CENTRAL EUROPEAN UNIVERSITY DEPARTMENT OF MATHEMATICS AND ITS APPLICATION

A Comparison Of Optimal Strategies Under Law Invariant Coherent Risk Measures and Utility Functionals

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2 Introduction

Our existence has been always based on the best allocation and utilization of available resources. The modern era of economics brought new tools and structure but at the same time new challenges that add complexity of choice. Tackling the problem started with first defining the value of the intrinsic physical or intangible quantity. This philosophy of quantification is the driver of the mathematics of price and the process of pricing. After defining such quantity it boils down to the best asset allocation based upon certain universal measures. In the language of financial mathematics the best allocation corresponds to the optimal asset investment strategy which achieves the maximum value of the agreed upon measure. The existence and uniqueness of an optimal strategy is a complex process which takes different tools of pure and applied mathematics that ensure a well-defined problem and results.

Optimal strategies are applied enormously in different arenas of finance and trade. Financial portfolios which are included in funds of multiple functionalities use such strategies to ensure a superlative allocation of wealth and efficient economics and markets. Hedge, ETF's and mutual funds are prime examples of such funds which together they constitute a big chunk of the world's assets. Financial institutions as well use these strategies to manage wealth and choose good investments. Mathematical finance approaches the questions of value into two ways. One way is by market pricing models that use stochastic calculus, probability and economic theory to capture market forces and functions. The other approach is by measuring risk of certain positions which in turn gives us a pricing capability of the underlying assets. And in between the two faces of asset pricing lies the problem of optimization represented in strategies or allocations.

In this paper I will compare optimal investment under two measures. First using the basic market assumptions (Axioms) to construct risk measures which are used to rank our investments. Second is by using the economics theory of utility to asses our investments and prioritize them accordingly. The two approaches are used to choose the best investments according to the optimal strategies that arise. The aim is to compare these strategies and see the driving forces behind their differences. Using simulations and multiple pricing models, progressive in complexity, the final result should give a picture on the intrinsic features of the two approaches. Literature for risk and utility measures mainly comes from mathematical finance papers published dominantly by Cherny and Madan. other resources include books such as Stochastic Finance by Follmer and Schied and Stochastic Calculus by Lamberton plus other supplementary papers.

Our setting is a probability space $\left(\Omega, \mathcal{F}, (\mathcal{F})_{t\in N}, P\right)$with the natural filtration in discrete time. let $X_t \in L^{\infty}$be a random variable on the probability space such that $X_t : \Omega \to \mathbb{R}$. The Random Variable represents a particular cash flow (i.e payoff) given the information until time t and $X \sim D$ with any distribution D . we define a risk measure $\rho : L^{\infty} \to \mathbb{R}$ and Utility Functional $U : L^{\infty} \to \mathbb{R}$ which will give us the benchmark for our rating. Under No arbitrage and risk free rate r we can define our portfolio cash flow as

$$X_t = V_t^{(\alpha,\beta,\gamma)} = \sum_{i=1}^{n} \gamma_{t+1}^{i} g(S_t^i) + \beta_{t+1} B_t + \sum_{i=1}^{n} \alpha_{t+1}^{i} S_t^i$$

where S_t^i risky assets$i$ price at time t , $g(S_t^i)$ are contingent claim on asset i (function of asset i), $B_t$is risk less asset price at time t, (α, β, γ) are the corresponding strategies of all assets (Quantity of each asset) and they are predictable processes (i.e $(\alpha, \beta, \gamma)_{t+1}$

are measurable w.r.t $\mathcal{F}_t$). we choose a specific stochastic model for S and then try to optimize ρ and U with the corresponding strategy i.e we need to find

$$\sup_{(\alpha,\beta,\gamma)\in\mathbb{R}^d} \rho(V_t^{(\alpha,\beta,\gamma)}) = \rho(V_t^{(\alpha^*,\beta^*,\gamma^*)})$$

and

$$\sup_{(\alpha,\beta,\gamma)\in\mathbb{R}^d} U(V_t^{(\alpha,\beta,\gamma)}) = U(V_t^{(\alpha^-,\beta^-,\gamma^-)})$$

and then compare $(\alpha^-,\beta^-,\gamma^-)$and $(\alpha^*,\beta^*,\gamma^*)$. Then we proceed with the study of their structure and behavior.

The main questions that I am trying to answer are 1) what is the structure of these optimal strategies (existence, uniqueness, form and behavior)? are the two measures optimal strategies similar? if yes, for which payoff class are they similar? if no, what are the advantages and disadvantages of both? and which one is a better representation of the market?.

To answer these questions, tools of stochastic analysis have to be used to derive an optimal solution. numerical simulation under algorithms is another approach that could be an indicator of structure which can be used preliminary. the main methodology is to use these tools to give an idea on the general direction or founded results if possible.

3 Arbitrage Pricing and Preferences

3.1 Arbitrage Pricing

For the following lets consider one risky asset with no contingent claim for simplicity (i.e $(S,B),(\alpha,\beta)\in\mathbb{R}^2$).

Definition 2.1.1 : A portfolio is called self-financing if $V_t^{(\alpha,\beta)} = \beta_{t+1}\cdot B_t + \alpha_{t+1}\cdot S_t = \beta_t\cdot B_t + \alpha_t\cdot S_t$ holds $\forall t\in N$.

Notation 2.1.1 : lets denote the family of all predictable and self-financing strategies as A.

Definition 2.1.2: A strategy$(\alpha,\beta)\in A$ is called an arbitrage opportunity if $\exists t\in N$ such that $V_0^{(\alpha,\beta)} = 0$ and $V_t^{(\alpha,\beta)} \geq 0$ almost surely and $p(V_t^{(\alpha,\beta)} > 0) > 0$. and called No Arbitrage $\forall(\alpha,\beta)$ if $V_t^{(\alpha,\beta)} \geq 0$ then $V_t^{(\alpha,\beta)} = 0$.

In our setting, the market is efficient which means no arbitrage opportunity exists. if such opportunities arise then the market will eliminate them instantaneously through the dynamics of market forces.

Definition2.1.3: A probability measure q is called equivalent (i.e $q \backsim p(q(A)=0 \iff p(A)=0)$) martingale measure (EMM) or risk neutral measure with numeraire (how value is quoted) B (could be any process, in our case we chose B) if $\frac{S_k}{B_k} = E_q[\frac{S_t}{B_t} \mid \mathcal{F}_k]$ $\forall t,k\in N$ and $k<t$.

i.e the expected cash flows of the future given the information till today is today's value which implies a fair game (martingale) dynamic. The need for a risk neutral measure arises due to the multiplicity of subjective risk return pricing. An investor will demand a price that fits within his or her risk aversion which would lead to discrepancies in price.

The neutral risk measure is a measure that takes into account every risk appetite of every individual which leads to a consistent pricing.

Theorem 2.1.1: No Arbitrage hold if and only if $\exists q$ equivalent martingale measure in the market (S, B).

Proof : $\Longleftarrow$ By contradiction, assume there exist an arbitrage opportunity (α, β), then $V_0^{(\alpha,\beta)} = 0$,$V_t^{(\alpha,\beta)} \geq 0$ a.s and $p(V_t^{(\alpha,\beta)} > 0) > 0 \Rightarrow q(V_t^{(\alpha,\beta)} > 0) > 0$ and $q(V_t^{(\alpha,\beta)} \geq 0) = 1$ $\Rightarrow E_q[\frac{V_t^{(\alpha,\beta)}}{B_t}] > 0$ and on the other hand $E_q[\frac{V_t^{(\alpha,\beta)}}{B_t}] = V_0^{(\alpha,\beta)} = 0$ contradiction.

$\Longrightarrow$using the stochastic integral proposition in discrete time (If for every α bounded and predictable process $G_t(\alpha, M) = \sum_{i=1}^{t} \alpha_i(M_i - M_{i-1})$ has zero expectation $\forall t \in N$ then M is a martingale) . Let $M = S^* = \frac{S_t}{B_t}$ then it is enough to prove the existence of $q \sim p$s.t $E_q[G_t(\alpha, S^*)] = 0$. $\sum_{i=1}^{t} \alpha_i(S_i^* - S_{i-1}^*)$

Denote the cardinality of Ω by m and for any random variable X on Ω we denote $X(\omega_i) = X_i$such that $E[X] = \sum_{i=1}^{m} X_i q(\omega_i)$. we see that the arbitrage free condition translates into $G(\alpha, S^*) \notin \mathbb{R}_+^m = \{X \mid X_i \geq 0, 0 \leq i \leq m\}$ for every predictable process α. Thus $\theta = \{G(\alpha, S^*) \mid \alpha \in A\}$is a linear subspace of $\mathbb{R}^m$s.t $\theta \cap \vartheta = \phi$ with $\vartheta = \{X \in \mathbb{R}_+^m \mid X_1 + \ldots + X_m = 1\}$. using a corollary that states (let ϑ be compact convex subset of $\mathbb{R}^m$and θ a linear subspace of $\mathbb{R}^m$with $\theta \cap \vartheta = \phi$ then $\exists c \in \mathbb{R}^m$s.t $< c, G >= 0, \forall G \in \theta$ and $< c, G >> 0, \forall X \in \vartheta$ or equivalently $\sum_{i=1}^{m} G_i(\alpha, S^*)c_i = 0$ and $\sum_{i=1}^{m} X_i c_i > 0$) which implies $c_i > 0, \forall i$ so we can normalize c to define q s.t $q(\omega_i) = \frac{c_i}{\sum_{j=1}^{m} c_j}$ then $\sum_{i=1}^{m} G_i(\alpha, S^*)c_i = 0 \Rightarrow E_q[G(\alpha, S^*)] = 0$.

The following Definitions and Theorems deals with the properties of the Contingent Claim Function $g(S_t)$

Definition 2.1.4: A contingent claim payoff $g(S_T)$ is a financial product that derives its value from the underlying asset/s S_T, i.e it is a function of the underlying asset/s.

Definition 2.1.5: A contingent claim is replaceable if $\exists(\alpha, \beta) \in A$ such that $V_T^{(\alpha,\beta)} = g(S_T)$ where T maturity date of the option.

Definition 2.1.6 : A super- and sub-replicating portfolios for a contingent claim $g(S_T)$ are $A_+ = \{(\alpha, \beta) \in A \mid V_T^{(\alpha,\beta)} \geq g(S_T)\}$ and $A_- = \{(\alpha, \beta) \in A \mid V_T^{(\alpha,\beta)} \leq g(S_T)\}$

Definition 2.1.7: H_t is a fair price of the contingent claim $g(S_T)$ at time t if and only if $\frac{H_t}{B_t} = E_q[\frac{g(S_T)}{B_T} \mid \mathcal{F}_t]$ (under risk neutral measure q).

Proposition 2.1.1: if $g(S_T)$ is replaceable then it has a unique fair price H_t.

Proof: let $q \in M$ risk neutral measure ($M = \{q : q \sim p\}$. then $E_q[\frac{g(S_T)}{B_T} \mid \mathcal{F}_t] = E_q[\frac{V_T^{(\alpha,\beta)}}{B_T} \mid \mathcal{F}_t] = \frac{V_t^{(\alpha,\beta)}}{B_t} = \frac{H_t}{B_t}$, since (α, β) is constant we can define $H_t = \alpha S_t + \beta B_t$.

Definition 2.1.8: A market is complete if every claim is replaceable (i.e everything has unique price).

Theorem 2.1.2: A market is called complete if and only if $\exists! q \sim p$ risk neutral measure, otherwise it is called incomplete.

This another characterization of the complete market. the existence of unique prices are equivalent to the existence of a unique risk neutral measure which is used to price them.

In case the market is incomplete then there exists a range of risk neutral measures.

$proof$: $\Longleftarrow$By contradiction. Suppose the market is not complete and we construct another $q^{'}$EMM different from q. let $\pi = \{\frac{V_T^{(\alpha,\beta)}}{B_T} \mid (\alpha,\beta) \in A\}$. we use the random elements X as in Theorem 2.1. Then the incompleteness of the market translates into $\pi \subset \mathbb{R}^m$. we define the scalar product as $[X,Y] = \sum_{i=1}^m X_i Y_i q(\omega_i) = E_q[XY]$. Then $\exists L \in \mathbb{R}^m$orthogonal to π, i.e $[L,X] = 0, \forall X \in \pi$, if we choose $X = 1 \Rightarrow E_q[L] = 0$. for a fixed parameter $\delta > 1$ we define $q_\delta(\omega_i) = (1 + \frac{L_i}{\delta \|L\|_\infty}) q(\omega_i)$ where $\| L \|_\infty = \max_i[| L_i |]$, so we showed we can construct different EMM for each $\delta > 1$ and $q_\delta(\omega_i) > 0$ since $(1 + \frac{L_i}{\delta\|L\|_\infty}) > 0$, furthermore $q_\delta(\Omega) = \sum_{i=1}^m (1 + \frac{L_i}{\delta\|L\|_\infty}) q(\omega_i) = q(\Omega) + \frac{1}{\delta\|L\|_\infty} E_q[L] = 1$ therefore q_δ is EMM. Then we calculate $E_{q_\delta}[G(\alpha, S^*)] = E_q[G(\alpha,S^*)] + \frac{1}{\delta\|L\|_\infty} E_q[LG(\alpha,S^*)] = E_q[G(\alpha,S^*)] = 0$ by the proposition (if $S^*$is a martingale and $\alpha$is a bounded and predictable process then $G(\alpha,S^*)$ is a martingale with null expectation).

In case of incomplete market we have a range of arbitrage free derivative prices which poses another question of which price should we choose. one way is the super replicating price which is the $price = \inf\{(\alpha,\beta) \in A_+ \mid V_t^{(\alpha,\beta)} \geq g(S_T)\}$or the sub replicating if on the opposite direction. Also we can use entropy function between the EMM's and take the minimal. Another approach is to use indifference price using utility functions which we will explore in the next section.

3.2 Preferences

Definition 2.2.1: A relation $\propto$is a preference order if we write $X \propto Y$ for $X, Y \in L^\infty$we mean Y is preferred to X and it satisfies the following axioms.

1) Completeness: $\forall X, Y \in L^\infty$then either $X \propto Y$ or $Y \propto X$ or both ($X \sim Y$ indifference)

2) Transitivity: If $X \underline{\propto} Y$ and $Y \underline{\propto} Z$ then $X \underline{\propto} Z$.

3) Independence : If $X \underline{\propto} Y$ and for any Z (independent random variable) and $\lambda \in (0,1)$ then $\lambda X + (1-\lambda) Z \underline{\propto} \lambda Y + (1-\lambda) Z$.

4) Continuity: If $X \underline{\propto} Y \underline{\propto} Z$ then there exists a probability $\varepsilon \in (0,1)$ such that $\varepsilon X + (1-\varepsilon) Z \underline{\propto} Y$.

all of these axioms characterize the rational behavior of the economic agent which is the underlying assumption of utility theory.

Definition 2.2.2: A preference relation is called to have Savage (Von Neumann - Morgenstein VNM) representation if $\exists u : \mathbb{R} \longrightarrow \mathbb{R}$such that $X \underline{\propto} Y \iff E_p[u(X)] \leq E_p[u(Y)]$. we call u utility function.

Proposition 2.2.1: An investor is risk averse if the utility function u is concave.

$proof$: Take any two points X, Y s.t $X, Y > 0, X < Y$ then by concavity $u(\lambda X + (1-\lambda)Y) \geq \lambda u(X) + (1-\lambda) u(Y)) \Rightarrow w \geq \lambda w_1 + (1-\lambda) w_2, w_1 \leq w \leq w_2 \forall \lambda \in (0,1)$,$\lambda$where is the probability X happening which means we prefer a sure thing to any gamble.

Theorem 2.2.1 :If $\underline{\propto}$ satisfies its axioms and $u : \mathbb{R} \longrightarrow \mathbb{R}$ concave (risk aversion) and monotonically increasing $\Longrightarrow \exists$a Savage representation.

$proof$: we will skip the proof of this theorem.

So $\exists u$ hidden utility function which drives the underlying preference relation.

In our aim of optimal strategies we define the utility functional as $U(X) = \inf_{q \in M_0} E_q[u(X)]$ for some set of subjective equivalent probability measures M_0. From now on we will assume that such minimum is attained and the robust utility maximization is reduced to the standard utility maximization i.e $U(X) = \inf_{q \in M_0} E_q[u(X)] = E_p[u(X)]$. so our Savage preference representation will be $X \propto\_\ Y \iff E_p[u(X)] = U(X) \leq E_p[u(Y)] = U(Y)$.

There exist multiple classes of utility functions but we will choose two main classes which will give a good idea about utility behavior. the first class is the **exponential utility functions** which are defined by:

$$u(c) = \begin{cases} \frac{(1-e^{-ac})}{a} & a \neq 0 \\ c & a = 0 \end{cases}$$

where a represents the degree of risk preference ($a > 0, a < 0, a = 0$ are risk aversion, risk seeking and risk neutral respectively).

In our case we only consider $a > 0$(risk aversion) which gives the rise to the Savage representation.

The second class is the **Power utility functions** which are defined by:

$$u(c) = \begin{cases} \frac{c^{1-v}-1}{1-v} & v \neq 1 \\ \ln(c) & v = 1 \end{cases}$$

where v is a constant and $\ln(c)$ is the limiting case. we take$v > 0$ as it represents the risk aversion property for VNM representation.

Now we will see how we can price contingent claims in incomplete markets.

Definition 2.2.3: The utility indifference price of a derivative is $price(g(S_T)) = pr^* = \inf\{pr \mid \sup_{(\alpha,\beta)} E_p[u(V_T^{(\alpha,\beta)+pr} - g(S_T))] \geq \sup_{(\alpha,\beta)} E_p[u(V_T^{(\alpha,\beta)})]\}$.

What this is basically saying is that entering the contract and receiving pr and paying $g(S_T)$at T should be preferred by the seller of this contract than not entering it. and it is usually less that the super replicating price.

So now we have a somewhat clear idea of complete and incomplete market characterization within risk neutral and utility constructions.

Finally, lets discover a nice uniting factor between the risk neutral and utility world.

Let X be a random variable then there is a unique $j(X) \in \mathbb{R}$ such that $u(j(X)) = E_p[u(X)]$. where $j(X)$ is a deterministic amount. since u is concave then by Jensen Theorem we have $E_p[u(X)] \leq u(E_p[X])$ since$j(X) \leq E_p[X]$ then $\xi(X) = E_p[X]) - j(X)) \geq 0$. where $\rho$is the risk premium the investor demands. or it is the amount the investor is willing to pay to change $j(X)$ to $E_p[X]$. Lets write the Taylor expansion for $u(j(X))$ around $E_p[X]$.

$$u(j(X)) = u(E_p[X]) + (j(X) - E_p[X]) \cdot u^{'}(E_p[X]) + \frac{1}{2}(j(X) - E_p[X])^2 u^{''}(E_p[X]) + small$$

From above we have $u(j(X)) = E_p[u(X)]{=}\int u(x)p(dx)$ we do 2nd order expansion around $E_p[X]$ and take expectation $\Longrightarrow E_p[E_p[u(X)]] = E_p[u(E_p[X]) + (X - E_p[X]) \cdot u^{'}(E_p[X]) + \frac{1}{2}(X - E_p[X])^2 u^{''}(E_p[X]) + small] \Longrightarrow E_p[u(X)] = u(E_p[X]) + 0 + \frac{1}{2}Var[X]u^{''}(E_p[X])$ by Mean Value Theorem we get $(j(X) - E_p[X])u^{'}(E_p[X]) = -\frac{1}{2}Var[X]u^{''}(E_p[X]) \Longrightarrow$

$\xi(X) = -\frac{1}{2}Var[X]\frac{u''(E_p[X])}{u'(E_p[X])}$. so the risk premium is a scaling of the quantitatively objective risk measure $Var[X]$ by the risk aversion $\frac{u''(E_p[X])}{u'(E_p[X])}$ of the investor. In risk neutral market scenario the utility is linear and the risk aversion is zero for all investors. Thus the market return for a position is solely dependent on the objective risk measure in the market. Furthermore, our $Var[X]$ or any other measure of risk in any market scenario could be replaced and defined by a more involved measure ρ and in the next chapter we will take a closer look at these measures and how they play an important role in optimal investments.

4 Coherent Risk Measures

4.1 Definitions, Representations and Examples

The risk measure we will talk about below are an indicator of cash requirements that is needed for a certain cash flow or position to be risk free according to the supervisory agency in the market.

Definition 3.1.1: A risk measure $\rho : L^\infty \to \mathbb{R}$ is called Monetary if it satisfies the following axioms $\forall X, Y \in L^\infty$:

1) Monotonicity: If $X \leq Y \Longrightarrow \rho(X) \geq \rho(Y)$. i.e a better performing cash flow always has lesser risk.

2) Cash Invariance: If $m \in \mathbb{R} \Longrightarrow \rho(X + m) = \rho(X) - m$ i.e the risk measure is translated by cash amount. From cash invariance we can deduce $\rho(m) = \rho(0) - m$. In our case we will assume normalization i.e $\rho(0) = 0$.

Lemma 3.1.1: Monetary risk measures are Lipschitz continuous with w.r.t the L^∞ norm.

proof : We can clearly see that $X \leq Y + \| X - Y \|_\infty \Longrightarrow \rho(X) - \rho(Y) \leq \| X - Y \|_\infty$by Monotonicity and Cash Invariance. Reversing the roles of X and Y lead to the assertion.

Definition 3.1.2: A Value at Risk V@R risk measure is defined as follows $V@R_\lambda(X) = \inf\{m \mid p(X + m < 0) \leq \lambda\} = \inf\{m \mid p(X \leq m) \geq \lambda\}, \lambda \in (0, 1)$. in other words V@R is the minimum cash requirement that we add to our position which keeps our negative outcomes below the threshold λ.

Example 3.1.1: Value at Risk V@R is a Monetary risk measure. Lets check if it satisfies the two axioms.

1) Monotonicity: Assume $X \leq Y$. we have $V@R_\lambda(X) = \inf\{m \mid p(X+m < 0) \leq \lambda\}$and $V@R_\lambda(Y) = \inf\{m \mid p(Y + m < 0) \leq \lambda\}$. let $m_1 \in \{m \mid p(X + m < 0) \leq \lambda\}$then $X+m_1 \leq Y+m_1 \Longrightarrow p(X+m_1 < 0) \leq p(Y+m_1 < 0) \leq \lambda \Longrightarrow m_1 \in \{m \mid p(Y+m < 0) \leq \lambda\} \Longrightarrow \{m \mid p(X+m < 0) \leq \lambda\} \subseteq \{m \mid p(Y+m < 0) \leq \lambda\} \Longrightarrow V@R_\lambda(X) \geq V@R_\lambda(Y)$.

2) Cash Invariance: Let $V@R_\lambda(X + k) = \inf\{m \mid p(X + (m + k) < 0) \leq \lambda\} = \inf\{m + k \mid p(X + (m + k) < 0) \leq \lambda\} - k = V@R_\lambda(X) - k$.

Definition 3.1.3 :A monetary risk measure is called convex if it satisfies the convexity axiom: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, $0 \leq \lambda \leq 1$.

The axiom of convexity gives a precise quantification of the idea of diversification. The risk of a diversified portfolio is always less than the two portfolios separated.

Example 3.1.2 :In this example we show that the V@R is not a convex risk measure since it does not satisfy the convexity axiom.

Consider two default-able corporate bonds with return $s > r \geq 0$ where r is the risk free return. then our discounted return on an initial investment w on bond i is

$$X_i = \begin{cases} -w & Default \\ \frac{w(s-r)}{1+r} & otherwise \end{cases}$$

if the probability of default of 1st bond is $p_d \leq \lambda$ then

$$p(X_1 - \frac{w(s-r)}{1+r} < 0) = p(default) = p_d \leq \lambda$$

$$\Rightarrow V@R_\lambda(X_1) = -\frac{w(s-r)}{1+r} < 0$$

Which means every position is acceptable regardless of the loss of our initial investment w since V@R is negative. Now consider investing $\frac{w}{2}$ in each bond then our portfolio payoff is $Y = \frac{X_1+X_2}{2}$, let the same default probability for each bond p_d then for an appropriate s we get that $p(Y < 0)$ is equal to at least one of the bonds default $p(Y < 0) = p_d(2 - p_d)$. letting $p_d = 0.009$ and $\lambda = 0.01$ then $p_d < \lambda < p_d(2 - p_d)$ which means $V@R_\lambda(Y) = \frac{w}{2}(1 - \frac{w(s-r)}{1+r})$ which is close to $\frac{w}{2}$. thus $V@R_\lambda(X_1) < V@R_\lambda(Y)$ which means that the diversified portfolio has higher risk thus V@R is not convex consequently not coherent in general.

Definition 3.1.4: A Convex risk measure is called coherent if it satisfies the positive homogeneity axiom : $\rho(\lambda X) = \lambda\rho(X), \forall\lambda \geq 0$.

The positive homogeneity axiom stresses on the fact that the risk is increase linearly with the increase of the position we are holding. This axiom is not always true as the relationship could be nonlinear but it is enough to assume it for our study purpose. A very easy consequence of positive homogeneity is subadditivity $\rho(X+Y) \leq \rho(X) + \rho(Y)$ which is a more general characterization of diversification. We prove this consequence as follows: $\rho(X+Y) = \rho(\frac{\lambda}{\lambda}X + \frac{1-\lambda}{1-\lambda}Y) \leq \lambda\rho(\frac{1}{\lambda}X) + (1-\lambda)\rho(\frac{1}{1-\lambda}Y) = \rho(X)+\rho(Y)$.

Definition 3.1.5 : A monetary risk measure ρ is called law invariant if $X =^d Y$ (in distribution) then $\rho(X) = \rho(Y)$.

Definition 3.1.6 : the average value at risk AV@R at level $\lambda \in (0, 1]$ is defined by

$$AV@R_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda V@R_\zeta(X)d\zeta$$

it is also called the Expected Shortfall.

Claim 3.1.1 : AV@R is a coherent risk measure.

proof: Monotonicity and cash invariance are inherited from V@R from example3.1.1. Lets check positive homogeneity. to do so we need to check if V@R is a positive homogeneous measure:

$$V@R_\zeta(\lambda X) = \inf\{m \mid p(\lambda X + m < 0) \leq \zeta\} = \inf\{\lambda m \mid p(\lambda(X+m) < 0) \leq \zeta\}$$

$$= \inf\{\lambda m \mid p(X+m<0) \le \zeta\} = \lambda \inf\{m \mid p(X+m<0) \le \zeta\} = \lambda V@R_\zeta(X)$$

Lastly we need to check sub additivity i.e $AV@R_\lambda(X+Y) \le AV@R_\lambda(X)+AV@R_\lambda(Y)$. first we need to state two lemmas required for the proof.

Lemma 3.1.2: Let X be any random variable then $\exists U_X \sim U[0,1]$ (uniform random variable on $[0,1]$) such that $X = F_X^{-1}(U_X)$ where F is the cdf of X.

Lemma 3.1.3: Let $B_\lambda \in L^\infty$be the set of $bernoulli(\lambda), \lambda \in (0,1)$ random variables and let $A_X = I_{U_X \le \lambda} \in B_\lambda$then $E[XA_X] \ge E[XB_\lambda]\ \forall B \in B_\lambda$.

Coming back to the proof, from the second definition of V@R we get $V@R_\zeta(X) = F_X^{-1}(\zeta)$ then using the two lemmas we get $AV@R_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda V@R_\zeta(X)d\zeta = \frac{1}{\lambda}\int_0^\lambda F_X^{-1}(\zeta)d\zeta = \frac{1}{\lambda}E[F_X^{-1}(U_X)I_{U_X\le\lambda}] = \frac{1}{\lambda}E[XA_X] = \frac{1}{\lambda}\sup\{E[XA_X] \mid B \in B_\lambda\}, X \in L^\infty$, using that the supremium is sub additive thus AV@R is sub additive.

Definition 3.1.7: A weighted value at risk is defined by:

$$WV@R_q(X) = \int_{\lambda\in(0,1]} AV@R_\lambda(X)q(d\lambda)$$

Where q is a probability measure.

We can clearly see that WV@R is coherent since it inherits the properties from AV@R. WV@R is a better coherent measure since it possess nice properties for optimization and financial aspects (considers the whole distribution of X). so we will focus solely on this measure as nice coherent risk representative.

Remark 3.1.1 :V@R, AV@R and WV@R are all law invariant risk measures, which can clearly seen as they depend on the distribution function F_x of X.

Definition 3.1.8: An acceptance set associated with the risk measure $\rho$is defined as $A_\rho = \{X \in L^\infty \mid \rho(X) \le 0\}$(i.e the positions for which we have non positive risk).

Theorem 3.1.1: If $\rho$is a monetary risk measure with an acceptance set $A_\rho$then:

a) $A_\rho$is non empty, closed w.r.t the L^∞norm and satisfies the two conditions: $\inf\{m \in \mathbb{R} \mid m \in A_\rho\} > -\infty$and if $X \in A_\rho, Y \ge X \Rightarrow Y \in A_\rho$.

b) If we start with the acceptance set $A_\rho$then we can recover $\rho$from $A_\rho$by $\rho(X) = \inf\{m \in \mathbb{R} \mid X+m \in A_\rho\}$(i.e the smallest amount of money which will make X acceptable).

c) $\rho$is convex risk measure if and only if $A_\rho$is a convex set.

d) $\rho$is positive homogeneous measure if and only if $A_\rho$is a cone. Thus $\rho$is coherent risk measure if and only if $A_\rho$is a convex cone.

proof:

a) $A_\rho$is non empty since it always contains all $X \ge 0$. It is closed w.r.t L^∞norm because of Lemma3.1.1, it is easy to see it, take any sequence in A_ρ, $X_n \to X$ then$\mid \rho(X)-\rho(X_n) \mid \le \parallel X - X_n \parallel_\infty \Rightarrow \rho(X) \le \rho(X_n) + \parallel X - X_n \parallel_\infty \le \parallel X - X_n \parallel_\infty \to 0 \Longrightarrow \rho(X) \le 0$. For the first condition it is obvious since we are considering bounded positions. The second condition is a direct consequence of monotonicity axiom.

b) Using Cash Invariance$\rho(X) = \inf\{m \in \mathbb{R} \mid X+m \in A_\rho\} = \inf\{m \in \mathbb{R} \mid \rho(X+m) \le 0\} = \inf\{m \in \mathbb{R} \mid \rho(X) \le m\} = \rho(X)$.

c) $\Longrightarrow$if $X, Y \in A_\rho$then$\rho(\lambda X+(1-\lambda)Y) \le \lambda\rho(X)+(1-\lambda)\rho(Y) \le 0 \Rightarrow \lambda X+(1-\lambda)Y \in A_\rho$. $\Longleftarrow$If $A_\rho$is convex then let $X, Y \in L^\infty$and let $m_1, m_2 \in \mathbb{R}$ such that $X+m_1, Y+m_2 \in$

$A_\rho \Rightarrow (\lambda(X+m_1)+(1-\lambda)(Y+m_2)) \in A_\rho \Rightarrow \rho(\lambda(X+m_1)+(1-\lambda)(Y+m_2)) \leq 0$ by cash invariance $\Rightarrow \rho(\lambda(X+m_1)+(1-\lambda)(Y+m_2)) = \rho(\lambda X+(1-\lambda)Y) - \lambda m_1 - (1-\lambda)m_2 \leq 0 \Rightarrow \rho(\lambda X+(1-\lambda)Y) \leq \lambda m_1 + (1-\lambda)m_2 = \lambda\rho(X)+(1-\lambda)\rho(Y)$.

d) $\Longrightarrow$Letρbe a positive homogeneous measure and $X \in A_\rho$ then $\rho(\lambda X) = \lambda\rho(X) \leq 0 \Rightarrow \lambda\rho(X) \in A_\rho$ which means $A_\rho$is a cone. $\Longleftarrow$Let A_ρ be a cone and $X \in L^\infty$and let $m_1 \in \mathbb{R} \mid X+m_1 \in A_\rho \Rightarrow \rho(\lambda X+\lambda m_1) \leq 0 \Rightarrow \rho(\lambda X) \leq \lambda m_1$. For the other inequality direction, let $m_1 < \rho(X) \Rightarrow X+m_1 \notin A_\rho \Rightarrow \lambda(X+m_1) \notin A_\rho \Rightarrow \lambda m_1 < \rho(\lambda X)$.

So any monetary convex and coherent risk measure is characterized by a corresponding acceptance set which carries the same information as our measure.

Definition 3.1.9: A measure $\varrho : L^\infty \to \mathbb{R}_+$is called a performance measure or acceptability index if it satisfies the following axioms:

1) Convexity : $\varrho(\lambda X+(1-\lambda)Y) \leq \lambda\varrho(X)+(1-\lambda)\varrho(Y)$ $\lambda \in [0,1]$,$\forall X, Y \in L^\infty$.

2) Monotonicity: If $X \leq Y \Rightarrow \varrho(X) \leq \varrho(Y)$ $\forall X, Y \in L^\infty$.

3) Scale Invariance: $\varrho(\lambda X) = \varrho(X)$ $\lambda \in [0,1]$,$\forall X \in L^\infty$.

4) Fatou Property: This defines a sense of continuity for the measure. if $X_n$is a sequence of random variables such that $X_n \to X$ in probability, $\mid X_n \mid \leq 1$ and $\varrho(X_n) \geq x$ then $\varrho(X) \geq x$.

5) Law Invariance: If $X =^d Y$ meaning X and Y have the same distribution, then $\varrho(X) = \varrho(Y)$.

6) Arbitrage Consistency: If $X \geq 0 \Rightarrow \varrho(X) = \infty$.

7) Consistency with the second order stochastic dominance: If $\varrho(X) \leq \varrho(Y) \Rightarrow E[u(X)] \leq E[u(Y)]$ where u is utility function.

8) Consistency with expectation: If$\begin{cases} E[X] > 0 \Rightarrow & \varrho(X) > 0 \\ E[X] < 0 \Rightarrow & \varrho(X) = 0 \end{cases}$

The acceptability index is similar to the coherent risk measure with notable difference in scale invariance. The coherent risk measures are sensitive to the scale of investment while performance measures only care about the direction of these invesments. Now we will link the acceptability index to our coherent risk measures.

Theorem 3.1.2: A map ϱ is called acceptability index if and only if $\exists (D_x)_{x\in\mathbb{R}_+} \subseteq P$ (family of subsets in the probability measures set) such that $D_x \subseteq D_y$ for $x \leq y$ and

$$\varrho(X) = \sup\{x \in \mathbb{R}_+ \mid \rho_x(X) \leq 0\}$$

where $\rho_x(X) = \sup_{q\in D_x} E_q[-X]$ (which we will prove later as representation theorem of coherent risk measures).

So our acceptability index is basically the maximum probability subset such that the position X is still acceptable. It measures the maximum consensus (acceptability) of the position by the market participants using their correspondent pricing probabilities.

Definition 3.1.10: The acceptance set associated with the performance measure ϱat level x is defined as follows:

$$A_x = \{X \in L^\infty \mid \varrho(X) \geq x\}$$

Lets compare the acceptance sets of both coherent risk measures and acceptability indices by the following example:

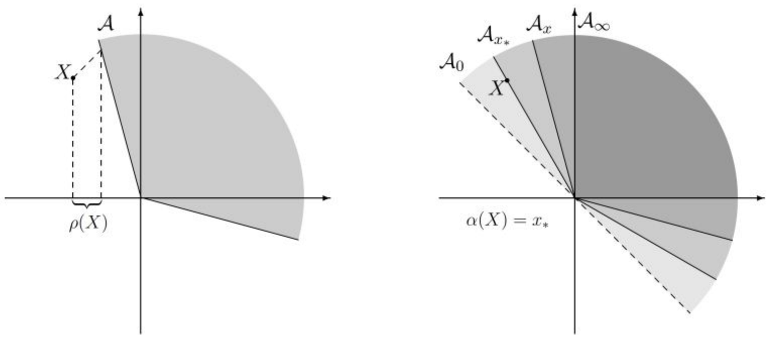


Figure 1. **(a)** Acceptability cones associated with coherent risks.
(b) Acceptability cones associated with acceptability indices.

Example 3.1.3: Let ρ be a coherent risk measure with a convex cone set Q (which we will discuss in detail later), ϱ is our acceptability index, and $X \in L^\infty$. Let $\mid \Omega \mid = 2, \Omega = \{\omega_1, \omega_2\}$Then our L^∞space is two dimensional Banach space. Our corresponding acceptance sets are:

$$A_\rho = \{X \in L^\infty \mid \rho(X) \leq 0\}$$

$$A_x = \{X \in L^\infty \mid \varrho(X) \geq x\}, x \in \mathbb{R}_+$$

Lets draw the following sets in our L^∞ space. we get:

So the performance measure ϱ is taking each $X \in L^\infty$and index it with respect to the maximum probability measure set. then the acceptance set at level x of $\varrho$is taking all the indexed $\{X \in L^\infty \mid \varrho(X) \geq x\}$. we notice that $A_x \subseteq Q^{max}, x \in \mathbb{R}_+$ where $Q^{max}$is the largest probability set for which the coherent risk measure representation holds (i.e we have a coherent risk measure which we will talk in detail later on). Our performance measure is basically indexing the positions w.r.t the probability measures sets which in return gives us a picture on where the particular position stands w.r.t the market acceptance. In conclusion, the coherent risk measures can be characterized in two levels, the first by their acceptance sets then by indexing the probability measure sets in which our coherent risk measure holds, which in return gives a family of indexed coherent measures $(\rho_x)_{x\in\mathbb{R}_+}$that describe the families of accepted positions $A_x, x \in \mathbb{R}_+$.

Definition 3.1.11: A finitely additive measure$q$ is defined as $q(\cup_{i=1}^n A_i) = \sum_{i=1}^n q(A_i), n < \infty$ for any disjoint subsets $A_{i's}$ of Ω .

Definition 3.1.12: A σadditive measure q is defined as $q(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty q(A_i)$ $\forall A_i$disjoint subsets of Ω.

Definition 3.1.13: A Total Variation TV norm of a finitely additive measure q is defined by $\parallel q \parallel_{TV} = \sup\{\sum_{i=1}^n \mid q(A_i) \mid, A_i \in \mathcal{F}, A_i \cap A_j = \phi, i \neq j\}$.

Definition 3.1.14: A Total Variation TV norm of a σ additive measure q is defined by $\parallel q \parallel_{TV} = \{\frac{1}{2}\sum_{\omega\in\Omega} \mid q(\omega) \mid\}$.

Theorem 3.1.3: Any convex risk measure can be represented by:

$$\rho(X) = \max_{q\in M_1}(E_q[-X] - \hbar(q))$$

where $M_1$is the set of all finitely additive functionals on $(\Omega,\mathcal{F})$ with finite total variation $\parallel q \parallel_{TV}<\infty$which are normalized by $q(\Omega)=1$. and $\hbar(q)=\sup_{X\in A_\rho}E_q[-X]$ is called the penalty function for the corresponding probability measure. $A_\rho$is the acceptance set of cash flows associated with the risk measure ρ.

proof: First we will prove $\rho(X)\geq\sup_{q\in M_1}(E_q[-X]-\hbar(q))$. Let $Y=X+\rho(X)\in A_\rho\Rightarrow\forall q\in M_1,\hbar(q)\geq E_q[-Y]=E_q[-X]-\rho(X)\Rightarrow\rho(X)\geq E_q[-X]-\hbar(q)\Rightarrow\rho(X)\geq\sup_{q\in M_1}(E_q[-X]-\hbar(q))$.

Second, for a given X we need to construct $q_X\in M_1$such that $\rho(X)\leq(E_{q_X}[-X]-\hbar(q_X))$. Let's prove this inequality for $X\in B^*=\{X\in L^\infty\mid\rho(X)=0\}$ it is clear that this set is convex, and by cash invariance we can extend the claim to any $X\in L^\infty$since $\forall Y\in L^\infty$it can be written as $\rho(Y)=\rho(X+k)=\rho(X)-k$ for some $k\in\mathbb{R}$. Let $B=\{X\in A_\rho\mid\rho(X)<0\}$ it is clear that B is an open convex set since its complement is closed by lipschitz continuity and $\rho$is convex. Then we apply the separation argument according to the following Theorem:

Thoerem 3.1.4: If $W=L^\infty$is a vector space with two disjoint convex sets B and $B^*$for which one has an interior point (B in our case) then they can be separated by a continuous non zero linear functional on L^∞i.e $\exists l$ such that $l(X)\leq l(Y)\ \forall X\in B^*,\forall Y\in B$.

Coming back to the proof and using the above Theorem we get $l(X)\leq l(Y)\Rightarrow l(X)\leq\inf_{Y\in B}l(Y)=b$. we claim that $l(Y)\geq 0$ if $Y\geq 0$. Using monotonicity and cash invariance we get that $1+\lambda Y\in B,\forall\lambda>0$ because$1+\lambda Y>0\Rightarrow\rho(1+\lambda Y)<0\Rightarrow 1+\lambda Y\in B$. Plugging it in the linear operator we get $l(X)\leq l(1+\lambda Y)=l(1)+\lambda l(Y),\forall\lambda>0$ not true if $l(Y)<0\ \Rightarrow l(Y)\geq 0$. The next claim is $l(1)>0$. since l is not identically zero then $\exists Y$ such that $l(Y)>0$, lets assume without loss of generality that$\parallel Y\parallel_\infty<1\Rightarrow 0<1-Y\leq 1\Rightarrow l(1-Y)\geq 0\Rightarrow l(1)\geq l(Y)>0\Rightarrow l(1)>0$. Using the below theorem we will get our desired q_X .

Theorem 3.1.5: $F\in L^\infty$ The Integral $l(F)=\int Fdq$ defines a one to one correspondence between the set of continuous linear functionals and the set of finitely additive probability measures M_1 on $(\Omega,\mathcal{F})$. Using the theorem we get that $E_{q_X}[Y]=\frac{l(Y)}{l(1)}\forall Y\in L^\infty$. Since $B\subset A_\rho\Rightarrow\hbar(q_x)=\sup_{Y\in A_\rho}E_{q_X}[-Y]\geq\sup_{Y\in B}E_{q_X}[-Y]=\sup_{Y\in B}-\frac{l(Y)}{l(1)}=-\inf_{Y\in B}\frac{l(Y)}{l(1)}=-\frac{b}{l(1)}\Rightarrow\hbar(q_x)\geq-\frac{b}{l(1)}$. Take any $Y\in A_\rho\Rightarrow Y+\epsilon\in B,\forall\epsilon>0\Rightarrow\hbar(q_x)=-\frac{b}{l(1)}$. it follows that $E_{q_X}[-X]-\hbar(q_x)=\frac{1}{l(1)}(b-l(X))\geq 0=\rho(X)\Rightarrow\rho(X)\leq E_{q_X}[-X]-\hbar(q_x)$ and this proves the other direction of inequality. The representation attains its supremium since M_1 is weak* compact in the dual space of L^∞ due to Banach-Alaoglu Theorem which is:

Theorem 3.1.6: Let E be a banach space with dual E^* then $\{l\in E\mid\parallel l\parallel_{E^*}\leq r\}$ is weak* compact for every $r\geq 0$.

And this concludes the proof of the representation.

Theorem 3.1.7: A risk measure ρ is coherent if and only if $\exists Q\subseteq M_1$ such that $\rho(X)=\sup_{q\in Q}E_q[-X],\ X\in L^\infty$. Moreover we can choose Q as a convex set such that the supremium is attained.

proof: To prove the result we need to prove the following corollary:

Corollary 3.1.1: For a coherent risk measure the penalty function takes on two values:

$$\hbar(q)=\begin{cases}0 & q\in Q\\ +\infty & otherwise\end{cases}$$

In particular $\rho(X) = \max_{q \in Q^{max}} E_q[-X],\ X \in L^\infty$ for the convex set $Q^{max} = \{q \in M_1 \mid \hbar(q) = 0\}$ and this set is the largest set for which the coherent risk measure representation holds.

proof : Since ρ is coherent then A_ρ is a cone, which means if $X \in A_\rho$ then $\lambda X \in A_\rho, \forall \lambda > 0 \Rightarrow \hbar(q) = \sup_{X \in A_\rho} E_q[-X] = \sup_{\lambda X \in A_\rho} E_q[-\lambda X] = \lambda \sup_{\lambda X \in A_\rho} E_q[-X] = \lambda \hbar(q) \Rightarrow \hbar(q) = \begin{cases} 0 \\ +\infty \end{cases}, \forall q \in M_1$Thus we can define the set $Q^{max} = \{q \in M_1 \mid \hbar(q) = 0\} \subseteq M_1$ as the maximal set for which the representation $\rho(X) = \max_{q \in Q^{max}} E_q[-X],\ X \in L^\infty$ holds, and $\forall Q \subset Q^{max}$ the representation $\rho(X) = \sup_{q \in Q} E_q[-X],\ X \in L^\infty$ follows. We have to note that the function $\hbar(q)$ is convex lower semi continuous function since it is the supremium of an affine continuous functions on M_1 , thus the set Q^{max} is convex, and since Q^{max} is the preimage of closed set then it is closed as well. And since Q^{max} is a subset of a compact set then it is compact and consequently the supreimum is attained.

Previously we considered M_1as the set of finitely additive functionals q with finite total variation which are normalized by $q(\Omega) = 1$. The dual space of our positions (cash flows) $L^\infty$is a space of bounded linear functionals $(L^\infty)^*$ which can be identified by a set of finitely additive functionals q with finite total variation, lets call the dual $\Gamma = (L^\infty)^*$. Thus if we consider the set of probability measures M_2 which are σadditive then we know that $M_2 \subseteq M_1 \subseteq \Gamma$. Also $\forall q \in M_2, \parallel q \parallel_{TV} = 1 < \infty$ but this set might be not closed in $\Gamma = (L^\infty)^*$ so we can't use Theorem3.1.5 and our convex risk measure might not achieve its supremium on the subset thus it is characterized by the general representation

$$\rho(X) = \sup_{q \in M_2} (E_q[-X] - \hbar(q))$$

Theorem 3.1.8 : A convex risk measure ρ is continuous from above and law invariant if and only if it admits the representation

$$\rho(X) = \sup_{q \in M_2((0,1])} (WV@R_q(X) - \hbar(q))$$

Where $M_2((0,1])$ is the set of probability measures on $(0,1]$.

Corollary 3.1.2: A coherent risk measure ρ is continuous from above and law invariant if and only if it admits the representation

$$\rho(X) = \sup_{q \in M_2((0,1])} WV@R_q(X)$$

A law invariant coherent risk measure is represented as $WV@R$ with some choice of probability measure. so in order to study the class of law invariant coherent risk measures we have to only focus on $WV@R$ as a prime example.

Definition 3.1.15: A Wang transform (or distortion function) of a cumulative density function F_X is an increasing function $\Psi : [0,1] \to [0,1]$ such that $\Psi(0) = 0, \Psi(1) = 1$. and its dual is defined as $\Psi^{\sim}(x) = 1 - \Psi(1-x)$. if Ψ is a concave increasing function with the above properties then it is called a concave distortion. The distortion function can be generally defined as:

$$\Psi_q(y) = \int_0^y \int_{(z,1]} \frac{1}{\lambda} q(d\lambda) dz$$

Now we will try to represent our WV@R law invariant coherent risk measure with respect to this transform.

Theorem 3.1.9 : Let $q \in M_2((0,1])$ and Ψ be a concave distortion of our cumulative density function F_X defined by q, then

$$WV@R_q(X) = -\int_{\mathbb{R}} yd(\Psi_q(F_X(y)))$$

proof: Lets define the Tail Value a Risk as $TV@R_\lambda(X) = E[-X \mid -X \geq V@R_\lambda(X)]$ and we use the fact that under a rich probability space (a space which supports a random variable with continuous distribution) $TV@R_\lambda(X) = AV@R_\lambda(X)$. Then $WV@R_q(X) = \int_{\lambda\in(0,1]} AV@R_\lambda(X)q(d\lambda) = \int_{\lambda\in(0,1]} TV@R_\lambda(X)q(d\lambda) = -\int_{\lambda\in(0,1]} \frac{1}{\lambda}\int_{(-\infty,-V@R_\lambda(X)]} yd(F_X(y)q(d\lambda) = -\int_{\mathbb{R}} y\int_{(F(y),1]} \frac{1}{\lambda}q(d\lambda)d(F_X(y)) = -\int_{\mathbb{R}} yd(\Psi_q(F_X(y)))$.

In order to study $WV@R_q(X) = -\int_{\mathbb{R}} yd(\Psi_q(F_X(y)))$ as a prime example of law invariant coherent risk measure we need to define a suitable choices of concave distortions which will represent our risk measure perfectly.

Example 3.1.4 : A suitable choices of concave distortions are as follows:

1. $\Psi_q(x) = 1-(1-x)^{q+1}, q \in \mathbb{R}_+, x \in [0,1]$, which define the law invariant coherent risk measure $MINV@R(X)$.

2. $\Psi_q(x) = x^{\frac{1}{q+1}}, q \in \mathbb{R}_+, x \in [0,1]$, which define the law invariant coherent risk measure $MAXV@R(X)$.

3. $\Psi_q(x) = (1-(1-x)^{q+1})^{\frac{1}{q+1}}, q \in \mathbb{R}_+, x \in [0,1]$, which define the law invariant coherent risk measure $MAXMINV@R(X)$.

4. $\Psi_q(x) = 1-(1-x^{\frac{1}{q+1}})^{q+1}, q \in \mathbb{R}_+, x \in [0,1]$, which define the law invariant coherent risk measure $MINMAXV@R(X)$.

MINMAXV@R will be our choice of law invariant risk measure in studying optimal strategies since it satisfies a nice limiting properties that deem useful.

4.2 Simulations

In this chapter we will test our optimal strategies under $MINMAXV@R$ (we will use $MINMAXV@R$ and ρ interchangeably) which is the prime benchmark of law invariant coherent risk measures and then analyze the results accordingly. In the following scenarios we will use parameters in each scenario to organize our settings. we will use the input as follows **(Model type in time horizon t, Number of risky assets S, Number of risk-less assets B, Number of Contingent Claims, Initial value S_0, Initial value B_0, Type of Contingent Claim, Probability measure q, Interest rate r, Risky asset model type in dynamics with parameters)**. Also the following assumptions hold for all the scenarios:

- No arbitrage
- Liquid market

- Predictable strategies
- Self-financed portfolio
- No extra costs (transaction cost,...)
- Bounded cash flows $X \in L^{\infty}$
- Risk-less assets follow the following model: $B_t = B_0(1+r)^t, \forall t \in N, r = constant.$
- Use of all initial investment z

In the following scenarios we will get ρ matrix for (α, β) or (α, β, γ) combinations. Then we will compute the mean numerical gradient on ρ i.e $\frac{d\rho}{d(\alpha,\beta,\gamma)}' = (\frac{\partial\rho}{\partial\beta}, \frac{\partial\rho}{\partial\alpha}, \frac{\partial\rho}{\partial\gamma})$ or $\frac{d\rho}{d(\alpha,\beta)}' = (\frac{\partial\rho}{\partial\beta}, \frac{\partial\rho}{\partial\alpha})$ where $\frac{\partial\rho}{\partial\beta} = mean(FX)$, $\frac{\partial\rho}{\partial\alpha} = mean(FY)\frac{\partial\rho}{\partial\gamma} = mean(FZ)$, $FX, FYFZ$ are the gradient matrices for ρ. The mean numerical gradient is taken with respect to one unit of α , β and γ.

4.2.1 Scenario 1 ($t = 0, 1$, 1, 1, 0, 1, 1, None, q, r , Log-normal(μ,σ^2))

Our initial investment is $z < \infty$. Then after initializing the prices we get $V_0^{(\alpha,\beta)} = \beta_1 B_0 + \alpha_1 S_0 = \alpha_1 + \beta_1 = z$.

Looking into $t = 1$, $X = V_1^{(\alpha,\beta)} = (1+r)\beta_1 + \alpha_1 e^{\mu+\sigma\varepsilon}$ then as soon as the prices are realized we re-balance our portfolio $X = (1+r)\beta_2 + \alpha_2 e^{\mu+\sigma\varepsilon}$, where $\varepsilon \sim Normal(0,1)$.

So the question is which (α_1, β_1) we should use to minimize our risk measure $MINMAXV@R_q(X)$?.

We know $S_1 \sim Lognormal(\mu, \sigma^2)$. then our risk measure becomes $MINMAXV@R_q(X) = \alpha_1 MINMAXV@R_q(S_1) - \beta_1(1+r) = -\alpha_1 \int_{\mathbb{R}} sd(1-(1-F_{S_1}(s)^{\frac{1}{q+1}})^{q+1}) - \beta_1(1+r)$. with F_{S_1} cdf of log-normal distribution of S_1 with $e^{\mu+\frac{\sigma^2}{2}}$ expectation and $(e^{2\mu+\sigma^2})(e^{\sigma^2}-1)$ variance.

We can clearly see that $\frac{d\rho}{d(\alpha,\beta)} = (\frac{\partial\rho}{\partial\beta_1}, \frac{\partial\rho}{\partial\alpha_1}) = (-(1+r),\ MINMAXV@R_q(S_1))$ so an optimal strategy for α_1 depends on the risk measure while $\beta_1$depends only on the interest rate.

To simulate our risk measure $MINMAXV@R_q(S_1)$ we need to tweak its representation. Taking the integration where the derivative of Ψ is bounded, we get that $\Psi_q(F_{S_1}(s)) = 1 - (1 - F_{S_1}(s)^{\frac{1}{q+1}})^{q+1} \Rightarrow \frac{d\Psi_q(F_{S_1}(s))}{ds} = (\frac{1-F_{S_1}(s)^{\frac{1}{q+1}}}{F_{S_1}(s)^{\frac{1}{q+1}}})^q \cdot f_{S_1}(s)$ where $f_{S_1}(s)$ is the pdf of S_1,$\Rightarrow MINMAXV@R_q(S_1) = -\int_{\mathbb{R}} s(\frac{1-F_{S_1}(s)^{\frac{1}{q+1}}}{F_{S_1}(s)^{\frac{1}{q+1}}})^q \cdot f_{S_1}(s)ds$.

4.2.2 Results of Scenario 1

This Scenario is done by taking 100 $\alpha_1's$ permuted with 100 $\beta_1's$ (10000 permutations in total).

The first result of the simulation is done by varying the interest rate r while keeping all other parameters fixed ($\mu = \sigma^2 = q = 1$).

r	$\frac{\partial \rho}{\partial \beta_1}$	$\frac{\partial \rho}{\partial \alpha_1}$
0.0	-1.0000	-1.1175
0.1	-1.1000	-1.1175
0.2	-1.2000	-1.1175
0.3	-1.3000	-1.1175
0.4	-1.4000	-1.1175
0.5	-1.5000	-1.1175
0.6	-1.6000	-1.1175
0.7	-1.7000	-1.1175
0.8	-1.8000	-1.1175
0.9	-1.9000	-1.1175
1.0	-2.0000	-1.1175

This table shows the gradient vector $(\frac{\partial \rho}{\partial \beta_1}, \frac{\partial \rho}{\partial \alpha_1})$ for different $r's$. It is clear that as r increases we get a lower risk by buying a riskless asset, especially if $r \geq 0.2$ then it is optimal to buy only riskless assets. It is also important to note that $\frac{\partial \rho}{\partial \beta_1}$ has a linear increase with respect to r as clearly shown from the gradient $(-(1+r),\ MINMAXV@R_q(S_1))$.

Now in the next simulation we will fix $r = \sigma^2 = q = 1$ and vary μ to see the effect. the results are summarized in the following table.

μ	$\frac{\partial \rho}{\partial \beta_1}$	$\frac{\partial \rho}{\partial \alpha_1}$
0.0	-2.0000	-0.4111
0.5	-2.0000	-0.6778
1.0	-2.0000	-1.1175
1.5	-2.0000	-1.8424
2.0	-2.0000	-3.0377
2.5	-2.0000	-5.0083
3.0	-2.0000	-8.2573
3.5	-2.0000	-13.6139
4.0	-2.0000	-22.4456
4.5	-2.0000	-37.0065
5.0	-2.0000	-61.0133

From the table above we can see that $\frac{\partial \rho}{\partial \alpha_1}$ decreases exponentially with respect to μ. and for $\mu > 1.5$ it is optimal to buy only risky assets.

Next we will vary σ^2over the constants $\mu = r = q = 1$.

σ^2	$\frac{\partial \rho}{\partial \beta_1}$	$\frac{\partial \rho}{\partial \alpha_1}$
0.00001	-2.0	-2.7182
0.0009	-2.0	-2.7148
0.0019	-2.0	-2.7110
0.0029	-2.0	-2.7071
0.0039	-2.0	-2.7033
0.0049	-2.0	-2.6995
0.0060	-2.0	-2.6956
0.0070	-2.0	-2.6918
0.0080	-2.0	-2.6880
0.0090	-2.0	-2.6842

σ^2	$\frac{\partial \rho}{\partial \beta_1}$	$\frac{\partial \rho}{\partial \alpha_1}$
0.01	-2.0	-2.6804
0.011	-2.0	-2.3383
0.22	-2.0	-2.0631
0.32	-2.0	-1.8405
0.42	-2.0	-1.6598
0.52	-2.0	-1.5126
0.62	-2.0	-1.3928
0.72	-2.0	-1.2955
0.83	-2.0	-1.2169
0.93	-2.0	-1.1542

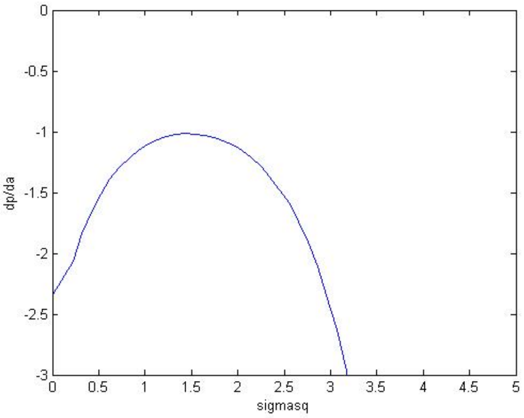
σ^2	$\frac{\partial \rho}{\partial \beta_1}$	$\frac{\partial \rho}{\partial \alpha_1}$
1.03	-2.0	-1.1051
1.13	-2.0	-1.0679
1.23	-2.0	-1.0414
1.33	-2.0	-1.0245
1.44	-2.0	-1.0168
1.54	-2.0	-1.0179
1.64	-2.0	-1.0275
1.74	-2.0	-1.0458
1.84	-2.0	-1.0731
1.95	-2.0	-1.1099

σ^2	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$
2.05	-2.0	-1.1570
2.15	-2.0	-1.2155
2.25	-2.0	-1.2867
2.35	-2.0	-1.3722
2.45	-2.0	-1.4742
2.56	-2.0	-1.5954
2.66	-2.0	-1.7389
2.76	-2.0	-1.9088
2.86	-2.0	-2.1099
2.96	-2.0	-2.3483

σ^2	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$
3.07	-2.0	-2.6314
3.17	-2.0	-2.9686
3.27	-2.0	-3.3712
3.37	-2.0	-3.8536
3.47	-2.0	-4.4337
3.57	-2.0	-5.1340
3.68	-2.0	-5.9828
3.78	-2.0	-7.0160
3.88	-2.0	-8.2792
3.98	-2.0	-9.8302

σ^2	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$
4.08	-2.0	-11.7437
4.19	-2.0	-14.1151
4.29	-2.0	-17.0681
4.39	-2.0	-20.7628
4.49	-2.0	-25.4078
4.59	-2.0	-31.2761
4.69	-2.0	-38.7263
4.80	-2.0	-48.2315
4.90	-2.0	-60.4190
5.0	-2.0	-76.1234

Varying σ^2 gives us a very interesting results. $\frac{\partial\rho}{\partial\alpha_1}$ increases then decreases in a concave fashion while $\frac{\partial\rho}{\partial\beta_1}$ starts remains constant as predicted. By the results, if $\sigma^2 \geq 2.86$ or $\sigma^2 \leq 0.22$ for $r = 1$ then it is optimal to buy only risky assets otherwise riskless asset give better $d\rho$ per unit if $2.86 > \sigma^2 > 0.22$.



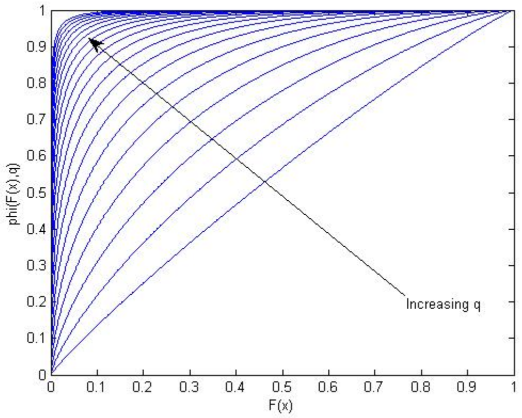
This concave behavior in the above figure is due to the fatter tail vs the leftward shift of the log-normal distribution at higher σ^2 which implies a higher probability for decreasing $[0, \epsilon)$ interval of S_1values which compensated by the fatter tail gives better risk score after some σ^2threshold.

The last result is investigating the effect of q (probability measure choice) on our risk measure. The results are as follows:

q	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$
0.1000	-2.0	-3.7456
0.3053	-2.0	-2.6965
0.5105	-2.0	-2.0179
0.7158	-2.0	-1.5528
0.9211	-2.0	-1.2204
1.1263	-2.0	-0.9751
1.3316	-2.0	-0.7898
1.5368	-2.0	-0.6469
1.7421	-2.0	-0.5349
1.9474	-2.0	-0.4459

q	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$
2.1526	-2.0	-0.3744
2.3579	-2.0	-0.3163
2.5632	-2.0	-0.2688
2.7684	-2.0	-0.2296
2.9737	-2.0	-0.1969
3.1789	-2.0	-0.1697
3.3842	-2.0	-0.1467
3.5895	-2.0	-0.1273
3.7947	-2.0	-0.1108
4.0000	-2.0	-0.0968

In the above table we see an negative inverse relation between $\frac{\partial\rho}{\partial\alpha_1}$ and q while $\frac{\partial\rho}{\partial\beta_1}$ remains constant.



Looking at above Figure of plots of different $q's$ for Ψ we see that as q increases Ψbecomes more concave and this explains the increasing $\frac{\partial\rho}{\partial\alpha_1}$ since we are giving negative outcomes higher measure and value (severe judging).

4.2.3 Scenario 2 ($t = 0, 1$, 1, 1, 0, 1, 1, None, q, r , Uniform$(0,b)$)

In this scenario we consider $S_1 \sim U[0, b]$ with expectation $\frac{b}{2}$ and variance $\frac{b^2}{12}$. We simulate $MINMAXV@R_q(S_1)$ same as Scenario 1 for different (α_1, β_1) and we get the following results.

4.2.4 Results of Scenario 2

Fixing $b = q = 1$ and varying r we get the following result:

r	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$
0.0	-1.0000	-0.1667
0.1	-1.1000	-0.1667
0.2	-1.2000	-0.1667
0.3	-1.3000	-0.1667
0.4	-1.4000	-0.1667
0.5	-1.5000	-0.1667
0.6	-1.6000	-0.1667
0.7	-1.7000	-0.1667
0.8	-1.8000	-0.1667
0.9	-1.9000	-0.1667
1.0	-2.0000	-0.1667

This expected result is similar to the Log-normal case. The only difference is in the constant value of $\frac{\partial\rho}{\partial\alpha_1} = -0.1667$ which is due to the choice of uniform distribution.

b	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$
1	-2.0000	-0.1667
2	-2.0000	-0.3333
3	-2.0000	-0.5000
4	-2.0000	-0.6667
5	-2.0000	-0.8333
6	-2.0000	-1.0000
7	-2.0000	-1.1667
8	-2.0000	-1.3333

b	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$
9	-2.0000	-1.5000
10	-2.0000	-1.6667
11	-2.0000	-1.8333
12	-2.0000	-2.0000
13	-2.0000	-2.1667
14	-2.0000	-2.3333
15	-2.0000	-2.5000
16	-2.0000	-2.6667

Varying b in the above result gives us a better $\frac{\partial\rho}{\partial\alpha_1}$ which is due the increasing expectation and variance (higher positive values are included) of S_1 that gives us a better risk score. The decreasing $\frac{\partial\rho}{\partial\alpha_1}$ is linear in nature which leads to demand a very high b ($b \geq 12$ for $r = 1$) in order to break even and consider only buying risky assets and for $b < 12$ an investor should only buy riskless assets.

In the next result we check different $q's$ for $b = r = 1$.

q	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$	q	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$
0.1000	-2.0	-0.4518	2.1526	-2.0	-0.0414
0.3053	-2.0	-0.3642	2.3579	-2.0	-0.0321
0.5105	-2.0	-0.2911	2.5632	-2.0	-0.0248
0.7158	-2.0	-0.2312	2.7684	-2.0	-0.0192
0.9211	-2.0	-0.1827	2.9737	-2.0	-0.0148
1.1263	-2.0	-0.1437	3.1789	-2.0	-0.0114
1.3316	-2.0	-0.1127	3.3842	-2.0	-0.0088
1.5368	-2.0	-0.0880	3.5895	-2.0	-0.0067
1.7421	-2.0	-0.0686	3.7947	-2.0	-0.0052
1.9474	-2.0	-0.0533	4.0000	-2.0	-0.0040

This table shows similar results as Log-normal case with value differences due to the uniform distribution choice .

4.2.5 Scenario 3 ($t = 0,1$, 1, 1, 1, 1, 1, Call Option, q, r , Log-normal(μ,σ^2))

In this Scenario we consider shorting a call option payout $g(S_1) = (S_1 - k)_+$ for a strike price k. Our cash flow at time $t = 1$ becomes $X = V_1^{(\alpha,\beta,\gamma)} = (1+r)\beta_1 + \alpha_1 S_1 - \gamma_1(S_1 - k)_+$. We see that $MINMAXV@R_q(X) = -\int_0^\infty(\alpha_1 s - \gamma_1(s-k)_+)d(1-(1-F_{S_1^*}(s)^{\frac{1}{q+1}})^{q+1}) - \beta_1(1+r)$ where $F_{S_1^*}$ is the cdf of $\alpha_1 S_1 - \gamma_1(S_1 - k)_+$. We can split the integral into two parts to deal with the contingent claim function. $MINMAXV@R_q(X) = -\alpha_1\int_0^k sd(1-(1-F_{S_1}(s)^{\frac{1}{q+1}})^{q+1}) - (\alpha_1-\gamma_1)\int_k^\infty sd(1-(1-F_{S_1}(s)^{\frac{1}{q+1}})^{q+1}) - \gamma_1 k\int_k^\infty d(1-(1-F_{S_1}(s)^{\frac{1}{q+1}})^{q+1}) - \beta_1(1+r) =$

$= -\alpha_1\int_0^\infty sd(1-(1-F_{S_1}(s)^{\frac{1}{q+1}})^{q+1}) + \gamma_1\int_k^\infty sd(1-(1-F_{S_1}(s)^{\frac{1}{q+1}})^{q+1}) - \gamma_1 k(1-F_{S_1}(k)^{\frac{1}{q+1}})^{q+1} - \beta_1(1+r)$. So we have two integrals with cutoff k , one constant dependent on β_1 and another constant dependent on β_1 and k. We can compute $\frac{d\rho}{d(\alpha_1,\beta_1,\gamma_1)} = (\frac{\partial\rho}{\partial\beta_1}, \frac{\partial\rho}{\partial\alpha_1}, \frac{\partial\rho}{\partial\gamma_1}) = (-(1+r), MINMAXV@R_q(S_1), \int_k^\infty sd(1-(1-F_{S_1}(s)^{\frac{1}{q+1}})^{q+1}) - k(1-F_{S_1}(k)^{\frac{1}{q+1}})^{q+1}) =$

$= (-(1+r), MINMAXV@R_q(S_1), -MINMAXV@R_q(S_1 1_{S_1\geq k}) - k(1-F_{S_1}(k)^{\frac{1}{q+1}})^{q+1})$ so $\frac{\partial\rho}{\partial\gamma_1}$ depends on k as well as $MINMAXV@R_q(S_1)$. Next we run our simulations accordingly by varying one parameter and keeping the rest constant to one and compare the results.

4.2.6 Results of Scenario 3

In the first run we fix $\mu = \sigma^2 = q = k = 1$ and vary r and the results are as follows:

r	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$	$\frac{\partial\rho}{\partial\gamma_1}$
0.0	-1.0000	-1.1175	0.4674
0.1	-1.1000	-1.1175	0.4674
0.2	-1.2000	-1.1175	0.4674
0.3	-1.3000	-1.1175	0.4674
0.4	-1.4000	-1.1175	0.4674
0.5	-1.5000	-1.1175	0.4674
0.6	-1.6000	-1.1175	0.4674
0.7	-1.7000	-1.1175	0.4674
0.8	-1.8000	-1.1175	0.4674
0.9	-1.9000	-1.1175	0.4674
1.0	-2.0000	-1.1175	0.4674

This result is as expected as it depicts the no effect of r on α_1, γ_1 while $\frac{\partial\rho}{\partial\beta_1}$ is the only decreasing value. Which implies at any $r \geq 0$ it is optimal to buy only risk less assets since a combination of optimal (α_1, γ_1) only gives us $\frac{\partial\rho}{\partial\alpha_1} + \frac{\partial\rho}{\partial\gamma_1} = -0.6501$ in risk reduction.

The next simulation is investigating the effect of μon our risk measure while keeping all other parameters equal to one.

μ	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$	$\frac{\partial\rho}{\partial\gamma_1}$	$\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}$
0.0	-2.0000	-0.4111	0.0556	-0.3555
0.5	-2.0000	-0.6778	0.1774	-0.5004
1.0	-2.0000	-1.1175	0.4674	-0.6501
1.5	-2.0000	-1.8424	1.0611	-0.7813
2.0	-2.0000	-3.0377	2.1589	-0.8788
2.5	-2.0000	-5.0083	4.0678	-0.9405
3.0	-2.0000	-8.2573	7.2832	-0.9741
3.5	-2.0000	-13.6139	12.6240	-0.9899
4.0	-2.0000	-22.4456	21.4490	-0.9966
4.5	-2.0000	-37.0065	36.0075	-0.999
5.0	-2.0000	-61.0133	60.0136	-0.9997
6.0	-2.0000	-165.9000	164.9000	-1.0000
7.0	-2.0000	-450.8000	449.8000	-1.0000
8.0	-2.0000	-1225.5000	1224.5000	-1.0000
9.0	-2.0000	-3331.2000	3330.2000	-1.0000
10.0	-2.0000	-9055.2000	9054.2000	-1.0000

In the above table we can clearly see that a combination strategy of contingent claim payoff and stock holding climbs asymptotically to $\frac{\partial\rho}{\partial\alpha_1} + \frac{\partial\rho}{\partial\gamma_1} \to -1$ as $\mu \to \infty$, while $\frac{\partial\rho}{\partial\beta_1} = -2$ when $r = 1$. thus it is optimal to only buy riskless assets. If $r = 0$ then $\frac{\partial\rho}{\partial\beta_1} = -1$ and a riskless strategy is still optimal.

The following table is concerned with varying σ^2 while other parameters are kept to constant one.

σ^2	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$	$\frac{\partial\rho}{\partial\gamma_1}$	$\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}$
0.0100	-2.0000	-2.6804	1.6804	-1.0000
0.2726	-2.0000	-1.9286	0.9311	-0.9975
0.5353	-2.0000	-1.4920	0.5866	-0.9054
0.7979	-2.0000	-1.2360	0.4822	-0.7537
1.0605	-2.0000	-1.0921	0.4693	-0.6228
1.3232	-2.0000	-1.0259	0.5034	-0.5225
1.5858	-2.0000	-1.0214	0.5743	-0.4470
1.8484	-2.0000	-1.0748	0.6849	-0.3899
2.1111	-2.0000	-1.1926	0.8467	-0.3459
2.3737	-2.0000	-1.3922	1.0808	-0.3115

σ^2	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$	$\frac{\partial\rho}{\partial\gamma_1}$	$\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}$
2.6363	-2.0000	-1.7067	1.4226	-0.2841
2.8989	-2.0000	-2.1930	1.9311	-0.2619
3.1616	-2.0000	-2.9493	2.7056	-0.2437
3.4242	-2.0000	-4.1454	3.9168	-0.2286
3.6868	-2.0000	-6.0822	5.8663	-0.2159
3.9495	-2.0000	-9.3052	9.1000	-0.2051
4.2121	-2.0000	-14.8300	14.6342	-0.1959
4.4747	-2.0000	-24.6009	24.4131	-0.1879
4.7374	-2.0000	-42.4458	42.2650	-0.1809
5.0000	-2.0000	-76.1234	75.9487	-0.1747

We can see a similar behavior of $\frac{\partial\rho}{\partial\beta_1}$, $\frac{\partial\rho}{\partial\alpha_1}$ as scenario 1 as expected. Also $\frac{\partial\rho}{\partial\gamma_1}$ follows a concave structure similar to $\frac{\partial\rho}{\partial\alpha_1}$. The interesting result is the combined strategy of (α_1,γ_1) which exhibits a property of $-1<\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}<0$ and asymptotic behavior on both ends. As $\sigma^2\to 0$ $\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}\to -1$ which is equal to $\frac{\partial\rho}{\partial\beta_1}$ at $r=0$ and $\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}\to 0$ as $\sigma^2\to\infty$ which means a riskless strategy is best regardless of σ^2 .

In This simulation we choose different $q's$ and keep the rest parameters equal to one.

q	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$	$\frac{\partial\rho}{\partial\gamma_1}$	$\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}$
0.1000	-2.0	-3.7456	2.8230	-0.9226
0.3053	-2.0	-2.6965	1.8251	-0.8713
0.5105	-2.0	-2.0179	1.2070	-0.8109
0.7158	-2.0	-1.5528	0.8079	-0.7449
0.9211	-2.0	-1.2204	0.5438	-0.6765
1.1263	-2.0	-0.9751	0.3667	-0.6084
1.3316	-2.0	-0.7898	0.2471	-0.5427
1.5368	-2.0	-0.6469	0.1661	-0.4808
1.7421	-2.0	-0.5349	0.1112	-0.4237
1.9474	-2.0	-0.4459	0.0741	-0.3718

q	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$	$\frac{\partial\rho}{\partial\gamma_1}$	$\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}$
2.1526	-2.0	-0.3744	0.0491	-0.3253
2.3579	-2.0	-0.3163	0.0324	-0.2839
2.5632	-2.0	-0.2688	0.0213	-0.2475
2.7684	-2.0	-0.2296	0.0139	-0.2157
2.9737	-2.0	-0.1969	0.0090	-0.1880
3.1789	-2.0	-0.1697	0.0058	-0.1639
3.3842	-2.0	-0.1467	0.0037	-0.1430
3.5895	-2.0	-0.1273	0.0024	-0.1249
3.7947	-2.0	-0.1108	0.0015	-0.1093
4.0000	-2.0	-0.0968	0.0009	-0.0958

Varying $q's$ gives a similar result as varying $q's$ in Scenario 1. Also $\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}$ exhibit a similar behavior as varying σ^2 in the previous simulation. The only difference is being $\frac{\partial\rho}{\partial\gamma_1}$ increase and $\frac{\partial\rho}{\partial\alpha_1}$ decrease monotonically as q increases while $\frac{\partial\rho}{\partial\gamma_1}$ decrease and $\frac{\partial\rho}{\partial\alpha_1}$ increase monotonically by varying σ^2.

In this last simulation we will vary k to measure its effect while keeping other parameters equal to one.

k	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$	$\frac{\partial\rho}{\partial\gamma_1}$	$\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}$
0.5	-2.0	-1.1175	0.7058	-0.4117
1	-2.0	-1.1175	0.4674	-0.6501
1.5	-2.0	-1.1175	0.3239	-0.7936
2	-2.0	-1.1175	0.2325	-0.8850
2.5	-2.0	-1.1175	0.1714	-0.9461
3	-2.0	-1.1175	0.1292	-0.9883
3.5	-2.0	-1.1175	0.0992	-1.0183
4	-2.0	-1.1175	0.0773	-1.0402
4.5	-2.0	-1.1175	0.0611	-1.0564
5	-2.0	-1.1175	0.0488	-1.0687

k	$\frac{\partial\rho}{\partial\beta_1}$	$\frac{\partial\rho}{\partial\alpha_1}$	$\frac{\partial\rho}{\partial\gamma_1}$	$\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}$
5.5	-2.0	-1.1175	0.0394	-1.0781
6	-2.0	-1.1175	0.0321	-1.0854
6.5	-2.0	-1.1175	0.0263	-1.0912
7	-2.0	-1.1175	0.0218	-1.0957
7.5	-2.0	-1.1175	0.0181	-1.0994
8	-2.0	-1.1175	0.0152	-1.1023
8.5	-2.0	-1.1175	0.0128	-1.1047
9	-2.0	-1.1175	0.0108	-1.1067
9.5	-2.0	-1.1175	0.0092	-1.1083
10	-2.0	-1.1175	0.0079	-1.1096

Looking at the above table the only variable is $\frac{\partial\rho}{\partial\gamma_1}$ since it is the only one dependent on k. A (α_1, γ_1) strategy is optimal when $r = 0$ if $k \geq 3.5$. Asymptotically $\frac{\partial\rho}{\partial\gamma_1} \to 0$ as $k \to \infty$ thus $\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1} \to \frac{\partial\rho}{\partial\alpha_1}$ and $\frac{\partial\rho}{\partial\gamma_1} \to -\frac{\partial\rho}{\partial\alpha_1}$ as $k \to 0$ thus $\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1} \to 0$.

Remark 1: In this scenario it is optimal to buy only one contingent claim (minimum) as it contributes positively to the risk measure.

Remark 2: Numerically for any k , $\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1} \to -k$ as $\mu \to \infty$ or $\sigma^2 \to 0$ or $q \to 0$ .Thus our optimal strategy for $\frac{\partial\rho}{\partial\alpha_1}+\frac{\partial\rho}{\partial\gamma_1}$ bound is dependent on k. Thus if $k = 1$ and $r = 0$

then our optimal strategy is only buying riskless assets since $\frac{\partial\rho}{\partial\beta_1} = -1$ and $\frac{\partial\rho}{\partial\alpha_1} + \frac{\partial\rho}{\partial\gamma_1} \leq 1$. If $k > 1$ then it is dependent on the combination of σ^2, μ and q.

5 Utility Functionals Simulations

In Section 2.2 we represented our preference relation as a Savage representation $X \underset{-}{\propto} Y \iff E_p[u(X)] \leq E_p[u(Y)]$ where $E_p[u(X)] = U(X)$ and $E_p[u(Y)] = U(Y)$. The utility functionals in this case is taken over a set of subjective probability measures M_0but we assumed such infimum exist and the functional is reduced to the expected utility function of the random variable over this infimum. To study the optimal strategies under these functionals we do our simulations to similar scenarios as in the coherent risk measure $MINMAXV@R$ and we choose the most prominent utility functions - Exponential function.

$$u(c) = \begin{cases} \frac{(1-e^{-ac})}{a} & a \neq 0 \\ c & a = 0 \end{cases}$$

where a represents the degree of risk preference ($a > 0, a < 0, a = 0$ are risk aversion, risk seeking and risk neutral respectively).

In our case we only consider $a > 0$(risk aversion).

5.1 Scenario 1 ($t = 0, 1$, 1, 1, 0, 1, 1, None, None, r , Lognormal(μ,σ^2))

We start by considering the payoff $X = V_1^{(\alpha,\beta)} = \alpha_1 S_1 + \beta_1(1+r)$ which gives $U(X) = \int_{\beta_1(1+r)}^{\infty} u(x) f_X(x)dx$ where f_X is the pdf of the payoff. Since $\alpha_1 S_1 \sim Lognormal(ln(\alpha_1) + \mu, \sigma^2)$ then $X = \alpha_1 S_1 + \beta_1(1+r) \sim Lognormal(ln(\alpha_1) + \mu, \sigma^2)$ *Shifted by* $\beta_1(1+r)$. The parameters in this scenario are $(\mu, \sigma^2, r, 1-\nu, a)$and we see the effect of each on the resulted utility functional. Similar to $MINMAXV@R$ simulations we want to know the gradient of alpha and beta $(\frac{\partial U}{\partial \alpha}, \frac{\partial U}{\partial \beta})$ so we can determine the corresponding optimal strategies.

First lets study the behavior of the utility functional with an example.

Lets fix $\mu = a = \sigma^2 = 1$ and $r = 0.1$ and run $U(X)$ for 40 $\alpha_1's$ permuted with 40 $\beta_1's$ each starts from 1 to 40(1600 permutations in total). Then we plot them against U and also plot $(\frac{\partial U}{\partial \alpha_1}, \frac{\partial U}{\partial \beta_1})$ each in a separate graph then we get the following:

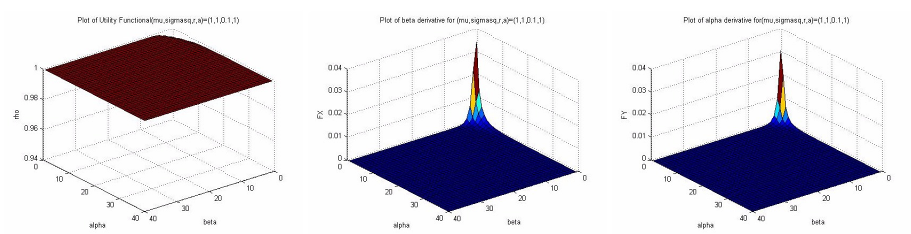
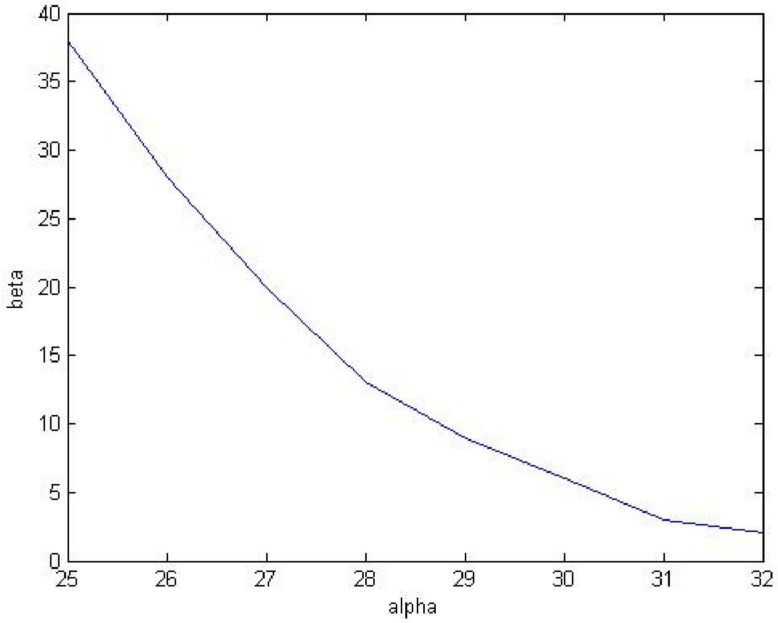


Figure 1: Utility Functional for $\mu = a = \sigma^2 = 1$ and $r = 0.1$

On the first plot we can clearly see that the general behavior for our utility functional is convergence to 1 since $\lim_{(\alpha,\beta)\to\infty} U(X) = \frac{1}{a}$ for $a > 0$ (risk averse). In our

Figure 2: Arc of optimal strategies



second and third plot $\frac{\partial U}{\partial \alpha_1}$ and $\frac{\partial U}{\partial \beta_1}$ leveled off to 0 . To investigate the optimal strategies ($\sup_{(\alpha_1,\beta_1)} U(X)$) we need to know the (α_1, β_1) on the light blue arc of the second and third plot (when the plot first touches the zero level set). To do this we will use the $U(X)$ matrix and find the first entry equal to one in each non all ones column. In our case it is equal to the following entries:

(α_1, β_1)	$(38, 25)$	$(28, 26)$	$(20, 27)$	$(13, 28)$	$(9, 29)$
	$(6, 30)$	$(3, 31)$	$(2, 32)$	$(1, 33)$	

and the below plot shows the arc of optimal strategies.

5.2 Results of Scenario 1

First we will vary r and hold all other parameters constant to one. we get the following results.

r	(α_1, β_1)	r	(α_1, β_1)	r	(α_1, β_1)
0.0	$\begin{pmatrix} 119 & 23 \\ 94 & 24 \\ 74 & 25 \\ 57 & 26 \\ 44 & 27 \\ 33 & 28 \\ 24 & 29 \\ 18 & 30 \\ 13 & 31 \\ 9 & 32 \\ 6 & 33 \\ 4 & 34 \\ 2 & 35 \end{pmatrix}$	0.4	$\begin{pmatrix} 99 & 17 \\ 70 & 18 \\ 49 & 19 \\ 33 & 20 \\ 22 & 21 \\ 13 & 22 \\ 8 & 23 \\ 4 & 24 \\ 2 & 25 \end{pmatrix}$	0.8	$\begin{pmatrix} 108 & 13 \\ 70 & 14 \\ 44 & 15 \\ 26 & 16 \\ 14 & 17 \\ 7 & 18 \\ 3 & 19 \end{pmatrix}$
0.1	$\begin{pmatrix} 116 & 21 \\ 90 & 22 \\ 69 & 23 \\ 52 & 24 \\ 38 & 25 \\ 28 & 26 \\ 20 & 27 \\ 13 & 28 \\ 9 & 29 \\ 6 & 30 \\ 3 & 31 \\ 2 & 32 \end{pmatrix}$	0.5	$\begin{pmatrix} 94 & 16 \\ 65 & 17 \\ 44 & 18 \\ 28 & 19 \\ 18 & 20 \\ 10 & 21 \\ 6 & 22 \\ 3 & 23 \end{pmatrix}$	0.9	$\begin{pmatrix} 80 & 13 \\ 49 & 14 \\ 28 & 15 \\ 15 & 16 \\ 8 & 17 \\ 3 & 18 \end{pmatrix}$
0.2	$\begin{pmatrix} 94 & 20 \\ 70 & 21 \\ 52 & 22 \\ 37 & 23 \\ 26 & 24 \\ 18 & 25 \\ 12 & 26 \\ 7 & 27 \\ 4 & 28 \\ 2 & 29 \end{pmatrix}$	0.6	$\begin{pmatrix} 94 & 15 \\ 64 & 16 \\ 41 & 17 \\ 26 & 18 \\ 15 & 19 \\ 9 & 20 \\ 4 & 21 \\ 2 & 22 \end{pmatrix}$	1.0	$\begin{pmatrix} 94 & 12 \\ 57 & 13 \\ 33 & 14 \\ 18 & 15 \\ 9 & 16 \\ 4 & 17 \end{pmatrix}$
0.3	$\begin{pmatrix} 108 & 18 \\ 80 & 19 \\ 57 & 20 \\ 40 & 21 \\ 28 & 22 \\ 18 & 23 \\ 12 & 24 \\ 7 & 25 \\ 4 & 26 \\ 2 & 27 \end{pmatrix}$	0.7	$\begin{pmatrix} 99 & 14 \\ 65 & 15 \\ 41 & 16 \\ 25 & 17 \\ 14 & 18 \\ 8 & 19 \\ 4 & 20 \end{pmatrix}$		

Figure 3: Optimal Strategies for different $r's$

Looking at the optimal strategies for different $r's$ we can clearly see when r increase we have smaller optimal strategies as we level off very fast (smaller $\beta's$ for the same combination of $\alpha's$). the following plot shows this relationship.

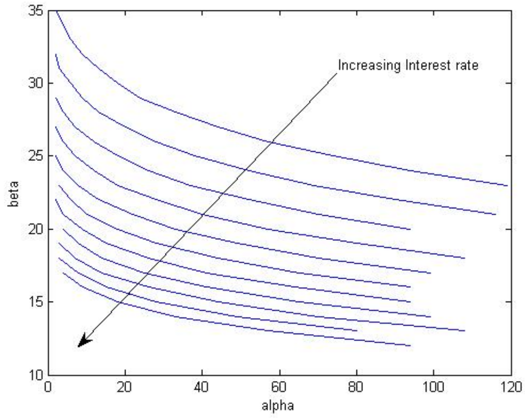


Figure 4: Optimal Strategies for Increasing r

For the next simulation we will only plot the optimal strategies as above for the different parameter variation.

In the next run we fix $r = a = \sigma^2 = 1$ and we vary μ accordingly. We get the following results.

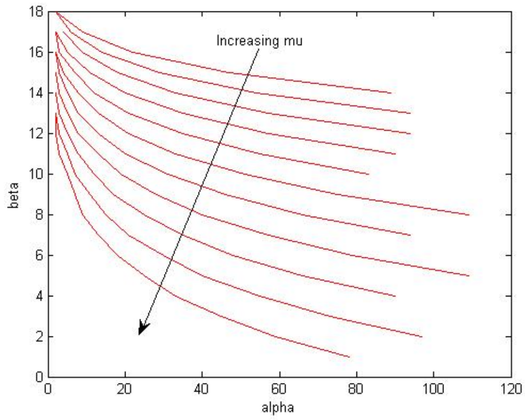


Figure 5: Optimal Strategies for Increasing μ

As we can see the relationship is similar to varying r. The higher μgives smaller optimal strategies as we level off very fast.

Next we will vary σ^2 and fix $r = a = \mu = 1$. And we get the following optimal strategies table and plot.

σ^2	(α_1, β_1)	σ^2	(α_1, β_1)	σ^2	(α_1, β_1)
0.0001	0	0.2633	$\begin{pmatrix} 32 & 1 \\ 29 & 2 \\ 26 & 3 \\ 23 & 4 \\ 21 & 5 \\ 19 & 6 \\ 16 & 7 \\ 14 & 8 \\ 13 & 9 \\ 11 & 10 \\ 9 & 11 \\ 8 & 12 \\ 6 & 13 \\ 5 & 14 \\ 4 & 15 \\ 3 & 16 \\ 2 & 17 \end{pmatrix}$	0.5264	$\begin{pmatrix} 111 & 3 \\ 93 & 4 \\ 78 & 5 \\ 65 & 6 \\ 53 & 7 \\ 43 & 8 \\ 35 & 9 \\ 27 & 10 \\ 21 & 11 \\ 16 & 12 \\ 12 & 13 \\ 9 & 14 \\ 6 & 15 \\ 4 & 16 \\ 2 & 17 \end{pmatrix}$
0.7896	$\begin{pmatrix} 118 & 9 \\ 85 & 10 \\ 60 & 11 \\ 41 & 12 \\ 28 & 13 \\ 18 & 14 \\ 11 & 15 \\ 6 & 16 \\ 3 & 17 \end{pmatrix}$	1.0527	$\begin{pmatrix} 116 & 12 \\ 69 & 13 \\ 39 & 14 \\ 20 & 15 \\ 10 & 16 \\ 4 & 17 \end{pmatrix}$	1.3158	$\begin{pmatrix} 91 & 14 \\ 41 & 15 \\ 16 & 16 \\ 5 & 17 \end{pmatrix}$
1.5790	$\begin{pmatrix} 87 & 15 \\ 30 & 16 \\ 8 & 17 \end{pmatrix}$	1.8421	$\begin{pmatrix} 54 & 16 \\ 12 & 17 \\ 2 & 18 \end{pmatrix}$	2.1053	$\begin{pmatrix} 101 & 16 \\ 18 & 17 \\ 2 & 18 \end{pmatrix}$
2.3684	$\begin{pmatrix} 27 & 17 \\ 2 & 18 \end{pmatrix}$	2.8947	$\begin{pmatrix} 42 & 17 \\ 2 & 18 \end{pmatrix}$	2.6316	$\begin{pmatrix} 66 & 17 \\ 2 & 18 \end{pmatrix}$
3.1579	$\begin{pmatrix} 104 & 17 \\ 3 & 18 \end{pmatrix}$	3.4210	$\begin{pmatrix} 3 & 18 \end{pmatrix}$	3.6842	$\begin{pmatrix} 4 & 18 \end{pmatrix}$
3.9473	$\begin{pmatrix} 4 & 18 \end{pmatrix}$	4.2105	$\begin{pmatrix} 5 & 18 \end{pmatrix}$	4.4736	$\begin{pmatrix} 6 & 18 \end{pmatrix}$
4.7368	$\begin{pmatrix} 7 & 18 \end{pmatrix}$	5.000	$\begin{pmatrix} 9 & 18 \end{pmatrix}$		

Figure 6: Table of Optimal Strategies for Increasing σ^2

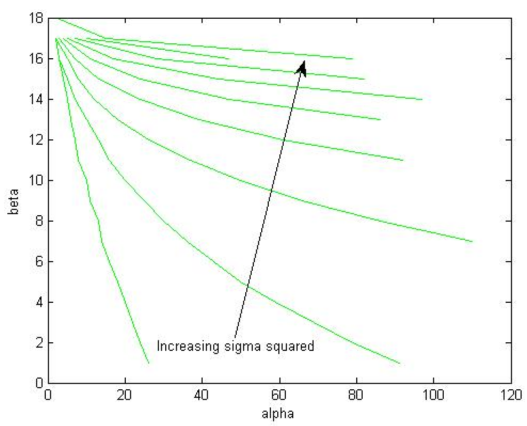


Figure 7: Plot of Optimal Strategies for Increasing σ^2

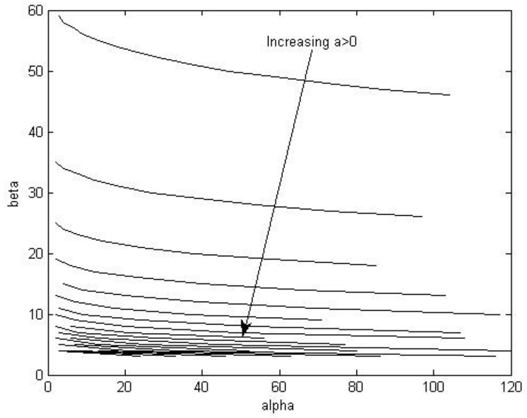


Figure 8: Optimal strategies for different risk aversion parameter $a > 0$

In this simulation we see that increasing σ^2 gives us bigger optimal strategies as shown in the plot. Looking at the table above we see only the smaller values appear as well as we increase σ^2. The fewer optimal strategies for high $\sigma^2$is due to our grid choice which has less points at smaller values (covers less points). The behavior of the log normal distribution with respect to μ and σ^2 conforms with our previous two results. As μincreases we have higher probability of high values which level off our $U(X)$ very fast thus the smaller valued optimal strategies. But when we increase $\sigma^2$the log normal distribution gives higher probability to smaller values thus our $U(X)$ slowly level off and we get higher values of optimal strategies.

In the next simulation we will fix $\sigma^2 = r = \mu = 1$ while varying the risk aversion parameter $a > 0$. We get the following results.

Looking at the plot we see a lower values for (α_1, β_1) for high a. This result is explained from the limit of $U(X)$ i.e $\lim_{(\alpha,\beta)\to\infty} \int_{\beta_1(1+r)}^{\infty} \frac{(1-e^{-ax})}{a} f_X(x)dx = \frac{1}{a}$ converges faster to $\frac{1}{a}$ for a higher a. Thus our $U(X)$ levels off faster with higher a.

5.3 Scenario 2 ($t = 0, 1$, 1, 1, 0, 1, 1, None, None, r , Uniform$(0,b)$)

In this scenario we will consider $S_1 \sim U[0, b] \Longrightarrow X \sim U[\beta_1(1 + r), \alpha_1 b + \beta_1(1 + r)]$. First we will compute $U(X)$.

$$U(X) = \int_{\beta_1(1+r)}^{\alpha_1 b+\beta_1(1+r)} \frac{(1-e^{-ax})}{a}\frac{1}{\alpha_1 b}dx = \frac{1}{a\alpha_1 b}(\alpha_1 b + \frac{1}{a}(e^{-a\alpha_1 b - a\beta_1(1+r)} - e^{-a\beta(1+r)}))$$

$$= \frac{1}{a} + \frac{e^{-a\alpha_1 b-a\beta_1(1+r)}}{a^2\alpha_1 b} - \frac{e^{-a\beta_1(1+r)}}{a^2\alpha_1 b} \Rightarrow \lim_{(\alpha_1,\beta_1)\to\infty} U(X) = \frac{1}{a}$$

Then lets compute$(\frac{\partial U}{\partial \alpha_1}, \frac{\partial U}{\partial \beta_1})$.

$$\frac{\partial U}{\partial \alpha_1} = -\frac{e^{-a\alpha_1 b-a\beta_1(1+r)}}{a\alpha_1}(\frac{1}{ab\alpha_1} - 1) + \frac{e^{-a\beta_1(1+r)}}{a^2\alpha_1^2 b}$$

$$\frac{\partial U}{\partial \beta_1} = \frac{-(1+r)}{ab\alpha_1}(e^{-a\alpha_1 b-a\beta_1(1+r)} - e^{-a\beta(1+r)})$$

We see that $\lim_{(\alpha_1,\beta_1)\to\infty} \frac{\partial U}{\partial \alpha_1} = 0$ and $\lim_{(\alpha_1,\beta_1)\to\infty} \frac{\partial U}{\partial \beta_1} = 0$ so our utility functional $U(X)$ level off at a very fast rate especially towards $\beta_1$direction.

5.4 Results of Scenario 2

Similar to our first scenario, we will vary one parameter and fix the others. Our expected results should be similar to Scenario 1 for varying a and r. $U(X)$ should exhibit the same decreasing behavior as b increases as we can see from $U(X)$ and $(\frac{\partial U}{\partial \alpha_1}, \frac{\partial U}{\partial \beta_1})$ functions (they decrease as b increases).

First we vary r and keep $b = a = 1$. Next we fix $r = b = 1$ and vary a . And lastly we vary b and fix $a = r = 1$. we get the following 3 plots of optimal strategies.

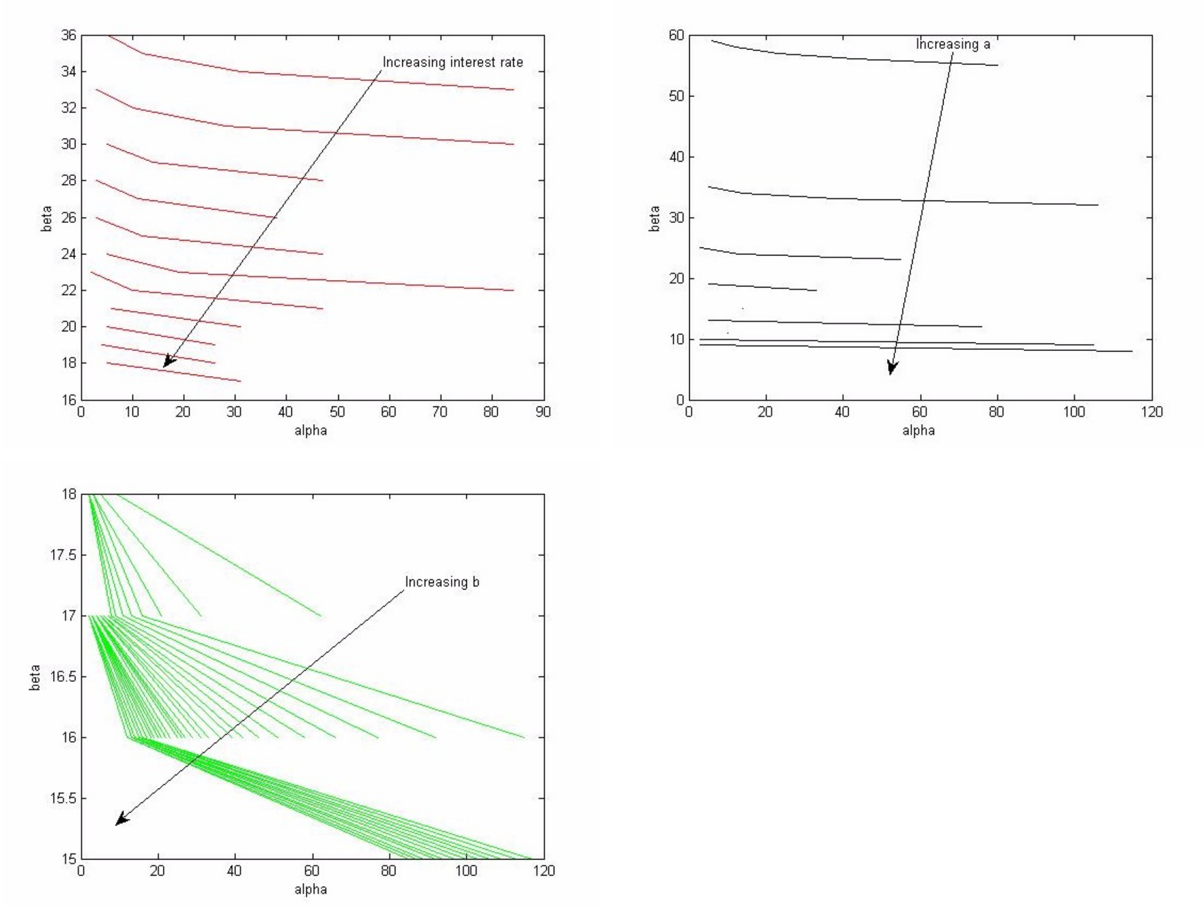


Figure 9: Plot of Optimal Strategies Under Different Varying Parameters

The Results are in conform with our predictions and previous limiting behavior.

5.5 Scenario 3 ($t = 0, 1$, 1, 1, 1, 1, 1, Call Option, None, r , Log-normal(μ,σ^2))

In this Scenario we consider shorting a call option payout $g(S_1) = (S_1 - k)_+$ for a strike price k. Our cash flow at time $t = 1$ becomes $X = V_1^{(\alpha,\beta,\gamma)} = (1 + r)\beta_1 + \alpha_1 S_1 - \gamma_1(S_1 - k)_+$. We see that $U(X) = \int \frac{(1-e^{-a(1+r)\beta_1 - a\alpha S_1 + a\gamma(S_1-k)_+})}{a} f_X(x)dx$ where f_X is the pdf of $(1 + r)\beta_1 + \alpha_1 S_1 - \gamma_1(S_1 - k)_+$. We can split the integral into two parts to deal with the contingent claim function. $U(X) = \int_{\beta_1(1+r)}^{\alpha_1 k+\beta_1(1+r)} \frac{(1-e^{-ax})}{a} f_{X^*}(x)dx + \int_{\alpha_1 k+\beta_1(1+r)}^{\infty} \frac{(1-e^{-ax})}{a} f_{X^-}(x)dx$ where $f_{X^*}$is the pdf of $(1 + r)\beta_1 + \alpha_1 S_1$ and $f_{X^-}$is the pdf of $(1+r)\beta_1 + \alpha_1 S_1 - \gamma_1(S_1 - k)$. Since $\alpha_1 S_1 \sim Lognormal(ln(\alpha_1) + \mu, \sigma^2)$ then $\alpha_1 S_1 + \beta_1(1+r) \sim Lognormal(ln(\alpha_1) + \mu, \sigma^2)\ Shifted\ by\ \beta_1(1+r)$ which is f_{X^*}. And f_X-is $\alpha_1 S_1 + \beta_1(1+r) - \gamma_1(S_1 - k) \sim Lognormal(ln(\alpha_1 - \gamma_1) + \mu, \sigma^2)\ Shifted\ by\ \beta_1(1+r) + \gamma_1 k$. For f_X-we choose $\gamma_1 < \alpha_1 \Rightarrow -\infty < ln(\alpha_1 - \gamma_1)$ (should always be finite).

So we have two integrals with cutoff $\alpha_1 k + \beta_1(1+r)$, one dependent on β_1 and α_1 and another is dependent on $\beta_1, \alpha_1, \gamma_1$ and k. Next we run our simulations accordingly by varying one parameter and keeping the rest constant to one and compare the results.

5.6 Results of Scenario 3

The parameters in this scenario are a, k, μ, σ^2, r.

First we vary r and set $a = k = \mu = \sigma^2 = 1$. We get the following results.

r	$(\beta_1, \alpha_1, \gamma_1)$	r	$(\beta_1, \alpha_1, \gamma_1)$	r	$(\beta_1, \alpha_1, \gamma_1)$	r	$(\beta_1, \alpha_1, \gamma_1)$
0.0	$\begin{pmatrix} 26 & 41 & 2 \\ 27 & 32 & 1 \\ 28 & 29 & 1 \\ 29 & 25 & 1 \\ 30 & 13 & 1 \\ 31 & 10 & 1 \\ 32 & 4 & 1 \\ 32 & 5 & 1 \\ 33 & 5 & 1 \end{pmatrix}$	0.1	$\begin{pmatrix} 23 & 40 & 1 \\ 24 & 37 & 1 \\ 25 & 15 & 1 \\ 26 & 17 & 1 \\ 27 & 15 & 1 \\ 28 & 29 & 1 \\ 30 & 4 & 1 \\ 30 & 5 & 1 \end{pmatrix}$	0.2	$\begin{pmatrix} 21 & 40 & 1 \\ 22 & 31 & 1 \\ 23 & 15 & 1 \\ 24 & 25 & 1 \\ 25 & 13 & 1 \\ 26 & 32 & 1 \end{pmatrix}$	0.3	$\begin{pmatrix} 20 & 41 & 2 \\ 21 & 16 & 1 \\ 22 & 17 & 1 \\ 24 & 10 & 1 \\ 25 & 5 & 1 \end{pmatrix}$
0.4	$\begin{pmatrix} 18 & 40 & 1 \\ 19 & 20 & 1 \\ 20 & 29 & 1 \\ 21 & 13 & 1 \\ 22 & 11 & 1 \\ 23 & 4 & 1 \end{pmatrix}$	0.5	$\begin{pmatrix} 17 & 41 & 2 \\ 18 & 32 & 1 \\ 19 & 28 & 1 \\ 20 & 13 & 1 \\ 21 & 10 & 1 \\ 22 & 5 & 1 \end{pmatrix}$	0.6	$\begin{pmatrix} 16 & 38 & 1 \\ 17 & 32 & 1 \\ 18 & 10 & 1 \\ 20 & 5 & 1 \end{pmatrix}$	0.7	$\begin{pmatrix} 15 & 35 & 1 \\ 15 & 37 & 1 \\ 15 & 38 & 1 \\ 15 & 39 & 1 \\ 15 & 40 & 1 \\ 15 & 41 & 2 \\ 16 & 35 & 1 \\ 16 & 36 & 1 \\ 17 & 13 & 1 \\ 17 & 14 & 1 \\ 17 & 15 & 1 \\ 18 & 5 & 1 \\ 19 & 4 & 1 \\ 19 & 5 & 1 \end{pmatrix}$
0.8	$\begin{pmatrix} 14 & 40 & 1 \\ 15 & 32 & 1 \\ 16 & 10 & 1 \\ 17 & 29 & 1 \end{pmatrix}$	0.9	$\begin{pmatrix} 14 & 20 & 1 \\ 15 & 28 & 1 \\ 16 & 30 & 1 \\ 17 & 5 & 1 \end{pmatrix}$	1.0	$\begin{pmatrix} 13 & 41 & 2 \\ 14 & 29 & 1 \\ 15 & 13 & 1 \\ 16 & 5 & 1 \end{pmatrix}$		

Table 2: Optimal Strategies under different $r's$

Next we will vary μ and fix $\sigma^2 = k = r = a = 1$ and we get the following table.

μ	$(\beta_1, \alpha_1, \gamma_1)$	μ	$(\beta_1, \alpha_1, \gamma_1)$	μ	$(\beta_1, \alpha_1, \gamma_1)$
0.0	$\begin{pmatrix} 22 & 41 & 1 \\ 23 & 41 & 1 \\ 24 & 41 & 1 \\ 25 & 23 & 1 \end{pmatrix}$	0.5	$\begin{pmatrix} 18 & 34 & 1 \\ 19 & 34 & 1 \\ 20 & 34 & 1 \\ 21 & 34 & 1 \\ 22 & 34 & 1 \\ 23 & 15 & 1 \end{pmatrix}$	1.0	$\begin{pmatrix} 13 & 41 & 2 \\ 14 & 29 & 1 \\ 15 & 13 & 1 \\ 16 & 5 & 1 \end{pmatrix}$
1.5	$\begin{pmatrix} 12 & 25 & 1 \\ 13 & 14 & 1 \\ 14 & 7 & 1 \end{pmatrix}$	2.0	0	2.5	0
3.0	0	3.5	0	4.0	0
4.5	0	5.0	0		

Table 3: Optimal strategies under different $\mu' s$

Next we will vary σ^2 and fix $\mu = k = r = a = 1$ and we get the following table.

σ^2	$(\beta_1, \alpha_1, \gamma_1)$	σ^2	$(\beta_1, \alpha_1, \gamma_1)$	σ^2	$(\beta_1, \alpha_1, \gamma_1)$
0.2231	$\begin{pmatrix} 2 & 41 & 1 \\ 3 & 41 & 1 \\ 4 & 41 & 1 \\ 5 & 41 & 1 \\ 6 & 41 & 1 \\ 7 & 41 & 1 \\ 8 & 40 & 1 \\ 9 & 38 & 1 \\ 10 & 41 & 2 \\ 11 & 39 & 1 \\ 12 & 37 & 1 \\ 13 & 35 & 1 \\ 14 & 33 & 1 \\ 15 & 31 & 1 \\ 16 & 29 & 1 \\ 17 & 27 & 1 \\ 18 & 25 & 1 \\ 19 & 23 & 1 \\ 20 & 21 & 1 \\ 21 & 19 & 1 \\ 22 & 17 & 1 \\ 23 & 15 & 1 \\ 24 & 13 & 1 \\ 25 & 11 & 1 \\ 26 & 9 & 1 \\ 27 & 7 & 1 \\ 28 & 5 & 1 \\ 29 & 3 & 1 \end{pmatrix}$	0.4452	$\begin{pmatrix} 5 & 41 & 1 \\ 6 & 41 & 1 \\ 7 & 41 & 1 \\ 8 & 41 & 1 \\ 9 & 41 & 1 \\ 10 & 41 & 1 \\ 11 & 41 & 1 \\ 12 & 41 & 1 \\ 13 & 41 & 1 \\ 14 & 41 & 1 \\ 15 & 41 & 1 \\ 16 & 41 & 1 \\ 17 & 41 & 1 \\ 18 & 41 & 1 \\ 19 & 41 & 1 \\ 20 & 41 & 1 \\ 21 & 41 & 1 \\ 22 & 41 & 1 \\ 23 & 41 & 1 \\ 24 & 41 & 1 \\ 25 & 41 & 1 \\ 26 & 41 & 1 \\ 27 & 41 & 1 \\ 28 & 41 & 1 \\ 29 & 41 & 1 \\ 30 & 41 & 1 \\ 31 & 41 & 1 \\ 32 & 41 & 1 \\ 33 & 41 & 1 \\ 34 & 41 & 1 \\ 35 & 41 & 1 \\ 36 & 41 & 1 \\ 37 & 41 & 1 \\ 38 & 41 & 1 \\ 39 & 41 & 1 \\ 40 & 39 & 1 \\ 41 & 41 & 1 \end{pmatrix}$	0.6673	$\begin{pmatrix} 9 & 39 & 1 \\ 10 & 40 & 1 \\ 11 & 41 & 1 \\ 12 & 41 & 1 \\ 13 & 41 & 1 \\ 14 & 41 & 1 \\ 15 & 41 & 1 \\ 17 & 3 & 1 \\ 17 & 41 & 1 \\ 18 & 41 & 1 \\ 19 & 41 & 1 \\ 20 & 41 & 1 \\ 21 & 41 & 1 \\ 22 & 41 & 1 \\ 23 & 41 & 1 \\ 24 & 41 & 1 \\ 25 & 41 & 1 \\ 26 & 41 & 1 \\ 27 & 41 & 1 \\ 28 & 41 & 1 \\ 29 & 41 & 1 \\ 30 & 41 & 1 \\ 31 & 41 & 1 \\ 32 & 41 & 1 \\ 33 & 41 & 1 \\ 34 & 41 & 1 \\ 35 & 41 & 1 \\ 36 & 41 & 1 \\ 37 & 41 & 1 \\ 38 & 41 & 1 \\ 39 & 41 & 1 \\ 40 & 41 & 1 \\ 41 & 41 & 1 \end{pmatrix}$

0.8894	$\begin{pmatrix} 12 & 41 & 1 \\ 13 & 41 & 2 \\ 14 & 41 & 2 \\ 15 & 41 & 2 \\ 16 & 41 & 2 \\ 17 & 41 & 2 \\ 18 & 41 & 2 \\ 19 & 41 & 2 \\ 20 & 41 & 2 \\ 21 & 41 & 2 \\ 22 & 41 & 2 \\ 23 & 41 & 2 \\ 24 & 41 & 2 \\ 25 & 41 & 2 \\ 26 & 41 & 2 \\ 27 & 41 & 2 \\ 28 & 41 & 2 \\ 29 & 41 & 2 \\ 30 & 41 & 1 \\ 31 & 41 & 1 \\ 32 & 41 & 1 \\ 33 & 41 & 1 \\ 34 & 41 & 1 \\ 35 & 41 & 1 \\ 36 & 41 & 1 \\ 37 & 41 & 1 \\ 38 & 41 & 1 \\ 39 & 41 & 1 \\ 40 & 40 & 1 \\ 41 & 41 & 2 \end{pmatrix}$	1.0000	$\begin{pmatrix} 13 & 33 & 1 \\ 13 & 36 & 1 \\ 13 & 38 & 1 \\ 13 & 41 & 2 \\ 14 & 19 & 1 \\ 14 & 20 & 1 \\ 14 & 21 & 1 \\ 14 & 22 & 1 \\ 14 & 23 & 1 \\ 14 & 25 & 1 \\ 14 & 27 & 1 \\ 14 & 28 & 1 \\ 14 & 29 & 1 \\ 15 & 11 & 1 \\ 15 & 12 & 1 \\ 15 & 13 & 1 \\ 16 & 4 & 1 \\ 16 & 5 & 1 \\ 41 & 7 & 1 \end{pmatrix}$		

Table 5: Optimal strategies under different σ^2

In this simulation we will vary a and fix $\mu = k = r = \sigma^2 = 1$ and we get the following table.

a	$(\beta_1, \alpha_1, \gamma_1)$	a	$(\beta_1, \alpha_1, \gamma_1)$	a	$(\beta_1, \alpha_1, \gamma_1)$

0.5875	$\begin{pmatrix} 24 & 41 & 2 \\ 25 & 32 & 1 \\ 26 & 30 & 1 \\ 27 & 30 & 1 \\ 28 & 30 & 1 \\ 29 & 30 & 1 \\ 30 & 30 & 1 \\ 31 & 30 & 1 \\ 32 & 30 & 1 \\ 33 & 30 & 1 \\ 34 & 30 & 1 \\ 35 & 30 & 1 \\ 40 & 41 & 2 \end{pmatrix}$	1.0750	$\begin{pmatrix} 12 & 41 & 2 \\ 13 & 31 & 1 \\ 14 & 31 & 1 \\ 15 & 28 & 1 \\ 16 & 27 & 1 \\ 17 & 27 & 1 \\ 18 & 25 & 1 \\ 19 & 17 & 1 \\ 20 & 17 & 1 \\ 21 & 17 & 1 \\ 22 & 17 & 1 \\ 23 & 13 & 1 \\ 24 & 13 & 1 \\ 25 & 11 & 1 \\ 26 & 9 & 1 \\ 27 & 7 & 1 \\ 28 & 5 & 1 \\ 37 & 41 & 2 \\ 38 & 41 & 2 \\ 39 & 41 & 2 \\ 40 & 41 & 2 \\ 41 & 41 & 2 \end{pmatrix}$	1.5625	$\begin{pmatrix} 8 & 39 & 1 \\ 9 & 39 & 1 \\ 10 & 36 & 1 \\ 11 & 34 & 1 \\ 12 & 32 & 1 \\ 13 & 31 & 1 \\ 14 & 29 & 1 \\ 15 & 27 & 1 \\ 16 & 25 & 1 \\ 17 & 23 & 1 \\ 18 & 20 & 1 \\ 19 & 17 & 1 \\ 20 & 17 & 1 \\ 21 & 15 & 1 \\ 22 & 13 & 1 \\ 23 & 11 & 1 \\ 24 & 9 & 1 \\ 25 & 7 & 1 \\ 26 & 5 & 1 \\ 27 & 3 & 1 \\ 40 & 41 & 2 \\ 41 & 39 & 1 \end{pmatrix}$

2.0500	$\begin{pmatrix} 7 & 40 & 1 \\ 8 & 40 & 1 \\ 9 & 40 & 1 \\ 10 & 41 & 2 \\ 11 & 41 & 2 \\ 12 & 39 & 1 \\ 13 & 36 & 1 \\ 14 & 32 & 1 \\ 15 & 31 & 1 \\ 16 & 32 & 1 \\ 17 & 32 & 1 \\ 18 & 32 & 1 \\ 19 & 32 & 1 \\ 20 & 32 & 1 \\ 21 & 32 & 1 \\ 22 & 32 & 1 \\ 23 & 32 & 1 \\ 24 & 32 & 1 \\ 25 & 32 & 1 \\ 26 & 32 & 1 \\ 27 & 32 & 1 \\ 28 & 32 & 1 \\ 29 & 32 & 1 \\ 30 & 32 & 1 \\ 31 & 32 & 1 \\ 32 & 32 & 1 \\ 33 & 32 & 1 \\ 34 & 32 & 1 \\ 35 & 32 & 1 \\ 36 & 32 & 1 \\ 37 & 32 & 1 \\ 38 & 30 & 1 \\ 39 & 41 & 2 \\ 40 & 41 & 2 \\ 41 & 39 & 1 \end{pmatrix}$	2.5375	$\begin{pmatrix} 5 & 41 & 2 \\ 6 & 39 & 1 \\ 7 & 38 & 1 \\ 8 & 36 & 1 \\ 9 & 34 & 1 \\ 10 & 31 & 1 \\ 11 & 32 & 1 \\ 12 & 32 & 1 \\ 13 & 30 & 1 \\ 14 & 29 & 1 \\ 15 & 32 & 1 \\ 16 & 30 & 1 \\ 17 & 29 & 1 \\ 18 & 16 & 1 \\ 19 & 13 & 1 \\ 20 & 12 & 1 \\ 21 & 10 & 1 \\ 22 & 8 & 1 \\ 23 & 6 & 1 \\ 24 & 4 & 1 \end{pmatrix}$	3.0250	$\begin{pmatrix} 4 & 41 & 2 \\ 5 & 41 & 2 \\ 6 & 41 & 2 \\ 7 & 41 & 2 \\ 8 & 38 & 1 \\ 9 & 38 & 1 \\ 10 & 36 & 1 \\ 11 & 34 & 1 \\ 12 & 32 & 1 \\ 13 & 30 & 1 \\ 14 & 30 & 1 \\ 15 & 30 & 1 \\ 16 & 30 & 1 \\ 17 & 30 & 1 \\ 18 & 30 & 1 \\ 19 & 30 & 1 \\ 20 & 30 & 1 \\ 21 & 30 & 1 \\ 22 & 30 & 1 \\ 23 & 30 & 1 \\ 24 & 30 & 1 \\ 25 & 30 & 1 \\ 26 & 30 & 1 \\ 27 & 30 & 1 \\ 28 & 29 & 1 \\ 29 & 29 & 1 \\ 30 & 29 & 1 \\ 31 & 29 & 1 \\ 32 & 30 & 1 \\ 33 & 30 & 1 \\ 34 & 30 & 1 \\ 35 & 30 & 1 \\ 36 & 30 & 1 \\ 37 & 30 & 1 \\ 38 & 30 & 1 \\ 39 & 41 & 2 \\ 40 & 38 & 1 \\ 41 & 41 & 2 \end{pmatrix}$
3.5125	$\begin{pmatrix} 3 & 41 & 2 \\ 4 & 32 & 1 \\ 5 & 32 & 1 \\ 6 & 32 & 1 \\ 7 & 32 & 1 \\ 8 & 32 & 1 \\ 9 & 32 & 1 \\ 10 & 32 & 1 \\ 11 & 30 & 1 \\ 12 & 29 & 1 \\ 40 & 41 & 2 \end{pmatrix}$	4.0000	$\begin{pmatrix} 3 & 32 & 1 \\ 4 & 32 & 1 \\ 5 & 32 & 1 \\ 6 & 32 & 1 \\ 7 & 32 & 1 \\ 8 & 32 & 1 \\ 9 & 32 & 1 \\ 10 & 32 & 1 \\ 11 & 32 & 1 \\ 12 & 32 & 1 \\ 13 & 30 & 1 \end{pmatrix}$		

Table 7: Optimal strategies under different $a's$

In the last part we fix $\mu = a = r = \sigma^2 = 1$ and vary k accordingly. We get the following results.

k	$(\beta_1, \alpha_1, \gamma_1)$	k	$(\beta_1, \alpha_1, \gamma_1)$	k	$(\beta_1, \alpha_1, \gamma_1)$	k	$(\beta_1, \alpha_1, \gamma_1)$
1	$\begin{pmatrix} 18 & 41 & 1 \\ 19 & 41 & 1 \\ 20 & 41 & 1 \\ 21 & 41 & 1 \\ 22 & 41 & 1 \\ 23 & 41 & 1 \\ 24 & 41 & 1 \\ 25 & 41 & 2 \\ 32 & 41 & 1 \\ 33 & 41 & 1 \\ 34 & 41 & 1 \\ 35 & 41 & 1 \\ 36 & 41 & 1 \\ 37 & 41 & 1 \\ 38 & 41 & 1 \\ 39 & 41 & 1 \\ 40 & 41 & 1 \\ 41 & 41 & 1 \end{pmatrix}$	2	$\begin{pmatrix} 13 & 41 & 1 \\ 14 & 30 & 1 \\ 15 & 13 & 1 \\ 16 & 11 & 1 \\ 17 & 11 & 1 \\ 18 & 11 & 1 \\ 19 & 41 & 1 \\ 20 & 41 & 1 \\ 21 & 41 & 1 \\ 22 & 5 & 1 \\ 23 & 41 & 1 \\ 24 & 41 & 1 \\ 25 & 41 & 1 \\ 26 & 41 & 1 \\ 27 & 41 & 2 \\ 28 & 30 & 1 \\ 28 & 41 & 1 \\ 29 & 32 & 1 \\ 30 & 34 & 1 \\ 31 & 34 & 1 \\ 32 & 34 & 1 \\ 33 & 34 & 1 \\ 34 & 34 & 1 \\ 35 & 34 & 1 \\ 36 & 34 & 1 \\ 37 & 34 & 1 \\ 38 & 34 & 1 \\ 39 & 34 & 1 \\ 40 & 34 & 1 \\ 41 & 34 & 1 \end{pmatrix}$	8	$\begin{pmatrix} 11 & 19 & 1 \\ 11 & 36 & 1 \\ 13 & 11 & 1 \end{pmatrix}$	9	$\begin{pmatrix} 9 & 33 & 1 \end{pmatrix}$

Table 8: Optimal strategies under different $k's$

In the above simulations we get similar results as Scenario 1. as we increase μ, a , r and k convergence is fast and we get smaller optimal strategies for (α_1, β_1). On the other hand when σ^2increase the convergence is slower due to the nature of the lognormal distribution (Higher σ^2implies higher probabilities for low S_1 values). Lastly γ_1 stays at 1 -which is the minimum amount- throughout all simulations. This is due the negative payout and optimally the minimum gives maximum expected utility.

6 Conclusion

From Chapter 3 and 4 we conclude the following remarks:

1. $MINMAXV@R$
   (a) The optimal strategies are determined by the Gradient of the resultant matrix.
   (b) The surface has constant gradient for all scenarios.
   (c) In Scenario 3 the optimal asset choice for $\gamma_1$is 1.
   (d) In Scenario 3 the choice of our $(\alpha_1, \gamma_1 = 1)$is capped by k.

(e) In all Scenarios increasing r favors riskless choice while increasing μ, b and k favors picking risky assets. Similarly a decreasing q gives favor to risky assets as well.

(f) An increasing σ^2gives two results. In Scenario 1 as it increases it favors riskless assets then level off and then exponentially favoring optimal strategy in risky assets only. In Scenario 3 due to the offset from the contingent claim an increasing σ^2 favors riskless assets only.

2. $U(X)$

(a) The optimal strategies follow an arc in the mesh of$U(X)$ surface.

(b) The $U(X)$ and its gradient varies and level off depending on the parameters.

(c) In Scenario 3 the optimal asset choice for $\gamma_1$is 1.

(d) In all Scenarios increasing r , μ, b, k and a gives a faster convergence and thus small values for both α_1 and β_1. While an increasing σ^2 gives higher values for the optimal strategies due to the nature of Log normal distribution.

We know that $U(X) = \inf_{q \in M_2} E_q[u(X)]$ as a robust representation. our utility function is defined by $u(c) = \begin{cases} \frac{(1-e^{-ac})}{a} & a \neq 0 \\ c & a = 0 \end{cases}$ and if we choose the special case of $a = 0$ we get $U(X) = \inf_{q \in M_0} E_q[X]$ and taking its negative we get $U(X) = -\inf_{q \in M_0} E_q[X]$ which is similar to the coherent risk measure representation $\rho(X) = -\inf_{q \in Q \subseteq M_2} E_q[X]$. if we assume that $M_0 = Q$ then our coherent risk measure is just a special case of $U(X)$ specifically $\rho(X) = -U(X)$ at $a = 0$. In our setting we chose $MINMAXV@R$ as a representative of the law invariant coherent risk measure and we know that any such measure can be represented as $\rho(X) = \sup_{q \in M_2((0,1])} WV@R_q(X)$ (q the concave distortions we choose) which $MINMAXV@R$ is an example of. so varying $q \in Q$ to get the infimum translates into varying the concave distortions q. This connection between $U(X)$ and $MINMAXV@R$ serves as an indicator into the behavior of optimal strategies under them.

Comparing the optimal strategies for both $U(X)$ and $MINMAXV@R$ we see differences. First $MINMAXV@R$ optimal strategies depend on the cash flow distribution only. The distribution and its parameters determine if we choose risky or riskless assets and this is due to the semi linear structure of $MINMAXV@R$ (inherited from positive homogeneity, subadditivity and translation invariance).

If we look into the optimal strategies under $U(X)$ we see a dependence on both the distribution parameters and risk aversion quantified by a. The dependence on both quantities shapes the$U(X)$ size of the optimal strategies as they control the functional convergence. In my simulations the distribution choice with its parameters and also of a gave no clear distinction between the preference of risky or riskless assets. It only controlled their size (the optimal arc in $U(X)$ domain mesh). This indifference could be a result of inadequate mesh size which should be further investigated in more involved simulations.

In conclusion, the optimal strategies tend to be convergence based as we follow $U(X)$ and as our risk aversion becomes neutral (Linearized) we cascade into a risk measure which depend solely on the distribution. This dependence gives a clear cut binary choice of optimal strategies as determined by the parameters.

The next interesting steps is first to further investigate $U(X)$ through more heavy simulations and second to study optimal strategies for multi step models.

A Appendix

A.1 MINMAXV@R Matlab Codes

```
function [rho,mbeta,malpha]=Sim1(mu,sigmasq,q,r)
alpha=linspace(1,100,100);
beta=linspace(1,100,100);
rho=zeros(length(alpha),length(beta));
i=1;j=1;
for a=alpha
   for b=beta
      pdf=@(s)lognpdf(s,mu,sigmasq);
      cdf=@(s)logncdf(s,mu,sigmasq);
      phi=@(s)s.*pdf(s).*((cdf(s).^(-1/(q+1))-1).^q);
      w=logninv(1e-100,mu,sigmasq);
      rho(i,j) =-1*a*integral(phi,w,Inf)-b*(1+r);
      j=j+1;
   end
   i=i+1;
   j=1;
end
[FX,FY]=gradient(rho);
surf(beta,alpha,rho);
view(142,36);
plottitle=strcat('Plot of (mu,sigmasq,q,r)=(',num2str(mu),',',num2str(sigmasq),',',num2str(q),',',num2str(r),')');
zlabel('rho');ylabel('alpha');xlabel('beta');title(plottitle);mbeta=mean2(FX);malpha=mean2(FY);
end
```

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```
r1=linspace(0,1,11);
mu1=linspace(0,5,11);
sigmasq1=linspace(0.0001,5,50);
q1=linspace(0.1,4,20);
y=@(x,q)(1-(1-x.^(1/(1+q))).^(q+1));
x=linspace(0,1,1000);
matrix1=zeros(11,3);matrix2=zeros(11,3);
matrix3=zeros(50,3);matrix4=zeros(20,3);
for q=q1
   yy=y(x,q);
   plot(x,yy);hold on;
end
i=1;
for r=r1
mu=1;q=1;sigmasq=1;
figure();
[rho,mbeta,malpha]=Sim1(mu,sigmasq,q,r);
matrix1(i,1)=r;
matrix1(i,2)=mbeta;
matrix1(i,3)=malpha;
i=i+1;
end
i=1;
for mu=mu1
r=1;q=1;sigmasq=1;
figure();
[rho,mbeta,malpha]=Sim1(mu,sigmasq,q,r);
matrix2(i,1)=mu;
matrix2(i,2)=mbeta;
matrix2(i,3)=malpha;
i=i+1;
end
i=1;
for sigmasq=sigmasq1
mu=1;q=1;r=1;
figure();
[rho,mbeta,malpha]=Sim1(mu,sigmasq,q,r);
matrix3(i,1)=sigmasq;
matrix3(i,2)=mbeta;
matrix3(i,3)=malpha;
i=i+1;
end
i=1;
for q=q1
mu=1;r=1;sigmasq=1;
figure();
[rho,mbeta,malpha]=Sim1(mu,sigmasq,q,r);
matrix4(i,1)=q;
matrix4(i,2)=mbeta;
matrix4(i,3)=malpha;
i=i+1;
end
```

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```
function [rho,mbeta,malpha]=Sim2(b,q,r)
alpha=linspace(1,100,100);
beta=linspace(1,100,100);
rho=zeros(length(alpha),length(beta));
i=1;j=1;
for at=alpha
   for bt=beta
       pdf=@(x)unifpdf(x,0,b);
       cdf=@(x)unifcdf(x,0,b);
       phi=@(x)x.*pdf(x).*(cdf(x).^(-1/(q+1))-1).^(q);
       w=unifinv(1e-100,0,b);
       rho(i,j) =-1*at*integral(phi,w,b)-bt*(1+r);
       j=j+1;
   end
   i=i+1;
   j=1;
end
[FX,FY]=gradient(rho);
surf(beta,alpha,rho);
view(142,36);
plottitle=strcat('Plot of (b,q,r)=(',num2str(b),',',num2str(q),',',num2str(r),')');
zlabel('rho');ylabel('alpha');xlabel('beta');title(plottitle);mbeta=mean2(FX);malpha=mean2(FY);
end
```

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```
r1=linspace(0,1,11);
b1=linspace(1,16,16);
q1=linspace(0.1,4,20);
matrix1=zeros(11,3);matrix2=zeros(11,3);
matrix3=zeros(20,3);
i=1;
for r=r1
b=1;q=1;
figure();
[rho,mbeta,malpha]=Sim2(b,q,r);
matrix1(i,1)=r;
matrix1(i,2)=mbeta;
matrix1(i,3)=malpha;
i=i+1;
end
i=1;
for b=b1
r=1;q=1;
figure();
[rho,mbeta,malpha]=Sim2(b,q,r);
matrix2(i,1)=b;
matrix2(i,2)=mbeta;
matrix2(i,3)=malpha;
i=i+1;
end
i=1;
for q=q1
b=1;r=1;
figure();
[rho,mbeta,malpha]=Sim2(b,q,r);
matrix3(i,1)=q;
matrix3(i,2)=mbeta;
matrix3(i,3)=malpha;
i=i+1;
end
```

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```
function [rho,mbeta,malpha,mgamma]=Sim3(mu,sigmasq,q,r,k)
alpha=linspace(1,25,25);
beta=linspace(1,25,25);
gamma=linspace(1,25,25);
rho=zeros(length(alpha),length(beta),length(gamma));
i=1;j=1;l=1;
for a=alpha
   for b=beta
      for c=gamma
         pdf=@(x)lognpdf(x,mu,sigmasq);
         cdf=@(x)logncdf(x,mu,sigmasq);
         phi=@(x)x.*pdf(x).*(cdf(x).^(-1/(q+1))-1).^(q);
         w=logninv(1e-100,mu,sigmasq);
         rho(i,j,l) =-1*a*integral(phi,w,Inf)+c*integral(phi,max(k,w),Inf)-b*(1+r)-c*k*(1-cdf(k).^(1/(q+1))).^(q+1);
         l=l+1;
      end
      j=j+1;
      l=1;
   end
   i=i+1;
   j=1;
   l=1;
end
[FX,FY,FZ]=gradient(rho);
mbeta=mean2(FX);malpha=mean2(FY);mgamma=mean2(FZ);
end
```

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```
r1=linspace(0,1,11);
mu1=linspace(0,5,11);
sigmasq1=linspace(0.01,5,20);
q1=linspace(0.1,4,20);
k1=linspace(0.5,10,20);
matrix1=zeros(11,4);matrix2=zeros(11,4);
matrix3=zeros(20,4);matrix4=zeros(20,4);
matrix5=zeros(20,4);
i=1;
for r=r1
mu=1;q=1;sigmasq=1;k=1;
[rho,mbeta,malpha,mgamma]=Sim3(mu,sigmasq,q,r,k);
matrix1(i,1)=r;
matrix1(i,2)=mbeta;
matrix1(i,3)=malpha;
matrix1(i,4)=mgamma;
i=i+1;
end
i=1;
for mu=mu1
r=1;q=1;sigmasq=1;k=1;
[rho,mbeta,malpha,mgamma]=Sim3(mu,sigmasq,q,r,k);
matrix2(i,1)=mu;
matrix2(i,2)=mbeta;
matrix2(i,3)=malpha;
matrix2(i,4)=mgamma;
i=i+1;
end
i=1;
for sigmasq=sigmasq1
mu=1;q=1;r=1;k=3;
[rho,mbeta,malpha,mgamma]=Sim3(mu,sigmasq,q,r,k);
matrix3(i,1)=sigmasq;
matrix3(i,2)=mbeta;
matrix3(i,3)=malpha;
matrix3(i,4)=mgamma;
i=i+1;
end
i=1;
for q=q1
mu=1;r=1;sigmasq=1;k=1;
[rho,mbeta,malpha,mgamma]=Sim3(mu,sigmasq,q,r,k);
matrix4(i,1)=q;
matrix4(i,2)=mbeta;
matrix4(i,3)=malpha;
matrix4(i,4)=mgamma;
i=i+1;
end
i=1;
for k=k1
mu=1;q=1;sigmasq=1;r=1;
[rho,mbeta,malpha,mgamma]=Sim3(mu,sigmasq,q,r,k);
matrix5(i,1)=k;
matrix5(i,2)=mbeta;
matrix5(i,3)=malpha;
matrix5(i,4)=mgamma;
i=i+1;
end
```

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A.2 U(X) Matlab Codes

```
function [UF,z] = Utility1(mu,sigmasq,r,a)
alpha=linspace(1,120,120);
beta=linspace(1,120,120);
UF=zeros(length(alpha),length(beta));
i=1;j=1;
for a1=alpha
for b1=beta
u=@(x)exp(-a*x).*lognpdf(x-b1*(1+r),mu+log(a1),sigmasq).*1;
U=(1/a).*(1-integral(u,b1*(1+r),Inf));
UF(i,j)=U;j=j+1;
end
i=i+1;
j=1;
end
z=zeros();j=1;A=UF-(1/a)*ones(size(UF));
for i=1:120
   if A(1,i)~=0
      [row]=find(not(A(:,i)),1);
      if isempty(row)==0
         z(j,1)=row;
         z(j,2)=i;
         j=j+1;
      end
   end
end
figure();surf(beta,alpha,UF);view(142,36);
plottitle=strcat('Plot of Utility Functional(mu,sigmasq,r,a)=(',num2str(mu),',',num2str(sigmasq),',',num2str(r),',',num2str(a),')');
zlabel('rho');ylabel('alpha');xlabel('beta');title(plottitle);
end
```

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```
r1=linspace(0,1,11);
mu1=linspace(0,5,11);
sigmasq1=linspace(0.001,2,10);
a1=linspace(0.1,4,20);
i=1;
for r=r1
mu=1;a=1;sigmasq=1;
[UF,z] = Utility1(mu,sigmasq,r,a);
latex_table = latex(sym(z));
surf(z(:,1),z(:,2),z(:,3));hold on;
i=i+1;
end
i=1;
for mu=mu1
r=1;a=1;sigmasq=1;
[UF,z] = Utility1(mu,sigmasq,r,a);
latex_table = latex(sym(z));
plot(z(:,1),z(:,2),'r');hold on;
i=i+1;
end
i=1;
for sigmasq=sigmasq1
mu=1;a=1;r=1;
[UF,z] = Utility1(mu,sigmasq,r,a);
 latex_table = latex(sym(z));
 if z~=0
    plot(z(:,1),z(:,2),'g');hold on;
 end
i=i+1;
end
i=1;
for a=a1
mu=1;r=1;sigmasq=1;
[UF,z] = Utility1(mu,sigmasq,r,a);
latex_table = latex(sym(z))
if z~=0
plot(z(:,1),z(:,2),'k');hold on;
end
i=i+1;
end
```

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```
function [UF,z] = Utility2(b,r,a)
alpha=linspace(1,120,120);
beta=linspace(1,120,120);
UF=zeros(length(alpha),length(beta));
i=1;j=1;
for a1=alpha
for b1=beta
u=@(x)exp(-a*x).*unifpdf(x,b1*(1+r),b*a1+b1*(1+r)).*1;
U=(1/a).*(1-integral(u,b1*(1+r),a1*b+b1*(1+r)));
UF(i,j)=U;j=j+1;
end
i=i+1;
j=1;
end
z=zeros();j=1;A=UF-(1/a)*ones(size(UF));
for i=1:120
    if A(1,i)~=0
        [row]=find(not(A(:,i)),1);
        if isempty(row)==0
            z(j,1)=row;
            z(j,2)=i;
            j=j+1;
        end
    end
end
figure();surf(beta,alpha,UF);view(142,36);
plottitle=strcat('Plot of Utility Functional(b,r,a)=(',num2str(b),',',num2str(r),',',num2str(a),')');
zlabel('rho');ylabel('alpha');xlabel('beta');title(plottitle);

end
```

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```
r1=linspace(0,1,11);
a1=linspace(0.1,2,10);
b1=linspace(0.5,20,40);
i=1;
for r=r1
b=1;a=1;
[UF,z] = Utility2(b,r,a);
latex_table = latex(sym(z))
if z~=0
plot(z(:,1),z(:,2),'r');hold on;
end
i=i+1;
end
i=1;
for b=b1
r=1;a=1;
[UF,z] = Utility2(b,r,a);
latex_table = latex(sym(z))
if z~=0
plot(z(:,1),z(:,2),'g');hold on;
end
i=i+1;
end
i=1;
for a=a1
b=1;r=1;
[UF,z] = Utility2(b,r,a);
latex_table = latex(sym(z))
if z~=0
plot(z(:,1),z(:,2),'k');hold on;
end
i=i+1;
end
```

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```
function [UF,z] = Utility3(mu,sigmasq,r,a,k)
alpha=linspace(2,41,40);
beta=linspace(2,41,40);
i=1;j=1;UF=zeros();l=1;
for b1=beta
for a1=alpha
for c1=1:(a1-1)
u1=@(x)((1-exp(-a*x))/a).*lognpdf(x-b1*(1+r),mu+log(a1),sigmasq);
u2=@(x)((1-exp(-a*x))/a).*lognpdf(x-b1*(1+r)-k*c1,mu+log(a1-c1),sigmasq);
U=integral(u1,b1*(1+r),b1*(1+r)+a1*k)+integral(u2,b1*(1+r)+a1*k,Inf);
UF(i,j,l)=U;l=l+1;
end
l=1;j=j+1;
end
j=1;l=1;
i=i+1;
end
z=zeros();A=UF-(1/a)*ones(size(UF));i=1;
for g=1:40
for h=1:40
if A(g,h,1)~=0
[row,col]=find(not(A(g,h,:)),1);
if isempty(row)==0
z(i,1)=g+1;
z(i,2)=h+1;
z(i,3)=floor(col/40)+1;
i=i+1;
end
end
end
end
%[FX1,FY1]=gradient(rho1);[FX11,FY11]=gradient(rho11);[FX2,FY2]=gradient(rho2);[FX22,FY22]=gradient(rho22);
%figure();surf(beta,alpha,rho1);
%figure();surf(beta,alpha,rho11);figure();surf(beta,alpha,rho2);figure();surf(beta,alpha,rho22);
end
```

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```
r1=linspace(0,1,11);
mu1=linspace(0,5,11);
sigmasq1=linspace(0.001,2,10);
a1=linspace(0.1,4,9);
k1=linspace(1,6,6);
i=1;
for r=r1
mu=1;a=1;sigmasq=1;k=1;
[UF,z] = Utility3(mu,sigmasq,r,a,k);
latex_table = latex(sym(z))
i=i+1;
end
i=1;
for mu=mu1
r=1;a=1;sigmasq=1;k=1;
[UF,z] = Utility3(mu,sigmasq,r,a,k);
latex_table = latex(sym(z))
i=i+1;
end
i=1;
for sigmasq=sigmasq1
mu=1;a=1;r=1;k=1;
[UF,z] = Utility3(mu,sigmasq,r,a,k);
 latex_table = latex(sym(z))
i=i+1;
end
i=1;
for a=a1
mu=1;r=1;sigmasq=1;k=1;
[UF,z] = Utility3(mu,sigmasq,r,a,k);
latex_table = latex(sym(z))
i=i+1;
end
i=1;
for k=k1
mu=1;r=1;sigmasq=1;a=1;
[UF,z] = Utility3(mu,sigmasq,r,a,k);
latex_table = latex(sym(z))
i=i+1;
end
```

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A.3 MINMAXV@R Figures

A.3.1 Scenario 1 Figures

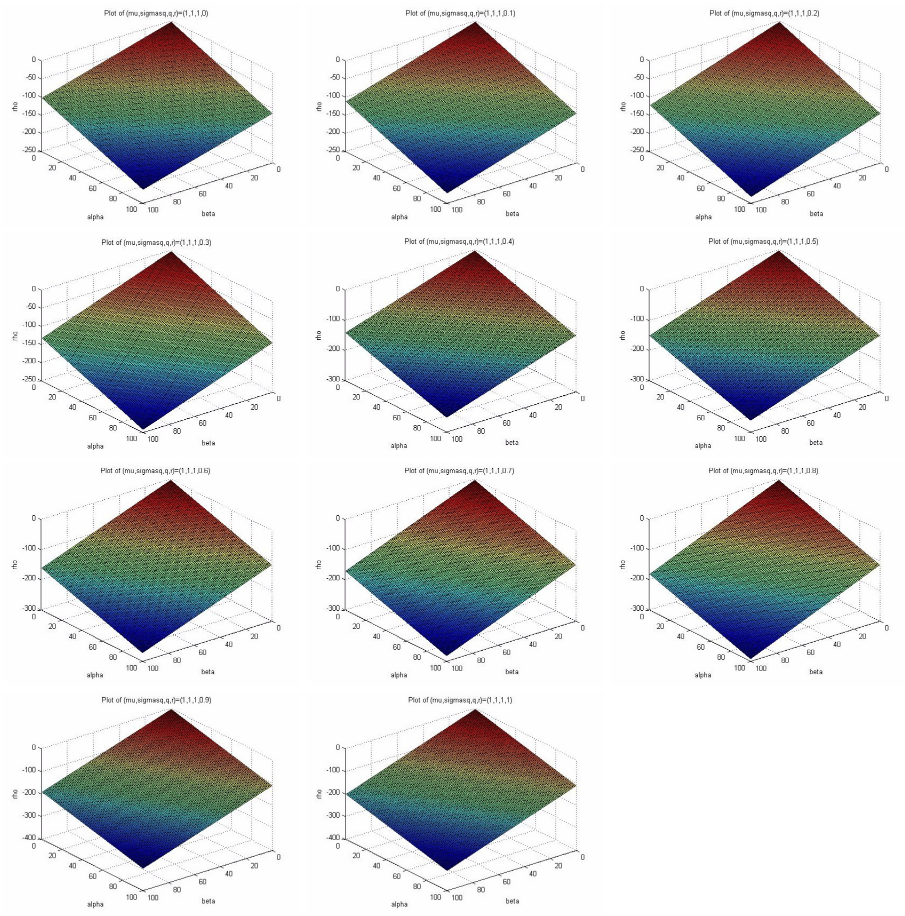

Figure 10: Varying r

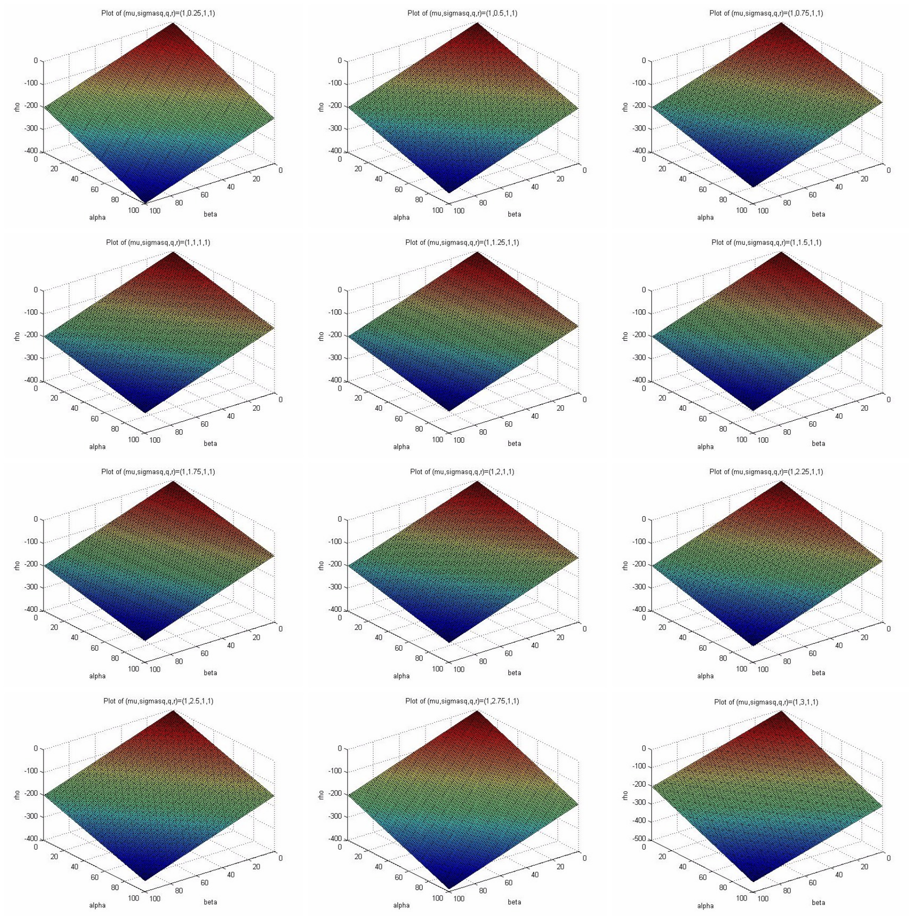

Figure 11: Varying σ^2

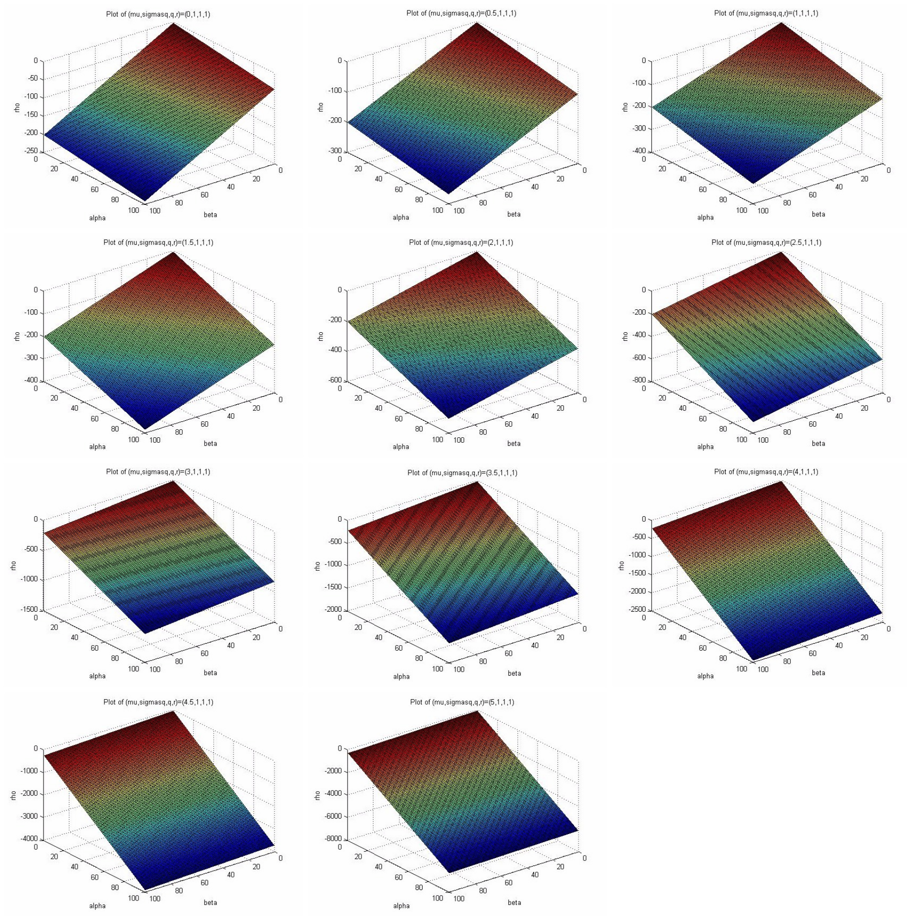

Figure 12: Varying μ

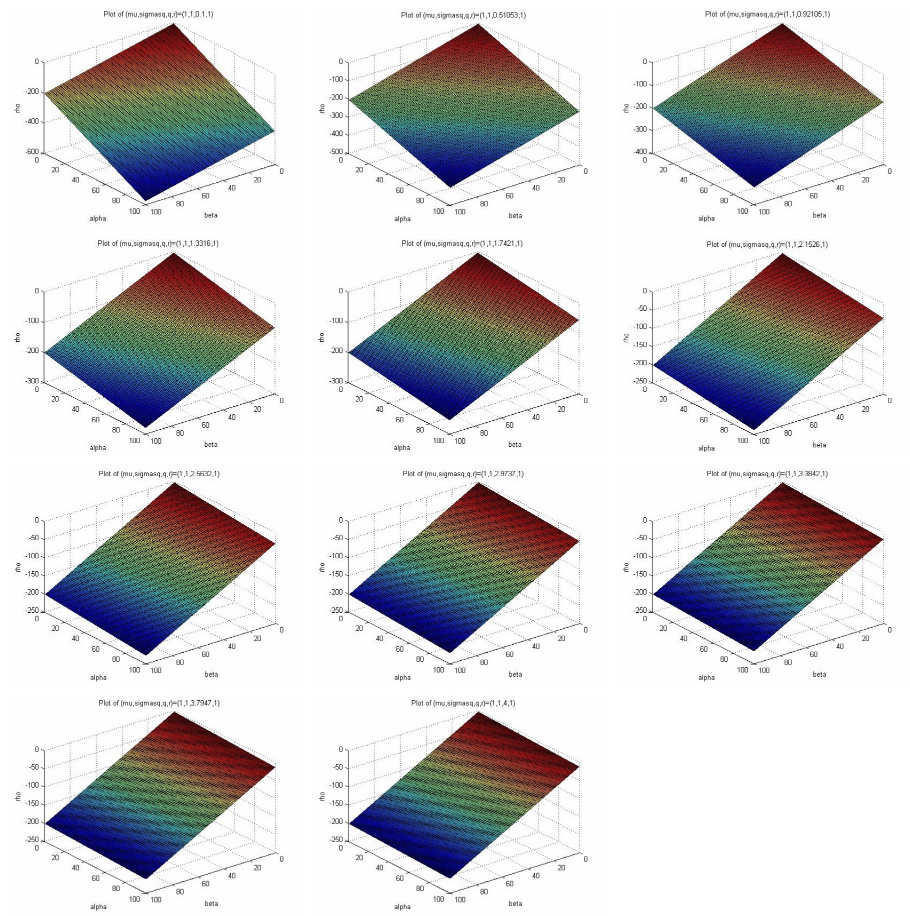

Figure 13: Varying q

A.3.2 Scenario 2 Figures

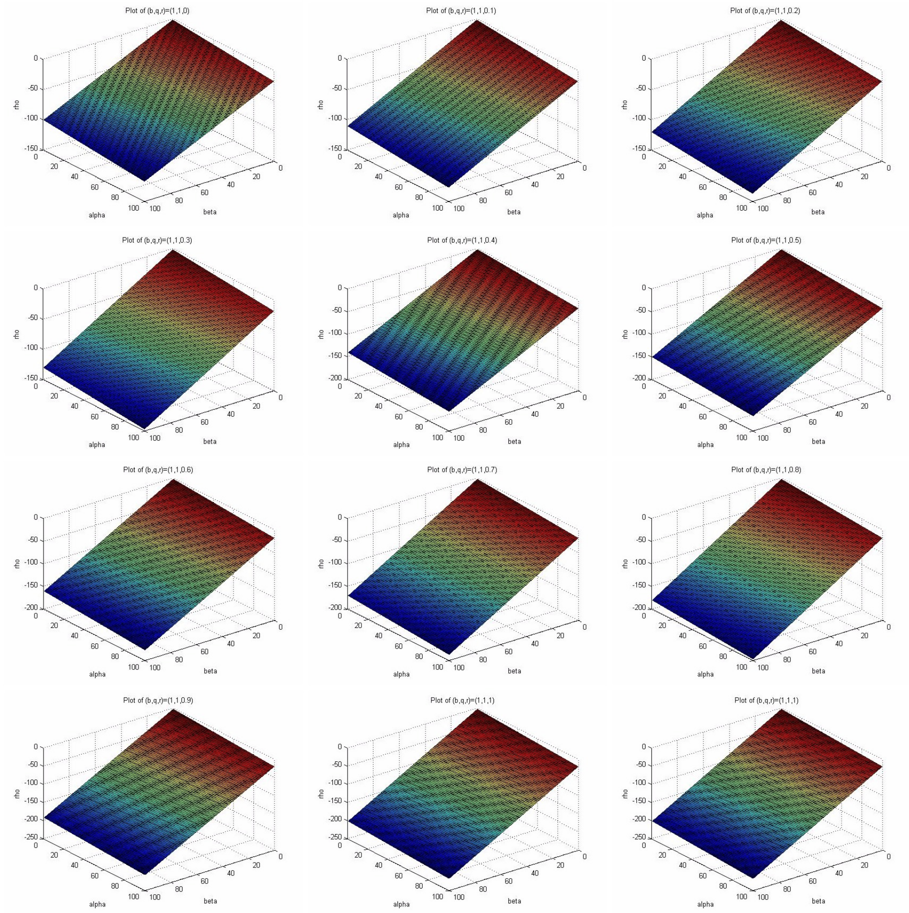

Figure 14: Varying r

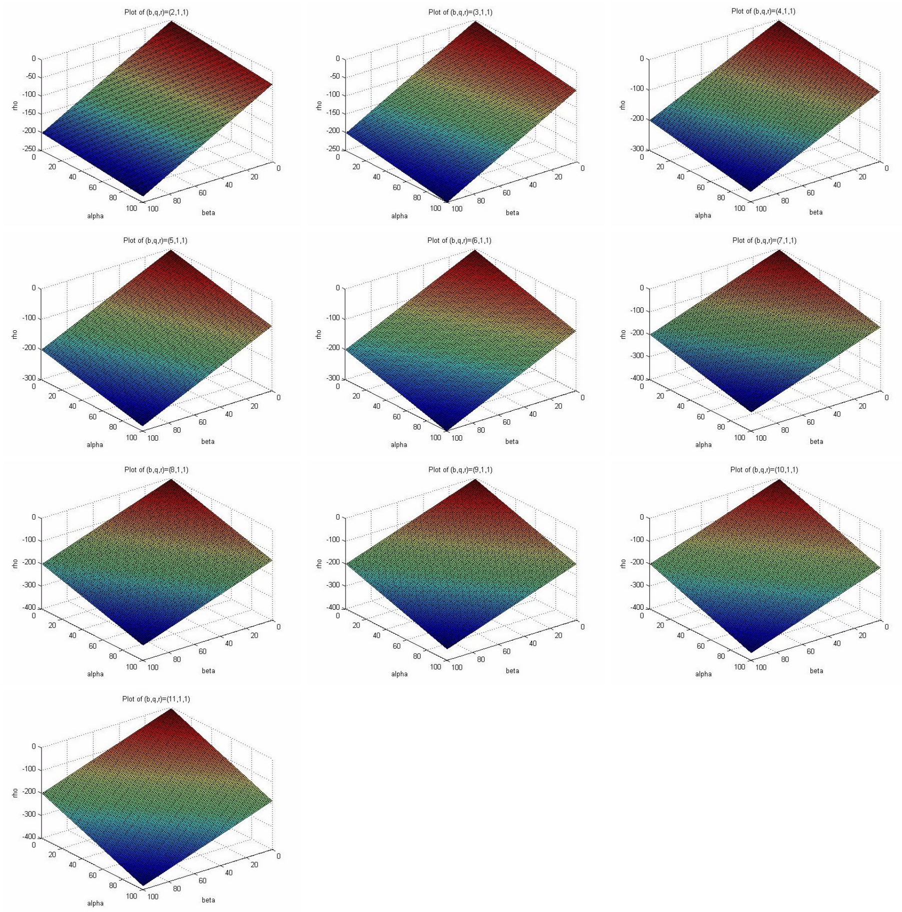

Figure 15: Varying b

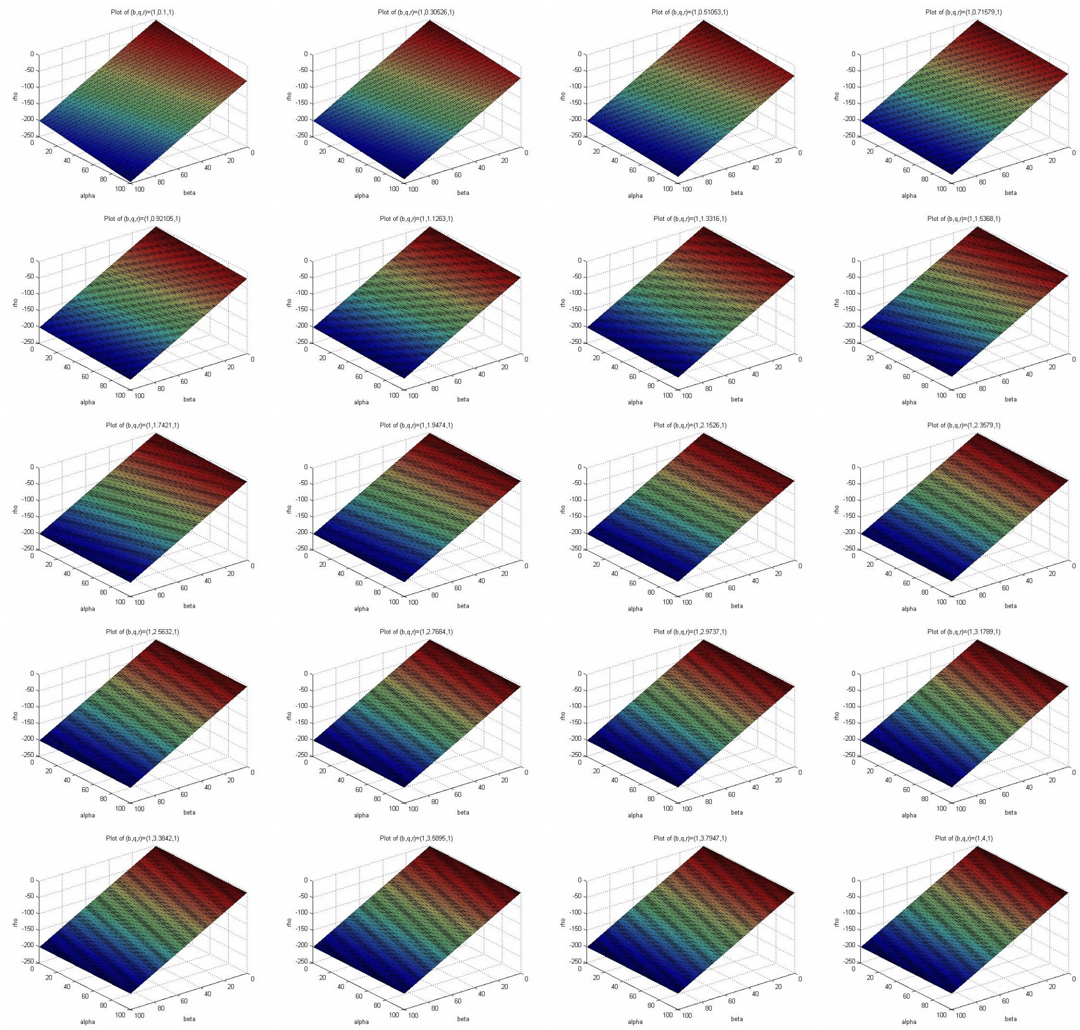

Figure 16: Varying q

A.4 U(X) Figures

A.4.1 Scenario 1 Figures

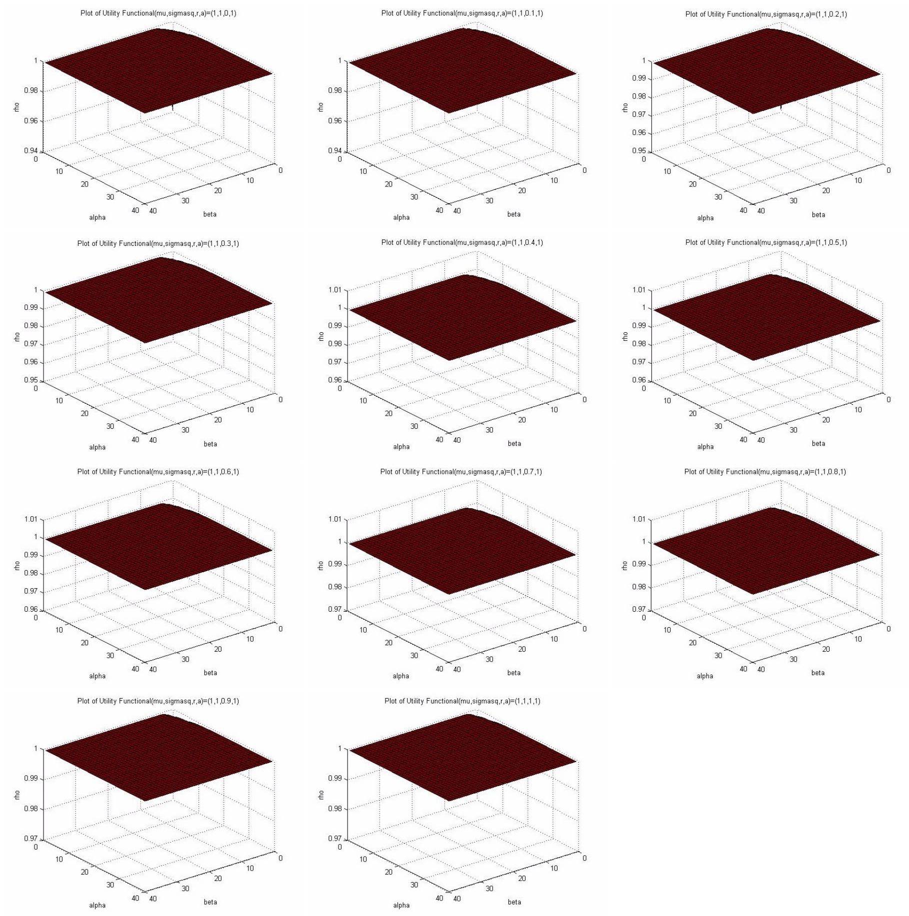

Figure 17: Varying r

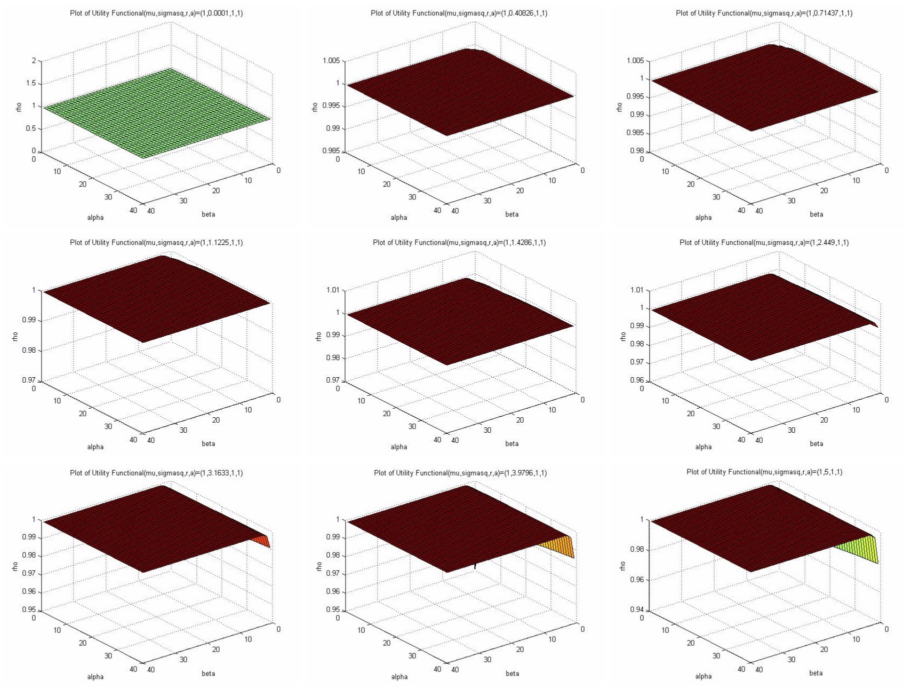

Figure 18: Varying σ^2

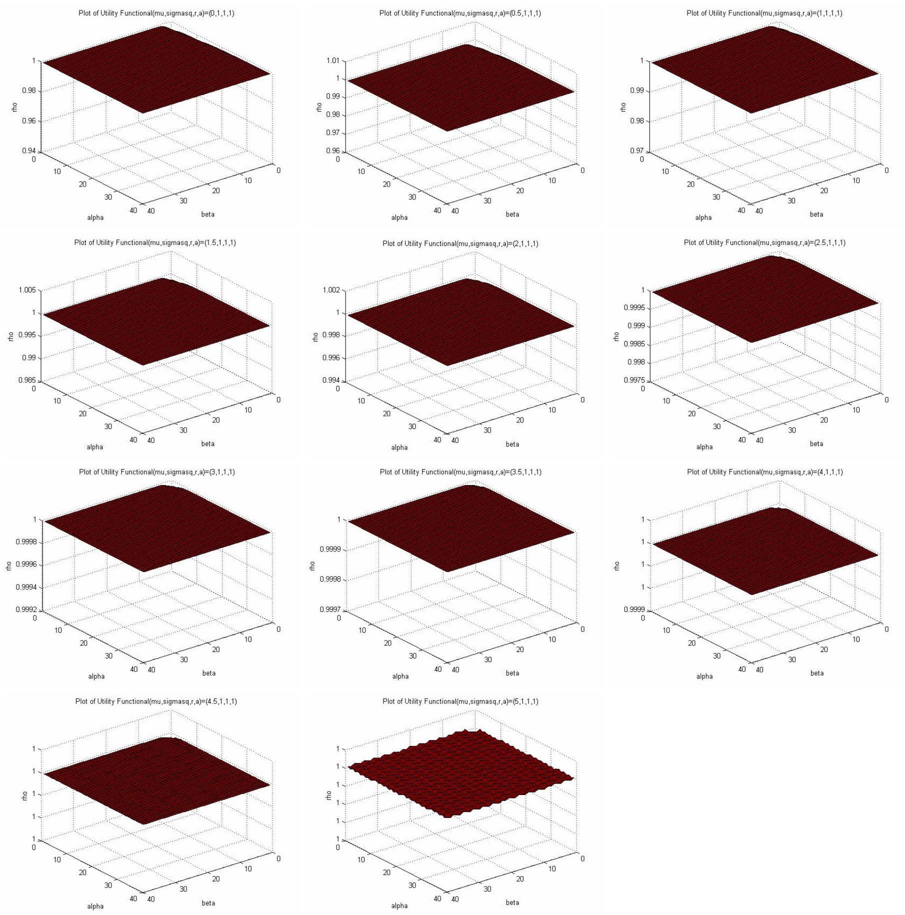

Figure 19: Varying μ

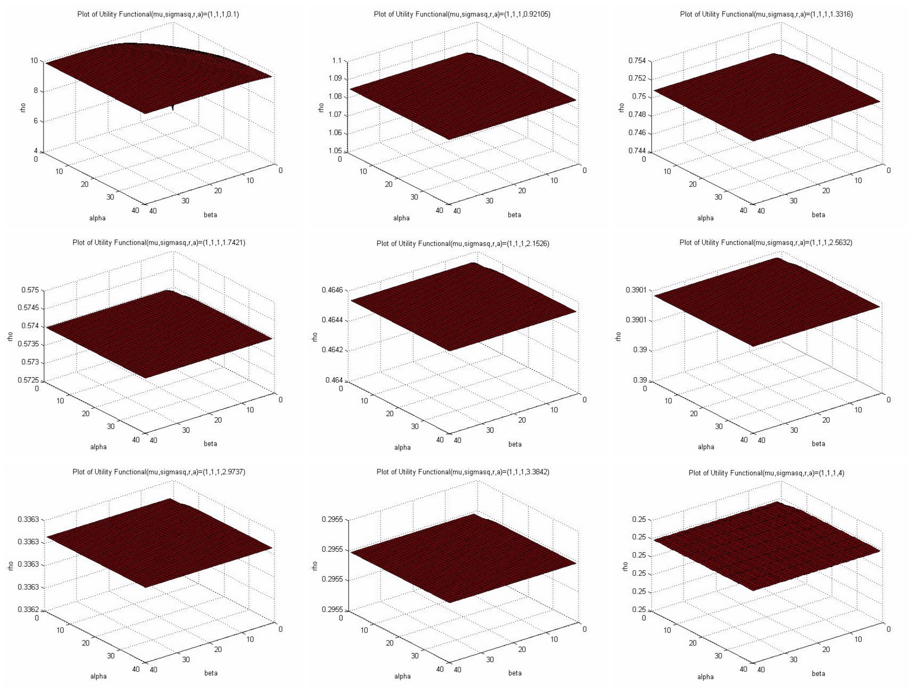

Figure 20: Varying a

A.4.2 Scenario 2 Figures

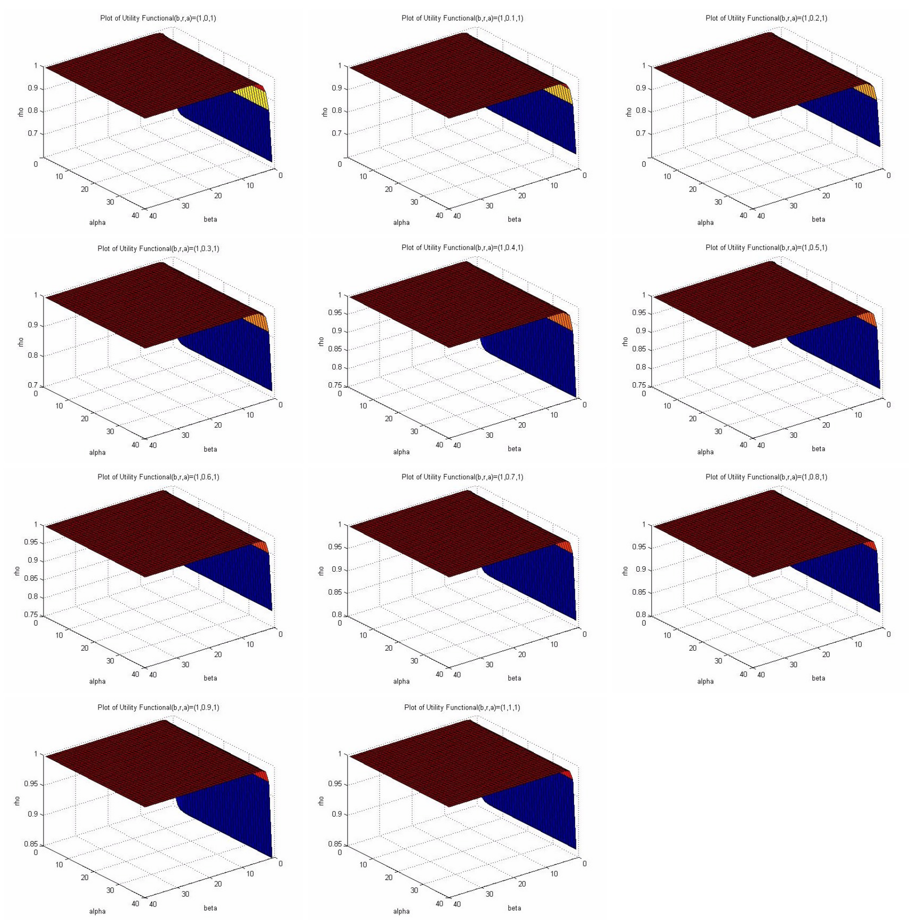

Figure 21: Varying r

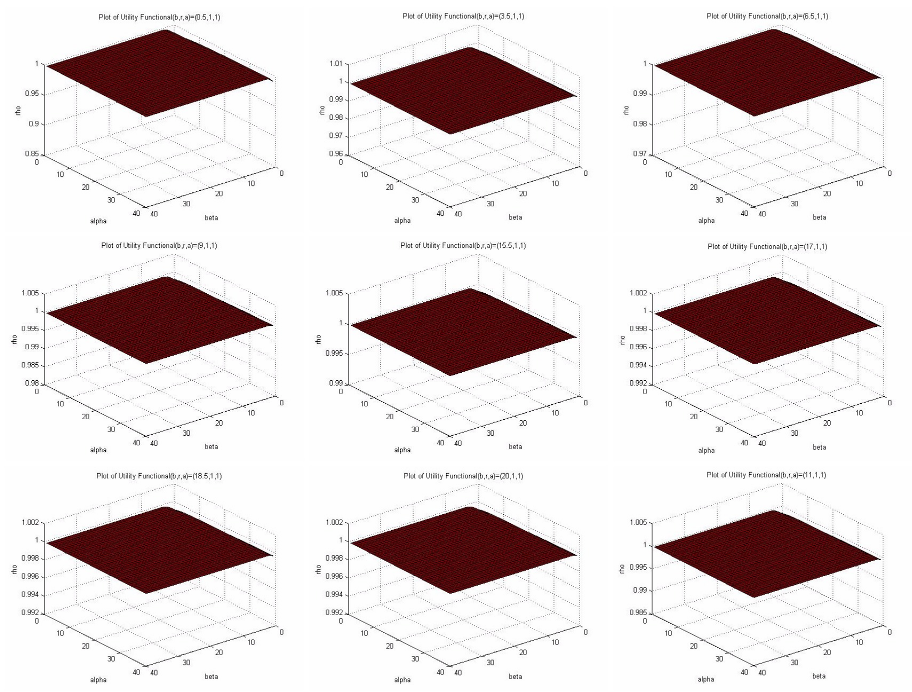

Figure 22: Varying b

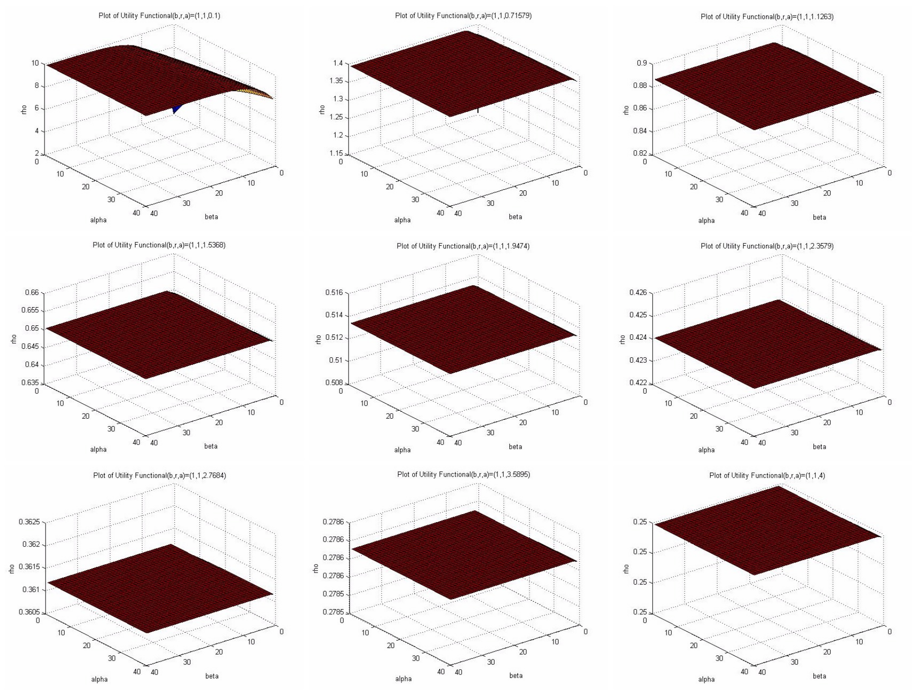

Figure 23: Varying a

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