pi < \theta \leq \frac{3\pi}{2}$ (especially near the Stokes lines $\theta = -\frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$), and they can be obtained from (3.107)–(3.112) using an argument similar to that given in Subsection 2.1.2:

$$\left|R_N^{(H)}(\nu)\right| \leq \tfrac{1}{3}\sqrt{3\mathrm{e}\left(2N+\tfrac{5}{2}\right)}\frac{2}{3\pi}\left|d_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}}$$
$$+\tfrac{1}{3}\sqrt{3\mathrm{e}\left(2N+\tfrac{9}{2}\right)}\frac{2}{3\pi}\left|d_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}},$$
$$\left|R_N^{(H')}(\nu)\right| \leq \tfrac{1}{3}\sqrt{3\mathrm{e}\left(2N+\tfrac{7}{2}\right)}\frac{2}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}}$$

when $N \equiv 0 \mod 3$ (with the additional condition that $N \geq 3$ in the case of $R_N^{(H)}(\nu)$),

$$\left|R_N^{(H)}(\nu)\right| \leq \tfrac{1}{3}\sqrt{3\mathrm{e}\left(2N+\tfrac{9}{2}\right)}\frac{2}{3\pi}\left|d_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}},$$
$$\left|R_N^{(H')}(\nu)\right| \leq \tfrac{1}{3}\sqrt{3\mathrm{e}\left(2N+\tfrac{7}{2}\right)}\frac{2}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}}$$
$$+\tfrac{1}{3}\sqrt{3\mathrm{e}\left(2N+\tfrac{11}{2}\right)}\frac{2}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}}$$

when $N \equiv 1 \mod 3$, and

$$\left|R_N^{(H)}(\nu)\right| \leq \tfrac{1}{3}\sqrt{3\mathrm{e}\left(2N+\tfrac{5}{2}\right)}\frac{2}{3\pi}\left|d_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}},$$
$$\left|R_N^{(H')}(\nu)\right| \leq \tfrac{1}{3}\sqrt{3\mathrm{e}\left(2N+\tfrac{11}{2}\right)}\frac{2}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}}$$

when $N \equiv 2 \mod 3$. These bounds may be used in conjunction with our earlier results (3.98)–(3.100) and (3.103)–(3.105), respectively.

Consider now the estimation of the remainder terms $R_N^{(J)}(\nu)$ and $R_N^{(J')}(\nu)$. A simple algebraic manipulation of (3.84) shows that

$$R_N^{(J)}(\nu) = \frac{(-1)^N}{3^{\frac{1}{2}}2\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty} u^{\frac{2N-2}{3}}\mathrm{e}^{-2\pi u}\frac{1-(u/\nu)^{\frac{4}{3}}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u, \quad (3.113)$$

$$R_N^{(J)}(\nu) = \frac{(-1)^N}{3^{\frac{1}{2}}2\pi} \frac{1}{\nu^{\frac{2N+3}{3}}} \int_0^{+\infty} u^{\frac{2N}{3}} \mathrm{e}^{-2\pi u} \frac{1+(u/\nu)^{\frac{2}{3}}}{1+(u/\nu)^2} \mathrm{i} H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u$$

and

$$R_N^{(J)}(\nu) = \frac{(-1)^{N+1}}{3^{\frac{1}{2}}2\pi} \frac{1}{\nu^{\frac{2N+1}{3}}} \int_0^{+\infty} u^{\frac{2N-2}{3}} \mathrm{e}^{-2\pi u} \frac{1+(u/\nu)^{\frac{2}{3}}}{1+(u/\nu)^2} \mathrm{i} H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u,$$

according to whether $N \equiv 0 \mod 3$, $N \equiv 1 \mod 3$ or $N \equiv 2 \mod 3$, respectively. (The special case $N = 0$ of (3.113) yields an expression for $J_\nu(\nu)$ which corrects a result of Dingle [35, exer. 18, p. 484].) Consider the case that $N \equiv 0 \mod 3$. We replace the function $H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ in (3.113) by its integral representation (3.92), make the change of integration variable from u and t to s and t by $s = u\mathrm{i}f(t)$ and use a simplification akin to (3.94). Thus we have

$$\begin{aligned} R_N^{(J)}(\nu) = {} & \frac{(-1)^{N+1}}{3^{\frac{1}{2}}2\pi} \frac{1}{\nu^{\frac{2N+1}{3}}} \int_0^{+\infty} s^{\frac{2N-2}{3}} \mathrm{e}^{-s} \\ & \times \frac{2}{\pi} \int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{1-(s/(\nu\mathrm{i}f(t)))^{\frac{4}{3}}}{1+(s/(\nu\mathrm{i}f(t)))^2} (\mathrm{i}f(t))^{-\frac{2N+1}{3}} \mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}} \mathrm{d}x \mathrm{d}s. \end{aligned} \tag{3.114}$$

We first estimate the right-hand side by using the inequality (3.89) in the case $|\theta| \le \frac{\pi}{4}$ and the inequality (2.36) in the case $\frac{\pi}{4} < |\theta| < \frac{\pi}{2}$. We then compare the result with (3.97), thereby obtaining the error bound

$$\begin{aligned} \left|R_N^{(J)}(\nu)\right| \le {} & \frac{1}{3\pi}|d_{2N}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}} \begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \le \frac{\pi}{4} \end{cases} \\ & + \frac{1}{3\pi}|d_{2N+4}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}} \begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 0 & \text{if } |\theta| \le \frac{\pi}{4}. \end{cases} \end{aligned} \tag{3.115}$$

We can prove in a similar manner that

$$\begin{aligned} \left|R_N^{(J)}(\nu)\right| \le {} & \left(\frac{1}{3\pi}|d_{2N+2}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}} + \frac{1}{3\pi}|d_{2N+4}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}} \right) \\ & \times \begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \le \frac{\pi}{4} \end{cases} \end{aligned} \tag{3.116}$$

when $N \equiv 1 \mod 3$, and

$$\left|R_N^{(J)}(\nu)\right| \le \left(\frac{1}{3\pi}\left|d_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}} + \frac{1}{3\pi}\left|d_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}}\right) \times \begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \le \frac{\pi}{4} \end{cases} \tag{3.117}$$

when $N \equiv 2 \mod 3$.

Let us now consider the estimation of the remainder $R_N^{(J')}(\nu)$. In this case, one finds that the expression corresponding to (3.114) is

$$\begin{aligned} R_N^{(J')}(\nu) = {} & \frac{(-1)^N}{3^{\frac{1}{2}}2\pi}\frac{1}{\nu^{\frac{2N+2}{3}}}\int_0^{+\infty} s^{\frac{2N-1}{3}}\mathrm{e}^{-s} \\ & \times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{1+(s/(\nu \mathrm{i} f(t)))^{\frac{2}{3}}}{1+(s/(\nu \mathrm{i} f(t)))^2}(\mathrm{i}f(t))^{-\frac{2N+2}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\sinh x\cos y\,\mathrm{d}y\,\mathrm{d}s \\ & + \frac{(-1)^N}{3^{\frac{1}{2}}2\pi}\frac{1}{\nu^{\frac{2N+2}{3}}}\int_0^{+\infty} s^{\frac{2N-1}{3}}\mathrm{e}^{-s} \\ & \times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{1+(s/(\nu \mathrm{i} f(t)))^{\frac{2}{3}}}{1+(s/(\nu \mathrm{i} f(t)))^2}(\mathrm{i}f(t))^{-\frac{2N+2}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\cosh x\sin y\,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{3.118}$$

when $N \equiv 0 \mod 3$. From (3.102), (3.118) and the inequality (2.36), we establish

$$\left|R_N^{(J')}(\nu)\right| \le \left(\frac{1}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}} + \frac{1}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}}\right) \times \begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \le \frac{\pi}{4}. \end{cases} \tag{3.119}$$

One may likewise show that

$$\begin{aligned} \left|R_N^{(J')}(\nu)\right| \le {} & \frac{1}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}}\begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \le \frac{\pi}{4} \end{cases} \\ & + \frac{1}{3\pi}\left|g_{2N+4}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}}\begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 0 & \text{if } |\theta| \le \frac{\pi}{4} \end{cases} \end{aligned} \tag{3.120}$$

when $N \equiv 1 \mod 3$, and

$$\left|R_N^{(J')}(\nu)\right| \leq \left(\frac{1}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}} + \frac{1}{3\pi}\left|g_{2N+4}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}}\right) \times \begin{cases} \left|\csc(2\theta)\right| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \leq \frac{\pi}{4} \end{cases} \tag{3.121}$$

when $N \equiv 2 \mod 3$, respectively.

In the special case when ν is positive, we have $0 < 1/(1+(s/(\nu i f(t)))^2) < 1$ in (3.114) and together with (3.97), the mean value theorem of integration shows that

$$R_N^{(J)}(\nu) = \frac{1}{3\pi}d_{2N}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{\nu^{\frac{2N+1}{3}}}\Theta_N(\nu) - \frac{1}{3\pi}d_{2N+4}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+5}{3}\right)}{\nu^{\frac{2N+5}{3}}}\widetilde{\Theta}_N(\nu), \tag{3.122}$$

when $N \equiv 0 \mod 3$ and with $0 < \Theta_N(\nu), \widetilde{\Theta}_N(\nu) < 1$ being appropriate numbers that depend on ν and N. We find in an analogous manner that

$$R_N^{(J)}(\nu) = -\frac{1}{3\pi}d_{2N+2}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{\nu^{\frac{2N+3}{3}}}\widehat{\Theta}_N(\nu) + \frac{1}{3\pi}d_{2N+4}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+5}{3}\right)}{\nu^{\frac{2N+5}{3}}}\widetilde{\Theta}_N(\nu)$$

when $N \equiv 1 \mod 3$, and

$$R_N^{(J)}(\nu) = -\frac{1}{3\pi}d_{2N}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{\nu^{\frac{2N+1}{3}}}\Theta_N(\nu) + \frac{1}{3\pi}d_{2N+2}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{\nu^{\frac{2N+3}{3}}}\widehat{\Theta}_N(\nu)$$

when $N \equiv 2 \mod 3$. Here $0 < \widehat{\Theta}_N(\nu) < 1$ is a suitable number that depends on ν and N. We note that, since $d_0, d_4 > 0$, formula (3.122) implies

$$J_\nu(\nu) < \frac{1}{3\pi}d_0\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{1}{3}\right)}{\nu^{\frac{1}{3}}} = \frac{\Gamma\left(\frac{1}{3}\right)}{2^{\frac{2}{3}}3^{\frac{1}{6}}\pi\nu^{\frac{1}{3}}}$$

for any $\nu > 0$. This upper bound was also established by Watson [117, pp. 258–259] using a method different from ours (for a lower bound of similar type, see [37]). Analogously, when ν is positive, we have $0 < 1/(1+(s/(\nu i f(t)))^2) < 1$ in (3.118) and together with (3.102), the mean value theorem of integration yields

$$R_N^{(J')}(\nu) = \frac{1}{3\pi}g_{2N}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{\nu^{\frac{2N+2}{3}}}\Xi_N(\nu) - \frac{1}{3\pi}g_{2N+2}\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{\nu^{\frac{2N+4}{3}}}\widehat{\Xi}_N(\nu),$$

when $N \equiv 0 \mod 3$ and with $0 < \Xi_N(\nu), \widehat{\Xi}_N(\nu) < 1$ being appropriate numbers that depend on ν and N. Similarly, one finds that

$$R_N^{(J')}(\nu) = -\frac{1}{3\pi} g_{2N} \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+2}{3}\right)}{\nu^{\frac{2N+2}{3}}} \Xi_N(\nu) + \frac{1}{3\pi} g_{2N+4} \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+6}{3}\right)}{\nu^{\frac{2N+6}{3}}} \widetilde{\Xi}_N(\nu)$$

when $N \equiv 1 \mod 3$, and

$$R_N^{(J')}(\nu) = \frac{1}{3\pi} g_{2N+2} \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+4}{3}\right)}{\nu^{\frac{2N+4}{3}}} \widehat{\Xi}_N(\nu) - \frac{1}{3\pi} g_{2N+4} \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+6}{3}\right)}{\nu^{\frac{2N+6}{3}}} \widetilde{\Xi}_N(\nu)$$

when $N \equiv 2 \mod 3$. Here $0 < \widetilde{\Xi}_N(\nu) < 1$ is a suitable number that depends on ν and N. We remark that by a result of Watson's [117, eq. (2), p. 260]

$$J'_\nu(\nu) < \frac{1}{3\pi} g_0 \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2}{3}\right)}{\nu^{\frac{2}{3}}} = \frac{3^{\frac{1}{6}} \Gamma\left(\frac{2}{3}\right)}{2^{\frac{1}{3}} \pi \nu^{\frac{2}{3}}}$$

for any $\nu > 0$ or, in other words, $R_1^{(J')}(\nu) < 0$ for positive values of ν.

The estimates (3.115)–(3.117) and (3.119)–(3.121) become singular as θ approaches $\pm\frac{\pi}{2}$ and are therefore not suitable near the Stokes lines $\theta = \pm\frac{\pi}{2}$. We now give alternative bounds that are appropriate for the sectors $\frac{\pi}{4} < |\theta| < \pi$ (which include the Stokes lines $\theta = \pm\frac{\pi}{2}$). We may use (3.82) and (3.86) to define the remainder terms in the sectors $\frac{\pi}{2} \le |\theta| < \pi$. These alternative bounds can be derived based on the representations (3.114) and (3.118) (or on the analogous formulae for the other two cases). Their derivation is similar to that of (2.43) discussed in Subsection 2.1.2, and the details are therefore omitted. One finds that in the sectors $\frac{\pi}{4} < |\theta| < \pi$, the remainder $R_N^{(J)}(\nu)$ can be estimated as follows:

$$\begin{aligned} \left|R_N^{(J)}(\nu)\right| \le\; & \frac{|\csc(2(\theta - \varphi^*))|}{\cos^{\frac{2N+1}{3}} \varphi^*} \frac{1}{3\pi} |d_{2N}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}} \\ & + \frac{|\csc(2(\theta - \varphi^{***}))|}{\cos^{\frac{2N+5}{3}} \varphi^{***}} \frac{1}{3\pi} |d_{2N+4}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}} \end{aligned} \tag{3.123}$$

when $N \equiv 0 \mod 3$,

$$\begin{aligned} \left|R_N^{(J)}(\nu)\right| \le\; & \frac{|\csc(2(\theta - \varphi^{**}))|}{\cos^{\frac{2N+3}{3}} \varphi^{**}} \frac{1}{3\pi} |d_{2N+2}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}} \\ & + \frac{|\csc(2(\theta - \varphi^{***}))|}{\cos^{\frac{2N+5}{3}} \varphi^{***}} \frac{1}{3\pi} |d_{2N+4}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}} \end{aligned} \tag{3.124}$$

when $N \equiv 1 \mod 3$, and

$$\left|R_N^{(J)}(\nu)\right| \le \frac{\left|\csc\left(2\left(\theta-\varphi^*\right)\right)\right|}{\cos^{\frac{2N+1}{3}}\varphi^*}\frac{1}{3\pi}\left|d_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}} + \frac{\left|\csc\left(2\left(\theta-\varphi^{**}\right)\right)\right|}{\cos^{\frac{2N+3}{3}}\varphi^{**}}\frac{1}{3\pi}\left|d_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}} \tag{3.125}$$

when $N \equiv 2 \mod 3$. Here φ^*, φ^{**} and φ^{***} are the minimizing values given by Lemma 2.1.2 with $\chi = \frac{2N+1}{3}$, $\chi = \frac{2N+3}{3}$ and $\chi = \frac{2N+5}{3}$, respectively. Similarly, for $\frac{\pi}{4} < |\theta| < \pi$, the remainder $R_N^{(J')}(\nu)$ satisfies the following bounds:

$$\left|R_N^{(J')}(\nu)\right| \le \frac{\left|\csc\left(2\left(\theta-\varphi^*\right)\right)\right|}{\cos^{\frac{2N+2}{3}}\varphi^*}\frac{1}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}} + \frac{\left|\csc\left(2\left(\theta-\varphi^{**}\right)\right)\right|}{\cos^{\frac{2N+4}{3}}\varphi^{**}}\frac{1}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}} \tag{3.126}$$

if $N \equiv 0 \mod 3$,

$$\left|R_N^{(J')}(\nu)\right| \le \frac{\left|\csc\left(2\left(\theta-\varphi^*\right)\right)\right|}{\cos^{\frac{2N+2}{3}}\varphi^*}\frac{1}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}} + \frac{\left|\csc\left(2\left(\theta-\varphi^{***}\right)\right)\right|}{\cos^{\frac{2N+6}{3}}\varphi^{***}}\frac{1}{3\pi}\left|g_{2N+4}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}} \tag{3.127}$$

if $N \equiv 1 \mod 3$, and

$$\left|R_N^{(J')}(\nu)\right| \le \frac{\left|\csc\left(2\left(\theta-\varphi^{**}\right)\right)\right|}{\cos^{\frac{2N+4}{3}}\varphi^{**}}\frac{1}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}} + \frac{\left|\csc\left(2\left(\theta-\varphi^{***}\right)\right)\right|}{\cos^{\frac{2N+6}{3}}\varphi^{***}}\frac{1}{3\pi}\left|g_{2N+4}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}} \tag{3.128}$$

if $N \equiv 2 \mod 3$. Here φ^*, φ^{**} and φ^{***} are the minimizing values given by Lemma 2.1.2 with the specific choices of $\chi = \frac{2N+2}{3}$, $\chi = \frac{2N+4}{3}$ and $\chi = \frac{2N+6}{3}$, respectively.

The following simple bounds are useful for the sectors $\frac{\pi}{4} < |\theta| \le \frac{\pi}{2}$ (especially near the Stokes lines $\theta = \pm\frac{\pi}{2}$) and $N \ge 4$, and they can be deduced from

(3.123)–(3.128) using an argument similar to that given in Subsection 2.1.2:

$$
\begin{aligned}
\left|R_N^{(J)}(\nu)\right| \le{}& \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{11}{2}\right)}\,\frac{1}{3\pi}\left|d_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}}\\
&+\tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{19}{2}\right)}\,\frac{1}{3\pi}\left|d_{2N+4}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}},\\
\left|R_N^{(J')}(\nu)\right| \le{}& \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{13}{2}\right)}\,\frac{1}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}}\\
&+\tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{17}{2}\right)}\,\frac{1}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}}
\end{aligned}
$$

when $N \equiv 0 \mod 3$,

$$
\begin{aligned}
\left|R_N^{(J)}(\nu)\right| \le{}& \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{15}{2}\right)}\,\frac{1}{3\pi}\left|d_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}}\\
&+\tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{19}{2}\right)}\,\frac{1}{3\pi}\left|d_{2N+4}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}},\\
\left|R_N^{(J')}(\nu)\right| \le{}& \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{13}{2}\right)}\,\frac{1}{3\pi}\left|g_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}}\\
&+\tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{21}{2}\right)}\,\frac{1}{3\pi}\left|g_{2N+4}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}}
\end{aligned}
$$

when $N \equiv 1 \mod 3$, and

$$
\begin{aligned}
\left|R_N^{(J)}(\nu)\right| \le{}& \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{11}{2}\right)}\,\frac{1}{3\pi}\left|d_{2N}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}}\\
&+\tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{15}{2}\right)}\,\frac{1}{3\pi}\left|d_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}},\\
\left|R_N^{(J')}(\nu)\right| \le{}& \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{17}{2}\right)}\,\frac{1}{3\pi}\left|g_{2N+2}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}}\\
&+\tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{21}{2}\right)}\,\frac{1}{3\pi}\left|g_{2N+4}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}}
\end{aligned}
$$

when $N \equiv 2 \mod 3$.

We end this subsection by considering the estimation of the remainder terms $R_N^{(Y)}(\nu)$ and $R_N^{(Y')}(\nu)$. A simple algebraic manipulation of (3.85) gives

$$R_N^{(Y)}(\nu) = \frac{(-1)^{N+1}}{2\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty} u^{\frac{2N-2}{3}}\mathrm{e}^{-2\pi u}\frac{1+(u/\nu)^{\frac{4}{3}}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u,$$

$$R_N^{(Y)}(\nu) = \frac{(-1)^{N}}{2\pi}\frac{1}{\nu^{\frac{2N+3}{3}}}\int_0^{+\infty} u^{\frac{2N}{3}}\mathrm{e}^{-2\pi u}\frac{1-(u/\nu)^{\frac{2}{3}}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u$$

and

$$R_N^{(Y)}(\nu) = \frac{(-1)^{N+1}}{2\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty} u^{\frac{2N-2}{3}}\mathrm{e}^{-2\pi u}\frac{1-(u/\nu)^{\frac{2}{3}}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u, \quad (3.129)$$

according to whether $N \equiv 0 \mod 3$, $N \equiv 1 \mod 3$ or $N \equiv 2 \mod 3$, respectively. Consider the case that $N \equiv 2 \mod 3$. We replace the function $H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ in (3.129) by its integral representation (3.92), make the change of variable from u and t to s and t by $s = u\mathrm{i}f(t)$ and use a simplification similar to (3.94). Hence we have

$$\begin{aligned} R_N^{(Y)}(\nu) = {} & \frac{(-1)^{N}}{2\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty} s^{\frac{2N-2}{3}}\mathrm{e}^{-s} \\ & \times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{1-(s/(\nu\mathrm{i}f(t)))^{\frac{2}{3}}}{1+(s/(\nu\mathrm{i}f(t)))^2}(\mathrm{i}f(t))^{-\frac{2N+1}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\mathrm{d}x\mathrm{d}s. \end{aligned} \quad (3.130)$$

We first estimate the right-hand side by using the inequality (3.90) and then compare the result with (3.97), thereby obtaining the error bound

$$\left|R_N^{(Y)}(\nu)\right| \le \frac{2}{3\pi}\left|d_{2N}\right|\frac{3}{4}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}}\begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \le \frac{\pi}{4}. \end{cases} \quad (3.131)$$

We can prove in a similar manner that

$$\begin{aligned} \left|R_N^{(Y)}(\nu)\right| \le {} & \left(\frac{2}{3\pi}\left|d_{2N}\right|\frac{3}{4}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}} + \frac{2}{3\pi}\left|d_{2N+4}\right|\frac{3}{4}\frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}}\right) \\ & \times \begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \le \frac{\pi}{4} \end{cases} \end{aligned} \quad (3.132)$$

when $N \equiv 0 \mod 3$, and

$$\left|R_N^{(Y)}(\nu)\right| \le \frac{2}{3\pi}\left|d_{2N+2}\right|\frac{3}{4}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}}\begin{cases} |\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } |\theta| \le \frac{\pi}{4} \end{cases} \quad (3.133)$$

when $N \equiv 1 \mod 3$.

Let us now consider the estimation of the remainder term $R_N^{(Y')}(\nu)$. In this case, one finds that the expression corresponding to (3.130) is

$$\begin{aligned} R_N^{(Y')}(\nu) &= \frac{(-1)^{N+1}}{2\pi}\frac{1}{\nu^{\frac{2N+4}{3}}}\int_0^{+\infty} s^{\frac{2N+1}{3}}\mathrm{e}^{-s} \\ &\times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{1-\left(s/\left(\nu \mathrm{i} f\left(t\right)\right)\right)^{\frac{2}{3}}}{1+\left(s/\left(\nu \mathrm{i} f\left(t\right)\right)\right)^{2}}\left(\mathrm{i}f\left(t\right)\right)^{-\frac{2N+4}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\sinh x\cos y\mathrm{d}y\mathrm{d}s \\ &\quad + \frac{(-1)^{N+1}}{2\pi}\frac{1}{\nu^{\frac{2N+4}{3}}}\int_0^{+\infty} s^{\frac{2N+1}{3}}\mathrm{e}^{-s} \\ &\times \frac{2}{\pi}\int_{\mathscr{C}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{1-\left(s/\left(\nu \mathrm{i} f\left(t\right)\right)\right)^{\frac{2}{3}}}{1+\left(s/\left(\nu \mathrm{i} f\left(t\right)\right)\right)^{2}}\left(\mathrm{i}f\left(t\right)\right)^{-\frac{2N+4}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\cosh x\sin y\mathrm{d}x\mathrm{d}s \end{aligned} \tag{3.134}$$

when $N \equiv 2 \mod 3$. From (3.102), (3.134) and the inequality (3.90), we establish

$$\left|R_N^{(Y')}(\nu)\right| \le \frac{2}{3\pi}\left|g_{2N+2}\right|\frac{3}{4}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{\left|\nu\right|^{\frac{2N+4}{3}}}\begin{cases}\left|\csc\left(2\theta\right)\right| & \text{if } \frac{\pi}{4} < \left|\theta\right| < \frac{\pi}{2}, \\ 1 & \text{if } \left|\theta\right| \le \frac{\pi}{4}.\end{cases} \tag{3.135}$$

One may likewise show that

$$\left|R_N^{(Y')}(\nu)\right| \le \frac{2}{3\pi}\left|g_{2N}\right|\frac{3}{4}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{\left|\nu\right|^{\frac{2N+2}{3}}}\begin{cases}\left|\csc\left(2\theta\right)\right| & \text{if } \frac{\pi}{4} < \left|\theta\right| < \frac{\pi}{2}, \\ 1 & \text{if } \left|\theta\right| \le \frac{\pi}{4}\end{cases} \tag{3.136}$$

if $N \equiv 0 \mod 3$, and

$$\begin{aligned} \left|R_N^{(Y')}(\nu)\right| &\le \left(\frac{2}{3\pi}\left|g_{2N}\right|\frac{3}{4}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{\left|\nu\right|^{\frac{2N+21}{3}}} + \frac{2}{3\pi}\left|g_{2N+4}\right|\frac{3}{4}\frac{\Gamma\left(\frac{2N+6}{3}\right)}{\left|\nu\right|^{\frac{2N+6}{3}}}\right) \\ &\qquad\times\begin{cases}\left|\csc\left(2\theta\right)\right| & \text{if } \frac{\pi}{4} < \left|\theta\right| < \frac{\pi}{2}, \\ 1 & \text{if } \left|\theta\right| \le \frac{\pi}{4}\end{cases} \end{aligned} \tag{3.137}$$

if $N \equiv 1 \mod 3$, respectively.

In the special case when ν is positive, we have $0 < 1/(1+(s/(\nu \mathrm{i} f(t)))^2) < 1$ in (3.130) and together with (3.97), the mean value theorem of integration shows that

$$R_N^{(Y)}(\nu) = -\frac{2}{3\pi}d_{2N}\frac{3}{4}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{\nu^{\frac{2N+1}{3}}}\Theta_N(\nu) - \frac{2}{3\pi}d_{2N+2}\frac{3}{4}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{\nu^{\frac{2N+3}{3}}}\widehat{\Theta}_N(\nu),$$

when $N \equiv 2 \mod 3$ and with $0 < \Theta_N(\nu), \widehat{\Theta}_N(\nu) < 1$ being appropriate numbers that depend on ν and N. We find in a similar way that

$$R_N^{(Y)}(\nu) = -\frac{2}{3\pi} d_{2N+2} \frac{3}{4} \frac{\Gamma\left(\frac{2N+3}{3}\right)}{\nu^{\frac{2N+3}{3}}} \widehat{\Theta}_N(\nu) - \frac{2}{3\pi} d_{2N+4} \frac{3}{4} \frac{\Gamma\left(\frac{2N+5}{3}\right)}{\nu^{\frac{2N+5}{3}}} \widetilde{\Theta}_N(\nu)$$

when $N \equiv 0 \mod 3$, and

$$R_N^{(Y)}(\nu) = -\frac{2}{3\pi} d_{2N} \frac{3}{4} \frac{\Gamma\left(\frac{2N+1}{3}\right)}{\nu^{\frac{2N+1}{3}}} \Theta_N(\nu) - \frac{2}{3\pi} d_{2N+4} \frac{3}{4} \frac{\Gamma\left(\frac{2N+5}{3}\right)}{\nu^{\frac{2N+5}{3}}} \widetilde{\Theta}_N(\nu)$$

when $N \equiv 1 \mod 3$. Here $0 < \widetilde{\Theta}_N(\nu) < 1$ is a suitable number that depends on ν and N. Similarly, when ν is positive, we have $0 < 1/(1 + (s/(\nu i f(t)))^2) < 1$ in (3.134) and together with (3.102), the mean value theorem of integration gives

$$R_N^{(Y')}(\nu) = \frac{2}{3\pi} g_{2N+2} \frac{3}{4} \frac{\Gamma\left(\frac{2N+4}{3}\right)}{\nu^{\frac{2N+4}{3}}} \widehat{\Xi}_N(\nu) + \frac{2}{3\pi} g_{2N+4} \frac{3}{4} \frac{\Gamma\left(\frac{2N+6}{3}\right)}{\nu^{\frac{2N+6}{3}}} \widetilde{\Xi}_N(\nu),$$

when $N \equiv 2 \mod 3$ and with $0 < \widehat{\Xi}_N(\nu), \widetilde{\Xi}_N(\nu) < 1$ being appropriate numbers that depend on ν and N. Analogously, one finds that

$$R_N^{(Y')}(\nu) = \frac{2}{3\pi} g_{2N} \frac{3}{4} \frac{\Gamma\left(\frac{2N+2}{3}\right)}{\nu^{\frac{2N+2}{3}}} \Xi_N(\nu) + \frac{2}{3\pi} g_{2N+2} \frac{3}{4} \frac{\Gamma\left(\frac{2N+4}{3}\right)}{\nu^{\frac{2N+4}{3}}} \widehat{\Xi}_N(\nu)$$

when $N \equiv 0 \mod 3$, and

$$R_N^{(Y')}(\nu) = \frac{2}{3\pi} g_{2N} \frac{3}{4} \frac{\Gamma\left(\frac{2N+2}{3}\right)}{\nu^{\frac{2N+2}{3}}} \Xi_N(\nu) + \frac{2}{3\pi} g_{2N+4} \frac{3}{4} \frac{\Gamma\left(\frac{2N+6}{3}\right)}{\nu^{\frac{2N+6}{3}}} \widetilde{\Xi}_N(\nu)$$

when $N \equiv 1 \mod 3$. Here $0 < \Xi_N(\nu) < 1$ is a suitable number that depends on ν and N.

The bounds (3.131)–(3.133) and (3.135)–(3.137) become singular as θ tends to $\pm\frac{\pi}{2}$ and are therefore not appropriate near the Stokes lines $\theta = \pm\frac{\pi}{2}$. We now provide alternative bounds that are suitable for the sectors $\frac{\pi}{4} < |\theta| < \pi$ (which include the Stokes lines $\theta = \pm\frac{\pi}{2}$). We can use (3.83) and (3.87) to define the remainder terms in the sectors $\frac{\pi}{2} \leq |\theta| < \pi$. These alternative bounds can be derived based on the representations (3.130) and (3.134) (or on the analogous expressions for the other two cases). Their derivation is similar to that of (2.43) discussed in Subsection 2.1.2, and the details are therefore omitted. It is found that in the sectors $\frac{\pi}{4} < |\theta| < \pi$, the remainder $R_N^{(Y)}(\nu)$ can be estimated as follows:

$$\begin{aligned}\left|R_N^{(Y)}(\nu)\right| \le {} & \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{\frac{2N+1}{3}}\varphi^*}\frac{2}{3\pi}|d_{2N}|\frac{3}{4}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}} \\ & + \frac{|\csc(2(\theta-\varphi^{***}))|}{\cos^{\frac{2N+5}{3}}\varphi^{***}}\frac{2}{3\pi}|d_{2N+4}|\frac{3}{4}\frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}}\end{aligned} \tag{3.138}$$

when $N \equiv 0 \mod 3$,

$$\left|R_N^{(Y)}(\nu)\right| \le \frac{|\csc(2(\theta-\varphi^{**}))|}{\cos^{\frac{2N+3}{3}}\varphi^{**}}\frac{2}{3\pi}|d_{2N+2}|\frac{3}{4}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}} \tag{3.139}$$

when $N \equiv 1 \mod 3$, and

$$\left|R_N^{(Y)}(\nu)\right| \le \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{\frac{2N+1}{3}}\varphi^*}\frac{2}{3\pi}|d_{2N}|\frac{3}{4}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}} \tag{3.140}$$

when $N \equiv 2 \mod 3$. Here φ^*, φ^{**} and φ^{***} are the minimizing values given by Lemma 2.1.2 with $\chi = \frac{2N+1}{3}$, $\chi = \frac{2N+3}{3}$ and $\chi = \frac{2N+5}{3}$, respectively. Similarly, for $\frac{\pi}{4} < |\theta| < \pi$, the remainder term $R_N^{(Y')}(\nu)$ satisfies the following bounds:

$$\left|R_N^{(Y')}(\nu)\right| \le \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{\frac{2N+2}{3}}\varphi^*}\frac{2}{3\pi}|g_{2N}|\frac{3}{4}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}} \tag{3.141}$$

if $N \equiv 0 \mod 3$,

$$\begin{aligned}\left|R_N^{(Y')}(\nu)\right| \le {} & \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{\frac{2N+2}{3}}\varphi^*}\frac{2}{3\pi}|g_{2N}|\frac{3}{4}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}} \\ & + \frac{|\csc(2(\theta-\varphi^{***}))|}{\cos^{\frac{2N+6}{3}}\varphi^{***}}\frac{2}{3\pi}|g_{2N+4}|\frac{3}{4}\frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}}\end{aligned} \tag{3.142}$$

if $N \equiv 1 \mod 3$, and

$$\left|R_N^{(Y')}(\nu)\right| \le \frac{|\csc(2(\theta-\varphi^{**}))|}{\cos^{\frac{2N+4}{3}}\varphi^{**}}\frac{2}{3\pi}|g_{2N+2}|\frac{3}{4}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}} \tag{3.143}$$

if $N \equiv 2 \mod 3$. Here φ^*, φ^{**} and φ^{***} are the minimizing values given by Lemma 2.1.2 with the specific choices of $\chi = \frac{2N+2}{3}$, $\chi = \frac{2N+4}{3}$ and $\chi = \frac{2N+6}{3}$, respectively.

The following simple estimates are suitable for the sectors $\frac{\pi}{4} < |\theta| \leq \frac{\pi}{2}$ (especially near the Stokes lines $\theta = \pm\frac{\pi}{2}$) and $N \geq 4$, and they can be obtained from (3.138)–(3.143) using an argument akin to that given in Subsection 2.1.2:

$$\begin{aligned}\left|R_N^{(Y)}(\nu)\right| &\leq \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{11}{2}\right)}\,\frac{2}{3\pi}\,|d_{2N}|\,\frac{3}{4}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}}\\ &\quad + \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{19}{2}\right)}\,\frac{2}{3\pi}\,|d_{2N+4}|\,\frac{3}{4}\frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}},\\ \left|R_N^{(Y')}(\nu)\right| &\leq \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{13}{2}\right)}\,\frac{2}{3\pi}\,|g_{2N}|\,\frac{3}{4}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}}\end{aligned}$$

when $N \equiv 0 \mod 3$,

$$\begin{aligned}\left|R_N^{(Y)}(\nu)\right| &\leq \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{15}{2}\right)}\,\frac{2}{3\pi}\,|d_{2N+2}|\,\frac{3}{4}\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}},\\ \left|R_N^{(Y')}(\nu)\right| &\leq \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{13}{2}\right)}\,\frac{2}{3\pi}\,|g_{2N}|\,\frac{3}{4}\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}}\\ &\quad + \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{21}{2}\right)}\,\frac{2}{3\pi}\,|g_{2N+4}|\,\frac{3}{4}\frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}}\end{aligned}$$

when $N \equiv 1 \mod 3$, and

$$\begin{aligned}\left|R_N^{(Y)}(\nu)\right| &\leq \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{11}{2}\right)}\,\frac{2}{3\pi}\,|d_{2N}|\,\frac{3}{4}\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}},\\ \left|R_N^{(Y')}(\nu)\right| &\leq \tfrac{1}{6}\sqrt{3\mathrm{e}\left(2N+\tfrac{17}{2}\right)}\,\frac{2}{3\pi}\,|g_{2N+2}|\,\frac{3}{4}\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}}\end{aligned}$$

when $N \equiv 2 \mod 3$.

3.2.3 Asymptotics for the late coefficients

In this subsection, we study the asymptotic nature of the coefficients d_{2n} and g_{2n} as $n \to +\infty$. A leading order approximation for the d_{2n}'s was stated, without proof, by Meissel [64] and was later proved rigorously by Watson [116, p. 233] using the method of Darboux. By employing integral methods, Olver [95, eq. (10.09), p. 315] gave higher approximations for the coefficients d_{2n} (see also [95, exer. 10.2, p. 315]). Formal expansions for both sets of coefficients were derived by Dingle [35, eqs. (62) and (66), pp. 171–172]; his expansions may be written, in our notation, as

$$\Gamma\left(\frac{2n+1}{3}\right) d_{2n} \approx \frac{(-1)^n 2}{3\pi (2\pi)^{\frac{2n}{3}}} \sum_{m=0}^{\infty} (-1)^m (2\pi)^{\frac{2m}{3}} d_{2m} \sin\left(\frac{\pi(2m+1)}{3}\right) \times \Gamma\left(\frac{2m+1}{3}\right) \Gamma\left(\frac{2n-2m}{3}\right) \tag{3.144}$$

and

$$\Gamma\left(\frac{2n+2}{3}\right) g_{2n} \approx \frac{(-1)^{n+1} 2}{3\pi (2\pi)^{\frac{2n}{3}}} \sum_{m=0}^{\infty} (-1)^m (2\pi)^{\frac{2m}{3}} g_{2m} \sin\left(\frac{\pi(2m+2)}{3}\right) \times \Gamma\left(\frac{2m+2}{3}\right) \Gamma\left(\frac{2n-2m}{3}\right). \tag{3.145}$$

We shall derive here the rigorous forms of Dingle's expansions by truncating them after a finite number of terms and constructing their error bounds.

We begin by considering the d_{2n}'s. We replace the function $\mathrm{i}H_{\mathrm{i}u}^{(1)}(ue^{\frac{\pi}{2}\mathrm{i}})$ in (3.96) by its truncated asymptotic expansion

$$\mathrm{i}H_{\mathrm{i}u}^{(1)}(ue^{\frac{\pi}{2}\mathrm{i}}) = \frac{2}{3\pi} \sum_{m=0}^{M-1} (-1)^m d_{2m} \sin\left(\frac{\pi(2m+1)}{3}\right) \frac{\Gamma\left(\frac{2m+1}{3}\right)}{u^{\frac{2m+1}{3}}} + \mathrm{i}R_M^{(H)}(ue^{\frac{\pi}{2}\mathrm{i}}) \tag{3.146}$$

where $M \geq 0$, and from (3.98)–(3.100),

$$\left|R_M^{(H)}(ue^{\frac{\pi}{2}\mathrm{i}})\right| \leq \frac{2}{3\pi} |d_{2M}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2M+1}{3}\right)}{u^{\frac{2M+1}{3}}} + \frac{2}{3\pi} |d_{2M+2}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2M+3}{3}\right)}{u^{\frac{2M+3}{3}}}, \tag{3.147}$$

$$\left|R_M^{(H)}(ue^{\frac{\pi}{2}\mathrm{i}})\right| \leq \frac{2}{3\pi} |d_{2M+2}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2M+3}{3}\right)}{u^{\frac{2M+3}{3}}} \tag{3.148}$$

and

$$\left|R_M^{(H)}(ue^{\frac{\pi}{2}\mathrm{i}})\right| \leq \frac{2}{3\pi} |d_{2M}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2M+1}{3}\right)}{u^{\frac{2M+1}{3}}}, \tag{3.149}$$

according to whether $M \equiv 0 \mod 3$, $M \equiv 1 \mod 3$ or $M \equiv 2 \mod 3$, respectively. Thus, from (3.96) and (3.146)–(3.149), and provided $n \geq 2$,

$$\Gamma\left(\frac{2n+1}{3}\right) d_{2n} = \frac{(-1)^n 2}{3\pi (2\pi)^{\frac{2n}{3}}} \left(\sum_{m=0}^{M-1} (-1)^m (2\pi)^{\frac{2m}{3}} d_{2m} \sin\left(\frac{\pi(2m+1)}{3}\right) \times \Gamma\left(\frac{2m+1}{3}\right) \Gamma\left(\frac{2n-2m}{3}\right) + A_M(n) \right) \tag{3.150}$$

where

$$\begin{aligned} |A_M(n)| \le\ & (2\pi)^{\frac{2M}{3}} |d_{2M}| \frac{3^{\frac{1}{2}}}{2} \Gamma\left(\frac{2M+1}{3}\right) \Gamma\left(\frac{2n-2M}{3}\right) \\ & + (2\pi)^{\frac{2M+2}{3}} |d_{2M+1}| \frac{3^{\frac{1}{2}}}{2} \Gamma\left(\frac{2M+3}{3}\right) \Gamma\left(\frac{2n-2M-2}{3}\right) \end{aligned} \tag{3.151}$$

provided $0 \le M \le n-2$ and $M \equiv 0 \mod 3$,

$$|A_M(n)| \le (2\pi)^{\frac{2M+2}{3}} |d_{2M+2}| \frac{3^{\frac{1}{2}}}{2} \Gamma\left(\frac{2M+3}{3}\right) \Gamma\left(\frac{2n-2M-2}{3}\right) \tag{3.152}$$

provided $1 \le M \le n-2$ and $M \equiv 1 \mod 3$,

$$|A_M(n)| \le (2\pi)^{\frac{2M}{3}} |d_{2M}| \frac{3^{\frac{1}{2}}}{2} \Gamma\left(\frac{2M+1}{3}\right) \Gamma\left(\frac{2n-2M}{3}\right) \tag{3.153}$$

provided $2 \le M \le n-1$ and $M \equiv 2 \mod 3$. For given large n, the least values of the bounds (3.151), (3.152) and (3.153) occur when $M \approx \frac{n}{2}$. With this choice of M, the ratio of the error bound to the leading term in (3.150) is $\mathcal{O}\left(n^{-\frac{1}{2}} 4^{-\frac{n}{3}}\right)$ in all three cases. This is the best accuracy available from truncating the expansion (3.150). Numerical examples illustrating the efficacy of (3.150), truncated optimally, are given in Table 3.2.

One may similarly show that for the coefficients g_{2n},

$$\begin{aligned} \Gamma\left(\frac{2n+2}{3}\right) g_{2n} = \frac{(-1)^{n+1} 2}{3\pi (2\pi)^{\frac{2n}{3}}} \Bigg(& \sum_{m=0}^{M-1} (-1)^m (2\pi)^{\frac{2m}{3}} g_{2m} \sin\left(\frac{\pi(2m+2)}{3}\right) \\ & \times \Gamma\left(\frac{2m+2}{3}\right) \Gamma\left(\frac{2n-2m}{3}\right) + B_M(n) \Bigg) \end{aligned} \tag{3.154}$$

where

$$|B_M(n)| \le (2\pi)^{\frac{2M}{3}} |g_{2M}| \frac{3^{\frac{1}{2}}}{2} \Gamma\left(\frac{2M+2}{3}\right) \Gamma\left(\frac{2n-2M}{3}\right) \tag{3.155}$$

provided $0 \le M \le n-1$ and $M \equiv 0 \mod 3$,

$$\begin{aligned} |B_M(n)| \le\ & (2\pi)^{\frac{2M}{3}} |g_{2M}| \frac{3^{\frac{1}{2}}}{2} \Gamma\left(\frac{2M+2}{3}\right) \Gamma\left(\frac{2n-2M}{3}\right) \\ & + (2\pi)^{\frac{2M+2}{3}} |g_{2M+2}| \frac{3^{\frac{1}{2}}}{2} \Gamma\left(\frac{2M+4}{3}\right) \Gamma\left(\frac{2n-2M-2}{3}\right) \end{aligned} \tag{3.156}$$

values of n and M	$n = 25$, $M = 13$
exact numerical value of $\left\lvert \Gamma\left(\frac{2n+1}{3}\right) d_{2n} \right\rvert$	$0.365998943362455445695153713414 \times 10^{0}$
approximation (3.150) to $\left\lvert \Gamma\left(\frac{2n+1}{3}\right) d_{2n} \right\rvert$	$0.365999204995926845752354038262 \times 10^{0}$
error	$-0.261633471400057200324848 \times 10^{-6}$
error bound using (3.152)	$0.912625604300875369350368 \times 10^{-6}$
values of n and M	$n = 50$, $M = 25$
exact numerical value of $\left\lvert \Gamma\left(\frac{2n+1}{3}\right) d_{2n} \right\rvert$	$0.186185539770426140010811239005 \times 10^{10}$
approximation (3.150) to $\left\lvert \Gamma\left(\frac{2n+1}{3}\right) d_{2n} \right\rvert$	$0.186185539771330169764915025615 \times 10^{10}$
error	$-0.904029754103786610 \times 10^{-2}$
error bound using (3.152)	$0.283805212193953 5246 \times 10^{-1}$
values of n and M	$n = 75$, $M = 37$
exact numerical value of $\left\lvert \Gamma\left(\frac{2n+1}{3}\right) d_{2n} \right\rvert$	$0.670515930675419436921950214548 \times 10^{23}$
approximation (3.150) to $\left\lvert \Gamma\left(\frac{2n+1}{3}\right) d_{2n} \right\rvert$	$0.670515930675419462694509569704 \times 10^{23}$
error	$-0.25772559355156 \times 10^{7}$
error bound using (3.152)	$0.78252106282231 \times 10^{7}$
values of n and M	$n = 100$, $M = 50$
exact numerical value of $\left\lvert \Gamma\left(\frac{2n+1}{3}\right) d_{2n} \right\rvert$	$0.737974366090019540631543787394 \times 10^{39}$
approximation (3.150) to $\left\lvert \Gamma\left(\frac{2n+1}{3}\right) d_{2n} \right\rvert$	$0.737974366090019540631782512745 \times 10^{39}$
error	$-0.238725351 \times 10^{18}$
error bound using (3.153)	$0.713047093 \times 10^{18}$

Table 3.2. *Approximations for* $\left\lvert \Gamma\left(\frac{2n+1}{3}\right) d_{2n} \right\rvert$ *with various n, using* (3.150).

provided $1 \le M \le n-2$ and $M \equiv 1 \mod 3$,

$$\left|B_M(n)\right| \le (2\pi)^{\frac{2M+2}{3}} \left|g_{2M+2}\right| \frac{3^{\frac{1}{2}}}{2} \Gamma\left(\frac{2M+4}{3}\right) \Gamma\left(\frac{2n-2M-2}{3}\right) \tag{3.157}$$

provided $2 \le M \le n-2$ and $M \equiv 2 \mod 3$. One readily establishes that the least values of the bounds (3.155)–(3.157) occur when $M \approx \frac{n}{2}$.

If we neglect the remainder terms $A_M(n)$ and $B_M(n)$ in (3.150) and (3.154), and we formally extend the sums to infinity, formulae (3.150) and (3.154) reproduce Dingle's expansions (3.144) and (3.145).

3.2.4 Exponentially improved asymptotic expansions

The aim of this subsection is to give exponentially improved asymptotic expansions for the Hankel and Bessel functions, and their derivatives, for equal order and argument. Re-expansions for the remainder terms of the asymptotic expansions of the functions $J_\nu(\nu)$, $J'_\nu(\nu)$, $Y_\nu(\nu)$ and $Y'_\nu(\nu)$ were derived, using formal methods, by Dingle [35, eqs. (50)–(55), pp. 470–471]. He divided each of the asymptotic expansions into two parts according to the value of the summation index n modulo 3 and considered the two remainders of these expansions separately. We shall derive here the rigorous forms of Dingle's formal re-expansions as well as the corresponding results for $H_\nu^{(1)}(\nu)$ and $H_\nu^{(1)\prime}(\nu)$.

Let us first consider the case of $H_\nu^{(1)}(\nu)$ and $H_\nu^{(1)\prime}(\nu)$. It is not possible to re-expand directly the remainders $R_N^{(H)}(\nu)$ and $R_N^{(H')}(\nu)$ in terms of terminant functions because of the presence of cube roots in the denominators of the integrands in their representations (3.73) and (3.79). To overcome this difficulty, we follow Dingle's idea and write both $H_\nu^{(1)}(\nu)$ and $H_\nu^{(1)\prime}(\nu)$ as a sum of two truncated asymptotic expansions plus a remainder thereby obtaining representations different from (3.69) and (3.77). The form of the remainders in these alternative expressions will be adequate for our purposes. Assuming $|\theta| < \frac{\pi}{2}$, the representation (3.73) for $H_\nu^{(1)}(\nu) = R_0^{(H)}(\nu)$ can be re-arranged in the form

$$H_\nu^{(1)}(\nu) = \frac{e^{-\frac{\pi}{3}i}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{\frac{1}{3}}}\int_0^{+\infty}\frac{u^{-\frac{2}{3}}e^{-2\pi u}}{1+(u/\nu)^2}iH_{iu}^{(1)}\left(ue^{\frac{\pi}{2}i}\right)du - \frac{e^{\frac{\pi}{3}i}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{\frac{5}{3}}}\int_0^{+\infty}\frac{u^{\frac{2}{3}}e^{-2\pi u}}{1+(u/\nu)^2}iH_{iu}^{(1)}\left(ue^{\frac{\pi}{2}i}\right)du.$$

Next, we expand the denominators of the integrands by means of (1.7) (with m and M in place of n and N in the second integral) and make use of the formula (3.96) to deduce

$$H_\nu^{(1)}(\nu) = \frac{e^{-\frac{\pi}{3}i}}{3^{\frac{1}{3}}\pi}\sum_{n=0}^{N-1}d_{6n}\frac{\Gamma\left(2n+\frac{1}{3}\right)}{\nu^{2n+\frac{1}{3}}} - \frac{e^{\frac{\pi}{3}i}}{3^{\frac{1}{3}}\pi}\sum_{m=0}^{M-1}d_{6m+4}\frac{\Gamma\left(2m+\frac{5}{3}\right)}{\nu^{2m+\frac{5}{3}}} + R_{N,M}^{(H)}(\nu) \quad (3.158)$$

(cf. [96, eq. 10.19.9, p. 232]). The remainder term $R_{N,M}^{(H)}(\nu)$ can be expressed as

$$\begin{aligned}R_{N,M}^{(H)}(\nu) = {} & (-1)^N\frac{e^{-\frac{\pi}{3}i}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{2N+\frac{1}{3}}}\int_0^{+\infty}\frac{u^{2N-\frac{2}{3}}e^{-2\pi u}}{1+(u/\nu)^2}iH_{iu}^{(1)}\left(ue^{\frac{\pi}{2}i}\right)du \\ & + (-1)^{M+1}\frac{e^{\frac{\pi}{3}i}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{2M+\frac{5}{3}}}\int_0^{+\infty}\frac{u^{2M+\frac{2}{3}}e^{-2\pi u}}{1+(u/\nu)^2}iH_{iu}^{(1)}\left(ue^{\frac{\pi}{2}i}\right)du\end{aligned} \quad (3.159)$$

for $|\theta| < \frac{\pi}{2}$ and $N, M \geq 0$. Equation (3.158) gives an expansion of $H_{\nu}^{(1)}(\nu)$ which has the convenient property that its remainder $R_{N,M}^{(H)}(\nu)$ can be expressed in a simple way using terminant functions.

In a similar manner, we write

$$H_{\nu}^{(1)\prime}(\nu) = \frac{\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}\pi} \sum_{n=0}^{N-1} g_{6n} \frac{\Gamma(2n+\frac{2}{3})}{\nu^{2n+\frac{2}{3}}} - \frac{\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}\pi} \sum_{m=0}^{M-1} g_{6m+2} \frac{\Gamma(2m+\frac{4}{3})}{\nu^{2m+\frac{4}{3}}} + R_{N,M}^{(H')}(\nu) \quad (3.160)$$

with

$$\begin{aligned} R_{N,M}^{(H')}(\nu) = {} & (-1)^{N+1} \frac{\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}\pi} \frac{1}{\nu^{2N+\frac{2}{3}}} \int_0^{+\infty} \frac{u^{2N-\frac{1}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2} H_{\mathrm{i}u}^{(1)\prime}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}u \\ & + (-1)^{M+1} \frac{\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}\pi} \frac{1}{\nu^{2M+\frac{4}{3}}} \int_0^{+\infty} \frac{u^{2M+\frac{1}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2} H_{\mathrm{i}u}^{(1)\prime}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}u \end{aligned}$$

for $|\theta| < \frac{\pi}{2}$, $N \geq 1$ and $M \geq 0$. From now on, we assume that $N, M \geq 0$ whenever we write $R_{N,M}^{(H)}(\nu)$ and $N \geq 1$, $M \geq 0$ whenever we write $R_{N,M}^{(H')}(\nu)$.

Now, we are able to formulate our re-expansions for the remainder terms $R_{N,M}^{(H)}(\nu)$ and $R_{N,M}^{(H')}(\nu)$, in Proposition 3.2.1 below. In this proposition, the functions $R_{N,M}^{(H)}(\nu)$ and $R_{N,M}^{(H')}(\nu)$ are extended to a sector larger than $|\theta| < \frac{\pi}{2}$ via (3.158) and (3.160) using analytic continuation.

Proposition 3.2.1. *Let K and L be arbitrary fixed non-negative integers. Suppose that $-2\pi + \delta \leq \theta \leq 3\pi - \delta$ with an arbitrary fixed small positive δ, $|\nu|$ is large and $N = \pi|\nu| + \rho$, $M = \pi|\nu| + \sigma$ with ρ and σ being bounded. Then*

$$\begin{aligned} R_{N,M}^{(H)}(\nu) = {} & -\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu} \frac{2}{3\pi} \sum_{k=0}^{K-1} d_{2k} \mathrm{e}^{\frac{2\pi(2k+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(2k+1)}{3}\right) \frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}} T_{2N-\frac{2k}{3}}(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}) \\ & + \frac{\mathrm{i}\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu} \frac{2}{3\pi} \sum_{k=0}^{K-1} d_{2k} \sin\left(\frac{\pi(2k+1)}{3}\right) \frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}} T_{2N-\frac{2k}{3}}(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}) \\ & + \frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu} \frac{2}{3\pi} \sum_{\ell=0}^{L-1} d_{2\ell} \mathrm{e}^{\frac{2\pi(2\ell+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(2\ell+1)}{3}\right) \frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}} T_{2M-\frac{2\ell-4}{3}}(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}) \\ & - \frac{\mathrm{i}\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu} \frac{2}{3\pi} \sum_{\ell=0}^{L-1} d_{2\ell} \sin\left(\frac{\pi(2\ell+1)}{3}\right) \frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}} T_{2M-\frac{2\ell-4}{3}}(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}) \\ & + R_{N,M,K,L}^{(H)}(\nu), \end{aligned} \quad (3.161)$$

$$
\begin{aligned}
R_{N,M}^{(H')}(\nu) &= \frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{K-1} g_{2k}\mathrm{e}^{\frac{2\pi(2k+2)}{3}\mathrm{i}}\sin\left(\frac{\pi(2k+2)}{3}\right)\frac{\Gamma\left(\frac{2k+2}{3}\right)}{\nu^{\frac{2k+2}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\\
&- \frac{\mathrm{i}\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{K-1} g_{2k}\sin\left(\frac{\pi(2k+2)}{3}\right)\frac{\Gamma\left(\frac{2k+2}{3}\right)}{\nu^{\frac{2k+2}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\\
&- \frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} g_{2\ell}\mathrm{e}^{\frac{2\pi(2\ell+2)}{3}\mathrm{i}}\sin\left(\frac{\pi(2\ell+2)}{3}\right)\frac{\Gamma\left(\frac{2\ell+2}{3}\right)}{\nu^{\frac{2\ell+2}{3}}}T_{2M-\frac{2\ell-2}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\\
&+ \frac{\mathrm{i}\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} g_{2\ell}\sin\left(\frac{\pi(2\ell+2)}{3}\right)\frac{\Gamma\left(\frac{2\ell+2}{3}\right)}{\nu^{\frac{2\ell+2}{3}}}T_{2M-\frac{2\ell-2}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\\
&+ R_{N,M,K,L}^{(H')}(\nu),
\end{aligned}
$$

where

$$
\begin{aligned}
R_{N,M,K,L}^{(H)}(\nu) &= \mathcal{O}_{K,\rho}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2K+1}{3}}}\right) + \mathcal{O}_{L,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2L+1}{3}}}\right),\\
R_{N,M,K,L}^{(H')}(\nu) &= \mathcal{O}_{K,\rho}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2K+2}{3}}}\right) + \mathcal{O}_{L,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2L+2}{3}}}\right)
\end{aligned}
\tag{3.162}
$$

for $|\theta| \le \frac{\pi}{2}$*, and*

$$
\begin{aligned}
R_{N,M,K,L}^{(H)}(\nu) &= \mathcal{O}_{K,\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathrm{m}(\nu)}}{|\nu|^{\frac{2K+1}{3}}}\right) + \mathcal{O}_{L,\sigma,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathrm{m}(\nu)}}{|\nu|^{\frac{2L+1}{3}}}\right),\\
R_{N,M,K,L}^{(H')}(\nu) &= \mathcal{O}_{K,\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathrm{m}(\nu)}}{|\nu|^{\frac{2K+2}{3}}}\right) + \mathcal{O}_{L,\sigma,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathrm{m}(\nu)}}{|\nu|^{\frac{2L+2}{3}}}\right)
\end{aligned}
\tag{3.163}
$$

for $\frac{\pi}{2} \le \pm\theta \le 2\pi - \delta$*. Moreover, if* $K = L$*, then the estimates* (3.162) *remain valid in the larger sector* $-\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ *and* (3.163) *holds in the range* $\frac{3\pi}{2} \le \theta \le 3\pi - \delta$ *with all the lower signs taken.*

Proposition 3.2.1 together with (3.158) and (3.160) yields the exponentially improved asymptotic expansions for the Hankel function and its derivative, for equal order and argument. In particular, if $K = L$, formula (3.161) together with (3.158) embraces the three asymptotic expansions (3.62) and

$$
\begin{aligned}
H_\nu^{(1)}(\nu) \sim &-\frac{2}{3\pi}\sum_{n=0}^{\infty} d_{2n}\mathrm{e}^{\frac{2\pi(2n+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2n+1)}{3}\right)\frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}}\\
&\pm \frac{\mathrm{i}\mathrm{e}^{\mp\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{\infty} d_{2k}\mathrm{e}^{-\frac{(1\mp1)\pi(2k+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2k+1)}{3}\right)\frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}}
\end{aligned}
$$

$$\mp \frac{\mathrm{i}\mathrm{e}^{\pm\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}} \mathrm{e}^{-2\pi \mathrm{i}\nu} \frac{2}{3\pi} \sum_{\ell=0}^{\infty} d_{2\ell} \mathrm{e}^{-\frac{(1\mp 1)\pi(2\ell+1)}{3}\mathrm{i}} \sin\left(\frac{\pi\,(2\ell+1)}{3}\right) \frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}}$$

which holds when $\nu \to \infty$ in the sectors $-\frac{\pi}{2} + \delta < \theta \mp \frac{3\pi}{2} < \frac{3\pi}{2} - \delta$ (see, e.g., [73]); furthermore, they give the smooth transition across the Stokes lines $\theta = -\frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.

The analogous results for the functions $H_\nu^{(2)}(\nu)$ and $H_\nu^{(2)\prime}(\nu)$ can be deduced directly from the functional relations

$$H_\nu^{(2)}(\nu) = \overline{H_{\bar{\nu}}^{(1)}(\bar{\nu})} \quad \text{and} \quad H_\nu^{(2)\prime}(\nu) = \overline{H_{\bar{\nu}}^{(1)\prime}(\bar{\nu})}$$

(see, e.g., [96, eq. 10.11.9, p. 226]); we do not pursue the details here.

In the following theorem, we give explicit bounds on the remainder terms $R_{N,M,K,L}^{(H)}(\nu)$ and $R_{N,M,K,L}^{(H')}(\nu)$. Note that in these results, N and M do not necessarily depend on ν. We assume $K, L \equiv 0 \mod 3$ merely for the sake of simplicity: estimations for $R_{N,M,K,L}^{(H)}(\nu)$ and $R_{N,M,K,L}^{(H')}(\nu)$ when K or L may not be divisible by 3 can be obtained similarly.

Theorem 3.2.2. *Let N, M, K and L be arbitrary fixed non-negative integers such that $K < 3N$, $L < 3M + 2$ and $K, L \equiv 0 \mod 3$. Then we have*

$$\begin{aligned}
\left|R_{N,M,K,L}^{(H)}(\nu)\right| \le{}& \frac{1}{3^{\frac{1}{2}}}\left|\mathrm{e}^{2\pi\mathrm{i}\nu}\right| \frac{2}{3\pi}\left|d_{2K}\right| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2K+1}{3}\right)}{|\nu|^{\frac{2K+1}{3}}} \left|T_{2N-\frac{2K}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&+ \frac{1}{3^{\frac{1}{2}}}\left|\mathrm{e}^{-2\pi\mathrm{i}\nu}\right| \frac{2}{3\pi}\left|d_{2K}\right| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2K+1}{3}\right)}{|\nu|^{\frac{2K+1}{3}}} \left|T_{2N-\frac{2K}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&+ \frac{2}{3\pi^2}\left|d_{2K}\right| \frac{\Gamma\left(\frac{2K+1}{3}\right)\Gamma\left(2N-\frac{2K}{3}\right)}{(2\pi)^{2N-\frac{2K}{3}}\,|\nu|^{2N+\frac{1}{3}}} \\
&+ \frac{1}{3^{\frac{1}{2}}}\left|\mathrm{e}^{2\pi\mathrm{i}\nu}\right| \frac{2}{3\pi}\left|d_{2L}\right| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2L+1}{3}\right)}{|\nu|^{\frac{2L+1}{3}}} \left|T_{2M-\frac{2L-4}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&+ \frac{1}{3^{\frac{1}{2}}}\left|\mathrm{e}^{-2\pi\mathrm{i}\nu}\right| \frac{2}{3\pi}\left|d_{2L}\right| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2L+1}{3}\right)}{|\nu|^{\frac{2L+1}{3}}} \left|T_{2M-\frac{2L-4}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&+ \frac{2}{3\pi^2}\left|d_{2L}\right| \frac{\Gamma\left(\frac{2L+1}{3}\right)\Gamma\left(2M-\frac{2L-4}{3}\right)}{(2\pi)^{2M-\frac{2L-4}{3}}\,|\nu|^{2M+\frac{5}{3}}}
\end{aligned} \tag{3.164}$$

provided that $|\theta| \le \frac{\pi}{2}$, and

$$\left|R_{N,M,K,L}^{(H')}(\nu)\right| \le \frac{1}{3^{\frac{1}{2}}}\left|\mathrm{e}^{2\pi\mathrm{i}\nu}\right| \frac{2}{3\pi}\left|g_{2K}\right| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2K+2}{3}\right)}{|\nu|^{\frac{2K+2}{3}}} \left|T_{2N-\frac{2K}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|$$

$$
\begin{aligned}
&+ \frac{1}{3^{\frac{1}{2}}} \left| \mathrm{e}^{-2\pi \mathrm{i}\nu} \right| \frac{2}{3\pi} \left| g_{2K} \right| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2K+2}{3}\right)}{\left|\nu\right|^{\frac{2K+2}{3}}} \left| T_{2N-\frac{2K}{3}}\left(2\pi\nu \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \right| \\
&+ \frac{2}{3\pi^2} \left| g_{2K} \right| \frac{\Gamma\left(\frac{2K+2}{3}\right)\Gamma\left(2N-\frac{2K}{3}\right)}{\left(2\pi\right)^{2N-\frac{2K}{3}} \left|\nu\right|^{2N+\frac{2}{3}}} \\
&+ \frac{1}{3^{\frac{1}{2}}} \left| \mathrm{e}^{2\pi \mathrm{i}\nu} \right| \frac{2}{3\pi} \left| g_{2L} \right| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2L+2}{3}\right)}{\left|\nu\right|^{\frac{2L+2}{3}}} \left| T_{2M-\frac{2L-2}{3}}\left(2\pi\nu \mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \right| \\
&+ \frac{1}{3^{\frac{1}{2}}} \left| \mathrm{e}^{-2\pi \mathrm{i}\nu} \right| \frac{2}{3\pi} \left| g_{2L} \right| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2L+2}{3}\right)}{\left|\nu\right|^{\frac{2L+2}{3}}} \left| T_{2M-\frac{2L-2}{3}}\left(2\pi\nu \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \right| \\
&+ \frac{2}{3\pi^2} \left| g_{2L} \right| \frac{\Gamma\left(\frac{2L+2}{3}\right)\Gamma\left(2M-\frac{2L-2}{3}\right)}{\left(2\pi\right)^{2M-\frac{2L-2}{3}} \left|\nu\right|^{2M+\frac{4}{3}}}
\end{aligned}
$$

provided that $|\theta| \leq \frac{\pi}{2}$ and $K, L \geq 3$. In the case when $K = L$, these bounds are also valid in the range $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$ with $\overline{\nu \mathrm{e}^{-\pi \mathrm{i}}}$ in place of ν on the right-hand sides.

We may derive the corresponding results for the functions $J_\nu(\nu)$, $J'_\nu(\nu)$, $Y_\nu(\nu)$ and $Y'_\nu(\nu)$ by substituting the expressions (3.158) and (3.160) into the right-hand sides of the functional relations

$$
\begin{aligned}
&2J_\nu(\nu) = H_\nu^{(1)}(\nu) + \overline{H_{\bar{\nu}}^{(1)}(\bar{\nu})},\ 2J'_\nu(\nu) = H_\nu^{(1)\prime}(\nu) + \overline{H_{\bar{\nu}}^{(1)\prime}(\bar{\nu})} \\
&2\mathrm{i}Y_\nu(\nu) = H_\nu^{(1)}(\nu) - \overline{H_{\bar{\nu}}^{(1)}(\bar{\nu})},\ 2\mathrm{i}Y'_\nu(\nu) = H_\nu^{(1)\prime}(\nu) - \overline{H_{\bar{\nu}}^{(1)\prime}(\bar{\nu})}
\end{aligned}
$$

(cf. [96, eqs. 10.4.4 and 10.11.9, pp. 222 and 226]). We thus arrive at

$$
\begin{aligned}
J_\nu(\nu) &= \frac{1}{3^{\frac{1}{3}} 2\pi} \sum_{n=0}^{N-1} d_{6n} \frac{\Gamma\left(2n+\frac{1}{3}\right)}{\nu^{2n+\frac{1}{3}}} - \frac{1}{3^{\frac{1}{3}} 2\pi} \sum_{m=0}^{M-1} d_{6m+4} \frac{\Gamma\left(2m+\frac{5}{3}\right)}{\nu^{2m+\frac{5}{3}}} + R_{N,M}^{(J)}(\nu), \\
J'_\nu(\nu) &= \frac{1}{3^{\frac{1}{3}} 2\pi} \sum_{n=0}^{N-1} g_{6n} \frac{\Gamma\left(2n+\frac{2}{3}\right)}{\nu^{2n+\frac{2}{3}}} - \frac{1}{3^{\frac{1}{3}} 2\pi} \sum_{m=0}^{M-1} g_{6m+2} \frac{\Gamma\left(2m+\frac{4}{3}\right)}{\nu^{2m+\frac{4}{3}}} + R_{N,M}^{(J')}(\nu), \\
Y_\nu(\nu) &= -\frac{1}{2\pi} \sum_{n=0}^{N-1} d_{6n} \frac{\Gamma\left(2n+\frac{1}{3}\right)}{\nu^{2n+\frac{1}{3}}} - \frac{1}{2\pi} \sum_{m=0}^{M-1} d_{6m+4} \frac{\Gamma\left(2m+\frac{5}{3}\right)}{\nu^{2m+\frac{5}{3}}} + R_{N,M}^{(Y)}(\nu)
\end{aligned}
$$

and

$$
Y'_\nu(\nu) = \frac{1}{2\pi} \sum_{n=0}^{N-1} g_{6n} \frac{\Gamma\left(2n+\frac{2}{3}\right)}{\nu^{2n+\frac{2}{3}}} + \frac{1}{2\pi} \sum_{m=0}^{M-1} g_{6m+2} \frac{\Gamma\left(2m+\frac{4}{3}\right)}{\nu^{2m+\frac{4}{3}}} + R_{N,M}^{(Y')}(\nu)
$$

(cf. [96, eq. 10.19.8, p. 232]), where the remainder terms are given by

$$
\begin{aligned}
2R_{N,M}^{(J)}(\nu) &= R_{N,M}^{(H)}(\nu) - \overline{R_{N,M}^{(H)}(\bar{\nu})},\\
2R_{N,M}^{(J')}(\nu) &= R_{N,M}^{(H')}(\nu) + \overline{R_{N,M}^{(H')}(\bar{\nu})},\\
2\mathrm{i}R_{N,M}^{(Y)}(\nu) &= R_{N,M}^{(H)}(\nu) + \overline{R_{N,M}^{(H)}(\bar{\nu})},\\
2\mathrm{i}R_{N,M}^{(Y')}(\nu) &= R_{N,M}^{(H')}(\nu) - \overline{R_{N,M}^{(H')}(\bar{\nu})}.
\end{aligned}
\tag{3.165}
$$

Now, a direct application of Proposition 3.2.1 to the right-hand sides yields the desired re-expansions which are summarized in the following proposition.

Proposition 3.2.3. *Let K and L be arbitrary fixed non-negative integers. Suppose that $|\theta| \le 2\pi - \delta$ with an arbitrary fixed small positive δ, $|\nu|$ is large and $N = \pi|\nu| + \rho$, $M = \pi|\nu| + \sigma$ with ρ and σ being bounded. Then*

$$
\begin{aligned}
R_{N,M}^{(J)}(\nu) = &-\frac{\mathrm{i}e^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}2}e^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{K-1} d_{2k}e^{\frac{2\pi(2k+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2k+1)}{3}\right)\frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu e^{\frac{\pi}{2}\mathrm{i}}\right)\\
&+\frac{\mathrm{i}}{3^{\frac{1}{2}}2}e^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{K-1} d_{2k}\sin\left(\frac{\pi(2k+1)}{3}\right)\frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu e^{-\frac{\pi}{2}\mathrm{i}}\right)\\
&+\frac{\mathrm{i}e^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}2}e^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} d_{2\ell}e^{\frac{2\pi(2\ell+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2\ell+1)}{3}\right)\frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}}T_{2M-\frac{2\ell-4}{3}}\left(2\pi\nu e^{\frac{\pi}{2}\mathrm{i}}\right)\\
&-\frac{\mathrm{i}}{3^{\frac{1}{2}}2}e^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} d_{2\ell}\sin\left(\frac{\pi(2\ell+1)}{3}\right)\frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}}T_{2M-\frac{2\ell-4}{3}}\left(2\pi\nu e^{-\frac{\pi}{2}\mathrm{i}}\right)\\
&+R_{N,M,K,L}^{(J)}(\nu),
\end{aligned}
\tag{3.166}
$$

$$
\begin{aligned}
R_{N,M}^{(J')}(\nu) = &\frac{\mathrm{i}e^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}2}e^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{K-1} g_{2k}e^{\frac{2\pi(2k+2)}{3}\mathrm{i}}\sin\left(\frac{\pi(2k+2)}{3}\right)\frac{\Gamma\left(\frac{2k+2}{3}\right)}{\nu^{\frac{2k+2}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu e^{\frac{\pi}{2}\mathrm{i}}\right)\\
&-\frac{\mathrm{i}}{3^{\frac{1}{2}}2}e^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{K-1} g_{2k}\sin\left(\frac{\pi(2k+2)}{3}\right)\frac{\Gamma\left(\frac{2k+2}{3}\right)}{\nu^{\frac{2k+2}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu e^{-\frac{\pi}{2}\mathrm{i}}\right)\\
&-\frac{\mathrm{i}e^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}2}e^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} g_{2\ell}e^{\frac{2\pi(2\ell+2)}{3}\mathrm{i}}\sin\left(\frac{\pi(2\ell+2)}{3}\right)\frac{\Gamma\left(\frac{2\ell+2}{3}\right)}{\nu^{\frac{2\ell+2}{3}}}T_{2M-\frac{2\ell-2}{3}}\left(2\pi\nu e^{\frac{\pi}{2}\mathrm{i}}\right)\\
&+\frac{\mathrm{i}}{3^{\frac{1}{2}}2}e^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} g_{2\ell}\sin\left(\frac{\pi(2\ell+2)}{3}\right)\frac{\Gamma\left(\frac{2\ell+2}{3}\right)}{\nu^{\frac{2\ell+2}{3}}}T_{2M-\frac{2\ell-2}{3}}\left(2\pi\nu e^{-\frac{\pi}{2}\mathrm{i}}\right)\\
&+R_{N,M,K,L}^{(J')}(\nu),
\end{aligned}
\tag{3.167}
$$

$$
\begin{aligned}
R_{N,M}^{(Y)}(\nu) &= \frac{\mathrm{i}\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{2}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{K-1} d_{2k}\mathrm{e}^{\frac{2\pi(2k+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2k+1)}{3}\right)\frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\
&- \frac{\mathrm{i}}{2}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{K-1} d_{2k}\sin\left(\frac{\pi(2k+1)}{3}\right)\frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
&+ \frac{\mathrm{i}\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{2}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} d_{2\ell}\mathrm{e}^{\frac{2\pi(2\ell+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2\ell+1)}{3}\right)\frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}}T_{2M-\frac{2\ell-4}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\
&- \frac{\mathrm{i}}{2}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} d_{2\ell}\sin\left(\frac{\pi(2\ell+1)}{3}\right)\frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}}T_{2M-\frac{2\ell-4}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
&+ R_{N,M,K,L}^{(Y)}(\nu)
\end{aligned}
\tag{3.168}
$$

and

$$
\begin{aligned}
R_{N,M}^{(Y')}(\nu) &= \frac{\mathrm{i}\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{2}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{K-1} g_{2k}\mathrm{e}^{\frac{2\pi(2k+2)}{3}\mathrm{i}}\sin\left(\frac{\pi(2k+2)}{3}\right)\frac{\Gamma\left(\frac{2k+2}{3}\right)}{\nu^{\frac{2k+2}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\
&- \frac{\mathrm{i}}{2}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=0}^{K-1} g_{2k}\sin\left(\frac{\pi(2k+2)}{3}\right)\frac{\Gamma\left(\frac{2k+2}{3}\right)}{\nu^{\frac{2k+2}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
&+ \frac{\mathrm{i}\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{2}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} g_{2\ell}\mathrm{e}^{\frac{2\pi(2\ell+2)}{3}\mathrm{i}}\sin\left(\frac{\pi(2\ell+2)}{3}\right)\frac{\Gamma\left(\frac{2\ell+2}{3}\right)}{\nu^{\frac{2\ell+2}{3}}}T_{2M-\frac{2\ell-2}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\
&- \frac{\mathrm{i}}{2}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} g_{2\ell}\sin\left(\frac{\pi(2\ell+2)}{3}\right)\frac{\Gamma\left(\frac{2\ell+2}{3}\right)}{\nu^{\frac{2\ell+2}{3}}}T_{2M-\frac{2\ell-2}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
&+ R_{N,M,K,L}^{(Y')}(\nu),
\end{aligned}
\tag{3.169}
$$

where

$$
R_{N,M,K,L}^{(J)}(\nu), R_{N,M,K,L}^{(Y)}(\nu) = \mathcal{O}_{K,\rho}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2K+1}{3}}}\right) + \mathcal{O}_{L,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2L+1}{3}}}\right),
$$

$$
R_{N,M,K,L}^{(J')}(\nu), R_{N,M,K,L}^{(Y')}(\nu) = \mathcal{O}_{K,\rho}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2K+2}{3}}}\right) + \mathcal{O}_{L,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2L+2}{3}}}\right)
$$

for $|\theta| \leq \frac{\pi}{2}$, *and*

$$
R_{N,M,K,L}^{(J)}(\nu), R_{N,M,K,L}^{(Y)}(\nu) = \mathcal{O}_{K,\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi\mathfrak{Im}(\nu)}}{|\nu|^{\frac{2K+1}{3}}}\right) + \mathcal{O}_{L,\sigma,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi\mathfrak{Im}(\nu)}}{|\nu|^{\frac{2L+1}{3}}}\right),
$$

$$R_{N,M,K,L}^{(J')}(\nu), R_{N,M,K,L}^{(Y')}(\nu) = \mathcal{O}_{K,\rho,\delta}\left(\frac{e^{\mp 2\pi \mathfrak{Im}(\nu)}}{|\nu|^{\frac{2K+2}{3}}}\right) + \mathcal{O}_{L,\sigma,\delta}\left(\frac{e^{\mp 2\pi \mathfrak{Im}(\nu)}}{|\nu|^{\frac{2L+2}{3}}}\right)$$

for $\frac{\pi}{2} \le \pm\theta \le 2\pi - \delta$.

Explicit bounds for $R_{N,M,K,L}^{(J)}(\nu)$, $R_{N,M,K,L}^{(J')}(\nu)$, $R_{N,M,K,L}^{(Y)}(\nu)$ and $R_{N,M,K,L}^{(Y')}(\nu)$ can be derived using Theorem 3.2.2 together with the inequalities

$$2\left|R_{N,M,K,L}^{(J)}(\nu)\right|, 2\left|R_{N,M,K,L}^{(Y)}(\nu)\right| \le \left|R_{N,M,K,L}^{(H)}(\nu)\right| + \left|R_{N,M,K,L}^{(H)}(\bar{\nu})\right|$$

and

$$2\left|R_{N,M,K,L}^{(J')}(\nu)\right|, 2\left|R_{N,M,K,L}^{(Y')}(\nu)\right| \le \left|R_{N,M,K,L}^{(H')}(\nu)\right| + \left|R_{N,M,K,L}^{(H')}(\bar{\nu})\right|,$$

which can be established readily from the expressions (3.165).

If we neglect the remainder terms in (3.166)–(3.169), and we formally extend the sums to infinity, formulae (3.166)–(3.169) reproduce Dingle's original expansions mentioned at the beginning of this subsection.

Proof of Proposition 3.2.1 and Theorem 3.2.2. We only prove the statements for $R_{N,M}^{(H)}(\nu)$ and $R_{N,M,K,L}^{(H)}(\nu)$; the remainders $R_{N,M}^{(H')}(\nu)$ and $R_{N,M,K,L}^{(H')}(\nu)$ can be handled similarly. Let N, M, K and L be arbitrary fixed non-negative integers such that $K < 3N$ and $L < 3M + 2$. Suppose further that $K, L \equiv 0 \mod 3$ and $|\theta| < \frac{\pi}{2}$. We begin by replacing the function $\mathrm{i}H_{\mathrm{i}u}^{(1)}(u e^{\frac{\pi}{2}\mathrm{i}})$ in (3.159) by its truncated asymptotic expansion (3.146) (with k and K in place of m and M in the first integral, and with ℓ and L in place of m and M in the second integral) and using the definition of the terminant function, in order to obtain

$$\begin{aligned}
R_{N,M}^{(H)}(\nu) = &-\frac{\mathrm{i}}{3^{\frac{1}{2}}} e^{2\pi \mathrm{i}\nu} \frac{2}{3\pi} \sum_{k=0}^{K-1} d_{2k} e^{\frac{2\pi(2k+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(2k+1)}{3}\right) \frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}} T_{2N-\frac{2k}{3}}\left(2\pi\nu e^{\frac{\pi}{2}\mathrm{i}}\right) \\
&+ \frac{\mathrm{i}e^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}} e^{-2\pi \mathrm{i}\nu} \frac{2}{3\pi} \sum_{k=0}^{K-1} d_{2k} \sin\left(\frac{\pi(2k+1)}{3}\right) \frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}} T_{2N-\frac{2k}{3}}\left(2\pi\nu e^{-\frac{\pi}{2}\mathrm{i}}\right) \\
&+ \frac{\mathrm{i}}{3^{\frac{1}{2}}} e^{2\pi \mathrm{i}\nu} \frac{2}{3\pi} \sum_{\ell=0}^{L-1} d_{2\ell} e^{\frac{2\pi(2\ell+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(2\ell+1)}{3}\right) \frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}} T_{2M-\frac{2\ell-4}{3}}\left(2\pi\nu e^{\frac{\pi}{2}\mathrm{i}}\right) \\
&- \frac{\mathrm{i}e^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}} e^{-2\pi \mathrm{i}\nu} \frac{2}{3\pi} \sum_{\ell=0}^{L-1} d_{2\ell} \sin\left(\frac{\pi(2\ell+1)}{3}\right) \frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}} T_{2M-\frac{2\ell-4}{3}}\left(2\pi\nu e^{-\frac{\pi}{2}\mathrm{i}}\right) \\
&+ R_{N,M,K,L}^{(H)}(\nu),
\end{aligned} \tag{3.170}$$

with

$$\begin{aligned}
R_{N,M,K,L}^{(H)}(\nu) &= (-1)^N \frac{e^{-\frac{\pi}{3}i}}{3^{\frac{1}{2}}\pi} \frac{1}{\nu^{2N+\frac{1}{3}}} \int_0^{+\infty} \frac{u^{2N-\frac{2}{3}} e^{-2\pi u}}{1+(u/\nu)^2} iR_K^{(H)}\left(u e^{\frac{\pi}{2}i}\right) du \\
&\quad + (-1)^{M+1} \frac{e^{\frac{\pi}{3}i}}{3^{\frac{1}{2}}\pi} \frac{1}{\nu^{2M+\frac{5}{3}}} \int_0^{+\infty} \frac{u^{2M+\frac{2}{3}} e^{-2\pi u}}{1+(u/\nu)^2} iR_L^{(H)}\left(u e^{\frac{\pi}{2}i}\right) du \\
&= (-1)^N \frac{e^{-\frac{\pi}{3}i}}{3^{\frac{1}{2}}2\pi} e^{-i\left(2N+\frac{1}{3}\right)\theta} \int_0^{+\infty} \frac{\tau^{2N-\frac{2}{3}} e^{-2\pi r\tau}}{1-i\tau e^{-i\theta}} iR_K^{(H)}\left(r\tau e^{\frac{\pi}{2}i}\right) d\tau \\
&\quad + (-1)^N \frac{e^{-\frac{\pi}{3}i}}{3^{\frac{1}{2}}2\pi} e^{-i\left(2N+\frac{1}{3}\right)\theta} \int_0^{+\infty} \frac{\tau^{2N-\frac{2}{3}} e^{-2\pi r\tau}}{1+i\tau e^{-i\theta}} iR_K^{(H)}\left(r\tau e^{\frac{\pi}{2}i}\right) d\tau \\
&\quad + (-1)^{M+1} \frac{e^{\frac{\pi}{3}i}}{3^{\frac{1}{2}}2\pi} e^{-i\left(2M+\frac{5}{3}\right)\theta} \int_0^{+\infty} \frac{\tau^{2M+\frac{2}{3}} e^{-2\pi r\tau}}{1-i\tau e^{-i\theta}} iR_L^{(H)}\left(r\tau e^{\frac{\pi}{2}i}\right) d\tau \\
&\quad + (-1)^{M+1} \frac{e^{\frac{\pi}{3}i}}{3^{\frac{1}{2}}2\pi} e^{-i\left(2M+\frac{5}{3}\right)\theta} \int_0^{+\infty} \frac{\tau^{2M+\frac{2}{3}} e^{-2\pi r\tau}}{1+i\tau e^{-i\theta}} iR_L^{(H)}\left(r\tau e^{\frac{\pi}{2}i}\right) d\tau.
\end{aligned} \tag{3.171}$$

In passing to the second equality, we have taken $\nu = re^{i\theta}$ and have made the change of integration variable from u to τ by $u = r\tau$. Let us consider the estimation of the first integral after the second equality in (3.171) which involves $R_K^{(H)}\left(r\tau e^{\frac{\pi}{2}i}\right)$. The remainder term $R_K^{(H)}\left(r\tau e^{\frac{\pi}{2}i}\right)$ is given by the integral representation (3.73), which can be re-expressed in the form

$$\begin{aligned}
R_K^{(H)}\left(r\tau e^{\frac{\pi}{2}i}\right) &= \frac{1}{3^{\frac{1}{2}}\pi} \frac{1}{(r\tau)^{\frac{2K+1}{3}}} \int_0^{+\infty} t^{\frac{2K-2}{3}} e^{-2\pi t} \frac{1-(t/r)^{\frac{4}{3}}}{1-(t/r)^2} H_{it}^{(1)}\left(t e^{\frac{\pi}{2}i}\right) dt \\
&\quad + \frac{1}{3^{\frac{1}{2}}\pi} \frac{\tau-1}{(r\tau)^{\frac{2K+1}{3}}} \int_0^{+\infty} t^{\frac{2K-2}{3}} e^{-2\pi t} \lambda(r,\tau,t) H_{it}^{(1)}\left(t e^{\frac{\pi}{2}i}\right) dt,
\end{aligned}$$

where the (real) quantity $\lambda(r,\tau,t)$ satisfies $|\lambda(r,\tau,t)| \leq 2$ (the proof of this technical claim can be found in [73, Appendix B]). Noting that

$$0 < \frac{1-(t/r)^{\frac{4}{3}}}{1-(t/r)^2} < 1$$

for positive r and t, substitution into (3.171) yields the upper bound

$$\begin{aligned}
&\left| (-1)^N \frac{e^{-\frac{\pi}{3}i}}{3^{\frac{1}{3}}2\pi} e^{-i\left(2N+\frac{1}{3}\right)\theta} \int_0^{+\infty} \frac{\tau^{2N-\frac{2}{3}} e^{-2\pi r\tau}}{1-i\tau e^{-i\theta}} iR_K^{(H)}\left(r\tau e^{\frac{\pi}{2}i}\right) d\tau \right| \leq \\
&\qquad \leq \frac{1}{3\pi} |d_{2K}| \frac{\Gamma\left(\frac{2K+1}{3}\right)}{|\nu|^{\frac{2K+1}{3}}} \frac{1}{2\pi} \left| \int_0^{+\infty} \frac{\tau^{2N-\frac{2K}{3}-1} e^{-2\pi r\tau}}{1-i\tau e^{-i\theta}} d\tau \right|
\end{aligned}$$

$$+\frac{1}{3\pi^2}\left|d_{2K}\right|\frac{\Gamma\left(\frac{2K+1}{3}\right)}{|\nu|^{\frac{2K+1}{3}}}\int_0^{+\infty}\tau^{2N-\frac{2K}{3}-1}\mathrm{e}^{-2\pi r\tau}\left|\frac{\tau-1}{\tau+\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}\right|\mathrm{d}\tau.$$

In arriving at this inequality, we have made use of the representation (3.96) of the coefficients d_{2K} and the fact that $\mathrm{i}H_{\mathrm{i}t}^{(1)}(t\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}) > 0$ for any $t > 0$ (see, for instance, [73]). Since $|(\tau-1)/(\tau+\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})| \leq 1$ for positive τ, after simplification we establish

$$\left|(-1)^N\frac{\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{3}}2\pi}\mathrm{e}^{-\mathrm{i}\left(2N+\frac{1}{3}\right)\theta}\int_0^{+\infty}\frac{\tau^{2N-\frac{2}{3}}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\mathrm{i}R_K^{(H)}(r\tau\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}\tau\right| \leq$$
$$\leq \frac{1}{3^{\frac{1}{2}}}\left|\mathrm{e}^{2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}\left|d_{2K}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2K+1}{3}\right)}{|\nu|^{\frac{2K+1}{3}}}\left|T_{2N-\frac{2K}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|$$
$$+\frac{1}{3\pi^2}\left|d_{2K}\right|\frac{\Gamma\left(\frac{2K+1}{3}\right)\Gamma\left(2N-\frac{2K}{3}\right)}{(2\pi)^{2N-\frac{2K}{3}}|\nu|^{2N+\frac{1}{3}}}.$$

We can estimate the other terms in (3.171) similarly, and we thus find

$$\begin{aligned}\left|R_{N,M,K,L}^{(H)}(\nu)\right| \leq{}& \frac{1}{3^{\frac{1}{2}}}\left|\mathrm{e}^{2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}\left|d_{2K}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2K+1}{3}\right)}{|\nu|^{\frac{2K+1}{3}}}\left|T_{2N-\frac{2K}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|\\
&+\frac{1}{3^{\frac{1}{2}}}\left|\mathrm{e}^{-2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}\left|d_{2K}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2K+1}{3}\right)}{|\nu|^{\frac{2K+1}{3}}}\left|T_{2N-\frac{2K}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|\\
&+\frac{2}{3\pi^2}\left|d_{2K}\right|\frac{\Gamma\left(\frac{2K+1}{3}\right)\Gamma\left(2N-\frac{2K}{3}\right)}{(2\pi)^{2N-\frac{2K}{3}}|\nu|^{2N+\frac{1}{3}}}\\
&+\frac{1}{3^{\frac{1}{2}}}\left|\mathrm{e}^{2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}\left|d_{2L}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2L+1}{3}\right)}{|\nu|^{\frac{2L+1}{3}}}\left|T_{2M-\frac{2L-4}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|\\
&+\frac{1}{3^{\frac{1}{2}}}\left|\mathrm{e}^{-2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}\left|d_{2L}\right|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2L+1}{3}\right)}{|\nu|^{\frac{2L+1}{3}}}\left|T_{2M-\frac{2L-4}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|\\
&+\frac{2}{3\pi^2}\left|d_{2L}\right|\frac{\Gamma\left(\frac{2L+1}{3}\right)\Gamma\left(2M-\frac{2L-4}{3}\right)}{(2\pi)^{2M-\frac{2L-4}{3}}|\nu|^{2M+\frac{5}{3}}}.\end{aligned}\tag{3.172}$$

By a continuity argument, this bound holds in the closed sector $|\theta| \leq \frac{\pi}{2}$. This proves Theorem 3.2.2 for $R_{N,M,K,L}^{(H)}(\nu)$.

From now on, we suppose that $|\nu|$ is large and that $N = 2\pi|\nu| + \rho$, $M = 2\pi|\nu| + \sigma$ with ρ and σ being bounded. Using these assumptions and Olver's estimate (1.90), the first two terms on the right-hand side of the inequality (3.172) are $\mathcal{O}_{K,\rho}(|\nu|^{-\frac{2K+1}{3}}\mathrm{e}^{-2\pi|\nu|})$. Similarly, the fourth and fifth terms are found to be $\mathcal{O}_{L,\sigma}(|\nu|^{-\frac{2L+1}{3}}\mathrm{e}^{-2\pi|\nu|})$. By employing Stirling's formula, the third and sixth terms are $\mathcal{O}_{K,\rho}(|\nu|^{-\frac{2K+1}{3}-\frac{1}{2}}\mathrm{e}^{-2\pi|\nu|})$ and $\mathcal{O}_{L,\sigma}(|\nu|^{-\frac{2L+1}{3}-\frac{1}{2}}\mathrm{e}^{-2\pi|\nu|})$, respectively. This establishes the estimate (3.162) for $R_{N,M,K,L}^{(H)}(\nu)$.

Consider now the sector $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. For such values of θ, the function $R_{N,M,K,L}^{(H)}(\nu)$ can be defined via (3.170). When ν enters this sector, the poles of the integrands in (3.171) cross the integration path. According to the residue theorem, we have

$$\begin{aligned} R_{N,M,K,L}^{(H)}(\nu) = {} & (-1)^N \frac{\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{2N+\frac{1}{3}}}\int_0^{+\infty}\frac{u^{2N-\frac{2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}R_K^{(H)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}u \\ & +(-1)^{M+1}\frac{\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}\pi}\frac{1}{\nu^{2M+\frac{5}{3}}}\int_0^{+\infty}\frac{u^{2M+\frac{2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}R_L^{(H)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}u \\ & +\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}(R_K^{(H)}(\nu)-R_L^{(H)}(\nu)) \\ = {} & -\overline{R_{N,M,K,L}^{(H)}(\overline{\nu\mathrm{e}^{-\pi\mathrm{i}}})}+\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}(R_K^{(H)}(\nu)-R_L^{(H)}(\nu)) \end{aligned} \tag{3.173}$$

for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Now, by continuity,

$$R_{N,M,K,L}^{(H)}(\nu) = -\overline{R_{N,M,K,L}^{(H)}(\overline{\nu\mathrm{e}^{-\pi\mathrm{i}}})}+\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}(R_K^{(H)}(\nu)-R_L^{(H)}(\nu)) \tag{3.174}$$

is true in the closed sector $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. (Due to the fact that the function on the right-hand side of (3.174) is not analytic in ν, a larger sector of validity can not be guaranteed.) It is readily seen that if $K = L$, the estimate (3.162) remains true in the above closed sector as well and that (3.164) holds in the range $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$ with $\nu\mathrm{e}^{-\pi\mathrm{i}}$ in place of ν on the right-hand sides. In the case $K \neq L$, first note that the asymptotic expansion (3.62) implies that $R_K^{(H)}(\nu) = \mathcal{O}_K(|\nu|^{-\frac{2K+1}{3}})$ and $R_L^{(H)}(\nu) = \mathcal{O}_L(|\nu|^{-\frac{2L+1}{3}})$ as $\nu \to \infty$ in $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. From (3.162), we infer $R_{N,M,K,L}^{(H)}(\overline{\nu\mathrm{e}^{-\pi\mathrm{i}}}) = \mathcal{O}_{K,\rho}(|\nu|^{-\frac{2K+1}{3}}\mathrm{e}^{-2\pi|\nu|}) + \mathcal{O}_{L,\sigma}(|\nu|^{-\frac{2L+1}{3}}\mathrm{e}^{-2\pi|\nu|})$ for large ν in the sector $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. Therefore the estimate (3.163) holds for $R_{N,M,K,L}^{(H)}(\nu)$ when $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. The proof for the conjugate sector $-\frac{3\pi}{2} \leq \theta \leq -\frac{\pi}{2}$ is completely analogous.

Let us now consider the sector $\frac{3\pi}{2} \leq \theta \leq 2\pi - \delta$. For such values of θ, we define $R_{N,M,K,L}^{(H)}(\nu)$ by (3.170) using analytic continuation. When ν enters this sector, the poles of the integrands in (3.173) cross the integration path. Thus, by the residue theorem, we obtain

$$\begin{aligned}
R_{N,M,K,L}^{(H)}(\nu) = (-1)^N \frac{\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}\pi} \frac{1}{\nu^{2N+\frac{1}{3}}} \int_0^{+\infty} \frac{u^{2N-\frac{2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2} \mathrm{i} R_K^{(H)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}u \\
+(-1)^{M+1} \frac{\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}\pi} \frac{1}{\nu^{2M+\frac{5}{3}}} \int_0^{+\infty} \frac{u^{2M+\frac{2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2} \mathrm{i} R_L^{(H)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}u \\
+\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}(R_K^{(H)}(\nu) - R_L^{(H)}(\nu)) \\
+\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu}(\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}} R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}) - \mathrm{e}^{\frac{\pi}{3}\mathrm{i}} R_L^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}})) \\
= - R_{N,M,K,L}^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}}) - \overline{R_{N,M,K,L}^{(H)}(\overline{\nu\mathrm{e}^{-2\pi\mathrm{i}}})} \\
+\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}(R_K^{(H)}(\nu) - R_L^{(H)}(\nu)) \\
+\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu}(\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}} R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}) - \mathrm{e}^{\frac{\pi}{3}\mathrm{i}} R_L^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}))
\end{aligned} \tag{3.175}$$

for $\frac{3\pi}{2} < \theta < \frac{5\pi}{2}$. Now, by a continuity argument,

$$\begin{aligned}
R_{N,M,K,L}^{(H)}(\nu) = - R_{N,M,K,L}^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}}) - \overline{R_{N,M,K,L}^{(H)}(\overline{\nu\mathrm{e}^{-2\pi\mathrm{i}}})} \\
+\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}(R_K^{(H)}(\nu) - R_L^{(H)}(\nu)) \\
+\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu}(\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}} R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}) - \mathrm{e}^{\frac{\pi}{3}\mathrm{i}} R_L^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}))
\end{aligned} \tag{3.176}$$

is true in the closed sector $\frac{3\pi}{2} \le \theta \le \frac{5\pi}{2}$. The asymptotic expansion (3.62) implies that $R_K^{(H)}(\nu) = \mathcal{O}_{K,\delta}(|\nu|^{-\frac{2K+1}{3}})$, $R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}) = \mathcal{O}_K(|\nu|^{-\frac{2K+1}{3}})$, $R_L^{(H)}(\nu) = \mathcal{O}_{L,\delta}(|\nu|^{-\frac{2L+1}{3}})$ and $R_L^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}) = \mathcal{O}_L(|\nu|^{-\frac{2L+1}{3}})$ as $\nu \to \infty$ in the sector $\frac{3\pi}{2} \le \theta \le 2\pi - \delta$. From (3.162), we infer that $R_{N,M,K,L}^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}}), R_{N,M,K,L}^{(H)}(\overline{\nu\mathrm{e}^{-2\pi\mathrm{i}}}) = \mathcal{O}_{K,\rho}(|\nu|^{-\frac{2K+1}{3}}\mathrm{e}^{-2\pi|\nu|}) + \mathcal{O}_{L,\sigma}(|\nu|^{-\frac{2L+1}{3}}\mathrm{e}^{-2\pi|\nu|})$ for large ν in $\frac{3\pi}{2} \le \theta \le 2\pi - \delta$. Thus, the estimate (3.163) holds for $R_{N,M,K,L}^{(H)}(\nu)$ when $\frac{3\pi}{2} \le \theta \le 2\pi - \delta$. The proof of the analogous estimate for the sector $-2\pi + \delta \le \theta \le -\frac{3\pi}{2}$ is similar.

Finally, consider the case when $\frac{3\pi}{2} \le \theta \le 3\pi - \delta$ and $K = L$. If $K = L$, (3.176) can be simplified to

$$R_{N,M,K,K}^{(H)}(\nu) = -R_{N,M,K,K}^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}}) - \overline{R_{N,M,K,K}^{(H)}(\overline{\nu\mathrm{e}^{-2\pi\mathrm{i}}})} + \mathrm{e}^{-2\pi\mathrm{i}\nu} R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}})$$

for all non-zero values of ν in the closed sector $\frac{3\pi}{2} \le \theta \le \frac{5\pi}{2}$. The asymptotic expansion (3.62) implies that $R_K^{(H)}(\nu \mathrm{e}^{-\pi\mathrm{i}}) = \mathcal{O}_K(|\nu|^{-\frac{2K+1}{3}})$ as $\nu \to \infty$ in $\frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{2}$. From (3.162), we infer $R_{N,M,K,K}^{(H)}(\nu \mathrm{e}^{-2\pi\mathrm{i}}), \overline{R_{N,M,K,K}^{(H)}(\overline{\nu \mathrm{e}^{-2\pi\mathrm{i}}})} = \mathcal{O}_{K,\rho,\sigma}(|\nu|^{-\frac{2K+1}{3}} \mathrm{e}^{-2\pi|\nu|})$ for large ν in the sector $\frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{2}$. Therefore, the estimate (3.163) holds for $R_{N,M,K,K}^{(H)}(\nu)$ when $\frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{2}$ with all the lower signs taken. If $\frac{5\pi}{2} \leq \theta \leq 3\pi - \delta$, we proceed as follows. For such values of θ, the function $R_{N,M,K,L}^{(H)}(\nu)$ can be defined via (3.170). When ν enters this sector, the poles of the integrands in (3.175) cross the integration path. Therefore, according to the residue theorem, we have

$$\begin{aligned}
R_{N,M,K,K}^{(H)}(\nu) = {} & (-1)^N \frac{\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}\pi} \frac{1}{\nu^{2N+\frac{1}{3}}} \int_0^{+\infty} \frac{u^{2N-\frac{2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2} \mathrm{i} R_K^{(H)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}u \\
& + (-1)^{M+1} \frac{\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}\pi} \frac{1}{\nu^{2M+\frac{5}{3}}} \int_0^{+\infty} \frac{u^{2M+\frac{2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2} \mathrm{i} R_K^{(H)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}u \\
& + \mathrm{e}^{-2\pi\mathrm{i}\nu} R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}) - \mathrm{e}^{2\pi\mathrm{i}\nu} R_K^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}}) \\
= {} & -R_{N,M,K,K}^{(H)}(\nu\mathrm{e}^{-3\pi\mathrm{i}}) + \mathrm{e}^{-2\pi\mathrm{i}\nu} R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}) - \mathrm{e}^{2\pi\mathrm{i}\nu} R_K^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}})
\end{aligned}$$

for $\frac{5\pi}{2} < \theta < \frac{7\pi}{2}$. Now, by analytic continuation, the equality

$$R_{N,M,K,K}^{(H)}(\nu) = -R_{N,M,K,K}^{(H)}(\nu\mathrm{e}^{-3\pi\mathrm{i}}) + \mathrm{e}^{-2\pi\mathrm{i}\nu} R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}) - \mathrm{e}^{2\pi\mathrm{i}\nu} R_K^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}})$$

holds for any complex ν, in particular for those lying in the sector $\frac{5\pi}{2} \leq \theta \leq 3\pi - \delta$. The asymptotic expansion (3.62) implies $R_K^{(H)}(\nu\mathrm{e}^{-\pi\mathrm{i}}) = \mathcal{O}_{K,\delta}(|\nu|^{-\frac{2K+1}{3}})$ and $R_K^{(H)}(\nu\mathrm{e}^{-2\pi\mathrm{i}}) = \mathcal{O}_K(|\nu|^{-\frac{2K+1}{3}})$ for large ν in $\frac{5\pi}{2} \leq \theta \leq 3\pi - \delta$. From (3.162), we infer $R_{N,M,K,K}^{(H)}(\nu\mathrm{e}^{-3\pi\mathrm{i}}) = \mathcal{O}_{K,\rho,\sigma}(|\nu|^{-\frac{2K+1}{3}} \mathrm{e}^{-2\pi|\nu|})$ as $\nu \to \infty$ in the sector $\frac{5\pi}{2} \leq \theta \leq 3\pi - \delta$. Thus, the estimate (3.163) holds for $R_{N,M,K,K}^{(H)}(\nu)$ when $\frac{5\pi}{2} \leq \theta \leq 3\pi - \delta$ with all the lower signs taken.

It remains to prove the estimates for $R_{N,M,K,L}^{(H)}(\nu)$ when K and L may not be divisible by 3. For this purpose, we choose, for any fixed non-negative integers K and L, two integers K' and L' such that $K \leq K'$, $L \leq L'$ and $K', L' \equiv 0 \mod 3$. Then for any complex number ν, we have

$$\begin{aligned}
& R_{N,M,K,L}^{(H)}(\nu) = \\
& = -\frac{\mathrm{i}}{3^{\frac{1}{2}}} \mathrm{e}^{2\pi\mathrm{i}\nu} \frac{2}{3\pi} \sum_{k=K}^{K'-1} d_{2k} \mathrm{e}^{\frac{2\pi(2k+1)}{3}\mathrm{i}} \sin\left(\frac{\pi(2k+1)}{3}\right) \frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}} T_{2N-\frac{2k}{3}}(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}) \\
& \quad + \frac{\mathrm{i}\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}} \mathrm{e}^{-2\pi\mathrm{i}\nu} \frac{2}{3\pi} \sum_{k=K}^{K'-1} d_{2k} \sin\left(\frac{\pi(2k+1)}{3}\right) \frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}} T_{2N-\frac{2k}{3}}(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}})
\end{aligned}$$

$$+\frac{\mathrm{i}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=L}^{L'-1} d_{2\ell}\mathrm{e}^{\frac{2\pi(2\ell+1)}{3}\mathrm{i}}\sin\left(\frac{\pi\left(2\ell+1\right)}{3}\right)\frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}}T_{2M-\frac{2\ell-4}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$$
$$-\frac{\mathrm{i}\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=L}^{L'-1} d_{2\ell}\sin\left(\frac{\pi\left(2\ell+1\right)}{3}\right)\frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}}T_{2M-\frac{2\ell-4}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$$
$$+R^{(H)}_{N,M,K',L'}\left(\nu\right). \tag{3.177}$$

The summands on the right-hand side of this equality can be estimated by Olver's result (1.90). To estimate $R^{(H)}_{N,M,K',L'}\left(\nu\right)$, we can use (3.162) and (3.163), which have been already proved in the case that $K', L' \equiv 0 \mod 3$. We thus establish

$$\begin{aligned}R^{(H)}_{N,M,K,L}\left(\nu\right) &= \sum_{k=K}^{K'-1}\mathcal{O}_{k,\rho}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2k+1}{3}}}\right)+\sum_{\ell=L}^{L'-1}\mathcal{O}_{\ell,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2\ell+1}{3}}}\right)\\&\quad+\mathcal{O}_{K',\rho}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2K'+1}{3}}}\right)+\mathcal{O}_{L',\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2L'+1}{3}}}\right)\\&= \mathcal{O}_{K,\rho}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2K+1}{3}}}\right)+\mathcal{O}_{L,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2L+1}{3}}}\right)\end{aligned}$$

as $\nu \to \infty$ in the sector $|\theta| \leq \frac{\pi}{2}$, and

$$\begin{aligned}R^{(H)}_{N,M,K,L}\left(\nu\right) &= \sum_{k=K}^{K'-1}\mathcal{O}_{k,\rho}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathfrak{m}(\nu)}}{|\nu|^{\frac{2k+1}{3}}}\right)+\sum_{\ell=L}^{L'-1}\mathcal{O}_{\ell,\sigma}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathfrak{m}(\nu)}}{|\nu|^{\frac{2\ell+1}{3}}}\right)\\&\quad+\mathcal{O}_{K',\rho}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathfrak{m}(\nu)}}{|\nu|^{\frac{2K'+1}{3}}}\right)+\mathcal{O}_{L',\sigma}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathfrak{m}(\nu)}}{|\nu|^{\frac{2L'+1}{3}}}\right)\\&= \mathcal{O}_{K,\rho}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathfrak{m}(\nu)}}{|\nu|^{\frac{2K+1}{3}}}\right)+\mathcal{O}_{L,\sigma}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathfrak{m}(\nu)}}{|\nu|^{\frac{2L+1}{3}}}\right)\end{aligned}$$

as $\nu \to \infty$ in the sectors $\frac{\pi}{2} \leq \pm\theta \leq 2\pi - \delta$. In the case that $K = L$, we apply the functional equation $T_p\left(w\right) = 1 + \mathrm{e}^{-2\pi\mathrm{i}p}T_p(w\mathrm{e}^{-2\pi\mathrm{i}})$ (see, e.g., [99, eq. (6.2.45), p. 260]) to the terms of the first and third sums in (3.177) and find

$$R^{(H)}_{N,M,K,K}\left(\nu\right) =$$
$$= -\frac{\mathrm{i}\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=K}^{K'-1} d_{2k}\mathrm{e}^{\frac{\pi(2k+1)}{3}\mathrm{i}}\sin\left(\frac{\pi\left(2k+1\right)}{3}\right)\frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{3\pi}{2}\mathrm{i}}\right)$$

$$+\frac{\mathrm{i}\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=K}^{K'-1}d_{2k}\sin\left(\frac{\pi\left(2k+1\right)}{3}\right)\frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}}T_{2N-\frac{2k}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$$
$$+\frac{\mathrm{i}\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=K}^{K'-1}d_{2k}\mathrm{e}^{\frac{\pi(2k+1)}{3}\mathrm{i}}\sin\left(\frac{\pi\left(2k+1\right)}{3}\right)\frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}}T_{2M-\frac{2k-4}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{3\pi}{2}\mathrm{i}}\right)$$
$$-\frac{\mathrm{i}\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3^{\frac{1}{2}}}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{k=K}^{K'-1}d_{2k}\sin\left(\frac{\pi\left(2k+1\right)}{3}\right)\frac{\Gamma\left(\frac{2k+1}{3}\right)}{\nu^{\frac{2k+1}{3}}}T_{2M-\frac{2k-4}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$$
$$+R_{N,M,K',K'}^{(H)}\left(\nu\right).$$

Proceeding in a similar way as above, we find that

$$R_{N,M,K,K}^{(H)}\left(\nu\right)=\sum_{k=K}^{K'-1}\mathcal{O}_{k,\rho,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2k+1}{3}}}\right)+\mathcal{O}_{K',\rho,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2K'+1}{3}}}\right)=\mathcal{O}_{K,\rho,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2K+1}{3}}}\right)$$

as $\nu\to\infty$ in the sector $\frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}$, and

$$R_{N,M,K,K}^{(H)}\left(\nu\right)=\sum_{k=K}^{K'-1}\mathcal{O}_{k,\rho,\sigma}\left(\frac{\mathrm{e}^{2\pi\mathfrak{Im}(\nu)}}{|\nu|^{\frac{2k+1}{3}}}\right)+\mathcal{O}_{K',\rho,\sigma}\left(\frac{\mathrm{e}^{2\pi\mathfrak{Im}(\nu)}}{|\nu|^{\frac{2K'+1}{3}}}\right)=\mathcal{O}_{K,\rho,\sigma}\left(\frac{\mathrm{e}^{2\pi\mathfrak{Im}(\nu)}}{|\nu|^{\frac{2K+1}{3}}}\right)$$

as $\nu\to\infty$ in the sector $\frac{3\pi}{2}\leq\theta\leq 3\pi-\delta$. ■

3.3 Anger, Weber and Anger–Weber functions of large order and argument

We considered in Section 2.2 asymptotic expansions for the Anger function $\mathbf{J}_\nu\left(z\right)$, the Weber function $\mathbf{E}_\nu\left(z\right)$ and the Anger–Weber function $\mathbf{A}_\nu\left(z\right)$ which hold when $z\to\infty$ and $\nu=o(|z|)$. In this section, we discuss the cases when $\nu\to\infty$ and $z/\nu>1$ or $z/\nu<-1$ is fixed, i.e., both the order and the argument are large. For this purpose, it is convenient to study the functions $\mathbf{J}_{\pm\nu}\left(\nu\sec\beta\right)$, $\mathbf{E}_{\pm\nu}\left(\nu\sec\beta\right)$ and $\mathbf{A}_{\pm\nu}\left(\nu\sec\beta\right)$ with β an arbitrary (fixed) acute angle. The asymptotic power series of the functions $\mathbf{A}_{\pm\nu}\left(\nu\sec\beta\right)$ were first established by Watson [116, Sec. 10.15] in 1922.

In modern notation, Watson's asymptotic power series can be written

$$\mathbf{A}_{\pm\nu}\left(\nu\sec\beta\right)\sim\pm\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{a_n\left(\pm\sec\beta\right)\Gamma\left(2n+1\right)}{\nu^{2n+1}},\qquad(3.178)$$

as $\nu \to \infty$ in the sector $|\theta| \leq \pi - \delta$, with δ being an arbitrary small positive constant and $\theta = \arg \nu$ (cf. [96, eqs. 11.11.8 and 11.11.10, p. 298]). The related asymptotic expansions for the Anger function $\mathbf{J}_{\pm\nu}(\nu \sec \beta)$ and the Weber function $\mathbf{E}_{\pm\nu}(\nu \sec \beta)$ may be obtained by using their relation with $\mathbf{A}_{\pm\nu}(\nu \sec \beta)$, and they are as follows:

$$\mathbf{J}_{\pm\nu}(\nu \sec \beta) \sim J_{\pm\nu}(\nu \sec \beta) + \frac{\sin(\pi\nu)}{\pi} \sum_{n=0}^{\infty} \frac{a_n(\pm \sec \beta)\,\Gamma(2n+1)}{\nu^{2n+1}} \tag{3.179}$$

and

$$\begin{aligned}\mathbf{E}_{\pm\nu}(\nu \sec \beta) \sim -Y_{\pm\nu}(\nu \sec \beta) \mp \frac{\cos(\pi\nu)}{\pi} \sum_{n=0}^{\infty} \frac{a_n(\pm \sec \beta)\,\Gamma(2n+1)}{\nu^{2n+1}} \\ \pm \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{a_m(\mp \sec \beta)\,\Gamma(2m+1)}{\nu^{2m+1}},\end{aligned} \tag{3.180}$$

as $\nu \to \infty$ in the sector $|\theta| \leq \pi - \delta$. The coefficients $a_n(x)$ in these expansions are rational functions of x; some expressions for them will be given in Subsection 3.3.1 below. (The reader should not confuse the $a_n(x)$'s with the coefficients $a_n(\nu)$ introduced in Section 2.1.)

We note that the asymptotic expansions of $\mathbf{A}_{-\nu}(\nu \sec \beta)$, $\mathbf{J}_{-\nu}(\nu \sec \beta)$ and $\mathbf{E}_{\pm\nu}(\nu \sec \beta)$ satisfy Poincaré's definition only if the requirement $\beta^{-1} = o(|\nu|^{\frac{1}{3}})$ holds. And so, these asymptotic expansions cease to be true as $\beta \to 0+$. The case when $\beta = 0$ will be discussed in the next section. There exist other types of asymptotic expansions which are uniformly valid for all $\beta \geq 0$ (see, for instance, [95, Ch. 9, Secs. 12 and 13]), but these expansions involve non-elementary functions and therefore our methods are not suitable for their investigation. Surprisingly, however, the asymptotic expansions of $\mathbf{A}_{\nu}(\nu \sec \beta)$ and $\mathbf{J}_{\nu}(\nu \sec \beta)$ remain true if the quantity $\sec \beta$ is replaced by any positive real number. (For the resurgence properties of $\mathbf{A}_{\nu}(\nu\lambda)$ with $0 < \lambda \leq 1$, the interested reader is referred to the paper [72] of the present author.)

This section is organized as follows. In Subsection 3.3.1, we prove resurgence formulae for the Anger, Weber and Anger–Weber functions, and their derivatives, for large order and argument. In Subsection 3.3.2, we derive error bounds for the asymptotic expansions of these functions. Subsection 3.3.3 deals with the asymptotic behaviour of the corresponding late coefficients. Finally, in Subsection 3.3.4, we derive exponentially improved asymptotic expansions for the above mentioned functions.

3.3.1 The resurgence formulae

In this subsection, we investigate the resurgence properties of the Anger, Weber and Anger–Weber functions, and that of their derivatives, for large order and argument. A detailed study is needed only in the case of the functions $\mathbf{A}_{\pm\nu}(\nu\sec\beta)$ and $\mathbf{A}'_{\pm\nu}(\nu\sec\beta)$, as the analogous results for the other functions can be deduced in a simple way through their relations with them.

We begin by considering the function $\mathbf{A}_{\nu}(\nu\sec\beta)$. Let β be a fixed acute angle. We substitute $z = \nu\sec\beta$ into the integral representation (2.69) to obtain

$$\mathbf{A}_{\nu}(\nu\sec\beta) = \frac{1}{\pi}\int_0^{+\infty} \mathrm{e}^{-\nu(\sec\beta\sinh t + t)}\mathrm{d}t, \tag{3.181}$$

for $|\theta| < \frac{\pi}{2}$. The function $\sec\beta\sinh t + t$ has infinitely many first-order saddle points in the complex t-plane situated at $t^{(r,k)} = (-1)^r\,\mathrm{i}\beta + (2k+1)\,\pi\mathrm{i}$ with $r = 0, 1$ and $k \in \mathbb{Z}$. The path of steepest descent $\mathscr{P}^{(o)}(0)$ issuing from the origin coincides with the positive real axis, and its orientation is chosen so that it leads away from 0. Hence we may write

$$\mathbf{A}_{\nu}(\nu\sec\beta) = \frac{1}{\pi\nu}T^{(o)}(\nu),$$

where $T^{(o)}(\nu)$ is given in (1.3) with the specific choices of $f(t) = \sec\beta\sinh t + t$ and $g(t) = 1$. The problem is therefore one of linear dependence at the endpoint, which we considered in Subsection 1.1.1. To determine the domain $\Delta^{(o)}$ corresponding to this problem, we have to identify the adjacent saddles and contours. When $\theta = -\frac{\pi}{2}$, the path $\mathscr{P}^{(o)}(\theta)$ connects to the saddle point $t^{(1,0)} = -\mathrm{i}\beta + \pi\mathrm{i}$, whereas when $\theta = \frac{\pi}{2}$, it connects to the saddle point $t^{(0,-1)} = \mathrm{i}\beta - \pi\mathrm{i}$. These are therefore adjacent to 0. Because the horizontal lines through the points $\frac{\pi}{2}\mathrm{i}$ and $-\frac{\pi}{2}\mathrm{i}$ are asymptotes of the corresponding adjacent contours $\mathscr{C}^{(1,0)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(0,-1)}\left(\frac{\pi}{2}\right)$, respectively (see Figure 3.3), there cannot be further saddles adjacent to 0 other than $t^{(1,0)}$ and $t^{(0,-1)}$. The domain $\Delta^{(o)}$ is formed by the set of all points between these adjacent contours.

Following the analysis in Subsection 1.1.1, we expand $T^{(o)}(\nu)$ into a truncated asymptotic power series with remainder,

$$T^{(o)}(\nu) = \sum_{n=0}^{N-1}\frac{a_n^{(o)}}{\nu^n} + R_N^{(o)}(\nu).$$

It is not difficult to verify that the conditions posed in Subsection 1.1.1 hold good for the domain $\Delta^{(o)}$ and the functions $f(t) = \sec\beta\sinh t + t$ and $g(t) = 1$ with any $N \geq 0$. We choose the orientation of the adjacent contours so that

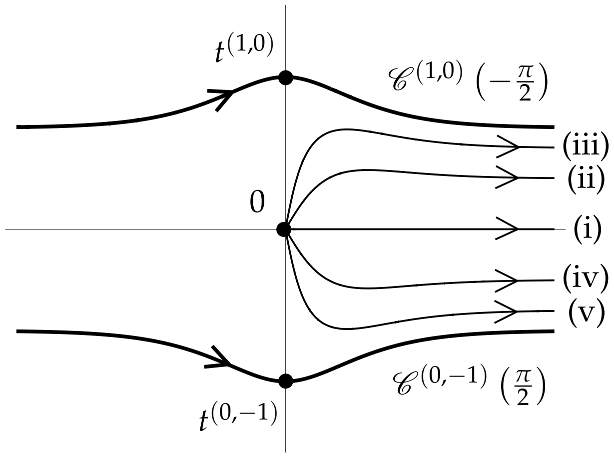

Figure 3.3. *The steepest descent contour $\mathscr{P}^{(o)}(\theta)$ associated with the Anger–Weber function of large order and argument emanating from the origin when (i) $\theta = 0$, (ii) $\theta = -\frac{\pi}{4}$, (iii) $\theta = -\frac{2\pi}{5}$, (iv) $\theta = \frac{\pi}{4}$ and (v) $\theta = \frac{2\pi}{5}$. The paths $\mathscr{C}^{(1,0)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(0,-1)}\left(\frac{\pi}{2}\right)$ are the adjacent contours for 0. The domain $\Delta^{(o)}$ comprises all points between $\mathscr{C}^{(1,0)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(0,-1)}\left(\frac{\pi}{2}\right)$.*

$\mathscr{C}^{(1,0)}\left(-\frac{\pi}{2}\right)$ is traversed in the negative direction and $\mathscr{C}^{(0,-1)}\left(\frac{\pi}{2}\right)$ is traversed in the positive direction with respect to the domain $\Delta^{(o)}$. Consequently the orientation anomalies are $\gamma_{o1,0} = 1$ and $\gamma_{o0,-1} = 0$, respectively. The relevant singulant pair is given by

$$\mathcal{F}_{o1,0} = \sec\beta\sinh\left(-\mathrm{i}\beta + \pi\mathrm{i}\right) - \mathrm{i}\beta + \pi\mathrm{i} - \sec\beta\sinh 0 - 0 = \mathrm{i}\left(\tan\beta - \beta + \pi\right),$$
$$\arg\mathcal{F}_{o1,0} = \sigma_{o1,0} = \frac{\pi}{2}$$

and

$$\mathcal{F}_{o0,-1} = \sec\beta\sinh\left(\mathrm{i}\beta - \pi\mathrm{i}\right) + \mathrm{i}\beta - \pi\mathrm{i} - \sec\beta\sinh 0 - 0 = -\mathrm{i}\left(\tan\beta - \beta + \pi\right),$$
$$\arg\mathcal{F}_{o0,-1} = \sigma_{o0,-1} = -\frac{\pi}{2}.$$

We thus find that

$$\begin{aligned} R_N^{(o)}(\nu) = &-\frac{(-\mathrm{i})^N \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}}{2\pi\mathrm{i}\nu^N}\int_0^{+\infty}\frac{u^{N-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta+\pi)u}}{1+\mathrm{i}u/\nu}T^{(1,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ &+\frac{\mathrm{i}^N\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}}{2\pi\mathrm{i}\nu^N}\int_0^{+\infty}\frac{u^{N-\frac{1}{2}}\mathrm{e}^{-(\tan\beta-\beta+\pi)u}}{1-\mathrm{i}u/\nu}T^{(0,-1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u, \end{aligned} \tag{3.182}$$

with $|\theta| < \frac{\pi}{2}$ and $N \geq 0$.

It is possible to arrive at a simpler result, by observing that we can express the functions $T^{(1,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$ and $T^{(0,-1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ in terms of the Hankel function $H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)$. Indeed, denoting by $\widetilde{\mathscr{C}}^{(1,0)}\left(-\frac{\pi}{2}\right)$ the contour which is congruent to $\mathscr{C}^{(1,0)}\left(-\frac{\pi}{2}\right)$ but is shifted downwards in the complex plane by $\pi\mathrm{i}$, we have

$$\begin{aligned}
T^{(1,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) &= u^{\frac{1}{2}}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}\int_{\mathscr{C}^{(1,0)}}\mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}(\sec\beta\sinh t+t-\sec\beta\sinh(-\mathrm{i}\beta+\pi\mathrm{i})+\mathrm{i}\beta-\pi\mathrm{i})}\mathrm{d}t\\
&= \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}u^{\frac{1}{2}}\mathrm{e}^{(\tan\beta-\beta)u}\int_{\widetilde{\mathscr{C}}^{(1,0)}}\mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}(t-\sec\beta\sinh t)}\mathrm{d}t\\
&= -\pi\mathrm{i}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}u^{\frac{1}{2}}\mathrm{e}^{(\tan\beta-\beta)u}H_{-\mathrm{i}u}^{(2)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\\
&= \pi\mathrm{i}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}u^{\frac{1}{2}}\mathrm{e}^{(\tan\beta-\beta)u}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right),
\end{aligned}$$

using an argument similar to (3.11). One can prove in an analogous manner that

$$T^{(0,-1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) = \pi\mathrm{i}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}u^{\frac{1}{2}}\mathrm{e}^{(\tan\beta-\beta)u}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right).$$

Therefore, the representation (3.182) simplifies to

$$\begin{aligned}
R_N^{(o)}(\nu) &= \frac{(-\mathrm{i})^N}{2\nu^N}\int_0^{+\infty}\frac{u^N\mathrm{e}^{-\pi u}}{1+\mathrm{i}u/\nu}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u\\
&\quad+\frac{\mathrm{i}^N}{2\nu^N}\int_0^{+\infty}\frac{u^N\mathrm{e}^{-\pi u}}{1-\mathrm{i}u/\nu}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u
\end{aligned} \tag{3.183}$$

for all non-zero values of ν in the sector $|\theta| < \frac{\pi}{2}$ with $N \geq 0$.

We may now connect the above results with the asymptotic power series (3.178) of $\mathbf{A}_\nu(\nu\sec\beta)$ by writing

$$\mathbf{A}_\nu(\nu\sec\beta) = \frac{1}{\pi}\sum_{n=0}^{N-1}\frac{a_n(\sec\beta)\Gamma(2n+1)}{\nu^{2n+1}} + R_N^{(\mathbf{A})}(\nu,\beta), \tag{3.184}$$

with the notation $a_n(\sec\beta) = a_{2n}^{(o)}/\Gamma(2n+1)$ and $R_N^{(\mathbf{A})}(\nu,\beta) = (\pi\nu)^{-1}R_{2N}^{(o)}(\nu)$. Hence, from (3.183), we deduce that

$$R_N^{(\mathbf{A})}(\nu,\beta) = \frac{(-1)^N}{\pi}\frac{1}{\nu^{2N+1}}\int_0^{+\infty}\frac{u^{2N}\mathrm{e}^{-\pi u}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u \tag{3.185}$$

under the same conditions which were required for (3.183) to hold. Equations (3.184) and (3.185) together provide the exact resurgence formula for $\mathbf{A}_\nu(\nu\sec\beta)$.

When deriving (3.184), we used implicitly the fact that $a_n^{(o)}$ vanishes for odd n. To prove this, first note that, by (1.12),

$$a_n^{(o)} = \left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\frac{t}{\sec\beta \sinh t + t} \right)^{n+1} \right]_{t=0} . \tag{3.186}$$

Because the quantity under the differentiation sign is an even function of t and therefore its odd-order derivatives at $t = 0$ are zero, the claim follows.

Taking $a_n(\sec\beta) = a_{2n}^{(o)} / \Gamma(2n+1)$ and (3.186) into account, we obtain the following representation for the coefficients $a_n(\sec\beta)$:

$$a_n(\sec\beta) = \frac{1}{\Gamma(2n+1)} \left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} \left(\frac{t}{\sec\beta \sinh t + t} \right)^{2n+1} \right]_{t=0} . \tag{3.187}$$

Although this formula expresses the coefficients $a_n(\sec\beta)$ in a closed form, it is not very efficient for their practical computation. A more useful expression for the rational functions $a_n(x)$ was given by the present author, in the form of a recurrence, using a method based on differential equations:

$$a_0(x) = \frac{1}{1+x} \quad \text{and} \quad a_n(x) = \frac{x}{1-x^2} \frac{x a''_{n-1}(x) + a'_{n-1}(x)}{2n(2n-1)}$$

for $n \geq 1$, see [71]. In this paper, the following formula involving the generalized Bernoulli polynomials is also shown:

$$a_n(x) = \frac{1}{(1+x)^{2n+1}} \sum_{k=0}^{n} \frac{2^{2n}\Gamma(2n+k+1)}{\Gamma^2(2n+1)} \left(\frac{x}{1+x} \right)^k \sum_{j=0}^{k} \frac{(-1)^j B_{2n}^{(-j)}\left(-\frac{j}{2}\right)}{\Gamma(k-j+1)\,\Gamma(j+1)} .$$

The reader may find further representations in the paper [71].

Consider now the resurgence properties of the derivative $\mathbf{A}'_\nu(\nu\sec\beta)$.[5] From (2.69), we infer that

$$\mathbf{A}'_\nu(\nu\sec\beta) = -\frac{1}{\pi} \int_0^{+\infty} \mathrm{e}^{-\nu(\sec\beta\sinh t + t)} \sinh t \,\mathrm{d}t \tag{3.188}$$

with $|\theta| < \frac{\pi}{2}$. Observe that the saddle point structure of the integrand in (3.188) is identical to that of (3.181). In particular, the problem is one of linear dependence at the saddle point, and the domain $\Delta^{(o)}$ corresponding to this problem is the same as that in the case of $\mathbf{A}_\nu(\nu\sec\beta)$. Since the derivation is essentially

[5] By this derivative, we mean $[\mathbf{A}'_\nu(z)]_{z=\nu\sec\beta}$.

the same as that of the resurgence formula for $\mathbf{A}_\nu(\nu\sec\beta)$, we omit the details and provide only the final result. We have

$$\mathbf{A}'_\nu(\nu\sec\beta) = \frac{1}{\pi}\sum_{n=0}^{N-1}\frac{b_n(\sec\beta)\,\Gamma(2n+1)}{\nu^{2n+2}} + R_N^{(\mathbf{A}')}(\nu,\beta), \tag{3.189}$$

where the remainder term $R_N^{(\mathbf{A}')}(\nu,\beta)$ is given by the integral formula

$$R_N^{(\mathbf{A}')}(\nu,\beta) = \frac{(-1)^{N+1}}{\pi}\frac{1}{\nu^{2N+2}}\int_0^{+\infty}\frac{u^{2N+1}\mathrm{e}^{-\pi u}}{1+(u/\nu)^2}H_{\mathrm{i}u}^{(1)\prime}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u, \tag{3.190}$$

provided that $|\theta| < \frac{\pi}{2}$ and $N \geq 0$.

The coefficients $b_n(\sec\beta)$ may be expressed in the form

$$b_n(\sec\beta) = \frac{1}{\Gamma(2n+1)}\left[\frac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}}\left(\sinh t\left(\frac{t}{\sec\beta\sinh t+t}\right)^{2n+2}\right)\right]_{t=0}.$$

However, they can be computed in a simpler way using the relation $b_n(x) = a'_n(x)$. (The reader should not confuse the $b_n(x)$'s with the coefficients $b_n(\nu)$ introduced in Section 2.1.)

Let us now turn our attention to the resurgence properties of the function $\mathbf{A}_{-\nu}(\nu\sec\beta)$. We substitute $-\nu$ in place of ν and $\nu\sec\beta$ in place of z in the integral representation (2.69) to obtain

$$\mathbf{A}_{-\nu}(\nu\sec\beta) = \frac{1}{\pi}\int_0^{+\infty}\mathrm{e}^{-\nu(\sec\beta\sinh t-t)}\mathrm{d}t, \tag{3.191}$$

for $|\theta| < \frac{\pi}{2}$. The function $\sec\beta\sinh t - t$ has infinitely many first-order saddle points in the complex t-plane located at $t^{(r,k)} = (-1)^r\,\mathrm{i}\beta + 2\pi\mathrm{i}k$ with $r = 0, 1$ and $k \in \mathbb{Z}$. The path of steepest descent $\mathscr{P}^{(o)}(0)$ emerging from the origin coincides with the positive real axis, and its orientation is chosen so that it leads away from 0. Hence we may write

$$\mathbf{A}_{-\nu}(\nu\sec\beta) = \frac{1}{\pi\nu}T^{(o)}(\nu),$$

where $T^{(o)}(\nu)$ is given in (1.3) with the specific choices of $f(t) = \sec\beta\sinh t - t$ and $g(t) = 1$. The problem is therefore one of linear dependence at the endpoint, which we discussed in Subsection 1.1.1. To identify the domain $\Delta^{(o)}$ corresponding to this problem, we have to determine the adjacent saddles and contours. When $\theta = -\frac{\pi}{2}$, the path $\mathscr{P}^{(o)}(\theta)$ connects to the saddle point $t^{(0,0)} = \mathrm{i}\beta$, whereas when $\theta = \frac{\pi}{2}$, it connects to the saddle point $t^{(1,0)} = -\mathrm{i}\beta$. These are therefore adjacent to 0. Because the horizontal lines through the points $\frac{\pi}{2}\mathrm{i}$ and $-\frac{\pi}{2}\mathrm{i}$ are asymptotes of the corresponding adjacent contours $\mathscr{C}^{(0,0)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(1,0)}\left(\frac{\pi}{2}\right)$, respectively (see Figure 3.4), there cannot be other saddles adjacent to 0 besides $t^{(0,0)}$ and $t^{(1,0)}$. The domain $\Delta^{(o)}$ is formed by the set of all points between these adjacent contours.

Now, we expand $T^{(o)}(\nu)$ into a truncated asymptotic power series with remainder,

$$T^{(o)}(\nu) = \sum_{n=0}^{N-1} \frac{a_n^{(o)}}{\nu^n} + R_N^{(o)}(\nu).$$

It can be shown that the conditions posed in Subsection 1.1.1 are satisfied by the domain $\Delta^{(o)}$ and the functions $f(t) = \sec\beta \sinh t - t$ and $g(t) = 1$ with any $N \geq 0$. We choose the orientation of the adjacent contours so that $\mathscr{C}^{(0,0)}\left(-\frac{\pi}{2}\right)$ is traversed in the negative sense and $\mathscr{C}^{(1,0)}\left(\frac{\pi}{2}\right)$ is traversed in the positive sense with respect to the domain $\Delta^{(o)}$. Consequently the orientation anomalies are $\gamma_{o0,0} = 1$ and $\gamma_{o1,0} = 0$. The relevant singulant pair is given by

$$\mathcal{F}_{o0,0} = \sec\beta \sinh(\mathrm{i}\beta) + \mathrm{i}\beta - \sec\beta \sinh 0 - 0 = \mathrm{i}(\tan\beta + \beta),$$
$$\arg \mathcal{F}_{o0,0} = \sigma_{o0,0} = \frac{\pi}{2}$$

and

$$\mathcal{F}_{o1,0} = \sec\beta \sinh(-\mathrm{i}\beta) - \mathrm{i}\beta - \sec\beta \sinh 0 - 0 = -\mathrm{i}(\tan\beta + \beta),$$
$$\arg \mathcal{F}_{o1,0} = \sigma_{o1,0} = -\frac{\pi}{2}.$$

We thus find that for $|\theta| < \frac{\pi}{2}$ and $N \geq 0$, the remainder term $R_N^{(o)}(\nu)$ may be written

$$\begin{aligned} R_N^{(o)}(\nu) = &-\frac{(-\mathrm{i})^N \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}}{2\pi\mathrm{i}\nu^N} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}} \mathrm{e}^{-(\tan\beta+\beta)u}}{1 + \mathrm{i}u/\nu} T^{(0,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u \\ &+ \frac{\mathrm{i}^N \mathrm{e}^{\frac{\pi}{4}\mathrm{i}}}{2\pi\mathrm{i}\nu^N} \int_0^{+\infty} \frac{u^{N-\frac{1}{2}} \mathrm{e}^{-(\tan\beta+\beta)u}}{1 - \mathrm{i}u/\nu} T^{(1,0)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \mathrm{d}u. \end{aligned} \tag{3.192}$$

A representation simpler than (3.192) is available. To derive it, we note that the functions $T^{(0,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)$ and $T^{(1,0)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ can both be written in terms of the Hankel function $H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}} \sec\beta\right)$. Indeed, we have

$$\begin{aligned} T^{(0,0)}\left(u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) &= u^{\frac{1}{2}} \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}} \int_{\mathscr{C}^{(0,0)}} \mathrm{e}^{-u\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}(\sec\beta \sinh t - t - \sec\beta \sinh(\mathrm{i}\beta) + \mathrm{i}\beta)} \mathrm{d}t \\ &= \mathrm{e}^{-\frac{\pi}{4}\mathrm{i}} u^{\frac{1}{2}} \mathrm{e}^{(\tan\beta+\beta)u} \int_{\mathscr{C}^{(0,0)}} \mathrm{e}^{-u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}(t - \sec\beta \sinh t)} \mathrm{d}t \\ &= \pi\mathrm{i}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}} u^{\frac{1}{2}} \mathrm{e}^{(\tan\beta+\beta)u} H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}} \sec\beta\right), \end{aligned}$$

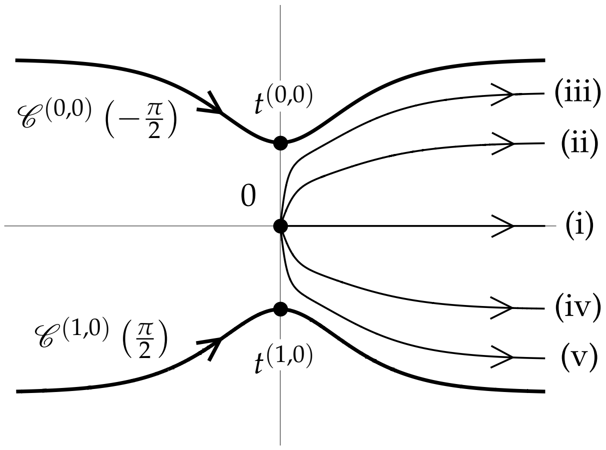

Figure 3.4. *The steepest descent contour $\mathscr{P}^{(o)}(\theta)$ associated with the Anger–Weber function of large order and argument emanating from the origin when (i) $\theta = 0$, (ii) $\theta = -\frac{\pi}{4}$, (iii) $\theta = -\frac{2\pi}{5}$, (iv) $\theta = \frac{\pi}{4}$ and (v) $\theta = \frac{2\pi}{5}$. The paths $\mathscr{C}^{(0,0)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(1,0)}\left(\frac{\pi}{2}\right)$ are the adjacent contours for 0. The domain $\Delta^{(o)}$ comprises all points between $\mathscr{C}^{(0,0)}\left(-\frac{\pi}{2}\right)$ and $\mathscr{C}^{(1,0)}\left(\frac{\pi}{2}\right)$.*

using an argument similar to (3.11). Likewise, one can show that

$$T^{(1,0)}\left(ue^{\frac{\pi}{2}\mathrm{i}}\right) = \pi \mathrm{i} e^{\frac{\pi}{4}\mathrm{i}} u^{\frac{1}{2}} e^{(\tan\beta+\beta)u} H^{(1)}_{\mathrm{i}u}\left(ue^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right).$$

Therefore, the representation (3.192) can be simplified to

$$\begin{aligned} R^{(o)}_N(\nu) = {} & \frac{(-\mathrm{i})^N}{2\nu^N}\int_0^{+\infty}\frac{u^N}{1+\mathrm{i}u/\nu}\mathrm{i}H^{(1)}_{\mathrm{i}u}\left(ue^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u \\ & + \frac{\mathrm{i}^N}{2\nu^N}\int_0^{+\infty}\frac{u^N}{1-\mathrm{i}u/\nu}\mathrm{i}H^{(1)}_{\mathrm{i}u}\left(ue^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u \end{aligned} \tag{3.193}$$

for all non-zero values of ν in the sector $|\theta| < \frac{\pi}{2}$ with $N \geq 0$.

We may now connect the above results with the asymptotic power series (3.178) of $\mathbf{A}_{-\nu}(\nu\sec\beta)$ by writing

$$\mathbf{A}_{-\nu}(\nu\sec\beta) = -\frac{1}{\pi}\sum_{n=0}^{N-1}\frac{a_n(-\sec\beta)\,\Gamma(2n+1)}{\nu^{2n+1}} + \widetilde{R}^{(\mathbf{A})}_N(\nu,\beta) \tag{3.194}$$

with $a_n(-\sec\beta) = -a^{(o)}_{2n}/\Gamma(2n+1)$ and $\widetilde{R}^{(\mathbf{A})}_N(\nu,\beta) = (\pi\nu)^{-1}R^{(o)}_{2N}(\nu)$. Formula (3.193) then imply

$$\widetilde{R}^{(\mathbf{A})}_N(\nu,\beta) = \frac{(-1)^N}{\pi}\frac{1}{\nu^{2N+1}}\int_0^{+\infty}\frac{u^{2N}}{1+(u/\nu)^2}\mathrm{i}H^{(1)}_{\mathrm{i}u}\left(ue^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u \tag{3.195}$$

under the same conditions which were required for (3.193) to hold. Equations (3.194) and (3.195) together give the exact resurgence formula for $\mathbf{A}_{-\nu}\left(\nu\sec\beta\right)$. When deducing (3.194), we used implicitly the fact that $a_n^{(o)}$ vanishes for odd n. This may be proved in the same way as the corresponding statement for the expression (3.184) of $\mathbf{A}_{\nu}\left(\nu\sec\beta\right)$. Taking $a_n\left(-\sec\beta\right) = -a_{2n}^{(o)}/\Gamma\left(2n+1\right)$ and (1.12) into account, we find

$$a_n\left(-\sec\beta\right) = \frac{1}{\Gamma\left(2n+1\right)}\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\left(\frac{t}{-\sec\beta\sinh t + t}\right)^{2n+1}\right]_{t=0},$$

in complete accord with (3.187).

The corresponding result for the derivative $\mathbf{A}'_{-\nu}\left(\nu\sec\beta\right)$ can be obtained starting from the integral representation

$$\mathbf{A}'_{-\nu}\left(\nu\sec\beta\right) = -\frac{1}{\pi}\int_0^{+\infty}\mathrm{e}^{-\nu\left(\sec\beta\sinh t - t\right)}\sinh t\,\mathrm{d}t, \qquad (3.196)$$

which is valid when $|\theta| < \frac{\pi}{2}$. Since the saddle point structure of the integrand in (3.196) is identical to that of (3.191), the derivation is analogous to that of the resurgence formula for $\mathbf{A}_{-\nu}\left(\nu\sec\beta\right)$ and so is omitted. The final result is

$$\mathbf{A}'_{-\nu}\left(\nu\sec\beta\right) = \frac{1}{\pi}\sum_{n=0}^{N-1}\frac{b_n\left(-\sec\beta\right)\Gamma\left(2n+1\right)}{\nu^{2n+2}} + \widetilde{R}_N^{(\mathbf{A}')}\left(\nu,\beta\right), \qquad (3.197)$$

where the remainder term $\widetilde{R}_N^{(\mathbf{A}')}\left(\nu,\beta\right)$ is given by the integral formula

$$\widetilde{R}_N^{(\mathbf{A}')}\left(\nu,\beta\right) = \frac{\left(-1\right)^{N+1}}{\pi}\frac{1}{\nu^{2N+2}}\int_0^{+\infty}\frac{u^{2N+1}}{1+\left(u/\nu\right)^2}H_{\mathrm{i}u}^{(1)\prime}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u, \qquad (3.198)$$

provided that $|\theta| < \frac{\pi}{2}$ and $N \geq 0$.

Let us now consider the resurgence properties of the Anger function $\mathbf{J}_{\pm\nu}\left(\nu\sec\beta\right)$ and the Weber function $\mathbf{E}_{\pm\nu}\left(\nu\sec\beta\right)$. These functions are related to $\mathbf{A}_{\pm\nu}\left(\nu\sec\beta\right)$ through the connection formulae

$$\mathbf{J}_{\pm\nu}\left(\nu\sec\beta\right) = J_{\pm\nu}\left(\nu\sec\beta\right) \pm \sin\left(\pi\nu\right)\mathbf{A}_{\pm\nu}\left(\nu\sec\beta\right)$$

and

$$\mathbf{E}_{\pm\nu}\left(\nu\sec\beta\right) = -Y_{\pm\nu}\left(\nu\sec\beta\right) - \cos\left(\pi\nu\right)\mathbf{A}_{\pm\nu}\left(\nu\sec\beta\right) - \mathbf{A}_{\mp\nu}\left(\nu\sec\beta\right)$$

(cf. [96, eqs. 11.10.15 and 11.10.16, p. 296]). We substitute (3.184) and (3.197) into the right-hand sides and match the notation with those of (3.179) and (3.180) in order to obtain

$$\mathbf{J}_\nu(\nu\sec\beta) = J_\nu(\nu\sec\beta) + \frac{\sin(\pi\nu)}{\pi}\left(\sum_{n=0}^{N-1}\frac{a_n(\sec\beta)\Gamma(2n+1)}{\nu^{2n+1}} + \pi R_N^{(\mathbf{A})}(\nu,\beta)\right), \tag{3.199}$$

$$\mathbf{J}_{-\nu}(\nu\sec\beta) = J_{-\nu}(\nu\sec\beta) + \frac{\sin(\pi\nu)}{\pi}\left(\sum_{n=0}^{N-1}\frac{a_n(-\sec\beta)\Gamma(2n+1)}{\nu^{2n+1}} - \pi\widetilde{R}_N^{(\mathbf{A})}(\nu,\beta)\right), \tag{3.200}$$

$$\begin{aligned}\mathbf{E}_\nu(\nu\sec\beta) = &- Y_\nu(\nu\sec\beta) \\ &- \frac{\cos(\pi\nu)}{\pi}\left(\sum_{n=0}^{N-1}\frac{a_n(\sec\beta)\Gamma(2n+1)}{\nu^{2n+1}} + \pi R_N^{(\mathbf{A})}(\nu,\beta)\right) \\ &+ \frac{1}{\pi}\sum_{m=0}^{M-1}\frac{a_m(-\sec\beta)\Gamma(2m+1)}{\nu^{2m+1}} - \widetilde{R}_M^{(\mathbf{A})}(\nu,\beta)\end{aligned} \tag{3.201}$$

and

$$\begin{aligned}\mathbf{E}_{-\nu}(\nu\sec\beta) = &- Y_{-\nu}(\nu\sec\beta) \\ &+ \frac{\cos(\pi\nu)}{\pi}\left(\sum_{n=0}^{N-1}\frac{a_n(-\sec\beta)\Gamma(2n+1)}{\nu^{2n+1}} - \pi\widetilde{R}_N^{(\mathbf{A})}(\nu,\beta)\right) \\ &- \frac{1}{\pi}\sum_{m=0}^{M-1}\frac{a_m(\sec\beta)\Gamma(2m+1)}{\nu^{2m+1}} - R_M^{(\mathbf{A})}(\nu,\beta).\end{aligned} \tag{3.202}$$

If $|\theta| < \frac{\pi}{2}$ and $M, N \geq 0$, equations (3.199)–(3.202), (3.185) and (3.195) together give the desired resurgence formulae for the functions $\mathbf{J}_{\pm\nu}(\nu\sec\beta)$ and $\mathbf{E}_{\pm\nu}(\nu\sec\beta)$.

We may derive the corresponding expressions for the derivatives $\mathbf{J}'_{\pm\nu}(\nu\sec\beta)$ and $\mathbf{E}'_{\pm\nu}(\nu\sec\beta)$, by substituting the results (3.189) and (3.197) into the right-hand sides of the functional relations

$$\mathbf{J}'_{\pm\nu}(\nu\sec\beta) = J'_{\pm\nu}(\nu\sec\beta) \pm \sin(\pi\nu)\,\mathbf{A}'_{\pm\nu}(\nu\sec\beta)$$

and

$$\mathbf{E}'_{\pm\nu}(\nu\sec\beta) = -Y'_{\pm\nu}(\nu\sec\beta) - \cos(\pi\nu)\,\mathbf{A}'_{\pm\nu}(\nu\sec\beta) - \mathbf{A}'_{\mp\nu}(\nu\sec\beta).$$

Thus we have

$$\begin{aligned}\mathbf{J}'_\nu(\nu\sec\beta) &= J'_\nu(\nu\sec\beta)\\ &\quad + \frac{\sin(\pi\nu)}{\pi}\left(\sum_{n=0}^{N-1}\frac{b_n(\sec\beta)\,\Gamma(2n+1)}{\nu^{2n+2}} + \pi R_N^{(\mathbf{A}')}(\nu,\beta)\right),\end{aligned} \tag{3.203}$$

$$\begin{aligned}\mathbf{J}'_{-\nu}(\nu\sec\beta) &= J'_{-\nu}(\nu\sec\beta)\\ &\quad - \frac{\sin(\pi\nu)}{\pi}\left(\sum_{n=0}^{N-1}\frac{b_n(-\sec\beta)\,\Gamma(2n+1)}{\nu^{2n+2}} + \pi \widetilde{R}_N^{(\mathbf{A}')}(\nu,\beta)\right),\end{aligned} \tag{3.204}$$

$$\begin{aligned}\mathbf{E}'_\nu(\nu\sec\beta) &= -Y'_\nu(\nu\sec\beta)\\ &\quad - \frac{\cos(\pi\nu)}{\pi}\left(\sum_{n=0}^{N-1}\frac{b_n(\sec\beta)\,\Gamma(2n+1)}{\nu^{2n+2}} + \pi R_N^{(\mathbf{A}')}(\nu,\beta)\right)\\ &\quad - \frac{1}{\pi}\sum_{m=0}^{M-1}\frac{b_m(-\sec\beta)\,\Gamma(2m+1)}{\nu^{2m+2}} - \widetilde{R}_M^{(\mathbf{A}')}(\nu,\beta)\end{aligned} \tag{3.205}$$

and

$$\begin{aligned}\mathbf{E}'_{-\nu}(\nu\sec\beta) &= -Y'_{-\nu}(\nu\sec\beta)\\ &\quad - \frac{\cos(\pi\nu)}{\pi}\left(\sum_{n=0}^{N-1}\frac{b_n(-\sec\beta)\,\Gamma(2n+1)}{\nu^{2n+2}} + \pi \widetilde{R}_N^{(\mathbf{A}')}(\nu,\beta)\right)\\ &\quad - \frac{1}{\pi}\sum_{m=0}^{M-1}\frac{b_m(\sec\beta)\,\Gamma(2m+1)}{\nu^{2m+2}} - R_M^{(\mathbf{A}')}(\nu,\beta).\end{aligned} \tag{3.206}$$

If $|\theta| < \frac{\pi}{2}$ and $M, N \geq 0$, equations (3.203)–(3.206) in conjunction with (3.190) and (3.198) give the required resurgence formulae for the functions $\mathbf{J}'_{\pm\nu}(\nu\sec\beta)$ and $\mathbf{E}'_{\pm\nu}(\nu\sec\beta)$.

Neglecting the remainder terms in (3.189), (3.197) and (3.203)–(3.206) and formally extending the sums to infinity, we obtain asymptotic expansions for the functions $\mathbf{A}'_{\pm\nu}(\nu\sec\beta)$, $\mathbf{J}'_{\pm\nu}(\nu\sec\beta)$ and $\mathbf{E}'_{\pm\nu}(\nu\sec\beta)$. These asymptotic expansions are valid in the sector $|\theta| \leq \pi - \delta$, with δ being an arbitrary small positive constant.

3.3.2 Error bounds

This subsection is devoted to obtaining computable bounds for the remainders $R_N^{(\mathbf{A})}(\nu,\beta)$, $R_N^{(\mathbf{A}')}(\nu,\beta)$, $\widetilde{R}_N^{(\mathbf{A})}(\nu,\beta)$ and $\widetilde{R}_N^{(\mathbf{A}')}(\nu,\beta)$. The procedure of deriving these error bounds is essentially the same as in the case of the Hankel and Bessel functions of large order and argument discussed in Subsection 3.1.2, and therefore we omit the details. Some of the results we give here coincide with those obtained by Meijer [61].

The following estimates are valid in the right half-plane and are useful when ν is bounded away from the Stokes lines $\theta = \pm\frac{\pi}{2}$:

$$\left|R_N^{(\mathbf{A})}(\nu,\beta)\right| \le \frac{1}{\pi}\frac{|a_N(\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+1}}\begin{cases}|\csc(2\theta)| & \text{if } \frac{\pi}{4}<|\theta|<\frac{\pi}{2},\\ 1 & \text{if } |\theta|\le\frac{\pi}{4},\end{cases} \tag{3.207}$$

$$\left|R_N^{(\mathbf{A}')}(\nu,\beta)\right| \le \frac{1}{\pi}\frac{|b_N(\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+2}}\begin{cases}|\csc(2\theta)| & \text{if } \frac{\pi}{4}<|\theta|<\frac{\pi}{2},\\ 1 & \text{if } |\theta|\le\frac{\pi}{4},\end{cases} \tag{3.208}$$

$$\left|\widetilde{R}_N^{(\mathbf{A})}(\nu,\beta)\right| \le \frac{1}{\pi}\frac{|a_N(-\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+1}}\begin{cases}|\csc(2\theta)| & \text{if } \frac{\pi}{4}<|\theta|<\frac{\pi}{2},\\ 1 & \text{if } |\theta|\le\frac{\pi}{4}\end{cases} \tag{3.209}$$

and

$$\left|\widetilde{R}_N^{(\mathbf{A}')}(\nu,\beta)\right| \le \frac{1}{\pi}\frac{|b_N(-\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+2}}\begin{cases}|\csc(2\theta)| & \text{if } \frac{\pi}{4}<|\theta|<\frac{\pi}{2},\\ 1 & \text{if } |\theta|\le\frac{\pi}{4}.\end{cases} \tag{3.210}$$

For the special case when ν is positive, one finds that

$$\begin{aligned} R_N^{(\mathbf{A})}(\nu,\beta) &= \frac{1}{\pi}\frac{a_N(\sec\beta)\,\Gamma(2N+1)}{\nu^{2N+1}}\Theta_N(\nu,\beta),\\ R_N^{(\mathbf{A}')}(\nu,\beta) &= \frac{1}{\pi}\frac{b_N(\sec\beta)\,\Gamma(2N+1)}{\nu^{2N+2}}\Xi_N(\nu,\beta),\\ \widetilde{R}_N^{(\mathbf{A})}(\nu,\beta) &= -\frac{1}{\pi}\frac{a_N(-\sec\beta)\,\Gamma(2N+1)}{\nu^{2N+1}}\widetilde{\Theta}_N(\nu,\beta) \end{aligned}$$

and

$$\widetilde{R}_N^{(\mathbf{A}')}(\nu,\beta) = \frac{1}{\pi}\frac{b_N(-\sec\beta)\,\Gamma(2N+1)}{\nu^{2N+2}}\widetilde{\Xi}_N(\nu,\beta).$$

Here $0 < \Theta_N(\nu,\beta), \widetilde{\Theta}_N(\nu,\beta), \Xi_N(\nu,\beta), \widetilde{\Xi}_N(\nu,\beta) < 1$ are suitable numbers that depend on ν, β and N. In particular, the remainder terms do not exceed the corresponding first neglected terms in absolute value, and they have the same sign provided that $\nu > 0$.

The estimates (3.207)–(3.210) become singular as θ approaches $\pm\frac{\pi}{2}$ and are therefore not suitable near the Stokes lines $\theta = \pm\frac{\pi}{2}$. We now give alternative bounds that are appropriate for the sectors $\frac{\pi}{4} < |\theta| < \pi$ (which include the

Stokes lines $\theta = \pm\frac{\pi}{2}$). We may use (3.184), (3.189), (3.194) and (3.197) to define the remainder terms in the sectors $\frac{\pi}{2} \le |\theta| < \pi$. The bounds are as follows:

$$\left|R_N^{(\mathbf{A})}(\nu,\beta)\right| \le \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{2N+1}\varphi^*}\frac{1}{\pi}\frac{|a_N(\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+1}}, \tag{3.211}$$

$$\left|R_N^{(\mathbf{A}')}(\nu,\beta)\right| \le \frac{|\csc(2(\theta-\varphi^{**}))|}{\cos^{2N+2}\varphi^{**}}\frac{1}{\pi}\frac{|b_N(\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+2}}, \tag{3.212}$$

$$\left|\widetilde{R}_N^{(\mathbf{A})}(\nu,\beta)\right| \le \frac{|\csc(2(\theta-\varphi^*))|}{\cos^{2N+1}\varphi^*}\frac{1}{\pi}\frac{|a_N(-\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+1}} \tag{3.213}$$

and

$$\left|\widetilde{R}_N^{(\mathbf{A}')}(\nu,\beta)\right| \le \frac{|\csc(2(\theta-\varphi^{**}))|}{\cos^{2N+2}\varphi^{**}}\frac{1}{\pi}\frac{|b_N(-\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+2}} \tag{3.214}$$

for $\frac{\pi}{4} < |\theta| < \pi$, were φ^* and φ^{**} are the minimizing values given by Lemma 2.1.2 with the specific choices of $\chi = 2N+1$ and $\chi = 2N+2$, respectively. Note that the ranges of validity of the bounds (3.207)–(3.210) and (3.211)–(3.214) together cover that of the asymptotic expansions of the Anger, Weber and Anger–Weber functions, and their derivatives, for large order and argument. We remark that bounds equivalent to (3.211)–(3.214) were proved by Meijer [61].

The following simple estimates are suitable near the Stokes lines $\theta = \pm\frac{\pi}{2}$, and they can be obtained from (3.211)–(3.214) using an argument similar to that given in Subsection 2.1.2:

$$\left|R_N^{(\mathbf{A})}(\nu,\beta)\right| \le \tfrac{1}{2}\sqrt{\mathrm{e}\left(2N+\tfrac{5}{2}\right)}\frac{1}{\pi}\frac{|a_N(\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+1}},$$

$$\left|R_N^{(\mathbf{A}')}(\nu,\beta)\right| \le \tfrac{1}{2}\sqrt{\mathrm{e}\left(2N+\tfrac{7}{2}\right)}\frac{1}{\pi}\frac{|b_N(\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+2}},$$

$$\left|\widetilde{R}_N^{(\mathbf{A})}(\nu,\beta)\right| \le \tfrac{1}{2}\sqrt{\mathrm{e}\left(2N+\tfrac{5}{2}\right)}\frac{1}{\pi}\frac{|a_N(-\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+1}}$$

and

$$\left|\widetilde{R}_N^{(\mathbf{A}')}(\nu,\beta)\right| \le \tfrac{1}{2}\sqrt{\mathrm{e}\left(2N+\tfrac{7}{2}\right)}\frac{1}{\pi}\frac{|b_N(-\sec\beta)|\,\Gamma(2N+1)}{|\nu|^{2N+2}}$$

with $\frac{\pi}{4} < |\theta| \le \frac{\pi}{2}$ and $N \ge 1$. These bounds might be used in conjunction with with the earlier results (3.211)–(3.214).

3.3.3 Asymptotics for the late coefficients

In this subsection, we consider the asymptotic behaviour of the coefficients $a_n(\pm\sec\beta)$ and $b_n(\pm\sec\beta)$ for large n. To our best knowledge, the only known result related the approximation of these coefficients is that of Dingle [35, exer. 11, p. 202]. He gave a formal expansion for the $a_n(-\sec\beta)$'s, which can be written, in our notation, as

$$\begin{aligned} a_n(-\sec\beta)\Gamma(2n+1) \approx \left(\frac{2\cot\beta}{\pi(\tan\beta-\beta)}\right)^{\frac{1}{2}} \frac{(-1)^{n+1}}{(\tan\beta-\beta)^{2n}} \\ \times \sum_{m=0}^{\infty} (\mathrm{i}(\tan\beta-\beta))^m U_m(\mathrm{i}\cot\beta)\Gamma\left(2n-m+\frac{1}{2}\right). \end{aligned} \tag{3.215}$$

We shall derive here the rigorous form of Dingle's expansion by truncating it after a finite number of terms and constructing its error bound. The corresponding results for the coefficients $a_n(\sec\beta)$ and $b_n(\pm\sec\beta)$ will also be provided.

We begin by considering the coefficients $a_N(\sec\beta)$. First, we substitute (3.185) into the right-hand side of $a_N(\sec\beta)\Gamma(2N+1) = \pi\nu^{2N+1}(R_N^{(\mathbf{A})}(\nu,\beta) - R_{N+1}^{(\mathbf{A})}(\nu,\beta))$, to establish

$$a_n(\sec\beta)\Gamma(2n+1) = (-1)^n \int_0^{+\infty} u^{2n}\mathrm{e}^{-\pi u}\mathrm{i}H_{\mathrm{i}u}^{(1)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta)\mathrm{d}u, \tag{3.216}$$

where we have written N in place of n. Next, we replace $\mathrm{i}H_{\mathrm{i}u}^{(1)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\sec\beta)$ on the right-hand side of (3.216) by its truncated asymptotic expansion (3.40) and use the error bound (3.41) to arrive at

$$\begin{aligned} a_n(\sec\beta)\Gamma(2n+1) = \left(\frac{2\cot\beta}{\pi(\tan\beta-\beta+\pi)}\right)^{\frac{1}{2}} \frac{(-1)^n}{(\tan\beta-\beta+\pi)^{2n}} \\ \times\left(\sum_{m=0}^{M-1}(\mathrm{i}(\tan\beta-\beta+\pi))^m U_m(\mathrm{i}\cot\beta)\Gamma\left(2n-m+\frac{1}{2}\right)+A_M(n,\beta)\right) \end{aligned} \tag{3.217}$$

where

$$|A_M(n,\beta)| \le (\tan\beta-\beta+\pi)^M |U_M(\mathrm{i}\cot\beta)|\,\Gamma\left(2n-M+\frac{1}{2}\right), \tag{3.218}$$

provided $n \ge 0$ and $0 \le M \le 2n$. If n is large and β is bounded away from zero, the least value of the bound (3.218) occurs when $M \approx \frac{4n+1}{3+\pi/(\tan\beta-\beta)}$. With this choice of M, the ratio of the error bound to the leading term in (3.217) is

$$\mathcal{O}_\beta\left(\left(n\frac{\tan\beta-\beta}{3(\tan\beta-\beta)+\pi}\right)^{-\frac{1}{2}}\left(1+2\frac{\tan\beta-\beta}{\tan\beta-\beta+\pi}\right)^{-2n}\right).$$

values of n, β and M	$n = 20, \beta = \frac{\pi}{6}, M = 2$
exact numerical value of $\lvert a_n(\sec\beta)\Gamma(2n+1)\rvert$	$0.4652719458110908340442716876380 \times 10^{27}$
approximation (3.217) to $\lvert a_n(\sec\beta)\Gamma(2n+1)\rvert$	$0.445959574102882590087019983736 \times 10^{27}$
error	$0.193123717082082439572517039020 \times 10^{26}$
error bound using (3.218)	$0.429717660548301062206094654350 \times 10^{26}$
values of n, β and M	$n = 20, \beta = \frac{6\pi}{13}, M = 23$
exact numerical value of $\lvert a_n(\sec\beta)\Gamma(2n+1)\rvert$	$0.151319595449560481258052890390 \times 10^{7}$
approximation (3.217) to $\lvert a_n(\sec\beta)\Gamma(2n+1)\rvert$	$0.151319595449560488113789097564 \times 10^{7}$
error	$-0.6855736207174 \times 10^{-10}$
error bound using (3.218)	$0.13499409317334 \times 10^{-9}$
values of n, β and M	$n = 40, \beta = \frac{\pi}{3}, M = 21$
exact numerical value of $\lvert a_n(\sec\beta)\Gamma(2n+1)\rvert$	$0.585927322062805440144383753339 \times 10^{71}$
approximation (3.217) to $\lvert a_n(\sec\beta)\Gamma(2n+1)\rvert$	$0.585927322063411800229861686075 \times 10^{71}$
error	$-0.606360085477932736 \times 10^{59}$
error bound using (3.218)	$0.119556239455041154 9 \times 10^{60}$
values of n, β and M	$n = 40, \beta = \frac{5\pi}{12}, M = 37$
exact numerical value of $\lvert a_n(\sec\beta)\Gamma(2n+1)\rvert$	$0.323330973889413509226587306318 4 \times 10^{58}$
approximation (3.217) to $\lvert a_n(\sec\beta)\Gamma(2n+1)\rvert$	$0.323330973889413509226587491522 0 \times 10^{58}$
error	$-0.1852036 \times 10^{35}$
error bound using (3.218)	0.3658063×10^{35}

Table 3.3. *Approximations for $\lvert a_n(\sec\beta)\,\Gamma(2n+1)\rvert$ with various n and β, using* (3.217).

Whence, the smaller β is the larger n has to be to get a reasonable approximation from (3.217). Numerical examples illustrating the efficacy of (3.217), truncated optimally, are given in Table 3.3.

One may similarly show that for the coefficients $b_n(\sec\beta)$,

$$b_n(\sec\beta)\,\Gamma(2n+1) = \left(\frac{\sin(2\beta)}{\pi(\tan\beta-\beta+\pi)}\right)^{\frac{1}{2}}\frac{(-1)^{n+1}}{(\tan\beta-\beta+\pi)^{2n+1}}$$
$$\times\left(\sum_{m=0}^{M-1}(\mathrm{i}(\tan\beta-\beta+\pi))^m V_m(\mathrm{i}\cot\beta)\,\Gamma\left(2n-m+\frac{3}{2}\right)+B_M(n,\beta)\right)$$

where

$$\left|B_M\left(n,\beta\right)\right| \le \left(\tan\beta-\beta+\pi\right)^M \left|V_M\left(\mathrm{i}\cot\beta\right)\right| \Gamma\left(2n-M+\frac{3}{2}\right), \tag{3.219}$$

as long as $n \ge 0$ and $1 \le M \le 2n+1$. One readily establishes that the least values of the bound (3.219) occurs again when $M \approx \frac{4n+1}{3+\pi/(\tan\beta-\beta)}$, provided that n is large and β is not too close to the origin.

Consider now the coefficients $a_N\left(-\sec\beta\right)$. We insert (3.195) into the right-hand side of $a_N\left(-\sec\beta\right)\Gamma\left(2N+1\right) = -\pi\nu^{2N+1}(\widetilde{R}_N^{(\mathbf{A})}\left(\nu,\beta\right) - \widetilde{R}_{N+1}^{(\mathbf{A})}\left(\nu,\beta\right))$ and replace N by n, to arrive at

$$a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right) = \left(-1\right)^n \int_0^{+\infty} u^{2n}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(ue^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)\mathrm{d}u. \tag{3.220}$$

Next, we replace $\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(ue^{\frac{\pi}{2}\mathrm{i}}\sec\beta\right)$ in (3.220) by its truncated asymptotic expansion (3.40) and use the error bound (3.41) to establish

$$\begin{aligned} a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right) &= \left(\frac{2\cot\beta}{\pi\left(\tan\beta-\beta\right)}\right)^{\frac{1}{2}} \frac{\left(-1\right)^{n+1}}{\left(\tan\beta-\beta\right)^{2n}} \\ &\times \left(\sum_{m=0}^{M-1}\left(\mathrm{i}\left(\tan\beta-\beta\right)\right)^m U_m\left(\mathrm{i}\cot\beta\right)\Gamma\left(2n-m+\frac{1}{2}\right) + C_M\left(n,\beta\right)\right) \end{aligned} \tag{3.221}$$

where

$$\left|C_M\left(n,\beta\right)\right| \le \left(\tan\beta-\beta\right)^M \left|U_M\left(\mathrm{i}\cot\beta\right)\right| \Gamma\left(2n-M+\frac{1}{2}\right), \tag{3.222}$$

provided $n \ge 0$ and $0 \le M \le 2n$. For given large n, the least value of the bound (3.222) occurs when $M \approx \frac{4n}{3}$. With this choice of M, the ratio of the error bound to the leading term in (3.221) is $\mathcal{O}_\beta(n^{-\frac{1}{2}}9^{-n})$. This is the best accuracy we can achieve using the truncated version of the expansion (3.221). Numerical examples illustrating the efficacy of (3.221), truncated optimally, are given in Table 3.4.

If we neglect the remainder term $C_M\left(n,\beta\right)$ in (3.221) and formally extend the sum to infinity, formula (3.221) reproduces Dingle's expansion (3.215).

One can similarly show that for the coefficients $b_n\left(-\sec\beta\right)$,

$$\begin{aligned} b_n\left(-\sec\beta\right)\Gamma\left(2n+1\right) &= \left(\frac{\sin\left(2\beta\right)}{\pi\left(\tan\beta-\beta\right)}\right)^{\frac{1}{2}} \frac{\left(-1\right)^{n+1}}{\left(\tan\beta-\beta\right)^{2n+1}} \\ &\times \left(\sum_{m=0}^{M-1}\left(\mathrm{i}\left(\tan\beta-\beta\right)\right)^m V_m\left(\mathrm{i}\cot\beta\right)\Gamma\left(2n-m+\frac{3}{2}\right) + D_M\left(n,\beta\right)\right) \end{aligned}$$

values of n, β and M	$n=15$, $\beta=\frac{\pi}{6}$, $M=20$
exact numerical value of $\left|a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right)\right|$	$0.26704739449812455367469821205 2\times 10^{71}$
approximation (3.221) to $\left|a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right)\right|$	$0.267047394498124495293385515582\times 10^{71}$
error	$0.58381312696470\times 10^{55}$
error bound using (3.222)	$0.115325952521316\times 10^{56}$
values of n, β and M	$n=15$, $\beta=\frac{6\pi}{13}$, $M=20$
exact numerical value of $\left|a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right)\right|$	$0.578182278411391748669067674043\times 10^{6}$
approximation (3.221) to $\left|a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right)\right|$	$0.578182278411391622513139516532\times 10^{6}$
error	$0.126155928157511\times 10^{-9}$
error bound using (3.222)	$0.249207238615031\times 10^{-9}$
values of n, β and M	$n=25$, $\beta=\frac{\pi}{3}$, $M=33$
exact numerical value of $\left|a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right)\right|$	$0.520979340670722090241522846652\times 10^{72}$
approximation (3.221) to $\left|a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right)\right|$	$0.520979340670722090241522872480\times 10^{72}$
error	-0.25828×10^{47}
error bound using (3.222)	0.50537×10^{47}
values of n, β and M	$n=25$, $\beta=\frac{5\pi}{12}$, $M=33$
exact numerical value of $\left|a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right)\right|$	$0.687614482335478651602822854016\times 10^{44}$
approximation (3.221) to $\left|a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right)\right|$	$0.687614482335478651602822888085\times 10^{44}$
error	-0.34069×10^{19}
error bound using (3.222)	0.66662×10^{19}

Table 3.4. *Approximations for* $\left|a_n\left(-\sec\beta\right)\Gamma\left(2n+1\right)\right|$ *with various* n *and* β*, using* (3.221).

holds, where

$$\left|D_M\left(n,\beta\right)\right| \le \left(\tan\beta-\beta\right)^M \left|V_M\left(\mathrm{i}\cot\beta\right)\right| \Gamma\left(2n-M+\frac{3}{2}\right), \tag{3.223}$$

as long as $n \ge 0$ and $1 \le M \le 2n+1$. It is readily established that the least value of the bound (3.223) occurs when $M \approx \frac{4n}{3}$.

3.3.4 Exponentially improved asymptotic expansions

The aim of this subsection is to give exponentially improved asymptotic expansions for the Anger, Weber and Anger–Weber functions, and their derivatives, for large order and argument. A re-expansion for the remainder term of the asymptotic power series of the function $\mathbf{A}_{-\nu}(\nu \sec \beta)$ was obtained, using non-rigorous methods, by Dingle [35, exer. 22, p. 485]. We shall derive here the rigorous form of Dingle's formal re-expansion as well as the corresponding results for $R_N^{(\mathbf{A})}(\nu,\beta)$, $R_N^{(\mathbf{A}')}(\nu,\beta)$ and $\widetilde{R}_N^{(\mathbf{A}')}(\nu,\beta)$. The proof of our results in Propositions 3.3.1 and 3.3.2 below is essentially the same as that of Proposition 3.1.1 on the analogous expansion for the Hankel function of large order and argument, and therefore the proof is omitted.

Proposition 3.3.1. *Let M be an arbitrary fixed non-negative integer. Suppose that $|\theta| \leq 2\pi - \delta$ with an arbitrary fixed small positive δ, $|\nu|$ is large and $N = \frac{1}{2}(\tan\beta - \beta + \pi)|\nu| + \rho$ with ρ being bounded. Then*

$$
\begin{aligned}
R_N^{(\mathbf{A})}(\nu,\beta) = {} & \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\xi}\mathrm{e}^{\pi\mathrm{i}\nu}}{\left(\frac{1}{2}\pi\nu\tan\beta\right)}\sum_{m=0}^{M-1}(-1)^m\frac{U_m(\mathrm{i}\cot\beta)}{\nu^m} \\
& \times T_{2N-m+\frac{1}{2}}\left((\tan\beta-\beta+\pi)\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\
& - \mathrm{i}\frac{\mathrm{e}^{-\mathrm{i}\xi}\mathrm{e}^{-\pi\mathrm{i}\nu}}{\left(\frac{1}{2}\pi\nu\tan\beta\right)}\sum_{m=0}^{M-1}\frac{U_m(\mathrm{i}\cot\beta)}{\nu^m} \\
& \times T_{2N-m+\frac{1}{2}}\left((\tan\beta-\beta+\pi)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
& + R_{N,M}^{(\mathbf{A})}(\nu,\beta)
\end{aligned}
$$

and

$$
\begin{aligned}
R_N^{(\mathbf{A}')}(\nu,\beta) = {} & -\left(\frac{\sin(2\beta)}{\pi\nu}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\xi}\mathrm{e}^{\pi\mathrm{i}\nu}\sum_{m=0}^{M-1}(-1)^m\frac{V_m(\mathrm{i}\cot\beta)}{\nu^m} \\
& \times T_{2N-m+\frac{3}{2}}\left((\tan\beta-\beta+\pi)\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\
& - \left(\frac{\sin(2\beta)}{\pi\nu}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\xi}\mathrm{e}^{-\pi\mathrm{i}\nu}\sum_{m=0}^{M-1}\frac{V_m(\mathrm{i}\cot\beta)}{\nu^m} \\
& \times T_{2N-m+\frac{3}{2}}\left((\tan\beta-\beta+\pi)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\
& + R_{N,M}^{(\mathbf{A}')}(\nu,\beta),
\end{aligned}
$$

where $\xi = (\tan\beta - \beta)\nu - \frac{\pi}{4}$ and

$$
R_{N,M}^{(\mathbf{A})}(\nu,\beta), R_{N,M}^{(\mathbf{A}')}(\nu,\beta) = \mathcal{O}_{M,\beta,\rho}\left(\frac{\mathrm{e}^{-|\xi|}\mathrm{e}^{-\pi|\nu|}}{|\nu|^{M+\frac{1}{2}}}\right)
$$

for $|\theta| \le \frac{\pi}{2}$, *and*

$$R_{N,M}^{(\mathbf{A})}(\nu,\beta), R_{N,M}^{(\mathbf{A}')}(\nu,\beta) = \mathcal{O}_{M,\beta,\rho,\delta}\left(\frac{\mathrm{e}^{\mp\Im\mathfrak{m}(\xi)}\mathrm{e}^{\mp\pi\Im\mathfrak{m}(\nu)}}{|\nu|^{M+\frac{1}{2}}}\right)$$

for $\frac{\pi}{2} \le \pm\theta \le 2\pi - \delta$.

Proposition 3.3.1 together with (3.184), (3.189), (3.199), (3.201), (3.203) and (3.205) gives the exponentially improved asymptotic expansions for the functions $\mathbf{A}_\nu(\nu\sec\beta)$, $\mathbf{J}_\nu(\nu\sec\beta)$, $\mathbf{E}_\nu(\nu\sec\beta)$ and that of the corresponding derivatives.

Proposition 3.3.2. *Let M be an arbitrary fixed non-negative integer. Suppose that $|\theta| \le 2\pi - \delta$ with an arbitrary fixed small positive δ, $|\nu|$ is large and $N = \frac{1}{2}(\tan\beta - \beta)|\nu| + \rho$ with ρ being bounded. Then*

$$\begin{aligned}\widetilde{R}_N^{(\mathbf{A})}(\nu,\beta) = {} & \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\xi}}{\left(\frac{1}{2}\pi\nu\tan\beta\right)}\sum_{m=0}^{M-1}(-1)^m\frac{U_m(\mathrm{i}\cot\beta)}{\nu^m}T_{2N-m+\frac{1}{2}}\left((\tan\beta-\beta)\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\\ & - \mathrm{i}\frac{\mathrm{e}^{-\mathrm{i}\xi}}{\left(\frac{1}{2}\pi\nu\tan\beta\right)}\sum_{m=0}^{M-1}\frac{U_m(\mathrm{i}\cot\beta)}{\nu^m}T_{2N-m+\frac{1}{2}}\left((\tan\beta-\beta)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\\ & + \widetilde{R}_{N,M}^{(\mathbf{A})}(\nu,\beta)\end{aligned} \tag{3.224}$$

and

$$\begin{aligned}\widetilde{R}_N^{(\mathbf{A}')}(\nu,\beta) = {} & -\left(\frac{\sin(2\beta)}{\pi\nu}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\xi}\sum_{m=0}^{M-1}(-1)^m\frac{V_m(\mathrm{i}\cot\beta)}{\nu^m}T_{2N-m+\frac{3}{2}}\left((\tan\beta-\beta)\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\\ & - \left(\frac{\sin(2\beta)}{\pi\nu}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\xi}\sum_{m=0}^{M-1}\frac{V_m(\mathrm{i}\cot\beta)}{\nu^m}T_{2N-m+\frac{3}{2}}\left((\tan\beta-\beta)\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\\ & + \widetilde{R}_{N,M}^{(\mathbf{A}')}(\nu,\beta),\end{aligned}$$

where $\xi = (\tan\beta - \beta)\nu - \frac{\pi}{4}$ *and*

$$\widetilde{R}_{N,M}^{(\mathbf{A})}(\nu,\beta), \widetilde{R}_{N,M}^{(\mathbf{A}')}(\nu,\beta) = \mathcal{O}_{M,\beta,\rho}\left(\frac{\mathrm{e}^{-|\xi|}}{|\nu|^{M+\frac{1}{2}}}\right)$$

for $|\theta| \le \frac{\pi}{2}$, *and*

$$\widetilde{R}_{N,M}^{(\mathbf{A})}(\nu,\beta), \widetilde{R}_{N,M}^{(\mathbf{A}')}(\nu,\beta) = \mathcal{O}_{M,\beta,\rho,\delta}\left(\frac{\mathrm{e}^{\mp\Im\mathfrak{m}(\xi)}}{|\nu|^{M+\frac{1}{2}}}\right)$$

for $\frac{\pi}{2} \le \pm\theta \le 2\pi - \delta$.

Proposition 3.3.2 in conjunction with (3.194), (3.197), (3.199), (3.202), (3.204) and (3.206) yields the exponentially improved asymptotic expansions for the functions $\mathbf{A}_{-\nu}(\nu\sec\beta)$, $\mathbf{J}_{-\nu}(\nu\sec\beta)$, $\mathbf{E}_{-\nu}(\nu\sec\beta)$ and that of the corresponding derivatives.

If we neglect the remainder term $\widetilde{R}_{N,M}^{(\mathbf{A})}(\nu,\beta)$ in (3.224) and formally extend the sums to infinity, formula (3.224) reproduces Dingle's original expansion which was mentioned at the beginning of this subsection.

In the following theorem, we give explicit bounds on the remainder terms $R_{N,M}^{(\mathbf{A})}(\nu,\beta)$, $R_{N,M}^{(\mathbf{A}')}(\nu,\beta)$, $\widetilde{R}_{N,M}^{(\mathbf{A})}(\nu,\beta)$ and $\widetilde{R}_{N,M}^{(\mathbf{A}')}(\nu,\beta)$. Note that in these results, N may not necessarily depend on ν and β.

Theorem 3.3.3. *Let N and M be arbitrary fixed non-negative integers such that $M\leq 2N$, and let β be a fixed acute angle. Then we have*

$$\begin{aligned}\left|R_{N,M}^{(\mathbf{A})}(\nu,\beta)\right| \leq{}& \frac{\left|\mathrm{e}^{\mathrm{i}\xi}\mathrm{e}^{\pi\mathrm{i}\nu}\right|}{\left(\frac{1}{2}\pi\left|\nu\right|\tan\beta\right)}\frac{\left|U_M(\mathrm{i}\cot\beta)\right|}{\left|\nu\right|^M}\left|T_{2N-M+\frac{1}{2}}\left((\tan\beta-\beta+\pi)\,\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|\\&+\frac{\left|\mathrm{e}^{-\mathrm{i}\xi}\mathrm{e}^{-\pi\mathrm{i}\nu}\right|}{\left(\frac{1}{2}\pi\left|\nu\right|\tan\beta\right)}\frac{\left|U_M(\mathrm{i}\cot\beta)\right|}{\left|\nu\right|^M}\left|T_{2N-M+\frac{1}{2}}\left((\tan\beta-\beta+\pi)\,\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|\\&+\frac{1}{\left(\frac{1}{2}\pi\tan\beta\right)}\frac{\left|U_M(\mathrm{i}\cot\beta)\right|\Gamma\left(2N-M+\frac{1}{2}\right)}{\pi\left(\tan\beta-\beta+\pi\right)^{2N-M+\frac{1}{2}}\left|\nu\right|^{2N+1}},\end{aligned}$$

$$\begin{aligned}\left|\widetilde{R}_{N,M}^{(\mathbf{A})}(\nu,\beta)\right| \leq{}& \frac{\left|\mathrm{e}^{\mathrm{i}\xi}\right|}{\left(\frac{1}{2}\pi\left|\nu\right|\tan\beta\right)}\frac{\left|U_M(\mathrm{i}\cot\beta)\right|}{\left|\nu\right|^M}\left|T_{2N-M+\frac{1}{2}}\left((\tan\beta-\beta)\,\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|\\&+\frac{\left|\mathrm{e}^{-\mathrm{i}\xi}\right|}{\left(\frac{1}{2}\pi\left|\nu\right|\tan\beta\right)}\frac{\left|U_M(\mathrm{i}\cot\beta)\right|}{\left|\nu\right|^M}\left|T_{2N-M+\frac{1}{2}}\left((\tan\beta-\beta)\,\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|\\&+\frac{1}{\left(\frac{1}{2}\pi\tan\beta\right)}\frac{\left|U_M(\mathrm{i}\cot\beta)\right|\Gamma\left(2N-M+\frac{1}{2}\right)}{\pi\left(\tan\beta-\beta\right)^{2N-M+\frac{1}{2}}\left|\nu\right|^{2N+1}}\end{aligned}$$

provided that $|\theta|\leq\frac{\pi}{2}$, and

$$\begin{aligned}&\left|R_{N,M}^{(\mathbf{A}')}(\nu,\beta)\right| \leq\\&\leq\left(\frac{\sin(2\beta)}{\pi\left|\nu\right|}\right)^{\frac{1}{2}}\left|\mathrm{e}^{\mathrm{i}\xi}\mathrm{e}^{\pi\mathrm{i}\nu}\right|\frac{\left|V_M(\mathrm{i}\cot\beta)\right|}{\left|\nu\right|^M}\left|T_{2N-M+\frac{3}{2}}\left((\tan\beta-\beta+\pi)\,\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|\\&+\left(\frac{\sin(2\beta)}{\pi\left|\nu\right|}\right)^{\frac{1}{2}}\left|\mathrm{e}^{-\mathrm{i}\xi}\mathrm{e}^{-\pi\mathrm{i}\nu}\right|\frac{\left|V_M(\mathrm{i}\cot\beta)\right|}{\left|\nu\right|^M}\left|T_{2N-M+\frac{3}{2}}\left((\tan\beta-\beta+\pi)\,\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|\\&+\left(\frac{\sin(2\beta)}{\pi}\right)^{\frac{1}{2}}\frac{\left|V_M(\mathrm{i}\cot\beta)\right|\Gamma\left(2N-M+\frac{3}{2}\right)}{\pi\left(\tan\beta-\beta+\pi\right)^{2N-M+\frac{3}{2}}\left|\nu\right|^{2N+1}},\end{aligned}$$

$$\left|\widetilde{R}_{N,M}^{(\mathbf{A}')}(\nu,\beta)\right| \leq \left(\frac{\sin(2\beta)}{\pi|\nu|}\right)^{\frac{1}{2}} \left|e^{i\xi}\right| \frac{|V_M(i\cot\beta)|}{|\nu|^M} \left|T_{2N-M+\frac{3}{2}}\left((\tan\beta-\beta)\,\nu e^{\frac{\pi}{2}i}\right)\right|$$
$$+ \left(\frac{\sin(2\beta)}{\pi|\nu|}\right)^{\frac{1}{2}} \left|e^{-i\xi}\right| \frac{|V_M(i\cot\beta)|}{|\nu|^M} \left|T_{2N-M+\frac{3}{2}}\left((\tan\beta-\beta)\,\nu e^{-\frac{\pi}{2}i}\right)\right|$$
$$+ \left(\frac{\sin(2\beta)}{\pi}\right)^{\frac{1}{2}} \frac{|V_M(i\cot\beta)|\,\Gamma\left(2N-M+\frac{3}{2}\right)}{\pi(\tan\beta-\beta)^{2N-M+\frac{3}{2}}|\nu|^{2N+1}}$$

for $|\theta| \leq \frac{\pi}{2}$ *and* $M \geq 1$.

The proof of Theorem 3.3.3 is essentially the same as that of Theorem 3.1.2 and so, we omit it.

3.4 Anger–Weber function of equally large order and argument

This section concerns the large-ν asymptotic expansions of the Anger–Weber function $\mathbf{A}_{-\nu}(\nu)$ and its derivative of equally large order and argument. The asymptotic expansion of the function $\mathbf{A}_{-\nu}(\nu)$, ν positive, was first established by Airey [1] in 1918 and subsequently by Watson [117, eq. (7), p. 319] in 1922. (In fact, both Airey and Watson dealt with the more general case of $\mathbf{A}_{-\nu}(\nu+\kappa)$ when $\kappa = o(|\nu|^{\frac{1}{3}})$, but, for the sake of simplicity, we restrict ourselves to the special case of $\kappa = 0$.)

In modern notation, the asymptotic expansion of $\mathbf{A}_{-\nu}(\nu)$ may be written

$$\mathbf{A}_{-\nu}(\nu) \sim \frac{1}{3\pi}\sum_{n=0}^{\infty} d_{2n} \frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}}, \tag{3.225}$$

as $\nu \to \infty$ in the sector $|\theta| \leq 2\pi - \delta$, where δ denotes an arbitrary small positive constant and $\theta = \arg\nu$ (see the paper [71] of the present author). The cube root in this expansion is defined to be positive when $\theta = 0$, and it is defined elsewhere by analytic continuation. The coefficients d_{2n} are the same as those appearing in the asymptotic expansion of the Hankel function $H_\nu^{(1)}(\nu)$ discussed in Section 3.2. To our best knowledge, the precise range of validity of (3.225) has not been determined in the literature prior to [71].

The structure of this section is as follows. In Subsection 3.4.1, we prove resurgence formulae for the Anger–Weber function and its derivative of equally large order and argument. In Subsection 3.4.2, we obtain error bounds for the asymptotic expansions of these functions. Finally, in Subsection 3.4.3, we derive exponentially improved asymptotic expansions for the above mentioned functions.

3.4.1 The resurgence formulae

In this subsection, we investigate the resurgence properties of the Anger–Weber function, together with its derivative, for equally large order and argument.

We begin by considering the function $\mathbf{A}_{-\nu}(\nu)$. We substitute $-\nu$ in place of ν and ν in place of z in the integral representation (2.69) to obtain

$$\mathbf{A}_{-\nu}(\nu) = \frac{1}{\pi}\int_0^{+\infty} \mathrm{e}^{-\nu(\sinh t - t)}\mathrm{d}t \tag{3.226}$$

for $|\theta| < \frac{\pi}{2}$. The function $\sinh t - t$ has infinitely many second-order saddle points in the complex t-plane situated at $t^{(k)} = 2\pi i k$ with $k \in \mathbb{Z}$. Let $\mathscr{P}^{(0)}(\theta)$ be the steepest descent path emerging from $t^{(0)} = 0$ which coincides with the positive real axis when $\theta = 0$. We set the orientation of $\mathscr{P}^{(0)}(0)$ so that it leads away from the origin. Hence we may write

$$\mathbf{A}_{-\nu}(\nu) = \frac{1}{3\pi\nu^{\frac{1}{3}}} T^{(0/3)}(\nu) \tag{3.227}$$

where $T^{(0/3)}(\nu)$ is given in (1.38) with the specific choices of $f(t) = \sinh t - t$ and $g(t) = 1$. The problem is therefore one of cubic dependence at the endpoint, which we considered in Subsection 1.1.3. To determine the domain $\Delta^{(0)}$ corresponding to this problem, we have to identify the adjacent saddles and contours. When $\theta = -\frac{3\pi}{2}$, the path $\mathscr{P}^{(0)}(\theta)$ connects to the saddle point $t^{(1)} = 2\pi\mathrm{i}$, whereas when $\theta = \frac{3\pi}{2}$, it connects to the saddle point $t^{(-1)} = -2\pi\mathrm{i}$. These are therefore adjacent to $t^{(0)} = 0$. Because the horizontal lines through the points $\frac{3\pi}{2}\mathrm{i}$ and $-\frac{3\pi}{2}\mathrm{i}$ are asymptotes of the corresponding adjacent contours $\mathscr{C}^{(1)}\left(-\frac{3\pi}{2}\right)$ and $\mathscr{C}^{(-1)}\left(\frac{3\pi}{2}\right)$, respectively (see Figure 3.5), there cannot be further saddles adjacent to $t^{(0)}$ other than $t^{(1)}$ and $t^{(-1)}$. The domain $\Delta^{(0)}$ is formed by the set of all points between these adjacent contours.

By analytic continuation, the representation (3.227) is valid in a considerably larger domain than (3.226), namely in $|\theta| < \frac{3\pi}{2}$. Following the analysis in Subsection 1.1.3, we expand $T^{(0/3)}(\nu)$ into a truncated asymptotic power series with remainder,

$$T^{(0/3)}(\nu) = \sum_{n=0}^{N-1} \frac{a_n^{(0/3)}}{\nu^{\frac{n}{3}}} + R_N^{(0/3)}(\nu).$$

It is not difficult to verify that the conditions posed in Subsection 1.1.3 hold good for the domain $\Delta^{(0)}$ and the functions $f(t) = \sinh t - t$ and $g(t) = 1$ with any $N \geq 0$. We choose the orientation of the adjacent contours so that $\mathscr{C}^{(1)}\left(-\frac{3\pi}{2}\right)$ is traversed in the negative direction and $\mathscr{C}^{(-1)}\left(\frac{3\pi}{2}\right)$ is traversed

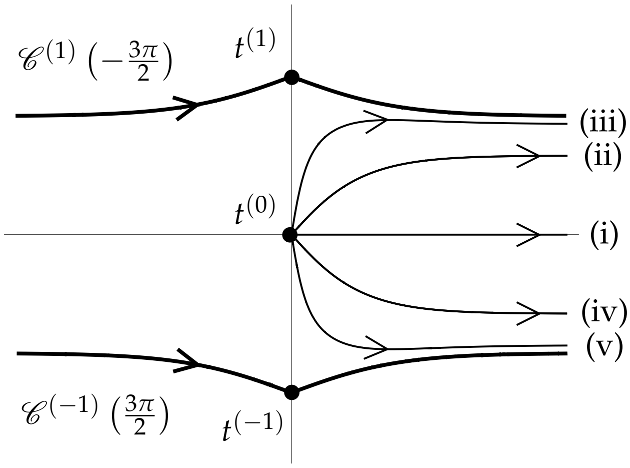

Figure 3.5. *The steepest descent contour $\mathscr{P}^{(0)}(\theta)$ associated with the Anger–Weber function of equal order and argument emanating from the saddle point $t^{(0)} = 0$ when (i) $\theta = 0$, (ii) $\theta = -\pi$ and (iii) $\theta = -\frac{7\pi}{5}$, (iv) $\theta = \pi$ and (v) $\theta = \frac{7\pi}{5}$. The paths $\mathscr{C}^{(1)}\left(-\frac{3\pi}{2}\right)$ and $\mathscr{C}^{(-1)}\left(\frac{3\pi}{2}\right)$ are the adjacent contours for $t^{(0)}$. The domain $\Delta^{(0)}$ comprises all points between $\mathscr{C}^{(1)}\left(-\frac{3\pi}{2}\right)$ and $\mathscr{C}^{(-1)}\left(\frac{3\pi}{2}\right)$.*

in the positive direction with respect to the domain $\Delta^{(0)}$. Consequently the orientation anomalies are $\gamma_{01} = 1$ and $\gamma_{0-1} = 0$, respectively. The relevant singulant pair is given by

$$\mathcal{F}_{0\pm1} = \sinh(\pm 2\pi\mathrm{i}) \mp 2\pi\mathrm{i} = \mp 2\pi\mathrm{i}, \quad \arg \mathcal{F}_{0\pm1} = \sigma_{0\pm1} = \pm\frac{3\pi}{2}.$$

We thus find that

$$\begin{aligned} R_N^{(0/3)}(\nu) = &-\frac{(-\mathrm{i})^N}{2\pi\mathrm{i}\nu^{\frac{N}{3}}}\int_0^{+\infty}\frac{u^{\frac{N}{3}-1}\mathrm{e}^{-2\pi u}}{1+\mathrm{i}(u/\nu)^{\frac{1}{3}}}T^{(2,1/3)}\left(u\mathrm{e}^{-\frac{3\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ &+\frac{\mathrm{i}^N}{2\pi\mathrm{i}\nu^{\frac{N}{3}}}\int_0^{+\infty}\frac{u^{\frac{N}{3}-1}\mathrm{e}^{-2\pi u}}{1-\mathrm{i}(u/\nu)^{\frac{1}{3}}}T^{(2,-1/3)}\left(u\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}\right)\mathrm{d}u \end{aligned} \tag{3.228}$$

with $|\theta| < \frac{3\pi}{2}$ and $N \geq 0$.

We may now connect the above results with the asymptotic expansion (3.225) of $\mathbf{A}_{-\nu}(\nu)$ by writing

$$\mathbf{A}_{-\nu}(\nu) = \frac{1}{3\pi}\sum_{n=0}^{N-1} d_{2n}\frac{\Gamma\left(\frac{2n+1}{3}\right)}{\nu^{\frac{2n+1}{3}}} + R_N^{(\mathbf{A})}(\nu), \tag{3.229}$$

with the notation $d_{2n} = a_{2n}^{(0/3)}/\Gamma\left(\frac{2n+1}{3}\right)$ and $R_N^{(\mathbf{A})}(\nu) = (3\pi\nu^{\frac{1}{3}})^{-1}R_{2N}^{(0/3)}(\nu)$. When deriving (3.229), we used implicitly the fact that $a_n^{(0/3)}$ vanishes for odd n. To prove this, first note that, by (1.46),

$$a_n^{(0/3)} = \frac{\Gamma\left(\frac{n+1}{3}\right)}{\Gamma(n+1)}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{t^3}{\sinh t - t}\right)^{\frac{n+1}{3}}\right]_{t=0}. \tag{3.230}$$

Because the quantity under the differentiation sign is an even function of t and therefore its odd-order derivatives at $t=0$ are zero, the claim follows. Formulae (3.230) and $d_{2n} = a_{2n}^{(0/3)}/\Gamma\left(\frac{2n+1}{3}\right)$ together reproduces the expression (3.74) for the coefficients d_{2n}.

It is possible to obtain a representation for the remainder term $R_N^{(\mathbf{A})}(\nu)$ simpler than (3.228) by observing that the functions $T^{(2,1/3)}(ue^{-\frac{3\pi}{2}\mathrm{i}})$ and $T^{(2,-1/3)}(ue^{\frac{3\pi}{2}\mathrm{i}})$ can be expressed in terms of the Hankel function $H_{\mathrm{i}u}^{(1)}(ue^{\frac{\pi}{2}\mathrm{i}})$. To see this, let us take $\widetilde{\mathscr{P}}^{(0)}(\theta) \stackrel{\text{def}}{=} \mathscr{P}^{(0)}(\theta+3\pi)$, which is exactly the steepest descent path that we have denoted by $\mathscr{P}^{(0)}(\theta)$ in Subsection 3.2.1 on the resurgence properties of the Hankel function $H_\nu^{(1)}(\nu)$. Then the contour $\mathscr{C}^{(1)}\left(-\frac{3\pi}{2}\right)$ is congruent to $\widetilde{\mathscr{P}}^{(0)}\left(-\frac{\pi}{2}\right) \cup \widetilde{\mathscr{P}}^{(0)}\left(\frac{7\pi}{2}\right)$ but is shifted upwards in the complex plane by $2\pi\mathrm{i}$ and has opposite orientation, whence[6]

$$\begin{aligned}
T^{(2,1/3)}(ue^{-\frac{3\pi}{2}\mathrm{i}}) &= u^{\frac{1}{3}}e^{-\frac{\pi}{2}\mathrm{i}}\int_{\mathscr{C}^{(1)}\left(-\frac{3\pi}{2}\right)} e^{-ue^{-\frac{3\pi}{2}\mathrm{i}}(\sinh t - t - \sinh(2\pi\mathrm{i}) + 2\pi\mathrm{i})}\mathrm{d}t \\
&= -u^{\frac{1}{3}}e^{-\frac{\pi}{2}\mathrm{i}}\int_{\widetilde{\mathscr{P}}^{(0)}\left(-\frac{\pi}{2}\right)\cup\widetilde{\mathscr{P}}^{(0)}\left(\frac{7\pi}{2}\right)} e^{-ue^{-\frac{\pi}{2}\mathrm{i}}(t-\sinh t)}\mathrm{d}t \\
&= -\pi u^{\frac{1}{3}}H_{-\mathrm{i}u}^{(2)}(ue^{-\frac{\pi}{2}\mathrm{i}}) = \pi u^{\frac{1}{3}}H_{\mathrm{i}u}^{(1)}(ue^{\frac{\pi}{2}\mathrm{i}}),
\end{aligned} \tag{3.231}$$

using an argument similar to (3.71). One can prove in an analogous manner that

$$T^{(2,-1/3)}(ue^{\frac{3\pi}{2}\mathrm{i}}) = -\pi u^{\frac{1}{3}}H_{\mathrm{i}u}^{(1)}(ue^{\frac{\pi}{2}\mathrm{i}}). \tag{3.232}$$

The desired expression for the remainder term $R_N^{(\mathbf{A})}(\nu)$ now follows from (3.231), (3.232) and $R_N^{(\mathbf{A})}(\nu) = (3\pi\nu^{\frac{1}{3}})^{-1}R_{2N}^{(0/3)}(\nu)$:

$$R_N^{(\mathbf{A})}(\nu) = \frac{(-1)^N}{3\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty}\frac{u^{\frac{2N-2}{3}}e^{-2\pi u}}{1+(u/\nu)^{\frac{2}{3}}}\mathrm{i}H_{\mathrm{i}u}^{(1)}(ue^{\frac{\pi}{2}\mathrm{i}})\mathrm{d}u \tag{3.233}$$

[6]We specify the orientation of $\widetilde{\mathscr{P}}^{(0)}\left(\frac{7\pi}{2}\right)$ so that it leads into the saddle point $t^{(0)}$.

for $|\theta| < \frac{3\pi}{2}$ and $N \geq 0$. Equations (3.229) and (3.233) together yield the exact resurgence formula for $\mathbf{A}_{-\nu}(\nu)$.

Consider now the resurgence properties of the derivative $\mathbf{A}'_{-\nu}(\nu)$. From (2.69), we infer that

$$\mathbf{A}'_{-\nu}(\nu) = -\frac{1}{\pi}\int_0^{+\infty} \mathrm{e}^{-\nu(\sinh t - t)} \sinh t \mathrm{d}t \tag{3.234}$$

with $|\theta| < \frac{\pi}{2}$. Observe that the saddle point structure of the integrand in (3.234) is identical to that of (3.226). In particular, the problem is one of cubic dependence at the saddle point, and the domain $\Delta^{(0)}$ corresponding to this problem is the same as that in the case of $\mathbf{A}_{-\nu}(\nu)$. Since the derivation is essentially the same as that of the resurgence formula for $\mathbf{A}_{-\nu}(\nu)$, we omit the details and provide only the final result. We have

$$\mathbf{A}'_{-\nu}(\nu) = -\frac{1}{3\pi}\sum_{n=0}^{N-1} g_{2n}\frac{\Gamma\left(\frac{2n+2}{3}\right)}{\nu^{\frac{2n+2}{3}}} + R_N^{(\mathbf{A}')}(\nu) \tag{3.235}$$

where the remainder term $R_N^{(\mathbf{A}')}(\nu)$ is given by the integral formula

$$R_N^{(\mathbf{A}')}(\nu) = \frac{(-1)^N}{3\pi}\frac{1}{\nu^{\frac{2N+2}{3}}}\int_0^{+\infty}\frac{u^{\frac{2N-1}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^{\frac{2}{3}}} H_{\mathrm{i}u}^{(1)\prime}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u, \tag{3.236}$$

provided that $|\theta| < \frac{3\pi}{2}$ and $N \geq 1$. The coefficients g_{2n} are the same as those appearing in the asymptotic expansion of the function $H_\nu^{(1)\prime}(\nu)$ and are given by (3.80).

By neglecting the remainder term in (3.235) and formally extending the sum to infinity, we obtain an asymptotic expansion for the function $\mathbf{A}'_{-\nu}(\nu)$. This asymptotic expansion is valid in the sector $|\theta| \leq 2\pi - \delta$, with δ being an arbitrary small positive constant.

3.4.2 Error bounds

This subsection is devoted to obtaining computable bounds for the remainders $R_N^{(\mathbf{A})}(\nu)$ and $R_N^{(\mathbf{A}')}(\nu)$. To our best knowledge, no explicit bounds for these remainder terms have been given in the literature. Unless otherwise stated, we assume that $N \geq 0$ when dealing with $R_N^{(\mathbf{A})}(\nu)$, and $N \geq 1$ is assumed in the case of $R_N^{(\mathbf{A}')}(\nu)$.

First, we shall obtain two different sets of bounds for $R_N^{(\mathbf{A})}(\nu)$ and $R_N^{(\mathbf{A}')}(\nu)$ which are valid in the sector $|\theta| < \frac{3\pi}{2}$ and are useful when ν is bounded away from the Stokes lines $\theta = \pm\frac{3\pi}{2}$. From (3.233) and the fact that $\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)$ is positive for positive u, one can infer

$$\left|R_N^{(\mathbf{A})}(\nu)\right| \leq \frac{1}{3\pi}\frac{1}{|\nu|^{\frac{2N+1}{3}}}\int_0^{+\infty}\frac{u^{\frac{2N-2}{3}}\mathrm{e}^{-2\pi u}}{\left|1+(u/\nu)^{\frac{2}{3}}\right|}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u.$$

We estimate $1/\left|1+(u/\nu)^{\frac{2}{3}}\right|$ via the inequality (2.36) and then compare the result with (3.96) in order to obtain the error bound

$$\left|R_N^{(\mathbf{A})}(\nu)\right| \leq \frac{1}{3\pi}\left|d_{2N}\right|\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}}\begin{cases}\left|\csc\left(\frac{2}{3}\theta\right)\right| & \text{if } \frac{3\pi}{4} < |\theta| < \frac{3\pi}{2},\\ 1 & \text{if } |\theta| \leq \frac{3\pi}{4}.\end{cases} \tag{3.237}$$

Likewise, one can show that

$$\left|R_N^{(\mathbf{A}')}(\nu)\right| \leq \frac{1}{3\pi}\left|g_{2N}\right|\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}}\begin{cases}\left|\csc\left(\frac{2}{3}\theta\right)\right| & \text{if } \frac{3\pi}{4} < |\theta| < \frac{3\pi}{2},\\ 1 & \text{if } |\theta| \leq \frac{3\pi}{4}.\end{cases}$$

Now let us assume that $\frac{3\pi}{4} \leq |\theta| < \frac{3\pi}{2}$. By making use of the identity

$$\frac{1}{1+(u/\nu)^{\frac{2}{3}}} = \frac{1}{1+(u/\nu)^2} - \frac{1}{\nu^{\frac{2}{3}}}\frac{u^{\frac{2}{3}}}{1+(u/\nu)^2} + \frac{1}{\nu^{\frac{4}{3}}}\frac{u^{\frac{4}{3}}}{1+(u/\nu)^2},$$

we remove the cube root in the denominator of the integrand in (3.233) and obtain

$$\begin{aligned}R_N^{(\mathbf{A})}(\nu) = {} & \frac{(-1)^N}{3\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty}\frac{u^{\frac{2N-2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u\\ & + \frac{(-1)^{N+1}}{3\pi}\frac{1}{\nu^{\frac{2N+3}{3}}}\int_0^{+\infty}\frac{u^{\frac{2N}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u\\ & + \frac{(-1)^N}{3\pi}\frac{1}{\nu^{\frac{2N+5}{3}}}\int_0^{+\infty}\frac{u^{\frac{2N+2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u.\end{aligned} \tag{3.238}$$

For convenience, we denote $f(t) = t - \sinh t$. Let $\widetilde{\mathscr{C}}^{(0)}(\theta)$ be the steepest descent contour through the saddle $t^{(0)} = 0$ which is the union $\widetilde{\mathscr{P}}^{(0)}(\theta) \cup \widetilde{\mathscr{P}}^{(0)}(\theta+2\pi)$, where $\widetilde{\mathscr{P}}^{(0)}(\theta)$ is as defined in the previous subsection. Note that $\widetilde{\mathscr{C}}^{(0)}(\theta)$ is a steepest descent contour for $f(t)$ and that it is exactly the doubly infinite steepest descent path that was denoted by $\mathscr{C}^{(0)}(\theta)$ in Subsection

3.2.1 (on the resurgence properties of the Hankel function $H_\nu^{(1)}(\nu)$). With this notation, we may write

$$H_{\mathrm{i}u}^{(1)}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) = -\frac{1}{\pi\mathrm{i}}\int_{\widetilde{\mathscr{C}}^{(0)}\left(\frac{\pi}{2}\right)} \mathrm{e}^{-u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}f(t)}\mathrm{d}t$$

for any $u > 0$ (cf. equation (3.92)). By replacing in (3.238) the function $H_{\mathrm{i}u}^{(1)}(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}})$ by the above integral representation and performing the change of integration variable from u and t to s and t via $s = u\mathrm{i}f(t)$, one finds

$$\begin{aligned} R_N^{(\mathbf{A})}(\nu) &= \frac{(-1)^{N+1}}{3\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty} s^{\frac{2N-2}{3}}\mathrm{e}^{-s}\frac{1}{\pi}\int_{\widetilde{\mathscr{C}}^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+1}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\mathrm{d}t\mathrm{d}s \\ &+ \frac{(-1)^{N}}{3\pi}\frac{1}{\nu^{\frac{2N+3}{3}}}\int_0^{+\infty} s^{\frac{2N}{3}}\mathrm{e}^{-s}\frac{1}{\pi}\int_{\widetilde{\mathscr{C}}^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+3}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\mathrm{d}t\mathrm{d}s \\ &+ \frac{(-1)^{N+1}}{3\pi}\frac{1}{\nu^{\frac{2N+5}{3}}}\int_0^{+\infty} s^{\frac{2N+2}{3}}\mathrm{e}^{-s}\frac{1}{\pi}\int_{\widetilde{\mathscr{C}}^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+5}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\mathrm{d}t\mathrm{d}s. \end{aligned} \tag{3.239}$$

Denote by $\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)$ the part of the steepest descent contour $\widetilde{\mathscr{C}}^{(0)}\left(\frac{\pi}{2}\right)$ which lies in the right half-plane. (The contour $\widetilde{\mathscr{C}}^{(0)}\left(\frac{\pi}{2}\right)$ is congruent to and has the opposite orientation as $\mathscr{C}^{(-1)}\left(\frac{3\pi}{2}\right)$ but is shifted upwards in the complex plane by $2\pi\mathrm{i}$, cf. Figure 3.5.) An argument similar to (3.26) shows that (3.239) may be written as

$$\begin{aligned} R_N^{(\mathbf{A})}(\nu) &= \frac{(-1)^{N+1}}{3\pi}\frac{1}{\nu^{\frac{2N+1}{3}}}\int_0^{+\infty} s^{\frac{2N-2}{3}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+1}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\mathrm{d}x\mathrm{d}s \\ &+ \frac{(-1)^{N}}{3\pi}\frac{1}{\nu^{\frac{2N+3}{3}}}\int_0^{+\infty} s^{\frac{2N}{3}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+3}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\mathrm{d}x\mathrm{d}s \\ &+ \frac{(-1)^{N+1}}{3\pi}\frac{1}{\nu^{\frac{2N+5}{3}}}\int_0^{+\infty} s^{\frac{2N+2}{3}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)} \frac{(\mathrm{i}f(t))^{-\frac{2N+5}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\mathrm{d}x\mathrm{d}s, \end{aligned} \tag{3.240}$$

where we have taken $x = \mathfrak{Re}(t)$. A formula for the coefficients d_{2N} analogous to (3.240) will be needed when deriving the error bounds; it can be obtained by inserting (3.240) into the relation $\Gamma\left(\frac{2N+1}{3}\right)d_{2N} = 3\pi\nu^{\frac{2N+1}{3}}\left(R_N^{(\mathbf{A})}(\nu) - R_{N+1}^{(\mathbf{A})}(\nu)\right)$. Hence we have

$$\Gamma\left(\frac{2N+1}{3}\right)d_{2N} = (-1)^{N+1}\int_0^{+\infty} s^{\frac{2N-2}{3}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)} (\mathrm{i}f(t))^{-\frac{2N+1}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\mathrm{d}x\mathrm{d}s \tag{3.241}$$

for any $N \geq 0$ (cf. equation (3.97)). We first estimate the right-hand side of (3.240) by using the inequality (2.36) and then compare the result with (3.241), thereby obtaining the error bound

$$\begin{aligned}\left|R_N^{(\mathbf{A})}(\nu)\right| \leq &\left(\frac{1}{3\pi}\left|d_{2N}\right|\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}}+\frac{1}{3\pi}\left|d_{2N+2}\right|\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}}\right.\\ &\left.+\frac{1}{3\pi}\left|d_{2N+4}\right|\frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}}\right)\begin{cases}\left|\csc\left(2\theta\right)\right| & \text{if } \frac{5\pi}{4}<|\theta|<\frac{3\pi}{2},\\ 1 & \text{if } |\theta|\leq\frac{5\pi}{4}.\end{cases}\end{aligned} \tag{3.242}$$

In arriving at this estimate, one uses the positivity of $\mathrm{i}f(t)$ and the monotonicity of x on the path $\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)$. Note that it is possible to extend the validity of the above estimate from $\frac{3\pi}{4}\leq|\theta|\leq\frac{5\pi}{4}$ to $|\theta|\leq\frac{5\pi}{4}$ due to our earlier bound (3.237).

Let us now turn our attention to the estimation of the remainder $R_N^{(\mathbf{A}')}(\nu)$. In this case, one finds that the expressions corresponding to (3.240) and (3.241) are

$$\begin{aligned}R_N^{(\mathbf{A}')}(\nu) = &\\ =&\frac{(-1)^{N+1}}{3\pi}\frac{1}{\nu^{\frac{2N+2}{3}}}\int_0^{+\infty}s^{\frac{2N-1}{3}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)}\frac{(\mathrm{i}f(t))^{-\frac{2N+2}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\sinh x\cos y\,\mathrm{d}y\mathrm{d}s\\ &+\frac{(-1)^{N+1}}{3\pi}\frac{1}{\nu^{\frac{2N+2}{3}}}\int_0^{+\infty}s^{\frac{2N-1}{3}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)}\frac{(\mathrm{i}f(t))^{-\frac{2N+2}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\cosh x\sin y\,\mathrm{d}x\mathrm{d}s\\ &+\frac{(-1)^{N}}{3\pi}\frac{1}{\nu^{\frac{2N+4}{3}}}\int_0^{+\infty}s^{\frac{2N+1}{3}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)}\frac{(\mathrm{i}f(t))^{-\frac{2N+4}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\sinh x\cos y\,\mathrm{d}y\mathrm{d}s\\ &+\frac{(-1)^{N}}{3\pi}\frac{1}{\nu^{\frac{2N+4}{3}}}\int_0^{+\infty}s^{\frac{2N+1}{3}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)}\frac{(\mathrm{i}f(t))^{-\frac{2N+4}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\cosh x\sin y\,\mathrm{d}x\mathrm{d}s\\ &+\frac{(-1)^{N+1}}{3\pi}\frac{1}{\nu^{\frac{2N+6}{3}}}\int_0^{+\infty}s^{\frac{2N+3}{3}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)}\frac{(\mathrm{i}f(t))^{-\frac{2N+6}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\sinh x\cos y\,\mathrm{d}y\mathrm{d}s\\ &+\frac{(-1)^{N+1}}{3\pi}\frac{1}{\nu^{\frac{2N+6}{3}}}\int_0^{+\infty}s^{\frac{2N+3}{3}}\mathrm{e}^{-s}\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)}\frac{(\mathrm{i}f(t))^{-\frac{2N+6}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}}{1+(s/(\nu\mathrm{i}f(t)))^2}\cosh x\sin y\,\mathrm{d}x\mathrm{d}s\end{aligned} \tag{3.243}$$

and

$$\begin{aligned}\Gamma\left(\frac{2N+2}{3}\right)g_{2N} = (-1)^N\int_0^{+\infty}s^{\frac{2N-1}{3}}\mathrm{e}^{-s}&\\ \times\frac{2}{\pi}\int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)}&(\mathrm{i}f(t))^{-\frac{2N+2}{3}}\mathrm{e}^{-2\pi\frac{s}{\mathrm{i}f(t)}}\sinh x\cos y\,\mathrm{d}y\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+(-1)^N \int_0^{+\infty} s^{\frac{2N-1}{3}} \mathrm{e}^{-s} \\
&\times \frac{2}{\pi} \int_{\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)} (\mathrm{i}f(t))^{-\frac{2N+2}{3}} \mathrm{e}^{-2\pi \frac{s}{\mathrm{i}f(t)}} \cosh x \sin y \mathrm{d}x\mathrm{d}s
\end{aligned}
$$

for any $N \geq 1$ with $y = \mathfrak{Im}(t)$. From these expressions and the inequality (2.36), we establish

$$
\begin{aligned}
\left|R_N^{(\mathbf{A}')}(\nu)\right| \leq &\left(\frac{1}{3\pi}|g_{2N}| \frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}} + \frac{1}{3\pi}|g_{2N+2}| \frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}}\right. \\
&\left.+\frac{1}{3\pi}|g_{2N+4}| \frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}}\right) \begin{cases} |\csc(2\theta)| & \text{if } \frac{5\pi}{4} < |\theta| < \frac{3\pi}{2}, \\ 1 & \text{if } |\theta| \leq \frac{5\pi}{4}, \end{cases}
\end{aligned} \tag{3.244}
$$

by making use of the additional facts that on the contour $\widetilde{\mathscr{C}}_2^{(0)}\left(\frac{\pi}{2}\right)$ the quantity $\mathrm{i}f(t)$ is positive, x is monotonic and y decreases monotonically from $\frac{\pi}{2}$ to 0.

In the special case when ν is positive, we have $0 < 1/(1+(u/\nu)^{\frac{2}{3}}) < 1$ in (3.233) and together with (3.96), the mean value theorem of integration shows that

$$
R_N^{(\mathbf{A})}(\nu) = \frac{1}{3\pi} d_{2N} \frac{\Gamma\left(\frac{2N+1}{3}\right)}{\nu^{\frac{2N+1}{3}}} \Theta_N(\nu),
$$

where $0 < \Theta_N(\nu) < 1$ is a suitable number that depends on ν and N. In other words, the remainder term $R_N^{(\mathbf{A})}(\nu)$ does not exceed the first neglected term in absolute value and has the same sign provided that $\nu > 0$. We can prove in a similar manner that

$$
R_N^{(\mathbf{A}')}(\nu) = -\frac{1}{3\pi} g_{2N} \frac{\Gamma\left(\frac{2N+2}{3}\right)}{\nu^{\frac{2N+2}{3}}} \Xi_N(\nu),
$$

where $0 < \Xi_N(\nu) < 1$ is an appropriate number that depends on ν and N.

In the case that ν lies in the closed sector $|\theta| \leq \frac{5\pi}{4}$, the bounds (3.242) and (3.244) are as sharp as one can reasonably expect. However, although acceptable in much of the sectors $\frac{5\pi}{4} < |\theta| < \frac{3\pi}{2}$, the bounds (3.242) and (3.244) become inappropriate near the Stokes lines $\theta = \pm\frac{3\pi}{2}$. We now provide alternative estimates that are suitable for the sectors $\frac{5\pi}{4} < |\theta| < 2\pi$ (which include the Stokes lines $\theta = \pm\frac{3\pi}{2}$). We may use (3.233) and (3.236) to define the remainder terms $R_N^{(\mathbf{A})}(\nu)$ and $R_N^{(\mathbf{A}')}(\nu)$ in the sectors $\frac{3\pi}{2} \leq |\theta| < 2\pi$. These alternative bounds can be derived based on the representations (3.240) and (3.243). Their derivation is similar to that of (2.43) discussed in Subsection 2.1.2, and the details are therefore omitted. One finds that in the sectors $\frac{5\pi}{4} < |\theta| < 2\pi$, the remainder $R_N^{(\mathbf{A})}(\nu)$ can be estimated as follows:

$$\begin{aligned}\left|R_N^{(\mathbf{A})}(\nu)\right| \leq & \frac{\left|\csc\left(2\left(\theta-\varphi^*\right)\right)\right|}{\cos^{\frac{2N+1}{3}}\varphi^*}\frac{1}{3\pi}\left|d_{2N}\right|\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}} \\ & + \frac{\left|\csc\left(2\left(\theta-\varphi^{**}\right)\right)\right|}{\cos^{\frac{2N+3}{3}}\varphi^{**}}\frac{1}{3\pi}\left|d_{2N+2}\right|\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}} \\ & + \frac{\left|\csc\left(2\left(\theta-\varphi^{***}\right)\right)\right|}{\cos^{\frac{2N+5}{3}}\varphi^{***}}\frac{1}{3\pi}\left|d_{2N+4}\right|\frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}}.\end{aligned} \tag{3.245}$$

Here φ^*, φ^{**} and φ^{***} are the minimizing values given by Lemma 3.4.1 below with $\chi = \frac{2N+1}{3}$, $\chi = \frac{2N+3}{3}$ and $\chi = \frac{2N+5}{3}$, respectively.

Lemma 3.4.1. *Let χ be a fixed positive real number, and let θ be a fixed angle such that $\frac{5\pi}{4} < |\theta| < 2\pi$. Consider the problem of minimizing $\left|\csc\left(2\left(\theta-\varphi\right)\right)\right| / \cos^{\chi}\varphi$ in φ with respect to the following conditions: $\frac{5\pi}{4} < |\theta-\varphi| < \frac{3\pi}{2}$, and $0 < \varphi < \frac{\pi}{2}$ when $\frac{5\pi}{4} < \theta < 2\pi$ while $-\frac{\pi}{2} < \varphi < 0$ when $-2\pi < \theta < -\frac{5\pi}{4}$. Under these conditions, the minimization problem has a unique solution φ^* that satisfies the implicit equation*

$$(\chi+2)\cos\left(2\theta-3\varphi^*\right) = (\chi-2)\cos\left(2\theta-\varphi^*\right),$$

and has the property that $0 < \varphi^* < -\frac{5\pi}{4}+\theta$ if $\frac{5\pi}{4} < \theta < \frac{3\pi}{2}$, $-\frac{3\pi}{2}+\theta < \varphi^* < -\frac{5\pi}{4}+\theta$ if $\frac{3\pi}{2} \leq \theta < \frac{7\pi}{4}$, $-\frac{3\pi}{2}+\theta < \varphi^* < \frac{\pi}{2}$ if $\frac{7\pi}{4} \leq \theta < 2\pi$, $\frac{5\pi}{4}+\theta < \varphi^* < 0$ if $-\frac{3\pi}{2} < \theta < -\frac{5\pi}{4}$, $\frac{5\pi}{4}+\theta < \varphi^* < \frac{3\pi}{2}+\theta$ if $-\frac{7\pi}{4} < \theta \leq -\frac{3\pi}{2}$ and $-\frac{\pi}{2} < \varphi^* < \frac{3\pi}{2}+\theta$ if $-2\pi < \theta \leq -\frac{7\pi}{4}$.

Lemma 3.4.1 follows easily from Lemma 2.1.2 of Meijer; the details are left to the reader. Similarly, for $\frac{5\pi}{4} < |\theta| < 2\pi$, the remainder $R_N^{(\mathbf{A}')}(\nu)$ satisfies the following bound:

$$\begin{aligned}\left|R_N^{(\mathbf{A}')}(\nu)\right| \leq & \frac{\left|\csc\left(2\left(\theta-\varphi^*\right)\right)\right|}{\cos^{\frac{2N+2}{3}}\varphi^*}\frac{1}{3\pi}\left|g_{2N}\right|\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}} \\ & + \frac{\left|\csc\left(2\left(\theta-\varphi^{**}\right)\right)\right|}{\cos^{\frac{2N+4}{3}}\varphi^{**}}\frac{1}{3\pi}\left|g_{2N+2}\right|\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}} \\ & + \frac{\left|\csc\left(2\left(\theta-\varphi^{***}\right)\right)\right|}{\cos^{\frac{2N+6}{3}}\varphi^{***}}\frac{1}{3\pi}\left|g_{2N+4}\right|\frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}},\end{aligned} \tag{3.246}$$

where φ^*, φ^{**} and φ^{***} are the minimizing values given by Lemma 3.4.1 with $\chi = \frac{2N+2}{3}$, $\chi = \frac{2N+4}{3}$ and $\chi = \frac{2N+6}{3}$, respectively.

The following simple estimates are suitable for the sectors $\frac{5\pi}{4} < |\theta| \leq \frac{3\pi}{2}$ (especially near the Stokes lines $\theta = \pm\frac{3\pi}{2}$) and $N \geq 4$ and can be obtained from (3.245) and (3.246) using an argument similar to that given in Subsection 2.1.2:

$$\begin{aligned}\left|R_N^{(\mathbf{A})}(\nu)\right| \leq{}& \frac{1}{6}\sqrt{3\mathrm{e}\left(2N+\frac{11}{2}\right)}\frac{1}{3\pi}\left|d_{2N}\right|\frac{\Gamma\left(\frac{2N+1}{3}\right)}{|\nu|^{\frac{2N+1}{3}}}\\ &+\frac{1}{6}\sqrt{3\mathrm{e}\left(2N+\frac{15}{2}\right)}\frac{1}{3\pi}\left|d_{2N+2}\right|\frac{\Gamma\left(\frac{2N+3}{3}\right)}{|\nu|^{\frac{2N+3}{3}}}\\ &+\frac{1}{6}\sqrt{3\mathrm{e}\left(2N+\frac{19}{2}\right)}\frac{1}{3\pi}\left|d_{2N+4}\right|\frac{\Gamma\left(\frac{2N+5}{3}\right)}{|\nu|^{\frac{2N+5}{3}}}\end{aligned}$$

and

$$\begin{aligned}\left|R_N^{(\mathbf{A}')}(\nu)\right| \leq{}& \frac{1}{6}\sqrt{3\mathrm{e}\left(2N+\frac{13}{2}\right)}\frac{1}{3\pi}\left|g_{2N}\right|\frac{\Gamma\left(\frac{2N+2}{3}\right)}{|\nu|^{\frac{2N+2}{3}}}\\ &+\frac{1}{6}\sqrt{3\mathrm{e}\left(2N+\frac{17}{2}\right)}\frac{1}{3\pi}\left|g_{2N+2}\right|\frac{\Gamma\left(\frac{2N+4}{3}\right)}{|\nu|^{\frac{2N+4}{3}}}\\ &+\frac{1}{6}\sqrt{3\mathrm{e}\left(2N+\frac{21}{2}\right)}\frac{1}{3\pi}\left|g_{2N+4}\right|\frac{\Gamma\left(\frac{2N+6}{3}\right)}{|\nu|^{\frac{2N+6}{3}}}.\end{aligned}$$

These bounds may be used in conjunction with our earlier results (3.242) and (3.244), respectively.

3.4.3 Exponentially improved asymptotic expansions

The aim of this subsection is to give exponentially improved asymptotic expansions for the Anger–Weber function and its derivative of equally large order and argument. Re-expansions for the remainder terms of the asymptotic expansions of these functions were derived, using formal methods, by Dingle [35, exer. 23, p. 485]. He divided each of the asymptotic expansions into three parts according to the value of $n \bmod 3$ and considered the three remainders of these expansions separately. We shall derive here the rigorous forms of Dingle's formal re-expansions by truncating them after a finite number of terms and constructing their error bounds.

It is not possible to re-expand directly the remainders $R_N^{(\mathbf{A})}(\nu)$ and $R_N^{(\mathbf{A}')}(\nu)$ in terms of terminant functions because of the presence of cube roots in the denominators of the integrands in their representations (3.233) and (3.236). To overcome this difficulty, we follow Dingle's idea and write both $\mathbf{A}_{-\nu}(\nu)$ and

$\mathbf{A}'_{-\nu}(\nu)$ as a sum of three truncated asymptotic expansions plus a remainder thereby obtaining representations different from (3.229) and (3.235). The form of the remainders in these alternative expressions will be suitable for our purposes. Assuming $|\theta| < \frac{\pi}{2}$, the representation (3.233) for $\mathbf{A}_{-\nu}(\nu) = R_0^{(\mathbf{A})}(\nu)$ can be re-arranged in the form

$$\begin{aligned}\mathbf{A}_{-\nu}(\nu) = & \frac{1}{3\pi}\frac{1}{\nu^{\frac{1}{3}}}\int_0^{+\infty}\frac{u^{-\frac{2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\big(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big)\mathrm{d}u \\ & - \frac{1}{3\pi}\frac{1}{\nu}\int_0^{+\infty}\frac{\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\big(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big)\mathrm{d}u \\ & + \frac{1}{3\pi}\frac{1}{\nu^{\frac{5}{3}}}\int_0^{+\infty}\frac{u^{\frac{2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\big(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big)\mathrm{d}u\end{aligned}$$

(cf. equation (3.238)). Next, we expand the denominators of the integrands by means of (1.7) (with m and M in place of n and N in the second integral, and with k and K in place of n and N in the third integral) and make use of the representation (3.96) to deduce

$$\begin{aligned}\mathbf{A}_{-\nu}(\nu) = & \frac{1}{3\pi}\sum_{n=0}^{N-1} d_{6n}\frac{\Gamma\left(2n+\frac{1}{3}\right)}{\nu^{2n+\frac{1}{3}}} + \frac{1}{3\pi}\sum_{m=0}^{M-1} d_{6m+2}\frac{\Gamma\left(2m+1\right)}{\nu^{2m+1}} \\ & + \frac{1}{3\pi}\sum_{k=0}^{K-1} d_{6k+4}\frac{\Gamma\left(2k+\frac{5}{3}\right)}{\nu^{2k+\frac{5}{3}}} + R_{N,M,K}^{(\mathbf{A})}(\nu)\,.\end{aligned} \tag{3.247}$$

The remainder term $R_{N,M,K}^{(\mathbf{A})}(\nu)$ can be expressed as

$$\begin{aligned}R_{N,M,K}^{(\mathbf{A})}(\nu) = & \frac{(-1)^N}{3\pi}\frac{1}{\nu^{2N+\frac{1}{3}}}\int_0^{+\infty}\frac{u^{2N-\frac{2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\big(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big)\mathrm{d}u \\ & + \frac{(-1)^{M+1}}{3\pi}\frac{1}{\nu^{2M+1}}\int_0^{+\infty}\frac{u^{2M}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\big(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big)\mathrm{d}u \\ & + \frac{(-1)^K}{3\pi}\frac{1}{\nu^{2K+\frac{5}{3}}}\int_0^{+\infty}\frac{u^{2K+\frac{2}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}\mathrm{i}H_{\mathrm{i}u}^{(1)}\big(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\big)\mathrm{d}u\end{aligned}$$

for $|\theta| < \frac{\pi}{2}$ and $N, M, K \geq 0$. Equation (3.247) gives an expansion of $\mathbf{A}_{-\nu}(\nu)$ which has the convenient property that its remainder term $R_{N,M,K}^{(\mathbf{A})}(\nu)$ can be expressed in a simple way using terminant functions.

In a similar manner, we write

$$\begin{aligned}\mathbf{A}'_{-\nu}(\nu) = &-\frac{1}{3\pi}\sum_{n=0}^{N-1} g_{6n}\frac{\Gamma\left(2n+\frac{2}{3}\right)}{\nu^{2n+\frac{2}{3}}} - \frac{1}{3\pi}\sum_{m=0}^{M-1} g_{6m+2}\frac{\Gamma\left(2m+\frac{4}{3}\right)}{\nu^{2m+\frac{4}{3}}} \\ &- \frac{1}{3\pi}\sum_{k=0}^{K-1} g_{6k+4}\frac{\Gamma\left(2k+2\right)}{\nu^{2k+2}} + R_{N,M,K}^{(\mathbf{A}')}(\nu)\end{aligned} \tag{3.248}$$

with

$$\begin{aligned}R_{N,M,K}^{(\mathbf{A}')}(\nu) = &\frac{(-1)^N}{3\pi}\frac{1}{\nu^{2N+\frac{2}{3}}}\int_0^{+\infty}\frac{u^{2N-\frac{1}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}H_{\mathrm{i}u}^{(1)\prime}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ &+ \frac{(-1)^{M+1}}{3\pi}\frac{1}{\nu^{2M+\frac{4}{3}}}\int_0^{+\infty}\frac{u^{2M+\frac{1}{3}}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}H_{\mathrm{i}u}^{(1)\prime}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u \\ &+ \frac{(-1)^{K}}{3\pi}\frac{1}{\nu^{2K+2}}\int_0^{+\infty}\frac{u^{2K+1}\mathrm{e}^{-2\pi u}}{1+(u/\nu)^2}H_{\mathrm{i}u}^{(1)\prime}\left(u\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\mathrm{d}u\end{aligned}$$

for $|\theta| < \frac{\pi}{2}$, $N \geq 1$ and $M, K \geq 0$.

Now, we are in the position to formulate our re-expansions for the remainder terms $R_{N,M,K}^{(\mathbf{A})}(\nu)$ and $R_{N,M,K}^{(\mathbf{A}')}(\nu)$, in Proposition 3.4.2 below. In this proposition, the functions $R_{N,M,K}^{(\mathbf{A})}(\nu)$ and $R_{N,M,K}^{(\mathbf{A}')}(\nu)$ are extended to a sector larger than $|\theta| < \frac{\pi}{2}$ via (3.247) and (3.248) using analytic continuation. The proof of Proposition 3.4.2 is essentially the same as that of Proposition 3.2.1 on the analogous expansion for the Hankel function of equal order and argument, and therefore the proof is omitted.

Proposition 3.4.2. *Let J, L and Q be arbitrary fixed non-negative integers. Suppose that $|\theta| \leq 3\pi - \delta$ with an arbitrary fixed small positive δ, $|\nu|$ is large and $N = \pi|\nu| + \rho$, $M = \pi|\nu| + \sigma$, $K = \pi|\nu| + \eta$ with ρ, σ and η being bounded. Then*

$$\begin{aligned}R_{N,M,K}^{(\mathbf{A})}(\nu) = &-\frac{\mathrm{i}\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{j=0}^{J-1} d_{2j}\mathrm{e}^{\frac{2\pi(2j+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2j+1)}{3}\right)\frac{\Gamma\left(\frac{2j+1}{3}\right)}{\nu^{\frac{2j+1}{3}}}T_{2N-\frac{2j}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\ &+ \frac{\mathrm{i}}{3}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{j=0}^{J-1} d_{2j}\sin\left(\frac{\pi(2j+1)}{3}\right)\frac{\Gamma\left(\frac{2j+1}{3}\right)}{\nu^{\frac{2j+1}{3}}}T_{2N-\frac{2j}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right) \\ &+ \frac{\mathrm{i}}{3}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} d_{2\ell}\mathrm{e}^{\frac{2\pi(2\ell+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2\ell+1)}{3}\right)\frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}}T_{2M-\frac{2\ell-2}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right) \\ &+ \frac{\mathrm{i}}{3}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} d_{2\ell}\sin\left(\frac{\pi(2\ell+1)}{3}\right)\frac{\Gamma\left(\frac{2\ell+1}{3}\right)}{\nu^{\frac{2\ell+1}{3}}}T_{2M-\frac{2\ell-2}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\mathrm{i}\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}{3}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{q=0}^{Q-1} d_{2q}\mathrm{e}^{\frac{2\pi(2q+1)}{3}\mathrm{i}}\sin\left(\frac{\pi\left(2q+1\right)}{3}\right)\frac{\Gamma\left(\frac{2q+1}{3}\right)}{\nu^{\frac{2q+1}{3}}}T_{2K-\frac{2q-4}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\\
&+ \frac{\mathrm{i}}{3}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{q=0}^{Q-1} d_{2q}\sin\left(\frac{\pi\left(2q+1\right)}{3}\right)\frac{\Gamma\left(\frac{2q+1}{3}\right)}{\nu^{\frac{2q+1}{3}}}T_{2K-\frac{2q-4}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\\
&+ R^{(\mathbf{A})}_{\substack{N,M,K\\J,L,Q}}(\nu),
\end{aligned}
\tag{3.249}
$$

$$
\begin{aligned}
R^{(\mathbf{A}')}_{N,M,K}(\nu) = &-\frac{\mathrm{i}\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{j=0}^{J-1} g_{2j}\mathrm{e}^{\frac{2\pi(2j+1)}{3}\mathrm{i}}\sin\left(\frac{\pi(2j+2)}{3}\right)\frac{\Gamma\left(\frac{2j+2}{3}\right)}{\nu^{\frac{2j+2}{3}}}T_{2N-\frac{2j}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\\
&+ \frac{\mathrm{i}}{3}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{j=0}^{J-1} g_{2j}\sin\left(\frac{\pi\left(2j+2\right)}{3}\right)\frac{\Gamma\left(\frac{2j+2}{3}\right)}{\nu^{\frac{2j+2}{3}}}T_{2N-\frac{2j}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\\
&- \frac{\mathrm{i}\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}}{3}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} g_{2\ell}\mathrm{e}^{\frac{2\pi(2\ell+2)}{3}\mathrm{i}}\sin\left(\frac{\pi\left(2\ell+2\right)}{3}\right)\frac{\Gamma\left(\frac{2\ell+2}{3}\right)}{\nu^{\frac{2\ell+2}{3}}}T_{2M-\frac{2\ell-2}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\\
&+ \frac{\mathrm{i}}{3}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{\ell=0}^{L-1} g_{2\ell}\sin\left(\frac{\pi\left(2\ell+2\right)}{3}\right)\frac{\Gamma\left(\frac{2\ell+2}{3}\right)}{\nu^{\frac{2\ell+2}{3}}}T_{2M-\frac{2\ell-2}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\\
&+ \frac{\mathrm{i}}{3}\mathrm{e}^{2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{q=0}^{Q-1} g_{2q}\mathrm{e}^{\frac{2\pi(2q+2)}{3}\mathrm{i}}\sin\left(\frac{\pi\left(2q+2\right)}{3}\right)\frac{\Gamma\left(\frac{2q+2}{3}\right)}{\nu^{\frac{2q+2}{3}}}T_{2K-\frac{2q-4}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\\
&+ \frac{\mathrm{i}}{3}\mathrm{e}^{-2\pi\mathrm{i}\nu}\frac{2}{3\pi}\sum_{q=0}^{Q-1} g_{2q}\sin\left(\frac{\pi\left(2q+2\right)}{3}\right)\frac{\Gamma\left(\frac{2q+2}{3}\right)}{\nu^{\frac{2q+2}{3}}}T_{2K-\frac{2q-4}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\\
&+ R^{(\mathbf{A}')}_{\substack{N,M,K\\J,L,Q}}(\nu),
\end{aligned}
\tag{3.250}
$$

where

$$
\begin{aligned}
R^{(\mathbf{A})}_{\substack{N,M,K\\J,L,Q}}(\nu) &= \mathcal{O}_{J,\rho}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2J+1}{3}}}\right) + \mathcal{O}_{L,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2L+1}{3}}}\right) + \mathcal{O}_{Q,\eta}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2Q+1}{3}}}\right),\\
R^{(\mathbf{A}')}_{\substack{N,M,K\\J,L,Q}}(\nu) &= \mathcal{O}_{J,\rho}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2J+2}{3}}}\right) + \mathcal{O}_{L,\sigma}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2L+2}{3}}}\right) + \mathcal{O}_{Q,\eta}\left(\frac{\mathrm{e}^{-2\pi|\nu|}}{|\nu|^{\frac{2Q+2}{3}}}\right)
\end{aligned}
\tag{3.251}
$$

for $|\theta| \leq \frac{\pi}{2}$, *and*

$$
R^{(\mathbf{A})}_{\substack{N,M,K\\J,L,Q}}(\nu) = \mathcal{O}_{J,\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathfrak{m}(\nu)}}{|\nu|^{\frac{2J+1}{3}}}\right) + \mathcal{O}_{L,\sigma,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathfrak{m}(\nu)}}{|\nu|^{\frac{2L+1}{3}}}\right) + \mathcal{O}_{Q,\eta,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi\Im\mathfrak{m}(\nu)}}{|\nu|^{\frac{2Q+1}{3}}}\right),
$$

$$R^{(\mathbf{A}')}_{\substack{N,M,K\\J,L,Q}}(\nu) = \mathcal{O}_{J,\rho,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi \mathfrak{Im}(\nu)}}{|\nu|^{\frac{2J+2}{3}}}\right) + \mathcal{O}_{L,\sigma,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi \mathfrak{Im}(\nu)}}{|\nu|^{\frac{2L+2}{3}}}\right) + \mathcal{O}_{Q,\sigma,\delta}\left(\frac{\mathrm{e}^{\mp 2\pi \mathfrak{Im}(\nu)}}{|\nu|^{\frac{2Q+2}{3}}}\right) \tag{3.252}$$

for $\frac{\pi}{2} \leq \pm\theta \leq 2\pi - \delta$. *Moreover, if* $J = L = Q$, *then the estimates* (3.251) *remain valid in the larger sector* $|\theta| \leq \frac{3\pi}{2}$ *and* (3.252) *holds in the range* $\frac{3\pi}{2} \leq \mp\theta \leq 3\pi - \delta$.

Proposition 3.4.2 together with (3.247) and (3.248) yields the exponentially improved asymptotic expansions for the Anger–Weber function and its derivative of equally large order and argument.

If we neglect the remainder terms in (3.249) and (3.250) and formally extend the sums to infinity, formulae (3.249) and (3.250) reproduce Dingle's original expansions which were mentioned at the beginning of this subsection.

In the following theorem, we give explicit bounds on the remainder terms

$$R^{(\mathbf{A})}_{\substack{N,M,K\\J,L,Q}}(\nu) \quad \text{and} \quad R^{(\mathbf{A}')}_{\substack{N,M,K\\J,L,Q}}(\nu).$$

Note that in these results, N, M and K may not necessarily depend on ν.

Theorem 3.4.3. *Let* N, M, K, J, L *and* Q *be arbitrary fixed non-negative integers such that* $J < 3N$, $L < 3M + 1$ $Q < 3K + 2$ *and* $J, L, Q \equiv 0 \mod 3$. *Then we have*

$$\begin{aligned}
\left|R^{(\mathbf{A})}_{\substack{N,M,K\\J,L,Q}}(\nu)\right| \leq{}& \frac{1}{3}\left|\mathrm{e}^{2\pi \mathrm{i}\nu}\right| \frac{2}{3\pi}|d_{2J}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2J+1}{3}\right)}{|\nu|^{\frac{2J+1}{3}}} \left|T_{2N-\frac{2J}{3}}\left(2\pi\nu \mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&+ \frac{1}{3}\left|\mathrm{e}^{-2\pi \mathrm{i}\nu}\right| \frac{2}{3\pi}|d_{2J}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2J+1}{3}\right)}{|\nu|^{\frac{2J+1}{3}}} \left|T_{2N-\frac{2J}{3}}\left(2\pi\nu \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&+ \frac{2}{3^{\frac{3}{2}}\pi^2}|d_{2J}| \frac{\Gamma\left(\frac{2J+1}{3}\right)\Gamma\left(2N - \frac{2J}{3}\right)}{(2\pi)^{2N-\frac{2J}{3}}\,|\nu|^{2N+\frac{1}{3}}} \\
&+ \frac{1}{3}\left|\mathrm{e}^{2\pi \mathrm{i}\nu}\right| \frac{2}{3\pi}|d_{2L}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2L+1}{3}\right)}{|\nu|^{\frac{2L+1}{3}}} \left|T_{2M-\frac{2L-2}{3}}\left(2\pi\nu \mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&+ \frac{1}{3}\left|\mathrm{e}^{-2\pi \mathrm{i}\nu}\right| \frac{2}{3\pi}|d_{2L}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2L+1}{3}\right)}{|\nu|^{\frac{2L+1}{3}}} \left|T_{2M-\frac{2L-2}{3}}\left(2\pi\nu \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right| \\
&+ \frac{2}{3^{\frac{3}{2}}\pi^2}|d_{2L}| \frac{\Gamma\left(\frac{2L+1}{3}\right)\Gamma\left(2M - \frac{2L-2}{3}\right)}{(2\pi)^{2M-\frac{2L-2}{3}}\,|\nu|^{2M+1}} \\
&+ \frac{1}{3}\left|\mathrm{e}^{2\pi \mathrm{i}\nu}\right| \frac{2}{3\pi}|d_{2Q}| \frac{3^{\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{2Q+1}{3}\right)}{|\nu|^{\frac{2Q+1}{3}}} \left|T_{2K-\frac{2Q-4}{3}}\left(2\pi\nu \mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|
\end{aligned}$$

$$+\frac{1}{3}\left|\mathrm{e}^{-2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}|d_{2Q}|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2Q+1}{3}\right)}{|\nu|^{\frac{2Q+1}{3}}}\left|T_{2K-\frac{2Q-4}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|$$
$$+\frac{2}{3^{\frac{3}{2}}\pi^2}|d_{2Q}|\frac{\Gamma\left(\frac{2Q+1}{3}\right)\Gamma\left(2K-\frac{2Q-4}{3}\right)}{(2\pi)^{2K-\frac{2Q-4}{3}}|\nu|^{2K+\frac{5}{3}}}$$

provided that $|\theta|\leq\frac{\pi}{2}$*, and*

$$\left|R^{(\mathbf{A}')}_{\substack{N,M,K\\J,L,Q}}(\nu)\right|\leq\frac{1}{3}\left|\mathrm{e}^{2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}|g_{2J}|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2J+2}{3}\right)}{|\nu|^{\frac{2J+2}{3}}}\left|T_{2N-\frac{2J}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|$$
$$+\frac{1}{3}\left|\mathrm{e}^{-2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}|g_{2J}|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2J+2}{3}\right)}{|\nu|^{\frac{2J+2}{3}}}\left|T_{2N-\frac{2J}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|$$
$$+\frac{2}{3^{\frac{3}{2}}\pi^2}|g_{2J}|\frac{\Gamma\left(\frac{2J+2}{3}\right)\Gamma\left(2N-\frac{2J}{3}\right)}{(2\pi)^{2N-\frac{2J}{3}}|\nu|^{2N+\frac{2}{3}}}$$
$$+\frac{1}{3}\left|\mathrm{e}^{2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}|g_{2L}|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2L+2}{3}\right)}{|\nu|^{\frac{2L+2}{3}}}\left|T_{2M-\frac{2L-2}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|$$
$$+\frac{1}{3}\left|\mathrm{e}^{-2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}|g_{2L}|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2L+2}{3}\right)}{|\nu|^{\frac{2L+2}{3}}}\left|T_{2M-\frac{2L-2}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|$$
$$+\frac{2}{3^{\frac{3}{2}}\pi^2}|g_{2L}|\frac{\Gamma\left(\frac{2L+2}{3}\right)\Gamma\left(2M-\frac{2L-2}{3}\right)}{(2\pi)^{2M-\frac{2L-2}{3}}|\nu|^{2M+\frac{4}{3}}}$$
$$+\frac{1}{3}\left|\mathrm{e}^{2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}|g_{2Q}|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2Q+2}{3}\right)}{|\nu|^{\frac{2Q+2}{3}}}\left|T_{2K-\frac{2Q-4}{3}}\left(2\pi\nu\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\right)\right|$$
$$+\frac{1}{3}\left|\mathrm{e}^{-2\pi\mathrm{i}\nu}\right|\frac{2}{3\pi}|g_{2Q}|\frac{3^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{2Q+2}{3}\right)}{|\nu|^{\frac{2Q+2}{3}}}\left|T_{2K-\frac{2Q-4}{3}}\left(2\pi\nu\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\right)\right|$$
$$+\frac{2}{3^{\frac{3}{2}}\pi^2}|g_{2Q}|\frac{\Gamma\left(\frac{2Q+2}{3}\right)\Gamma\left(2K-\frac{2Q-4}{3}\right)}{(2\pi)^{2K-\frac{2Q-4}{3}}|\nu|^{2K+2}}$$

provided that $|\theta|\leq\frac{\pi}{2}$ *and* $J,L,Q\geq 3$*. In the case when* $J=L=Q$*, these bounds are also valid in the range* $\frac{\pi}{2}\leq\pm\theta\leq\frac{3\pi}{2}$ *with* $\overline{\nu\mathrm{e}^{\mp\pi\mathrm{i}}}$ *in place of* ν *on the right-hand sides.*

The proof of Theorem 3.4.3 is essentially the same as the proof of Theorem 3.2.2 and is therefore omitted.

Bibliography

[1] J. R. Airey, *The Lommel–Weber Ω function and its application to the problem of electric waves on a thin anchor ring*, Proc. Roy. Soc. London Ser. A **94** (1918), no. 661, 307–314.

[2] ______, *The "converging factor" in asymptotic series and the calculation of Bessel, Laguerre and other functions*, Phil. Mag. **24** (1937), no. 7, 521–552.

[3] G. B. Airy, *On the intensity of light in the neighbourhood of a caustic*, Trans. Camb. Phil. Soc. **6** (1838), 379–403.

[4] H. Anders, *A History of Probability and Statistics and Their Applications before 1750*, John Wiley, New Jersey, 1989.

[5] M. V. Berry, *Stokes' phenomenon: smoothing a Victorian discontinuity*, Inst. Hautes Études Sci. Publ. Math. **68** (1988), no. 1, 211–221.

[6] ______, *Uniform asymptotic smoothing of Stokes' discontinuities*, Proc. Roy. Soc. London Ser. A **422** (1989), no. 1862, 7–21.

[7] M. V. Berry and C. J. Howls, *Hyperasymptotics*, Proc. Roy. Soc. London Ser. A **430** (1990), no. 1880, 653–668.

[8] ______, *Hyperasymptotics for integrals with saddles*, Proc. Roy. Soc. London Ser. A **434** (1991), no. 1892, 657–675.

[9] ______, *Divergent series: taming the tails*, The Princeton Companion to Applied Mathematics (N. Higham, ed.), Princeton University Press, Princeton, 2015.

[10] W. G. Bickley, *Formulæ relating to Bessel functions of moderate or large argument and order*, Philos. Mag. **34** (1943), no. 228, 37–49.

[11] J. P. Boyd, *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series*, Acta Appl. Math. **56** (1999), no. 1, 1–98.

[12] W. G. C. Boyd, *Stieltjes transforms and the Stokes phenomenon*, Proc. Roy. Soc. London Ser. A **429** (1990), no. 1876, 227–246.

[13] ______, *Error bounds for the method of steepest descents*, Proc. Roy. Soc. London Ser. A **440** (1993), no. 1910, 493–518.

[14] ______, *Gamma function asymptotics by an extension of the method of steepest descents*, Proc. Roy. Soc. London Ser. A **447** (1994), no. 1931, 609–630.

[15] ______, *Approximations for the late coefficients in asymptotic expansions arising in the method of steepest descents*, Methods Appl. Anal. **2** (1995), no. 4, 475–489.

[16] ______, *Recent developments in asymptotics obtained via integral representations of Stieltjes-transform type*, Algebraic Analysis of Singular Perturbations, RIMS Kokyuroku, no. 968, 1996, pp. 1–30.

[17] S. Brassesco and M. A. Méndez, *The asymptotic expansion for $n!$ and the Lagrange inversion formula*, Ramanujan J. **24** (2011), no. 2, 219–234.

[18] A.-L. Cauchy, *Sur la transformation des variables qui déterminent les mouvements d'une planète ou même d'une comète en fonction explicite du temps, et sur le développement de ces fonctions en séries convergentes*, Comptes Rendus **38** (1854), 990–993.

[19] ______, *Sur la résolution des équations et sur le développement de leurs racines en séries convergentes*, Comptes Rendus **38** (1854), 1104–1107.

[20] E. T. Copson, *Asymptotic Expansions*, Cam. Tracts Math. Math. Phys., no. 55, Cambridge University Press, London, 1965.

[21] G. M. Darboux, *Mémoire sur l'approximation des fonctions de très-grandes nombres, et sur une classe étendue de développements en série*, J. Math. Pures Appl. **4** (1878), no. 3, 6–56, 377–416.

[22] V. De Angelis, *Asymptotic expansions and positivity of coefficients for large powers of analytic functions*, Int. J. Math. Math. Sci. **2003** (2003), no. 16, 1003–1025.

[23] ______, *Stirling's series revisited*, Amer. Math. Monthly **116** (2009), no. 9, 839–843.

[24] P. S. de Laplace, *Théorie Analytique des Probabilités*, Courcier, Paris, 1812.

[25] P. Debye, *Näherungsformeln für die Zylinderfunktionen für große Werte des Arguments und unbeschränkt veränderliche Werte des Index*, Math. Annal. **67** (1909), no. 4, 535–558.

[26] ______, *Semikonvergente Entwickelungen für die Zylinderfunktionen und ihre Ausdehnung ins komplexe*, Sitzungsberichte der Akad. der Wiss. zu München **40** (1910), no. 5, 29 pp.

[27] E. Delabaere and C. J. Howls, *Global asymptotics for multiple integrals with boundaries*, Duke Math. J. **112** (2002), no. 2, 199–264.

[28] O. Diekmann, *Asymptotic expansion of certain numbers related to the gamma function*, Tech. Report TN 80/75, Mathematisch Centrum, 1975.

[29] R. B. Dingle, *Asymptotic expansions and converging factors I. General theory and basic converging factors*, Proc. Roy. Soc. London Ser. A **244** (1958), no. 1239, 456–475.

[30] ______, *Asymptotic expansions and converging factors II. Error, Dawson, Fresnel, exponential, sine and cosine, and similar integrals*, Proc. Roy. Soc. London Ser. A **244** (1958), no. 1239, 476–483.

[31] ______, *Asymptotic expansions and converging factors III. Gamma, psi and polygamma functions, and Fermi–Dirac and Bose–Einstein integrals*, Proc. Roy. Soc. London Ser. A **244** (1958), no. 1239, 484–490.

[32] ______, *Asymptotic expansions and converging factors IV. Confluent hypergeometric, parabolic cylinder, modified Bessel, and ordinary Bessel functions*, Proc. Roy. Soc. London Ser. A **249** (1959), no. 1257, 270–283.

[33] ______, *Asymptotic expansions and converging factors V. Lommel, Struve, modified Struve, Anger and Weber functions, and integrals of ordinary and modified Bessel functions*, Proc. Roy. Soc. London Ser. A **249** (1959), no. 1257, 284–292.

[34] ______, *Asymptotic expansions and converging factors VI. Application to physical prediction*, Proc. Roy. Soc. London Ser. A **249** (1959), no. 1257, 293–295.

[35] ______, *Asymptotic Expansions: Their Derivation and Interpretation*, Academic Press, New York and London, 1973.

[36] B. Döring, *Über Fehlerschranken zu den Hankelschen asymptotischen Entwicklungen der Besselfunktionen für komplexes Argument und reellen Index*, Z. Angew. Math. Mech. **42** (1962), no. 1-2, 62–76.

[37] Á. Elbert and A. Laforgia, *A lower bound for $J_\nu(\nu)$*, Appl. Anal. **19** (1985), no. 2-3, 137–145.

[38] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Tables of Integral Transforms Volume I*, Bateman Manuscript Project, McGraw-Hill, New York, 1954.

[39] ______, *Tables of Integral Transforms Volume II*, Bateman Manuscript Project, McGraw-Hill, New York, 1954.

[40] L. Gatteschi, *Sulla rappresentazione asintotica delle funzioni di Bessel di uguale ordine ed argomento*, Ann. Mat. **38** (1955), no. 1, 267–280.

[41] V. P. Gurariĭ, *Error bounds, duality, and the Stokes phenomenon. I*, St. Petersburg Math. J. **21** (2010), no. 6, 903–956.

[42] H. Hankel, *Die Cylinderfunctionen erster und zweiter Art*, Math. Ann. **1** (1869), no. 3, 467–501.

[43] P. A. Hansen, *Schriften der Sternwarte Seeberg: Ermittlung der Absoluten Störungen in Ellipsen von beliebiger Excentricität und Neigung*, Gotha bei Carl Gläser, 1843.

[44] Z. Heitman, J. Bremer, V. Rokhlin, and B. Vioreanu, *On the asymptotics of Bessel functions in the Fresnel regime*, Appl. Comput. Harmon. Anal. **39** (2015), no. 2, 347–356.

[45] C. J. Howls, *Hyperasymptotics for integrals with finite endpoints*, Proc. Roy. Soc. London Ser. A **439** (1992), no. 1906, 373–396.

[46] ______, *Hyperasymptotics for multidimensional integrals, exact remainder terms and the global connection problem*, Proc. Roy. Soc. London Ser. A **453** (1997), no. 1966, 2271–2294.

[47] C. J. Howls and A. B. Olde Daalhuis, *Hyperasymptotic solutions of inhomogeneous linear differential equations with a singularity of rank one*, Proc. Roy. Soc. London Ser. A **459** (2003), no. 2038, 2599–2612.

[48] C. G. J. Jacobi, *Versuch einer Berechnung der grossen Ungleichheit des Saturns nach einer strengen Entwickelung*, Astr. Nach. **28** (1849), no. 6, 81–94.

[49] D. S. Jones, *Uniform asymptotic remainders*, Asymptotic and Computational Analysis (R. Wong, ed.), Lecture Notes in Pure and Appl. Math., no. 124, Marcel Dekker, New York, 1990, pp. 241–264.

[50] G. Kirchhoff, *Ueber den inducirten Magnetismus eines unbegrenzten Cylinders von weichem Eisen*, J. Reine Angew. Math. **48** (1854), 348–376.

[51] E. E. Kummer, *De integralibus quibusdam definitis et seriebus infinitis*, J. Reine Angew. Math. **17** (1837), 228–242.

[52] A. M. Legendre, *Exercices De Calcul Intégral Sur Divers Ordres De Transcendantes Et Sur Les Quadratures*, Courcier, Paris, 1811.

[53] E. Lindelöf, *Le Calcul des Résidus*, Gauthier-Villars, Paris, 1905.

[54] R. Lipschitz, *Ueber ein Integral der Differentialgleichung*, J. Reine Angew. Math. **56** (1859), 189–196.

[55] E. Lommel, *Zur Theorie der Bessel'schen Functionen*, Math. Ann. **16** (1880), no. 3, 183–208.

[56] J. L. López and P. Pagola, *An explicit formula for the coefficients of the saddle point method*, Constr. Approx. **33** (2011), no. 2, 145–162.

[57] L. Lorenz, *Sur la lumière refléchie et refractée par une sphère transparente*, K. Danske Videnskabernes Selskabs Skrifter **6** (1890), no. 6, 1–62.

[58] Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill, New York, 1962.

[59] ______, *The Special Functions and Their Approximations Volume I*, Academic Press, New York, 1969.

[60] C. S. Meijer, *Asymptotische Entwicklungen von Besselschen, Hankelschen und verwandten Funktionen I–IV*, Proc. Kon. Akad. Wet. Amsterdam **35** (1932), 656–667,852–866, 948–958, 1079–1090.

[61] ______, *Über die asymptotische Entwicklung von* $\int_0^{\infty-\mathrm{i}(\arg w-\mu)} \mathrm{e}^{\nu z - w\sinh z}\mathrm{d}z$ $(-\frac{\pi}{2} < \mu < \frac{\pi}{2})$ *für grosse Werte von* $|w|$ *und* $|\nu|$, Proc. Kon. Akad. Wet. Amsterdam **35** (1932), 1170–1180, 1291–1303.

[62] ______, *Asymptotische Entwicklungen von Besselschen und Hankelschen Funktionen für große Werte des Arguments und der Ordnung*, Math. Ann. **108** (1933), no. 1, 321–359.

[63] E. Meissel, *Beitrag zur Theorie der allgemeinen Bessel'schen Function*, Astr. Nach. **128** (1891), no. 8, 145–154.

[64] ______, *Einige Entwickelungen, die Bessel'schen I-functionen betreffend*, Astr. Nach. **127** (1891), no. 22, 359–362.

[65] ———, *Weitere Entwickelungen über die Bessel'schen Functionen*, Astr. Nach. **130** (1892), no. 23-24, 364–368.

[66] J. C. P. Miller, *A method for the determination of converging factors, applied to the asymptotic expansions for the parabolic cylinder functions*, Proc. Cambridge Phil. Soc. **48** (1952), no. 2, 243–254.

[67] L. M. Milne-Thomson, *The Calculus of Finite Differences*, AMS Chelsea Publishing, New York, 1981, reprint of the original (1933) edition.

[68] F. D. Murnaghan and J. W. Wrench, *The converging factor for the exponential integral*, Tech. Report 1535, Dept. of the Navy, David Taylor Model Basin, Applied Mathematics Laboratory, 1963.

[69] J. D. Murray, *Asymptotic Analysis*, Appl. Math. Sci., no. 48, Springer-Verlag, New York, 1984.

[70] G. Nemes, *An explicit formula for the coefficients in Laplace's method*, Constr. Approx. **38** (2013), no. 3, 471–487.

[71] ———, *The resurgence properties of the large order asymptotics of the Anger–Weber function I*, J. Classical Anal. **4** (2014), no. 1, 1–39.

[72] ———, *The resurgence properties of the large order asymptotics of the Anger–Weber function II*, J. Classical Anal. **4** (2014), no. 2, 121–147.

[73] ———, *The resurgence properties of the large-order asymptotics of the Hankel and Bessel functions*, Anal. Appl. (Singap.) **12** (2014), no. 4, 403–462.

[74] ———, *On the large argument asymptotics of the Lommel function via Stieltjes transforms*, Asymptotic Anal. **91** (2015), no. 3-4, 265–281.

[75] ———, *Error bounds and exponential improvements for the asymptotic expansions of the gamma function and its reciprocal*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 3, 571–596.

[76] ———, *The resurgence properties of the Hankel and Bessel functions of nearly equal order and argument*, Math. Ann. (2015), to appear.

[77] ———, *The resurgence properties of the incomplete gamma function I*, Anal. Appl. (Singap.) (2015), to appear.

[78] ———, *The resurgence properties of the incomplete gamma function II*, Stud. Appl. Math. **135** (2015), no. 1, 86–116.

[79] J. W. Nicholson, *On Bessel functions of equal argument and order*, Phil. Mag. **16** (1908), no. 92, 271–279.

[80] N. Nielsen, *Handbuch der Theorie der Cylinderfunktionen*, B. G. Teubner, Leipzig, 1904.

[81] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Die Grundlehren der Mathematischen Wissenschaften, vol. 13, Springer-Verlag, Berlin, 1927.

[82] A. B. Olde Daalhuis, *Hyperasymptotic expansions of confluent hypergeometric functions*, IMA J. Appl. Math. **49** (1992), no. 3, 203–216.

[83] ———, *Hyperasymptotics and the stokes phenomenon*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), no. 4, 731–743.

[84] ———, *Hyperasymptotic solutions of second-order linear differential equations II*, Methods Appl. Anal. **2** (1995), no. 2, 198–211.

[85] A. B. Olde Daalhuis and F. W. J. Olver, *Exponentially improved asymptotic solutions of ordinary differential equations. II Irregular singularities of rank one*, Proc. Roy. Soc. London Ser. A **445** (1994), no. 1923, 39–56.

[86] ———, *Hyperasymptotic solutions of second-order linear differential equations I*, Methods Appl. Anal. **2** (1995), no. 2, 173–197.

[87] F. W. J. Olver, *Error bounds for the Laplace approximation for definite integrals*, J. Approx. Theory **1** (1968), no. 3, 293–313.

[88] ———, *Why steepest descents?*, SIAM Rev. **12** (1970), no. 2, 228–247.

[89] ———, *Asymptotic approximations and error bounds*, SIAM Rev. **22** (1980), no. 2, 188–203.

[90] ———, *On Stokes' phenomenon and converging factors*, Asymptotic and Computational Analysis (R. Wong, ed.), Lecture Notes in Pure and Appl. Math., no. 124, Marcel Dekker, New York, 1990, pp. 329–355.

[91] ———, *Uniform, exponentially improved, asymptotic expansions for the generalized exponential integral*, SIAM J. Math. Anal. **22** (1991), no. 5, 1460–1474.

[92] ———, *Uniform, exponentially improved, asymptotic expansions for the confluent hypergeometric function and other integral transforms*, SIAM J. Math. Anal. **22** (1991), no. 5, 1475–1489.

[93] ———, *Exponentially improved asymptotic solutions of ordinary differential equations I: The confluent hypergeometric function*, SIAM J. Math. Anal. **24** (1993), no. 3, 756–767.

[94] ______, *Asymptotic expansions of the coefficients in asymptotic series solutions of linear differential equations*, Methods Appl. Anal. **1** (1994), no. 1, 1–13.

[95] ______, *Asymptotics and Special Functions*, AKP Class., A. K. Peters Ltd., Wellesley, MA., 1997, reprint of the original (1974) edition.

[96] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010.

[97] R. B. Paris, *Smoothing of the Stokes phenomenon using Mellin–Barnes integrals*, J. Comput. Appl. Math. **41** (1992), no. 1-2, 117–133.

[98] ______, *Hadamard Expansions and Hyperasymptotic Evaluation: An Extension of the Method of Steepest Descents*, Encyclopedia Math. Appl., no. 141, Cambridge University Press, Cambridge, 2011.

[99] R. B. Paris and D. Kaminski, *Asymptotics and Mellin–Barnes Integrals*, Encyclopedia Math. Appl., vol. 85, Cambridge University Press, Cambridge, 2001.

[100] R. B. Paris and A. D. Wood, *Exponentially-improved asymptotics for the gamma function*, J. Comput. Appl. Math. **41** (1992), no. 1-2, 135–143.

[101] J. H. Poincaré, *Sur les intégrales irrégulières des equations linéaires*, Acta Math. **8** (1886), no. 1, 295–344.

[102] S.-D. Poisson, *Suite du mémoire sur la distribution de la chaleur dans les corps solides*, Journal de l'École R. Polytechnique **12** (1823), no. 19, 249–403.

[103] R. Remmert and L. D. Kay, *Classical Topics in Complex Function Theory*, Grad. Texts in Math., vol. 172, Springer, New York, 1997.

[104] F. W. Schäfke and A. Sattler, *Restgliedabschätzungen für die Stirlingsche Reihe*, Note. Mat. **10** (1990), no. 2, 453–470.

[105] T. J. Stieltjes, *Recherches sur quelques séries semi-convergentes*, Ann. Sci. École Norm. Sup. **3** (1886), no. 3, 201–258.

[106] ______, *Sur le développement de* $\log \Gamma(a)$, J. Math. Pures Appl. **5** (1889), no. 9, 425–444.

[107] G. G. Stokes, *On the numerical calculation of a class of definite integrals and infinite series*, Trans. Camb. Phil. Soc. **9** (1847), 379–407.

[108] ______, *On the discontinuity of arbitrary constants which appear in divergent developments*, Trans. Camb. Phil. Soc. **10** (1864), 106–128.

[109] Lord Rayleigh (J. W. Strutt), *On point-, line-, and plane-sources of sound*, Proc. London Math. Soc. **19** (1887), no. 1, 504–507.

[110] ______, *The problem of the whispering gallery*, Phil. Mag. **20** (1910), no. 120, 1001–1004.

[111] H. Struve, *Beitrag zur Theorie der Diffraction an Fernröhren*, Ann. der Phys. und Chemie **253** (1882), no. 13, 1008–1016.

[112] N. M. Temme, *Asymptotic Methods for Integrals*, Ser. Analysis, vol. 6, World Scientific, Singapore, 2015.

[113] E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, London, 1975, reprint of the second (1939) edition.

[114] J. G. van der Corput, *Asymptotic Expansions I*, Tech. report, Department of Mathematics, University of California, Berkeley, 1954.

[115] J. Walker, *The Analytical Theory of Light*, Cambridge University Press, Cambridge, 1904.

[116] G. N. Watson, *Theorems stated by Ramanujan (V): Approximations connected with* e^x, Proc. London Math. Soc. **29** (1929), no. 2, 293–308.

[117] ______, *A Treatise on the Theory of Bessel Functions*, Cambridge Math. Lib., Cambridge University Press, Cambridge, 1995, reprint of the second (1944) edition.

[118] H. F. Weber, *Die wahre Theorie der Fresnel'schen Interferenz-Erscheinungen*, Vierteljahrsschr. Naturf. Ges. Zürich **24** (1879), 33–76.

[119] J. Wojdylo, *Computing the coefficients in Laplace's method*, SIAM Rev. **48** (2006), no. 1, 76–96.

[120] ______, *On the coefficients that arise from Laplace's method*, J. Comput. Appl. Math. **196** (2006), no. 1, 241–266.

[121] R. Wong, *Asymptotic Approximations of Integrals*, Classics Appl. Math., no. 34, SIAM, Philadelphia, PA, 2001, reprint of the original (1989) edition.

[122] R. Wong and Y.-Q. Zhao, *Smoothing of Stokes's discontinuity for the generalized Bessel function. II*, Proc. Roy. Soc. London Ser. A **455** (1999), no. 1988, 3065–3084.

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