A COMPUTATION OF KNOT FLOER HOMOLOGY OF SPECIAL (1,1)-KNOTS

Jiangnan Yu

Department of Mathematics
Central European University

Advisor: András Stipsicz

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Abstract

We will introduce Heegaard decompositions and Heegaard diagrams for three-manifolds and for three-manifolds containing a knot. We define (1,1)-knots and explain the method to obtain the Heegaard diagram for some special (1,1)-knots, and prove that torus knots and 2-bridge knots are (1,1)-knots. We also define the knot Floer chain complex by using the theory of holomorphic disks and their moduli space, and give more explanation on the chain complex of genus-1 Heegaard diagram. Finally, we compute the knot Floer homology groups of the trefoil knot and the (-3,4)-torus knot.

1 Introduction

Knot Floer homology is a knot invariant defined by P. Ozsváth and Z. Szabó in [6], using methods of Heegaard diagrams and moduli theory of holomorphic discs, combined with homology theory.

Given a closed, connected, oriented three manifold Y, the Heegaard decomposition of Y is a decomposition into two handlebodies U_0, U_1 such that $\partial U_0 = -\partial U_1 = \Sigma$ and $Y = U_0 \cup_\Sigma U_1$. The decomposition is determined by specifying a connnected, closed, oriented two-manifold Σ of genus g and two collections of curves $\{\alpha_1, ..., \alpha_g\}$, $\{\beta_1, ..., \beta_g\}$.

In the second section we will give the definition of a (1,1)-knot, describe how to get Heegaard diagram for (1,1)-knots using method from [3], and give parametrization for (1,1)-knots. Later, we will prove that torus knots are (1,1)-knots, and define another kind of knots called 2-bridge knots, and prove that they are (1,1)-knots.

Then we will use the Heegaard diagram to define the chain complex and the differential map. The differential relies on counting holomorphic disks in the symmetric product $\mathrm{Sym}^g(\Sigma)$. After these definition we will compute the homology of the trefoil knot and (-3,4)-torus knot.

2 Heegaard Decomposition

Definition 2.1. *Equip* Y *with a self indexing Morse function* $f : Y \to [0, 3]$ *with one minimum and one maximum, and let* $\Sigma = f^{-1}(\frac{3}{2})$ *be a genus g surface. Then* f *induces a Heegaard decomposition* $Y = U_0 \cup_\Sigma U_1$ *along* $\Sigma = \partial U_0 = -\partial U_1$*, where* $U_0 = f^{-1}[0, 3/2]$ *and* $U_1 = f^{-1}[3/2, 3]$*. There are two sets of* ***attaching circles*** $\boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_g\}$ *and* $\boldsymbol{\beta} = \{\beta_1, ..., \beta_g\}$*, which are intersections of the descending manifold of index 2 critical points and ascending manifold of index 1 critical points with* Σ*. The triple* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *is called a* ***Heegaard diagram****.*

U_0 can be obtained by attaching 2-handles to Σ along the $\boldsymbol{\alpha}$ curves and then attaching a 3-handle, U_1 can be obtained by attaching 2-handles to Σ along the $\boldsymbol{\beta}$ curves and then attaching a 3-handle. The attaching curves are not unique for the decomposition, we have the following operation.

Definition 2.2. *Let* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *and* $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ *be two Heegaard diagrams for the three-manifold Y. We say they are* ***isotopic*** *if there are one-parameter families* $\boldsymbol{\alpha_t}$ *and* $\boldsymbol{\beta_t}$ *of g-tuples of curves, connecting the two pairs of curves, moving by isotopies so that for each t, both the* $\boldsymbol{\alpha_t}$*,* $\boldsymbol{\beta_t}$ *are g-tuple of smoothly embedded, pairwise disjoint curves.*

Definition 2.3. *Assume* γ_1 *and* γ_2 *are two curves on a genus-g surface. The curve* γ_1' *is a* ***handleslide*** *of* γ_1 *over* γ_2*, if* γ_1' *is obtained by* *the following process: choose two points on γ_1 and γ_2, connect them by a curve, open γ_1 and γ_2 along the two chosen points and turn the curve connecting them into two curves, these two curves connect the end points of the opened γ_1 and γ_2 , so all these curve become one curve. We move this curve a little on the surface and get a new curve which does not intersect γ_1 and γ_2. The resulting curve is γ_1', as shown in Figure 2.*

*We say that $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is obtained by **handleslide** on $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ if $\Sigma' = \Sigma$ and $\boldsymbol{\alpha}'$, $\boldsymbol{\beta}'$ is obtained by handleslide from $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$.*

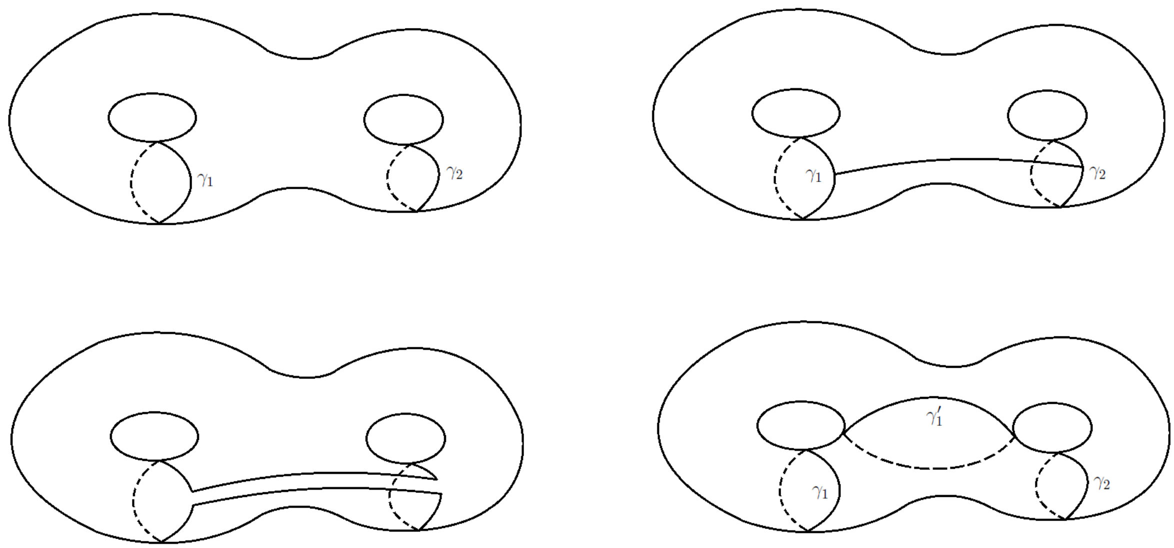


Figure 1: Example of a handleslide

Definition 2.4. *$(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is called the **stablization** of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, if*

- *$\Sigma' = \Sigma \# E$ where E is the 2-torus;*
- *$\boldsymbol{\alpha}' = \boldsymbol{\alpha} \cup \{\alpha_{g+1}\}$ and $\boldsymbol{\beta}' = \boldsymbol{\beta} \cup \{\beta_{g+1}\}$ where α_{g+1} and β_{g+1} are two curves in E which meet tranversely in a single point on the torus.*

*Conversely, we say $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is obtained from $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ by **destabliza-***

tion.

We have the following result (from [8, 9]):

Proposition 2.5. *Any two Heegaard diagrams* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *and* $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ *specifying the same three-manifold* Y *are isotopic after a finite sequence of isotopies, handleslides, stabilizations and destabilizations.*

The proof can be found in [5, Chapter 2].

We would like to add a point $z \in \Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g$ on the torus and set it as a base point for the diagram, in need to count holomorphic disks later.

Definition 2.6. $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ *is called a* **pointed Heegaard diagram** *with a point* z *on the Heegaard surface without intersecting the* α*-curves and* β*-curves. During the isotopy, stablization and destabliza-tion operations, the curves do not intersect the point* z.

The knot complement $Y - nd(K)$ for a knot $K \subset Y$ can also be represented by a Heegaard diagram

$$(\Sigma, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_{g-1}).$$

Assume now that μ is a curve on Σ which is disjoint from $\boldsymbol{\beta_0} = \{\beta_1, ..., \beta_{g-1}\}$ representing the meridian of the knot in Y, so $(\Sigma, \boldsymbol{\alpha}, \{\mu\} \cup \boldsymbol{\beta}_0)$ is a Heegaard diagram of Y. Let $m \in \mu \cap (\Sigma - \alpha_1 - ... - \alpha_g)$ and let δ be an arc that meets μ transversely in m, which is disjoint from all $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}_0$. Let z be the initial point of δ and w be the final point of δ, as shown in Figure 2.

Definition 2.7. $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ *is called a* ***doubly pointed Heegaard diagram***.

To reconstruct the knot, we connect z to w without intersecting the α-curves and push it into the first handlebody, and connect z to w

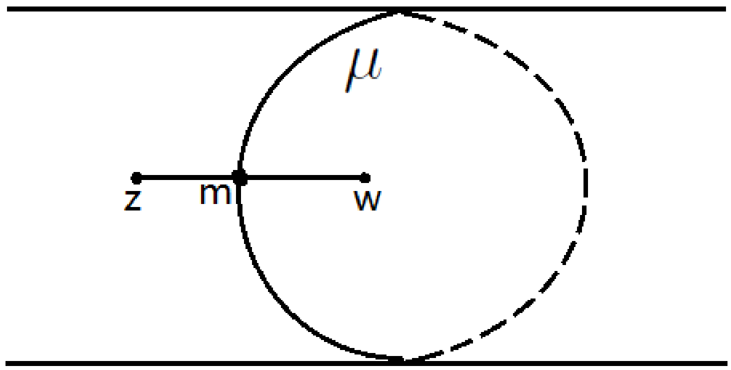


Figure 2: Doubly-pointed diagram

without intersecting β curves and push it into the second handlebody. These two curves form the knot.

For the rest of this paper, the doubly pointed Heegaard diagrams only correspond to the knot in S^3.

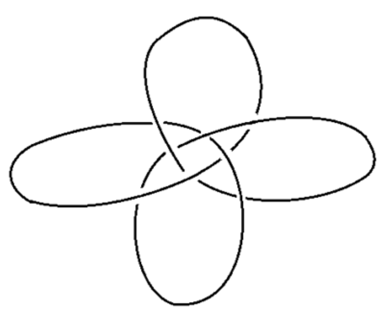
3 (1,1)-knots

In this section we discuss (1,1)-knots, including their definition, a method to get Heegaard diagrams for such knots and their parameterization. We will also introduce some specific kinds of (1,1)-knots, for example torus knots and 2-bridge knots.

Definition 3.1. *A arc in the solid torus is a **trivially embedded arc** if there is an embedded disk in the solid torus such that one part of the boundary of the disk is the arc, and the rest of the boundary is on the boundary of the solid torus.*

Definition 3.2. *A knot K is said to be a **(1,1)-knot** if there is a genus-1 Heegaard splitting $S^3 = H_1 \cup_{T_2} H_2$ of S^3 such that $K \cap H_i$ is a single trivially embedded arc.*

The knot Floer homology of these knots is calculated in [6, 3]. By following the method in [3, Figure 1] we can obtain the Heegaard diagram for such knots. We use the (-3,4)-torus knot as an example to illustrate this method.



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Figures 3 show the construction process. As Figure 3(b) shows, the knot intersects the torus at two points, which separate the knot into

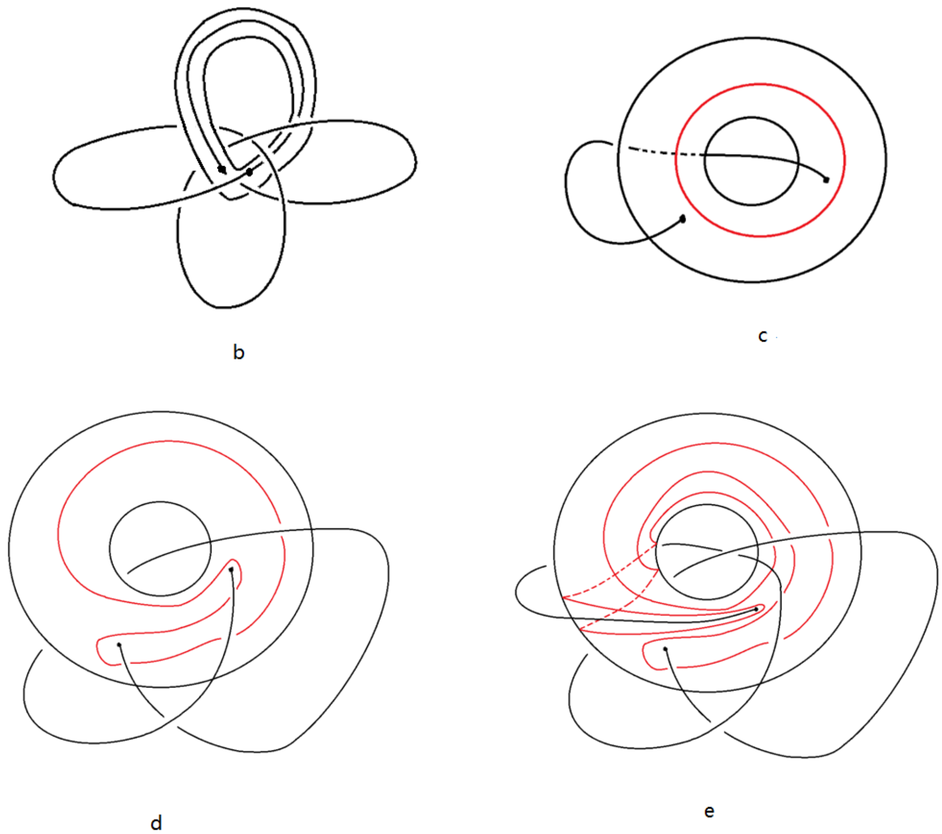

Figure 3: The process to get the Heegaard diagram for (-3,4)-torus knot

two parts, one part is inside the torus, and it is not hard to see it is a trivial arc, another part is outside the torus. Then we need to move the outer part to form the shape of the knot. As Figure 3(c) shows, we start with a trivial arc attached to the torus, and with a longitude curve on the torus. The reason we add this longitude curve is because by following the perturbation the longitude curve is also moved on the torus. Without touching the arc it form a contour of the arc, we can project the arc onto the torus inside the perturbed longitude curve. From (d) to (e) we get the perturbed longitude curve corresponding to the outer part of the (-3,4)-torus knot, we call it the β-curve.

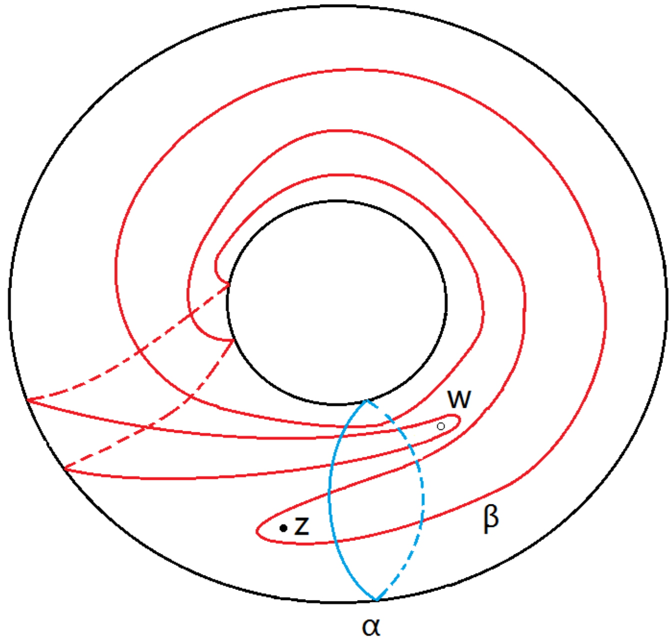

Figure 4: Heegaard diagram for (-3,4)-torus knot

Figure 3 gives the doubly pointed Heegaard diagram for the knot. If we cut along the α curve, we get the tubular representation of the diagram. If we choose γ curve connecting the left and right boundary of the tube below one part of the β curve, as shown in the figure 3, we cut along γ, we get a plane Heegaard diagram for (-3,4)-torus knot, as shown in Figure 3.

Definition 3.3. *In [7] Rasmussen gave a* ***parametrization*** *for such a diagram with four non-negative integers p, q, r and s. The number p is the total number of intersection points of α with β, q is the number of strands in each "rainbow," r is the number of strands running from below the left-hand rainbow to above the right-hand one, and "s" is the "twist parameter": if we label the intersection points on either side of the diagram starting from the top, then the i-th point on the right-hand side is identified with the $(i + s)$-th point on the*

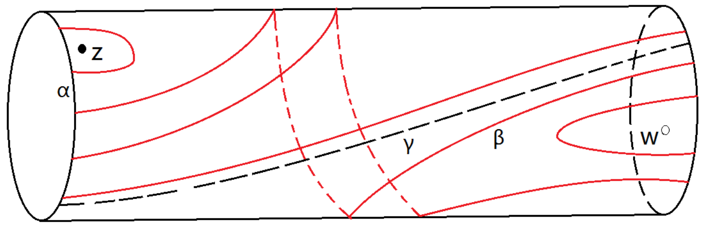

Figure 5: Tubular Heegaard diagram for (-3,4)-torus knot

left-hand side.

For example, for the (-3,4)-torus knot we have $(p, q, r, s) = (5, 1, 1, 1)$. Conversely, suppose we are given $p, q, r, s \geqslant 0$ satisfying $2q + r \leqslant p$ and $s < p$, and with the property that the resulting curve α has intersection number 1 with β. We can construct the Heegaard diagram according to the number of strands in the rainbow and outside the rainbow. To recover the knot, we add z and w inside the two bigons, and attach the opposite sides of the rectangle with the points attached according to the twist parameter, giving us a torus diagram. Then we connect z to w without intersecting the α curves and perturb it inside the torus, and connect z to w without intersecting β curves. These two curves form the knot.

(1,1)-knots form a wide and important class in knot theory; for example:

Theorem 3.4. *Torus knots are (1,1)-knots.*

Proof. For each torus knot choose two points in the knot which separate the knot into two parts. We can perturb one of these parts inside the torus, and perturb the other part outside the torus. If we consider this process in $S^3 = H_1 \cup_{T^2} H_2$ where T^2 is the torus, then each part

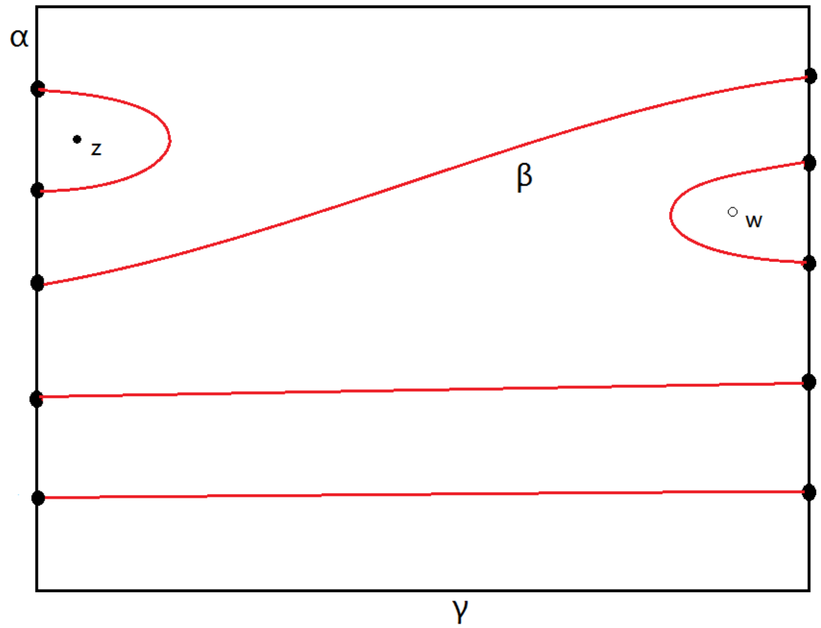

Figure 6: Plane Heegaard diagram for (-3,4)-torus knot

of the knot is trivially embedded into the handlebody hence the torus knot is a (1,1)-knot. □

We apply the method of the proof for the left-handed trefoil knot on the torus and get the Heegaard diagram for it, as shown in Figures 3.

Another source of (1,1)-knots is provided by 2-bridge knots. We will need the following definition.

Definition 3.5. *Assume Σ is a genus-g surface in S^3. We say that the knot K in S^3 is in* ***n-bridge position*** *with respect to the surface Σ if K intersects the closure of each component of $S^3 \setminus \Sigma$ in n trivially embedded arcs. The* ***genus-g bridge number*** *of K, denoted by $\mathbf{b_g(K)}$, is the smallest integer n for which K can be in n-bridge posi-*

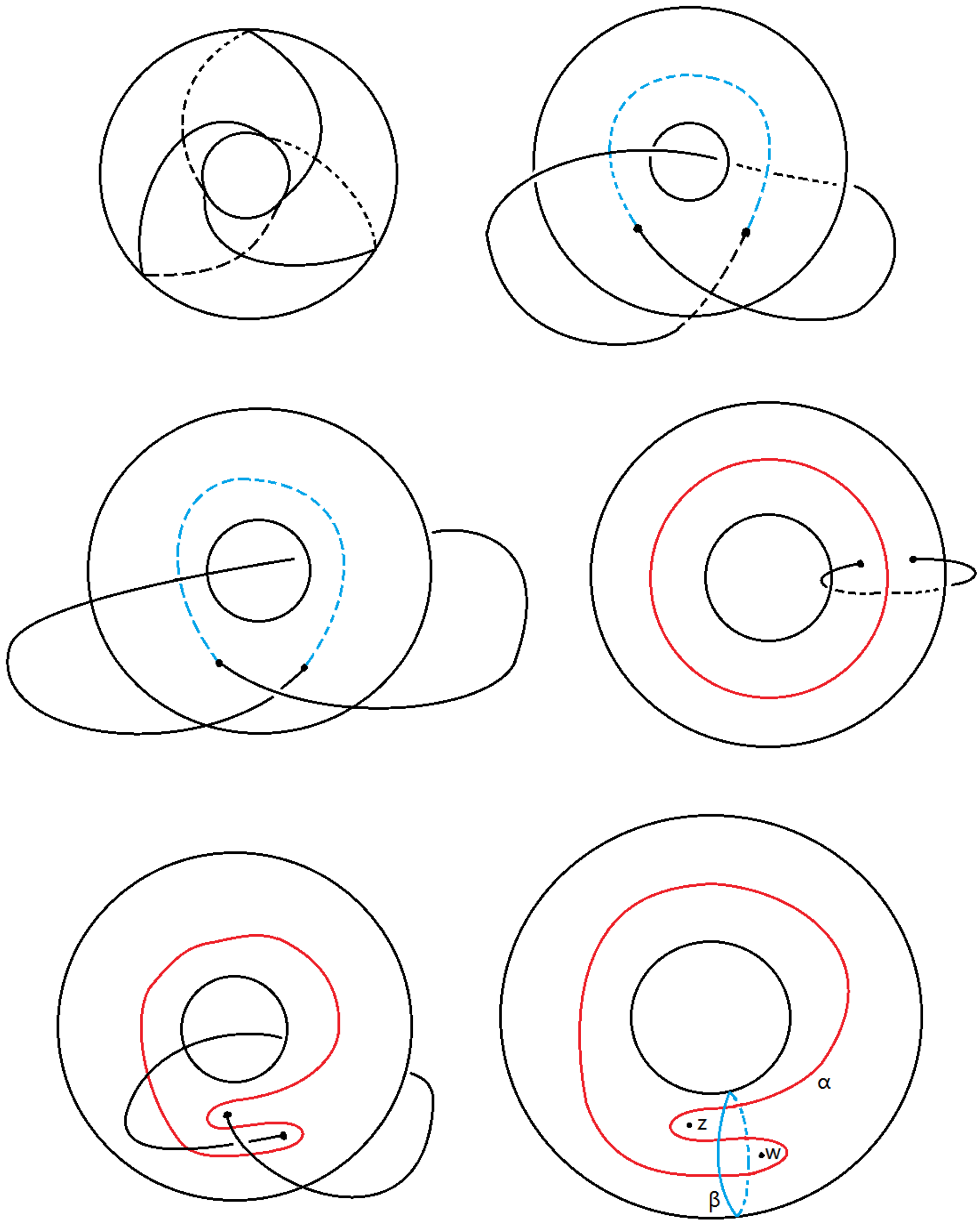

Figure 7: The process to get the Heegaard diagram for left-handed trefoil knot

tion with respect to Σ. When the genus of Σ is 0, we call the number $\mathbf{b_0(K)}$ the bridge number of the knot.

Definition 3.6. ***2-bridge knots*** *are the knots with bridge number 2.*

Theorem 3.7. *2-bridge knots are (1,1)-knots.*

Proof. Assume K is a 2-bridge knot. We can find a sphere S^2 embedded in S^3, such that K can be put in a position which intersect the inner and outer part of S^2 in two arcs, the boundary points of the arcs are on the sphere. We choose one arc inside S^2, delete a tubular neighbourhood of it, then the inner part of S^2 become a genus-1 handlebody, the boundary of the handlebody is a torus T^2, and there is only one arc inside the torus and one arc outside the torus. Since 1 is the smallest number that the knot can intersect the components of $S^3 \setminus T^2$, we see that the genus-1 bridge number of K is 1, hence K is a (1,1)-knot. $\square$

Figure 3 gives an application of this proof for the trefoil knot. The blue curve is inside the sphere, the green curve is outside the sphere, the red tube is the tubular neighbourhood of one inner arc. After taking out the tubular neighbourhood, we get a torus intersecting the knot in one arc inside and one arc outside.

Besides torus knots and 2-bridge knots, there are other (1,1)-knots. For example, the $(2, m, n)$-pretzel knots (as shown by Figure 3) are such knots.

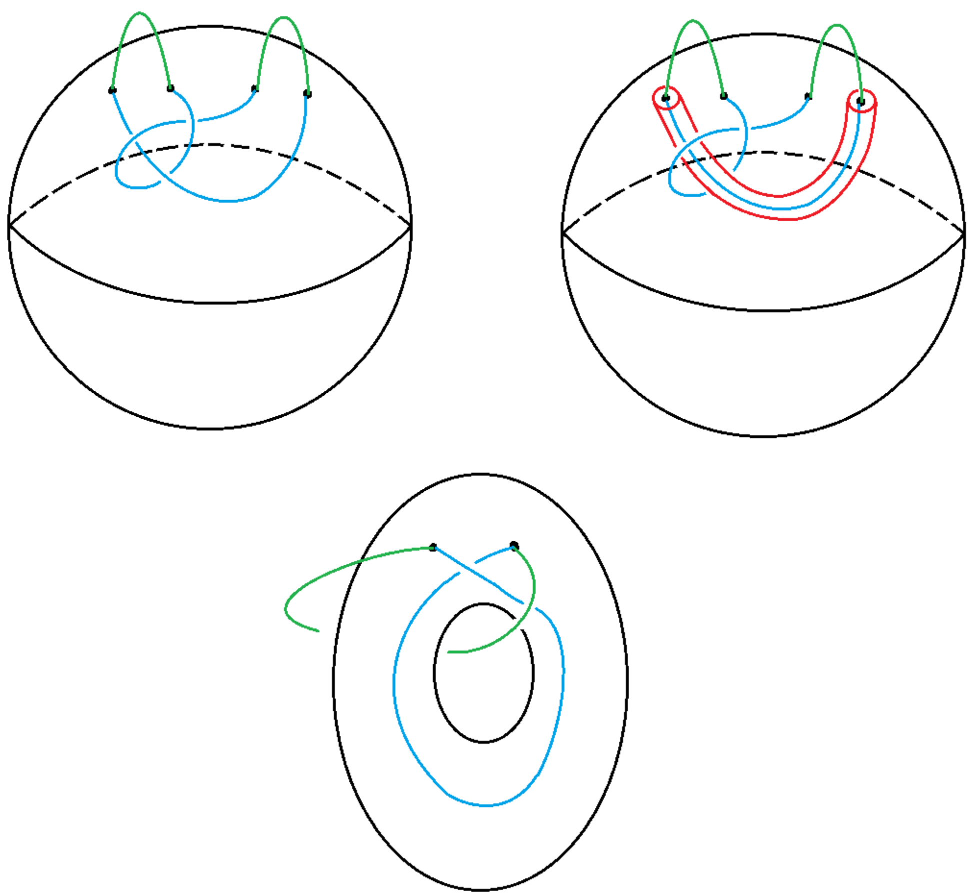

Figure 8: Proof for left-handed trefoil knot is (1,1)-knot

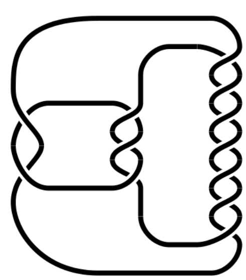

Figure 9: The (2,-3,-7)-pretzel knot.

4 Holomorphic disks and knot Floer chain complex

Having the doubly pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ for a knot $K \subset Y$, we construct the symmetric product space $Sym^g(\Sigma) = \Sigma \times ... \times \Sigma / S_n$ and the two subspaces $\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g \subset Sym^g(\Sigma)$ and $\mathbb{T}_\beta = \beta_1 \times ... \times \beta_g \subset Sym^g(\Sigma)$. Next we define almost complex structures and almost complex manifolds. Then we prove that $Sym^g(\Sigma)$ is a manifold, and we will study the holomorphic disks inside $Sym^g(\Sigma)$ connecting two points $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Definition 4.1. *Let M be a smooth manifold. An* ***almost complex structure*** *J on M is a linear complex structure on each tangent space of the manifold, that is, it is a map $J : TM \to TM$ on the tangent bundle of M satisfying $J^2 = -1$.*

An almost complex structure on Σ induces an almost complex structure on $Sym^g(\Sigma)$. Every 2-dimensional surface (in particular, every Heegaard surface) admits a complex structure and can be turned into a Riemann surface, in which every point has a neighbourhood homeomorphic $\mathbb{C}$.

Proposition 4.2. *$Sym^g(\Sigma)$ is a manifold.*

Proof. Choose a point $\mathbf{x} = \{x_1, ..., x_g\} \in Sym^g(\Sigma)$. Each point x_i is on Σ, and since Σ is a complex manifold, we can find a neighbourhood U_i of it so that $x_i \in U_i \subset \Sigma$ and U_i is homeomorphic to $\mathbb{C}$. So we could take x_i's as points in $\mathbb{C}$, and these g points corresponds to a polynomial in $\mathbb{C}[z]$ which is $f(z) = (z - x_1) \cdots (z - x_g)$. After expansion we get $f(z) = z^g + a_{g-1} z^{g-1} + \cdots + a_1 z + a_0$, and $(a_0, a_1, \cdots, a_{g-1})$

is a point in $\mathbb{C}^g$. Furthermore, for any point $(b_0, ..., b_{g-1}) \in \mathbb{C}^g$ the polynomial $g(z) = z^g + b_{g-1}z^{g-1} + \cdots + b_1 z + b_0$ can be factorized in $\mathbb{C}$ as $g(z) = (z - y_1) \cdots (z - y_g)$, and $\{y_1, ..., y_g\}$ can be taken as a point in $U_1 \times U_2 \times \cdots \times U_g / S_g$. So there is a bijection between $(U_1 \times \cdots \times U_g)/S_g$ and $\mathbb{C}^g$. This map is also continuous, hence it is a homeomorphism, implying that $Sym^g(\Sigma)$ is a manifold. $\square$

Definition 4.3. *Consider the unit disk $\mathbb{D}$ in $\mathbb{C}$, and let $e_1 \subset \partial\mathbb{D}$ denote the arc where $Re(z) \geqslant 0$, and $e_2 \subset \partial\mathbb{D}$ denote the arc where $Re(z) \leqslant 0$. Futhermore, let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Then we denote by $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ the set of homotopy classes of maps*

$$\{u : \mathbb{D} \to Sym^g(\Sigma) | u(-i) = \boldsymbol{x}, u(i) = \boldsymbol{y}, u(e_1) \subset \mathbb{T}_\alpha, u(e_2) \subset \mathbb{T}_\beta\}$$

An element of this set is called a Whitney disks connecting $\boldsymbol{x}$ to $\boldsymbol{y}$.

There is a splicing action between two different Whitney disks, defined as follows. Let ϕ_1 be a Whitney disk connecting $\boldsymbol{x}$ and $\boldsymbol{y}$, and ϕ_2 be a Whitney disk connecting $\boldsymbol{y}$ to $\boldsymbol{z}$. We can "splice" them to be a Whitney disk connecting $\boldsymbol{x}$ to $\boldsymbol{z}$. This operation gives a general splicing operation

$$* : \pi_2(\boldsymbol{x}, \boldsymbol{y}) \times \pi_2(\boldsymbol{y}, \boldsymbol{z}) \to \pi_2(\boldsymbol{x}, \boldsymbol{z}).$$

For the given basepoint $z \in \Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g$, we define the algebraic intersection number

$$n_z = \#u^{-1}(\{z\} \times Sym^{g-1}(\Sigma))$$

for the Whitney disk. If we consider complex structure for $Sym^g(\Sigma)$, we can define pseudo-holomorphic representatives for ϕ which is a map u mapping from $\mathbb{D}$ to $Sym^g(\Sigma)$ satisfying the non-linear Cauchy-Riemann equation for a family of almost complex structure $J = (J_s)_{s\in[0,1]}$.

Definition 4.4. *Let $\mathbb{D} = [0,1] \times i\mathbb{R} \subset \mathbb{C}$ be the strip in the complex plane, and J_s be a path of almost complex structures on $Sym^g(\Sigma)$. We define $\mathcal{M}_{J_s}(\boldsymbol{x}, \boldsymbol{y})$ to be the moduli space of pseudo-holomorphic curves satisfying the following conditions:*

$$M_{J_s}(\boldsymbol{x}, \boldsymbol{y}) = \{u : \mathbb{D} \to Sym^g(\Sigma)|\, u(\{1\}\times\mathbb{R}) \subset \mathbb{T}_\alpha, u(\{0\}\times\mathbb{R}) \subset \mathbb{T}_\beta,$$

$$\lim_{t\to-\infty} u(s+it) = \boldsymbol{x}, \lim_{t\to+\infty} u(s+it) = \boldsymbol{y}, \frac{du}{ds} + J(s)\frac{du}{dt} = 0\}$$

The equation included is the Cauchy-Riemann equation. Given an element $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, we define the space $\mathcal{M}_{J_s}(\phi)$ as the subset of $M_{J_s}(\boldsymbol{x}, \boldsymbol{y})$ consisting those holomorphic maps that are homotopic to ϕ. There is a translation action T_a on $\mathbb{D}$ for any $a \in \mathbb{R}$ such that $T : \mathbb{D} \to \mathbb{D}$ and $T_a(s+it) = s + i(t+a)$. This induces an $\mathbb{R}$ action on $\mathcal{M}_{J_s}(\phi)$. By taking the quotient of this $\mathbb{R}$-action, we get a new space $\widehat{\mathcal{M}}_{J_s}(\phi) = \frac{\mathcal{M}_{J_s}(\phi)}{\mathbb{R}}$.

From the analytic theory of holomorphic disks described in [5, Chapter 3] we know that there is a energy bound for holomorphic disks and we can use the compactification theorem for holomorphic curves proved by Gromov in [4] to show that $\widehat{\mathcal{M}}_{J_s}(\phi)$ admits a geometric compactification. In some cases this argument also shows that (for suitable choices of ϕ) the space $\widehat{\mathcal{M}}_{J_s}(\phi)$ itself is compact.

There is a quantity $\mu(\phi)$ associated to ϕ, called the Maslov index (see [2]). For a generic complex structure, it equals the dimension of the moduli space $\mathcal{M}_{J_s}(\phi)$. Since $\widehat{\mathcal{M}}_{J_s}(\phi)$ is obtained by taking the quotient of $\mathcal{M}_{J_s}(\phi)$ by $\mathbb{R}$, its dimension is $\mu(\phi)-1$. When $\mu(\phi)=1$, the dimension of $\widehat{\mathcal{M}}_{J_s}(\phi)$ is 0, and since this space is also compact, we get that $\widehat{\mathcal{M}}_{J_s}(\phi)$ consists of a finite number of points.

Definition 4.5. *A* ***chain complex*** *is a sequence of abelian groups A_n with homomorphisms ∂_n connecting them,*

$$\cdots \longrightarrow A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} 0$$

satisfying $\partial_{n-1}\circ\partial_n=0$. From $\partial_{n-1}\circ\partial_n=0$ we know that Im $\partial_n \subset$ Ker ∂_{n-1}. By taking the quotient group $H_n=$ Ker $\partial_n/$Im ∂_{n+1}, we get the ***n-th homology group*** *of the chain complex.*

Definition 4.6. *Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z,w)$ be a doubly pointed Heegaard diagram for (S^3,K). We define $CFK^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z,w)$ to be the free module over the ring $\mathbb{Z}[U,U^{-1}]$ generated by the intersection points of $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$. The differential is defined as:*

$$\partial^\infty\mathbf{x}=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y})|\mu(\phi)=1\}}\#\widehat{\mathcal{M}}(\phi)U^{n_w(\phi)}\cdot\mathbf{y}.$$

We equip the chain complex $CFK^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z,w)$ with a filtartion A where the filtration difference of two intersection points $\mathbf{x}$ and $\mathbf{y}$ is given

$$A(\mathbf{x})-A(\mathbf{y})=n_z(\phi)-n_w(\phi)$$

for a domain $\phi\in\pi_2(\mathbf{x},\mathbf{y})$. We normalize the filtration by requiring that

$$\#\{\mathbf{x}|A(\mathbf{x})=i\}\equiv\#\{\mathbf{x}|A(\mathbf{x})=-i\}(\mod 2)$$

for every $i\in\mathbb{Z}$. Multiplication by U lowers the filtration level by 1. In this way we get a ***filtered chain complex****, which can also be written as $CFK^\infty(S^3,K)$.*

An important consequence of the geometric compactification of the moduli space is the following statement.

Theorem 4.7. *For a knot K in S^3 with doubly-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ and chain complex $CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ and ∂^∞, the differential satisfies $(\partial^\infty)^2 = 0$.*

Theorem 4.8. *The filtered chain complex $CFK^\infty(S^3, K)$ is a topological invariant of the knot K; i.e. for different doubly pointed Heegaard diagrams corresponding to the same knot K in S^3, the two filtered chain complexes are (filtered) chain homotopy equivalent.*

The proof can be found in paper [6], we will omit it here. We will prove $(\partial^\infty)^2 = 0$ for genus-1 Heegaard diagram in the next chapter.

Definition 4.9. *We define the sub-complex $CFK^-(S^3, K)$ as the $\mathbb{Z}_2[U]$-subcomplex of $CFK^\infty(S^3, K)$ generated by the intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Therefore $CFK^-(S^3, K)$ is a finitely generated filtered chain complex over $\mathbb{Z}_2[U]$. The boundary map ∂^- on $CFK^-(S^3, K)$ is simply the restriction of ∂^∞; in detail:*

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) | \mu(\phi) = 1\}} \# \widehat{\mathcal{M}}(\phi) U^{n_w(\phi)} \cdot \mathbf{y}.$$

Definition 4.10. *We define*

$$CFK^+(S^3, K) = CFK^\infty(S^3, K) / CFK^-(S^3, K).$$

It is a ***quotient complex*** *of $CFK^\infty(S^3, K)$. We define*

$$\widehat{CFK}(S^3, K) = CFK^-(S^3, K) / U \cdot CFK^-(S^3, K)$$

as the ***quotient complex*** *of $CFK^-(S^3, K)$.*

The homology of the associated graded object to $CFK^-(S^3, K)$, that is, the same module equipped with the differential

$$\partial_K^- \mathbf{x} = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y})|\mu(\phi)=1, n_z(\phi)=0\}} \#\widehat{\mathcal{M}}(\phi) U^{n_w(\phi)} \cdot \mathbf{y}.$$

which respects the Alexander filtration, provides, through taking homology, the invariant the knot Floer homology group of K:

$$HFK^-(S^3, K) = H_*(CFK^-(S^3, K), \partial_K^-).$$

Considering the differential ∂_K^- induces on the quotinet complex $\widehat{CFK}(S^3, K)$ we get

$$\widehat{\partial}_K \mathbf{x} = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y})|\mu(\phi)=1, n_z(\phi)=n_w(\phi)=0\}} \#\widehat{\mathcal{M}}(\phi) \cdot \mathbf{y}.$$

The corresponding homology is denoted by $\widehat{HFK}(S^3, K)$.

Obviously there exist exact sequences relating these chain complexes. The natural short exact sequences of chain complexes below induce long exact sequences on their homologies.

$$0 \longrightarrow CFK^- \xrightarrow{i} CFK^\infty \longrightarrow CFK^+ \longrightarrow 0$$
$$0 \longrightarrow \widehat{CFK} \xrightarrow{i} CFK^+ \xrightarrow{U} CFK^+ \longrightarrow 0,$$

here i is the inclusion map.

Following Theorem 4.8, we get the corollary below.

Corollary 4.11. *The knot homology groups $HFK^+(S^3, K)$, $\widehat{HFK}(S^3, K)$ and $HFK^-(S^3, K)$ are topological invariants for the knot $K \subset S^3$, meaning that for two knots K_1 and K_2 in S^3, if any of these three groups corresponding to the two knots are different, then the knot K_1 is not equivalent to K_2.*

5 Genus-1 Heegaard diagram and the chain complex

In this section we will focus on applying the concepts of the previous section to a genus-1 doubly pointed Heegaard diagram for a knot K. In this case, $\Sigma = T$ is the torus, $Sym^g(\Sigma) = T$, $\boldsymbol{\alpha} = \{\alpha_1\}$ and $\boldsymbol{\beta} = \{\beta_1\}$, $\mathbb{T}_\alpha = \alpha_1$ and $\mathbb{T}_\beta = \beta_1$. The chain group $CFK^\infty(S^3, K)$ is generated by points from $\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \alpha_1 \cap \beta_1$.

If we choose two points $\boldsymbol{x}, \boldsymbol{y}$ from $\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \alpha_1 \cap \beta_1$, we claim:

Theorem 5.1. *For Heegaard diagram on torus T, the holomorphic disks connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ are the bigons on T connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ with boundary on the α_1 and β_1 curve. For each bigon $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, $\widehat{\mathcal{M}}(\phi) = \{pt\}$.*

Before the proof, we give a lemma which is needed.

Lemma 5.2. *The Möbius transformations preserving the unit disk $\mathbb{D}$ in $\mathbb{C}$ are precisely those of the form*

$$T(z) = \lambda \frac{z - a}{\bar{a}z - 1},$$

where $|\lambda| = 1$ and $|a| < 1$. Denote the set of Möbius transformations of this form by $A = Aut(\mathbb{D})$.

Proof of Theorem 5.1. We consider the universal cover of T, which is the complex plane $\mathbb{C}$ with the covering map $\pi : \mathbb{C} \longrightarrow T$. The lift of α_1 and β_1 are $\pi^{-1}(\alpha_1)$ and $\pi^{-1}(\beta_1)$, they are embedded submanifold of $\mathbb{C}$ and each of them is homeomorphic to $\mathbb{R} \times \mathbb{Z}$. The bigons in T are lifted to bigons in $\mathbb{C}$, for one bigon $B \subset T$, its lift $\pi^{-1}(B)$ contains infinitely many bigons in $\mathbb{C}$. Since each bigon is a simply connected subset in the complex plane, according to the Riemann mapping theorem, for one bigon $\tilde{B} \subset \pi^{-1}(B)$, we can find a holomorphic map $f : \mathbb{D} \longrightarrow \tilde{B}$, where $\mathbb{D}$ is a unit disk in the complex plane. Combining this with the projection map we get a map $\pi \circ f : \mathbb{D} \longrightarrow B$ from the unit disk to the bigon in T, and it is also a holomorphic map.

We have already proved that $\widehat{\mathcal{M}}(\phi)$ is not empty. Now we want to prove the second part, which says that up to an equivalence relation, i.e. Möbius transformation, for a given bigon ϕ connecting $\boldsymbol{x}$ and $\boldsymbol{y}$, there is only one holomorphic disk in the moduli space of ϕ. Assume the holomorphic map from unit disk $\mathbb{D}$ in the complex plane to ϕ is u, then u induces a trivial map in the fundamental group, since $\mathbb{D}$ is simply connected. According to the lifting criterion for covering space, u can be lifted to a map $\tilde{u}$ which maps $\mathbb{D}$ to one bigon $\tilde{\phi}$ in $\mathbb{C}$, $\tilde{\phi} \subset \pi^{-1}(\phi)$, and $\mathbb{C}$ is the universal cover of T. If there is another holomorphic map v from $\mathbb{D}$ to ϕ, we can also lift it to $\tilde{v}$ which maps $\mathbb{D}$ to $\tilde{\phi}$. Since u and v are both bijective maps, they are actually biholomorphic. Combining v^{-1} with u we get $g = v^{-1} \circ u : \mathbb{D} \longrightarrow \mathbb{D}$, and $g(-i) = -i; g(i) = i$. According to Lemma 5.2 and after some computation we get that a Möbius map from $\mathbb{D}$ to $\mathbb{D}$ preserving i and $-i$ has the form

$$T_y(z) = \frac{z - iy}{1 + iy},$$

where y can be any real number. This implies that $u = v \circ T_y$. From this we know any two holomorphic maps from $\mathbb{D}$ to ϕ differ by a Möbius map with the above form, hence $\dim \mathcal{M}(\phi) = 1$ and $\dim \widehat{\mathcal{M}}(\phi) = 0$, $\widehat{\mathcal{M}}(\phi) = \{pt\}$. $\square$

Now we know the differential map ∂^∞ can be reduced to

$$\partial^\infty(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y})|\mu(\phi)=1\}} U^{n_w(\phi)}\mathbf{y}.$$

We will discuss in detail about the property $(\partial^\infty)^2 = 0$ for the case of genus-1 Heegaard diagram. To simplify matters, we notice that we can assume that the diagram contains only two bigons: one containing w and another one containing z. Indeed, if there is an empty bigon (i.e. one without w or z) then isotoping the β-curve we can easily eliminate it. Continuing in this manner, we reach the case when we have two bigons.

Proposition 5.3. $(\partial^\infty)^2 = 0$.

Proof. We want to argue that for any $x \in \mathbb{T}_\alpha\cap\mathbb{T}_\beta$, $(\partial^\infty)^2(x) = 0$. The idea is that if we can find a bigon D_1 connecting x and y, and a bigon D_2 connecting y and u, then we can find a point $v \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with a bigon D_1' connecting x and v, and a bigon D_2' connecting v and u in such a way that $D_1+D_2 = D_1'+D_2'$. Therefore a is counted two times in $(\partial^\infty)^2(x)$, so it becomes zero. There are mainly three cases, we will use ∂ to represent ∂^∞ and omit to count multiplicities of z and w in computation for convenience. (Indeed, since $D_1 + D_2 = D_1' + D_2'$, the two count give coinciding results.)

In the first case, there are four bigons, $D_1 = \phi_1 + \phi_2, D_2 = \phi_3$. D_1 connects x and y, D_2 connects y and v, we find $D_1' = \phi_1 + \phi_3$ connecting x and u and $D_2' = \phi_2$ connecting u and v, see Figure 5. So we do the computation:

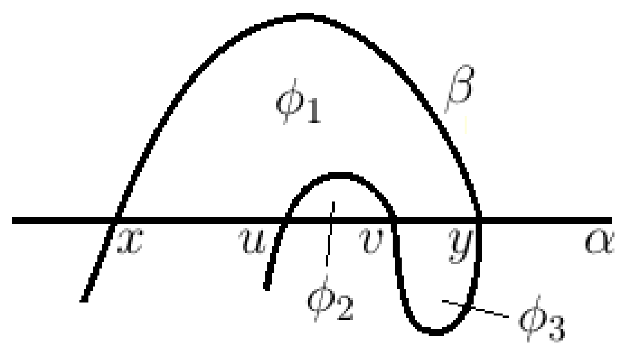

Figure 10: Case 1

$$\begin{aligned}
\partial x &= u + y + \cdots \\
\partial u &= v + \cdots \\
\partial v &= 0 + \cdots \\
\partial y &= v + \cdots.
\end{aligned}$$

Thus, we get

$$\partial^2 x = \partial(u + y) + \cdots = 2v = 0 + \cdots.$$

For the second case, the bigon $D_1 = \phi_1 + \phi_3 + \phi_5$ connect x to y, $D_2 = \phi_2 + \phi_4$ connect y to u. We can find $D_1' = \phi_1 + \phi_2 + \phi_3$ connect x to t and $D_2' = \phi_4 + \phi_5$ connect t to u. So u appear twice in ∂^2, which equals zero.

$$\begin{aligned}
\partial x &= y + t + \cdots \\
\partial y &= u + \cdots \\
\partial t &= u + \cdots.
\end{aligned}$$

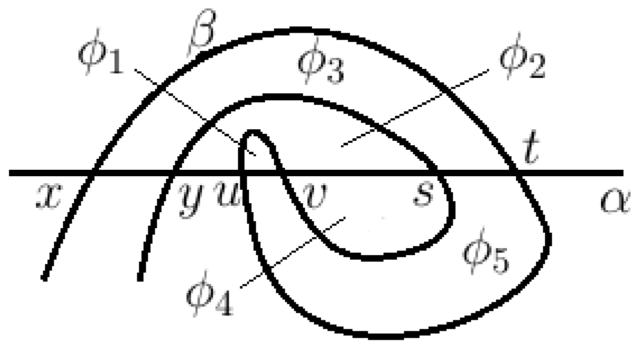

Figure 11: Case 2

It is not hard to verify $\partial^2 = 0$.

For the third case, $D_1 = \phi_1 + \phi_2$ connect x to y, $D_2 = \phi_3$ connect y to v, then $D'_1 = \phi_2 + \phi_3$ connect x to u, $D'_2 = \phi_1$ connect u to v. v appears twice in $\partial^2 x$, so it equals zero. We have the computation:

$$\begin{aligned}
\partial x &= y + u + \cdots \\
\partial y &= v + \cdots \\
\partial v &= 0 + \cdots \\
\partial u &= v + \cdots .
\end{aligned}$$

So that $(\partial^\infty)^2 = 0$.

□

Claim 5.4. $\widehat{\partial} = 0$ *for genus-1 doubly pointed Heegaard diagram.*

Proof. As mentioned in the above proof, for any bigon which does not contain w and z, we can isotope the curves to eliminate such bigon, so the bigons which remain contain at least one of the base-points. From the definition of $\widehat{\partial}$ we know it is equal to zero. □

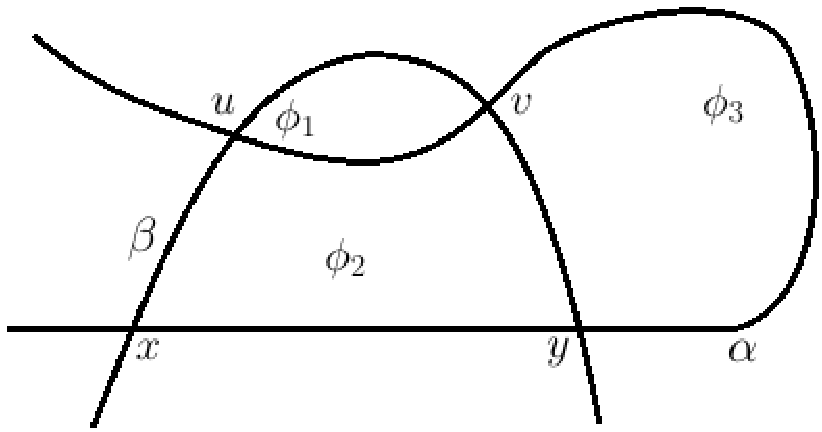

Figure 12: Case 3

6 Some Computations

In this section we will define the Alexander grading and the Maslov grading, and will compute knot Floer homology for the trefoil knot and for the (-3,4)-torus knot with the doubly pointed Heegaard diagram given above, following the method used in [6, Chapter 6] [3, Chapter 3]. Since ∂^- and ∂^∞ gives the same computation, we will write ∂^- here representing the two differentials.

6.1 Alexander grading and Maslov grading

Definition 6.1. *For a given intersection point* $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$*, the* ***Alexander grading*** *and* ***Maslov grading*** *are integers associated to* $\mathbf{x}$*. We denote them to be* $A(\mathbf{x})$ *and* $M(\mathbf{x})$*. For any points* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *and* $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$*, they satisfy the following formula:*

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2n_w(\phi) \tag{1}$$

$$A(\mathbf{x}) - A(\mathbf{y}) = n_z(\phi) - n_w(\phi) \tag{2}$$

$$\sum_{\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} (-1)^{M(x)} q^{A(x)} = \Delta_K(q). \tag{3}$$

In Equation (3) Δ_K *is the* ***Alexander-Conway polynomial*** *of the knot* K.

6.2 The trefoil knot

We have already obtained the doubly pointed Heegaard diagram for the trefoil knot as show in Figure 3. Now we cut along the β-curve and a meridian curve of the torus which does not intersect the α-curve, and convert this diagram into a diagram on a square, as shown in Figure 6.2.

So the α and β curves intersect at three points x_1, x_2, x_3, and there are two bigons ϕ and ψ, ϕ connects x_2 and x_3, ψ connects x_2 and x_1. The boundary of these two bigons both have two parts, one part is on the α curve, the other part is on β curve. There exist holomorphic maps from $\mathbb{D}$ to ϕ and ψ. The bigon from x_2 to x_3 contains z, while the bigon from x_2 to x_1 contains w. Therefore the differential ∂^- vanishes on x_1 and x_3 and $\partial^- x_2 = x_3 + Ux_1$. On the other hand, since the Alexander gradings of x_2 and x_3 are different (but $A(x_2) = A(Ux_1)$), for the differential ∂_K^- of the associated graded object we have

$$\partial_K^- x_2 = Ux_1.$$

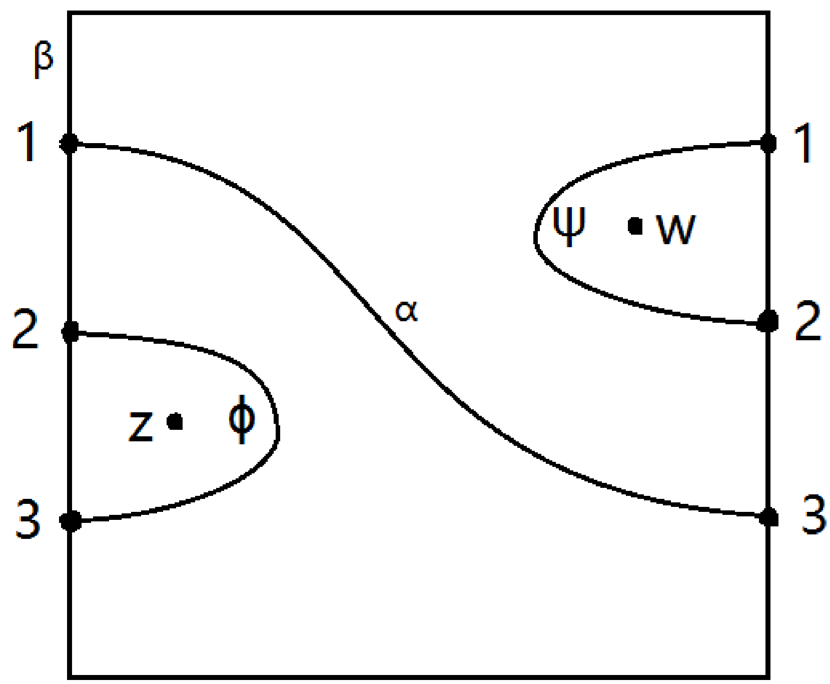

Figure 13: Plane Heegaard diagram for trefoil knot

$CFK^\infty(S^3,K)$ is generated by x_1,x_2,x_3 over $\mathbb{Z}_2[U]$. If we consider $\widehat{CFK}(S^3,K)$, the generators are still x_1,x_2,x_3 and the differential $\widehat{\partial}=0$. So they also form generators of $\widehat{HFK}(S^3,K)$. We get $\widehat{HFK}(S^3,K)=\mathbb{Z}_2^3$.

The homology $HFK^-(S^3,K)$ for the trefoil knot K can be easily computed:

$$HFK^-(S^3,K)=\mathbb{Z}_2[U]\oplus\mathbb{Z}_2$$

where $\mathbb{Z}_2[U]$ is generated by x_3 and $\mathbb{Z}_2=\mathbb{Z}_2[U]/U\mathbb{Z}_2[U]$ is generated by x_1 (and Ux_1 is zero in homology, since it is the boundary of x_2).

6.3 (-3,4)-torus knot

From Section 2 we have already obtained the Heegaard diagram for the (-3,4)-torus knot. We construct its universal cover, as shown below.

There is a holomorphic disk connecting x_1 and x_2 with one z point inside, one holomorphic disk connecting x_5 and x_3 with two z point inside, one disk connecting x_5 and x_2 with two w points inside, one disk connecting x_4 and x_3 with one w points inside.

Using the earlier formulas for the gradings (1) and (2) we have:

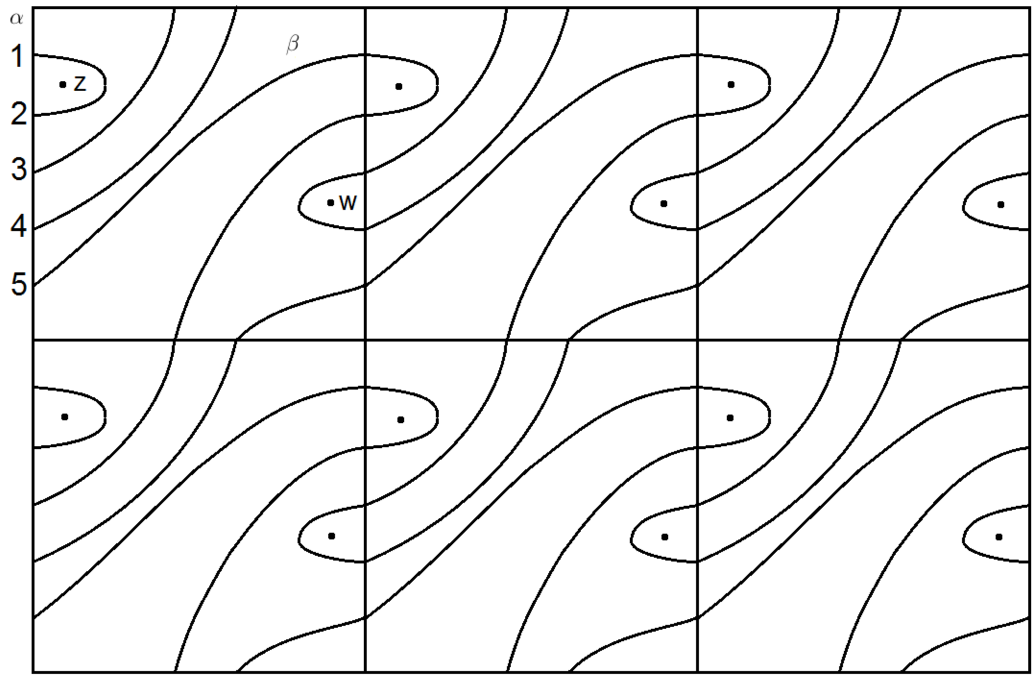

Figure 14: Universal cover for Heegaard diagram for trefoil knot

$$
\begin{aligned}
M(x_1) &= 6, \quad A(x_1) = 3, \\
M(x_2) &= 5, \quad A(x_2) = 2, \\
M(x_3) &= 2, \quad A(x_3) = 0, \\
M(x_4) &= 1, \quad A(x_4) = -2, \\
M(x_5) &= 0, \quad A(x_5) = -3.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\partial^- x_1 &= 0 \\
\partial^- x_2 &= x_1 + U^2 x_5 \\
\partial^- x_3 &= U x_4 + x_5 \\
\partial^- x_4 &= 0 \\
\partial^- x_5 &= 0
\end{aligned}
$$

From this description the map ∂_K^- of the associated graded object can be easily computed:

$$
\begin{aligned}
\partial_K^- x_1 &= 0 \\
\partial_K^- x_2 &= U^2 x_5 \\
\partial_K^- x_3 &= U x_4 \\
\partial_K^- x_4 &= 0 \\
\partial_K^- x_5 &= 0.
\end{aligned}
$$

Therefore $\widehat{HFK}(S^3, K)$ for the (-3,4) torus knot K is isomorphic to $\mathbb{Z}_2^5$, while $HFK^-(S^3, K)$ is isomorphic to $\mathbb{Z}_2[U] \oplus \mathbb{Z}_2^3$, where $\mathbb{Z}_2[U]$ is generated by x_1, one copy of $\mathbb{Z}_2$ is generated by x_4 (and it can be viewed as $\mathbb{Z}_2[U]/U\mathbb{Z}_2[U]$), and the two further copies of $\mathbb{Z}_2$ are generated by x_5 and Ux_5 and as a $\mathbb{Z}_2[U]$-module these two copies of $\mathbb{Z}_2$ are equal to $\mathbb{Z}_2[U]/U^2\mathbb{Z}_2[U]$.

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