Incomplete Pairwise Comparison Matrices in Multi-Criteria Decision Making and Ranking

by

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Contents

Introduction

Multi-criteria decision making (MCDM) is a branch of Operations Research that deal with decision problems under the existence of many conflicting criteria. It plays a very important role in many real life problems. For instance, industries, governmental agencies, and business activities involve evaluating a set of alternatives with respect to multiple criteria. To determine the best alternative or criteria, there are different MCDM methodologies like AHP (Analytical Hierarchy Process), ELECTRE (ELimination and Choice Expressing REality), and many others [41].

Since Thomas L. Saaty proposed the Analytic Hierarchy Process (AHP) [36], *pairwise comparison matrices* are often used as basic tools in multi-criteria decision making. It is assumed that a decision maker doesn't know the weights of criteria or values of the alternatives precisely while the comparison is possible. When a decision maker wants to compare n objects (criteria or alternatives with respect to a given criteria), a pairwise comparison matrix is introduced as $\mathbf{A} = [\alpha_{ij}]_{i,j=1,\dots,n}$, where α_{ij} is the numerical answer for the question 'How many times is Criterion i better than Criterion j?', or 'How many times is Alternative i preferred to Alternative j?'. It is always positive and reciprocal, i.e. $\alpha_{ij} > 0$ and $\alpha_{ij} = \frac{1}{\alpha_{ji}}$ $\forall i, j$. The incomplete version of $\mathbf{A}$ is known as *incomplete pairwise comparison matrix*.

The thesis considers analysis of incomplete pairwise comparison matrices and its application for ranking top table tennis players by taking large-scale data (54×54 size) from International Table Tennis Federation (ITTF) database [26]. The aim is to find an appropriate positive priority vector that demonstrates the decision maker's preference and ranking problems in the incomplete case. There are a number of estimation methods for determining the weights from the pairwise comparison matrix filled in by the decision maker [2,12]. The thesis emphasizes the two methods:

Eigenvector Method (EM) and Logarithmic Least Squares Method (LLSM) [29].

My thesis is organized as follows. Chapter 1 describes the basic definitions and theorems that are connected to the topic. In Chapter 2, three algorithms are presented in order to minimize inconsistency ratio in the case of incomplete pairwise comparison matrix. Implementation of *gradient descent method* for eigenvalue minimization problem regarding incomplete pairwise comparison matrices is also provided. This is one of the main research parts of the thesis for which the eigenvalue minimization problem is of the form

$$\min_{\mathbf{x}\in\mathbb{R}_+^k} \lambda_{max}(\mathbf{A}(\mathbf{x})),$$

where $\mathbf{x} = (x_1, x_2, ..., x_k) \in \mathbb{R}_+^k$ represents the missing elements in the matrix, and $\mathbb{R}_+^k$ denotes the positive orthant of the k-dimensional Euclidean space whereas $\lambda_{max}(\mathbf{A}(\mathbf{x}))$ is the maximum eigenvalue of a given incomplete matrix $\mathbf{A}$.

Chapter 3 observes an application of incomplete pairwise comparison of matrices for ranking top table tennis players from large-scale historical data. The expected and unexpected results have also been presented. Chapter 4 is the conclusion of the thesis. Moreover, Appendix A comprises basic Harker's formulas, Tables for a ranking problem, and ITTF world ranking basic description.

Here, I have listed my own contributions to the thesis so far as follows:

- collecting the proper sources (books, papers, lecture notes) of information;
- my effort to run several algorithms (Cyclic coordinates algorithm, Newton's method for univariate and multivariate case, Logarithmic Least Squares Method, and other weighting methods);
- implementation of the gradient descent algorithm for the eigenvalue minimization problem regarding incomplete pairwise comparison matrices;
- analyzing my own ranking problem in Chapter 3 including data collection, problem definition, calculations, discussion of the results and recommendations.

Last but not least, my thesis addresses aspects of Operations Research, Decision Theory, Optimization and Ranking.

Chapter 1

Preliminaries

The aim of this chapter is to present the basic definitions with theorems and its proofs. Here, the theory is explored in general incomplete pairwise comparison matrices. The global existence and uniqueness of optimal solution for eigenvalue minimization problem is also stated.

Throughout this thesis, we shall use notations λ_{max}, *EM* and *LLSM* for Perron eigenvalue, Eigenvector Method and Logarithmic Least Squares Method, respectively. Similarly, weight vector as $\mathbf{w} = (w_1, ..., w_n)^T$, the set of positive real numbers as $\mathbb{R}_+$, and the positive orthant of the n-dimensional Euclidean space as $\mathbb{R}^n_+$.

1.1 Pairwise Comparison Matrices

Definition 1.1.1 (Pairwise comparison matrix)**.** A real matrix $\mathbf{A} = [\alpha_{ij}]_{n\times n)}$ is called *pairwise comparison matrix* if $\alpha_{ij} > 0$, $\alpha_{ii} = 1$ and $\alpha_{ij} = \frac{1}{\alpha_{ji}}$ for all $i, j = 1, ..., n$.

The goal is to determine a positive weight vector $\mathbf{w} = (w_1, w_2, ..., w_n)^T \in \mathbb{R}^n_+$ such that $\alpha_{ij} \approx \frac{w_i}{w_j}$ for all i, j. That means, if the entries precisely represent ratios between weights, then the given matrix $\mathbf{A}$ can be expressed as $\mathbf{A} = [\frac{w_i}{w_j}]_{(n\times n)}$.

With reciprocity condition, $\mathbf{A}$ can be rewritten as

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ \frac{1}{\alpha_{12}} & 1 & \alpha_{23} & \cdots & \alpha_{2n} \\ \frac{1}{\alpha_{13}} & \frac{1}{\alpha_{23}} & 1 & \cdots & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha_{1n}} & \frac{1}{\alpha_{2n}} & \frac{1}{\alpha_{3n}} & \cdots & 1 \end{pmatrix}. \tag{1.1.1}$$

Definition 1.1.2 (Consistency)**.** A pairwise comparison matrix is $\mathbf{A} = [\alpha_{ij}]_{(n\times n)}$ called *consistent* if the transitivity $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ holds for all $i, j, k = 1, 2, ..., n$. Otherwise, the matrix is called *inconsistent*.

If $\mathbf{A}$ is consistent, then the rank of $\mathbf{A}$ is one and the weights are unique up to a positive scalar multiple.

There are a lot of ways of measuring or indexing the level of inconsistency, see Subsection 1.1.4. Here, the consistency conditions are formulated in some equivalent ways as follows:

- $\alpha_{ik} = \alpha_{ij}\alpha_{jk}$ for all i, j, k,
- There exists a weight vector $(w_1, ..., w_n)^T$ such that $\alpha_{ij} = \frac{w_i}{w_j}$ for all i, j,
- Rank($\mathbf{A}$) =1.
- The pairwise comparison matrix $\mathbf{A}$ has its λ_{max}, equal to n.

Remark 1.1. The number of independent transitivities (i, j, k) in a maximum of order n is $\binom{n}{3}$. See ([10], page 27).

1.1.1 Eigenvector Method

Definition 1.1.3 (Eigenvector Method (EM))**.** The Eigenvector Method provides the real weight vector $\mathbf{w}^{EM}$ (right eigenvector) of a pairwise comparison matrix $\mathbf{A}$ such that

$$\mathbf{A}\mathbf{w}^{EM} = \lambda_{max}\mathbf{w}^{EM},$$

where λ_{max} denotes the maximal eigenvalue, also known as Perron eigenvalue, of $\mathbf{A}$.

By the following Theorem 1.1.1, $\mathbf{w}^{EM}$ is positive and unique up to a scalar multiplication. A usual normalization is $\sum_{i=1}^{n} w_i^{EM} = 1$.

Theorem 1.1.1 (Perron-Frobenius Theorem)**.** *A positive square matrix has a unique largest real eigenvalue with multiplicity one and an associated strictly positive right eigenvector.*

1.1.2 The Logarithmic Least Squares Method

Definition 1.1.4 (The Logarithmic Least Squares Method (LLSM))**.** The Logarithmic Least Squares Method provides a positive real weight vector $\mathbf{w}^{LLSM}$ as the optimal solution of the following optimization problem:

$$\min_{(w_1,\ldots,w_n)} \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\log \alpha_{ij} - \log \frac{w_i}{w_j}\right]^2 \tag{1.1.2a}$$

$$\sum_{i=1}^{n} w_i = 1, w_i > 0, i = 1, 2, ..., n. \tag{1.1.2b}$$

Problem (1.1.2a)–(1.1.2a) can be solved by the geometric mean method. It is supposed that the solutions are geometrically normalized and constitutes a unique optimal solution.

Theorem 1.1.2 ([6, 29])**.** *The optimal solution of the* LLSM *problem* (1.1.2*a*) − (1.1.2*b*) *is unique and can be explicitly written as the geometric mean of the pairwise comparison matrix's row elements*

$$w_i^{LLSM} = \left(\prod_{j=1}^{n} \alpha_{ij}\right)^{\frac{1}{n}} \tag{1.1.3}$$

for all $i = 1, 2, ...n$ *after the appropriate renormalization..*

Proof. Let $\alpha > 0$. Then the objective function's value at any $\mathbf{w}$ is the same as to $\alpha\mathbf{w}$. Now let us rewrite the normalization in 1.1.2b as

$$\prod_{i=1}^{n} w_i = 1. \tag{1.1.4}$$

Putting $x_{ij} = \log \alpha_{ij}$ and $y_i = \log w_i$, we can find the following from 1.1.2a:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \left[x_{ij} + y_j - y_i\right]^2. \tag{1.1.5}$$

From $y_i = \log w_i$ and the transformation of reciprocity condition as $x_{ij} = -x_{ji}$ for $i, j = 1, ..., n$, we have $\prod_{i=1}^{n} w_i = e^{\sum_{i=1}^{n} y_i} = 1$. So, $\sum_{i=1}^{n} y_i = 0$.

Thus, we can write problem $(1.1.2a) - (1.1.2b)$ as:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left[x_{ij} + y_j - y_i \right]^2 \tag{1.1.6a}$$

$$\sum_{i=1}^{n} y_i = 0. \tag{1.1.6b}$$

Since the objective function 1.1.6a is strictly convex, it is enough to apply the first optimality conditions:

$$\frac{\partial}{\partial y_i} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_{ij} + y_j - y_i)^2 = 0, \; i = 1, ..., n.$$

Then,

$$\frac{\partial}{\partial y_i} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_{ij} + y_j - y_i)^2 = -2 \sum_{j=1}^{n} (x_{ij} + y_j - y_i) = -2 \sum_{j=1}^{n} x_{ij} + 2ny_i - 2 \sum_{j=1}^{n} y_j = 0.$$

Since $\sum_{j=1}^{n} y_j = 0$ in 1.1.6b, then $2 \sum_{j=1}^{n} x_{ij} = 2ny_i$. Hence, we get the solution

$$y_i = \frac{\sum_{j=1}^{n} x_{ij}}{n}, \; i = 1, ..., n.$$

Rewriting this again to the original variables, we obtain $\log w_i = \frac{\sum_{j=1}^{n} \log \alpha_{ij}}{n}$. That means,

$$w_i^{LLSM} = \left(\prod_{j=1}^{n} \alpha_{ij} \right)^{\frac{1}{n}}, \; i = 1, ..., n.$$

□

1.1.3 Other Weighting Methods

There are many estimation methods for the priority values other than the above two methods. However, none of the weighting methods are totally superior over the others. Each of them works best on its own standard of effectiveness.

Some estimating methods for deriving ratio scaled weight vectors are presented in a short form in Table 1.1 with their formulas. Most of them have the following required properties [2, 12]:

- keeps ranks robustly: if $\alpha_{ik} \geq \alpha_{jk}\,(k = 1, ..., n)$, then $w_i \geq w_j$
- correctness in the case of consistency: if $\mathbf{A}$ is consistent, then the estimated priority vector is equal to the true priority vector;
- smoothness of outcomes: small changes in $\mathbf{A}$ do not cause big changes in the estimated priority vector;
- comparison order independence: the estimated weight vector is independent of the order of the attributes compared in $\mathbf{A}$.

Remark 1.2. The EM, LLSM and all other weighting methods bring about the same weights when the matrix is consistent.

1.1.4 Inconsistency Ratio

In solving real decision problems, the priority vector $\mathbf{w}$ can be approximated by using the inconsistent pairwise comparison matrix though the levels of inconsistency are different. Some of which are acceptable by the real decision maker and some are not in Saaty's sense [8, 36, 37].

There are different consistency indices in many literatures. Saaty [36, 37] proposed the *Consistency Index* as

$$CI = \frac{\lambda_{max} - n}{n - 1}.$$

Note that the logical name would be inconsistency index, but customarily it is often called consistency index.

However, CI is not guaranteed to compare matrices of different orders due to the variations of expectations of CI values [10]. So, it requires to be rescaled as follows.

Definition 1.1.5 (Inconsistency Ratio)**.** According to Saaty [36, 37], the inconsistency ratio CR is defined as

$$CR = \frac{CI}{RI_n}$$

where $RI_n = \frac{\overline{\lambda}_{max} - n}{n-1}$, and $\overline{\lambda}_{max}$ is an average value of the Perron eigenvalues of randomly generated $n \times n$ pairwise comparison matrices. Some estimated values of RI_n are given in *Table* 1.2 [10, 36, 37].

Table 1.1: Weighting Methods

Method	The problem to be solved
The Geometric Mean Method (GM)	$\min \sum_{i \neq j} \left[\log \alpha_{ij} - \log \frac{w_i}{w_j} \right]^2$ $\prod_{i=1}^{n} w_i = 1,$ $w_i > 0,\ i = 1, 2, ..., n$
The Normalized Column Mean Method (NCM)	$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\alpha_{ij}}{\sum_{k=1}^{n} \alpha_{kj}} - w_i \right]^2$ $\sum_{i=1}^{n} w_i = 1,$ $w_i > 0,\ i = 1, 2, ..., n$
Least Squares Method (LSM)	$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\alpha_{ij} - \frac{w_i}{w_j} \right]^2$ $\sum_{i=1}^{n} w_i = 1,$ $w_i > 0,\ i = 1, 2, ..., n$
Chi Squares Method (χ^2 M)	$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\left(\alpha_{ij} - \frac{w_i}{w_j} \right)^2}{\frac{w_i}{w_j}}$ $\sum_{i=1}^{n} w_i = 1,$ $w_i > 0,\ i = 1, 2, ..., n$
Singular Value Decomposition Method (SVDM)	$w_i^{SVD} = \frac{u_i + \frac{1}{v_i}}{\sum_{j=1}^{n} u_i + \frac{1}{v_i}}$ $i = 1, ..., n$ *where* $\mathbf{A}_{[1]} = \alpha_1 \mathbf{u}\mathbf{v}^T$, *approximation of* $\mathbf{A}$ *with the best* 1 *rank in* *Frobenius norm*

n	3	4	5	6	7	8	9	10	11
RI_n	0.5247	0.8816	1.1086	1.2479	1.3417	1.4057	1.4499	1.4854	1.5156

Table 1.2: Some values for RI_n

It is shown in Proposition 1.1.3 that $\lambda_{max} \geq n$, and equals to n if and only if the matrix is consistent. In practice one should accept matrices with values CR less than or equal to 0.1 and reject values greater than 0.1 [36, 37].

Example 1.1.1. Consider a pairwise comparison matrix $\mathbf{A}$ as follows.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 8 & 1 \\ 1/3 & 1 & 1/2 & 1/7 \\ 1/8 & 2 & 1 & 3 \\ 1 & 7 & 1/3 & 1 \end{pmatrix}.$$

Here, the maximum eigenvalue, $\lambda_{max}(\mathbf{A}) = 5.3612$. Using the formula for CR, we have

$$CR = \frac{CI}{RI_4} = \frac{(5.3612 - 4)/3}{0.8816} \approx 0.5146,$$

which is significantly greater than 0.1. Hence, in Saaty sense, the judgement matrix $\mathbf{A}$ is too inconsistent. So, a decision maker has to revise his decision until $CR < 0.1$.

Proposition 1.1.3 ([10])**.** *Let* $\mathbf{A}$ *be a pairwise comparison matrix. Then* $\lambda_{max} = n$ *if and only if* $\mathbf{A}$ *is consistent, and* $\lambda_{max} > n$ *otherwise.*

Proposition 1.1.4 ([39])**.** *Let* $\mathbf{A} = [\alpha_{ij}]_{n\times n}$ *be a pairwise comparison matrix. Then* $\mathbf{A}$ *is consistent if and only if its characteristic polynomial can be written as*

$$p_{\mathbf{A}}(t) = t^n - nt^{n-1}.$$

Proof. We consider the characteristic polynomial of $\mathbf{A}$,

$$p_{\mathbf{A}}(t) = \det(\mathbf{A} - tI).$$

Let $p_{\mathbf{A}}(t) = t^n - nt^{n-1}$. Then $p_{\mathbf{A}}(t)$ has solutions $t = 0$, and $t = n$. So, the maximum eigenvalue is equal to n, i.e. $t^* = \lambda_{max} = n$. Hence, by Proposition 1.1.3, $\mathbf{A}$ is consistent.

Conversely, let $\mathbf{A}$ be consistent. Then using the relation $\alpha_{1j}\alpha_{jk} = \alpha_{1k}$ for all $j, k = 1, ..., n$, we obtain

$$p_{\mathbf{A}}(t) = \det(\mathbf{A} - tI) = \det \begin{pmatrix} 1-t & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ \alpha_{21} & 1-t & \alpha_{23} & \cdots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & 1-t & \cdots & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \cdots & 1-t \end{pmatrix}$$

$$= \alpha_{21} \cdots \alpha_{n1} \cdot \det \begin{pmatrix} 1-t & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ 1 & \alpha_{12}(1-t) & \alpha_{13} & \cdots & \alpha_{1n} \\ 1 & \alpha_{12} & \alpha_{13}(1-t) & \cdots & \alpha_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n}(1-t), \end{pmatrix}$$

$$= \det \begin{pmatrix} 1-t & 1 & 1 & \cdots & 1 \\ 1 & 1-t & 1 & \cdots & 1 \\ 1 & 1 & 1-t & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1-t \end{pmatrix}$$

$$= \det \begin{pmatrix} n-t & n-t & n-t & \cdots & n-t \\ 1 & 1-t & 1 & \cdots & 1 \\ 1 & 1 & 1-t & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1-t \end{pmatrix}$$

$$= (n-t) \det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1-t & 1 & \cdots & 1 \\ 1 & 1 & 1-t & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1-t \end{pmatrix}$$

$$= (n-t)\det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -t & 0 & \cdots & 0 \\ 0 & 0 & -t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -t \end{pmatrix}$$

$$= (n-t)(-t)^{n-1}$$

$$= t^n - nt^{n-1}.$$

□

Remark 1.3. Any 2×2 pairwise comparison matrix is *consistent*.

Now we consider the coefficients of characteristic polynomial of $\mathbf{A}$ for the following theorem. That is, $p_{\mathbf{A}}(t) = t^n + c_1 t^{n-1} + ... + c_{n-1}t + c_n$. Thus, we can decide the consistency of $\mathbf{A}$ from the coefficient c_3 alone.

Theorem 1.1.5 ([39], Theorem 1)**.** *Let $n \geq 3$ and $\mathbf{A}$ be a pairwise comparison matrix. Then $\mathbf{A}$ is* consistent *if and only if $c_3 = 0$.*

1.2 Incomplete Pairwise Comparison Matrices

In the case of complete pairwise comparison matrix, i.e. when all its entries are known, the two well known methods are defined in Sections 1.1.1 and 1.1.2.

It may happen that one or more entries are not given by the decision expert for several reasons: when the decision maker does not complete the judgment matrix due to lack of time or some of the data have been lost, or s/he is not able to make comparisons about ranking of certain pairs. Incompleteness may indeed be the case if the number n of items is a huge task to give as many as $\binom{n}{2}$ considerable estimates.

From now on, we shall assume our pairwise comparison matrices are not completely filled in by the decision maker.

Definition 1.2.1. Pairwise comparison matrix with one or more missing elements is called *Incomplete Pairwise Comparison Matrix (IPCM). IPCM* is similar to (1.1.1)

in form, except the missing elements. Missing elements of pairwise comparison matrices are denoted by *. It is structured as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha_{12} & * & \cdots & \alpha_{1n} \\ \frac{1}{\alpha_{12}} & 1 & \alpha_{23} & \cdots & * \\ * & \frac{1}{\alpha_{23}} & 1 & \cdots & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha_{1n}} & * & \frac{1}{\alpha_{3n}} & \cdots & 1 \end{pmatrix}. \tag{1.2.7}$$

Generally, by introducing variables x_i's instead of the missing elements, we can rewrite the *incomplete pairwise comparison matrix* $\mathbf{A}$ in (1.2.7) as

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}(x_1, x_2, ..., x_k) = \begin{pmatrix} 1 & \alpha_{12} & x_1 & \cdots & \alpha_{1n} \\ \frac{1}{\alpha_{12}} & 1 & \alpha_{23} & \cdots & x_k \\ \frac{1}{x_1} & \frac{1}{\alpha_{23}} & 1 & \cdots & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha_{1n}} & \frac{1}{x_k} & \frac{1}{\alpha_{3n}} & \cdots & 1 \end{pmatrix} \tag{1.2.8}$$

where $\mathbf{x} = (x_1, x_2, ..., x_k) \in \mathbb{R}^k_+$. There are k variables in the upper triangular part and another k number of variables in its lower triangular part. That means, totally $2k$ number of variables occur in $\mathbf{A}$. Furthermore, linear algebraic manipulations in $\mathbf{A}$ are defined after the completion of the missing entries.

1.2.1 Graph Representation

In multi-criteria decision making process, it is a natural phenomenon to describe pairwise comparison matrices with the corresponding associated graph structures. Two criteria that have no direct relation may be described as an indirect relation through its associated undirected graph.

Definition 1.2.2 (Undirected Graph)**.** For a given $n \times n$ (in)complete pairwise comparison matrix $\mathbf{A}$, its associated *undirected graph* G is defined by

$$G := (V, E),$$

where $V = \{1, 2, ..., n\}$ denotes the nodes (vertices) corresponding to matrix order n, and E= $\{e(i,j)|\alpha_{ij}$ (and $\alpha_{ji})$ is given, $i \neq j\}$ represents the undirected edges corresponding to the matrix entries. The edge is assigned exactly from node i to j if the comparison element α_{ij} is already given. No edges are assigned for the missing elements in the matrix.

Remark 1.4. For a complete pairwise comparison matrix, its corresponding undirected graph G is a complete graph K_n [6].

1.2.2 Eigenvector Method for Incomplete Pairwise Comparison Matrices

Based on Saaty's inconsistency ratio CR [36], Shiraishi, Obata and Daigo [38, 39] considered the eigenvalue minimization problem for a given incomplete matrix $\mathbf{A}$ as

$$\min_{\mathbf{x}\in\mathbb{R}_+^k} \lambda_{max}(\mathbf{A}(\mathbf{x})), \tag{1.2.9}$$

where $\mathbf{x} = (x_1, x_2, ..., x_k) \in \mathbb{R}_+^k$ represents the missing elements in the matrix, and $\mathbb{R}_+^k$ denotes the positive orthant of the k-dimensional Euclidean space whereas $\lambda_{max}\mathbf{A}(\mathbf{x})$ is the maximum eigenvalue that can be obtained from $(\mathbf{A}(\mathbf{x}))$ in (1.2.8).

The aim is to solve this eigenvalue optimization problem by applying different suitable algorithms that are presented in Chapter 2. That means, equivalently, we need to find the best completion of the given matrix in order to get the minimum inconsistency ratio (in Saaty sense).

In this section, we present some basic concepts and state the main theorems that are applicable to the next chapters.

Definition 1.2.3 (Logarithmically Convex Function)**.** Let $S \subset \mathbb{R}^k$ be a convex set. A function $f : S \to \mathbb{R}_+$ is called log-convex or superconvex if the composition function $\log f : S \to \mathbb{R}$ is a convex function.

Basically, the logarithm extremely makes to decrease the growth of the original function f, hence we call it superconvex if the composition still holds convexity. It is possible to show that log-convexity implies convexity, but the converse is not necessarily true.

Parameterizing the elements of the incomplete pairwise comparison matrix $\mathbf{A}(\mathbf{x}) = \mathbf{A}(x_1, x_2, ..., x_k)$ of (1.2.8) as $x_i = e^{y_i}$ for $i = 1, 2, ..., k$, we can establish a matrix of the form [6]

$$\mathbf{A}(\mathbf{x}) = \mathbf{B}(\mathbf{y}) = \mathbf{B}(y_1, y_2, ..., y_k). \tag{1.2.10}$$

Proposition 1.2.1 ([6]). *For the parametrized matrix* $\mathbf{B}(\mathbf{y})$ *from (1.2.10) the Perron eigenvalue* $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ *is a log-convex (hence convex) function of* y.

Proof. See (pages 321–323). □

Proposition 1.2.2 ([6]). *The Perron eigenvalue* $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ *is either strictly convex or constant along any line in the* $\mathbf{y}$ *space.*

Theorem 1.2.3 ([6], Theorem 2). *The optimal solution of the eigenvalue minimization problem (1.2.9) is unique if and only if the graph* G *corresponding to the incomplete pairwise comparison matrix is connected.*

Theorem 1.2.4 ([6], Theorem 3). *The function* $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ *attains its minimum over* $\mathbb{R}^k$, *and the optimal solutions constitute an* $(s-1)$ *dimensional affine set of* $\mathbb{R}^k$, *where* s *is the number of the connected components in* G.

Remark 1.5. The Perron eigenvalue minimization problem (1.2.9) is a non-convex function of its entries. However, if graph G corresponding to the incomplete pairwise comparison matrix is connected, then by using the exponential parametrization $x_1 = e^{t_1}, x_2 = e^{t_2}, ..., x_k = e^t_k$, it can be transformed into a strictly convex minimization problem.

Remark 1.6. Based on Tummala and Ling [42], the formula of the maximum eigenvalue λ_{max} for 3×3 incomplete pairwise comparison matrix $\mathbf{M}$ (where x, y, z are the elements in the upper triangular):

$$\mathbf{M}(x, y, z) = \begin{pmatrix} 1 & x & y \\ \frac{1}{x} & 1 & z \\ \frac{1}{y} & \frac{1}{z} & 1 \end{pmatrix}$$

is given as

$$\lambda_{max}(\mathbf{M}(x, y, z)) = 1 + \sqrt[3]{\frac{y}{xz}} + \sqrt[3]{\frac{xz}{y}}. \tag{1.2.11}$$

Example 1.2.1. Let $\mathbf{M}$ be a 3×3 pairwise comparison matrix with one missing element x as follows:

$$\mathbf{M} = \begin{pmatrix} 1 & x & 3 \\ \frac{1}{x} & 1 & 6 \\ \frac{1}{3} & \frac{1}{6} & 1 \end{pmatrix}.$$

Now using formula (1.2.11) and exponential parametrization $x = e^t$, we obtain $\lambda_{max}(\mathbf{M}(x)) = 1 + \sqrt[3]{\frac{1}{2x}} + \sqrt[3]{2x}$ and $\lambda_{max}(\mathbf{M}(e^t)) = 1 + \sqrt[3]{\frac{1}{2e^t}} + \sqrt[3]{2e^t}$, respectively. Hence, one can see the transformation of non-convexity of the function $\lambda_{max}(\mathbf{M}(x))$ in Fig. 1.1 into convexity of the function $\lambda_{max}(\mathbf{M}(e^t))$ in Fig. 1.2.

1.2.3 The Logarithmic Least Squares Method for Incomplete Pairwise Comparison Matrices

In this section, we consider the extension of Logarithmic Least Squares Method (LLSM) (1.1.2) for incomplete pairwise comparison matrix $\mathbf{A}$. Considering only the the given entries α_{ij}, we can define *LLSM* as

$$\min_{(w_1,\ldots,w_n)} \sum_{1 \le i < j \le n} \left[\log \alpha_{ij} - \log \frac{w_i}{w_j} \right]^2 + \left[\log \alpha_{ji} - \log \frac{w_j}{w_i} \right]^2 \tag{1.2.12a}$$

$$\sum_{i=1}^{n} w_i = 1, w_i > 0, i = 1, 2, ..., n. \tag{1.2.12b}$$

The objective function (1.2.12a) consists of each pair of elements related to α_{ij} and α_{ji}. The entries related to $i = j$ are 0 and eliminated from the objective function.

Theorem 1.2.5 ([6], Theorem 4)**.** *The optimal solution of the incomplete LLSM problem (1.2.12a)–(1.2.12b) is unique if and only if graph G corresponding to the incomplete pairwise comparison matrix is connected.*

Remark 1.7. The uniqueness of the optimal solution depends on the structure of the graph G, but not the values of comparisons.

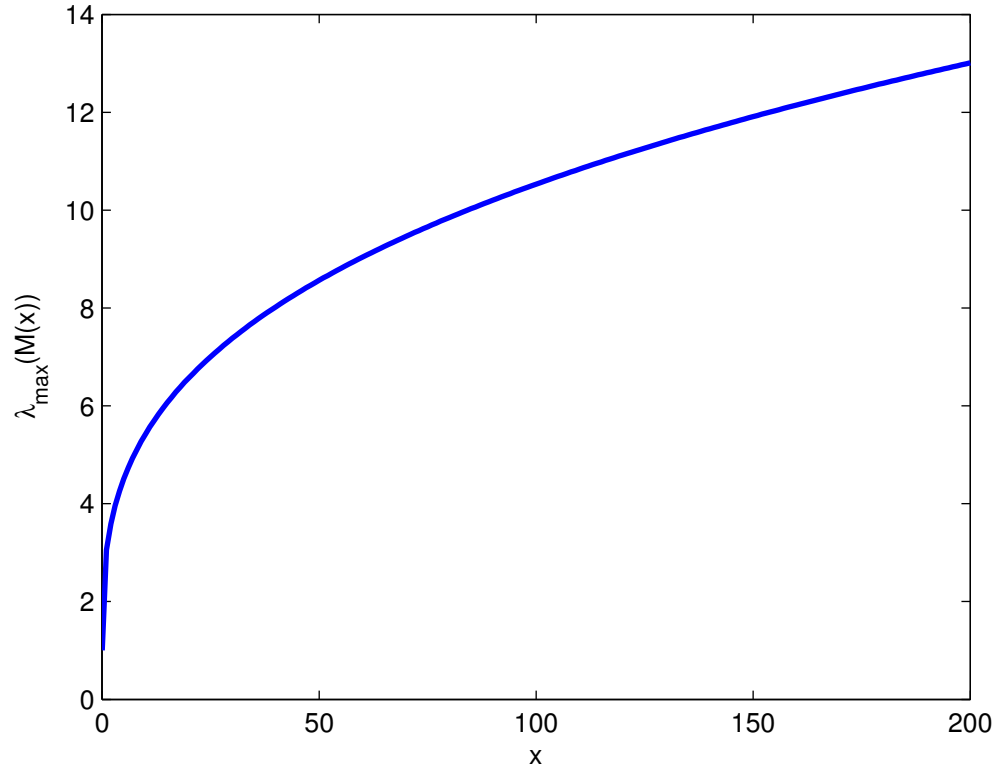

Figure 1.1: Non-convexity of the function $x \mapsto \lambda_{max}(\mathbf{M}(x))$ in Example 1.2.1

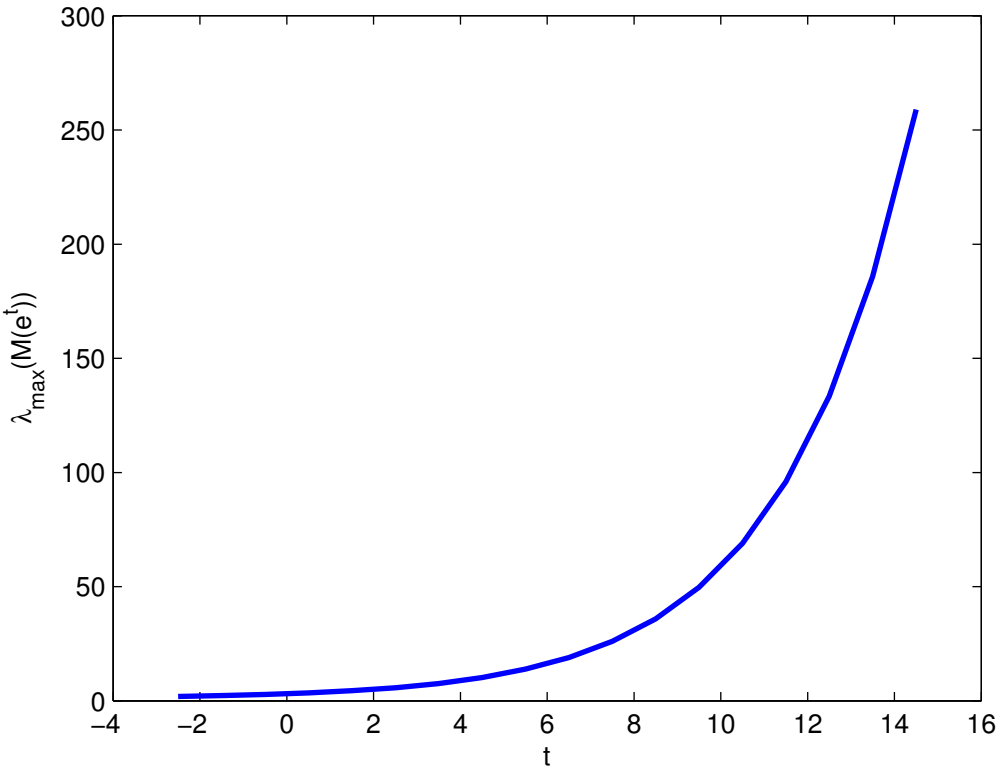

Figure 1.2: Strict convexity of the function $t \mapsto \lambda_{max}(\mathbf{M}(e^t))$ in Example 1.2.1

Chapter 2

Algorithms for Perron Eigenvalue Minimization Problem

In this chapter, we observe three practical Perron eigenvalue minimization algorithms that are used to minimize the inconsistency ratio CR, or equivalently, to find the best (λ_{max}-optimal) completion to the eigenvalue minimization problem (1.2.9). The generalization of the eigenvector method is also applied here as proposed by Shiraishi, Obata and Diago [38, 39].

It is also shown that the Perron eigenvalue minimization problem can be transformed into convex minimization problem using exponential parametrization. Hence, the existence of optimal solution is guaranteed [6].

One can provide an algorithm with or without derivative information for filling in the gap of the incomplete pairwise comparison matrix as good as possible (i.e. $CR < 0.1$). Bozóki et al. [6] proposed an algorithm which is called *cyclic coordinates*. They used a general optimization function *fminbnd* in MATLAB for determining the best optimal completion. Further, Ábele-Nagy [1] suggested *Newton's Method* for univariate and multivariate case to compute the Perron eigenvalue optimization problem. He applied Harker's [24] first and second derivatives in his iterations procedure. In our approach, a *gradient descent method* is proposed in Section 2.3.

2.1 Cyclic Coordinates Algorithm

The iterative method of *cyclic coordinates* minimizes the function λ_{max} cyclically with respect to the coordinate variables. The method of cyclic coordinates is independent of the derivatives. It can be used with derivatives, but also without derivatives. The main idea of cyclic coordinates is that the problem is divided to many, single variable optimization problems. In each step, a single variable function is minimized. The procedure is, with arbitrary or algorithmic initial value $x_i^{(0)}, i = 1, 2, ..., k$, where k denotes the number of missing elements (current variables) by considering the upper triangular matrix only. Each iteration of the method comprises k steps. Thus, x_1 is changed first by fixing the rest associated variables, then x_2 and so forth through x_k. The process is then repeated starting with x_1 again until it achieves the stopping criteria.

The steps of the algorithm is stated as follows in reference to [1,6,30]. Here, we consider the incomplete pairwise comparison matrix $\mathbf{A}$. The aim is to find a complete pairwise comparison matrix $\mathbf{A}$ such that λ_{max} is minimal. Let $\mathbf{x} = (x_1, x_2, ..., x_k)$, where $x_i \in \mathbf{R}_+$ are the missing elements for finite i. Also let $x_i^{(m)}$ denote the value of x_i in the m^{th} step of the iteration.

Algorithm 1 Cyclic coordinates algorithm for $\min_{\mathbf{x}} \lambda_{max}(\mathbf{A}(\mathbf{x}))$

1: **procedure** LET $x_i^{(m)}$ DENOTE THE VALUE OF x_i IN THE m^{th} STEP OF THE ITERATION AND $t_i = \log x_i \ \forall i = 1, 2, ..., k$.
2: *Set* $m \leftarrow 0$, *and* $x_i^{(0)} \leftarrow 1 \ \forall i$
3: **while** $\max_{i=1,2,..,k} ||x_i^{(m)} - x_i^{(m-1)}|| > Tolerance$ **do**
4: choose $i \in \{1, 2, .., k\}$
5: $x_i^{(m)} \leftarrow \arg\min_{x_i} \lambda_{max} A(x_1^{(m)}, ..., x_{i-1}^{(m)}, x_i, x_{i+1}^{(m-1)}, ..., x_k^{(m-1)})$
6: repeat step 5 for all $i \in \{1, 2, .., k\}$, then go to step 7
7: $m \leftarrow m + 1$
8: **end while**.

The global convergence for *cyclic coordinate descent algorithm* is stated and proved in ([30], pages 266–267). Regarding to the rates of convergence, it is difficult

to compare cyclic coordinates with that of *steepest descent* and Newton's method, since it is not affected by scale factor changes. However, it is affected by rotation of coordinates. Nonetheless, some comparison is doable.

2.2 Newton's Method

Newton's method is the popular algorithm that is used to find an optimal solution using first derivative (in univariate case) and second derivative (in multivariate case) information. In univariate case, it endeavors to construct a sequence x^n from an initial value (guess) x^0 that converges to a certain value x^* that makes the first derivate to vanish at x^*.

In multivariate case, the gradient vector $\nabla F(\mathbf{x})$ and the inverse of Hessian matrix $\mathbf{H}(\mathbf{x})$ are required provided that the function F is twice differentiable.

In this section, we give multivariate Newton's algorithm for solving Perron eigenvalue minimization problem for a given incomplete pairwise comparison matrix. Following Bozóki et al. [6] paper, Ábele-Nagy [1] proposed the univariate and multivariate Newton methods for his optimal completion procedure. In the univariate case, he used the method of cyclic coordinates with Newton iteration to optimize only one variable at a time.

Applying Newton's Multivariate method, one can optimize in all of the variables at the same time instead of optimizing one variable at a time cyclically [1]. The procedure for multivariate Newton's method is provided as follows [1]. Let $x = e^t$ be in the position of (i, j) in the matrix and $F(t) = \lambda_{max}(e^{t_1}, e^{t_2}, ..., e^{t_k})$, where k is the number of elements in the matrix (upper triangular at a time). By Harker [21], the derivatives $\frac{\partial \lambda_{max}(x)}{\partial x}$ and $\frac{\partial^2 \lambda_{max}(x)}{(\partial x)^2}$ are known. Here, we want to minimize F. To do so, we use Newton's iteration

$$t^{(m+1)} = t^{(m)} - \gamma[HF(t^{(m)})]^{-1}\nabla F((t^{(m)}), \tag{2.2.1}$$

where $HF(t^{(m)})$ denotes the Hessian matrix of $F(t)$, $\nabla F(t^{(m)})$ is the gradient vector of $F(t)$, and the usual Newton's step size γ .

The gradient vector $\nabla F(t) = (\frac{\partial F}{\partial t_1}, ..., \frac{\partial F}{\partial t_k})$ can be found from the following

method [1, 21] and it is also given in the Appendix (A.1):

$$\frac{\partial F(t)}{\partial t} = \frac{\partial \lambda_{max}(e^t)}{\partial t} = \frac{\partial \lambda_{max}(x)}{\partial x} \cdot \frac{\partial (e^t)}{\partial t} = \frac{\partial \lambda_{max}(x)}{\partial x} \cdot e^t. \tag{2.2.2}$$

Similarly, the derivation for the Hessian matrix has been given from the paper ([1], page 65). The stopping criteria must be given for x, but not for t as small changes in t results in large differences in x.

Algorithm 2 Newton's Multivariate Algorithm for $\min_{\mathbf{x}} \lambda_{max}(\mathbf{A(x)})$

1: **procedure** LET $t^{(m)}$ DENOTE THE VALUE OF t IN THE m^{th} ITERATION, AND $x_i^{(m)}$ BE THE VALUE OF x_i IN THE m^{th} ITERATION AND $t_i = \log x_i \; \forall i = 1, 2, ..., k$.
2: *Starting values of* t, where $t = \log x$, and *step size* γ
3: *Set* $m \leftarrow 0$
4: **while** $\max_{i=1,2,..,k} ||x_i^{(m)} - x_i^{(m-1)}|| > Tolerance$ **do**
5: $t^{(m+1)} \leftarrow t^{(m)} - \gamma [HF(t^{(m)})]^{-1} \nabla F(t^{(m)})$
6: $m \leftarrow m + 1$.
7: **end while**.

Assuming the Jacobian matrix at the current point is invertible, we can say that Newton's method has a quadratic convergence. Otherwise, the global converge may fail. Newton's method provides a better finite iteration relative to the cyclic coordinates and gradient descent methods. However, evaluation of the Hessian at every point can be costly. Its convergence is stated and proved in Numerical Optimization, Nocedal and Wright's book ([34], pages 44–45).

2.3 Gradient Descent Method

Gradient descent algorithm (also called *steepest descent*) is a first-order derivative iterative optimization method by taking steps proportional to the negative of the gradient function at the current point.

In this section, we are interested in solving the eigenvalue minimization problem (1.2.9) by *gradient descent method*.

2.3.1 The Algorithm

With parametrization $x = e^t$, let $F(t) = \lambda_{max}(e^{t_1}, e^{t_2}, ..., e^{t_k})$, where k denotes the number of missing elements in the pairwise comparison matrix (upper triangular). The aim is to minimize $F(t)$. The gradient function $\nabla F(t) = (\frac{\partial F}{\partial t_1}, ..., \frac{\partial F}{\partial t_k})$ is obtained from $F(t)$ in equation (2.2.2). (See also the appendix for Harker's [21] first derivative formula).

Since $F(t)$ is convex and differentiable, for a given point $t = \bar{t}$, the approximation of $F(t)$ can be presented as a linear approximation

$$F(\bar{t} + d) \approx F(\bar{t}) + \nabla F(\bar{t})^T d,$$

if the norm $||d||$ is small. Now we choose d so as to make the scalar product $\nabla F(\bar{t})^T d$ is as small as possible. Normalizing d with $||d|| = 1$, a particular direction $d^* = \frac{-\nabla F(\bar{t})}{||\nabla F(\bar{t})||}$ gives the smallest scalar product with the gradient $\nabla F(\bar{t})$. The following inequality asserts this reality [18]:

$$\nabla F(\bar{t})^T d \geq -||\nabla F(\bar{t})|| \cdot ||d|| = \nabla F(\bar{t})^T \left(\frac{-\nabla F(\bar{t})}{||\nabla F(\bar{t})||} \right) = \nabla F(\bar{t})^T d^*.$$

Following this, the un-normalized direction $\bar{d} = -\nabla F(\bar{t})$ is called the *direction of gradient descent*, or simply *descent direction*, at the current point $\bar{t}$.

It is worth noting that $\bar{d}^T \nabla F(\bar{t}) = -\nabla F(\bar{t})^T \nabla F(\bar{t}) < 0$ only if $\nabla F(\bar{t}) \neq 0$, i.e. $\bar{d}$ is the descent direction provided that $\nabla F(\bar{t}) \neq 0$. It follows that $F(\bar{t}^{m+1}) < F(\bar{t}^m)$.

The stopping criteria is given for x, but not for t as small changes in t results in large differences in x. The locally optimal step size γ maybe chosen by an exact or inexact *line search algorithm* in each iteration or just fixed *gamma*. However, carrying out *line search* can be time-taking. On the other hand, using a fixed γ can result in poor convergence. Using Newton's method and Hessian versions of conjugate gradient methods can be better options to get fewer iterations. But, its each iteration cost is higher. Nonetheless, the algorithm works well in any number of dimension. Its global and rate of convergence is stated and proved in ([18], pages 3–5). The algorithm is given as follows.

Algorithm 3 Gradient Descent Algorithm for $\min_{\mathbf{x}} \lambda_{max}(\mathbf{A}(\mathbf{x}))$

procedure LET t^m DENOTE THE VALUE OF t IN THE m^{th} ITERATION, AND x_i^m BE THE VALUE OF x_i IN THE m^{th} ITERATION WHERE $t_i = \log x_i \ \forall i = 1, 2, ..., k$.
  Choose starting values of t and step size γ
  Set $m \leftarrow 0$
  while $\max_{i=1,2,..,k} ||x_i^{(m)} - x_i^{(m-1)}|| > Tolerance$ **do**
  $d^m \leftarrow -\nabla F(t^m)$
  If $d^m = 0$, *then* stop. Otherwise, go to step 7
  $t^{m+1} \leftarrow t^m + \gamma d^m$
  $m \leftarrow m + 1$
  end while.

2.3.2 Numerical Example

Let us take an 8×8 incomplete pairwise comparison matrix as follows:

$$\mathbf{A}(\mathbf{x}) = \begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & x_1 & 5 & x_2 & 3 & x_3 & 1/7 \\ 1/3 & 1/x_1 & 1 & x_4 & 3 & x_5 & 6 & x_6 \\ 1/7 & 1/5 & 1/x_4 & 1 & x_7 & 1/4 & x_8 & 1/8 \\ 1/6 & 1/x_2 & 1/3 & 1/x_7 & 1 & x_9 & 1/5 & x_10 \\ 1/6 & 1/3 & 1/x_5 & 4 & 1/x_9 & 1 & x_{11} & 1/6 \\ 3 & 1/x_3 & 1/6 & 1/x_8 & 5 & 1/x_{11} & 1 & x_{12} \\ 4 & 7 & 1/x_6 & 8 & 1/x_{10} & 6 & 1/x_{12} & 1 \end{pmatrix}.$$

This is an example of Saaty's 'buying a house' incomplete version [36]. Moreover, this matrix has been used as an example in both Bozóki et al. [6] and Ábele-Nagy [1] papers. It is easy to see that the graph associated to the matrix is connected [6]: the first row is completely filled in, node 1 is directly connected to all nodes.

The aim is to find the complete matrix $\mathbf{A}$ for which λ_{max} is minimal by using gradient descent method. In this matrix, the missing elements are denoted by vector $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \in \mathbb{R}_+^{12}$.

Table 2.1: The 21 iterations of the gradient descent algorithm to the given incomplete pairwise comparison matrix **A**.

m	$x_1^{(m)}$	$x_2^{(m)}$	$x_3^{(m)}$	$x_4^{(m)}$	$x_5^{(m)}$	$x_6^{(m)}$	$x_7^{(m)}$	$x_8^{(m)}$	$x_9^{(m)}$	$x_{10}^{(m)}$	$x_{11}^{(m)}$	$x_{12}^{(m)}$
0	0.3823	1.8428	0.4758	8.9924	4.2688	0.5228	0.5361	0.1384	0.8855	0.1085	0.2916	0.4200
1	0.3338	1.7890	0.4866	9.7728	4.7893	0.5609	0.5426	0.1495	0.8975	0.1061	0.3045	0.3834
2	0.3384	1.7603	0.4746	9.6902	4.7370	0.5565	0.5396	0.1445	0.9084	0.1063	0.2961	0.3979
3	0.3341	1.7529	0.4727	9.7944	4.7905	0.5626	0.5344	0.1446	0.9142	0.1075	0.2953	0.3967
4	0.3333	1.7403	0.4710	9.8250	4.8064	0.5641	0.5320	0.1437	0.9199	0.1079	0.2941	0.3999
5	0.3322	1.7351	0.4692	9.8566	4.8199	0.5297	0.5297	0.1434	0.9231	0.1084	0.2930	0.4002
6	0.3315	1.7298	0.4686	9.8752	4.8313	0.5669	0.5285	0.1431	0.9257	0.1086	0.2926	0.4015
7	0.3311	1.7270	0.4677	9.8899	4.8365	0.5680	0.5274	0.1429	0.9273	0.1088	0.2921	0.4017
8	0.3307	1.7246	0.4674	9.8987	4.8425	0.5683	0.5267	0.1427	0.9285	0.1090	0.2919	0.2918
9	0.3305	1.7232	0.4670	9.9059	4.8447	0.5688	0.5263	0.1427	0.9293	0.1091	0.2916	0.4024
10	0.3303	1.7221	0.4669	9.9100	4.8478	0.5690	0.5260	0.1426	0.9299	0.1092	0.2915	0.4027
11	0.3303	1.7214	0.4667	9.9136	4.8487	0.5693	0.5257	0.1425	0.9303	0.1092	0.2914	0.4028
12	0.3302	1.7209	0.4666	9.9155	4.8503	0.5693	0.5256	0.1425	0.9306	0.1092	0.2914	0.4029
13	0.3301	1.7205	0.4665	9.9173	4.8507	0.5694	0.5255	0.1425	0.9308	0.1092	0.2913	0.4029
14	0.3301	1.7203	0.4665	9.9182	4.8515	0.5695	0.5254	0.1425	0.9309	0.1093	0.2913	0.4030
15	0.3301	1.7201	0.4664	9.9190	4.8516	0.5695	0.5254	0.1425	0.9310	0.1093	0.2913	0.4030
16	0.3300	1.7199	0.4664	9.9194	4.8520	0.5695	0.5253	0.1424	0.9310	0.1093	0.2912	0.4030
17	0.3300	1.7199	0.4663	9.9198	4.8520	0.5695	0.5253	0.1424	0.9311	0.1093	0.2912	0.4030
18	0.3300	1.7198	0.4663	9.9200	4.8522	0.5695	0.5253	0.1424	0.9311	0.1093	0.2912	0.4030
19	0.3300	1.7198	0.4663	9.9202	4.8523	0.5695	0.5252	0.1424	0.9311	0.1093	0.2912	0.4030
20	0.3300	1.7198	0.4663	9.9204	4.8524	0.5695	0.5253	0.1424	0.9312	0.1093	0.2912	0.4030
21	0.3300	1.7198	0.4663	9.9205	4.8524	0.5696	0.5253	0.1424	0.9312	0.1093	0.2912	0.4031

The approximated values (up to 4 digits) for all variables in each iteration are provided in Table 2.1 based on the algorithm. Further, we have used the tolerance $T = 10^{-4}$ and initial value $x_k^{(0)}$ using the weights from Incomplete Logarithmic Least Squares Method (ILLSM) in [1, 9] depending on the position of $x_k^{(0)}$ in the matrix (i, j) such that $x_k^{(0)} = w_i/w_j$ for $k = 1, ..., 12$. With several numerical experiments, we have applied the value for step size $\gamma = 6.77$ in order to get the minimum number of iterations. Thus, the least possible 21 iterations were found which can be seen in Table 2.1.

When the starting value $x_k^{(0)} = 1 \, \forall k = 1, ..., 12$, the gradient algorithm yields 25 iterations with $\gamma = 6.55$. However, with $\gamma = 6.77$, its number of iterations gets higher which is 37.

In the case of Newton's method with ILLSM for $x_k^{(0)}$, $\gamma = 0.45$ resulted in the least number of iterations $m = 14$ in both univariate and multivariate case. But, with starting value $x_k^{(0)} = 1$, again $\gamma = 0.45$ yielded $m = 15$ in the univariate case

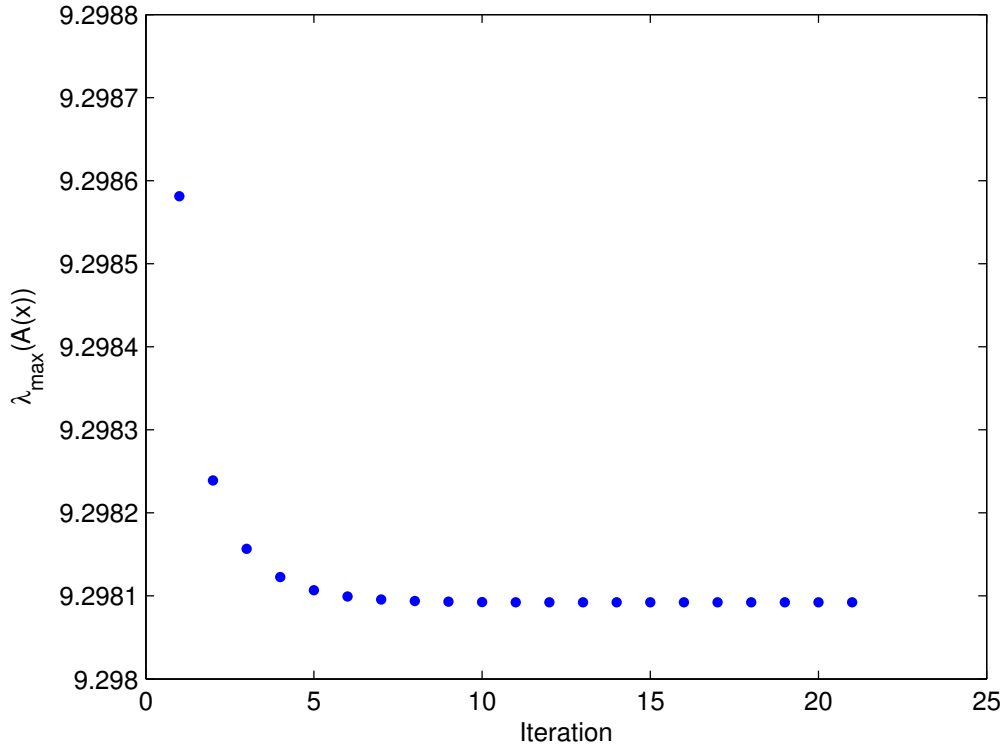

Figure 2.1: $\lambda_{max}(\mathbf{A}(\mathbf{x}))$ values for the first 21 iterations

while $m = 26$ iterations in the multivariate case [1]. However, Newton's multivariate method has provided the least possible iterations ($m = 20$) with $\gamma = 1.77$ and $x_k^{(0)} = 1$ for all $k = 1, 2, .., 12$.

In addition to this, one can observe that in Figure 2.1 the objective function's value decreases in the first few iterations (with 4 decimal place approximation). However, $\lambda_{max}(\mathbf{A}(\mathbf{x}))$ values decrease throughout the iterations if we consider 15 decimal places. For instance, λ_{max} value for the 20th and 21st iterations is 9.298092250616087 and 9.298092250477843, respectively.

Based on the algorithm results from Table 2.1, the optimal solution of the eigenvalue minimization problem for the incomplete matrix $\mathbf{A}$ is given as follows. Here, the missing elements are completed with four digits accuracy. The algorithm yields the same result with that of Bozóki et al. [6] and Ábele-Nagy [1], except few variants in the third and fourth decimal places (missing values are taken from the last iteration):

$$\mathbf{A}(\mathbf{x}^*) = \begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & \mathbf{0.3300} & 5 & \mathbf{1.7198} & 3 & \mathbf{0.4664} & 1/7 \\ 1/3 & \mathbf{3.0303} & 1 & \mathbf{9.9205} & 3 & \mathbf{4.8524} & 6 & \mathbf{0.5696} \\ 1/7 & 1/5 & \mathbf{0.1008} & 1 & \mathbf{0.5253} & 1/4 & \mathbf{0.1424} & 1/8 \\ 1/6 & \mathbf{0.5815} & 1/3 & \mathbf{1.9040} & 1 & \mathbf{0.9312} & 1/5 & \mathbf{0.1093} \\ 1/6 & 1/3 & \mathbf{0.2061} & 4 & \mathbf{1.0740} & 1 & \mathbf{0.2912} & 1/6 \\ 3 & \mathbf{2.1445} & 1/6 & \mathbf{7.0225} & 5 & \mathbf{3.4341} & 1 & \mathbf{0.4030} \\ 4 & 7 & \mathbf{1.7556} & 8 & \mathbf{9.1491} & 6 & \mathbf{2.4814} & 1 \end{pmatrix}.$$

The minimal objective function $\lambda_{max}(\mathbf{A}(\mathbf{x}^*))$ value is 9.2981. The corresponding normalized right eigenvector is

$$\mathbf{w}^{EM} = (0.1894, 0.0567, 0.2116, 0.0175, 0.0319, 0.0354, 0.1509, 0.3066)^T.$$

Its inconsistency ratio is computed as

$$CR = \frac{CI}{RI_8} = \frac{(9.2981 - 8)/7}{1.4057} \approx 0.1319,$$

which is above the threshold (i.e. $CR > 0.1$). Hence, the matrix is a bit above acceptable inconsistency according to Saaty's criteria [36, 37].

Table 2.2 represents the comparison of three algorithms (Newton's multivariate, Gradient descent and Cyclic coordinates) using MATLAB programming with regard to iteration times (*tic* and *toc*) and iteration numbers from the same incomplete matrix **A** 2.3.2. In the table 2.2: m, $\lambda_{max}(N)$, $\lambda_{max}(G)$, $\lambda_{max}(C)$, iterTime(N), iterTime(G), and iterTime(C) denote number of iterations, λ_{max} from Newton's method, λ_{max} from Gradient descent method, λ_{max} from Cyclic coordinates method, iteration time from Newton's method, iteration time from Gradient descent, and iteration time from Cyclic coordinates, respectively. Here, we need to observe which algorithm is costly in each step of the iterations (with fixed $x_k^{(0)} = 1$, $\gamma = 6.55$ for gradient descent, and $\gamma = 1.77$ for Newton's Method). We have run the algorithms for several times on the same computer. Hence, one can see that each iteration time of Cyclic coordinates method is higher than the others. In the case of gradient

Table 2.2: Comparison of Newton's, Gradient Descent and Cyclic Coordinates Algorithms with respect to iteration time

m	λ_{max}(N)	λ_{max}(G)	λ_{max}(C)	iterTime(N)	iterTime(G)	iterTime(C)
5	9.3793	9.2998	9.2981	684.8107	317.2281	799.4947
8	9.3046	9.2983	9.2981	684.8344	317.2295	799.616
10	9.2991	9.2981	9.2981	684.8444	317.2318	799.6974
12	9.2982	9.2981	9.2981	684.8543	317.2341	799.7758
14	9.2981	9.2981	9.2981	684.8747	317.2361	799.8768
20	9.2981	9.2981	9.2981	684.9389	317.242	800.125

descent, each iteration time is much less. It is noted that each corresponding λ_{max} value is relatively similar for all algorithms.

In general, it is important to note that if the objective function is convex, Newton's method provides a better finite iteration. But, each steps of the iteration is costly relative to gradient descent method.

Chapter 3

An Application of Incomplete Pairwise Comparison Matrices for Ranking Top Table Tennis Players

In this chapter, we observe an application of incomplete pairwise comparison matrices for ranking of top table tennis players for the last 90 years. Literature review and main results are also discussed.

3.1 Data Analysis

The ranking idea is based on the players that have played in a pairwise way in the long run. The data is taken from a freely available International Table Tennis Federation (ITTF) database [26] beginning from 1926. Our alternatives are the players. Thus, the comparison considers the results of the first and the second top players of each year. In this case, the reason of missing of elements is: when it is impossible to compare those players who have never played against each other on the court [5]. This yields a 54×54 incomplete pairwise comparison matrix.

The claim is to assert that we can produce a ranking of players by generating the weight vectors using one of the estimation methods which leads to the ITTF-ranking for a long period of any length. That means, ranking of players can be done based on the weight vectors, and the player with maximum weight is considered as the

'top' player.

In the ranking process, one of the main purpose of the ITTF-ranking is to provide an assistance for professional organizers depending on the winner regarding to the basic principles in Appendix A.3. In this case, the person who was defeated is not relevant. Our ranking process is different from ITTF-ranking. Both situations will be considered. Unlike ITTF-ranking, our pairwise comparison considers both the final and the match results in Round 16 equally if the players themselves are the same.

The challenging question is 'who is the best player ever?' in the 'historical' sport competition. Many literatures have been proposed regarding to the question. In the case of ranking tennis players, Radicchi (2011) computed the priority vector and constructed directed preference graph by considering all matches played between 1968 and 2010. He found that his new rankings has a better predictive power than the ATP (Professional Tennis Association) rankings. Dingle et al. [17] used PageRank-based tennis rankings instead of ATP and WTA rankings. Motegi and Masuda [33] proposed a network-based dynamic ranking system taking into account that the strength of a player depends on time. Csató [14] considered ranking by pairwise comparisons for Swiss-system tournaments. He used the two methods (Eigenvector and Logarithmic Least Squares) for his ranking procedure. However, results were constructed regarding to different scaling techniques. The proposed rankings were almost the same, except the one estimation method he applied. Bozóki et al. [5] suggested the possibility of ranking the top tennis players by using the two mathematical estimation methods and constructed directed preference graph. They applied the two estimation methods (EM and LLSM) along with correction and transformation methods. Their rankings were almost the same in both weighting methods.

Our paper discusses the results applying the correction methods in [5,40]. Based on our data, we have obtained a 54×54 incomplete matrix. The corresponding graph with its associated matrix is connected as it can be shown in Figure 3.1. Consequently, from Theorems 1.2.3 and 1.2.5, we can confirm that the optimal solution exists, and it is unique. The thesis first considers a 5×5 sub-matrix for the ranking

procedure as it can be seen in Table 3.1.

In order to give calculations from our initial data [26], we first state the definition as follows.

Definition 3.1.1 (Pairwise Comparison Matrix of Top Table Tennis Players [5])**.** We consider our initial data as

- $z_{ij}\,(i = 1, ..., n, i \neq j)$: the number of matches that have played between players P_i and P_j, where $z_{ij} = z_{ji}$;
- $x_{ij}\,(i > j)$: the number of matches between player P_i and P_j, where P_i was the winner;
- $y_{ij} = z_{ij} - x_{ij}\,(i > j)$: the number of matches between player P_i and P_j, where P_i lost against P_j.

  Then the pairwise comparison matrix $\mathbf{R} = [p_{ij}]_{n\times n}$ is calculated as follows:

- $p_{ij} = \frac{x_{ij}}{y_{ij}}$ if $i, j = 1, ..., n, i > j$ and $x_{ij} \neq 0, y_{ij} \neq 0$;
- $p_{ji} = \frac{y_{ij}}{x_{ij}}$ if $i, j = 1, ..., n, i < j$ and $x_{ij} \neq 0, z_{ij} \neq 0$;
- $p_{ii} = 1$ for all $i = 1, ..., n$;
- p_{ij} and p_{ji} elements are missing otherwise.

In addition to Definition 3.1.1, it may happen that P_i won many times over P_j without being beaten by his his opponent. At this time, one may use artificial p_{ij} when $z_{ij} \neq 0$ with either $x_{ij} = 0$ or $y_{ij} = 0$ in order to keep unbiasedness [5,40]. As a result, we can form the weight vectors by using different estimation methods. In this thesis, we have used Logarithmic Least Squares Method for our data manipulation with different correction methods in [5, 40].

The pairwise comparison from the matches that have played between players P_i and P_j, i.e. number of wins per number of matches, is given in the Appendix A, Table (A.1)–(A.4), in partition form from a 54×54 table. The blue colored cells in the first and the last tables indicate the indifferent value 1.

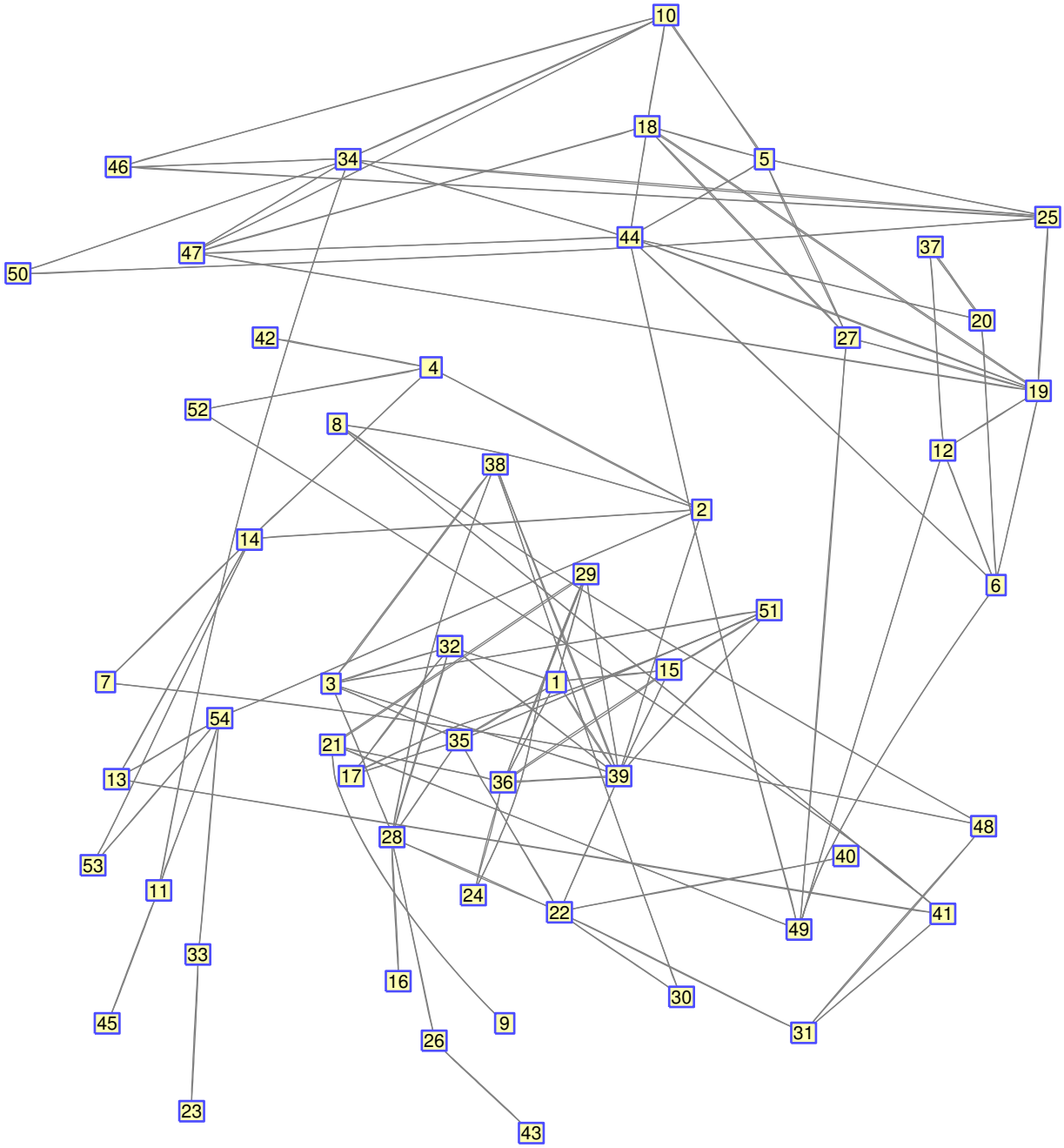

Figure 3.1: Graph representation for the corresponding 54×54 adjacent matrix.

Table 3.1: Number of wins per number of matches

	Hao	Jike	Liqin	Long	Schlager
Hao		1/2	1/2	3/3	
Jike	1/2		1/1		
Liqin	1/2	0/1			0/1
Long	0/3				
Schlager			1/1		

3.2 Main Result

In this section, the expected result is provided by taking a 5×5 incomplete matrix which appears in Section 3.2.1. We have also put our observations from our calculations in Table (A.1)–(A.4). As a consequence, our unexpected result will be provided in Section 3.2.2 .

3.2.1 Expected Result

Let us now consider our incomplete data (3.1). The data was taken from the International Table Tennis Federation (ITTF) freely available database since 2001 by considering only Men's single World Championship [26, 44].

In our 5×5 data, we have made some correction methods by considering a re-scalings of the $n : 0$ matches [5, 40] . We have carried out re-scalings $+0.1, +0.5, +1$ and $+2$. For instance, with re-scaling $+0.5$, we have used the calculations $1 : 0 \to 1.5$, $2 : 0 \to 2.5$, $3 : 0 \to 3.5$ [40]. It is worth noting that these methods have no significant change in the rankings as suggested by Temesi et al. [40]. We have also found that these methods yield an insignificant result. This might happen because of few values in the data 3.1 that were used these re-scalings. But, in the case of 54×54 matrix, there is an obvious change in our final rankings. Our recommendations are included in Section 3.2.2.

Since the corresponding graph to the incomplete matrix (3.1) is connected, it has a unique solution. Thus, using the weight estimation method *LLSM* with three re-scaling of $n : 0$ matches, let *A-LLSM*, *B-LLSM*, *C-LLSM*, and *D-LLSM* denote

Table 3.2: Rankings of 5×5 pairwise comparisons

	A-LLSM	B-LLSM	C-LLSM	D-LLSM	W/L
Hao	1	1	1	2	2
Jike	2	2	2	1	3
Long	3	3	3	4	4
Liqin	4	4	4	3	5
Schlager	5	5	5	5	1

Table 3.3: Spearman rank correlation coefficients

	A-LLSM	B-LLSM	C-LLSM	D-LLSM
A-LLSM	1	1	1	0.8
B-LLSM	1	1	1	0.8
C-LLSM	1	1	1	0.8
D-LLSM	0.8	0.8	0.8	1

the priority vectors obtained from $n : 0 \to n + 0.1$, $n : 0 \to n + 0.5$, $n : 0 \to n + 1$, and $n : 0 \to n + 2$ re-scalings, respectively. We can also consider the win to loss ratio (W/L). Their respective ratios are $5 : 2, 2 : 1, 1 : 3, 0 : 3$, and $1 : 0$. Their rankings are also given in Table 3.2. Here, rankings were practically the same for all correction methods, except a few variants in *D-LLSM*. However, it doesn't make a substantial change in its rankings. These results valid the suggestion of Temesi et al. [40] by handling a balance. Because huge values of re-scalings for $n : 0$ matches may cause a huge difference in their rankings.

The above statements are also reinforced by the values of the Spearman rank correlation coefficients as it can be shown in Table 3.3. It is one of the most known index for the rankings of pairwise comparisons [5, 14, 40]. The values in the table are closer to unity except the bottom and the right corner parts. This is from the fact that the first three re-scalings were taken smaller in value. The result has brought about almost the same rankings.

Player "Jike" is one of the first top players of the 2017 world ranking lists [44]. He is also ranked in the second place in our most priority vectors estimation results.

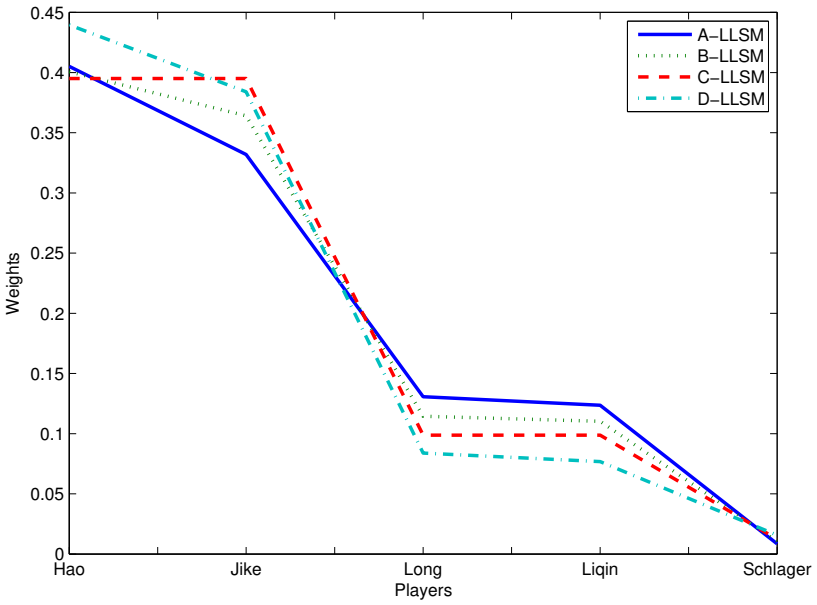

Figure 3.2: Weight vectors derived from LLSM

Even though player "Hao" has got the first place in our rankings, player "Long" has been assigned in the first position in the ITTF list according to the ITTF basic rule: "4 years after the last appearance on the *ITTF* ranking list, the players ranking points will be completely deleted" [27]. For the reader's convenience, only two players are listed on the ITTF rankings based on this rule among the pairs. The first three top players have been included in the ITTF world rankings, except "Hao". This is because of the rule. He was off for the last 4 years. Even though "Long" was listed in the first place of ITTF world rankings, it doesn't mean that he is the best player in the Men's Single World Championship.

The weight vectors derived from *LLSM* are presented in Figure 3.2. For instance, in this graph, "Jike" has got weights 0.33 from *A-LLSM*, 0.36 from *B-LLSM*, 0.39 from *C-LLSM*, and 0.38 from *D-LLSM*. Similarly, it is easy to see the weights for other players. We can also compare its overall rankings with Table 3.2 which is exactly the same.

From our data [26], it can be seen that the players that have received gold medals starting from the year 2001 were listed with their respective years as: Hao [2009], Jike [2011, 2013], Liqin [2001,2005, 2007], Long [2015] and Schlager [2003].

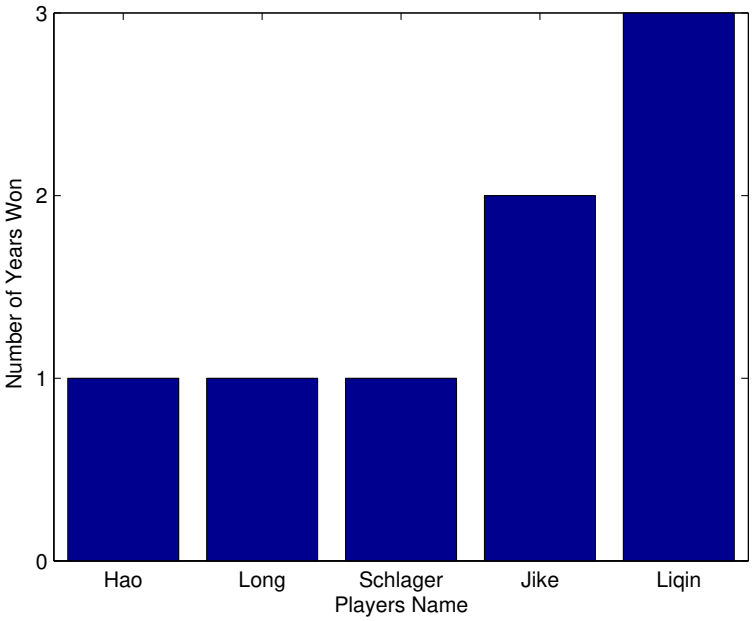

Figure 3.3: Number of years for which players have won gold medals

It's noted again that there is a two years gap in this competition according to the Champions game rule. Moreover, the 2017 competition event has not been included in our data until the time we have done our research because of the event didn't happen yet. Following this, Figure 3.3 demonstrates the number of years versus list of players who have won Men's Single World Championship since 2001. One can also observe that "Hao" has won the champion only once in these time intervals. Again, "Liqin" has won gold medals for three times. However, in our calculation 3.2, "Hao" is better than "Liqin" based on our previous estimation method. It is important to note that matches are not distinguished in the pairwise comparison matrix. A match in Round 16 and another match in the final have the same effect unlike in the world championship.

3.2.2 Unexpected Result

In our 54×54 incomplete pairwise comparison matrix, one can observe in Table (A.1)–(A.4) that the data values (Number of wins/number of matches) are very sparse. For our calculations, we have used different re-scalings for many values in it.

In the paper [5,40], there were few modifications in their data based on the correction methods in comparison to our data. Moreover, the transformations method [4] were not applicable in our paper as the maximum number of matches in our data was not far from the other values.

We have also tried to see other numerical calculations by ignoring the players with degree one using the re-scalings in Section 3.2.1. These methods produced a substantial change of orders in our rankings.

In our 54×54 corresponding graph 3.1, player 23 is connected to player 33, while player 33 is connected to players 54 (and 23). For instance, using *D-LLSM*, player 23 is 3 times better than player 33, and player 33 is 3 times better than player 54. Since this subset of comparisons forms a tree (the path $54-33-23$), all these ratios will be estimated perfectly:

$3 \times 0.000000029959154$ (54's weight) $= 0.000000089877462$ (33 's weight);

$3 \times 0.000000089877462$ (33 's weight) $= 0.000000269632386$ (23 's weight).

So, $9 \times 0.000000029959154$ (54's weight) $= 0.000000269632386$ (23's weight).

This is a kind of drawback of the method. Player 23 is probably not 9 times better than player 54. The result is undoubtedly sensitive to the elements along the path. If both of them were 4, then player 23 would be 16 times better than player 54. It is also noted that longer paths would result in even larger differences. For example, a 4-path will end up with a multiplier $3^4 = 81$ (if every value is 3). In this case, the multipliers 3^4 cause the huge differences.

It seems that the method implicitly requires a kind of balance in the graph, not just connectedness. Very long paths may cause bias in the final result. For example, although the graph is connected, but some nodes are connected to each other by one path only. Thus, from our data analysis and results, our recommendation will be: eliminating the nodes with degree 1 such as 45, 43, (26), 42, 23, (33), 16 and 9. Connectivity will not be lost. Another recommendation is the modification of the scale. That means, 3 times better can be too large to express the difference.

In general, from our point of view for ranking problems, a small subset of the comparisons may lead to a very different rankings.

Chapter 4

Conclusion

The connectedness of an associated graph in Theorems 1.2.3 and 1.2.5 is a necessary and sufficient condition for the optimal completion of a given incomplete pairwise comparison matrix.

Since the eigenvalue minimization problem can be transformed into convex problem (strictly convex in the case of connected graph), the global convergence of the algorithms (cyclic coordinates method, Newton's method, and gradient descent method) is guaranteed. However, the choice of step size γ still affects the stability of the algorithms. The future research can be the choice of step size γ for Gradient Descent and Newton's methods, and comparative analysis of the algorithms.

In the case of ranking, incomplete pairwise comparison matrices can be applied to give response for the questions: 'Who is the best player ever?' and 'What is the rank of players for any length of time?' from the given historical data. Different kinds of correction and transformation methods can modify the original data in order to form unbiasedness among the pairs. Further, the sparsity of pairwise comparison matrix perturbs the rankings by applying results from correction methods.

Having had several numerical calculations with table tennis results, we have realized that a small subset of comparisons may lead to an extremely different rankings. The future research could be the impact of the degrees of vertices for ranking top table tennis players.

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Appendix A

A.1 Harker's Formulas

The following Harker's [21] formulas are related to Chapter 2. Harker's formulas for the derivatives of the Perron eigenvalue are given as follows [1, 21]. Let $\mathbf{A}$ denote a pairwise comparison matrix, and let $x = x(\mathbf{A})$ denote the left Perron eigenvectors, $y = y(\mathbf{A})$ denote its right Perron eigenvector and $\lambda_{max} = \lambda_{max}(\mathbf{A})$ its Perron eigenvalue, so $\mathbf{A}x = \lambda_{max}x$ and $y^T\mathbf{A} = \lambda_{max}y^T$. The normalization for the eigenvectors in this case is $y^Tx = 1$. Let $Q = \lambda_{max}I - \mathbf{A}$. Also let Q^+ denote the pseudoinverse of Q, with properties: $QQ^+Q = Q$, $Q^+QQ^+ = Q^+$, $Q^+Q = QQ^+$. Further, $\partial\alpha_{ij}$ represents differentiation with respect to the entry in position (i, j) in $\mathbf{A}$, and similarly $\partial\alpha_{kl}$ represents differentiation with respect to the entry in position (k, l). Applying these notations, the formulas are given as follows:

- **Harkers formula for the first derivative** $\frac{\partial\lambda_{max}(x)}{\partial x}$:

$$\left(\frac{\partial\lambda_{max}}{\partial_{ij}}\|i > j\right) = \left([y_ix_j] - \frac{[y_jx_i]}{[\alpha_{ij}]^2}\right)$$

  where vectors $x = x(\mathbf{A}), y = y(\mathbf{A})$ are the right-hand side and left-hand side eigenvectors of $\mathbf{A}$, respectively.

- **Harker's formula for the second derivative (when** $i \neq k$ **or** $j \neq l$**):**

$$\frac{\partial^2\lambda_{max}}{\partial_{ij}\partial_{kl}} = (xy^T)_{li}Q^+_{jk} + (xy^T)_{jk}Q^+_{li}$$

$$-\frac{(xy^T)_{ki}Q^+_{jl}+(xy^T)_{jl}Q^+_{ki}}{[\alpha_{kl}]^2}$$
$$-\frac{(xy^T)_{lj}Q^+_{ik}+(xy^T)_{ik}Q^+_{lj}}{[\alpha_{ij}]^2}$$
$$-\frac{(xy^T)_{kl}Q^+_{il}+(xy^T)_{il}Q^+_{kj}}{[\alpha_{ij}]^2[\alpha_{kl}]^2}.$$

- **Harker's formula for the second derivative (when $i \neq k$ or $j \neq l$):**

$$\frac{\partial^2\lambda_{max}}{\partial_{ij}\partial_{kl}} = \frac{2(xy^T)_{ij}}{[\alpha_{ij}]^3} + 2(xy^T)_{ji}Q^+_{ii}$$
$$-2\frac{(xy^T)_{ii}Q^+_{jj}+(xy^T)_{jj}Q^+_{ii}}{[\alpha_{ij}]^2}$$
$$+2\frac{(xy^T)_{ij}Q^+_{ij}}{[\alpha_{ij}]^4}$$

A.2 Pairwise Comparisons: Number of wins per number of matches

There are four tables in this section which are a partition of 54×54 incomplete matrix whose values represent Number of wins per number of matches. These are related to Chapter 3.

Table A.1: Number of wins/total number of matches: The table continues to to the next page.

		Barna	Bengtsson	Bergmann	En-ting His	Guoliang	Hao	Hasegawa	Itoh	Jacobi	Jean-Philippe	Jialiang	Jike	Jonyer	Kohno	Kolar	Kuo-tuan	Leach	Linghui	Liqin	Long	Mechlovits	Ogimura	Ono	Perry	Persson	Satoh	Schlager
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	Barna															2/2												
2	Bengtsson				1/1				1/1						0/2													
3	Bergmann																											
4	E.His		0/1												1/1													
5	Guoliang										1/1								0/1							1/1		1/1
6	Hao												1/2							1/1	2/2							
7	Hasegawa														1/1													
8	Itoh		0/1																									
9	Jacobi																					1/1						
10	J-Philippe					0/1													0/1									
11	Jialiang																											
12	Jike						1/2													1/1								
13	Jonyer														1/1													
14	Kohno		2/2		0/1			0/1						0/1														
15	Kolar	0/2																										
16	Kuo-tuan																											
17	Leach																											
18	Linghui					1/1					1/1									1/1								1/2
19	Liqin						0/1						0/1						0/1							1/1		0/1
20	Long						0/2																					
21	Mechlovits									0/1																		
22	Ogimura																											
23	Ono																											
24	Perry																											
25	Persson					0/1														0/1								
26	Satoh																											
27	Schlager					1/1													1/2	1/1								

Table A.2: Number of wins/total number of matches: Continued from Table A.1

		Sido	Szabados	Tanaka	Tse-Tung	Vana	Yuehua	Waldner	Andreadis	Bellak	Bo	Dolinar	Ehrlich	Flisberg	Fu-Jung	Johansson	Koczian	Lin	Longcan	Saive	Samsonov	Schler	Se Hyuk	Sen	Soos	Stipancic	Yao-hua	Zhenhua
		28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
1	Barna		2/3			0/1			0/1	2/2			1/2															
2	Bengtsson												1/1															0/1
3	Bergmann	1/1				3/4			1/1			0/1	1/2												1/3			
4	E. His															1/1										1/1		
5	Guoliang																	1/1										
6	Hao																	0/1					1/1					
7	Hasegawa																					1/1						
8	Itoh														1/1							1/1						
9	Jacobi																											
10	J-Philippe							0/2												1/1	1/1							
11	Jialiang							1/1											1/1									0/1
12	Jike										0/1												1/1					
13	Jonyer														0/1													0/1
14	Kohno																										1/1	
15	Kolar									0/1			1/1												1/1			
16	Kuo-tuan	1/1																										
17	Leach					1/1			1/1																1/1			
18	Linghui																	1/1			1/2							
19	Liqin																	1/1			0/1							
20	Long										1/1							1/1										
21	Mechlovits		0/1							1/1													0/1					
22	Ogimura	0/1		1/2	0/1				1/1				1/1	1/1														
23	Ono						1/1																					
24	Perry		1/1							1/3																		
25	Persson							1/2												1/1				1/1				
26	Satoh																1/1											
27	Schlager																						1/1					

Table A.3: Number of wins/total number of matches: Continued from Table A.2

		Barna	Bengtsson	Bergmann	En-ting His	Guoliang	Hao	Hasegawa	Itoh	Jacobi	Jean-Philippe	Jialiang	Jike	Jonyer	Kohno	Kolar	Kuo-tuan	Leach	Linghui	Liqin	Long	Mechlovits	Ogimura	Ono	Perry	Persson	Satoh	Schlager
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
28	Sido			0/1													0/1						1/1				0/1	
29	Szabados	0/1																				1/1			0/1			
30	Tanaka																						1/2					
31	Tse-Tung																						1/1					
32	Vana	1/1		1/4														1/2										
33	Yuehua																							0/1				
34	Waldner										2/2	0/1														1/2		
35	Andreadis	1/1		0/1														0/1					0/1					
36	Bellak	0/2														1/1						0/1			1/1			
37	Bo												1/1								0/1							
38	Dolinar			1/1																								
39	Ehrlich	1/2	0/1	1												0/1							0/1					
40	Flisberg																						0/1					
41	Fu-Jung								0/1					1/1														
42	Johansson				0/1																							
43	Koczian																										0/1	
44	Lin					0/1	1/1												0/1	0/3	0/1							
45	Longcan Chen											0/1																
46	Saive										0/1															0/1		
47	Samsonov										0/1								1/2	1/1								
48	Schler							0/1	0/1																			
49	Se Hyuk						0/1						0/1									1/1						0/1
50	Sen																									0/1		
51	Soos			2/3												0/1		0/1										
52	Stipancic				0/1																							
53	Yao-hua														0/1													
54	Cai Zhenhua		1/1									1/1		1/1														

Table A.4: Number of wins/total number of matches: Continued from Table A.3

		Sido	Szabados	Tanaka	Tse-Tung	Vana	Yuehua	Waldner	Andreadis	Bellak	Bo	Dolinar	Ehrlich	Flisberg	Fu-Jung	Johansson	Koczian	Lin	Longcan	Saive	Samsonov	Schler	Se Hyuk	Sen	Soos	Stipancic	Yao-hua	Zhenhua
		28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
28	Sido					0/1			1/2			0/1																
29	Szabados									0/1			1/1															
30	Tanaka											1/1																
31	Tse-Tung														3/3							1/1						
32	Vana	1/1											0/1															
33	Yuehua																											1/1
34	Waldner																	0/1		0/1	1/2			0/1				
35	Andreadis	1/2																							0/1			
36	Bellak		1/1										0/1															
37	Bo																											
38	Dolinar	1/2		0/1									1/2															
39	Ehrlich		0/1			1/1				1/1		1/2													1/1			
40	Flisberg																											
41	Fu-Jung				0/3																					0/1		
42	Johansson																											
43	Koczian																											
44	Lin							1/1													1/1		1/2					
45	Longcan Chen																											
46	Saive							1/1																				
47	Samsonov							1/2										0/1										
48	Schler				0/1																							
49	Se Hyuk																	1/2										
50	Sen							1/1																				
51	Soos								1/1				0/1															
52	Stipancic														1/1													
53	Yao-hua																											1/1
54	Cai Zhenhua						0/1																				0/1	

A.3 ITTF World Ranking Basic Description

This section is related to Chapter 3. It is also available online [27].

1. BASIC PRINCIPLES

   1.1. For each match, the winner gains a certain number of points whilst, at the same time, points are deducted from the rating of the loser. The number of points won or lost depends on the relative strength of the two players involved.

   1.2. Points are added or deducted in accordance with the Rating points table. "Expected results" are those of matches where the winner had a higher rating than the loser. "Unexpected results" are those of matches where the winner had a lower rating than the loser.

   1.3. Wins and losses against unrated players give no points to the winner and loser.

   1.4. It has to be noted that these calculations are carried out only on the basis of the rating points, without taking into account the Bonus Points.

   1.5. The rating points are calculated in bulk and not after each match.

2. WEIGHTINGS

   2.1. Depending on the event, the basic points gained for each win are increased by multiplying them by a weighting factor.

   2.2. The ITTFs Competition Department has classified all the events considered for Ratings into tiers. There are three (3) tiers of events, each with a specific weight ranging from the normal rate of 1, or 1.5 (points are multiplied by a factor of 1.5) or 2 (points are multiplied by a factor of 2).

   2.3. Weightings for losses are all calculated at half of the winners points.

   2.4. Values in the Rating points table are rounded so as to avoid fractions.

3. BONUS POINTS

   3.1. Bonus Points are generally awarded on the basis of the final positions in knockout singles events. However, in competitions where alternative formats (to knockout) are used, up to 20 players may be given bonus points. Additionally, in the same way as for 3rd place (see table in Bonus Points are awarded whenever possible when other matches for intermediate positions are played.

3.2. The number of Bonus Points awarded is related to the event. These Bonus Points remain valid for a period of 12 months only from the date of each Bonus point earned.

3.3. Bonus Points are not awarded for Team events, U21 events, Qualifying events or Ranking (Consolation) events except as listed in 3.4.

3.4. Last two (2) qualification rounds of the Mens and Womens Singles at the ITTF World Tour Platinum events and at the ITTF World Tour events with K.O. playing system (see table in [27]).

4. FIRST RANKING

4.1 A player enters the ranking list with a 1st Rating only when he/she has at least 4 significant wins against already ranked players.

4.2 The 1st Rating is calculated as follows

4.2.1 As soon as a player has 4 significant wins, he/she will get a Rating equal to the Rating of his/her best win less 10 points.

4.2.2 The 4 significant wins are valid from the entire database of matches over the past 4 years.

4.2.3 All losses within the previous 12 rating periods (from the date of the 4th win) would then be taken into consideration and points deducted based on the normal exchange points table for each event played.

4.2.4 The 1st Rating is then equal to the rating of the highest win, less 10 points, less points deducted for losses over the previous 12 periods.

4.2.5 If a player has 4 wins and no losses, then the 1st Rating is equal to his/her best win less 10 points.

4.2.6 The 1st Rating can never exceed the best win less 10 points.

4.2.7 The lowest 1st Rating can never be less than 200 points, regardless of the number of points lost in the calculation.

4.2.8 Bonus points are added as usual. Markers for ratings yellow shade Yellow background means that this is the ranking list on which the player appears for the first time with the first rating.

5. DELETED PLAYERS

5.1 Players who have no recorded international results for more than four (4) months are marked with grey shades and they appear on a separate list but their ranking points are used for seeding purposes, but not for selections, qualifications and invitations for specific tournaments.

5.2 Players who do not have any recorded international results (ITTF Events, ITTF Recognized Events or ITTF Approved Events) for more than eight (8) months are excluded from the published ranking lists. However such players will retain their rating points until they become active again, meaning that in the interim, they still have rating points, and these points could still be used for seeding purposes at non ITTF events.

5.3 4 years after the last appearance on the ITTF ranking list, the players ranking points will be completely deleted, and if the player becomes active again, this player will be treated as a new player and subject to first rating procedures.