# A general theory of solution algebras in differential and difference Galois theory

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# Contents

# Introduction

Differential Galois theory is an analogue of Galois theory for studying solutions of linear differential equations in a purely algebraic way. It originates in work of Kolchin carried out from the 1940's and has undergone considerable development since.

In the classical setting, the object of study is a linear differential equation of the form

$$y^{(n)} + a_{n-1}y^{(n)} + \dots a_1 y' + a_0 y = 0$$

where the coefficients $a_i$ are taken from a field $K$ of characteristic 0 equipped with a derivation $\partial$, the basic example being the rational function field $\mathbb{C}(t)$. To such an equation one associates an analogue of the splitting field in Galois theory called the Picard–Vessiot extension. It is a field extension $L|K$ generated by a system of solutions of the equation and their derivatives and is equipped with a natural extension of the derivation of $K$. As Kolchin showed, the group $G$ of relative automorphisms of $L|K$ respecting the derivation $\partial$ has a natural structure of a linear algebraic group over the field of constants $k \subset K$ (defined as the subfield of elements killed by $\partial$). Moreover, there is a Galois correspondence between intermediate fields of $L|K$ carrying an extension of $\partial$ and closed subgroups of $G$. For these classical results we refer to the book of van der Put and Singer [**30**].

In his recent paper [**4**], Yves André introduced a refinement of the differential Galois correspondence over algebraically closed constant fields $k$ of characteristic zero. Namely, he obtained a characterization of closed subgroups corresponding to intermediate extensions generated by *some* but not necessarily all solutions of the differential equations. He called these subfields solution fields and showed that they correspond to observable subgroups of the differential Galois group, i.e. closed subgroups $H \subset G$ with quasi-affine quotient $G/H$. Using the more refined Tannakian approach to differential Galois theory, André also showed that solution fields arise as fraction fields of so-called solution algebras which are generalizations of the classical Picard–Vessiot algebras, and established a correspondence between solution algebras and affine quasi-homogeneous varieties under the differential Galois group.

At the end of his paper ([**4**], Remark 6.5 (3)), André writes that he expects a similar theory of solution algebras in characteristic $p > 0$ using iterated derivations and a similar theory for difference equations. In this thesis we confirm his expectations. In fact, we develop a theory of solution algebras within a general Tannakian framework and prove a correspondence with affine quasi-homogeneous varieties. For the theory of solution fields, however, we shall work in the specialized settings of iterative differential or difference equations (the latter in characteristic 0). One important feature of the characteristic $p$ theory is that nonreduced Galois group schemes may occur. In particular, we shall exhibit an example of a solution field corresponding to a non-reduced closed subgroup scheme of a reduced differential differential Galois group which is moreover not a Picard–Vessiot extension.

Let us now describe the contents of the thesis in more detail.

The first chapter is devoted to an abstract theory of Picard–Vessiot ring objects in Tannakian categories. We work in a $k$-linear tensor category $\mathcal{C}$ equipped with a faithful tensor functor $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ into the category $\mathrm{QCoh}(S)$ of quasi-coherent sheaves on some scheme $S$; we will call such tensor categories *pointed*. (In the applications we shall mainly have $S = \mathrm{Spec}(k)$ but for technical reasons it is useful to allow this flexibility.) In Proposition 1.2 we shall establish that the functor $X \mapsto \mathrm{Hom}_{\mathcal{C}}(1, X)$ (where 1 denotes the unit object) from $\mathcal{C}$ to $k$-vector spaces has a left adjoint $\tau$; we call objects of $\mathcal{C}$ in the essential image of $\tau$ *trivial*. For instance, in the category of differential modules over a differential rings the trivial objects are the trivial differential modules, and in the category of representations of a group $G$ the trivial objects are those with trivial $G$-action. Roughly speaking, a *Picard–Vessiot ring object* for an object $X$ of $\mathcal{C}$ is a faithfully flat simple ring object $\mathcal{P}$ in $\mathcal{C}$ such that $\mathcal{P} \otimes X$ is a trivial object and $\mathcal{A}$ is minimal with respect to this property (see Definition 1.13 for details). Our main result is then the following abstract generalization of the Tannakian characterization of Picard–Vessiot extensions in [**11**].

For a dualizable object $X$ of a tensor category $\mathcal{C}$ we will denote by $\langle X \rangle_{\otimes}$ the full essential subcategory of $\mathcal{C}$ consisting of subquotients of finite direct sums of objects of the form $X^{\otimes i} \otimes (X^{\vee})^{\otimes j}$.

Theorem 0.1. (= Theorem 1.21.) *Let $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ be a pointed tensor category over an algebraically closed field $k$ with simple unit object, and let $X$ be a dualizable object of $\mathcal{C}$. The subcategory $\langle X \rangle_{\otimes}$ has the structure of a neutral Tannakian category if and only if there exists a Picard-Vessiot ring for $X$ in $\mathcal{C}$.*

*Moreover, there is a bijective correspondence between Picard-Vessiot rings of $X$ in $\mathcal{C}$ and $k$-valued fibre functors on $\langle X \rangle_{\otimes}$.*

As usual, given a fibre functor $\omega$ on $\langle X \rangle_{\otimes}$, we define the associated differential Galois group scheme as the group of tensor automorphisms of $\omega$.

This part of the thesis was written during the winter of 2014/15. Theorem 1.21. was independently proven by Maurischat in [**27**] by a somewhat different method.

We next consider an abstract version of the notion of solution algebras introduced in [**4**]. In the situation of the above theorem, a *solution algebra* for $\langle X \rangle_{\otimes}$ is a ring object $\mathcal{B}$ in $\mathcal{C}$ such that

(1) there exists an injective ring homomorphism $\iota\colon \mathcal{B} \to \mathcal{P}$ in $\mathcal{C}$,
(2) there exists an object $Y$ of $\langle X \rangle_\otimes$ and a morphism $\sigma\colon Y \to \mathcal{B}$ in $\mathcal{C}$ such that the induced ring homomorphism $\mathrm{Sym}^*(Y) \to \mathcal{B}$ is surjective.

Note that $\mathcal{B}$ is an ind-object in $\langle X \rangle_\otimes$, so given a fibre functor $\omega$ on $\langle X \rangle_\otimes$, it makes sense to consider its value $\omega(\mathcal{B})$.

With this definition we have a generalization of ([**4**], Theorem 1.4.2 (3)). Recall that given a group scheme $G$ over our algebraically closed field $k$, a $k$-scheme $X$ is a quasi-homogeneous $G$-scheme over $k$ there exists a $G$-equivariant quasi-compact schematically dominant morphism $G \to X$. The unit section of $G$ yields a $k$-point $z$ of the schematically dominant $G$-orbit in $X$. We will assume $G$ and $X$ to be finite type over $k$.

THEOREM 0.2. (= Theorem 1.32.) *Fix a fibre functor $\omega$ on $\langle X \rangle_\otimes$, and let $G$ be the associated differential Galois group.*

*The map $(\mathcal{B}, \iota) \mapsto (\mathrm{Spec}(\omega(\mathcal{B})), z)$ gives an anti-equivalence between the category of solution algebras and the category of affine quasi-homogeneous $G$-schemes of finite type over $k$ with a given $k$-point of the schematically dominant orbit. Ideals of a solution algebra $\mathcal{B}$ correspond to closed $G$-subschemes of $\mathrm{Spec}(\omega(\mathcal{B}))$.*

In characteristic zero, applying the theorem to the category of usual differential modules gives back the result in [**4**]. In characteristic $p > 0$, the theorem is applicable to the category of iterative differential modules (or ID-modules for short) developed in Matzat–van der Put [**25**] and Maurischat [**26**]; see the beginning of Chapter 2 for more details. The former reference implicitly works under a separability assumption while the latter does not. In particular, in the more general theory of [**26**] differential Galois groups may be non-reduced. In this more general context, Maurischat proves a differential Galois correspondence analogous to the classical correspondence of Kolchin. Combining it with the above theorem, we are able to extend André's theory of solution fields to positive characteristic.

Namely, given an ID-module $\mathcal{M}_\mathcal{K}$ over an ID-field $\mathcal{K}$ with constant field $k$, we may associate with it a Picard–Vessiot extension $\mathcal{J}|\mathcal{K}$ of ID-fields by the theory of [**25**] and [**26**]. On the other hand, we say that an extension $\mathcal{L}|\mathcal{K}$ is a solution field for $\langle \mathcal{M}_\mathcal{K} \rangle_\otimes$ if the constant field of $\mathcal{L}$ is $k$ and there exists an ID-module $\mathcal{N}_\mathcal{K}$ in $\langle \mathcal{M}_\mathcal{K} \rangle_\otimes$ and a morphism of ID-modules $\mathcal{N}_\mathcal{K} \to \mathcal{L}$ whose image generates the underlying field extension of $\mathcal{L}|\mathcal{K}$. The main result of Chapter 2 is then the following analogue of ([**4**], Theorem 1.2.1 (2)):

THEOREM 0.3. (= Theorem 2.13.) *An intermediate ID-extension $\mathcal{L}$ of $\mathcal{J}|\mathcal{K}$ is a solution field for $\langle \mathcal{M}_\mathcal{K} \rangle_\otimes$ if and only if the corresponding subgroup scheme $H$ is an observable subgroup scheme of the Galois group scheme $G$.*

Recall that $H$ is called an observable subgroup scheme if the quotient $G/H$ is quasi-affine. Note that in contrast to [**4**], the group schemes involved may be reduced. Indeed, we have the following example.

EXAMPLE 0.4. (= Example 2.15.) Let $k$ be an algebraically closed field of characteristic $p > 0$. Consider $k(t)$ equipped with its natural ID-structure, and the ID-module over $k(t)$ corresponding to the system of equations

$$\partial_{p^n} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & a_n t^{-p^n} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $a_n \in \{1, \ldots, p-1\}$. For a suitable choice of the $a_i$, the associated differential Galois group will be a semi-direct product $G := \mathbb{G}_m \ltimes \mathbb{G}_a$. The composite embedding $\mu_p \subset \mathbb{G}_m \subset G$ realizes $\mu_p$ as an non-reduced, non-normal but observable subgroup scheme, which thus corresponds to a solution field that is not a Picard–Vessiot extension.

In Chapter 3 we apply the general theory of Chapter 1 to difference modules over difference rings. Difference rings are rings equipped with an endomorphism $\sigma$ and difference modules are modules over them equipped with a $\sigma$-linear endomorphism (see the first two sections of the chapter for basic definitions). For difference modules satisfying certain mild restrictive conditions we obtain a Tannakian theory to which the considerations of Chapter 1 can be applied, whence a correspondence between solution algebras for difference modules and quasi-homogeneous affine schemes.

The theory of solution fields works in a less general setting as for ID-modules because the difference Galois correspondence of van der Put–Singer [**30**] only works in characteristic zero and for certain difference subrings of the Picard–Vessiot ring. Also, simple difference rings are not necessarily integral domains, so we have to work with their total quotient rings.

We thus consider a difference field $\mathcal{K} = (K, \sigma)$ with bijective endomorphism $\sigma$ and a finite dimensional difference module $\mathcal{M} = (M, \Sigma)$ over $\mathcal{K}$ such that $\Sigma$ is bijective as well. We say that a difference ring $\mathcal{L} \supset \mathcal{K}$ is a total solution ring for $\langle \mathcal{M} \rangle_\otimes$ if every non-zerodivisor of the underlying ring $L$ is a unit, the constant ring $k$ of $\mathcal{L}$ is the same as that of $\mathcal{K}$, and there exists a difference module $\mathcal{N}$ in $\langle \mathcal{M} \rangle_\otimes$ and a morphism of difference modules $\mathcal{N} \to \mathcal{L}$ such that the total fraction ring of the image of this homomorphism is $L$.

With this definition we have:

THEOREM 0.5. (= Theorem 3.19) *Assume $k$ is an algebraically closed field of characteristic 0, and denote by $T(\mathcal{P})$ the Picard-Vessiot ring associated with $\mathcal{M}$; it is a semisimple $K$-algebra. Let $\mathcal{L}$ be an intermediate difference ring of $T(\mathcal{P})|\mathcal{K}$ in which every non-zerodivisor is a unit.*

*The ring $\mathcal{L}$ is a total solution ring for $\langle \mathcal{M} \rangle_\otimes$ if and only if the corresponding subgroup $H$ is an observable subgroup of the Galois group $G(k)$.*

The thesis closes with an appendix assembling category-theoretical constructions needed for the theory of Chapter 1.

CHAPTER 1

# Picard-Vessiot theory in tensor categories

In this chapter we develop an abstract version of the Tannakian approach to the theory of Picard–Vessiot extensions that will be applied in the concrete situation of iterative differential modules and difference modules in the next chapters. We also introduce a generalization of Yves André's notion of a solution algebra and establish an abstract variant of his correspondence between solution algebras and quasi-homogeneous varieties.

## 1. Tensor categories

We begin with some reminders concerning tensor categories.

In this text a *tensor category* $\mathcal{C}$ is a cocomplete (i.e. admits all small colimits) abelian symmetric monoidal category such that the tensor product of $\mathcal{C}$ is additive and commutes with small colimits (hence right exact) in both variables. A tensor category over a commutative ring $k$ is a tensor category which is $k$-linear such that the tensor product is $k$-linear in both variables.

A *tensor functor* between tensor categories is a cocontinuous (i.e. commutes with all small colimits) additive symmetric monoidal functor. If the tensor categories are over a commutative ring $k$, then a tensor functor is also assumed to be $k$-linear.

The endomorphism ring $\mathrm{End}_{\mathcal{C}}(1)$ of the unit object of a tensor category $\mathcal{C}$ is commutative by an Eckmann-Hilton type argument (see [**29**] I.1.3.3.1., p.21). Furthermore, this endomorphism ring acts on the Hom-sets of $\mathcal{C}$ and with this action $\mathcal{C}$ can be seen as a tensor category over the endomorphism ring. We assume that $k = \mathrm{End}_{\mathcal{C}}(1)$ for every tensor category $\mathcal{C}$.

EXAMPLE 1.1.

(1) Let $k$ be a commutative ring and $R$ a commutative $k$-algebra. The category $\mathrm{Mod}(R)$ of $R$-modules is a tensor category over $k$. Generally, if $S$ is a scheme over $k$, then the category $\mathrm{QCoh}(S)$ is a tensor category over $k$. Given a $k$-morphism $f\colon S_1 \to S_2$ of schemes, the pullback functor $f^*\colon \mathrm{QCoh}(S_2) \to \mathrm{QCoh}(S_1)$ is a tensor functor.

(2) Let $k$ be a field and $G$ be an affine group scheme over $k$. Then the category $\mathrm{Rep}_k(G)$ of arbitrary dimensional $k$-representations of $G$ is a tensor category over

$k$. The forgetful functor $\mathrm{Rep}_k(G) \to \mathrm{Vec}(k)$ to the category of $k$-vector spaces is a tensor functor.

(3) Let $\mathcal{R}$ be an (iterative) differential ring with constant ring $k$. The category $\mathrm{Diff}(\mathcal{R})$ of (iterative) differential modules is a tensor category over $k$ and the forgetful functor $\mathrm{Diff}(\mathcal{R}) \to \mathrm{Mod}(R)$ is a tensor functor.

Let $\mathcal{C}$ be a tensor category over a commutative ring $k$ and $S \neq \emptyset$ be a $k$-scheme. An *$S$-valued point* of $\mathcal{C}$ is a *faithful exact* tensor functor (over $k$) from $\mathcal{C}$ to the category $\mathrm{QCoh}(S)$ of quasi-coherent $\mathcal{O}_S$-modules. Forgetful functors (e.g. as in the case of representations and differential modules) provide natural examples of $S$-valued points. A tensor category $\mathcal{C}$ (over $k$) with an $S$-valued point $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ will be called a pointed tensor category (over $k$).

One can think of an $S$-point of a tensor category as a "faithfully flat cover" of the tensor category: in the case of $\mathcal{C} = \mathrm{QCoh}(T)$, where $T$ is a $k$-scheme, the pullback functor of a faithfully flat $k$-morphism $S \to T$ gives an $S$-valued point of $\mathrm{QCoh}(T)$.

## 2. Trivial objects

In this section we generalize the notion of trivial (iterative) differential modules of [**3**] III.1.1. to an abstract tensor categorical framework.

PROPOSITION 1.2. *Let $\mathcal{C}$ be a tensor category with endomorphism ring $k = \mathrm{End}_{\mathcal{C}}(1)$. Then there exists a unique (up to unique isomorphism) $k$-linear tensor functor*

$$\tau\colon \mathrm{Mod}(k) \to \mathcal{C}$$

*such that it is a left adjoint of the functor*

$$(-)^{\nabla} := \mathrm{Hom}_{\mathcal{C}}(1, -)\colon \mathcal{C} \to \mathrm{Mod}(k).$$

PROOF. The existence and the uniqueness of $\tau$ follows from the following general result (where $k$ can be an arbitrary commutative ring, not just the endomorphism ring of the tensor category).

PROPOSITION 1.3 ([**9**] Proposition 2.2.3., [**20**] Theorem 4.51.). *Let $k$ be a commutative ring, $A$ be a commutative $k$-algebra and $\mathcal{C}$ be a tensor category over $k$. Then there exists an equivalence of categories*

$$\mathrm{Hom}_{c,\otimes,/k}(\mathrm{Mod}(A), \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Alg}(k)}(A, \mathrm{End}_{\mathcal{C}}(1)).$$

*between the category of $k$-linear tensor functors from* $\mathrm{Mod}(A)$ *to* $\mathcal{C}$ *and the discrete category of $k$-algebra homomorphisms from $A$ to* $\mathrm{End}_{\mathcal{C}}(1)$.

*Moreover, every cocontinuous $k$-linear symmetric monoidal functor $F\colon \mathrm{Mod}(A) \to \mathcal{C}$ has a right adjoint, given by*

$$\mathcal{C} \to \mathrm{Mod}(A)$$
$$X \mapsto \mathrm{Hom}_{\mathcal{C}}(1, X),$$

*where $A$ acts on $\mathrm{Hom}_{\mathcal{C}}(1, X)$ via the $k$-algebra homomorphism $A \to \mathrm{End}_{\mathcal{C}}(1)$ corresponding to the functor $F$ in the equivalence above.*

The required result follows now immediately: the category of functors from $\mathrm{Mod}(k)$ to $\mathcal{C}$ is equivalent to the discrete category of $k$-algebra homomorphisms from $k$ to $k = \mathrm{End}_{\mathcal{C}}(1)$, but the latter consists of only one object, namely the identity homomorphism. Thus we get the existence and uniqueness of $\tau\colon \mathrm{Mod}(k) \to \mathcal{C}$. Furthermore, the existence of a right adjoint and its explicit description also follows. □

An object $X$ of $\mathcal{C}$ will be called *trivial* if $X$ is isomorphic to an object of the form $\tau(Q)$ for a $k$-module $Q$. The functor $\tau$ will be called the *trivial object functor* of $\mathcal{C}$ over $k$. The category $\mathrm{Triv}(\mathcal{C})$ of trivial objects of $\mathcal{C}$ is defined as the full subcategory of $\mathcal{C}$ consisting of trivial objects.

As the trivial object functor is a tensor functor, we immediately get that $\mathrm{Triv}(\mathcal{C})$ is a cocomplete $k$-linear additive symmetric monoidal subcategory of $\mathcal{C}$. Moreover, the trivial object functor $\tau$ is right exact as it is a left adjoint.

EXAMPLE 1.4.

(1) For a commutative $k$-algebra $R$, the functor $\mathrm{Hom}_R(R, -)$ is just the restriction functor from $R$-modules to $k$-modules and the trivial object functor is the base change functor from $k$ to $R$. More generally, for a scheme $S$ over a ring $k$, the adjoint pair of pullback and pushforward via the structure morphism $S \to \mathit{Spec}(k)$ is precisely the adjoint pair of trivial object functor and the global sections functor.

(2) Let $G$ be an affine group scheme over a field $k$ and $V$ be a vector space over $k$. Let $V_\tau$ denote the trivial representation associated to a vector space $V$. For a representation $V_\rho$, denote by $V_\rho^G = \mathrm{Hom}_G(k_\tau, V_\rho)$ the $G$-invariant elements. The adjoint isomorphism

$$\mathrm{Hom}_G(V_\tau, W_\rho) \cong \mathrm{Hom}_k(V, W_\rho^G)$$

shows that $V \mapsto V_\tau$ is the trivial object functor.

(3) Let $\mathcal{R}$ be an (iterative) differential ring with constant ring $k$. For any $k$-module $Q$, we can endow the $R$-module $R \otimes_k Q$ with the connection $\partial \otimes \mathrm{id}_Q$, since $\partial$ is $k$-linear, and we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{R} \otimes_k Q, \mathcal{M}) \cong \mathrm{Hom}_k(Q, \mathcal{M}^\nabla),$$

showing that $\mathcal{R} \otimes_k -$ is the trivial object functor.

Let $Q$ be a $k$-module and $X$ be an object of $\mathcal{C}$. We will write

$$\eta_Q \colon Q \to \tau(Q)^{\nabla}$$

and

$$\varepsilon_X \colon \tau(X^{\nabla}) \to X$$

for the adjoint morphisms corresponding to the adjoint pair $(\tau, (-)^{\nabla})$.

The unit object of a tensor category is called simple if it has no proper, non-trivial subobject. For a more general definition for ring objects in a tensor category, see Definition A.2. If we assume that the unit object is simple, then by Proposition A.3 of the Appendix the endomorphism ring $k = \mathrm{End}_{\mathcal{C}}(1)$ is a field.

*Until the end of this section we will consider a pointed tensor category $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ with simple unit object.*

PROPOSITION 1.5. *An object $X$ of $\mathcal{C}$ is trivial if and only if the adjoint morphism $\varepsilon_X \colon \tau(X^{\nabla}) \to X$ is an isomorphism.*

PROOF. If the adjoint morphism is an isomorphism, then $X$ is trivial by definition. Conversely, let $X$ be a trivial object. We will prove that $\varepsilon_X$ is a (split) epimorphism and a monomorphism.

Since $\varepsilon$ is a functor morphism, we can assume that $X = \tau(Q)$. The general theory of adjoint pairs tells us that the composition

$$\tau(Q) \xrightarrow{\tau(\eta_Q)} \tau(\tau(Q)^{\nabla}) \xrightarrow{\varepsilon_{\tau(Q)}} \tau(Q)$$

is the identity map, hence $\varepsilon_{\tau(Q)}$ is a split epimorphism.

We now prove that the adjoint morphism is a monomorphism. First we note that the coproduct of monomorphisms in $\mathcal{C}$ is a monomorphism. Indeed, $\vartheta$ is a faithful and exact functor, hence the claim follows from the fact that the coproduct of monomorphisms is a monomorphism in $\mathrm{QCoh}(S)$.

The $k$-vector space $X^{\nabla}$ can be written as a coproduct $k^{\oplus I}$. The functor $\tau$ commutes with small colimits and sends the unit object to unit object, therefore $\tau(X^{\nabla})$ is isomorphic to $1^{\oplus I}$. Now the adjoint morphism $\varepsilon_X$ can be written as the coproduct of morphisms of the form $1 \to X$. But 1 is a simple ring, thus these are monomorphisms and using the previous observation, we get that $\varepsilon_X$ is a monomorphism. □

REMARK 1.6. The last part of the proof even shows that the adjoint morphism $\varepsilon_X \colon \tau(X^{\nabla}) \to X$ is a monomorphism for *every object* of $\mathcal{C}$ (under the conditions of the proposition), a fact we will use later.

PROPOSITION 1.7. *The trivial object functor $\tau\colon \mathrm{Mod}(k) \to \mathcal{C}$ is a $k$-linear faithful exact tensor functor.*

Proof. By [**9**] Corollary 2.2.4., the composite functor

$$\vartheta \circ \tau\colon \operatorname{Mod}(k) \to \operatorname{QCoh}(S)$$

is isomorphic to the pullback functor induced by a $k$-morphism $S \to \operatorname{Spec}(k)$. But there is only one such morphism, the structure morphism $\sigma\colon S \to \operatorname{Spec}(k)$. As the unit object is simple, the endomorphism ring $k$ is a field and therefore the structure morphism $\sigma$ is faithfully flat. It follows that the pullback functor $\sigma^* \cong \vartheta \circ \tau$ is faithful exact and in conclusion, $\tau$ is faithful exact. □

Proposition 1.8. *Subquotients of trivial objects are again trivial and thus* $\operatorname{Triv}(\mathcal{C})$ *is an abelian subcategory of* $\mathcal{C}$.

Proof. Let $X_2$ be a trivial object and consider a short exact sequence in $\mathcal{C}$

$$0 \to X_1 \to X_2 \to X_3 \to 0.$$

We have to show that $X_1$ and $X_3$ are trivial objects. Since $\tau$ is an exact functor and $(-)^\nabla$ is a left exact functor, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tau(X_1^\nabla) & \xrightarrow{\tau(\phi^\nabla)} & \tau(X_2^\nabla) & \xrightarrow{\tau(\psi^\nabla)} & \tau(X_3^\nabla) & & \\
 & & \downarrow{\scriptstyle \varepsilon_1} & & \downarrow{\scriptstyle \varepsilon_2} & & \downarrow{\scriptstyle \varepsilon_3} & & \\
0 & \longrightarrow & X_1 & \xrightarrow{\phi} & X_2 & \xrightarrow{\psi} & X_3 & \longrightarrow & 0.
\end{array}$$

By assumption, $\varepsilon_2$ is an isomorphism and Remark 1.6. tells us that $\varepsilon_1$ and $\varepsilon_3$ are monomorphisms. Hence, it is enough to show that $\varepsilon_1$ and $\varepsilon_3$ are epimorphisms.
Let $g$ be a morphism such that $g \circ \varepsilon_3 = 0$. Then $g \circ \varepsilon_3 \circ \tau(\psi^\nabla) = 0$ and using the commutativity, we get that $g \circ \psi \circ \varepsilon_2 = 0$. But $\psi \circ \varepsilon_2$ is an epimorphism (composition of two epimorphisms), hence $g = 0$ and thus $\varepsilon_3$ is an epimorphism.
It follows that $\tau(\psi^\nabla)$ is an epimorphism, too. Using now the snake lemma, we can conclude that $\varepsilon_1$ is an epimorphism. □

Proposition 1.9. *The trivial object functor*

$$\tau\colon \operatorname{Mod}(k) \to \operatorname{Triv}(\mathcal{C})$$

*induces an equivalence of tensor categories over $k$ with a quasi-inverse given by the restriction of the functor* $(-)^\nabla$ *to* $\operatorname{Triv}(\mathcal{C})$.

Proof. We already know that $\tau$ is a $k$-linear faithful exact tensor functor and by definition, the essential image of $\tau$ is the category $\operatorname{Triv}(\mathcal{C})$ of trivial objects. To prove that $\tau$ is an equivalence, we only have to show that $\tau$ is also full. The general theory of adjoint functors tells us that for every $k$-module $Q$, the composition

$$\tau(Q) \xrightarrow{\tau(\eta_Q)} \tau(\tau(Q)^\nabla) \xrightarrow{\varepsilon_{\tau(Q)}} \tau(Q)$$

is the identity. We have seen that $\varepsilon_{\tau(Q)}$ is an isomorphism, hence $\tau(\eta_Q)$ is also an isomorphism. Using that $\tau$ is faithful exact, we get that $\eta_Q$ is an isomorphism. But is well-known that the adjoint morphism $\eta_Q$ is an isomorphism if and only if the left adjoint functor $\tau$ is fully faithful (c.f. [**21**] IV.3. Theorem 1.).

Furthermore, the adjoint morphisms $\eta_Q$ and $\tau_X$ are isomorphisms for any $k$-module $Q$ and trivial object $X$, hence the right adjoint functor $(-)^\nabla$ is a quasi-inverse of the trivial object functor. □

We can use this equivalence for proving the following result. Recall the notion of dualizable objects from Section 3 of the Appendix.

Proposition 1.10. *Let $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ be a pointed tensor category over a commutative ring $k$ with simple unit object (hence $k$ is a field).*

(1) *A trivial object $X$ is dualizable in $\mathcal{C}$ if and only if $X^\nabla$ is a finite dimensional $k$-vector space.*
(2) *The dual of a dualizable trivial object is also trivial and is isomorphic to $\tau((X^\nabla)^\vee)$, where $^\vee$ denotes the $k$-dual of a vector space.*
(3) *If $X$ is a trivial object and $\vartheta(X)$ is a quasi-coherent $\mathcal{O}_S$-module of finite type, then $X$ has a dual.*

Proof. The equivalence between the category of trivial objects in $\mathcal{C}$ and the category of $k$-vector spaces implies the first two parts of the proposition.

Let $X$ be a trivial object in $\mathcal{C}$ such that the quasi-coherent $\mathcal{O}_S$-module $\vartheta(X)$ is of finite type. As $X$ is trivial, we may write $\vartheta(X)$ as $(\vartheta \circ \tau)(X^\nabla)$. We know that $\vartheta \circ \tau$ is isomorphic to the pullback of the structure morphism $\sigma\colon S \to \mathrm{Spec}(k)$, hence $\vartheta(X) \cong \sigma^*(X^\nabla)$. The structure morphism $\sigma$ is faithfully flat because $k$ is a field. The property of being of finite type descends via faithfully flat morphisms, hence we get that $X^\nabla$ is a finite dimensional vector space over $k$. But finite dimensional vector spaces are dualizable and tensor functors preserve dualizable objects, hence $X \cong \tau(X^\nabla)$ is dualizable. □

## 3. Solvable objects

We now come to an abstract version of the notion of solvability for differential modules. For a general exposition of ring and module objects, we refer to Section 1 of the Appendix.

Definition 1.11. Let $\mathcal{C}$ be a tensor category, $X$ be an object of $\mathcal{C}$ and $\mathcal{A}$ be a ring in $\mathcal{C}$. We say that the object $X$ is solvable in $\mathcal{A}$ if the $\mathcal{A}$-module $\mathcal{A} \otimes X$ is a trivial object in $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$.

For a ring $\mathcal{A}$ in the tensor category $\mathcal{C}$, we will denote by $k_{\mathcal{A}}$ the endomorphism ring $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$. The base change functor $\mathcal{A} \otimes -$ induces a homomorphism between the endomorphism rings

$k \to k_{\mathcal{A}}$. Via this ring homomorphism, the category $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$ can be viewed as a tensor category over $k$.

Let $\tau\colon \mathrm{Mod}(k) \to \mathcal{C}$ and $\tau_{\mathcal{A}}\colon \mathrm{Mod}(k_{\mathcal{A}}) \to \mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$ be the trivial object functors of $\mathcal{C}$ and $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$. We have two functors from $\mathrm{Mod}(k)$ to $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$: the first one is the composition $(\mathcal{A} \otimes -) \circ \tau$ and the second one is the composition $\tau_{\mathcal{A}} \circ (k_{\mathcal{A}} \otimes_k -)$. By Proposition 1.3., these functors correspond to $k$-algebra homomorphisms $k \to k_{\mathcal{A}}$, but there is only one such $k$-algebra homomorphism, thus these functors are isomorphic. In other words the following diagram is commutative (up to functor isomorphism):

$$\begin{array}{ccc} \mathrm{Mod}(k) & \xrightarrow{\quad\tau\quad} & \mathcal{C} \\ \big\downarrow{\scriptstyle k_{\mathcal{A}}\otimes_k -} & & \big\downarrow{\scriptstyle -\otimes\mathcal{A}} \\ \mathrm{Mod}(k_{\mathcal{A}}) & \xrightarrow{\tau_{\mathcal{A}}} & \mathrm{Mod}_{\mathcal{C}}(\mathcal{A}) \end{array}$$

Furthermore, if $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ is pointed for some scheme $S$, then the category $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$ will be pointed as well. Denote by $S_{\mathcal{A}}$ the relative spectrum of the quasi-coherent $\mathcal{O}_S$-algebra $\vartheta(\mathcal{A})$ over $S$ and by

$$\mu\colon S_{\mathcal{A}} \to S$$

the structure morphism. It is known (see Appendix, Section 2) that there exists a faithful exact tensor functor

$$\vartheta_{\mathcal{A}}\colon \mathrm{Mod}_{\mathcal{C}}(\mathcal{A}) \to \mathrm{QCoh}(S_{\mathcal{A}})$$

such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\vartheta\quad} & \mathrm{QCoh}(S) \\ \big\downarrow{\scriptstyle \mathcal{A}\otimes -} & & \big\downarrow{\scriptstyle \mu^*} \\ \mathrm{Mod}_{\mathcal{C}}(\mathcal{A}) & \xrightarrow{\vartheta_{\mathcal{A}}} & \mathrm{QCoh}(S_{\mathcal{A}}). \end{array} \tag{1}$$

We will denote by $\omega_{\mathcal{A}}$ the composite functor $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, -) \circ (\mathcal{A} \otimes -)$.

We will investigate the properties of those objects of $\mathcal{C}$ that are solvable in a faithfully flat simple ring. The existence of a faithfully flat simple ring in a tensor category implies that the unit object of the tensor category is also simple (see Proposition A.6.), therefore we may assume that the unit object of $\mathcal{C}$ is simple. This implies that the endomorphism ring $k = \mathrm{End}_{\mathcal{C}}(1)$ is a field.

Proposition 1.12. *Let $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ be a pointed tensor category with simple unit object. Let $\mathcal{A}$ be a faithfully flat simple ring in $\mathcal{C}$.*

(1) *Objects of $\mathcal{C}$ solvable in $\mathcal{A}$ are closed under arbitrary small colimits, tensor products and subquotients. The unit object of $\mathcal{C}$ is solvable in $\mathcal{A}$.*
(2) *An $\mathcal{A}$-solvable object $X$ is dualizable if and only if the $k_{\mathcal{A}}$-vector space $\omega_{\mathcal{A}}(X)$ is finite dimensional.*
(3) *The dual of an $\mathcal{A}$-solvable dualizable object is also $\mathcal{A}$-solvable.*

(4) *If $X$ is solvable in $\mathcal{A}$ and $\vartheta(X)$ is a quasi-coherent $\mathcal{O}_S$-module of finite type, then $X$ is dualizable in $\mathcal{C}$.*

(5) *For an $\mathcal{A}$-solvable object $X$, we have an isomorphism*

$$\mu^*\vartheta(X) \cong \sigma_{\mathcal{A}}^*\omega_{\mathcal{A}}(X)$$

*of quasi-coherent $\mathcal{O}_{S_{\mathcal{A}}}$-modules, where $\sigma_{\mathcal{A}}\colon S_{\mathcal{A}} \to \mathrm{Spec}(k_{\mathcal{A}})$ is the structure morphism.*

Proof.

(1) Base change to $\mathcal{A}$ commutes with small colimits (resp. tensor product) and sends the unit object to the unit object, hence small colimits (resp. tensor product) of solvable objects are solvable and the unit object of $\mathcal{C}$ is solvable. Since $\mathcal{A}$ is faithfully flat and thus the base change functor is exact, we see that subquotients are mapped to subquotients. We can now use Proposition 1.8. to conclude that subquotients of solvable objects are solvable.

(2)-(4) By faithfully flat descent (Proposition A.9.), we know that an object is dualizable in $\mathcal{C}$ if and only if its base change is dualizable in $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$. Using this fact and the definition of solvability, parts (2)-(4) follow from Proposition 1.10.

(5) By definition of solvability we have an isomorphism

$$X \otimes \mathcal{A} \cong \tau_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{A} \otimes X)) = \tau_{\mathcal{A}}\omega_{\mathcal{A}}(X)$$

of $\mathcal{A}$-modules. We get the desired isomorphism after applying functor $\vartheta_{\mathcal{A}}$, and using the commutativity of diagram (1) on the left side and isomorphism $\sigma_{\mathcal{A}}^* \cong \vartheta_{\mathcal{A}}\tau_{\mathcal{A}}$ (we get this as in Proposition 1.7.) on the right side.

□

## 4. Picard-Vessiot rings

We now come to the first key definition in this chapter.

Definition 1.13. Let $\mathcal{C}$ be a tensor category with simple unit object and $X$ be a dualizable object of $\mathcal{C}$. A ring $\mathcal{P}$ in $\mathcal{C}$ is called a Picard-Vessiot ring for $X$ in $\mathcal{C}$ if it satisfies the following properties:

(1) $\mathcal{P}$ is a faithfully flat simple ring in $\mathcal{C}$,
(2) the homomorphism $k = \mathrm{End}_{\mathcal{C}}(1) \to \mathrm{End}_{\mathcal{P}}(\mathcal{P})$, induced by the morphism $1 \to \mathcal{P}$, is an isomorphism,
(3) the object $X$ is solvable in $\mathcal{P}$,
(4) the ring $\mathcal{P}$ is minimal with these properties, i.e. if $\mathcal{P}'$ is another ring in $\mathcal{C}$ satisfying the previous properties and $\mathcal{P}' \to \mathcal{P}$ is a ring homomorphism, then it is an isomorphism.

Let $X$ be a dualizable object of a tensor category $\mathcal{C}$. We will denote by $\langle X\rangle_\otimes$ the full essential subcategory of $\mathcal{C}$ consisting of subquotients of finite direct sums of objects of the form $X^{\otimes i}\otimes(X^\vee)^{\otimes j}$. The category of finite dimensional vector spaces over a field $k$ will be denoted by $\mathrm{Vecf}(k)$.

PROPOSITION 1.14. *Let $\vartheta\colon\mathcal{C}\to\mathrm{QCoh}(S)$ be a pointed tensor category with simple unit object. Let $X$ be a dualizable object of $\mathcal{C}$ and assume that there exist a Picard-Vessiot ring $\mathcal{P}$ for $X$ in $\mathcal{C}$. Then $\langle X\rangle_\otimes$ is a rigid $k$-linear abelian symmetric monoidal category and*

$$\omega_{\mathcal{P}}=\mathrm{Hom}_{\mathcal{P}}(\mathcal{P},\mathcal{P}\otimes -)\colon\langle X\rangle_\otimes\to\mathrm{Vecf}(k)$$

*is a $k$-linear faithful exact symmetric monoidal functor. Moreover, we have a functorial isomorphism*

$$\mu^*\vartheta(Y)\cong\sigma^*\omega_{\mathcal{P}}(Y)$$

*for every object $Y$ of $\langle X\rangle_\otimes$, where $\sigma\colon S_{\mathcal{P}}\to\mathrm{Spec}(k)$ is the structure morphism.*

PROOF. First, $X$ is solvable in $\mathcal{P}$ and hence its dual is also solvable in $\mathcal{P}$. This implies that every object of $\langle X\rangle_\otimes$ is solvable since they can be written as subquotients of direct sums of tensor products of solvable objects.

The base change functor maps objects of $\langle X\rangle_\otimes$ to the subcategory of trivial objects of $\mathrm{Mod}_{\mathcal{C}}(\mathcal{P})$, hence by Proposition 1.9. the functor $\omega_{\mathcal{P}}=\mathrm{Hom}_{\mathcal{P}}(\mathcal{P},-)\circ(\mathcal{P}\otimes -)$ is a faithful exact tensor functor, implying that $\omega_{\mathcal{P}}(Y)$ will be a finite dimensional $k$-vector space for every object $Y$ of $\langle X\rangle_\otimes$. Now all the claims follow from Proposition 1.12. □

In other words, $\langle X\rangle_\otimes$ is a neutral Tannakian category with $\omega_{\mathcal{P}}$ as a fibre functor.

DEFINITION 1.15. The Tannakian fundamental group scheme $G=\mathrm{Aut}^\otimes(\omega_{\mathcal{P}})$ of the neutral Tannakian category $\langle X\rangle_\otimes$ with fibre functor $\omega_{\mathcal{P}}$ is called the *Galois group scheme of $X$* (pointed in $\mathcal{P}$ or $\omega_{\mathcal{P}}$).

The isomorphism

$$\mu^*\vartheta(Y)\cong\sigma^*\omega_{\mathcal{P}}(Y)$$

implies that the isomorphism scheme $\mathrm{Isom}^\otimes(\omega_{\mathcal{P}},\vartheta)$, which is a $G$-torsor, has an $S_{\mathcal{P}}$-valued point.

PROPOSITION 1.16. *Let $\mathcal{C}$ be a tensor category with simple unit object and $X$ be a dualizable object of $\mathcal{C}$. Assume that there exists a Picard-Vessiot ring $\mathcal{P}$ for $X$ in $\mathcal{C}$. Then for every $Y\in\langle X\rangle_\otimes$, the elements of $\omega_{\mathcal{P}}(Y)$ can be identified with $\mathcal{C}$-morphisms from $Y^\vee$ to the Picard-Vessiot ring $\mathcal{P}$.*

In other words, the vector space $\omega_{\mathcal{P}}(Y)$ can be thought of as the vector space of "solutions" of the dual of $Y$ in $\mathcal{P}$, and hence we could call $\omega_{\mathcal{P}}$ as the functor of solutions in $\mathcal{P}$.

Proof. The identification goes via the isomorphism

$$\mathrm{Hom}_{\mathcal{P}}(\mathcal{P}, \mathcal{P} \otimes -) \cong \mathrm{Hom}_{\mathcal{C}}(1, \mathcal{P} \otimes -)$$

and the following (adjoint) isomorphism, induced by a dualizable object $Y_3$,

$$\mathrm{Hom}_{\mathcal{C}}(Y_1, Y_2 \otimes Y_3) \cong \mathrm{Hom}_{\mathcal{C}}(Y_1 \otimes Y_3^{\vee}, Y_2).$$

□

We state now the converse of Proposition 1.14.

Proposition 1.17. *Let $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ be a pointed tensor category with simple unit object. Let $X$ be a dualizable object of $\mathcal{C}$ and assume that the subcategory $\langle X\rangle_{\otimes}$ is a rigid $k$-linear abelian symmetric monoidal category equipped with a $k$-valued fibre functor $\omega$. Then there exists a Picard-Vessiot ring for $X$ in $\mathcal{C}$ and the induced functor of solutions of this Picard-Vessiot ring is isomorphic to the given fibre functor $\omega$.*

The proof will be given in the next section.

## 5. Existence of Picard–Vessiot rings

In order to prepare for the proof of Proposition 1.17, we first examine the fundamental example for a Picard-Vessiot ring, namely the regular representation of an affine algebraic $k$-group scheme $G$, where $k$ is a field. Let $\mathrm{Rep}_k(G)$ (resp. $\mathrm{Repf}_k(G)$) be the category of arbitrary (resp. finite) dimensional $G$-representations over $k$. We make some observations about these categories:

(1) any finite dimensional faithful representation $V_\rho$ $\otimes$-generates the category $\mathrm{Repf}_k(G)$ of finite dimensional representations, meaning that $\mathrm{Repf}_k(G) = \langle V_\rho\rangle_{\otimes}$, hence we will simply say that the regular representation is a Picard-Vessiot ring for $\mathrm{Repf}_k(G)$ in $\mathrm{Rep}_k(G)$ (instead of choosing a specific faithful finite dimensional representation);
(2) every representation of the affine algebraic group scheme $G$ over $k$ can be written as the union of its finite dimensional subrepresentations. In other words, the Ind-category of $\mathrm{Repf}_k(G)$ is $\mathrm{Rep}_k(G)$;
(3) the trivial object functor $\tau\colon \mathrm{Vec}(k) \to \mathrm{Rep}_k(G)$ is simply the functor equipping a vector space $V$ with the trivial representation, which will be denoted by $V_\tau$, and the functor $(-)^{\nabla}$ is just sending a representation to the vector space of its $G$-invariant elements.

Proposition 1.18. *Let $G$ be an affine algebraic group scheme over a field $k$ and consider the tensor category $\mathrm{Rep}_k(G)$ of (arbitrary dimensional) $k$-representations of $G$ with the forgetful functor $\mathrm{Rep}_k(G) \to \mathrm{Vec}(k)$.*

*The regular representation* $\mathcal{O}(G)_\rho$ *is a Picard-Vessiot ring for* $\mathrm{Repf}_k(G)$ *in* $\mathrm{Rep}_k(G)$ *and the fibre functor induced by the regular representation is isomorphic to the natural forgetful functor* $\omega\colon \mathrm{Repf}_k(G) \to \mathrm{Vecf}(k)$.

*Moreover, if* $\mathcal{P}$ *is a Picard-Vessiot ring for* $\mathrm{Repf}_k(G)$ *in* $\mathrm{Rep}_k(G)$ *such that the functor of solutions* $\omega_\mathcal{P}$ *is isomorphic to the forgetful functor* $\omega$*, then* $\mathcal{P}$ *is isomorphic to the regular representation* $\mathcal{O}(G)_\rho$.

Proof. The regular representation is a faithfully flat ring in $\mathrm{Rep}_k(G)$ since its image under the forgetful functor is a faithfully flat ring in $\mathrm{Vec}(k)$. Denote by ${}_GG$ the scheme $G$ equipped with $G$-action induced by the regular representation. As there are no non-trivial proper closed $G$-subsets of ${}_GG$, we get that the regular representation is simple. The elements of the endomorphism ring of the regular representation can be identified with the $G$-invariant regular functions of ${}_GG$ and hence they are just the constants $k$.

Let $V_\rho$ be a finite dimensional representation and denote by $\mathbb{V}_\rho$ the associated vector bundle $\mathrm{Spec}(\mathrm{Sym}^*(V^\vee))$. Solvability is equivalent to the existence of a $G$-equivariant isomorphism

$$ {}_GG \times_k \mathbb{V}_\rho \to {}_GG \times_k \mathbb{V}_\tau, $$

which can be explicitly given on scheme-theoretic points by $(g, v) \mapsto (g, g^{-1}v)$.

By taking $G$-invariant elements and taking into account the graded structure, we can deduce from the isomorphism

$$ V_\rho \otimes_k \mathcal{O}(G)_\rho \cong V_\tau \otimes \mathcal{O}(G)_\rho $$

that the fibre functor induced by the regular representation is isomorphic to the natural forgetful functor $\omega$.

Lastly, we have to show that the regular representation is minimal: let $\mathcal{P}$ be a ring in $\mathrm{Rep}_k(G)$ having the necessary properties and let $\mathcal{P} \to \mathcal{O}(G)_\rho$ be a ring homomorphism in $\mathrm{Rep}_k(G)$. Since $\mathcal{P}$ is simple, this homomorphism is injective, hence we only have to show that it is surjective. The homomorphism induces a functor morphism

$$ \omega_\mathcal{P} \cong \mathrm{Hom}_G(k_\tau, \mathcal{P} \otimes_k -) \to \mathrm{Hom}_G(k_\tau, \mathcal{O}(G)_\rho \otimes_k -) \cong \omega $$

between the fibre functors. But $\mathrm{Repf}_k(G)$ is rigid, hence this functor morphism is in fact a functor isomorphism. This implies that for any finite dimensional subrepresentation of the regular representation, the embedding $V_\rho \to \mathcal{O}(G)_\rho$ can be extended to $\mathcal{P}$ such that the following diagram is commutative

$$ \begin{array}{ccc} & & V_\rho \\ & \swarrow & \downarrow \\ \mathcal{P} & \longrightarrow & \mathcal{O}(G)_\rho \end{array} $$

This implies that the ring homomorphism is surjective and we get the minimal property of the regular representation.

Let $\mathcal{P}$ be a Picard-Vessiot ring for $\mathrm{Repf}_k(G)$ in $\mathrm{Rep}_k(G)$ whose functor of solutions $\omega_{\mathcal{P}}$ is isomorphic to $\omega$ and hence, as we just have seen, to the fibre functor induced by the regular representation. We have to prove that $\mathcal{P}$ is isomorphic to $\mathcal{O}(G)_\rho$ and it is enough to show that there exists a homomorphism $\mathcal{O}(G)_\rho \to \mathcal{P}$ by the minimal property of Picard-Vessiot rings. We can write $\mathcal{O}(G)_\rho$ as a colimit $\varinjlim V_i$ of finite dimensional representations $V_i$. By assumption, $\omega_{\mathcal{P}}$ is naturally isomorphic to $\omega_{\mathcal{O}(G)_\rho}$. Using the same trick as before, we can associate to every morphism $V_i \to \mathcal{O}(G)_\rho = \varinjlim V_i$ a morphism $V_i \to \mathcal{P}$. By naturality, the morphisms $V_i \to \mathcal{P}$ are compatible with the morphism $V_i \to V_j$ of the inductive system, hence they give rise to a homomorphism $\mathcal{O}(G)_\rho \to \mathcal{P}$. Moreover, as the natural isomorphism of the fibre functors respects the symmetric monoidal structure, the homomorphism $\mathcal{O}(G)_\rho \to \mathcal{P}$ commutes with multiplication and thus, it is a ring homomorphism. □

REMARK 1.19. Let $\vartheta\colon \mathrm{Rep}_k(G) \to \mathrm{QCoh}(S)$ be a $k$-linear faithful exact symmetric monoidal functor, where $S \neq \emptyset$ is a k-scheme. Then by [**11**] Theorem 3.2., the relative spectrum $S_\vartheta$ of the quasi-coherent $\mathcal{O}_S$-algebra $\vartheta(\mathcal{O}(G)_\rho)$ is faithfully flat over $S$, it represents the functor $\underline{\mathrm{Isom}}^{\otimes}(\omega, \vartheta)$ and it is a $G$-torsor over $S$ in the fpqc-topology.

Let now $(\mathcal{T}, \omega)$ be a neutral Tannakian category over a field $k$. As recalled in the Appendix, the main theorem of neutral Tannakian categories (Theorem A.10) says that the $k$-group functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is representable by an affine group scheme $G$ over $k$ and the functor $\omega$ induces a tensor equivalence $\mathcal{T} \cong \mathrm{Repf}_k(G)$.

By Proposition A.13 the pair $(\mathrm{Ind}(\mathcal{T}), J(\omega))$ is a tensor category over $k$ equipped with an exact $k$-linear faithful tensor functor. Moreover, the above equivalence can be extended to an equivalence $\mathrm{Ind}(\mathcal{T}) \cong \mathrm{Ind}(\mathrm{Repf}_k(G)) = \mathrm{Rep}_k(G)$ of Ind-categories.

Denote by $\mathcal{P}_\omega$ the image of $\mathcal{O}(G)_\rho$ under this equivalence. Proposition 1.18 and the main theorem of neutral Tannakian categories now imply the following.

COROLLARY 1.20. *Let $(\mathcal{T}, \omega)$ be a neutral Tannakian category over a field $k$ that can be $\otimes$-generated by an object. Let $\mathcal{P}_\omega$ be the previously defined ring in* $\mathrm{Ind}(\mathcal{T})$. *Then $\mathcal{P}_\omega$ is a Picard-Vessiot ring for $\mathcal{T}$ in* $\mathrm{Ind}(\mathcal{T})$ *and the fibre functor induced by this Picard-Vessiot ring is isomorphic to the fibre functor $\omega$.*

We finally come to:

PROOF OF PROPOSITION 1.17. By the previous corollary we know that there exists a Picard-Vessiot ring in the ind-category of $\langle X\rangle_{\otimes}$. By a general category-theoretical construction (Proposition A.19) we can embed this ind-category in $\mathcal{C}$.

As the ind-category is closed under subquotients in $\mathcal{C}$, the object $\mathcal{P}_\omega$ will satisfy all the required properties of Definition 1.13 in $\mathcal{C}$, except perhaps for the faithful flatness. For this latter property, it is enough to show that the image of $\mathcal{P}_\omega$ under the functor $\vartheta$ is a faithfully flat quasi-coherent algebra (by Example A.5.(3)). This follows from Remark 1.19. □

## 6. Picard-Vessiot rings and fibre functors

Let $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ be a pointed tensor category with simple unit object and $X$ be a dualizable object of $\mathcal{C}$.

We have seen in Proposition 1.14. that the existence of a Picard-Vessiot ring for $X$ in $\mathcal{C}$ implies that $\langle X\rangle_\otimes$ is a neutral Tannakian category with the functor of solutions as fibre functor. In particular, we can associate a fibre functor to a Picard-Vessiot ring.

In the other direction, we have proved in Proposition 1.17. that if the subcategory $\langle X\rangle_\otimes$ of $\mathcal{C}$ is neutral Tannakian, then we can construct a Picard-Vessiot ring for $X$ in $\mathcal{C}$ using the fibre functor such that the fibre functor induced by this Picard-Vessiot ring is isomorphic to the original one. By combining these two results we get the following.

THEOREM 1.21. *Let $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ be a pointed tensor category with simple unit object. Let $X$ be a dualizable object of $\mathcal{C}$ such that $\langle X\rangle_\otimes$ is a rigid $k$-linear abelian symmetric monoidal subcategory of $\mathcal{C}$. The previously defined maps are inverse bijections between Picard-Vessiot rings of $X$ in $\mathcal{C}$ and $k$-valued fibre functors on $\langle X\rangle_\otimes$.*

Suppose that there exists a Picard-Vessiot ring $\mathcal{P}$ for $X$ in $\mathcal{C}$. The Tannakian fundamental group $\underline{\mathrm{Aut}}^\otimes(\omega_\mathcal{P})$ is called the Galois group scheme of $X$ w.r.t. the Picard-Vessiot ring $\mathcal{P}$ or w.r.t. the fibre functor $\omega_\mathcal{P}$. It will be denoted by $\mathrm{Gal}(\mathcal{T},\mathcal{P})$ or $\mathrm{Gal}(\mathcal{T},\omega_\mathcal{P})$.

The Galois group scheme can be interpreted as the group of automorphisms of the Picard-Vessiot ring object over the unit object as follows. Let $\tau\colon\ \mathrm{Mod}(k) \to \mathcal{C}$ be the trivial object functor. For any $k$-algebra $k'$, we can take the ring object $\mathcal{P}\otimes\tau(k')$ over $\tau(k')$ and consider the group of relative automorphisms of the former ring object over the latter one. This way we obtained a group functor

$$\underline{\mathrm{Aut}}(\mathcal{P}|1)\colon\ \mathrm{Alg}(k) \to \mathrm{Group}$$

over $k$.

PROPOSITION 1.22. *The $k$-group functor $\underline{\mathrm{Aut}}(\mathcal{P}|1)$ is representable by the Galois group scheme $G$.*

PROOF. Applying the functor $\omega$, we may identify the $k$-group functor $\underline{\mathrm{Aut}}(\mathcal{P}|1)$ with $\underline{\mathrm{Aut}}_G(\mathcal{O}(G))$, which is nothing else but the functor that represents $G$. □

General results about torsors and Tannakian categories tell us the following:

(1) the relative spectrum $S_\mathcal{P}$ of the quasi-coherent $\mathcal{O}_S$-algebra $\vartheta(\mathcal{P})$ is a faithfully flat scheme over $S$,
(2) the scheme $S_\mathcal{P}$ represents the functor $\underline{\mathrm{Isom}}^\otimes(\omega_\mathcal{P},\vartheta)$ and thus it is an fpqc-torsor under the Galois group scheme over $S$,

(3) the set of isomorphism classes of Picard-Vessiot rings is in bijection with the first non-abelian cohomology set $H^1_{\mathrm{fpqc}}(k, G)$,
(4) there exists a closed immersion $G \to \mathrm{GL}(\omega(X))$ of affine group schemes,
(5) the Galois group scheme is smooth (resp. étale resp. finite) over $k$ if and only if $S_{\mathcal{P}}$ is smooth (resp. étale resp. finite) over $S$.

We mention now some basic observations about the existence and uniqueness of Picard-Vessiot rings. A natural source for Picard-Vessiot rings is the set $S(k)$ of $k$-points of $S$ (when nonempty): indeed, pulling back the $S$-valued fibre functor $\vartheta\colon \langle X\rangle_{\otimes} \to \mathrm{QCoh}(S)$ via the morphism $\mathrm{Spec}(k) \to S$, we get a $k$-valued fibre functor on $\langle X\rangle_{\otimes}$ and a Picard-Vessiot ring for $X$ in $\mathcal{C}$. In particular, there certainly exists a Picard-Vessiot ring in the following cases:

(1) the field $k$ is algebraically closed and $S$ is locally of finite type over $k$,
(2) the field $k$ is pseudo-algebraically closed and $S$ is a geometrically integral separated scheme of finite type over $k$.

The question of uniqueness of Picard-Vessiot rings is translated to the question of cardinality of the non-abelian cohomology set $H^1(k, G)$. There are many results concerning the first cohomology set, we just mention a trivial one: if the field $k$ is algebraically closed and a Picard-Vessiot ring exists, then it is unique.

## 7. Quasi-homogeneous $G$-schemes

In the next section we shall generalize André's correspondence between solution algebras and quasi-homogeneous varieties. As a preliminary, we assemble here some facts concerning the latter.

We first recall the definition of schematically dominant morphisms: a quasi-compact morphism is called schematically dominant if and only if the scheme-theoretic closed image of the morphism equals the target.

Definition 1.23. Let $k$ be a field, $G$ be a group scheme over $k$ and $X$ be a $G$-scheme over $k$. We say that $X$ is a quasi-homogeneous $G$-scheme over $k$ (for the fppf-topology of $k$) if there exists a finite extension $k'|k$ and a $G_{k'}$-equivariant quasi-compact schematically dominant $G_{k'} \to X_{k'}$ morphism.

Before proving an equivalent characterization of quasi-homogeneous $G$-schemes, we review the basic facts of quotients of affine algebraic group schemes. The classical theorem about the existence of quotients is the following.

Theorem 1.24 ([**12**] III.3.5.4 Theorem). *Let $k$ be a field, $G$ be an affine algebraic group scheme over $k$ and $H$ be a closed subgroup scheme of $G$. Then the fppf-quotient $G/H$ is representable by an algebraic $k$-scheme with a canonical $G$-equivariant ample line bundle.*

*Furthermore, if $N$ is a closed normal subgroup scheme, then $G/N$ has the structure of an affine group scheme and the quotient morphism $G \to G/N$ is a homomorphism of affine group schemes.*

The proof of this theorem uses the following special case.

PROPOSITION 1.25 ([**12**] III.3.5.2 Proposition). *Let $k$ be a field, $G$ be an affine algebraic group scheme over $k$ and $X$ be an algebraic $G$-scheme over $k$. If $x \in X(k)$ is a $k$-valued point of $X$, then the stabilizer $\operatorname{Stab}_G(x)$ is a closed subgroup scheme of $G$, the fppf-quotient $G/\operatorname{Stab}_G(x)$ is representable by an algebraic $k$-scheme and the induced morphism $\phi_x \colon G \to X$ factors through a $G$-equivariant* immersion $G/\operatorname{Stab}_G(x) \to X$*:*

$$\begin{array}{ccc} G & \xrightarrow{\quad\phi_x\quad} & X \\ & {\scriptstyle q}\searrow \quad \nearrow & \\ & G/\operatorname{Stab}_G(x) & \end{array}$$

Now we can state and prove a second characterization of quasi-homogeneous $G$-schemes.

PROPOSITION 1.26. *Let $k$ be a field, $G$ be an affine algebraic group scheme over $k$ and $X$ be an algebraic $G$-scheme over $k$. Then $X$ is quasi-homogeneous if and only if there exists a finite field extension $k'|k$ such that $X_{k'}$ has a $G_{k'}$-invariant subscheme $U'$ such that*

(1) *the scheme $U'$ represents the fppf-quotient $G_{k'}/H$ for a closed subgroup scheme $H$ of $G_{k'}$,*
(2) *the morphism $U' \to X_{k'}$ is quasi-compact and schematically dominant.*



PROOF. Assume that $X$ satisfies the second condition, i.e. we have a composition

$$G_{k'} \xrightarrow{q} G_{k'}/H \cong U' \hookrightarrow X_{k'},$$

where every morphism is $G_{k'}$-equivariant. Fppf-quotients are also geometric quotients, hence it follows that the quotient morphism $q$ is schematically dominant. The second morphism is schematically dominant by definition, hence the composition is also quasi-compact and schematically dominant.

Conversely, assume that there exists a finite extension $k'|k$ and a $G_{k'}$-equivariant quasi-compact schematically dominant morphism $f \colon G_{k'} \to X_{k'}$. Let

$$x \colon \operatorname{Spec}(k') \xrightarrow{e'} G_{k'} \xrightarrow{f} X_{k'}$$

be the $k'$-point of $X_{k'}$ obtained from the unit of $G_{k'}$. We will call the composition

$$G_{k'} \xrightarrow{\cong} G_{k'} \times_{k'} \operatorname{Spec}(k') \xrightarrow{\mathrm{id}_{G_{k'}} \times e'} G_{k'} \times_{k'} G_{k'} \xrightarrow{\mathrm{id}_{G_{k'}} \times f} G_{k'} \times_{k'} X_{k'} \xrightarrow{h_{X_{k'}}} X_{k'}$$

the morphism induced by this point. This induced morphism is just the original morphism $f$, indeed, this follows from the commutative diagram

$$\begin{array}{ccccccc} G_{k'} & \xrightarrow{\cong} & G_{k'} \times_{k'} \operatorname{Spec}(k') & \xrightarrow{\mathrm{id}_{G_{k'}} \times e'} & G_{k'} \times_{k'} G_{k'} & \xrightarrow{\mathrm{id}_{G_{k'}} \times f} & G_{k'} \times_{k'} X_{k'} \\ & & & \searrow^{\cong} & \downarrow{\scriptstyle m_{G_{k'}}} & & \downarrow{\scriptstyle h_{X_{k'}}} \\ & & & & G_{k'} & \xrightarrow{f} & X_{k'} \end{array}$$

Proposition 1.25. tells us now that the stabilizer $H := \mathrm{Stab}_{G_{k'}}(x)$ is a closed subgroup scheme of $G_{k'}$, the fppf-quotient $G_{k'}/H$ is representable by an algebraic $k'$-scheme and we have a $G_{k'}$-equivariant *immersion* $G_{k'}/H \to X_{k'}$ and a commutative diagram

$$\begin{array}{ccc} G_{k'} & \xrightarrow{f} & X_{k'} \\ & {\scriptstyle q}\searrow & \nearrow \\ & G_{k'}/H & \end{array}$$

We only have to show that the morphism $G_{k'}/H \to X_{k'}$ is quasi-compact and schematically dominant. First, this morphism is quasi-compact, since $G_{k'}/H$ is quasi-compact and $X_{k'}$ is quasi-separated ([**14**] Prop. 10.3.(2) and Remark 10.4.).

Since this morphism is quasi-compact, it is schematically dominant if and only if its schematic image is $X_{k'}$ (for a quasi-compact morphism $f\colon X \to Y$, the schematic image is given by the closed subscheme corresponding to the quasi-coherent ideal $\ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$). The schematic image of the quotient morphism $q\colon G_{k'} \to G_{k'}/H$ is $G_{k'}/H$, hence by the transitivity of schematic image ([**16**] Prop. (9.5.5)) we get that the schematic image of $G_{k'}/H \to X_{k'}$ is the same as the schematic image of the morphism $f\colon G_{k'} \to X_{k'}$, which is the desired $X_{k'}$ by assumption. □

In the classical setting of an algebraically closed field of characteristic 0, the above definition gives the usual notion of quasi-homogeneous space, i.e. a reduced scheme with a dense $G$-orbit.

PROPOSITION 1.27. *Let $k$ be an algebraically closed field of characteristic* 0*, $G$ be an affine algebraic group scheme over $k$ and $X$ be an algebraic scheme over $k$. Then $X$ is a quasi-homogeneous $G$-scheme over $k$ if and only if $X$ is a reduced scheme and there exists a point $x \in X(k)$ such that the induced morphism $\phi_x\colon G \to X$ is dominant.*

PROOF. If $X$ is quasi-homogeneous under $G$ over $k$, then there exists a schematically dominant (hence dominant) $G$-equivariant morphism $G \to X$ (finite extensions do not appear as $k$ is algebraically closed) and which also gives us the required point $x \in X(k)$. Moreover, since $k$ is of characteristic 0, we know that $G$ is a reduced scheme. By [**14**] Remark 10.32., the scheme theoretic image of $G \to X$, which is $X$ by assumption, is the reduced induced scheme on $\overline{f(G)}$, hence $X$ is reduced.

Conversely, a dominant morphism $G \to X$ to a reduced scheme is schematically dominant by [**18**] Proposition 11.10.4. □

## 8. Solution algebras

Let now $\vartheta\colon \mathcal{C} \to \mathrm{QCoh}(S)$ be a pointed tensor category with simple unit object such that the endomorphism ring $k$ is an algebraically closed field. Let $X$ be a dualizable object of $\mathcal{C}$, and let $\mathcal{P}$ be the Picard-Vessiot ring $\mathcal{P}$ for $X$ in $\mathcal{C}$. Denote by $\omega = \omega_{\mathcal{P}}$ the fibre functor induced by $\mathcal{P}$ and by $G$ the Galois group scheme of $X$.

Definition 1.28. A solution algebra for $\langle X\rangle_\otimes$ is a ring $\mathcal{B}$ in $\mathcal{C}$ such that

(1) there exists an injective ring homomorphism $\iota\colon \mathcal{B} \to \mathcal{P}$ (i.e. this morphism is a monomorphism in $\mathcal{C}$),
(2) there exists an object $Y$ of $\langle X\rangle_\otimes$ and a morphism $\sigma\colon Y \to \mathcal{B}$ in $\mathcal{C}$ such that the induced ring homomorphism $\mathrm{Sym}^*(Y) \to \mathcal{B}$ is surjective.

The first observation we make is that a solution algebra is an object of $\mathrm{Ind}\langle X\rangle_\otimes$: indeed, $\mathrm{Ind}(\langle X\rangle_\otimes)$ is closed under subquotients in $\mathcal{C}$, hence we get from item (2) of the definition that $\mathcal{B}$ is an object of $\mathrm{Ind}\langle X\rangle_\otimes$.

Example 1.29. The first example for a solution algebra is the Picard-Vessiot ring itself: by definition, $\mathcal{P}$ is generated by the covariant and contravariant solutions of $X$ in $\mathcal{P}$, that is the morphism $X^{\oplus r} \oplus (X^\vee)^{\oplus r} \to \mathcal{P}$ induces a surjective ring homomorphism, where $r$ is the $k$-dimension of the solutions of $X$ in $\mathcal{P}$.

The relative spectrum $S_{\mathcal{B}} = \mathrm{Spec}_S(\vartheta(\mathcal{B}))$ of the image of a solution algebra $\mathcal{B}$ under $\vartheta$ is a finite type scheme over $S$: the surjective ring homomorphism $\mathrm{Sym}^*(Y) \to \mathcal{B}$ gives a surjective homomorphism $\mathrm{Sym}^*(\vartheta(Y)) \to \vartheta(\mathcal{B})$, where $\vartheta(Y)$ is a locally free $\mathcal{O}_S$-module of finite rank (as $Y$ is dualizable, its image under a tensor functor is also dualizable, but dualizable quasi-cohenernt modules are locally free). Taking relative spectrum over $S$, we get a closed immersion $S_{\mathcal{B}} \to \mathbb{V}(\vartheta(Y))$ and $\mathbb{V}(\vartheta(Y))$ is of finite type over $S$.

The converse statement provides us a second source of examples for solution algebras.

Proposition 1.30. *If $\mathcal{B}$ is a subring of $\mathcal{P}$ such that the morphism $S_{\mathcal{B}} \to S$ is of finite type, then $\mathcal{B}$ is a solution algebra.*

Proof. By Proposition 1.14. we have an isomorphism of quasi-coherent $\mathcal{O}_{S_{\mathcal{P}}}$-algebras

$$\mu_{\mathcal{P}}^*\vartheta(\mathcal{B}) \cong \sigma_{\mathcal{P}}^*\omega(\mathcal{B})$$

or after taking relative spectrum

$$S_{\mathcal{P}} \times_S S_{\mathcal{B}} \cong S_{\mathcal{P}} \times_k \mathrm{Spec}(\omega(\mathcal{B})).$$

Since $S_{\mathcal{P}}$ is faithfully flat over the field $k$ and $S$ as well, we can deduce that $\omega(\mathcal{B})$ is a $G$-algebra which is finitely generated over $k$. We have the following general result.

LEMMA 1.31. *Let $G$ be an affine group scheme over a field $k$ and $Z$ be an affine $G$-scheme finite type over $k$. Then there exists a finite dimensional $G$-representation $V_\rho$ over $k$ and a $G$-equivariant closed immersion $Z \to \operatorname{Spec}(\operatorname{Sym}^*(V_\rho))$.*

PROOF. Let $f_1, \dots, f_n$ be a set of generators of the coordinate ring $\mathcal{O}_Z(Z)$ over $k$. It is known that each $f_i$ is contained in a finite dimensional subrepresentation of $\mathcal{O}_Z(Z)$, hence there is a finite dimensional subrepresentation $V_\rho$ containing all the $f_i$-s. By assumption, the induced morphism $\operatorname{Sym}^* V_\rho \to \mathcal{O}_Z(Z)$ is surjective. □

Using the previous lemma, there exists a finite dimensional $G$-subrepresentation $V_\rho$ of $\omega(\mathcal{B})$ such that the induced morphism is surjective. Using the equivalence $\omega$, we get an object $Y$ of $\langle X \rangle_\otimes$ and a morphism $\sigma \colon Y \to \mathcal{B}$ such that $\operatorname{Sym}^*(\sigma)$ is surjective. □

The category of solution algebras for $\langle X \rangle_\otimes$ in $\mathcal{C}$ consists of pairs $(\mathcal{B}, \iota)$, where $\iota$ is the given embedding of $\mathcal{B}$ into $\mathcal{P}$ with the obvious morphisms. The (extension of the) functor $\omega$ sends a solution algebra to a $G$-algebra over $k$, after taking the spectrum, we get an affine $G$-scheme over $k$. We have also seen in the previous reasoning that $\omega(\mathcal{B})$ is of finite type over $k$. The given embedding $\iota \colon \mathcal{B} \to \mathcal{P}$ is mapped to a $G$-equivariant morphism $G \to \operatorname{Spec}(\omega(\mathcal{B}))$ under $\omega$. Taking the composition with the identity element morphism $\operatorname{Spec}(k) \to G$, we get a $k$-point $z$ of $\operatorname{Spec}(\omega(\mathcal{B}))$.

The following theorem is the generalization of the equivalence proved by André for (generalized) differential rings in [**4**] 3.2.1. Theorem to the general, tensor categorical framework.

THEOREM 1.32. *The map $(\mathcal{B}, \iota) \mapsto (\operatorname{Spec}(\omega(\mathcal{B})), z)$ is anti-equivalence between the category of solution algebras and the category of affine quasi-homogeneous $G$-schemes of finite type over $k$ with a given $k$-point of the schematically dominant orbit. Ideals of a solution algebra $\mathcal{B}$ correspond to closed $G$-subschemes of* $\operatorname{Spec}(\omega(\mathcal{B}))$.

PROOF. Let $\mathcal{B}$ be a solution algebra for $\langle X \rangle_\otimes$. Apply the functors $\omega$ and Spec on the ring homomorphisms

$$\operatorname{Sym}^*(Y) \twoheadrightarrow \mathcal{B} \hookrightarrow \mathcal{P}.$$

Using that $\omega(\mathcal{P})$ is just the regular representation of $G$, we get the following $G$-equivariant morphisms:

$${}_G G \to \operatorname{Spec}(\omega(\mathcal{B})) \to \operatorname{Spec}(\operatorname{Sym}^*(\omega(Y))).$$

The first morphism is schematically dominant and quasi-compact, as it comes from an injection of rings. Let $z$ be the $k$-point given by the composition

$$\operatorname{Spec}(k) \xrightarrow{e} G \to \operatorname{Spec}(\omega(\mathcal{B})).$$

Proposition 1.25. tells us that the fppf-quotient $G/\operatorname{Stab}_G(z)$ exists and the morphism $G \to \operatorname{Spec}(\omega(\mathcal{B}))$ factors through this quotient such that $G/\operatorname{Stab}_G(z) \to \operatorname{Spec}(\omega(\mathcal{B}))$ is an immersion.

We only have to show that the morphism $G/\operatorname{Stab}_G(z) \to \operatorname{Spec}(\omega(\mathcal{B}))$ is quasi-compact and schematically dominant. First, this morphism is quasi-compact, since $G/\operatorname{Stab}_G(z)$ is quasi-compact and $\operatorname{Spec}(\omega(\mathcal{B}))$ is affine, hence quasi-separated ([**14**] Prop. 10.3.(2) and Remark 10.4.).

By definition, this morphism is schematically dominant if and only if its schematic image is $\operatorname{Spec}(\omega(\mathcal{B}))$. The schematic image of the quotient morphism is $G/\operatorname{Stab}_G(z)$, hence by [**16**] Prop. (9.5.5) we get that the schematic image of $G/\operatorname{Stab}_G(z) \to \operatorname{Spec}(\omega(\mathcal{B}))$ is the same as the schematic image of the morphism $f\colon G \to \operatorname{Spec}(\omega(\mathcal{B}))$, which is the desired $\operatorname{Spec}(\omega(\mathcal{B}))$ by assumption.

We also see that the second map is a closed immersion into a finite type affine scheme since the $G$-representation $\omega(Y)$ is finite dimensional and the morphism comes from a surjection.

In conclusion, we get that $\operatorname{Spec}(\omega(\mathcal{B}))$ is an affine quasi-homogeneous $G$-scheme of finite type.

The composite functor $\operatorname{Spec} \circ \omega$ is fully faithful as it is the composition of fully faithful functors.

It is left to prove that it is also essentially surjective. Let $Z$ be an affine quasi-homogeneous $G$-scheme of finite type over $k$ with a given $k$-point $z$ of the schematically dominant orbit. First, we have a schematically dominant $G$-morphism ${}_G G \to Z$ (given by $z$), or on the level of $G$-algebras, an injection

$$\mathcal{O}(Z) \hookrightarrow \mathcal{O}(G)_{\mathrm{reg}}.$$

Second, there exists a finite dimensional $G$-representation $V$ with a homomorphism $V \to \mathcal{O}(Z)$ of $G$-representations such that the induced homomorphism

$$\operatorname{Sym}^*(V) \twoheadrightarrow \mathcal{O}(Z)$$

is surjective. The equivalence $\omega$ can be used to translate objects and morphism between $\operatorname{Rep}_k(G)$ and $\operatorname{Ind}\langle X\rangle_\otimes$ as follows:

| $\operatorname{Rep}_k(G)$ | $\operatorname{Ind}\langle X\rangle_\otimes$ |
|---|---|
| $V$ | $Y$ |
| $\mathcal{O}(Z)$ | $\mathcal{B}$ |
| $V \to \mathcal{O}(Z)$ | $Y \to \mathcal{B}$ |
| $\mathcal{O}(Z) \to \mathcal{O}(G)_{\mathrm{reg}}$ | $\mathcal{B} \to \mathcal{P}$ |

The functor $\omega$ is faithful and exact, hence it reflects monomorphisms and epimorphisms, thus $\mathcal{B} \to \mathcal{P}$ is injective and the induced morphism $\operatorname{Sym}^*(Y) \to \mathcal{B}$ is surjective. In conclusion, $\mathcal{B}$ is a solution algebra.

We can associate to an ideal $\mathcal{I}$ of a solution algebra $\mathcal{B}$ the closed $G$-subscheme $\mathrm{Spec}(\omega(\mathcal{B}/\mathcal{I}))$.

$\square$

CHAPTER 2

# Iterative differential rings and modules

In this chapter we apply the theory of the previous one to the iterative differential modules of Matzat and van der Put [**25**]. After recalling the necessary preliminaries, we state our abstract theorems on Picard–Vessiot and solution algebras in this special context. Afterwards, we discuss the analogue of André's solution fields for iterative differential modules and gives some examples.

## 1. Iterative derivations on rings

A general exposition on iterative differential rings can be found in [**25**].

An iterative differential ring (ID-ring for short) is a pair $\mathcal{R} = (R, \{\partial_i\}_{i\geq 0})$, where $R$ is a commutative ring and $\partial_i \colon R \to R$ are additive maps for all $i \geq 0$ such that

(1) $\partial_0 = id_R$,
(2) $\partial_i(r_1 r_2) = \sum_{j+j'=i} \partial_j(r_1)\partial_{j'}(r_2)$,
(3) $\partial_i \circ \partial_j = \binom{i+j}{i}\partial_{i+j}$.

The set $\{\partial_j\}_{j\geq 0}$ of maps is called an iterative derivation on $\mathcal{R}$.

A homomorphism $f \colon \mathcal{R}_1 \to \mathcal{R}_2$ of ID-rings is a ring homomorphism $f \colon R_1 \to R_2$ such that $f \circ \partial_i^{R_1} = \partial_i^{R_2} \circ f$ for every $i \geq 0$, that is, the following diagram is commutative for all $i \geq 0$:

$$\begin{array}{ccc} R_1 & \xrightarrow{f} & R_2 \\ \big\downarrow{\scriptstyle \partial_i^{R_1}} & & \big\downarrow{\scriptstyle \partial_i^{R_2}} \\ R_1 & \xrightarrow{f} & R_2 \end{array}$$

We say that $\mathcal{K} = (K, \{\partial_i\}_{i\geq 0})$ is an iterative differential field (or ID-field) if the underlying ring $K$ is a field.

An iterative derivation on $R$ gives a ring homomorphism

$$\begin{aligned} \partial \colon R &\to R[[t]] \\ r &\mapsto \sum_{i=0}^{\infty} \partial_i(r) t^i, \end{aligned}$$

and for every $k \geq 0$ a ring homomorphism

$$\partial_{\leq k} \colon R \to R[t]/(t^{k+1})$$
$$r \mapsto \sum_{i=0}^{k} \partial_i(r) t^i$$

Example 2.1.

(1) For every commutative ring $R$, there is the trivial iterative derivation, given by $\partial_0 = id_R$ and $\partial_i = 0$ for every $i \geq 1$.
(2) Let $R$ be a ring containing the field $\mathbb{Q}$ of rational numbers as a subring. Then every derivation $\delta$ on $R$ can be extended to an iterative derivation on $R$ by $\partial_i = \frac{1}{i!}\delta^i$. In particular, if $R$ is a simple differential ring such that the field of constants has characteristic 0, then we can extend the derivation of $R$ to an iterative derivation of $R$.
(3) Let $R$ be a commutative ring. There is an iterative derivation on the polynomial ring $R[t]$ defined in the following way: if $\sum_{j=0}^{n} r_j t^j$ is a polynomial, then
$$\partial_i(\sum_{j=0}^{n} r_j t^j) = \sum_{j=0}^{n} r_j \binom{j}{i} t^{j-i},$$
where $\binom{j}{i} = 0$ if $j < i$. We note that $\partial_1$ is just the usual (formal) derivation of polynomials. The same definition gives an iterative derivation on the ring $R[[t]]$ of formal power series and we will see that the iterative derivations on $R[t]$ and $R[[t]]$ can be extended to iterative derivations on the fraction fields $R(t)$ and $R((t))$.

Let $f, g \colon \mathcal{R}_1 \to \mathcal{R}_2$ be homomorphisms of ID-rings. Then the equalizer $\mathrm{eq}(f,g) = \{r \in R_1 \mid f(r) = g(r)\}$ of the pair $(f,g)$ is an ID-ring with the restriction of the iterative derivation of $\mathcal{R}_1$ and the embedding $\mathrm{eq}(f,g) \to \mathcal{R}_1$ is a homomorphism of ID-rings.

Let $\mathcal{R}$ be an ID-ring and $I$ be an ideal in $R$. We say that $I$ is an iterative differential ideal (ID-ideal) if for all $i \geq 0$ we have $\partial_i(I) \subseteq I$. An iterative differential ring $\mathcal{R}$ is called simple if the only iterative differential ideals of $\mathcal{R}$ are 0 and $R$. Trivially, an ID-field is simple. The kernel of a homomorphism of ID-rings is an ID-ideal. In particular, an ID-ring homomorphism from a simple ID-ring must be injective.

Iterative derivations can be extended through formally étale ring homomorphisms. In particular, any localization of an ID-ring is again an ID-ring such that the natural ring homomorphism commutes with the iterative derivations. The proof for the general case can be found in [**23**] Theorem 27.1; we only present here the proof for localizations.

Proposition 2.2. *Let $\mathcal{R}$ be an ID-ring, $S \subseteq R$ be a multiplicatively closed subset and $f \colon R \to S^{-1}R$ be the induced localization homomorphism. Then we can extend the iterative derivation of $\mathcal{R}$ uniquely to an iterative derivation on $S^{-1}R$ such that $f$ becomes a homomorphism of ID-rings.*

PROOF. We can define the iterative derivation on $S^{-1}R$ iteratively: for $i = 0$, $\partial_0$ is just the identity on $S^{-1}R$ . Let us assume that we have defined the iterative derivations $\partial_i$ for $i = 0, \ldots, k$ on $S^{-1}R$ that satisfy the required properties. Then we have a ring homomorphism

$$\partial_{\leq k} \colon S^{-1}R \to S^{-1}R[t]/(t^{k+1}).$$

Since if an element is a unit modulo a nilpotent ideal, then it is itself a unit, the diagonal homomorphism exists in the following diagram, and it determines the map $\partial_{k+1}$ on $S^{-1}R$:

$$\begin{array}{ccc} S^{-1}R & \xrightarrow{\partial_{\leq k}} & S^{-1}R[t]/(t^{k+1}) \\ {\scriptstyle f}\uparrow & \searrow & \uparrow{\scriptstyle \mathrm{proj}} \\ R & \xrightarrow{\partial_{\leq k+1}} & S^{-1}R[t]/(t^{k+2}) \end{array}$$

It remains to check that the maps $\partial_{k+1}$ satisfy the properties of an iterative derivation. As we got the maps $\partial_{k+1}$ from ring homomorphism, we have the additivity of the map. The definition of multiplication in the truncated polynomial rings $S^{-1}R[t]/(t^{k+2})$ implies the multiplicative property. □

An element $c$ of an ID-ring $\mathcal{R}$ is called constant if $\partial_i(c) = 0$ for all $i \geq 1$. The set of constants of $\mathcal{R}$ is a ring, called the ring of constants of $\mathcal{R}$ and denoted by $k_{\mathcal{R}}$ (or simply $k$). A homomorphism $f \colon \mathcal{R}_1 \to \mathcal{R}_2$ of ID-rings induces a ring homomorphism $k_{\mathcal{R}_1} \to k_{\mathcal{R}_2}$.

PROPOSITION 2.3.

(1) *If $\mathcal{R}$ is a simple ID-ring, then $R$ is an integral domain.*
(2) *If $\mathcal{R}$ is a simple ID-ring, then the ring of constants $k$ is a field.*
(3) *If $\mathcal{R}$ is a simple ID-ring with constant ring $k$ and $K$ is the fraction field of $R$, then there is a unique iterative derivation on $K$ extending the iterative derivation on $\mathcal{R}$. Moreover, the field of constants of the iterative differential field $\mathcal{K}$ is $k$.*

PROOF.

(1) Let $P$ be a prime ideal of $R$. We prove first that the kernel of the composition

$$R \xrightarrow{\partial} R[[t]] \to (R/P)[[t]]$$

is an iterative differential ideal of $\mathcal{R}$. By construction, an element $r \in R$ is in the kernel if and only if $\partial_i(r) \in P$ for all $i \geq 0$. Let $r$ be in the kernel, we have to prove that $\partial_j(r)$ is in the kernel, or equivalently: $\partial_i \circ \partial_j(r)$ is in $P$ for all $i \geq 0$. But $\partial_i \circ \partial_j(r) = \binom{i+j}{i}\partial_{i+j}(r)$ is in $P$ (as $r$ is in the kernel). By assumption, $\mathcal{R}$ is simple, so the kernel can only be 0 or $R$. But 1 is not in the kernel, hence the kernel is trivial and the map $R \to (R/P)[[t]]$ is an injection. In other words, $R$ is a subring of an integral domain, therefore $R$ is an integral domain.

(2) Let $c$ be an element of the constant ring of $\mathcal{R}$. The ideal $Rc$ is an iterative differential ideal:

$$\partial_i(rc) = \sum_{j+j'=i} \partial_j(r)\partial_{j'}(c) = \partial_i(r)c \in Rc.$$

As $\mathcal{R}$ is simple and $Rc$ is non-empty ($c \in Rc$), we have that $Rc = R$, i.e. $c$ is invertible in $R$. Using induction on $i$, one can show that $\partial_i(c^{-1})$ is 0 for $i \geq 1$, hence $c^{-1}$ is in the constant ring and $k$ is a field.

(3) We can extend the iterative derivation of $\mathcal{R}$ to its fraction field by Prop. 2.2. Let $\gamma$ be in the constant field of $\mathcal{K}$. Let $I$ be the following set

$$I := \{r \in R \mid r\gamma \in R\}.$$

It is a non-empty ideal in $R$, moreover, we show that $I$ is an iterative differential ideal. If $r$ is an element of $I$, we have to prove that $\partial_i(r)$ is in $I$. We have

$$\partial_i(r)\gamma = \sum_{j+j'=i} \partial_j(r)\partial_{j'}(\gamma) = \partial_i(r\gamma)$$

and $r\gamma \in R$ by the definition of $r$, so $\partial_i(r)\gamma$ is an element of $R$. Therefore $I$ must be $R$, meaning that $1 \cdot \gamma = \gamma$ is in $R$ and hence it is in $k$.

□

## 2. Iterative connections on modules

Let $\mathcal{R}$ be an ID-ring. An iterative differential module (or ID-module) $\mathcal{M}$ over $\mathcal{R}$ is a pair $(M, \{\nabla_i\}_{i\geq 0})$, where $M$ is an $R$-module and $\nabla_i \colon M \to M$ are additive maps for $i \geq 0$ such that

(1) $\nabla_0 = id_M$,
(2) $\nabla_i(rm) = \sum_{j+j'=i} \partial_j(r)\nabla_{j'}(m)$,
(3) $\nabla_i \circ \nabla_j = \binom{i+j}{i}\nabla_{i+j}$.

The set of maps $\{\nabla_i\}_{i\geq 0}$ is called an iterative connection on $M$ over $\mathcal{A}$.

A homomorphism $\phi \colon \mathcal{M} \to \mathcal{N}$ of ID-modules over $\mathcal{R}$ is an $R$-module homomorphism such that $\phi \circ \partial_i^M = \partial_i^N \circ \phi$ for every $i \geq 0$, that is, the following diagram is commutative for all $i \geq 0$:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow{\scriptstyle \partial_i^M} & & \downarrow{\scriptstyle \partial_i^N} \\ M & \xrightarrow{\phi} & N \end{array}$$

ID-modules over $\mathcal{R}$ (with ID-module homomorphisms as morphisms) form a category, which will be denoted by $\mathrm{Diff}(\mathcal{R})$.

An ID-submodule of an ID-module $\mathcal{M}$ is a submodule stable under the maps of the iterative connection. If $\mathcal{N}$ is an ID-submodule of $\mathcal{M}$, then there exists a unique iterative connection on the quotient module $M/N$. The kernel (resp. cokernel) of the underlying module homomorphism of an ID-module homomorphism is an ID-module and it satisfies the universal property in the category of ID-modules over $\mathcal{R}$.

If we have an inductive system $\mathcal{M}_i$ of ID-modules, then there exists an ID-module structure on the inductive limit $\varinjlim M_i$ such that the natural morphisms $M_i \to \varinjlim M_i$ are homomorphisms of ID-modules. In particular, we can take arbitrary direct sums of iterative differential modules.

The set $\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ of ID-module homomorphisms over $\mathcal{R}$ is a module over the constant ring $k$ of $\mathcal{R}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be ID-modules over an ID-ring $\mathcal{R}$. The tensor product $M \otimes_R N$ can be endowed with an iterative connection, given by $\partial_i(m \otimes n) = \sum_{j+j'=i} \partial_j(m) \otimes \partial_{j'}(n)$.

The maps

$$f \mapsto \sum_{j+j'=i} (-1)^{j'} \nabla_j^N \circ f \circ \nabla_{j'}^M$$

define an iterative connection on the module $\mathrm{Hom}_R(M, N)$ of $R$-linear homomorphisms from $M$ to $N$, called the internal hom of the ID-modules $\mathcal{M}$ and $\mathcal{N}$.

PROPOSITION 2.4. *Let $\mathcal{R}$ be an ID-ring. An ID-module $\mathcal{M}$ over $\mathcal{R}$ has a dual (in the sense of symmetric monoidal categories) if and only if the underlying $R$-module $M$ is finitely generated and projective.*

PROOF. The forgetful functor is monoidal and it is known that monoidal functors commute with duals, hence if the iterative differential module has a dual, then its underlying module has also, or in other words, it is finitely generated and projective. Conversely, we take the inner hom $\mathcal{M}^\vee = \mathcal{H}om_R(M, R)$ of $\mathcal{M}$ and $\mathcal{R}$. As $M$ is finitely generated projective, we have evaluation and coevaluation maps for $\mathcal{M}$ and $\mathcal{M}^\vee$ which will satisfy the necessary diagrams. The only thing to be checked is that the evaluation and coevaluation maps are really homomorphisms of differential modules, which is a short calculation. □

In summary, the category $\mathrm{Diff}(\mathcal{R})$ of ID-modules over an ID-ring $\mathcal{R}$ is a tensor category over the constant ring $k$ of $\mathcal{R}$ and the forgetful functor $\vartheta\colon \mathrm{Diff}(\mathcal{R}) \to \mathrm{Mod}(R)$ is a faithful exact tensor functor over $k$.

If $\mathcal{R}_1 \to \mathcal{R}_2$ is a homomorphism of iterative differential rings, then the base change functor $\mathrm{Diff}(\mathcal{R}_1) \to \mathrm{Diff}(\mathcal{R}_2)$ is a tensor functor over $k$.

The following proposition is a direct generalization of [**4**], Theorem 2.2.1 to the iterative differential setup.

PROPOSITION 2.5. *Let $\mathcal{R}$ a simple iterative differential ring and denote by $\mathcal{K}$ the quotient field of $\mathcal{R}$ with its canonical ID-structure. Let $\mathcal{M}$ be a finitely generated ID-module over $\mathcal{R}$. We have the following:*

(1) *the underlying module of $\mathcal{M}$ and its ID-subquotients are all projective modules,*
(2) *the category consisting of objects that are ID-subquotients of finite direct sums of tensor products of the form $\mathcal{M}^{\otimes i} \otimes (\mathcal{M}^{\vee})^{\otimes j}$ form a Tannakian category $\langle \mathcal{M} \rangle_{\otimes}$ over the constant field $k$ of $\mathcal{R}$,*
(3) *the base change functor $\langle \mathcal{M} \rangle_{\otimes} \to \langle \mathcal{M}_{\mathcal{K}} \rangle_{\otimes}$ is an equivalence.*

Proof. We first prove item (3): let $\mathcal{M}$ and $\mathcal{N}$ be finitely generated ID-modules over $\mathcal{R}$ and consider the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{N}) & \dashrightarrow & \mathrm{Hom}_{\mathcal{K}}(\mathcal{M}_{\mathcal{K}}, \mathcal{N}_{\mathcal{K}}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{R}(M, N) & \longrightarrow & \mathrm{Hom}_{K}(M_K, N_K) \end{array}$$

The map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_K(M_K, N_K)$ is injective since $K$ is flat over $A$ and $M$ is finitely generated ([**7**] Proposition §2.10. Prop. 11.). This implies that the base change functor is faithful. For surjectivity, let $\phi \colon \mathcal{M}_{\mathcal{K}} \to \mathcal{N}_{\mathcal{K}}$ be a homomorphism of ID-modules over $\mathcal{K}$. The set $\phi(\mathcal{M})$ is an ID-submodule of $\mathcal{N}_{\mathcal{K}}$ over $\mathcal{R}$ and the quotient

$$\phi(\mathcal{M})/(\phi(\mathcal{M}) \cap \mathcal{N})$$

is a finitely generated torsion ID-module over $\mathcal{R}$. Hence its annihilator is a differential ideal in $\mathcal{R}$, therefore it is either 0 or $R$. But since $M$ is finitely generated, the annihilator cannot be 0, thus it is $R$, meaning that $\phi(\mathcal{M})/(\phi(\mathcal{M}) \cap \mathcal{N}) = 0$, therefore $\phi(\mathcal{M}) \subseteq \mathcal{N}$.

The previous observations show that the base change functor

$$\langle \mathcal{M} \rangle_{\otimes} \to \langle \mathcal{M}_{\mathcal{K}} \rangle_{\otimes},$$

is fully faithful. To see that it is an equivalence, it is enough to show essential surjectivity, which follows from the fact that any subobject $\mathcal{N}'$ of $\mathcal{N}_{\mathcal{K}} \in \langle \mathcal{M}_{\mathcal{K}} \rangle_{\otimes}$ comes from the subobject $\mathcal{N}' \cap \mathcal{N}$ of $\mathcal{N}$.

We can now prove item (1). First we note that $M_K$ is a finitely generated projective $K$-module, as $K$ is a field. Therefore $\mathcal{M}_{\mathcal{K}}$ has a dual in $\mathrm{Diff}(\mathcal{K})$ and the dual is just

$$\mathcal{H}om_K(M_K, K).$$

The evaluation and coevaluation morphisms of $\mathcal{M}_{\mathcal{K}}$ are homomorphisms of ID-modules and we have just proved that the base change functor is fully faithful in this situation, thus we get homomorphisms of ID-modules

$$\mathcal{H}om_R(M, R) \otimes_R \mathcal{M} \to \mathcal{R}$$
$$\mathcal{R} \to \mathcal{M} \otimes_R \mathcal{H}om_R(M, R).$$

These maps satisfy the identities of the dual object since they satisfy them after applying the fully faithful base change functor. This shows that $\mathcal{M}$ has a dual and in particular, $M$ is a finitely generated projective $R$-module.

The quotient module of a finitely generated module is again finitely generated, hence ID-quotients of $\mathcal{M}$ are again projective. For an ID-submodule $\mathcal{M}'$ of $\mathcal{M}$, we use that the quotient ID-module is finitely generated, hence projective, therefore the submodule $M'$ is a direct summand in $M$ and in particular finitely generated, thus projective.

The previous considerations show that $\langle\mathcal{M}\rangle_{\otimes}$ is a rigid $k$-linear abelian symmetric monoidal category and the forgetful functor to the category of $R$-modules is a fibre functor, whence we have item (2). □

## 3. Picard-Vessiot theory of iterative connections

Let $\mathcal{R}$ be a simple ID-ring with constant field $k$ and denote by $\mathcal{K}$ the quotient field of $\mathcal{R}$ with its canonical iterative connection. Let $\mathcal{M}$ be a finitely generated ID-module over $\mathcal{R}$. By Proposition 2.5. we know that $\langle\mathcal{M}\rangle_{\otimes}$ is a Tannakian category with the forgetful functor $\vartheta\colon \langle\mathcal{M}\rangle_{\otimes} \to \mathrm{Mod}(R)$, furthermore, the base change functor $\langle\mathcal{M}\rangle_{\otimes} \to \langle\mathcal{M}_{\mathcal{K}}\rangle_{\otimes}$ is an equivalence.

A Picard-Vessiot ring for $\mathcal{M}$ in the category $\mathrm{Diff}(\mathcal{R})$ defined as in Definition 1.13. is a generalization of the classical definition (see [**25**] Definition 3.3) of Picard-Vessiot rings, allowing non-free ID-modules.

We assume from now on that the constant field $k$ is *algebraically closed*. It follows that there exists a fibre functor $\langle\mathcal{M}\rangle \to \mathrm{Vecf}(k)$ and hence, a Picard-Vessiot ring $\mathcal{P}$ for $\mathcal{M}$ by Proposition 1.17. The fibre functor will be denoted by $\omega = \omega_{\mathcal{P}}$. The Tannakian fundamental group $\underline{Aut}^{\otimes}(\omega)$ is called the Galois group scheme of $\mathcal{M}$ and will be denoted by $G = \mathrm{Gal}(\mathcal{M}, \omega)$. We have an equivalence of categories $\langle\mathcal{M}\rangle_{\otimes} \cong \mathrm{Repf}_k(G)$, which can be extended to an equivalence of Ind-categories $\mathrm{Ind}\langle\mathcal{M}\rangle_{\otimes} \cong \mathrm{Rep}_k(G)$.

The underlying ring $P$ of the Picard-Vessiot ring is faithfully flat over $R$ and it represents the $G$-torsor (the torsor of solutions) $\underline{\mathrm{Isom}}^{\otimes}(\omega, \vartheta)$. The Galois group scheme $G$ is a closed subgroup scheme of $\mathrm{GL}(\omega(\mathcal{M}))$. The canonical $\mathcal{P}$-point of the torsor of solutions gives an isomorphism of ID-modules over $\mathcal{P}$

$$\omega(\mathcal{N}) \otimes_k \mathcal{P} \cong \mathcal{N} \otimes_R \mathcal{P}$$

for every $\mathcal{N}$ of $\langle\mathcal{M}\rangle_{\otimes}$. This isomorphism holds as well for the objects of $\mathrm{Ind}\langle\mathcal{M}\rangle_{\otimes}$ via the extension of $\omega$ to the equivalence of Ind-categories, therefore we have the following.

Corollary 2.6. *The underlying $R$-module of every object of* $\mathrm{Ind}\langle\mathcal{M}\rangle_{\otimes}$ *is faithfully flat over $R$.*

Consider finally the localization $\mathcal{M}_{\mathcal{K}}$. Using Proposition 2.5 (3), we obtain:

Corollary 2.7. *With notations as above, the ring $\mathcal{P}_{\mathcal{K}}$ is a Picard-Vessiot ring for $\mathcal{M}_{\mathcal{K}}$ and the Galois group scheme of $\mathcal{M}_{\mathcal{K}}$ is naturally isomorphic to the Galois group scheme of $\mathcal{M}$.*

## 4. Solution algebras of iterative connections

The general theory of solution algebras (see Proposition 1.32) tells us that there exists an anti-equivalence between the category of solution algebras for $\langle\mathcal{M}\rangle_\otimes$ and the category of affine quasi-homogeneous $G$-schemes of finite type over $k$. The Picard-Vessiot ring $\mathcal{P}$ is a solution algebra itself, whose corresponding quasi-homogeneous $G$-scheme is simply $G$.

As a solution algebra is an object in the Ind-category of $\langle\mathcal{M}\rangle_\otimes$, we know by Corollary 2.6. that the underlying ring of a solution algebra is faithfully flat over $R$.

The equivalence $\langle\mathcal{M}\rangle_\otimes \to \langle\mathcal{M}_\mathcal{K}\rangle_\otimes$ given by the base change functor in Proposition 2.5 (3) induces an equivalence between the categories of solution algebras for $\langle\mathcal{M}\rangle_\otimes$ and $\langle\mathcal{M}_\mathcal{K}\rangle_\otimes$. The quasi-inverse assigns the intersection $\mathcal{S}'\cap\mathcal{P}$ to a solution algebra $\mathcal{S}'$ for $\langle\mathcal{M}_\mathcal{K}\rangle_\otimes$.

Finally, we present an equivalent characterization of solution algebras, which is taken from [**4**] Definition 3.1.1.

PROPOSITION 2.8. *An ID-ring $\mathcal{S}$ over $\mathcal{R}$ is a solution algebra for $\langle\mathcal{M}\rangle_\otimes$ if and only if the underlying ring $S$ is an integral domain, the constant field of the quotient field of $S$ is $k$ and there exists a morphism $\mathcal{N}\to\mathcal{S}$ of ID-modules over $\mathcal{R}$ whose image generates $S$ as an $R$-algebra.*

PROOF. Let first $\mathcal{S}$ be a solution algebra. We only have to check that $S$ is a domain and that the constant field of its quotient field is $k$, as the third condition is satisfied by definition. Since $\mathcal{S}$ can be embedded into the Picard-Vessiot ring and as simple ID-rings are integral domains, it follows that $S$ is an integral domain, too. Moreover, we have the following sequence of injective homomorphisms of ID-rings (the injectivity of the first one follows from the fact that $\mathcal{R}$ is a simple ID-ring):

$$\mathcal{R}\to\mathcal{S}\to\mathcal{P}.$$

Pasing to the level of quotient fields and than taking constant fields, we get the following sequence of injective homomorphism:

$$k\to \mathrm{Quot}(\mathcal{S})^\nabla\to k,$$

where the composition is an isomorphism. This implies that the constant field of the quotient field of $\mathcal{S}$ is $k$.

In the other direction, we can assume that $\mathcal{N}$ is $\mathcal{M}$ (as the Picard-Vessiot ring of $\mathcal{N}$ embeds in the Picard-Vessiot ring of $\mathcal{M}$). Let $\mathcal{L}$ be the quotient field of $\mathcal{S}$ and denote by $\mathcal{S}'$ the Picard-Vessiot ring for $\mathcal{M}_\mathcal{L}$. It is a simple ID-ring with constant field $k$ (the constant field of $\mathcal{L}$ is $k$ by assumption) and contains $\mathcal{L}$, and hence $\mathcal{S}$. The ID-module $\mathcal{M}$ is solvable in $\mathcal{S}'$, since $\mathcal{M}_\mathcal{L}$ is solvable. It follows that the Picard-Vessiot ring $\mathcal{P}$ of $\mathcal{M}$ is contained in $\mathcal{S}'$. Since $\mathcal{P}$ contains all of the solutions of $\mathcal{M}$, it follows by the third condition that $\mathcal{S}$ is contained in $\mathcal{P}$. □

## 5. Solution fields for iterative connections

Consider now the ID-module $\mathcal{M}_{\mathcal{K}}$ over the ID-field $\mathcal{K}$. The quotient field $\mathcal{J}$ of the Picard-Vessiot ring $\mathcal{P}_{\mathcal{K}}$ is called the Picard-Vessiot field of $\mathcal{M}_{\mathcal{K}}$.

More generally, we have the following analogue of André's notion of solution fields for differential modules.

DEFINITION 2.9. Let $\mathcal{L}|\mathcal{K}$ be an extension of ID-fields. We say that $\mathcal{L}$ is a solution field for $\langle \mathcal{M}_{\mathcal{K}} \rangle_{\otimes}$ if the constant field of $\mathcal{L}$ is $k$ and there exists an ID-module $\mathcal{N}_{\mathcal{K}}$ in $\langle \mathcal{M}_{\mathcal{K}} \rangle_{\otimes}$ and a morphism of ID-modules $\mathcal{N}_{\mathcal{K}} \to \mathcal{L}$ whose image generates the field extension $L|K$.

PROPOSITION 2.10. *We have the following properties:*

(1) *the quotient field of a solution algebra $\mathcal{S}$ for $\langle \mathcal{M} \rangle_{\otimes}$ is a solution field for $\langle \mathcal{M}_{\mathcal{K}} \rangle_{\otimes}$,*
(2) *every solution field $\mathcal{L}$ for $\langle \mathcal{M}_{\mathcal{K}} \rangle_{\otimes}$ is the quotient field of a solution algebra $\mathcal{S}$ for $\langle \mathcal{M} \rangle_{\otimes}$,*
(3) *every solution field $\mathcal{L}$ for $\langle \mathcal{M}_{\mathcal{K}} \rangle_{\otimes}$ embeds as an intermediate ID-extension of $\mathcal{J}|\mathcal{K}$.*

PROOF.

(1) This follows from the definition of solution fields and Proposition 2.8.
(2) Let $S$ be the $R$-subalgebra of $L$ generated by the image of the ID-morphism $\mathcal{N}_{\mathcal{K}} \to \mathcal{L}$. It is an ID-ring with quotient field $L$ and the conditions of being a solution algebra are satisfied by construction.
(3) This follows from the previous point using the fact that solution algebras are embedded into the Picard-Vessiot ring.

□

We next develop the Galois theory of solution fields. We begin by recalling the Galois correspondence for Picard-Vessiot extensions. The Galois group scheme $G$ represents the $k$-group functor $\underline{\mathrm{Aut}}^{\mathrm{ID}}(\mathcal{P}_{\mathcal{K}}|\mathcal{K})$ and using this representation of the Galois group scheme, one can naturally extend the action of $G$ to the Picard-Vessiot field $\mathcal{J}$. An element $p/q \in \mathcal{J}$ is called invariant under a closed subgroup scheme $H$ if for all $k$-algebras $k'$ and for all $h \in H(k')$ we have an equality

$$h(p \otimes 1) \cdot (q \otimes 1) = (p \otimes 1) \cdot h(q \otimes 1) \in \mathcal{P}_{\mathcal{K}} \otimes_k k'.$$

The set of invariant elements of $\mathcal{J}$ under $H$ is denoted by $\mathcal{J}^H$.

The iterative differential Galois correspondence is now stated as follows.

THEOREM 2.11 ([**26**] Theorem 11.5.). *Let $\mathcal{M}_{\mathcal{K}}$ be a finite dimensional ID-module over the ID-field $\mathcal{K}$. Let $\mathcal{J}$ be the quotient field of the Picard-Vessiot ring of $\mathcal{M}_{\mathcal{K}}$. Denote by $G$ the Galois group scheme of $\mathcal{M}_{\mathcal{K}}$. The previously defined map $H \mapsto \mathcal{J}^H$ gives an order-reversing bijection between the closed subgroup schemes $H$ of $G$ and the intermediate ID-fields of $\mathcal{J}|\mathcal{K}$.*

We note that the above theorem is stated for $\theta$-rings/fields in the reference, i.e. for rings/fields equipped with an iterable higher derivation (see [**26**] Definition 10.4.). The most natural example of an iterable higher derivation is an iterative derivation, and hence the general result given in [**26**] Theorem 11.5. can be specialised to our iterative differential framework. Note the important point that [**26**] makes no separability assumption on Picard–Vessiot extensions.

In characteristic zero André proved that solution fields correspond to observable subgroups of the Galois group. Here we need a slightly more general notion allowing non-reduced group schemes. Namely, we call a closed subgroup scheme $H$ of an affine group scheme $G$ over a field $k$ observable if every finite dimensional $H$-representation is an $H$-subrepresentation of a finite dimensional $G$-representation.

We have the following equivalent characterizations of observable subgroups.

THEOREM 2.12. *Let $G$ be an affine algebraic group scheme over a field $k$ and $H$ be a closed subgroup scheme of $G$. Then the following are equivalent:*

(1) *$H$ is an observable subgroup of $G$,*
(2) *the quotient $G/H$ is quasi-affine over $k$,*
(3) *there exists a finite dimensional $G$-representation $V$ and a vector $v \in V$ such that $H$ is the stabilizer subgroup scheme of the vector $v$ in $G$.*

The proof of equivalence (1) $\Leftrightarrow$ (2) is Theorem 1.3. in [**2**]. The proof of (1) $\Rightarrow$ (3) goes the same as the proof of (7) $\Rightarrow$ (2) in Theorem 2.1. of [**15**]: the crucial point is that for any closed subgroup scheme $H$ of an affine group scheme $G$, there exists a finite dimensional $G$-representation $V$, a vector $v \in V$ and a character $\chi$ of $H$ such that $H$ acts via the character $\chi$ on $v$ and only $H$ stabilizes the line $k \cdot v$. For the proof of (3) $\Rightarrow$ (2), we can use Prop. 1.25. to see that there exists an immersion of $G/H$ to the vector bundle $\mathbb{V}$, which is an algebraic affine scheme, hence we can conclude that $G/H$ is quasi-affine.

We can now state the following generalization of [**4**] 4.2.3. Theorem (3) to iterative differential rings.

THEOREM 2.13. *An intermediate ID-extension $\mathcal{L}$ of $\mathcal{J}|\mathcal{K}$ is a solution field for $\langle \mathcal{M}_{\mathcal{K}} \rangle_{\otimes}$ if and only if the corresponding subgroup scheme $H$ is an observable subgroup scheme of the Galois group scheme $G$.*

PROOF. Let $H$ be an observable subgroup scheme of the Galois group scheme. There exists a finite dimensional $G$-representation $V$ and a vector $v \in V$ such that $H$ is the isotropy subgroup scheme of $v$ in $V$. Using the equivalence given by $\omega$, we can write $V$ as $\omega(\mathcal{N}_{\mathcal{K}}^{\vee})$ for some ID-module $\mathcal{N}_{\mathcal{K}}$ in $\langle \mathcal{M}_{\mathcal{K}} \rangle_{\otimes}$. The vector $v$ determines a ID-homomorphism $v \colon \mathcal{N}_{\mathcal{K}} \to \mathcal{P}_{\mathcal{K}} \to \mathcal{J}$. Let $\mathcal{L}$ be the subfield of $\mathcal{J}$ generated by the image of this ID-homomorphism and let $H'$ be the closed subgroup scheme corresponding to $\mathcal{L}$ by the Galois correspondence (Theorem 2.11), i.e. $\mathcal{L} = \mathcal{J}^{H'}$.

$H$ is the isotropy subgroup scheme of $v$, hence for all $k$-algebra $k'$ and $h \in H(k')$ we have $h(v(n) \otimes 1) = (h \cdot v)(n) \otimes 1 = v(n) \otimes 1$ and thus $H \leq H'$. Conversely, $\mathcal{L} = \mathcal{J}^{H'}$ means that for any $k$-algebra $k'$ and any $h' \in H'(k')$ we have $(h \cdot v)(n) \otimes 1 = h(v(n) \otimes 1) = v(n) \otimes 1$ for all $n$, in other words $h' \cdot v = v$ and hence $H' \leq H$. This show that for every observable subgroup scheme $H$, the intermediate ID-field $\mathcal{L} = \mathcal{J}^H$ is a solution field.

Let $\mathcal{L}$ be a solution field that is generated by a solution $v$ of $\mathcal{N}_{\mathcal{K}}$ and denote by $H$ the subgroup scheme attached to $\mathcal{L}$ in $G$. Just as in the previous calculation, we see that $H$ is the isotropy subgroup scheme of the solution $v$ in $\omega(\mathcal{N}_{\mathcal{K}}^{\vee})$ and hence, $H$ is observable. □

Remark 2.14. The following general group theoretic observations show that the relationship between solution algebras and solution fields is not at all unexpected. For the ease of exposition, assume that the base field $k$ is algebraically closed.

Let $X$ be an affine quasi-homogeneous $G$-scheme with given $G$-invariant subscheme $U \cong G/H$. Then $G/H$ is quasi-affine, since by definition the morphism $G/H \cong U \to X$ is a quasi-compact immersion and hence, it is quasi-affine. This shows that $H$ is an observable subgroup. In other words, when $X$ is an affine quasi-homogeneous $G$-scheme and $x \in X(k)$ is a $k$-point of the schematically dense orbit, then we can associate an observable subgroup scheme $H$ of $G$ to the pair $(X, x)$.

In the other direction, let $H$ be an observable subgroup scheme of $G$. By definition, the quotient $G/H$ is a quasi-affine algebraic scheme. Proposition 13.80. of [**14**] tells that the canonical morphism (also called the canonical embedding) $G/H \to \mathrm{Spec}(\mathcal{O}(G/H))$ is a quasi-compact schematically dominant immersion, hence the affine scheme $\mathrm{Spec}(\mathcal{O}(G/H))$ with its natural $G$-action is quasi-homogeneous. The $k$-point $\tilde{1}$ given by the image of the identity element of $G$ in $\mathrm{Spec}(\mathcal{O}(G/H))$ is in the schematically dense orbit.

We also note that if we start with an observable subgroup scheme $H$, then the observable subgroup scheme associated to the pair consisting of the affine quasi-homogeneous $G$-scheme $\mathrm{Spec}(\mathcal{O}(G/H))$ and its natural $k$-point $\tilde{1}$ is $H$ itself. However, this latter pair is not unique with this property, as the different affine embeddings of the quasi-affine scheme $G/H$ all give rise to $H$ as their associated observable subgroup schemes.



## 6. Examples

In the following examples, $k$ will denote an algebraically closed field of characteristic $p > 0$. We can define an iterative derivation on the polynomial ring $k[t]$ by setting $\partial_i(t^k) = \binom{k}{i} t^{k-i}$ and extending it linearly to the whole polynomial ring. Since $\partial_k(t^k) = 1$, we get that $k[t]$ is a simple ID-ring with this iterative derivation. Furthermore, the field of constants is precisely $k \subseteq k[t]$. The iterative derivation can be extended to the quotient field $k(t)$ of the polynomial ring $k[t]$.

Iterative derivations (resp. connections) are determined by the $p$-th power maps $\partial_{p^n}$ (resp. $\nabla_{p^n}$): if we write $n$ as the sum $a_0 + a_1 p + \ldots + a_m p^m$, where $a_i \in \{0, 1, \ldots, p-1\}$, then

$$(\partial_1)^{a_0} \circ (\partial_p)^{a_1} \circ \ldots \circ (\partial_{p^m})^{a_m} = c \cdot \partial_n,$$

where $c$ is a non-zero element of $\mathbb{F}_p$.

Example 2.15.

(1) Let $\mathcal{M}$ be the ID-module corresponding to the sequence of equations

$$\partial_{p^n}(y) = a_n t^{-p^n} y,$$

where $a_n \in \{1, \ldots, p-1\}$. Over the ID-ring $k[t]$, this ID-module has a Picard-Vessiot ring $\mathcal{P} = k[t][s, s^{-1}]$ and Picard-Vessiot field $k[t](s)$ for a solution $s$ of the system, after base change to $k(t)$, the Picard-Vessiot ring is $k(t)[s, s^{-1}]$ and the Picard-Vessiot field is $k(t)(s)$. This implies that the Galois group scheme is a closed subscheme of the multiplicative group $\mathbb{G}_m$. A suitable choice of the coefficients $a_n$ (see [**25**] Section 4. and [**24**] Theorem 3.13.) guarantees that the Galois group scheme is the whole $\mathbb{G}_m$.

The closed subgroup schemes of $\mathbb{G}_m$ are given by the $n$-th roots of unity for $n \geq 0$. The intermediate field of the Picard-Vessiot field extension $k(t)(s)|k(t)$ corresponding to the subgroup scheme of $n$-th roots of unity is $k(t)(s^n)$. All subgroup schemes of $\mathbb{G}_m$ are observable, since $\mathbb{G}_m$ is commutative, hence all intermediate fields are solution fields. The explicit description of the intermediate fields also shows that the intermediate fields are solution fields: they are generated by the solutions $s^n$.

(2) Let $\mathcal{M}$ be the ID-module corresponding to the sequence of equations

$$\partial_{p^n} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & a_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $a_n \in \{1, \ldots, p-1\}$. By [**24**] Theorem 3.15., we know that for a solution $\mathbf{s} = (s_1, s_2)^T$, $s_2$ is a constant, $s_1$ can be written as $s_2 \cdot \sum a_i t^{p^i}$, thus the Picard-Vessiot ring (resp. Picard-Vessiot field) is generated by $s_1$. Moreover, the Galois group scheme acts by translation on $s_1$, hence it is a subgroup of the additive group $\mathbb{G}_a$ and with a suitable choice of the coefficients $a_n$, the Galois group scheme will be whole additive group $\mathbb{G}_a$.

The closed subgroup schemes of $\mathbb{G}_a$ correspond to the kernels of additive polynomials $\phi$ ([**12**] p. 483). The intermediate field corresponding to kernel($\phi$) for an additive polynomial $\phi$ is generated by $\phi(s_1)$. Again, all intermediate fields are solution fields: one can say either that the Galois group scheme $\mathbb{G}_a$ is commutative or that we know the generating solution of the intermediate field.

(3) Let $\mathcal{M}$ be the ID-module corresponding to the sequence of equations

$$\partial_{p^n}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & a_n t^{-p^n} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $a_n \in \{1, \dots, p-1\}$. Let $\mathbf{s} = (s_1, s_2)^T$ be a non-trivial solution of this iterative differential equation. First, we see that $s_2$ is a solution of the iterative differential equation

$$\partial_{p^n}(y) = a_n t^{-p^n} y,$$

hence after a suitable choice of the coefficients $a_n$, we see that $s_2$ is transcendental over $k(t)$ and the multiplicative group $\mathbb{G}_m = \underline{Aut}^{\mathrm{ID}}(k[t][s_2]|k[t])$ is a quotient of the differential Galois group

$$G = \underline{Aut}^{\mathrm{ID}}(k[t][s_1, s_2]|k[t]).$$

Moreover, for every element $h$ of the differential Galois group that fixes $s_2$, the element $h(s_1) - s_1$ is a constant, since

$$\partial_{p^n}(h(s_1) - s_1) = h(\partial_{p^n}(s_1)) - \partial_{p^n}(s_1) = h(s_2) - s_2 = 0.$$

This implies that $\underline{Aut}^{\mathrm{ID}}(k[t][s_1, s_2]|k[t][s_2])$ is a subgroup of the additive group $\mathbb{G}_a$.

We can explicitly describe the elements of the Galois group scheme. It is enough to give the image of the solutions $s_1$ and $s_2$ under an ID-automorphism as the generate the Picard-Vessiot ring. The general form of an ID-automorphism is the following:

$$s_1 \mapsto \beta s_1 + \alpha$$
$$s_2 \mapsto \beta s_2,$$

where $\alpha$ is an arbitrary and $\beta$ is an invertible element. We will denote this element of the Galois group scheme by $(\alpha, \beta)$. We note that multiplication is given by $(\alpha', \beta')(\alpha, \beta) = (\beta'\alpha + \alpha', \beta'\beta)$ and $(\alpha, \beta)^{-1}$ is just $(-\alpha\beta^{-1}, \beta^{-1})$.

We can extend an ID-automorphism of $k[t][s_2]|k[t]$ given by multiplication of $s_2$ with an invertible element $\beta$ to an ID-automorphism of $k[t][s_1, s_2]|k[t]$ by simply multiplying both $s_1$ and $s_2$ with $\beta$, with the notation above, it is the element $(0, \beta)$. This gives us a homomorphism from $\mathbb{G}_m = \underline{Aut}^{\mathrm{ID}}(k[t][s_2]|k[t])$ to $G = \underline{Aut}^{\mathrm{ID}}(k[t][s_1, s_2]|k[t])$, whose composition with the quotient homomorphism is the identity on $\mathbb{G}_m$. In other words, that extension splits and the differential Galois group is a closed subgroup scheme of $\mathbb{G}_m \ltimes \mathbb{G}_a$.

The explicit description of the elements of the Galois group scheme also shows that the group $\mu_{p^n}$ of $p^n$-th roots of unity is not a normal sungroup scheme in the Galois group scheme: over a $k$-algebra $k'$, where $\mu_{p^n}(k') \neq 1$, the subgroup $\mu_{p^n}(k')$

is not normal, since it is not stable under the conjugation with an element $(\alpha, \beta)$, where $\alpha \neq 0$.

The only subgroups of $\mathbb{G}_a$ that are stable under the action of the multiplicative group are the Frobenius kernels, but the differential Galois group must be reduced by [**26**] Corollary 11.7., thus $G$ is either $\mathbb{G}_m$ or $\mathbb{G}_m \ltimes \mathbb{G}_a$.

We show that $s_1$ is transcendental over $k(t)(s_2)$, which implies that the differential Galois group is indeed the whole $\mathbb{G}_m \ltimes \mathbb{G}_a$. Let $s_1 = b_0 + b_1 s_2 + b_2 s_2^2 + \ldots$ be a polynomial with coefficients in $k(t)$. Applying $\partial_1$, we get that

$$\begin{aligned} &s_2 = \partial_1(s_1) = \partial_1(b_0 + b_1 s_2 + b_2 s_2^2 + \ldots) = \\ &\quad = \partial_1(b_0) + \partial_1(b_1) s_2 + b_1 a_0 t^{-1} s_2 + \text{second or higher order terms in } s_2. \end{aligned}$$

We get that the element $b_1 \in k(t)$ must satisfy the following equation:

$$\partial_1(b_1) = 1 - a_0 t^{-1} b_1.$$

Write $b_1$ as $f/g$, where $f, g \in k[t]$ are relatively prime polynomials. The previous equation can now be rewritten as

$$t\partial_1(f)g - tf\partial_1(g) = tg^2 - a_0 fg. \qquad (2)$$

We get that $g$ must divide the term $tf\partial_1(g)$, hence $g$ divides $t\partial_1(g)$, as $f$ and $g$ are relatively prime. This can happen either if $g = t\partial_1(g)$ or $\partial_1(g) = 0$. We show that neither case can happen, which means that $s_1$ is transcendental over $s_2$.

Let $g = t\partial_1(g)$: substituting back this identity to Equation 2, we get that

$$\begin{aligned} t\partial_1(f)g - fg &= tg^2 - a_0 fg \\ t\partial_1(f) - f &= tg - a_0 f \\ t(\partial_1(f) - g) &= (1 - a_0) f. \end{aligned}$$

If we assume that $a_0 \neq 1$, then we see that $t$ divides $f$, but it also divides $g$, which is a contradiction as $f$ and $g$ are relatively prime.

Let $\partial_1(g) = 0$: this implies that $g(t) = g'(t^p)$. Equation 2 now becomes

$$\begin{aligned} t\partial_1(f)g &= tg^2 - a_0 fg \\ t\partial_1(f) &= tg - a_0 f \\ t(\partial_1(f) - g) &= -a_0 f. \end{aligned}$$

It follows that $f$ can be divided by $t$, write $f$ as $t \cdot f'$:

$$\begin{aligned} t(\partial_1(t \cdot f') - g) &= -a_0 t f' \\ f' + t\partial_1(f') - g &= -a_0 f' \\ g &= (1 + a_0) f' + t\partial_1(f'). \end{aligned}$$

We apply $\partial_1$ now and use that $\partial_1(g) = 0$

$$0 = (1 + a_0)\partial_1(f') + \partial_1(f') + 2t\partial_2(f') =$$
$$= (2 + a_0)\partial_1(f') + 2t\partial_2(f')$$

From now on we assume that $\mathrm{char}(k) \neq 2$. The previous identity implies that the coefficients $c_i$ of $f'$ must satisfy the following:

$$i \cdot c_i \cdot (a_0 + 2 + i - 1) = 0.$$

If we let $a_0$ be $p - 1$, then we see that the non-zero coefficients $c_i$ must satisfy $p|i$, or in other words, $f'(t) = f''(t^p)$. In summary, $b_1$ is of the form $t\frac{f''(t^p)}{g'(t^p)}$, but the first iterative derivative of such an element is $\frac{f''(t^p)}{g'(t^p)}$, not the required $1 + (p-1)t^{-1}t\frac{f''(t^p)}{g'(t^p)} = 1 + \frac{f''(t^p)}{g'(t^p)}$, a contradiction.

In conclusion: if $\mathrm{char}(k) \neq 2$ and $a_0 = p - 1$ and the other $a_i$-s are chosen suitably, then the differential Galois group scheme is $\mathbb{G}_m \ltimes \mathbb{G}_a$. In this case we get solution fields corresponding to non-normal, non-reduced observable subgroup schemes, namely the $p^n$-th roots of unity $\mu_{p^n}$ in $\mathbb{G}_m \subseteq \mathbb{G}_m \ltimes \mathbb{G}_a$.

CHAPTER 3

# Difference rings and modules

In this chapter we apply the general theory developed in Chapter 1 to difference Galois theory. After recalling some basics on difference rings and modules, we study Tannakian properties of categories of difference modules. The theory of invertible difference rings and modules is detailed in [**30**], a summary about étale difference modules over $\sigma$-flat difference rings can be found in [**13**] Section A1.1. The theory of solution algebras and solution fields is given in the last two sections.

## 1. Difference rings

A difference ring $\mathcal{A}$ is a pair $(A, \sigma)$ where $A$ is a commutative ring and $\sigma\colon A \to A$ is a ring endomorphism. We say that $\mathcal{K}$ is a difference field if the underlying ring is a field.

A difference ideal of $\mathcal{A}$ is an ideal $I$ such that $\sigma(I) \subseteq I$. A simple difference ring is a difference ring with only the trivial difference ideals: 0 and $A$. Trivially, a difference field is simple.

A homomorphism $f\colon \mathcal{A} \to \mathcal{A}'$ of difference rings is a ring homomorphism $f\colon A \to A'$ such that $f \circ \sigma = \sigma' \circ f$. The kernel of a difference homomorphism is a difference ideal. In particular, a difference homomorphism from a simple difference ring is always injective.

As the endomorphism $\sigma$ commutes with itself, we see that $\sigma$ is also a *difference* ring endomorphism. Hence, if $\mathcal{A}$ is simple, then $\sigma$ is injective.

An element $c \in A$ is called constant if $\sigma(c) = c$. The set of constant elements is a ring, it will be denoted by $k_{\mathcal{A}}$. A morphism $f\colon \mathcal{A} \to \mathcal{A}'$ of difference rings induces a homomorphism $k_{\mathcal{A}} \to k_{\mathcal{A}'}$ of constant rings.

PROPOSITION 3.1. *Let $\mathcal{A} = (A, \sigma)$ be a difference ring.*

(1) *If $\mathcal{A}$ is simple, then the constant ring $k$ is a field.*
(2) *The nilradical of a difference ring is always a difference ideal. More generally, the radical of a difference ideal is a difference ideal.*
(3) *If $\mathcal{A}$ is simple, then it is reduced.*
(4) *A maximal difference ideal $I$ satisfies $a \in I$ if and only if $\sigma(a) \in I$.*

PROOF.

(1) Let $c$ be a constant element. Then the ideal $(c)$ is a difference ideal of $A$: indeed, $\sigma(ac) = \sigma(a)\sigma(c) = \sigma(a)c \in (c)$. Therefore $(c)$ must be the whole ring (as it is a non-empty ideal), hence there exists an element $c'$ such that $cc' = 1$. Now $1 = \sigma(1) = \sigma(cc') = \sigma(c)\sigma(c') = c\sigma(c')$ shows that $\sigma(c') = c'$, thus the inverse of $c$ is a constant element, too.

(2) If $a$ is an element of the nilradical, i.e. $a^n = 0$, then $\sigma(a)^n = \sigma(a^n) = \sigma(0) = 0$ and hence $\sigma(a)$ is an element of the nilradical, too. The second claim can be proven similarly.

(3) Trivial from the previous item.

(4) If $a \in I$, then $\sigma(a) \in I$. Conversely, the ideal $\{a \in A \mid \sigma(a) \in I\}$ is a difference ideal containing $I$, but not containing 1.

□

We can extend the difference ring structure through special localizations.

Proposition 3.2. *Let $\mathcal{A}$ be a difference ring and $S \subseteq A$ be a $\sigma$-stable multiplicatively closed subset. Then there exists an endomorphism $\sigma'$ of $S^{-1}A$ such that the natural homomorphism $A \to S^{-1}A$ is a homomorphism of difference rings. If $\mathcal{A}$ is a simple difference ring, then $S^{-1}\mathcal{A}$ will be a simple difference ring, too.*

Proof. We have a ring homomorphism $A \to S^{-1}A$, given by $a \mapsto \sigma(a)/1$, which is just the composition of $\sigma\colon A \to A$ and the natural homomorphism $A \to S^{-1}A$.

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & A \\ \downarrow & & \downarrow \\ S^{-1}A & \dashrightarrow & S^{-1}A \end{array}$$

As $S$ is $\sigma$-stable, every element of $S$ is mapped to an invertible element, hence by the universal property of the localization, we get a ring homomorphism $\sigma'\colon S^{-1}A \to S^{-1}A$. The commutativity of the previous diagram proves that the natural localization homomorphism is a homomorphism of difference rings.

The inverse image of a difference ideal $I$ of $S^{-1}\mathcal{A}$ via $\mathcal{A} \to S^{-1}\mathcal{A}$ is a difference ideal in $\mathcal{A}$, hence it is either 0 or $A$. But then $I$ is either 0 or $S^{-1}A$. □

Corollary 3.3. *Let $\mathcal{A}$ be a difference ring.*

(1) *If the endomorphism $\sigma$ is an isomorphism, then there exists a unique difference ring structure on the total fraction ring $T(A)$ of $A$ such that the natural ring homomorphism $A \to T(A)$ is a homomorphism of difference rings. The endomorphism of $T(A)$ will be again an automorphism. Moreover, if $\mathcal{A}$ is a simple difference ring, then the constant field of $T(\mathcal{A})$ will be the same as the constant field of $\mathcal{A}$.*

(2) *If the endomorphism $\sigma$ is injective and $A$ is an integral domain, then there exists difference ring structure on the fraction field $K = \mathrm{Quot}(A)$ such that the natural ring homomorphism $A \to K$ is a homomorphism of difference rings. Moreover, if $\mathcal{A}$ is a simple difference ring, then the constant field of $\mathcal{K}$ will be the same as the constant field of $\mathcal{A}$.*

Proof. In both cases, the defining multiplicatively closed subsets are $\sigma$-stable under the assumptions. To see that the endomorphism of $T(A)$ is an automorphism, it is enough to prove that the multiplicatively closed subset is closed under $\sigma^{-1}$, or in other words, if $s \in S$ is a non-zero-divisor, then $\sigma(s)$ is a non-zero-divisor.

It is left to check that the constant field does not change if $\mathcal{A}$ is simple. In both cases, one checks that for a constant element $c$ of $T(\mathcal{A})$ (or $\mathcal{K}$), the set

$$I_c = \{a \in A \mid a \cdot c \in \mathcal{A}\}$$

is a non-empty difference ideal, hence $I_c = A$ and thus $c$ is in the constant field of $\mathcal{A}$. □

## 2. Difference modules

Let $\mathcal{A}$ be a difference ring. A difference module $\mathcal{M}$ over $\mathcal{A}$ is pair $(M, \Sigma)$ where $M$ is an $R$-module over $R$ and $\Sigma\colon M \to M$ is an additive map such that $\Sigma$ is $\sigma$-semilinear, i.e. $\Sigma(rm) = \sigma(r)\Sigma(m)$.

A trivial example for a difference module is the difference ring itself.

A difference submodule of $\mathcal{M}$ is an $A$-submodule $N$ which is stable under $\Sigma$, that is $\Sigma(N) \subseteq N$. In this case, there exists a unique difference module structure on $M/N$ induced by $\Sigma$.

A homomorphism of difference modules $\phi\colon \mathcal{M} \to \mathcal{M}'$ is an $A$-module homomorphism such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M' \\ \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \Sigma'} \\ M & \xrightarrow{\phi} & M' \end{array}$$

The kernel, cokernel and image of the underlying homomorphism of difference modules are again difference modules with the natural semilinear maps.

If we have an inductive system $\mathcal{M}_i$ of difference modules, then there exists a difference module structure on the inductive limit $\varinjlim M_i$ such that the natural morphisms $M_i \to \varinjlim M_i$ become homomorphisms of difference modules. In particular, we can take direct sums of difference modules.

For two difference modules $\mathcal{M}$ and $\mathcal{M}'$, the maps $m \otimes m' \mapsto \Sigma(m) \otimes \Sigma'(m')$ define semilinear maps on the tensor product $M \otimes_A M'$. This tensor product $\mathcal{M} \otimes_A \mathcal{N}$ of difference modules inherits nice properties from the plain tensor product, e.g. it is associative, commutative, has a unit object (the difference module $\mathcal{A}$) and commutes with small inductive limits in both variables).

For a special case, let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of difference rings and $\mathcal{M}$ be a difference module over $\mathcal{A}$. Then the tensor product $M \otimes_A B$ is not just a difference module over $\mathcal{A}$, but also over $\mathcal{B}$. We call it the base change (or pullback) of $\mathcal{M}$ w.r.t. the difference ring homomorphism $\mathcal{A} \to \mathcal{B}$. This base change again has the expected nice properties.

In general there does not exist a good notion of inner hom of difference modules. However, we will see later that under some restrictions on the difference ring or the difference modules, we indeed have inner homs.

The set $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ of difference module homomorphisms $\mathcal{M} \to \mathcal{N}$ has a natural $k$-linear structure ($k$ is the constant ring of $\mathcal{A}$) and this $k$-linear structure behaves nicely on compositions.

We note that all constructions above used the underlying modules of the differential modules.

To summarize, we rephrase these properties as follows.

Proposition 3.4. *Let $\mathcal{A}$ be a difference ring with constant ring $k$. Then the category* $\mathrm{Diff}(\mathcal{A})$ *of difference modules over $\mathcal{A}$ is a cocomplete $k$-linear abelian symmetric monoidal category with unit object the trivial difference module $\mathcal{A}$. The forgetful functor $\vartheta\colon$* $\mathrm{Diff}(\mathcal{A}) \to \mathrm{Mod}(A)$ *is a cocontinuous $k$-linear faithful exact symmetric monoidal functor.*

*If $\mathcal{A} \to \mathcal{B}$ is a homomorphism of difference rings, then the base change functor* $\mathrm{Diff}(\mathcal{A}) \to \mathrm{Diff}(\mathcal{B})$ *is a cocontinuous $k$-linear additive symmetric monoidal functor.*

A constant element of a difference module $\mathcal{M}$ over a difference ring $\mathcal{A}$ is an element $m \in M$ such that $\Sigma(m) = m$. The set $\mathcal{M}^{\Sigma} = \{m \in M \mid \Sigma(m) = m\}$ of constant elements is a module over the constant ring $k$ of $\mathcal{A}$. For the unit object $\mathcal{A}$, the constant elements are precisely the constant elements defined previously. A homomorphism $\mathcal{M} \to \mathcal{N}$ of difference modules over $\mathcal{A}$ induces a $k$-linear homomorphism $\mathcal{M}^{\Sigma} \to \mathcal{N}^{\Sigma}$.

As $\sigma(1) = 1$ in the difference ring $\mathcal{A}$, we get that for any homomorphism $\phi\colon \mathcal{A} \to \mathcal{M}$ of difference modules, the element $\phi(1)$ is constant. Conversely, any constant element gives rise to a homomorphism of difference modules, hence we have the following.

Proposition 3.5. *Let $\mathcal{A}$ be a difference ring and $\mathcal{M}$ be a difference module over $\mathcal{A}$. Then we have a natural isomorphism*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{M}) &\to \mathcal{M}^{\Sigma} \\ \phi &\mapsto \phi(1). \end{aligned}$$

In particular, we see that the endomorphism ring of the unit difference module $\mathcal{A}$ is the constant ring $k$ and the $k$-linear structure of the category $\mathrm{Diff}(\mathcal{A})$ is just the $k$-linear structure induced by the endomorphism ring of the unit object.

Let us mention two much-studied examples of difference rings and difference modules.

EXAMPLE 3.6.

(1) Let $\kappa$ be a perfect field of characteristic $p > 0$. Denote by $W$ the ring of Witt vectors of $\kappa$ and by $K$ the fraction field of $W$. We note that $W$ is a complete discrete valuation ring with residue field $\kappa$. Let $F$ be the Frobenius automorphism of $\kappa$. It extends to automorphisms of $W$ and $K$, which will be denoted by $F$, also. Denote by $\mathcal{W}$ and $\mathcal{K}$ the corresponding difference rings.

The classical definition of an $F$-crystal (resp. $F$-isocrystal) over $\kappa$ is the following: it is a finitely generated free $W$-module (resp. finite dimensional $K$-vector space) $M$ (resp. $V$) with an injective $F$-semilinear endomorphism $M \to M$. If $V$ is an $F$-isocrystal, then the injective endomorphism is in fact a bijection, since the dimension of $V$ is finite. So $F$-isocrystals are precisely the finite dimensional difference modules over $\mathcal{K}$ with invertible endomorphism. But the endomorphism of an $F$-crystal may not be bijective, hence an $F$-crystal is just an object of $\mathrm{Diff}(\mathcal{A})$. See e.g. [**6**] for the use of crystals in $p$-adic cohomology theories.

(2) Consider the finite field $\mathbb{F}_q$ for some $p$-power $q$. Let $|\cdot|_\infty$ be the infinite place of the rational function field $\kappa = \mathbb{F}_q(\theta)$ normalized such that $|\theta|_\infty = q$. Let $\kappa_\infty = \mathbb{F}_q((1/\theta))$ be the $\infty$-adic completion of $\mathbb{F}_q(\theta)$, $\overline{\kappa_\infty}$ be the algebraic closure and $\mathbb{K}$ be the $\infty$-adic completion of $\overline{\kappa_\infty}$. Finally, let $\overline{\kappa}$ be the algebraic closure of $\kappa$ in $\mathbb{K}$.

There is an automorphism $\sigma\colon \overline{\kappa}(t) \to \overline{\kappa}(t)$, which is just taking $q$-th root in the coefficients. The constant ring of $\overline{\kappa}(t)$ with respect to this automorphism is $\mathbb{F}_q(t)$. We can restrict the automorphism $\sigma$ to an automorphism of the polynomial ring $\overline{\kappa}[t]$. The constant ring of this difference ring is $\mathbb{F}_q[t]$.

An *Anderson t-motive* is a difference module $\mathcal{M}$ over the difference ring $\overline{\kappa}[t]$ such that $M$ is finitely generated and free as a $\overline{\kappa}[t]$-module and as a $\overline{\kappa}[\sigma]$-module and for all large enough $n$, we have $(t-\theta)^n M \subseteq \Sigma(M)$. For more on these, see e.g. [**28**].

Now we turn our attention to a special case where we have inner homs for any pair of difference modules. Let $\mathcal{A} = (A, \sigma)$ be a difference ring. Semilinear maps give rise to special $A$-module homomorphisms: let $M_\sigma$ be the tensor product $A_\sigma \otimes_A M$, where $A_\sigma$ is the $A$-$A$-bimodule on which $A$ acts regularly from the left, but via $\sigma$ from the right. A $\sigma$-semilinear map $\Sigma\colon M \to M$ gives an $A$-module homomorphism $\Psi\colon M_\sigma \to M$, given by $a \otimes m \mapsto a\Sigma(m)$. Conversely, such an $A$-module homomorphism $M_\sigma \to M$ gives is $\sigma$-semilinear endomorphism of $M$.

DEFINITION 3.7. Let $\mathcal{A}$ be a difference ring. A difference module $\mathcal{M}$ over $\mathcal{A}$ is called étale if the induced morphism

$$\Psi\colon M_\sigma \to M$$

is an isomorphism.

The trivial difference module $\mathcal{A}$ is étale, since in this case the $\Psi$ is the identity. Another example is given by:

EXAMPLE 3.8. Let $\mathcal{A}$ be a difference ring such that $\sigma$ is an automorphism. Then a difference module $\mathcal{M}$ is étale if and only if its endomorphism $\Sigma$ is bijective.

Indeed, let first let $\mathcal{M}$ be étale, i.e. the map

$$\Psi\colon A_\sigma \otimes_A M \to M$$
$$a \otimes m \mapsto a\Sigma(m)$$

is a bijection. From this it follows immediately that $\Sigma$ is surjective, since any element $m \in M$ can be written as

$$m = \Psi(\sum_i a_i m_i) = \sum_i a_i \Sigma(m_i) = \Sigma(\sum_i \sigma^{-1}(a_i) m_i).$$

The map $\iota\colon M \to A_\sigma \otimes_A M\colon m \mapsto 1 \otimes m$ is injective, since it has a left inverse $A_\sigma \otimes_A M \to M$ defined by $a \otimes m \mapsto \sigma^{-1}(a)m$. Therefore the composition

$$M \xrightarrow{\iota} A_\sigma \otimes_A M \xrightarrow{\Psi} M$$

is injective, but this is just $\Sigma$.

Conversely, assume that $\Sigma$ is bijective. Then we can define an inverse to $\Psi$ as follows: send an element $m \in M$ to $1 \otimes \Sigma^{-1}(m)$. It is a trivial calculation that it is indeed an inverse, thus $\mathcal{M}$ is étale.

$F$-isocrystals are étale difference modules of the kind discussed in this example.

Now back to general étale difference modules. As tensor product over $A$ commutes with small inductive limits, we see that small inductive limits of étale difference modules are étale. Similarly, the tensor product of two étale difference modules is again étale. For kernels and cokernels we have to assume that $\sigma\colon A \to A$ is a flat ring homomorphism. In this case we will call $\mathcal{A}$ a $\sigma$-flat difference ring.

PROPOSITION 3.9. *Let $\mathcal{A}$ be a $\sigma$-flat difference ring. Let $\phi\colon \mathcal{M} \to \mathcal{M}'$ be a homomorphism of étale difference modules. Then the kernel and cokernel difference module is also étale.*

PROOF. Let $\phi\colon \mathcal{M} \to \mathcal{M}'$ be a homomorphism of étale difference modules and consider the exact sequence

$$0 \to \text{kernel}(\phi) \to \mathcal{M} \to \mathcal{M}' \to \text{coker}(\phi) \to 0.$$

Since $\sigma\colon A \to A$ is flat, after tensoring with $A_\sigma$ we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{kernel}(\phi)_\sigma & \longrightarrow & \mathcal{M}_\sigma & \longrightarrow & \mathcal{M}'_\sigma & \longrightarrow & \mathrm{coker}(\phi)_\sigma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{kernel}(\phi) & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathrm{coker}(\phi) & \longrightarrow & 0 \end{array}$$

Since the two middle vertical maps are isomorphism, it follows that the other two vertical arrows must be isomorphisms, too, hence the kernel and cokernel are also étale difference modules. □

In conclusion, we get the following.

PROPOSITION 3.10. *Let $\mathcal{A}$ be a $\sigma$-flat difference ring and $k$ be the constant ring of $\mathcal{A}$. Then the full subcategory* $\mathrm{Diff}^{\text{ét}}(\mathcal{A})$ *of the category of all difference modules consisting of étale difference modules over $\mathcal{A}$ is a cocomplete $k$-linear abelian symmetric monoidal category and the forgetful functor* $\vartheta\colon \mathrm{Diff}^{\text{ét}}(\mathcal{A}) \to \mathrm{Mod}(A)$ *is a cocontinuous $k$-linear faithful exact symmetric monoidal functor.*

*If $\mathcal{B}$ is an other $\sigma$-flat difference ring and $\mathcal{A} \to \mathcal{B}$ is a homomorphism of difference rings, then the base change functor* $\mathrm{Diff}^{\text{ét}}(\mathcal{A}) \to \mathrm{Diff}^{\text{ét}}(\mathcal{B})$ *is a cocontinuous $k$-linear additive symmetric monoidal functor.*

For the base change, the fact that we have a homomorphism of difference rings is important to establish the étaleness of the base change module, since then we have an isomorphism $B_{\sigma'} \otimes_B (B \otimes_A M) \cong B \otimes_A (A_\sigma \otimes_A M)$.

Unfortunately, we do not necessarily have inner hom for any two étale difference modules, but we have the following result.

PROPOSITION 3.11. *Let $\mathcal{A}$ be a $\sigma$-flat difference module. Let $\mathcal{M}$ and $\mathcal{M}'$ be étale difference modules such that $M$ is a finitely presented $A$-module. Then there exists a difference module structure on* $\mathrm{Hom}_A(M, M')$ *such that it becomes the inner hom of $\mathcal{M}$ and $\mathcal{M}'$ and which is étale.*

PROOF. Since $A$ is $\sigma$-flat and $M$ is a finitely presented $A$-module, by [**7**] §2.10. Prop.11. the natural homomorphism

$$A_\sigma \otimes_A \mathrm{Hom}_A(M, M') \to \mathrm{Hom}_{A_\sigma}(M_\sigma, M'_\sigma)$$

is an isomorphism. We can define a homomorphism

$$\Psi\colon A_\sigma \otimes_A \mathrm{Hom}_A(M, M') \cong \mathrm{Hom}_{A_\sigma}(M_\sigma, M'_\sigma) \to \mathrm{Hom}_A(M, M')$$

by $\Psi(\phi) = \Psi_{\mathcal{M}'} \circ \phi \circ \Psi_{\mathcal{M}}^{-1}$, where $\phi \in \mathrm{Hom}_{A_\sigma}(M_\sigma, M'_\sigma)$. This induces an $\sigma$-semilinear endomorphism on $\mathrm{Hom}_A(M, M')$ and $\Psi$ is bijective, since $\Psi_{\mathcal{M}}$ and $\Psi_{\mathcal{M}'}$ are bijective. □

Applying the above with $\mathcal{M}' = \mathcal{A}$ and using that a finitely generated projective module is necessarily finitely presented, we get the following in the same way as in Proposition 2.4.

Corollary 3.12. *Let $\mathcal{A}$ $\sigma$-flat difference module and $\mathcal{M}$ be an étale difference module over $\mathcal{A}$. Then $\mathcal{M}$ has a dual in* $\mathrm{Diff}^{\text{ét}}(\mathcal{A})$ *if and only if $M$ is a finitely generated projective $A$-module.*

## 3. Tannakian categories of difference modules

We now prove an analogue of Proposition 2.5 for étale difference modules over a $\sigma$-flat difference ring.

First we need a lemma.

Lemma 3.13. *Let $\mathcal{A} = (\mathcal{A}, \sigma)$ be a difference ring and $\mathcal{M}$ be an étale difference module over $\mathcal{A}$. Then the annihilator* $\mathrm{Ann}_A(M)$ *of $M$ in $A$ is a difference ideal.*

Proof. We only have to show that if $a \in \mathrm{Ann}_A(M)$, then $\sigma(a) \in \mathrm{Ann}_A(M)$. As $\mathcal{M}$ is an étale difference module, we know that every $m \in M$ can be written as $\sum_i a_i \Sigma(m_i)$. Then

$$\sigma(a) \sum_i a_i \Sigma(m_i) = \sum_i a_i \sigma(a) \Sigma(m_i) = \sum_i a_i \Sigma(a m_i) = 0$$

since $aM = 0$. □

We can now state:

Proposition 3.14. *Let $\mathcal{A} = (A, \sigma)$ be a $\sigma$-flat simple difference ring such that $A$ is a noetherian ring and let $\mathcal{M}$ be a finitely genereted étale difference module over $\mathcal{A}$. Denote by $k$ the constant field of $\mathcal{A}$. We have the following:*

1. *The underlying module of $\mathcal{M}$ and its difference subquotients are projective modules,*
2. *The category consisting of objects that are difference subquotients of finite direct sums of tensor products of the form $\mathcal{M}^{\otimes i} \otimes (\mathcal{M}^{\vee})^{\otimes j}$ form a Tannakian category $\langle \mathcal{M} \rangle_{\otimes}$ over the constant field $k$ of $\mathcal{A}$,*
3. *the natural base change functor*

$$\langle \mathcal{M} \rangle_{\otimes} \to \langle \mathcal{M}_{T(\mathcal{A})} \rangle_{\otimes}$$

*is an equivalence of categories.*

Proof. As $\mathcal{A}$ is simple, we get that $A$ is reduced. Moreover, since $A$ is assumed to be noetherian, we see that the total ring $T(A)$ of fractions is a semisimple commutative ring, i.e. the finite product of fields. By Proposition 3.2. and Corollary 3.3., the ring $T(A)$ is a simple difference ring, its endomorphism is an automorphism and the natural map $A \to T(A)$ is a homomorphism of difference rings. Moreover, since $A$ is assumed to be noetherian, we have that finitely generated modules are also finitely presented.

Let $\mathcal{M}$ and $\mathcal{M}'$ be finitely generated étale difference modules over $\mathcal{A}$. We know that their inner hom $\mathcal{H}om_A(M, M')$ exists in $\mathrm{Diff}^{\text{ét}}(\mathcal{A})$. We first prove item (3), i.e. that the base change functor is fully faithful for finitely generated difference modules:

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M}') \cong \mathrm{Hom}_{T(\mathcal{A})}(\mathcal{M}_{T(\mathcal{A})}, \mathcal{M}'_{T(\mathcal{A})}).$$

To see injectivity, we consider the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M}') & \dashrightarrow & \mathrm{Hom}_{T(\mathcal{A})}(\mathcal{M}_{T(\mathcal{A})}, \mathcal{M}'_{T(\mathcal{A})}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_A(M, M') & \longrightarrow & \mathrm{Hom}_{T(A)}(M_{T(A)}, M'_{T(A)}) \end{array}$$

The map $\mathrm{Hom}_A(M, M') \to \mathrm{Hom}_{T(A)}(M_{T(A)}, M'_{T(A)})$ is injective since $T(A)$ is flat over $A$ and $M$ is finitely generated ([**7**] Proposition §2.10. Prop. 11.). This implies the injectivity of the dotted map.

For surjectivity, let $\phi\colon \mathcal{M}_{T(\mathcal{A})} \to \mathcal{M}'_{T(\mathcal{A})}$ be a homomorphism of difference modules over $T(\mathcal{A})$. The set $\phi(\mathcal{M})$ is a difference $\mathcal{A}$-submodule of $\mathcal{M}'_{T(\mathcal{A})}$ and the quotient

$$\phi(\mathcal{M})/(\phi(\mathcal{M}) \cap \mathcal{M}')$$

is a finitely generated torsion étale difference module over $\mathcal{A}$. Hence its annihilator is a difference ideal in $\mathcal{A}$, therefore it is either 0 or $A$. But since $M$ is finitely generated, the annihilator cannot be 0, thus it is $A$, meaning that $\phi(\mathcal{M})/(\phi(\mathcal{M}) \cap \mathcal{M}') = 0$, therefore $\phi(\mathcal{M}) \subseteq \mathcal{M}'$. We have just proved that base change functor

$$\langle \mathcal{M} \rangle_{\otimes} \to \langle \mathcal{M}_{T(\mathcal{A})} \rangle_{\otimes},$$

is fully faithful. For proving that the base change is an equiavalence, it is enough to show essential surjectivity, which follows from the fact that any subobject $\mathcal{N}$ of $\mathcal{M}'_{T(\mathcal{A})} \in \langle \mathcal{M}_{T(\mathcal{A})} \rangle_{\otimes}$ comes from the subobject $\mathcal{N} \cap \mathcal{M}'$ of $\mathcal{M}'$.

We can now prove that the $A$-module $M$ is projective. First we note that $M_{T(A)}$ is a finitely generated projective $T(A)$-module, as $T(A)$ is a semisimple commutative ring. Therefore $\mathcal{M}_{T(\mathcal{A})}$ has a dual in $\mathrm{Diff}^{\text{ét}}(T(\mathcal{A}))$ and the dual is just

$$\mathcal{H}om_{T(A)}(M_{T(A)}, T(A)).$$

The assumption that $A$ is noetherian implies that any finitely generated $A$-module is finitely presented, therefore the base change of $\mathrm{Hom}_A(M, A)$ via $A \to T(A)$ is $\mathrm{Hom}_{T(A)}(M_{T(A)}, T(A))$. In particular, the base change of the inner hom difference module $\mathcal{H}om_A(M, A)$ is the dual of $\mathcal{M}_{T(\mathcal{A})}$. Moreover, $\mathrm{Hom}_A(M, A)$ is finitely generated over $A$, too.

The evaluation and coevaluation morphisms of $\mathcal{M}_{T(\mathcal{A})}$ are homomorphisms of difference modules and we have just proved that the base change functor is fully faithful in this situation, thus we get homomorphisms of difference modules

$$\mathcal{H}om_A(M,A) \otimes_A \mathcal{M} \to \mathcal{A}$$
$$\mathcal{A} \to \mathcal{M} \otimes_A \mathcal{H}om_A(M,N).$$

These maps satisfy the identities of the dual object since they satisfy them after applying the fully faithful base change functor. This shows that $\mathcal{M}$ has a dual and in particular, $M$ is a finitely generated projective $A$-module. The quotient module of a finitely generated module is again finitely generated, hence quotients of $\mathcal{M}$ are again projective. For a difference submodule $\mathcal{N}$ of $\mathcal{M}$, we use that the quotient difference module is finitely generated, hence projective, therefore the submodule $N$ is a direct summand in $M$ and in particular finitely generated, thus projective. This proves item (1).

For item (2), the previous shows that $\langle \mathcal{M} \rangle_{\otimes}$ is a rigid $k$-linear abelian symmetric monoidal category and the forgetful functor to the category of $A$-modules is a fibre functor. □

## 4. Picard-Vessiot theory and solution algebras of difference modules

The general theory of Picard-Vessiot rings and solution algebras in tensor categories developed in Chapter 1 can be applied to difference modules as well. From now on, we will assume that $\mathcal{A}$ is a $\sigma$-flat simple difference ring such that $A$ is a noetherian ring and $\mathcal{M}$ is a finitely generated étale difference module over $\mathcal{A}$.

The category $\langle \mathcal{M} \rangle_{\otimes}$ generated by $\mathcal{M}$ is a Tannakian category, hence by Theorem 1.21 the question of the existence of a Picard-Vessiot ring for $\mathcal{M}$ is equivalent to the existence of $k$-valued fibre functors on $\langle \mathcal{M} \rangle_{\otimes}$, where $k$ is the constant field of $\mathcal{A}$.

*From now on we assume that $k$ is algebraically closed.* We get that there exists a Picard-Vessiot ring $\mathcal{P}$ for $\mathcal{M}$ and the category $\langle \mathcal{M} \rangle_{\otimes}$ is equivalent to the category $\mathrm{Repf}_k(G)$, where $G$ is the Galois group scheme of $\mathcal{M}$ pointed at the Picard-Vessiot ring. We will denote by $\omega$ the fibre functor given by the Picard-Vessiot ring. Furthermore, Proposition 3.14 (3) implies that the base change of the Picard-Vessiot ring to the total ring of fractions is the Picard-Vessiot ring of the difference module $\mathcal{M}_{T(\mathcal{A})}$, and the associated Galois group schemes are isomorphic.

We also note that the Picard-Vessiot ring $\mathcal{P}$ has the properties satisfied by the base ring $\mathcal{A}$: it is a noetherian ring and the endomorphism of $\mathcal{P}$ is flat. The first property follows from that $\mathrm{Spec}(P) \to \mathrm{Spec}(A)$ is an fppf-cover and the fact that locally noetherian is a local property of schemes in the fppf-topology. The Picard-Vessiot ring can be written as the colimit of étale difference modules, hence it is étale as a difference module over $\mathcal{A}$. The étale property implies that the endomorphism of $\mathcal{P}$ is the base change of the endomorphism of $\mathcal{A}$ via the homomorphism $A \to P$, and since the base change of a flat morphism is flat, we are done.

The general definition of solution algebras given in Chapter 1 applies in this concrete context, and by Theorem 1.32 we obtain a correspondence between solution algebras and quasi-homogeneous schemes over the Galois group scheme.

We even have the following analogue of Proposition 2.8 for difference rings.

Proposition 3.15. *A $\sigma$-flat difference ring $\mathcal{S}$ over $\mathcal{A}$ is a solution algebra for $\langle\mathcal{M}\rangle_\otimes$ if and only if the $S$ is contained in a $\sigma$-flat noetherian simple difference ring whose constant field is $k$ and there exists a morphism $\mathcal{N} \to \mathcal{S}$ of étale difference modules over $\mathcal{A}$ whose image generates $S$ as an $A$-algebra.*

Proof. We have just seen that the Picard-Vessiot ring is a $\sigma$-flat noetherian simple difference ring and by definition, its field of constants is $k$.

In the other direction, we can assume that $\mathcal{N}$ is $\mathcal{M}$. Let $\mathcal{S}'$ be the $\sigma$-flat noetherian simple difference ring containing $\mathcal{S}$. Let $\mathcal{P}'$ be the Picard-Vessiot ring of $\mathcal{M}_{\mathcal{S}'}$: this difference ring contains $\mathcal{S}'$ and thus $\mathcal{S}$. Moreover, $\mathcal{M}$ is solvable in $\mathcal{P}'$, hence the Picard-Vessiot ring $\mathcal{P}$ of $\mathcal{M}$ is also contained in $\mathcal{P}'$. By definition, $\mathcal{P}$ is generated by the solutions of $\mathcal{M}$, hence by the last assumption on $\mathcal{S}$, the Picard-Vessiot ring $\mathcal{P}$ will contain $\mathcal{S}$. □

## 5. Solution fields for difference modules

In this section we will consider a difference field $\mathcal{K}$ with bijective endomorphism $\sigma$ and a finite dimensional étale difference module $\mathcal{M}$ over $\mathcal{K}$. In this case the endomorphism $\Sigma$ of $\mathcal{M}$ is also bijective by Example 3.8. Denote by $k$ the constant field of $\mathcal{K}$. The total ring of fractions $T(\mathcal{P})$ of the Picard-Vessiot ring $\mathcal{P}$ for $\mathcal{M}$ is called the total Picard-Vessiot ring of the difference module. This is a semisimple commutative ring, i.e. a finite product of fields.

The notion corresponding to solution fields in the difference setting is defined as follows.

Definition 3.16. Let $\mathcal{L}|\mathcal{K}$ be an extension of difference rings. We say that $\mathcal{L}$ is a total solution ring for $\langle\mathcal{M}\rangle_\otimes$ if

(1) every non-zerodivisor of $L$ is a unit in $L$,
(2) the constant ring of $\mathcal{L}$ is $k$,
(3) there exists a difference module $\mathcal{N}$ in $\langle\mathcal{M}\rangle_\otimes$ and a morphism of difference modules $\mathcal{N} \to \mathcal{L}$ such that the total fraction ring of the image of this homomorphism is $L$.

Proposition 3.17. *We have the following properties:*

(1) *the total fraction ring of a solution algebra $\mathcal{S}$ for $\langle\mathcal{M}\rangle_\otimes$ is a total solution ring for $\langle\mathcal{M}\rangle_\otimes$,*
(2) *every total solution ring $\mathcal{L}$ for $\langle\mathcal{M}\rangle_\otimes$ is the total fraction ring of a solution algebra $\mathcal{S}$ for $\langle\mathcal{M}\rangle_\otimes$,*
(3) *every total solution ring $\mathcal{L}$ for $\langle\mathcal{M}\rangle_\otimes$ embeds as an intermediate difference extension of $T(\mathcal{P})|\mathcal{K}$.*

Proof.

(1) This follows from the definition of total fraction ring, solution algebras and total solution rings.
(2) Let $S$ be the $K$-subalgebra of $L$ generated by the image of the difference morphism $\mathcal{N} \to \mathcal{L}$. It is a difference ring with total fraction ring $L$ and the conditions of being a solution algebra are satisfied by construction.
(3) This follows from the previous point using the fact that solution algebras are embedded into the Picard-Vessiot ring.

□

There exists a Galois correspondence for the total Picard-Vessiot rings in characteristic 0 which establishes a bijection between certain difference subrings of the total Picard-Vessiot ring and closed subgroups of the Galois group scheme.

Proposition 3.18 ([**30**] Thm. 1.29). *Let $\mathcal{K}$ be an invertible difference field of characteristic 0 with algebraically closed constant field $k$, and let $\mathcal{M}$ be a finite dimensional invertible difference module over $\mathcal{K}$. Denote by $T(\mathcal{P})$ the total Picard-Vessiot ring of $\mathcal{M}$ over $\mathcal{K}$ and by $G$ the Galois group scheme. The maps $H \mapsto T(\mathcal{P})^H$ and $\mathcal{L} \mapsto \mathrm{Aut}(T(\mathcal{P})|\mathcal{L})$ define an order-reversing bijection between the set of closed subgroups of $G(k)$ and those intermediate difference rings of $\mathrm{Aut}(T(\mathcal{P})|\mathcal{K})$ where every non-zerodivisor is a unit.*

Having this Galois correspondence, we have a similar characterization of total solution rings in the difference case as for (iterative) connections.

Theorem 3.19. *Let $\mathcal{L}$ be an intermediate difference ring of $T(\mathcal{P})|\mathcal{K}$ in which every non-zerodivisor is a unit.*

*The ring $\mathcal{L}$ is a total solution ring for $\langle\mathcal{M}\rangle_\otimes$ if and only if the corresponding subgroup $H$ is an observable subgroup of the Galois group $G(k)$.*

Proof. Let $H$ be an observable subgroup of the Galois group. There exists a finite dimensional $G$-representation $V$ and a vector $v \in V$ such that $H$ is the isotropy subgroup of $v$ in $V$. Using the equivalence given by $\omega$, we can write $V$ as $\omega(\mathcal{N}^\vee)$ for some difference module $\mathcal{N}$ in $\langle\mathcal{M}\rangle_\otimes$. The vector $v$ determines a difference homomorphism $v\colon \mathcal{N} \to \mathcal{P} \to T(\mathcal{P})$. Let $\mathcal{L}$ be the total fraction ring of the ring generated by the image of this difference homomorphism in $T(\mathcal{P})$ and let $H'$ be the closed subgroup corresponding to $\mathcal{L}$ by the Galois correspondence (Theorem 3.18), i.e. $\mathcal{L} = T(\mathcal{P})^{H'}$.

$H$ is the isotropy subgroup of $v$, hence for all $h \in H$ we have $h(v(n) \otimes 1) = (h \cdot v)(n) \otimes 1 = v(n) \otimes 1$ and thus $H \le H'$. Conversely, $\mathcal{L} = T(\mathcal{P})^{H'}$ means that for any $h' \in H'$ we have $(h \cdot v)(n) \otimes 1 = h(v(n) \otimes 1) = v(n) \otimes 1$ for all $n$, in other words $h' \cdot v = v$ and hence $H' \le H$. This show that for every observable subgroup $H$, the intermediate difference ring $\mathcal{L} = \mathcal{J}^H$ is a total solution ring.

Let $\mathcal{L}$ be a total solution ring that is generated by a solution $v$ of $\mathcal{N}$ and denote by $H$ the subgroup scheme attached to $\mathcal{L}$ in $G$. Just as in the previous calculation, we see that $H$ is the isotropy subgroup of the solution $v$ in $\omega(\mathcal{N}^\vee)$ and hence, $H$ is observable. $\square$

APPENDIX A

# Tools from category theory

## 1. Rings and modules in tensor categories

We briefly review the basics of commutative algebra in tensor categories. Let $\mathcal{C}$ be a tensor category. A (commutative unitary) ring in $\mathcal{C}$ is a a triple $(\mathcal{A}, m, u)$, where $\mathcal{A}$ is an object of $\mathcal{C}$, $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $u\colon 1 \to \mathcal{A}$ are morphisms in $\mathcal{C}$ that satisfy the usual commutative diagrams expressing associativity and commutativity of multiplication with $u$ being the unit morphism. All rings in tensor categories will be commutative and unitary.

A homomorphism of rings in $\mathcal{C}$ is a $\mathcal{C}$-morphism that commutes with multiplication and sends the unit morphism to the unit morphism. We will call a ring homomorphism a monomorphism (resp. an epimorphism) if the underlying $\mathcal{C}$-morphism is monic (resp. epic).

A module over a ring $\mathcal{A}$ in a tensor category $\mathcal{C}$ is a pair $(\mathcal{M}, a)$, where $\mathcal{M}$ is an object of $\mathcal{C}$ and $a\colon \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ is a morphism that satisfies the usual commutative diagrams of the associativity of the action and such that the action is unitary.

An $\mathcal{A}$-module homomorphism is a $\mathcal{C}$-morphism that commutes with the action of the ring. Naturally, we have the category $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$ of $\mathcal{A}$-modules in $\mathcal{C}$.

Proposition A.1. *Let $\mathcal{C}$ be a tensor category and $\mathcal{A}$ be a ring in $\mathcal{C}$.*

(1) *The category $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$ of $\mathcal{A}$-modules in $\mathcal{C}$ is a tensor category with unit object $\mathcal{A}$.*
(2) *The base change functor $\mathcal{A} \otimes -\colon \mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$ is a left adjoint of the forgetful functor $\mathrm{res}\colon \mathrm{Mod}_{\mathcal{C}}(\mathcal{A}) \to \mathcal{C}$.*
(3) *The base change functor is a cocontinuous right exact symmetric monoidal functor. The forgetful functor is a cocontinuous faithful exact functor.*

Proof. The tensor product $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ of two $\mathcal{A}$-modules is defined as the cokernel of the morphisms $\mathcal{M} \otimes \mathcal{A} \otimes \mathcal{N} \rightrightarrows \mathcal{M} \otimes \mathcal{N}$, where the maps are given by the action of $\mathcal{A}$ on $\mathcal{M}$ and $\mathcal{N}$ respectively. Now, using that $\mathcal{C}$ is cocomplete and that the tensor product is cocontinuous in both variables, one can prove that $- \otimes_{\mathcal{A}} -$ is indeed an associative, commutative tensor product on the category $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$ of $\mathcal{A}$-modules. For details, see [**8**] Proposition 4.1.10. or [**22**] Proposition 1.2.15.

The small colimit of $\mathcal{A}$-modules is constructed by defining an $\mathcal{A}$-module structure on the small colimit of the underlying objects of $\mathcal{A}$-modules, we can do this since $-\otimes-$ is co-continuous in both variables. These facts also show that the tensor product of $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$ is cocontinuous in both variables.

For a homomorphism of $\mathcal{A}$-modules, we can define an $\mathcal{A}$-module structure on the kernel and cokernel of the underlying $\mathcal{C}$-morphism and they will be the kernel and cokernel in the category of $\mathcal{A}$-modules in $\mathcal{C}$. For details, see [**5**] Theorem 3.6. So far we know that $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$ is a cocomplete abelian symmetric monoidal category and that the forgetful functor is a cocontinuous faithful exact functor. The base change functor is clearly cocontinuous and symmetric monoidal. If we prove that it is the left adjoint of the forgetful functor, then we get that it is right exact. Let $X$ be an object of $\mathcal{C}$. If $\mathcal{A}\otimes X\to\mathcal{N}$ is a homomorphism of $\mathcal{A}$-modules, then the composition $X\to\mathcal{A}\otimes X\to\mathcal{N}$ is a morphism in $\mathcal{C}$. This way we obtained a map

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}\otimes X,\mathcal{N})\to\mathrm{Hom}_{\mathcal{C}}(X,\mathcal{N}).$$

Conversely, if $X\to\mathcal{N}$ is a morphism in $\mathcal{C}$, then we can tensor it with $\mathcal{A}$ and use the morphism defining the $\mathcal{A}$-action on $\mathcal{N}$, getting a homomorphism of $\mathcal{A}$-modules

$$\mathcal{A}\otimes X\to\mathcal{A}\otimes\mathcal{N}\to\mathcal{N}.$$

These maps are the inverses of each other, hence the base change functor $\mathcal{A}\otimes-$ is a left adjoint of the forgetful functor. □

Using that $\mathcal{A}\cong\mathcal{A}\otimes 1$ as $\mathcal{A}$-modules, we get the following special case of the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{A}\otimes-)\cong\mathrm{Hom}_{\mathcal{C}}(1,\mathcal{A}\otimes-).$$

Definition A.2. Let $\mathcal{C}$ be a tensor category and $\mathcal{A}$ be a ring in $\mathcal{C}$. We say that $\mathcal{A}$ is simple ring if it has no proper, non-trivial $\mathcal{A}$-submodules. Equivalently, every $\mathcal{A}$-module homomorphism $\mathcal{A}\to\mathcal{M}$, to an $\mathcal{A}$-module $\mathcal{M}$ in $\mathcal{C}$, is either 0 or a monomorphism.

We have the following analogue of Schur's lemma.

Proposition A.3. *Let $\mathcal{C}$ be a tensor category and $\mathcal{A}$ be a simple ring in $\mathcal{C}$. Then the ring $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$ of $\mathcal{A}$-module endomorphisms of $\mathcal{A}$ is a field.*

Proof. The ring $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$ is the endomorphism ring of the unit object of the tensor category $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$, hence it is commutative. An non-zero endomorphism $\mathcal{A}\to\mathcal{A}$ must be a monomorphism by definition and it has full image $\mathcal{A}$, since the image is an $\mathcal{A}$-submodule, hence it is an epimorphism, thus it is an automorphism. □

## 2. Image of rings and modules under tensor functors

Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a tensor functor between tensor categories and $\mathcal{A}$ be a ring in $\mathcal{C}$. The object $\mathcal{A}' := F(\mathcal{A})$ inherits a ring structure from $\mathcal{A}$, furthermore, the image of an $\mathcal{A}$-module $\mathcal{M}$ can be considered as a module over $\mathcal{A}'$. In other words, the functor $F$ induces a functor $F_{\mathcal{A}}\colon \operatorname{Mod}_{\mathcal{C}}(\mathcal{A}) \to \operatorname{Mod}_{\mathcal{C}'}(\mathcal{A}')$ and the following diagrams are commutative

$$\begin{CD} \mathcal{C} @>{F}>> \mathcal{C}' \\ @V{\mathcal{A}\otimes -}VV @VV{\mathcal{A}'\otimes -}V \\ \operatorname{Mod}_{\mathcal{C}}(\mathcal{A}) @>{F_{\mathcal{A}}}>> \operatorname{Mod}_{\mathcal{C}'}(\mathcal{A}') \end{CD} \qquad \begin{CD} \operatorname{Mod}_{\mathcal{C}}(\mathcal{A}) @>{F_{\mathcal{A}}}>> \operatorname{Mod}_{\mathcal{C}'}(\mathcal{A}') \\ @V{\text{res}}VV @VV{\text{res}}V \\ \mathcal{C} @>{F}>> \mathcal{C}' \end{CD}$$

Using the commutativity of the right-side diagram and the faithful exactness of the forgetful functors, we see that $F_{\mathcal{A}}$ is a faithful exact functor if $F$ is faithful exact.

We will use this most frequently in the following situation: let $\vartheta\colon \mathcal{C} \to \operatorname{QCoh}(S)$ be a pointed tensor category over a commutative ring $k$ and let $\mathcal{A}$ be a ring in $\mathcal{C}$. Then $\vartheta(\mathcal{A})$ is a quasi-coherent $\mathcal{O}_S$-algebra and we can take the relative spectrum

$$\mu\colon S_{\mathcal{A}} = \operatorname{Spec}_S(\vartheta(\mathcal{A})) \to S$$

over $S$. The category of quasi-coherent modules over $S_{\mathcal{A}}$ is equivalent to the category of quasi-coherent $\vartheta(\mathcal{A})$-modules in $\operatorname{QCoh}(S)$, thus we have a commutative diagram

$$\begin{CD} \mathcal{C} @>{\vartheta}>> \operatorname{QCoh}(S) \\ @V{\mathcal{A}\otimes -}VV @VV{\mu^*}V \\ \operatorname{Mod}_{\mathcal{C}}(\mathcal{A}) @>{\vartheta_{\mathcal{A}}}>> \operatorname{QCoh}(S_{\mathcal{A}}) \end{CD}$$

By definition, $\vartheta$ is faithful exact, hence $\vartheta_{\mathcal{A}}$ is an $\mathcal{S}_{\mathcal{A}}$-valued point of the tensor category $\operatorname{Mod}_{\mathcal{C}}(\mathcal{A})$.

## 3. Faithfully flat descent in tensor categories

Definition A.4. Let $\mathcal{C}$ be a tensor category and $\mathcal{A}$ be a ring in $\mathcal{C}$. We say that $\mathcal{A}$ is flat (resp. faithfully flat) (over 1) in $\mathcal{C}$ if the base change functor

$$\mathcal{A} \otimes -\colon \mathcal{C} \to \operatorname{Mod}_{\mathcal{C}}(\mathcal{A})$$

is an exact (resp. a faithful exact) functor.

Example A.5.

(1) Let $R$ be a commutative ring. In the category $\operatorname{Mod}(R)$ of $R$-modules, the flat (resp. faithfully flat) rings in the above sense are the same as flat (resp. faithfully flat) $R$-algebras.

(2) For a general scheme $S$, there is a little difference between flatness in the above sense and flat morphisms. A morphism $Y \to X$ is called flat (resp. faithfully flat) if the functor

$$- \boxtimes_X \mathcal{O}_Y \colon \mathrm{QCoh}(X') \to \mathrm{QCoh}(X' \times_X Y)$$

is exact (resp. faithful and exact) for every morphism $X' \to X$ (for more details, see [**17**] §2.). But our definition only requires the exactness for the identity morphism $X \to X$, hence a (faithfully) flat affine morphism $Y \to X$ gives rise to (faithfully) flat quasi-coherent $\mathcal{O}_X$-algebra, but the converse is not necessarily true.

(3) Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a faithful exact tensor functor between tensor categories. If $\mathcal{A}$ is a ring in $\mathcal{C}$ and $F(\mathcal{A})$ is a flat (resp. faithfully flat) ring in $\mathcal{C}'$, then $\mathcal{A}$ is also flat (resp. faithfully flat) in $\mathcal{C}$: indeed, this follows from the fact that faithful exact functor preserves and reflects short exact sequences.

Proposition A.6. *Let $\mathcal{C}$ be a tensor category. Then the unit object is simple if and only if there exists a faithfully flat simple ring $\mathcal{A}$ in $\mathcal{C}$.*

Proof. If the unit object 1 is simple, then the unit object itself is the faithfully flat simple ring in $\mathcal{C}$. Conversely, let $\mathcal{A}$ be a faithfully flat simple ring in $\mathcal{C}$ and let $X \to 1$ be a proper subobject of the unit object. As the base change is faithful exact, we get that $\mathcal{A} \otimes X$ is a proper $\mathcal{A}$-submodule of $\mathcal{A}$, hence it must be 0 and therefore $X$ is 0, too. ☐

Let $\mathcal{C}$ be a tensor category and $\mathcal{A}$ be a ring $\mathcal{C}$. A descent datum on an $\mathcal{A}$-module $\mathcal{M}$ is an isomorphism $\mathcal{A} \otimes \mathcal{M} \cong \mathcal{M} \otimes \mathcal{A}$ of $\mathcal{A} \otimes \mathcal{A}$-modules that satisfies the cocycle condition. A morphism between modules with descent data is an $\mathcal{A}$-module homomorphism such that the natural diagram is commutative. With these morphisms, there is the category $\mathrm{Desc}_{\mathcal{C}}(\mathcal{A})$ of $\mathcal{A}$-modules with descent data.

If $X$ is an object of $\mathcal{C}$, then the base change $\mathcal{A} \otimes X$ has a natural descent datum as in the classical case and thus we obtain a functor $\mathcal{C} \to \mathrm{Desc}_{\mathcal{C}}(\mathcal{A})$. The classical theorem of faithfully flat descent holds in this general situation, too.

Proposition A.7 (Barr-Beck theorem). *Let $\mathcal{C}$ be a tensor category and $\mathcal{A}$ be a faithfully flat ring in $\mathcal{C}$. Then the functor $\mathcal{C} \to \mathrm{Desc}_{\mathcal{C}}(\mathcal{A})$ is an equivalence of categories.*

A proof can be found in [**11**] 4.1.

Our aim now is to show that faithfully flat descent preserves dualizable objects. Recall that an object $X$ of a tensor category $\mathcal{C}$ is called dualizable if there exists an object $X^\vee$ in $\mathcal{C}$ and morphisms $\mathrm{ev} \colon X \otimes X^\vee \to 1$, $\mathrm{coev} \colon 1 \to X^\vee \otimes X$ such that the following diagrams are commutative:

$$\begin{array}{ccc} X & \xrightarrow{\cong} & 1 \otimes X \\ \downarrow{\scriptstyle\cong} & & \uparrow{\scriptstyle \mathrm{ev}\otimes \mathrm{id}} \\ X \otimes 1 & \xrightarrow{\mathrm{id}\otimes \mathrm{coev}} & X \otimes X^\vee \otimes X \end{array} \qquad \begin{array}{ccc} X^\vee & \xrightarrow{\cong} & X^\vee \otimes 1 \\ \downarrow{\scriptstyle\cong} & & \uparrow{\scriptstyle \mathrm{id}\otimes \mathrm{ev}} \\ 1 \otimes X^\vee & \xrightarrow{\mathrm{coev}\otimes \mathrm{id}} & X^\vee \otimes X \otimes X^\vee \end{array}$$

The dual object $X^\vee$ is unique up to isomorphism. Tensor functors commute with duals, i.e. the image of a dualizable object is again dualizable and the dual of the image is the image of the dual.

Example A.8.

(1) The dualizable objects in the category $\mathrm{Mod}(R)$ of $R$-modules are precisely the finitely generated projective modules. More generally, the dualizable quasi-coherent modules over a scheme are precisely the locally free modules of finite rank.
(2) Let $G$ be an affine group scheme over a field $k$. A representation of $G$ over $k$ is dualizable if and only if it is finite dimensional: if it is dualizable, then the underlying vector space is a dualizable $k$-vector space, i.e. it is finite dimensional. Conversely, if the representation is finite dimensional, then it can be checked that the dual representation is the dual (in the above sense).

Let $\mathcal{C}$ be a tensor category, $\mathcal{A}$ be a ring and $X$ be a dualizable object in $\mathcal{C}$. As the base change functor $\mathcal{A} \otimes -$ is symmetric monoidal, we know that the $\mathcal{A}$-module $\mathcal{A} \otimes X$ is also dualizable. We show that the converse is also true if the ring $\mathcal{A}$ is faithfully flat.

Proposition A.9. *Let $\mathcal{C}$ be a tensor category, $\mathcal{A}$ be a faithfully flat ring in $\mathcal{C}$ and $X$ be an object of $\mathcal{C}$. If the $\mathcal{A}$-module $\mathcal{A} \otimes X$ is dualizable in $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$, then $X$ is dualizable in $\mathcal{C}$.*

Proof. Let $\mathcal{N}$ be the dual of $\mathcal{M} = \mathcal{A} \otimes X$ in $\mathrm{Mod}_{\mathcal{C}}(\mathcal{A})$. We first show that there is a descent datum on $\mathcal{N}$: the $\mathcal{A} \otimes \mathcal{A}$-module $\mathcal{A} \otimes \mathcal{M}$ is isomorphic to $(\mathcal{A} \otimes \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$, hence it has a dual in the category of $\mathcal{A} \otimes \mathcal{A}$-modules, namely $\mathcal{A} \otimes \mathcal{N} \cong (\mathcal{A} \otimes \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{N}$. The same can be said of $\mathcal{M} \otimes \mathcal{A}$. Therefore we can dualize the isomorphism giving the descent datum on $\mathcal{M}$ and obtain an isomorphism

$$\mathcal{A} \otimes \mathcal{N} \cong (\mathcal{A} \otimes \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{N} \cong \mathcal{N} \otimes_{\mathcal{A}} (\mathcal{A} \otimes \mathcal{A}) \cong \mathcal{N} \otimes \mathcal{A}.$$

We can also dualize the cocycle condition and hence we see that $\mathcal{N}$ has a descent datum. By the Barr-Beck theorem, $\mathcal{N}$ is of the form $\mathcal{A} \otimes Y$ for an object $Y$ of $\mathcal{C}$. The evaluation and coevaluation maps of $\mathcal{M}$ and $\mathcal{N}$ commute with the descent data, hence there are evaluation and coevaluation maps for $X$ and $Y$. These maps satisfy the triangle identities since they satisfy it after the base change to the faithfully flat ring $\mathcal{A}$. □

## 4. The Ind-category of a Tannakian category

Let $k$ be a field and $(\mathcal{T}, \omega)$ be a Tannakian category over $k$, that is, $\mathcal{T}$ is a rigid $k$-linear abelian symmetric monoidal category with $\mathrm{End}_{\mathcal{T}}(1) \cong k$ and

$$\omega\colon \mathcal{T} \to \mathrm{QCoh}(S)$$

is a $k$-linear faithful exact symmetric monoidal functor, where $S \neq \emptyset$ is a $k$-scheme. The functor $\omega$ is called an $S$-valued fibre functor on $\mathcal{T}$. We say that a Tannakian category $(\mathcal{T}, \omega)$ is neutral if $\omega$ is $\mathrm{Spec}(k)$-valued.

Tannakian categories have the following properties:

(1) the fibre functor $\omega$ factors through the category $\mathrm{LF}(S)$ of finite locally free $\mathcal{O}_S$-modules,

(2) if $f\colon S' \to S$ is a morphism of schemes, then the composition

$$f^* \circ \omega\colon \mathcal{T} \to \mathrm{QCoh}(S) \to \mathrm{QCoh}(S')$$

is an $S'$-valued fibre functor,

(3) if $\omega'\colon \mathcal{T} \to \mathrm{QCoh}(S)$ is an other $S$-valued fibre functor, then any functor homomorphism $\omega \to \omega'$ is an isomorphism,

(4) every object $X$ of $\mathcal{T}$ has finite length, in particular, every object is noetherian.

The main theorem of neutral Tannakian categories is the following.

Proposition A.10 ([**11**],Theorem 2.11). *Let $(\mathcal{T}, \omega)$ be a neutral Tannakian category over a field $k$. Then the automorphism group functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is representable by an affine group scheme $G$ over $k$ and $\omega$ induces an equivalence of categories $\mathcal{C} \cong \mathrm{Repf}_k(G)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\cong} & \mathrm{Repf}_k(G) \\ & {\scriptstyle\omega}\searrow \quad \swarrow{\scriptstyle\omega} & \\ & \mathrm{Vecf}(k) & \end{array}$$

The representing affine group scheme is called the Tannakian fundamental group of the neutral Tannakian category $(\mathcal{T}, \omega)$.

Proposition A.11 ([**11**] Prop. 2.20., [**29**] II.4.3.2.). *Let $(\mathcal{T}, \omega)$ be a neutral Tannakian category over a field $k$ and let $G$ be its Tannakian fundamental group.*

(1) *$G$ is of finite type over $k$ if and only if the category $\mathcal{T}$ has a tensor generator, i.e. an object $X$ such that every object of $\mathcal{T}$ is isomorphic to a subquotient of $X^{\oplus n} \otimes (X^{\vee})^{\oplus m}$ for some $n, m \geq 0$. In this case, we have a closed immersion $G \to \mathrm{GL}(\omega(X))$, where $X$ is a tensor generator.*

(2) *$G$ is finite over $k$ if and only if $\mathcal{T}$ has a generator, i.e. an object $X$ such that every object of $\mathcal{T}$ is isomorphic to a subquotient of $X^{\oplus n}$ for some $n \geq 0$.*

The full subcategory consisting of objects isomorphic to a subquotient of $X^{\oplus n} \otimes (X^{\vee})^{\oplus m}$ for some $n, m \geq 0$ for a fixed object $X$ of $\mathcal{T}$ will be denoted by $\langle X \rangle_{\otimes}$. Similarly, the full subcategory consisting of objects isomorphic to a subquotient of $X^{\oplus n}$ for some $n \geq 0$ will be denoted by $\langle X \rangle_{\oplus}$.

PROPOSITION A.12 ([**11**] Prop. 2.21., [**29**] II.4.3.2.). *Let $(\mathcal{T}, \omega)$ and $(\mathcal{T}', \omega')$ be neutral Tannakian categories over a field k with Tannakian fundamental groups $G$ and $G'$. There is a bijection between the set of symmetric monoidal functors $F \colon \mathcal{T} \to \mathcal{T}'$ such that $\omega' \circ F = \omega$ and the set of homomorphisms $f \colon G' \to G$ of affine $k$-group schemes.*

*Moreover, we have the following properties:*

(1) *The homomorphism $f$ is a closed immersion if and only if every object $X'$ of $\mathcal{T}'$ is the subquotient of an object of the form $F(X)$ for some $X$ in $\mathcal{T}$.*
(2) *The homomorphism $f$ is faithfully flat if and only if $F$ is fully faithful and $F$ induces an equivalence*
$$\langle X \rangle_{\otimes} \to \langle F(X) \rangle_{\otimes}$$
*for every object $X$ of $\mathcal{T}$.*

Our aim is to show that the indization a Tannakian category is a tensor category with a scheme-valued point. We state the result now, but postpone the proof until later.

PROPOSITION A.13. *Let $(\mathcal{T}, \omega \colon \mathcal{T} \to \mathrm{QCoh}(S))$ be a Tannakian category over a field $k$. Then the category $\mathrm{Ind}(\mathcal{T})$ is a cocomplete $k$-linear abelian symmetric monoidal category and the functor $J(\omega) \colon \mathrm{Ind}(\mathcal{T}) \to \mathrm{QCoh}(S)$ is a cocontinuous $k$-linear faithful exact symmetric monoidal functor, i.e. $\mathrm{Ind}(\mathcal{T})$ is a tensor category and $J(\omega)$ is an $S$-valued point of this category.*

Before we can prove this, we first need to recollect the basics of Ind-categories and the process of indization. For a complete treatment, see [**1**] Exp.I. §8. or [**19**] Chapter 6 and Section 8.6.

Let $\mathcal{C}$ be a category. An ind-object of $\mathcal{C}$ is a covariant functor $\alpha \colon \mathcal{I} \to \mathcal{C}$ (i.e. an inductive system), where $\mathcal{I}$ is a small filtered category. We will denote an ind-object corresponding to a small filtered inductive system $\alpha$ by $(C_\alpha)$. The category $\mathrm{Ind}(\mathcal{C})$ of ind-objects is the category consisting of ind-objects of $\mathcal{C}$ as objects and where the morphisms from an ind-object $(C_\alpha)$ to an ind-object $(D_\beta)$ are given by
$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}((C_\alpha), (D_\beta)) = \varprojlim_{\alpha} \varinjlim_{\beta} \mathrm{Hom}_{\mathcal{C}}(C_\alpha, D_\beta).$$

We have a natural functor $\iota_{\mathcal{C}}$ from $\mathcal{C}$ to $\mathrm{Ind}(\mathcal{C})$, where we map an object $C$ of $\mathcal{C}$ to the small filtered inductive system $(C)$ consisting only of $C$.

PROPOSITION A.14 ([**19**] Cor. 6.1.6., Thm. 6.1.8., Prop. 6.1.9., Prop. 6.1.10., Prop. 6.1.11.,Prop. 6.1.12.). *Let $\mathcal{C}$ be a category.*

(1) *The category* $\mathrm{Ind}(\mathcal{C})$ *admits small filtered colimits.*
(2) *The functor* $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ *commutes with finite colimits that exist in* $\mathcal{C}$.
(3) *Let* $F \colon \mathcal{C} \to \mathcal{D}$ *be a functor. Then there exists a unique functor*

$$\mathrm{Ind}(F) \colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$$

*such that* $\mathrm{Ind}(F)$ *commutes with small filtered colimits and the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle \iota_{\mathcal{C}}} & & \downarrow{\scriptstyle \iota_{\mathcal{D}}} \\ \mathrm{Ind}(\mathcal{C}) & \xrightarrow{\mathrm{Ind}(F)} & \mathrm{Ind}(\mathcal{D}) \end{array}$$

(4) *If the functor* $F$ *is faithful (resp. fully faithful), then* $\mathrm{Ind}(F)$ *is faithful (resp. fully faithful).*
(5) *Let* $G \colon \mathcal{D} \to \mathcal{D}'$ *be a functor. Then* $\mathrm{Ind}(G \circ F)$ *is isomorphic to the composite functor* $\mathrm{Ind}(G) \circ \mathrm{Ind}(F)$.
(6) *The projection functors from* $\mathcal{C} \times \mathcal{D}$ *to the factors induce a natural equivalence of categories* $\mathrm{Ind}(\mathcal{C} \times \mathcal{D}) \cong \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{D})$.

In case when a category $\mathcal{C}$ admits small filtered colimits, we have a left adjoint to the inclusion functor.

PROPOSITION A.15 ([**19**] Prop. 6.3.1.). *Let* $\mathcal{D}$ *be a category admitting small filtered colimits. Then the inclusion functor* $\iota_{\mathcal{D}} \colon \mathcal{D} \to \mathrm{Ind}(\mathcal{D})$ *has a left adjoint* $\sigma_{\mathcal{D}} \colon \mathrm{Ind}(\mathcal{D}) \to \mathcal{D}$ *such that* $\sigma_{\mathcal{D}} \circ \iota_{\mathcal{D}} \cong \mathrm{id}_{\mathcal{D}}$ *and if* $(D_\alpha)$ *is an ind-object, then* $\sigma_{\mathcal{D}}((D_\alpha)) \cong \varinjlim D_\alpha$.

Let now $F \colon \mathcal{C} \to \mathcal{D}$ be a functor, where the category $\mathcal{D}$ admits all small filtered colimits. Using the left adjoint $\sigma_{\mathcal{D}}$, we can define the functor $J(F) \colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ as the composite functor $\sigma_{\mathcal{D}} \circ \mathrm{Ind}(F)$. The functor $J(F)$ commutes with all small filtered colimits and $J(F) \circ \iota_{\mathcal{C}}$ is isomorphic to the original functor $F$ (c.f. [**19**] Cor. 6.3.2.).

Let $\mathcal{C}$ be a category admitting small filtered colimits. An object $C$ of $\mathcal{C}$ is called of finite presentation if for any filtered inductive system $\alpha \colon I \to \mathcal{C}$, the natural map

$$\varinjlim \mathrm{Hom}_{\mathcal{C}}(C, \alpha_i) \to \mathrm{Hom}_{\mathcal{C}}(C, \varinjlim \alpha_i)$$

is an isomorphism.

PROPOSITION A.16. *Let* $\mathcal{C}$ *be a category and* $\mathcal{D}$ *be a category that admits all small filtered colimits. Let* $F \colon \mathcal{C} \to \mathcal{D}$ *be a faithful (resp. fully faithful) functor such that* $F(X)$ *is of finite presentation for every object* $X$ *of* $\mathcal{C}$. *Then* $J(F) \colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ *is a faithful (resp. fully faithful) functor, too.*

A proof in the case when $F$ is fully faithful can be found in [**19**] Prop. 6.3.4., but it can be applied to the faithful case as well: we only need that taking small filtered limits (resp.

colimits) is a left exact (resp. exact) functor in the category of sets or in the category of modules over a ring, hence it takes monomorphisms to monomorphisms.

What structures does the ind-category inherit from the original category? Naturally, we are interested in the inheritance of $k$-linearity (for some commutative ring $k$), symmetric monoidal structure and abelianness.

Using the explicit description of Hom-sets in the ind-category, we see that $\mathrm{Ind}(\mathcal{C})$ is a $k$-linear category if $\mathcal{C}$ is and the functor $\iota_{\mathcal{C}}$ becomes a $k$-linear functor. One can argue similarly for the indization of a $k$-linear category.

The tensor product on $\mathcal{C}$ can be extended to a tensor product on $\mathrm{Ind}(\mathcal{C})$ using the natural equivalence of categories $\mathrm{Ind}(\mathcal{C} \times \mathcal{C}) \cong \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C})$. More explicitly, the tensor product of two ind-objects $(C_\alpha)$ and $(D_\beta)$ is just the inductive system $(C_\alpha \otimes D_\beta)$ on $I \times J$ (we note that the product of small filtered categories is again small and filtered). The unit object is just the constant ind-object $(1)$. To check that the necessary diagrams are commutative, one can use the following result which says that finite loopless diagrams in $\mathrm{Ind}(\mathcal{C})$ are essentially small filtered systems of diagrams in $\mathcal{C}$.

The explicit description of objects also shows that the functor $\iota_{\mathcal{C}}$ is symmetric monoidal. One can reason similarly to show that the indization of a symmetric monoidal functor is symmetric monoidal.

For abelian categories, we have the following:

Proposition A.17 ([**19**] Thm. 8.6.5., Cor. 8.6.8., Prop. 8.6.11.). *Let $\mathcal{C}$ be an abelian category.*

(1) *The category $\mathrm{Ind}(\mathcal{C})$ is an abelian category and the natural functor $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is fully faithful and exact.*
(2) *The category $\mathrm{Ind}(\mathcal{C})$ is cocomplete (i.e. admits all small colimits) and taking small filtered colimits is an exact functor.*
(3) *Let $F \colon \mathcal{C} \to \mathcal{D}$ be an additive functor of abelian categories. If $F$ is a right exact (resp. left exact) functor, then $\mathrm{Ind}(F)$ is a right exact (resp. left exact) functor, too.*
(4) *The category $\mathcal{C}$ is closed by kernels, cokernels and extension in $\mathrm{Ind}(\mathcal{C})$ (i.e. the kernel in $\mathrm{Ind}(\mathcal{C})$ of any morphism of $\mathcal{C}$ is isomorphic to an object of $\mathcal{C}$, and similarly for the others).*

In conclusion, we get that the Ind-category of a $k$-linear abelian symmetric monoidal category is a tensor category. Moreover, the indization of a $k$-linear (faithful exact) symmetric monoidal functor is a (faithful exact) tensor functor.

Let $\mathcal{D}$ be a cocomplete $k$-linear abelian symmetric monoidal category. We know that the inclusion functor $\iota_{\mathcal{D}} \colon \mathcal{D} \to \mathrm{Ind}(\mathcal{D})$ has a left adjoint $\sigma_{\mathcal{D}}$, that maps a small filtered inductive system $(X_i)$ to the colimit $\varinjlim X_i$. We shall need the properties of this functor.

Proposition A.18. *Let $\mathcal{D}$ be a tensor category. Then $\sigma_{\mathcal{D}} \colon \mathrm{Ind}(\mathcal{D}) \to \mathcal{D}$ is a right exact tensor functor. If small filtered colimits are exact in $\mathcal{D}$, then $\sigma_{\mathcal{D}}$ is an exact functor.*

Since $\sigma_{\mathcal{D}}$ is a left adjoint, it is cocontinuous and right exact. The explicit description of $\sigma_{\mathcal{D}}$ shows that it is $k$-linear and symmetric monoidal (we use here the assumption on $\mathcal{D}$ that the tensor product of $\mathcal{D}$ commutes with colimits). The explicit description of $\sigma_{\mathcal{D}}$ shows that it is exact if small filtered colimits are exact in $\mathcal{D}$.

We can now prove Proposition A.13.

Proof of Proposition A.13. We have the following diagram:

$$\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\quad\omega\quad} & \mathrm{QCoh}(S) \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\
\mathrm{Ind}(\mathcal{T}) & \xrightarrow{\mathrm{Ind}(\omega)} & \mathrm{Ind}(QCoh(S)) \\
& \overset{J(\omega)}{\searrow} & \downarrow{\scriptstyle \sigma} \\
& & \mathrm{QCoh}(S)
\end{array}$$

We have seen that the indization of a $k$-linear abelian symmetric monoidal category is a tensor category, hence $\mathrm{Ind}(\mathcal{T})$ is a tensor category. Moreover, the indization of a $k$-linear faithful exact symmetric monoidal functor is a faithful exact tensor functor, hence $\mathrm{Ind}(\omega)$ is a faithful exact tensor functor. By definition, $J(\omega)$ is the composite of $\sigma$ and $\mathrm{Ind}(\omega)$. By Proposition A.18., we get almost every required property, except the faithfulness of $J(\omega)$. But this will follow from Proposition A.16. if we can show $\omega(X)$ is of finite presentation (in the categorical sense) for every object $X$ of $\mathcal{T}$. We know that $\omega(X)$ is a locally free sheaf of finite rank, hence it is of finite presentation (in the algebro-geometrical sense).

Thus we need to prove that for every scheme $S$, a quasi-coherent $\mathcal{O}_S$-module $\mathcal{M}$ of finite presentation is a finitely presented object in the category of quasi-coherent modules, i.e. the natural map

$$\varinjlim \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{M}, \alpha_i) \to \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{M}, \varinjlim \alpha_i)$$

is an isomorphism for every small filtered inductive system $(\alpha_i)$. This map is the global section of the map of sheaves

$$\varinjlim \mathcal{H}om_{\mathcal{O}_S}(\mathcal{M}, \alpha_i) \to \mathcal{H}om_{\mathcal{O}_S}(\mathcal{M}, \varinjlim \alpha_i),$$

therefore it is enough to prove that it is an isomorphism of sheaves. But we can check this locally, meaning that it is sufficient to prove that a finitely presented module over a ring is finitely presented in the categorical sense, which is an easy calculation. □

## 5. Embedding the Ind-category of a neutral Tannakian category

Suppose now a neutral Tannakian category $(\mathcal{T}, \omega)$ is contained (as an essentially full $k$-linear abelian symmetric monoidal subcategory) in a tensor category $\mathcal{C}$. As explained before Proposition A.16, the embedding functor $i : \mathcal{T} \to \mathcal{C}$ extends to a functor $J(i)\colon \mathrm{Ind}(\mathcal{T}) \to \mathcal{C}$.

PROPOSITION A.19. *In the above situation assume moreover that $\mathcal{T}$ is closed under subquotients in $\mathcal{C}$.*

*Then the functor $J(i)\colon \mathrm{Ind}(\mathcal{T}) \to \mathcal{C}$ is a fully faithful exact $k$-linear tensor functor and* $\mathrm{Ind}(\mathcal{T})$ *is closed under subquotients $\mathcal{C}$.*

The proof is based on [**10**], Lemme 4.2.2.

PROOF. Results from the previous section show that $J(i)$ is a exact tensor functor (exactness follows from the fact that small filtered colimits in $\mathcal{C}$ are exact).

An object of $\mathcal{T}$ is noetherian in $\mathcal{T}$ (using the fibre functor), but it is also noetherian considered as an object of $\mathcal{C}$, since we assumed that $\mathcal{T}$ is closed under subquotients in $\mathcal{C}$.

We claim that

(1) every small filtered colimit $\varinjlim X_\alpha$ of objects of $\mathcal{T}$ is the colimit of an ind-object of $\mathcal{T}$ whose transition morphisms are monomorphisms,
(2) if $(X_\alpha)$ is an ind-object of $\mathcal{T}$ whose transition morphisms are monomorphisms, then the natural map $X_\alpha \to \varinjlim X_\alpha$ is a monomorphism in $\mathcal{C}$.

Let $(X_\alpha)$ be an ind-object of $\mathcal{T}$. For a fixed $\alpha$, the objects $\mathrm{kernel}(X_\alpha \to X_\beta)$ $(\alpha < \beta)$ form an increasing system of subobjects in $X_\alpha$, hence it stabilizes. Denote by $K_\alpha$ the largest one. Then $\varinjlim X_\alpha \cong \varinjlim X_\alpha/K_\alpha$ and the transition morphisms of the ind-object $(X_\alpha/K_\alpha)$ are monomorphisms.

The natural morphism $X_\alpha \to \varinjlim X_\alpha$ is the filtered colimit of the morphisms $X_\alpha \to X_\beta$ $(\alpha < \beta)$, since filtered colimits in $\mathcal{C}$ are exact, it is a monomorphism.

Using these two properties, we can show that the Ind-category is closed under subquotients in $\mathcal{C}$: first, let $Z$ be a subobject of $\varinjlim X_\alpha$ in $\mathcal{C}$. We can assume that the transition morphisms of $(X_\alpha)$ are monomorphisms and that $X_\alpha$ is a subobject of $\varinjlim X_\alpha$. Consider the exact sequence

$$0 \to Z \cap X_\alpha \to X_\alpha \to (\varinjlim X_\alpha)/Z.$$

Taking the colimit and using that filtered colimits are exact in $\mathcal{C}$, we get that $Z$ is the filtered colimit of its subobjects $Z \cap X_\alpha$. Since $\mathcal{T}$ is closed under subquotients, we have that $Z \cap X_\alpha$ is in $\mathcal{T}$ and hence $Z$ is in $\mathrm{Ind}(\mathcal{T})$.

If $Y$ is now a quotient of $\varinjlim X_\alpha$, then we just saw that the kernel $Z$ of the quotient map comes from $\mathrm{Ind}(\mathcal{T})$ and we can write $Y$ as the filtered colimit of $X_\alpha/Z_\alpha$.

Let again $(X_\alpha)$ be an ind-object whose transition morphisms are monomorphisms. We can adapt the previous proof to show that the subobjects $X_\alpha$ in $\varinjlim X_\alpha$ are cofinal in the directed set of $\mathcal{T}$-subobjects of $\varinjlim X_\alpha$: let $Z \in \mathcal{T}$ be a subobject of $\varinjlim X_\alpha$ and consider again the exact sequence

$$0 \to Z \cap X_\alpha \to X_\alpha \to (\varinjlim X_\alpha)/Z.$$

Taking colimit again, we see that $Z$ is the filtered colimit of the subobjects $Z \cap X_\alpha$, but $Z$ is noetherian, hence the subobjects must stabilize at some $\alpha$: $Z \subseteq X_\alpha$. In particular, this implies that $\varinjlim X_\alpha$ is the filtered colimit of its $\mathcal{T}$-subobjects.

Let now $f\colon \varinjlim X_\alpha \to \varinjlim Y_\beta$ be a morphism in $\mathcal{C}$. For every $\mathcal{T}$-subobject $Z$ of $\varinjlim X_\alpha$, the morphism $f$ maps $Z$ to a $\mathcal{T}$-subobject of $\varinjlim Y_\beta$. Since $\varinjlim X_\alpha$ is the filtered colimit of its $\mathcal{T}$-subobjects, the previous restrictions induce a morphism $\varinjlim X_\alpha \to \varinjlim Y_\beta$, which is just the original morphism, hence $J(i)$ is full.

For faithfulness, let $f\colon (X_\alpha)_{\alpha \in I} \to (Y_\beta)_{\beta \in I'}$ be a morphism of ind-objects whose colimit is 0. We can assume that the indexing set is the same ([**19**] Prop. 6.1.13.) and we can also assume that the transition morphisms are monomorphisms for both ind-objects, moreover, there exists a small filtered indexing category $I''$ and cofinal functors $I'' \to I$ and $I'' \to I'$ such that the morphism $f$ is $\varinjlim_{\alpha \in I''} f_\alpha$. Thus we have a commutative diagram, where the vertical morphisms are monomorphisms

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \\ \big\downarrow & & \big\downarrow \\ \varinjlim X_\alpha & \xrightarrow{0} & \varinjlim Y_\beta \end{array}$$

and we can deduce that $f_\alpha$ has to be 0 and thus $f$ is 0. □

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