The Quantified Argument Calculus:

An Inquiry into Its Logical Properties and Applications

by

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I hereby declare that the dissertation contains no material accepted for the completion of any other degrees in any other institutions and no materials previously written and/or published by another person unless appropriate acknowledgment is made in the form of bibliographical reference.

Chapters 2 and 3 of this dissertation contain co-authored material. Footnotes at the beginning on those chapters disclose the respective contributions of the authors.

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Abstract

The topic of this dissertation is the Quantified Argument Calculus, or Quarc, and its goal to explore its formal properties, and to investigate its application to issues in philosophy.

Chapter 1 briefly introduces the motivation for the forthcoming inquiry and lays out the plan of the rest of the dissertation.

Chapter 2 presents the formal system of Quarc and demonstrates the completeness of it, as well as some additional features.

Chapter 3 presents the sequent-calculus representation of Quarc, the LK-Quarc. It demonstrates that Quarc and LK-Quarc are deductively equivalent, and establishes the cut elimination property and its corollaries, as well as some additional features, for a series of subsystems, and finally for the full system LK-Quarc.

Chapter 4 follows up on the previous chapter by demonstrating that the Craig interpolation property holds of a system closely related to LK-Quarc, and outlines venues of further research.

Chapter 5 discusses the modal expansions of Quarc and LK-Quarc, as well as their relation. Cut elimination property and its corollaries are established for a range of modal systems.

Chapter 6 applies some of the lessons of previous chapters to a case study of a part of Aristotle's modal syllogistic. Quarc is shown to be an appropriate tool for study of Aristotle, and then applied to establish some indicative difficulties for the modal syllogistic.

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Chapter 1

Introduction

The topic of this dissertation, as its title suggests, is the Quantified Argument Calculus, its logical properties and its applications. The idea of the Quantified Argument Calculus, or Quarc for short, was first presented by Hanoch Ben-Yami in his 2004 book *Logic and Natural Language*, where he discusses the motivation, stemming from considerations on natural language, philosophy and history of logic, for moving away from the standard Predicate Calculus. Importantly, Ben-Yami also offered an alternative system to complement his criticism. This system was further developed and given formal rigor in Ben-Yami's 2014 paper *The Quantified Argument Calculus* (Review of Symbolic Logic 7(1), pp. 120-146). In that paper Quarc is given its current form, with truth-valuational semantics and a Suppes-Lemmon natural deduction system. Moreover, soundness of the system is demonstrated, and some considerations on its completeness are offered. Finally, Ben-Yami discusses a possible extension of the system, in a way that is both natural and in the spirit of Quarc, with modal operators. The primary driving force of this dissertation is the continuation of that project, and while some headway is made, it is far from finished. Throughout the dissertation, possible venues of continued research will be pointed out.

1.1 Motivation

Before giving a more precise plan of what this dissertations will be concerned with, a brief outline of the motivation for this work is in order.[1] First, it should be noted that Quarc is arguably closer to natural language, at least in its surface structure, than the Predicate Calculus. A thorough discussion of this can be found in the 2004 book, and here we will offer just an illustration of that. Note first and foremost that the quantified phrases are embedded, in the argument position, within a clause, as they appear in natural language. While this is the titular feature of Quarc and the most notable similarity to natural language, several other closely follow.

Consider the example of anaphora. In natural language, a term, mostly a noun, be it common or proper, is referred to by means of a anaphoric expression (e.g. John thinks *he* is going to win, Every dog has *its* day). So, there is an

[1] Since the formal system has not been introduced yet, any reader unfamiliar with Quarc is advised to skip this section and return to it after finishing Chapter 2.

asymmetry between the anaphoric expressions and the source of an anaphor. In the Predicate Calculus, while the individual variables function somewhat like anaphors, the asymmetry is not present. Moreover, variables can only occur in a quantified context, losing the possibility of referring to a proper noun. It is left to the readers' natural language ability to render the anaphors correctly in terms of variables and vice versa. In other words, that feature of language is, in fact, absent from the formal language of Predicate Calculus. Take the example of

(i) Charles is bald and he is bold.

This is rendered in the Predicate Calculus as

(ii) $Bald(c) \wedge Bold(c)$

Sentence (ii) is sufficient to conclude that

(iii) $Bold(c)$

which, if the correspondence with (i) was preserved, would read as

(iv) He is bold.

But (i) is not sufficient to conclude that, since the sentence (iv) does not function as well as (i) when it's self-standing. For instance, there might not be anyone around us for the pronoun 'he' to pick up, in which case we would deem it meaningless. Instead, we should read (iii) as

(v) Charles is bold.

This, of course, is done using the aforementioned readers' natural language ability. By contrast, in Quarc, (i) is rendered as

(vi) $(c_\alpha)Bald \wedge (\alpha)Bold.$

And from this we cannot derive

(vii) $(\alpha)Bold.$

as this is again does not function self-standing. Of course, we can derive that Charles is bold, which (i) entails as well, if we remove the anaphor, just like we would going from (i) to (v). So, the inferential patterns of natural language anaphora are more closely modeled by Quarc, at no loss of derivational force.

A full discussion of why anaphora, as well as such specific elements of Quarc as reordered predicates or negative predications were introduced into the system can be found in the 2004 book. But this brief aside should serve to illustrate why Quarc is taken to be (at least arguably) closer to natural language on one side, while at the same time being an appropriate system of formal logic.

Given this, it is the goal of this dissertation to demonstrate in the subsequent chapters that Quarc has the logical properties of a well-behaved logical system, somewhat extend it, and then apply Quarc to obtain some results in philosophy (or history of philosophy as the case may be).

1.2 Plan of the Dissertation

The subsequent chapters will be concerned with, in order, establishing the completeness of Quarc, presenting the sequent-calculus representation of Quarc and exploring its properties, most notably cut elimination, demonstrating that it possesses the Craig interpolation property, expanding it with modal operators and then applying that to the study of Aristotle's modal syllogistic.

Chapter 2: Completeness of the Quantified Argument Calculus is concerned with establishing the completeness of Quarc, as well as some minor additional features. The proof is an adaptation of the seminal proof by Henkin to the truth-valuational approach used by Ben-Yami in his 2014 paper. It is worth noting that here the system used is the one from that paper, extended with the identity predicate.

Chapter 3: The Proof-Theoretic Analysis of the Quantified Argument Calculus presents LK-Quarc, a sequent-calculus representation of Quarc. The system is presented as a series of expanding systems, first without identity and instantial import, which are added in the course of the chapter. What is established is that the system is deductively equivalent to Quarc, thus transferring all the properties established here back to Ben-Yami's version of Quarc. We then demonstrate that LK-Quarc possesses the cut elimination property. The proof, as well as the system LK-Quarc itself, is an adaptation of the seminal work by Gentzen.

After establishing the cut elimination theorem, we demonstrate that LK-Quarc possesses the subformula property, and is therefore also consistent. Moreover, we prove conservativity for the versions of LK-Quarc containing identity.

Chapter 4: Craig Interpolation Property follows up on the previous chapter by using the cut elimination theorem to prove that LK-Quarc possesses the Craig interpolation. It establishes the significant result that LK-Quarc°, the system extended with a unary predicate τ, possesses the property and shows a constructive method, following Maehara's method, of constructing an appropriate interpolant for every derivable sequent of LK-Quarc°. Given the timeline of this dissertation, it is left for future research to determine whether and how this result can be extended to LK-Quarc, but the final sections of the chapter offer some headway on that problem, with discussing some possibilities for eliminating τ.

Chapter 5: Modal Extension of LK-Quarc is a brief exploration of an extension of Quarc with modal operators. It establishes that everything provable in the modal extension of Quarc along the lines suggested in Ben-Yami's 2014 paper is likewise derivable in the modal extension of LK-Quarc, for any of a range of modal systems (K, T, 4, B, S4, S5). Moreover, the cut elimination theorem, subformula property and consistency are established for the entire range of modal systems, to be used in the following, final, chapter.

Chapter 6: On a Mismatch between Aristotle's Modal Syllogistic and Modern Modal Logic explores Aristotle's Modal Syllogistic from the perspective of Quarc. It is shown why Quarc is a very fruitful approach for investigating Aristotle's syllogistic, providing streamlined proofs of the validity of its assertoric segment, while at the same time providing sound and independently motivated reasons for all the concessions needed for a success of such an endeavor.

We then turn our attention to the modal, or more specifically apodictic, syllogistic. Quarc is here used to demonstrate the shortcomings of a popular recent attempt, by Marko Malink, of proving its validity. This result is used to illustrate an existence of a fundamental mismatch between Aristotle's and modern approaches to modality which makes it unlikely that any attempt of that sort will be successful.

Chapter 2

Completeness of the Quantified Argument Calculus

2.1 Introduction[1]

The presentation of the Quantified Argument Calculus in [Ben-Yami, 2014] makes a mention of the completeness of the system, but does not demonstrate it. In this chapter we will prove that Quarc is complete, alongside a few additional facts about it.

Structure of the chapter: in the second section of this chapter we present Quarc, closely following [Ben-Yami, 2014]. A significant addition, though, is the introduction into the system of identity, which was not systematically considered in that paper. We add formation rules for formulas containing identity, rules for truth value assignments and derivation rules. Corresponding adjustments have been made in the ordering and numbering of the definitions and rules. We also prove that any truth value assignment uniquely determines the truth value of any formula, a fact stated without proof in the earlier paper.

In the third section we provide a completeness proof for Quarc. Note that the soundness of the system was proved in the earlier paper, and like that paper, this one doesn't use a model-theoretic semantics but a truth-valuational approach. The proof is an adaptation of the completeness proof from [Henkin, 1949], with some changes to accommodate that difference in approach.

2.2 The Formal System

In this section we present the formal system of Quarc. As noted in the previous section, this presentation follows [Ben-Yami, 2014], mostly verbatim, with the

[1]The author's research in this chapter contributed to a joint paper with Hanoch Ben-Yami [Pavlovic and Ben-Yami, 2013]. The structure of the proof of the uniqueness of truth-value assignment is due to Ben-Yami, as is the idea of a Henkin assignment and a non-model-theoretic completeness proof, taken from [Ben-Yami, 2011].

addition of identity to the vocabulary of the language, and the corresponding adjustments to truth-value assignments and addition of the derivation rules.

2.2.1 Vocabulary of Quarc

Definition 1 (Vocabulary of Quarc) The language of Quarc contains the following symbols:

1. Predicates: $P, Q, R, ...$, denumerably many and with a fixed arity, including a binary predicate of identity, $=$.
2. Singular arguments (SAs): $a, b, c, ...$
3. Anaphors: $\alpha, \beta, \gamma, ...$
4. Sentential operators: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$.
5. Quantifiers: $\forall, \exists$.
6. Numerals used as indices, comma, parentheses.

If P is a unary predicate, then $\forall P$ and $\exists P$ will be called *quantified arguments* (QAs). An argument is only a singular argument or a quantified one (anaphors are not considered arguments).

An occurrence of an argument A is the *source* of the anaphor α if A is to the left of α, α is written as a subscript to A (i.e. A_α), and α is not a subscript to any argument occurring between it and A. In A_α, only A is considered an argument.

For every n-ary ($n > 1$) predicate R, R^π, where π is any permutation of $1, ..., n$ (including identity permutation) is called a *reordered* form of R.

2.2.2 Formulas of Quarc

Definition 2 (Formula) The following rules specify all the ways in which formulas can be generated.

1. (Basic formula) If P is an n-ary predicate and $t_1, ..., t_n$ SAs, then $(t_1, ..., t_n)P$ is a formula, called a *basic formula.*
2. (Reorder) If P is a reordered n-ary predicate ($n > 1$) and $t_1, ..., t_n$ SAs, then $(t_1, ..., t_n)P$ is a formula.
3. (Identity) If a and b are singular arguments, not necessarily different, then $a = b$ is a formula. $a = b$ is an alternative way of writing $(a, b) =$, which is a basic formula. We shall not use the latter form of the formula.
4. (Negative predication) If P is an n-ary predicate or a reordered n-ary predicate and $t_1, ..., t_n$ SA's, then $(t_1, ..., t_n)\neg P$ is a formula.
5. (Sentential operators) If A and B are formulas, so are $\neg(A), (A)\wedge(B), (A)\vee(B), (A) \rightarrow (B)$. The parentheses surrounding formulas are called *sentential* parentheses.

6. (Anaphora) If A is a formula containing, from left to right, $t_1, ..., t_n$ $(n > 1)$ occurences of SA t, none of which is a source of any anaphor, and A does not contain α, then $A\left[t_\alpha/t_1, \alpha/t_2, ..., \alpha/t_n\right]$ is a formula. We call t_α the *source* of the anaphora α.

   Before proceeding to define the rules for quantifiers, a notion of governance, related to that of scope in the Predicate Calculus, needs to be introduced:

7. (Governance) An occurrence qP of a QA governs a formula A just in case qP is the leftmost QA in A and A does not contain any other string of symbols (B) in which the parentheses are a pair of sentential parentheses, such that B contains qP and all the anaphors of all the QAs in B.

8. (Quantification) If A is a formula containing an occurrence of an SA t, and substituting a QA qP for t will result in qP governing A, then $A\left[qP/t\right]$ is a formula.

For examples of formulas and explanation of the ideas behind some of the definitions, see [Ben-Yami, 2014].

2.2.3 Truth-Value Assignments

We next list the rules for assigning truth values to formulas.

Definition 3 (Truth-Value Assignments) The following holds for any truth-value assignment:

1. (Basic formula) Every basic formula is assigned the truth-value of true or false, but not both.

2. (Reorder) Let P be an n-ary predicate and $\pi = \pi_1, ..., \pi_n$ a permutation of $1, ..., n$. Then, the truth-value assigned to $(t_{\pi 1}, ..., t_{\pi n})P^\pi$ is that assigned to $(t_1, ..., t_n)P$.

3. (Law of Identity) Every formula of the form $t = t$ is true.

4. (Indiscernibility of Identicals) If $t = c$ is true and the formula $A\left[t_1, ..., t_n\right]$ is a basic formula containing the instances $t_1, ..., t_n$ of an SA t, then $A\left[c/t_1, ..., c/t_n\right]$ is true if $A\left[t_1, ..., t_n\right]$ is true. [2]

5. (Instantiation) For every unary predicate P there is some SA t such that $(t)P$ is true.

6. (Sentential operators) Let A and B be formulas. Then, $\neg(A)$ is true just in case A is false. Etc.

7. (Negative predication) Let P be an n-ary predicate and $t_1, ..., t_n$ SA's. The truth value of $(t_1, ..., t_n)\neg P$ is that of $\neg(t_1, ..., t_n)P$.

[2] The *only if* part follows from Definition 3.3 and Definition 3.4. This rule could have been defined for only a single occurence of t, but this formulation will save us some unnecessary longer derivations.

8. (Anaphora) If A is a formula containing, from left to right, occurrences $t_1, ..., t_n$ of SA t, none of which is the source of any anaphors, and A does not contain α, then the truth-value of $A\,[t_\alpha/t_1, \alpha/t_2, ..., \alpha/t_n]$ is that of A.

9. (Quantification) Let $A\,[\forall P]$ $(A\,[\exists P])$ be a formula governed by an occurrence of $\forall P$ $(\exists P)$. If for every (some) SA t for which $(t)P$ is true $A\,[t/\forall P]$ $(A\,[t/\exists P])$ is true, then $A\,[\forall P]$ $(A\,[\exists P])$ is true. If for some (every) t for which $(t)P$ is true $A\,[t/\forall P]$ $(A\,[t/\exists P])$ is false, then $A\,[\forall P]$ $(A\,[\exists P])$ is false.

For a discussion of these definitions, and in particular of Instantiation, see [Ben-Yami, 2014].

Uniqueness of Truth-Value Assignments

On any assignment, every basic formula is assigned a truth-value, and any formula that is not basic is assigned a truth-value according to the way it is generated. It follows that on any assignment, every formula has a truth-value. However, since some formulas can be generated in more than a single way, we have to show that on any assignment all these ways determine the same truth-value.

Any element of a formula is introduced by a unique rule. Accordingly, any formula is generated by a unique combination of rules. However, some formulas can be generated by applying the rules in more than one order. Consider for instance the formula $(a_\alpha, \alpha)R \wedge (b)Q$. It can be generated from $(a_\alpha, \alpha)R$ and $(b)Q$ according to Definition 2.5 (Sentential operator), or from $(a, a)R \wedge (b)Q$ by Definition 2.6 (Anaphora).

In general, if two consecutive applications of rules can be transposed without affecting the generated formula, then a concatenation of such rules can sometimes yield more than two ways of generating the same formula. For instance, the formula $(a_\alpha, \alpha)R \wedge (b_\beta, \beta)Q$ can be generated by two applications of Definition 2.6 and one of Definition 2.5 to the formulas $(a, a)R$ and $(b, b)Q$ in five different orders.

Examination of Definition 2 shows that the order of two applications of Anaphora can always be transposed, and that the order of application of Sentential Operators or Quantification can sometimes be transposed with the application of Anaphora. The consecutive application of any two other rules can never be transposed.

We will prove the following:

Theorem 4 If some formula can be generated by applying two definitions in two different sequences, then for any truth-value assignment each sequence assigns to that formula the same truth-value.

Note that if a longer sequence of rules can be arranged in several ways to generate the same formula, then recurrent application of the theorem for the transposition of two rules will show that for any truth-value assignment, any way of generating the same formula determines the same truth-value for it.

Proof. By cases. As an illustration, let us examine just the proof for Definition 2.6 (Anaphora) and Definition 2.8 (Quantification), and more specifically, for the case of the universal quantifier. Consider the formula $A\,[a_1, ..., a_n, b]$. By applying the Definition 2.6 and then Definition 2.8, we obtain the formula (i) $A\,[a_1, ..., a_n, b]\,[a_\alpha/a_1, ..., \alpha/a_n]\,[\forall P/b]$, and by applying the Definition 2.8 and then Definition 2.6 we obtain the formula (ii) $A\,[a_1, ..., a_n, b]\,[\forall P/b]\,[a_\alpha/a_1, ..., \alpha/a_n]$. These two formulas are identical in case $\forall P$ governs both $A\,[a_1, ..., a_n, b]\,[\forall P/b]$ and (i).

Now, (i) is true just in case for every c for which cP is true, the formula $A\,[a_1, ..., a_n, c]\,[a_\alpha/a_1, ..., \alpha/a_n]$ is true, and this is true just in case $A\,[a_1, ..., a_n, c]$ is true. Therefore, (i) is true just in case for every c for which cP is true, $A\,[a_1, ..., a_n, c]$ is true.

Similarly, (ii) is true just in case $A\,[a_1, ..., a_n, b]\,[\forall P/b]$ is true, and this is true just in case for every c for which cP is true, $A\,[a_1, ..., a_n, c]$ is true. Therefore, (ii) is true just in case for every c for which cP is true, $A\,[a_1, ..., a_n, c]$ is true.

It follows that (i) is true just in case (ii) is. Similarly for other cases.

2.2.4 Derivation Rules

We proceed to define the proof system of Quarc. We use a natural deduction system, in which proofs are written as follows:

Definition 5 (Proof) A proof is a list of lines of the form $\langle L, (i), A, R\rangle$, where L is a (possibly empty) list of line numbers of premises; (i) the line number; A a formula; and R the justification, an element of a set of derivation rules, such that the line is written in accordance with it. A is said to depend on the premises listed in L. The line numbers in L are written without repetitions and in ascending order. The formula in the last line of the proof is its *conclusion.*

We next list the derivation rules of the system.

Definition 6 (Derivation Rules)

1. (Premise) At any stage in a proof, any formula can be written, depending on itself, its justification being Premise:

   i (i) A Premise

2. (Propositional Calculus Rules, **PCR**) We allow the usual derivation rules of the Propositional Calculus, but with the constraint that for each rule, all the appropriate symbols in its formulation stand for formulas.

3. (Sentence negation to Predication negation, **SP**) Let P be an n-ary predicate or a reordered n-ary predicate, and $t_1, ..., t_n$ singular arguments.

   L (i) $\neg(t_1, \ldots t_n)P$
   L (j) $(t_1, \ldots t_n)\neg P$ SP, i

4. (Predication negation to Sentence negation, **PS**) Let P be an n-ary predicate or a reordered n-ary predicate, and $t_1, \ldots t_n$ singular arguments.

   L (i) $(t_1, \ldots t_n)\neg P$
   L (j) $\neg(t_1, \ldots t_n)P$ PS, i

5. (Reorder, **R**) Let P be an n-ary predicate and $\pi = \pi 1, \dots \pi n$ and $\rho = \rho 1, \dots \rho n$ two permutations of $1, 2, \dots n$ (the identity permutation included).

L	(i)	$(t_{\pi 1}, \dots t_{\pi n})P^{\pi}$	
L	(j)	$(t_{\rho 1}, \dots t_{\rho n})P^{\rho}$	R, i

6. (Anaphora Introduction, **AI**) Let A be a formula containing, from left to right, occurrences $t_1, \dots t_n$ of the singular argument t, none of which has any anaphors, and suppose α does not occur in A.

L	(i)	A	
L	(j)	$A\,[t_\alpha/t_1, \alpha/t_2 \dots \alpha/t_n]$	AI, i

7. (Anaphora Elimination, **AE**) Let A be a formula containing, from left to right, occurrences $t_1, \dots t_n$ of the singular argument t, none of which has any anaphors, and suppose α does not occur in A.

L	(i)	$A\,[t_\alpha/t_1, \alpha/t_2 \dots \alpha/t_n]$	
L	(j)	A	AE, i

8. (Universal Introduction, **UI**) Let $A\,[\forall P]$ be a formula governed by $\forall P$. Assume that neither $A\,[\forall P]$ nor the formulas in lines L apart from $(t)P$ contain any occurrence of the singular argument t.

i	(i)	$(t)P$	Premise
L	(j)	$A\,[t/\forall P]$	
$L - i$	(k)	$A\,[\forall P]$	UI, i, j

Where $L - i$ is the (possibly empty) list of numbers occurring in L apart from i.

9. (Universal Elimination, **UE**) Let $A\,[\forall P]$ be a formula governed by $\forall P$.

L_1	(i)	$A\,[\forall P]$	
L_2	(j)	$(t)P$	
L_1, L_2	(k)	$A\,[t/\forall P]$	UE, i, j

Where L_1, L_2 is the list of numbers occurring either in L_1 or in L_2.

10. (Particular Introduction, **PI**) Let $A\,[\exists P]$ be a formula governed by $\exists P$.

L_1	(i)	$A\,[t/\exists P]$	
L_2	(j)	$(t)P$	
L_1, L_2	(k)	$A\,[\exists P]$	PI, i, j

11. (Instantial Import, **Ins**) Let q stand for either $\forall$ or $\exists$, and $A\,[qP]$ be governed by qP. Assume t does not occur in $A\,[qP]$, B or L_1 and in no formula in L_2 apart from j and k.

L_1	(i)	$A\,[qP]$	
j	(j)	$(t)P$	Premise
k	(k)	$A\,[t/qP]$	Premise
L_2	(l)	B	
$L_1, L_2 - j - k$	(m)	B	Ins, i, j, k, l

12. (Identity Introduction, **=I**) In any line of the proof a formula of the form $t = t$ can be written, depending on no premises, with its justification being =I.

    (i) $t = t$ =I

13. (Identity Elimination, **=E**) Let A be a basic formula containing occurrences $t_1, \dots t_n$ of the singular argument t (A may also contain additional occurrences of t).

L_1	(i)	A	
L_2	(j)	$t = c$	
L_1, L_2	(k)	$A\,[c/t_1, \dots c/t_n]$	=E, i, j

For examples of proofs, see [Ben-Yami, 2014] §3.5.

2.3 Completeness of Quarc

Completeness is a metalogical property of a formal system, a relation between validity and derivability. Therefore, before proving completeness, let us define these two:

Definition 7 (Validity) An argument whose premises are all and only the formulas in the set of formulas Γ, and whose conclusion is the formula A is valid (written $\Gamma \vDash A$) just in case every truth-value assignment that makes all the formulas in Γ true also makes A true, even if we add or eliminate singular arguments from our language (of course, only singular arguments not occurring in Γ and A can be eliminated).[3]

Definition 8 (Derivability) Formula A is derivable from a (possibly empty) set of formulas Γ (written $\Gamma \vdash A$) just in case there is a proof whose conclusion is A, such that A depends only on premises $\gamma_1 \dots \gamma_n$, where $\{\gamma_1 \dots \gamma_n\} \subseteq \Gamma$.

We can now define completeness:

Definition 9 (Completeness) A formal proof system is complete just in case for every valid argument $\Gamma \vDash A$, it holds that $\Gamma \vdash A$. Completeness can also be formulated as $\Gamma \vDash A \Rightarrow \Gamma \vdash A$.

We proceed to prove the completeness of Quarc. The proof is an adaptation of Leon Henkin's proof [Henkin, 1949]. A standard Henkin-style proof consists of the following stages: adding witnessing constants; constructing the Henkin theory; defining the Henkin construction; proving the elimination theorem; and some final steps. The structure of the proof below, which applies to the truth-valuational approach, departs from the standard structure in its replacement of the Henkin construction, which is the step in which models are introduced, by the Henkin *assignment*.

[3]The need to make validity on the truth-valuational approach independent of a specific individual constant list has long been recognized. For more details see [Ben-Yami, 2011].

2.3.1 Henkin Theory

The Henkin Theory is a set of formulas that fall under given schemas, which we will use to establish a connection between Quarc and the Propositional Calculus, a calculus that we know to be complete. The Henkin Theory uses a language L_H, which is an extension of the language of Quarc with new singular arguments, the *witnessing constants*. We first define these.

Definition 10 (Witnessing Constant) For every formula of Quarc of the form $A[\exists P]$, where $\exists P$ governs the formula, we introduce the witnessing constant $w_{A[\exists P]}$.

Extending our language with witnessing constants will generate a language L_1, which will contain new formulas, some of them once again of the form $A[\exists P]$, where $\exists P$ governs the formula. We repeat the same process for L_1 to obtain L_2, and so on for any subsequent language L_n.

Definition 11 (Henkin Language L_H) The Henkin language L_H is the union of all the languages L_i, $i \in \mathbb{N}$ produced by the extensions of Quarc by witnessing constants.

Definition 12 (Date of Birth) Date of birth of a witnessing constant is the i of the L_i in which it was introduced.

With these definitions in place, we can now define the Henkin Theory (where P is a unary predicate and R an n-ary predicate, R^π and R^ϱ reorderings of R, c and t singular arguments, with c with numbered indices the same argument, but t a possibly different singular argument, C a formula of Quarc in which the anaphor α does not occur, and where $c_1 \ldots c_n$ are not the source of any anaphor, B a basic formula, and $A[qP]$ a formula of Quarc governed by the quantified argument qP).

Definition 13 (Henkin Theory) Henkin Theory H consists of all the formulas that fall under one of the following schemas:

H1. $A[\exists P] \to ((w_{A[\exists P]})P \land A[w_{A[\exists P]}/\exists P])$

H2. $(tP \land A[t/\exists P]) \to A[\exists P]$

H3.1. $\neg A\,[\forall P] \leftrightarrow (\exists P_\alpha P \land \neg A\,[\alpha/\forall P])$, if $\forall P$ is source of no anaphors
H3.2. $\neg A\,[\forall P] \leftrightarrow (\exists P_\alpha P \land \neg A\,[\alpha/\forall P_\alpha])$, if $\forall P$ is source of anaphor α

H4. $c = c$

H5. $c = t \to (B\,[c_1 \ldots c_n] \to B\,[t/c_1 \ldots t/c_n])$

H6. $(t_1 \ldots t_n)\neg R \leftrightarrow \neg(t_1 \ldots t_n)R$

H7. $C\,[c_1, c_2 \ldots c_n] \leftrightarrow C\,[c_\alpha/c_1, \alpha/c_2 \ldots \alpha/c_n]$

H8. $(t_{\pi 1} \ldots t_{\pi n})R^\pi \leftrightarrow (t_{\varrho 1} \ldots t_{\varrho n})R^\varrho$

H9. $(\exists P)P$

We next define the Henkin Assignments and examine their properties.

2.3.2 Henkin Assignment

Definition 14 (Henkin Assignment) A Henkin Assignment χ is a truth-value assignment that assigns truth-values to all formulas of L_H, such that χ respects the truth-value assignment rules for the connectives of the propositional calculus (Definition 3), while also making all the formulas of the Henkin Theory true.

We now prove the following important Lemma:

Lemma 15 Any Henkin assignment χ respects all the truth-value assignment rules of Quarc. Notably, it respects the truth-value assignment rules for the particular quantifier, negative predication, anaphora, the universal quantifier, identity and instantiation.

Proof. First note that the truth-value assignment rule for basic formulas, that each is either true or false, is observed by χ, since any formula is either true or false on χ.

Particular Quantifier. Suppose that a formula of the form $A\,[\exists P]$ governed by $\exists P$ is true. By $H1$ it follows that so is a formula of the form $A\,[t/\exists P]$, namely $A[w_{A[\exists P]}/\exists P]$, and moreover, that the formula $(w_{A[\exists P]})P$ is true.

Likewise, if a formula of the form $A[t/\exists P]$ and the formula $(t)P$ are true, then by $H2$ $A[\exists P]$ is true as well. Therefore, χ satisfies the truth-value assignment rule for the particular quantifier.

Negative Predication. Let $(t_1 \ldots t_n)\neg R$ be true (false). Then, by $H6$, the formula $\neg(t_1 \ldots t_n)R$ is true (false). Therefore, the truth-value assignment for $(t_1 \ldots t_n)\neg R$ is the same as $\neg(t_1 \ldots t_n)R$, and χ satisfies the truth-value assignment rule for negative predication.

Anaphora. Suppose $C\,[c_1, c_2 \ldots c_n]$ is true (false). Then by $H7$, the formula $C\,[c_\alpha/c_1, \alpha/c_2 \ldots \alpha/c_n]$ is true (false), and χ satisfies the truth-value assignment rule for anaphora.

Reorder. As above, since χ satisfies $H8$, it satisfies the truth-value assignment rule for reorder.

Universal Quantifier. Let $A\,[\forall P]$ be a formula governed by the quantified argument $\forall P$, which does not contain the singular argument c. Assume first $\forall P$ is not a source of any anaphor.

Assume (i) that for every singular argument t for which $(t)P$ is true, $A\,[t/\forall P]$ is also true. We need to show that $A\,[\forall P]$ is then true. Now assume for *reductio* that (ii) $\exists P_\alpha P \wedge \neg A\,[\alpha/\forall P]$ is true.

Since it has already been established that χ satisfies the truth-value assignment rule for particular quantification, it follows from (ii) that for some a, $a_\alpha P \wedge \neg A\,[\alpha/\forall P]$ is true, and from that, since it has already been established that χ satisfies the truth-value assignment rule for anaphora, $aP \wedge \neg A\,[a/\forall P]$ is true. But this is contrary to assumption (i). So, $\exists P_\alpha P \wedge \neg A\,[\alpha/\forall P]$ is false, and by $H3.1$, $\neg A\,[\forall P]$ is also false, and therefore $A\,[\forall P]$ is true. Similar for falsity.

Now assume $\forall P$ is a source of anaphor α. The case is *mutatis mutandis* same as above, using $H3.2$. Therefore, χ satisfies the truth-value assignment rule for the particular quantifier.

Identity. Suppose $c = t$ is true and $B[c_1 \ldots c_n]$ a basic formula containing occurrences $c_1 \ldots c_n$ of a singular argument c. Since χ satisfies $H5$, the formula $B[t/c_1 \ldots t/c_n]$ obtained by substituting some or all occurrences of c by t in B is true if $B[c_1 \ldots c_n]$ is true. Moreover, χ satisfies $H4$. Therefore, χ satisfies the truth-value assignment rules for identity.

Insantiation. Since χ satisfies $H9$, $(\exists P)P$ is true, and since it satisfies $H1$, $(w_{(\exists P)P})P$ is true. Therefore, for every unary predicate P there is a singular argument t for which $(t)P$ is true, namely $w_{(\exists P)P}$, and χ satisfies the truth-value assignment rule for instantiation, which concludes the proof of Lemma 15.

Now take any argument $\Gamma \vDash A$, valid in Quarc, where χ and all the formulas of Γ belong to L (i.e. they contain no witnessing constants). Since L_H is simply an extension of Quarc with additional singular arguments, by the definition of validity, $\Gamma \vDash A$ is valid in Quarc with L_H as its language as well. And, since adding premises to a valid argument does not affect its validity, the argument $\Gamma, H \vDash A$ is likewise valid in Quarc with L_H as its language. Now, given that any truth-value assignment that respects the rules of the Propositional Calculus and makes all the formulas of Γ and H true will be a Henkin Assignment, it follows by Lemma 15 that it will likewise respect all the other truth-value assignment rules of Quarc, and therefore that it will make χ true as well. It follows that $\Gamma, H \vDash A$ also in the Propositional Calculus.

Given the completeness of the Propositional Calculus, this yields that $\Gamma, H \vdash A$ in the Propositional Calculus. What needs to be shown now is that, if $\Gamma, H \vdash A$ in the Propositional Calculus, then $\Gamma \vdash A$ in Quarc – the Elimination Theorem.

2.3.3 Elimination Theorem

Before proceeding with the proof of this theorem, several preliminary results need to be established. The proofs that involve only the derivation rules of the Propositional Calculus will be omitted.

Let Γ be a set of formulas of Quarc and A, B and C formulas of Quarc.

Theorem 16 (Deduction Theorem) If $\Gamma, A \vdash B$ then $\Gamma \vdash A \to B$.

Proposition 17 If $\Gamma, A_1 \ldots A_n \vdash B$ and for every $i \leq n$, $\Gamma \vdash A_i$ then $\Gamma \vdash B$.

Lemma 18 If $\Gamma \vdash A \to B$ and $\Gamma \vdash \neg A \to B$, then $\Gamma \vdash B$.

Lemma 19 If $\Gamma \vdash (A \to B) \to C$, then $\Gamma \vdash \neg A \to C$ and $\Gamma \vdash B \to C$.

Lemma 20 If $A[\exists P]$ is governed by $\exists P$ and t is a singular argument appearing nowhere in Γ, $A[\exists P]$ or B, then if $\Gamma \vdash (tP \wedge A[t/\exists P]) \to B$, then $\Gamma \vdash A[\exists P] \to B$.

Proof. Since $\Gamma \vdash (tP \wedge A[t/\exists P]) \rightarrow B$, then there is a finite number of formulas of Γ, say k, such that $C_1 \ldots C_k \vdash (tP \wedge A[t/\exists P]) \rightarrow B$. Let us now write the proof of $(tP \wedge A[t/\exists P]) \rightarrow B$, with $C_1 \ldots C_k$ as premises. Suppose that the proof is i lines long. The proof then continues as follows:

...

L	(i)	$(tP \wedge A[t/\exists P]) \rightarrow B$	
$i+1$	(i+1)	$A[\exists P]$	Premise
$i+2$	(i+2)	$A[t/\exists P]$	Premise
$i+3$	(i+3)	tP	Premise
$i+2$, $i+3$	(i+4)	$tP \wedge A[t/\exists P]$	$\wedge$I, $i+2, i+3$
L, $i+2$, $i+3$	(i+5)	B	$\rightarrow$E, i, $i+4$
L, $i+1$	(i+6)	B	Ins, $i+1$, $i+2$, $i+3$, $i+5$
L	(i+7)	$A[\exists P] \rightarrow B$	$\rightarrow$I, $i+1$, $i+6$

Since all the formulas in L are in Γ, it follows that $\Gamma \vdash A[\exists P] \rightarrow B$.

Next we prove Lemma 21, needed so we can eliminate the formulas containing witnessing constants from $\Gamma, H \vdash A$:

Lemma 21 If $A[\exists P]$ is governed by $\exists P$ and t is a singular argument nowhere in Γ, $A[\exists P]$ or B, then if $\Gamma, A[\exists P] \rightarrow (tP \wedge A[t/\exists P]) \vdash B$ then $\Gamma \vdash B$

Proof. By applying Theorem 16 to $\Gamma, A[\exists P] \rightarrow (tP \wedge A[t/\exists P]) \vdash B$, we obtain $\Gamma \vdash (A[\exists P] \rightarrow (tP \wedge A[t/\exists P])) \rightarrow B$. Then by Lemma 19 we get (i) $\Gamma \vdash \neg A[\exists P] \rightarrow B$ and (ii) $\Gamma \vdash (tP \wedge A[t/\exists P]) \rightarrow B$. Applying Lemma 20 (and given our assumptions) to (ii) we get (iii) $\Gamma \vdash A[\exists P] \rightarrow B$. Finally, applying Lemma 19 to (i) and (iii), we get $\Gamma \vdash B$.

The final preliminary consideration we need for the proof of the Elimination Theorem is to show that the formulas $H2$-$H9$ of Henkin Theory are theorems of Quarc. Let us consider the straightforward cases first.

$H2$ corresponds to the derivation rule PI and can be proved by it and the derivation rules of the Propositional Calculus; $H4$ corresponds to the rule =I; $H5$ corresponds to =E; the two directions of $H6$ correspond to the rules PS and SP; the two directions of $H7$, to AI and AE; and $H8$ to Reorder. This leaves us with $H3.1$, $H3.2$ and $H9$ to prove.

We first prove $H3.1$. Let $\forall P$ be the source of no anaphora.

(i) $\vdash \neg A[\forall P] \rightarrow (\exists P_\alpha P \wedge \neg A[\alpha/\forall P])$

1	(1)	$\neg A\,[\forall P]$	Premise
2	(2)	$\neg(\exists P_\alpha P \wedge \neg A\,[\alpha/\forall P])$	Premise
3	(3)	tP	Premise
4	(4)	$\neg A\,[t/\forall P]$	Premise
$3,4$	(5)	$tP \wedge \neg A\,[t/\forall P]$	$\wedge$I: 3, 4
$3,4$	(6)	$t_\alpha P \wedge \neg A\,[\alpha/\forall P]$	AI: 5
$3,4$	(7)	$\exists P_\alpha P \wedge \neg A\,[\alpha/\forall P]$	PI: 3, 6
$2,3$	(8)	$\neg\neg A\,[t/\forall P]$	$\neg$I:4, 2, 7
$2,3$	(9)	$A\,[t/\forall P]$	$\neg$E: 8
2	(10)	$A\,[\forall P]$	UI: 3, 9
1	(11)	$\neg\neg(\exists P_\alpha P \wedge \neg A\,[\alpha/\forall P])$	$\neg$I: 2, 1, 10
1	(12)	$\exists P_\alpha P \wedge \neg A\,[\alpha/\forall P]$	$\neg$E: 11
	(13)	$\neg A\,[\forall P] \rightarrow (\exists P_\alpha P \wedge \neg A\,[\alpha/\forall P])$	$\rightarrow$I: 1, 12

In line (3) we choose a t that does not occur in the string $\neg A\,[\alpha/\forall P]$. This is necessary for the use of UI in line (10).

(ii) $\vdash (\exists P_\alpha P \wedge \neg A\,[\alpha/\forall P]) \rightarrow \neg A\,[\forall P]$

1	(1)	$\exists P_\alpha P \wedge \neg A\,[\alpha/\forall P]$	Premise
2	(2)	$A\,[\forall P]$	Premise
3	(3)	tP	Premise
4	(4)	$t_\alpha P \wedge \neg A\,[\alpha/\forall P]$	Premise
4	(5)	$tP \wedge \neg A\,[t/\forall P]$	AE: 4
4	(6)	$\neg A\,[t/\forall P]$	$\wedge$E: 5
$2,3$	(7)	$A\,[t/\forall P]$	UE: 2, 3
$3,4$	(8)	$\neg A\,[\forall P]$	$\neg$I: 2, 6, 7
1	(9)	$\neg A\,[\forall P]$	Ins: 1, 3, 4, 8
	(10)	$(\exists P_\alpha P \wedge \neg A\,[\alpha/\forall P]) \rightarrow \neg A\,[\forall P]$	$\rightarrow$I: 1, 9

We now prove $H3.2$. Let $\forall P$ be the source of anaphora α. The proof differs only slightly from the one for $H3.1$ and will be omitted here.

And finally, we prove $H9$:

1	(1)	tP	Premise
	(2)	$(\forall P)P$	UI: 1, 1
1	(3)	$(\exists P)P$	PI: 1, 1
	(4)	$(\exists P)P$	Ins: 2, 1, 1, 3

We should keep in mind that whatever is derivable in the Propositional Calculus is derivable in Quarc as well, as the derivation rules or the latter include those of the former.

With all this preliminary work done, we can now proceed with the proof of the Elimination Theorem itself. We have previously indicated what the Elimination Theorem is, but let us now formulate it precisely:

Theorem 22 (Elimination Theorem) If A is a formula of Quarc derivable from formulas of Quarc $C_1 \ldots C_n$ together with formulas of Henkin Theory (written as $C_1 \ldots C_n, h_1 \ldots h_m \vdash A$), and A, $C_1 \ldots C_n$ belong to L, then A is derivable from $C_1 \ldots C_n$ alone ($C_1 \ldots C_n \vdash A$).

Proof. By induction on the number of Henkin formulas k from which A is derivable.

Basic step. If $k = 0$, then Theorem 22 vacuously holds as there are no Henkin formulas to eliminate.

Inductive step. We now wish to show that, if the Elimination Theorem holds for any derivation with k or fewer formulas of the Henkin theory H, it also holds for any formula A derivable from $k+1$ formulas of the Henkin theory and formulas $C_1 \dots C_n$ of L. There are two cases to consider here.

First, suppose one of the formulas of H is of the form $H2 - H9$, say h_{k+1}. Since all of these formulas are theorems of Quarc, so $\vdash h_{k+1}$ and therefore $C_1 \dots C_n, h_1 \dots h_k \vdash h_{k+1}$. Therefore, by Proposition 17, $C_1 \dots C_n, h_1 \dots h_k \vdash A$, and by inductive hypothesis these can be eliminated.

The second case is that in which all the $k+1$ Henkin formulas are of the form $H1$. In that case, we choose one instance of $H1$ the witnessing constant of which is of the same or greater date of birth than any witnessing constant of any other instance of H1. Since this witnessing constant does not occur in any of the other schema instances, and neither does it occur $C_1 \dots C_n$ or A (as they are formulas of the non-extended language of Quarc), by Lemma 21 it can be eliminated. So, A is derivable from $C_1 \dots C_n$ and k formulas of H, and by inductive hypothesis these can be eliminated.

Since Quarc includes all the derivation rules of the Propositional Calculus, we can now conclude that if $\Gamma, H \vdash A$ in the propositional calculus, then $\Gamma, H \vdash A$ in Quarc as well. Since the conclusion in any proof depends on a finite number of premises, it follows that $C_1 \dots C_n, H \vdash A$. If Γ, A are all formulas of Quarc that belong to L, then by the Elimination Theorem, $C_1 \dots C_n \vdash A$, and therefore $\Gamma \vdash A$ in Quarc.

To summarize, what we have established is that, if $\Gamma \vDash A$ in Quarc, then $\Gamma, H \vDash A$ in Quarc as well. But, given the definition of a Henkin assignment, it follows $\Gamma, H \vDash A$ in the Propositional Calculus, and since this calculus is complete, $\Gamma, H \vdash A$ in it. We have further established by means of the Elimination Theorem that if $\Gamma, H \vdash A$ in the Propositional Calculus, then $\Gamma \vdash A$ in Quarc. This finally enables us to conclude that if $\Gamma \vDash A$ in Quarc, then $\Gamma \vdash A$ in Quarc, and therefore Quarc is complete.

Chapter 3

Proof-Theoretic Analysis of The Quantified Argument Calculus

3.1 Introduction[1]

This chapter investigates the proof theory of the Quantified Argument Calculus (Quarc) as developed and systematically studied by Hanoch Ben-Yami [Ben-Yami, 2014], [Ben-Yami, 2004]. Ben-Yami makes use of a natural deduction (Suppes-Lemmon-style), we, however, have chosen a sequent calculus presentation; which allows for the proofs of a multitude of significant meta-theoretic results with minor modifications to the Gentzen's original framework, i.e. LK. LK, although it has been developed in the 1930ies serves still (as a basis) for proof theoretic investigations [Baaz and Leitsch, 2011], [Buss, 1998], [Takeuti, 2013].

The way the research on Quarc is conducted here is as follows: we observe first that Ben-Yami's Quarc is a rather rich system. In our analysis we split up Quarc into three distinct sub-systems, namely (1) LK-Quarc$_B$, (2) LK-Quarc$_2$, (3) LK-Quarc$_3$, and finally, LK-Quarc – which is Ben-Yami's (full) Quarc. LK-Quarc$_B$ does not contain either the rules for identity or instantiation (to be explained later). LK-Quarc$_2$ is an extension of LK-Quarc$_B$ with identity, and LK-Quarc$_3$ an extension of LK-Quarc$_B$ with the rule for instantiation. Finally, LK-Quarc is obtained by combining LK-Quarc$_2$ and LK-Quarc$_3$. As will be made clear in the course of the chapter LK-Quarc will enjoy Cut elimination and its corollaries (including subformula property and thus consistency which is outlined but not proven in [Ben-Yami, 2014]).

Plan of the chapter: In section 2 we present Quarc$_B$, consisting of its language, truth-value assignments and derivation rules (natural deduction – following Ben-Yami) with appropriate modifications for the purposes of this chapter. Section 3 sets out with the sequent calculus formulation of Quarc$_B$. Section 4

[1] The author's research in this chapter contributed to a joint paper with Norbert Gratzl [Pavlovic and Gratzl, 2016]. The system LK-Quarc, as well as its susbsystems, were jointly constructed.

proves the deductive equivalence of the two formulations of Quarc$_B$. The central section of this chapter, 5, prove the Cut elimination theorem and its corollaries (subformula property and consistency) for LK-Quarc$_B$. Section 6 expands LK-Quarc$_B$ with the rules for identity, proves again deductive equivalence, Cut elimination and its corollaries and furthermore conservativity over LK-Quarc$_B$. Section 7 extends LK-Quarc$_B$ with a rule of instantiation and once again proves all the results from above for LK-Quarc$_3$.

3.2 Quarc$_B$

The system presented here will be Quarc$_B$, which differs from the full Quarc in containing no rules for identity and instantiation. Moreover, the definition of Reorder has been slightly altered to not include identity permutation. Finally, the truth value assignment rule for quantifiers is modified.

3.2.1 Language

Definition 23 Our language has the same symbols as Definition 1, except as noted:

1. Predicates: $P, Q, R, ...$, denumerably many and with a fixed arity, not including identity.
2. Reordered predicates: For every n-ary ($n > 1$) predicate R, reordered predicates R^π, where π is any permutation of $1, ..., n$ except identity permutation.
3. Sentential operators: $\neg, \vee, \wedge, \rightarrow$.

3.2.2 Formula

Definition 24 (Formula) The rules for formula formation of Quarc$_B$ are the same as in Definition 2, *mutatis mutandis* for changes in Definition 23.

As noted in the previous chapter, an inspection of the rules shows that some of these can be applied in multiple orders. Namely, applications of the anaphora rule can be transposed with one or more applications of the quantifier, sentential operator, or anaphora rules. Here, whenever such a situation occurs, as a matter of convention, anaphora rules are applied first. Among the anaphora rules, first applied is that which has then rightmost argument as its source. Given that every anaphor has a single source, and no two anaphors have the same source, this convention produces a unique order of applications of formula-generation rules.

Definition 25 (Terminal Symbol) The symbol introduced, for any formula, by the last application of a formula-generation rule is called a *terminal symbol* of that formula.

3.2.3 Truth-Value Assignments

Definition 26 (Truth-Value Assignments) Any truth-value assignment is the same as in Definition 3, except as noted, and *mutatis mutandis* for the changes in vocabulary (chiefly for identity):

9. (Quantification) Let $A\,[\forall P]$ $(A\,[\exists P])$ be formula A governed by the QA $\forall P$ $(\exists P)$. If for every (some) SA t for which $(t)P$ is true $A\,[t/\forall P]$ $(A\,[t/\exists P])$ is true, then A is true, and false otherwise.

   In addition to these, one of the rules needed for full Quarc is that of instantiation, which is **not** a part of Quarc$_B$:

5. (Instantiation) For any unary predicate P there is an SA t such that $(t)P$ is true.

3.2.4 Derivation Rules

The rules presented here are taken from [Ben-Yami, 2014]. We only present the rules specific to Quarc; the rules for propositional connectives are standard and will be omitted. We begin by a definition of a proof, slightly modified from Definition 5:

Definition 27 (Proof) A *proof* is a list of lines of the form $\langle L, (i), A, R\rangle$, where L is a possibly empty sequence of formulas (represented by the number of the line in which they are introduced as premises), (i) the line number, A a formula and R a *justification*, an element of the set of the derivation rules. We write $L \vdash A$ to indicate the existence of a proof in Quarc with the sequence of formulas L and the formula A in its final line. The formula A is called the *conclusion* of that proof and the fomulas in L its *premises*.

Definition 28 (Derivation Rules) The derivation rules of Quarc$_B$ are the same as those in Definition 6, except that they do not contain Identity Introduction, Identity Elimination and Instantial Import. The latter is replaced by the rule Particular Elimination (PE):

10. (Particular Elimination, PE)

$$\begin{array}{rcll} L_1 & \text{(i)} & A\,[\exists P] & \\ j & \text{(j)} & (t)P & \text{Premise} \\ k & \text{(k)} & A\,[t/\exists P] & \text{Premise} \\ L_2 & \text{(l)} & B & \\ L_1 \cup L_2 - \{j,k\} & \text{(m)} & B & \text{PE, i, j, k, l} \end{array}$$

   Where t does not appear in $A\,[\exists P]$ or B.

3.2.5 Instantial Import

Note that PE is a deviation from [Ben-Yami, 2014], since it is a rule of Quarc$_B$, but not full Quarc, which uses a rule called Instantial Import in order to distinguish it from the semantic rule and stress the relation to Existential Import.

This rule resembles PE but is defined for either quantifier. Let q be either $\exists$ or $\forall$, and again let t not appear in $A[qP]$ or B.

1. (Instantial Import, Ins)

L_1	(i)	$A[qP]$	
j	(j)	$(t)P$	Premise
k	(k)	$A[t/qP]$	Premise
L_2	(l)	B	
$L_1 \cup L_2 - \{j,k\}$	(m)	B	Ins, i, j, k, l

Consequently, the following is a theorem of full Quarc, but not (as we will see) Quarc$_B$:

Theorem 29 (Particular Import in Quarc): $(\forall M)P \vdash (\exists M)P$

Proof.

1	(1)	$(\forall M)P$	Premise
2	(2)	$(a)M$	Premise
3	(3)	$(a)P$	Premise
2,3	(4)	$(\exists M)P$	PI, 2, 3
1	(5)	$(\exists M)P$	Ins, 1, 2, 3, 4

Examples

In this section we provide several examples of the uses of (full) Quarc, namely to prove the syllogism Barbara and several instances of the DeMorgan laws.

Example 30 Syllogism Barbara

$(\forall M)P, (\forall S)M \vdash (\forall S)P$

Proof.

1	(1)	$(\forall M)P$	Premise
2	(2)	$(\forall S)M$	Premise
3	(3)	$(a)S$	Premise
2,3	(4)	$(a)M$	UE, 2, 3
1,2,3	(5)	$(a)P$	UE, 1, 4
1,2	(6)	$(\forall S)P$	UI, 3, 5

Example 31 DeMorgan Laws

$(\exists M)P \vdash \neg(\forall M)\neg P$

Proof.

1	(1)	$(\exists M)P$	Premise
2	(2)	$(\forall M)\neg P$	Premise
3	(3)	$(a)M$	Premise
4	(4)	$(a)P$	Premise
2,3	(5)	$(a)\neg P$	UE, 2, 3
2,3	(6)	$\neg(a)P$	PS, 5
3,4	(7)	$\neg(\forall M)\neg P$	¬I, 2, 4, 6
1	(8)	$\neg(\forall M)\neg P$	PE, 1, 3, 4, 7

$\neg(\forall M)\neg P \vdash (\exists M)P$

Proof.

1	(1)	$\neg(\forall M)\neg P$	Premise
2	(2)	$\neg(\exists M)P$	Premise
3	(3)	$(a)M$	Premise
4	(4)	$(a)P$	Premise
3,4	(5)	$(\exists M)P$	PI, 3, 4
2,3	(6)	$\neg(a)P$	¬I, 4, 2, 5
2,3	(7)	$(a)\neg P$	SP, 6
2	(8)	$(\forall M)\neg P$	UI, 3, 7
1	(9)	$\neg\neg(\exists M)P$	¬I, 2, 1, 8
1	(10)	$(\exists M)P$	¬E, 9

3.3 LK-Quarc$_B$

We now move to the presentation of the sequent-calculus version of Quarc$_B$, called LK-Quarc$_B$. LK-Quarc$_B$ is an adaptation of the Gerhard Gentzen's system LK from [Szabo, 1969], along the lines of the presentation in [Baaz and Leitsch, 2011]. The system presented here consists of *sequents* of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are possibly empty sequences of formulas, connected into *derivations* via derivation rules. These rules take one or more (usually two), sequents, called the *upper sequent(s)* and produce a single sequent, called the *lower sequent*. A single application of a derivation rule will be referred to as an *inference*.

Derivation rules are divided into five types: (i) axioms, (ii) structural, (iii) propositional, (iv) quantificational and (v) special. *Axioms* are the initial sequents of a derivation. *Structural* rules concern the addition, removal or transposition of formulas in a sequent. *Propositional* rules concern the addition or removal of propositional (truth-functional) connectives from the lower sequent of an inference, *quantification* rules do he same for quantifiers, and *special* for reordered predicates, anaphora and negative predication. Finally, the Cut rule, although a structural rule, is listed separately, as it will be a rule we will eliminate in subsequent sections.

Every rule of LK-Quarc$_B$, with the exception of Cut operates either on the left (marked by L before the relevant symbol), or the right (R) side of the arrow in the lower sequent. As we will see later, LK-Quarc$_2$ and LK-Quarc$_3$ will offer further exceptions to this convention.

The sequent which is not an upper sequent of an inference is called an *endsequent* of a derivation it belongs to. A derivation can have only one endsequent, as will be obvious from the structure of the derivation rules. We now proceed to define them.

Definition 32 (LK-Quarc$_B$) The following are the rules of LK-Quarc$_B$. In all but the Cut rule, the formula occurring in the lower sequent of a rule other than Γ and Δ is called the *principal formula* of that rule.

3.3.1 Axioms

An axiom is a formula of the form $(t_1, ..., t_n)P \Longrightarrow (t_1, ..., t_n)P$, where $t_1, ..., t_n$ are singular arguments and P is a n-ary predicate. Axioms are also called *initial* sequents.

3.3.2 Structural

We next define the structural rules. As stated previously, these rules govern the addition (*weakening, W*), removal (*contraction, C*), and transposition (*exchange, P*) of formulas in the lower sequent.

1. $\dfrac{\Gamma \Longrightarrow \Delta}{A, \Gamma \Longrightarrow \Delta}$ (LW) $\dfrac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, A}$ (RW)

2. $\dfrac{A, A, \Gamma \Longrightarrow \Delta}{A, \Gamma \Longrightarrow \Delta}$ (LC) $\dfrac{\Gamma \Longrightarrow \Delta, A, A}{\Gamma \Longrightarrow \Delta, A}$ (RC)

3. $\dfrac{\Gamma', A, B, \Gamma \Longrightarrow \Delta}{\Gamma', B, A, \Gamma \Longrightarrow \Delta}$ (LP) $\dfrac{\Gamma \Longrightarrow \Delta, A, B, \Delta'}{\Gamma \Longrightarrow \Delta, B, A, \Delta'}$ (RP)

3.3.3 Propositional

The rules in this section do not introduce anything unfamiliar to those acquainted with standard LK. Therefore, in a number of subsequent section segments concerning these rules will be omitted or presented only schematically.

1. $\dfrac{\Gamma \Longrightarrow \Delta, A}{\neg A, \Gamma \Longrightarrow \Delta}$ (L$\neg$) $\dfrac{A, \Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg A}$ (R$\neg$)

2. $\dfrac{A, \Gamma \Longrightarrow \Delta}{A \wedge B, \Gamma \Longrightarrow \Delta}$ (L$\wedge$)* $\dfrac{\Gamma \Longrightarrow \Delta, A \qquad \Gamma \Longrightarrow \Delta, B}{\Gamma \Longrightarrow \Delta, A \wedge B}$ (R$\wedge$)

3. $$\frac{A, \Gamma \Longrightarrow \Delta \qquad B, \Gamma \Longrightarrow \Delta}{A \vee B, \Gamma \Longrightarrow \Delta} \text{(L}\vee\text{)} \qquad \frac{\Gamma \Longrightarrow \Delta, A}{\Gamma \Longrightarrow \Delta, A \vee B} \text{(R}\vee\text{)*}$$

4. $$\frac{B, \Gamma \Longrightarrow \Delta \qquad \Gamma \Longrightarrow \Delta, A}{A \rightarrow B, \Gamma \Longrightarrow \Delta} \text{(L}\rightarrow\text{)} \qquad \frac{A, \Gamma \Longrightarrow \Delta, B}{\Gamma \Longrightarrow \Delta, A \rightarrow B} \text{(R}\rightarrow\text{)}$$

* - the rules L$\wedge$ and R$\vee$ can also, respectively, produce the formula $B \wedge A$ and $B \vee A$.

3.3.4 Quantificational

The primary novelty of Quarc is in its treatment of Quantified Arguments. Therefore, the rules in this section will constitute (along with the Cut rule) the primary focus of this chapter.

1. $$\frac{A\,[a/\forall M]\,, \Gamma \Longrightarrow \Delta \qquad \Gamma \Longrightarrow \Delta, aM}{A\,[\forall M]\,, \Gamma \Longrightarrow \Delta} \text{(L}\forall\text{)} \qquad \frac{aM, \Gamma \Longrightarrow \Delta, A\,[a/\forall M]}{\Gamma \Longrightarrow \Delta, A\,[\forall M]} \text{(R}\forall\text{)*}$$

2. $$\frac{aM, A\,[a/\exists M]\,, \Gamma \Longrightarrow \Delta}{A\,[\exists M]\,, \Gamma \Longrightarrow \Delta} \text{(L}\exists\text{)*} \qquad \frac{\Gamma \Longrightarrow \Delta, aM \qquad \Gamma \Longrightarrow \Delta, A\,[a/\exists M]}{\Gamma \Longrightarrow \Delta, A\,[\exists M]} \text{(R}\exists\text{)}$$

* - the Singular Argument a does not occur anywhere in Γ, Δ, $A\,[\forall M]$ or $A\,[\exists M]$.

3.3.5 Special

This section introduces further rules (in addition to those for quantification) specific to Quarc, those for anaphora, reorder and negative predication.

1. $$\frac{A\,[...a_1...a_n...]\,, \Gamma \Longrightarrow \Delta}{A\,[...a_\alpha/a_1...\alpha/a_n...]\,, \Gamma \Longrightarrow \Delta} \text{(L}A\text{)} \qquad \frac{\Gamma \Longrightarrow \Delta, A\,[...a_1...a_n...]}{\Gamma \Longrightarrow \Delta, A\,[...a_\alpha/a_1...\alpha/a_n...]} \text{(R}A\text{)}$$

2. $$\frac{(t_1, ..., t_n)R, \Gamma \Longrightarrow \Delta}{(t_{\pi 1}, ..., t_{\pi n})R^\pi, \Gamma \Longrightarrow \Delta} \text{(L}Rd\text{)} \qquad \frac{\Gamma \Longrightarrow \Delta, (t_1, ..., t_n)R}{\Gamma \Longrightarrow \Delta, (t_{\pi 1}, ..., t_{\pi n})R^\pi} \text{(R}Rd\text{)}$$

3. $$\frac{\neg(t_1, ..., t_n)P, \Gamma \Longrightarrow \Delta}{(t_1, ..., t_n)\neg P, \Gamma \Longrightarrow \Delta} \text{(L}NP\text{)} \qquad \frac{\Gamma \Longrightarrow \Delta, \neg(t_1, ..., t_n)P}{\Gamma \Longrightarrow \Delta, (t_1, ..., t_n)\neg P} \text{(R}NP\text{)}$$

3.3.6 Cut

Finally, we have the Cut rule. The formula A in the schema below is called the *Cut formula* of the application of the rule.

1. $$\dfrac{\Gamma \Rightarrow \Theta, A \qquad A, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Theta, \Delta}$$

3.3.7 Axiom Generalization

Before proceeding, let us demonstrate a simple and useful lemma – that the axiom rule, which has been defined only for the basic sentences, can be generalized for any formula A.

Lemma 33 Sequent of the form $A \Rightarrow A$ is derivable in LK-Quarc$_B$.

Proof. By induction on the terminal symbol of A.

Basic step. Every initial sequent is derivable.

Inductive step.

1. Negation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\text{(ind. hyp.)}}{\neg A, A \Rightarrow}\,(\mathrm{L}\neg)}{A, \neg A \Rightarrow}\,(\mathrm{L}P)}{\neg A \Rightarrow \neg A}\,(\mathrm{R}P)$$

2. Conjunction

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\text{(ind. hyp.)}}{A \wedge B \Rightarrow A}\,(\mathrm{L}\wedge) \qquad \dfrac{\dfrac{}{B \Rightarrow B}\,\text{(ind. hyp.)}}{A \wedge B \Rightarrow B}\,(\mathrm{L}\wedge)}{A \wedge B \Rightarrow A \wedge B}\,(\mathrm{R}\wedge)$$

3. Disjunction

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\text{(ind. hyp.)}}{A \Rightarrow A \vee B}\,(\mathrm{R}\vee) \qquad \dfrac{\dfrac{}{B \Rightarrow B}\,\text{(ind. hyp.)}}{B \Rightarrow A \vee B}\,(\mathrm{R}\vee)}{A \vee B \Rightarrow A \vee B}\,(\mathrm{L}\vee)$$

4. Conditional

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,\text{(ind. hyp.)}}{A, B \Rightarrow B}\,(\mathrm{L}W)}{B, A \Rightarrow B}\,(\mathrm{L}P) \qquad \dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\text{(ind. hyp.)}}{A \Rightarrow A, B}\,(\mathrm{R}W)}{A \Rightarrow B, A}\,(\mathrm{R}P)}{A \to B, A \Rightarrow B}\,(\mathrm{L}\to)}{A, A \to B \Rightarrow B}\,(\mathrm{L}P)}{A \to B \Rightarrow A \to B}\,(\mathrm{R}\to)$$

5. Universal Quantifier

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\,[a/\forall M] \Rightarrow A\,[a/\forall M]}\,(\text{ind. hyp.})}{aM, A\,[a/\forall M] \Rightarrow A\,[a/\forall M]}\,(\text{L}W)}{A\,[a/\forall M]\,, aM \Rightarrow A\,[a/\forall M]}\,(\text{L}P) \qquad \dfrac{\dfrac{aM \Rightarrow aM}{aM \Rightarrow aM, A\,[a/\forall M]}\,(\text{R}W)}{aM \Rightarrow A\,[a/\forall M]\,, aM}\,(\text{R}P)}{\dfrac{aM, A\,[\forall M] \Rightarrow A\,[a/\forall M]}{A\,[\forall M] \Rightarrow A\,[\forall M]}\,(\text{R}\forall)}\,(\text{L}\forall)$$

Where a is some singular argument such that $A\,[\forall M]$ does not contain it.

6. Particular Quantifier

$$\dfrac{\dfrac{\dfrac{aM \Rightarrow aM}{A\,[a/\exists M]\,, aM \Rightarrow aM}\,(\text{L}W)}{aM, A\,[a/\exists M] \Rightarrow aM}\,(\text{L}P) \qquad \dfrac{\dfrac{}{A\,[a/\exists M] \Rightarrow A\,[a/\exists M]}\,(\text{ind. hyp.})}{aM, A\,[a/\exists M] \Rightarrow A\,[a/\exists M]}\,(\text{L}W)}{\dfrac{aM, A\,[a/\exists M] \Rightarrow A\,[\exists M]}{A\,[\exists M] \Rightarrow A\,[\exists M]}\,(\text{L}\exists)}\,(\text{R}\exists)$$

As above, where a is some singular argument such that $A\,[\exists M]$ does not contain it.

7. Special

$$\dfrac{\dfrac{\dfrac{}{A\,[...a_1...a_n...] \Rightarrow A\,[...a_1...a_n...]}\,(\text{ind. hyp.})}{A\,[...a_\alpha/a_1...\alpha/a_n...] \Rightarrow A\,[...a_1...a_n...]}\,(\text{L}A)}{A\,[...a_\alpha/a_1...\alpha/a_n...] \Rightarrow A\,[...a_\alpha/a_1...\alpha/a_n...]}\,(\text{R}A)$$

Obviously, the steps for the special symbols are trivial and the remaining ones will thus be skipped.

3.4 Deductive Equivalence

In this section we will demonstrate the deductive equivalence of LK-Quarc$_B$ and Quarc$_B$. Note that we will make full use of the Cut rule (even though the Cut Elimination Theorem will later guarantee that for each derivation presented here, there is a Cut-free derivation).

Before proceeding, a note on the structure of this section may perhaps be helpful. Theorem 35 is demonstrated by proving two auxiliary lemmas, Lemma 36 and Lemma 37, each corresponding to one direction of the biconditional in Theorem 35. The proof of the basic step of Lemma 37 is Lemma 33 and the inductive step of Lemma 37 for the Universal Elimination requires the (trivial) Lemma 38.

First we need to be explain the correspondence between the lines of a proof and sequents. To do that, we define the *standard translation*:

Definition 34 (Standard Translation) *Standard translation* of a sequent $\Gamma \Rightarrow \Delta$ of LK-Quarc, where $\Gamma = \gamma_1, ..., \gamma_n$ and $\Delta = \delta_1, ..., \delta_m$ is the derivation in Quarc $\gamma_1 \wedge ... \wedge \gamma_n \vdash \delta_1 \vee ... \vee \delta_m$. Conversely, standard translation of a line of a proof in Quarc $\langle \Gamma, (i), \delta, R \rangle$ is the sequent $\Gamma \Rightarrow \delta$.

We now wish to show the following:

Theorem 35 LK-Quarc$_B$ and Quarc$_B$ are deductively equivalent. Namely, the standard translation of every endsequent of any derivation of LK-Quarc$_B$ is derivable in Quarc$_B$, and the standard translation of any line (i) of any proof in Quarc$_B$ can be derived in LK-Quarc$_B$ from trivial lemmas and the standard translations of the lines of a proof (i) is derived from in Quarc$_B$.

The proof of the theorem proceeds through proof of two lemmas, one going from the LK-Quarc$_B$ to Quarc$_B$, and the other in the opposite direction.

3.4.1 From LK-Quarc to Quarc

The proof in this direction goes by the following lemma:

Lemma 36 Every endsequent $\Gamma \Rightarrow \Delta$ of some derivation in LK-Quarc$_B$ is, given standard translation, derivable in Quarc$_B$.

Proof. By induction on applications of rules of LK-Quarc$_B$.

Basic step. Every initial sequent is derivable in Quarc$_B$. Follows trivially from the Premise rule of Quarc.

Inductive step. We outline the important steps. We will abbreviate $\gamma_1 \wedge ... \wedge \gamma_n$ as Γ and $\delta_1 \vee ... \vee \delta_m$ as Δ.

1. (L$\forall$) Assume that in Quarc$_B$ (i) $A\,[a/\forall M] \wedge \Gamma \vdash \Delta$ and (ii) $\Gamma \vdash \Delta \vee (a)M$. We need to show that $A\,[\forall M] \wedge \Gamma \vdash \Delta$.

1	(1)	$A\,[\forall M] \wedge \Gamma$	Premise
1	(2)	$A\,[\forall M]$	$\wedge$E, 1
1	(3)	Γ	$\wedge$E, 1
1	(4)	$\Delta \vee (a)M$	by (ii)
5	(5)	Δ	Premise
6	(6)	$(a)M$	Premise
1,6	(7)	$A\,[a/\forall M]$	UE, 2, 6
1,6	(8)	$A\,[a/\forall M] \wedge \Gamma$	$\wedge$I, 7, 3
1,6	(9)	Δ	by (i)
1	(10)	Δ	$\vee$E, 4, 5, 5, 6, 9

2. (R$\forall$) Assume that in Quarc$_B$ (i) $(a)M \wedge \Gamma \vdash \Delta \vee A\,[a/\forall M]$ and (ii) a does not appear anywhere in Γ, Δ or $A\,[\forall M]$. We need to show that $\Gamma \vdash \Delta \vee A\,[\forall M]$.

1	(1)	Γ	Premise
2	(2)	$(a)M$	Premise
1,2	(3)	$(a)M \wedge \Gamma$	$\wedge$I, 1, 2
1,2	(4)	$\Delta \vee A\,[a/\forall M]$	by (i)
	(5)	$\Delta \vee \neg\Delta$	Prop.
6	(6)	Δ	Premise
6	(7)	$\Delta \vee A\,[\forall M]$	$\vee$I, 6
8	(8)	$\neg\Delta$	Premise
1,2,8	(9)	$A\,[a/\forall M]$	Prop. 4, 8
1,8	(10)	$A\,[\forall M]$	UI, 2, 9, given (ii)
1,8	(11)	$\Delta \vee A\,[\forall M]$	$\vee$I, 10
1	(12)	$\Delta \vee A\,[\forall M]$	$\vee$E, 5, 6, 7, 8, 11

3. (L$\exists$) Assume that (i) $(a)M \wedge A\,[a/\exists M] \wedge \Gamma \vdash \Delta$ and (ii) a does not appear anywhere in Γ, Δ or $A\,[\exists M]$. We need to show that $A\,[\exists M] \wedge \Gamma \vdash \Delta$.

1	(1)	$A\,[\exists M] \wedge \Gamma$	Premise
1	(2)	$A\,[\exists M]$	$\wedge$E, 1
1	(3)	Γ	$\wedge$E, 1
4	(4)	$(a)M$	Premise
5	(5)	$A\,[a/\exists M]$	Premise
4,5	(6)	$(a)M \wedge A\,[a/\exists M]$	$\wedge$I, 4, 5
1,4,5	(7)	$(a)M \wedge A\,[a/\exists M] \wedge \Gamma$	$\wedge$I, 6, 3
1,4,5	(8)	Δ	by (i)
1	(9)	Δ	PE, 2, 4, 5, 8, given (ii)

4. (R$\exists$) Assume (i) $\Gamma \vdash \Delta \vee (a)M$ and (ii) $\Gamma \vdash \Delta \vee A\,[a/\exists M]$. We need to show that $\Gamma \vdash \Delta \vee A\,[\exists M]$.

1	(1)	Γ	Premise
1	(2)	$\Delta \vee (a)M$	by (i)
3	(3)	Δ	Premise
3	(4)	$\Delta \vee A\,[\exists M]$	$\vee$I, 3
5	(5)	$(a)M$	Premise
1	(6)	$\Delta \vee A\,[a/\exists M]$	by (ii)
7	(7)	$A\,[a/\exists M]$	Premise
5,7	(8)	$A\,[\exists M]$	PI, 5, 7
5,7	(9)	$\Delta \vee A\,[\exists M]$	$\vee$I, 8
1,5	(10)	$\Delta \vee A\,[\exists M]$	$\vee$E, 6, 3, 4, 7, 9
1	(11)	$\Delta \vee A\,[\exists M]$	$\vee$E, 2, 3, 4, 5, 10

5. (LA) Assume (i) $A\,[...a_1...a_n...] \wedge \Gamma \vdash \Delta$. We need to show that $A\,[a_\alpha/a_1...\alpha/a_n] \wedge \Gamma \vdash \Delta$.

1	(1)	$A\,[a_\alpha/a_1...\alpha/a_n...] \wedge \Gamma$	Premise
1	(2)	$A\,[a_\alpha/a_1...\alpha/a_n...]$	$\wedge$E, 1
1	(3)	$A\,[...a_1...a_n...]$	AE, 2
1	(4)	Γ	$\wedge$E, 1
1	(5)	$A\,[a_1...a_n...] \wedge \Gamma$	$\wedge$I, 3, 4
1	(6)	Δ	by (i)

Obviously, this is straightforward.

6. Similarly for other Special rules.

This concludes the proof of Lemma 36. We now turn to the proof of the other Lemma.

3.4.2 From Quarc to LK-Quarc

In this direction the proof relies on the following lemma:

Lemma 37 For any line (i) of any proof in Quarc$_B$ there exists a corresponding sequent in LK-Quarc$_B$ which can be derived from trivial lemmas and sequents corresponding to the lines of a proof (i) is derived from in Quarc$_B$.

Before proceeding with the proof, perhaps a slight clarification of this lemma is in order. Keep in mind that every step of a proof in Quarc is derived from previous step or steps (or none for Premise and, as we shall see, Identity Introduction) via the application of a certain rule. What this lemma does is construct a segment of a derivation in LK-Quarc$_B$ (not a full derivation because it does not necessarily have an initial sequent in all of its topmost places) that begins with (the standard translation of) the steps the application of the rule of Quarc$_B$ relies on, and ends with (the standard translation of) the step that the rule produces.

Since any proof in Quarc consist of a finite number of steps each produced by a rule, by "stacking" the segments of the derivation one after the other (one segment for each step, according to the rule used in that step), we produce a derivation for which the endsequent is the standard translation of the conclusion of the proof in Quarc$_B$. We now proceed with the proof of the lemma.

Proof. By induction on the applications of the rules of derivation of Quarc$_B$.

Basic step. Since we are dealing with Quarc$_B$, which does not include the identity rules, a proof can only begin with an application of a Premise rule. For any application of the Premise rule, the corresponding sequent is $A \Rightarrow A$. That such a sequent exists is shown by Lemma 33.

Inductive step.

1. ($\neg$I) The rule for the Negation Introduction has the following form:

k	(k)	A	Premise
L$_1$	(m)	B	
L$_2$	(n)	$\neg B$	
L$_1$*, L$_2$*	(i)	$\neg A$	$\neg$I, k, m, n

Here L_n* stands for the sequence of formulas L_n with all the occurrences of k omitted.

The corresponding segment of a derivation in LK-Quarc$_B$ is as follows (part separated out for legibility):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,(\text{Lemma 33})}{B, \neg B \Rightarrow}\,(\text{L}\neg)}{B \wedge \neg B, \neg B \Rightarrow}\,(\text{L}\wedge)}{\neg B, B \wedge \neg B \Rightarrow}\,(\text{L}P)}{B \wedge \neg B, B \wedge \neg B \Rightarrow}\,(\text{L}\wedge)}{B \wedge \neg B \Rightarrow}\,(\text{L}C)$$

We now use this part in the top right and provide the rest of the segment:

$$\dfrac{\dfrac{(\text{k})\ A \Rightarrow A \qquad \dfrac{\dfrac{\dfrac{\dfrac{(\text{m})\ L_1 \Rightarrow B \qquad (\text{n})\ L_2 \Rightarrow \neg B}{L_1, L_2 \Rightarrow B \wedge \neg B}\,(\text{R}\wedge) \qquad B \wedge \neg B \Rightarrow}{L_1, L_2 \Rightarrow}\,(\text{Cut})}{L_1, L_2, A \Rightarrow}\,(\text{maybe L}W)}{L_1*, L_2*, A \Rightarrow}\,(\text{maybe some L}C)}{L_1*, L_2*, A \Rightarrow}\,(\text{Cut})}{(\text{i})\ L_1*, L_2* \Rightarrow \neg A}\,(\text{R}\neg)$$

The use of 'maybe' in this derivation will be explained below. Obviously, the sequent corresponding to the step (k) is unnecessary here, but we use all the steps that are listed in the justification of the application of the rule in Quarc, regardless of whether they are premises or not.

These derivations are schematic. For instance, the inference between the sequents $L_1, L_2 \Rightarrow$ and $L_1, L_2, A \Rightarrow$ may require a use of the left weakening rule (LW) in case neither L_1 nor L_2 contain A. If they do, this step can be omitted. Similarly, if either L_1 or L_2 contain A, one or more applications of the left contraction (LC) rule may be required to obtain the sequent $L_1*, L_2*, A \Rightarrow$. Again, in case neither L_1 nor L_2 contains A these steps can be omitted.

2. Similarly for other propositional rules.

3. (UE) The rule for the Universal Elimination has the following form:

L_1	(k)	$A\,[\forall M]$	
L_2	(m)	$(a)M$	
L_1, L_2	(i)	$A\,[a/\forall M]$	UE, k, m

Before proceeding with the corresponding segment of a derivation, we need to prove the following (easy) lemma:

Lemma 38 The sequent $A\,[\forall M]\,, aM \Rightarrow A\,[a/\forall M]$ is derivable in LK-Quarc$_B$.

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\,[a/\forall M] \Rightarrow A\,[a/\forall M]}\,(\text{Lemma } 33)}{aM, A\,[a/\forall M] \Rightarrow A\,[a/\forall M]}\,(\text{L}W)}{A\,[a/\forall M]\,, aM \Rightarrow A\,[a/\forall M]}\,(\text{L}P) \qquad \dfrac{\dfrac{aM \Rightarrow aM}{aM \Rightarrow aM, A\,[a/\forall M]}\,(\text{R}W)}{aM \Rightarrow A\,[a/\forall M]\,, aM}\,(\text{R}P)}{A\,[\forall M]\,, aM \Rightarrow A\,[a/\forall M]}\,(\text{L}\forall)$$

The corresponding segment of a derivation in LK-Quarc$_B$ for the rule UE is as follows:

$$\dfrac{(\text{k})L_1 \Rightarrow A\,[\forall M] \qquad \dfrac{(\text{m})L_2 \Rightarrow aM \qquad \dfrac{}{A\,[\forall M]\,, aM \Rightarrow A\,[a/\forall M]}\,(\text{Lemma } 38)}{L_2, A\,[\forall M] \Rightarrow A\,[a/\forall M]}\,(\text{Cut})}{L_1, L_2 \Rightarrow A\,[a/\forall M]}\,(\text{Cut})$$

4. (UI) The rule for the Universal Introduction has the following form:

k	(k)	$(a)M$	Premise
L_1	(m)	$A\,[a/\forall M]$	
L_1*	(i)	$A\,[\forall M]$	UI, k, m

Here L_1* stands for the sequence of formulas L_1 with all the occurrences of k omitted. By rule, L_1 contains no occurrences of the SA a apart from that in k, and therefore L_1* contains no occurrences of a.

The corresponding segment of a derivation in LK-Quarc$_B$ for the rule UI is as follows:

$$\dfrac{\dfrac{(\text{k})aM \Rightarrow aM \qquad \dfrac{\dfrac{(\text{m})L_1 \Rightarrow A\,[a/\forall M]}{aM, L_1 \Rightarrow A\,[a/\forall M]}\,(\text{L}W)}{aM, L_1* \Rightarrow A\,[a/\forall M]}\,(\text{maybe L}C)}{aM, L_1* \Rightarrow A\,[a/\forall M]}\,(\text{Cut})}{L_1* \Rightarrow A\,[\forall M]}\,(\text{R}\forall)$$

Since L_1* contains no occurrences of a, this is an appropriate use of the rule R∀.

5. (PI) The rule for the Particular Introduction has the following form:

L_1	(k)	$A\,[a/\exists M]$	
L_2	(m)	$(a)M$	
L_1, L_2	(i)	$A\,[\exists M]$	PI, i, j

The corresponding segment of a derivation in LK-Quarc$_B$ for the rule PI is as follows:

$$\dfrac{\dfrac{(m)L_2 \Rightarrow aM}{L_1, L_2 \Rightarrow aM}\,(\text{some L}W) \qquad \dfrac{(k)L_1 \Rightarrow A\,[a/\exists M]}{L_1, L_2 \Rightarrow A\,[a/\exists M]}\,(\text{some L}W,\ \text{L}P)}{(i)L_1, L_2 \Rightarrow A\,[\exists M]}\,(\text{R}\exists)$$

6. (PE) The rule for the Particular Elimination has the following form:

$$\begin{array}{rcll} L_1 & \text{(k)} & A\left[\exists M\right] & \\ j & \text{(l)} & (a)M & \text{Premise} \\ k & \text{(m)} & A\left[a/\exists M\right] & \text{Premise} \\ L_2 & \text{(n)} & B & \\ L_1, L_2 - \{j,k\} & \text{(i)} & B & \text{PE, k, l, m, n} \end{array}$$

The singular argument a occurs nowhere in L_1, $A\left[\exists M\right]$ or B, and nowhere in L_2 except j or k.

The corresponding segment of a derivation in LK-Quarc$_B$ for the rule PE is as follows (broken into two parts for legibility):

$$\dfrac{(l)aM \Rightarrow aM \qquad \dfrac{\dfrac{(m)A\left[a/\exists M\right] \Rightarrow A\left[a/\exists M\right] \qquad \dfrac{(n)L_2 \Rightarrow B}{A\left[a/\exists M\right], aM, L_2{*} \Rightarrow B}\text{(some L}W\text{, L}C\text{)}}{A\left[a/\exists M\right], aM, L_2{*} \Rightarrow B}\text{(Cut)}}{aM, A\left[a/\exists M\right], L_2{*} \Rightarrow B}\text{(L}P\text{)}}{\dfrac{aM, A\left[a/\exists M\right], L_2{*} \Rightarrow B}{A\left[\exists M\right], L_2{*} \Rightarrow B}\text{(L}\exists\text{)}}\text{(Cut)}$$

where $L_2{*}$ stands for the sequence of formulas L_2 with all instances of aM and $A\left[a/\exists M\right]$ removed. Since $L_2{*}$ and B contain no instances of SA a, this is an appropriate use of the rule L$\exists$. Now, having obtained the sequent $A\left[\exists M\right], L_2{*} \Rightarrow B$, we combine it with the step (k) and obtain the desired sequent:

$$\dfrac{(k)L_1 \Rightarrow A\left[\exists M\right] \qquad A\left[\exists M\right], L_2{*} \Rightarrow B}{(i)L_1, L_2{*} \Rightarrow B}\text{(Cut)}$$

7. The rule for Reorder has the following form:

$$\begin{array}{lll} L & \text{(i)} & (t_{\pi 1}, ..., t_{\pi n})P^{\pi} \\ L & \text{(j)} & (t_{\rho 1}, ..., t_{\rho n})P^{\rho} \quad \text{R, i} \end{array}$$

The corresponding segment of a derivation in LK-Quarc$_B$ for the Reorder rule is as follows:

$$\dfrac{(i)L \Rightarrow (t_{\pi 1}, ..., t_{\pi n})P^{\pi} \qquad \dfrac{\dfrac{(t_1, ..., t_n)P \Rightarrow (t_1, ..., t_n)P}{(t_{\pi 1}, ..., t_{\pi n})P^{\pi} \Rightarrow (t_1, ..., t_n)P}\text{(L}Rd\text{)}}{(t_{\pi 1}, ..., t_{\pi n})P^{\pi} \Rightarrow (t_{\rho 1}, ..., t_{\rho n})P^{\rho}}\text{(R}Rd\text{)}}{L \Rightarrow (t_{\rho 1}, ..., t_{\rho n})P^{\rho}}\text{(Cut)}$$

8. The remaining derivations of sequents corresponding to the special symbols of Quarc are trivial and will be omitted here

This concludes the proof of Lemma 37 and thus of Theorem 35.

3.5 Cut Elimination Theorem

We finally arrive at the central section of this chapter, the demonstration of the Cut elimination theorem for LK-Quarc$_B$. This, in turn, will allow us to arrive at the subformula property for our system and motivate some further considerations in the following sections.

3.5.1 Preliminaries

The proof presented in this section is modified of Gentzen's from [Szabo, 1969]. It is a double induction on the grade and rank of the Cut formula.

Cut and Mix

Since LK-Quarc$_B$ contains the contraction rules, there might be multiple instances of the Cut formula occurring. In order to be able to Cut on all of those, let us also define the Mix rule:

Definition 39 (Mix rule)

$$\frac{\Gamma \Rightarrow \Theta \qquad \Pi \Rightarrow \Delta}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}$$

Where some formula M, called the *Mix formula* occurs at least once in Π and Θ, and Π^* and Θ^* are obtained by removing all instances of M from Π and Θ, respectively.

Definition 40 ($LK - Quarc^\dagger$) $LK - Quarc^\dagger$ is a sequent calculus obtained from LK-Quarc by replacing the Cut rule by the Mix rule.

Lemma 41 For any sequent S, S is provable in $LK - Quarc^\dagger$ just in case it is provable in LK-Quarc.

Proof. By showing Cut is derivable in $LK - Quarc^\dagger$

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta, A \qquad A, \Pi \Rightarrow \Delta}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}\ (\text{Mix})}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\ (\text{some L}W\text{, R}W)$$

and conversely that Mix is derivable in LK-Quarc.

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta}{\Gamma \Rightarrow \Theta^*, A}\ (\text{some R}P\text{, R}C) \qquad \dfrac{\Pi \Rightarrow \Delta}{A, \Pi^* \Rightarrow \Delta}\ (\text{some L}P\text{, L}C)}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}\ (\text{Cut})$$

Since LK-Quarc$_B$ contains all the rules used, this lemma will hold for it. We will call LK-Quarc$^\dagger_B$ the sequent calculus obtained by substituting the Mix rule for the Cut rule in LK-Quarc$_B$.

Grade and Rank

Definition 42 (Grade, γ) Let A, B and C be formulas, R an n-ary predicate, P an n-ary predicate or a reordered n-ary predicate, $t_1, ..., t_n$ SA's and $\pi 1, ..., \pi n$ some permutation of $1, ..., n$ except identity permutation. Then, the grade $\gamma(A)$ of the formula A is:

1. $\gamma(A) = 0$ if A is basic.
2. $\gamma(A) = 1$ if A is $(t_{\pi 1}, ..., t_{\pi n})R^{\pi}$.
3. $\gamma(A) = \gamma((t_1, ..., t_n)P) + 1$ if A is $(t_1, ..., t_n)\neg P$.
4. $\gamma(A) = \gamma(B) + 1$ if A is $\neg B$.
5. $\gamma(A) = \gamma(B) + \gamma(C) + 1$ if A is $B \wedge C$, $B \vee C$ or $B \to C$.
6. $\gamma(A) = \gamma(B\,[t/\forall P]) + 1$ if A is $B\,[\forall P]$.
7. $\gamma(A) = \gamma(B\,[t/\exists P]) + 1$ if A is $B\,[\exists P]$.
8. $\gamma(A) = \gamma(B\,[..., t_1, ..., t_n, ...]) + 1$ if A is $B\,[..., t_\alpha/t_1, ..., \alpha/t_n]$.

The order of application of the rule for anaphora can sometimes be transposed with the application of the rules for sentential operators, quantifiers, or another anaphora. It can be shown by induction that all of those transpositions assign the same grade to a formula. For a similar proof, see [Pavlovic and Ben-Yami, 2013].

Definition 43 (Rank, ρ) *Rank* of a derivation is the sum of the left and right rank of a Mix formula. *Left rank* (*right rank*) is the maximal number of sequents in a branch, starting from the upper left (right) sequent of the Mix rule, such that each sequent of the branch contains the Mix formula in the succedent (antecedent).

Re-designating the Proper Singular Arguments

Before proceeding to the Cut elimination theorem, we shall prove an auxiliary lemma. Again, this is due to Gentzen from [Szabo, 1969].

Definition 44 (Re-designation procedure) Call the Singular Argument a occurring in the Definition 6 of the rules R∀ and L∃ the *proper* singular argument of the respective rules. To re-designate the proper singular arguments, we alter a derivation according to the following procedure. First, for every occurrence of a rule R∀ or L∃ above which no other occurrence of these rules is present (to have a unique procedure we can start with the leftmost and move right), we replace their proper singular argument in all the sequents above the lower sequent of the occurrence of the rule with a singular argument that has so far not occurred anywhere in the derivation. Second, we apply the same procedure to all the occurrences of the rules R∀ or L∃ which are such that the procedure has already been applied to any other occurrence of said rules in all the sequents above their lower sequents.

We need to prove the following auxiliary lemma:

Lemma 45 If **In** is an initial sequent or a correct inference which contains a singular argument a, which is not the proper singular argument of **In**, and if the singular argument b is likewise not the proper singular argument of **In**, then **In′**, obtained from **In** by uniformly substituting b for a is an initial sequent or a correct inference.

Proof. By induction on the rules of LK-Quarc$_B$.

Next we prove the following lemma:

Lemma 46 If we re-designate the proper singular arguments of a derivation, it will yield a derivation of the same grade and rank and of the same endsequent.

That the two derivations end in the same endsequent is obvious from the definition of the re-designation procedure. We now need to show this is a correct derivation of the said sequent.

Proof. By induction on the steps of the re-designation procedure. For every occurrence of a rule R∀ or L∃, every sequent above its lower sequent is derived correctly, by Lemma 45 and the inductive hypothesis. Moreover, replacing the proper singular argument of a correct application of R∀ or L∃ with a singular argument that occurs nowhere above its lower sequent will likewise produce a correct instance of R∀ or L∃.

3.5.2 Cut Elimination

We want to show the following:

Theorem 47 (Cut Elimination) For any sequent S, if S is provable in LK-Quarc$_B$, then it is provable in LK-Quarc$_B$ without using the Cut rule.

Given Lemma 41, it will suffice to show:

Lemma 48 For any sequent S, if S is provable in LK-Quarc$^\dagger_B$, then it is provable in LK-Quarc$^\dagger_B$ without using the Mix rule.

Proof. By induction on grade and rank.

Case $\rho = 2$

Obviously, the lowest rank of an application of a Mix rule is 2. So, suppose $\rho(M) = 2$. We will omit all the familiar cases and focus on the symbols of Quarc. Moreover, for legibility the rule labels will be omitted (it will always be the L and the R rule for a given symbol).

Special

We start with the special symbols of LK-Quarc$_B$ as those have the lowest grade.

1. Reorder:

$$\dfrac{\dfrac{\Gamma \Longrightarrow \Theta, (t_1, ..., t_n)R}{\Gamma \Longrightarrow \Theta, (t_{\pi 1}, ..., t_{\pi n})R^{\pi}} \qquad \dfrac{(t_1, ..., t_n)R, \Pi \Longrightarrow \Delta}{(t_{\pi 1}, ..., t_{\pi n})R^{\pi}, \Pi \Longrightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Theta, \Delta} \text{(Mix)}$$

This can be transformed into:

$$\dfrac{\dfrac{\Gamma \Longrightarrow \Theta, (t_1, ..., t_n)R \qquad (t_1, ..., t_n)R, \Pi \Longrightarrow \Delta}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta} \text{(Mix)}}{\Gamma, \Pi \Rightarrow \Theta, \Delta} \text{(some R}W\text{, R}P\text{, L}W\text{, L}P\text{)}$$

Since the Mix formula is of a lower grade, by inductive hypothesis, it can be eliminated.

2. Anaphora:

$$\dfrac{\dfrac{\Gamma \Longrightarrow \Theta, A\,[...a_1...a_n...]}{\Gamma \Longrightarrow \Theta, A\,[...a_\alpha/a_1...\alpha/a_n...]} \qquad \dfrac{A\,[...a_1...a_n...]\,, \Pi \Longrightarrow \Delta}{A\,[...a_\alpha/a_1...\alpha/a_n...]\,, \Pi \Longrightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Theta, \Delta} \text{(Mix)}$$

This can be transformed into:

$$\dfrac{\dfrac{\Gamma \Longrightarrow \Theta, A\,[...a_1...a_n...] \qquad A\,[...a_1...a_n...]\,, \Pi \Longrightarrow \Delta}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta} \text{(Mix)}}{\Gamma, \Pi \Rightarrow \Theta, \Delta} \text{(some R}W\text{, R}P\text{, L}W\text{, L}P\text{)}$$

Again, since the Mix formula is of a lower grade, by inductive hypothesis, it can be eliminated.

3. Negative Predication:

$$\dfrac{\dfrac{\Gamma \Longrightarrow \Theta, \neg(t_1, ..., t_n)P}{\Gamma \Longrightarrow \Theta, (t_1, ..., t_n)\neg P} \qquad \dfrac{\neg(t_1, ..., t_n)P, \Pi \Longrightarrow \Delta}{(t_1, ..., t_n)\neg P, \Pi \Longrightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Theta, \Delta} \text{(Mix)}$$

This can be transformed into:

$$\dfrac{\dfrac{\Gamma \Longrightarrow \Theta, \neg(t_1, ..., t_n)P \qquad \neg(t_1, ..., t_n)P, \Pi \Longrightarrow \Delta}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}\ (\text{Mix})}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\ (\text{some R}W,\ \text{R}P,\ \text{L}W,\ \text{L}P)$$

This Mix formula can be eliminated according to the procedure for negation below.

Propositional

Cut elimination theorem for the propositional symbols is a familiar result and will be omitted here, apart from negation, which is required to finalize the Cut elimination for negative predication above:

$$\dfrac{\dfrac{A, \Gamma \Rightarrow \Theta}{\Gamma \Rightarrow \Theta, \neg A} \qquad \dfrac{\Pi \Rightarrow \Delta, A}{\neg A, \Pi \Rightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\ (\text{Mix})$$

This can be transformed into:

$$\dfrac{\dfrac{\Pi \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Theta}{\Pi, \Gamma^* \Rightarrow \Delta^*, \Theta}\ (\text{Mix})}{\Pi, \Gamma \Rightarrow \Delta, \Theta}\ (\text{some L}W,\ \text{L}P,\ \text{R}W,\ \text{R}P)$$

Since the Mix formula A is of lesser grade than $\neg A$, by inductive hypothesis, it can be eliminated.

Quantification - Universal

Let the terminal symbol of the Mix formula be a universal quantifier:

$$\dfrac{\dfrac{aM, \Gamma \Rightarrow \Theta, A\left[a/\forall M\right]}{\Gamma \Rightarrow \Theta, A\left[\forall M\right]} \qquad \dfrac{A\left[b/\forall M\right], \Pi \Rightarrow \Delta \qquad \Pi \Rightarrow \Delta, bM}{A\left[\forall M\right], \Pi \Rightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\ (\text{Mix})$$

This can be transformed into:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi \Rightarrow \Delta, bM \qquad bM, \Gamma \Rightarrow \Theta, A\left[b/\forall M\right]}{\Pi, \Gamma^* \Rightarrow \Delta^*, \Theta, A\left[b/\forall M\right]}\ (\text{Mix})}{\Pi, \Gamma \Rightarrow \Delta, \Theta, A\left[b/\forall M\right]}\ (\text{some R}W,\ \text{R}P,\ \text{L}W,\ \text{L}P) \qquad A\left[b/\forall M\right], \Pi \Rightarrow \Delta}{\Pi, \Gamma, \Pi^* \Rightarrow \Delta^*, \Theta^*, \Delta}\ (\text{Mix})}{\Pi, \Gamma \Rightarrow \Theta, \Delta}\ (\text{some L}C,\ \text{L}P,\ \text{R}W,\ \text{R}C,\ \text{L}P)$$

The change from the the sequent $aM, \Gamma \Rightarrow \Theta, A\left[a/\forall M\right]$ to the sequent $bM, \Gamma \Rightarrow \Theta, A\left[b/\forall M\right]$ in the transformation above is justified by Lemma 46.

Quantification - Particular

Let the terminal symbol of the Mix formula be a particular quantifier:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta, bM \qquad \Gamma \Rightarrow \Theta, A\,[b/\exists M]}{\Gamma \Rightarrow \Theta, A\,[\exists M]} \qquad \dfrac{aM, A\,[a/\exists M]\,, \Pi \Rightarrow \Delta}{A\,[\exists M]\,, \Pi \Rightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\ (\text{Mix})$$

This can be transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta, A\,[b/\exists M] \qquad \dfrac{\dfrac{\Gamma \Rightarrow \Theta, bM \qquad bM, A\,[b/\exists M]\,, \Pi \Rightarrow \Delta}{\Gamma, A\,[b/\exists M]\,, \Pi \Rightarrow \Theta^{*}, \Delta}\ (\text{Mix})}{\Gamma, A\,[b/\exists M]\,, \Pi \Rightarrow \Theta, \Delta}\ (\text{some R}W,\ \text{R}P)}{\Gamma, \Gamma^{**}, \Pi \Rightarrow \Theta^{**}, \Theta, \Delta}\ (\text{Mix})}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\ (\text{some L}C,\ \text{L}P,\ \text{R}C,\ \text{R}P)$$

For both quantifiers, the grade of the Mix formula of the upper Mix rule is 0, and of the lower is 1 less. Therefore, by induction hypothesis, those can be eliminated.

Case $\rho > 2$

Here we will assume left rank is 1 and right is greater than 1. Again, the majority of cases here are familiar results, and we focus on LK-Quarc$_B$. The only part that is not a familiar result here is L∀ and R∃, which fall under the case of two-sequent rules. However, before proceeding we should at least mention the case where the mix formula is obtained by the Contraction rule, since that is precisely what motivates the use of Mix instead of the Cut rule in the first place.

1. (LC)

   $$\dfrac{\Gamma \Rightarrow \Theta, A \qquad \dfrac{A, A, \Pi \Rightarrow \Delta}{A, \Pi \Rightarrow \Delta}}{\Gamma, \Pi^{*}, \Rightarrow \Theta, \Delta}\ (\text{Mix})$$

   This is transformed into:

   $$\dfrac{\Gamma \Rightarrow \Theta, A \qquad A, A, \Pi \Rightarrow \Delta}{\Gamma, \Pi^{*}, \Rightarrow \Theta, \Delta}\ (\text{Mix})$$

   Right rank was reduced by 1, while left remains the same, and so by inductive hypothesis, Mix can be eliminated.

2. (L∀)

   $$\dfrac{\Gamma \Rightarrow \Theta \qquad \dfrac{A\,[a/\forall M]\,, \Pi \Rightarrow \Delta \qquad \Pi \Rightarrow \Delta, aM}{A\,[\forall M]\,, \Pi \Rightarrow \Delta}}{\Gamma, \Pi^{*}, A\,[\forall M] \Rightarrow \Theta^{*}, \Delta}\ (\text{Mix})$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta \qquad A\,[a/\forall M]\,, \Pi \Rightarrow \Delta}{\Gamma, \Pi^*, A\,[a/\forall M] \Rightarrow \Theta^*, \Delta}\,(\text{Mix}) \quad \dfrac{\Gamma \Rightarrow \Theta \qquad \Pi \Rightarrow \Delta, aM}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta, aM}\,(\text{Mix})}{\Gamma, \Pi^*, A\,[\forall M] \Rightarrow \Theta^*, \Delta}$$

As each instance of a Mix rule has rank lowered by 1, so by the inductive hypothesis both can be eliminated. We now proceed to examine the case of R$\exists$.

3. (R$\exists$)

$$\dfrac{\Gamma \Rightarrow \Theta \qquad \dfrac{\Pi \Rightarrow \Delta, aM \qquad \Pi \Rightarrow \Delta, A\,[a/\exists M]}{\Pi \Rightarrow \Delta, A\,[a/\exists M]}}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta, A\,[\exists M]}\,(\text{Mix})$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta \qquad \Pi \Rightarrow \Delta, aM}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta, aM}\,(\text{Mix}) \quad \dfrac{\Gamma \Rightarrow \Theta \qquad \Pi \Rightarrow \Delta, A\,[a/\exists M]}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta, A\,[a/\exists M]}\,(\text{Mix})}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta, A\,[\exists M]}$$

Again, the rank of each instance of a Mix rule has been lowered by 1, and by the inductive hypothesis both can be eliminated.

Similarly if the left rank is greater than 1 or if both are greater. This concludes the proof of the Cut elimination theorem.

3.5.3 Subformula Property

Definition 49 (Subformula)

1. Every formula is a subformula of itself.
2. The formula $(t_1, ..., t_n)R$ is a subformula of $(t_{\pi 1}, ..., t_{\pi n})R^{\pi}$.
3. The formula $\neg(t_1, ..., t_n)P$ is a subformula of $(t_1, ..., t_n)\neg P$.
4. Every formula A and B mentioned in the antecedent of the rules for generation of a formula in Definition 24 is a subformula of that formula. Moreover, any formula tM is likewise the subformula of the formula $A\,[qM]$.
5. If a formula A is a subformula of any subformula of B, then it is a subformula of B.

Theorem 50 (Subformula property) Any formula appearing in any Cut-free proof of LK-Quarc$_B$, is a subformula of some formula in its endsequent.

Proof. We only need to show that the subformula property holds for all rules of LK-Quarc$_B$, except Cut, which can be eliminated. Since this is a familiar result for the propositional and structural rules, what remains to be shown is that it holds for the quantification and special rules of LK-Quarc$_B$.

Observing the rules for the universal quantifier:

$$\frac{A\,[t/\forall M]\,,\Gamma \Longrightarrow \Delta \qquad \Gamma \Longrightarrow \Delta, tM}{A\,[\forall M]\,,\Gamma \Longrightarrow \Delta}\ (\mathrm{L}\forall) \qquad \frac{tM, \Gamma \Longrightarrow \Delta, A\,[t/\forall M]}{\Gamma \Longrightarrow \Delta, A\,[\forall M]}\ (\mathrm{R}\forall)^*$$

We can see that any formula of Γ and Δ will be a subformula of some formula of Γ and Δ in the lower sequent, namely itself. Moreover, tM and $A\,[t/\forall M]$ are both subformulas of $A\,[\forall M]$. Therefore, the subformula property holds for this derivation. The proof for the particular quantifier proceeds in the same manner, and is straightforward for the special symbols of Quarc.

Consistency

Given the definition of consistency,

Definition 51 (Consistency) A sequent calculus is consistent just in case the sequent $\cdots \Rightarrow \cdots$ is not derivable.

An important corollary from Theorem 50 immediately follows:

Corollary 52 LK-Quarc$_B$ is consistent.

To see this, one need only observe that no formula is a subformula of an empty sequent.

3.6 Identity

In this section we expand LK-Quarc$_B$ into LK-Quarc$_2$ by adding the two identity rules.

3.6.1 Identity Rules

The identity rules in Quarc are as follows [Pavlovic and Ben-Yami, 2013]:

Identity Introduction, =I

(k) $a = a$ =I

Identity Elimination, =E

Let $A\,[b]$ be a basic formula containing occurrences $b_1, ..., b_n$ of a singular argument b (A might also contain further occurrences of b).

L_1	(k)	$A[b]$
L_2	(m)	$a = b$
L_1, L_2	(n)	$A[a/b_1, ..., a/b_n]$

To expand LK-Quarc$_B$ into LK-Quarc$_2$ we add the following rules. The rules here are modified for Quarc from [Negri and von Plato, 2001].

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(=_1) \qquad \frac{A[b], a = b, A[a/b], \Gamma \Rightarrow \Delta}{a = b, A[a/b], \Gamma \Rightarrow \Delta}\,(=_2)$$

Where A is a basic formula and $A[a/b]$ is a formula produced by substituting any number of occurrences of the singular argument b by a.

Before proceeding, let us prove a simple and useful lemma.

Lemma 53 $\dfrac{a = b, \Gamma \Rightarrow \Delta}{b = a, \Gamma \Rightarrow \Delta}$

Proof.

$$\cfrac{\cfrac{\cfrac{\cfrac{a = b, \Gamma \Rightarrow \Delta}{a = b, b = a, b = b, \Gamma \Rightarrow \Delta}}{b = a, b = b, \Gamma \Rightarrow \Delta}\,(=_2)}{b = b, b = a, \Gamma \Rightarrow \Delta}}{b = a, \Gamma \Rightarrow \Delta}\,(=_1)$$

3.6.2 Deductive Equivalence

The proof of deductive equivalence proceeds with the expansion of the proof of Theorem 35 with the appropriate steps for the identity rules.

LK-Quarc to Quarc

1. $(=_1)$ Assume that in Quarc$_2$ (i) $a = a \wedge \Gamma \vdash \Delta$. We need to show that $\Gamma \vdash \Delta$.

1	(1)	Γ	Premise
	(2)	$a = a$	=I
1	(3)	$a = a \wedge \Gamma$	$\wedge$I, 1, 2
1	(4)	Δ	by (i)

2. $(=_2)$ Assume that in Quarc$_2$ (i) $A[b] \wedge a = b \wedge A[a/b] \wedge \Gamma \vdash \Delta$. We need to show that $a = b \wedge A[a/b] \wedge \Gamma \vdash \Delta$.

1	(1)	$a = b \wedge A[a/b] \wedge \Gamma$	Premise
1	(2)	$a = b$	$\wedge$E, 1
1	(3)	$A[a/b]$	$\wedge$E, 1
1	(4)	$A[b]$	=E, 2, 3
1	(5)	$A[b] \wedge a = b \wedge A[a/b] \wedge \Gamma$	$\wedge$I, 1, 4
1	(6)	Δ	by (i)

The derivation rule used in the steps (2) and (3) is a straightforward generalization of $\wedge$E.

Quarc to LK-Quarc

1. The segment of a derivation corresponding to the rule =I is as follows:

$$\dfrac{a = a \Rightarrow a = a}{\Rightarrow a = a}\;(=_1)$$

2. The segment of a derivation corresponding to the rule =E is as follows:

$$\dfrac{(\mathrm{k})\ L_1 \Rightarrow A[b] \qquad \dfrac{\dfrac{(\mathrm{m})\ L_2 \Rightarrow a = b \qquad \dfrac{\dfrac{\dfrac{A[a/b] \Rightarrow A[a/b]}{A[a/b], b = a, A[b] \Rightarrow A[a/b]}\;(\mathrm{L}W,\ \mathrm{L}P)}{b = a, A[b] \Rightarrow A[a/b]}\;(=_2)}{a = b, A[b] \Rightarrow A[a/b]}\;(\text{Lemma } 53)}{L_2, A[b] \Rightarrow A[a/b]}\;(\mathrm{Cut})}{A[b], L_2 \Rightarrow A[a/b]}\;(\mathrm{L}P)}{L_1, L_2 \Rightarrow A[a/b]}\;(\mathrm{Cut})$$

This concludes the proof of deductive equivalence of LK-Quarc$_2$ and Quarc$_2$.

3.6.3 Cut Elimination

We prove the Cut elimination theorem for LK-Quarc$_2$:

Theorem 54 For any sequent S, if S is provable in LK-Quarc$_2$, then it is provable in LK-Quarc$_2$ without using the Cut rule.

Proof. By expanding the proof for LK-Quarc$_B$. Clearly, in both rules for identity all the formulas appearing in the lower sequent also appear in the upper sequent. Therefore, we only need to expand the proof for $\rho > 2$.

The Rule $(=_1)$

The rule $=_1$ fits into the general proof for one-sequent derivations in the case $\rho > 2$ with no modification, since Π does not contain the Mix formula (it is empty). However, we will examine the rule $=_2$ more closely, since it, in fact, has two principal formulas, which also occur as the side formulas, and either of which could be the Mix formula.

The Rule $(=_2)$

The case that needs to be examined here is when either $a = b$ or $A[a/b]$ is the Mix formula. Assume it is $a = b$. The application of the Mix rule then looks as follows:

$$\dfrac{\Gamma \Rightarrow \Theta \qquad \dfrac{A[b], a = b, A[a/b], \Pi \Rightarrow \Delta}{a = b, A[a/b], \Pi \Rightarrow \Delta}\;(=_2)}{\Gamma, A[a/b], \Pi^* \Rightarrow \Theta^*, \Delta}\;(\mathrm{Mix})$$

The formula $a = b$ either occurs or it doesn't occur in Γ. Suppose it does. Then the derivation is transformed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A[b], a = b, A[a/b], \Pi \Rightarrow \Delta}{a = b, A[a/b], \Pi \Rightarrow \Delta}(=_2)}{a = b, \Pi, A[a/b] \Rightarrow \Delta}(\text{some L}P)}{a = b, \Pi^*, A[a/b] \Rightarrow \Delta}(\text{some L}C)}{\Gamma, a = b, \Pi^*, A[a/b] \Rightarrow \Delta}(\text{some L}W)}{\Gamma, \Pi^*, A[a/b] \Rightarrow \Delta}(\text{some L}C)}{\Gamma, A[a/b], \Pi^* \Rightarrow \Theta^*, \Delta}(\text{some L}P,\ \text{R}W,\ \text{R}P)$$

Now suppose $a = b$ does not occur in Γ. Since $a = b$ is a basic formula, and by assumption the left rank is 1, the sequent $\Gamma \Rightarrow \Theta$ is obtained by RW from $\Gamma \Rightarrow \Theta^*$. The derivation is then transformed as follows:

$$\dfrac{\Gamma \Rightarrow \Theta^*}{\Gamma, A[a/b], \Pi^* \Rightarrow \Theta^*, \Delta}(\text{some L}W,\ \text{L}P,\ \text{R}W)$$

Since $A[a/b]$ is likewise a basic formula, the same considerations will apply there. The remainder of the proof runs in parallel. This concludes the proof of the Theorem 54.

3.6.4 Subformula Property

Here we can adopt a slightly weaker definition of subformula property, due to [Negri and von Plato, 2001]:

Theorem 55 Any formula appearing in any Cut-free proof of LK-Quarc$_2$ is a subformula of some formula in its endsequent or a basic formula.

Proof. We only need to expand the proof of Theorem 50 with the cases for $=_1$ and $=_2$. However, these only remove basic formulas. Therefore, Theorem 55 holds.

Now, using this we can show consistency:

Corollary 56 LK-Quarc$_2$ is consistent.

Proof. From Theorem 55, by noting that basic formulas can only disappear from the left side of a sequent. Therefore, the empty sequent is not derivable.

3.6.5 Conservativity

Theorem 57 LK-Quarc$_2$ is conservative expansion of LK-Quarc$_B$. Namely, if $\Gamma \Rightarrow \Delta$ is derivable in LK-Quarc$_2$, and Γ and Δ contain no identity, then $\Gamma \Rightarrow \Delta$ is derivable in LK-Quarc$_B$.

Proof. Assume $\Gamma \Rightarrow \Delta$ is derivable in LK-Quarc$_2$, and Γ and Δ contain no identity. By weak subformula property, it follows that

Corollary 58 Any formula in the derivation of $\Gamma \Rightarrow \Delta$ that contains identity is a basic formula.

Moreover, it follows that

Corollary 59 No formula containing identity occurs on the right side of any sequent in the derivation.

Furthermore, given that the rule $=_2$ can never reduce the number of formulas containing identity below 1, and that the rule $=_1$ can only reduce the number of such formulas below 1 if they are of the form $a = a$, it follows that

Corollary 60 Any identity formula in the derivation of $\Gamma \Rightarrow \Delta$ is of the form $a = a$.

Take a (Cut-free) derivation of $\Gamma \Rightarrow \Delta$. It is then transformed in two step.

First step. Any occurrence of the rule $=_2$, given Corollary 60, is of the form:

$$\frac{A\left[a\right], a = a, A\left[a/a\right], \Gamma' \Rightarrow \Delta'}{a = a, A\left[a/a\right], \Gamma' \Rightarrow \Delta'}\ (=_2)$$

Since $A\left[a\right]$ and $A\left[a/a\right]$ are the same formula, this is transformed into

$$\frac{A\left[a\right], a = a, A\left[a/a\right], \Gamma' \Rightarrow \Delta'}{a = a, A\left[a/a\right], \Gamma' \Rightarrow \Delta'}\ (\mathrm{L}C)$$

Second step. Any occurrence of the rule $\mathbf{L}C$, where $a = a$ is the principal formula,

$$\frac{a = a, a = a, \Gamma \Rightarrow \Delta}{a = a, \Gamma \Rightarrow \Delta}\ (\mathrm{L}C)$$

Is transformed into an occurrence of the rule $=_1$:

$$\frac{a = a, a = a, \Gamma \Rightarrow \Delta}{a = a, \Gamma \Rightarrow \Delta}\ (=_1)$$

Observation 61 Obviously, both these transformations yield correct derivations. After completing both, the rule $=_2$ does not occur, and the formula $a = a$ is the principal formula of either the rule $=_1$ or $\mathbf{L}W$ (since by Corollary 59 it cannot occur in an initial sequent).

We now proceed to prove the above theorem by proving the following lemma:

Lemma 62 Any occurrence of the formula $a = a$ in the derivation of $\Gamma \Rightarrow \Delta$ can be eliminated.

Proof. Given Observation 61, every formula of the form $a = a$ will form a chain of sequents, such that the first sequent of the chain is the lower sequent of a $\mathbf{L}W$ rule with $a = a$ as a principal formula, and the last sequent of the chain the upper sequent of a $=_1$ rule with $a = a$ as a principal formula. Let the length of such a chain be the number of sequents in the chain.

We only need to show that such a chain ending with the topmost leftmost occurrence of $=_1$ can be eliminated. The proof is by induction on the length of the chain.

Basic step. The shortest chain has length 1, and is of the following form:

$$\dfrac{\dfrac{\Gamma' \Rightarrow \Delta'}{a = a, \Gamma' \Rightarrow \Delta'}\,(\mathrm{L}W)}{\Gamma' \Rightarrow \Delta'}\,(=_1)$$

This is transformed into the derivation of the upper sequent $\Gamma' \Rightarrow \Delta'$, which by Corollary 59 does not contain the formula $a = a$.

Inductive step. Let the end of a chain be

$$\dfrac{\dfrac{a = a, \Gamma'' \Rightarrow \Delta''}{a = a, \Gamma' \Rightarrow \Delta'}\,(\mathrm{Inf})}{\Gamma' \Rightarrow \Delta'}\,(=_1)$$

Since $a = a$ is not principal in **Inf**, this can be transformed into

$$\dfrac{\dfrac{a = a, \Gamma'' \Rightarrow \Delta''}{\Gamma'' \Rightarrow \Delta''}\,(=_1)}{\Gamma' \Rightarrow \Delta'}\,(\mathrm{Inf})$$

Where the length of the chain is reduced by one. Similarly for the two-sequent rules. This concludes the proof of Lemma 62.

By Corollary 60 and Lemma 62 if follows that the derivation transformed in this manner contains no identity. Moreover, it contains no rule $=_2$ (Observation 61) nor $=_1$ (Lemma 62). Therefore, it is a derivation of LK-Quarc$_B$. This concludes the proof of Theorem 57.

3.6.6 Generalization of Identity Rules

As we can see, the rules in LK-Quarc$_2$, just like in Quarc, are defined only for the basic formulas. We will now show that these rules generalize to any formula. Similar result is shown in [Negri and von Plato, 2001], and the approach here is also independently motivated by [Ben-Yami, 2011].

Theorem 63 (Identity Generalization) For any formula S of Quarc, it is an admissible rule of LK-Quarc$_2$ that

$$\dfrac{S\left[b\right], a = b, S\left[a/b\right], \Gamma \Rightarrow \Delta}{a = b, S\left[a/b\right], \Gamma \Rightarrow \Delta}$$

Proof. By induction on the terminal symbol of S. Basic step is trivial, so we proceed to the inductive step, and only examine the interesting step of the universal quantifier. In the following section A need not stand for a basic formula.

Let S be $A\left[\forall M\right]$. Assume (i) that the sequent $A\left[\forall M\right]\left[b\right], a = b, A\left[\forall M\right]\left[a/b\right], \Gamma \Rightarrow$

Δ is derivable. From (i) it follows[2] that the sequents (ii) $A\left[c/\forall M\right]\left[b\right], a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta'$ and (iii) $a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta', cM$ are derivable. We need to show the sequent $a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta'$ is derivable. The derivation proceeds as follows, broken into parts for legibility:

$$\dfrac{\dfrac{A\left[c/\forall M\right]\left[b\right] \Rightarrow A\left[c/\forall M\right]\left[b\right]}{A\left[c/\forall M\right]\left[b\right], a = b, A\left[c/\forall M\right]\left[a/b\right] \Rightarrow A\left[c/\forall M\right]\left[b\right]}\,(\text{some L}W\text{, L}P)}{a = b, A\left[c/\forall M\right]\left[a/b\right] \Rightarrow A\left[c/\forall M\right]\left[b\right]}\,(\text{ind. hyp.})$$

We now proceed by using this sequent as the upper left sequent of the following Cut, also utilizing (ii):

$$\dfrac{\dfrac{a = b, A\left[c/\forall M\right]\left[a/b\right] \Rightarrow A\left[c/\forall M\right]\left[b\right] \qquad \text{(ii) } A\left[c/\forall M\right]\left[b\right], a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta'}{a = b, A\left[c/\forall M\right]\left[a/b\right], a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta'}\,(\text{Cut})}{A\left[c/\forall M\right]\left[a/b\right], a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta'}\,(\text{L}C)$$

Next, we use this sequent as the upper left sequent of L$\forall$, also utilizing (iii):

$$\dfrac{\dfrac{A\left[c/\forall M\right]\left[a/b\right], a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta' \qquad \text{(iii) } a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta', cM}{A\left[\forall M\right]\left[a/b\right], a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta'}\,(\text{L}\forall)}{a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta'}\,(\text{L}C)$$

This concludes the proof of the Theorem 63.

3.7 Particular Import in LK-Quarc$_B$

Having proven the Cut elimination theorem in Section 3.5, we now proceed to use it in further considerations. The application here will be to demonstrate that particular import is not derivable in LK-Quarc$_B$ and therefore, given the deductive equivalence result of Theorem 35, it is likewise not derivable in Quarc$_B$.

As we have seen in a simplified version in Example 2 in Section 3.2, DeMorgan laws hold in Quarc$_B$ (and consequently in Quarc as well). In LK-Quarc$_B$, the DeMorgan laws, in their general form, hold as well:

Lemma 64 (DeMorgan) $A\left[\exists S\right] \Leftrightarrow \neg(\forall S_\alpha S \wedge \neg A\left[\alpha/\exists S\left\langle\alpha\right\rangle\right])$

A note on notation – the quantified argument $\exists S$ may or may not be a source of an anaphor. If it is, since an anaphoric expression cannot itself be a source of an anaphor, in the right-hand side of the lemma we need to replace not only the

[2]Given Cut elimination and the fact initial sequents contain nothing but basic formulas, the quantified formula $A\left[\forall M\right]\left[b\right]$ is introduced either via LW, or via L$\forall$. Let the sequent $A\left[\forall M\right]\left[b\right], \Gamma'' \Rightarrow \Delta''$ be the topmost sequent in which $A\left[\forall M\right]\left[b\right]$ occurs.

The sequent $A\left[\forall M\right]\left[b\right], a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta'$ is derivable from it without $A\left[\forall M\right]\left[b\right]$ being principal, and $A\left[\forall M\right]\left[b\right], a = b, A\left[\forall M\right]\left[a/b\right], \Gamma \Rightarrow \Delta$ from it without any of the principal formulas of $=_2$ being principal.

The sequents (ii$'$) $A\left[c/\forall M\right]\left[b\right], \Gamma'' \Rightarrow \Delta''$ and (iii$'$) $\Gamma'' \Rightarrow \Delta'', cM$ are either trivially derivable if formula $A\left[\forall M\right]\left[b\right]$ is introduced via LW, or are upper sequents of L$\forall$. (ii) is derivable from (ii$'$) and (iii) from (iii$'$), applying the above reasoning repeatedly if the formulas in question feature in any Contractions. Finally, $a = b, A\left[\forall M\right]\left[a/b\right], \Gamma \Rightarrow \Delta$ is derivable from $a = b, A\left[\forall M\right]\left[a/b\right], \Gamma' \Rightarrow \Delta'$.

quantified argument $\exists S$, but both it and the index (assume it is α) indicating the source of an anaphor. The notation $A\left[\alpha/\exists S\langle\alpha\rangle\right]$ indicates that in A, $\exists S_\alpha$ is replaced by α in case $\exists S$ is the source of an anaphor, and $\exists S$ is replaced by α otherwise.

Given that this is a sizable derivation, it will be presented here broken down in two parts, each corresponding to one direction of the equivalence. Moreover, the derivation rules will be omitted.

Proof L - R.

$$\begin{array}{c}
\begin{array}[b]{c}
A\left[a/\exists S\right] \Rightarrow A\left[a/\exists S\right] \\ \hline
A\left[a/\exists S\right] \Rightarrow A\left[\alpha/\exists S_{\langle\alpha\rangle}\right]\left[a/\alpha_1, ..., a/\alpha_n\right] \\ \hline
aS, A\left[a/\exists S\right] \Rightarrow A\left[\alpha/\exists S_{\langle\alpha\rangle}\right]\left[a/\alpha_1, ..., a/\alpha_n\right] \\ \hline
\neg A\left[\alpha/\exists S_{\langle\alpha\rangle}\right]\left[a/\alpha_1, ..., a/_n\right], aS, A\left[a/\exists S\right] \Rightarrow \\ \hline
aS \wedge \neg A\left[\alpha/\exists S_{\langle\alpha\rangle}\right]\left[a/\alpha_1, ..., a/_n\right], aS, A\left[a/\exists S\right] \Rightarrow \\ \hline
a_\alpha S \wedge \neg A\left[\alpha/\exists S_{\langle\alpha\rangle}\right], aS, A\left[a/\exists S\right] \Rightarrow
\end{array}
\quad
\begin{array}[b]{c}
aS \Rightarrow aS \\ \hline
A\left[a/\exists S\right], aS \Rightarrow aS \\ \hline
aS, A\left[a/\exists S\right] \Rightarrow aS
\end{array} \\ \hline
\forall S_\alpha S \wedge \neg A\left[\alpha/\exists S_{\langle\alpha\rangle}\right], aS, A\left[a/\exists S\right] \Rightarrow \\ \hline
aS, A\left[a/\exists S\right] \Rightarrow \neg(\forall S_\alpha S \wedge \neg A\left[\alpha/\exists S_{\langle\alpha\rangle}\right]) \\ \hline
A\left[\exists S\right] \Rightarrow \neg(\forall S_\alpha S \wedge \neg A\left[\alpha/\exists S_{\langle\alpha\rangle}\right])
\end{array}$$

Proof R - L.

Again, in the interest of legibility, the derivation of the sequent $aS \Rightarrow aS \wedge \neg A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right], A\left[a/\exists S\right]$ is here shown separately.

$$\begin{array}{c}
\begin{array}[b]{c}
aS \Rightarrow aS \\ \hline
aS \Rightarrow aS, A\left[a/\exists S\right] \\ \hline
aS \Rightarrow A\left[a/\exists S\right], aS
\end{array}
\quad
\begin{array}[b]{c}
A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right] \Rightarrow A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right] \\ \hline
A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right] \Rightarrow A\left[a/\exists S\right] \\ \hline
\Rightarrow A\left[a/\exists S\right], \neg A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right] \\ \hline
aS \Rightarrow A\left[a/\exists S\right], \neg A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right]
\end{array} \\ \hline
aS \Rightarrow A\left[a/\exists S\right], aS \wedge \neg A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right] \\ \hline
aS \Rightarrow aS \wedge \neg A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right], A\left[a/\exists S\right]
\end{array}$$

We now use this endsequent as the upper right sequent of the full derivation:

$$\begin{array}{c}
\begin{array}[b]{c}
aS \Rightarrow aS \\ \hline
aS \Rightarrow aS, aS \wedge \neg A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right] \\ \hline
aS \Rightarrow aS \wedge \neg A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right], aS
\end{array}
\quad
\begin{array}[b]{c}
aS \Rightarrow aS \wedge \neg A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right], A\left[a/\exists S\right]
\end{array} \\ \hline
aS \Rightarrow aS \wedge \neg A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right], A\left[\exists S\right] \\ \hline
aS \Rightarrow A\left[\exists S\right], aS \wedge \neg A\left[a/\exists S_{\langle\alpha\rangle}, a/\alpha_1, ..., a/\alpha_n\right] \\ \hline
aS \Rightarrow A\left[\exists S\right], a_\alpha S \wedge \neg A\left[\alpha/\exists S_{\langle\alpha\rangle}\right] \\ \hline
\Rightarrow A\left[\exists S\right], \forall S_\alpha S \wedge \neg A\left[\alpha/\exists S_{\langle\alpha\rangle}\right] \\ \hline
\neg(\forall S_\alpha S \wedge \neg A\left[\alpha/\exists S_{\langle\alpha\rangle}\right]) \Rightarrow A\left[\exists S\right],
\end{array}$$

This gives us the general form of DeMorgan laws for LK-Quarc$_B$. We now wish to demonstrate the invalidity of Particular Import for it. In order to generate a counterexample to it, we will opt for the simplest sufficient sequent. Therefore, the following simplification will be of use:

Lemma 65 (Simplification) $\forall M_\alpha M \wedge \neg \alpha P \Leftrightarrow \forall M \neg P$

This equivalence is again presented in two parts.

Proof L - R.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg aP \Rightarrow \neg aP}{aM, \neg aP \Rightarrow \neg aP}}{\neg aP, aM \Rightarrow \neg aP}}{aM \wedge \neg aP, aM \Rightarrow \neg aP}}{a_\alpha M \wedge \neg \alpha P, aM \Rightarrow \neg aP} \quad \dfrac{\dfrac{aM \Rightarrow aM}{aM \Rightarrow aM, a\neg P}}{aM \Rightarrow a\neg P, aM}}{\forall M_\alpha M \wedge \neg \alpha P, aM \Rightarrow a\neg P}}{aM, \forall M_\alpha M \wedge \neg \alpha P \Rightarrow a\neg P}}{\forall M_\alpha M \wedge \neg \alpha P \Rightarrow \forall M \neg P}$$

Proof R - L.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{aM \Rightarrow aM}{\forall M\neg P, aM \Rightarrow aM} \quad \dfrac{\dfrac{\dfrac{\dfrac{\neg aP \Rightarrow \neg aP}{aM, \neg aP \Rightarrow \neg aP}}{\neg aP, aM \Rightarrow \neg aP}}{a\neg P, aM \Rightarrow \neg aP} \quad \dfrac{\dfrac{aM \Rightarrow aM}{aM \Rightarrow aM, \neg aP}}{aM \Rightarrow \neg aP, aM}}{\forall M\neg P, aM \Rightarrow \neg aP}}{\forall M\neg P, aM \Rightarrow aM \wedge \neg aP}}{aM, \forall M\neg P \Rightarrow aM \wedge \neg aP}}{aM, \forall M\neg P \Rightarrow a_\alpha M \wedge \neg \alpha P}}{\forall M \neg P \Rightarrow \forall M_\alpha M \wedge \neg \alpha P}$$

Observation 66 $\forall MP \vdash \exists MP$ holds in Quarc (Theorem 29), and given standard translation, that is $\forall MP \Rightarrow \exists MP$ in LK-Quarc. Given Lemma 64, this sequent is derivable just in case $\forall MP \Rightarrow \neg(\forall M_\alpha M \wedge \neg(\alpha)P)$ is derivable. Furthermore, given Lemma 65, this sequent is derivable just in case $\forall MP \Rightarrow \neg\forall M \neg P$ is derivable.

We will demonstrate that

Theorem 67 The sequent $\forall MP \Rightarrow \neg\forall M\neg P$, and therefore particular import, is not derivable in LK-Quarc$_B$.

Proof. Suppose there is a Cut-free proof of $\forall MP \Rightarrow \neg\forall M\neg P$ in LK-Quarc$_B$. Then, there is also a Cut-free proof of $\forall M\neg P, \forall MP \Rightarrow$. Before the end of the derivation, this sequent may have undergone any number of applications of LW, LC and LP, resulting in a sequent $(\forall MP)_1, ..., (\forall MP)_n, (\forall M\neg P)_1, ..., (\forall M\neg P)_m \Rightarrow$, with formulas in any order.

Since every initial sequent contains only basic formulas, each of the formulas must have been first introduced either by LW or by L$\forall$. Focusing on those introduced via L∀, we see that for each application of the rule (where $\sim$ is $\neg$ or empty),

$$\frac{a \sim P, \Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, aM}{\forall M \sim P, \Gamma \Rightarrow \Delta}\ (\mathrm{L}\forall)$$

there is an upper right sequent with one quantified formula less, one occurrence of the predicate P less, and one formula aM more on the right. Let us consider the right upper sequent of the topmost rightmost application of L∀. Γ could either be empty or contain only formulas introduced via LW, but in any case that sequent is derivable only if $\Rightarrow \Delta, aM$ is. By subformula property Δ can contain no quantified formulas or formulas with any connective and no formulas containing P, as none of those can be rightmost. Therefore, Δ can only contain formulas aM. But for no i is $\Rightarrow aM_1 \ldots aM_i$ an initial sequent.

Therefore, there is no Cut-free proof of $\forall M \neg P, \forall MP \Rightarrow$ in LK-Quarc$_B$, and so no Cut-free proof of $\forall MP \Rightarrow \neg\forall M\neg P$. Given the Cut elimination theorem, this means there is no proof of $\forall MP \Rightarrow \neg\forall M\neg P$ in LK-Quarc$_B$.

This concludes the proof of Theorem 67. This is the greatest obstacle to expanding LK-Quarc$_B$ into a sequent calculus deductively equivalent with full Quarc. In the following subsection, we will see a way to expand LK-Quarc$_B$ with a rule that will give the resulting system equivalence with Quarc which includes Instantiation (Quarc$_3$).

3.7.1 Instantial Import Rule

To expand LK-Quarc$_B$ into LK-Quarc$_3$, we add the rule for Instantial Import:

$$\frac{tM, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{Ins})^*$$

* - where neither Γ nor Δ contain the singular argument t.

This rule allows for the derivation of a particular sentence from a corresponding sentence governed by the universal quantified argument:

Theorem 68 $A\,[\forall S] \Rightarrow A\,[\exists S/\forall S]$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{aS \Rightarrow aS}{A\,[\forall S]\,, aS \Rightarrow aS}\ (\mathrm{L}W)}{aS, A\,[\forall S] \Rightarrow aS}\ (\mathrm{L}P) \qquad \dfrac{\dfrac{}{A\,[\forall S]\,, aS \Rightarrow A\,[a/\forall S]}\ (\text{Lemma 38})}{aS, A\,[\forall S] \Rightarrow A\,[a/\forall S]}\ (\mathrm{R}P)}{aS, A\,[\forall S] \Rightarrow A\,[\exists S/\forall S]}\ (\mathrm{R}\exists)}{A\,[\forall S] \Rightarrow A\,[\exists S/\forall S]}\ (\mathrm{Ins})$$

Moreover, it allows for the derivation of a theorem

Theorem 69 $\Rightarrow (\exists S)S$

Proof.

$$\dfrac{\dfrac{aS \Rightarrow aS \qquad aS \Rightarrow aS}{aS \Rightarrow (\exists S)S}}{\Rightarrow (\exists S)S}\,(\text{Ins})$$

However, this rule will not allow the derivation of the problematic sequent '$\Rightarrow aM$' from the proof of Theorem 67, since the following is not a permissible application of this rule:

$$\dfrac{aM \Rightarrow aM}{\Rightarrow aM}\,{*}(\text{Ins})$$

So, this sequent calculus is, at least *prima facie*, powerful enough, without being too powerful. We now formalize this result.

3.7.2 Deductive Equivalence

Theorem 70 Quarc$_3$ and LK-Quarc$_3$ are deductively equivalent.

Proof. In addition to the proof of Lemma 36, we need to show that

1. (Ins) Assume (i) $(t)S \wedge \Gamma \Rightarrow \Delta$ and (ii) Γ and Δ do not contain t. We need to show that $\Gamma \vdash \Delta$.

1	(1)	Γ	Premise
2	(2)	$(t)S$	Premise
	(3)	$(\forall S)S$	UI, 2, 2
4	(4)	$(t)S$	Premise
5	(5)	$(t)S$	Premise
1,4	(6)	$(t)S \wedge \Gamma$	$\wedge$I, 1, 4
1,4	(7)	Δ	by (i)
1	(8)	Δ	Ins, 3, 4, 5, 7 given (ii)

Note that the justification of Instantial Import in line (8) could use the line (2) twice instead of the lines (4) and (5), and this slightly longer proof is presented for clarity.

In addition to the proof of Lemma 37 we need to construct a corresponding segment of a derivation for the Instantial Import rule of Quarc$_3$.

1. (Ins) The Instantial Import rule has the following form:

L_1	(i)	$A\,[qP]$	
j	(j)	$(t)P$	Premise
k	(k)	$A\,[t/qP]$	Premise
L_2	(l)	B	
$L_1 \cup L_2 - \{j,k\}$	(m)	B	Ins, i, j, k, l

where L$_1$ and A do not contain the singular argument t, and in L_2 the only occurrences of t are in (j) and (k).

Since we have already demonstrated Lemma 37 for the particular quantifier, we need to concern ourselves only with the cases where q stands for the universal quantifier $\forall$. The corresponding segment of that derivation is as follows (let L_2* be the list L_2 with (j) and (k) omitted – it thus contains no singular argument t):

$$\dfrac{\dfrac{\text{(i) } L_1 \Rightarrow A\,[\forall P] \qquad \dfrac{\dfrac{\dfrac{}{A\,[\forall P]\,,tP \Rightarrow A\,[t/\forall P]}\,\text{(Lemma 38)} \qquad \text{(l) } A\,[t/\forall P]\,,tP, L_2* \Rightarrow B}{A\,[\forall P]\,,tP,tP,L_2* \Rightarrow B}\,\text{(Cut)}}{A\,[\forall P]\,,tP,L_2* \Rightarrow B}}{\dfrac{L_1, tP, L_2* \Rightarrow B}{tP, L_1, L_2* \Rightarrow B}\,(\mathrm{L}P)}\,\text{(Cut)}}{L_1, L_2* \Rightarrow B}\,\text{(Ins)}$$

Since neither L_1, L_2* nor B contain the singular argument t, this is an appropriate use of the Ins rule of LK-Quarc$_3$. Of course, for this segment to have the appropriate form of using all the steps listed in the justification in Quarc, the segment above the application of Lemma 38 should have the following form:

$$\dfrac{\dfrac{\dfrac{\text{(j) } tP \Rightarrow tP \qquad \dfrac{\dfrac{}{A\,[\forall P]\,,tP \Rightarrow A\,[t/\forall P]}\,\text{(Lemma 38)}}{tP, A\,[\forall P] \Rightarrow A\,[t/\forall P]}\,(\mathrm{L}P)}{tP, A\,[\forall P] \Rightarrow A\,[t/\forall P]}\,\text{(Cut)} \qquad \text{(k) } A\,[t/\forall P] \Rightarrow A\,[t/\forall P]}{tP, A\,[\forall P] \Rightarrow A\,[t/\forall P]}\,\text{(Cut)}}{A\,[\forall P]\,,tP \Rightarrow A\,[t/\forall P]}\,(\mathrm{L}P)$$

However, since steps (j) and (k) are always premises, the above segment will suffice on its own.

3.7.3 Cut Elimination

Here we need to check only the cases where $\rho > 2$. Let the right rank be greater than 1. So, the application of the Mix rule will be:

$$\dfrac{\Gamma \Rightarrow \Theta \qquad \dfrac{\Psi \Rightarrow \Delta}{\Pi \Rightarrow \Delta}\,\text{(Ins)}}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}\,\text{(Mix)}$$

(suppose the Mix formula A is not in Γ). Then this transforms into (re-designating the SA if it occurs in Γ or Θ):

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Theta \qquad \Psi \Rightarrow \Delta}{\Gamma, \Psi^* \Rightarrow \Theta^*, \Delta}}{\Psi^*, \Gamma \Rightarrow \Theta^*, \Delta}\,\text{(some L}P\text{)}}{\Pi^*, \Gamma \Rightarrow \Theta^*, \Delta}\,\text{(Ins)}}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}\,\text{(some L}P\text{)}$$

Since the right rank was reduced by 1, while the left remains the same, the rank of the resulting Mix rule is one less and, by inductive hypothesis, it can be eliminated. Similarly when the left rank is greater than 1.

3.7.4 Subformula Property

The reasoning here runs in parallel to Theorem 55:

Theorem 71 Any formula appearing in any Cut-free proof of LK-Quarc$_3$ is a subformula of some formula in its endsequent or a basic formula.

Proof. We only need to expand the proof of Theorem 50 with the case for Ins. However, it only removes basic formulas. Therefore, Theorem 71 holds.

And consistency follows:

Corollary 72 LK-Quarc$_3$ is consistent.

Proof. Same as Corollary 56.

3.8 Concluding remarks

In this chapter we have provided a concise proof-theoretic study of Quarc within LK-systems. An obvious next step would naturally be completeness which follows from the deductive equivalences [Pavlovic and Ben-Yami, 2013]. Moreover, there is also a more direct way of establishing this important theorem, by adopting a proof of completeness that is typical for sequent calculi [Schütte, 1960], [Buss, 1998].

Topics for further research include an interpolation theorem for the various LK-Quarc systems covered in the next chapter; thereby we could also examine Beth's definability theorem, an investigation left for the future. On a more philosophical side, Quarc enriched by modalities – as suggested by [Ben-Yami, 2014] – and correspondingly with its expansion of expressive power, provides ample opportunity for exploration. Some initial examination of the possibilities is conducted in Chapter 5.

Chapter 4

Interpolation Theorem for LK-Quarc

In this chapter we aim to prove the Craig interpolation property, first proved in [Craig, 1957], for LK-Quarc. The proof uses some elements of the proof in [Boolos et al., 2007], but is primarily an adaptation of Maehara's method [Maehara, 1960], as found in [Ono, 1998]:

Definition 73 (Craig Interpolation Property) LK-Quarc has the *interpolation property* just in case that, if a formula $A \Rightarrow C$ is derivable in LK-Quarc, then there is a formula B such that $A \Rightarrow B$, $B \Rightarrow C$ are derivable in LK-Quarc and $V(B) \subseteq V(A) \cap V(C)$ if $V(A) \cap V(C) \neq \emptyset$, and otherwise either $\Rightarrow \neg A$ or $\Rightarrow C$ is derivable (where $V(A)$ is a set of all the non-logical predicate constants in A).

We first expand the language of Quarc with the (logical) predicate τ[1] and constants $\top$ and $\bot$, and the system LK-Quarc with the corresponding initial sequents, to obtain the system LK-Quarc°

$$\Rightarrow \top \qquad \bot \Rightarrow \qquad \text{and} \qquad \Rightarrow a\tau$$

It is easy to check that this expanded system still has the cut elimination and all the associated properties. To prove that LK-Quarc° has the Interpolation property, we will prove the following Theorem:

Theorem 74 If $\Gamma \Rightarrow \Delta$ is derivable in LK-Quarc° and $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ is any partition of Γ, Δ, then there is a formula C such that $\Gamma_1 \Rightarrow \Delta_1, C$, $C, \Gamma_2 \Rightarrow \Delta_2$ and $V(C) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.

If Theorem 74 holds, and we can eliminate the constants and the predicate τ, then taking Γ as Γ_1, with Δ_1 empty, and Δ as Δ_2, with Γ_2 empty, LK-Quarc has the Interpolation property.

In the proof below, for any sequent $\Gamma \Rightarrow \Delta$, we take some cut-free derivation of it, δ.

Proof. By induction on the length n of derivation δ.

[1] Intuitively this predicate can be read as "thing" (cf. [Lanzet and Ben-Yami, 2004]).

4.1 Initial

If the length of the derivation is 1, then $\Gamma \Rightarrow \Delta$ is an initial sequent $A \Rightarrow A$. The partitions to be considered here are $\langle (A : A); (:)\rangle$, $\langle (:); (A : A)\rangle$, $\langle (A :); (: A)\rangle$ and $\langle (: A); (A :)\rangle$.

4.1.1 $\langle (A : A); (:)\rangle$

The interpolant here is $\perp$:

$$\frac{A \Rightarrow A}{A \Rightarrow A, \perp} \quad \text{and} \quad \perp \Rightarrow$$

and $V(\perp) = \emptyset$, so therefore $V(\perp) \subseteq \emptyset \cap V(A, A)$.

4.1.2 $\langle (:); (A : A)\rangle$

The interpolant here is $\top$:

$$\Rightarrow \top \quad \text{and} \quad \frac{A \Rightarrow A}{\top, A \Rightarrow A}$$

and $V(\top) = \emptyset$, so therefore $V(\top) \subseteq V(A, A) \cap \emptyset$.

4.1.3 $\langle (A :); (: A)\rangle$ and $\langle (: A); (A :)\rangle$

In the first case the interpolant is A, since $A \Rightarrow A$ holds, and $V(A) = V(A) \cap V(A)$. In the second case the interpolant is $\neg A$, since $\Rightarrow A, \neg A$ and $A, \neg A \Rightarrow$ hold and $V(\neg A) = V(A) \cap V(A)$

Proceeding to the inductive step, we will examine the rule used in the last inference of δ. We start with the rules for the universal quantifier.

4.2 Universal

We inspect the rules for the universal quantifier starting with the easier case of L$\forall$.

4.2.1 L$\forall$

The rule has the following form:

$$\frac{A\,[a/\forall M]\,, \Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, aM}{A\,[\forall M]\,, \Gamma \Rightarrow \Delta}$$

So, by inductive hypothesis, there is a C (for any partition $\langle (\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ of Γ, Δ, same in all the cases below) such that

1. $A\,[a/\forall M]\,, \Gamma_1 \Rightarrow \Delta_1, C$
2. $C, \Gamma_2 \Rightarrow \Delta_2$

3. $V(C) \subseteq V(A[a/\forall M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

   and moreover, there is a D such that

4. $\Gamma_1 \Rightarrow \Delta_1, aM, D$
5. $D, \Gamma_2 \Rightarrow \Delta_2$
6. $V(D) \subseteq V(\Gamma_1, \Delta_1, aM) \cap V(\Gamma_2, \Delta_2)$

   We want to show that there is a formula K, such that:

   (a) $A[\forall M], \Gamma_1 \Rightarrow \Delta_1, K$
   (b) $K, \Gamma_2 \Rightarrow \Delta_2$
   and
   (c) $V(K) \subseteq V(A[\forall M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Such a formula is $C \vee D$.

Proof. (a)

$$\dfrac{\dfrac{A[a/\forall M], \Gamma_1 \Rightarrow \Delta_1, C}{A[a/\forall M], \Gamma_1 \Rightarrow \Delta_1, C \vee D} \qquad \dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1, aM, D}{\Gamma_1 \Rightarrow \Delta_1, aM, C \vee D}}{\Gamma_1 \Rightarrow \Delta_1, C \vee D, aM}}{A[\forall M], \Gamma_1 \Rightarrow \Delta_1, C \vee D}$$

Proof. (b)

$$\dfrac{C, \Gamma_2 \Rightarrow \Delta_2 \qquad D, \Gamma_2 \Rightarrow \Delta_2}{C \vee D, \Gamma_2 \Rightarrow \Delta_2}$$

Proof. (c)

Clearly, $V(C \vee D) = V(C) \cup V(D)$.

Since by inductive hypothesis (3) $V(C) \subseteq V(A[a/\forall M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$ and $V(A[a/\forall M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2) \subseteq V(A[\forall M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$ given that the formula $A[\forall M]$ is identical to $A[a/\forall M]$ except for containing the quantified argument $\forall M$, it follows that $V(C) \subseteq V(A[\forall M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.

Likewise, since by inductive hypothesis (6) $V(D) \subseteq V(\Gamma_1, \Delta_1, aM) \cap V(\Gamma_2, \Delta_2)$ and $V(\Gamma_1, \Delta_1, aM) \cap V(\Gamma_2, \Delta_2) \subseteq V(A[\forall M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$ it follows that $V(D) \subseteq V(A[\forall M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.

Therefore, $V(C \vee D) \subseteq V(A[\forall M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Mutatis mutandis for partition $\langle(\Gamma_1 : \Delta_1); (A[\forall M], \Gamma_2 : \Delta_2)\rangle$ with the interpolant formula $C \wedge D$.

4.2.2 R$\forall$

We now move to the rule R$\forall$, which has the following form:

$$\dfrac{aM, \Gamma \Rightarrow \Delta, A[a/\forall M]}{\Gamma \Rightarrow \Delta, A[\forall M]}$$

Where Γ, Δ and $A[\forall M]$ do not contain a.

By inductive hypothesis, there is a C such that

1. $aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], C$
2. $C, \Gamma_2 \Rightarrow \Delta_2$
3. $V(C) \subseteq V(aM, \Gamma_1, \Delta_1, A[a/\forall M]) \cap V(\Gamma_2, \Delta_2)$

We want to show that there is a formula K, such that:

(a) $\Gamma_1 \Rightarrow \Delta_1, A[\forall M], K$

(b) $K, \Gamma_2 \Rightarrow \Delta_2$

and

(c) $V(K) \subseteq V(\Gamma_1, \Delta_1, A[\forall M]) \cap V(\Gamma_2, \Delta_2)$

Case 75 If C does not contain a, such a formula is C.

Proof. (a)

$$\dfrac{\dfrac{\dfrac{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], C}{aM, \Gamma_1 \Rightarrow \Delta_1, C, A[a/\forall M]}}{\Gamma_1 \Rightarrow \Delta_1, C, A[\forall M]}\text{R}\forall}{\Gamma_1 \Rightarrow \Delta_1, A[\forall M], C}$$

Proof. (b) Trivial

Proof. (c) Clearly, $V(aM, \Gamma_1, \Delta_1, A[a/\forall M]) = V(\Gamma_1, \Delta_1, A[\forall M])$. Therefore, $V(C) \subseteq V(\Gamma_1, \Delta_1, A[\forall M]) \cap V(\Gamma_2, \Delta_2)$

Case 76 However, if C contains the SA a (labeled as $C[a]$), the application of R$\forall$ in proof (a) will not be permissible. In that case we move to the following, replacing C with $(\exists\tau_\alpha)\tau \wedge C[\alpha/a_1...\alpha/a_n]$:

Proof. (a)

$$\dfrac{\dfrac{\dfrac{\Rightarrow a\tau}{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], a\tau} \quad \dfrac{\dfrac{\dfrac{\Rightarrow a\tau}{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], a\tau} \quad aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], C[a]}{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], a\tau \wedge C[a]}}{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], a_\alpha\tau \wedge C[\alpha/a_1...\alpha/a_n]}}{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], (\exists\tau_\alpha)\tau \wedge C[\alpha/a_1...\alpha/a_n]}}{\Gamma_1 \Rightarrow \Delta_1, A[\forall M], (\exists\tau_\alpha)\tau \wedge C[\alpha/a_1...\alpha/a_n]}$$

Proof. (b)

$$\dfrac{\dfrac{\dfrac{\dfrac{C[a], \Gamma_2 \Rightarrow \Delta_2}{a\tau \wedge C[a], \Gamma_2 \Rightarrow \Delta_2}}{a_\alpha\tau \wedge C[\alpha/a_1...\alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}}{a\tau, a_\alpha\tau \wedge C[\alpha/a_1...\alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}}{(\exists\tau_\alpha)\tau \wedge C[\alpha/a_1...\alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}$$

Proof. (c) Since τ is a logical predicate, $V((\exists\tau_\alpha)\tau \wedge C[\alpha/a_1...\alpha/a_n]) = V(C[a])$, and thus the proof is the same as Case 75.

Mutatis mutandis for partition $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2, (A[\forall M])\rangle$ with the interpolant C for Case 75 and $(\forall\tau_\alpha)\tau \wedge C[\alpha/a_1...\alpha/a_n]$ for Case 76.

4.3 Particular

We now move on to inspecting the particular quantifier, once again starting with the easier case, this time, of R$\exists$.

4.3.1 R$\exists$

This rule has the following form:

$$\frac{\Gamma \Rightarrow \Delta, aM \qquad \Gamma \Rightarrow \Delta, A[a/\exists M]}{\Gamma \Rightarrow \Delta, A[\exists M]}$$

So, by inductive hypothesis, there is a C such that

1. $\Gamma_1 \Rightarrow \Delta_1, C$
2. $C, \Gamma_2 \Rightarrow \Delta_2, aM$
3. $V(C) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, aM)$

   and moreover, there is a D such that

4. $\Gamma_1 \Rightarrow \Delta_1, D$
5. $D, \Gamma_2 \Rightarrow \Delta_2, A[a/\exists M]$
6. $V(D) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, A[a/\exists M])$

   We want to show that there is a formula K, such that:

   (a) $\Gamma_1 \Rightarrow \Delta_1, K$
   (b) $K, \Gamma_2 \Rightarrow \Delta_2, A[\exists M]$

   and

   (c) $V(K) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, A[\exists M])$

Such a formula is $C \wedge D$.

Proof. (a)

$$\frac{\Gamma_1 \Rightarrow \Delta_1, C \qquad \Gamma_1 \Rightarrow \Delta_1, D}{\Gamma_1 \Rightarrow \Delta_1, C \wedge D}$$

Proof. (b)

$$\frac{\dfrac{C, \Gamma_2 \Rightarrow \Delta_2, aM}{C \wedge D, \Gamma_2 \Rightarrow \Delta_2, aM} \qquad \dfrac{D, \Gamma_2 \Rightarrow \Delta_2, A[a/\exists M]}{C \wedge D, \Gamma_2 \Rightarrow \Delta_2, A[a/\exists M]}}{C \wedge D, \Gamma_2 \Rightarrow \Delta_2, A[\exists M]}$$

Proof. (c)

Clearly, $V(C \wedge D) = V(C) \cup V(D)$.

Since by inductive hypothesis (3) $V(C) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, aM)$ and $V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, aM) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, A[\exists M])$, it follows that $V(C) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, A[\exists M])$.

Likewise, since by inductive hypothesis (6) $V(D) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, A[a/\exists M])$ and $V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, A[a/\exists M]) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, A[\exists M])$ it follows that $V(D) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2, , A[\exists M])$.

Therefore, $V(C \wedge D) \subseteq V(A[\forall M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Mutatis mutandis for partition $\langle(\Gamma_1 : \Delta_1, A[\exists M]); (\Gamma_2 : \Delta_2)\rangle$ with the interpolant $C \vee D$.

4.3.2 L$\exists$

This rule has the following form:

$$\frac{aM, A[a/\exists M], \Gamma \Rightarrow \Delta}{A[\exists M], \Gamma \Rightarrow \Delta}$$

Where Γ, Δ and $A[\exists M]$ do not contain a.

By inductive hypothesis, there is a C such that

1. $aM, A[a/\exists M], \Gamma_1 \Rightarrow \Delta_1, C$
2. $C, \Gamma_2 \Rightarrow \Delta_2$
3. $V(C) \subseteq V(aM, A[a/\exists M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

   We want to show that there is a formula K, such that:

   (a) $A[\exists M], \Gamma_1 \Rightarrow \Delta_1, K$

   (b) $K, \Gamma_2 \Rightarrow \Delta_2$

   and

   (c) $V(K) \subseteq V(A[\exists M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Case 77 If C does not contain a, such a formula is C.

Proof. (a)

$$\frac{aM, A[a/\exists M], \Gamma_1 \Rightarrow \Delta_1, C}{A[\exists M], \Gamma_1 \Rightarrow \Delta_1, C}\ \mathrm{L}\exists$$

Proof. (b) Trivial

Proof. (c)

Since $V(aM, A[a/\exists M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2) = V(A[\exists M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$, it follows from inductive hypothesis (3) that $V(C) \subseteq V(A[\exists M], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.

Case 78 Of course, the application of the rule L$\exists$ in Proof (a) will not be permissible if C contains a. In that case we move to the following, replacing C with $\exists\tau_\alpha \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]$:

Proof. (a)

$$\dfrac{\dfrac{\dfrac{\Rightarrow a\tau}{aM, A\,[a/\exists M]\,, \Gamma_1 \Rightarrow \Delta_1, a\tau} \qquad \dfrac{\dfrac{\dfrac{\Rightarrow a\tau}{aM, A\,[a/\exists M]\,, \Gamma_1 \Rightarrow \Delta_1, a\tau} \qquad aM, A\,[a/\exists M]\,, \Gamma_1 \Rightarrow \Delta_1, C\,[a]}{aM, A\,[a/\exists M]\,, \Gamma_1 \Rightarrow \Delta_1, a\tau \wedge C\,[a]}}{aM, A\,[a/\exists M]\,, \Gamma_1 \Rightarrow \Delta_1, a\tau_\alpha \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]}}{aM, A\,[a/\exists M]\,, \Gamma_1 \Rightarrow \Delta_1, \exists\tau_\alpha \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]}}{A\,[\exists M]\,, \Gamma_1 \Rightarrow \Delta_1, \exists\tau_\alpha \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]}$$

Proof. (b)

$$\dfrac{\dfrac{\dfrac{\dfrac{C\,[a]\,, \Gamma_2 \Rightarrow \Delta_2}{a\tau \wedge C\,[a]\,, \Gamma_2 \Rightarrow \Delta_2}}{a_\alpha\tau \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]\,, \Gamma_2 \Rightarrow \Delta_2}}{a\tau, a_\alpha\tau \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]\,, \Gamma_2 \Rightarrow \Delta_2}}{(\exists\tau)_\alpha\tau \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]\,, \Gamma_2 \Rightarrow \Delta_2}$$

Proof. (c) Since τ is a logical predicate, $V((\exists\tau_\alpha)\tau \wedge C\,[\alpha/a_1...\alpha/a_n]) = V(C\,[a])$, and thus the proof is the same as Case 77.

Mutatis mutandis for partition $\langle(\Gamma_1 : \Delta_1); (\Gamma_2, (A\,[\exists M] : \Delta_2)\rangle$ with the interpolant C for Case 77 and $(\forall\tau_\alpha)\tau \wedge C\,[\alpha/a_1...\alpha/a_n]$ for Case 78.

4.4 Special Rules

Next, we inspect the special rules of Quarc, starting with Anaphora.

4.4.1 Anaphora

The rules for anaphora have the following form:

$$\dfrac{A\,[\ldots a_1 \ldots a_n \ldots]\,, \Gamma \Rightarrow \Delta}{A\,[\ldots a_\alpha/a_1 \ldots \alpha/a_n \ldots]\,, \Gamma \Rightarrow \Delta}\,\mathrm{L}A \qquad \dfrac{\Gamma \Rightarrow \Delta, A\,[\ldots a_1 \ldots a_n \ldots]}{\Gamma \Rightarrow \Delta, A\,[\ldots a_\alpha/a_1 \ldots \alpha/a_n \ldots]}\,\mathrm{R}A$$

Starting with LA, by inductive hypothesis there is a C such that

1. $A\,[\ldots a_1 \ldots a_n \ldots]\,, \Gamma_1 \Rightarrow \Delta_1, C$
2. $C, \Gamma_2 \Rightarrow \Delta_2$

   and

3. $V(C) \subseteq V(A\,[\ldots a_1 \ldots a_n \ldots]\,, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

   We want to show there is a formula K such that

   (a) $A\,[\ldots a_\alpha/a_1 \ldots \alpha/a_n \ldots]\,, \Gamma_1 \Rightarrow \Delta_1, K$

(b) $K, \Gamma_2 \Rightarrow \Delta_2$

and

(c) $V(K) \subseteq V(A[\ldots a_\alpha/a_1 \ldots \alpha/a_n \ldots], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Such a formula is C.

Proof. (a)

$$\frac{A[\ldots a_1 \ldots a_n \ldots], \Gamma_1 \Rightarrow \Delta_1, C}{A[\ldots a_\alpha/a_1 \ldots \alpha/a_n \ldots], \Gamma_1 \Rightarrow \Delta_1, C}$$

Proof. (b) Trivial

Proof. (c) Since $V(A[\ldots a_1 \ldots a_n \ldots], \Gamma_1, \Delta_1) = V(A[\ldots a_\alpha/a_1 \ldots \alpha/a_n \ldots], \Gamma_1, \Delta_1)$, it follows from $V(C) \subseteq V(A[\ldots a_1 \ldots a_n \ldots], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$ that $V(C) \subseteq V(A[\ldots a_\alpha/a_1 \ldots \alpha/a_n \ldots], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.

Mutatis mutandis for the partition $\langle(\Gamma_1 : \Delta_1); (A[\ldots a_\alpha/a_1 \ldots \alpha/a_n \ldots], \Gamma_2 : \Delta_2)\rangle$ and similarly for the rule RA.

4.4.2 Reorder

An important thing to note with the Reorder rules is that here we will treat them as logical operations on predicates, and not independent predicates (remember that we, correspondingly, treated them as increasing grade of a formula for the purposes of the cut elimination theorem).

The rules for reorder have the following form:

$$\frac{(t_1 \ldots t_n)R, \Gamma \Rightarrow \Delta}{(t_{\pi 1} \ldots t_{\pi n})R^\pi, \Gamma \Rightarrow \Delta}\,\mathrm{L}Rd \qquad \frac{\Gamma \Rightarrow \Delta, (t_1 \ldots t_n)R}{\Gamma \Rightarrow \Delta, (t_{\pi 1} \ldots t_{\pi n})R^\pi}\,\mathrm{R}Rd$$

Starting with LRd, by inductive hypothesis there is a C such that

1. $(t_1 \ldots t_n)R, \Gamma_1 \Rightarrow \Delta_1, C$
2. $C, \Gamma_2 \Rightarrow \Delta_2$

   and

3. $V(C) \subseteq V((t_1 \ldots t_n)R, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

   We want to show there is a formula K such that

   (a) $(t_{\pi 1} \ldots t_{\pi n})R^\pi, \Gamma_1 \Rightarrow \Delta_1, K$
   (b) $K, \Gamma_2 \Rightarrow \Delta_2$

   and

   (c) $V(K) \subseteq V((t_{\pi 1} \ldots t_{\pi n})R^\pi, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Such a formula is C.

Proof. (a)

$$\frac{(t_1 \dots t_n)R, \Gamma_1 \Rightarrow \Delta_1, C}{(t_{\pi 1} \dots t_{\pi n})R^\pi, \Gamma_1 \Rightarrow \Delta_1, C}$$

Proof. (b) Trivial

Proof. (c) Since we treat reordered predicates as operations on predicates rather than separate predicates, it follows that $V((t_1 \dots t_n)R, \Gamma_1, \Delta_1) = V((t_{\pi 1} \dots t_{\pi n})R^\pi, \Gamma_1, \Delta_1)$, from this and $V(C) \subseteq V((t_1 \dots t_n)R, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$ it follows that $V(C) \subseteq V((t_{\pi 1} \dots t_{\pi n})R^\pi, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Mutatis mutandis for the partition $\langle(\Gamma_1 : \Delta_1); ((t_{\pi 1} \dots t_{\pi n})R^\pi, \Gamma_2 : \Delta_2)\rangle$ and similarly for the rule RRd.

4.4.3 Negative Predication

Since the rules LNP and RNP precisely mirror the respective rules for reorder, they will be omitted here.

4.5 Instantiatial Import

We now move on to LK-Quarc$_3$ and show that the inductive step holds for the rule of Instantial Import:

$$\frac{aM, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Where Γ, Δ do not contain a.

By inductive hypothesis, there is a C such that

1. $aM, \Gamma_1 \Rightarrow \Delta_1, C$
2. $C, \Gamma_2 \Rightarrow \Delta_2$
3. $V(C) \subseteq V(aM, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

And moreover, D such that

4. $\Gamma_1 \Rightarrow \Delta_1, D$
5. $D, aM, \Gamma_2 \Rightarrow \Delta_2$
6. $V(D) \subseteq V(\Gamma_1, \Delta_1) \cap V(aM, \Gamma_2, \Delta_2)$

We want to show that there is a formula K, such that:

(a) $\Gamma_1 \Rightarrow \Delta_1, K$
(b) $K, \Gamma_2 \Rightarrow \Delta_2$

and

(c) $V(K) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

There are several cases to consider here.

Case 79 If $M \notin V(\Gamma_2, \Delta_2)$ and C does not contain a, then K is C.

Proof. (a)

$$\frac{aM, \Gamma_1 \Rightarrow \Delta_1, C}{\Gamma_1 \Rightarrow \Delta_1, C}$$

Proof. (b) Trivial

Proof. (c) Since $M \notin V(\Gamma_2, \Delta_2)$, it follows that $V(aM, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2) = V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$, and therefore $V(C) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

If C does contain a, then we move to the following case:

Proof. (a)

$$\cfrac{\cfrac{\cfrac{\Rightarrow a\tau}{aM, \Gamma_1 \Rightarrow \Delta_1, a\tau} \qquad \cfrac{\cfrac{\cfrac{\Rightarrow a\tau}{aM, \Gamma_1 \Rightarrow \Delta_1, a\tau} \qquad aM, \Gamma_1 \Rightarrow \Delta_1, C[a]}{aM, \Gamma_1 \Rightarrow \Delta_1, a\tau \wedge C[a]}}{aM, \Gamma_1 \Rightarrow \Delta_1, a_\alpha\tau \wedge C[\alpha/a_1 \ldots \alpha/a_n]}}{aM, \Gamma_1 \Rightarrow \Delta_1, \exists\tau_\alpha\tau \wedge C[\alpha/a_1 \ldots \alpha/a_n]}}{\Gamma_1 \Rightarrow \Delta_1, \exists\tau_\alpha\tau \wedge C[\alpha/a_1 \ldots \alpha/a_n]}$$

Proof. (b)

$$\cfrac{\cfrac{\cfrac{\cfrac{C[a], \Gamma_2 \Rightarrow \Delta_2}{a\tau \wedge C[a], \Gamma_2 \Rightarrow \Delta_2}}{a_\alpha\tau \wedge C[\alpha/a_1 \ldots \alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}}{a\tau, a_\alpha\tau \wedge C[\alpha/a_1 \ldots \alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}}{\exists\tau_\alpha\tau \wedge C[\alpha/a_1 \ldots \alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}$$

Proof. (c) Since $M \notin V(\Gamma_2, \Delta_2)$, same as before.

Case 80 If $M \in V(\Gamma_2, \Delta_2)$, then K is D or $\exists\tau_\alpha\tau \wedge C[\alpha/a_1 \ldots \alpha/a_n]$ as in Case 79, except noting that if $M \in V(\Gamma_2, \Delta_2)$, then $V(aM, \Gamma_2, \Delta_2) = V(\Gamma_2, \Delta_2)$ in Proof (c).

4.6 Identity

Finally, we examine the rules for identity - LK-Quarc$_2$. The rules are:

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} =_1 \qquad \frac{A[b], a = b, A[a/b], \Gamma \Rightarrow \Delta}{a = b, A[a/b], \Gamma \Rightarrow \Delta} =_2$$

We first focus on the $=_1$ rule. By inductive hypothesis, there is a C such that

1. $a = a, \Gamma_1 \Rightarrow \Delta_1, C$
2. $C, \Gamma_2 \Rightarrow \Delta_2$

   and

3. $V(C) \subseteq V(a = a, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

   We want to show there is a formula K such that:

   (a) $\Gamma_1 \Rightarrow \Delta_1, K$

   (b) $K, \Gamma_2 \Rightarrow \Delta_2$

   and

   (c) $V(K) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Such a formula is C.

Proof. (a)

$$\frac{a = a, \Gamma_1 \Rightarrow \Delta_1, C}{\Gamma_1 \Rightarrow \Delta_1, C}$$

Proof. (b) Trivial.

Proof. (c) Since $=$ is a logical predicate, $V(a = a, \Gamma_1, \Delta_1) = V(\Gamma_1, \Delta_1)$, and therefore $V(C) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.

We now focus on the rule $=_2$. We need to inspect several partitions here.

Case 81 Partition $\langle (a = b, A\,[a/b]\,, \Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2) \rangle$

By inductive hypothesis, there is a C such that

1. $A\,[b]\,, a = b, A\,[a/b]\,, \Gamma_1 \Rightarrow \Delta_1, C$

2. $C, \Gamma_2 \Rightarrow \Delta_2$

   and

3. $V(C) \subseteq V(A\,[b]\,, a = b, A\,[a/b]\,, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

   We want to show there is a formula K such that:

   (a) $a = b, A\,[a/b]\,, \Gamma_1 \Rightarrow \Delta_1, K$

   (b) $K, \Gamma_2 \Rightarrow \Delta_2$

   and

   (c) $V(K) \subseteq V(a = b, A\,[a/b]\,, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Such a formula is C.

Proof. (a)

$$\frac{A\,[b]\,, a = b, A\,[a/b]\,, \Gamma_1 \Rightarrow \Delta_1, C}{a = b, A\,[a/b]\,, \Gamma_1 \Rightarrow \Delta_1, C}$$

Proof. (b) Trivial.

Proof. (c) Since $V(A[a]) = V(A[b/a])$ it follows that, $V(A[a], a = b, A[a/b]) = V(a = b, A[a/b])$ and therefore $V(C) \subseteq V(a = b, A[a/b], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Mutatis mutandis for partition $\langle(\Gamma_1 : \Delta_1); (a = b, A[a/b], \Gamma_2 : \Delta_2)\rangle$

Case 82 Partition $\langle(A[a/b], \Gamma_1 : \Delta_1); (a = b, \Gamma_2 : \Delta_2)\rangle$

By inductive hypothesis, there is a C such that

1. $A[b], A[a/b], \Gamma_1 \Rightarrow \Delta_1, C$
2. $C, a = b, \Gamma_2 \Rightarrow \Delta_2$

   and

3. $V(C) \subseteq V(A[b], A[a/b], \Gamma_1, \Delta_1) \cap V(a = b, \Gamma_2, \Delta_2)$

   We want to show there is a formula K such that:

   (a) $A[a/b], \Gamma_1 \Rightarrow \Delta_1, K$

   (b) $K, a = b, \Gamma_2 \Rightarrow \Delta_2$

   and

   (c) $V(K) \subseteq V(A[a/b], \Gamma_1, \Delta_1) \cap V(a = b, \Gamma_2, \Delta_2)$

Such a formula is $C \vee \neg a = b$.

Proof. (a)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A[b], A[a/b], \Gamma_1 \Rightarrow \Delta_1, C}{A[b], a = b, A[a/b], \Gamma_1 \Rightarrow \Delta_1, C}}{a = b, A[a/b], \Gamma_1 \Rightarrow \Delta_1, C}}{A[a/b], \Gamma_1 \Rightarrow \Delta_1, C, \neg a = b}}{A[a/b], \Gamma_1 \Rightarrow \Delta_1, C \vee \neg a = b, C \vee \neg a = b}}{A[a/b], \Gamma_1 \Rightarrow \Delta_1, C \vee \neg a = b}$$

Proof. (b)

$$\cfrac{C, a = b, \Gamma_2 \Rightarrow \Delta_2 \qquad \cfrac{\cfrac{\cfrac{}{a = b \Rightarrow a = b}}{\neg a = b, a = b \Rightarrow}}{\neg a = b, a = b, \Gamma_2 \Rightarrow \Delta_2}}{C \vee \neg a = b, a = b, \Gamma_2 \Rightarrow \Delta_2}$$

Proof. (c) Since $V(A[b]) = V(A[a/b])$, it follows that $V(A[b], A[a/b], \Gamma_1, \Delta_1) = V(A[a/b], \Gamma_1, \Delta_1)$. Since "=" is a logical predicate, it follows that $V(a = b, \Gamma_2, \Delta_2) = V(\Gamma_2, \Delta_2)$. Therefore, it follows from $V(C) \subseteq V(A[b], A[a/b], \Gamma_1, \Delta_1) \cap V(a = b, \Gamma_2, \Delta_2)$ that $V(C) \subseteq V(A[a/b], \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

Similarly for partition $\langle(a = b, \Gamma_1 : \Delta_1); (A[a/b], \Gamma_2 : \Delta_2)\rangle$

This concludes the proof of Theorem 74 and therefore, we have established that the interpolation property holds for LK-Quarc°. We next need to show we can transfer that result to LK-Quarc by eliminating constants and the predicate τ. We start with the former.

4.7 Eliminating Constants

When an interpolant contains constants $\top$ or $\bot$, we can simplify it using the following equivalences, which we will not prove here ("$\equiv$" means "is logically equivalent in LK-Quarc") whenever A is a formula containing either no anaphors or sources of anaphors, or both a source and all the anaphors it is a source of:

1. $\top \to A \equiv A$; $\bot \to A \equiv \top$; $A \to \top \equiv \top$; $A \to \bot \equiv \neg A$
2. $\top \wedge A \equiv A$; $\bot \wedge A \equiv \bot$
3. $\top \vee A \equiv \top$; $\bot \vee A \equiv A$
4. $\neg\top \equiv \bot$; $\neg\bot \equiv \top$

However, these transformations will not do if the formula A contains a source, but not all the anaphors it is a source of. E.g. $((\forall S_\alpha)P \vee \top) \to (\alpha)Q$ would become $\top \to (\alpha)Q$, which is not a formula.

In these cases we perform the previous substitutions when A is (i) a string of symbols such that for all anaphors $\alpha_1, \ldots, \alpha_n, \ldots, \beta_1, \ldots \beta_m$ in A there is a formula $C[c_1, \ldots, c_n, \ldots d_1, \ldots, d_m]$, such that $C[\alpha_1/c_1, \ldots, \alpha_n/c_n, \ldots \beta_1/d_1 \ldots \beta_m/d_m]$ is a string of symbols identical to A (A differs from a formula only in containing anaphors), or (ii) a formula, while also making sure to:

5. If an argument stops being source of any anaphors, remove source marking.
6. If a source of an anaphor is a singular argument that disappears in the transformation, replace the leftmost anaphor by its source with a source marking.
7. If a source of an anaphor is a quantified argument that would disappear in the transformation, replace $\top$ with $(\forall S)S$ and $\bot$ with $\neg(\forall S)S$, where S is the unary predicate appearing in the quantified argument.

It should be clear that, as above, we are replacing formulas with their equivalent formulas.

After these transformations we are left with an interpolant C' which is either free of constants, or a constant itself. In the first case the elimination is complete. Let us now consider the second case.

4.7.1 Replacing Constants

If $V(\Gamma) \cap V(\Delta) \neq \emptyset$, then, for some predicate $R \in V(\Gamma) \cap V(\Delta)$ of arity n and some singular arguments $t_1 \ldots t_n$ which do not appear in Γ, Δ, it holds that $\top \equiv (t_1 \ldots t_n)R \to (t_1 \ldots t_n)R$, and $\bot \equiv \neg((t_1 \ldots, t_n)R \to (t_1 \ldots t_n)R)$ and therefore the constants can be replaced by an expression not containing them.

If $V(\Gamma) \cap V(\Delta) = \emptyset$, then either (i) $\Gamma \Rightarrow \bot$ and $\bot \Rightarrow \Delta$ or (ii) $\Gamma \Rightarrow \top$ and $\top \Rightarrow \Delta$. If (i), then $\Rightarrow \neg\Gamma$, and if (ii) then $\Rightarrow \Delta$. Therefore, either $\Rightarrow \neg\Gamma$ or $\Rightarrow \Delta$.

We have now demonstrated that the interpolation property holds of an intermediate system LK-Quarc†, which does not contain the constants but does contain the predicate τ. In the following section we will make some headway towards eliminating it.

4.8 Eliminating τ

We now eliminate the predicate τ. The order of presentation here notwithstanding, it will be obvious from the proceeding that τ needs to be eliminated prior to the elimination of the constants if we wish to show that the interpolation property holds for LK-Quarc. The three places where τ was used were the rules Ins, R$\forall$ and L$\exists$. We will examine how to eliminate it in the first case, and offer some thoughts how this reflects on the latter two.

Note that here we are dealing only with LK-Quarc$_3$, since some of the procedures below will not apply to identity.

Before proceeding, we prove a lemma, beginning with introducing a definition. This is adopted from [Wansing, 1996], omitting the part containing Cut, as here we are dealing with Cut-free derivations.

Definition 83 Two formulas are *immediately connected* just in case they are

1. two sides of an initial sequent
2. non-parametric in an inference (i.e. either side or principal/main formula of the inference)
3. parametric and congruent (i.e. appearing on the same side of the sequents) in an inference.

All immediately connected formulas are *connected* and the connection relation is transitive and symmetric.

Lemma 84 If $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ and no formula of Γ_1, Δ_1 is connected to any formula of Γ_2, Δ_2, then either $\Gamma_1 \Rightarrow \Delta_1$ or $\Gamma_2 \Rightarrow \Delta_2$.

Proof. By induction on inference steps.

Basic step. If $A \Rightarrow A$ then either $A \Rightarrow A$ or (since Γ_2, Δ_2 are empty) $\ldots \Rightarrow \ldots$ Holds because the first disjunct holds.

Inductive step. We want to show that for every rule of inference

$$\frac{\Gamma_1', \Gamma_2' \Rightarrow \Delta_1', \Delta_2'}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,\text{Inf}$$

if either $\Gamma_1' \Rightarrow \Delta_1'$ or $\Gamma_2' \Rightarrow \Delta_2'$, and no formula of Γ_1, Δ_1 is connected to any formula of Γ_2, Δ_2, then either

$$\frac{\Gamma_1' \Rightarrow \Delta_1'}{\Gamma_1 \Rightarrow \Delta_1}\,\text{Inf} \quad \text{or} \quad \frac{\Gamma_2' \Rightarrow \Delta_2'}{\Gamma_2 \Rightarrow \Delta_2}\,\text{Inf}'$$

for rules with one upper sequent, and for the rules with two upper sequents

$$\frac{\Gamma_1', \Gamma_2' \Rightarrow \Delta_1', \Delta_2' \qquad \Gamma_1'', \Gamma_2'' \Rightarrow \Delta_1'', \Delta_2''}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,\text{Inf}$$

if either ($\Gamma_1' \Rightarrow \Delta_1'$ and $\Gamma_1'' \Rightarrow \Delta_1''$) or ($\Gamma_2' \Rightarrow \Delta_2'$ and $\Gamma_2'' \Rightarrow \Delta_2''$), then either

$$\frac{\Gamma_1' \Rightarrow \Delta_1' \qquad \Gamma_1'' \Rightarrow \Delta_1''}{\Gamma_1 \Rightarrow \Delta_1}\text{Inf} \quad \text{or} \quad \frac{\Gamma_2' \Rightarrow \Delta_2' \qquad \Gamma_2'' \Rightarrow \Delta_2''}{\Gamma_2 \Rightarrow \Delta_2}\text{Inf}'$$

We will assume that the principal formula is always connected to Γ_1 and Δ_1.

The cases for most rules are similar, so we will only illustrate on a couple of examples.

(LW)

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{A, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\text{L}W$$

Assume $\Gamma_1 \Rightarrow \Delta_1$ or $\Gamma_2 \Rightarrow \Delta_2$. Then either

$$\frac{\Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\text{L}W \quad \text{or} \quad \frac{\Gamma_2 \Rightarrow \Delta_2}{\Gamma_2 \Rightarrow \Delta_2}$$

Clearly holds. Similarly for other structural rules.

(L$\wedge$)

$$\frac{A, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{A \wedge B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\text{L}\wedge$$

Assume $A, \Gamma_1 \Rightarrow \Delta_1$ or $\Gamma_2 \Rightarrow \Delta_2$. Then either

$$\frac{A, \Gamma_1 \Rightarrow \Delta_1}{A \wedge B, \Gamma_1 \Rightarrow \Delta_1}\text{L}\wedge \quad \text{or} \quad \frac{\Gamma_2 \Rightarrow \Delta_2}{\Gamma_2 \Rightarrow \Delta_2}$$

Clearly holds. Similarly for other one-sequent rules.

(R$\wedge$)

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A \qquad \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, B}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A \wedge B}\text{R}\wedge$$

Assume that either $\Gamma_1 \Rightarrow \Delta_1, A$ *and* $\Gamma_1 \Rightarrow \Delta_1, B$ (note that if A were connected to Γ_1, Δ_1, but B to Γ_2, Δ_2 then, contrary to our assumption, Γ_1, Δ_1 and Γ_2, Δ_2 would be connected), or $\Gamma_2 \Rightarrow \Delta_2$. Then either

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad \Gamma_1 \Rightarrow \Delta_1, B}{\Gamma_1 \Rightarrow \Delta_1, A \wedge B}\text{R}\wedge \quad \text{or} \quad \frac{\Gamma_2 \Rightarrow \Delta_2 \qquad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_2 \Rightarrow \Delta_2}$$

Clearly holds. Similarly for other two-sequent rules. This concludes the proof of Lemma 84.

4.8.1 Instantiation

We now move on to elimination of the predicate τ for the case of Ins:

$$\frac{aM, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}(\text{Ins})^*$$

* - where neither Γ nor Δ contain the singular argument a.

By inductive hypothesis, there are C and D such that

1. $aM, \Gamma_1 \Rightarrow \Delta_1, C$
2. $C, \Gamma_2 \Rightarrow \Delta_2$
3. $V(C) \subseteq V(aM, \Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$

   and

4. $\Gamma_1 \Rightarrow \Delta_1, D$
5. $D, aM, \Gamma_2 \Rightarrow \Delta_2$
6. $V(D) \subseteq V(\Gamma_1, \Delta_1) \cap V(aM, \Gamma_2, \Delta_2)$

   if either of those do not contain a, the problem of eliminating τ does not arise, so assume both contain a (marked as $C[a]$ and $D[a]$). There are multiple possibilities here:

   (a) $M \in V(\Gamma_1, \Delta_1)$ and $M \in V(\Gamma_2, \Delta_2)$
   Whenever M is both in $V(\Gamma_1, \Delta_1)$ and $V(\Gamma_2, \Delta_2)$, we can quantify over C:

$$\dfrac{\dfrac{\dfrac{\dfrac{aM \Rightarrow aM}{aM, \Gamma_1 \Rightarrow \Delta_1, aM} \qquad aM, \Gamma_1 \Rightarrow \Delta_1, C[a]}{aM, \Gamma_1 \Rightarrow \Delta_1, aM \wedge C[a]}}{aM, \Gamma_1 \Rightarrow \Delta_1, a_\alpha M \wedge C[\alpha/a_1 \dots \alpha/a_n]}}{\Gamma_1 \Rightarrow \Delta_1, \forall M_\alpha M \wedge C[\alpha/a_1 \dots \alpha/a_n]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C[a], \Gamma_2 \Rightarrow \Delta_2}{aM \wedge C[a], \Gamma_2 \Rightarrow \Delta_2}}{aM \wedge C[a], aM, \Gamma_2 \Rightarrow \Delta_2}}{a_\alpha M \wedge C[\alpha/a_1 \dots \alpha/a_n], aM, \Gamma_2 \Rightarrow \Delta_2} \qquad \dfrac{aM \Rightarrow aM}{aM, \Gamma_2 \Rightarrow \Delta_2, aM}}{\forall M_\alpha M \wedge C[\alpha/a_1 \dots \alpha/a_n], aM, \Gamma_2 \Rightarrow \Delta_2}\,\mathrm{L}\forall}{aM, \forall M_\alpha M \wedge C[\alpha/a_1 \dots \alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}}{\forall M_\alpha M \wedge C[\alpha/a_1 \dots \alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}\,\mathrm{Ins}$$

   Moreover, since $M \in V(\Gamma_1, \Delta_1)$, $V(aM, \Gamma_1, \Delta_1) = V(\Gamma_1, \Delta_1)$, and therefore $V(C) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$. Similar proof is available for D. Now assume either $M \notin V(\Gamma_1, \Delta_1)$ or $M \notin V(\Gamma_2, \Delta_2)$

   (b) $M \notin V(C)$ and $M \notin V(D)$
   From $M \notin V(\Gamma_1, \Delta_1)$ and $M \notin V(C)$ it follows by subformula property that aM in $aM, \Gamma_1 \Rightarrow \Delta_1, C$ was introduced by LW, and from there and the fact it is a basic formula it follows by Definition 83 that it is not connected to anything. Therefore, by Lemma 84 (and since $aM \Rightarrow$ does not hold), it follows that $\Gamma_1 \Rightarrow \Delta_1, C$. Moreover, since $M \notin V(C)$, $V(C) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$. A similar proof is available for D and $M \notin V(\Gamma_2, \Delta_2)$.

Let us take an overview of the cases covered and solved so far. Here '+' indicates $\in$ and '-' indicates $\notin$. That the solution is imp(ossible) indicates that it violates some inductive hypotheses. For example, in line (2), we have a situation where $V(C)$ contains M, but $V(\Gamma_2, \Delta_2)$ does not, which contradicts inductive hypothesis 3.

	$M \in V(C)$	$M \in V(D)$	$M \in V(\Gamma_1, \Delta_1)$	$M \in V(\Gamma_2, \Delta_2)$	Solution
1	+	+	+	+	a
2	+	+	+	-	imp
3	+	+	-	+	imp
4	+	+	-	-	imp
5	+	-	+	+	a
6	+	-	+	-	imp
7	+	-	-	+	?
8	+	-	-	-	imp
9	-	+	+	+	a
10	-	+	+	-	?
11	-	+	-	+	imp
12	-	+	-	-	imp
13	-	-	+	+	a,b
14	-	-	+	-	b
15	-	-	-	+	b
16	-	-	-	-	b

Of course, this table is only the top quarter of the full table that also takes into consideration whether C and D contain a (remember we assumed they do because otherwise the solution is covered by Case 79 and 80). This clearly illustrates the complexity of the problem.

In light of this, in order to further approximate the solution in the remainder of this section we will consider some further options for τ elimination, but exploring whether they are exhaustive is left for future work. We will examine the case in row 7, but 10 is similar.

Further Possibilities

If in inductive hypothesis (5), D and aM are not connected, then by Lemma 84, either

(i) $D, \Gamma_2' \Rightarrow \Delta_2'$ or (ii) $aM, \Gamma_2'' \Rightarrow \Delta_2''$

If (i), then and if (ii), then

$$\dfrac{D, \Gamma_2' \Rightarrow \Delta_2'}{D, \Gamma_2 \Rightarrow \Delta_2}\,\mathrm{L}W,\,\mathrm{R}W \qquad \dfrac{\dfrac{aM, \Gamma_2'' \Rightarrow \Delta_2''}{\Gamma_2'' \Rightarrow \Delta_2''}\,\mathrm{Ins}}{D, \Gamma_2 \Rightarrow \Delta_2}\,\mathrm{L}W,\,\mathrm{R}W$$

Moreover, since $M \notin V(\Gamma_1, \Delta_1)$, $V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2) = V(\Gamma_1, \Delta_1) \cap V(aM, \Gamma_2, \Delta_2)$ and therefore $V(D) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.

Alternatively, given $\Gamma_2'' \Rightarrow \Delta_2''$ we can derive

$$\frac{\Rightarrow \top}{\Gamma_1 \Rightarrow \Delta_1, \top}\ \mathrm{L}W, \mathrm{R}W \quad \text{and} \quad \frac{\Gamma_2'' \Rightarrow \Delta_2''}{\top, \Gamma_2 \Rightarrow \Delta_2}\ \mathrm{L}W, \mathrm{R}W$$

Obviously, $V(\top) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.

Furthermore, if D is not connected to some Γ_1'', Δ_1'' in inductive hypothesis (4), then either

(iii) $\Gamma_1' \Rightarrow \Delta_1', D$ or (iv) $\Gamma_1'' \Rightarrow \Delta_1''$

and if (iv), then

$$\frac{\Gamma_1'' \Rightarrow \Delta_1''}{\Gamma_1 \Rightarrow \Delta_1, \bot} \quad \text{and} \quad \frac{\bot \Rightarrow}{\bot, aM, \Gamma_2 \Rightarrow \Delta_2}$$

As above, $V(\bot) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.

This can be further strengthened by "severing" connections in the inductive hypotheses. We first re-designate the entire derivation, leaving a intact, and then apply the following procedure.

Definition 85 (Maximization Procedure)
We start with the topmost leftmost instance of an inference, except weakening and initial sequents, where a-formula is non-parametric. If any occurrence of a congruent formula in the upper sequent(s) of the rule was introduced via weakening rules, then the inference is transformed in one of the following ways, depending on the rule used.

(L$\wedge$)

$$\frac{A[a], \Gamma \Rightarrow \Delta}{A[a] \wedge B, \Gamma \Rightarrow \Delta} \quad \text{is transformed into} \quad \frac{\Gamma \Rightarrow \Delta}{A[a] \wedge B, \Gamma \Rightarrow \Delta}\ \mathrm{L}W$$

Since A is introduced via (left) weakening and then parametric down to this inference, there is a previous inference in the derivation

$$\frac{\Gamma' \Rightarrow \Delta'}{A, \Gamma' \Rightarrow \Delta'}$$

Moreover, since A is parametric in any interceding line, it follows (by removing A in those lines) that

$$\frac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}$$

Therefore, the derivation above the sequent $\Gamma \Rightarrow \Delta$ is correct. Likewise, the derivation below the sequent $A[a] \wedge B, \Gamma \Rightarrow \Delta$ is correct because nothing there has changed, and the rule of inference is used correctly. Therefore, this is a correct derivation. Similarly for other one sequent rules. Note that due to re-designation, a-formulas are only principal in R$\forall$ or L$\exists$ if the proper singular argument is some other SA c.

Then, the following transformation occurs:

$$\frac{cS, A[c/\exists S][a], \Gamma \Rightarrow \Delta}{A[\exists S][a], \Gamma \Rightarrow \Delta} \quad \text{is transformed into} \quad \cfrac{\cfrac{cS, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{Ins}}{A[\exists S][a], \Gamma \Rightarrow \Delta}\ \mathrm{L}W$$

And similarly for R$\forall$.

(R$\wedge$)

If $A[a]$ is introduced via weakening, then

$$\frac{\Gamma \Rightarrow \Delta, A[a] \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A[a] \wedge B} \quad \text{is transformed into} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A[a] \wedge B}\,\mathrm{R}W$$

And same for $B[a]$. Applying the same considerations as above, we can show this is a correct derivation. Similarly for other two sequent derivations.

Obviously, any time the same sequent occurs multiple times in the same branch, the lowest occurrence can be substituted for the highest one, keeping the proof above the latter intact.

We apply this procedure repeatedly to a derivation. Since the procedure either pushes an instance of weakening lower into a derivation or removes it completely, and does not extend the derivation below that weakening, it will terminate in a finite number of steps.

This procedure will maximize the complexity of a-formulas introduced via weakening, thereby reducing the need for them to be non-parametric in instances of rules, and consequently reducing the number of formulas they are connected to.

Consider an example:

Example 86

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \Rightarrow C}{C, B \Rightarrow C} \qquad \dfrac{B \Rightarrow B}{B \Rightarrow C, B}}{B \to C, B \Rightarrow C}}{B \to C, B \Rightarrow C, A[a]}}{B, B \to C \Rightarrow C, A[a]} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C^4 \Rightarrow C^3}{C, \neg C \Rightarrow}}{C \Rightarrow \neg\neg C}}{B, C \Rightarrow \neg\neg C}}{C, B \Rightarrow C, \neg\neg C} \qquad \dfrac{B \Rightarrow B}{B \Rightarrow C, \neg\neg C, B}}{B, B \to C^5 \Rightarrow C, \neg\neg C^2}}{B, B \to C \Rightarrow C, \neg\neg C \vee \neg C}}{B, B \to C^6 \Rightarrow C, A[a] \wedge (\neg\neg C \vee \neg C)^1}$$

Where the formula labeled 1 is connected to the one labeled 2 etc. and by transitivity, formula 1 is connected to formula 6. This is transformed (in two steps) into

$$\dfrac{\dfrac{\dfrac{\dfrac{C \Rightarrow C}{C, B \Rightarrow C} \qquad \dfrac{B \Rightarrow B}{B \Rightarrow C, B}}{B \to C, B \Rightarrow C}}{B, B \to C \Rightarrow C}}{B, B \to C \Rightarrow C, A[a] \wedge (\neg\neg C \vee \neg C)^1}$$

Where formula 1 is no longer connected to any other formula, and therefore by Lemma 84 it follows that either $B, B \to C \Rightarrow C$, in which case $\perp$ is an acceptable interpolant (as in (iv) above), or $\Rightarrow A[a] \wedge (\neg\neg C \vee \neg C)$, in which case $\top$ is an acceptable interpolant (as in (ii) above).

As noted at the beginning of this section, this procedure will not work for LK-Quarc$_2$, since it does not apply to identity rules. Consider the following example:

Example 87

$$\dfrac{\dfrac{bS \Rightarrow bS}{bS, a=b, aS \Rightarrow bS}\,\mathrm{L}W,\ \mathrm{L}P}{a=b, aS \Rightarrow bS}\,=_2$$

Both $a = b$ and aS are introduced via weakening, but transplacing the weakening and the rule will not work here:

$$\dfrac{\dfrac{bS \Rightarrow bS}{\Rightarrow bS}\,\mathrm{Inf}^*}{a=b, aS \Rightarrow bS}\,\mathrm{L}W$$

Clearly, Inf* corresponds to no rule of LK-Quarc$_2$.

We've so far dealt with cases where $D[a]$ is, in some sense, unnecessary. Another option is that we could have a solution using only some part of D which does not contain a. Consider the following Lemma:

Lemma 88 If every a-formula is introduced via weakening, L$\wedge$ or R$\vee$, by a substitution procedure which replaces every first occurrence of an a-formula on the left by $\top$, and on the right by $\bot$, we can, by making corresponding changes in the rest of the proof (substituting the formula A by its *replacement formula* A' according to the rule used), produce a correct derivation $\Gamma \Rightarrow \Delta, A'$ such that $A' \Rightarrow A$ (or, if A' is on the left, $A \Rightarrow A'$).

Proof. By induction on the rules of LK-Quarc.

Basic step.

(LW)

$$\dfrac{\Gamma \Rightarrow \Delta}{A[a], \Gamma \Rightarrow \Delta} \quad \text{is transformed into} \quad \dfrac{\Gamma \Rightarrow \Delta}{\top, \Gamma \Rightarrow \Delta}$$

And $A[a] \Rightarrow \top$

(RW)

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A[a]} \quad \text{is transformed into} \quad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \bot}$$

And $\bot \Rightarrow A[a]$

(L$\wedge$)

$$\dfrac{B, \Gamma \Rightarrow \Delta}{A[a] \wedge B, \Gamma \Rightarrow \Delta} \quad \text{is transformed into} \quad \dfrac{B, \Gamma \Rightarrow \Delta}{\top \wedge B, \Gamma \Rightarrow \Delta}$$

And $A[a] \wedge B \Rightarrow \top \wedge B$

(L$\vee$)

$$\dfrac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, B \vee A[a]} \quad \text{is transformed into} \quad \dfrac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, B \vee \bot}$$

And $B \vee \top \Rightarrow B \vee A$

Inductive step. We first consider the cases where the replacement formula is non-parametric.

(L$\wedge$)

$\dfrac{A', \Gamma \Rightarrow \Delta}{A' \wedge B, \Gamma \Rightarrow \Delta}$ is correct, and moreover if $A \Rightarrow A'$, then $A \wedge B \Rightarrow A' \wedge B$.

(R$\wedge$)

$\dfrac{\Gamma \Rightarrow \Delta, A' \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A' \wedge B}$ is correct, and moreover if $A' \Rightarrow A$, then $A' \wedge B \Rightarrow A \wedge B$.

(L$\neg$)

$\dfrac{\Gamma \Rightarrow \Delta, A'}{\neg A', \Gamma \Rightarrow \Delta}$ is correct, and moreover if $A' \Rightarrow A$, then $\neg A \Rightarrow \neg A'$.

Similarly for other propositional rules.

(L$\forall$)

$\dfrac{A[b], \Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, bS}{A[\forall S/b], \Gamma \Rightarrow \Delta}$ is transformed into $\dfrac{A[b]', \Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, bS}{A[\forall S/b]', \Gamma \Rightarrow \Delta}$

or, if A' does not contain b, $\dfrac{A', \Gamma \Rightarrow \Delta}{A', \Gamma \Rightarrow \Delta}$

and if $A[b] \Rightarrow A[b]'$, then $A[\forall S/b] \Rightarrow A[\forall S/b]'$ (or trivially, $A' \Rightarrow A'$):

$$\dfrac{\dfrac{\dfrac{\dfrac{A[b] \Rightarrow A[b]'}{bS, A[b] \Rightarrow A[b]'}}{A[b], bS \Rightarrow A[b]'} \qquad \dfrac{\dfrac{bS \Rightarrow bS}{bS \Rightarrow bS, A[b]'}}{bS \Rightarrow A[b]', bS}}{A[\forall S/b], bS \Rightarrow A[b]'}}{\dfrac{bS, A[\forall S/b] \Rightarrow A[b]'}{A[\forall S/b] \Rightarrow A[\forall S/b]'}}$$

(R$\forall$)

$\dfrac{bS, \Gamma \Rightarrow \Delta, A[b]}{\Gamma \Rightarrow \Delta, A[\forall S/b]}$ is transformed into $\dfrac{bS, \Gamma \Rightarrow \Delta, A[b]'}{\Gamma \Rightarrow \Delta, A[\forall S/b]'}$

or, if A' does not contain b, $\dfrac{bS, \Gamma \Rightarrow \Delta, A'}{\Gamma \Rightarrow \Delta, A'}\,\text{Ins}$

and, if $A[b]' \Rightarrow A[b]$, then $A[\forall S/b]' \Rightarrow A[\forall S/b]$ (or trivially, $A' \Rightarrow A'$): *mutatis mutandis*, as the previous rule.

(R$\exists$)

$$\frac{\Gamma \Rightarrow \Delta, bS \qquad \Gamma \Rightarrow \Delta, A\,[b]}{\Gamma \Rightarrow \Delta, A\,[\exists S/b]} \quad \text{is transformed into} \quad \frac{\Gamma \Rightarrow \Delta, bS \qquad \Gamma \Rightarrow \Delta, A\,[b]'}{\Gamma \Rightarrow \Delta, A\,[\exists S/b]'}$$

$$\text{or, if } A' \text{ does not contain } b, \quad \frac{\Gamma \Rightarrow \Delta, A'}{\Gamma \Rightarrow \Delta, A'}$$

and, if $A\,[b]' \Rightarrow A\,[b]$, then $A\,[\exists S/b]' \Rightarrow A\,[\exists S/b]$ (or trivially, $A' \Rightarrow A'$):

$$\cfrac{\cfrac{\cfrac{\cfrac{bS \Rightarrow bS}{A\,[b]', bS \Rightarrow bS}}{bS, A\,[b]' \Rightarrow bS} \qquad \cfrac{A\,[b]' \Rightarrow A\,[b]}{bS, A\,[b]' \Rightarrow A\,[b]}}{bS, A\,[b]' \Rightarrow A\,[\exists S/b]}}{A\,[\exists S/b]' \Rightarrow A\,[\exists S/b]}$$

(L∃)

$$\frac{bS, A\,[b]\,, \Gamma \Rightarrow \Delta}{A\,[\exists S/b]\,, \Gamma \Rightarrow \Delta} \quad \text{is transformed into} \quad \frac{bS, A\,[b]'\,, \Gamma \Rightarrow \Delta}{A\,[\exists S/b]'\,, \Gamma \Rightarrow \Delta}$$

$$\text{or, if } A' \text{ does not contain } b, \quad \frac{bS, A', \Gamma \Rightarrow \Delta}{A', \Gamma \Rightarrow \Delta}\,\text{Ins}$$

and, if $A\,[b] \Rightarrow A\,[b]'$, then $A\,[\exists S/b] \Rightarrow A\,[\exists S/b]'$ (or trivially, $A' \Rightarrow A'$): *mutatis mutandis*, as the previous rule.

Due to re-designation, any instance of Ins remains unchanged.

The case where the replacement formula is parametric in a one-sequent inference is trivial. In a two sequent inference, it can occur that the congruent formulas A are replaced by different replacement formulas A' and A'':

$$\frac{A, \Gamma' \Rightarrow \Delta' \qquad A, \Gamma'' \Rightarrow \Delta''}{A, \Gamma \Rightarrow \Delta} \quad \text{is replaced by} \quad \frac{A', \Gamma' \Rightarrow \Delta' \qquad A'', \Gamma'' \Rightarrow \Delta''}{???, \Gamma \Rightarrow \Delta}$$

In such a case the derivation is transformed in the following way:

$$\cfrac{\cfrac{A', \Gamma' \Rightarrow \Delta'}{A' \wedge A'', \Gamma' \Rightarrow \Delta'} \qquad \cfrac{A'', \Gamma'' \Rightarrow \Delta''}{A' \wedge A'', \Gamma'' \Rightarrow \Delta''}}{A' \wedge A'', \Gamma \Rightarrow \Delta}$$

Clearly, this is again a correct derivation, and moreover, since both A' and A'' are replacement formulas of the same formula A, it holds by inductive hypothesis that $A \Rightarrow A'$ and $A \Rightarrow A''$, and therefore

$$\frac{A \Rightarrow A' \qquad A \Rightarrow A''}{A \Rightarrow A' \wedge A''}$$

Mutatis mutandis on the right with the formula $A' \vee A''$. This concludes the proof of Lemma 88.

This procedure will again not work for LK-Quarc$_2$. Coming back to Example 87, we can see that the endsequent would be (underivable) $\top, \top \Rightarrow bS$.

As noted at the beginning of this section, it is left for future work to determine whether these procedures give an exhaustive method of eliminating τ, or if indeed one exists at all.

4.8.2 Quantification

A lot of the same procedures from the previous section will apply here as well. As an illustration, let us examine them for the case of R$\forall$. Remember that by inductive hypothesis, there is a C such that

1. $aM, \Gamma_1 \Rightarrow \Delta_1, A\,[a/\forall M]\,, C$
2. $C, \Gamma_2 \Rightarrow \Delta_2$
3. $V(C) \subseteq V(aM, \Gamma_1, \Delta_1, A\,[a/\forall M]) \cap V(\Gamma_2, \Delta_2)$

Whenever C does not contain a, the problem of eliminating τ does not arise, so assume it does. We can again quantify over $C\,[a]$ whenever $M \in V(\Gamma_1, \Delta_1, A\,[\forall M]) \cap V(\Gamma_2, \Delta_2)$ (which holds just in case $M \in V(\Gamma_2, \Delta_2)$):

Proof. (a)

$$\dfrac{\dfrac{\dfrac{aM \Rightarrow aM}{aM, \Gamma_1 \Rightarrow \Delta_1, A\,[a/\forall M]\,, aM} \qquad \dfrac{\dfrac{\dfrac{aM \Rightarrow aM}{aM, \Gamma_1 \Rightarrow \Delta_1, A\,[a/\forall M]\,, aM} \quad aM, \Gamma_1 \Rightarrow \Delta_1, A\,[a/\forall M]\,, C\,[a]}{aM, \Gamma_1 \Rightarrow \Delta_1, A\,[a/\forall M]\,, aM \wedge C\,[a]}}{aM, \Gamma_1 \Rightarrow \Delta_1, A\,[a/\forall M]\,, a_\alpha M \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]}}{aM, \Gamma_1 \Rightarrow \Delta_1, A\,[a/\forall M]\,, \exists M_\alpha M \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]}}{\Gamma_1 \Rightarrow \Delta_1, A\,[\forall M]\,, \exists M_\alpha M \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]}$$

Proof. (b)

$$\dfrac{\dfrac{\dfrac{\dfrac{C\,[a]\,, \Gamma_2 \Rightarrow \Delta_2}{aM \wedge C\,[a]\,, \Gamma_2 \Rightarrow \Delta_2}}{a_\alpha M \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]\,, \Gamma_2 \Rightarrow \Delta_2}}{aM, a_\alpha M \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]\,, \Gamma_2 \Rightarrow \Delta_2}}{\exists M_\alpha M \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]\,, \Gamma_2 \Rightarrow \Delta_2}$$

Proof. (c) Since $V(aM, \Gamma_1, \Delta_1, A\,[a/\forall M]) = V(\Gamma_1, \Delta_1, A\,[\forall M])$, $M \in V(\Gamma_2, \Delta_2)$, it follows that $V(\exists M_\alpha M \wedge C\,[\alpha/a_1 \ldots \alpha/a_n]) \subseteq V(\Gamma_1, \Delta_1, A\,[\forall M]) \cap V(\Gamma_2, \Delta_2)$.

Furthermore, whenever $C\,[a]$ is connected to neither aM nor $A\,[a/\forall M]$, by Lemma 84 either

(i) $\Gamma_1' \Rightarrow \Delta_1', C\,[a]$ or (ii) $aM, \Gamma_1'' \Rightarrow \Delta_1'', A\,[a/\forall M]$

If (i), then by re-designation and if (ii), then

$$\dfrac{\Gamma_1' \Rightarrow \Delta_1', C\,[b/a]}{\Gamma_1 \Rightarrow \Delta_1, A\,[\forall M]\,, C\,[b/a]} \qquad \dfrac{\dfrac{\dfrac{aM, \Gamma_1'' \Rightarrow \Delta_1'', A\,[a/\forall M]}{\Gamma_1'' \Rightarrow \Delta_1'', A\,[\forall M]}}{\Gamma_1'' \Rightarrow \Delta_1'', A\,[\forall M]\,, C\,[b/a]}}{\Gamma_1 \Rightarrow \Delta_1, A\,[\forall M]\,, C\,[b/a]}$$

In both cases, by re-designation it also holds that $C\,[b/a]\,, \Gamma_2 \Rightarrow \Delta_2$, and moreover it holds that $V(C\,[a]) = V(C\,[b/a])$, and therefore this reduces to the case where C does not contain a.

Another point of similarity with instantiation is that whenever there are some $\Gamma_1' \subseteq \Gamma_1$ and some $\Delta_1' \subseteq \Delta_1$ ($\Gamma_2' \subseteq \Gamma_2$ and $\Delta_2' \subseteq \Delta_2$), such that $\Gamma_1' \Rightarrow \Delta_1'$ ($\Gamma_2' \Rightarrow \Delta_2'$) then $C[a]$ is replaceable by $\perp$ ($\top$).

A separate method for eliminating a in $C[a]$ occurs when for some unary predicate S it holds that $C[a] \Rightarrow aS$. In the derivation of this sequent, aS is either introduced via weakening and therefore not connected to anything, in which case $C[a]$ can be replaced with $\perp$, or $S \in V(C[a])$. In the latter case it holds that $aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], aS$ and we can once again quantify over $C[a]$:

Proof. (a)

$$\dfrac{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], aS \qquad \dfrac{\dfrac{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], aS \qquad aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], C[a]}{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], aS \wedge C[a]}}{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], a_\alpha S \wedge C[\alpha/a_1 \dots \alpha/a_n]}}{aM, \Gamma_1 \Rightarrow \Delta_1, A[a/\forall M], \exists S_\alpha S \wedge C[\alpha/a_1 \dots \alpha/a_n]}$$

Proof. (b)

$$\dfrac{\dfrac{\dfrac{\dfrac{C[a], \Gamma_2 \Rightarrow \Delta_2}{aS \wedge C[a], \Gamma_2 \Rightarrow \Delta_2}}{a_\alpha S \wedge C[\alpha/a_1 \dots \alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}}{aS, a_\alpha S \wedge C[\alpha/a_1 \dots \alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}}{\exists S_\alpha S \wedge C[\alpha/a_1 \dots \alpha/a_n], \Gamma_2 \Rightarrow \Delta_2}$$

Proof. (c) Since $S \in V(C[a])$, it follows that $V(C[a]) = V(\exists S_\alpha S \wedge C[\alpha/a_1 \dots \alpha/a_n])$ and therefore $V(\exists S_\alpha S \wedge C[\alpha/a_1 \dots \alpha/a_n]) \subseteq V(\Gamma_1, \Delta_1, A[\forall M]) \cap V(\Gamma_2, \Delta_2)$.

Finally, the inductive hypotheses can undergo the transformations in Definition 85 and Lemma 88. As above, checking whether these are exhaustive and whether τ can be fully eliminated is left for future work.

Chapter 5

Modal Expansion of LK-Quarc

5.1 Introduction

In this chapter we take a closer look at the final section of [Ben-Yami, 2014], in which Ben-Yami touches on the possibility of the modal extension of Quarc. We will develop a corresponding system by transforming LK-Quarc into a labeled sequent calculus, and then by expanding it with the rules for modal operators. We will follow the outline presented in [Negri and von Plato, 2011]. The relational rules are retained in full, and the rules for operators are simplified since at this point we are not interested in invertibility.

Since this is an expansion of LK-Quarc, the resulting system will be a quantified modal sequent calculus. Notice that unlike [Negri and von Plato, 2011], we do not need to add a condition of a constant being in the domain of a label, since Quarc has a condition of a unary predicate holding of a singular argument. Therefore, the modal expansion of a quantified system presented here is more straightforward.

We now proceed with the expansion, after which we demonstrate cut elimination theorem and its associated properties. Since the proofs of those for LK-Quarc are all by induction on rules, we will only demonstrate them for the new rules added to that system.

5.2 M-Quarc

We first transform Quarc into M-Quarc. This modal system is not quite identical to the one presented in [Ben-Yami, 2014], but it is easy to see that anything that holds in the latter will likewise hold sin the former.

5.2.1 Language of M-Quarc

To obtain the language of M-Quarc, we expand the language of Quarc with:

1. Labels: $x, y, z, \ldots,$

2. Assignment set function symbol, σ,
3. Element function symbol, $\in$,
4. Modal operators: $\Box$, $\Diamond$, which function both as sentential and predication operators.

5.2.2 Formula of M-Quarc

We next expand the definition of a formula:

1. If A is a (basic) formula of Quarc and x a label, then $x : A$ is a (basic) formula of M-Quarc.
2. If $x : (t_1 \dots t_n) \sim S$ is a formula of M-Quarc, where S is an n-ary predicate or a reordered n-ary predicate, $t_1 \dots t_n$ are SA's and $\sim$ a possibly empty string of operators which function both as sentential and predication operators, then $x : (t_1 \dots t_n)\Box \sim S$, $x : (t_1 \dots t_n)\Diamond \sim S$ and $x : (t_1 \dots t_n)\neg \sim S$ are formulas of M-Quarc.
3. If A is a formula of M-Quarc, then $(\Box A)$ and $(\Diamond A)$ are formulas of M-Quarc.
4. All other formula formation rules apply normally.
5. If x, y are labels, then $y \in \sigma(x)$ is a *relational formula* of M-Quarc. Formula formation rules do not apply to relational formulas, and therefore all relational formulas are basic.

5.2.3 Value Assignments

1. For any formula A of Quarc, the formula $x : A$ of M-Quarc is assigned the same value as A.
2. Every relational formula is assigned either $\top$ or $\perp$.
3. The formula $x : \Box A$ ($x : \Diamond A$) is assigned $\top$ if for every (some) y, such that $y \in \sigma(x)$, $y : A$ is assigned $\top$, and $\perp$ otherwise.
4. The formula $x : (t_1 \dots t_n) \sim S$ is assigned the same value as $x :\sim (t_1 \dots t_n)S$.

5.3 LK-Quarc$_K$

We now transform LK-Quarc into LK-Quarc$_K$. Every formula A in all the rules of LK-Quarc is replaced by $x : A$. Moreover, relational formulas can be non-parametric in structural rules and parametric in all rules of LK-Quarc. Furthermore, the rules LNP and RNP are replaced by rules LPS and RSP, respectively:

$$\frac{x :\sim (t_1, \dots, t_n)P, \Gamma \Rightarrow \Delta}{x : (t_1, \dots, t_n) \sim P, \Gamma \Rightarrow \Delta}\,\mathrm{L}PS \qquad \frac{\Gamma \Rightarrow \Delta, x :\sim (t_1, \dots, t_n)P}{\Gamma \Rightarrow \Delta, x : (t_1, \dots, t_n) \sim P}\,\mathrm{R}SP$$

where $\sim$ is any string of operators which can function both as sentential and predication operators (i.e. $\neg$, $\Box$ or $\Diamond$). We now add the modal rules:

5.3.1 Modal

$$\frac{y : A, y \in \sigma(x), \Gamma \Rightarrow \Delta}{x : \Box A, y \in \sigma(x), \Gamma \Rightarrow \Delta}\text{L}\Box \qquad \frac{y \in \sigma(x), \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\text{R}\Box^*$$

$$\frac{y : A, y \in \sigma(x), \Gamma \Rightarrow \Delta}{x : \Diamond A, \Gamma \Rightarrow \Delta}\text{L}\Diamond^* \qquad \frac{y \in \sigma(x), \Gamma \Rightarrow \Delta, y : A}{y \in \sigma(x), \Gamma \Rightarrow \Delta, x : \Diamond A}\text{R}\Diamond$$

* - where y does not occur in the lower sequent.

5.3.2 Re-designating the Labels

First, we show that everything derivable in LK-Quarc will still (*mutatis mutandis*) be derivable in LK-Quarc$_K$:

Lemma 89 If $\Gamma \Rightarrow \Delta$ is derivable in LK-Quarc, then for any label x it holds that $\Gamma' \Rightarrow \Delta'$ is derivable in LK-Quarc$_K$, where Γ' and Δ' are the result of substituting every A in Γ and Δ, respectively, by $x : A$.

Proof. Simple. By substituting every occurrence of every formula A in the derivation of $\Gamma \Rightarrow \Delta$ by $x : A$.

We now define the re-designation procedure:

Definition 90 Call the label y appearing in the definition of the rules R$\Box$ and L$\Diamond$ the *proper* label of the respective rules. To re-designate the labels, we start with the topmost occurrence of one of these rules (going left to right) to which this procedure has not been applied. We replace every occurrence of their proper label in all the sequents above the lower sequent of the rule with a label that has so far not appeared in the derivation, and we continue doing so until all the instances of these rules have been treated in this manner.

We prove an auxiliary lemma:

Lemma 91 If **Inf** is an initial sequent or a correct inference which contains label y which is not the proper label of **Inf**, and if z is likewise not a proper label of **Inf**, then **Inf′**, obtained by uniformly substituting z for y is an initial sequent or a correct inference.

Proof. By induction on the rules of LK-Quarc$_K$.

Basic step. Follows from Lemma 89.

Inductive step. Easy for most cases. We will illustrate on the example of L$\Box$:

If

$$\frac{y : A, y \in \sigma(x), \Gamma \Rightarrow \Delta}{x : \Box A, y \in \sigma(x), \Gamma \Rightarrow \Delta}$$

is a correct inference, then so is

$$\frac{z : A, z \in \sigma(x), \Gamma \Rightarrow \Delta}{x : \Box A, z \in \sigma(x), \Gamma \Rightarrow \Delta}$$

And similarly for other rules.

Lemma 92 If we re-designate the proper labels of a derivation, it will yield a correct (beginning with permissible initial sequents and consisting only of correct inferences) derivation of the same endsequent. Moreover, the derivation will be of the same grade (defined below) and rank.

Proof. That the derivation is of a same endsequent is obvious form Definition 90. That the derivation begins with permissible initial sequents follows from Lemma 89. That every inference, except R$\Box$ and L$\Diamond$ is correct follows from Lemma 91, and replacing the proper label of a correct application of R$\Box$ and L$\Diamond$ with one that has not appeared in the derivation so far will likewise produce a correct inference. Finally, that the derivation will be of the same grade and rank is clear from Definition 90.

We can extend axiom generalization to LK-Quarc$_K$ by extending the proof for the cases of the modal operators:

Lemma 93 If the terminal symbol[1] of $x : B$ is $\Box$ or $\Diamond$, $x : B \Rightarrow x : B$ is derivable in LK-Quarc$_K$.

Proof. Consider the $\Box$ first, and let B be $\Box A$. If $x : A \Rightarrow x : A$ is derivable in LK-Quarc$_K$, then $x : \Box A \Rightarrow x : \Box A$ is as well.

If $x : A \Rightarrow x : A$ is derivable, then, by Lemma 91, so is $y : A \Rightarrow y : A$, for some y which does not appear in the derivation of $x : A \Rightarrow x : A$. Then,

$$\cfrac{\cfrac{\cfrac{y : A \Rightarrow y : A}{y \in \sigma(x), y : A \Rightarrow y : A}\ \mathrm{L}W}{y \in \sigma(x), x : \Box A \Rightarrow y : A}\ \mathrm{L}\Box}{x : \Box A \Rightarrow x : \Box A}\ \mathrm{R}\Box$$

Similarly for the case of $\Diamond$. Cases for LPS and RSP will follow from cases for $\neg$, $\Box$ and $\Diamond$ according to their order in $\sim$.

Observation 94 Necessitation is admissible in LK-Quarc$_K$ (keeping in mind that, by Lemma 91, if $\Rightarrow x : A$ is derivable, so is $\Rightarrow y : A$ for some y as above)

$$\cfrac{\cfrac{\Rightarrow y : A}{y \in \sigma(x) \Rightarrow y : A}\ \mathrm{L}W}{\Rightarrow x : \Box A}\ \mathrm{R}\Box$$

[1] As previously, the symbol introduced, for any formula, by the last application of a formula-generation rule is called a *terminal symbol* of that formula.

Observation 95 These rules allow us to derive the K axiom:
$\Rightarrow x : \Box(P \to Q) \to (\Box P \to \Box Q)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : Q \Rightarrow y : Q}{y : Q, y : P \Rightarrow y : Q}\text{ L}W\text{, L}P \qquad \dfrac{y : P \Rightarrow y : P}{y : P \Rightarrow y : Q, y : P}\text{ R}W\text{, R}P}{y : P \to Q, y : P \Rightarrow y : Q}\text{ L}\to}{y \in \sigma(x), y : P \to Q, y : P \Rightarrow y : Q}\text{ L}W}{y \in \sigma(x), x : \Box(P \to Q), y : P \Rightarrow y : Q}\text{ L}\Box}{y \in \sigma(x), x : \Box(P \to Q), x : \Box P \Rightarrow y : Q}\text{ L}\Box}{x : \Box(P \to Q), x : \Box P \Rightarrow x : \Box Q}\text{ R}\Box}{x : \Box(P \to Q) \Rightarrow x : \Box P \to \Box Q}\text{ R}\to}{\Rightarrow x : \Box(P \to Q) \to (\Box P \to \Box Q)}\text{ R}\to$$

5.3.3 LK-Quarc$_T$

To obtain LK-Quarc$_T$, we expand LK-Quarc$_K$ with the following rule:

$$\dfrac{x \in \sigma(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\text{ Ref}$$

Observation 96 Now we can derive the T axiom:
$\Rightarrow x : \Box P \to P$

$$\dfrac{\dfrac{\dfrac{\dfrac{x : P \Rightarrow x : P}{x \in \sigma(x), x : P \Rightarrow x : P}\text{ L}W}{x \in \sigma(x), x : \Box P \Rightarrow x : P}\text{ L}\Box}{x : \Box P \Rightarrow x : P}\text{ Ref}}{\Rightarrow x : \Box P \to P}\text{ R}\to$$

5.3.4 LK-Quarc$_B$

To obtain LK-Quarc$_B$, we expand LK-Quarc$_K$ with the following rule:

$$\dfrac{x \in \sigma(y), y \in \sigma(x), \Gamma \Rightarrow \Delta}{y \in \sigma(x), \Gamma \Rightarrow \Delta}\text{ Sym}$$

Observation 97 Now we can derive the B axiom:
$\Rightarrow x : P \to \Box\Diamond P$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x : P \Rightarrow x : P}{x \in \sigma(y), y \in \sigma(x), x : P \Rightarrow x : P}\text{ several L}W}{x \in \sigma(y), y \in \sigma(x), x : P \Rightarrow y : \Diamond P}\text{ R}\Diamond}{y \in \sigma(x), x : P \Rightarrow y : \Diamond P}\text{ Sym}}{x : P \Rightarrow x : \Box\Diamond P}\text{ R}\Box}{\Rightarrow x : P \to \Box\Diamond P}\text{ R}\to$$

5.3.5 LK-Quarc$_4$

To obtain LK-Quarc$_4$, we expand LK-Quarc$_K$ with the following rule:

$$\frac{z \in \sigma(x), y \in \sigma(x), z \in \sigma(y), \Gamma \Rightarrow \Delta}{y \in \sigma(x), z \in \sigma(y), \Gamma \Rightarrow \Delta}\ \text{Trans}$$

Observation 98 Now we can derive the 4 axiom:
$\Rightarrow x : \Box P \to \Box\Box P$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : P \Rightarrow z : P}{z \in \sigma(x), z \in \sigma(y), y \in \sigma(x), z : P \Rightarrow z : P}\ \text{several L}W}{z \in \sigma(x), z \in \sigma(y), y \in \sigma(x), x : \Box P \Rightarrow z : P}\ \text{L}\Box}{z \in \sigma(y), y \in \sigma(x), x : \Box P \Rightarrow z : P}\ \text{Trans}}{y \in \sigma(x), x : \Box P \Rightarrow y : \Box P}\ \text{R}\Box}{x : \Box P \Rightarrow x : \Box\Box P}\ \text{R}\Box}{\Rightarrow x : \Box P \to \Box\Box P}\ \text{R}\to$$

5.3.6 LK-Quarc$_{S4}$

To obtain LK-Quarc$_{S4}$, we expand LK-Quarc$_K$ with rules **Ref** and **Trans**. Obviously, here both T and 4 axioms will hold.

5.3.7 LK-Quarc$_{S5}$

Finally, we obtain LK-Quarc$_{S5}$ by adding to LK-Quarc$_K$ the rules **Ref**, **Sym** and **Trans**.

Observation 99 Now we can derive the 5 axiom:
$\Rightarrow x : \Diamond P \to \Box\Diamond P$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : P \Rightarrow y : P}{y \in \sigma(z), y : P \Rightarrow y : P}\ \text{L}W}{y \in \sigma(z), y : P \Rightarrow z : \Diamond P}\ \text{R}\Diamond}{y \in \sigma(z), x \in \sigma(z), y \in \sigma(x), y : P \Rightarrow z : \Diamond P}\ \text{several L}W\text{, L}P}{x \in \sigma(z), y \in \sigma(x), y : P \Rightarrow z : \Diamond P}\ \text{Trans}}{x \in \sigma(z), z \in \sigma(x), y \in \sigma(x), y : P \Rightarrow z : \Diamond P}\ \text{L}W\text{, L}P}{z \in \sigma(x), y \in \sigma(x), y : P \Rightarrow z : \Diamond P}\ \text{Sym}}{y \in \sigma(x), y : P \Rightarrow x : \Box\Diamond P}\ \text{R}\Box}{x : \Diamond P \Rightarrow x : \Box\Diamond P}\ \text{L}\Diamond}{\Rightarrow x : \Diamond P \to \Box\Diamond P}\ \text{R}\to$$

And obviously, the T axiom will hold as well.

5.4 Cut Elimination

In this section we prove that the Cut elimination theorem holds for LK-Quarc$_K$ (and later for its expansions). The proof is an expansion of that for LK-Quarc and therefore still a modification of the proof in [Szabo, 1969]. We first define the grade of modal formulas (the definition of the rank, ρ, stays the same):

Definition 100 Grade of formula $x : A$ in LK-Quarc$_K$ is the same as the grade of formula A in LK-Quarc. Moreover,

- $\gamma(A) = 0$ if A is a relational formula,
- $\gamma(A) = \gamma(B) + 1$ if A is $\Box B$ or $\Diamond B$,
- $\gamma(A) = \gamma(\sim (t_1 \dots t_n)P)$ if A is $(t_1 \dots t_n) \sim P$.

Theorem 101 For any sequent S, if S is derivable in LK-Quarc$_K$, then it is derivable in LK-Quarc$_K$ without using the cut rule.

Proof. By expanding the proof of the Cut elimination theorem for LK-Quarc with the cases for rules of LK-Quarc$_K$. We start with the case of $\rho = 2$.

5.4.1 Case $\rho = 2$

Relational formulas are handled the same way as other formulas of grade 0. We now move on to the formulas in which the terminal symbols are modal operators.

1. We first tackle the expanded rules RSP and LPS:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta, x :\sim (t_1 \dots t_n)P}{\Gamma \Rightarrow \Theta, x : (t_1 \dots t_n) \sim P} \quad \dfrac{x :\sim (t_1 \dots t_n)P, \Pi \Rightarrow \Delta}{x : (t_1 \dots t_n) \sim P, \Pi \Rightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\text{Mix}$$

   This is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta, x :\sim (t_1 \dots t_n)P \qquad x :\sim (t_1 \dots t_n)P, \Pi \Rightarrow \Delta}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}\text{Mix}}{\Gamma, \Pi \Rightarrow \Theta, \Delta}$$

   This mix formula can be eliminated according to the procedure for the modal operators below and previously established procedure for negation, according to their order in $\sim$.

2. We now consider the rules for $\Box$:

$$\dfrac{\dfrac{y \in \sigma(x), \Gamma \Rightarrow \Theta, y : A}{\Gamma \Rightarrow \Theta, x : \Box A} \quad \dfrac{y : A, y \in \sigma(x), \Pi \Rightarrow \Delta}{x : \Box A, y \in \sigma(x), \Pi \Rightarrow \Delta}}{y \in \sigma(x), \Gamma, \Pi \Rightarrow \Theta, \Delta}\text{Mix}$$

   This is transformed into:

$$\dfrac{\dfrac{y \in \sigma(x), \Gamma \Rightarrow \Theta, y : A \qquad y : A, y \in \sigma(x), \Pi \Rightarrow \Delta}{y \in \sigma(x), \Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}\text{Mix}}{y \in \sigma(x), \Gamma, \Pi \Rightarrow \Theta, \Delta}$$

3. And finally we consider the rules for $\Diamond$:

$$\dfrac{\dfrac{y \in \sigma(x), \Gamma \Rightarrow \Theta, y : A}{y \in \sigma(x), \Gamma \Rightarrow \Theta, x : \Diamond A} \quad \dfrac{y : A, y \in \sigma(x), \Pi \Rightarrow \Delta}{x : \Diamond A, \Pi \Rightarrow \Delta}}{y \in \sigma(x), \Gamma, \Pi \Rightarrow \Theta, \Delta}\text{Mix}$$

This is transformed into:

$$\dfrac{\dfrac{y \in \sigma(x), \Gamma \Rightarrow \Theta, y : A \qquad y : A, y \in \sigma(x), \Pi \Rightarrow \Delta}{y \in \sigma(x), \Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}\ \text{Mix}}{y \in \sigma(x), \Gamma, \Pi \Rightarrow \Theta, \Delta}$$

Both new mix formulas are of lesser grade and can by inductive hypothesis be eliminated.

5.4.2 Case $\rho > 2$

Suppose left rank is 1 and right rank is greater than 1. We will examine situations when the last rule on the right is each of our new rules.

1. Suppose first the last rule on the right is L$\Box$. The application of the mix rule then has the following form:

$$\dfrac{\Gamma \Rightarrow \Theta \qquad \dfrac{y : A, y \in \sigma(x), \Pi \Rightarrow \Delta}{x : \Box A, y \in \sigma(x), \Pi \Rightarrow \Delta}}{\Gamma, x : \Box A, y \in \sigma(x), \Pi^* \Rightarrow \Theta^*, \Delta}\ \text{Mix}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta \qquad y : A, y \in \sigma(x), \Pi \Rightarrow \Delta}{\Gamma, y : A, y \in \sigma(x), \Pi^* \Rightarrow \Theta^*, \Delta}\ \text{Mix}}{\Gamma, x : \Box A, y \in \sigma(x), \Pi^* \Rightarrow \Theta^*, \Delta}$$

2. Now suppose the last rule on the right is R$\Box$. The application of the mix rule then has the following form:

$$\dfrac{\Gamma \Rightarrow \Theta \qquad \dfrac{y \in \sigma(x), \Pi \Rightarrow \Delta, y : A}{\Pi \Rightarrow \Delta, x : \Box A}}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta, x : \Box A}\ \text{Mix}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta \qquad z \in \sigma(x), \Pi \Rightarrow \Delta, z : A}{\Gamma, z \in \sigma(x), \Pi^* \Rightarrow \Theta^*, \Delta, z : A}\ \text{Mix}}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta, x : \Box A}$$

The change from the sequent $y \in \sigma(x), \Pi \Rightarrow \Delta, y : A$ to the sequent $z \in \sigma(x), \Pi \Rightarrow \Delta, z : A$ is justified by Lemma 92.

3. Suppose the last rule on the right is L$\Diamond$. The application of the mix rule then has the following form:

$$\dfrac{\Gamma \Rightarrow \Theta \qquad \dfrac{y : A, y \in \sigma(x), \Pi \Rightarrow \Delta}{x : \Diamond A, \Pi \Rightarrow \Delta}}{\Gamma, x : \Diamond A, \Pi^* \Rightarrow \Theta^*, \Delta}\ \text{Mix}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta \qquad z : A, z \in \sigma(x), \Pi \Rightarrow \Delta}{z : A, z \in \sigma(x), \Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}\text{Mix}}{\Gamma, x : \Diamond A, \Pi^* \Rightarrow \Theta^*, \Delta}$$

As above, the change from the sequent $y \in \sigma(x), \Pi \Rightarrow \Delta, y : A$ to the sequent $z \in \sigma(x), \Pi \Rightarrow \Delta, z : A$ is justified by Lemma 92.

4. Suppose the last rule on the right is R$\Diamond$. The application of the mix rule then has the following form:

$$\dfrac{\Gamma \Rightarrow \Theta \qquad \dfrac{y \in \sigma(x), \Pi \Rightarrow \Delta, y : A}{y \in \sigma(x), \Pi \Rightarrow \Delta, x : \Diamond A}}{\Gamma, y \in \sigma(x), \Pi^* \Rightarrow \Theta^*, \Delta, x : \Diamond A}\text{Mix}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta \qquad y \in \sigma(x), \Pi \Rightarrow \Delta, y : A}{\Gamma, y \in \sigma(x), \Pi^* \Rightarrow \Theta^*, \Delta, y : A}\text{Mix}}{\Gamma, y \in \sigma(x), \Pi^* \Rightarrow \Theta^*, \Delta, x : \Diamond A}$$

5. Finally, suppose the last rule on the right is LPS. The application of the mix rule then has the following form:

$$\dfrac{\Gamma \Rightarrow \Theta \qquad \dfrac{x :\sim (t_1, \ldots, t_n)P, \Pi \Rightarrow \Delta,}{x : (t_1, \ldots, t_n) \sim P, \Pi \Rightarrow \Delta}}{\Gamma, x : (t_1, \ldots, t_n) \sim P, \Pi^* \Rightarrow \Theta^*, \Delta}\text{Mix}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta \qquad x :\sim (t_1, \ldots, t_n)P, \Pi \Rightarrow \Delta,}{\Gamma, x :\sim (t_1, \ldots, t_n)P, \Pi^* \Rightarrow \Theta^*, \Delta}\text{Mix}}{\Gamma, x : (t_1, \ldots, t_n) \sim P, \Pi^* \Rightarrow \Theta^*, \Delta}$$

And *mutatis mutandis* for RSP.

In all the cases, the rank of the mix formula is reduced by 1 and can therefore by inductive hypothesis be eliminated. Similarly for left rank, and both left and right rank, being higher than 1. This concludes the proof of Theorem 101.

5.4.3 Subformula Property

Definition 102 The definition of a subformula stays the same as in LK-Quarc, apart from the fact we need to substitute every formula A with $x : A$, and add the following clauses:

6. $x :\sim (t_1, \ldots, t_n)P$ is a subformula of $x : (t_1, \ldots, t_n) \sim P$.

7. $y : A$ is a subformula of $x : \Box A$ and $x : \Diamond A$ for any y.

We want to show that subformula property holds of LK-Quarc$_K$:

Theorem 103 Any formula appearing in any cut-free derivation of LK-Quarc$_K$ is either a subformula of some formula in its endsequent or a basic formula.

Proof. We extend the proof for LK-Quarc with the new rules of LK-Quarc$_K$. Let us examine the example of rules for $\Box$:

$$\frac{y : A, y \in \sigma(x), \Gamma \Rightarrow \Delta}{x : \Box A, y \in \sigma(x), \Gamma \Rightarrow \Delta}\ \mathrm{L}\Box \qquad \frac{y \in \sigma(x), \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\ \mathrm{R}\Box^*$$

We can see that any formula in Γ and Δ will be subformula of some formula in the lower sequent, namely itself. Furthermore, $y : A$ is a subformula of $x : \Box A$. Finally, $y \in \sigma(x)$ is basic. Similarly for other rules.

Corollary 104 LK-Quarc$_K$ is consistent.

Proof. From Theorem 103, by noting no rule removes basic formulas on the right.

5.4.4 Cut Elimination for Ref, Sym and Trans

With all three rules, all the formulas that appear in the lower sequent appear in the upper sequent as well, so we only need to extend the cut elimination procedure for the case where $\rho > 2$. As before, assume left rank is 1 and right rank is greater than 1.

1. The Rule **Ref**

   Suppose the last rule on the right is **Ref**. The application of the mix rule then has the following form:

$$\frac{\Gamma \Rightarrow \Theta \qquad \dfrac{x \in \sigma(x), \Pi \Rightarrow \Delta}{\Pi \Rightarrow \Delta}}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}$$

   This is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta \qquad x \in \sigma(x), \Pi \Rightarrow \Delta}{\Gamma, x \in \sigma(x), \Pi^* \Rightarrow \Theta^*, \Delta}}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}$$

   Or, if $x \in \sigma(x)$ is the mix formula, into:

$$\frac{\Gamma \Rightarrow \Theta \qquad x \in \sigma(x), \Pi \Rightarrow \Delta}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}$$

   In either case, the rank is reduced by 1 and by inductive hypothesis can be eliminated. Note also that only basic formulas are eliminated via this rule.

2. The Rule **Sym**

   The case we need to examine is when $y \in \sigma(x)$ is the mix formula. The application of the mix rule then has the following form:

$$\dfrac{\Gamma \Rightarrow \Theta \qquad \dfrac{x \in \sigma(y), y \in \sigma(x), \Pi \Rightarrow \Delta}{y \in \sigma(x), \Pi \Rightarrow \Delta}}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}$$

Suppose $y \in \sigma(x)$ is in Γ. Then the derivation is transformed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{x \in \sigma(y), y \in \sigma(x), \Pi \Rightarrow \Delta}{y \in \sigma(x), \Pi \Rightarrow \Delta}\text{Sym}}{\Gamma, y \in \sigma(x), \Pi \Rightarrow \Delta}\text{some L}W}{\Gamma, \Pi^* \Rightarrow \Delta}\text{some L}C}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}\text{some R}W\text{, R}P$$

Now suppose $y \in \sigma(x)$ is not in Γ. Then, since left rank is 1, $\Gamma \Rightarrow \Theta$ is derived by RW from $\Gamma \Rightarrow \Theta^*$, and the derivation is transformed as follows:

$$\dfrac{\Gamma \Rightarrow \Theta^*}{\Gamma, \Pi^* \Rightarrow \Theta^*, \Delta}$$

In either case, the mix rule is eliminated. Note again that only basic formulas are eliminated via this rule.

3. The Rule **Trans**

   The cases for this rule are the same, *mutatis mutandis*, as the previous rule, for both $y \in \sigma(x)$ and $z \in \sigma(y)$ the principal formula. Again, only basic formulas are eliminated.

This establishes:

Theorem 105 For any sequent S, if it is derivable in LK-Quarc$_T$, LK-Quarc$_B$, LK-Quarc$_4$, LK-Quarc$_{S4}$ or LK-Quarc$_{S5}$, it is derivable in its respective system without using the cut rule.

and (given all the rules only remove basic formulas)

Theorem 106 Any formula appearing in any cut-free derivation of LK-Quarc$_T$, LK-Quarc$_B$, LK-Quarc$_4$, LK-Quarc$_{S4}$ or LK-Quarc$_{S5}$ is a subformula of some formula in its endsequent or a basic formula.

Again, this leads to the corollary (since no rule removes the basic formulas on the right):

Corollary 107 LK-Quarc$_T$, LK-Quarc$_B$, LK-Quarc$_4$, LK-Quarc$_{S4}$ and LK-Quarc$_{S5}$ are consistent.

5.4.5 From Modal Quarc to LK-Quarc$_K$ and Others

For our present purposes we do not need to show full deductive equivalence, but simply that for any line of a proof in Modal Quarc there is a derivation with an endsequent corresponding to that line. Note that Modal Quarc presented in [Ben-Yami, 2014] is not the same as M-Quarc presented here, but we only need to show the result for the former anyway. We first expand the definition of standard translation of a line of a proof:

Definition 108 Standard translation of a line $\langle L, i, A, R\rangle$ of Modal Quarc (on any variety) is a sequent $L \Rightarrow x : A$ of the corresponding variety of LK-Quarc.

We want to show that

Lemma 109 For any line (i) of any proof in any version of Modal Quarc there exists a corresponding sequent in the respective version of LK-Quarc, such that it is derivable from trivial lemmas and sequents corresponding to lines (i) relies on.

Proof. Note that Modal Quarc in [Ben-Yami, 2014] consists of rule of necessitation, which is admissible in any of LK-Quarc$_K$ to LK-Quarc$_{S5}$, and a rule introducing the characteristic axiom with no premises, which is derivable for its respective system, as noted in Observations 95 - 99. For any other rule the deductive equivalence has already been established (exchanging the formula A for $x : A$). This concludes the proof of Lemma 109.

It follows that

Corollary 110 Modal Quarc, on any of its versions, is consistent.

Proof. From Lemma 109 and Corollary 104 and 107.

Chapter 6

On a Mismatch between Aristotle's Modal Syllogistic and Modern Modal Logic

6.1 Introduction

The dominant opinion on Aristotle's modal syllogistic is perhaps best expressed by Jan Łukasiewicz, who was also instrumental in shaping it:

> There are two reasons why Aristotle's modal logic is so little known. The first is due to the author himself: in contrast to the assertoric syllogistic which is perfectly clear and nearly free of errors, Aristotle's modal syllogistic is almost incomprehensible because of its many faults and inconsistencies. [Łukasiewicz, 1957]

The most prominent defender of Aristotle today is Marko Malink[1]. In addition to other features, and crucially for the proceeding, Malink presents his work in a symbolism useful for understanding the subject matter, especially for someone who is not an Aristotle or ancient scholar. Therefore, when approaching Aristotle in this chapter, we will utilize the assistance provided by Malink.

In his [Malink, 2006] and [Malink, 2013], Malink tries to offer a single consistent model for Aristotle's modal syllogistic. The goal of this chapter is to approach Aristotle and Malink's results from a perspective of a modern modal quantified calculus. The goal here is not to determine what Aristotle's opinion on various modal syllogisms was, but rather, given the interpretations by Malink, himself following other Aristotle scholars, to evaluate the conditions of their validity and supply some possible formal reconstructions of Aristotle's modal logic, given current views of modality. The guiding criterion in doing so will be to avoid what Robin Smith calls "tinkering" [Smith, 1995], which in this instance we take to be the introduction of principles and procedures for the sole purpose of validating Aristotle. An aim of this chapter is to show that there is a mismatch between modern and Aristotelian systems of modal logic which the

[1] Malink follows in the footsteps of Richard Patterson [Patterson, 1995], and explicitly addresses criticism raised against him by Tad Brennan [Brennan, 2007].

defenders of Aristotle cannot overcome without being charged of tinkering. We focus on one approach from each side, but those will be indicative of the entire discussion.

6.2 The Assertoric Syllogistic

On the modern side of things, we will be using the Quantified Argument Calculus and its modal extension, M-Quarc presented in [Ben-Yami, 2014]. There are several reasons to use Quarc when dealing with Aristotle's syllogistic. First, it validates all the Aristotelian assertoric syllogisms, as well as the relations of the Square of Opposition, in a tinkering-free manner. Most notably, given that a quantified phrase occurs in the argument position, instantial import is introduced to make Quarc a non-free logic.[2] Second, Quarc simplifies the proofs of syllogisms compared to standard Predicate Calculus, as we will shortly see in an example. Finally, it clearly distinguishes what options are available when we incorporate modal logic.

One feature of Quarc we should recall is that it allows for both sentential and predication operators. This is notably a *syntactic* feature, and that point will be important in the following discussion. Let us therefore remember that:

If '$\sim$' stands for a string, possibly empty, of operators which can function both as sentential and predication operators, S and P for unary predicates and 'q' for a quantifier $\{\forall, \exists\}$, then:

Definition 111 $\sim (qS)P$ is a sentence of Quarc, with the string of symbols $\sim$ said to be in *sentential* position,

Definition 112 $(qS) \sim P$ is a sentence of Quarc, with the string of symbols $\sim$ said to be in *predication* position.

The predication position can be seen as representing the copula, where an empty string of operators stands for the positive copula 'is' or 'are', and the negative sentences are obtained by placing a single negation into this position:

Proposition 113 $(\forall S)\neg P$ - No S is P.

Proposition 114 $(\exists S)\neg P$ - Some S are not P.

One can see the notational simplicity of representing the categorical propositions in Quarc. In the following section we will expand this idea by demonstrating how the whole of the assertoric syllogistic is valid in Quarc. We will also briefly contrast it with the Predicate Calculus.

6.2.1 Assertoric Syllogistic in Quarc

In this section, we demonstrate the validity of Aristotle's assertoric syllogistic, as well as some other supplementary results. Given that Quarc is sound [Ben-Yami, 2014] and complete [Pavlovic and Ben-Yami, 2013], we will freely move

[2] Note that since [Ben-Yami, 2014] uses truth-valuational semantics, existential import is there substituted by instantial import. The diferrence between the two will not, however, be crucial for anything in the proceeding.

between the considerations of semantics, truth-value and entailment (the latter symbolized by '$\models$') on the one hand, and syntax, proof and derivability (the latter symbolized by '$\vdash$') on the other. Unless otherwise noted, the entailment and derivability are in Quarc. We aim to show that:

Theorem 115 All of Aristotle's assertoric syllogistic is valid.

Since the proofs of the syllogistic ultimately rely upon the "perfect" syllogisms (Barbara, Celarent, Darii, Ferio), the validity of those will be demonstrated first. In this section we also briefly compare Quarc to the Predicate Calculus. After the perfect syllogisms, we demonstrate the auxiliary theorems needed to prove the remaining ones. We do not offer proof of the entire remainder of the syllogistic, but merely provide some examples.

Perfect Syllogisms

The four perfect syllogisms are the starting proof for the derivation of all valid syllogisms. In the following proofs, let '$\sim$' be a metaoperator standing for either the empty string of operators or the negation operator '$\neg$', which allows us to tackle the proofs two at the time. We can see that this metaoperator, standing in the predication position, basically represents the copula, with the empty string representing 'is', and the negation representing 'is not'.[3] After each of the proofs, we will offer the corresponding proofs in the Predicate Calculus to demonstrate the relative simplicity of using Quarc.

Lemma 116 (Barbara: AAA-1; Celarent: EAE-1):
$(\forall M) \sim P, (\forall S)M \vdash (\forall M) \sim P$

Proof.

1	(1)	$(\forall M) \sim P$	Premise
2	(2)	$(\forall S)M$	Premise
3	(3)	$(a)S$	Premise
2,3	(4)	$(a)M$	UE, 2, 3
1,2,3	(5)	$(a) \sim P$	UE, 1, 4
1,2	(6)	$(\forall M) \sim P$	UI, 3, 5

Compare this proof with the proof of Barbara, using the standard presentation in the Predicate Calculus (obviously, adding the '$\sim$' metaoperator, this would serve as the proof of Celarent as well, but that need not concern us at the moment). One can easily see that the additional syntactic element, the conditional arrow (highlighted), makes even the simplest proof of Barbara more cumbersome. The rest of the proof completely matches the previous one:

Observation 117 (Barbara: AAA-1):
$\forall x(Mx \rightarrow Px), \forall x(Sx \rightarrow Mx) \vdash_{PC} \forall x(Sx \rightarrow Px)$

Proof.

[3] In fact, a related system, presented in [Lanzet and Ben-Yami, 2004], had an explicit copula

1	(1)	$\forall x(Mx \rightarrow Px)$	Premise
2	(2)	$\forall x(Sx \rightarrow Mx)$	Premise
3	(3)	Sa	Premise
2	(4)	$Sa \rightarrow Ma$	$\forall$E: 2
2,3	(5)	Ma	$\rightarrow$E: 3, 4
1	(6)	$Ma \rightarrow Pa$	$\forall$E: 1
1,2,3	(7)	Pa	$\rightarrow$E: 5, 6
1,2	(8)	$Sa \rightarrow Pa$	$\rightarrow$I: 3, 7
1,2	(9)	$\forall x(Sx \rightarrow Px)$	$\forall$I: 8

Next we move to the proofs of the perfect syllogisms containing the particular quantifier, Darii and Ferio:

Lemma 118 (Darii: AII-1; Ferio: EIO-1):
$(\forall M) \sim P, (\exists S)M \vdash (\exists S) \sim P$

Proof.

1	(1)	$(\forall M) \sim P$	Premise
2	(2)	$(\exists S)M$	Premise
3	(3)	$(a)S$	Premise
4	(4)	$(a)M$	Premise
1,4	(5)	$(a) \sim P$	UE, 1, 4
1,3,4	(6)	$(\exists S) \sim P$	PI, 3, 5
1,2	(7)	$(\exists S) \sim P$	Ins, 2, 3, 4, 6

Compare once more with the corresponding proof in the Predicate Calculus. Again, the additional syntactic elements, in this case the conditional and the conjunction, prolong the proof:

Observation 119 (Darii: AII-1):
$\forall x(Mx \rightarrow Px), \exists x(Sx \wedge Mx) \vdash_{PC} \exists x(Sx \wedge Px)$

Proof.

1	(1)	$\forall x(Mx \rightarrow Px)$	Premise
2	(2)	$\exists x(Sx \wedge Mx)$	Premise
3	(3)	$Sa \wedge Ma$	Premise
3	(4)	Ma	$\wedge$E: 3
1	(5)	$Ma \rightarrow Pa$	$\forall$E: 1
1,3	(6)	Pa	$\rightarrow$E: 4, 5
3	(7)	Sa	$\wedge$E: 3
1,3	(8)	$Sa \wedge Pa$	$\wedge$I: 6, 7
1,3	(9)	$\exists x(Sx \wedge Px)$	$\exists$I: 8
1,2	(10)	$\exists x(Sx \wedge Px)$	$\exists$E: 2, 3, 9

Of course, the greatest problem in the Predicate Calculus is not the length of proofs, but the special provisions that need to be made for the existential import. We will see that Quarc does not run into similar problems in the next section, in which we demonstrate the validity of conversions needed to demonstrate the rest of the syllogistic.

Conversions

A conversion is an inference in which the subject and the predicate switch positions. The three demonstrated here are: the conversion of the universal negative ($MeP \Rightarrow PeM$), particular affirmative ($MiP \Rightarrow PiM$), and universal affirmative into the particular affirmative ($MaP \Rightarrow PiM$):

Lemma 120 (Conversion $MeP \Rightarrow PeM$):
$(\forall M)\neg P \vdash (\forall P)\neg M$

Proof.

1	(1)	$(\forall M)\neg P$	Premise
2	(2)	$(a)M$	Premise
3	(3)	$(a)P$	Premise
1,2	(4)	$(a)\neg P$	UE, 1, 2
1,2	(5)	$\neg(a)P$	PS, 4
1,3	(6)	$\neg(a)M$	$\neg$I, 2, 3, 5
1,3	(7)	$(a)\neg M$	SP, 6
1	(8)	$(\forall P)\neg M$	UI, 3, 7

Lemma 121 (Conversion $MiP \Rightarrow PiM$):
$(\exists M)P \vdash (\exists P)M$

Proof.

1	(1)	$(\exists M)P$	Premise
2	(2)	$(a)M$	Premise
3	(3)	$(a)P$	Premise
2,3	(4)	$(\exists P)M$	PI, 2, 3
1	(5)	$(\exists P)M$	Ins, 1, 2, 3, 4

Note that the next conversion differs only in having the universal quantifier where the previous one had a particular. Given that the rule of Instantial Import is defined for either quantifier, it is to be expected that the proof of conversions will likewise be similar:

Lemma 122 (Conversion $MaP \Rightarrow PiM$):
$(\forall M)P \vdash (\exists P)M$

Proof.

1	(1)	$(\forall M)P$	Premise
2	(2)	$(a)M$	Premise
3	(3)	$(a)P$	Premise
2,3	(4)	$(\exists P)M$	PI, 2, 3
1	(5)	$(\exists P)M$	Ins, 1, 2, 3, 4

Obviously, the final conversion would normally be a more controversial one, but not so in Quarc which validates all the entailments of the Square of opposition, as we will see in the next section.

This concludes the proof of Theorem 115. Let us now observe a couple of examples of how the perfect syllogisms and conversions are used to demonstrate the other syllogisms. The first, slightly more complex proof of the third figure syllogism Disamis will just serve as an illustration of this procedure, while the simpler proof of the syllogism Cesare will be useful in the further discussion. Obviously, both proofs are in metalanguage.

Example 123 (Disamis: IAI-3):
$(\exists M)P, (\forall M)S \vdash (\exists S)P$

Proof. Take as premises (1) $(\exists M)P$ and (2) $(\forall M)S$. From (1) by Conversion $MiP \Rightarrow PiM$ follows (3) $(\exists P)M$. From (2) and (3) if follows by Darii (4) $(\exists P)S$. From (4) again by Conversion $MiP \Rightarrow PiM$ follows (5) $(\exists S)P$:

1	(1)	$(\exists M)P$	Premise
2	(2)	$(\forall M)S$	Premise
1	(3)	$(\exists P)M$	$MiP \Rightarrow PiM$, 1
1,2	(4)	$(\exists P)S$	*Darii*, 2, 3
1,2	(5)	$(\exists S)P$	$MiP \Rightarrow PiM$, 4

Therefore $(\exists M)P, (\forall M)S \vdash (\exists S)P$.

Example 124 (Cesare: EAE-2):
$(\forall P)\neg M, (\forall S)M \vdash (\forall S)\neg P$

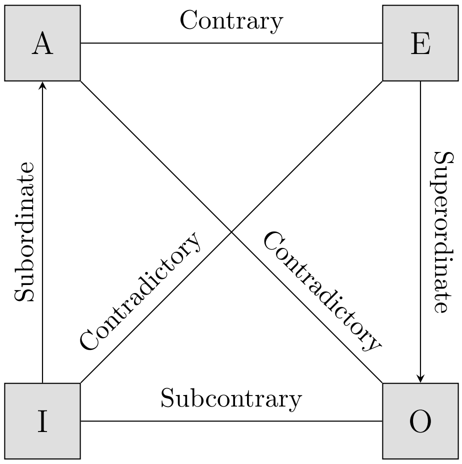
Proof. Take as premises (1) $(\forall P)\neg M$ and (2) $(\forall S)M$. From (1) by Conversion $PeM \Rightarrow MeP$ follows (3) $(\forall M)\neg P$, and from (2) and (3) it follows by Celarent (4) $(\forall S)\neg P$.

1	(1)	$(\forall P)\neg M$	Premise
2	(2)	$(\forall S)M$	Premise
1	(3)	$(\forall M)\neg P$	$PeM \Rightarrow MeP$, 1
1,2	(4)	$(\forall S)\neg P$	*Celarent*, 2, 3

Therefore, $(\forall P)\neg M, (\forall S)M \vdash (\forall S)\neg P$.

Square of Opposition

The relations of the Square of opposition that hold between the categorical sentences are as shown on the following diagram:



Where *contraries* cannot both be true, *subcontraries* cannot both be false, *superordinate* entails its *subordinate* sentence and each sentence entails the negation of its *contradictory* sentence. We will now show that

Theorem 125 All the relations of the Square of Opposition are valid in Quarc.

We prove this by considering all the relations in turn. To again merge the proofs, let the operator '$\sim$' stand for either an empty string of operators or the operator '$\neg$', with '$\neg\neg$' read as an empty string of operators, and the following special cases of SP and PS, respectively:

1	(1)	$\neg(a)\neg P$	Premise
2	(2)	$\neg(a)P$	Premise
2	(3)	$(a)\neg P$	SP, 2
1	(4)	$\neg\neg(a)P$	$\neg$I, 1, 2, 3
1	(5)	$(a)P$	$\neg$E, 4

1	(1)	$(a)P$	Premise
2	(2)	$(a)\neg P$	Premise
2	(3)	$\neg(a)P$	PS, 2
1	(4)	$\neg(a)\neg P$	$\neg$I, 1, 2, 3

1. Contradictions

Let us start with the more mundane relation of contradictions or DeMorgan's laws.

Lemma 126 (Contradiction):
$\neg(\forall S) \sim P \vdash (\exists S)\neg \sim P$

Proof.

1	(1)	$\neg(\forall S) \sim P$	Premise
2	(2)	$\neg(\exists S)\neg \sim P$	Premise
3	(3)	$(a)S$	Premise
4	(4)	$\neg(a) \sim P$	Premise
4	(5)	$(a)\neg \sim P$	SP, 4
3,4	(6)	$(\exists S)\neg \sim P$	PI, 3, 5
2,3	(7)	$\neg\neg(a) \sim P$	¬I, 4, 2, 6
2,3	(8)	$(a) \sim P$	¬E, 7
2	(9)	$(\forall S) \sim P$	PI, 3, 8
1	(10)	$\neg\neg(\exists S)\neg \sim P$	¬I, 2, 1, 9
1	(11)	$(\exists S)\neg \sim P$	¬E, 10

Lemma 127 (Contradiction):
$(\exists S)\neg \sim P \vdash \neg(\forall S) \sim P$

Proof.

1	(1)	$(\exists S)\neg \sim P$	Premise
2	(2)	$(\forall S) \sim P$	Premise
3	(3)	$(a)S$	Premise
4	(4)	$(a)\neg \sim P$	Premise
4	(5)	$\neg(a) \sim P$	PS, 4
2,3	(6)	$(a) \sim P$	UE, 2, 3
3,4	(7)	$\neg(\forall S) \sim P$	¬I, 2, 5, 6
1	(8)	$\neg(\forall S) \sim P$	Ins, 1, 3, 4, 7

2. Subordination

We move on to the proof of subordination. It is a straightforward consequence of the rule of Instantial Import. We will discuss in more detail the justification and implications of this rule in the following sections.

Lemma 128 (Subordination):
$(\forall S) \sim P \vdash (\exists S) \sim P$

Proof.

1	(1)	$(\forall S) \sim P$	Premise
2	(2)	$(a)S$	Premise
3	(3)	$(a) \sim P$	Premise
2,3	(4)	$(\exists S) \sim P$	PI, 2, 3
1	(5)	$(\exists S) \sim P$	Ins, 1, 2, 3, 4

With these two relations of Square of opposition in place, others follow.

3. Contrariness

Lemma 129 (Contrariness):
$(\forall S) \sim P \vdash \neg(\forall S)\neg \sim P$

Proof. Take as premises (1) $(\forall S) \sim P$ and (2) $(\forall S)\neg \sim P$. It follows from (1) by Contradiction that (3) $\neg(\exists S)\neg \sim P$. It follows from (2) by Subordination that (4) $(\exists S)\neg \sim P$. But this is a contradiction. So, (5) $\neg(\forall S)\neg \sim P$.

1	(1)	$(\forall S) \sim P$	Premise
2	(2)	$(\forall S)\neg \sim P$	Premise
1	(3)	$\neg(\exists S)\neg \sim P$	Lemma 6.2, 1
2	(4)	$(\exists S)\neg \sim P$	Lemma 7, 2
1	(5)	$\neg(\forall S)\neg \sim P$	$\neg$I, 2, 3, 4

Therefore, $(\forall S) \sim P \vdash \neg(\forall S)\neg \sim P$.

4. Subcontrariness

Lemma 130 (Subcontrariness):
$\neg(\exists S) \sim P \vdash (\exists S)\neg \sim P$.

Proof. Take as a premise (1) $\neg(\exists S) \sim P$. It follows from (1) by Contradiction that (2) $(\forall S)\neg \sim P$. From (2) by Subordination it follows that (3) $(\exists S)\neg \sim P$.

1	(1)	$\neg(\exists S) \sim P$	Premise
1	(2)	$(\forall S)\neg \sim P$	Lemma 6.1, 1
1	(3)	$(\exists S)\neg \sim P$	Lemma 7, 2

Therefore, $\neg(\exists S) \sim P \vdash (\exists S)\neg \sim P$.

This concludes the proof of Theorem 125.

6.2.2 Malink's Formalization of the Assertoric Syllogistic

In this section we will briefly present and then comment on some of the features of Malink's formalization of the assertoric syllogistic. These carry over into the modal syllogistic and are therefore relevant to the overall discussion. To distinguish Quarc from Malink, we retain his notation. Therefore, it will need to be explained before proceeding.

Preliminaries

First, a few notes on the mode of presentation employed by Malink in [Malink, 2013]. The copulas he uses combine the four letters for the categorical propositions (a, e, i, o) with an indexed modality of the proposition (of interest to us are 'X' for assertoric propositions and 'N' for necessity). So, the set of copulas used in this chapter is $\{a_N, a_X, e_N, e_X, i_N, i_X, o_N, o_X\}$. These combine the subject and predicate terms, in reverse order as Malink reads the copula as "belongs to".

So, the set of propositions we will examine, and the natural language renderings preferred by Malink, are as follows:

ASSERTORIC PROPOSITIONS	FORMALIZATION	READ AS
Universal Affirmative	$Pa_X S$	P belongs to all S
Universal Negative	$Pe_X S$	P belongs to no S
Particular Affirmative	$Pi_X S$	P belongs to some S
Particular Negative	$Po_X S$	P does not belong some S

APODICTIC PROPOSITIONS	FORMALIZATION	READ AS
Universal Affirmative	$Pa_N S$	P necessarily belongs to all S
Universal Negative	$Pe_N S$	P necessarily belongs to no S
Particular Affirmative	$Pi_N S$	P necessarily belongs to some S
Particular Negative	$Po_N S$	P necessarily does not belong some S

Note that proofs which use this notation are in this chapter meant to be taken as proof in Malink's system, not in Quarc.

Existential Import

Obviously, the most controversial element required to demonstrate the validity of assertoric syllogistic is the existential import, needed for one of the conversions. Malink discusses several possible ways one can go about securing it before presenting his own account. The first and most plausible, given various remarks by Aristotle, is to simply introduce a stipulation that only terms that apply to at least one element of the domain (or something to that effect) are permissible.

To this Malink objects, accurately, that "the assertoric syllogistic would then rely on a tacit extralogical presupposition about the nature of admissible terms which Aristotle failed to make explicit. It would not be the universally applicable system of formal logic that it is often thought to be." [Malink, 2013, p. 43]. In the terminology of this chapter, such an attempt would invariably be open to the charge of tinkering.

Malink instead opts for a variant of this strategy, where he does not impose a general requirement, but specifies his semantics in a way that secures existential import, by introducing the *heterodox dictum semantics.* In the next section we examine it in more detail.

The Heterodox *Dictum* Semantics

In a nutshell, the heterodox dictum semantics[4] takes the universal affirmative predication as a primitive, and uses it to provide the semantics for the four categorical propositions.

> C[5] is a member of a plurality associated with A if and only if A is a_X-predicated of C. [Malink, 2013, p. 63]

In other word, something is an A just in case A is universally affirmatively predicated of it. The heterodox dictum semantics of the four categorical propositions then goes as (lettering adjusted):

[4]As opposed to the *orthodox dictum* semantics, as Malink calls, following [Barnes, 2007], the familiar way of rendering the four Aristotelian categorical sentences in the Predicate Calculus.

[5]C here need not be an individual, as Malink argues in [Malink, 2013, pp. 52–53].

$Pa_X S$	iff	$\forall Z(Sa_X Z \Rightarrow Pa_X Z)$
$Pe_X S$	iff	$\forall Z(Sa_X Z \Rightarrow \neg Pa_X Z)$
$Pi_X S$	iff	$\exists Z(Sa_X Z \wedge Pa_X Z)$
$Po_X S$	iff	$\exists Z(Sa_X Z \wedge \neg Pa_X Z)$

Malink, naturally, concedes that this is not a definition of the four categorical propositions (in virtue of not being a definition of a_X predication, since it can offer only a circular definition), and is thus less informative than an explicit definition. However, he contends these suffice to establish the validity of the four perfect syllogisms and three conversions, and thereby validate the assertoric syllogistic. The biggest problem, that of existential import, is solved by reflexivity of the universal affirmative predication – there is always something A is a_X-predicated of, namely A.

Problems for Malink

By way of an argument for his position, Malink offers two considerations, a negative and a positive one. The negative comes in response to an exegetic challenge from Barnes, where Malink states that

> it is not implausible that what [Aristotle] meant in his formulation of the dictum is adequately captured by the heterodox interpretation,

although he accepts that

> if he wanted, Aristotle could have explicitly stated the heterodox dictum de omni. [Malink, 2013, p. 65][6]

The second argument, in reply to the circularity objection, is to point out the usefulness of this formulation in validating the assertoric syllogistic.

The problem with both of these responses is that they are not sufficient to justify the concessions of Malink's system. And if we consider the results of the previous section, we see that Quarc likewise achieves all the positive results, but with two significant advantages. First, Quarc does not risk circularity, and offers full-fledged definitions of all the categorical propositions. Second, it also validates the assertoric syllogistic, and the requirement of "existential" import. Moreover, the motivation for the latter is widely accepted – avoiding a free logic (cf. [Pavlovic and Gratzl, 2016]). Therefore, it is not necessary to take the route Malink does.

Combined with the previous section, this should establish that Quarc is more appropriate formalization of Aristotle's assertoric syllogistic than either the standard Predicate Calculus or Malink's system. The consequences of this become apparent once we move to the modal syllogistic.

6.3 Aristotle's Modal Syllogistic

We now move on to a discussion of Aristotle's modal syllogistic. Following the pattern from the previous section, we will first present the Quarc approach, and

[6] In the interest of full disclosure, the rhetorical force of these two quotes has been changed from the source by placing them in a different order. This is as justified as Malink is in using the other ordering to the opposite effect, but the reader should be aware of the discrepancy.

then move on to compare and contrast it with Malink's. The modal extension of Quarc is called M-Quarc, and it will presented first.

6.3.1 The M-Quarc

When we move to M-Quarc, we expand the list of operators with the modal operators $\{\Box, \Diamond\}$. Like negation, these can function either as the sentential or the predication operator. Then, the sentential position of a (modal) operator corresponds to what is commonly called *de dicto* modality, while the predication position corresponds[7] to *de re* modality. Note that we are concerned here only with the syntactic issue of the position of modal operators, and will therefore prefer the terms introduced in Definitions 111 and 112, as these carry no metaphysical implications. Note also that rules PS and SP of Quarc will apply to these operators as well.

The semantics for both Quarc and M-Quarc used here is substitutional, in keeping with [Ben-Yami, 2014]. For example, $(\forall S)P$ is true iff every substitution of the quantified argument '$\forall S$' by a constant 'a' for which $(a)S$ is true, the resulting sentence $(a)P$ is also true. Compare to Aristotle:

> We use the expression 'predicated of every' when none of the subject can be taken of which the other term cannot be said. [Aristotle, 1989, 24b28]

However, nothing in this chapter hinges on this approach to semantics. Specifically, anyone more familiar with the possible world semantics for modal logic can treat truth-value assignments below as possible worlds, and an assignments set of an assignment as a set of possible worlds accessible to that world.

Preliminaries

In this section we lay the groundwork for what is to come. We will establish rules and definitions for the modal expansion of Quarc used here.

One adjustment of the system provided in [Ben-Yami, 2014] is that for brevity, instead of modal axioms for K and T, we use rules of derivation:

Definition 131 (K): For any sentences A and B, if on some line of the proof (i) we have a sentence $\Box A$ and on some line of the proof (j) we have $\Box(A \to B)$, we can in any subsequent line (k) derive the sentence $\Box B$. The line (k) depends on all premises that (i) and (j) depend on, and its justification is written as 'K i, j'.

L_1	(i)	$\Box A$	
L_2	(j)	$\Box(A \to B)$	
L_1, L_2	(k)	$\Box B$	K, i, j

Definition 132 (T): For any sentence A, if at some line (i) of the proof we have a sentence $\Box A$, then in any subsequent line (j) we can derive the sentence

[7] To an extent. In this context, *de praedicationi* might be a more fortunate label, but it is not overly important, given that we mostly use the terms 'sentential' and 'predication' instead.

A. The line (k) depends on whatever (i) depends on, and its justification is written as 'T, i'.

L_1	(i)	$\Box A$	
L_1	(j)	A	T, i

The proof system used in this chapter, unless noted otherwise, is the minimal M-Quarc with added rules of derivation from Definitions 131 and 132.

6.3.2 Validity of Modal Syllogisms in M-Quarc

The valid syllogisms of the first figure remain valid when the sentences are replaced with sentences with necessary predication. This was also Aristotle's view, as can be seen in the following passage from his *Prior Analytics*:

> In the case of necessary premises, then, the situation is almost the same as with premises of belonging: that is, there either will or will not be a deduction with the terms put in the same way, both in the case of belonging and in the case of belonging or not belonging of necessity, except that they will differ in the addition of "belonging (or not belonging) of necessity" to the terms. [Aristotle, 1989, 29b36]

What Aristotle seems to be saying is that these syllogisms will come out as (in)valid in the same manner as their non-modal counterparts. Given that all of the listed (non-modal) syllogisms come out valid in Quarc, we should expect the NNN syllogisms to follow suit, which they indeed do, as we will see in the following section.

NNN Syllogisms

The four perfect assertoric syllogisms all come out valid when both the premises and the conclusion are necessitated, on either the sentential or the predication reading. However, the proofs differ somewhat for the two readings, so we will tackle them in turn. First, the sentential reading.

1. Sentential Reading ("*de dicto*")

Although we are only concerned with the first figure here, one proof in fact demonstrates the validity of all the NNN syllogisms. Before providing the proof of the validity of the NNN syllogisms on the sentential reading, we prove a simple auxiliary lemma:

Lemma 133 :
$\vdash \Box(A_1 \to (A_2 \to (A_1 \land A_2)))$

Proof.

1	(1)	A_1	Premise
2	(2)	A_2	Premise
1,2	(3)	$A_1 \land A_2$	$\land$I, 1, 2
1	(4)	$A_2 \to (A_1 \land A_2)$	$\to$I, 2, 3
	(5)	$A_1 \to (A_2 \to (A_1 \land A_2))$	$\to$I, 1, 4
	(6)	$\Box(A_1 \to (A_2 \to (A_1 \land A_2)))$	Nec, 5

Now we can demonstrate the validity of all NNN syllogisms for which the corresponding assertoric syllogism is valid all at once. The proof will provide the general schema for proving any one particular syllogism.

Lemma 134 (NNN Validity on Sentential Reading):
If $A_1, A_2 \vdash B$, then $\Box A_1, \Box A_2 \vdash \Box B$.

Proof.

1	(1)	$\Box A_1$	Premise
2	(2)	$\Box A_2$	Premise
3	(3)	$A_1 \wedge A_2$	Premise
3	(4)	A_1	$\wedge$E, 3
3	(5)	A_2	$\wedge$E, 3
3	(6)	B	Assumption, 4, 5
	(7)	$(A_1 \wedge A_2) \to B$	$\to$I, 3, 6
	(8)	$\Box((A_1 \wedge A_2) \to B)$	Nec, 7
	(9)	$\Box(A_1 \to (A_2 \to (A_1 \wedge A_2)))$	Lemma 133
1	(10)	$\Box(A_2 \to (A_1 \wedge A_2))$	K, 1, 9
1,2	(11)	$\Box(A_1 \wedge A_2)$	K, 2, 10
1,2	(12)	$\Box B$	K, 8, 11

Obviously, the import of this theorem extends far beyond the syllogistic, but that need not concern us at the moment. Note, though, that a simplified version of this proof will guarantee that the necessitated conversions likewise hold: if $A \vdash B$ then $\Box A \vdash \Box B$.

2. Predication Reading

The proof of validity on the predication reading does not follow as straightforwardly from the corresponding assertoric syllogisms, but the proofs of the two are nonetheless similar, with only two added lines (use of the PS and T rule in the proof below) to accommodate the addition of the necessity operator to the minor premise. Let '$\sim$' stand for either an empty string of operators or the negation, same as above.

Lemma 135 (Barbara-NNN, Celarent-NNN):
$(\forall M)\Box \sim P, (\forall S)\Box M \vdash (\forall S)\Box \sim P$

Proof.

1	(1)	$(\forall M)\Box \sim P$	Premise
2	(2)	$(\forall S)\Box M$	Premise
3	(3)	$(a)S$	Premise
2,3	(4)	$(a)\Box M$	UE, 2, 3
2,3	(5)	$\Box(a)M$	PS, 4
2,3	(6)	$(a)M$	T, 5
1,2,3	(7)	$(a)\Box \sim P$	UE, 1, 6
1,2	(8)	$(\forall S)\Box \sim P$	UI, 3, 7

The proofs of Darii-NNN and Ferio-NNN on the predication reading are related in the same way, *mutatis mutandis*, to their non-modal versions. As we can see, the NNN syllogisms of the first figure are relatively unproblematic.

XNN Syllogisms

None of these syllogisms will be valid, and therefore we will proceed to construct assignments sets that invalidate them. But first, let us observe the textual support for Aristotle holding them invalid. Given that in all of these cases the major premise is not necessary, Aristotle writes:

> It sometimes results that the deduction becomes necessary when only one of the premises is necessary (not whatever premise it might be, however, but only the premise in relation to the major extreme). [Aristotle, 1989, 30a15]

Since the conclusion can be necessary only when the major premise is necessary, and that is not the case in any of these premises, we should expect all of them to be invalid. Let us now demonstrate their invalidity, again starting with the sentential reading.

1. Sentential Reading

Invalidity 136 (Barbara XNN, Sentential):
$(\forall M)P, \Box(\forall S)M \nvDash \Box(\forall S)P$

Proof. Let $\sigma(s_1) = \{s_1, s_2\}$.
Let $s_1 = \{\langle(a)S, \top\rangle, \langle(a)M, \top\rangle, \langle(a)P, \top\rangle, \langle(b)S, \bot\rangle, \langle(b)M, \bot\rangle, \langle(b)P, \bot\rangle\}$.
Let $s_2 = \{\langle(a)S, \top\rangle, \langle(a)M, \top\rangle, \langle(a)P, \bot\rangle, \langle(b)S, \top\rangle, \langle(b)M, \top\rangle, \langle(b)P, \top\rangle\}$.

(i) The major premise, $(\forall M)P$, is true on s_1: every M, namely a, is P.

(ii) The minor premise, $\Box(\forall S)M$, is true on s_1: on s_1 every S, namely a, is M and so $(\forall S)M$ is true, and on s_2 every S, namely a and b, are M and so $(\forall S)M$ is true. Therefore, $\Box(\forall S)M$ is true.

(iii) But the conclusion, $\Box(\forall S)P$, is not true on s_1: on s_2, a is S but not P, so $(\forall S)P$ is false on s_2, and therefore $\Box(\forall S)P$ is false on s_1.

Invalidity 137 (Cesare XNN, Sentential):
$(\forall M)\neg P, \Box(\forall S)M \nvDash \Box(\forall S)\neg P$

Proof. Same as Invalidity 136, with all values of P switched.

Invalidity 138 (Darii XNN, Sentential):
$(\forall M)P, \Box(\exists S)M \nvDash \Box(\exists S)P$

Proof. Let $\sigma(s_1) = \{s_1, s_2\}$.
Let $s_1 = \{\langle(a)S, \top\rangle, \langle(a)M, \top\rangle, \langle(a)P, \top\rangle, \langle(b)S, \bot\rangle, \langle(b)M, \bot\rangle, \langle(b)P, \bot\rangle\}$.
Let $s_2 = \{\langle(a)S, \top\rangle, \langle(a)M, \top\rangle, \langle(a)P, \bot\rangle, \langle(b)S, \bot\rangle, \langle(b)M, \top\rangle, \langle(b)P, \top\rangle\}$.

(i) The major premise, $(\forall M)P$, is true on s_1: every M, namely a, is P.

(ii) The minor premise, $\Box(\exists S)M$, is true on s_1: on s_1 some S, namely a, is M and so $(\exists S)M$ is true, and on s_2 some S, namely a, is M and so $(\exists S)M$ is true. Therefore, $\Box(\exists S)M$ is true.

(iii) But the conclusion, $\Box(\exists S)P$, is not true on s_1: on s_2, the only S, namely a is not P, so $(\exists S)P$ is false on s_2, and therefore $\Box(\exists S)P$ is false on s_1.

Invalidity 139 (Ferio XNN, Sentential):
$(\forall M)\neg P, \Box(\exists S)M \nvDash \Box(\exists S)\neg P$

Proof. Same as Invalidity 138, with all values of P switched.

2. Predication Reading

Invalidity 140 (Barbara XNN, Predication):
$(\forall M)P, (\forall S)\Box M \nvDash (\forall S)\Box P$

Proof. $\sigma(s_1)$ from Invalidity 136 will invalidate this syllogism as well.

Invalidity 141 (Cesare XNN, Predication):
$(\forall M)\neg P, (\forall S)\Box M \nvDash (\forall S)\Box\neg P$

Proof. $\sigma(s_1)$ from Invalidity 137 will invalidate this syllogism as well.

Invalidity 142 (Darii XNN, Predication):
$(\forall M)P, (\exists S)\Box M \nvDash (\exists S)\Box P$

Proof. $\sigma(s_1)$ from Invalidity 136 will invalidate this syllogism as well.

Invalidity 143 (Ferio XNN, Predication):
$(\forall M)\neg P, (\exists S)\Box M \nvDash (\exists S)\Box\neg P$

Proof. $\sigma(s_1)$ from Invalidity 137 will invalidate this syllogism as well.

This concludes the examination of the XNN necessity syllogisms from the first figure. As we have seen, for both the NNN and the XNN syllogisms, all those and only those syllogisms Aristotle believed to be valid are in fact valid, on both the predication and the sentential versions. In the following section, we will encounter some complications, as that result does not hold for the NXN syllogisms.

NXN Syllogisms

In this section we will see that in Quarc a familiar problem for Aristotle's modal syllogistic arises from an ambiguity with respect to the position of the modal operator. Namely, Aristotle is often charged with conflating the *de re* version, which validates some first figure syllogisms, with the *de dicto* one, required to validate the proofs of the second figure syllogisms[8]. The problem arises for the

[8] [Malink, 2013, p. 10] lists a number of authors

NXN syllogisms (where the minor premise is assertoric), which we focus on in this section.

Here is what Aristotle says on the first two of these syllogisms, which he obviously takes to be valid:

> If P has been taken to belong or not to belong of necessity to M, and M merely to belong to S: for if the premises have been taken in this way, then P will belong or not belong to S of necessity. [Aristotle, 1989, 30a18]

Let us therefore proceed with the proofs. The *de re* or predication version of the proof is the same as for Barbara and Celarent, as the above quote suggests.

Lemma 144 (Barbara NXN; Celarent NXN, Predication): $(\forall M)\Box \sim P, (\forall S)M \vdash (\forall S)\Box \sim P$

Proof. Same as Lemma 116, if we substitute '$\Box \sim$' for '$\sim$' in that proof.[9]

Notice, however, that neither of these come out valid on the sentential or *de dicto* reading.[10]

Invalidity 145 (Barbara NXN, Sentential):
$\Box(\forall M)P, (\forall S)M \nvDash \Box(\forall S)P$

Proof. Let $\sigma(s_1) = \{s_1, s_2\}$.
Let $s_1 = \{\langle(a)S, \bot\rangle, \langle(a)M, \bot\rangle, \langle(a)P, \top\rangle, \langle(b)S, \top\rangle, \langle(b)M, \top\rangle, \langle(b)P, \top\rangle\}$.
Let $s_2 = \{\langle(a)S, \top\rangle, \langle(a)M, \bot\rangle, \langle(a)P, \bot\rangle, \langle(b)S, \bot\rangle, \langle(b)M, \top\rangle, \langle(b)P, \top\rangle\}$.

(i) The major premise, $\Box(\forall M)P$, is true on s_1: on s_1 every M, namely b, is P, so $(\forall M)P$ is true on s_1. Likewise on s_2 every M, namely b, is P, so $(\forall M)P$ is true on s_2. Therefore, $\Box(\forall M)P$ is true on s_1.

(ii) The minor premise, $(\forall S)M$, is true on s_1: every S, namely b, is M.

(iii) But the conclusion, $\Box(\forall S)P$, is not true on s_1: on s_2 some S, namely a, is not P, so $(\forall S)P$ is false on s_2 and therefore $\Box(\forall S)P$ is false on s_1.

Invalidity 146 (Cesare NXN, Sentential):
$\Box(\forall M)\neg P, (\forall S)M \nvDash \Box(\forall S)\neg P$

Proof. Same as Invalidity 145, with all values of P switched.

As we can see, similarly to the validity proofs of Barbara and Celarent for the predication reading, their invalidity proofs for sentential versions run in parallel.

In the following section, we will demonstrate that the two readings are not interchangeable when it comes to modality (and that we therefore must opt for one). The principles which state that one reading entails the other are known as the Barcan formula and the Converse Barcan formula.

[9] Compare to the quote 29b36 above

[10] One could trivialize the difference between the sentential and the predication version by deriving either one from the other with the help of Barcan formulas. However, as we will soon see, neither the Barcan formula nor its converse are valid in M-Quarc.

6.3.3 Barcan Formula and the Converse Barcan Formula in M-Quarc

Barcan Formula and its Converse involve the "traveling" of the modal operator over the quantifier. Let us label their respective versions in M-Quarc as Barcan Formula′ and Converse Barcan Formula′.

Definition 147 (Barcan Formula′, BF′):
BF′ is an M-Quarc formula of the form $(\forall M)\Box P \rightarrow \Box(\forall M)P$.

Definition 148 (Converse Barcan Formula′, CBF′):
CBF′ is an M-Quarc formula of the form $\Box(\forall M)P \rightarrow (\forall M)\Box P$.

As we shall now see, neither of these are valid in M-Quarc. We consider the BF′ first.

Invalidity 149 (BF′):
$(\forall M)\Box P \nvDash \Box(\forall M)P$

Proof. Let $\sigma(s_1) = \{s_1, s_2\}$.
Let $s_1 = \{\langle(a)M, \top\rangle, \langle(a)P, \top\rangle, \langle(b)M, \bot\rangle, \langle(b)P, \top\rangle\}$.
Let $s_2 = \{\langle(a)M, \bot\rangle, \langle(a)P, \top\rangle, \langle(b)M, \top\rangle, \langle(b)P, \bot\rangle\}$.

(i) The premise, $(\forall M)\Box P$, is true on s_1: on s_1, every M, namely a, is P both on s_1 and on s_2, so on s_1 $(a)\Box P$ is true, and therefore $(\forall M)\Box P$ is true.

(ii) But the conclusion, $\Box(\forall M)P$, is not true on s_1: on s_2, some M, namely b is not P, so $(\forall M)P$ is false on s_2, and therefore $\Box(\forall M)P$ is false on s_1.

Invalidity 150 (CBF′):
$\Box(\forall M)P \nvDash (\forall M)\Box P$

Proof. Let $\sigma(s_1) = \{s_1, s_2\}$.
Let $s_1 = \{\langle(a)M, \top\rangle, \langle(a)P, \top\rangle, \langle(b)M, \bot\rangle, \langle(b)P, \bot\rangle\}$.
Let $s_2 = \{\langle(a)M, \bot\rangle, \langle(a)P, \bot\rangle, \langle(b)M, \top\rangle, \langle(b)P, \top\rangle\}$.

(i) The premise, $\Box(\forall M)P$, is true on s_1: on s_1 every M, namely a, is P and on s_2 every M, namely b, is P. So $(\forall M)P$ is true on both s_1 and s_2, and therefore $\Box(\forall M)P$ is true on s_1.

(ii) But the conclusion, $(\forall M)\Box P$, is not true on s_1: some M, namely a, is not P on s_2, so $(a)\Box P$ is false on s_1, and therefore $(\forall M)\Box P$ is false on s_1.

As we can see, in Quarc neither the Barcan formula nor its Converse are valid. The reason for this is that here we are dealing with quantified arguments. However, this is not only not detrimental when dealing with Aristotle, but arguably an advantage of Quarc. It allows, as we have seen, more elegant proofs, while also more neatly lining up with the structure of categorical sentences, both in Aristotle and natural language.

Another way of putting this is that with Quarc, it does not hold that every M is necessarily M, and therefore the Converse Barcan formula is not valid. To make this clear, observe the following lemma:

Lemma 151 :
$(\forall M)\Box M \vdash \Box(\forall M)P \rightarrow (\forall M)\Box P$

Proof.

1	(1)	$(\forall M)\Box M$	Premise
2	(2)	$\Box(\forall M)P$	Premise
3	(3)	$(\forall M)P$	Premise
4	(4)	$(a)M$	Premise
3,4	(5)	$(a)P$	UE, 3, 4
3	(6)	$(a)M \rightarrow (a)P$	$\rightarrow$I, 4, 5
	(7)	$(\forall M)P \rightarrow ((a)M \rightarrow (a)P)$	$\rightarrow$I, 3,6
	(8)	$\Box((\forall M)P \rightarrow ((a)M \rightarrow (a)P))$	Nec, 7
2	(9)	$\Box((a)M \rightarrow (a)P)$	K, 2, 8
1,4	(10)	$(a)\Box M$	UE, 1, 4
1,4	(11)	$\Box(a)M$	PS, 10
1,2,4	(12)	$\Box(a)P$	K, 9, 11
1,2,4	(13)	$(a)\Box P$	SP, 12
1,2	(14)	$(\forall M)\Box P$	UI, 4, 13
1	(15)	$\Box(\forall M)P \rightarrow (\forall M)\Box P$	$\rightarrow$I, 2, 14

Likewise, the Converse Barcan formula′ will entail $(\forall M)\Box M$, from $\Box(\forall M)M$ and the instance of the CBF′: $\Box(\forall M)M \rightarrow (\forall M)\Box M$. As we can see, adding the provision that every M is necessarily M,[11] makes the Converse Barcan formula′ valid. A close observation of the above invalidity proof will reveal this was precisely what made it work. This condition is analogous to the condition of non-contracting domain (if a possible world w_2 is accessible to a world w_1, then the domain of w_2, $D(w_2) \subseteq D(w_1)$), which makes the CBF valid, but for the predicates contained in the quantified argument. Let us therefore define it as:

Definition 152 (Non-Contracting Predicate):
A unary predicate P is *non-contracting* just in case for every singular argument a, if $(a)P$ then $(a)\Box P$.

The situation is similar with the Barcan formula′. Let us define the analogous condition of the predicate contained within the quantified argument for validity of BF′:

Definition 153 (Non-Expanding Predicate):
A unary predicate P is *non-expanding* just in case for every singular argument a, if $(a)\neg P$ then $\Box(a)\neg P$.

We can now demonstrate that

Lemma 154 The Barcan formula′ is valid for formulas with non-expanding predicates in their governing quantified arguments.

Proof.

For our present purposes it will suffice to examine the case of BF′ as presented in Definition 147, even though expanding this to the case for any formula

[11]With deliberate word order, not to be mistaken with necessarily, every M is M, i.e. $\Box(\forall M)M$, which is a theorem of Quarc, but will not do.

governed by $\forall M$ is straightforward. Let M be a non-expanding predicate, and let $(\forall M)\Box P$ be a formula governed by $\forall M$. Assume that on s_1 ($s_1 \in \sigma(s_1)$), (1) $(\forall M)\Box P$ is true and (2), for *reductio*, that $\Box(\forall M)P$ is false. Therefore, on some $s_i \in \sigma(s_1)$, there is a c, such that (3) $(c)M$ is true on s_i and (4) $(c)P$ is false on s_i.

Assume (6) $(c)M$ is true on s_1. If follows from (1) that $(c)\Box P$ is true on s_1, and so $\Box(c)P$ is true on s_1 and therefore $(c)P$ is true on s_i. But this is a contradiction with (4).

Assume (7) $(c)M$ is false on s_1. It follows that $\neg(c)M$ is true on s_1, and therefore by Definition 153, that $\Box(c)\neg M$ is true on s_1. But then $(c)\neg M$ is true on s_i, and therefore $(c)M$ is false on s_i. But this is a contradiction with (3).

Therefore, by *reductio*, $\Box(\forall M)P$ is true on s_1.

This property of a predicate is analogous to the condition of non-expanding domain for the validity of BF. Combining the previous two definitions, we can say that the Barcan formula$'$ and its Converse hold for predicates with the property:

Definition 155 (Constant Predicate):
A predicate is *constant* just in case it is non-contracting and non-expanding.

Of course, given there is no obvious reason to limit the governing quantified arguments to constant predicates, BF$'$ and CBF$'$ remain invalid in Quarc.

We will revisit the topic later in the chapter, but the takeaway now should be that the two positions of operators, sentential and predicative, are not interchangeable. Given this, and the fact that the NXN syllogisms only come out valid on the predicative version, it seems to be the preferable reading.[12] However, as we move to the second figure we see that this cannot be the reading throughout. Let us examine Cesare NXN to demonstrate this point.

6.3.4 A Study of Cesare NXN

Aristotle considers this syllogism valid:

> For first let the privative be necessary, and let it not be possible for M to belong to any P, but let M merely belong to S. Then, since the privative converts, neither is it possible for P to belong to any M. But M belongs to every S; consequently, it is not possible for P to belong to any S, for S is below M. [Aristotle, 1989, 30b9](letters adjusted)

He derives it from Celarent NXN by conversion of the major premise. However, the conversion does not hold on predication reading (here reading 'not possible' as 'necessarily not'):

Invalidity 156 :
$(\forall P)\Box\neg M \nvDash (\forall M)\Box\neg P$

Proof. Let $\sigma(s_1) = \{s_1, s_2\}$.
Let $s_1 = \{\langle(a)M, \top\rangle, \langle(a)P, \bot\rangle, \langle(b)M, \bot\rangle, \langle(b)P, \top\rangle\}$.
Let $s_2 = \{\langle(a)M, \top\rangle, \langle(a)P, \top\rangle, \langle(b)M, \bot\rangle, \langle(b)P, \top\rangle\}$.

[12] As we have seen, it makes no difference for the first figure NNN syllogisms, and likewise for the XNN syllogisms. Constraints of space prevent us from tackling the syllogisms with other modalities, as they run into a number of issues peculiar just to them.

(i) The premise, $(\forall P)\Box\neg M$, is true on s_1: for every P, namely b, it holds both on s_1 and s_2 that $(b)\neg M$, and so it holds on s_1 that $(b)\Box\neg M$, and therefore it holds on s_1 that $(\forall P)\Box\neg M$.

(ii) But the conclusion, $(\forall M)\Box\neg P$, is false on s_1: there is an M, namely a, such that $(a)P$ is true on s_2, and so $(a)\neg P$ is false on s_2. Therefore $(a)\Box\neg P$ is false on s_1 and so $(\forall M)\Box\neg P$ is false on s_1.

This conversion *does* hold on the sentential reading:

Lemma 157 :
$\Box(\forall P)\neg M \vdash \Box(\forall M)\neg P$

Proof. From Lemma 120 by propositional modal logic.

Since Celarent NXN is valid only on the predicative reading but the conversion only on the sentential one, we cannot derive Cesare NXN from Celarent NXN by this conversion. This results in the familiar charge against Aristotle mentioned above.

Furthermore, one should not expect that some other procedure will be more effective, given that in any case Cesare NXN comes out invalid on either reading:

Lemma 158 (Cesare NXN):
$(\forall P)\Box\neg M, (\forall S)M \nvDash (\forall S)\Box\neg P$; $\Box(\forall P)\neg M, (\forall S)M \nvDash \Box(\forall S)\neg P$

Proof. For the predication version, an assignment set $\sigma(s_1) = \{s_1, s_2\}$,
with $s_1 = \{\langle(a)S, \top\rangle, \langle(a)M, \top\rangle, \langle(a)P, \bot\rangle, \langle(b)S, \bot\rangle, \langle(b)M, \bot\rangle, \langle(b)P, \top\rangle\}$.
and $s_2 = \{\langle(a)S, \bot\rangle, \langle(a)M, \top\rangle, \langle(a)P, \top\rangle, \langle(b)S, \top\rangle, \langle(b)M, \bot\rangle, \langle(b)P, \bot\rangle\}$.

For the sentential version, an assignment set $\sigma(s_1) = \{s_1, s_2\}$,
with $s_1 = \{\langle(a)S, \top\rangle, \langle(a)M, \top\rangle, \langle(a)P, \bot\rangle, \langle(b)S, \bot\rangle, \langle(b)M, \bot\rangle, \langle(b)P, \top\rangle\}$.
and $s_2 = \{\langle(a)S, \top\rangle, \langle(a)M, \bot\rangle, \langle(a)P, \top\rangle, \langle(b)S, \top\rangle, \langle(b)M, \top\rangle, \langle(b)P, \bot\rangle\}$.

To answer this objection, Malink points out that this relies on the idea that *de dicto* and *de re* are the only two possible understandings of modality in these sentences. Instead, he offers an alternative, one that is, he argues, akin to the source material and which nullifies the objection. Again, in this section we will use Malink's notation to indicate that these are not theorems of M-Quarc.

The idea behind the modal system Malink suggests in [Malink, 2013] relies on the concepts developed in his treatment of the assertoric syllogistic, once again rooting it in the a_X-predication (universal affirmative assertoric predication). Namely, the analysis of the necessary affirmative assertoric predication (a_N-predication) is as follows:

Definition 159 (Necessary Universal Affirmative Predication, a_N):
P is a_N-predicated of S if and only if P is a_N-predicated of everything of which S is a_X-predicated. [Malink, 2013, p. 112](letters adjusted):

Malink gives the truth conditions of a necessary universal-negative (signified by e_N, while the assertoric universal negative is signified by e_X) sentence in the following way:

Definition 160 :
$Pe_N M$ if and only if $Pe_X M$ and P and M are essence terms. [Malink, 2013, p. 170](letters adjusted)

This means that P is universally-negatively necessarily predicated of M just in case $(\forall M)\neg P$ is true and M and P are both *essence terms*. Essence terms are those that are subjects of essential predication [Malink, 2013, p. 141], and the latter has the following properties:

P1: If there is a P that is predicated essentially of M, and M is predicated of S, then M is predicated essentially of S [Malink, 2013, p. 124](letters adjusted)

P2: if P is predicated essentially of S, then $Pa_N S$ [Malink, 2013, p. 125](letters adjusted)

So, if an essence term is predicated of something, it is predicated of it essentially and thus, necessarily.

On this reading, the conversion between $Me_N P$ and $Pe_N M$ is valid – by Definition 160, M and P are both essence terms, and the assertoric conversion is valid (HDS here stands for heterodox *dictum* semantics):

Lemma 161 :
$Me_N P \vDash_{HDS} Pe_N M$

Proof. Assume $Me_N P$ is true. By Definition 160, (1) M and P are essence terms and (2) $Me_X P$. From (2) by Lemma 120, *mutatis mutandis* for Malink's system, it follows (3) $Pe_X M$. From (1) and (3) by Definition 160 it follows that $Pe_N M$.

Therefore, the conversion holds as well. Moreover, it does not mean we have proved something invalid, since Cesare NXN is valid under this definition:

Lemma 162 :
$Me_N P, Ma_X S \vDash_{HDS} Pe_N S$

Proof. From $Me_N P$ it follows that M and P are essence terms as above, and from that and $Ma_X S$ it follows that $Ma_N S$ by P1 and P2 above. So S is an essence term. From $Me_N P$ it follows that $Me_X P$, and from that and $Ma_X S$ it follows by assertoric syllogistic Cesare (Example 2 previously) that $Pe_X S$. So finally, since P and S are essence terms, it follows by Definition 160 that $Pe_N S$.

So Malink does provide us with a solution that accounts for the validity of the desired syllogisms, one that, according to him, blurs the distinction between *de dicto* and *de re* reading [Malink, 2013, p. 176].

6.3.5 Problems for Malink

There are several problems with Malink's account. First of all, the question that he answered might not be the same one, or even in the same field, as the one that was raised (remember, the terms "sentential" and "predicative" were used to clearly delineate the questions of syntax from those of metaphysics). Moreover, the textual support for Aristotle agreeing with him seems to be tenuous, which raises the question of tinkering.

Furthermore, it seems that when we consider the logical ramifications of some of his definitions, they turn out to rely on principles that lack independent plausibility. Finally, it seems Malink, at least in part, commits something he himself spoke against with regards to the assertoric syllogistic.

Does the Answer Match the Question?

Malink is right in pointing out that *de re* and *de dicto* might not be the only two possible readings of modal sentences. But the issue here is not merely whether there is a consistent system that validates all those and only those syllogisms Aristotle held as valid, but also to do so in a manner that is free of the charge of tinkering. Otherwise, one will only convince those already inclined to consent.

With this in mind, observe that the solutions presented here, and in Malink's book, introduce notions like predicables, essence and essential predication, substance and genus, mostly taken from Aristotle's *Topics*. These do, perhaps, offer a reply to his harsher critics, but they also rely on his specific metaphysical views, while the problem here stems from syntax. Syntactically, there are only two possibilities – the sentential or the predication position. Since he moves beyond the only independently plausible options and the very field in which the problem arises, Malink can avoid the charge of tinkering only if he can produce solid evidence that that is indeed what Aristotle had intended. But, Aristotle himself does not handle the issue in such a manner – as Malink points out, the relations needed for his solution do not appear in the part of the *Prior Analytics* concerned with these issues [Malink, 2013, p. 114].

The textual support Malink does offer for the use of predicables is from the following passages:

> So one must select the premises about each subject in this way, assuming first the subject itself, and both its definitions and whatever is peculiar to the subject; next after this, whatever follows the subject; next, whatever the subject follows; and then, whatever cannot belong to it. (Those to which it is not possible for the subject to belong need not be selected, because the privative converts). The terms which follow the subject must also be divided into those which are predicated of it essentially, those which are peculiar to it, and those which are predicated incidentally. And these, again, ⟨should be divided⟩ into such as are matters of opinion and such as are according to the truth. [Aristotle, 1989, 43b1]

However, this is not an analysis of modal syllogistic, but merely instructions on what one needs to keep in mind when choosing the terms for syllogisms. Nowhere does Aristotle say we need to limit ourselves to terms predicated essentially, so the evidence this passage provides is circumstantial at best. Likewise for the next passage:

> For it is possible for A to belong neither to any B nor to any C, nor B to any C, as the genus does to species from another genus. [Aristotle, 1989, 54b36]

The mention of the predicable "genus" here is meant merely as an example and as such offers little by way of definitive proof. Of course, these considerations do not prove Malink wrong, nor are they meant to. The point is to make a distinction between raising a charge of tinkering against Malink and against Aristotle, the latter being much less plausible. But since nothing Malink offers compels us to accept his view as Aristotle's, it becomes much less persuasive.

In short, the issue here was whether to understand the modal operators as fulfilling the role of a sentential or a predication operator, a question of syntax. These notions, unlike *de re* and *de dicto* labels, do not involve metaphysics. On the other hand, Malink's answer crucially involved metaphysical notions. Moreover, there is no evidence that in this respect, Aristotle intended for his metaphysics to be reflected in the syntax. Therefore, it seems Malink introduces these notions as a way of validating Aristotle, and that would classify it as tinkering.

De Re – De Dicto **Distinction**

Malink in effect limits the subjects of necessary propositions to essence terms. And those are such that they are "a_N-predicated of everything of which they are a_X-predicated." [Malink, 2013, p. 152]. In effect, Malink limits the modal syllogistic to *non-contracting* predicates.

Moreover, although the situation with the non-expanding predicates is not as clear, it is obvious that at least as far as the necessary universal-negative is concerned (which will suffice for the purposes of this chapter, as it only focuses on Cesare NXN), it also limits the predicates to *non-expanding*. Therefore, Malink's treatment of Cesare NXN is such that it limits it to *constant* predicates. In other words, he does not do away with the *de re – de dicto* distinction, it's just that his formalization validates the principles required for the Barcan formula and it converse to be valid, thus making the two readings equivalent.

However, those principles are not universally acceptable, so the solution can be said, in Malink's own words, to "rely on a tacit extralogical presupposition about the nature of admissible terms which Aristotle failed to make explicit. It would not be the universally applicable system of formal logic" [Malink, 2013, p. 43]. Recall that these same words were previously quoted when discussing the strategies for ensuring the existential import in the assertoric syllogistic, and here Malink seems to fall victim of his own objection. Of course, an important difference is that unlike the assertoric syllogistic, Aristotle's modal syllogistic is not often thought to be a "universally applicable system of formal logic". If anything, the opposite holds true. So, even this limited result would be an improvement. However, it simply fails to match up to the modern modal logic, which *is* considered a universally applicable system of formal logic.

6.4 Conclusions

The issues presented in this chapter are not peculiar to the system used here, as the questions discussed here predate not only Quarc, but the majority of modern logic. However, the way in which these problems arise in Quarc and M-Quarc is particularly informative. In the closing section of this chapter, we gather all the result obtained so far into a systematic argument.

One fundamental difference between the Aristotelian modal syllogistic and modern modal systems is that the most natural way of extending a quantified logic to incorporate modalities is by adding modal operators as distinct syntactic units. These can serve two functions – sentential, i.e. ranging over the whole sentence, or predication, i.e. modify the mode of predication. Again, this is not peculiar to the modern system at hand, but the syntactic idiosyncrasies of Quarc make this readily apparent – it is at the same time closer to both the natural language and to Aristotle.

Proposition 163 Quarc has exactly two available positions where one can introduce the modal operators (as is evident from Definitions 111 and 112) – the sentential and the predication position.

As we have seen in Section 6.3.3, in general these two options are not equivalent, and to make them equivalent, one would need to accept counter-intuitive extralogical limitations on the kinds of predicates we use.

Proposition 164 The two options are not equivalent when it comes to quantified sentences.

Therefore, one needs to choose one of the options (tacit assumption being that we want a uniform reading throughout). However, as we have seen in Section 6.3.2, the NXN syllogisms of the first figure only come out valid on the predication reading. But, Section 6.3.4 demonstrates that the inferences needed for the proof of the second figure syllogism are only valid on the sentential reading. Moreover, some syllogisms are invalid no matter the reading we take. So:

Proposition 165 Neither option satisfies all of the required inferences.

It follows that there is no independently plausible way, in M-Quarc, of reconstructing Aristotle's modal syllogistic that validates it in its entirety. The fact that modern modal systems have independent reasons for introducing modal operators as distinct syntactic units (expressive power, compositionality, simplicity), and that those are precisely the cause of problems for Aristotle's modal syllogistic, seems to suggest that

Proposition 166 There is a fundamental mismatch between Aristotle's modal syllogistic and modern modal logic that makes it highly unlikely the latter will ever treat the former as valid.

Solutions like that of Malink seem to offer a solution, by way of a compromise. In the example we have seen in this chapter, Malink limits the scope of the universal necessary propositions. However, due to Proposition 166, any such attempt can be construed as "tinkering". Therefore, even if one were to come up with a system that validates all those and only those modal syllogisms Aristotle considers valid, one is unlikely to change the dominant opinion on Aristotle's modal syllogistic.

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