# *Gödel's incompleteness properties and the guarded fragment: An algebraic approach*

by

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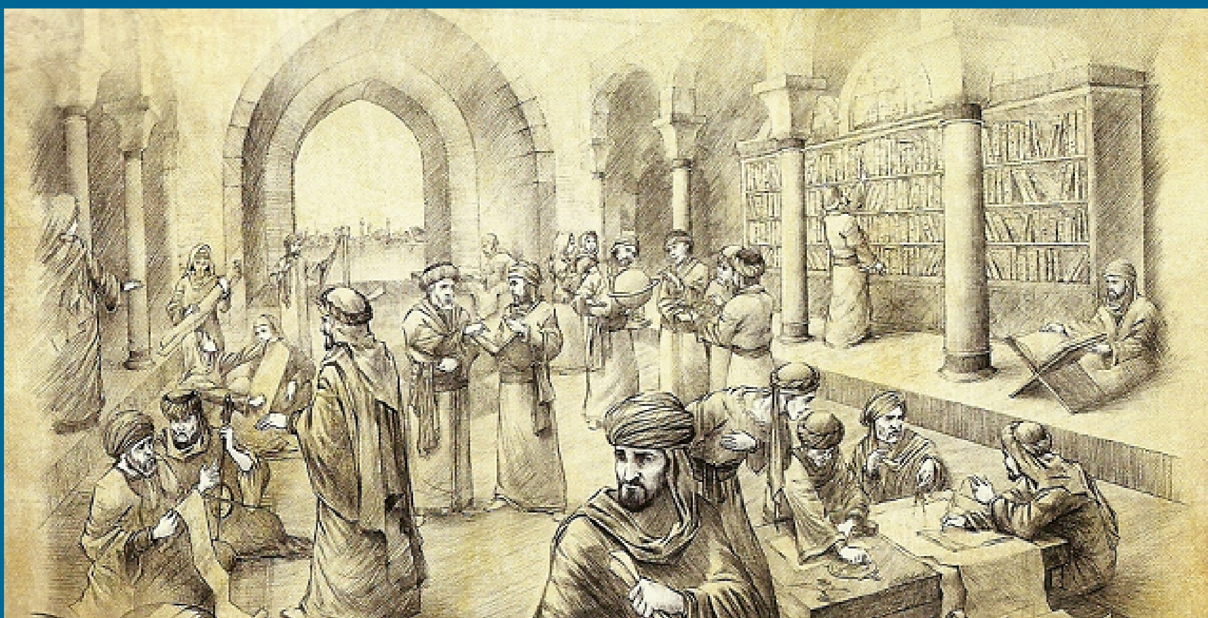
(PhD thesis)



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# GÖDEL'S INCOMPLETENESS PROPERTIES AND THE GUARDED FRAGMENT: AN ALGEBRAIC APPROACH

by

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Budapest, 8 April 2016

Signature

*To the soul of my mother ,,,*

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قَالُوا سُبْحَانَكَ لَا عِلْمَ لَنَا إِلَّا مَا عَلَّمْتَنَا ۖ إِنَّكَ أَنتَ الْعَلِيمُ الْحَكِيمُ ﴿٣٢﴾

سورة البقرة

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# ABSTRACT

The guarded fragment (GF), introduced by H. Andréka, J. van Benthem and I. Németi in [AvBN98], is a large decidable fragment of first order logic that has a wide range of applications in computer science, linguistics, among others, because of its good properties. Its logical properties were investigated by many logicians. By being decidable, GF gives up some expressive power relative to first order logic. A measure of expressive power of a logic is Gödel's Incompleteness Property (GIP). A logic is said to have GIP (wGIP, respectively) if there is a satisfiable formula in this logic which cannot be extended to a recursively (finitely, respectively) axiomatized complete theory of the logic in question.

GIP fails for the guarded fragment, but we prove that wGIP holds for GF on infinite languages while for finite languages wGIP does not hold, either. On the other hand, we prove that the so-called solo-guarded fragment of GF has wGIP on finite languages, too.

To prove the above, we use algebraic methods. Namely, we use the theorem from [Ném86] that a logic has wGIP if and only if its Lindenbaum-Tarski formula algebras are not atomic. The latter in the case of GF are closely related to the free cylindric relativized set algebras. We solve a long-standing open problem first asked in 1985 and then published in [Ném86], [AMN91] and [AFN13] by proving that most of the free cylindric relativized set algebras are non-atomic. We also give structural descriptions of these free algebras from the point of view of atoms.

# CONTENTS

# INTRODUCTION

One of the main interests of mathematical logic is to study several versions of predicate logic. This helps in understanding the reasons for the particular properties of the predicate logic, but also helps in finding versions of predicate logic with desirable properties. Here is an example of an excellently behaving version of predicate logic. Basic modal logic is concerned with the intensional operators *possibly* $\Diamond$ and *necessarily* $\Box$. It can be seen as a fragment of first order logic via the well-known translation $t$ that sends the modalities $\Diamond\varphi$ and $\Box\varphi$ to $\exists v(Ruv \wedge t(\varphi))$ and $\forall v(Ruv \to t(\varphi))$, respectively. The image of basic modal logic under this translation is referred to as the modal fragment. It was shown that modal fragment shares nice properties with the standard first order logic, e.g., Craig interpolation and Beth definability. In addition, modal fragment has some nice properties that fail for the predicate logic, e.g., decidability.

In [AvBN98], it is argued that the distinguishing characteristic of the modal fragment is its restriction on quantifier patterns. This brings H. Andréka, J. van Benthem and I. Németi to investigate the question of what extent we can loosen these quantifier restrictions while retaining the attractive modal behavior. The outcome of this investigation is the guarded fragment which allows quantifications of the form $\exists\bar{u}(R(\bar{u},\bar{v}) \wedge \varphi(\bar{u},\bar{v}))$ and $\forall\bar{u}(R(\bar{u},\bar{v}) \to \varphi(\bar{u},\bar{v}))$, where $\bar{u}$ and $\bar{v}$ are finite sequences of variables and $\varphi$ is a guarded formula with free variables among $\bar{u},\bar{v}$ which all must appear in the atomic formula $R(\bar{u},\bar{v})$. Other more liberal versions of guarded fragment are the loosely guarded fragment and the packed fragment. In these versions the quantifications can be guarded with conjunctions of some special formulas. Clearly, guarded fragments extend the modal fragment.

Another way of having nice versions of first order logic is to keep the set of formulas as it is but consider generalized models when giving meaning for these formulas. Such a move was first taken by Henkin in [Hen50]. The general assignment models for first order logic, where the set of assignments of the variables into a model is allowed to be an arbitrary subset of the usual one, was introduced by I. Németi [Ném86]. With selecting a subset of the assignments, dependence between the variables can be introduced into the semantics. For a survey on generalized semantics see [AvBBN14]. There is an important connection between first order logic with general assignment models and the guarded fragment mentioned above. This connection was pointed out by the creators of guarded fragment in [AvBN98].

The above versions of predicate logic attracted many logicians and were shown to have several desirable properties, e.g., decidability through finite model property. These logics are considered to be the most important decidable versions of first order logic among the large number that have been introduced over the years. They are widely applied in various areas of computer science and linguistics (e.g., description logics, database theory, combining logics), see [AMdNdR99], [BMP13], [GHKL14], [PH04]. But having a decidable version of first order logic has a price: we give up expressive power either by banning the use of some formulas (in the case of guarded fragment), or by changing the meaning of the formulas (in the case of the general assignment models). An interesting question arises here: How much expressive power did we give up? One way of measuring the expressive power of a logic is to investigate whether it has Gödel's incompleteness property or not.

Gödel's (first) incompleteness theorem is among the most important results in modern logic. This discovery revolutionized the understanding of mathematics and logic, and had strong impacts in mathematics, physics, psychology, theology and some applications in other fields of philosophy. It also plays a part in modern linguistic theories, which emphasize the power of language to come up with new ways to express ideas. The first incompleteness theorem states that no formal system within first order logic (i.e., effectively axiomatized first order theory) capable of expressing elementary arithmetic can be both consistent and complete. A formal system is basically a set of axioms together with a set of rules of reasoning that can be expressed in some formal language. The existence of an incomplete formal system in itself is not particularly surprising. A system may be incomplete simply because not all the necessary axioms have been discovered. Gödel's theorem shows that a complete and consistent finite list of axioms for arithmetic can never be created, nor even an infinite list that can be enumerated by an effective method (an algorithm or a computer program).

To investigate the analogous property for any logic, first we abstract from expressing arithmetic. Indeed, not every logic can speak about numbers and their arithmetic and Gödel's incompleteness theorem in fact speaks about a phenomenon of formal systems that is important without being able to speak about arithmetic. A logic is said to have *Gödel's incompleteness property (GIP)* if there is a formula that cannot be extended to an *effectively axiomatized* theory that is both consistent and complete. A logic is said to have *weak Gödel's incompleteness property (wGIP)* if there is a formula that cannot be extended to a *finitely axiomatized* theory that is both consistent and complete. These properties were introduced, and named, by I. Németi in [Ném86] and investigated in [Gye11]. GIP says about a logic that it is capable to state a statement which is strong in the sense that no model in which it is true can be recursively axiomatized. wGIP says the same, but "recursive" replaced with "finite". Under mild assumptions, no decidable logic has GIP.

No logic we deal with in this thesis has GIP (except for a few cases when we do not know whether GIP holds or not). This still leaves wGIP possible. Let us now restrict attention to finite languages. We show that guarded fragments do not have wGIP either, but both their solo-quantifier fragments, where quantifiers can only occur in the form $\exists u(R(u,\bar{v})\wedge\varphi(u,\bar{v}))$ with $u$ a single variable, and first order logic with generalized assignment models do have wGIP. With this we also provide natural logics distinguishing the two properties GIP and wGIP on finite languages. In proving or disproving wGIP, we rely on the fact that wGIP has a natural algebraic equivalent, namely non-atomicity of the formula-algebras (Lindenbaum-Tarski algebras), and we devise novel algebraic methods for proving or disproving atomicity of a free algebra.

The results presented here can be classified as part of algebra or algebraic logic. That is the science which is concerned with the ways of algebraizing logics and with the ways of investigating the algebras of logics. Algebraic logic effectively began with G. Boole, A. De Morgan, C. S. Peirce and E. Schröder in the mid-nineteenth century. Today, the framework of algebraic logic is universal algebra. Universal algebra is the field which investigates classes of algebras in general, interconnections, fundamental properties and so on. In other words, universal algebra is a unifying framework for investigating properties of the algebras of logics. Algebraic logic in the modern sense can be said to have begun with A. Tarski in 1935. The state of algebraic logic owes more to Tarski and his followers than to the nineteenth century founders.

The main idea of algebraic logic is that it states equivalences of important and intuitive algebraic and logical properties, and then uses these equivalences to transfer results and methods from one field to the other. For example, we often solve problems in logic by first translating them to algebra, then using the powerful methodology of algebra for solving the translated problems, and then we translate the solution back to logic. In [Hal85, on page 212], P. Halmos raised the question of what the algebraic counterpart of Gödel's incompleteness theorem was. I. Németi argued that GIP is connected to the atomicity of the so-called Lindenbaum-Tarski formula algebras. He showed that if the Lindenbaum-Tarski formula algebra of some logic is atomic then GIP fails for this logic, cf., [Ném86, proposition 8]. The property wGIP was introduced by Németi as the property that is equivalent to the non-atomicity of the Lindenbaum-Tarski algebras. Lindenbaum-Tarski formula algebras of a logic are the free algebras of the corresponding class of algebras. The algebraic counterpart of first order logic with generalized assignment models (GAM) is the class of cylindric relativized set algebras. Our main concern in this work is to investigate the atomicity of free algebras of this class with algebraic methods. Then we apply what we found to GAM and to the related guarded fragment logics.

Cylindric algebras were introduced by A. Tarski in 1930-1940. These are Boolean algebras of relations enriched with some natural operations such as the operations of creating cylinders in the $n$-dimensional space. The notion of a relativized algebra has been introduced in the theory of Boolean algebras and then it was extended to algebras of logics by L. Henkin. Relativization of an algebra amounts to intersecting all its elements with a fixed set and to defining the new operations as the restrictions of the old operations on this set. Relativized algebras gained their own interest at the end of the twentieth century. Indeed, relativization in many cases turns the negative results to positive ones. Several relativized versions of algebras do have most of the nice properties which their standard counterparts lack. See [AT88],[AvBN95], [AvBN98], [Fer12], [Mik95], [Mon93] and [Ném91].

The free algebras of a variety play an essential role in understanding this variety. These free algebras are algebras that live at the frontier of syntax and semantics. On the one hand, they are semantic by virtue of being members of the variety. On the other hand, they are syntactic in that their elements behave like terms or formulas. The free algebras of a variety represent the structure of the different concepts that can be expressed in its variety. The intrinsic structures of the free relativized cylindric algebras are very involved. Some problems concerning these algebras still remained open. For example, the problem addressing the atomicity of these algebras was still open. This problem dates back to 1985 when I. Németi posed it in his Academic Doctoral Dissertation [Ném86, Remark 18 (i), p. 97]. In 1991, Németi posed the same problem again in [AMN91, Problem 38, p. 738]. Then, it was posed again as an open problem in 2013 in the most recent book on algebraic logic [AFN13, Problem 1.3.3, p. 34]. We solve this problem in the present dissertation by showing that almost none of these free algebras is atomic.

The above problem presented some difficulties. The relevant free algebras are infinite (except only very few of them) and showing atomicity (or non-atomicity) of an infinite Boolean algebra is not so easy. Either we had to find a nonzero element below which there is no distinct nonzero element, or else we had to show that below each nonzero element there is another nonzero element. A proof showing atomicity of an infinite free algebra, due to A. Tarski, can be found in [HMT85, 2.5.7]. In [AN16], H. Andréka and I. Németi generalized Tarski's proof and showed that the (finitely generated) free algebras of any discriminator variety (of finite similarity type) which is generated by its finite members are atomic. The varieties of the relativized algebras are generated by their finite members, this points to the free algebras being atomic. On the other hand, none of these varieties is discriminator, this points to the free algebras being non-atomic. The question basically was, which property was more decisive in these varieties. Below, we summarize our answer to this problem.

Let $n, m$ be finite numbers, $n > 1$. Let $Crs_n$ denote the class of $n$-dimensional cylindric-relativized set algebras. These are natural algebras of subrelations of a relation $V$. The subclasses $D_n, P_n, G_n$ of $Crs_n$ are defined by posing various natural restrictions on $V$, for concrete definition see definition 1.0.1. An element is called zero-dimensional if it is a fixed-point to all of the cylindrifications. They correspond in logic to formulas with no free variables. For a class $K$ of algebras, let $\mathfrak{Fr}_m K$ denote the $m$-generated free algebra of $K$. Let $K \in \{Crs, D, P, G\}$. We prove the following.

(a) The free algebra $\mathfrak{Fr}_0 G_2$ is finite, hence, atomic. Some of its atoms are zero-dimensional while there are other atoms that are not zero-dimensional.

(b) The free algebras $\mathfrak{Fr}_{m+1} G_2$, $\mathfrak{Fr}_m G_{n+1}$, $\mathfrak{Fr}_m D_n$, $\mathfrak{Fr}_m P_2$ and $\mathfrak{Fr}_m Crs_2$ are not atomic and each of them contains only finitely many atoms. Every atom in these free algebras is zero-dimensional. Each of these free algebras can be decomposed as a direct product of an atomless algebra and a finite, hence atomic, algebra.

(c) The free algebras $\mathfrak{Fr}_m Crs_{n+2}$ and $\mathfrak{Fr}_m P_{n+2}$ are not atomic but each of them contains infinitely many atoms. In these free algebras only finitely many atoms are zero-dimensional while infinitely many atoms are not zero-dimensional.

(d) The free algebras $\mathfrak{Fr}_m Crs_3$ and $\mathfrak{Fr}_m P_3$ are not atomic and each of them contains only finitely many atoms. There are atoms in these free algebras that are zero-dimensional and there are other atoms that are not zero-dimensional.

(e) There are only finitely many zero-dimensional elements in the free algebra $\mathfrak{Fr}_m K_n$.

We note that $\mathfrak{Fr}_m Crs_3$ contains finitely many atoms while $\mathfrak{Fr}_m Crs_n$, for $n \geq 4$, contains infinitely many atoms. This is quite surprising because usually, in cylindric-like algebras, dimension $3$ shares the characteristic properties with the higher dimensions but with harder proofs. Here, dimension $3$ and the higher dimensions behave differently. The reason is combinatorial: in dimension $3$ "there is not enough room" for the techniques that work in higher dimensions.

It is worth mentioning that similar results concerning related classes of relativized relation algebras were shown in [Kha15b]. The classes of relativized relation algebras correspond to decidable fragments of the so-called arrow logic. It is a two-dimensional modal logic and it has various applications, e.g., in linguistics (dynamic semantics of natural language, relational semantics of Lambek Calculus), and in computer science (dynamic algebra, dynamic logic). For more about arrow logic as modal logic see [vB91], [vB94], [vB96], [GKWZ03] and [MV97]. Similar results concerning the free algebras of syntactical variants of cylindric algebras were shown in [Kha15a].

In the present thesis, we assume familiarity with the basic notions and concepts of both universal algebra and logic. The thesis consists of two chapters plus two appendices:

- Chapter 1: Here, we consider the class of relativized cylindric algebras and its above-mentioned subclasses. We investigate the atomicity of the free algebras of these classes and we show that almost none of them is atomic. We also give results about the number of the atoms in each of these free algebras.

- Chapter 2: Here, we apply the results of chapter 1 and/or we modify the method used there a little bit. We prove that guarded fragment of first order logic has neither GIP nor wGIP on finite languages. We also show that the solo-fragment of guarded fragment and first order logic with general assignment models have wGIP but not GIP.

- Appendix A: In the above chapters, we make a novel use of the disjunctive normal forms introduced by Kit Fine in [Fin75]. In this appendix, we prove disjunctive normal forms for any class of Boolean algebras with operators.

- Appendix B: Here, we investigate the atomicity of the free algebras of the classes of relativized cylindric algebras when the dimension is infinite or the number of free generators is infinite.

Algebraic logic is a living and lively subject. We hope that this will be conveyed by the large sequence of works that leads to the work in the present thesis.

# NOTATION

We define only those notions and notation which are not universally adopted in the literature.

**Set-theoretic notions.** Throughout, we use the von Neumann ordinals. The smallest infinite ordinal (the set of natural numbers) is denoted by $\omega$. The empty set is denoted by $\emptyset$. Let $A, B$ be any two sets.

- $\mathcal{P}(A)$ is the powerset (set of all subsets) of $A$.
- $A \subseteq_\omega B \stackrel{\text{def}}{\iff} A \subseteq B$ and $A$ is finite.
- $f \circ g \stackrel{\text{def}}{=} \{(a,b) : \exists c\,[(a,c) \in g \;\&\; (c,b) \in f]\}$; the composition of the functions $f$ and $g$.
- ${}^{A}B \stackrel{\text{def}}{=} \{f : f : A \to B\}$; the set of all functions mapping $A$ into $B$.

Let $R \subseteq A \times B$ be any relation, we define the domain and the range of $R$ as follows.

- $Dom(R) \stackrel{\text{def}}{=} \{a \in A : \exists b \in B\ (a,b) \in R\}$.
- $Rng(R) \stackrel{\text{def}}{=} \{b \in B : \exists a \in A\ (a,b) \in R\}$.

Let $\alpha$ be any ordinal. A function $f$ with domain $\alpha$ is called a sequence of length $\alpha$. We do not distinguish the sequences of length $2$ from pairs. Let $i, j \in \alpha$.

- $[i/j] \in {}^{\alpha}\alpha$ is the sequence that sends $i$ to $j$ and fixes any element in $\alpha \setminus \{i\}$.
- $[i,j] \in {}^{\alpha}\alpha$ is the sequence that interchanges $i$ and $j$ and fixes any element in $\alpha \setminus \{i,j\}$.

**Algebraic notions.** An algebraic similarity type, or for short a type, is a set of function symbols each of which is associated with a finite rank. The function symbols of rank $2$, $1$ and $0$ are called binary function symbols, unary function symbols and constant, respectively. Let $t$ be any type and let $K$ be a class of algebras of type $t$.

- $\mathbf{I}K$, $\mathbf{S}K$, $\mathbf{P}K$ and $\mathbf{H}K$ denote the classes of isomorphic copies, subalgebras, direct products and homomorphic images of the members of $K$, respectively.
- $Tm_{\alpha,t}$ denotes the set of terms of type $t$ built up using $\alpha$-many variable symbols.
- $\mathfrak{Tm}_{\alpha,t}$ denotes the natural $t$-type algebra with universe $Tm_{\alpha,t}$.
- $[\tau]^{\mathfrak{A}}_{\iota}$ denotes the interpretation of the term $\tau$ in the algebra $\mathfrak{A}$ under the evaluation $\iota$.
- $\mathfrak{Fr}_{\alpha}K$ denotes the free algebra of the class (most likely variety) $K$ that is generated by $\alpha$-many free variables.

**Boolean algebras.** A boolean algebra is an algebraic structure of the form

$$\mathfrak{A} := \langle A, +, \cdot, -, 0, 1\rangle$$

such that $+$ and $\cdot$ are binary operations, $-$ is a unary operations, $0$ (called the zero of $\mathfrak{A}$) and $1$ (called the unit of $\mathfrak{A}$) are constants in $A$ and the followings are true for every $x, y, z \in A$:

BA1 $x + y = y + x$ and $x \cdot y = y \cdot x$.

BA2 $x + (y \cdot z) = (x + y) \cdot (x + z)$ and $x \cdot (y + z) = x \cdot y + x \cdot z$.

BA3 $x + 0 = x$ and $x \cdot 1 = x$.

BA4 $x + -x = 1$ and $x - \cdot - x = 0$.

An algebra $\mathfrak{A}$ is said to be an algebra with Boolean reduct if and only if its type contains the type of Boolean algebras $\{+, \cdot, -, 0, 1\}$ and the Boolean part $\mathfrak{Bf}\mathfrak{A} = \langle A, +, \cdot, -, 0, 1\rangle$ is Boolean algebra. One can define a relation (pre-order) on $\mathfrak{A}$ as follows. For every $x, y \in A$, $x \leq y \iff x \cdot y = x$. An atom in $a \in \mathfrak{A}$ is a minimal non zero element, i.e., $a \neq 0$ and, for every $b \in \mathfrak{A}$, $b \leq a \implies b = 0$ or $b = a$. $\mathfrak{A}$ is said to be atomic if for every non zero $b \in \mathfrak{A}$ there is an atom $a \in \mathfrak{A}$ such that $a \leq b$. The set of all atoms in $\mathfrak{A}$ is denoted by $At(\mathfrak{A})$.

1

CHAPTER

# Free relativized cylindric set algebras

The notion of a relativized algebra has been introduced in the theory of Boolean algebras and was extended to algebras of logics by L. Henkin. Intuitively, relativized cylindric (set) algebras are Boolean algebras of sets of sequences where the non-Boolean operations are derived from the structure of these sequences in a natural way. Let $n \geq 2$ be any ordinal. Let $V$ be an arbitrary set of sequences of length $n$, that is $V \subseteq {}^nU$ for some set $U$. Then the set $V$ is said to be a *concrete $Crs_n$-atom structure*. If, in addition, $f \circ [i/j] \in V$ for every $f \in V$ and every $i, j \in n$ then we say that $V$ is a *concrete $D_n$-atom structure*. The set $V$ is said to be a *concrete $P_n$-atom structure* if $f \circ [i, j] \in V$ for every $f \in V$ and every $i, j \in n$. The set $V$ is said to be a *concrete $G_n$-atom structure* if it is both concrete $D_n$-atom structure and concrete $P_n$-atom structure. For every $i \in n$ and every two sequences $f, g \in V$, we write $f \equiv_i g$ if and only if $g = f^i_u$ for some $u \in U$. Where, $f^i_u$ is the sequence which is like $f$ except that its value at $i$ equals to $u$. For every $i, j \in n$ and every $X \subseteq V$, let

$$D^{[V]}_{ij} := \{f \in V : f(i) = f(j)\} \quad \text{and} \quad C^{[V]}_i X := \{f \in V : (\exists g \in X) f \equiv_i g\}.$$

When no confusion is likely, we merely omit the superscript $[V]$ from the above defined objects.

**Definition 1.0.1** (Henkin and Németi)**.** Let $n$ be any ordinal and let $K \in \{Crs, D, P, G\}$. Let $V$ be a concrete $K_n$-atom structure. The complex algebra over $V$ is defined to be the structure

$$\mathfrak{Cm}V = \langle \mathcal{P}(V), \cup, \cap, \setminus, \emptyset, V, C^{[V]}_i, D^{[V]}_{ij} \rangle_{i,j \in n}.$$

The smallest $U$ with $V \subseteq {}^nU$ is called *the base of* $V$. Both $V$ and $U$ are also called *the unit and the base of the complex algebra* $\mathfrak{Cm}V$, respectively. Define $K_n$ to be the class of all isomorphic copies of the subalgebras of the complex algebras over the concrete $K_n$-atom structures.

Thus, $K_n = \mathbf{IS}\{\mathfrak{Cm}V : V$ is a concrete $K_n$-atom structure$\}$. Note that the equational theory of $K_n$ coincides with the equational theory of the class of the complex algebras over concrete $K_n$-atom structures. We make use of this fact so often. For instance, if $K_n \not\models \tau = 0$ then we suppose a concrete $K_n$-atom structure $V$ and an evaluation $\iota$ of the free variables into $\mathcal{P}(V)$ such that $[\tau]_\iota^{\mathfrak{Cm}V} \neq \emptyset$.

Relativized algebras were not really studied in their own right, but as tools to obtain results for the standard algebras. At the end of the twentieth century, H. Andréka, J. van Benthem, J. D. Monk and I. Németi started promoting relativized algebras as structures which are interesting independently of their classical versions, see e.g. [AT88],[AvBN95], [AvBN98], [Mon93], [Ném91] and [Mik95]. Indeed, relativization in many cases turns the negative results to positive ones. Several relativized versions of algebras do have most of the nice properties which their standard counterparts lack. The standard (representable) cylindric algebras of dimension $n$ can be defined as follows: $RCA_n := \mathbf{ISP}\{\mathfrak{Cm}^n U :$ for some set $U\}$. Here, we are interested only in the algebras of finite dimensions. Fix a finite ordinal $n \geq 2$ and fix $K \in \{Crs, D, P, G\}$.

| Property | When $n = 2$ | | When $n > 2$ | |
|---|---|---|---|---|
| | $RCA_n$ | $K_n$ | $RCA_n$ | $K_n$ |
| variety if $K \in \{Crs, D, G\}$ | ✓ | ✓ | ✓ | ✓ |
| finitely axiomatizability if $K \in \{D, G\}$ | ✓ | ✓ | ✗ | ✓ |
| finite variables axiomatizability if $K \in \{Crs, D, G\}$ | ✓ | ✓ | ✗ | ✓ |
| Decidability | ✓ | ✓ | ✗ | ✓ |
| finite base property | ✓ | ✓ | ✗ | ✓ |
| Discriminator variety | ✓ | ✗ | ✓ | ✗ |
| AP & SAP & SUPAP if $K \in \{Crs, G\}$ | ✗ | ✓ | ✗ | ✓ |

The class $RCA_n$ is a variety by [HMT85, 3.1.103] and it is well known that it is a discriminator variety, e.g., [BS81, Examples (2) on pages 186-187]. It was shown that the class $RCA_n$ has any of the following properties if and only if $n = 2$: finite axiomatizability, finite variable axiomatizability, decidability, finite algebra property and finite base property. Moreover, $RCA_n$ does not have the amalgamation property AP (and the stronger versions SAP, SUPAP). See [Mon69], [HMT71], [HMT85], [And91],[ANS94] and [AKN$^+$96].

In [Ném81], I. Németi proved that the class $Crs_n$ is a variety, but not finitely axiomatizable if $n \geq 3$. We guess that his proof can be applied to show similar results for the class $P_n$, but this has not been thoroughly checked yet. An elegant infinite-axiomatization for $Crs_n$ that uses only finitely many variables, due to D. Resek and R. J. Thompson, can be found in [Mon00]. In [AT88], it was shown that the class $D_n$ is finitely axiomatizable. Then, in [And01], Andréka used the finite axiomatization of $D_n$ to give a finite axiomatization for the class $G_n$. The same method, but using the axioms of $Crs_n$, may give finite variable axiomatization for the class $P_n$, but we did not check this yet.

Decidability of the equational theories of the classes $Crs_n, D_n, P_n$ and $G_n$ was shown by Németi in [Ném86] and [Ném95, Theorem 4.2 (3)]. All the above relativized classes were shown to have the finite base property (and, consequently, the finite algebra property) in [AHN99]. M. Marx in [Mar95, Appendix 5] showed that the classes $Crs_n$ and $D_n$ have the super amalgamation property, hence they have the strong amalgamation and the amalgamation properties. For $G_n$ and $P_n$, we don't know even if they have the amalgamation property or not. In [AN], it was shown that the class $K_n$ is not discriminator by constructing subdirectly irreducible algebras that are not simple. We note that the results in this chapter together with [AN16, Theorem 2] imply that these classes are not discriminator.

From the universal algebra, the free algebras of the variety $K_n$ play an essential role in understanding this variety. The $K_n$-free algebras represent the structure of the different concepts that can be expressed in $K_n$ by using terms in the language of $K_n$. In this chapter, we investigate the atomicity of the $K_n$-free algebras and we prove the following.

**Theorem 1.0.2.** *Let* $n \in\geq 2$, $m \in \omega$ *and* $K \in \{Crs, D, G, P\}$. *The free algebra* $\mathfrak{Fr}_m K_n$ *is atomic if and only if* $n = 2$, $m = 0$ *and* $K = G$. *Moreover,* $\mathfrak{Fr}_m K_n$ *contains infinitely many atoms if and only if* $n \geq 4$ *and* $K \in \{Crs, P\}$. *In more detail, we have the followings.*

- $\mathfrak{Fr}_m Crs_n$ *is not atomic, it contains finitely many atoms for* $n \leq 3$ *and it contains infinitely many atoms for* $n \geq 4$. *The same are true for* $\mathfrak{Fr}_m P_n$.

- $\mathfrak{Fr}_m D_n$ *is not atomic and contains finitely many atoms.*

- $\mathfrak{Fr}_0 G_2$ *is finite, hence atomic.* $\mathfrak{Fr}_m G_n$ *is not atomic but still contains finitely many atoms if either* $n \geq 3$, *or,* $n = 2$ *and* $m \geq 1$.

Proving the above theorem answers an open problem. The atomicity problem of the finitely generated free relativized cylindric algebras goes back to 1985 when it was posed by I. Németi in his DSc dissertation [Ném86, Remark 18 (i), p. 97]. This problem presents some difficulties, that is why it was posed again in [AMN91, Problem 38, p. 738] and in the most recent book in algebraic logic [AFN13, Problem 1.3.3, p. 34]. These free algebras are infinite (except only $\mathfrak{Fr}_m K_0$, $\mathfrak{Fr}_0 K_1$ and $\mathfrak{Fr}_0 G_2$) and we cannot use the method in [HMT85, 2.5.7] or use [AN16, Theorem 2] because none of the above varieties is discriminator.

## §1.1 Disjunctive normal forms for the cylindric type

We shall use the disjunctive normal forms. Disjunctive normal forms can provide elegant and constructive proofs of many standard results, cf. [And54] and [Fin75]. Fix finite ordinals $n \geq 2$ and $m \geq 0$. The algebras in $K_n$ are of type $cyl_n$, the cylindric type, that consists of binary operation symbols $\cdot, +$, unary operation symbols $-, c_i$ $(i \in n)$ and constant symbols $0, 1, d_{ij}$ $(i, j \in n)$.

For every $k \in \omega$, we define a set $F_k^{n,m} \subseteq \mathfrak{Tm}_{m,cyl_n}$ of normal forms of degree $k$ such that every normal form contains complete information about the cylindrifications of the normal forms of the smaller degrees. Then, we show that every term in $\mathfrak{Tm}_{m,cyl_n}$ can be rewritten as a disjunction of some normal forms of the same degree. First we need the following conventions: Let $\prod, \sum$ be the grouped versions of $\cdot, +$, respectively. Let $T \subseteq \mathfrak{Tm}_{m,cyl_n}$ be a finite set of terms and let $\alpha \in {}^T\{-1, 1\}$. For every $\tau \in T$, let $\tau^\alpha = \tau$ if $\alpha(\tau) = 1$ and $\tau^\alpha = -\tau$ otherwise. Define, $CT := \{c_i\tau : i \in n, \tau \in T\}$ and $T^\alpha := \prod\{\tau^\alpha : \tau \in T\}$.

**Definition 1.1.1.** Set $D_{n,m} = \{d_{ij} : i, j \in n\} \cup \{x_0, \ldots, x_{m-1}\}$, where $x_0, \ldots, x_{m-1}$ are the $m$ free variables that generate $\mathfrak{Tm}_{m,cyl_n}$. For every $k \in \omega$, we define the followings inductively.

- The normal forms of degree 0, $F_0^{n,m} = \{D_{n,m}^\beta : \beta \in {}^{D_{n,m}}\{-1, 1\}\}$.

- The set of normal forms of degree $k + 1$,

$$F_{k+1}^{n,m} = \{D_{n,m}^\beta \cdot (CF_k^{n,m})^\alpha : \beta \in {}^{D_{n,m}}\{-1, 1\} \text{ and } \alpha \in {}^{CF_k^{n,m}}\{-1, 1\}\}.$$

- The set of all forms, $F^{n,m} = \bigcup_{k\in\omega} F_k^{n,m}$.

**Theorem 1.1.2.** *Let $n \geq 2$ and $m \geq 0$ be finite ordinals, $K \in \{Crs, D, P, G\}$ and $k \in \omega$. The followings are true:*

*(i)* $K_n \models \sum F_k^{n,m} = 1$.

*(ii) For every* $\tau, \sigma \in F_k^{n,m}$*, if* $\tau$ *and* $\sigma$ *are different then* $K_n \models \tau \cdot \sigma = 0$.

*(iii) There exists an effective method (a finite algorithm) to find, for every* $\tau \in \mathfrak{Tm}_{m,cyl_n}$*, a non-negative integer* $q \in \omega$ *and a finite set* $S_\tau \subseteq F_q^{n,m}$ *such that* $K_n \models \tau = \sum S_\tau$.

*Proof.* Let $CBA_n$ be the class of all Boolean algebras with operators of type $cyl_n$. In appendix A, it is shown that the above theorem is true for $CBA_n$ instead of $K_n$. But since $K_n \subseteq CBA_n$, then the statement above is true for $K_n$ as well. $\square$

Thus, any satisfiable term that is not equal to any normal form (in $K_n$) can be broken into at least two disjoint satisfiable normal forms. Hence, to find the atoms in $\mathfrak{Fr}_m K_n$, it is enough to search for them among the normal forms. Recall that every form in $F^{n,m}$ is determined by some information given on the diagonals, free variables and the cylindrifications of the forms of the first smaller degree. So, we need to introduce some notions that allow us to handle this information easily.

**Definition 1.1.3.** Define the colors of the normal forms, $color^{n,m} : F^{n,m} \to D_{n,m}$, as follows. For every $k \in \omega$, every $\beta \in {}^{D_{n,m}}\{-1, 1\}$ and every $\alpha \in {}^{CF_k^{n,m}}\{-1, 1\}$, define

$$color^{n,m}(D_{n,m}^\beta) := color^{n,m}(D_{n,m}^\beta \cdot (CF_k^{n,m})^\alpha) := \{\sigma \in D_{n,m} : \beta(\sigma) = 1\}.$$

**Definition 1.1.4.** For every $i \in n$, define $sub_i^{n,m} : F^{n,m} \to \mathcal{P}(F^{n,m})$ as follows.

- For every form $\tau \in F_0^m$ of degree $0$, let $sub_i^{n,m}(\tau) = \emptyset$.
- For every $k \in \omega$, every $\beta \in {}^{D_{n,m}}\{-1, 1\}$ and every $\alpha \in {}^{CF_k^{n,m}}\{-1, 1\}$, let

$$sub_i^{n,m}(D_{n,m}^\beta \cdot (CF_k^{n,m})^\alpha) = \{\sigma \in F_k^{n,m} : \alpha(c_i\sigma) = 1\}.$$

Theorem 1.1.2 can be also used to label the elements of any concrete $K_n$-atom structure by normal forms of any desired degree. Let $S$ be any concrete $K_n$-atom structure and let $\iota : \{x_0, \ldots, x_{m-1}\} \to \mathcal{P}(S)$ be any evaluation. For every $f \in S$ and every $\tau \in \mathfrak{Tm}_{m,cyl_n}$, we write $(S, f, \iota) \models \tau$ if and only if $f \in [\tau]_\iota^{\mathfrak{Cm}S}$. Let $k \in \omega$. For every $f \in S$, the unique term $\tau \in F_k^{n,m}$ for which $(S, f, \iota) \models \tau$ is called the label of $f$ of degree $k$.

**Remark.** Let $S$ and $\iota$ be as above. For every $k \in \omega$ and every $f \in S$, let $tag_k(f)$ be the label of $f$ of degree $k$. Now, fix $k \in \omega$, $i \in n$ and $f \in S$ such that $k \geq 1$. Set the $i$-th neighbors of $f$ as follows, $Nbr_i(f) := C_i^{[S]}(\{f\}) = \{h \in S : h \equiv_i f\}$. Note that in order to determine $tag_{k+1}(f)$, we need also to determine the label $tag_k(g)$ for every $g \in Nbr_i(f)$. For example, let $\tau \in F_{k+1}^{n,m}$ and suppose that we need to show that $tag_{k+1}(f) = \tau$. Then we need to check that $tag_k(g) \in sub_i^{n,m}(\tau)$ for every $g \in Nbr_i(f)$. For any $g \in Nbr_i(f)$, the information carried by the label $tag_{k-1}(g)$ might be consistent with the information carried by $\tau$, however, it is not enough to prove that $tag_{k+1}(f) = \tau$. We call this observation the "degree consistency" of the atom structure $S$.

## §1.2 The free algebra $\mathfrak{Fr}_m Crs_2$ is not atomic.

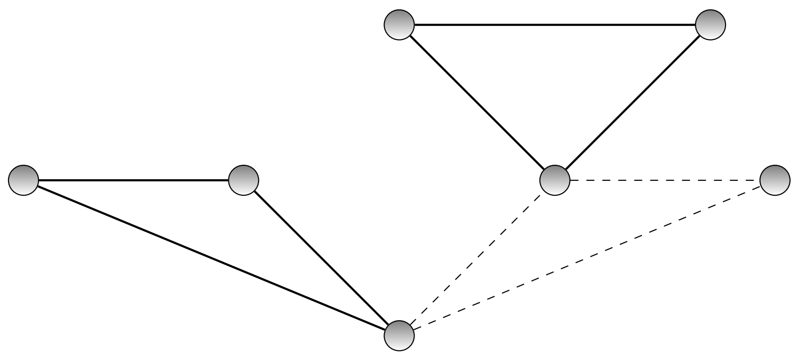
In this section, we show that $\mathfrak{Fr}_m Crs_2$ is not atomic. Our strategy goes as follows. Fix a finite ordinal $k \in \omega$ and a normal form $\tau \in F_k^{2,m}$ such that $\mathfrak{Fr}_m Crs_2 \not\models \tau = 0$. First, we associate to $\tau$ an algebra $\mathfrak{A}^\tau \in Crs_2$, with finite base, that witnesses $\mathfrak{Fr}_m Crs_2 \not\models \tau = 0$. Then, in most of the cases, we use $\mathfrak{A}^\tau$ to show that $\tau$ is not an atom in $\mathfrak{Fr}_m Crs_2$ by constructing a "sister" algebra $\mathfrak{A}^{\tau'}$ and a term $\sigma$ such that $\tau \cdot \sigma \neq 0$ in $\mathfrak{A}^\tau$ while $\tau \cdot -\sigma \neq 0$ in $\mathfrak{A}^{\tau'}$. Here is a convention: For every $i \in 2$, let $j \in 2$ denote the unique non-negative integer satisfying $i + j = 1$.

To construct $\mathfrak{A}^\tau$, we construct a concrete $Crs_2$-atom structure $V^\tau$ and we let $\mathfrak{A}^\tau$ to be the complex (full) algebra over $V^\tau$. We construct, inductively, a sequence $V_0 \subseteq \cdots \subseteq V_k$ of concrete $Crs_2$-atom structures. For $V_0$: Let $f_0, f_1$ be two elements such that $f_0 = f_1$ if and only if $d_{01} \in color^{2,m}(\tau)$. Set $V_0 := V_0^0 := V_0^1 := \{(f_0, f_1)\}$ and define $tag(f_0, f_1) = \tau$. The label $tag(f_0, f_1) = \tau$ means that the pair $f := (f_0, f_1)$ is responsible for validating $\tau$ at the end of the construction. To guarantee this, we need to extend $V_0$ by the information given by $sub_i^{2,m}(\tau)$, $i \in 2$, as follows. Let $U$ be an infinite set disjoint from $\{f_0, f_1\}$. For each $i \in 2$, construct an injective function, $\psi_f^i : \{\sigma \in sub_i^{2,m}(\tau) : d_{01} \notin color^{2,m}(\sigma)\} \longrightarrow U$, such that $Rng(\psi_f^0) \cap Rng(\psi_f^1) = \emptyset$ and $U \setminus (Rng(\psi_f^0) \cup Rng(\psi_f^1))$ is infinite. Set

$$\begin{aligned} V_1^i &= \{(f_0, f_1)_u^i : u \in Rng(\psi_f^i)\} \\ &\cup \{(f_j, f_j) : f_0 \neq f_1 \text{ and } \exists! \sigma \in sub_i^{2,m}(\tau) \text{ with } d_{01} \in color^{2,m}(\sigma)\}. \end{aligned}$$

We omitted the terms in $sub_i^{2,m}$ that contain $d_{01}$ in their colors from the domain of $\psi_f^i$ because we do not need new bases to represent these terms, in other words these terms have to be represented by $(f_j, f_j)$. Moreover, if there is such a term then it is unique because every two different normal forms are disjoint. Define the labels as follows. Let $u \in Rng(\psi_f^i)$ and $\sigma \in sub_i^{2,m}(\tau)$ such that $\psi_f^i(\sigma) = u$, define $tag(f_u^i) = \sigma$. If $f_0 \neq f_1$ and there exists a unique $\sigma \in sub_i^{2,m}(\tau)$ with $d_{01} \in color^{2,m}(\sigma)$, define $tag(f_j, f_j) = \sigma$. Finish this round by setting $V_1 := V_1^0 \cup V_1^1$.

It is not hard to see that, under a suitable evaluation, if every element in $V_1$ satisfies its label then $f$ satisfies $\tau$. Hence, we need to extend $V_1$ by adding more elements according to the information carried by the functions $sub_i^{2,m}$, $i \in 2$, to guarantee that each element of $V_1$ satisfies its label. Here, we note that every two elements $g, h \in V_1^0$ (similarly in $V_1^1$) are 0-connected, i.e., $g \equiv_0 h$. In the class $Crs_2$, the operator $c_0$ is a complemented closure operator. Therefore, the information given by $sub_0^{2,m}$ of any element in $V_1^0$ is already guaranteed by the other elements in $V_1^0$. So, we need to consider the information given by $sub_1^{2,m}$ only. See the figure below (the dotted edges represent the 0-connections while the thick edges represent the 1-connections.)



More generally, suppose that $V_l$,$V_l^0$ and $V_l^1$ have been constructed and the labeling $tag$ has been extended to cover $V_l$, for some $0 < l < k$. Let $i \in 2$, for every $g \in V_l^j$ create an injective function,

$$\psi_g^i : \{\sigma \in sub_i^{2,m}(tag(g)) : d_{01} \notin color^{2,m}(\sigma)\} \longrightarrow (U \setminus \bigcup\{Rng(V_q) : q \in l+1\}),$$

such that $Rng(\psi_g^i)$'s are pairwise disjoint and $(U^* \setminus \bigcup\{Rng(\psi_g^i) : i \in 2, g \in V_l^j\})$ is still infinite, where $U^* := (U \setminus \bigcup\{Rng(V_q) : q \in l+1\})$. Set

$$\begin{aligned} V^i_{l+1} &= \{(g_0, g_1)^i_u : g = (g_0, g_1) \in V^j_l \text{ and } u \in Rng(\psi^i_g)\} \\ &\cup \{(g_j, g_j) : g = (g_0, g_1) \in V^j_l, g_0 \neq g_1 \text{ and } \exists! \sigma \in sub^{2,m}_i(tag(g)), d_{01} \in color^{2,m}(\sigma)\}. \end{aligned}$$

The labels are extended in a natural way. Let $g = (g_0, g_1) \in V^j_l$, $u \in Rng(\psi^i_g)$ and let $\sigma \in sub^{2,m}_i(tag(g))$ such that $\psi^i_g(\sigma) = u$, define $tag(g^i_u) = \sigma$. If $g_0 \neq g_1$ and there exists a unique $\sigma \in sub^{2,m}_i(tag(g))$ with $d_{01} \in color^{2,m}(\sigma)$, define $tag(g_j, g_j) = \sigma$. Finally, set $V_{l+1} := V^0_{l+1} \cup V^1_{l+1}$.

Set $V^\tau = V_0 \cup \cdots \cup V_k$ and let $\mathfrak{A}^\tau = \mathfrak{Cm} V^\tau$ be the complex algebra over $V^\tau$. For an evaluation, define $\iota^\tau(x_i) = \{g \in V^\tau : x_i \in color^{2,m}(tag(g))\}$ for every $i \in m$. The next lemma proves that $\mathfrak{A}^\tau \not\models \tau = 0$.

**Lemma 1.2.1.** *For every $g \in V^\tau$, we have $(V^\tau, g, \iota^\tau) \models tag(g)$.*

*Proof.* Let $l, q \in k+1$ and let $g \in V_l$. Note that the label $tag(g) \in F^{2,m}_{k-l}$. If $q \leq k-l$, then define $tag_q(g)$ to be the unique term in $F^{2,m}_q$ such that $Crs_2 \models tag(g) \leq tag_q(g)$. If $q \geq k-l$, then let $tag_q(g) = tag(g)$. To prove the statement above, it suffices to prove the following. For every $g \in V^\tau$ and every $q \in k+1$,

$$(V^\tau, g, \iota^\tau) \models tag_q(g). \tag{1.1}$$

We use induction on $q$. By the construction of $V^\tau$ and the special choice of the evaluation $\iota^\tau$, it is clear that $(V^\tau, g, \iota^\tau) \models tag_0(g)$, for every $g \in V^\tau$. Suppose that $k > 0$ and, for some $q \in k$, $(V^\tau, g, \iota^\tau) \models tag_q(g)$, for every $g \in V^\tau$. We need to step the induction up to $q+1$. Let $g = (g_0, g_1) \in V^\tau$ be arbitrary, then there exists $l \in k+1$ such that $g \in V_l$. If $q \geq k-l$ then $tag_{q+1}(g) = tag_q(g)$ and, by induction hypothesis, $(V^\tau, g, \iota^\tau) \models tag_q(g) = tag_{q+1}(g)$.

So, we may suppose that $q < k-l$. Let $i \in 2$ and let $\sigma \in sub^{2,m}_i(tag_{q+1}(g))$. Remember $Crs_2 \models tag(g) \leq tag_{q+1}(g)$, so there exists $\chi \in sub^{2,m}_i(tag(g))$ such that $Crs_2 \models \chi \leq \sigma$.

- Suppose that $g \in V^j_l$ and $d_{01} \notin color^{2,m}(\chi)$. By the construction of $V^\tau$, there exists $u \in Rng(\psi^i_g)$ such that $g^i_u \in V^i_{l+1}$ and $tag(g^i_u) = \chi$. Then by induction hypothesis, $(V^\tau, g^i_u, \iota^\tau) \models tag_q(g^i_u) = \sigma$. Hence, $(V^\tau, g, \iota^\tau) \models c_i\sigma$.

- Suppose that $g \in V_l^j$ and $d_{01} \in color^{2,m}(\chi)$. If $g_0 \neq g_1$, then $(g_j, g_j) \in V_{l+1}^i$ and $tag(g_j, g_j) = \chi$. We have $(V^\tau, (g_j, g_j), \iota^\tau) \models tag_q(g_j, g_j) = \sigma$, by the induction hypothesis. Consequently, $(V^\tau, g, \iota^\tau) \models c_j\sigma$.

- Suppose that $g \in V_l^j$ and $d_{01} \in color^{2,m}(\chi)$. If $g_0 = g_1$, then $d_{01} \in tag(g)$. That implies that $Crs_2 \models tag(g) \leq \chi \leq \sigma$, otherwise $\mathfrak{Fr}_m Crs_2 \models tag(g) = 0$ which contradicts the assumption $\mathfrak{Fr}_m Crs_2 \not\models \tau = 0$. By induction, we have $(V^\tau, g, \iota^\tau) \models tag_q(g) = \sigma$. Hence, $(V^\tau, g, \iota^\tau) \models c_i\sigma$.

- Suppose that $l \neq 0$ and $g \in V_l^i$. Then there exists an element $h \in V_{l-1}^j$ such that $h \equiv_i g$ and $tag(g) \in sub_i^{2,m}(tag(h))$. Then, there exists $\gamma \in sub_i^{2,m}(tag(h))$ with $Crs_n \models \gamma \leq \chi \leq \sigma$. By a similar arguments to the ones used in the above items, one can easily find $\bar{h} \in V_l^i$ such that $\bar{h} \equiv_i h$ and $tag(\bar{h}) = \gamma$. Hence, $\bar{h} \equiv_i g$ and, by the induction hypothesis, $(V^\tau, \bar{h}, \iota^\tau) \models tag_q(\bar{h}) = \sigma$. Therefore, $(V^\tau, g, \iota^\tau) \models c_i\sigma$.

Conversely, let $\sigma \in (F_q^{2,m} \backslash sub_i^{2,m}(tag_{q+1}(g)))$. Assume toward a contradiction that there exists $h \in V^\tau$ such that $g \equiv_i h$ and $(V^\tau, h, \iota^\tau) \models \sigma$. It is not hard to see that, by the construction and the induction hypothesis, there exists $\gamma \in sub_i^{2,m}(tag_{q+1}(g)))$ such that $(V^\tau, h, \iota^\tau) \models \gamma$. Then $\gamma \neq \sigma$ but $Crs_2 \not\models \sigma \cdot \gamma = 0$. This contradicts theorem 1.1.2, (ii). Hence, $(V^\tau, g, \iota^\tau) \not\models c_i\sigma$. Now, recall that $(V^\tau, g, \iota^\tau) \models tag_0(g)$. Therefore, $(V^\tau, g, \iota^\tau) \models tag_{q+1}(g)$. We have proved (1.1) for every $g \in V^\tau$ and every $q \in k+1$, as desired. $\square$

Now we are ready to show that $\mathfrak{Fr}_m Crs_2$ is not atomic. The idea basically is as follows. For the term $\tau$, if we can find a zigzag in $V^\tau$ that is starting from the unique point in $V_0$ and goes up one level at each step then, at the $k$-th level, we have freedom to decide whether to extend this zigzag and construct a new algebra or to leave it as it is. This may give two different non zero disjoint terms each of which is below $\tau$, hence $\tau$ is not an atom in $\mathfrak{Fr}_m Crs_2$.

**Proposition 1.2.2.** *Let $m \in \omega$. There is no atom in $\mathfrak{Fr}_m Crs_2$ that is below $c_0 - d_{01} + c_1 - d_{01}$.*

*Proof.* Let $k \geq 1$ and let $\tau \in F_k^{2,m}$ be such that $\mathfrak{Fr}_m Crs_2 \models 0 \neq \tau \leq c_0 - d_{01} + c_1 - d_{01}$. Consider the algebra $\mathfrak{A}^\tau$, its unit $V^\tau = V_0 \cup \cdots V_k$ and the labeling $tag$ as constructed before. Without loss of generality, we may assume that $\mathfrak{Fr}_m Crs_2 \not\models \tau \cdot c_0 - d_{01} = 0$. Our aim is to find a (zigzag) sequence, $h_0 \ldots, h_k$, of elements from $V^\tau$ such that, for every $1 \le q \le k$, $d_{01} \notin \mathit{color}^{2,m}(tag(h_q))$ and $h_{q-1} \equiv_i h_q$ where $i \cong q-1$ $(mod\, 2)$.

Let $h_0 = (f_0, f_1)$ be the unique element in $V_0$ whose label is $\tau$. The assumption on $\tau$ that $\mathfrak{Fr}_m Crs_2 \not\models \tau \cdot c_0 - d_{01} = 0$ implies that there is a form $\sigma_0 \in sub_0^{2,m}(tag(h_0))$ such that $d_{01} \notin \mathit{color}^{2,m}(\sigma_0)$. By the construction of $V^\tau$, there exists $h_1 \in V_1^0$ such that $h_0 \equiv_0 h_1$ and $tag(h_1) = \sigma_0$. Suppose that we have already found $h_q$, for some $0 < q < k$, such that $h_q \in V_q^i$, $h_{q-1} \equiv_i h_q$ and $d_{01} \notin \mathit{color}^{2,m}(tag(h_q))$, where $i \in 2$ and $i \cong q-1$ $(mod\, 2)$. By the assumption $d_{01} \notin \mathit{color}^{2,m}(tag(h_q))$, there exists $\sigma_q \in sub_j^{2,m}(tag(h_q))$ such that $d_{01} \notin \mathit{color}^{2,m}(\sigma_q)$. Therefore, there exists $h_{q+1} \in V_{q+1}^j$ such that $h_q \equiv_j h_{q+1}$ and $tag(h_{q+1}) = \sigma_q$. Recall that by lemma 1.2.1, we have the following.

$$(\forall q \in k+1)\ \ (V^\tau, h_q, \iota^\tau) \models tag(h_q). \tag{1.2}$$

Suppose that $h_k = (u_0, u_1)$ and assume that $h_k \in V_k^i$, for some $i \in 2$. Note that $tag(h_k)$ is a normal form of degree 0 and $d_{01} \notin \mathit{color}^{2,m}(tag(h_k))$. Therefore, by the construction of $V^\tau$, $(u_i, u_i) \notin V^\tau$. Set $W^\tau = V^\tau \cup \{(u_i, u_i)\}$ and define $\mathfrak{D}^\tau := \mathfrak{Cm} W^\tau$. Define an evaluation $\nu^\tau : \{x_0, \ldots, x_{m-1}\} \to \mathcal{P}(W^\tau)$ such that $\nu^\tau(x_i) = \iota^\tau(x_i)$, for every $i \in m$. By a similar argument to the one used in the proof of lemma 1.2.1, one can easily prove the following.

$$(\forall q \in k+1)\ \ (W^\tau, h_q, \nu^\tau) \models tag(h_q). \tag{1.3}$$

For each $q \in k+1$, let $\chi_q$ and $\gamma_q$ be the unique forms in $F_{q+1}^{2,m}$ satisfy $(V^\tau, h_{k-q}, \tau^\tau) \models \chi_q$ and $(W^\tau, h_{k-q}, \nu^\tau) \models \gamma_q$, respectively. Hence, both $\chi_q, \gamma_q$ are not zeros in $\mathfrak{Fr}_m Crs_2$. Moreover, by (1.2) and (1.3), we have,

$$\mathfrak{Fr}_m Crs_2 \models 0 \neq \chi_q \le tag(h_{k-q})\ \ and\ \ \mathfrak{Fr}_m Crs_2 \models 0 \neq \gamma_q \le tag(h_{k-q}). \tag{1.4}$$

It remains to show that, for every $q \in k+1$, $\mathfrak{Fr}_m Crs_2 \models \chi_q \cdot \gamma_q = 0$. We use induction on $q$. Remember, we have $h_k = (u_0, u_1) \in V_k^i$, $(u_i, u_i) \notin V^\tau$ and $(u_i, u_i) \in W^\tau$ for some $i \in 2$. Hence, $Crs_2 \models \chi_0 \le -c_j d_{01}$ while $Crs_2 \models \gamma_0 \le c_j d_{01}$. Therefore, $\mathfrak{Fr}_m Crs_2 \models \chi_0 \cdot \gamma_0 = 0$. The induction step is going in a similar way. Suppose that $\mathfrak{Fr}_m Crs_2 \models \chi_q \cdot \gamma_q = 0$, for some $q \in k$. Consider the elements $h_{k-q}$ and $h_{k-q-1}$. Let $i \in 2$ be such that $i = k-q-1$ $(mod\, 2)$.

Remember that $h_{k-q-1} \equiv_i h_{k-q}$. Also, remember that $\chi_q, \gamma_q, tag(h_{k-q-1}) \in F^{2,m}_{q+1}$. But $\chi_q$ and $\gamma_q$ are disjoint in $\mathfrak{Fr}_m Crs_2$, hence $\chi_q$ or $\gamma_q$ is disjoint from $tag(h_{k-q-1})$. Without loss of generality, assume that $Crs_2 \models \chi_q \cdot tag(h_{k-q-1}) = 0$. By the construction of $V^\tau$ we have, for every $g \in V^\tau \setminus \{h_{k-q-1}, h_{k-q}\}$, if $g \equiv_i h_{k-q-1}$ then $Crs_2 \models tag(g) \cdot tag(h_{k-q}) = 0$. Therefore, by (1.4), we have the following. For every $g \in V^\tau \setminus \{h_{k-q}\}$, if $g \equiv_i h_{k-q-1}$ then $(V^\tau, g, \iota^\tau) \not\models \chi_q$ and $(W^\tau, g, \nu^\tau) \not\models \chi_q$. Remember that $\chi_q$ and $\gamma_q$ were chosen such that $(V^\tau, h_{k-q}, \iota^\tau) \models \chi_q$ and $(W^\tau, h_{k-q}, \nu^\tau) \models \gamma_q$. Hence, $(W^\tau, h_{k-q-1}, \nu^\tau) \models \gamma_{q+1} \cdot -c_i\chi_q$ and $(V^\tau, h_{k-q-1}, \iota^\tau) \models \chi_{q+1} \cdot c_i\chi_q$. Therefore, $Crs_2 \models \chi_{q+1} \leq c_i\chi_q$ but $Crs_2 \models \gamma_{q+1} \leq -c_i\chi_q$. By the induction principle, we have the following. For every $q \in k+1$,

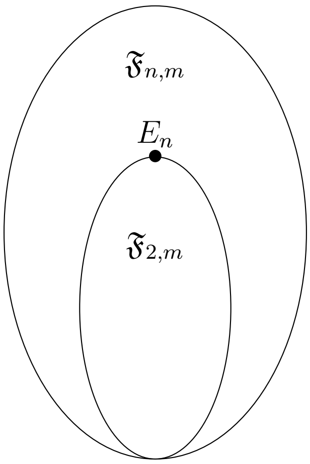
$$\mathfrak{Fr}_m Crs_2 \models \chi_q \cdot \gamma_q = 0. \tag{1.5}$$

Combining (1.4) and (1.5) with the fact $tag(h_0) = \tau$ shows that $\tau$ is not an atom in the free algebra $\mathfrak{Fr}_m Crs_2$, as desired. □

## §1.3 The free algebra $\mathfrak{Fr}_m Crs_n$ is not atomic.

Suppose that $n \geq 3$. The aim of this section is to prove the non-atomicity of the free algebra $\mathfrak{Fr}_m Crs_n$. Unfortunately, at least if we apply it verbatim, the idea used in the previous section cannot help for this purpose. To explain why it doesn't work, let $\tau \in F^{n,m}_5$ and suppose that $Crs_n \models 0 \neq \tau \leq \prod_{i \in j \in n} -d_{ij}$. Suppose that an atom structure $V^\tau$ is constructed in the very similar way to the construction in the previous section. Suppose that $f := (f_0, \ldots, f_{n-1})$ has label $\tau$ and suppose that the elements $(f_0, f_0, f_2, \ldots, f_{n-1})$ and $g := (f_2, f_1, f_2, \ldots, f_{n-1})$ were added in $V_1$. Suppose that for building $V^2$, one needed to add $(u, f_0, f_2, \ldots, f_{n-1})$, for some brand new $u$. Then in $V^3$, the element $h := (f_2, f_0, f_2, \ldots, f_{n-1})$ might be added. Since $h$ was added in $V^3$, then its label has to be from $F^{n,m}_2$. But $h \equiv_1 g$ and $g$ carries a label from $F^{n,m}_4$. Therefore, $h$ carries a consistent information with the label of $g$ but it is not complete. Recall the remark on page 20 and note that yet we don't know which normal form in $F^{n,m}_3$ would be satisfied by $h$ at the end of the construction. To overcome this problem, $h$ has to get a label from $F^{n,m}_3$. Actually there is a way to do that, but now it is clear that $V^\tau$ has to be changed and it cannot be constructed in the same way as before. Moreover, we may lose one of the key properties that we used in our proof. Namely, we cannot guarantee that there is a node that is connected by a finite zigzag to $f$ and whose label is a normal form of degree $0$.

However, there is a nice way to show that the free algebra $\mathfrak{Fr}_m Crs_n$ is not atomic using the fact that the free algebra $\mathfrak{Fr}_m Crs_2$ is not atomic. Roughly, we do this by showing that the Boolean part of $\mathfrak{Fr}_m Crs_2$ is isomorphic to the Boolean part of some relativized subalgebra of $\mathfrak{Fr}_m Crs_n$. To be self contained, we recall the definition of relativized subalgebras of the members of $Crs_n$. Let $\mathfrak{A} = \langle A, \cup, \cap, \setminus, \emptyset, V, C_i^{[V]}, D_{ij}^{[V]} \rangle_{i,j \in n} \in Crs_n$. Let $X \in A$ and let $Rl_X \mathfrak{A} = \{B \cap X : B \in A\}$. Then the structure $\mathfrak{Rl}_X \mathfrak{A} = \langle Rl_X \mathfrak{A}, \cup, \cap, \setminus, \emptyset, X, C_i^{[X]}, D_{ij}^{[X]} \rangle_{i,j \in n}$ is called the $X$-relativization of $\mathfrak{A}$. Clearly, $\mathfrak{Rl}_X \mathfrak{A} \in Crs_n$. Note that $\mathfrak{Tm}_{m,cyl_2} \subseteq \mathfrak{Tm}_{m,cyl_n}$.




Let $E_n := \prod\{-c_k - d_{1j} : 1 < k < n, 1 < j < n\} \in \mathfrak{Tm}_{m,cyl_n}$. The Boolean algebra $\mathfrak{Bf}\mathfrak{Fr}_m Crs_2$ is not atomic by proposition 1.2.2. We show that $\mathfrak{Bf}\mathfrak{Fr}_m Crs_2$ is Boolean isomorphic to $\mathfrak{Bf}\mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n$. Therefore, the algebra $\mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n$ is not atomic and, consequently, $\mathfrak{Fr}_m Crs_n$ is not atomic. We need the following lemmas.

**Lemma 1.3.1.** *For every $\tau \in \mathfrak{Tm}_{m,cyl_2}$, if $Crs_2 \not\models \tau = 0$ then $Crs_n \not\models \tau \cdot E_n = 0$.*

*Proof.* Let $\tau \in \mathfrak{Tm}_{m,cyl_2}$ be such that $Crs_2 \not\models \tau = 0$. Then there is a concrete $Crs_2$-atom structure $V$, an evaluation of the free variables $\iota : \{x_0, \ldots, x_{m-1}\} \to \mathcal{P}(V)$ and $(u, v) \in V$ such that $(V, (u, v), \iota) \models \tau$. For every $f \in V$, let $f^*$ be the sequence that extends $f$ such that $f^*(i) = v$ for every $2 \leq i < n$. Let $V^* = \{f^* : f \in V\}$. Let $\iota^* : \{x_0, \ldots, x_{m-1}\} \to \mathcal{P}(V^*)$ be the evaluation defined as follows: For every $i \in m$, $\iota^*(x_i) = \{f^* : f \in V \cap \iota(x_i)\}$. Now, we prove the following. For every $t \in \mathfrak{Tm}_{m,cyl_2}$,

$$(\forall f \in V)[(V, f, \iota) \models t \iff (V^*, f^*, \iota^*) \models t]. \tag{1.6}$$

We prove (1.6) by induction on terms. For $t \in \{d_{01}, x_0, \ldots, x_{m-1}\}$, it is clear that (1.6) holds. Suppose that (1.6) hold for $\sigma_1$ and $\sigma_2$. It is easy to see that (1.6) holds for $\sigma_1 \cdot \sigma_2$, $\sigma_1 + \sigma_2$ and $-\sigma_1$. Let $k \in 2$ and let $f \in V$, then

$$\begin{aligned}
(V, f, \iota) \models c_k\sigma_1 \quad &\iff \quad (\exists g \in V) g \equiv_k f \text{ and } (V, g, \iota) \models \sigma_1 \\
&\iff \quad (\exists g^* \in V^*) g^* \equiv_k f^* \text{ and } (V^*, g^*, \iota^*) \models \sigma_1 \\
&\iff \quad (V^*, f^*, \iota^*) \models c_k\sigma_1.
\end{aligned}$$

We have proved (1.6). In particular, $(V^*, (u, v)^*, \iota^*) \models \tau$. Moreover, $(V^*, (u, v)^*, \iota^*) \models E_n$. Therefore, $Crs_n \not\models \tau \cdot E_n = 0$, as desired. □

**Lemma 1.3.2.** *For every $\tau \in \mathfrak{Tm}_{m,cyl_2}$, if $Crs_n \not\models \tau \cdot E_n = 0$ then $Crs_2 \not\models \tau = 0$.*

*Proof.* Let $\tau \in \mathfrak{Tm}_{m,cyl_2}$ be such that $Crs_n \not\models \tau \cdot E_n = 0$. Then there is a concrete $Crs_n$-atom structure $V$, an evaluation $\iota : \{x_0, \ldots, x_{m-1}\} \to \mathcal{P}(V)$ and $(u, v, \ldots, v) \in V$ such that $(V, (u, v, \ldots, v), \iota) \models \tau \cdot E_n$. Let $W = \{f \in V : (\forall\, 2 \leq i < n) f(i) = v\}$. For every $f \in W$, let $f_*$ be the sequence $(f(0), f(1))$. Let $V_* = \{f_* : f \in W\}$ and, for each $i \in m$, define $\iota_*(x_i) = \{f_* \in V_* : f \in \iota(x_i) \cap W\}$. Now we prove the following. For every $t \in \mathfrak{Tm}_{m,cyl_2}$,

$$(\forall f \in W)[(V, f, \iota) \models t \iff (V_*, f_*, \iota_*) \models t]. \tag{1.7}$$

We use induction on terms. It is easy to check that (1.7) holds for $\{d_{01}, x_0, \ldots, x_{m-1}\}$. Suppose that (1.7) hold for $\sigma_1$ and $\sigma_2$. One can easily check that (1.7) holds for $\sigma_1 \cdot \sigma_2$, $\sigma_1 + \sigma_2$ and $-\sigma_1$. It remains to show that, for every $k \in 2$, (1.7) holds for $c_k\sigma_1$. Let $k \in 2$ and let $f \in W$, then

$$\begin{aligned}
(V, f, \iota) \models c_k\sigma_1 \quad &\iff \quad (\exists g \in W) g \equiv_k f \text{ and } (V, g, \iota) \models \sigma_1 \\
&\iff \quad (\exists g_* \in V_*) g_* \equiv_k f_* \text{ and } (V_*, g_*, \iota_*) \models \sigma_1 \\
&\iff \quad (V_*, f_*, \iota_*) \models c_k\sigma_1.
\end{aligned}$$

We have proved (1.7). Therefore, $Crs_2 \not\models \tau = 0$, as desired. □

Now, we are ready to prove the non-atomicity of $\mathfrak{Fr}_m Crs_n$. For the following proposition, it is worth noting that $\mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n \models E_n = 1$.

**Proposition 1.3.3.** *The Boolean algebras $\mathfrak{Bf}\mathfrak{Fr}_m Crs_2$ and $\mathfrak{Bf}\mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n$ are Boolean isomorphic.*

*Proof.* We show this isomorphism via $\psi : (\forall \tau \in \mathfrak{Tm}_{m,cyl_2})\tau \mapsto \tau \cdot E_n$. Note that, by lemma 1.3.1 and lemma 1.3.2, we have the following. For every $\tau, \sigma \in \mathfrak{Tm}_{m,cyl_2}$,

$$\begin{aligned} Crs_2 \models \tau = \sigma \quad &\iff \quad Crs_2 \models ((\tau \cdot -\sigma) + (-\tau \cdot \sigma)) = 0 \\ &\iff \quad Crs_n \models ((\tau \cdot -\sigma) + (-\tau \cdot \sigma)) \cdot E_n = 0 \\ &\iff \quad Crs_n \models \tau \cdot E_n = \sigma \cdot E_n. \end{aligned} \tag{1.8}$$

Therefore, by (1.8), $\psi$ is well defined injective function. It is straightforward to check that $\psi$ preserves the Boolean-operations. Hence, $\psi$ is a Boolean-monomorphism. To show that $\psi$ is onto, it is enough to prove the following. For every $\sigma \in \mathfrak{Tm}_{m,cyl_n}$, there exists $\tau \in \mathfrak{Tm}_{m,cyl_2}$ such that

$$\mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n \models \tau \cdot E_n = \sigma \cdot E_n. \tag{1.9}$$

We prove (1.9) by induction on terms. Let $i, j \in n$ and let $k \in m$. If $\sigma$ is $d_{01}$, $d_{ij}$ (and $i, j \notin \{0, 1\}$) or $x_k$, let $\tau$ be $d_{01}$, $1$ or $x_k$, respectively. Then (1.9) holds for those $\sigma$ and $\tau$. Let $\sigma_1, \sigma_2 \in \mathfrak{Tm}_{m,cyl_n}$ and suppose that there exist $\tau_1, \tau_2 \in \mathfrak{Tm}_{m,cyl_2}$ such that

$$\mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n \models \tau_1 \cdot E_n = \sigma_1 \cdot E_n \quad \text{and} \quad \mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n \models \tau_2 \cdot E_n = \sigma_2 \cdot E_n.$$

Then we have the followings. Let $j \in 2$ and let $1 < k < n$,

$$\begin{aligned} \mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n \ &\models \ (\tau_1 + \tau_2) \cdot E_n = (\tau_1 \cdot E_n) + (\tau_2 \cdot E_n) = (\sigma_1 + \sigma_2) \cdot E_n \\ \mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n \ &\models \ (\tau_1 \cdot \tau_2) \cdot E_n = (\tau_1 \cdot E_n) \cdot (\tau_2 \cdot E_n) = (\sigma_1 \cdot \sigma_2) \cdot E_n \\ \mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n \ &\models \ -\tau_1 \cdot E_n = E_n \cdot -(\tau_1 \cdot E_n) = E_n \cdot -(\sigma_1 \cdot E_n) = -\sigma_1 \cdot E_n \\ \mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n \ &\models \ c_j\tau_1 \cdot E_n = c_j(\tau_1 \cdot E_n) \cdot E_n = c_j(\sigma_1 \cdot E_n) \cdot E_n = c_j\sigma_1 \cdot E_n \\ \mathfrak{Rl}_{E_n}\mathfrak{Fr}_m Crs_n \ &\models \ \tau_1 \cdot E_n = c_k(\tau_1 \cdot E_n) \cdot E_n = c_k(\sigma_1 \cdot E_n) \cdot E_n = c_k\sigma_1 \cdot E_n. \end{aligned}$$

The first $=$ in the last line is because $Crs_n \models c_k(d_{ij} \cdot x) \cdot d_{ij} = x \cdot d_{ij}$. Thus, we have proved

(1.9). Therefore, $\psi$ is an isomorphism as desired. $\square$

**Theorem 1.3.4.** *Let $n \geq 2$ and $m \geq 0$ be finite ordinals. The free algebra $\mathfrak{Fr}_m Crs_n$ is not atomic. Let $i \in j \in n$, then neither the part below $d_{ij}$ nor the part below $-d_{ij}$ in $\mathfrak{Fr}_m Crs_n$ is atomic.*

*Proof.* For $n = 2$, we are done by proposition 1.2.2. Suppose that $n \geq 3$. If $i = 0$ and $j = 1$, the statement follows from propositions 1.2.2 and 1.3.3. For arbitrary $i \in j \in n$, it is enough to find an isomorphism $\psi : \mathfrak{Fr}_m Crs_n \to \mathfrak{Fr}_m Crs_n$ such that $\psi(d_{01}) = d_{ij}$.

Let $\rho \in {}^n n$ be any permutation on $n$ (bijection from $n$ onto $n$). For every $\tau \in \mathfrak{Tm}_{m,cyl_n}$, define $\tau^\rho$ inductively as follows. For every $i, j \in n$, $k \in m$ and $\tau_1, \tau_2, \tau \in \mathfrak{Tm}_{m,cyl_n}$, define

- $d_{ij}^\rho = d_{\rho(i)\rho(j)}$, $x_k^\rho = x_k$,
- $(\tau_1 + \tau_2)^\rho = \tau_1^\rho + \tau_2^\rho$, $(\tau_1 \cdot \tau_2)^\rho = \tau_1^\rho \cdot \tau_2^\rho$,
- $(-\tau)^\rho = -(\tau^\rho)$ and $(c_i\tau)^\rho = c_{\rho(i)}\tau^\rho$.

Clearly, for every $\tau \in \mathfrak{Tm}_{m,cyl_n}$,

$$(\tau^\rho)^{\rho^{-1}} = (\tau^{\rho^{-1}})^\rho = \tau \text{ where } \rho^{-1} \text{ is the inverse of } \rho. \tag{1.10}$$

Let $V$ be any concrete $Crs_n$-atom structure and let $\iota : \{x_0, \ldots, x_{m-1}\} \to \mathcal{P}(V)$ be an evaluation of the free variables. Set $V^\rho = \{f \circ \rho : f \in V\}$. Clearly, $V^\rho$ is concrete $K_n$-atom structure. Let $\iota^\rho : \{x_0, \ldots, x_{m-1}\} \to \mathcal{P}(V^\rho)$ be the evaluation defined as follows. For every $i \in m$, $\iota^\rho(x_i) = \{f \circ \rho : f \in V \cap \iota(x_i)\}$. By a simple induction argument on terms, one can easily check the following. For every $f \in V$ and every $\tau \in \mathfrak{Tm}_{m,cyl_n}$,

$$(V, f, \iota) \models \tau \iff (V^\rho, f \circ \rho, \iota^\rho) \models \tau^\rho. \tag{1.11}$$

The $\iff$ above is because of (1.10). Therefore, for every $\tau, \sigma \in \mathfrak{Tm}_{m,cyl_n}$, we have

$$\begin{aligned} Crs_n \models \tau = \sigma &\iff Crs_n \models ((\tau \cdot -\sigma) + (-\tau \cdot \sigma)) = 0 \\ &\iff Crs_n \models ((\tau \cdot -\sigma) + (-\tau \cdot \sigma))^\rho = 0 \\ &\iff Crs_n \models \tau^\rho = \sigma^\rho. \end{aligned} \tag{1.12}$$

Now, let $\rho \in {}^n n$ be a permutation such that $\rho(0) = i$ and $\rho(1) = j$. Define the isomorphism $\psi : \mathfrak{Fr}_m Crs_n \to \mathfrak{Fr}_m Crs_n$ as follows. For every $\tau \in \mathfrak{Tm}_{m,cyl_n}$, define $\psi(\tau) := \tau^\rho$. By (1.12) and (1.10), $\psi$ is a well defined bijection. By the definition of $\psi$, it is clear that $\psi$ is a homomorphism. Therefore, $\psi$ is an isomorphism and $\psi(d_{01}) = d_{\rho(0)\rho(1)} = d_{ij}$, as desired. $\square$

In this section, we showed the non-atomicity of $\mathfrak{Fr}_m Crs_n$ by reducing the case of $n \geq 3$ to the case of $n = 2$, which was treated in § 1.2. In the next section, we generalize the method used to show the non-atomicity for $n = 2$ to $n \geq 3$, and with this we get more information about the structure of the free algebras $\mathfrak{Fr}_m K_n$ for $K \in \{Crs, D, P, G\}$.

## §1.4 Almost no free algebra $\mathfrak{Fr}_m K_n$ is atomic

The method used in § 1.2 can be modified to obtain the same results but with $D_2$ in the place of $Crs_2$. Let $k \in \omega$ and let $\tau \in F_k^{2,m}$. We have to add extra elements to the structure $V^\tau$, namely to close the elements added in the last level $V_k$ under the substitutions $[0/1]$ and $[1/0]$. This guarantees that the resulting atom structure is indeed a concrete $D_2$-atom structure. Suppose that $D_2 \models \tau \leq c_0 - d_{01} + c_1 - d_{01}$. To prove $\tau$ is not an atom in $\mathfrak{Fr}_m D_2$, find the zigzag $h_0, \ldots, h_k$ as in the proof of proposition 1.2.2. Suppose that $h_k = (u_0, u_1) \in V_k^i$ for some $i \in 2$. We extend the structure $V^\tau$ by picking a brand new node $w$ and adding $(u_1, w), (w, w)$ if $i = 1$, or $(w, u_0), (w, w)$ if $i = 0$, to $W^\tau$. Then by choosing suitable labels for these new elements, one can mimic the proof of proposition 1.2.2 but to find two disjoint forms in $F_{k+2}^{2,m}$ (instead of $F_{k+1}^{2,m}$) each of which is satisfiable and below $\tau$ in the free algebra $\mathfrak{Fr}_m D_2$. Unfortunately, we cannot use the idea of the previous section to jump to the higher dimensional free $D$-algebras. Indeed, for $n \geq 3$, lemmas 1.3.1 and 1.3.2 are not true if we replace $Crs$ by $D$. Instead, we introduce a new idea that is, basically, developed from the idea used in § 1.2. Using this new idea we prove following.

**Theorem 1.4.1.** *Let $n \geq 2$ and $m \geq 0$ be finite ordinals and let $K \in \{Crs, D, P, G\}$. Then the free algebra $\mathfrak{Fr}_m K_n$ is not atomic if and only if $m \neq 0$, $n \neq 2$ or $K \neq G$.*

In fact we prove more than this, see lemma 1.4.2 and proposition 1.4.10 below. We start with the following lemma which proves one direction of the above theorem.

**Lemma 1.4.2.** *The free algebra $\mathfrak{Fr}_0 G_2$ is finite, hence atomic.*

*Proof.* Let $\lambda := d_{01} \cdot -c_0 - d_{01} \cdot -c_1 - d_{01}$, $\beta := d_{01} \cdot c_0 - d_{01} \cdot c_1 - d_{01}$ and $\delta := -d_{01}$. One can easily check that for any arbitrary satisfiable form $\tau \in F_1^{2,0}$ there exists $\sigma \in \{\lambda, \beta, \delta\}$ such that $G_2 \models \tau = \sigma$. Let $i \in 2$, the followings are straightforward.

$$G_2 \models \lambda \cdot c_i\beta = 0, \qquad G_2 \models \lambda \cdot c_i\delta = 0,$$

$$G_2 \models \beta \cdot c_i\lambda = 0, \qquad G_2 \models \beta \cdot c_i\delta = \beta,$$

$$G_2 \models \delta \cdot c_i\lambda = 0, \qquad G_2 \models \delta \cdot c_i\beta = \delta.$$

Hence, by theorem 1.1.2 (iii) and by the construction of the normal forms $F^{2,0}$, the free algebra $\mathfrak{Fr}_0 G_2$ is $+$-generated by the terms $\lambda, \beta$ and $\delta$. Therefore, $\mathfrak{Fr}_0 G_2$ is finite and, consequently, atomic. $\square$

For the other direction, we need to analyze the idea used in § 1.2 and to highlight what was really essential for the proofs there. To prove the non-atomicity of the free algebra $\mathfrak{Fr}_m Crs_2$, we constructed for every satisfiable $\tau \in F^{2,m}$ a concrete $Crs_2$-atom structure $S_\tau$ associated with labels for its elements that are consistent with the desired "degree consistency". We showed that each of the elements of $S_\tau$ satisfies its label under some natural evaluation.

**Definition 1.4.3.** Let $q \in \omega$ and let $S$ be any concrete $K_n$-atom structure associated with a degree function, $deg : S \to \{0, \ldots, q\}$, and a labeling function, $tag : S \to \bigcup\{F_i^{n,m} : i \in q\}$.

1. $(S, deg)$ is said to be a degree consistent concrete $(K_n, q)$-atom structure if:

   (a) For every $f \in S$, $tag(f) \in F_k^{n,m}$ where $k = deg(f)$.

   (b) For every $f, g \in S$ and every $i \in n$, if $f \equiv_i g$ then $\mid deg(f) - deg(g) \mid \leq 1$.

2. $(S, tag)$ is said to be a label consistent concrete $(K_n, q)$-atom structure if: For every element $f \in S$, $(S, f) \models tag(f)$. Where $(S, f) \models tag(f)$ means that $f$ satisfies the normal from $tag(f)$ in the algebra $\mathfrak{Cm}S$ under the evaluation $\iota$ which is defined as follows: For every $i \in m$, $\iota(x_i) = \{g \in S : x_i \in color^{n,m}(tag(g))\}$.

3. $(S, deg, tag)$ is said to be a consistent concrete $(K_n, q)$-atom structure if $(S, deg)$ is a degree consistent concrete $(K_n, q)$-atom structure and $(S, tag)$ is a label consistent concrete $(K_n, q)$-atom structure.

For the sake of simplicity, when we speak about the type of consistency of some concrete $K_n$-atom structure, we may use only $S$ instead of $(S, deg)$, $(S, tag)$ or $(S, deg, tag)$. But, in this case, we indicate the type of consistency we are talking about. Also, we may use the same name for the labeling functions (same for the degree functions) of two different concrete $K_n$-atom structures if these functions agree on the intersection of their domains. We hope these simplifications don't cause any confusion.

Then we found two elements $f, g \in S_\tau$ with $(S_\tau, f) \models \tau$ such that $f$ and $g$ are connected by some zigzag and the element $g$ has one (only one) degree of freedom in $S_\tau$, i.e., there exists (unique) $i \in 2$ such that no matter how we extend $S_\tau$ through an $i$-connection between $g$ and a new element $h$, the new extension still satisfies $\tau$ at $f$. Finally, we used the freedom of $g$ to get two disjoint satisfiable forms each of which is below $\tau$. We give formal definition for zigzags and then we define "free elements" that have full freedom ($n$-degrees of freedom).

**Definition 1.4.4.** Let $S$ be any concrete $K_n$-atom structure and let $k \in \omega$ be such that $k \geq 1$. For every $f, g \in S$, we say that $f$ and $g$ are connected by a zigzag of length $k$ if and only if there exist $\beta \in {}^k n$ and $h_0, h_1, \ldots, h_{k-1}, h_k \in S$ such that $f = h_0$, $g = h_k$ and, for every $j \in k+1$, $h_j \equiv_{\beta_j} h_{j+1}$. In this case we write $f \equiv_\beta g$ and we say that $\beta$ is a zigzag of finite length connecting $f$ and $g$.

**Definition 1.4.5.** Let $q \in \omega$ and let $(S, deg)$ be any degree consistent concrete $(K_n, q)$-atom structure. An element $f \in S$ is said to be

- a regular element if and only if all the elements in $\{f \circ \rho : \rho \in {}^n n\} \cap S$ have the same degree.
- a free element if it is regular of degree $0$.

Let $\delta := \prod_{i \in j \in n} -d_{ij}$ be the co-diagonal. Lemma 1.4.7 below shows that, for every $q \in \omega$ and every $\tau \in F_q^{n,m}$ with $K_n \models 0 \neq \tau \leq \delta$, the form $\tau$ can be associated with a consistent concrete $(K_n, q)$-atom structure that witnesses the satisfiability of $\tau$ at some element which is connected by a finite length zigzag to some free element. We need the following definition.

**Definition 1.4.6.** Let $S_1$ and $S_2$ be two concrete $K_n$-atom structures. We say that $S_1$ and $S_2$ are isomorphic if there is a bijection $\psi : S_1 \to S_2$ such that the bijection given by, for every $A \subseteq S_1$, $A \mapsto \{\psi(e) : e \in A\}$ is an isomorphism between $\mathfrak{Cm}S_1$ and $\mathfrak{Cm}S_2$.

Let $S$ be a concrete $K_n$-atom structure with base $U$. Let $\psi : U \to U^*$ be any bijection for some set $U^*$. Define $\psi(S) = \{(\psi(a_0), \ldots, \psi(a_{n-1})) : (a_0, \ldots, a_{n-1}) \in S\}$. Thus, $\psi(S)$ is a concrete $K_n$-atom structure, and, $S$ and $\psi(S)$ are isomorphic via the bijection $\widehat{\psi} : S \to \psi(S)$ given as follows: $\widehat{\psi}(a_0, \ldots, a_{n-1}) = (\psi(a_0), \ldots, \psi(a_{n-1}))$, for every $(a_0, \ldots, a_{n-1}) \in S$. $\widehat{\psi}$ is called the point wise extension of $\psi$.

**Lemma 1.4.7.** *Let $q \in \omega$ and let $\tau \in F_q^{n,m}$ be a normal form such that $K_n \models 0 \neq \tau \leq \delta$. Then there exist a consistent concrete $(K_n, q)$-atom structure $(S, deg, tag)$, $f, g \in S$ and a finite length zigzag $\beta$ such that $tag(f) = \tau$, $f \equiv_\beta g$ and $g$ is a free element in $S$.*

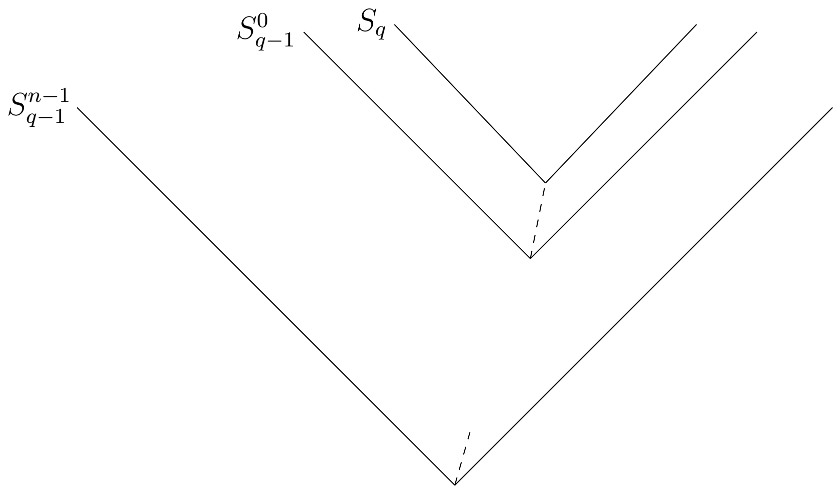
*Proof.* Let $q \in \omega$ and let $\tau \in F_q^{n,m}$ be a satisfiable normal form. Then there is a concrete $K_n$-atom structure $S_q$ and an evaluation of the free variables $\iota$ such that $(S_q, f_q, \iota) \models \tau$, for some $f_q \in S_q$. For every element $g \in S_q$, let $deg(g) = q$ and let $tag(g)$ be the unique term in $F_q^{n,m}$ such that $(S_q, g, \iota) \models tag(g)$. Therefore, $S_q$ is a consistent concrete $(K_n, q)$-atom structure and there exists a regular element, $f_q$, with degree $q$ and label $\tau$. Suppose that $U$ is the base of $S_q$.

Suppose that $f_q = (r_0, \ldots, r_{n-1})$. Let $U_0, \ldots, U_{n-1}$ be mutually disjoint sets such that each of which is disjoint from $U$ and has the same size of $U \setminus Rng(f_q)$. Pick brand new $n$-many different nodes $s_0, \ldots, s_{n-1}$. Let $j \in n$ be arbitrary. Let

$$\psi_{q-1}^j : U \to U_j \cup \{s_0, \ldots, s_j\} \cup \{r_{j+1}, \ldots, r_{n-1}\}$$

be any bijection such that, for every $i \in n$, $\psi_{q-1}^j(r_i) = s_i$ if $i \leq j$ and $\psi_{q-1}^j(r_i) = r_i$ if $i > j$. Set $S_{q-1}^j = \{(\psi_{q-1}^j(u_0), \ldots, \psi_{q-1}^j(u_{n-1})) : (u_0, \ldots, u_{n-1}) \in S_q\}$.

Clearly, $S_{q-1}^j$ is a concrete $K_n$-atom structure that is isomorphic to $S_q$ via the component wise extension $\widehat{\psi_{q-1}^j}$ of the bijection $\psi_{q-1}^j$. Extend the functions $deg$ and $tag$ as follows. Let $g \in S_{q-1}^j$, suppose that $h \in S_q$ is the inverse image of $g$ under the isomorphism $\widehat{\psi_{q-1}^j}$. If $g \in S_q$, keep its label and degree, i.e., $deg(g) = deg(h) = q$ and $tag(g) = tag(h)$. If $g \notin S_q$ then let $deg(g) = q - 1$ and let $tag(g)$ be the unique term in $F^{n,m}_{q-1}$ such that $(S_q, h, \iota) \models tag(g)$.




Finally, let $S_{q-1} = S_q \bigcup \{S^j_{q-1} : j \in n\}$. Clearly, $S_{q-1}$ is a degree consistent concrete $(K_n, q)$-atom structure. Indeed, every element of $S_{q-1}$ has degree either $q$ or $q - 1$. Moreover, the element $f_{q-1} := \widehat{\psi^{n-1}_{q-1}}(f_q) = (s_0, \ldots, s_{n-1}) \in S^{n-1}_{q-1}$ is a regular element of degree $q - 1$ that is connected to $f_q$ via the zigzag $(0, \ldots, n - 1)$.

Let $U_{q-1}$ be the base of $S^{n-1}_{q-1}$ and let $V$ be the base of $S_{q-1}$. We extend $S_{q-1}$ to $S_{q-2}$ by adding $n$-many isomorphic copies of $S^{n-1}_{q-1}$ (copies of $S_q$) in a similar way. Let $V_0, \ldots, V_{n-1}$ be mutually disjoint sets such that each of which is disjoint from $V$ and has the same size of $U_{q-1} \setminus Rng(f_{q-1})$. Pick brand new $n$-many different nodes $z_0, \ldots, z_{n-1}$. Let $j \in n$ be arbitrary. Let

$$\psi^j_{q-2} : U_{q-1} \to V_j \cup \{z_0, \ldots, z_j\} \cup \{s_{j+1}, \ldots, s_{n-1}\}$$

be any bijection such that, for every $i \in n$, $\psi^j_{q-1}(s_i) = z_i$ if $i \leq j$ and $\psi^j_{q-1}(s_i) = s_i$ if $i > j$. Set $S^j_{q-2} = \{(\psi^j_{q-2}(u_0), \ldots, \psi^j_{q-2}(u_{n-1})) : (u_0, \ldots, u_{n-1}) \in S^{n-1}_{q-1}\}$. Again, $S^j_{q-2}$ is a concrete $K_n$-atom structure that isomorphic to $S^{n-1}_{q-1}$ via the component wise extension $\widehat{\psi^j_{q-2}}$ of the bijection $\psi^j_{q-2}$ and, of course, isomorphic to $S_q$ via the composition $\varphi^j_{q-2} := \widehat{\psi^j_{q-2}} \circ \widehat{\psi^{n-1}_{q-1}}$. Extend $deg$ and $tag$ as follows. Let $g \in S^j_{q-2}$, suppose that $h \in S_q$ is the inverse image of $g$ under the isomorphism $\varphi^j_{q-2}$. If $g \in S^{n-1}_{q-1}$, keep its label and degree. If $g \notin S^{n-1}_{q-1}$ then let $deg(g) = q - 2$ and let $tag(g)$ be the unique term in $F^{n,m}_{q-2}$ such that $(S_q, h, \iota) \models tag(g)$. The structure $S_{q-2} = S_{q-1} \bigcup \{S^j_{q-2} : j \in n\}$ is a degree consistent concrete $(K_n, q)$-atom structure, because the bases of $S_q$ and the base of $S_{q-1}^{n-1}$ are disjoint, so there are no neighboring elements of degree $q$ and $q-2$. The element $f_{q-2} := \widehat{\psi_{q-2}^{n-1}}(f_{q-1}) = (z_0, \ldots, z_{n-1}) \in S_{q-2}^{n-1}$ is a regular element of degree $q-2$ that is connected to $f_q$ via the zigzag $(0, \ldots, n-1, 0, \ldots, n-1)$.

Inductively, we get a sequence of concrete $K_n$-atom structures $S_q \subseteq \ldots \subseteq S_0$ with a sequence of elements $(f_q, \ldots, f_0)$ such that: For every $k \in q+1$,

(a) $S_k$ is a degree consistent concrete $(K_n, q)$-atom structure and $tag(f_q) = \tau$.

(b) $f_k \in S_k$ is a regular element of degree $k$ and there is a zigzag of finite length that connects both $f_q$ and $f_k$.

(c) For every $i, j \in n$ and every $l \in q+1$, if $base(S_k^i) \cap base(S_l^j) \neq \emptyset$ then there exists a bijection $\psi : base(S_k^i) \to base(S_l^j)$ that fixes the elements in this intersection such that the point-wise extension of $\psi$, $\hat{\psi} : S_k^i \to S_l^j$ is an isomorphism and, for every $g \in S_k^i$, $(S_l^j, \hat{\psi}(g)) \models tag(g)$.

It remains to prove that $S_0$ is the desired atom structure, i.e., it remains to prove that it is label consistent with the labeling $tag$. Toward this end, we introduce the $tag_l$'s of the nodes in $S_0$ as follows. For every $l \in q+1$ and every $g \in S_0$, define $tag_l(g) = tag(g)$ if $l \geq deg(g)$ and define $tag_l(g)$ to be the unique form in $F_l^{n,m}$ such that $K_n \models tag(g) \leq tag_l(g)$ if $l \leq deg(g)$. To prove the label consistency, it is enough to prove the following. For every $l \in q+1$ and every $g \in S_0$, $(S_0, g) \models tag_l(g)$. We use induction on $l$. For $l = 0$, it is clear that $(S_0, g) \models tag_0(g)$, for every $g \in S_0$. Suppose that for some $l \in q$ and for every $g \in S_0$, $(S_0, g) \models tag_l(g)$.

Let $g \in S_0$. If $deg(g) \leq l$, then $(S_0, g) \models tag_l(g) = tag_{l+1}(g)$. So suppose that $deg(g) > l$. Let $i \in n$ and let $\sigma \in F_l^{n,m}$. We have to show that

$$\sigma \in sub_i^{n,m}(tag_{l+1}(g)) \iff \exists h \in S_0 \;\; (g \equiv_i h \text{ and } (S_0, h) \models \sigma).$$

Suppose that $deg(g) = k$, that means that $g$ appeared for the first time in $S_k^j$, for some $j \in n$. Suppose that $\sigma \in sub_i^{n,m}(tag_{l+1}(g))$. Recall that $S_k^j$ is isomorphic to $S_q$ (the original atom structure satisfying $\tau$), then there exists a node $h \in S_k^j$ such that $h \equiv_i g$ and $tag_l(h) = \sigma$. Then we are done by the induction hypothesis. Conversely, suppose that $\exists h \in S_0$ such that $g \equiv_i h$ and $(S_0, h) \models \sigma$. Suppose that $h$ first appears in some $S_p^z$ for some $p \in q+1$ and some $z \in n$. Then $Rng(g) \setminus \{g_i\} = Rng(h) \setminus \{h_i\} \subseteq base(S^j_{k-1}) \cap base(S^z_p)$. Therefore, by (c) above, there exists a bijection $\psi : base(S^z_p) \to base(S^j_{k-1})$ that fixes the elements of the intersection $base(S^j_{k-1}) \cap base(S^z_p)$ and the point-wise extension $\hat{\psi} : S^z_p \to S^j_{k-1}$ is an isomorphism with $(S^j_{k-1}, \hat{\psi}(h)) \models tag(h)$. Then $\bar{h} \equiv_i h \equiv_i g$ and $tag_l(\bar{h}) = tag_l(h) = \sigma$, where $\bar{h} := \hat{\psi}(h)$. But $tag_l(\bar{h}) \in sub^{n,m}_i(tag_{l+1}(g))$, otherwise $(S^j_{k-1}, g) \not\models tag(g)$. Therefore, $\sigma \in sub^{n,m}_i(tag_{l+1}(g))$, as desired. $\square$

Thus, we can use the above lemma to prove that there are no atoms below $\delta$ in the non-atomic free algebras. From now on throughout this section, suppose that $m \geq 1$, $n \geq 3$ or $K \in \{Crs, D, P\}$. Before proving that the part below the co-diagonal in the free algebra $\mathfrak{Fr}_m K_n$ contains no atoms, we prove that it contains infinitely many different elements.

**Proposition 1.4.8.** *Let $n \geq 2$ and $m \geq 0$ be finite ordinals and let $K \in \{Crs, D, P, G\}$. Suppose that $n \neq 2$, $m \neq 0$ or $K \neq G$, then there are infinitely many (mutually disjoint) non zero elements below the co-diagonal $\delta$ in the free algebra $\mathfrak{Fr}_m K_n$.*

*Proof.* We divide the proof into four cases as follows.

**Case 1:** Suppose that $m \geq 1$. Then one can find two normal forms $\delta_0, \delta_1 \in F^{n,m}_0$ such that $K_n \models 0 \neq \delta_0 \leq \delta$, $K_n \models 0 \neq \delta_1 \leq \delta$ and $K_n \models \delta_0 \cdot \delta_1 = 0$. Inductively, we define infinitely many distinct terms $\sigma_0, \sigma_1, \ldots$ as follows. Define $\sigma_0 = \delta_0 \cdot -c_0\delta_1$. Let $i \in \omega$ be such that $i \geq 1$. If $i$ is even, define $\sigma_i = \delta_0 \cdot c_0\sigma_{i-1}$. If $i$ is odd, define $\sigma_i = \delta_1 \cdot c_1\sigma_{i-1}$. Then, we define infinitely many disjoint elements below $\delta$ as follows. Define $\tau_0 = \sigma_0$ and, for every finite $k \geq 1$, define $\tau_k = \prod\{-\sigma_i : i \in k\} \cdot \sigma_k$. Hence, for every $i, j \in \omega$, $\mathfrak{Fr}_m K_n \models \tau_i \cdot \tau_j = 0$ if $i \neq j$. Therefore, it remains to prove that $\mathfrak{Fr}_m K_n \not\models \tau_k = 0$, for every $k \in \omega$.

Let $u_0, u_1, \ldots$ be infinitely many distinct positive numbers and let $v_0, \ldots, v_{n-3}$ be a string of distinct negative numbers of length $n-2$ (note that this string is empty if $n = 2$). For every $i \in \omega$, set $V_i = {}^nRng(f_i)$ where $f_i := (u_i, u_{i+1}, v_0, \ldots, v_{n-3})$ if $i$ is even and $f_i := (u_{i+1}, u_i, v_0, \ldots, v_{n-3})$ if $i$ is odd. Let $k \in \omega$ be arbitrary but fixed. Let $S_k := V_0 \cup \cdots \cup V_k$ and define the evaluation $\iota_k$ as follows. For every $i \in m$, let

$$\iota_k(x_i) = \{f_j : j \text{ is odd and } x_i \in color^{n,m}(\delta_1), \text{ or, } j \text{ is even and } x_i \in color^{n,m}(\delta_0)\}.$$

It is easy to check that, $S_k$ is concrete $K_n$-atom structure and $(S_k, f_k, \iota_k) \models \tau_k$ (See page 39). Therefore, $\tau_k$ is a non-zero element in the free algebra $\mathfrak{Fr}_m K_n$.

**Case 2:** Suppose that $K \in \{Crs, D\}$. Let $\lambda := d_{01} \cdot -c_1\delta$, $\sigma_0 = \delta \cdot c_0\lambda$ and $\lambda_0 = d_{01} \cdot c_1\sigma_0$. For every finite $i \geq 1$, define $\sigma_i = \delta \cdot c_0\lambda_{i-1}$ and $\lambda_i = d_{01} \cdot c_1\sigma_i$. The terms $\sigma_i$'s are distinct in $\mathfrak{Fr}_m K_n$, we give the disjoint terms as follows. Let $\tau_0 = \sigma_0$ and, inductively, for every finite $k \geq 1$, let $\tau_k = \prod\{-\sigma_i : i \in k\} \cdot \sigma_k$. Clearly, the terms $\tau_0, \tau_1, \ldots$ are disjoint in the free algebra $\mathfrak{Fr}_m K_n$. It remains to show that, for every $k \in \omega$, $\mathfrak{Fr}_m K_n \not\models \tau_k = 0$.

Let $u_0, u_1, \ldots$ be infinitely many distinct positive numbers and let $v_0, \ldots, v_{n-3}$ be a string of distinct negative numbers of length $n-2$ (empty string if $n = 2$). For every $i \in \omega$, let $f_i := (u_i, u_{i+1}, v_0, \ldots, v_{n-3})$, let $g_i := (u_i, u_i, v_0, \ldots, v_{n-3})$ and set

$$V_i = \{f_i \circ \rho : \rho \in {}^n n \text{ is not a bijection}\}.$$

Let $k \in \omega$ be arbitrary but fixed. Let $S_k = V_0 \cup \cdots V_k$ and let $\iota_k$ be any evaluation of the free variables (if any). It is not hard to check that, $S_k$ is concrete $K_n$-atom structure and $(S_k, f_k, \iota_k) \models \tau_k$ (See page 39). Therefore, $\mathfrak{Fr}_m K_n \not\models \tau_k = 0$, as desired.

**Case 3:** Suppose that $K \in \{Crs, P\}$. Let $\chi_0 := \delta \cdot -c_0 d_{01} \cdot -c_1 d_{01}$, $\chi_1 := \delta \cdot c_0 d_{01} \cdot -c_1 d_{01}$, $\chi_2 := \delta \cdot c_0 d_{01} \cdot c_1 d_{01}$ and $\chi_3 := \delta \cdot -c_0 d_{01} \cdot c_1 d_{01}$. Let $\sigma_0 := \chi_0 \cdot -c_0(\delta \cdot c_1 d_{01})$ and, for every finite $k \geq 1$, define $\sigma_k := \chi_i \cdot c_j \sigma_{k-1}$, where $i = k \ (mod\, 4)$ and $j = k \ (mod\, 2)$. The infinitely many disjoint terms are given as follows. Let $\tau_0 = \sigma_0$ and, inductively, for every finite $k \geq 1$, let $\tau_k = \prod\{-\sigma_i : i \in k\} \cdot \sigma_k$. We need to show that, for every $k \in \omega$, $\mathfrak{Fr}_m K_n \not\models \tau_k = 0$.

Let $u_0, u_1, \ldots$ be infinitely many distinct positive numbers and let $v_0, \ldots, v_{n-3}$ be a string of distinct negative numbers of length $n-2$ (empty if $n = 2$). Let $i \in \omega$. If $i = 0 \ (mod\, 2)$, define $f_i := (u_i, u_{i+1}, v_0, \ldots, v_{n-3})$. If $i = 1 \ (mod\, 2)$, define $f_i := (u_i, u_{i-1}, v_0, \ldots, v_{n-3})$. If $i = 2, 3 \ (mod\, 4)$, define $g_i := (u_i, u_i, v_0, \ldots, v_{n-3})$. If $i = 0, 1 \ (mod\, 4)$, define $g_i = \emptyset$. Set $V_i = \{g_i \circ \rho, g_{i+1} \circ \rho, f_i \circ \rho : \rho \in {}^n n \text{ is a bijection}\}$. Let $k \in \omega$ be arbitrary but fixed. Let $S_k = V_0 \cup \cdots V_k$ and let $\iota_k$ be any evaluation of the free variables (if any). It is not hard to check that, $S_k$ is concrete $K_n$-atom structure and

$(S_k, f_k, \iota_k) \models \tau_k$ (See page 39). Therefore, $\mathfrak{Fr}_m K_n \not\models \tau_k = 0$.

**Case 4:** Suppose that $n \geq 3$. Let $\lambda_0 := d_{02} \cdot -c_2\delta$, $\beta_0 := d_{02} \cdot c_1\lambda_0$ and $\sigma_0 := \delta \cdot c_0\beta_0$. For every finite $i \geq 1$, define $\lambda_i = d_{02} \cdot c_2\sigma_{i-1}$, $\beta_i = d_{02} \cdot c_1\lambda_i$ and $\sigma_i = \delta \cdot c_0\beta_i$. We purpose the following infinitely many elements below $\delta$. Let $\tau_0 = \sigma_0$ and, inductively, for every finite $k \geq 1$, let $\tau_k = \prod\{-\sigma_i : i \in k\} \cdot \sigma_k$. We need to show that, for every $k \in \omega$, $\mathfrak{Fr}_m K_n \not\models \tau_k = 0$.

Let $u_0, v_0, u_1, v_1, \ldots$ be infinitely many distinct positive numbers and let $w_0, \ldots, w_{n-1}$ be $n$ many distinct negative numbers. Let $f_0 := (w_0, \ldots, w_{n-1})$, $g_0 = f_0 \circ [0/2]$ and $h_0 = [g_0]^1_{u_0}$. For every finite $i \geq 1$, let $f_i = [h_0]^2_{v_{i-1}}$, $g_i = f_i \circ [0/2]$ and $h_i = [g_i]^1_{u_i}$. Let $k \in \omega$ be arbitrary but fixed. Set $S_k = {}^nRng(f_0) \cup \cdots \cup {}^nRng(f_k) \cup {}^nRng(h_k)$. One can easily see that $S_k$ is concrete $K_n$-atom structure and for any evaluation of the free variables (if any) $\iota_k$, $(S_k, f_0, \iota_k) \models \tau_k$ (See page 39). Therefore, $\mathfrak{Fr}_m K_n \not\models \tau_k = 0$.

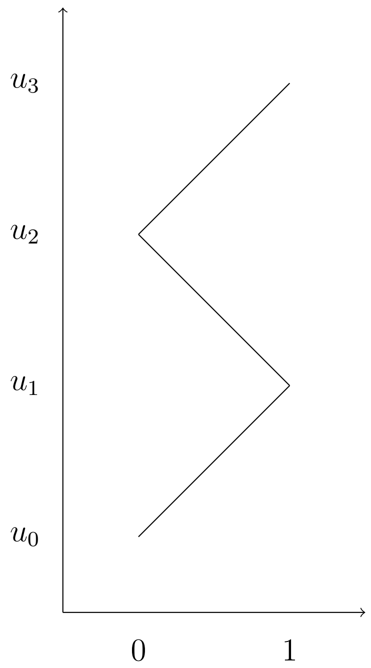
$\square$

**Definition 1.4.9.** For every finite $k \geq 1$ and every $\beta \in {}^kn$, define a new operator $C_\beta$ in the language $cyl_n$ such that $C_\beta := c_{\beta_0}c_{\beta_1}\cdots c_{\beta_{k-1}}$.

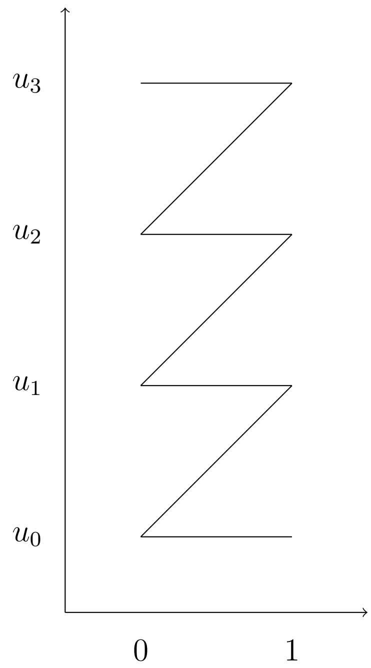
We are ready now to prove that the free algebra $\mathfrak{Fr}_m K_n$ is not atomic. The idea goes as follows. For any normal form $K_n \models 0 \neq \tau \leq \delta$, we find a different term and a sequence of finite length $\beta$ such that $K_n \models 0 \neq \sigma \leq \delta$ and $K_n \not\models \tau \cdot -c_\beta\sigma = 0$. Then we use lemma 1.4.7 to build a structure $\mathfrak{A} \in K_n$ in which $\mathfrak{A} \not\models \tau \cdot c_\beta\sigma = 0$. We use the finite base property for the class $K_n$ that was proved in [AHN99].

**Proposition 1.4.10.** *Let $n \geq 2$, $m \geq 0$ be finite ordinals and let $K \in \{Crs, D, P, G\}$ be such that $n \neq 2$, $m \neq 0$ or $K \neq G$. There is no atom below the co-diagonal $\delta$ in the free algebra $\mathfrak{Fr}_m K_n$.*

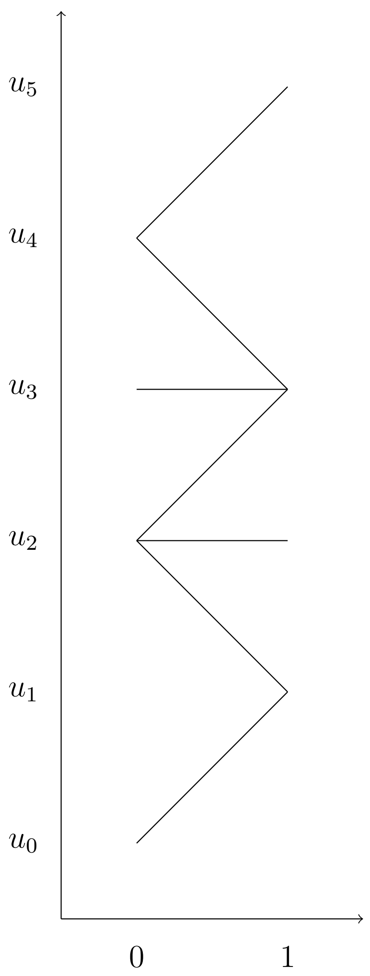
*Proof.* Let $q \in \omega$ and let $\tau \in F^{n,m}_q$ be a satisfiable normal form such that $K_n \models 0 \neq \tau \leq \delta$. By [AHN99], there exists a finite concrete $K_n$-atom structure $S$ and an evaluation $\iota$ of the free variables such that $(S, f, \iota) \models \tau$ for some $f \in S$. Since $S$ is finite, by theorem 1.1.2 (iii) and proposition 1.4.8, there exists a satisfiable normal form $\sigma$ such that $K_n \models 0 \neq \sigma \leq \delta$ and



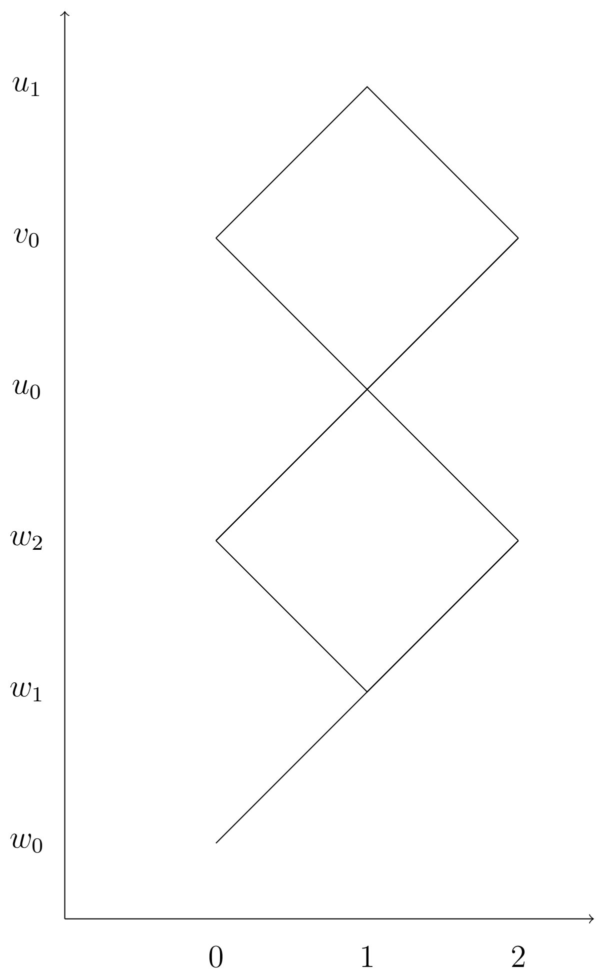

Case 1: $(S_3, f_3, \iota_3) \models \tau_3$




Case 2: $(S_3, f_3, \iota_3) \models \tau_3$




Case 3: $(S_5, f_5, \iota_5) \models \tau_5$



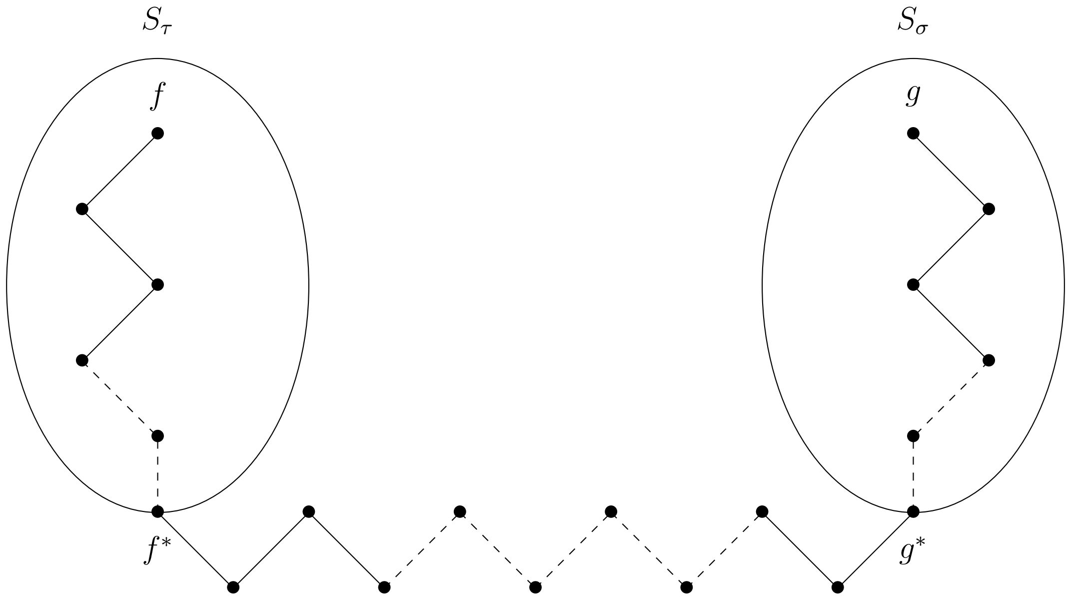

Case 4: $(S_2, f_0, \iota_2) \models \tau_2$

$(S, g, \iota) \not\models \sigma$ for every $g \in S$. Therefore, for every finite length sequence $\alpha$ of elements from $n$, we have $(S, f, \iota) \models \tau. - C_\alpha\sigma$, i.e.,

$$K_n \not\models \tau. - C_\alpha\sigma = 0. \tag{1.13}$$

Suppose that $\sigma \in F^{n,m}_{q^*}$ for some $q^* \in \omega$. By lemma 1.4.7, there are a consistent concrete $(K_n, q)$-atom structure $S_\tau$ and a consistent concrete $(K_n, q^*)$-atom structure $S_\sigma$ (without loss of generality we can assume that they are disjoint) such that there exist $f, f^* \in S_\tau$ and $g, g^* \in S_\sigma$ satisfy the followings.

- $(S_\tau, f) \models \tau$ and $(S_\sigma, g) \models \sigma$.
- $f^*$ is free element and it is connected to $f$ by a finite zigzag in $S_\tau$.
- $g^*$ is free element and it is connected to $g$ by a finite zigzag in $S_\sigma$.




Suppose that $f^* = (f_0, \dots, f_{n-1})$ and $g^* = (g_0, \dots, g_{n-1})$. Let $h_0 = [f^*]^{g_0}_0$ and, for every $i \in \{1, \dots, n-1\}$, let $h_i = [h_{i-1}]^{g_i}_i$. Note that $h_{n-1} = g^*$. Define

$$S_{bridge} = \bigcup\{{}^nRng(h_i) : i \in n-1\}.$$

Let $S' = S_\tau \cup S_{bridge} \cup S_\sigma$. Since both $S_\tau$ and $S_\sigma$ are concrete $K_n$-atom structures, then $S$ is concrete $K_n$-atom structure as well. Define the evaluation $\iota$ as follows. For every $i \in m$, let

$\iota(x_i) = \{h \in S_\tau \cup S_\sigma : x_i \in color^{n,m}(tag(h))\}$. Then (By a very similar argument to the one used in lemma 1.4.7. The nodes that don't have degrees and labels will not disturb the prove, indeed they are connected to nodes of degree 0 only.) we have $(S', f, \iota) \models \tau$ and $(S', g, \iota) \models \sigma$. Moreover, by the construction there is a finite length sequence $\beta$ of elements from $n$ such that $f \equiv_\beta g$. That means, $(S', f, \iota) \models \tau \cdot C_\beta \sigma$. Hence,

$$K_n \not\models \tau \cdot C_\beta \sigma = 0. \tag{1.14}$$

By equations 1.13 and 1.14, $\tau$ is not an atom in $\mathfrak{Fr}_m K_n$. Recall that $\tau$ was arbitrary satisfiable normal form below the co-diagonal $\delta$, therefore there is no atom below $\delta$ in the free algebra $\mathfrak{Fr}_m K_n$. □

In the next section, we give structural descriptions of these free algebras from the point of view of atoms.

## §1.5 On the atoms in the free algebra $\mathfrak{Fr}_m K_n$

In the theory of cylindric algebras, it is well known that there is a connection between the zero-dimensional elements and the atoms in the free cylindric algebras. Indeed, many zero-dimensional elements in the free cylindric algebras are in fact atoms (c.f. [HMT71, 1.10.3(i)]). Here, we try to investigate the analogous connection for the class $K_n$. Let $\mathfrak{A} \in K_n$, an element $a \in \mathfrak{A}$ is said to be zero-dimensional element if $\Delta(a) := \{i \in n : \mathfrak{A} \models c_i a \neq a\} = \emptyset$, i.e., if it is closed under all cylindrifications.

**Proposition 1.5.1.** *Let $n \geq 2$ be finite and let $K \in \{Crs, D, P, G\}$. Suppose that $\mathfrak{A} \in K_n$ is generated by $Y$. Let $a \in \mathfrak{A}$ be a non-zero zero-dimensional element such that either $a \leq d_{ij}$ or $a \leq -d_{ij}$ and either $a \leq y$ or $a \leq -y$ for all $i, j \in n$ and all $y \in Y$. Then $a$ is an atom in $\mathfrak{A}$.*

*Proof.* Suppose that $\mathfrak{A}$, $a \in \mathfrak{A}$ are as required above. To prove that $a$ is an atom in $\mathfrak{A}$, it is enough to prove the following. For every $b \in \mathfrak{A}$, we have

$$a \leq b \quad \text{or} \quad a \leq -b. \tag{1.15}$$

We prove (1.15) by induction. If $b \in \{d_{ij} : i, j \in n\} \cup Y$, then (1.15) holds by the assumptions. Suppose that (1.15) holds for some $b_1, b_2 \in \mathfrak{A}$. Clearly, the induction step goes smoothly for the Boolean operations, that is (1.15) holds for $b_1 \cdot b_2, b_1 + b_2$ and $-b_1$. Let $k \in n$, we need only to check that (1.15) holds for $c_k b_1$. The operator $c_k$ is complemented operator, that is $c_k - c_k a = -c_k a$. But $a$ is zero-dimensional, i.e., $c_k a = a$. Thus, $c_k - a = -a$. Suppose that $a \leq b_1$, then $a = c_k a \leq c_k b_1$ because $c_k$ is an additive operator. Suppose that $a \leq -b_1$, or equivalently, $b_1 \leq -a$. Therefore, $c_k b_1 \leq c_k - a = -a$ and $a \leq -c_k b_1$, as desired. $\square$

Now, we give some zero-dimensional elements in $\mathfrak{Fr}_m K_n$ that are also atoms. This shows that the free algebra $\mathfrak{Fr}_m K_n$ is not atomless. An $\alpha \in {}^{D_{n,m}}\{-1, 1\}$ is said to be equalizer if $\alpha(d_{ij}) = 1$ for every $i, j \in n$. Let $\alpha \in {}^{D_{n,m}}\{-1, 1\}$ be an equalizer, define

$$a_\alpha := \prod\{-c_k - D^\alpha_{n,m} : k \in n\}.$$

**Proposition 1.5.2.** *Let $n \geq 2$ and $m \geq 0$ be finite and let $K \in \{Crs, D, P, G\}$. Then, for any equalizer $\alpha \in {}^{D_{n,m}}\{-1, 1\}$, the term $a_\alpha$ defined above is an atom in the free algebra $\mathfrak{Fr}_m K_n$.*

*Proof.* Let $\alpha \in {}^{D_{n,m}}\{-1, 1\}$ be an equalizer and let $a_\alpha$ be as defined above. It is not hard to see that $\mathfrak{Fr}_m K_n \models a_\alpha \neq 0$. Let $i, j \in n$ and let $k \in \{i, j\}$. Since $\alpha$ is an equalizer, then we have $\mathfrak{Fr}_m K_n \models D^\alpha_{n,m} \leq d_{ij}$. Consequently, $\mathfrak{Fr}_m K_n \models a_\alpha \leq -c_k - D_{n,m} \leq -c_k - d_{ij} \leq d_{ij}$. That means, $\mathfrak{Fr}_m K_n \models a_\alpha \cdot d_{ij} = a_\alpha$. It is well known that $K_n \models (\forall x)\ c_k(x \cdot d_{ij}) \cdot d_{ij} = x \cdot d_{ij}$ (it is one of the axioms proposed by Andréka for the class $Crs_n$ that contains $K_n$ by definition, moreover, one can easily check that it is true by the same argument used in proposition 1.5.3 below). Thus, $\mathfrak{Fr}_m K_n \models c_k a_\alpha = c_k a_\alpha \cdot d_{ij} + c_k a_\alpha \cdot -d_{ij} = a_\alpha + 0 = a_\alpha$. Therefore, $a_\alpha$ is an atom in $\mathfrak{Fr}_m K_n$, by proposition 1.5.1, as desired. $\square$

The following proposition is new in the theory of relativized cylindric algebras. We know that the algebras in the class $K_n$ don't need to obey the commutativity axiom of the cylindrifications, however, the following proposition shows that they obey a weaker version of this axiom.

**Proposition 1.5.3.** *Let $n \geq 2$ be finite and let $K \in \{Crs, D, P, G\}$. Let $i, j \in n$, then $K_n \models c_i c_j(-d_{ij} + x) = c_j c_i(-d_{ij} + x)$.*

*Proof.* Let $\mathfrak{A} \in K_n$. Without loss of generality, we can assume that $\mathfrak{A}$ is a concrete algebra, i.e., a subalgebra of a complex algebra over a concrete $K_n$-atom structure. Let $x \in A$ and let $f = (f_0, \dots, f_{n-1}) \in c_i c_j(-d_{ij} + x)$. If $f(i) \neq f(j)$, then $f \in c_j c_i(-d_{ij} + x)$. Assume that $f(i) = f(j)$, then there exists $u$ such that $f_u^i \in c_j(-d_{ij}+x)$. If $u = f(i)$, then $f \in c_j(-d_{ij}+x)$ and we are done. If $u \neq f(i)$, then $f_i^u \in -d_{ij}$, hence $f \in c_i(-d_{ij}+x)$, i.e., $f \in c_j c_i(-d_{ij}+x)$ as desired. $\square$

Let $d := \prod\{-c_k - d_{ij} : i \in j \in n, k \in n\}$. By proposition 1.5.3, for every $\tau \in \mathfrak{Tm}_{m,cyl_n}$, one can see that both $d \cdot \tau$ and $-(d \cdot \tau)$ are zero-dimensional elements in $\mathfrak{Fr}_m K_n$. Therefore, both $\mathfrak{Rl}_d \mathfrak{Fr}_m K_n$ and $\mathfrak{Rl}_{-d} \mathfrak{Fr}_m K_n$ are homomorphic images of $\mathfrak{Fr}_m K_n$ and

$$\mathfrak{Fr}_m K_n \cong \mathfrak{Rl}_d \mathfrak{Fr}_m K_n \oplus \mathfrak{Rl}_{-d} \mathfrak{Fr}_m K_n.$$

This isomorphism can be easily checked via $\psi : (\forall \tau \in \mathfrak{Tm}_{m,cyl_n})\tau \mapsto (\tau \cdot d) + (\tau \cdot -d)$. The algebra $\mathfrak{Rl}_d \mathfrak{Fr}_m K_n$ is finite, by proposition 1.5.2, and $+$-generated by the terms $a_\alpha$'s, for the equalizers $\alpha$'s, because $d = \sum\{a_\alpha : \alpha \in {}^{\{-1,1\}}D_m$ is an equalizer$\}$. By proposition 1.2.2, $\mathfrak{Fr}_m Crs_2$ is decomposed into a finite algebra $\mathfrak{Rl}_d \mathfrak{Fr}_m Crs_2$ and an atomless algebra $\mathfrak{Rl}_{-d} \mathfrak{Fr}_m Crs_2$. We show that the same also holds for non atomic $\mathfrak{Fr}_m K_n$ when $K \in \{D, G\}$ ($\mathfrak{Fr}_0 G_2$ is excluded).

**Proposition 1.5.4.** *Let $n \geq 2$ and $m \geq 0$ be finite and let $K \in \{D, G\}$. Suppose that $n \neq 2$, $n \neq 0$ or $K \neq G$, then there is no atom below $-d$ in the free algebra $\mathfrak{Fr}_m K_n$. Hence, $\mathfrak{Fr}_m K_n$ contains only finitely many atoms.*

*Proof.* Suppose that $K \in \{D, G\}$ and the free algebra $\mathfrak{Fr}_m K_n$ is not atomic. Let $q \in \omega$ be such that $q \geq 1$ and let $\tau \in F_q^{n,m}$ be a satisfiable normal form such that $K_n \models 0 \neq \tau \leq -d$. We prove that $\tau$ is not an atom in $\mathfrak{Fr}_m K_n$. Let $S$ be a finite concrete $K_n$-atom structure and let $\iota$ be an evaluation such that $(S, f, \iota) \models \tau$ for some $f \in S$. Since $S$ is finite, then (by proposition 1.4.8) there exists a satisfiable normal form $\sigma$ such that $K_n \models 0 \neq \sigma \leq \delta$ and, for every finite length sequence $\alpha$ of elements from $n$, we have $(S, g, \iota) \not\models \tau \cdot C_\alpha \sigma$ for every $g \in S$, i.e.,

$$K_n \not\models \tau. - C_\alpha \sigma = 0. \tag{1.16}$$

We define another concrete $K_n$-atom structure, in the same spirit of lemma 1.4.7, as follows. Define $S_q := S$ and for every $h \in S_q$, define $deg(h) = q$ and let $tag(h)$ be the unique normal form of degree $q$ such that $(S, h, \iota) \models tag(h)$. Clearly $S_q$ is a consistent concrete $(K_n, q)$-atom structure. Without loss of generality, we may assume that $K \models \tau \leq c_0 - d_{01}$. Therefore, there exists $g \in S_q$ such that $g \equiv_0 f$ and $g_0 \neq g_1$. Since $K \in \{D, G\}$, then both $(g_0, \ldots, g_0)$ and $(g_1, \ldots, g_1)$ are in $S_q$. We may assume that $f \neq (g_0, \ldots, g_0)$. Then $g_q := (g_0, \ldots, g_0)$ is a regular elements of degree $q$ that is connected to $f$ by a zigzag of finite length.

Let $U_q$ be the unit of $S_q$ and let $V$ be any set such that $V \cap U_q = \emptyset$ and $\mid V \mid = \mid U_q \setminus \{g_0\} \mid$. Let $\psi_{q-1}$ be a bijection between $U_q$ and $V \cup \{g_0\}$ such that $\psi_{q-1}(g_0) = g_0$. Set,

$$S_{q-1} := \{(\psi_{q-1}(h_0), \ldots, \psi_{q-1}(h_{n-1})) : (h_0, \ldots, h_{n-1}) \in S_q\}.$$

Clearly, $S_{q-1}$ is a concrete $K_n$-atom structure that is isomorphic to $S_q(= S)$ via the component wise extension $\widehat{\psi_{q-1}}$ of $\psi_{q-1}$. Extend $deg$ and $tag$ as follows. Let $h \in S_{q-1}$, suppose that $\bar{h} \in S_q$ is the inverse image of $h$ under the isomorphism $\widehat{\psi_{q-1}}$. If $h \in S_q$, keep its label and degree, i.e., $deg(h) = deg(\bar{h}) = q$ and $tag(h) = tag(\bar{h})$. If $h \notin S_q$ then let $deg(h) = q - 1$ and let $tag(h)$ be the unique term in $F^{n,m}_{q-1}$ such that $(S, \bar{h}, \iota) \models tag(h)$. Thus, $S_{q-1}$ is a consistent concrete $(K_n, q)$-atom structure and $\widehat{\psi_{q-1}}(g_1, \ldots, g_1)$ is a regular elements of degree $q - 1$ that is connected to $f$ by a finite length zigzag.

Let $U_{q-1}$ be the unit of $S_{q-1}$ and suppose that $\widehat{\psi_{q-1}}(g_1, \ldots, g_1) = (w, \ldots, w)$. Let $W$ be any set such that $W \cap U_q = \emptyset$, $W \cap U_{q-1} = \emptyset$ and $\mid W \mid = \mid U_{q-1} \setminus \{w\} \mid$. Let $\psi_{q-2}$ be a bijection between $U_{q-1}$ and $W \cup \{w\}$ such that $\psi_{q-2}(w) = w$. Set,

$$S_{q-2} := \{(\psi_{q-2}(h_0), \ldots, \psi_{q-2}(h_{n-1})) : (h_0, \ldots, h_{n-1}) \in S_{q-1}\}.$$

Clearly, $S_{q-2}$ is a concrete $K_n$-atom structure that is isomorphic to $S_{q-1}$ via the component wise extension $\widehat{\psi_{q-2}}$ of $\psi_{q-2}$ and isomorphic to $S$ via the composition $\widehat{\psi_{q-2}} \circ \widehat{\psi_{q-1}}$. Extend $deg$ and $tag$ as follows. Let $h \in S_{q-2}$, suppose that $\bar{h} \in S_q$ is the inverse image of $h$ under the isomorphism $\widehat{\psi_{q-2}} \circ \widehat{\psi_{q-1}}$. If $h \in S_{q-1}$, keep its label and degree, i.e., $deg(h) = deg(\bar{h}) = q-1$ and $tag(h) = tag(\bar{h})$. If $h \notin S_{q-1}$ then let $deg(h) = q - 2$ and let $tag(h)$ be the unique term in $F^{n,m}_{q-2}$ such that $(S, \bar{h}, \iota) \models tag(h)$ satisfies. Thus, $S_{q-2}$ is a consistent concrete $(K_n, q)$-atom structure and $\widehat{\psi_{q-2}}(\widehat{\psi_{q-1}}(g_0, \ldots, g_0))$ is a regular elements of degree $q-2$ that is connected to $f$ by a finite length zigzag.

Inductively, we get a sequence of consistent $(K_n, q)$-atom structures $S_q, \ldots, S_0$. Finally, let $S_\tau = \bigcup\{S_k : k \in q+1\}$. One can easily check that $S_\tau$ is a consistent concrete $(K_n, q)$-atom structure and there exist $f, f^* \in S_\tau$ such that $(S_\tau, f) \models \tau$, $f$ is connected to $f^*$ via a finite length zigzag and $f^*$ is a free element. Remember the form $\sigma$ and (1.16). Suppose that $\sigma \in F^{n,m}_{q^*}$, for some $q^* \in \omega$. By lemma 1.4.7, let $S_\sigma$ be a consistent concrete $(K_n, q^*)$-atom structure such that $S_\tau \cap S_\sigma = \emptyset$, there exist $h, h^* \in S_\sigma$ such that $(S_\sigma, h) \models \sigma$, $h$ is connected to $h^*$ via a finite length zigzag and $h^*$ is a free element.

Suppose that $f^* = (u, \ldots, u)$ and $h^* = (h_0, \ldots, h_{n-1})$. Let $y_0 = [f^*]^{h_0}_0$ and, for every $i \in \{1, \ldots, n-1\}$, let $y_i = [y_{i-1}]^{h_i}_i$. Note that $y_{n-1} = h^*$. Define

$$S_{bridge} = \bigcup\{{}^nRng(y_i) : i \in n-1\}.$$

Let $S^* = S_\tau \cup S_{bridge} \cup S_\sigma$. Since both $S_\tau$ and $S_\sigma$ are concrete $K_n$-atom structures, then $S$ is concrete $K_n$-atom structure as well. Also, it is not hard to check that, for some evaluation $\iota^*$, $(S^*, f, \iota^*) \models \tau$ and $(S^*, h, \iota^*) \models \sigma$ and there is a finite length zigzag connecting $f$ and $h$. Hence, there exists a finite length sequence $\beta$ such that

$$K_n \not\models \tau \cdot C_\beta \sigma = 0. \tag{1.17}$$

By equations 1.16 and 1.17, $\tau$ is not an atom in $\mathfrak{Fr}_m K_n$ as desired. □

Surprisingly, although the analogue of the above proposition is true for $\mathfrak{Fr}_m Crs_2$, it is not true for $\mathfrak{Fr}_m K_n$ when $n \geq 4$ and $K \in \{Crs, P\}$. Indeed, in this case, the subalgebra $\mathfrak{Nl}_{-d}\mathfrak{Fr}_m K_n$ contains infinitely many atoms. The reason for this is that the assumption $n \geq 4$ that allows us to construct a zigzag of elements such that in all of them two specific entries, say the third and the forth entries, are identically equal to some entity and, no matter how we continue this zigzag, we cannot add a co-diagonal sequence (that is a sequence whose entries are all different from each other). To prove this, we need the following definitions.

**Definition 1.5.5.** Let $S$ be a concrete $K_n$-atom structure and let $\iota$ be an evaluation of the free variables. For every $k \in \omega$ and every $f \in S$, let $tag_k(S, f, \iota)$ be the unique term in $F^{n,m}_k$ such that $(S, f, \iota) \models tag_k(S, f, \iota)$.

**Definition 1.5.6.** Let $S_1, S_2$ be two concrete $K_n$-atom structures and let $\iota_1, \iota_2$ be two evaluations of the free variables into $S_1, S_2$, respectively. A relation $\Theta \subseteq S_1 \times S_2$ is said to be a tag-homomorphism between $(S_1, \iota_1)$ and $(S_2, \iota_2)$ if and only if the followings hold for every $(f, g) \in \Theta$, every $i, j \in n$ and every $k \in m$.

1. $(f(i) = f(j) \iff g(i) = g(j))$ and $(f \in \iota_1(x_k) \iff g \in \iota_2(x_k))$.

2. If there is $f^* \in S_1$ such that $f \equiv_i f^*$, then there exists $g^* \in S_2$ such that $g \equiv_i g^*$ and $(f^*, g^*) \in \Theta$.

3. If there is $g^* \in S_2$ such that $g \equiv_i g^*$, then there exists $f^* \in S_1$ such that $f \equiv_i f^*$ and $(f^*, g^*) \in \Theta$.

**Lemma 1.5.7.** *Let $S_1, S_2$ be two concrete $K_n$-atom structures and let $\iota_1, \iota_2$ be two evaluations of the free variables into $S_1, S_2$, respectively. Let $\Theta \subseteq S_1 \times S_2$ be a tag-homomorphism between $(S_1, \iota_1)$ and $(S_2, \iota_2)$. Then for every $(f, g) \in \Theta$ and every $k \in \omega$, we have*

$$tag_k(S_1, f, \iota_1) = tag_k(S_2, f, \iota_2).$$

*Proof.* By induction on the degrees of the normal forms. The base of the induction follows from condition 1 of the above definition. For the induction step, it follows directly from conditions 2 and 3 together with the induction hypothesis. $\square$

**Proposition 1.5.8.** *Let $n \geq 4$ and $m \geq 0$ be finite and let $K \in \{Crs, P\}$, then there are infinitely many atoms below $-d$ in the free algebra $\mathfrak{Fr}_m K_n$.*

*Proof.* Let $h_0 := (0, \dots, 0) \in {}^n\{0\}$. For every finite $j \geq 1$, define $h_j$ inductively as follows. If $j$ is even, let $h_j = [h_{j-1}]^0_j$. If $j$ is odd, let $h_j = [h_{j-1}]^1_j$. Let $k \in \omega$ be arbitrary but fixed. let

$$S_k = \{h_j : j \in k+1\} \cup \{h_j \circ \rho : j \in k+1, \rho \in {}^n n \text{ is bijection and } K = P\}.$$

(The condition $K = P$ in the second part of $S_k$ is not necessary, i.e., if we omit it we get more atoms in $\mathfrak{Fr}_m Crs_n$.) Fix any arbitrary evaluation $\iota_k$ and define $\tau_k := tag_{2k+1}(S_k, h_0, \iota_k)$. One can easily check that, for every $j \in \omega \setminus \{k\}$, $(S_k, f_k, \iota_k) \models \tau_k \cdot -\tau_j$. Therefore, $K \models \tau_k \cdot \tau_j = 0$, for every $j \in \omega$.

We claim that the term $\tau_k$ is an atom in the free algebra $\mathfrak{Fr}_m K_n$. Let $S$ be a concrete $K_n$-atom structure and let $\iota$ be an evaluation of the free variables. Suppose that there is $h \in S$ such that

$$(S, h, \iota) \models \tau_k = tag_{2k+1}(S_k, h_0, \iota_k). \tag{1.18}$$

It is enough to prove the following. For every $j \in \omega$, $tag_j(S, h, \iota) = tag_j(S_k, h_0, \iota_k)$. We use lemma 1.5.7, so we need to construct a tag-homomorphism $\Theta$ between $(S_k, \iota_k)$ and $(S, \iota)$ such that $(h_0, h) \in \Theta$. Inductively, define $\Theta_0 = \{(h_0, h)\}$ and suppose that we are given $\Theta_j$ for some $j \in k$. Define $\Theta_{j+1}$ as follows. Let $V_{j+1} := \{f \in S_k : f \notin Dom(\bigcup_{i \in j+1} \Theta_i)\}$. For every $(f, g) \in V_{j+1} \times S$,

$$(f, g) \in \Theta_{j+1} \iff \exists (f^*, g^*) \in \Theta_j \text{ and } i \in n \text{ such that } f \equiv_i f^*, g \equiv_i g^* \text{ and}$$
$$tag_{2k-j}(S_k, f, \iota_k) = tag_{2k-j}(S, g, \iota).$$

Let $\Theta = \bigcup\{\Theta_j : j \in k+1\}$. Clearly, conditions 1 and 2 of the tag-homomorphisms are guaranteed for $\Theta$ by (1.18). It remains to check condition 3. Let $j \in k+1$ and let $(f, g) \in \Theta_j$. Suppose that there exists $g^* \in S$ and $i \in n$ such that $g^* \equiv_i g$ and $g^* \notin Rng(\bigcup_{l \leq j} \Theta_l)$. By the definition of $\Theta_j$, we have $tag_{2k-j+1}(S, g, \iota) = tag_{2k-j+1}(S_k, f, \iota_k)$. Therefore, there exists $f^* \in S_k$ such that $f \equiv_i f^*$ and

$$tag_{2k-j}(S, g^*, \iota) = tag_{2k-j}(S_k, f^*, \iota_k). \tag{1.19}$$

If $f^* \notin Dom(\bigcup_{l \leq j} \Theta_l)$ then $j < k$ and, by the definition of $\Theta_{j+1}$, $(f^*, g^*) \in \Theta_{j+1}$. Suppose that $f^* \in Dom(\Theta_l)$ for some $l \leq j$. Thus, for every $s < l$, there is $f_s \in S_k$ such that $f_0 = h_0$ and (without loss of generality) $h_0 \equiv_1 f_1 \equiv_0 f_2 \equiv_1 f_3 \equiv \cdots \equiv f^*$. Here is the secrete of choosing $\tau_k$ to be of degree $2k+1$. By (1.19), for every $s < l$, there is $g_s \in S$ such that $g_0 = h$ and $h \equiv_1 g_1 \equiv_0 g_2 \equiv_1 g_3 \equiv \cdots \equiv g^*$. It is not hard to check that, for every $s < l$, $(f_s, g_s) \in \Theta_s$. Hence, $(f^*, g^*) \in \Theta_l$ which makes a contradiction. Therefore, $\Theta$ is a tag-homomorphism between $(S_k, \iota_k)$ and $(S, \iota)$, and, $(h_0, h) \in \Theta$, as desired. $\square$

The assumption $n \geq 4$ is essential in the proof of the above proposition. If $n = 3$ and we construct $\tau_k$, for some $k \geq 2$, as above, one can easily find a satisfiable term below the co-diagonal that is connected to $\tau_k$ by a sequence of finite length. Then the method used in lemma 1.4.7 can be applied to prove that $\tau_k$ is not an atom in $\mathfrak{Fr}_m K_n$.

**Proposition 1.5.9.** *Let $m \geq 0$ be finite. In the free algebra $\mathfrak{Fr}_m P_2$, there is no atom below $-d$. In each of the free algebras $\mathfrak{Fr}_m Crs_3$ and $\mathfrak{Fr}_m P_3$, there are only finitely many atoms below $-d$.*

*Proof.* The method used in proposition 1.4.10 (and lemma 1.4.7) can be used again to show that there are no atoms below $-d$ in the free algebra $\mathfrak{Fr}_m P_2$. For the other free algebras, we need to consider all the possible cases of constructing a sub-unit that does not contain a co-diagonal. Let $V = \{(0,0,0)\}$, $V_0 = \{(1,0,0)\}$, $V_1 = \{(0,1,0)\}$ and $V_2 = \{(0,0,1)\}$. For every $i, j \in 3$, let $V_{ij} = V_i \cup V_j$ and let $V_{012} = V_0 \cup V_1 \cup V_2$. We claim the followings.

$$\begin{aligned}
At(\mathfrak{Fr}_m Crs_3) &= \{tag_1(V, f, \iota) : f \in V \text{ and } \iota \text{ any evaluation}\} \\
&\cup \{tag_2(V_i, f, \iota) : i \in 3, f \in V_i \text{ and } \iota \text{ any evaluation}\} \\
&\cup \{tag_2(V_{ij}, f, \iota) : i, j \in 3, f \in V_{ij} \text{ and } \iota \text{ any evaluation}\} \\
&\cup \{tag_2(V_{012}, f, \iota) : f \in V_{012} \text{ and } \iota \text{ any evaluation}\} \\
&\cup \{tag_3(V \cup V_i, f, \iota) : i \in 3, f \in V \cup V_i \text{ and } \iota \text{ any evaluation}\} \\
&\cup \{tag_3(V \cup V_{ij}, f, \iota) : i, j \in 3, f \in V \cup V_{ij} \text{ and } \iota \text{ any evaluation}\} \\
&\cup \{tag_3(V \cup V_{012}, f, \iota) : f \in V \cup V_{012} \text{ and } \iota \text{ any evaluation}\}. \\
At(\mathfrak{Fr}_m P_3) &= \{tag_1(V, f, \iota) : f \in V \text{ and } \iota \text{ any evaluation}\} \\
&\cup \{tag_2(V_{012}, f, \iota) : f \in V_{012} \text{ and } \iota \text{ any evaluation}\} \\
&\cup \{tag_3(V \cup V_{012}, f, \iota) : f \in V \cup V_{012} \text{ and } \iota \text{ any evaluation}\}.
\end{aligned}$$

To prove the inclusions $\supseteq$, one can use the tag-homomorphism method used in proposition 1.5.8. To prove the other inclusions, let $k \in \omega$ be such that $k \geq 3$. Take any term $\tau \in F_3^{n,m}$ that is not equal to any of the terms in the sets given on the right hand side. Then one can easily see that there should be terms $\sigma_k = \sigma, \ldots, \sigma_0$ such that $\sigma_0$ is below the co-diagonal and, for every $l \in k-1$, $\sigma_l \in sub_j(\sigma_{l+1})$ for some $j \in n$. Hence, one can build a structure satisfying $\tau$ and has a free element as we did in lemma 1.4.7. Therefore, proposition 1.4.10 can be extended to show that $\tau$ is not an atom. $\square$

Let $n \geq 2$ and $m \geq 0$ be arbitrary finite ordinals and let $K \in \{Crs, D, P, G\}$. To describe all the atoms in the free algebra $\mathfrak{Fr}_m K_n$, one has to find all the possible ways of constructing a zigzag such that, no matter how we extend it, it cannot contain a sequence that all of its entries are different, i.e., it cannot contain a sequence that satisfies the co-diagonal.

We end this chapter by summarizing our results in this chapter. First, we would like to indicate that the method used in this chapter was not used before to investigate the atomicity of the free algebras. The secret key in this chapter is the normal forms and the structures that satisfy the "gradually fading" constructions of these normal forms. This method is powerful and we believe that it can be used to write simpler proofs of some known facts and/or to answer some open problems. The following interesting result follows as a consequence of our method in the former results. It might seem not so hard for the classes $D_n$ and $G_n$, but it is surprising for the classes $Crs_n$ and $P_n$.

**Theorem 1.5.10.** *Let $n \in \omega$ and $m \in \omega$ be such that $n \geq 2$. Let $K \in \{Crs, D, P, G\}$. The free algebra $\mathfrak{Fr}_m K_n$ contains only finitely many zero-dimensional elements.*

*Proof.* Recall the equalizers $\alpha$'s and the atoms $a_\alpha$'s defined before proposition 1.5.2. We claim that the set of all zero-dimensional elements in the free algebra $\mathfrak{Fr}_m K_n$ is:

$$\{0, 1\} \cup \{a_\alpha, -a_\alpha : \alpha \in {}^{D_{n,m}}\{-1, 1\} \text{ is an equalizer}\}.$$

In the proof of proposition 1.5.2, we proved that each of $a_\alpha$, for any equalizer $\alpha$, is zero-dimensional. The complementarity of the cylindrifications implies that $-a_\alpha$ is zero-dimensional too. It remains to prove that no other element in the free algebra $\mathfrak{Fr}_m K_n$ is zero dimensional. Remember the co-diagonal element $\delta = \prod\{-d_{ij} : i \in j \in n\}$. Let $\tau \in \mathfrak{Tm}_{m,cyl_n}$, then we have one of the following cases:

(a) Suppose that $\mathfrak{Fr}_m K_n \not\models \delta \cdot \tau = 0$ and $\mathfrak{Fr}_m K_n \not\models \delta \cdot -\tau = 0$. Then there are normal forms $\sigma_1$ and $\sigma_2$ such that $\mathfrak{Fr}_m K_n \models \sigma_1 \leq \delta \cdot \tau$ and $\mathfrak{Fr}_m K_n \models \sigma_2 \leq \delta \cdot -\tau$. By the proof of proposition 1.4.10, there is a concrete $K_n$-atom structure $V$, an evaluation $\iota$ and $f, g \in V$ such that $(V, f, \iota) \models \delta \cdot \tau$, $(V, g, \iota) \models \delta \cdot -\tau$ and $f, g$ are connected by a finite length zigzag $f = h_0, \ldots, h_l = g$. Let $k := min = \{j \in l + 1 : (V, h_j, \iota) \not\models \tau\}$. Clearly, such $k$

exists because $(V, h_l, \iota) \not\models \tau$ and $k \geq 1$. Since $f = h_0, \ldots, h_l = g$ is a zigzag, then there exists $i \in n$ such that $h_{k-1} \equiv_i h_k$. Hence, $(V, h_{k-1}, \iota) \models \tau \cdot c_i - \tau$. Therefore, $\tau$ is not zero-dimensional in $\mathfrak{Fr}_m K_n$ as desired.

(b) Suppose that $\mathfrak{Fr}_m K_n \models \delta \cdot \tau = 0$. Then $\mathfrak{Fr}_m K_n \models \tau \leq -\delta = \sum\{d_{ij} : i \in j \in n\}$. Suppose that there are $i \in j \in n$ and $k \in n$ such that $\mathfrak{Fr}_m K_n \not\models \tau \cdot c_k - d_{ij} = 0$, then $\tau$ is not zero-dimensional as desired. Suppose that, for every $i \in j \in n$ and every $k \in n$, $\mathfrak{Fr}_m K_n \models \tau \cdot c_k - d_{ij} = 0$. Hence, $\mathfrak{Fr}_m K_n \models \tau \leq d := \prod\{-c_k - d_{ij} : i \in j \in n, k \in n\}$. But, $\mathfrak{Fr}_m K_n \models d = \sum\{a_\alpha : \alpha \in {}^{D_{n,m}}\{-1, 1\}$ is an equalizer$\}$. By proposition 1.5.2, since all $a_\alpha$'s are atoms, there is an equalizer $\beta \in {}^{D_{n,m}}\{-1, 1\}$ such that $\mathfrak{Fr}_m K_n \models \tau = a_\beta$ as desired.

(c) Suppose that $\mathfrak{Fr}_m K_n \models \delta \cdot -\tau = 0$. By (b), there is an equalizer $\alpha \in {}^{D_{n,m}}\{-1, 1\}$ such that $\mathfrak{Fr}_m K_n \models \tau = -a_\alpha$ as desired.

□

Let $n \geq 2$ and $m \geq 0$ be any two finite ordinals. In this chapter, we have shown the followings.

- There are only finitely many zero-dimensional elements in the free algebra $\mathfrak{Fr}_m K_n$.
- The free algebra $\mathfrak{Fr}_0 G_2$ is finite, hence, atomic. Some of its atoms are zero-dimensional while there are other atoms that are not zero-dimensional.
- The free algebras $\mathfrak{Fr}_{m+1} G_2$, $\mathfrak{Fr}_m G_{n+1}$, $\mathfrak{Fr}_m D_n$, $\mathfrak{Fr}_m P_2$ and $\mathfrak{Fr}_m Crs_2$ are not atomic and each of them contains only finitely many atoms. Every atom in these free algebras is zero-dimensional. Each of these free algebras can be decomposed into a finite algebra and an atomless algebra.
- The free algebras $\mathfrak{Fr}_m Crs_{n+2}$ and $\mathfrak{Fr}_m P_{n+2}$ are not atomic but each of them contains infinitely many atoms. In these free algebras only finitely many atoms are zero-dimensional while infinitely many atoms are not zero-dimensional. Each of these algebras can be decomposed into a finite algebra and a non atomic algebra that contains infinitely many atoms.

- The free algebras $\mathfrak{Fr}_m Crs_3$ and $\mathfrak{Fr}_m P_3$ are not atomic and each of them contains only finitely many atoms. There are atoms in these free algebras that are zero-dimensional and there are other atoms that are not zero-dimensional. Each of these free algebras can be decomposed into a finite algebra and a non atomic algebra that contains only finitely many atoms atoms.

2

CHAPTER

# Guarded fragment and FO with general assignment models

The guarded fragment (GF) of first order logic was introduced by H. Andréka, J. van Benthem and I. Németi in [AvBN98]. It is closely connected to first order logic (FO) with general assignment models introduced by I. Németi [Ném86]. These logics were investigated by many logicians and it was shown that they have a number of desirable properties, e.g. decidability through finite model property. These logics are considered to be the most important decidable versions of first order logic among the large number that have been introduced over the years. They have applications in linguistics (dynamic semantics of natural language) and computer science.

Gödel's incompleteness property (GIP) fails for most of these logics (because they are decidable) on countable languages. Weak Gödel's incompleteness property (wGIP) holds for most of these logics, too, but on infinite languages. We show that guarded fragment do not have wGIP on finite languages, but both its solo-quantifier fragment and FO with general assignment models do have wGIP. On the other hand, we prove that FO with general assignment models enriched with polyadic-quantifiers does not have wGIP on finite languages.

We assume familiarity with first order logic, but here we briefly recall the definitions that we will use. By a *(first order) language* we understand a set of relation symbols each of which is associated with a finite (positive) rank, together with a set of variables (of size at least 2). That is, a language is $\mathcal{L} := \langle \mathcal{R}, \mathcal{V} \rangle$ where $\mathcal{R} = \{(R_i, rank(R_i)) : i \in \beta, 0 \in rank(R_i) \in \omega \cap \alpha\}$ and $\mathcal{V} = \{x_i : i \in \alpha\}$ for some ordinals $\alpha \geq 2, \beta$. We put the restriction $\alpha \geq 2$ because the results in this chapter are already known for the propositional logic, so we need to deal with logics that are different than the propositional logic. The restriction of having relation symbols only of positive rank is not important, the relation symbols of rank $0$ do not affect (at least semantically) the logics under question. We say that $\mathcal{L}$ *is finite, infinite, countable or uncountable* if $\alpha \cup \beta$ is finite, infinite, countable or uncountable, respectively.

A *relational FO-atom* on $\mathcal{L}$ is a string of the form $R(v_0, \ldots, v_{k-1})$ for some $(R, k) \in \mathcal{R}$ and some $v_0, \ldots, v_{k-1} \in \mathcal{V}$. An *equational FO-atom* on $\mathcal{L}$ is a string of the form $u = v$ (where $=$ is the equality logical symbol) for some variables $u, v \in \mathcal{V}$. A *FO-atom* on $\mathcal{L}$ is a relational or an equational FO-atom on $\mathcal{L}$. The set of the *first order formulas (FO-formulas)* on $\mathcal{L}$ is defined recursively to be the smallest set that contains the FO-atoms on $\mathcal{L}$ such that, for any FO-formulas $\varphi, \psi$ and any variable $u$, the conjunction $\varphi \wedge \psi$, the disjunction $\varphi \vee \psi$, the negation $\neg\varphi$ and the existential quantification $\exists u\varphi$ are all FO-formulas. Let $\varphi$ and $\psi$ be any FO-formulas on a language $\mathcal{L}$. The formulas $\forall v_i\varphi$, $\varphi \to \psi$ and $\varphi \leftrightarrow \psi$ denote the usual abbreviations.

A *(first order) model* for the language $\mathcal{L}$ is a pair $\mathfrak{M} = (M, \mathcal{R}^{\mathfrak{M}})$ where $M$ is a nonempty set and $\mathcal{R}^{\mathfrak{M}} = \{R^{\mathfrak{M}} \subseteq {}^{rank(R)}M : (R, rank(R)) \in \mathcal{R}\}$. Usually the old capital German letters are used to denote the models and the corresponding capital Latin letters are used to denote their underlying sets. An *assignment* of the variables $\mathcal{V}$ into the model $\mathfrak{M}$ is a function $f \in {}^{\mathcal{V}}M$ that assigns for each variable an element of $M$. The *interpretations* of the FO-formulas are defined by induction on the their complexity as follows.

$$\mathfrak{M}, f \models u = v \iff f(u) = f(v),$$

$$\mathfrak{M}, f \models R(v_0, \ldots, v_{k-1}) \iff (f(v_0), \ldots, f(v_{k-1})) \in R^{\mathfrak{M}},$$

$$\mathfrak{M}, f \models \varphi \vee \psi \iff \mathfrak{M}, f \models \varphi \text{ or } \mathfrak{M}, f \models \psi,$$

$$\mathfrak{M}, f \models \varphi \wedge \psi \iff \mathfrak{M}, f \models \varphi \text{ and } \mathfrak{M}, f \models \psi,$$

$$\mathfrak{M}, f \models \neg\varphi \iff \mathfrak{M}, f \not\models \varphi,$$

$$\mathfrak{M}, f \models \exists u\varphi \iff (\exists g \in {}^{\mathcal{V}}M)\ [f \equiv_u g \text{ and } \mathfrak{M}, g \models \varphi].$$

Where, $f \equiv_u g$ means $f(v) = g(v)$ for all $v \in \mathcal{V} \setminus \{u\}$. Let $\varphi$ be any FO-formula on $\mathcal{L}$. The formula $\varphi$ is said to be *valid in* $\mathfrak{M}$, in symbols $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, f \models \varphi$ for any assignment $f \in {}^{\mathcal{V}}M$. The formula $\varphi$ is said to be *valid* if $\mathfrak{N}, g \models \varphi$ for any model $\mathfrak{N}$ and any assignment

$g$. The formula $\varphi$ is said to be *satisfiable* if there is a model $\mathfrak{N}$ for the language $\mathcal{L}$ and an assignment $g$ such that $\mathfrak{N}, g \models \varphi$. A FO-formula $\psi$ is said to be a *consequence* of $\varphi$, in symbols $\varphi \models \psi$, if $\mathfrak{N}, g \models \varphi \implies \mathfrak{N}, g \models \psi$, for every model $\mathfrak{N}$ and every assignment $g$.

Let $n \geq 2$ be any ordinal and let $FO(n)$ be the collection of first order logics on languages with exactly $n$-many variables. It was shown that $FO(n)$ has several undesirable properties. See [Göd29], [Chu36], [Mon69], [HMT71], [HMT85], [Sai90], [AvBN98] and [ACM$^+$09].

(1) There is a finitely axiomatizable Hilbert-style system for $FO(n)$ iff $n = 2$ or $n \geq \omega$.

(2) $FO(n)$ has the finite model property if and only if $n = 2$.

(3) $FO(n)$ is decidable if and only if $n = 2$.

(4) Craig's interpolation and Beth definability fail for $FO(n)$ if and only if $n \in \omega$.

## §2.1 Guarded fragment of first order logic

Roughly, the guarded fragment is a large part of first order logic in which quantification is allowed only when it is bounded with atomic formulas. Let $\varphi$ and $\psi$ be any two FO-formulas on $\mathcal{L}$. We write $\varphi(v_1, \ldots, v_n)$ to indicate that the free variables of $\varphi$ are among the variables $v_1, \ldots, v_n$. Let $free(\varphi)$ denote the set of the variables that occur free in $\varphi$. For any finite tuple of variables $\bar{v} = v_1, \ldots, v_k$, let $\exists\bar{v}\varphi := \exists v_1 \cdots \exists v_k \varphi$.

**Definition 2.1.1** (*Guarded fragment (GF)*)**.** Let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be any language. The set of *GF-formulas* on $\mathcal{L}$ is defined recursively to be the smallest subset of the set of FO-formulas on $\mathcal{L}$ that satisfies the followings.

(a) Any FO-atom on $\mathcal{L}$ is a GF-formula on $\mathcal{L}$.

(b) If $\varphi$ and $\psi$ are GF-formulas on $\mathcal{L}$, then $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\neg\varphi$ are GF-formulas on $\mathcal{L}$.

(c) Let $\varphi$ be a GF-formula on $\mathcal{L}$ and $G$ be a FO-atom on $\mathcal{L}$ such that $free(\varphi) \subseteq free(G)$. Then, for any finite tuple $\bar{v} \subseteq free(G)$, $\exists\bar{v}(G \wedge \varphi)$ is a GF-formula on $\mathcal{L}$ and such $G$ is called a *suitable GF-guard for* $\varphi$.

As a dual of GF-guarded existential quantification we also get GF-guarded universal quantification of the form $\forall\bar{v}(\gamma \to \psi)$, where $\gamma$ is a suitable GF-guard for $\psi$. The semantics of the guarded logic GF on $\mathcal{L}$ is the standard semantics of FO on $\mathcal{L}$.

Let $n \geq 2$ be any ordinal and let $GF(n)$ be the collection of the guarded fragments of FO logics on languages that have exactly $n$-many variables. It turns out that in $GF(n)$ many of the undesirable properties of $FO(n)$ disappear, see [AvBN95], [Grä99], [Mar01], [HO03], [HM02] and [Hoo01].

(1) $GF(n)$ has a finitely axiomatizable Hilbert-style system.

(2) $GF(n)$ has finite model property.

(3) $GF(n)$ is decidable.

(4) $GF(n)$ has Craig's interpolation if and only if $n = 2$. The modal interpolation and Beth definability hold for $GF(n)$.

A *GF-sentence* on a language $\mathcal{L}$ is a GF-formula on $\mathcal{L}$ that contains no free variable. For the sake of simplicity, if a language $\mathcal{L}$ is specified then we rather use GF-formula and GF-sentence instead of GF-formula on $\mathcal{L}$ and GF-sentence on $\mathcal{L}$.

### 2.1.1 wGIP fails for GF on finite languages

Let $\mathcal{L}$ be any language. The notions satisfiable, valid, consequence, etc, are identical with the standard corresponding notions for first order logic. For any GF-formula $\varphi$, we say that $\varphi$ has a *GF-complete extension* if there is GF-formula $\psi$ such that $\varphi \wedge \psi$ is satisfiable and, for any GF-formula $\chi$, either $\varphi \wedge \psi \models \chi$ or $\varphi \wedge \psi \models \neg\chi$.

**Theorem 2.1.2.** *Let $\mathcal{L}$ be a finite language. Each satisfiable GF-formula $\varphi$ on $\mathcal{L}$ has a GF-complete extension.*

The above theorem says that GF has neither GIP nor wGIP on finite languages. The proof of theorem 2.1.2 relies on the finite model property of guarded fragment [Grä99] and on the presence of polyadic quantifiers $\exists v_1 \dots v_n$. In § 2.1.2, we show that in the "non-polyadic" fragment (or the solo-fragment) of GF, where only "monadic" quantifiers $\exists v_1$ are available, the theorem is not true anymore. We begin with some lemmas that seem to be interesting in their own. Fix a finite language $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$.

Let $\mathfrak{M}$ be a finite model for $\mathcal{L}$. The *meaning* of any GF-formula $\varphi$ in $\mathfrak{M}$, denoted by $\varphi^{\mathfrak{M}}$, is the set of assignments $f$ for which $\varphi$ is true in $\mathfrak{M}$ at, i.e., $\varphi^{\mathfrak{M}} = \{f \in {}^{\mathcal{V}}M : \mathfrak{M}, f \models \varphi\}$. Let us call two formulas *equivalent* if they have the same meaning. For each $X \subseteq \mathcal{V}$ and GF-formula $\varphi$ let us select a GF-formula $\rho(\varphi, X)$ such that $\rho(\varphi, X)^{\mathfrak{M}} = \varphi^{\mathfrak{M}}$, and further $free(\rho(\varphi, X)) \subseteq X$ if there is such a GF-formula. Let rep$(\varphi)$ denote $\rho(\varphi, free(\varphi))$, we call it the *representative* of $\varphi$. The main idea of the proof of Theorem 2.1.2 is to write up a sentence $\Delta(\mathfrak{M})$ which contains enough information for proving that each GF-formula is equivalent to its representative, below any suitable GF-guard.

**Lemma 2.1.3.** *Let $\mathfrak{M}$ be a finite model. There is a GF-sentence $\Delta(\mathfrak{M})$ that is valid in $\mathfrak{M}$ and $\Delta(\mathfrak{M}) \models \gamma \to (\varphi \leftrightarrow rep(\varphi))$ for any GF-formula $\varphi$ and suitable GF-guard $\gamma$ for $\varphi$.*

*Proof.* The set of representatives $\mathbb{R}(\mathfrak{M}) = \{\text{rep}(\varphi) : \varphi \text{ is a GF-formula}\}$ is finite because both $M$ and $\mathcal{V}$ are finite, and so $\{\varphi^{\mathfrak{M}} : \varphi \text{ is a GF-formula}\}$ is also finite. We define the GF-sentence $\Delta(\mathfrak{M})$ as the conjunction of all the GF-sentences listed below, where $G$ is any FO-atom, $\rho, \rho_1, \rho_2, \rho_3, \rho_4 \in \mathbb{R}(\mathfrak{M})$ are any representatives such that $\rho_3$ and $\rho_4$ are equivalent, $\bar{u}$ is a finite tuple of variables, $\gamma'$ and $\gamma$ are GF-guards such that $\gamma'$ is a suitable GF-guard in the GF-formula $\exists \bar{u}(\gamma' \wedge \rho)$, $\gamma$ is a suitable GF-guard for the formula that it guards, e.g., in the first line $\gamma$ is a suitable GF-guard for $G \leftrightarrow \text{rep}(G)$, and $\bar{v}$ is any enumeration of $free(\gamma)$.

- $\forall \bar{v}[\gamma \to (G \leftrightarrow \text{rep}(G))]$,
- $\forall \bar{v}[\gamma \to (\neg\rho \leftrightarrow \text{rep}(\neg\rho))]$,
- $\forall \bar{v}[\gamma \to ((\rho_1 \wedge \rho_2) \leftrightarrow \text{rep}(\rho_1 \wedge \rho_2))]$,
- $\forall \bar{v}[\gamma \to ((\rho_1 \vee \rho_2) \leftrightarrow \text{rep}(\rho_1 \vee \rho_2))]$,
- $\forall \bar{v}[\gamma \to (\exists \bar{u}(\gamma' \wedge \rho) \leftrightarrow \text{rep}(\exists \bar{u}(\gamma \wedge \rho)))]$,
- $\forall \bar{v}[\gamma \to (\rho_3 \leftrightarrow \rho_4)]$.

Each one of the above GF-formulas is indeed a GF-sentence (recall that $\bar{v}$ is an enumeration of $free(\gamma)$ and $\gamma$ contains all the free variables of the formula that it guards). There are only finitely many such GF-sentences, because there are only finitely many GF-guards on the finite language $\mathcal{L}$ and $\mathbb{R}(\mathfrak{M})$ is finite. Hence, $\Delta(\mathfrak{M})$ is a finite conjunction of GF-sentences, thus it is also a GF-sentence. It is valid in $\mathfrak{M}$ by the definition of the representative formulas. Now we prove that $\Delta(\mathfrak{M})$ is the desired GF-sentence. For every GF-formula $\varphi$ we need to show that

$$\Delta(\mathfrak{M}) \models \gamma \to (\varphi \leftrightarrow \text{rep}(\varphi)), \quad \text{for all suitable GF-guard } \gamma \text{ for } \varphi. \tag{2.1}$$

We prove this by induction on the complexity of the GF-formula $\varphi$. When $\varphi$ is a FO-atom $G$, then (2.1) holds by the first line in the definition of $\Delta(\mathfrak{M})$.

Let $\varphi = \neg\psi$ and let $\gamma$ be any suitable GF-guard for $\varphi$. Note that $\gamma$ is a suitable GF-guard for $\psi$ also. By the induction hypothesis (2.1) holds for $\psi$ and $\gamma$, thus $\Delta(\mathfrak{M}) \models \gamma \to (\psi \leftrightarrow \text{rep}(\psi))$. Then, by $\varphi = \neg\psi$,

$$\Delta(\mathfrak{M}) \models \gamma \to (\varphi \leftrightarrow \neg\text{rep}(\psi)).$$

By the second line in the definition of $\Delta(\mathfrak{M})$ we have

$$\Delta(\mathfrak{M}) \models \gamma \to (\neg\text{rep}(\psi) \leftrightarrow \text{rep}(\neg\text{rep}(\psi)))$$

since $\gamma$ is a suitable GF-guard for $\neg\text{rep}(\psi)$ and $\text{rep}(\psi) \in \mathbb{R}(\mathfrak{M})$. We have, by the definition of the rep function, that $\text{rep}(\neg\text{rep}(\psi))$ and $\text{rep}(\neg\psi)$ are equivalent, both are representatives and $\gamma$ is a suitable GF-guard for both, so by the last line in the definition of $\Delta(\mathfrak{M})$ we have

$$\Delta(\mathfrak{M}) \models \gamma \to (\text{rep}(\neg\text{rep}(\psi) \leftrightarrow \text{rep}(\varphi))).$$

Putting together the three displayed formulas show that (2.1) holds for $\varphi$. The induction step goes in a similar way for all the other cases, too. We write out the proof for the last case.

Let $\varphi$ be $\exists\bar{u}(\gamma' \wedge \psi)$ and let $\gamma$ be a suitable GF-guard for $\varphi$. Note that $\gamma$ is not necessarily a suitable GF-guard for $\psi$. However, by the induction hypothesis (2.1) holds for $\psi$ and $\gamma'$, thus $\Delta(\mathfrak{M}) \models \gamma' \to (\psi \leftrightarrow \text{rep}(\psi))$, so

$$\Delta(\mathfrak{M}) \models \gamma \to (\varphi \leftrightarrow \exists\bar{u}(\gamma' \wedge \text{rep}(\psi))).$$

By the fifth line in the definition of $\Delta(\mathfrak{M})$ we have

$$\Delta(\mathfrak{M}) \models \gamma \to (\exists \bar{u}(\gamma' \wedge \text{rep}(\psi)) \leftrightarrow \text{rep}(\exists \bar{u}(\gamma' \wedge \text{rep}(\psi))))$$

since $\gamma$ is a suitable GF-guard for the GF-formula $\exists \bar{u}(\gamma' \wedge \text{rep}(\psi))$. Now, $\text{rep}(\exists \bar{u}(\gamma' \wedge \text{rep}(\psi)))$ and $\text{rep}(\exists \bar{u}(\gamma' \wedge \psi))$ are equivalent, both are representatives and $\gamma$ is a suitable GF-guard for both, so by the last line in the definition of $\Delta(\mathfrak{M})$ we have

$$\Delta(\mathfrak{M}) \models \gamma \to (\text{rep}(\exists \bar{u}(\gamma' \wedge \text{rep}(\psi))) \leftrightarrow \text{rep}(\varphi)).$$

Thus we get that (2.1) holds for $\varphi$. □

**Lemma 2.1.4.** *Let $\mathfrak{M}$ be a finite model for the language $\mathcal{L}$ and let $f$ be an assignment of the variables $\mathcal{V}$ into $M$. There is a GF-formula $\Gamma(\mathfrak{M}, f)$ on $\mathcal{L}$ such that $\mathfrak{M}, f \models \Gamma(\mathfrak{M}, f)$ and, for any GF-formula $\chi$ on $\mathcal{L}$, either $\Gamma(\mathfrak{M}, f) \models \chi$ or $\Gamma(\mathfrak{M}, f) \models \neg\chi$.*

*Proof.* Let $\mathfrak{M}$ be a finite model for the language $\mathcal{L}$ and let $f$ be an assignment of the variables $\mathcal{V}$ into $M$. Recall the set $\mathbb{R}(\mathfrak{M})$ of representatives in $\mathfrak{M}$. Let $\Sigma$ be the set of all FO-atoms together with all GF-formulas of form $\exists \bar{u}(\gamma \wedge \rho)$ where $\rho \in \mathbb{R}(\mathfrak{M})$ and $\gamma$ is a suitable GF-guard for $\rho$, and their negations. Then $\Sigma$ is finite. Let $\Delta(\mathfrak{M})$ be the GF-sentence given in lemma 2.1.3 for the model $\mathfrak{M}$. Define

$$\Gamma(\mathfrak{M}, f) = \Delta(\mathfrak{M}) \wedge \bigwedge\{\sigma \in \Sigma : \mathfrak{M}, f \models \sigma\}.$$

It remains to show that for any GF-formula $\varphi$, either $\Gamma(\mathfrak{M}, f) \models \varphi$ or $\Gamma(\mathfrak{M}, f) \models \neg\varphi$. We use induction on the complexity of the GF-formulas. This is true for any FO-atom $\varphi$ because $\varphi, \neg\varphi \in \Sigma$ and either $\mathfrak{M}, f \models \varphi$ or $\mathfrak{M}, f \models \neg\varphi$. Assume this true for some GF-formulas $\varphi, \psi$. Then clearly it is true for $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\neg\varphi$. Let $\gamma$ be any suitable GF-guard for $\varphi$ and let $\chi = \exists \bar{u}(\gamma \wedge \varphi)$. We need to show that either $\Gamma(\mathfrak{M}, f) \models \chi$ or $\Gamma(\mathfrak{M}, f) \models \neg\chi$. Let $\chi' := \exists \bar{u}(\gamma \wedge \text{rep}(\varphi))$. By lemma 2.1.3, $\Delta(\mathfrak{M}) \models \gamma \to (\chi \leftrightarrow \chi')$ and $\chi', \neg\chi' \in \Sigma$. Thus either $\Gamma(\mathfrak{M}, f) \models \chi$ or $\Gamma(\mathfrak{M}, f) \models \neg\chi$ and we are done. □

Lemma 2.1.4 gives a GF-formula as desired in theorem 2.1.2, but to use lemma 2.1.4 we need to handle finite GF-models. In [Grä99], it was shown that every satisfiable GF-formula is satisfiable in a finite model for $\mathcal{L}$.

*Proof of theorem 2.1.2.* Let $\varphi$ be a satisfiable GF-formula on a finite language $\mathcal{L}$. Let $\mathfrak{M}$ be a finite GF-model for $\mathcal{L}$ and $f$ be an assignment of the variables such that $\mathfrak{M}, f \models \varphi$. The required GF-formula $\psi$ is suggested by lemma 2.1.4, i.e., let $\psi := \Gamma(\mathfrak{M}, f)$. By lemma 2.1.4 $\varphi \wedge \psi$ is satisfiable GF-formula (it is satisfied in $\mathfrak{M}$ at $f$) and, for any GF-formula $\chi$ on $\mathcal{L}$, either $\varphi \wedge \psi \models \chi$ or $\varphi \wedge \psi \models \neg\chi$. $\square$

We give some corollaries. First, we need to define the Lindenbaum-Tarski algebras of the propositional part of the guarded fragment GF on $\mathcal{L}$.

**Definition 2.1.5.** Let $\mathcal{L}$ be any language. The GF-formula algebra on $\mathcal{L}$ is defined as follows. $\mathfrak{F} =: \langle F, \vee, \wedge, \neg, \bot, \top \rangle$, where $F$ is the set of all GF-formulas on $\mathcal{L}$, $\vee, \wedge, \neg$ are the propositional connectives, $\top$ is a fixed GF-formula on $\mathcal{L}$ that is valid in every model for $\mathcal{L}$ and $\bot = \neg\top$. Clearly, $\mathfrak{F}$ is Boolean-type algebra. Define the congruence $\equiv$ on $\mathfrak{F}$ as follows. For all GF-formulas $\varphi, \psi$ on $\mathcal{L}$,

$$\varphi \equiv \psi \iff (\forall \text{ model } \mathfrak{M})\ \mathfrak{M} \models \varphi \leftrightarrow \psi.$$

This is a congruence relation on the GF-formula algebra. The *Lindenbaum-Tarski formula algebra* of the propositional part of GF on $\mathcal{L}$ is defined to be the quotient algebra $\mathfrak{F}/_{\equiv}$. The *Lindenbaum-Tarski sentence algebra* $\mathfrak{S}/_{\equiv}$ of the propositional part of GF on $\mathcal{L}$ is defined in a very similar way where $\mathfrak{S} =: \langle S, \vee, \wedge, \neg, \bot, \top \rangle$ and $S$ is the set of all GF-sentences on $\mathcal{L}$.

**Theorem 2.1.6.** *Let $\mathcal{L}$ be any finite language. Then*

*(i) Both the Lindenbaum-Tarski algebras $\mathfrak{F}/_{\equiv}$ and $\mathfrak{S}/_{\equiv}$ of the propositional part of GF on $\mathcal{L}$ are atomic.*

*(ii) The GF-theory of any finite model $\mathfrak{M}$ for the language $\mathcal{L}$ is finitely axiomatizable, i.e., there exists GF-sentence $\sigma$ on $\mathcal{L}$ such that any GF-sentence on $\mathcal{L}$ is valid in $\mathfrak{M}$ if and only if it is a consequence of $\sigma$.*

*Proof.* First we prove (ii). To prove (ii), we modify the proof of lemma 2.1.4 a bit. Let $\mathfrak{M}$ be a finite model for $\mathcal{L}$ and let $\mathbb{R}(\mathfrak{M})$ be the set of representatives in $\mathfrak{M}$. Let $\Delta(\mathfrak{M})$ be the GF-sentence defined in lemma 2.1.3. Let $\Sigma$ be the set of GF-sentences of form $\exists\bar{u}(\gamma \wedge \rho)$ where $\gamma$

is a suitable GF-guard for $\rho$ and $\rho \in \mathbb{R}(\mathfrak{M})$, and their negations. Define

$$\Lambda(\mathfrak{M}) = \Delta(\mathfrak{M}) \wedge \bigwedge\{\sigma \in \Sigma : \mathfrak{M} \models \sigma\}.$$

Then $\Lambda(\mathfrak{M})$ is a GF-sentence that is valid in $\mathfrak{M}$. To show that it is an atom among the GF-sentences, let first $\sigma$ be a GF-sentence of form $\exists\bar{u}(\gamma\wedge\delta)$ where $\delta$ is any GF-sentence and $\gamma$ is any GF-guard. By lemma 2.1.3, $\Delta(\mathfrak{M}) \models \gamma \to (\delta \leftrightarrow \text{rep}(\delta))$, so $\Delta(\mathfrak{M}) \models (\sigma \leftrightarrow \exists\bar{u}(\gamma \wedge \text{rep}(\delta))$. By $\Lambda(\mathfrak{M}) \models \Delta(\mathfrak{M})$ and $\exists\bar{u}(\gamma \wedge \text{rep}(\delta)) \in \Sigma$, we have either $\Lambda(\mathfrak{M}) \models \sigma$ or $\Lambda(\mathfrak{M}) \models \neg\sigma$ according to whether $\sigma$ or $\neg\sigma$ is valid in $\mathfrak{M}$. Now, let $\sigma$ be an arbitrary GF-sentence. Then $\sigma$ is a Boolean (propositional) combination of GF-sentences of the form $\exists\bar{u}(\gamma \wedge \delta)$, thus again $\Lambda(\mathfrak{M}) \models \sigma$ or $\Lambda(\mathfrak{M}) \models \neg\sigma$ according to whether $\sigma$ or $\neg\sigma$ is valid in $\mathfrak{M}$, by using the proven previous case.

To prove (i), next we prove that the Lindenbaum-Tarski formula algebra $\mathfrak{F}/_{\equiv}$ (of the propositional part of $GF$) is atomic. This follows from theorem 2.1.2 as follows. Let $\varphi$ be any GF-formula such that $\mathfrak{F}/_{\equiv} \not\models \varphi = \bot$. Then $\varphi$ is satisfiable and, by theorem 2.1.2, there exists a GF-formula $\psi$ such that $\varphi \wedge \psi$ is an atom in $\mathfrak{F}/_{\equiv}$ below $\varphi$ in $\mathfrak{F}/_{\equiv}$. For the second part, the atomicity of the Lindenbaum-Tarski sentence algebra $\mathfrak{S}/_{\equiv}$ follows from (ii). Indeed, assume that $\sigma$ is a GF-sentence such that $\mathfrak{S}/_{\equiv} \not\models \sigma = \bot$. Then there exists a finite model $\mathfrak{M}$ for $\mathcal{L}$ such that $\mathfrak{M} \models \sigma$ (for any assignment $f$ because $\sigma$ is GF-sentence). Take the GF-sentence $\rho$ that implies the GF-theory of $\mathfrak{M}$, then $\sigma \wedge \rho$ is an atom below $\sigma$ in $\mathfrak{S}/_{\equiv}$. $\square$

### 2.1.2 wGIP holds for solo-GF on finite languages

We show that the polyadic-quantifiers are responsible for not having Gödel's incompleteness property for the guarded fragment. We define the *solo-fragment of GF (sGF)* as follows. Let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be any language. The set of the sGF-formulas on $\mathcal{L}$ is defined analogously to the set of the GF-formulas on $\mathcal{L}$ except that the sGF-guarded existential quantification $\exists\bar{u}(\gamma \wedge \varphi)$ is allowed only if the block of quantifiers $\bar{u}$ is of length $\leq 1$.

The semantics of the solo fragment sGF is same as the semantics of guarded fragment GF. The notions satisfiable, valid, consequence, etc, are defined as usual.

For any sGF-formula $\varphi$, we say that $\varphi$ has a *sGF-complete extension* if there is an sGF-formula $\psi$ such that $\varphi \wedge \psi$ is satisfiable and, for any sGF-formula $\chi$, either $\varphi \wedge \psi \models \chi$ or $\varphi \wedge \psi \models \neg\chi$. We show that the use of longer blocks of quantifiers in GF is essential in proving Theorems 2.1.2, 2.1.6. We note that if $rank(R) = 1$ for every $(R, rank(R)) \in \mathcal{R}$, let us call these *unary languages*, then the solo-fragment sGF on $\mathcal{L}$ coincides with GF on $\mathcal{L}$, and therefore the opposite of the conclusion of theorem 2.1.7 below holds for it.

**Theorem 2.1.7.** *Let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be any finite non-unary language such that there is a relation symbol $(R, rank(R)) \in \mathcal{R}$ with $rank(R) =\mid \mathcal{V} \mid$. Then, there is a satisfiable sGF-formula $\varphi$ that has No sGF-complete extension.*

Suppose $\mathcal{L}$ and $R$ are as required in the above theorem and let $\bar{v}$ be any enumeration of the variables $\mathcal{V}$. We adapt the method used in § 1.4 to show that there are no atoms in the Lindenbaum-Tarski sGF-formula algebra that is below the sGF-formula

$$R(\bar{v}) \wedge \bigwedge\{\neg(u = v) : u, v \in \mathcal{V} \text{ are different variables}\}.$$

We need to introduce normal forms for the sGF-formulas. Fix a finite language $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$. For every $k \in \omega$ and every $X \subseteq \mathcal{V}$, we define a set $F(k, X)$ of sGF-formulas and we call it the set of sGF-normal forms of degree $k$ and free variables in $X$. We need the following convention: Let $\Sigma$ be any finite set of sGF-formulas and let $\alpha \in {}^{\Sigma}\{-1, 1\}$, define $\Sigma^\alpha := \bigwedge\{\varphi^\alpha : \varphi \in \Sigma\}$ where, for every $\varphi \in \Sigma$, $\varphi^\alpha = \varphi$ if $\alpha(\varphi) = 1$ and $\varphi^\alpha = \neg\varphi$ otherwise.

**Definition 2.1.8.** Let $X \subseteq \mathcal{V}$ be any set of variables in the language $\mathcal{L}$. Let $At(X)$ denote the set of all FO-atoms whose free variables are members of the set $X$. Let $G(X)$ be the set of all GF-guards whose free variables are exactly the variables in $X$. For any $k \in \omega$, we define the followings recursively.

- $F(0, X) := \{(At(X))^\alpha : \alpha \in {}^{At(X)}\{-1, 1\}\}$.

- $F(k + 1, X) := \{(At(X))^\alpha \wedge (\exists F(k, X))^\beta : \alpha \in {}^{At(X)}\{-1, 1\}$ and $\beta \in {}^{\exists F(k,X)}\{-1, 1\}\}$, where $\exists F(k, X) = \{\exists u(\gamma \wedge \varphi) : \gamma \in G(Y), \varphi \in F(k, Y), u \in Y$ and $Y \setminus \{u\} \subseteq X\}$.

- $F(X) := \bigcup\{F(k, X) : k \in \omega\}$.

By definition, every normal form contains complete information about the guarded quantifications of all the normal forms of the first smaller degree. As before, we introduce the following notions that focus on this information.

**Definition 2.1.9.** Let $X \subseteq \mathcal{V}$ be any set of variables in $\mathcal{L}$. Let $k \in \omega$, let $\alpha \in {}^{At(X)}\{-1, 1\}$ and let $\beta \in {}^{\exists F(k,X)}\{-1, 1\}$. Define

- $\mathit{color}_X((At(X))^\alpha) := \mathit{color}_X((At(X))^\alpha \wedge (\exists F(k, X))^\beta) := \{\psi \in At(X) : \alpha(\psi) = 1\}$.

Let $\gamma$ be any sGF-guard and let $u \in free(\gamma)$ be such that $free(\gamma) \setminus u \subseteq X$, define

- $sub_X^{u,\gamma}(At(X)^\alpha) := \emptyset$ and
- $sub_X^{u,\gamma}((At(X))^\alpha \wedge (\exists F(k, X))^\beta) := \{\varphi \in F(k, free(\gamma)) : \beta(\exists u(\gamma \wedge \varphi)) = 1\}$.

We prove that every sGF-formula can be rewritten in an equivalent form as a disjunction of some normal forms in $F(k, X)$ for some $k, X$. The following lemma follows immediately from definition 2.1.8.

**Lemma 2.1.10.** *Let $k \in \omega$, let $X \subseteq \mathcal{V}$ and let $\varphi, \psi \in F(k, X)$ be two different normal forms. Then*

$$\models \varphi \leftrightarrow \varphi \wedge \neg\psi \quad \textit{and} \quad \models \bigvee F(k, X).$$

**Theorem 2.1.11.** *Let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be any finite language. Let $\varphi$ be any sGF-formula on $\mathcal{L}$ and suppose that $d$ is the maximum depth of quantifier nesting in $\varphi$. Then, for any $k \geq d$ and any $Y \supseteq free(\varphi)$, there is $\Sigma(k, \varphi, Y) \subseteq F(k, Y)$ such that $\models \varphi \leftrightarrow \bigvee \Sigma$.*

*Proof.* We use induction on the complexity of the sGF-formula $\varphi$.

- Suppose that $\varphi$ is an FO-atom. Let $k \geq 0$ and let $Y \supseteq free(\varphi)$. Set

  $$\Sigma(k, \varphi, Y) = \{\psi \in F(k, Y) : \varphi \in \mathit{color}_Y(\psi)\}.$$

  Then, it is easy to see that $\models \varphi \leftrightarrow \bigvee \Sigma(k, \varphi, Y)$.

- Suppose that $\varphi = \varphi_1 \wedge \varphi_2$ for some sGF-formulas $\varphi_1$ and $\varphi_2$. Let $d_1, d_2$ and $d$ be the the maximum depths of quantifiers nesting in $\varphi_1$, $\varphi_2$ and $\varphi$, respectively. Let $k \geq d$ and let $Y \supseteq free(\varphi)$. Then, for every $i \in \{1, 2\}$, $k \geq d_i$, $Y \supseteq free(\varphi_i)$ and, by induction hypothesis, there is $\Sigma(k, \varphi_i, Y) \subseteq F(k, Y)$ such that $\models \varphi_i \leftrightarrow \bigvee \Sigma(k, \varphi_i, Y)$. Set,

$$\Sigma(k, \varphi, Y) = \{\psi_1 \wedge \psi_2 : \psi_1 \in \Sigma(k, \varphi_1, Y) \text{ and } \psi_2 \in \Sigma(k, \varphi_2, Y)\}.$$

Therefore, by lemma 2.1.11, $\Sigma(k, \varphi, Y) \subseteq F(k, Y)$ and $\models \varphi \leftrightarrow \bigvee \Sigma(k, \varphi, Y)$. The induction step goes in a similar way for the disjunction $\varphi_1 \vee \varphi_2$.

- Suppose that $\varphi = \neg\psi$ for some sGF-formula $\psi$. Let $d$ be the maximum depths of quantifiers nesting in $\varphi$, then $d$ is the maximum depths of quantifiers nesting in $\psi$ as well. Moreover, we have $free(\varphi) = free(\psi)$. Let $k \geq d$ and let $Y \supseteq free(\varphi)$. By induction hypothesis, there is $\Sigma(k, \psi, Y) \subseteq F(k, Y)$ such that $\models \psi \leftrightarrow \bigvee \Sigma(k, \psi, Y)$. Set, $\Sigma(k, \varphi, Y) = F(k, Y) \setminus \Sigma(k, \psi, Y)$. Now, it is clear that $\models \varphi \leftrightarrow \Sigma(k, \varphi, Y)$.

- Suppose that $\varphi = \exists u(\gamma \wedge \psi)$ for some sGF-formula $\psi$, some sGF-guard $\gamma$ suitable for $\psi$ and some $u \in free(\gamma)$. Let $d$ be the maximum depths of quantifiers nesting in $\varphi$, then $d - 1$ is the maximum depths of quantifiers nesting in $\psi$. Let $X = free(\gamma)$ and let $k \geq d$. Then there is a set of normal forms $\Sigma(k - 1, \psi, X) \subseteq F(k - 1, X)$ such that $\models \psi \leftrightarrow \bigvee \Sigma(k - 1, \psi, X)$. Note that $free(\sigma) \subseteq X = free(\gamma)$ for every $\sigma \in \Sigma(k - 1, \psi, X)$. Hence, $\models \varphi \leftrightarrow \bigvee\{\exists u(\gamma \wedge \sigma) : \sigma \in \Sigma(k - 1, \psi, X)\}$. Let $Y \supseteq free(\varphi) = X \setminus \{u\}$. Set,

$$\Sigma(k, \varphi, Y) = \{\chi \in F(k, Y) : sub_Y^{u,\gamma}(\chi) \cap \Sigma(k - 1, \psi, X) \neq \emptyset\}.$$

  Therefore, $\models \varphi \leftrightarrow \bigvee\{\exists u(\gamma \wedge \sigma) : \sigma \in \Sigma(k - 1, \psi, X)\} \leftrightarrow \Sigma(k, \varphi, Y)$ as desired.

$\square$



**Definition 2.1.12.** Let $\mathfrak{M}$ be any model for $\mathcal{L}$. For any assignment $f \in {}^{\mathcal{V}}M$, any $X \subseteq \mathcal{V}$ and any $k \in \omega$, let $tag_{(k,X)}(\mathfrak{M}, f)$ denote the unique normal form in $F(k, X)$ that is satisfiable at $\mathfrak{M}, f$, i.e., $\mathfrak{M}, f \models tag_{(k,X)}(\mathfrak{M}, f)$. Lemma 2.1.10 ensures the existence and the uniqueness of such normal form.

**Definition 2.1.13.** Let $\mathfrak{M} = (M, \mathcal{R}^{\mathfrak{M}})$ and $\mathfrak{N} = (N, \mathcal{R}^{\mathfrak{N}})$ be two models for $\mathcal{L}$ and suppose that $\pi : M \to N$ is a bijection. We say that $\pi$ is a tag-isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$ if, for any $(R, k) \in \mathcal{R}$ and any $f \in {}^k M$, we have $f \in R^{\mathfrak{M}} \iff \pi \circ f \in R^{\mathfrak{N}}$.

**Lemma 2.1.14.** *Let $\mathfrak{M} = (M, \mathcal{R}^{\mathfrak{M}})$ and $\mathfrak{N} = (N, \mathcal{R}^{\mathfrak{N}})$ be two models for $\mathcal{L}$ and suppose that $\pi : M \to N$ is a tag-isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$. Then for any assignment $f \in {}^{\mathcal{V}} M$, any $X \subseteq \mathcal{V}$ and any $k \in \omega$, we have $tag_{(k,X)}(\mathfrak{M}, f) = tag_{(k,X)}(\mathfrak{N}, \pi \circ f)$.*

*Proof.* Let $\mathfrak{M}$, $\mathfrak{N}$ and $\pi$ be as above. We need to prove the following. For every $X \subseteq \mathcal{V}$, every $k \in \omega$ and every $\varphi \in F(k, X)$,

$$(\forall f \in {}^{\mathcal{V}} M)\ [\mathfrak{M}, f \models \varphi \iff \mathfrak{N}, \pi \circ f \models \varphi]. \tag{2.2}$$

We use induction on the degrees of the normal forms. By the definition of tag-isomorphisms, it is clear that (2.2) holds for any $X \subseteq \mathcal{V}$ and any $\varphi \in F(0, X)$. Let $k \in \omega$ and suppose that (2.2) holds for every $X \subseteq \mathcal{V}$ and every $\varphi \in F(k, X)$. Let $X \subseteq \mathcal{V}$, let $\varphi \in F(k+1, X)$ and let $f \in {}^{\mathcal{V}} M$. By the definition of tag-isomorphisms, we have

$$(\forall \psi \in At(X))\ [\mathfrak{M}, f \models \psi \iff \mathfrak{N}, \pi \circ f \models \psi]. \tag{2.3}$$

Let $u \in \mathcal{V}$, $Y \subseteq X$, $\gamma \in G(Y \cup \{u\})$ and $\psi \in F(k, Y)$. Suppose that $\mathfrak{M}, f \models \exists u(\gamma \wedge \psi)$. Then there exists $f' \in {}^{\mathcal{V}} M$ such that $f' \equiv_u f$ and $\mathfrak{M}, f' \models \gamma \wedge \psi$. Since $\pi$ is bijection, then $\pi \circ f \equiv_u \pi \circ f'$ and, by induction hypothesis, we have $\mathfrak{N}, \pi \circ f' \models \gamma \wedge \psi$. Hence, $\mathfrak{N}, f' \models \exists u(\gamma \wedge \psi)$. Conversely, Suppose that $\mathfrak{N}, \pi \circ f \models \exists u(\gamma \wedge \psi)$. Then there exists $f' \in {}^{\mathcal{V}} N$ such that $f' \equiv_u \pi \circ f$ and $\mathfrak{N}, f' \models \gamma \wedge \psi$. Since $\pi$ is bijection then $\pi^{-1} \circ f' \equiv_u f$, where $\pi^{-1}$ is the inverse of $\pi$. By induction hypothesis, we have $\mathfrak{M}, \pi^{-1} \circ f' \models \gamma \wedge \psi$. Hence, $\mathfrak{M}, f \models \exists u(\gamma \wedge \psi)$. Thus,

$$(\forall \chi \in \exists F(k, X))\ [\mathfrak{M}, f \models \chi \iff \mathfrak{N}, \pi \circ f \models \chi]. \tag{2.4}$$

By (2.4) and (2.3), it follows that $\mathfrak{M}, f \models \varphi \iff \mathfrak{N}, \pi \circ f \models \varphi$. Therefore, we have proved (2.2) as desired. $\square$

From now on, suppose that $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ is as required in theorem 2.1.7. Fix an enumeration $v_0, \dots, v_{n-1}$ of $\mathcal{V}$, where $n = \mid \mathcal{V} \mid$. For the sake of simplicity, we write any assignment of the variables into any model as $f = (f_0, \dots, f_{n-1})$, where $f_i = f(v_i)$ for every $i \in n$. Fix a relation symbol $(R, rank(R)) \in \mathcal{R}$ of rank $rank(R) = n$. Consider the following sGF-formula

$$\vartheta := R(v_0, \dots, v_{n-1}) \bigwedge\{\neg(v_i = v_j) : i, j \in n \text{ and } i \neq j\}.$$

One can see that $\vartheta$ is satisfiable by constructing a model $\mathfrak{M}$ for $\mathcal{L}$ and an assignment $f$ as follows. Let $a_0, \dots, a_{n-1}$ be different numbers, let $f = (a_0, \dots, a_{n-1})$, let $M = Rng(f)$, let $R^{\mathfrak{M}} = \{f\}$ and $S^{\mathfrak{M}} = \emptyset$ for any other relation symbol in $\mathcal{R}$ that is different than $R$. Clearly, $\mathfrak{M}, f \models \vartheta$. We prove that $\vartheta$ cannot be extended to a sGF-complete extension. We still follow the strategy used in § 1.4. We prove the following preliminary lemmas.

**Lemma 2.1.15.** *There are infinitely many satisfiable sGF-formulas $\{\vartheta \wedge \varphi_i : i \in \omega\}$ such that, for every $i, j \in \omega$, if $i \neq j$ then $\models \neg(\vartheta \wedge \varphi_i) \vee \neg(\vartheta \wedge \varphi_j)$.*

*Proof.* Let $\chi := (v_0 = v_1) \wedge \neg\exists v_0((R(v_0, \dots, v_{n-1}) \wedge \vartheta)$, $\psi_0 := \vartheta \wedge \exists v_1(R(v_0, \dots, v_{n-1}) \wedge \chi)$ and $\chi_0 := (v_0 = v_1) \wedge \exists v_0((R(v_0, \dots, v_{n-1}) \wedge \psi_0)$. Inductively, for every finite $i \geq 1$, define $\psi_i := \vartheta \wedge \exists v_1((R(v_0, \dots, v_{n-1}) \wedge \chi_{i-1})$ and $\chi_i := (v_0 = v_1) \wedge \exists v_0((R(v_0, \dots, v_{n-1}) \wedge \psi_i)$. Now, we are ready to give the desired infinitely many sGf-formulas. Define $\varphi_0 := \psi_0$ and, for every finite $k \geq 1$, $\varphi_k := \bigwedge\{\neg\psi_i : i \in k\} \wedge \psi_k$. Clearly, for any $i, j \in n$, if $i \neq j$ then $\models \neg(\vartheta \wedge \varphi_i) \vee \neg(\vartheta \wedge \varphi_j)$.

It remains to prove that the formulas constructed above are satisfiable. Let $a_0, a_1, \dots$ be infinitely many distinct positive numbers and let $b_0, \dots, b_{n-3}$ be a string of distinct negative numbers of length $n-2$ (empty string if $n = 2$). Let $k \in \omega$. Let $f_k := (a_{k+1}, a_k, b_0, \dots, b_{n-3})$, let $g_k := (a_k, a_k, b_0, \dots, b_{n-3})$ and set $M_k = \{a_0, \dots, a_k\} \cup \{b_0, \dots, b_{n-3}\}$. Define the model $\mathfrak{M}_k = (M_k, \mathcal{R}^{\mathfrak{M}_k})$ such that $R^{\mathfrak{M}_k} = \{f_0, g_0, \dots, f_k, g_k\}$ and $S^{\mathfrak{M}_k} = \emptyset$ for every relation symbol $S \in \mathcal{R}$ that is different from $R$. Therefore, $\mathfrak{M}_k, f_k \models \vartheta \wedge \varphi_k$ as desired. □

Now, we prove that there is no normal form in $F(\mathcal{V})$ that is an atom below $\vartheta$ in the Lindenbaum-Tarski sGF-formula algebra. We define zigzags as follows.

**Definition 2.1.16.** Let $\mathfrak{M}$ be any model and let $k \in \omega$ be such that $k \geq 1$. Any $f, g \in {}^{\mathcal{V}}M$ are said to be connected by a zigzag of length $k$ if there exist $\beta \in {}^{k}n$ and $h_0, h_1, \dots, h_{k-1}, h_k \in {}^{\mathcal{V}}M$ such that $f = h_0$, $g = h_k$, $(\forall j \in k)\ h_j \equiv_{v_{\beta(j)}} h_{j+1}$ and $(\forall j \in k+1)\ \mathfrak{M}, h_j \models \vartheta$. In this case we write $f \equiv_\beta g$ and we say that $\beta$ is a zigzag of finite length (of length $k$) connecting $f$ and $g$.

**Definition 2.1.17.** Let $k \in \omega$ be such that $k \geq 1$ and let $\varphi$ be any sGF-formula. Let $\beta \in {}^k n$, we define a sGF-formula $\exists_\beta \varphi$ as follows. Let $\exists_{\beta(0)}\varphi = \exists v_{\beta(0)}(R(v_0, \ldots, v_{n-1}) \wedge \varphi)$. Inductively, for every $i \in k \setminus \{0\}$, let $\exists_{\beta(i)}\varphi = \exists v_{\beta(i)}(R(v_0, \ldots, v_{n-1}) \wedge \exists_{\beta(i-1)}\varphi)$. Finally, we define $\exists_\beta \varphi := \exists_{\beta(k-1)}\varphi$.

**Lemma 2.1.18.** *Let $q \in \omega$ and let $\varphi \in F(q, \mathcal{V})$ be any normal form such that $\vartheta \wedge \varphi$ is satisfiable. Then there exists sGF-formula $\varphi'$ such that both $\vartheta \wedge \varphi \wedge \varphi'$ and $\vartheta \wedge \varphi \wedge \neg\varphi'$ are satisfiable.*

*Proof.* Let $q \in \omega$ and let $\varphi \in F(q, \mathcal{V})$ be any normal form such that $\vartheta \wedge \varphi$ is satisfiable. By [Grä99] and [HO03], there is a finite model $\mathfrak{M}$ and an assignment $f \in {}^{\mathcal{V}}M$ such that $\mathfrak{M}, f \models \vartheta \wedge \varphi$. Since $\mathfrak{M}$ and $\mathcal{V}$ are finite then, by theorem 2.1.11 and lemma 2.1.15, there is $q' \in \omega$ and $\chi \in F(q', \mathcal{V})$ such that $\vartheta \wedge \chi$ is satisfiable and $\mathfrak{M}, g \not\models \chi$ for every $g \in {}^{\mathcal{V}}M$. Hence,

$$(\forall \text{ finite } k \geq 1)\ (\forall \beta \in {}^k n) \quad \mathfrak{M}, f \models \vartheta \wedge \varphi \wedge \neg\exists_\beta \chi. \tag{2.5}$$

Since $\vartheta \wedge \chi$ is satisfiable, then there is a model $\mathfrak{N}$ and an evaluation $g$ such that $\mathfrak{N}, g \models \vartheta \wedge \chi$.

**Step 1:** We construct a sequence of models and a sequence of assignments as follows. Let $\mathfrak{M}_q^0 := \cdots := \mathfrak{M}_q^{n-1} := \mathfrak{M}$ and let $f_q^0 := \cdots := f_q^{n-1} := f$; the assignment for which $\mathfrak{M}, f \models \vartheta \wedge \varphi$. Set $M_q := M_q^0 \cup \cdots \cup M_q^{n-1} = M$ and suppose that $f = f_q^{n-1} = (r_0, \ldots, r_{n-1})$.

Let $U_0, \ldots, U_{n-1}$ be mutually disjoint sets such that each of which is disjoint from $M$ and has the same size of $M \setminus Rng(f_q^{n-1})$. Pick brand new $n$-many different nodes $s_0, \ldots, s_{n-1}$. Let $i \in n$ be arbitrary. Set $M_{q-1}^i := U_i \cup \{s_0, \ldots, s_i\} \cup \{r_{i+1}, \ldots, r_{n-1}\}$ and let $\pi_{q-1}^i : M \to M_{q-1}^i$ be any bijection such that, for all $j \in n$, $\pi_{q-1}^i(r_j) = s_j$ if $j \leq i$ and $\pi_{q-1}^i(r_j) = r_j$ if $j > i$. Define $\mathfrak{M}_{q-1}^i := (M_{q-1}^i, \mathcal{R}^{\mathfrak{M}_{q-1}^i})$, where for every $(S, k) \in \mathcal{R}$,

$$S^{\mathfrak{M}_{q-1}^i} = \{(\pi_{q-1}^i(a_0), \ldots, \pi_{q-1}^i(a_{k-1})) : (a_0, \ldots, a_{k-1}) \in S^{\mathfrak{M}}\}.$$

Let $f_{q-1}^i = (s_0, \ldots, s_i, r_{i+1}, \ldots, r_{n-1})$. We have constructed the models $\mathfrak{M}_{q-1}^0, \ldots, \mathfrak{M}_{q-1}^{n-1}$ and the assignments $f_{q-1}^0, \ldots, f_{q-1}^{n-1}$. By construction, it is easy to see the following. For any $s, t \in \{q, q-1\}$ and any $i, j \in n$: $\mathfrak{M}_s^i, f_s^i \models \vartheta$ and there is a tag-isomorphism $\pi : M_s^i \to M_t^j$ such that $\pi(a) = a$ for every $a \in M_s^i \cap M_t^j$. Set $M_{q-1} := M_q \cup M_{q-1}^0 \cup \cdots M_{q-1}^{n-1}$. We repeat what we did above but for $\mathfrak{M}_{q-1}^{n-1}$ and $f_{q-1}^{n-1}$ in place of $\mathfrak{M}_q^{n-1}$ and $f_q^{n-1}$.

Let $V_0, \ldots, V_{n-1}$ be mutually disjoint sets such that each of which is disjoint from $M_{q-1}$ and has the same size of $M_{q-1}^{n-1} \setminus Rng(f_{q-1}^{n-1})$. Pick brand new $n$-many different nodes $z_0, \dots, z_{n-1}$. Let $i \in n$ be arbitrary. Set $M_{q-2}^i := V_i \cup \{z_0, \dots, z_i\} \cup \{s_{i+1}, \dots, s_{n-1}\}$ and consider any bijection $\pi_{q-2}^i : M_{q-1}^{n-1} \to M_{q-2}^i$ such that, for every $j \in n$, $\pi_{q-1}^i(s_j) = z_j$ if $j \leq i$ and $\pi_{q-1}^i(s_j) = s_j$ if $j > i$. Define $\mathfrak{M}_{q-2}^i := (M_{q-2}^i, \mathcal{R}^{\mathfrak{M}_{q-2}^i})$, where for every $(S, k) \in \mathcal{R}$,

$$S^{\mathfrak{M}_{q-2}^i} = \{(\pi_{q-2}^i(a_0), \dots, \pi_{q-2}^i(a_{k-1})) : (a_0, \dots, a_{k-1}) \in S^{\mathfrak{M}}\}.$$

Let $f_{q-2}^i = (z_0, \dots, z_i, s_{i+1}, \dots, s_{n-1})$. Set $M_{q-2} := M_{q-1} \cup M_{q-2}^0 \cup \cdots M_{q-2}^{n-1}$. We continue in the same way. For every $j \in q$, we construct $\mathfrak{M}_j^0, \dots, \mathfrak{M}_j^{n-1}$ by adding $n$-many copies of $\mathfrak{M}_{j+1}^{n-1}$ by changing the coordinates of $f_{j+1}^{n-1}$ on by one. At the end, we get a sequence of models and a sequence of assignments

$$\mathfrak{M}_q^0, \dots, \mathfrak{M}_q^{n-1}, \dots, \mathfrak{M}_{q-1}^0, \dots, \mathfrak{M}_{q-1}^{n-1}, \dots, \mathfrak{M}_0^0, \dots, \mathfrak{M}_0^{n-1}$$

$$f_q^0, \dots, f_q^{n-1}, \dots, f_{q-1}^0, \dots, f_{q-1}^{n-1}, \dots, f_0^0, \dots, f_0^{n-1}$$

such that, for any $i, j \in n$ and any $s, t \in q+1$, $\mathfrak{M}_s^i, f_s^i \models \vartheta$ and the following holds.

(M) There is a tag-isomorphism $\pi : \mathfrak{M}_s^i \to \mathfrak{M}_t^j$ such that $\pi(a) = a$ for every $a \in M_s^i \cap M_t^j$.

**Step 2:** We repeat what we did for $\mathfrak{M}$ and $f$ but for $\mathfrak{N}$ and $g$ to get a sequence of models and a sequence of assignments

$$\mathfrak{N}_{q'}^0, \dots, \mathfrak{N}_{q'}^{n-1}, \dots, \mathfrak{N}_{q'-1}^0, \dots, \mathfrak{N}_{q'-1}^{n-1}, \dots, \mathfrak{N}_0^0, \dots, \mathfrak{N}_0^{n-1}$$

$$g_{q'}^0, \dots, g_{q'}^{n-1}, \dots, g_{q'-1}^0, \dots, g_{q'-1}^{n-1}, \dots, g_0^0, \dots, g_0^{n-1}$$

such that, for any $i, j \in n$ and any $s, t \in q'+1$, $\mathfrak{N}_s^i, g_s^i \models \vartheta$ and the followings holds.

(N) There is a tag-isomorphism $\pi : \mathfrak{N}_s^i \to \mathfrak{N}_t^j$ such that $\pi(a) = a$ for every $a \in N_s^i \cap N_t^j$.

Without loss of generality, we can assume that $N \cap M = \emptyset$. So we can suppose the following.

(MN) For every $i, j \in n$, every $s \in q+1$ and every $t \in q'+1$, we have $M_s^i \cap N_t^j = \emptyset$.

Now, we connect $f_{n-1}^0$ and $g_{n-1}^0$ by a zigzag and we construct a model that contains this zigzag together with all the models constructed above. For every $i \in n$, let $g_i$ denotes $g_0^{n-1}(v_i)$.

**Step 3:** Let $h_0 = [f_q^{n-1}]_{v_0}^{g_0}$ and, for every $i \in \{1, \ldots, n-1\}$, let $h_i = [h_{i-1}]_{v_i}^{g_i}$. Note that $h_{n-1} = g_{q'}^{n-1}$. Define $\mathfrak{K} = (K, \mathcal{R}^{\mathfrak{K}})$ where $K = \bigcup\{M_s^i, N_t^j : i, j \in n, s \in q+1 \text{ and } t \in q'+1\}$,

$$R^{\mathfrak{K}} = \bigcup\{R^{\mathfrak{M}_s^i}, R^{\mathfrak{N}_t^j} : i, j \in n, s \in q+1 \text{ and } t \in q'+1\} \cup \{h_0, \ldots, h_{n-1}\}$$

and, for every $(S, k) \in \mathcal{R}$ with $R \neq S$, $S^{\mathfrak{K}} = \bigcup\{S^{\mathfrak{M}_s^i}, S^{\mathfrak{N}_t^j} : i, j \in n, s \in q+1 \text{ and } t \in q'+1\}$. By the construction of $\mathfrak{K}$, we have the followings.

(K) For any relational FO-atom $\psi$ and any $h \in {}^{\mathcal{V}}K$, if $\mathfrak{K}, h \models \psi$ then we have one of the followings:

   (a) $h = h_i$ for some $i \in n$.

   (b) $h \in {}^{\mathcal{V}}M_s^i$ for some $i \in n$ and some $s \in q+1$.

   (c) $h \in {}^{\mathcal{V}}N_t^j$ for some $j \in n$ and some $t \in q'+1$.

(KM) For every $s \in q+1$ and every $i \in n$, if $s \geq 1$ then $Rng(h_j) \cap M_s^i = \emptyset$ for every $j \in n$.

(KN) For every $t \in q'+1$ and every $j \in n$, if $t \geq 1$ then $Rng(h_i) \cap N_s^j = \emptyset$ for every $i \in n$.

Now, we prove the following. For every $s \in q+1$ and every $k \leq s$,

$$(\forall i \in n)\ (\forall X \subseteq \mathcal{V})\ (\forall h \in {}^{\mathcal{V}}M_s^i)\ \ \mathfrak{K}, h \models tag_{(k,X)}(\mathfrak{M}_s^i, h). \tag{2.6}$$

We use double induction on $s$ and $k$. By the constructions of $\mathfrak{M}_0^i$'s and $\mathfrak{K}$, it is clear that (2.6) holds for $s = k = 0$. Suppose that (2.6) holds for some $s \in q$ and for all $k \leq s$. We need to show that (2.6) holds for $s+1$ and every $k \leq s+1$. By the constructions of $\mathfrak{M}_{s+1}^i$'s and $\mathfrak{K}$, (2.6) hold for $s+1$ and $k = 0$. Now suppose that (2.6) holds for $s+1$ and some $k \in s$. So, we need to show that (2.6) holds for $s+1$ and $k+1$. Let $i \in n$, let $X \subseteq \mathcal{V}$ and let $h \in {}^{\mathcal{V}}M_s^i$. By the definition of $\mathfrak{K}$, it is clear that

$$(\forall \psi \in At(X))\ \ [\mathfrak{K}, h \models \psi \iff \psi \in color_X(tag_{(k,X)}(\mathfrak{M}_{s+1}^i, h))].$$

Let $\gamma$ be an sGF-guard and let $u \in free(\gamma)$ such that $free(\gamma) \setminus \{u\} \subseteq X$. We need to show the following.

$$(\forall \psi \in F(k, free(\gamma)))\ \ [\mathfrak{K}, h \models \exists u(\gamma \wedge \psi) \iff \psi \in sub_X^{u,\gamma}(tag_{(k,X)}(\mathfrak{M}_{s+1}^i, h))].$$

Let $\psi \in F(k, free(\gamma))$. Suppose that $\psi \in sub_X^{u,\gamma}(tag_{(k,X)}(\mathfrak{M}_{s+1}^i, h))$, then there exists an assignment $h' \in {}^{\mathcal{V}}M_{s+1}^i$ such that $h \equiv_u h'$ and $\mathfrak{M}_{s+1}^i, h' \models \gamma \wedge \psi$. By the constriction of $\mathfrak{K}$, we should have $\mathfrak{K}, h' \models \gamma$ (since $\gamma$ is an atomic formula). Moreover, by induction we should have $\mathfrak{K}, h' \models \psi$. Hence, $\mathfrak{K}, h' \models \gamma \wedge \psi$ and $\mathfrak{K}, h \models \exists u(\gamma \wedge \psi)$. Conversely, Suppose that $\mathfrak{K}, h \models \exists u(\gamma \wedge \psi)$, then there exists $h' \in {}^{\mathcal{V}}K$ such that $h \equiv_u h'$ and $\mathfrak{K}, h \models \gamma \wedge \psi$. Then $Rng(h) \cap Rng(h') \neq \emptyset$. By condition (K) and since $\mathfrak{K}, h' \models \gamma$, we have one of the following cases.

- $h' = h_i$ for some $i \in n$. But $M_{s+1}^i \supseteq Rng(h') \neq \emptyset$, then this contradicts condition (KM).
- $h' \in {}^{\mathcal{V}}N_t^j$ for some $j \in n$ and some $t \in q' + 1$, again this contradicts condition (MN).
- $h' \in {}^{\mathcal{V}}M_t^j$ for some $j \in n$ and some $t \in q + 1$. Then by condition (M), there is a tag-isomorphism $\pi : M_{s+1}^i \to M_t^j$ such that $\pi(a) = a$ for every $a \in Rng(h) \cap Rng(h')$. Then by lemma 2.1.14, there is $\bar{h} \in M_{s+1}^i$ such that $tag_{(k,X)}(M_{s+1}^i, h') = tag_{(k,X)}(M_t^j, \bar{h})$. Thus, $\mathfrak{M}_{s+1}^i, \bar{h} \models \gamma \wedge \psi$. Hence, $\mathfrak{M}_{s+1}^i, h \models \exists u(\gamma \wedge \psi)$ as desired.

Therefore, we have proved (2.6). Similarly, one can prove the following. For every $t \in q' + 1$ and every $k \leq t$,

$$(\forall j \in n)\ (\forall X \subseteq \mathcal{V})\ (\forall h \in {}^{\mathcal{V}}N_t^j)\ \ \mathfrak{K}, h \models tag_{(k,X)}(\mathfrak{N}_t^j, h). \tag{2.7}$$

**Step 4:** By (2.6) and (2.7), we have $\mathfrak{K}, f_q^0 \models \vartheta \wedge \varphi$ and $\mathfrak{K}, g_{q'}^0 \models \vartheta \wedge \chi$. Moreover, the assignments $f_q^1, \ldots, f_q^{n-1}, \ldots, f_0^0, \ldots, f_0^{n-1}, h_0, \ldots, h_{n-2}, g_0^{n-1}, \ldots, g_0^0, \ldots, g_{q'}^{n-1}, g_{q'}^1$ form a zigzag between $f_q^0$ and $g_{q'}^0$ in $\mathfrak{K}$. Thus, there exist finite $k \geq 1$ and $\alpha \in {}^k n$ such that

$$\mathfrak{K}, f_q^0 \models \vartheta \wedge \varphi \wedge \exists_\alpha \chi. \tag{2.8}$$

Put $\varphi' := \exists_\alpha \chi$. Therefore, by (2.5) and (2.8), both $\vartheta \wedge \varphi \wedge \varphi'$ and $\vartheta \wedge \varphi \wedge \neg\varphi'$ are satisfiable sGF-formulas as desired. □

*Proof of theorem 2.1.7.* Recall the sGF-formula $\vartheta$ define in page 66. We prove that $\vartheta$ has no sGF-extension. Let $\psi$ be any sGF-formula such that $\vartheta \wedge \psi$ is satisfiable. We prove that $\vartheta \wedge \psi$ is incomplete, i.e., we find a sGF-formula $\chi$ such that $\vartheta \wedge \psi \not\models \chi$ and $\vartheta \wedge \psi \not\models \neg\chi$. By theorem

2.1.11, there is some $k \in \omega$ and some $\Sigma(k, \vartheta \wedge \psi, \mathcal{V}) \subseteq F(k, \mathcal{V})$ such that

$$\models \vartheta \wedge \psi \leftrightarrow \bigvee \Sigma(k, \vartheta \wedge \psi, \mathcal{V}).$$

Suppose that there are $\varphi_1, \varphi_2 \in \Sigma(k, \vartheta \wedge \psi, \mathcal{V})$ such that both $\varphi_1$ and $\varphi_2$ are satisfiable. Therefore, by lemma 2.1.10, we have $\vartheta \wedge \psi \not\models \varphi_1$ and $\vartheta \wedge \psi \not\models \neg\varphi_1$ as desired.

Suppose that there is only one satisfiable $\varphi \in \Sigma(k, \vartheta \wedge \psi, \mathcal{V})$. Then, $\models \vartheta \wedge \psi \leftrightarrow \vartheta \wedge \varphi$. By lemma 2.1.18, there is sGF-formula $\varphi'$ such that both $\vartheta \wedge \varphi \wedge \varphi'$ and $\vartheta \wedge \varphi \wedge \neg\varphi'$ are satisfiable and $\models \vartheta \wedge \psi \leftrightarrow \vartheta \wedge \varphi \leftrightarrow (\vartheta \wedge \varphi \wedge \varphi') \vee (\vartheta \wedge \varphi \wedge \neg\varphi')$. Therefore, we have $\vartheta \wedge \psi \not\models \vartheta \wedge \varphi \wedge \varphi'$ and $\vartheta \wedge \psi \not\models \neg(\vartheta \wedge \varphi \wedge \varphi')$ as desired. $\square$

Let $\mathcal{L}$ be a as required in theorem 2.1.7. We have shown that GF on $\mathcal{L}$ do have neither Gödel's incompleteness property nor weak Gödel's incompleteness property. However, the solo-fragment of GF on $\mathcal{L}$ does have weak Gödel's incompleteness property. On the language $\mathcal{L}$, we have a chain $FO \supseteq GF \supseteq sGF$ of logics in which weak Gödel's incompleteness property alternates.

### 2.1.3 wGIP holds for both GF and solo-GF on infinite langauges

Now, we show that wGIP holds for GF and sGF on real infinite languages. This because any formula is built up from finitely many atomic formulas, but if the language is infinite then there are infinitely many atomic formulas, thus any formula can be refined by some atom that does not appear in it.

**Theorem 2.1.19.** *Let* $\mathbf{L} \in \{GF, sGF\}$ *and let* $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ *be an infinite language. Then there is a satisfiable* $\mathbf{L}$*-formula that has no* $\mathbf{L}$*-complete extension.*

*Proof.* Let $\mathcal{R} = \langle \mathcal{R}, \mathcal{V} \rangle$ be an infinite language. Suppose that $\mathcal{R} = \{(R_j, rank(R_j)) : j \in m\}$ and $\mathcal{V} = \{v_i : i \in n\}$ for some ordinals $n \geq 2, m \geq 0$. For every $\mathbf{L}$-formula $\varphi$, let $index(\varphi)$ be the set of all $i \in n$ such that $v_i$ appears in $\varphi$ and let $rel(\varphi)$ be the set of all relation symbols that appear in $\varphi$. Hence, both $index(\varphi)$ and $rel(\varphi)$ are finite.

- Suppose that $n$ is infinite. We prove that the $\mathbf{L}$-formula $\vartheta := \neg(v_0 = v_1)$ has no complete extensions. Let $\varphi$ be any $\mathbf{L}$-formula such that $\vartheta \wedge \varphi$ is satisfiable, we need to show that $\vartheta \wedge \varphi$ not complete. There exist a model $\mathfrak{M}$ and an evaluation $f \in {}^{\mathcal{V}}M$ such that $\mathfrak{M}, f \models \vartheta \wedge \varphi$. For any two evaluations $g, h \in {}^{\mathcal{V}}M$ and any $\gamma \subseteq n$, we write $g \equiv_{\Gamma} h$ if and only if $g(v_k) = h(v_k)$ for every $k \in n \setminus \Gamma$. Let $\Gamma := index(\vartheta \wedge \varphi)$. The following can be easily checked by an induction argument on the complexity of the **L**-formulas: For every $g, h \in {}^{\mathcal{V}}M$ and every **L**-formula $\psi$,

$$[index(\psi) \subseteq \Gamma \;\&\; g \equiv_{n \setminus \Gamma} h] \implies [\mathfrak{M}, g \models \psi \iff \mathfrak{M}, h \models \psi] \tag{2.9}$$

$\Gamma$ is finite subset of $n$, fix $i, j \in n \setminus \Gamma$ such that $i \neq j$. Let $g \in {}^{\mathcal{V}}M$ be such that $g \equiv_{n \setminus \Gamma} f$ and $g(i) = g(j) \iff f(i) \neq f(j)$. Such $g$ exists because $f(0) \neq f(1)$ and $f(0), f(1) \in M$. Then, by (2.9), we have $\mathfrak{M}, g \models \vartheta \wedge \varphi$ and

$$\mathfrak{M}, g \models (v_i = v_j) \iff \mathfrak{M}, f \models \neg(v_i = v_j).$$

Hence, both $\varphi_1 := \vartheta \wedge \varphi \wedge (v_i = v_j)$ and $\varphi_2 := \vartheta \wedge \varphi \wedge \neg(v_i = v_j)$ are satisfiable. Therefore, $\vartheta \wedge \varphi \not\models \varphi_1$ and $\vartheta \wedge \varphi \not\models \varphi_2$ as desired.

- Suppose that $m$ is infinite. We prove that no **L**-formula on $\mathcal{L}$ has complete extension. Let $\varphi, \psi$ be any **L**-formulas such that $\varphi \wedge \psi$ is satisfiable. Then there is a model $\mathfrak{M}$ and an assignment $f \in {}^{\mathcal{V}}M$ such that $\mathfrak{M}, f \models \varphi \wedge \psi$. Let $(R, k) \in \mathcal{R}$ be such that $k \geq 1$ and $R \notin rel(\varphi \wedge \psi)$. Define the model $\mathfrak{N} = (N, \mathcal{R}^{\mathfrak{N}})$ as follows: $N = M$, $S^{\mathfrak{N}} = S^{\mathfrak{M}}$ for every $S \neq R$ and

$$(f(0), \ldots, f(k-1)) \in R^{\mathfrak{M}} \implies R^{\mathfrak{N}} = R^{\mathfrak{M}} \setminus \{(f(0), \ldots, f(k-1))\}$$
$$(f(0), \ldots, f(k-1)) \notin R^{\mathfrak{M}} \implies R^{\mathfrak{N}} = R^{\mathfrak{M}} \cup \{(f(0), \ldots, f(k-1))\}.$$

By induction on the complexity of **L**-formulas, one can prove the following. For every $g \in \mathcal{V}N$ and for every **L**-formula $\chi$,

$$rel(\chi) \subseteq rel(\varphi \wedge \psi) \implies [\mathfrak{M}, g \models \chi \iff \mathfrak{N}, g \models \chi]. \tag{2.10}$$

Hence, in particular, $\mathfrak{N}, f \models \varphi \wedge \psi$. By the construction of $\mathfrak{N}$, we have

$$\mathfrak{N}, f \models R(v_0, \ldots, v_{k-1}) \iff \mathfrak{M}, f \models \neg R(v_0, \ldots, v_{k-1}).$$

Therefore, $\varphi \wedge \psi \not\models R(v_0, \ldots, v_{k-1})$ and $\varphi \wedge \psi \not\models \neg R(v_0, \ldots, v_{k-1})$ as desired.

$\square$

We note that definition 2.1.1, contrary to [AvBN98], allows for equational atoms as guards. This issue does not affect any of the results presented above, i.e., same statements eventually hold for guarded fragment defined in [AvBN98]. This is also place us in line with [Grä99], [HM02] and [Hoo01].

The loosely guarded fragment [vB97] and the packed fragment [Mar01] are more liberal versions of guarded fragment in the sense that the guards can be conjunctions of some special formulas.

**Definition 2.1.20** (Loosely guarded fragment (LGF))**.** Let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be a language. The set of *LGF-formulas* on $\mathcal{L}$ is defined analogously to the GF-formulas by relaxing clause (c) as follows:

(c') Let $\bar{v}$ be a finite tuple of variables, $\varphi$ be a LGF-formula on $\mathcal{L}$ and $\gamma$ be a conjunction of relational FO-atoms on $\mathcal{L}$ such that $free(\varphi) \cup \bar{v} \subseteq free(\gamma)$ and, for any $v \in \bar{v}$ and any $u \in free(\gamma)$, there exists a conjunct in $\gamma$ in which both $u$ and $v$ occur free. Then $\exists \bar{v}(\gamma, \varphi)$ is a $LGF$-formula on $\mathcal{L}$ and such $\gamma$ is called a *suitable LGF-guard for* $\varphi$.

**Definition 2.1.21** (Packed fragment (PF))**.** Let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be a language. The set of *PF-formulas* on $\mathcal{L}$ is defined analogously to the GF-formulas by relaxing clause (c) as follows:

(c”) Let $\gamma$ be a conjunction of FO-formulas on $\mathcal{L}$ each of which is either a FO-atom or of the form $\exists \bar{u} R(\bar{u})$ (where $R(\bar{u})$ is a relational FO-atom). Let $\varphi$ be a PF-formula on $\mathcal{L}$ such that $free(\varphi) \subseteq free(\gamma)$ and, for every $u, v \in free(\gamma)$, there is a conjunct in $\gamma$ in which $u$ and $v$ booth occur free. Then, for any finite tuple $\bar{v} \subseteq free(\gamma)$, $\exists \bar{v}(\gamma, \phi)$ is a PF-formula on $\mathcal{L}$ and such $\gamma$ is called a *suitable $PF$-guard for* $\phi$.

On the one hand, PF is more liberal. The FO-formulas of the form $\exists \bar{u} R(\bar{u})$ and the equational FO-atoms may occur in the PF-guards but not in the LGF-guards. On the other hand, the loosely guarded fragment is more liberal as in a LGF-formula $\exists \bar{v}(\gamma, \psi)$ a pair of variables from $free(\gamma) \setminus \bar{v}$ doesn't need to occur free in one of the conjuncts of $\gamma$.

In [HO03], it was shown that LGF and PF have the finite model property and by definition both contain polyadic-quantifiers. Thus the proof of theorem 2.1.2 can be applied verbatim to show that both wGIP and GIP fail for LGF and PF on finite languages. If we define the solo-quantifiers fragment sLGF of the loosely guarded fragment in the very similar way to sGF, i.e., by allowing bounded solo-quantifiers only, then one can easily see that the method used in § 2.1.2 works to proving theorem 2.1.7 for sLGF in place of sGF. This is not the case for the packed fragment. Indeed the PF-guards may contain polyadic-quantifiers in their conjuncts, so (2.6) and (2.7) in the proof of lemma 2.1.18 are no longer true in this case. But, if we define the solo-fragment sPF of the packed fragment such that we allow only solo-quantifiers even in the conjuncts of the PF-guards then theorem 2.1.7 is true for sPF in place of sGF for the languages that have at least three variables. It is easy to see that the proof of theorem 2.1.19 works to show that LGF, sLGF, PF and sPF do have wGIP on infinite languages.

## §2.2 FO with $K$-general assignment models

The guarded fragment is closely connected to first order logic with general assignment models introduced by I. Németi [Ném86]. The general assignments present intermediate semantics that take away the existential assumption of "fullness" from the standard models for first order logic. The algebraic counterparts of FO with general assignment models are the classes of relativized cylindric algebras. Therefore, the nice properties of these classes imply several nice properties for first order logic with general assignment models.



**Definition 2.2.1** (FO with $K$-general assignment models (GAM($K$))). Let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be any language and let $K \in \{Crs, D, P, G\}$. Suppose that $\mathcal{V} = \{v_i : i \in n\}$ for some ordinal $n$. First order logic with $K$-general assignment models (GAM($K$)) on $\mathcal{L}$ is defined as follows. The GAM($K$)-formulas are the usual FO-formulas. A GAM($K$)-model for $\mathcal{L}$ is an ordered pair $(\mathfrak{M}, V)$ such that $\mathfrak{M} = (M, \mathcal{R}^{\mathfrak{M}})$ is a standard FO-model for $\mathcal{L}$ and $\emptyset \neq V \subseteq {}^{\mathcal{V}}M$ is a concrete $K_n$-atom structure. The GAM($K$)-formulas on $\mathcal{L}$ are interpreted as usual, now at triples $\mathfrak{M}, V, f$ with $f \in V$-with the following clause for quantifiers:

$$\mathfrak{M}, V, f \models \exists u\varphi \text{ iff for some } g \in V\text{: } g \equiv_u f \text{ and } \mathfrak{M}, V, g \models \varphi,$$

where $g \equiv_u f$ means that, for every $v \in \mathcal{V}$, if $v$ is different than $u$ then $g(v) = f(v)$.

Fix $K \in \{Crs, D, P, G\}$ and let $\mathcal{L}$ be any language. Suppose that $\mathcal{L}$ contains exactly $n$-many variables, for some ordinal $n \geq 2$. The Lindenbaum-Tarski formula algebras of first order logic with GAM($K$) models are very close to the free $K$-algebras, we sketch this connection in the paragraph below. We investigated the atomicity of the free $K_n$-algebras in the previous chapter. In this section we use the results of the previous chapter to investigate wGIP for first order logic with $K$-general models GAM($K$).

We now turn to sketching the connection between formula algebras and free algebras. Let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be a language, let $n = |\mathcal{V}|$ and let $\mathfrak{F}(\mathcal{L}, K)$ denote the Lindenbaum-Tarski formula algebra of first order logic with GAM($K$)-semantics for this language as defined in definition 2.1.5. Let $R(\mathcal{L})$ denote the set of relational atomic formulas of $\mathcal{L}$ and let $m = |R(\mathcal{L})|$. There is a strong connection between $\mathfrak{F}(\mathcal{L}, K)$ and $\mathfrak{Fr}_m K_n$ as follows. By the definition of satisfiability, the function assigning $\varphi^{\mathfrak{M},V} = \{f \in V : \mathfrak{M}, V, f \models \varphi\}$, the meaning of $\varphi$ in $(\mathfrak{M}, V)$, to $\varphi$ is a homomorphism from $\mathfrak{F}(\mathcal{L}, K)$ to the complex algebra $\mathfrak{Cm}V$ over $V$. Not every homomorphism from $\mathfrak{F}(\mathcal{L}, K)$ to $\mathfrak{Cm}V$ is of this form, though, because the meanings of the atomic formulas $R(v_1, \ldots, v_k)$ have to be $k$-regular in the sense that they do not distinguish sequences that agree on the first $k$ indices, and there are dependencies between the meanings of the atomic formulas, e.g., the meaning of $R(v_1, v_0)$ is determined by that of $R(v_0, v_1)$. Thus, the formula algebra $\mathfrak{F}(\mathcal{L}, K)$ is a homomorphic image of $\mathfrak{Fr}_m K_n$, but not necessarily isomorphic to it. In the literature, investigating regular $K$-algebras and so-called restricted languages is used to fill this gap, see e.g., [HMT85, sec.4.3], [**?**], [**?**]. In this section, instead of using only the statements of the precise bridge-theorems of algebraic logic for formula algebras and free algebras, we sketched above informally the connection between formula algebras and free algebras in order to illuminate why the proofs of atomicity/non-atomicity of the free algebras $\mathfrak{Fr}_m K_n$ can almost always be adapted in a more-or-less straightforward way to show atomicity/non-atomicity of $\mathfrak{F}(\mathcal{L}, K)$.

For any GAM($K$)-formula $\varphi$ on $\mathcal{L}$, we say that $\varphi$ is GAM($K$)-satisfiable if and only if there is GAM($K$)-model $(\mathfrak{M}, V)$ such that $\mathfrak{M}, V, f \models \varphi$ for some $f \in V$. For any GAM($K$)-formulas $\varphi, \psi$ on $\mathcal{L}$, we say that $\psi$ is a *GAM($K$)-consequence* of $\varphi$, in symbols $\varphi \models \psi$, if and only if $\mathfrak{M}, V, f \models \varphi \to \psi$, for every GAM($K$)-model $(\mathfrak{M}, V)$ and every assignment $f \in V$. For any GAM($K$)-formula $\varphi$, we say that $\varphi$ has a *GAM($K$)-complete extension* if there is GAM($K$)-formula $\psi$ such that $\varphi \wedge \psi$ is GAM($K$)-satisfiable and, for any GAM($K$)-formula $\chi$, either $\varphi \wedge \psi \models \chi$ or $\varphi \wedge \psi \models \neg\chi$.

**Theorem 2.2.2.** *Let $K \in \{Crs, D, P, G\}$ and let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be any language. Let $n \geq 2$ be the number of the variables in $\mathcal{V}$ and let $m \geq 0$ be the number of the relation symbols in $\mathcal{L}$. Then,*

- *Suppose that both $n$ and $m$ are finite. If $n \geq 3$, $m \geq 1$, or, $n = 2$, $m = 0$ and $K \neq G$, then there is GAM($K$)-satisfiable formula $\varphi$ on $\mathcal{L}$ that has no GAM($K$)-complete extension. Otherwise, each GAM($K$)-satisfiable formula has a GAM($K$)-complete extension.*
- *Suppose that $m$ is infinite. Then there is no GAM($K$)-satisfiable formula $\varphi$ on $\mathcal{L}$ that has GAM($K$)-complete extension.*
- *Suppose that $n$ is infinite. If $K = Crs$, or, $m \geq 1$ and $K \neq P$ then there is GAM($K$)-satisfiable formula $\varphi$ on $\mathcal{L}$ that has no GAM($K$)-complete extension. Otherwise, we do not know whether wGIP holds or not.*

*Proof.* Let $K$, $\mathcal{L}$, $n$ and $m$ be as above. One can prove the above statements by repeating the algebraic proofs of theorem 1.0.2 (we only need the fact if $n$ is finite then GAM($K$) on $\mathcal{L}$ has the finite model property [AHN99]), theorem B.0.5 and theorem B.0.6 in the present logical setting, analogously as we did for sGF in the previous section. $\square$



### 2.2.1 Polyadic FO with $K$-general assignment models

Let $K \in \{Crs, D, P, G\}$. The polyadic version of first order logic with $K$-general assignment models (pGAM($K$)) is defined by allowing quantifiers of the form $\exists\bar{v}\varphi$, for any tuple of variables $\bar{v}$, together with the following clause: At a pGAM($K$)-model $(\mathfrak{M}, V)$ (that is a GAM($K$)-model) and an assignment $f \in V$,

$$\mathfrak{M}, V, f \models \exists\bar{v}\varphi \text{ iff for some } g \in V\text{: } g \equiv_{\bar{v}} f \text{ and } \mathfrak{M}, V, g \models \varphi$$

where $g \equiv_{\bar{v}} f$ means that $g(v) = f(v)$ for every variable $v \in \mathcal{V} \setminus \bar{v}$. To investigate wGIP for the pGAM($K$), we need to investigate the atomicity of its Lindenbaum-Tarski formula algebras. We do this by investigating the atomicity of the free algebras of the algebraic classes that correspond to the polyadic version of FO with $K$-general assignment models.

**Polyadic relativized cylindric set algebras.** Let $n \geq 2$ be any ordinal and let $V \subseteq {}^nU$ for some set $U$. Let $\Gamma \subseteq_\omega n$ and let $X \subseteq V$, define

$$C_{(\Gamma)}X = \{f \in V : (\exists g \in X)(\forall i \in n \setminus \Gamma) f(i) = g(i)\}.$$

Define the polyadic complex algebra $\mathfrak{Cm}^{\text{poly}}V$ over $V$ to be the structure of the form

$$\mathfrak{Cm}^{\text{poly}}V = \langle \mathcal{P}(V), \cap, \cup, \setminus, \emptyset, V, C_{(\Gamma)}, D_{ij} \rangle_{i,j \in n, \Gamma \subseteq_\omega n},$$

where $\cup, \cap, \setminus$ are the Boolean set-theoretic operations, $D_{ij}$ ($i, j \in n$) is as given in chapter 1 and $C_{(\Gamma)}$ ($\Gamma \subseteq_\omega n$) is as defined above.

**Definition 2.2.3.** Let $n \geq 2$ be any ordinal and let $K \in \{Crs, D, P, G\}$. The class $K_n^{\text{poly}}$ is defined to be the class of all subalgebras of the polyadic complex algebras over concrete $K_n$-atom structures, i.e., $K_n^{\text{poly}} = \mathbf{S}\{\mathfrak{Cm}^{\text{poly}}V : V \text{ is a concrete } K_n\text{-atom structure}\}$.

In [AHN99], it was shown that the same equations hold in $K_n^{\text{poly}}$ as in the finite members of this class, for finite $n \geq 2$. We use this fact together with [AN16, theorem 2] to investigate the atomicity of the free algebra of the class $K_n^{\text{poly}}$.

**Theorem 2.2.4.** *Let $n \geq 2$ and $m \geq 0$ be any finite ordinals and let $K \in \{Crs, D, P, G\}$. The free algebra $\mathfrak{Fr}_m K_n^{poly}$ is atomic.*

*Proof.* We show that $K_n^{\text{poly}}$ is a discriminator class, i.e., we show that there is a term $\tau(x)$ such that $K_n^{\text{poly}} \models \forall x[x \neq 0 \leftrightarrow \tau(x) = 1]$. Indeed, let $V$ be a concrete $K_n$-atom structure and let $A \subseteq V$, one can easily see that $\mathfrak{Cm}^{poly}V \models A \neq \emptyset \iff \mathfrak{Cm}^{poly}V \models C_{(n)}A = 1$. Thus, the following term $\tau(x) := c_{(n)}(x)$ will do in $K_n^{\text{poly}}$. Moreover, the same equations hold in $K_n^{\text{poly}}$ as in the finite members of this class as it was shown in [AHN99]. Therefore the variety generated by $K_n^{\text{poly}}$ is a discriminator variety generated by its finite members. In [AN16, theorem 2], it is shown that any class of algebras with Boolean reducts has atomic finitely generated free algebras if it is both discriminator and generated by its finite members. Therefore, $\mathfrak{Fr}_m K_n^{\text{poly}}$ is atomic as desired. $\square$

**wGIP for the polyadic version pGAM($K$).** Let $\mathcal{L}$ be any language. The notions of pGAM($K$)-satisfiability, pGAM($K$)-validity, pGAM($K$)-consequences, etc, are defined as expected. For any pGAM($K$)-formula $\varphi$, we say that $\varphi$ has a *pGAM($K$)-complete extension* if there is pGAM($K$)-formula $\psi$ such that $\varphi \wedge \psi$ is pGAM($K$)-satisfiable and, for any pGAM($K$)-formula $\chi$, either $\varphi \wedge \psi \models \chi$ or $\varphi \wedge \psi \models \neg\chi$. The same argument that is used in the proof of theorem 2.2.2 can be used again to prove the following. We note that the proofs of theorems B.0.5 and B.0.6 work if we replace each occurrence of $K$ by $K^{poly}$ in them.

**Theorem 2.2.5.** *Let $K \in \{Crs, D, P, G\}$ and let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be any language. Let $n \geq 2$ be the number of the variables in $\mathcal{V}$ and let $m \geq 0$ be the number of the relation symbols in $\mathcal{L}$. Suppose that both $n$ and $m$ are finite. Then, each pGAM($K$)-satisfiable formula has a pGAM($K$)-complete extension. Suppose that $m$ is infinite. Then no pGAM($K$)-satisfiable formula $\varphi$ on $\mathcal{L}$ has pGAM($K$)-complete extension. Suppose that $n$ is infinite. If $K = Crs$, or, $m \geq 1$ and $K \neq P$. Then there is pGAM($K$)-satisfiable formula $\varphi$ on $\mathcal{L}$ that has no pGAM($K$)-complete extension.*

**Problem 1.** *Let $\mathcal{L} = \langle \mathcal{R}, \mathcal{V} \rangle$ be any language. Let $n \geq \omega$ be the number of the variables in $\mathcal{V}$ and let $m \geq 0$ be the number of the relation symbols in $\mathcal{L}$. Does GAM($P$) or pGAM($P$) have wGIP? Suppose that $m = 0$, does any of GAM(D), GAM(G), pGAM(D) or pGAM(G) have wGIP?*



## §2.3 GIP fails for any of the above logics

Let $\mathcal{L}$ be any countable language and suppose that $n \geq 2$ is the number of variables in $\mathcal{L}$. Let $\mathbf{L} \in \{GF, sGF, LGF, sLGF, PF, sPF, GAM(K), pGAM(K)\}$, where $K \in \{Crs, D, P, G\}$ if $n$ is finite and $K \in \{Crs, G\}$ if $n$ is infinite. Here, we show that Gödel's incompleteness property fails for $\mathbf{L}$ because of the finite model property, cf. [Grä99], [AHN99], [Ném86, lemma 10.10], [Ném95, lemma 4.2] and [HO03], as we will see below.

For the sake of unification, we say that $\varphi$ is $\mathbf{L}$-satisfiable formula to mean that $\varphi$ is $\mathbf{L}$-formula and it is satisfiable with respect to the semantics of the logic $\mathbf{L}$. Similarly for all the other notions, e.g., $\mathbf{L}$-models and $\mathbf{L}$-assignments. Let $\Sigma$ be a set of $\mathbf{L}$-formulas. We say that $\Sigma$ is *recursively enumerable* if $\Sigma$ is countable and there is an algorithm that correctly decides when a $\mathbf{L}$-formula is in the set $\Sigma$; the algorithm may give no answer (but not the wrong answer) for $\mathbf{L}$-formulas not in $\Sigma$. We say that $\Sigma$ is *consistent* if $\Sigma \not\models \varphi \wedge \neg\varphi$ for any $\mathbf{L}$-formula $\varphi$, where $\Sigma \models \varphi$ means that $\mathfrak{M}, V, f \models \psi$ for all $\psi \in \Sigma$ implies $\mathfrak{M}, V, f \models \varphi$, for all $\mathfrak{M}, V, f$. We say that $\Sigma$ is *complete* if for every $\mathbf{L}$-formula $\varphi$ we have that either $\Sigma \models \varphi$ or $\Sigma \models \neg\varphi$.

**Theorem 2.3.1.** *Let $\mathcal{L}$ be any countable language and let $n \geq 2$ be the number of variables in $\mathcal{L}$. Let $\mathbf{L} \in \{GF, sGF, LGF, sLGF, PF, sPF, GAM(K), pGAM(K)\}$, where $K \in \{Crs, D, P, G\}$ if $n$ is finite and $K \in \{Crs, G\}$ if $n$ is infinite. For every $\mathbf{L}$-satisfiable formula $\varphi$ on $\mathcal{L}$, there is a set $\Sigma$ of $\mathbf{L}$-formulas on $\mathcal{L}$ such that $\Sigma$ is recursively enumerable, $\varphi \in \Sigma$, $\Sigma$ is consistent and $\Sigma$ is complete.*

*Proof.* Let $\varphi$ be any $\mathbf{L}$-satisfiable formula. By [Grä99], [Ném95, lemma 4.2], [AHN99] and [HO03], there is a finite $\mathbf{L}$-model $\mathfrak{M}$ and an $\mathbf{L}$-assignment $f$ such that $\mathfrak{M}, f \models \varphi$. Set,

$$\Sigma := \{\psi : \psi \text{ is } \mathbf{L}\text{-formula and } \mathfrak{M}, f \models \psi\}.$$

Clearly, $\Sigma$ is consistent, complete and contains $\varphi$. it remains to prove that $\Sigma$ is recursively enumerable. The set of all $\mathbf{L}$-formulas on $\mathcal{L}$ is countable (because the set of all FO-formulas is countable), hence $\Sigma$ is countable. Now, we describe an algorithm that recursively enumerates the $\mathbf{L}$-formulas in $\Sigma$.

- Remember that $\mathfrak{M}$ is finite, so for every $R \in \mathcal{R}$ the interpretation $R^{\mathfrak{M}}$ is finite (because its rank is finite). Thus, there is an algorithm **SATATOMS** that takes any $R \in \mathcal{R}$ with rank $k$ and any $a_0, \ldots, a_{k-1} \in Rng(f)$ then tells YES if $(a_0, \ldots, a_{k-1}) \in R^{\mathfrak{M}}$ and tells NO otherwise.

- The set of the $\mathbf{L}$-formulas is defined recursively such that all these formulas are built up from the FO-atoms. So, we can define an algorithm **FINDATOMS** that takes any

**L**-formula $\varphi$ as an input and gives as an output the set of all the atomic formulas that the formula $\varphi$ is built up from.

- Combining the above algorithm, we get an algorithm **SAT** that decides for every **L**-formula $\varphi$ whether $\mathfrak{M}, f \models \varphi$ or not.

Therefore, the set $\Sigma$ is recursively enumerable (in fact, it is decidable) as desired. $\square$

We do not know whether GAM($K$), for $K \in \{D, P\}$, has GIP on languages that have countably infinite many variables or not, because we even do not whether GAM($K$) on such languages is decidable or not. We note that it was accurate to name the property wGIP as a weaker version of GIP. By definition, it is clear that wGIP is weaker than GIP but there was no known example until now showing that wGIP is strictly weaker than GIP. Our results in this chapter show that wGIP is strictly weaker than GIP, indeed, we showed, for instance, GF on infinite languages and sGF on finite languages are such examples that differentiate wGIP and GIP.

# Appendices

# A

APPENDIX

# Disjunctive normal form for BAO's

Here, we give disjunctive normal forms for any class of Boolean algebras with operators. Let $I$ and $J$ be any two index sets and let $t = \{+, \cdot, -, 0, 1, f_i, d_j : i \in I \text{ and } j \in J\}$ be a similarity type such that $\{+, \cdot, -, 0, 1\}$ is the type of Boolean algebras, $f_i$ is an operator symbol of positive rank $rank(f_i) \geq 1$ (for any $i \in I$) and $d_j$ is a constant symbol (for any $j \in J$). Let $K$ be the class of all Boolean algebras with operators of type $t$, that is the class of all algebras of type $t$, $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_i, d_j : i \in I, j \in J\rangle$, that satisfy the following conditions:

K0 The Boolean part $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra.

K1 The operators of positive ranks are additive, i.e., for any $i \in I$, any $k \in n_i := rank(f_i)$ and any $a_0, \ldots, a_{k-1}, a, b, a_{k+1}, \ldots, a_{n_i-1}$,

$$\begin{aligned} f_i(a_0, \ldots, a_{k-1}, a+b, a_{k+1}, \ldots, a_{n_i-1}) &= f_i(a_0, \ldots, a_{k-1}, a, a_{k+1}, \ldots, a_{n_i-1}) \\ &+ f_i(a_0, \ldots, a_{k-1}, b, a_{k+1}, \ldots, a_{n_i-1}). \end{aligned}$$

Fix a finite number $m \in \omega$. Let $\mathfrak{Tm}_{m,t}$ denote the term algebra of type $t$ generated by $m$-many free variables. For every $n \in \omega$, we define a finite set of terms $F_n^{m,t} \subseteq \mathfrak{Tm}_{m,t}$ and we call it the set of normal forms of degree $n$. Every normal form is defined as a term that is built up from from the normal forms of the first smaller degree. We note that the normal forms we define here are similar to the ones introduced by Kit Fine in [Fin75]

Let $\prod$ and $\sum$ be the grouped versions of $\cdot$ and $+$, respectively. That is, for any set of terms $Y \subseteq \mathfrak{Tm}_{m,t}$, we fix any enumeration of $Y$ and then we define $\prod Y$ and $\sum Y$ inductively according to this enumeration. The algebras in the class $K$ are Boolean algebras with operators, hence $\cdot$ and $+$ are both commutative on the elements of these algebras. Thus, it doesn't matter which enumeration we use to define $\prod$ and $\sum$, all are equivalent terms in the class $K$.

**Definition A.0.2.** Let $Y \subseteq \mathfrak{Tm}_{m,t}$ be finite and let $\alpha \in {}^{Y}\{-1,1\}$. Define

1. $CY = \{f_i\tau : \tau \in Y, i \in I\}$, the one-step closure of $Y$ by the operations $\langle f_i : i \in I\rangle$.
2. $Y^\alpha = \prod\{\tau^\alpha : \tau \in Y\}$, where for every $\tau \in Y$, $\tau^\alpha = \tau$ if $\alpha(\tau) = 1$ and $\tau^\alpha = -\tau$ otherwise.

**Definition A.0.3.** Let $D_m = \{d_j : j \in J\} \cup \{x_0, \ldots, x_{m-1}\}$, where $x_0, \ldots, x_{m-1}$ are the $m$ free variables that generate $\mathfrak{Tm}_{m,t}$. For every $n \in \omega$, we define the followings inductively.

- The normal forms of degree $0$, $F_0^{m,t} = \{D_m^\beta : \beta \in {}^{D_m}\{-1,1\}\}$.

- The set of normal forms of degree $n+1$,

$$F_{n+1}^{m,t} = \{D_m^\beta \cdot (CF_n^{m,t})^\alpha : \beta \in {}^{D_m}\{-1,1\} \text{ and } \alpha \in {}^{CF_n^{m,t}}\{-1,1\}\}.$$

- The set of all forms, $F^{m,t} = \bigcup_{k\in\omega} F_k^{m,t}$.

Every form in $F_0^{m,t}$ is determined by the information telling whether its below any constant or its complement and whether it is below any free variable or its complement. Every form of degree $n+1$, $n \in \omega$, is determined by the same information plus information telling whether this term is below (or below the complement of) the elements in the closure $CF_n^{m,t}$ of the set of the normal forms of the first smaller degree.

**Theorem A.0.4.** *Let $n \in \omega$. Then the followings are true:*

*(i)* $K \models \sum F_n^{m,t} = 1$.

*(ii) For every $\tau, \sigma \in F_n^{m,t}$, if $\tau \neq \sigma$ then $K \models \tau \cdot \sigma = 0$.*

*(iii) There exists an effective method to find, for every $\tau \in F_n^{m,t}$, a finite $S \subseteq F_{n+1}^{m,t}$ such that $K \models \tau = \sum S$.*

*(iv) There exists an effective method to find, for every $\tau \in \mathfrak{Tm}_{m,t}$, an $k \in \omega$ and a finite $S_\tau \subseteq F_k^{m,t}$ such that $K \models \tau = \sum S_\tau$.*

*Proof.*

(i) Remember that the Boolean part of every member of $K$ is Boolean algebra. Then, for any $S \subseteq \mathfrak{Tm}_{m,t}$, we have $K \models \sum\{S^\beta : \beta \in {}^S\{-1,1\}\} = 1$. Therefore, in particular, we have $K \models \sum F_0^{m,t} = \sum\{D_m^\alpha : \alpha \in {}^{D_m}\{-1,1\}\} = 1$ and, for any $n \in \omega$,

$$\begin{aligned} K \models \sum F_{n+1}^{m,t} &= \sum\{D_m^\alpha \cdot (CF_n^{m,t})^\beta : \alpha \in {}^{D_m}\{-1,1\} \text{ and } \beta \in {}^{CF_n^{m,t}}\{-1,1\}\} \\ &= \sum\{D_m^\alpha \cdot \sum\{(CF_n^{m,t})^\beta : \beta \in {}^{CF_n^{m,t}}\{-1,1\}\} : \alpha \in {}^{D_m}\{-1,1\}\} \\ &= \sum\{D_m^\alpha : \alpha \in {}^{D_m}\{-1,1\}\} \\ &= 1. \end{aligned}$$

(ii) Let $\tau, \sigma \in F_0^{m,t}$ be two different normal forms. There exists $\alpha_1, \alpha_2 \in {}^{D_m}\{-1,1\}$ such that $\tau = D_m^{\alpha_1}$ and $\sigma = D_m^{\alpha_2}$. Since $\tau$ and $\sigma$ are different then, without loss of generality, we can assume that there is $x_i \in D_m$ such that $\alpha_1(x_i) = 1$ and $\alpha_2(x_i) = -1$. Hence, $K \models \tau \leq x_i$, $K \models \sigma \leq -x_i$. Therefore, $K \models \tau \cdot \sigma = 0$ as desired. Suppose that $n \geq 1$ and let $\tau, \sigma \in F_n^{m,t}$ be two different normal forms. Similarly and without loss of generality, we can assume that there is $y \in D_m \cup CF_{n-1}^{m,t}$ such that $K \models \tau \leq y$ and $K \models \sigma \leq -y$. Therefore, $K \models \tau \cdot \sigma = 0$, as desired.

(iii) By induction on $n$. Let $\tau \in F_0^{m,t}$, then there is $\alpha \in {}^{D_m}\{-1,1\}$ such that $\tau = D_m^\alpha$. Since the Boolean parts of the members of $K$ are Boolean algebras, we have

$$K \models \tau = \sum\{D_m^\alpha \cdot (CF_0^{m,t})^\beta : \beta \in {}^{F_0^{m,t}}\{-1,1\}\}.$$

Suppose that $n \geq 1$ and, by induction, suppose that for each $\sigma \in F_{n-1}^{m,t}$ there is $S_\sigma \subseteq F_n^{m,t}$ such that $K \models \sigma = \sum S_\sigma$. Let $i \in I$ and suppose that $f_i$ is of rank $k_i \geq 1$. For every forms $\sigma_0, \ldots, \sigma_{k_i-1} \in F_{n-1}^{m,t}$, set

$$S_{f_i(\sigma_0,\ldots,\sigma_{k_i-1})} := \{f_i(\gamma_0, \ldots, \gamma_{k_i-1}) : (\forall l \in k_i)\ \gamma_l \in S_{\sigma_l}\}.$$

Let $\tau = D_m^\alpha \cdot (CF_{n-1}^{m,t})^\beta \in F_n^{m,t}$. For every $\beta' \in {}^{CF_n^{m,t}}\{-1,1\}$, we say that $\beta'$ is compatible with $\beta$ if for each term $\sigma \in CF_{n-1}^{m,t}$, we have $\beta(\sigma) = 1 \iff (\exists \gamma \in S_\sigma)\ \beta'(\gamma) = 1$. Set, $S_\tau = \{D_m^\alpha \cdot (CF_n^{m,t})^{\beta'} : \beta' \in {}^{CF_n^{m,t}}\{-1,1\}, \beta' \text{ is compatible with } \beta\} \subseteq F_{n+1}^{m,t}$.

Recall that $K$ is a class of Boolean algebras with operators, therefore $K \models \tau = \sum S_\tau$.

(iv) By induction on terms. For every $\tau \in D_m$, we have

$$K \models \tau = \sum\{D_m^\alpha : \alpha \in {}^{D_m}\{-1, 1\}, \alpha(\tau) = 1\}.$$

Let $i \in I$ and suppose that the rank of $f_i$ is $k_i$. Let $\sigma_1, \sigma_2, \gamma_0, \ldots, \gamma_{k_i-1} \in \mathfrak{Tm}_{m,t}$ be such that there is an effective method to find $n_1, n_2, q_0, \ldots, q_{k_i-1}$ and finite

$$S_1 \subseteq F_{n_1}^m, S_2 \subseteq F_{n_2}^m, S_0' \subseteq F_{q_0}^{m,t}, \ldots, S_{k_i-1}' \subseteq F_{q_{k_i-1}}^{m,t}$$

such that $K \models \sigma_1 = \sum S_1$, $K \models \sigma_2 = \sum S_2$ and $K \models \gamma_l = \sum S_l'$ for all $l \in k_i$. By item (iii) we may assume that $n_1 = n_2 = q_0 = \cdots = q_{k_i-1} =: n$.

- If $\tau = \sigma_1 + \sigma_2$ then $K \models \tau = \sum(S_1 \cup S_2)$.
- If $\tau = \sigma_1 \cdot \sigma_2$ then $K \models \tau = \sum\{\lambda \cdot \beta : \lambda \in S_1, \beta \in S_2\}$. By item (ii), it is clear that $\{\lambda \cdot \beta : \lambda \in S_1, \beta \in S_2\} \subseteq F_n^{m,t}$.
- If $\tau = -\sigma_1$, then $K \models \tau = \sum(F_n^{m,t} \setminus S_1)$.
- If $\tau = f_i(\gamma_0, \ldots, \gamma_{k_i-1})$ then, for every $w \in S := \{f_i(\lambda_0, \ldots, \lambda_{k_i-1}) : \lambda_l \in S_l'\}$, let

$$S_w = \{D_m^\alpha \cdot (CF_n^{m,t})^\beta : \alpha \in {}^{D_m}\{-1, 1\}, \beta \in {}^{F_n^{m,t}}\{-1, 1\}, \beta(w) = 1\}.$$

Therefore, $K \models \tau = \sum \bigcup\{S_w : w \in S\}$.

□

Thus, every term in $\mathfrak{Tm}_{m,t}$ can be rewritten in an equivalent form (in $K$) as a disjunction of normal forms of the same degree. Disjunctive normal forms are often used to give elegant and constructive proofs for many standard results. They can be used to prove finite model property and decidability as in [And54] and [Fin75]. In chapter 1, we used them to show non-atomicity of the free relativized cylindric algebras and we also used them to specify the atoms, e.g., in the proof of proposition 1.5.8. Without the use of the normal forms, specifying the terms that are atoms would be quite lengthy. Disjunctive normal forms were used to show non-atomicity of other similar free algebras in [Kha15b] and [Kha15a].

# B

APPENDIX

# Infinite dimensional free algebras

The infinite dimensional relativized cylindric algebras are more interesting, many problems are still open for the infinite dimensions. Let $n$ be any infinite ordinal. I. Németi [Ném86] showed that $Crs_n$ is a variety, hence it follows that $D_n$ is a variety too. It is not known yet whether $G_n$ is a variety or not, but H. Andréka showed that $\mathbf{H}G_n$ is a finite-variable axiomatizable variety. In [Ném86], it was shown that both $Crs_n$ and $G_n$ have the finite base property and have decidable equational theories. Németi, also in [Ném86], showed that $D_n$ does not have the finite model property, and the following problem was posed as an open problem.

**Problem 2.** *Let $n \geq \omega$ be any infinite ordinal. Is the equational theory of $D_n$ decidable?*

We know much less about the class $P_n$. We even don't know wether $P_n$ is a variety or not and we don't know whether the variety generated by $P_n$ is finite-variable axiomatizable or not. Note that the variety generated by $P_n$ is $\mathbf{H}P_n$ because $P_n$ is closed under both $\mathbf{S}$ and $\mathbf{P}$.

**Problem 3.** *Let $n \geq \omega$ be any infinite ordinal.*

*(1) Is $P_n$ a variety? Is $\mathbf{H}P_n$ finite-variable axiomatizable?*

*(2) Does $P_n$ have the finite base property?*

*(3) Does $P_n$ have decidable equational theory?*

Throughout, let $n$ and $m$ be any two ordinals and let $K \in \{Crs, D, P, G\}$. We note that the equational theory of the variety generated by $K_n$ coincides with the equational theory of the class of complex algebras over concrete $K_n$-atom structures. Let $V$ be any concrete $K_n$-atom structure, let $f \in V$ and $\iota : \{x_i : i \in m\} \to \mathcal{P}(V)$ be any evaluation. For every term

$\tau \in \mathfrak{Tm}_{m,cyl_n}$ ($cyl_n$ is the signature of the class $K_n$), we write $(V, f, \iota) \models \tau$ if and only if $f \in [\tau]_\iota^{\mathfrak{Cm}V}$. In chapter 1, we investigated the atomicity of $\mathfrak{Fr}_m K_n$ if both $n$ and $m$ are finite. Suppose that $n$ or $m$ is infinite, here we investigate the atomicity of the free algebra $\mathfrak{Fr}_m K_n$.

**Non-atomicity of $\mathfrak{Fr}_m K_n$ for infinite $m$'s.** In [HMT71, 2.5.13], it was shown that if $m$ is infinite then $\mathfrak{Fr}_m CA_n$ is atomless, i.e., contains no atom. The proof of [HMT71, 2.5.13] depends only on the universal mapping property of the free algebras, so it can be used again to prove the following.

**Theorem B.0.5.** *Let $n \geq 2$ and $m \geq \omega$ be any two ordinals and let $K \in \{Crs, D, P, G\}$. The free algebra $\mathfrak{Fr}_m K_n$ contains no atom.*

*Proof.* Let $\{x_i : i \in m\}$ be the set of the free generators of $\mathfrak{Fr}_m K_n$. Let $\tau \in \mathfrak{Tm}_{m,cyl_n}$ be such that $\mathfrak{Fr}_m K_n \not\models \tau = 0$. Then we can assume that there is a concrete $K_n$-atom structure $V$, $f \in V$ and an evaluation $\iota : \{x_i : i \in m\} \to \mathcal{P}(V)$ such that $(V, f, \iota) \models \tau$. But, $\tau$ is built up only from finitely many free variables, hence there is $j \in m$ such that $x_j$ never appears in the syntactical construction of $\tau$, fix such a $j \in m$. We define two evaluations $\iota_1$ and $\iota_2$ as follows. For every $i \in m \setminus \{j\}$, let $\iota_1(x_i) := \iota(x_i) =: \iota_2(x_i)$, $\iota_1(x_j) := V$ and $\iota_2(x_j) := \emptyset$. One can easily check that,

$$(V, f, \iota_1) \models \tau \cdot x_j \quad \text{and} \quad (V, f, \iota_2) \models \tau \cdot -x_j.$$

Therefore, both $\tau \cdot x_j$ and $\tau \cdot -x_j$ are non-zero elements below $\tau$ in the free algebra $\mathfrak{Fr}_m K_n$, i.e., $\tau$ is not an atom in $\mathfrak{Fr}_m K_n$ as desired. $\square$

**Non-atomicity of $\mathfrak{Fr}_m K_n$ for infinite $n$'s.** We begin with some notation. For every term $\tau \in \mathfrak{Tm}_{m,cyl_n}$, let $index(\tau)$ be the set of all $i \in n$ such that $d_{ij}$, $d_{ji}$ or $c_i$ appears in the syntactical construction of $\tau$, for some $j \in n$. Let $f, g$ be any sequences of length $n$ and let $\Gamma \subseteq n$. We write $f \equiv_\Gamma g$ if and only if $f$ and $g$ agree on $n \setminus \Gamma$.

**Theorem B.0.6.** *Let $n \geq \omega$ and $m \geq 1$ be any ordinals. Let $K \in \{Crs, D, G\}$, then the free algebras $\mathfrak{Fr}_m K_n$ and $\mathfrak{Fr}_0 Crs_n$ are not atomic.*

*Proof.* Suppose that $n \geq \omega$, $m \geq 1$ and $K \in \{Crs, D, G\}$. We prove that there is no atom in $\mathfrak{Fr}_m K_n$ that is below $-d_{01}$. Let $\tau \in \mathfrak{Tm}_{m,cyl_n}$ be such that $\mathfrak{Fr}_m K_n \not\models -d_{01} \cdot \tau = 0$. Let $V$ be a concrete $K_n$-atom structure, $f \in V$ and $\iota$ be an evaluation such that $(V, f, \iota) \models -d_{01} \cdot \tau$. Let $\Gamma := index(\tau) \cup \{0, 1\}$. Every term is built up from finitely many variables and finitely many symbols in the signature of $K_n$, thus $\Gamma$ is finite. Fix $i, j \in n \setminus \Gamma$ such that $i \neq j$. We divide the proof into the following two cases.

**Case 1:** Suppose that $f(i) = f(j)$. By the assumption $(V, f, \iota) \models -d_{01} \cdot \tau$ and without loss of generality, we may assume that $f(0) \neq f(j)$. Let $T = \{g \in V : g(i) = g(j)\}$. Set $V' = V \cup \{f \circ [i/f(0)]\}$. It is easy to see that $V'$ is a concrete $K_n$-atom structure because $V$ is a concrete $K_n$-atom structure (we note that this is not true if $K = P$). Define the following evaluations on $\mathcal{P}(V')$. For every $k \in m$, let

$$\iota_1(x_k) = \iota(x_k) \cap T \qquad \text{and} \qquad \iota_2(x_k) = \iota_1(x_k) \cup \{f \circ [i/f(0)]\}.$$

A simple induction argument on the complexity of terms can show the following. For every $\sigma \in \mathfrak{Tm}_{m,cyl_n}$ and every $g \in V$,

$$[index(\sigma) \subseteq \Gamma \;\&\; g \equiv_\Gamma f] \implies [(V, g, \iota) \models \sigma \iff (V', g, \iota_1) \models \sigma]. \tag{B.1}$$

$$[index(\sigma) \subseteq \Gamma \;\&\; g \equiv_\Gamma f] \implies [(V, g, \iota) \models \sigma \iff (V', g, \iota_2) \models \sigma]. \tag{B.2}$$

By (B.1) and (B.2), we have the followings:

$$(V', f, \iota_1) \models -d_{01} \cdot \tau \cdot -c_i(x_0 \cdot -d_{ij}) \qquad \text{and} \qquad (V', f, \iota_2) \models -d_{01} \cdot \tau \cdot c_i(x_0 \cdot -d_{ij})$$

Thus, $-d_{01} \cdot \tau$ is not an atom in the free algebra $\mathfrak{Fr}_m K_n$ as desired.

**Case 2:** Suppose that $f(i) \neq f(j)$. Set $V' = V \cup \{f \circ [i/f(j)]\}$. It is easy to see that $V'$ is a concrete $K_n$-atom structure. Let $T = \{g \in V : g(i) \neq g(j)\}$. Define the following evaluations on $\mathcal{P}(V')$. For every $k \in m$, let

$$\iota_1(x_k) = \iota(x_k) \cap T \qquad \text{and} \qquad \iota_2(x_k) = \iota_1(x_k) \cup \{f \circ [i/f(j)]\}.$$

Similarly to the above case, one can show the followings:

$$(V', f, \iota_1) \models -d_{01} \cdot \tau \cdot -c_i(x_0 \cdot d_{ij}) \qquad \text{and} \qquad (V', f, \iota_2) \models -d_{01} \cdot \tau \cdot c_i(x_0 \cdot d_{ij}).$$

Thus, $-d_{01} \cdot \tau$ is not an atom in the free algebra $\mathfrak{Fr}_m K_n$ as desired.

Now, we show that the free algebra $\mathfrak{Fr}_0 Crs_n$ is not atomic, more precisely, contains no atom. Let $\tau \in \mathfrak{Tm}_{0,cyl_n}$ be any term such that $\mathfrak{Fr}_0 Crs_n \not\models \tau = 0$. Then there is a concrete $Crs_n$-atom structure $V$ and $f \in V$ such that $(V, f, \iota) \models \tau$ where $\iota = \emptyset$. Let $\Gamma = index(\tau)$, then $\Gamma$ is finite. Fix $i, j \in n \setminus \Gamma$ such that $i \neq j$. Pick brand new elements $a, b$ that are not in the base of $V$ such that $a = b \iff f(i) \neq f(j)$. For every $h \in V$ with $h \equiv_\Gamma f$, let $h^*$ be the sequence given as follows: $h^*(k) = h(k)$ for every $k \in n \setminus \{i, j\}$, $h^*(i) = a$ and $h^*(j) = b$. Set $V^* = \{h^* : h \in V \text{ and } h \equiv_\Gamma f\}$. By induction on terms, one can easily check the following. For every $\sigma \in \mathfrak{Tm}_{m,cyl_n}$ and every $h \in V$,

$$[index(\sigma) \subseteq \Gamma \ \& \ h \equiv_\Gamma f] \implies [(V, h, \iota) \models \sigma \iff (V^*, h^*, \iota) \models \sigma]. \tag{B.3}$$

Thus, $(V^*, f^*, \iota) \models \tau$. But, by the choice of $a, b$, we have

$$(V, f, \iota) \models d_{ij} \iff (V^*, f^*, \iota) \models -d_{ij}.$$

Therefore, both $\tau \cdot d_{ij}$ and $\tau \cdot -d_{ij}$ are not zero in the free algebra $\mathfrak{Fr}_0 Crs_n$, i.e., $\tau$ is not an atom in $\mathfrak{Fr}_0 Crs_n$ as desired. $\square$

Let $n \geq \omega$ and $m \in \omega$ be any ordinals. We do not know anything about the atomicity of the free algebras $\mathfrak{Fr}_m P_n$, $\mathfrak{Fr}_0 D_n$ and $\mathfrak{Fr}_0 G_n$. We could not modify the above proof to show non-atomicity of these free algebras. I. Németi in [Ném86] proved the non-atomicity of the free algebras $\mathfrak{Fr}_m CA_n$ and $\mathfrak{Fr}_m Gs_n$, his proofs depend on (the syntactical and semantical) wGIP of first order logic. We don't know anything about wGIP for the logics that correspond to the free algebras $\mathfrak{Fr}_m P_n$, $\mathfrak{Fr}_0 D_n$ and $\mathfrak{Fr}_0 G_n$.

**Problem 4.** *Let $n \geq \omega$ and $m \in \omega$ be any ordinals. Is any of the free algebras $\mathfrak{Fr}_m P_n$, $\mathfrak{Fr}_0 D_n$ and $\mathfrak{Fr}_0 G_n$ atomic?*

One may think that the free algebras mentioned in the above problem should be non-atomic. We believe that $\mathfrak{Fr}_m P_n$ is not atomic. But we would not be surprised if the free algebras $\mathfrak{Fr}_0 D_n$ and $\mathfrak{Fr}_0 G_n$ are atomic.

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