A GEOMETRIC RELATION BETWEEN THE ROOTS AND CRITICAL POINTS OF FINITE BLASCHKE PRODUCTS: GAUSS-LUCAS THEOREM IN HYPERBOLIC GEOMETRY

OĞUZHAN YÜRÜK

Supervisor: RÓBERT SZŐKE

A thesis submitted for the degree of

Master in Science

Department of Mathematics,

CEU

May 2017

Contents

OGUZHAN YURUK

This paper is dedicated to every single mathematician who contributed my education.

Date: May 16, 2017.

Key words and phrases. Complex Analysis, Geometry.

To everyone who contributed to my life as a mathematician.

Abstract. This paper is mainly on extending the results on geometry of complex polynomials in Euclidean geometry to hyperbolic geometry. The aim of this thesis is to investigate the Blaschke products, which mimic the polynomials for Hyperbolic geometry and present results on finite Blaschke products analogous to the Gauss-Lucas theorem.

1. Introduction

1.1. Motivation.

. Mathematics is conceived as a form of art among the pure mathematicians, because it is a way of expressing the ideas in a formal and structured manner which mathematician learns through meditation of his/her mind. Considering this piece of work with such a point of view, asking why a mathematical result is important is as meaningless as asking why a painting or a song is important. Even though we all have different tastes in life, in terms of art there is a common perception of aesthetics. Thus, instead of importance, one should question what makes this mathematical result beautiful? The main results that will be discussed in this paper are known for decades for Euclidean geometry. These results uncovers the great harmony between the roots and the critical points of the polynomials, more rigorous statements of these results will be provided in the next sections. Observing the similar scenario in non-Euclidean geometries is remarkable because in a way this extends our understanding of geometry in general by providing some clues about the hierarchy of the truths within the geometry.

. Apart from the pure mathematicians side of the story, there is also a scientific motivation behind this work. Study of roots and critical points of polynomials doubtlessly play a great role in science. These do not merely help us with expending our understanding of geometry but at some point these facts could come handy in computations. For example, Gauss-Lucas theorem restricts the solution of critical points to a convex hull formed by the roots. Moreover, Marden's theorem tells us the exact location of critical points when the roots are given and describes the possible roots when critical points are given. Unfortunately this holds only for cubic polynomials. These results indeed provides an ease of calculation, nonetheless these hold only in Euclidean geometry. Considering that we are existing in a non-Euclidean universe, one cannot stop himself to think about if similar results hold for non-Euclidean geometries as well?

1.2. **What do we know in Euclidean case and how to go further.**

. Gauss-Lucas theorem is a well-known theorem in complex analysis. Throughout the years many proofs of this theorem have been discovered by fellow mathematicians. The theorem is as follows:

Theorem 1.1 (Gauss-Lucas)**.** *Let $p(z) \in \mathbb{C}[z]$ be a degree n polynomial with roots $z_1, \dots z_n$. If we denote the convex hull of these points in the complex plane as H, then for any critical point of $p(z)$,i.e. $w \in \mathbb{C}$ such that $p'(w) = 0$, we have that $w \in H$.*

This result gives us a bounded area where the critical points can be. Furthermore if we restrict ourselves to the degree three case, even more remarkable result exists:

Theorem 1.2 (Marden)**.** *Given a cubic polynomial $p(z)$ with non-collinear roots z_1, z_2, z_3, there exists a unique ellipse $\mathcal{E}$ passing through the midpoints of the triangle formed by z_1, z_2, z_3. The focus points of $\mathcal{E}$ are exactly the critical points of $p(z)$.*

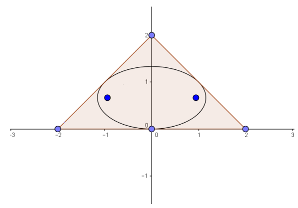

FIGURE 1. This figure both the Gauss-Lucas theorem for degree 3 and Marden's theorem. Vertices of the triangle are representing roots and the points in the triangle are critical points.

Even though it was not proven by Morris Marden first, the theorem is named after him [8]. There are various proofs of this result, however in this work the ideas given in [9] and [8] will have the priority. A reason for these choices is that the ideas presented in these papers provide a good geometric intuition. Moreover Northshield gives a complete picture of what is going on in the background of Marden's theorem while Kalman provides a neat elementary proof of the theorem which can be followed by any reader with a basic notion of complex polynomials.

. In this work, we will pursue these results in the hyperbolic geometry. Of course this raises serious questions; One way the visualization technique for hyperbolic geometry is Poincaré's disk model which models the whole hyperbolic geometry with the unit disk. The roots and critical points of a given polynomial can be outside of the unit disk. So we have to come up with a concept which will mimic the polynomials in the hyperbolic case. This is where Blaschke products come into play and [7] will come handy to understand this concept. In this survey, the authors provide general facts about finite Blaschke products as well as some insight in the geometry of their roots and critical points. They even provide a proof for the hyperbolic case of Gauss-Lucas Theorem in this survey. However in the provided proof, some arguments are missing and some of them are just mentioned without proof. One of the aims of this work will be to clarify and simplify some of the arguments presented in [7] with implementing the Northshield's ideas in [9] to the hyperbolic geometry.

1.3. **Outline.**

. There will be two main aims of this thesis; the first one is to familiarize the reader with the study of polynomial-like structures within other geometries. The other one is to point out some of the analogous results in hyperbolic geometry that have been known in Euclidean geometry for many years, namely Gauss-Lucas Theorem. While the second will be the main focus point of this work, the first aim has to be achieved at some level in order to actually achieve the second aim. However further research can be done on other beautiful and useful results from Euclidean geometry.

. In order to achieve these aims, the flow of topics will be as follows. The second chapter will begin with a brief introduction to non-euclidean geometries. The main aim is to give the idea of how non-euclidean geometry is discovered and progressed, also a reader who is familiar with the idea of non-euclidean geometries may skip this part. In the third chapter the Gauss-Lucas theorem will be stated and proved, a proof inspired by [9] will be given. The main ideas behind the proof will come handy in the next chapters. The chapter 4 will give us the technical background concerning the hyperbolic geometry and Blaschke products, the required definitions and facts will be provided in this chapter. Chapter 5 will be entirely on the hyperbolic version of Gauss-Lucas' theorem. Lastly the thesis will conclude with finishing remarks to point out in what direction a further research could be pursued in this area.

2. A Brief Introduction to non-Euclidean Geometries

. Geometry is one of the oldest fields in mathematics. First axiomatic book that we know in history, Euclid's Elements is probably written around 300 B.C. For centuries the system proposed by Euclid was presumed as geometry. As the study of mathematics progressed into a better system, fellow mathematicians started to question these axioms given by Euclid. Especially the fifth axiom was one of the main questions of debate.

> **Fifth (Parallel) Postulate:** If a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself) less than two right-angles, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (internal angles) are less than two right-angles (and do not meet on the other side)[4]

To make it more meaningful, the postulate can be rephrased as: If two line segments intersect a third one such that the sum of the inner intersection angles is less than 180 degrees, then these two line segments intersect at the same side where the two inner angles are taken.

. Some people tried to prove it using other axioms but in vain. Some mathematicians worked on simplifying the axiom to make it more *intuitive*. The fifth axiom is named as "Parallel Axiom" even though it does not say anything about being parallel. A well-known equivalent of this postulate, is given by the Scottish mathematician John Playfair and also named after him.

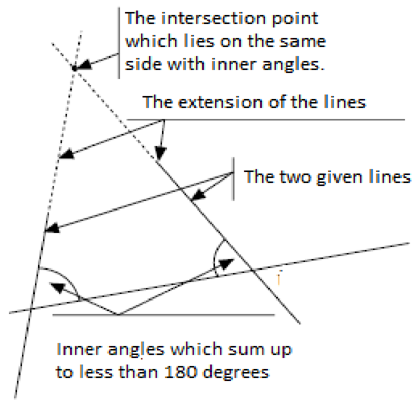

FIGURE 2. Illustration of the fifth axiom.

> **Playfair's Axiom:** In a plane, given a line and a point not on it, there is one and only one parallel line to the given line that can be drawn through the point.[5]

Sometimes the statement of the axiom is given as there exists at most one parallel line. However, it is possible to prove the uniqueness from the other axioms. As a side note instead of existence of a parallel line to a given line, it would be a better idea to say existence of a line that does not intersect the given line. Of course in the Euclidean sense this coincides with the line being parallel.

. Another approach was to disprove the postulate. It did not took so long for people to realize that it is possible to come up with a consistent system of geometry without the fifth axiom. In [4] as Fitzpatrick explains, the original version of the fifth axiom in a way is specifying that the surface that we study is actually flat. As it was realized later when this constraint of flatness is removed there are other possible geometries that can be studied. Therefore it was wrong to consider Euclid's proposed system as the unique geometry, so it turned into one of the geometries named as

Euclidean geometry. János Bolyai, Nikolai Lobachevsky and Carl Friedrich Gauss come up with a model where infinitely many parallel lines through a point to a given line existed on a surface, which we know now as hyperbolic geometry. On the other hand in Bernhard Riemann's spherical geometry, no such parallel lines exists on the surface. Later on with the application of these mathematical ideas to science, non-euclidean geometries secured its academic popularity to this day.

3. Gauss-Lucas Theorem in Euclidean Geometry

. This chapter will be fully reserved for the Euclidean version of the Gauss-Lucas theorem. This result is on complex polynomials, however well before complex analysis was discovered a similar statement was already found in real analysis.

Theorem 3.1 (Rolle)**.** *Given a real valued continuous function $f : [a, b] \longrightarrow \mathbb{R}$ where $a, b \in \mathbb{R}$ are different real numbers and f is differentiable on (a, b); if $f(a) = f(b) = 0$ then there exists $c \in [a, b]$ such that $f'(c) = 0$.*

Considering a real valued polynomial $f(x)$ with roots $x_1, \ldots x_n \in \mathbb{R}$, the above theorem is implies that the critical points of f are between the maximal and minimal root of f. In other words, in one dimensional space the critical points of the polynomial in the convex hull of the roots since in this case the convex hull is just the line segment between $\min\limits_{1 \leq i \leq n} x_i$ and $\max\limits_{1 \leq i \leq n} x_i$. Gauss-Lucas' theorem generalizes this to the case of complex polynomials. Roots of a complex polynomial $p(z) : \mathbb{C} \longrightarrow \mathbb{C}$ are in the complex plane. In the generic case the roots do not lie on the same line and produce a closed region that encloses the critical points. Let us state the theorem here one more time for the ease of the reader and prove it using an idea provided in [9].

Theorem 3.2 (Gauss-Lucas Theorem for Euclidean Geometry)**.** *Let $p(z) \in \mathbb{C}[z]$ be a degree n polynomial with roots $z_1, \ldots z_n$. If we denote the convex hull of these points in complex plane as H, then for any critical point of $p(z)$,i.e. $w \in \mathbb{C}$ such that $p'(w) = 0$, we have that $w \in H$*

Let us first consider an easy example;

Example 3.3. Consider the polynomial with roots $-1 + i, 1 + 3i, -i$:

$$p(z) = (z + 1 - i)(z - 1 - 3i)(z + i)$$

Set $z_1 = -1 + i, z_2 = 1 + 3i, z_3 = -i$. With an easy calculation we get

$$p'(z) = 3z^2 - 6iz - 2i$$

Finally with a routine discriminant calculation we get the critical points as $w_1 = i - \sqrt{-1 + \frac{2i}{3}}$ and $w_2 = i + \sqrt{-1 + \frac{2i}{3}}$. It is not hard to see that these two points are included in the convex hull of $-1 + i$, $1 + 3i$ and $-i$. Let us clarify this with a figure.

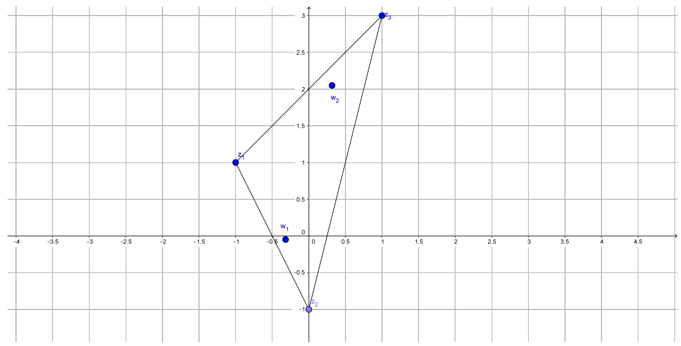

FIGURE 3. z_1, z_2, z_3 denotes the roots, w_1, w_2 denotes the critical points and the gray triangle is the convex hull of z_1, z_2, z_3.

. Before the actual proof of the theorem, we make few remarks about the proof that has to be underlined. Northshield mentions[8] the main idea that we are going to present, however he does not present a rigorous proof of why this works and how this critical idea leads to the proof of Gauss-Lucas' theorem. This will be presented in a lemma before the actual proof. The proof of the theorem will cover only the case of degree three polynomials, higher degree cases can be proven similarly. Case of degree 1 is trivial. For degree two; let $p(z)$ be a polynomial with two distinct roots z_1, z_2.So,

$$p(z) = c(z - z_1)(z - z_2) = c(z^2 - (z_1 + z_2)z + z_1 z_2)$$

for some non-zero complex constant c. From here it is easy to find the zero of the derivative:

$$p'(z) = c(2z - (z_1 + z_2)) = 0 \Longleftrightarrow z = \frac{z_1 + z_2}{2}$$

The convex hull of z_1 and z_2 is the line segment between them and in this case, the critical point lies in the midpoint of this line segment.

Lemma 3.4. *Let z_1, z_2, z_3 be three complex numbers and $\mathcal{H}$ be their convex hull, that is;*

$$\mathcal{H} = \{n_1 z_1 + n_2 z_2 + n_3 z_3 : n_i \in [0,1],\ n_1 + n_2 + n_3 = 1\}$$

Then for any $u \notin \mathcal{H}$ there exists $\theta \in [0, \pi]$ such that $e^{i\theta}(u-z_1), e^{i\theta}(u-z_2), e^{i\theta}(u-z_3)$ all have strictly positive real part.

Proof. Given a $u \notin \mathcal{H}$, define the function $f(z) = u - z$. Basically we want to find a line passing through origin such that $u - z_1, u - z_2, u - z_3$ are all on the same side of the line. To follow with the proof let's draw a basic figure where z_1, z_2, z_3 denotes the three complex numbers.

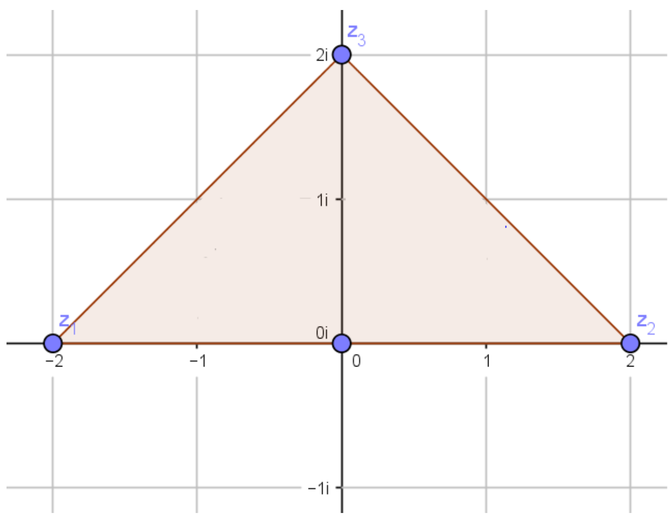

FIGURE 4.

Note that f is a composition of translation and rotation, thus it is just a conformal map from $\mathbb{C}$ to itself. This gives us that the image of $\mathcal{H}$ under f has to be a convex set as well. Moreover the following holds for all $z \in \mathbb{C}$;

$$z \in \mathcal{H} \iff f(z) \in f(\mathcal{H})$$

Thus we must have that $f(u) = 0 \notin f(\mathcal{H})$. Let's denote $f(z) = u - z = z'$ for all $z \in \mathbb{C}$ and use the Figure 4 to visualize $f(\mathcal{H})$.

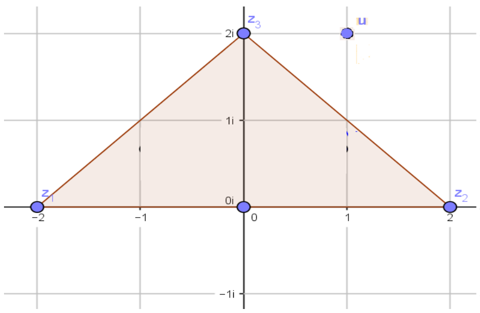

FIGURE 5. The arbitrary point u outside the convex hull is chosen as $1 + 2i$

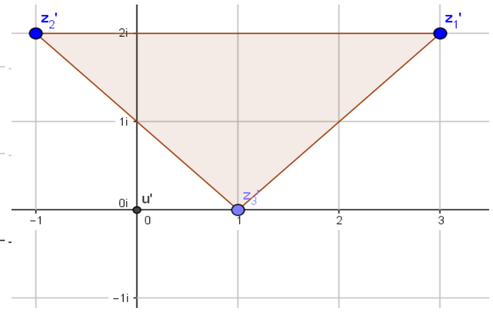

FIGURE 6. u', z_1', z_2', z_3' are the images of u, z_1, z_2, z_3 respectively under f. The triangle formed is $f(\mathcal{H})$

Now among z_1', z_2' and z_3', we will chose two points such that the difference of their arguments is maximal,i.e. $\arg(z_i') - \arg(z_j')$ is maximal. In order to do this, first we have to know such a maximal choice exists but this is trivial. Second we show that there is essentially one choice of difference of arguments exist. This means if z_a', z_b' and z_c', z_d' are two different choices then arguments of z_a' and z_b' are actually either equal to argument of z_c' or z_d'. Such a situation won't interfere with our future arguments in this proof. Let's say that we found one such pair with maximal difference z_a', z_b'. We can parametrize the lines that contains origin in the Euclidean plane with their argument angles $\theta \in [0, \pi)$. Let's say that l_a with the argument angle $\theta_a \in [0, \pi)$ is the line that contains z_a' and similarly l_b with the argument angle $\theta_b \in [0, \pi)$ is the line that contains z_b'.Without loss of generality let's say that $\theta_b < \theta_a$. These lines divide the plane into four parts, denoted as

A, B, C, D in the figure. Now one of the points z'_c or z'_d has to be equal to either

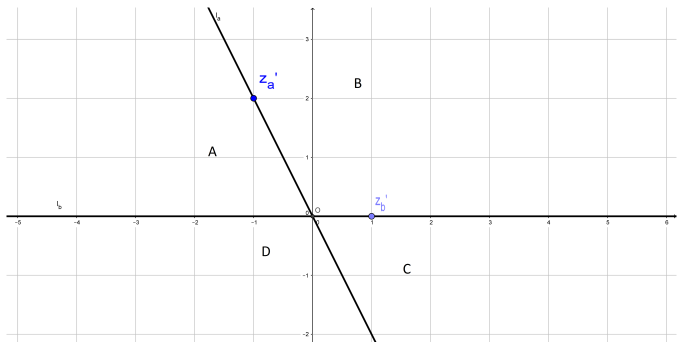

FIGURE 7. The choice of z_b and z_a following from Figure 6

z'_a or z'_b in our degree three case. But for the sake of higher degree cases let's assume that we are in the most generic case where z'_c and z'_d are different from z'_a,z'_b. Let's see in which region these two points can be. First of all observe that none of them can be in the region D since otherwise zero would fall into the convex hull. Similarly none of them can be in A or C since it would contradict $|\theta_a - \theta_b|$ being maximal.The argument difference between z'_a and the a point in the region C is higher than $|\theta_a - \theta_b|$, similar reasoning works for A and the point z'_b as well. This leaves the region B as the last option, however both z'_c and z'_d cannot be in the region B otherwise we would $|\theta_c - \theta_d| < |\theta_a - \theta_b|$ contradictory to $|\theta_c - \theta_d|$ being maximal. We had parameterized the lines containing zero with $\theta \in (0, \pi]$, among these lines we can choose any with argument not in (θ_b, θ_a)

Let's say we have chosen a line with argument θ explained as above. All that is left to do is to rotate our picture with $-\theta$ degrees,i.e. multiply all z'_1, z'_2, z'_3 with $e^{i(-\theta)}$. Thus for all i we have that $e^{i(-\theta)}z'_i = e^{i(-\theta)}(u - z_i)$ has strictly positive real part. $\square$

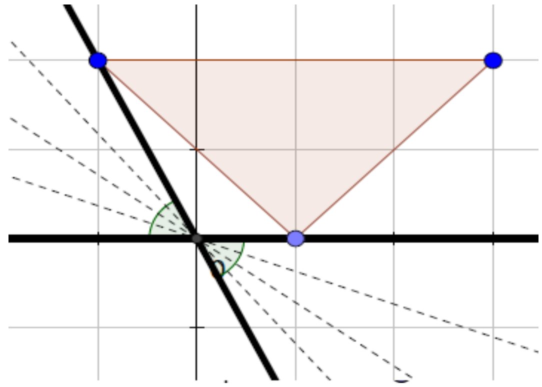

FIGURE 8. Bold lines denote l_b and l_a, dashed lines denote the possible choices for our desired line.

Remark 3.5. Note that even though the proof and the lemma above is stated for specific case of three points, it can be easily generalized using the same proof. So in general we can say given n points $z_1, \ldots, z_n$ and their convex hull $\mathcal{H}$, then for any $u \notin \mathcal{H}$ there exists a θ such that all $e^{i\theta}(u - z_i)$ has strictly positive real parts.

Remark 3.6. Note that in the above lemma u can be chosen as one of the z_i with losing the strictness of the positivity. In that case due to convexity the the roots with maximal difference in their arguments, z'_a and z'_b, are simply where z_{i-1} and z_{i+1} are mapped under f.

Proof of the Gauss-Lucas' Theorem. Let $p(z) = c(z - z_1)(z - z_2)(z - z_3)$ be a complex polynomial with non-collinear roots z_1, z_2, z_3, where c is a complex constant. Let $\mathcal{H}$ denote the convex hull of these roots. We want to see that the roots of $p'(z)$ are in $\mathcal{H}$. First of all, let's make an observation about the logarithmic derivative of

$p(z)$.

$$\begin{aligned}
\overline{\Big(\frac{p'(z)}{p(z)}\Big)} &= \overline{\Big(\frac{c((z-z_1)(z-z_2)+(z-z_1)(z-z_3)+(z-z_1)(z-z_2))}{c(z-z_1)(z-z_2)(z-z_3)}\Big)} \\
&= \overline{\Big(\frac{1}{z-z_1}\Big)} + \overline{\Big(\frac{1}{z-z_2}\Big)} + \overline{\Big(\frac{1}{z-z_3}\Big)} \qquad (3.1) \\
&= \frac{z-z_1}{|z-z_1|^2} + \frac{z-z_2}{|z-z_2|^2} + \frac{z-z_3}{|z-z_3|^2}
\end{aligned}$$

Let w be a critical point of $p(z)$, thus we have $p'(w) = 0$ and Equation 3.1 reduces to:

$$\overline{\Big(\frac{p'(w)}{p(w)}\Big)} = 0 \qquad (3.2)$$

Now on the contrary to our statement if we assume that $w \notin \mathcal{H}$ then Lemma 3.4 tells us that there exists θ such that $e^{i\theta}(w-z_1), e^{i\theta}(w-z_2), e^{i\theta}(w-z_3)$ all has positive real part. Combining this with Equation 3.2 we get a contradiction:

$$\begin{aligned}
0 &= \overline{\Big(\frac{p'(w)}{p(w)}\Big)} e^{i\theta} \\
&= \frac{w-z_1}{|w-z_1|^2} e^{i\theta} + \frac{w-z_2}{|w-z_2|^2} e^{i\theta} + \frac{w-z_3}{|w-z_3|^2} e^{i\theta}
\end{aligned}$$

The right hand side of the equation surely has a modulus greater than zero due to the positive real part while the left hand side is equal to zero. Therefore we conclude that w is actually in $\mathcal{H}$ as we wanted. □

4. Preliminary Information on The Poincaré Disk Model and Finite Blaschke Products

4.1. **Poincaré Disk Model of Hyperbolic Geometry.**

. This section will cover preliminary facts about hyperbolic geometry. Hyperbolic geometry shares the same first four axioms with Euclidean geometry, that means it is an absolute geometry like Euclidean geometry. Thus, Euclidean geometry and hyperbolic geometry have many common properties.On the other hand they have significant differences, due to the difference in the fifth axiom. In chapter two we mentioned Playfair's axiom, basically it is equivalent to the fifth axiom. In contrast to Playfair's axiom, in hyperbolic geometry the following is true.

> In the hyperbolic plane, given a line l and a point P there exists two distinct lines through the point P that does not intersect line l.

It is hard work to visualize the hyperbolic plane in a Euclidean space. But throughout the years several models were discovered. In this thesis, we will use one of the methods introduced by Henri Poincaré, the Poincaré disk model. In this method, hyperbolic plane is modeled as a small subset of the Euclidean plane. To be more specific, this subset is the unit disk $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} = \{z \in \mathbb{C} : |z| < 1\}$. Points in the disk are the points of the hyperbolic plane. Points in the boundary of the unit disk, i.e. the points in the unit circle are not in the model. However, they are named as "*ideal points*". As we change our point of view from Euclidean geometry to hyperbolic geometry, the notion of distance will also change. The ideal points' distances to the origin will be infinity with this distance definition and hence it helps intuitively to see the ideal points as the points in the infinity. The actual formula of the distance won't be useful for our purposes thus it will be skipped. More information can be found in various text books, one of such is [1].

. The lines and line segments will be one of our main concern. They also have different properties than they did in Euclidean case of course. In general lines in Poincaré disk model are circular arcs within $\mathbb{D}$ which intersect with the unit circle with a right angle and Euclidean lines that pass through origin. Let's clarify what do we mean by the first one. If Γ is the circle that the arc belongs and say

Γ intersects unit disc at the point P then the line segments from the center of the unit disk to P is perpendicular to the line from center of Γ to P. Just like

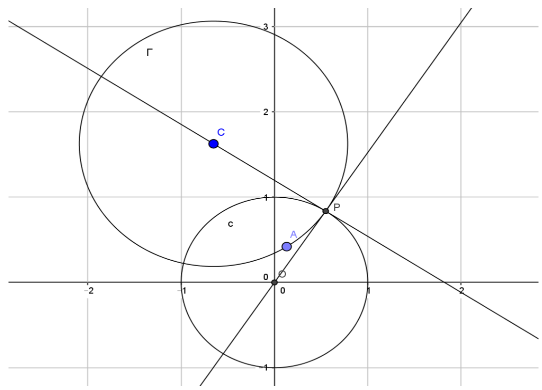

FIGURE 9. Here is an example of hyperbolic line that passes through A and intersects with the unit circle at point P.

in the Euclidean case, given two points, there exists a unique line passing through them. In order to visualize the line passing through points A and B in the Poincaré disk model, we have to draw the circular arc that passes through A and B at the same time is orthogonal to the unit circle. Basically what we have to do is first construct the line OA and then draw the perpendicular line to OA at point A. Denote the intersection points of this perpendicular and unit circle with D and E. Draw the tangents from D and E to the unit circle and name the intersection points of these tangents as F. Now the circle Γ that passes through A,B and F is the circle that is perpendicular to unit circle. This is not a trivial fact but with some elementary geometry it is easy to prove, hence will be skipped. Of course since Γ is circumcircle of the triangle formed by A,B and F; the center of gamma will be the intersection points of the perpendicular bisectors of the triangle ABF. Two intersecting lines have pretty much the same properties, they can have only one intersection point. However when we try to add another line we end up with

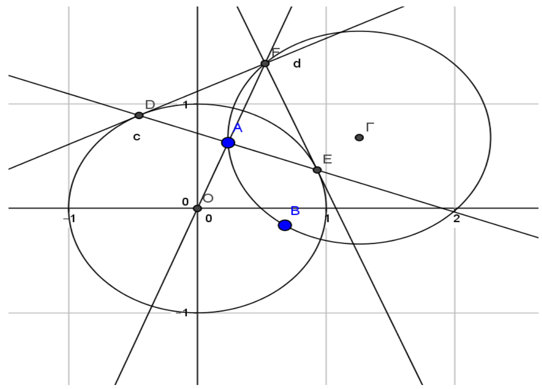

FIGURE 10. Geometric illustration of how to draw the hyperbolic line that passes through the points A and B

some differences to Euclidean geometry. Unlike the Euclidean case, this new line doesn't have to intersect with these two lines. These all emerge from the difference in the fifth axiom because now we know that there exists at least two distinct lines that pass through a point not on the line and does not intersect the line. In a way it makes sense to call them as parallel lines but essentially they are divided into two groups. The first group is called limiting parallel lines, in the disk model these lines correspond to the ones that never meet inside the unit disk but converge to same point on the unit circle. The second type is called ultra-parallel lines which actually diverge from each other, or do not intersect even on the unit circle. The following figure includes an example for both types.

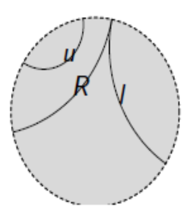

FIGURE 11. R is limiting parallel to l and ultra-parallel to u.

. It is useful to think $\mathbb{D}$ in $\mathbb{C}$ rather than in R^2, then given any two complex numbers z_1,z_2 from the unit disk the line segment between them can be parametrized as follows[6]:

$$[0,1] \longrightarrow \mathbb{D}$$

$$t \longrightarrow \frac{z_1 - \frac{z_1 - z_2}{1-\overline{z_1}z_2}t}{1 - \overline{z_1}\frac{z_1 - z_2}{1-\overline{z_1}z_2}t} \tag{4.1}$$

Any hyperbolic line in $\mathbb{D}$ can be parametrized as,

$$\rho\frac{\omega - z}{1-\overline{\omega}z} = t,\ \ t \in [-1,1] \tag{4.2}$$

This special expression on the left hand side of the equation will be the main concern of the next chapter, and then we will see why the statement above is true.

4.2. **What is a Finite Blaschke Product?** If we consider the Poincaré disk model, there is an ambiguity with the roots and critical points of the polynomials. Usually these roots and critical points are not in the unit disk and therefore these points are not in the model. So instead of looking at the functions that are analytic in the whole complex plane we have to restrict ourselves to the analytic functions on $\mathbb{D}$. The following definition will describe such functions which will take the role of polynomials in the hyperbolic version of Gauss-Lucas theorem.

Definition 4.1. (Finite Blaschke Product) A Finite Blaschke Product is a function of the following form:

$$B(z) = e^{i\alpha}z^K\prod_{i=1}^{n}\frac{|z_i|}{z_i}\frac{z_i - z}{1-\overline{z_i}z}$$

where $\alpha \in \mathbb{R}$, K is a non-negative integer and z_i are just some complex numbers which are in $\{0 < |z| < 1\}$.

Immediately from the definition we can see that all the zeros are in the unit disk. Furthermore any pole of $B(z)$ is outside of the unit disk since;

$$1 - \overline{z_i} z = 0 \;\Leftrightarrow\; z = \frac{1}{\overline{z_i}}$$

A single factor of this product is named as Blaschke factor. Here is a more rigorous definition:

Definition 4.2. (Blaschke Factor) Given $z_0 \in \mathbb{D}$, the Blaschke factor with a zero in z_0 is given as,

$$b_{z_0}(z) = \begin{cases} \frac{|z_0|}{z_0}\frac{z_0 - z}{1-\overline{z_0}z} \text{ if } z_0 \neq 0 \\ \\ z \text{ if } z_0 = 0 \end{cases}$$

Considering any finite Blaschke product $B(z)$ as the multiplication of Blaschke factors, now it is not hard to see that $B(z)$ is an analytic function from $\mathbb{D}$ to itself. Consider the automorphisms of the unit disk, i.e. bijective conformal maps on $\mathbb{D}$, namely Aut($\mathbb{D}$). Recall that this set can be characterized as follows. Let $\omega \in \mathbb{D}$, $\rho \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and define the functions

$$\tau_\omega(z) = \frac{\omega - z}{1 - \overline{\omega} z} \quad \text{and} \quad \rho_\gamma(z) = \gamma z$$

Then,

$$Aut\mathbb{D} = \{\rho_\gamma \circ \tau_\omega : \omega \in \mathbb{D}, \gamma \in \mathbb{T}\}$$

So every Blaschke factor corresponds to an automorphism of $\mathbb{D}$. Therefore if we consider $b_{z_0}(z) : \mathbb{D} \longrightarrow \mathbb{D}$, then we can say that $b_{z_0}(z)$ is actually an analytic function from $\mathbb{D}$ to itself with a single zero. In more general sense if we consider $B(z)$ a finite Blaschke product of degree n, we can easily see that it is an analytic function from $\mathbb{D}$ to itself with n zeros within the unit disk.

. This is a good moment to remark the resemblance to the Euclidean side of the story. Instead of focusing on the automorphisms of the disk if we consider the conformal automorphisms of the whole $\mathbb{C}$, we have:

$$Aut\mathbb{C} = \{az + b : a, b \in \mathbb{C}, a \neq 0\}$$

With the enough complex analytic tools this is an easy fact to prove. Let $f \in Aut\mathbb{C}$, so f is a bijective analytic function on $\mathbb{C}$. Then f cannot have an essential singularity around infinity otherwise Big Picard theorem would ensure that in any neighborhood of infinity f takes every value with at most a single exception infinitely often, therefore f cannot be an automorphism in this case. Also if f has a removable singularity, then f would be bounded by its value at infinity and Liouville's theorem gives us that f has to be constant in this case, so not an automorphism. Therefore f has a pole at infinity, therefore it must be a polynomial. Now if f has degree higher than 1, f would have more than one root meaning it cannot be injective. Therefore if f is in $Aut\mathbb{C}$ then f has to be a degree one polynomial. Again, any element in this set can be considered as a composition rotation and a bijective analytic function on $\mathbb{C}$, or an analytic function on $\mathbb{C}$ with a single zero, or a bijective conformal map on $\mathbb{C}$. Further in the case of polynomials we follow the exact same pattern, any degree n polynomial is actually multiplication of n such bijective analytic functions.

. Recall the equation (4.2) from the previous chapter, now we can see that it actually takes the line [-1,1] in the Poincaré disk and conformally maps it to a curve. It makes sense to call the image of $[-1, 1]$ as a parameterization of the hyperbolic line. Now we can actually prove that equation (4.1) is true. As starting step let's take

$z_2 = 0$, then the line segment between 0 and z_1 is parameterized as:

$$[0,1] \longrightarrow \mathbb{D}$$

$$t \longrightarrow tz_1$$

Now move on to the general case, let z_1, z_2 be two complex numbers from the unit disk. Then use the automorphism of $\mathbb{D}$, $\tau_{z_1}(z) = \frac{z_1 - z}{1-\overline{z_1}z}$ to move z_1 to zero and z_2 to some non-zero element of $\mathbb{D}$.

$$\tau_{z_1}(z_2) = \frac{z_1 - z_2}{1 - \overline{z_1}z_2} := z_3$$

Now we can parameterize the line from 0 to z_3 as before:

$$[0,1] \longrightarrow \mathbb{D}$$

$$t \longrightarrow tz_3 \tag{4.3}$$

However this is not the parameterization of the line segment that we want, but it's conformal image of it under the automorphism $\tau_{z_1}(z)$. We know that,

$$\tau_{z_1}(z)^{-1} = \frac{z_1 - z}{1 - \overline{z_1}z}$$

So if we compose this with parameterization in equation (4.3) we will get the parametrization of the desired line segment:

$$[0,1] \longrightarrow \mathbb{D}$$

$$t \longrightarrow \frac{tz_3 - z_1}{1 - \overline{z_1}tz_3}$$

As the last step if we write back z_3 in terms of z_1 and z_2 we get the parametrization of the line segment between z_1 and z_2 the same thing with equation (4.1):

$$t \longrightarrow \frac{t\frac{z_1 - z_2}{1-\overline{z_1}z_2} - z_1}{1 - \overline{z_1}t\frac{z_1-z_2}{1-\overline{z_1}z_2}}$$

. In the following chapters we will try to formalize the statement of Gauss-Lucas theorem for hyperbolic case. And as we have mentioned before, the Poincaré disk model will be our main model. As a result of this, we had to restrict ourselves to $\mathbb{D}$ where the hyperbolic space is modeled. Considering the pattern above, finite Blaschke products is a natural choice of functions to consider instead of polynomials. Therefore they will be our main concern in the statement of the hyperbolic case of the theorem. The next chapter will provide more information about finite Blaschke products.

4.3. **Preliminary Results about Finite Blaschke Products.**

. This section will be entirely denoted to some significant properties of finite Blaschke products. The aim will be to show the similarities between finite Blaschke products and polynomials, in addition to prepare some foundations to generalize the Gauss-Lucas theorem to hyperbolic case. First of all let us clarify the notion of degree for a finite Blaschke product. In the Definition 4.1, we simply regarded a degree n Blaschke product as multiplication of n Blaschke factors. In the polynomial sense we know that degree corresponds to number of zeros of the polynomial. Similarly for finite Blaschke products, degree directly corresponds to the number of zeros in the unit disk as it was mentioned previously. From another point of view, we can see any finite Blaschke product as a rational function;

$$B(z) = e^{i\alpha} z^K \prod_{i=1}^{n} \frac{|z_i|}{z_i} \frac{z_i - z}{1 - \overline{z_i} z} = \frac{P(z)}{Q(z)}$$

then the degree would be given as $\max\{deg(P(z)), deg(Q(z))\}$ if $P(z), Q(z)$ are coprime as polynomials, that is they don't have a common non-constant divisor. We know that both $P(z)$ and $Q(z)$ are polynomials of degree n. Moreover we know that $P(z)$ and $Q(z)$ cannot have a common factor. This is because for every root $z_0 \in \mathbb{D}$ of $P(z)$, $z_0' = \frac{1}{\overline{z_0}}$ is a root of $Q(z)$. Of course any z_0' is outside of $\mathbb{D}$. Therefore,

$$deg(B(z)) = \max\{n, n\} = n$$

So the degree of a Blaschke product can be both regarded as the number of Blaschke factors that it contains(or the number of zeros within $\mathbb{D}$) and degree of the rational function. These two different interpretation of degree actually coincides and we will use both interchangeably.

. Regarding a finite Blaschke product $B(z)$ as multiplication of Blaschke factors is very useful for many reasons. For example in order to calculate the modulus or argument of the $B(z)$, it is enough to focus on a single Blaschke factor, which is actually a conformal automorphism of the unit disk. Therefore any finite Blaschke product is actually multiplication of conformal automorphisms of the disk. Modulus of a single factor is fairly easy to calculate. Recalling from the Definition 4.2, for a Blaschke factor with a zero in $z_0 \in \mathbb{D}$ the modulus is calculated as follows:

$$\begin{aligned}
|b_{z_0}(z)|^2 = b_{z_0}(z)\overline{b_{z_0}(z)} &= \frac{|z_0|}{z_0}\frac{z_0 - z}{1 - \overline{z_0}z}\frac{|z_0|}{\overline{z_0}}\frac{\overline{z_0} - \overline{z}}{1 - z_0\overline{z}} \\
&= \frac{|z_0|^2 - z_0\overline{z} - \overline{z_0}z + |z|^2}{|1 - \overline{z_0}z|^2} \\
&= \frac{1 - z_0\overline{z} - \overline{z_0}z + |z|^2|z_0|^2 - 1 + |z_0|^2 + |z|^2 - |z|^2|z_0|^2}{|1 - \overline{z_0}z|^2} \qquad (4.4) \\
&= \frac{(1 - \overline{z_0}z)(1 - z_0\overline{z})}{|1 - \overline{z_0}z|^2} - \frac{(1 - |z_0|^2)(1 - |z|^2)}{|1 - \overline{z_0}z|^2} \\
&= 1 - \frac{(1 - |z_0|^2)(1 - |z|^2)}{|1 - \overline{z_0}z|^2}
\end{aligned}$$

In general given the degree n Blaschke product $B(z)$,

$$B(z) = e^{i\alpha} \prod_{j=1}^{n} b_{z_j}(z)$$

modulus of $B(z)$ can be calculated as follows.

$$|B(z)| = \prod_{j=1}^{n} |b_{z_j}(z)| \tag{4.5}$$

Remark 4.3. Following from the above equation, if $z \in \mathbb{T}$ then $|b_{z_0}(z)|^2 = 1$. Since $|z| = 1$, it follows that $(1-|z|) = 0$ which reduces the right-hand side to 1. Therefore we have that for any $z_0 \in \mathbb{D}$, $b_{z_0}(z)$ maps $\mathbb{T}$ to itself. In general, any finite Blaschke product actually maps $\mathbb{T}$ to $\mathbb{T}$.

. The argument of $b_{z_0}(z)$ is a little harder to calculate than the modulus. This part is not going to be used in the following parts but it is included for the sake of completeness. We will first investigate how the argument of $b_{z_0}(z)$ behaves on the boundary of the unit disk, $\mathbb{T}$. As an aside, for $z_0 = 0$ case trivially we have $\arg(b_{z_0}(z)) = \arg(z)$. So let $z = e^{i\theta}$ and $z_0 = r_0 e^{i\theta_0}$ for some $r_0 \in (0, 1)$. Then we have the following lemma to describe the behavior of the Blaschke factor's argument in the unit disk.

Lemma 4.4. *$b_{z_0}(z)$ maps $\mathbb{T}$ to itself, so if we denote $z = e^{i\theta}$ for some $\theta \in [0, 2\pi)$ we can write:*

$$b_{z_0}(z) = e^{i\arg(b_{z_0}(z))}$$

where $\arg(b_{z_0}(z)) \in [-\pi, \pi)$. *Then*

$$\arg(b_{z_0}(z)) = \begin{cases} -\pi \ \textit{if}\ \theta = \theta_0 \\ -2\arctan\left(\frac{(1-r_0)}{(1+r_0)\tan(\frac{\theta+\theta_0}{2})}\right) \ \textit{if}\ \theta \neq \theta_0 \end{cases}$$

Here we use the principal branch of the arctan function which has range $(-\frac{\pi}{2}, \frac{\pi}{2})$*, so indeed* $\arg(b_{z_0}(z)) \in [-\pi, \pi)$.

This lemma will be useful to investigate the argument of the Blaschke factor for any given point $z = re^{i\theta} \in \mathbb{D}$. If we normalize $b_{z_0}(z)$ we have:

$$\frac{b_{z_0}(re^{i\theta})}{|b_{z_0}(re^{i\theta})|} = e^{i \arg(b_{z_0}(re^{i\theta})}$$

Clearly if $r < |z_0| = r_0$ then $b_{z_0}(z)$ has no zeros inside the disk $|z| < r$. Therefore when r is given, $e^{i \arg(b_{z_0}(re^{i\theta})}$ is actually a continuous function of $\theta \in \mathbb{R}$. So, $\arg(b_{z_0}(re^{i\theta})$ is actually a 2π-periodic continuous function of $\theta \in \mathbb{R}$. On the other hand, if $r > |z_0| = r_0$ argument principle gives us that $\arg(b_{z_0}(re^{i\theta})$ has a jump discontinuity in every interval larger than 2π. These observations can be turned into a more specific description of the argument of $b_{z_0}(z)$, even though it will not be useful for this thesis' purposes. The following theorem gives us the total description of the argument of a Blaschke factor, the proof will be omitted however more detailed information can be found in [7].

Theorem 4.5. *Let* $r_0 \in (0,1)$ *and* $z_0 = r_0 e^{i\theta_0}$*, write:*

$$b_{z_0}(z) = |b_{z_0}(z)| e^{i \arg b_{z_0}(z)}$$

where $-\pi \leq \arg b_{z_0}(z) < \pi$ *. Then the following holds,*

$$\arg b_{z_0}(z) = \arcsin \frac{\Im(z_0 \overline{z})(1 - |z_0|^2)}{|z_0||z_0 - z||1 - \overline{z_0} z|} \tag{4.6}$$

where $\Im(z_0\overline{z})$ *denotes the imaginary part of* $z_0\overline{z}$.

. Following from the fact that $B(z)$ maps $\mathbb{D}$ and $\mathbb{T}$ to itself, we can say $B(z)$ maps $\mathbb{C}\backslash\mathbb{D}$ to itself. If we consider Poincaré disk model this fact means that every Blaschke product gives us a well-defined map of the model to itself. Every point of the model is mapped to some point inside the model and similarly points that stay out of the model are sent to points that stay out of the model. By our construction $B(z)$ has n zeros inside the unit disk, this can be interpreted as the equation

$$B(z) = 0$$

has n solutions. In fact $B(z)$ is a n-to-1 map from $\hat{C}$ to itself, where $\hat{C}$ denotes the extended complex plane. Since $B(z) = w$ actually gives a polynomial equation of degree n, which obviously have n solutions in $\hat{C}$. The following theorem will describe how these solutions are located in the extended complex plane for a given w.

Theorem 4.6. *Let $\hat{C}$ denote the extended complex plane and $B(z)$ be a finite Blaschke product of degree n. Then given a $w \in \hat{C}$, there exists exactly n solutions to the equation*

$$B(z) = w$$

Moreover if $w \in \mathbb{D}$ then all n solutions are in $\mathbb{D}$, if $w \in \mathbb{T}$ all solutions are in $\mathbb{T}$ and if all $w \in \hat{C}\backslash\overline{\mathbb{D}}$ then all the solutions are in $\hat{C}\backslash\overline{\mathbb{D}}$ where $\overline{\mathbb{D}}$ is the closed unit disk.

Proof. First of all we have already shown that this equation has n solutions in $\hat{C}$ above. Now let's assume first that $w \in \mathbb{D}$. We know that $B(z)$ maps $\hat{C}\backslash\overline{\mathbb{D}}$ to $\hat{C}\backslash\overline{\mathbb{D}}$, therefore if z is mapped to $w \in DD$ then z has to be within the unit disk or on the unit circle. Similarly $\mathbb{T}$ is mapped to $\mathbb{T}$, so z cannot be on $\mathbb{T}$. Therefore all n solutions of the equation is indeed in $\mathbb{D}$.

Second, assume that $w \in \mathbb{T}$. Following similar argument, any $z \in \hat{C} \backslash \overline{\mathbb{D}}$ would be mapped to something in $\hat{C} \backslash \overline{\mathbb{D}}$ and any $z \in \mathbb{D}$ would be mapped to something in $\mathbb{D}$. Therefore the complex number that was sent to w by $B(z)$ has no choice but to be in $\mathbb{T}$. Lastly for $w \in \hat{C} \backslash \overline{\mathbb{D}}$ case with a similar argument we can see that all solutions have to be in $\hat{C} \backslash \overline{\mathbb{D}}$. $\square$

. Our main concern in the above theorem is the case where $w \in \mathbb{D}$. Of course there may be repeated solutions, for example simply in the $w = 0$ case any finite Blaschke factor with repeated Blaschke factor would result in repeated solutions. An interesting side note is if $w \in \mathbb{T}$ there exists no repeated solutions[7], however these kinds of equations won't be our concern. A more important remark about the Theorem 4.6 is, it shows us that once again why finite Blaschke products are a natural choice instead of polynomials in the Poincaré disk model. Any equation of a degree n Blaschke product in the Poincaré disk, actually gives us n solutions in the disk. Recall from the proof of Lemma 3.4 in Chapter 3, we have defined a conformal map of $\mathbb{C}$ to move the roots of the polynomial so that they all had positive real parts. Luckily it was possible to do that since polynomials are conformally invariant, that is when a polynomial is composed with a conformal map the resulting map is again a polynomial. In the next theorem we will show a similar property for finite Blaschke products.

Theorem 4.7. *Let $B(z)$ be a finite Blaschke product of degree n and $\tau_w(z) = \frac{w-z}{1-\overline{w}z}$ for $w \in \mathbb{D}$. Then both $\tau_w \circ B(z)$ and $B(z) \circ \tau_w$ are finite Blaschke products of degree n.*

Before the proof of theorem we will present a characterization of finite Blaschke products within the analytic functions on $\mathbb{D}$. It is an important result proven by

Fatou, see [7]. Instead of calling it a theorem, we will call it a lemma since it will be one of the main arguments of the proof.

Lemma 4.8. *If f is analytic on $\mathbb{D}$ and*

$$\lim_{|z|\to 1} |f(z)| = 1$$

then f is a finite Blaschke product.

Remark 4.9. Note that if f is a finite Blaschke product we know f is analytic on $\mathbb{D}$ and ,

$$\lim_{|z|\to 1} |f(z)| = 1$$

Proof of Lemma 4.8. First of all note that $|f(z)| \to 1$ uniformly as $|z| \to 1$, thus we can say that there has to be an annulus $\{z : r_0 < |z| < 1\}$ where f does not vanish. Therefore as a result of analytic continuity f can have only finitely many zeros inside $\mathbb{D}$. Let $B(z)$ be the Blaschke product with the exact same roots of f within $\mathbb{D}$ with the same multiplicity. So we must have that both $\frac{f}{B}$ and $\frac{B}{f}$ are analytic within the unit disk and their moduli goes to 1 as $|z| \to 1$. Maximum modulus principle gives that as analytic functions on $\mathbb{D}$, both of these functions have moduli of at most 1. Therefore, $|\frac{f}{B}| \leq 1$,$|\frac{B}{f}| \leq 1$. But this tells us that $|\frac{f}{B}| = 1$ is just a unimodular and f is just a unimodular multiple of $B(z)$, meaning that we have to multiply $B(z)$ with a complex number of modulus 1 to get f. Thus f is a finite Blaschke product. □

Now as a corollary to this lemma, we can say that if the disk algebra $\mathcal{A}(\mathbb{D})$ denotes the set of analytic functions on $\mathbb{D}$ that extend continuously to $\overline{\mathbb{D}}$, then the finite Blaschke products are precisely those elements in $\mathcal{A}(\mathbb{D})$ that map $\mathbb{T}$ to $\mathbb{T}$. Now we can give the proof of Theorem 4.7

Proof of Theorem 4.7. Let $B(z) = e^{i\alpha} z^K \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{z_k - z}{1-\overline{z_k}z}$ be a finite Blaschke product of degree n and define $\tau_w(z) = \frac{w-z}{1-\overline{w}z}$ for $w \in \mathbb{D}$. First let's see that $B(z) \circ \tau_w(z)$ is a finite Blaschke product of degree n.

$$
\begin{aligned}
B(z) \circ \tau_w(z) &= e^{i\alpha} z^K \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{z_k - \frac{w-z}{1-\overline{w}z}}{1 - \overline{z_k}\frac{w-z}{1-\overline{w}z}} \\
&= e^{i\alpha} z^K \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{z_k - z_k\overline{w}z - w - z}{1 - \overline{w}z - \overline{z_k}w + \overline{z_k}z} \\
&= e^{i\alpha} z^K \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{1 - z_k\overline{w}}{1 - \overline{z_k}w} \frac{\frac{w-z_k}{1-\overline{w}z_k} - z}{1 - \frac{\overline{w}-\overline{z}}{1-w\overline{z_k}}z} \qquad (4.7) \\
&= e^{i\alpha'} z^K \prod_{k=1}^{n} \frac{|\tau_w(z_k)|}{\tau_w(z_k)} \frac{\tau_w(z_k) - z}{1 - \overline{\tau_w(z_k)}z}
\end{aligned}
$$

Note that in the last step we carried out the required unimodular constant outside so now we have α' instead of α. Therefore indeed we have that $B(z) \circ \tau_w(z)$ is a Blaschke product of degree n with zeros $\tau_w(z_k)$

Now we have to show that $\tau_w(z) \circ B(z)$ is also a finite Blaschke product of degree n. Observe that $\tau_w(z) \circ B(z)$ is analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. But more importantly it maps $\mathbb{T}$ to $\mathbb{T}$ since,

$$|\tau_w(z)| = |\frac{w - e^{i\theta}}{1 - \overline{w}e^{i\theta}}| = | - e^{i\theta}||\frac{1 - we^{-i\theta}}{1 - \overline{w}e^{i\theta}}| = 1$$

and $B(e^{i\theta}) = 1$ as we know. Therefore conclusion of the previous lemma gives us that $\tau_w(z) \circ B(z)$ is a finite Blaschke product. Let's try to verify it's degree now, we know:

$$\tau_w(z) \circ B(z) = 0 \Longleftrightarrow B(z) = w$$

and $w \in \mathbb{D}$. Now theorem 4.6 gives us that the equation on the left has exactly n solutions all inside $\mathbb{D}$. Therefore $\tau_w(z) \circ B(z)$ has exactly n zeros in $\mathbb{D}$ and therefore it is of degree n. $\square$

Note that Theorem 4.7 focuses on one kind of automorphism of disk, but it is easy to show that composition of a rotation with a Blaschke product is also a Blaschke product. Given $\rho_\theta(z) = e^{i\theta}z$ for $\theta \in [0, 2\pi)$ and

$$B(z) = e^{i\alpha}z^K \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k}z}$$

for $z_k \in \mathbb{D}$, we can easily calculate $B(z) \circ \rho_\theta(z)$ and $\rho_\theta(z) \circ B(z)$.

$$\begin{aligned} B(z) \circ \rho_\theta(z) &= e^{i\alpha}z^K \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{z_k - ze^{i\theta}}{1 - \overline{z_k}ze^{i\theta}} \\ &= e^{i\alpha}z^K \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{e^{i\theta}(z_k e^{-i\theta} - z)}{1 - \overline{z_k e^{-i\theta}}z} \\ &= e^{i\alpha}z^K \prod_{k=1}^{n} \frac{|z_k||e^{-i\theta}|}{z_k e^{-i\theta}} \frac{z_k e^{-i\theta} - z}{1 - \overline{z_k e^{-i\theta}}z} \end{aligned} \tag{4.8}$$

And this corresponds to a finite Blaschke product with zeros $z_k e^{i\theta}$. Similarly if we compute this from the other way:

$$\rho_\theta(z) \circ B(z) = e^{i\theta}\left(e^{i\alpha}z^K \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k}z}\right) \tag{4.9}$$

The result is trivially another degree n Blaschke product. As a further step, using Theorem 4.7 it is not hard to show that composition of two finite Blaschke products of degree m and n is again a Blaschke product of degree mn. Let $B_1(z) = e^{i\alpha}\prod_{k=1}^{n} b_k(z)$, $B_2(z) = e^{i\alpha'}\prod_{k=1}^{m} b'_k(z)$. Then we have,

$$B_1(z) \circ B_2(z) = e^{i\alpha} \prod_{k=1}^{n} (b_k(z) \circ B_2(z)) \tag{4.10}$$

From Theorem 4.7 we know $b_k(z) \circ B'(z)$ are Blaschke products of degree m and hence the resulting product is indeed a finite Blaschke product of degree nm.

. Before moving on to the Gauss-Lucas theorem there is one more topic that we have to elaborate on, the derivative of a Blaschke product. Let's say that we are in the generic case where $B(0) \neq 0$, so write $B(z) = e^{i\alpha} \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k} z}$. Since $B(z)$ is a product of many factor, we just have to take a derivative of products. In order to do so let us define:

$$B_j(z) = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{z_k - z}{1 - \overline{z_k} z} \tag{4.11}$$

Also note that for single Blaschke factor $b_{z_k}(z)$, we can write the derivative as:

$$(b_{z_k}(z))' = \frac{|z_k|^2 - 1}{(1 - \overline{z_k} z)^2} \tag{4.12}$$

Now using (4.11) and (4.12) we can write $B'(z)$ easily as follows.

$$B'(z) = \sum_{k=1}^{n} (b_{z_k}(z))' B_k(z) = -\sum_{k=1}^{n} \frac{1 - |z_k|^2}{(1 - \overline{z_k} z)^2} B_k(z) \tag{4.13}$$

Again in parallel with how we proceeded in the proof of the Euclidean Gauss-Lucas' theorem, logarithmic derivative of $B(z)$ is a powerful tool that we can use. If we divide both sides of (4.13) by $B(z)$ we can easily get the logarithmic derivative.

$$\begin{aligned} \frac{B'(z)}{B(z)} &= -\sum_{k=1}^{n} \frac{1 - |z_k|^2}{(1 - \overline{z_k} z)^2} \frac{B_k(z)}{B(z)} \\ &= -\sum_{k=1}^{n} \frac{1 - |z_k|^2}{(1 - \overline{z_k} z)^2} \frac{1}{b_{z_k}(z)} = \sum_{k=1}^{n} \frac{1 - |z_k|^2}{(1 - \overline{z_k} z)(z - z_k)} \end{aligned} \tag{4.14}$$

As a finishing remark for this chapter we will see another fascinating property of the derivative of Blaschke products. Let $z \in \mathbb{T}$ and write $z = e^{i\theta}$, we will manipulate

(4.14) as follows:

$$\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \frac{1-|z_k|^2}{(1-\overline{z_k}z)e^{i\theta}(1-e^{-i\theta}z_k)}$$
$$= \sum_{k=1}^{n} \frac{1-|z_k|^2}{(1-\overline{z_k}z)e^{i\theta}(1-\overline{z}z_k)} = \sum_{k=1}^{n} \frac{1-|z_k|^2}{|1-\overline{z_k}z|^2 e^{i\theta}}$$

Now if we take the modulus of both sides, since $|B(e^{i\theta})| = 1$ and $|z_k| < 1$:

$$\frac{|B'(z)|}{|B(z)|} = |B'(z)| = \sum_{k=1}^{n} \frac{1-|z_k|^2}{|1-\overline{z_k}z|^2} \tag{4.15}$$

Note that the right-handside is always positive and gives us the following remark.

Remark 4.10. If $B(z)$ is a finite Blaschke product then for any $e^{i\theta} \in \mathbb{T}$ we have $B'(e^{i\theta}) \neq 0$.

5. Hyperbolic Gauss-Lucas Theorem

. After some brief introduction to the Poincaré disk model of hyperbolic geometry and finite Blaschke products we are almost ready to state and prove the hyperbolic version of Gauss-Lucas theorem. Before we start recall the equation of the hyperbolic line segment (4.1), we will use this to define the notion of convexity in the hyperbolic sense. We call a set convex if given any two points in the set, all the points on the geodesic connecting these two points must also be in the convex set. In the light of (4.1) a set $A \subset \mathbb{D}$ is convex if for any two points $z_1, z_2 \in A$ we have that

$$\frac{z_1 - \frac{z_1 - z_2}{1-\overline{z_1}z_2}t}{1 - \overline{z_1}\frac{z_1 - z_2}{1-\overline{z_1}z_2}t} \in A \text{ for all } t \in [0,1] \tag{5.1}$$

Using this, the hyperbolic convex hull of n points $z_1, \ldots, z_n$ can be defined as the hyperbolic convex set that contains $z_1, \ldots, z_n$. Next we state the main theorem of this work.

Theorem 5.1 (Hyperbolic Gauss-Lucas Theorem)**.** *If $B(z)$ is a finite Blaschke product, then the roots of $B'(z)$ lies in the hyperbolic convex hull of $B(z)$.*

Before proceeding with the proof, let's give an example and illustrate what is happening.

Example 5.2. For the ease of computation, we will consider an example of degree three case. Let's take three points from the unit disk, $z_1 = 0.2 - 0.2i$, $z_2 = 0.4i$ and $z_3 = -0.6 - 0.4i$. Consider the finite Blaschke product with $\alpha = 0$ and z_1, z_2, z_3 as roots:

$$
\begin{aligned}
B(z) &= \frac{0.2 - 0.2i - z}{1 - \overline{(0.2 - 0.2i)}z} \frac{0.4i - z}{1 - \overline{(0.4i)}z} \frac{-0.6 - 0.4i - z}{1 - \overline{(-0.6 - 0.4i)}z} \\
&= \frac{0.2 - 0.2i - z}{1 - (0.2 + 0.2i)z} \frac{0.4i - z}{1 + (0.4i)} z \frac{-0.6 - 0.4i - z}{1 - (-0.6 + 0.4i)z}
\end{aligned} \tag{5.2}
$$

After calculating the critical roots using any basic software such as Mathematica or Wolfram Alpha, we can write them as $z = 0.0592106 + 0.104897i$ and $w_2 = z = -0.353041 - 0.258281i$. Let's put all these into a picture, see Figure 12.

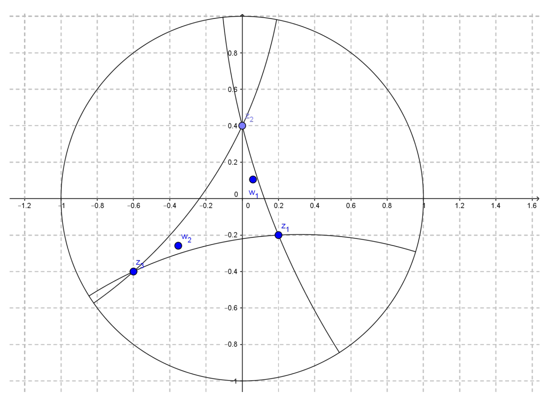

FIGURE 12. Picture of Example 5.2

Let's proceed with the proof of the hyperbolic version of Gauss-Lucas theorem.

Proof of 5.1. Let $B(z)$ be a finite Blaschke product of degree n with zeros $z_1, \ldots, z_n \in \mathbb{D}$ and $\mathcal{H}$ denote the hyperbolic convex hull of these points. The proof of the hyperbolic version will follow similar steps to what we have done in the Theorem 3.2. Recall that in order to prove the theorem we had to go through a lemma first, namely Lemma 3.4. We will proceed in a similar way, if $a \in \mathbb{D}$ is any point not inside the convex hull $\mathcal{H}$ then with the transformation $\tau_a(z) = \frac{a-z}{1-\bar{a}z}$ we can move a to zero. After that we had that since a is not in the hyperbolic convex hull then it is possible to choose an appropriate rotation such that each $\tau_a(z_k)$ is mapped to a complex number with a non-negative imaginary part. Note that in the proof of Lemma 3.4 we have done the same thing except we had moved the roots so that all of them had non-negative real part, in essence we just fit all the points into one half-plane. In addition, we had done all these for Euclidean case in Lemma 3.4 but same arguments holds in hyperbolic case as well since we did not use any property specific to the Euclidean geometry. So let's say that $f = B(z) \circ \tau_a(z) \circ \rho_\theta(z)$, where a is one of the roots of $B(z)$ and $\theta \in [0, 2\pi)$. We have already seen that f is a finite Blaschke product of degree n. Since $\tau_a^2 = id$, and $\rho_\theta^{-1}(z) = \rho_{-\theta}$ the zeros of f are located in $u_1 := \rho_{-\theta}(\tau_a(z_1)), \ldots, u_n := \rho_{-\theta}(\tau_a(z_n))$ and if we let $w_1, \ldots, w_{n-1}$ be the roots of $B'(z)$ then the roots of f' are $v_1 := rho_{-\theta}(\tau_a(w_1)), \ldots, v_n := \rho_{-\theta}(\tau_a(w_{n-1}))$. See Figure 13

Next step is to make use of this non-negativity of the imaginary parts. Let's divide $\mathbb{D}$ into three parts as follows.

$$\mathbb{D}_- = \mathbb{D} \cap \{z : \Im(z) < 0\} \text{ and } \mathbb{D}_+ = \{z : \Im(z) > 0\}$$

and lastly we have $[-1, 1]$. Basically positioning all z_k such that they have all non-negative imaginary parts means that they will stay on the same side of the line

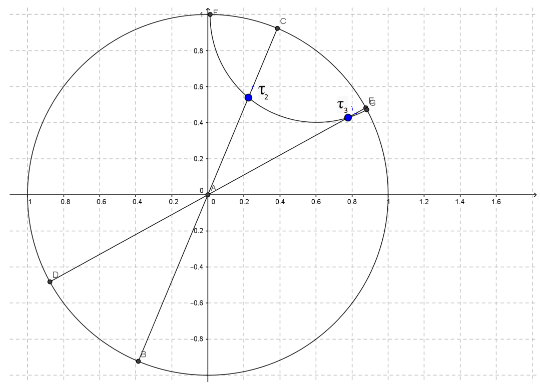

FIGURE 13. The transformation $\tau_{z_1}(z)$ is applied in the case of Example 5.2. z_1 is mapped to 0, τ_2 and τ_3 denotes the points $\tau_{z_1}(z_2)$ and $\tau_{z_1}(z_3)$ respectively. In this case $\theta = 0$

formed by any mapping of $[-1, 1]$ with $\tau_a(z)$,i.e. the line parameterized as:

$$\tau_a(z) = \frac{a - z}{1 - \overline{a}z} = t, \text{ for } t \in [-1, 1]$$

Our aim is to show that the roots of $B'(z)$ falls are on the same side of the line for all such a. Now referring back to equation (4.14), we have that putting f instead of $B(z)$

$$\Im\Big(\frac{f'(z)}{f(z)}\Big) = \Im\Big(\sum_{k=1}^{n} \frac{1 - |u_k|^2}{(1 - \overline{u_k}z)(z - u_k)}\Big) = \sum_{k=1}^{n} \Im\Big(\frac{1 - |u_k|^2}{(1 - \overline{u_k}z)(z - u_k)}\Big) \tag{5.3}$$

where $\Im$ denotes the imaginary part. Now instead of focusing the whole sum in the (5.3), just focus on one of the summands. Let's define

$$\varphi(z) = \frac{1 - |w|^2}{(1 - \overline{w}z)(z - w)} \tag{5.4}$$

for a fixed $w \in \mathbb{D}_+$. We want to consider the case where w has non-negative imaginary part in particular. Because we know that the apart from the origin remaining zeros of f are in the upper half-plane by construction. In order to investigate where the $\mathbb{D}_-$ is mapped, let's consider it's boundary under φ. That means we have to consider the image of $\mathbb{T}_- = \{e^{i\theta} : -\pi \leq \theta \leq 0\}$ and $[-1,1]$. For $\mathbb{T}_-$, we have

$$\begin{aligned}\varphi(e^{i\theta}) &= \frac{1-|w|^2}{(1-\overline{w}e^{i\theta})(e^{i\theta}-w)} \\ &= \frac{1-|w|^2}{(1-\overline{w}e^{i\theta})e^{i\theta}\overline{(1-\overline{w}e^{i\theta})}} = \frac{1-|w|^2}{|1-\overline{w}e^{i\theta}|^2}e^{-i\theta}\end{aligned} \tag{5.5}$$

Note that result of the equation (5.5) has non-negative imaginary part since $w \in \mathbb{D}_+$ and $\theta \in [-\pi, 0]$. Similarly for the interval $[-1,1]$, we have:

$$\begin{aligned}\varphi(x) &= \frac{1-|w|^2}{(1-\overline{w}x)(x-w)} \\ &= \frac{1-|w|^2}{(1-\overline{w}x)(x-w)\overline{(1-\overline{w}x)(x-w)}}\overline{(1-\overline{w}x)(x-w)} \\ &= \frac{1-|w|^2}{|(1-\overline{w}x)(x-w)|^2}(1-wx)(x-\overline{w})\end{aligned} \tag{5.6}$$

Observe that the first fraction part is actually just a real coefficient. If we calculate last part:

$$(1-wx)(x-\overline{w}) = x - \overline{w} - wx^2 + |w|^2 x$$

So imaginary part of this expression is basically $\Im(\overline{-w} - x^2 w) = \Im(w - x^2 w)$, hence in general combining this with (5.6) we get that:

$$\begin{aligned}\Im(\varphi(x)) &= \frac{1-|w|^2}{|(1-\overline{w}x)(x-w)|^2}(1-wx)\Im(w-x^2w) \\ &= \frac{1-|w|^2}{|(1-\overline{w}x)(x-w)|^2}(1-wx)(1-x^2)\Im(w)\end{aligned} \tag{5.7}$$

Note that we already had that $w \in \mathbb{D}_+$ hence this is a non-negative number.Therefore we can say that the boundary of the half-disk $\mathbb{D}_-$ is mapped to a curve in $\mathbb{C}_+ \cup \mathbb{R}$ where $\mathbb{C}_+ = \{z : \Im(z) > 0\}$. Note that φ is analytic on $\overline{\mathbb{D}_-} := \mathbb{D}_- \cup \mathbb{T}_- \cup [-1,1]$ since $w \in \mathbb{D}_+$. Therefore by continuity we can deduce that φ maps $\mathbb{D}_-$ to $\mathbb{C}_+$. This actually tells us that any $z \in \mathbb{D}_-$ we have that $\Im(\varphi(z)) > 0$. In total if we sum all $\varphi(w)$ while w runs over u_k we have that $\Im\left(\frac{f'(z)}{f(z)}\right) > 0$ for any $z \in \mathbb{D}_-$. Therefore $f'(z)$ has no zeros in $\mathbb{D}_-$. In other words $\Im(v_k) \geq 0$, therefore all $v_k \in \mathbb{D}_+ \cup (-1,1)$. Now we have shown that the roots of f and f' lie on the same side of the hyperbolic line parametrized by $\rho_\theta \circ \tau_a(z) = t$ for $t \in [-1,1]$. Now recall the Remark 3.6, if a is taken equal to z_k then $u_{k-1} = \rho_{-\theta}(\tau_a(z_{k-1}))$ and $u_{k+1} = \rho_{-\theta}(\tau_a(z_{k+1}))$ will have the maximal difference of arguments for all $u_1, \ldots, u_n$. If we let $\theta_0 = \arg(u_{k+1})$, then $g = \rho_{-\theta_0} \circ f(z)$ is another Blaschke product of degree n with one of its roots in zero and another one on real axis. In general the zeros of g are $\rho_{\theta_0}(u_k)$ and g' is of degree $n-1$ with zeros $\rho_{-\theta_0}(v_k)$. Moreover all the zeros of g and g' are on the same side of the line passing through $\rho_{-\theta_0}(u_k)$ and $\rho_{-\theta_0}(u_k+1)$. This gives us that all the zeros of $B(z)$ and $B'(z)$ are actually on the same side of the line containing z_k and z_{k+1}. Now going through the same process with every root, we get that actually the roots of $B'(z)$, w_k, are in the hyperbolic convex hull of $z_1, \ldots, z_k$ $\square$

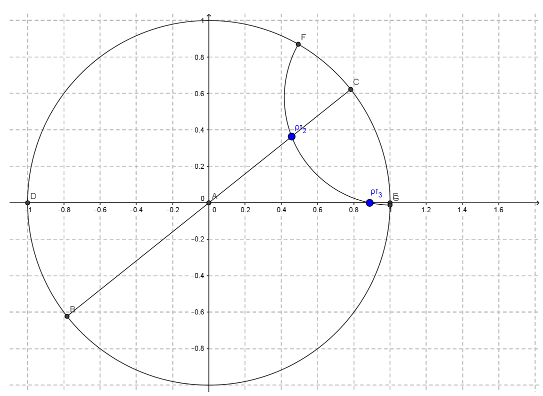

FIGURE 14. Following from the Figure 13, this figure shows the result of applying the required rotation. In this case we used $\rho(z) = e^{i*-0.503}$, since the angle formed by $\tau_3, 0$ and x axis was approximately 0.503 radians.

. Let's have a concluding remark on the proof. In essence the proof of hyperbolic version and Euclidean version are pretty similar. A key geometric step is to see that given a point outside of the convex hull, then it is possible to find a transformation which takes takes this particular point to zero and leaves all vertices of the hull on the same half-plane. After restricting ourselves to a certain half-plane, the rest of the job is done by the complex analytic theory. Even though there are certain differences, in essence these two proofs use closely related ideas.

6. Concluding Remarks

6.1. **What has been done in this work?** As stated before, there were two main aims of this work. First to familiarize the reader with the notion of the hyperbolic geometry. Second, to state and prove the well-known Gauss-Lucas theorem with the hyperbolic geometric tools instead of Euclidean ones. Of course the first one was just a milestone for the second goal and second and fourth chapters served specifically for this purpose. Since these were elementary steps that were taken by fellow mathematicians decades ago, most of these chapters gives references to various papers. In order to achieve the second goal, first the Euclidean case of the Gauss-Lucas theorem is introduced in chapter 3. As explained before, provided proof for the theorem is significant because similar arguments appear in the hyperbolic case of the theorem. The main idea of the proof, which is named as Lemma 3.4, is mentioned in [9]. However no proof for this argument was provided, so in this work a full proof of Gauss-Lucas theorem using the Lemma 3.4 is given. Finally in chapter 5, with the proof of the hyperbolic version of the Gauss-Lucas theorem the second aim of the work is fulfilled. The provided proof is actually a completed and more elaborated version of [7]. More specifically, the idea of Lemma 3.4 is implemented to hyperbolic case which was lacking in the original proof. Thus in addition to achieving two main goals, this work also gives a new and more complete perspective to the proof of the hyperbolic version of Gauss-Lucas theorem.

6.2. **Further Research Topics.** After reading this paper, many questions arise as well as a few is answered. First of all, after realizing how Blaschke products mimics the polynomials in hyperbolic sense, one cannot stop to wonder about other possible geometries. One way to move on with the research could be to try to achieve similar results to Gauss-Lucas in different geometries such as spherical. On the other hand, if we don't want to leave waters of the finite Blaschke products there are still many possible directions one can go. There are many results and conjectures relating the roots of the polynomial with critical points of the polynomial. Another possible direction for research is to focus on conjectures rather than already proven theorems in Euclidean geometry. An example of many such results is Marden's theorem

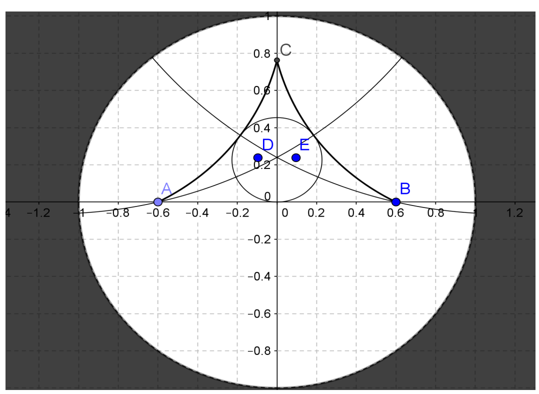

FIGURE 15. An illustration of a special case of Marden's theorem in hyperbolic geometry with roots located in 0.6, -0.6 and $0.76i$, which gives the resulting equilateral triangle. D and E are the critical points calculated as $-0.096 + 0.238i$ and $0.096 + 0.238i$ respectively.

which was mentioned in the first chapter. It could be a possible further research to study this theorem in hyperbolic geometry. For example the Sendov's conjecture mentioned in [9] is still open since 1958:

Theorem 6.1 (Sendov's Conjecture)**.** *Let p be the polynomial with roots $z_1, \ldots, z_k$ and let p' have roots at $w_1, \ldots, w_m$ without multiplicity. Then,*

$$\max_k \min_j |w_j - z_k| \leq \max_k |z_k|$$

The degree three case of the theorem is just a corollary of Marden's theorem. Also from a geometric point of view this conjecture is saying that if the roots of the polynomial are in the unit disk, then every root of $p'(z)$ is in the unit disk around one of the z_k's.

References

1. Anderson, James W. *Hyperbolic Geometry*, Second Edition, Springer-Verlag, 2005.
2. Bcher, Maxime. *Some Propositions Concerning The Geometric Representation Of Imaginaries.* The Annals Of Mathematics 7 (1/5): 70. doi:10.2307/1967882. 1892.
3. Daepp, Ulrich, Pamela Gorkin, and Raymond Mortini. *Ellipses And Finite Blaschke Products.*,The American Mathematical Monthly 109 (9): 785. doi:10.2307/3072367. 2002.
4. Euclides, J. L. Heiberg, and Richard Fitzpatrick. 2005. *Euclid's Elements in Greek: From Euclidis Elementa*. S.l.: Richard Fitzpatrick, 2005. Print.
5. Euclid, and John Playfair. *Elements of Geometry: Containing the First Six Books of Euclid, with a Supplement on the Quadrature of the Circle, and the Geometry of Solids: To Which Are Added, Elements of Plane and Spherical Trigonometry.* . MT: Kessinger, 1819. Print.
6. Fricain, Emmanuel, Javad Mashreghi. *On a characterization of Finite Blaschke Products* Arxiv.Org. 2011. https://arxiv.org/abs/1101.2296v1.
7. Garcia, Stephan Ramon, Javad Mashreghi, and William T. Ross. *Finite Blaschke Products: A Survey.* Arxiv.Org. 2017. https://arxiv.org/abs/1512.05444.
8. Kalman, Dan. 2008. *An Elementary Proof Of Marden's Theorem*. The American Mathematical Monthly 115 (4): 330-338.
9. Northshield, Sam. *Geometry of Cubic Polynomials*. Mathematics Magazine 86, no. 2 : 136-43. doi:10.4169/math.mag.86.2.136. 2013
10. Singer, David A. *The Location of Critical Points of Finite Blaschke Products.* Conformal Geometry And Dynamics Of The American Mathematical Society 10 (06): 117-125. doi:10.1090/s1088-4173-06-00145-7. 2006.

Department of Mathematics and Its Applications, Central European University, Budapest, Hungary

E-mail address: `Yuruk_Oguzhan@student.ceu.edu`