CENTRAL EUROPEAN UNIVERSITY

Introduction to Khovanov homology

by

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Abstract

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In this thesis, we study the definition of Khovanov homology. Before doing that, we describe some basic information about knot theory and homological algebra, which we need in the definition of Khovanov homology. We also describe the definition of the unnormalized Jones polynomial of a knot or link and then extend this construction (based on the paper of Bar Natan) to the definition of Khovanov homology. We describe detailed definition of Khovanov homology, as a bigraded vector space invariant of a link, which has the unnormalized Jones polynomial as its Euler characteristic. Moreover, we study slice genus via Khovanov homology. We study new knot invariant to study slice genus and prove main theorems about slice genus. At the last chapter, we study relation of Khovanov homology and topological quantum field theory.

Acknowledgements

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Contents

Chapter 1

Khovanov homology

In this chapter we define Khovanov homology and we show that it is independent under the Reidemeister moves. To define it, we would first introduce some basic concepts such as knot theory, Jones polynomial and some information about homological algebra. We start with knot theory.

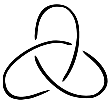
1.1 Short introduction to knot theory

Knot and link diagrams will be crucial for our Khovanov homology definition, so to understand the definition of the Khovanov homology we need some basic definitions and facts about knots and links.

Definition 1.1. A Knot is an embedding of a circle S^1 in 3-dimensional Euclidean space or in S^3.

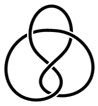
If we link or tangle more than one knot, we call it link. Generally, we are interested in regular projection of knots (links) onto the 2-dimensional Euclidean space, meaning that the projection is injective everywhere except finitely many points, called crossing points, where the knot projection crosses itself once. To understand the knot (link), we need to clarify crossing points, so the over-strand must be distinguished from the under-strand. We can do it with creating break in the strand going underneath. The resulting diagram is a knot (link) projection with the additional data about crossing points. We call this diagram knot (link) diagram. To become familiar with knots and links, in following we give some depictions of most well-known knots and link.

right-handed trefoil

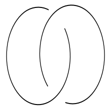


left-handed trefoil

Figure eight knot



Hopf link



Generally, when we study on knots or links, we have an orientation on them and orientation of a knot or a link is crucial to study them. Now we have some basic definitions and notions about knots.

Definition 1.2. The reverse rK of an oriented knot K is simply the same knot with the opposite orientation.

Definition 1.3. The mirror-image $m(K)$ of a knot K is obtained by reflecting it in a plane in R^3.

To obtain the diagram of the mirror image of knot K we change all the crossing points of diagram of K. For example, right-handed and left-handed trefoil are mirror image of each other. Crossing points are significant for our knot definition. We can have two kind of crossing point either one segment pass under of other segment or vice versa. Based on these positions, we assign sign to crossing points so either we have positive or negative crossing point. We can see definition of positive and negative crossing point at following figures.

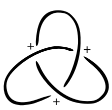
$--->$ positive crossing

$--->$ negative crossing

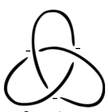
We denote n to be the number of crossings of knot or link, n_+ to be the number of positive crossings and n_- to be the number of negative crossings.

Definition 1.4. The writhe $w(D)$ of a diagram D of an oriented knot or link is the number of total number of positive crossings minus total number of negative crossings i.e $w(D) = n_+ - n_-$.

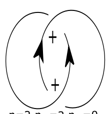
We can see these numbers on some different knots and links as in the example below.



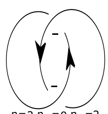
n=3 n_+=3 n_-=0



n=3 n_+=0 n_-=3



n=2 n_+=2 n_-=0



n=2 n_+=0 n_-=2

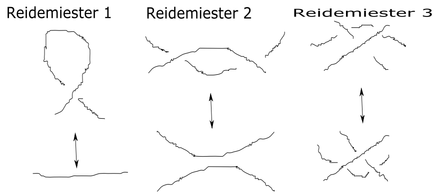
Equivalence between objects is widely used in mathematics. We generally use isomorphism class to study objects . For instance, we call two groups are the equivalent when we have isomorphism between them. Similarly, we have equivalences between knots. Although we have some different kind of equivalence between knots, in this paper we use only one of them.

Definition 1.5. Two knots K_1 and K_2 are ambient isotopic, if there is a smooth map $F : S^3x[0,1] \to S^3$ such that $F_x = F_{|S^3x\{x\}}$ is diffeomorphism for each $x \in [0,1]$, $F_0 = id$ and $F_1(K_1) = K_2$.

Since we are interested in knot diagrams more than knot itself, we need to understand by just studying on knot diagrams whether two given knots are equivalent or not. Before goint to see theorem about this, we need some definitions which we will use at the theorem.

Definition 1.6. Planar isotopy of knot projection is continuous deformation of projection plane.

We have three local moves on link diagram which are the crucial for the following theorem. We call these moves as Reidemeister moves and they can be seen in the figure below.



In order to investigate the equivalence in links the following theorem will be used.

Theorem 1.7. *We call two links are ambient isotopic in S^3 if and only if they are related by the finite number of Reidemeister moves and planar isotopy.*

Definition 1.8. A knot invariant is a property of a knot diagram that does not change under Reidemeister moves, hence it depends only on the knot (and not the chosen diagram).

For instance, the writhe number is not a knot invariant whereas we will see that the Jones polynomial and Khovanov homology will be knot invariants .

1.2 The Jones Polynomial

In this section, we define the Jones polynomial which is one of the famous knot polynomials. For defining the Jones polynomial we need to define bracket polynomial.

Definition 1.9. The Kauffman bracket is a function from unoriented link diagrams in the oriented plane to Laurent polynomial with integer coefficients in q. It maps a diagram D to $\langle D\rangle \in \mathbb{Z}[q,q^{-1}]$ and it is characterised by

1. $\langle \emptyset \rangle = 1$
2. $\langle D \sqcup \bigcirc \rangle = (q^{-1}+q)\langle D\rangle$
3. $\langle \times \rangle = \langle \asymp \rangle - q\langle)\,(\rangle$

In this definition , D is a diagram , $\emptyset$ is empty diagram , $\langle D\rangle$ is polynomial of the knot and the circle is the unknot without any crossings.

Now we can check what happens to this bracket definition when we apply Reidemeister moves . We start with $(R1)$.

$$\langle \gamma \rangle = \langle \overset{\smile}{\bigcirc} \rangle - q\langle \smile \rangle = (q+q^{-1})\langle \smile \rangle - q\langle \smile \rangle = q^{-1}\langle \smile \rangle$$

$$\langle\ \rangle = \langle\ \rangle - q\langle\ \rangle = \langle\ \rangle - q(q+q^{-1})\langle\ \rangle = -q^2\langle\ \rangle$$

For ($R2$) we have

$$\langle\ \rangle = \langle\ \rangle - q\langle\ \rangle - q\langle\ \rangle + q^2\langle\ \rangle = \langle\ \rangle - q(q+q^{-1})\langle\ \rangle + q^2\langle\ \rangle - q\langle\ \rangle = -q\langle\ \rangle$$

.

For (R3) we show.
$\langle\ \rangle = \langle\ \rangle$
We have
$\langle\ \rangle = \langle\ \rangle$ -q $\langle\ \rangle$ and $\langle\ \rangle = \langle\ \rangle$ -q$\langle\ \rangle$
If we apply (R2) to the $\langle\ \rangle$ and $\langle\ \rangle$ we get $\langle\ \rangle$ so we can say that $\langle\ \rangle = \langle\ \rangle$.
Regarding $-q$ $\langle\ \rangle$ and -q$\langle\ \rangle$, since we can create planar isotopy between these two we can say -q $\langle\ \rangle$ = -q$\langle\ \rangle$. That gives us with the following equality.
$\langle\ \rangle = \langle\ \rangle$
As a result, we see that our bracket definition is not knot invariant under Reidemeister moves but actually it is almost knot invariant. In order to make this definition knot invariant, we have to get rid of q^{-1} and q^2. So, multiplying $\langle D\rangle$ with $(-1)^{n}_{-}q^{n_+-2n_-}$ cancels out these factors and the remaining polynomial becomes a knot invariant. Now let us write this definition properly.

Definition 1.10. The unnormalized Jones polynomial of link is

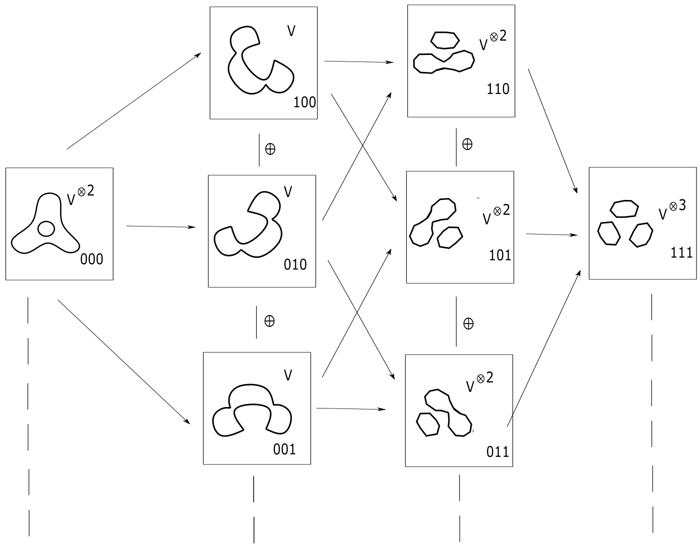
$$\hat{J}(D) = (-1)^{n-}q^{n_+-2n_-}\langle D\rangle$$

For this paper we need the unnormalized Jones polynomial, but we also have the normalized Jones polynomial $J(D) = \hat{J}(D)(q+q^{-1})^{-1}$. Now we defined the Jones polynomial but how can we find the Jones polynomial of any knot D ? We have 0-smoothing and 1-smoothing of . We have smoothings $\alpha \in \{0,1\}^n$, when we apply these different smoothings to our diagram, we have n-dimensional cube . We have just union of circles at the vertices of our cubes at the final stage. To compute the unnormalized Jones polynomial, we replace each union of k-circles with a term $(-1)^{r_\alpha}q^{r_\alpha}(q+q^{-1})^{k_\alpha}$, where r_α is the "height" of a α, the number of 1-smoothings, and k_α is the number of circles in smoothing α. We then sum all these terms over all $\alpha \in \{0,1\}^n$. In the end, we have the following formula.

$$\hat{J}(D) = \sum_{\alpha\in\{0,1\}^n}(-1)^{r_\alpha+n_-}q^{n_+-2n_-}q^{r_\alpha}(q+q^{-1})^{k_\alpha}$$

Let us do one example about finding the normalized Jones polynomial of trefoil knot :

Example 1.11. Let us find $J(\ )$ of (right-trefoil)

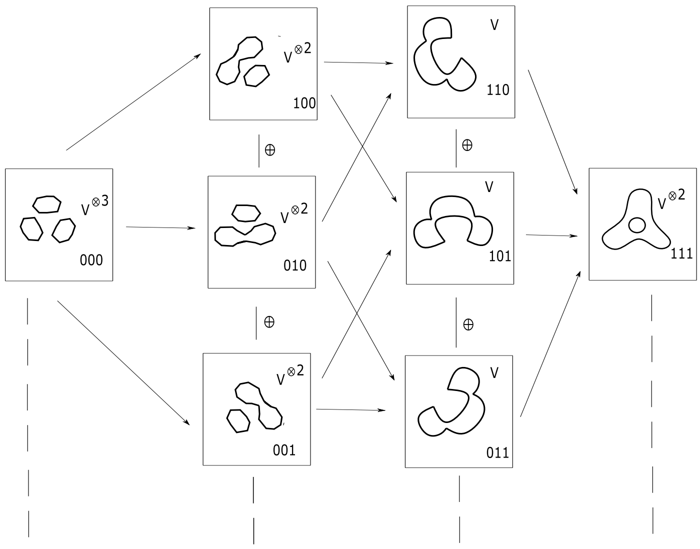


$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3$

$= \ q^{-2}+1+q^2-q^6 \ \rightarrow \hat{J}(\circledcirc) = \langle\circledcirc\rangle \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} = \ q+q^3+q^5-q^9$

$J(\circledcirc) = q+q^3+q^5-q^9 \ \xrightarrow{\cdot(q+q^{-1})^{-1}} \ J(\circledcirc) = q^2+q^6-q^8.$

Also it be good to see the Jones polynomial of the left-hand trefoil (mirror image of the right-hand trefoil). Let us see it:



$(q+q^{-1})^3 \quad - \quad 3q(q+q^{-1})^2 \quad + \quad 3q^2(q+q^{-1}) \quad - \quad q^3(q+q^{-1})^2$

$= (q+q^{-1})(q^{-2}-q^4-1) \rightarrow \hat{J}(\circledcirc) = \langle\circledcirc\rangle \xrightarrow[\text{(with } (n_+,n_-)=(0,3))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} = \ (q+q^{-1})(q^{-2}-q^4-1)q^{-6}$

$J(\circledcirc) = (q+q^{-1})(q^{-2}-q^4-1)q^{-6} \ \xrightarrow{\cdot(q+q^{-1})^{-1}} \ J(\circledcirc) = q^{-2}+q^{-6}-q^{-8}.$

As we know the right-handed and the left-handed trefoil are mirror image of each other and we found their Jones polynomials. It is easy to observe that if we put q^{-1} in the right-handed trefoil's Jones polynomial, we get the left-handed trefoil's Jones polynomial. Actually this is not coincidence we have the following lemma.

Lemma 1.12. *Suppose we have a knot K and its mirror image $m(K)$, then $J(K)(q) = J(m(K)(q^{-1})$*

Proof. See [1] Lemma 3.2. □

After these basic information about the Jones polynomial now we turn to homological algebra.

1.3 Introduction to homological algebra

Homological algebra is a branch of mathematics that helps to study topology with algebraic tools. In particular, we are using it in homology. Now we see some basic concepts of homological algebra which we use later . We begin with chain complexes.

Definition 1.13. A chain complex $(C_\bullet, d_\bullet)$ is a sequence of modules $\cdots C_{-2}, C_{-1}, C_0, C_1, C_2 \cdots$ connected by homomorphism $d_n : C_n \to C_{n-1}$ where $d_{n-1} \circ d_n = 0$. We call $(C'_\bullet, d_\bullet)$ subcomplex of $(C_\bullet, d_\bullet)$, if C'_i submodule of C_i and $d_n(C'_n) \subset C'_{n-1}$.

As all in other mathematical objects, we have maps between chain complexes.

Definition 1.14. Let $(C_\bullet, d_\bullet)$ and $(C'_\bullet, d'_\bullet)$ be chain complexes, a chain map $F : C_\bullet \to C'_\bullet$ is a sequence of map $\{F_n : C_n \to C'_n\}$ such that $F_{n-1} \circ d_n = d'_n \circ F_n$. In other words, the following diagram should be commutative.

$$\begin{tikzcd} C_n \arrow[r, "d_n"] \arrow[d, "F_n"'] & C_{n-1} \arrow[d, "F_{n-1}"] \\ C'_n \arrow[r, "d'_n"] & C'_{n-1} \end{tikzcd}$$

Chain complexes are crucial in defining and studying homology. The proposition given below make our proofs easy in this paper .

Proposition 1.15. *A Chain map $F : C_\bullet \to C'_\bullet$ induces a homomorphism between the homology groups of these two complexes.*

In mathematics generally, we have isomorphism between objects and we can define similar concepts here, we can define chain homotopy between two chain maps.

Definition 1.16. Let f and g be chain maps between $(C_\bullet, d_\bullet)$ and $(C'_\bullet, d'_\bullet)$. Chain homotopy ϕ between $f and\ g$ is a sequence of morphism $\phi_n : C_n \to C'_{n+1}$ such that $f_n - g_n = d'_{n+1} \circ \phi_n + \phi_{n-1} \circ d_n$. We call f and g are chain-homotopic chain maps and denote it $f \simeq g$.

We have equivalence between chain complexes also.

Definition 1.17. Suppose we have chain complexes A and B, we call chain complexes A and B are homotopy equivalent if and only if we have chain maps $f : (A_\bullet, d_\bullet) \to (B_\bullet, d'_\bullet)$ and $g : (B_\bullet, d'_\bullet) \to (A_\bullet, d_\bullet)$ such that $f \circ g \simeq id_{B_\bullet}$ and $g \circ f \simeq id_{A_\bullet}$.

Two chain homotopic maps induce homomorphism between homology of these chains. Do we have any relation between induced map f_* and g_* where chain maps are chain homotopic ? Next proposition shows us this relation.

Proposition 1.18. *If we have f and g chain-homotopic chain maps, their induced maps f_* and g_* are the same on homology groups (i.e $f_* = g_*$).*

Now we have one important theorem which is crucial in many proofs.

Definition 1.19. Suppose $M_1, M_2, \cdots, M_n$ are modules over the fixed ring R and $P_1, P_2, \cdots, P_n$ are module homomorphisms. We say that

$$M_1 \xrightarrow{P_1} M_2 \xrightarrow{P_2} M_3 \cdots \xrightarrow{P_{n-1}} M_n$$

is an exact sequence if $Im(P_{n-1}) = Ker(P_n)$.

Definition 1.20. Suppose A, B and C are modules over the fixed ring R we say that

$$0 \to A \xrightarrow{\text{i}} B \xrightarrow{\text{p}} C \to 0$$

is a short exact sequence, if i is monomorphism , p is epimorphism and $Im(i) = Ker(p)$.

In particular, we can define short exact sequence for chain complex category when we take A, B and C as chain complexes.

Definition 1.21. Suppose A, B and C are chain complexes and i and p are chain maps. We say that sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is short exact sequence if the induced sequence of maps

$$0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n \to 0$$

is a short exact sequence of modules.

In a similar way we can define long exact sequence for modules.

Definition 1.22. Long exact sequence is a exact sequence where we have infinitely many modules in sequence.

We can define exact sequence for cochain complexes similarly, we just need to replace chain complexes with cochain complexes. Now we can use these definitions to state the theorem.

Theorem 1.23. *Suppose A, B and C are chain complexes and we have short exact sequence of complexes given by:*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0,$$

then we obtain a long homology sequence of homology groups

$$\cdots H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \xrightarrow{\delta} \cdots$$

Proof. Alan Hatcher Algebraic Topology [2], Theorem 2.16 . □

Actually this theorem is also true for cochain complexes and the proof is similar to 1.23.

Theorem 1.24. *Suppose A, B and C are cochain complexes and we have a short exact sequence of complexes given by*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

then we can obtain long cohomology sequence of cohomology groups

$$\cdots H^n(A) \xrightarrow{i_*} H^n(B) \xrightarrow{j_*} H^n(C) \xrightarrow{\delta} H^{n+1}(A) \xrightarrow{i_*} H^{n+1}(B) \xrightarrow{j_*} H^{n+1}(C) \xrightarrow{\delta} \cdots$$

Now assume that we have two graded cochain complexes E and F can we write new cochain complex with using E and F ?

Definition 1.25. Let E and F be graded cochain complexes and let $E \xrightarrow{\mathrm{f}} F$ be a chain map that preserves gradings. The mapping cone is a chain complex given in a degree k by
$Cone(f)_k = E_k \bigoplus F_{k-1}$
with differential

$$\partial_{Cone(f)} = \begin{pmatrix} -\partial_E & 0 \\ f & \partial_F \end{pmatrix} : Cone(f)_k \to Cone(f)_{k+1}.$$

Now we can write the following lemma.

Lemma 1.26. *We have a short exact sequence which include $Cone(f)$*

$$0 \to F[1] \xrightarrow{i} Cone(f) \xrightarrow{p} E \to 0$$

where $F[1]_n = F_{n-1}$, $i(a) = (0, a)$ for $a \in F$ and $p(e', a') = -e'$, so we can get a long exact sequence by the theorem 1.24

$$\cdots \to H^d(E) \xrightarrow{H(f)} H^d(F) \xrightarrow{i_*} H^d(Cone(f)) \xrightarrow{p_*} H^{d+1}(E) \to \cdots$$

Now we see the definition of graded and bigraded vector spaces which is crucial for the definition of the Khovanov homology.

Definition 1.27. We say that the vector space V is a graded vector space, if V can be decomposed into the direct sum of the form $V = \bigoplus_{n \in \mathbb{N}} V_n$ where V_n is a vector space for any n. Elements of V_n are called homogeneous element of degree n.

The q dimension for this new vector space is $qdimV := \sum_m q^m dimV_m$. Note that $qdimV$ is a polynomial in q.

Example 1.28. Suppose we have field F and we have grading vector space $F_{-1} \bigoplus F_1$, then $qdim(F_{-1} \bigoplus F_1) = q + q^{-1}$.

Note that since $0 \in V_n$ for all n, we can say that it has any degree. Most famous example for the graded vector spaces is set of all polynomials in one variable. As we know we have some operations on vector spaces. We can define these operations on graded vector space too.

Definition 1.29. Given two graded vector space V and W, we can define their direct sum with grading

$$(V \bigoplus W)_i = V_i \bigoplus W_i$$

Similarly, we can define tensor product on graded vector space.

Definition 1.30. Given two graded vector space V and W, we can define their tensor product with grading

$$(V \otimes W)_n = \bigoplus_{k+l=n} V_k \otimes W_l$$

where $k, l, n \in \mathbb{N}$.

Proposition 1.31. *Given $\mathbb{Z}$graded vector spaces V and W we have three properties :*

1. $qdim(V \bigoplus W) = qdim(V) + qdim(W)$
2. $qdim(V \otimes W) = qdim(V).qdim(W)$
3. $qdim(V\{l\}) = q^l.qdim(V)$ *where* $V\{l\}_k = V_{k-l}$

Proof. 1.

$$\begin{aligned} qdim(V \bigoplus W) &= \sum_k q^k dim(V \bigoplus W)_k \\ &= \sum_k q^k dim(V_k \bigoplus W_k) \\ &= \sum_k q^k dim(V_k) + \sum_k q^k dim(W_k) \\ &= qdim(V) + qdim(W) \end{aligned}$$

2.

$$\begin{aligned} qdim(V\otimes W) &= \sum_k q^k dim(V\otimes W)_k \\ &= \sum_k q^k dim(\bigoplus_{j+l=k} V_j \otimes W_l) \\ &= \sum_k q^k \sum_{j+l=k} dim(V_j).dim(W_l) \\ &= \sum_k \sum_{j+l=k} q^j.q^l dim(V_j).dim(W_l) \\ &= \sum_j \sum_l q^j dim(V_j).q^l dim(W_l) \\ &= qdim(V).qdim(W) \end{aligned}$$

3.

$$\begin{aligned} qdim(V\{l\}) &= \sum_k q^k dim(V_{k-l}) \\ &= q^l \sum_k q^{k-l} dim(V_{k-l}) \\ &= q^l qdim(V) \end{aligned}$$

□

In this paper we generally use bigraded vector space. In the following, we see its definition.

Definition 1.32. A bigraded vector space V is a vector space that can be written as a direct sum of vector spaces which are indexed by $\mathbb{Z}\bigoplus\mathbb{Z}$:
$V = \bigoplus_{(i,j)\in\mathbb{Z}\bigoplus\mathbb{Z}} V_{i,j}$.

1.4 Definiton of Khovanov homology

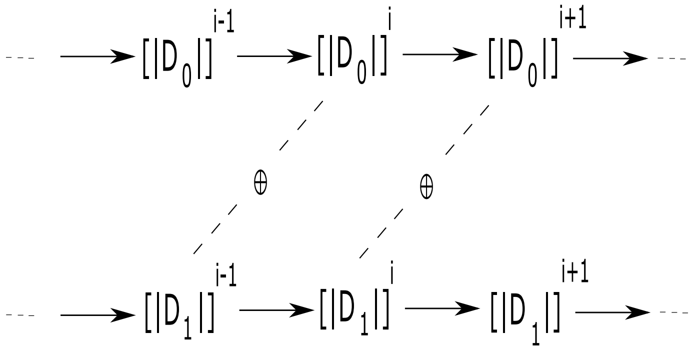
Remark 1.33. Although we call it Khovanov homology, actually it is cohomology .

Before going to the definition of the Khovanov homology, we define Khovanov bracket. The definition is similar to Kaufmann bracket definition. By using Kaufmann Bracket definition, we define the Jones polynomial and to define the Khovanov homology we will use the Khovanov Bracket.

Definition 1.34. Given an oriented link diagram L, Khovanov Bracket $[\![L]\!]$ is a chain complex of graded vector space and can be axiomatized by three axioms as follows.

1. $[\![\emptyset]\!] = 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$
2. $[\![\bigcirc \sqcup L]\!] = V \otimes [\![L]\!]$
3. $[\![\times]\!] = F\left(0 \to [\![\asymp]\!] \xrightarrow{d} [\![)(]\!]\{1\} \to 0\right)$

Here V is a graded vector space with q dimension $q + q^{-1}$, the $\{1\}$ operator is the degree shift operation which means in short $V\{l\}_m = V_{m-l}$. First axioms says that Khovanov bracket of empty diagram is complex which just has 0 and $\mathbb{Z}/2\mathbb{Z}$. Second axiom says that if we have diagram D which can be written as a disjoint sum of circle and diagram L then we just need to find diagram of L to find complex of diagram D. Third axiom is about flatten operation F which takes a double complex to single complex by taking direct sum along diagonals. More preciesly, third one says that if we have diagram D_1 and D_0 which can be obtained by respectively for 1 and 0 smoothing of ith crossing of D then we have $C^{i,*}(D) = C^{i,*}(D_0) + C^{i-1,*}(D_1)\{1\}$ and we can see this in the following figure.



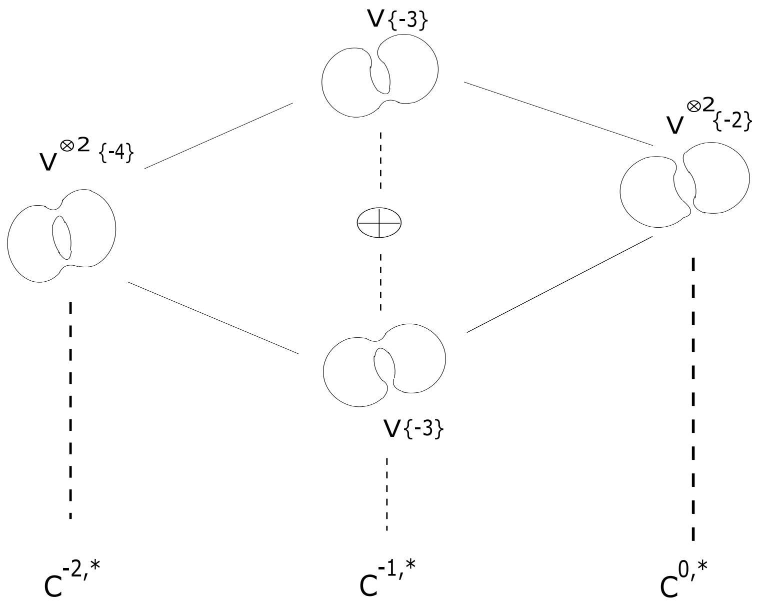
d^i will be defined later, which will be the boundary map in the Khovanov homology.

In our Khovanov homology definition, we define V as a two dimensional graded vector space generated by v_+ and v_- where deg $v_+ = 1$ and deg $v_- = -1$. Recall that we have 2^n smoothings and n crossing points in our diagram. We associate the graded vector space
$V_\alpha = V^{\otimes k_\alpha}\{r_\alpha + n_+ - 2n_-\}$ to each $\alpha \in \{0,1\}^n$ where,
k_α = the number of circles in the smoothing α,
r_α = the number of 1 in α,
n_+ = number of positive crossings in L,
n_- = number of negative crossings in L,
Now we can define our modules of our new chain complexes.

$$C^{i,*}(D) = \bigoplus_{\substack{\alpha\in\{0,1\}^n \\ i=r_\alpha-n_-}} V_\alpha$$

With this definition of $C^{i,*}(D)$ we have a chain complex. Now let us see one example. How can we get $C^{i,*}(D)$ when D is the Hopf link?

Example 1.35. For the Hopf link  with its orientation, we know $n_+ = 0$, $n_- = 2$.



As we can see $C^{-2,*} = V^{\otimes_2}\{-4\}$, $C^{-1,*} = V\{-3\} \oplus V\{-3\}$ and $C^{0,*} = V^{\otimes_2}\{-2\}$.
Later we will define boundary map from $C^{i,*}(D)$ to $C^{i+1,*}(D)$.
Now we talk about why we have $'*'$at $C^{i,*}(D)$. Let us put j instead of $'*'$ so we have $C^{i,j}(D)$. An element of $C^{i,j}(D)$ is said to have homological grading i and q-grading j. In other words, for $v \in V_\alpha \subset C^{i,j}(D)$ we say v has homological grading i and $q-$grading j where $i = r_\alpha - n_-$, $j = deg(v) + i + n_+ - n_-$.

Warning. Since not every $v \in C^{i,*}$ have degree we can say that not every $v \in C^{i,*}(D)$ have q-grading . For example, $v_- \oplus v_+ \in C^{i,*}$ does not have any q- grading. Only homogeneous elements have q-grading.

1.4.1 Definition of boundary map for Khovanov homology

In this section, we define boundary map and we check it is indeed boundary map. From now on for simplicity we study over the field F_2.
We define map d_ϵ where ϵ edge of our cube which we obtained from smoothings of our diagram. ϵ can be labeled by sequences in $\{0,1,*\}$ where height of the ϵ is denoted by $|\epsilon|$ and is defined by the number of '1' in domain of the d_ϵ. We can turn edges into arrows by the rule $* = 0$ gives the tail and $* = 1$ gives the head.

Definition 1.36. Now for ϵ, we define d_ϵ to be function :

$$V_\alpha \xrightarrow{d_\epsilon} V_{\alpha'}$$

where α and α' is just different by one digit where α has 0 whereas α' has 1 at this different digit. d_ϵ is id on the tensor factor corresponding to the circles that are not effected from smoothing. If two circles merge to one circle, d_ϵ is linear maps $'m'$ on these two circles whereas if we divide one circle into two circles, d_ϵ is linear map $'\Delta'$ on this circle.

From each vertex of the cube we go to vertex at next column at our cube. For example, we define $d_{001*000}$ from $V_{0010000}$ to $V_{0011000}$. On the other hand, we do not have maps between two vertex if $'1'$ goes to $'0'$ at the other vertex. For instance, we we do not have a map from V_{100} to V_{011}.

Now let us define m on $V \otimes V$. The map m is linear map and since we do not have canonical order on circles, we can define it commutatively i.e $m(a \otimes b) = m(b \otimes a)$.

Definition 1.37. The map ϵ defined by the following rules
$m : V \otimes V \to V$
$v_+ \otimes v_+ \to v_+$
$v_+ \otimes v_- \to v_-$
$v_- \otimes v_+ \to v_-$
$v_- \otimes v_- \to 0$
and it can be extended linearly on $V \otimes V$.

Here we can see v_+ is the identity element for the map m.
Now let us define Δ on V. Again since we have no canonical order on circles, we want Δ to be co-commutative.

Definition 1.38. The map Δ can be defined by the following rules
$\Delta : V \to V \otimes V$
$v_+ \to v_+ \otimes v_- + v_- \otimes v_+$
$v_- \to v_- \otimes v_-$
and it can be extended linearly on V.

Now we can define $d^i : C^{i,*}(D) \to C^{i+1,*}(D)$.

Definition 1.39. For $v \in V_\alpha \subset C^{i,*}(D)$

$$d^i(v) = \sum_{\substack{\epsilon \\ tail(\epsilon)=\alpha}} d_\epsilon(v)$$

From this definition we can observe that for any $v \in V_\alpha \subset C^{i,*}(D)$, $i(d^{\hat{1}}(v)) = i(v) + 1$. Now we check what happens to q gradings of elements of V_α, when we apply m and Δ. Note that all base elements $\{v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-\}$ have homology grading i at this computation. For $v_+ \otimes v_+$ we have $j(v_+ \otimes v_+) = deg(v_+ \otimes v_+) + i + n_+ - n_-$. When we apply m to $v_+ \otimes v_+$, we get $i + 1$. On the other hand $deg(v_+) = 1$ whereas $deg(v_+ \otimes v_+) = 2$. Besides, we have $deg(v_+ \otimes v_+) - 1 = deg(v_+)$ so

$$j(m(v_+ \otimes v_+)) = i + 1 + deg(v_+ \otimes v_+) - 1 + n_+ - n_- = i + deg(v_+ \otimes v_+) + n_+ - n_- = j(v_+ \otimes v_+)$$

Similarly, we can see that for other three elements $\{v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-\}$. m protects q-degree because everytime when we apply m, we have $i + 1$ and $deg(v) - 1$. Hence we get $j(m(v)) = i + 1 + deg(v) - 1 + n_+ - n_- = j(v)$. Note that deg(0) can be anything. We can do the same computation for Δ.

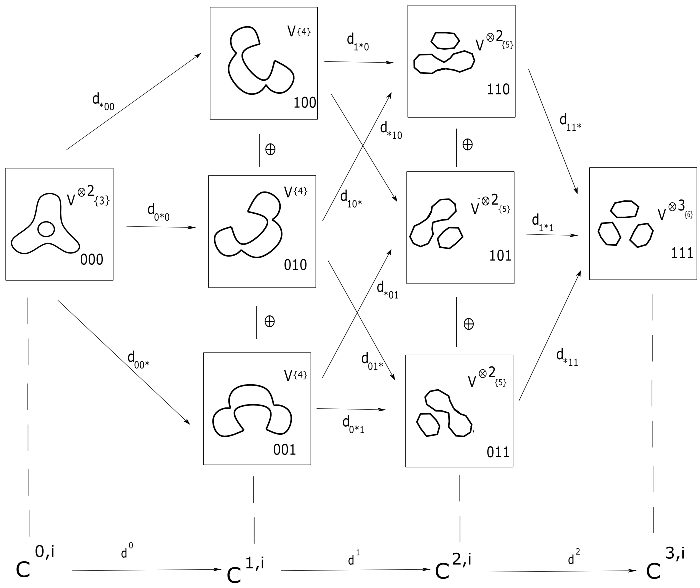
$$j(\Delta(v_+)) = i + 1 + deg(v_+) - 1 + n_+ - n_- = j(v_+)$$

$$j(\Delta(v_-)) = i + 1 + deg(v_-) - 1 + n_+ - n_- = j(v_-)$$

Furthermore, $d^r(v)$ is direct sum of $m(v)$ and $\Delta(v)$. Since these maps do not change q grading, their direct sum have the same q grading as $m(v)$ and $\Delta(v)$.

Proposition 1.40. *Suppose V is a graded vector space and $v \in V_n^{\otimes}$ where v is homogeneous element, then $j(d^r(v)) = j(v)$.*

Example 1.41. At this example we can see domain and range of d_c for right-handed trefoil.

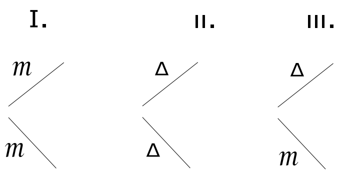


Now we need to check that d^i is indeed boundary map. In other words, we need to show $d^r \circ d^{r-1} = 0$. We prove it case by case. Again for simplicity let us do it in F_2.

Warning. We show that d^r is sum of id function, m and Δ. In proof we will not show id on the figures. It can be easily seen from figure where it is id.

Lemma 1.42. $d^r \circ d^{r-1} = 0$.

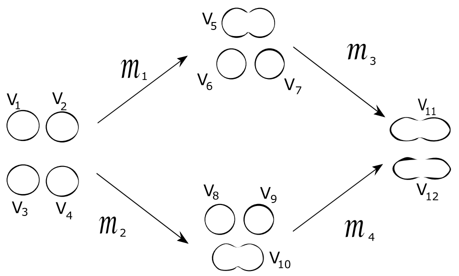
Proof. There are three options. We have two function m and Δ so we have positions as below.



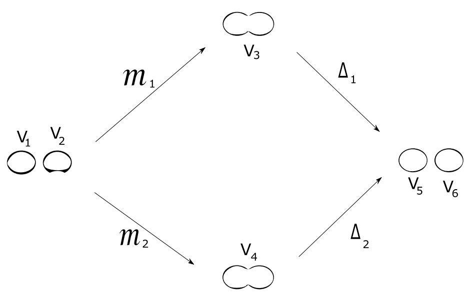
For the first position, we have three cases. In the first case, we have four disjoint circles in a domain of two m. For the second case, we have two circles in the domain of two m so they have same domain. For the final case, we have three circles. One of the circle is in domain of both m.
For the second position, we have two cases. For the first case, we have two circles in the domain of two Δ. For the second case, two Δ have the same domain they have just one circle.
For the third position, we have two cases. For the first case, we have three circles and we have disjoint domain. For the second case, m and Δ have one common circle in their domains.In other words, we have two circles where one of them is domain of Δ. In total, we have seven

cases. Now we start to investigate each of them separately. In the proof, we give number to each circles and each functions. In each case, at the end of case we can see figure which helps us to understand case better.

case 1. : In this case, we have 4 circles. $V_1 \otimes V_2$ is domain of m_1 whereas $V_3 \otimes V_4$ is domain of m_2. Now let us see what we have at V_{11}. Since m is commutative (i.e $m(v_- \otimes v_+) = m(v_+ \otimes v_-)$), it is enough to check three bases elements $\{v_+^1 \otimes v_+^2, v_-^1 \otimes v_+^2, v_-^1 \otimes v_-^2\}$. Let $v_+^1 \otimes v_+^2 \in V_1 \otimes V_2$ we have
$(id \circ m_1\ (v_+^1 \otimes v_+^2) + m_4 \circ\ id\ (v_+^1 \otimes v_+^2) = id\ (v_+^5) + m_4(v_+^8 \otimes v_+^9) = v_+^{12} + v_+^{12} = 0$ on V_{11}.
Similarly, for $v_-^1 \otimes v_+^2$ we have
$(id \circ m_1\ (v_-^1 \otimes v_+^2) + m_4 \circ\ id\ (v_-^1 \otimes v_+^2)) = id\ (v_-^5) + m_4(v_-^8 \otimes v_+^9) = v_-^{12} + v_-^{12} = 0$ at V_{11}.
Lastly, since $m(v_- \otimes v_-) = 0$ we have 0 at V_{11} for $(v_-^1 \otimes v_-^2)$.
We can say the same thing for V_{12} as we get the same result just by changing the numbers for circles.

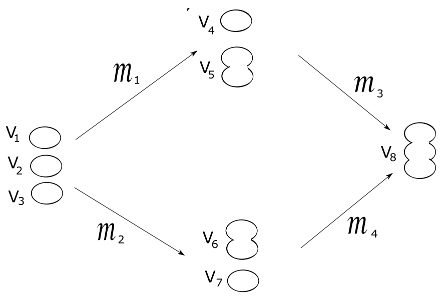


case 2. In this case, we have just 2 circles so our two functions have the same domain. Again since m is commutative, it is enough to check whether $\Delta_1 \circ m_1 + \Delta_2 \circ m_2 = 0$ or not for $\{v_+^1 \otimes v_+^2, v_-^1 \otimes v_+^2, v_-^1 \otimes v_-^2\}$. Let us begin with $v_+^1 \otimes v_+^2$
$\Delta_1 \circ m_1(v_+^1 \otimes v_+^2) + \Delta_2 \circ m_2(v_+^1 \otimes v_+^2) = \Delta_1(v_+^3) + \Delta_2(v_+^4) = 2(v_+^5 \otimes v_-^6 + v_-^5 \otimes v_+^6) = 0.$
Similarly, for $v_-^1 \otimes v_+^2$ we have
$\Delta_1 \circ m_1(v_-^1 \otimes v_+^2) + \Delta_2 \circ m_2(v_-^1 \otimes v_+^2) = \Delta_1(v_+^3) + \Delta_2(v_+^4) = 0$ as we shoved above .
For $v_-^1 \otimes v_-^2$, since $m(v_- \otimes v_-) = 0$ any $(\Delta \circ m)(v_- \otimes v_-) = 0$.

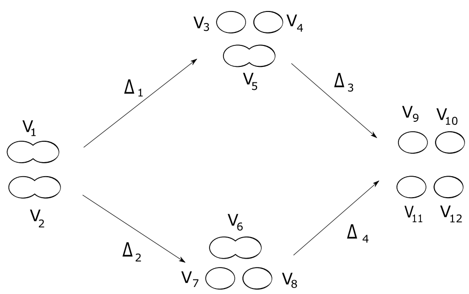


case 3. : In this case, we have 3 circles but one of them is in both domain. $V_1 \otimes V_2$ are domain of m_1 and $V_2 \otimes V_3$ are domain of m_2. Let us begin with basis in the form of $(a^1 \otimes v_+^2 \otimes b^2)$ where $a, b \in \{v_+, v_-\}$. Since v_+ is identity element for m we have
$m_3(id(a^1) \otimes m_1(v_+^2 \otimes b^2)) + m_4(m_2(a^1 \otimes v_+^2) \otimes b^2) = m_3(a^4 \otimes b^5) + m_4(a^6 \otimes b^7)$ this equation can be equal to $2v_+^8, 2v_-^8$ or $0+0$ but as a result it is equal to 0 in all cases which depend on a^1 and b^2.

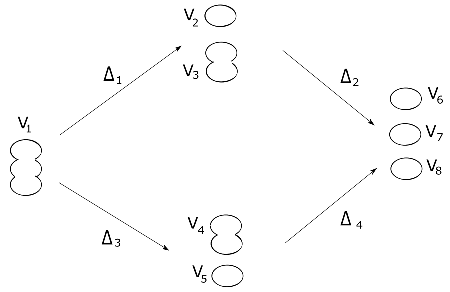
Now we check bases which we have v_-^2 in the middle. For $v_-^1 \otimes v_-^2 \otimes v_-^3$ we have
$m_3(id(v_-^1) \otimes m_1(v_-^2 \otimes v_-^3)) + m_4(m_2(v_-^1 \otimes v_-^2) \otimes v_-^3) = m_3(v_-^4 \otimes 0) + m_4(0 \otimes v_-^7) = 0$
For $v_+^1 \otimes v_-^2 \otimes v_-^3$ again we have
$m_3(id(v_+^1) \otimes m_1(v_-^2 \otimes v_-^3)) + m_4(m_2(v_+^1 \otimes v_-^2) \otimes v_-^3) = m_3(v_+^4 \otimes 0) + m_4(v_-^6 \otimes v_-^7) = 0 + 0 = 0$ and it is the same for $v_-^1 \otimes v_-^2 \otimes v_+^3$.
Finally, we can check $v_+^1 \otimes v_-^2 \otimes v_+^3$,
$m_3(id(v_+^1) \otimes m_1(v_-^2 \otimes v_+^3)) + m_4(m_2(v_+^1 \otimes v_-^2) \otimes v_+^3) = m_3(v_-^4 \otimes v_+^5) + m_4(v_-^6 \otimes v_+^7) = v_-^8 + v_-^8 = 0$



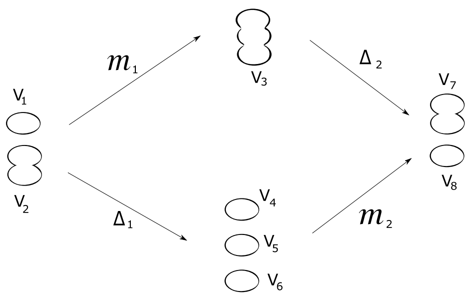
case 4. :In this case, we have two circles. V_1 is domain of Δ_1 and V_2 is domain of Δ_2. We check the result on just $V_9 \otimes V_{10}$ because on $V_{11} \otimes V_{12}$ computation is the same .
Let us begin with v_-^1.
$\mathrm{id} \circ (\Delta_1(v_-^1)) + \Delta_4 \circ \mathrm{id}\ (v_-^1) = \mathrm{id}(v_-^3 \otimes v_-^4) + \Delta_4(v_-^6) = v_-^9 \otimes v_-^{10} + v_-^9 \otimes v_-^{10} = 0.$
Next one is v_+^1.
$\mathrm{id} \circ (\Delta_1(v_+^1)) + \Delta_4 \circ \mathrm{id}\ (v_+^1) = \mathrm{id}(v_+^3 \otimes v_-^4 + v_-^3 \otimes v_+^4) + \Delta_4(v_+^6) = (v_+^9 \otimes v_-^{10} + v_-^9 \otimes v_+^{10}) + (v_+^9 \otimes v_-^{10} + v_-^9 \otimes v_+^{10}) = 0.$



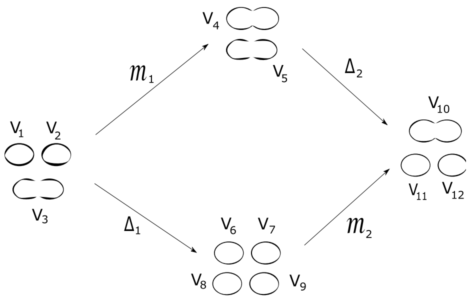
case 5. In this case, we have just one circle and again we begin with v_-^1.
$\Delta_2(\Delta_1(v_-^1)) + \Delta_4(\Delta_3(v_-^1)) = \Delta_2(v_-^2 \otimes v_-^3) + \Delta_4(v_-^4 \otimes v_-^5) = \mathrm{id}(v_-^2) \otimes \Delta_2(v_-^3) + \Delta_4(v_-^4) \otimes \mathrm{id}(v_-^5) = v_-^6 \otimes v_-^7 \otimes v_-^8 + v_-^6 \otimes v_-^7 \otimes v_-^8 = 0.$
Now we compute result for v_+^1.
$\Delta_2(\Delta_1(v_+^1)) + \Delta_4(\Delta_3(v_+^1)) = \Delta_2(v_+^2 \otimes v_-^3 + v_-^2 \otimes v_+^3) + \Delta_4(v_+^4 \otimes v_-^5 + v_-^4 \otimes v_+^5) = \mathrm{id}(v_+^2) \otimes \Delta_2(v_-^3) + \mathrm{id}(v_-^2) \otimes \Delta_2(v_+^3) + \Delta_4(v_+^4) \otimes \mathrm{id}(v_-^5) + \Delta(v_-^4) \otimes \mathrm{id}(v_+^5) = v_+^6 \otimes v_-^7 \otimes v_-^8 + v_-^6 \otimes v_+^7 \otimes v_-^8 + v_-^6 \otimes v_-^7 \otimes v_+^8 + v_+^6 \otimes v_-^7 \otimes v_-^8 + v_-^6 \otimes v_+^7 \otimes v_-^8 + v_-^6 \otimes v_-^7 \otimes v_+^8 = 0.$



case 6. : In this case, we have 2 circles. $V_1 \otimes V_2$ is domain of m_1 and V_2 is domain of Δ_1. We check four bases separately. Let us begin with $v_-^1 \otimes v_-^2$.
$\Delta_2(m_1(v_-^1 \otimes v_-^2)) + m_2(id(v_-^1) \otimes \Delta_1(v_-^2)) = \Delta_2(0) + m_2(v_-^4 \otimes v_-^5) \otimes id(v_-^6) = 0 + 0 \otimes v_-^8 = 0$.
Secondly, let us check $v_+^1 \otimes v_-^2$. We have
$\Delta_2(m_1(v_+^1 \otimes v_-^2)) + m_2(id(v_+^1) \otimes \Delta_1(v_-^2)) = \Delta_2(v_-^3) + m_2(v_+^4 \otimes v_-^5 \otimes v_-^6) = v_-^7 \otimes v_-^8 + m_2(v_+^4 \otimes v_-^5) \otimes id(v_-^6) = v_-^7 \otimes v_-^8 + v_-^7 \otimes v_-^8 = 0$.
Thirdly , next base element is $v_-^1 \otimes v_+^2$.
$\Delta_2(m_1(v_-^1 \otimes v_+^2)) + m_2(v_-^4 \otimes \Delta_1(v_+^2)) = \Delta_2(v_-^3) + m_2(v_-^4 \otimes v_+^5 \otimes v_-^6 + v_-^4 \otimes v_-^5 \otimes v_+^6) = v_-^7 \otimes v_-^8 + m(v_-^4 \otimes v_+^5) \otimes id(v_-^6) + m(v_-^4 \otimes v_-^5) \otimes id(v_+^6) = v_-^7 \otimes v_-^8 + v_-^7 \otimes v_-^8 + 0 \otimes v_+^8 = v_-^7 \otimes v_-^8 + v_-^7 \otimes v_-^8 = 0$.
Finally, let us check $v_+^1 \otimes v_+^2$.
$\Delta_2(m_1(v_+^1 \otimes v_+^2)) + m_2(v_+^4 \otimes \Delta_1(v_+^2)) + = \Delta_2(v_+^3) + m_2(v_+^4 \otimes v_+^5) \otimes id(v_-^6) + m(v_+^4 \otimes v_-^5) \otimes id(v_+^6)$ $= 2(v_+^7 \otimes v_-^8) + 2(v_-^7 \otimes v_+^8) = 0$.



case 7. : In this case, we have 3 circles. $V_1 \otimes V_2$ is domain of m_1 whereas $V_3$is domain of Δ_1.Firstly, We compute result just on V_{10} so we just have $m_2 + m_1$. For $v_- \otimes v_-$ since $m(v_- \otimes v_-) = 0$, result is 0 on $V_1 0$. Furthermore, m is commutative so we can just compute $v_+ \otimes v_-, v_+ \otimes v_+$.
Firstly , $v_+^1 \otimes v_-^2$ so
$id \circ m_1(v_+^1 \otimes v_-^2) + m_2 \circ id(v_+^1 \otimes v_-^2) = id(v_-^4) + m_2(v_+^6 \otimes v_-^7) = v_-^{10} + v_-^{10} = 0$. Next one is $v_+^1 \otimes v_+^2$.
$id \circ m_1(v_+^1 \otimes v_+^2) + m_2 \circ id(v_+^1 \otimes v_+^2) = id(v_+^4) + m_2(v_+^6 \otimes v_+^7) = v_+^{10} + v_+^{10} = 0$.
Redarding to $V_{11} \otimes V_{12}$, we have $\Delta_1 + \Delta_2$. Now let us check for v_-^3
$id(\Delta_1(v_-^3)) + \Delta_2(id(v_-^3)) = id(v_-^6 \otimes v_-^7) + \Delta_2(v_-^5) = v_-^{11} \otimes v_-^{12} + v_-^{11} \otimes v_-^{12} = 0$.
Now we compute v_+^3
$id(\Delta_1(v_+^3)) + \Delta_2(id(v_+^3)) = id(v_+^6 \otimes v_-^7 + v_-^6 \otimes v_+^7) + \Delta_2(v_+^5) = v_+^{11} \otimes v_-^{12} + v_-^{11} \otimes v_+^{12} + v_+^{11} \otimes v_-^{12} + v_-^{11} \otimes v_+^{12} = 0$.



□

This proof shows that d^i is indeed boundary map. We have already defined sequence of $C^{i,*}(L)$ for a link diagram L, hence we have a chain complex. We can compute the Khovanov homology anymore. At this Khovanov homology, we have some important definition. For example, Euler characteristic. The graded Euler characteristic for this complex is

$$\sum_r (-1)^r qdimH^r(D)$$

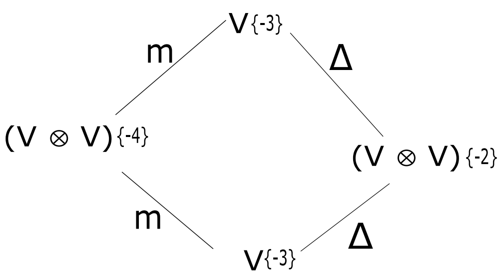
which is equal to the unnormalized Jones polynomial of the knot diagram (D).

Proof. [3] Theorem 1.

□

Now it is time to see one example about the Khovanov homology. How can we find Khovanov homology groups of Hopf link.

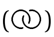
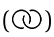
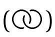
Example 1.43. Let us compute the homology of $C^{*,*}()$. As we can see at example 1.35, we have just only three non-trivial terms :
$0 \xrightarrow{d} C^{-2,*}() \xrightarrow{d} C^{-1,*}() \xrightarrow{d} C^{0,*}() \xrightarrow{d} 0$
More explicitly this can be rewritten as a chain :



Now we provide table to show cycles and boundaries and homology of the Hopf link.

Homologies	-2	-1	0
cycles	$\{v_- \otimes v_+ + v_+ \otimes v_-, v_- \otimes v_-\}$	$\{v_- \otimes v_-\}, \{v_+ \otimes v_+\}$	$\{v_- \otimes v_-\}, \{v_- \otimes v_+\}$ $\{v_+ \otimes v_-\}, \{v_+ \otimes v_+\}$
Boundaries	none	$\{v_- \otimes v_-\}, \{v_+ \otimes v_+\}$	$\{v_- \otimes v_-\}, \{v_- \otimes v_+ + v_+ \otimes v_-\}$
Homology	$\{v_- \otimes v_+ + v_+ \otimes v_-\}, \{v_- \otimes v_-\}$	none	$\{v_+ \otimes v_+\}, \{v_- \otimes v_+\}$
q- degrees	-4,-6		0.-2

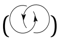
As a result, we have $H_{-2,-4} = H_{-2,-6} = H_{0,0} = H_{0,-2} = F$ where F is field we study in. Now we can think about what happens if we change orientation on one component of the Hopf link. We have the Hopf link with different orientation as in the following figure . Since we have $n_+ = 2$ and $n_- = 0$ for this orientation, we have chain complex as below

$0 \xrightarrow{d} C^{0,*}($$) \xrightarrow{d} C^{1,*}($$) \xrightarrow{d} C^{2,*}($$) \xrightarrow{d} 0$

where $C^{0,*} = V \otimes V$, $C^{1,*} = V \bigoplus V$ and $C^{2,*} = V \otimes V$ and maps between them are the same as in the above 1.43. Homology table is similar to the table above.

Homologies	0	1	2
cycles	$\{v_- \otimes v_+ + v_+ \otimes v_-, v_- \otimes v_-\}$	$\{v_- \otimes v_-\}, \{v_+ \otimes v_+\}$	$\{v_- \otimes v_-\}, \{v_- \otimes v_+\}$ $\{v_+ \otimes v_-\}, \{v_+ \otimes v_+\}$
Boundaries	none	$\{v_- \otimes v_-\}, \{v_+ \otimes v_+\}$	$\{v_- \otimes v_-\}, \{v_- \otimes v_+ + v_+ \otimes v_-\}$
Homology	$\{v_- \otimes v_+ + v_+ \otimes v_-\}, \{v_- \otimes v_-\}$	none	$\{v_+ \otimes v_+\}, \{v_- \otimes v_+\}$
q- degrees	2,0		6.4

Now let‘s compute euler characteristic of Hopf link at 1.43.

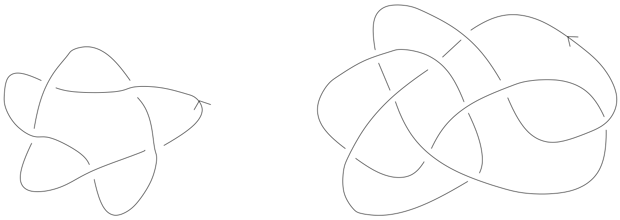
$\chi($$) = (-1)^{-2} qdimH^{-2}(D) + (-1)^0 qdimH^0(D) = q^{-6} + q^{-4} + q^{-2} + q^0$

whereas we have

$$q^6 + q^4 + q^2 + q^0$$

as the Euler characteristic of the Hopf link . Actually, if we have link with 2 components and if we change orientation of one component, we had relation between their Euler characteristic polynomial. We can explain this relation like that if we replace q^{-1} with q in one polynomial, we get other polynomial. On the other hand, for a knot this is not true because when we change orientation on the knot we are not changing n_+ or n_- so we do not change any grading.
So far we have seen the Jones polynomial and the Khovanov homology. Both of these are knot invariants but the Khovanov homology is stronger knot invariant than the Jones polynomial. This means that we can find two knots which have the same Jones polynomial whereas their Khovanov homology groups are different. In the following figure, we see two knots which they have the same Jones polynomial but their Khovanov homology are different. For computation of their Jones polynomials and Khovanov homology groups see [4] page 17 Example 3.2.

Example 1.44.



1.4.2 Reiedemeister moves

We know that if we have X and Y topological space and if they are homeomorphic then homology group of them is the same. We are expecting the same thing here also, but in knot theory we have the Reidemeister moves instead of homeomorphism. Now we show that if we have two equivalent knots, their Khovanov homology is the same.

1.4.2.1 Invariance of Khovanov homology under R1

At this section we use some figures but instead of using figures again and again we assign them letters and we use these letters instead of figures. Say $A =$  , $B =$  and $C =$ . Actually, diagram A is just local part of diagram link. In the proof, I will say A instead of link diagram. We can say the same for B and C.

Now we show $H(A) \cong H(C)$

Lemma 1.45. $H(A) \cong H(C)$

To begin with , if $[|A|]$ represented by n-dimensional cube then $[|B|]$ and $[|C|]$ represented by $(n-1)$- dimensional cube. If we apply "0" smoothing to the A it will be $[|B|]$ and if we apply "1" smoothing to $[|A|]$, it will be $[|C|]$. Between $[|B|]$ and $[|C|]$ we always have a map m because $[|B|]$ always have one more circle than $[|C|]$. Now we show that actually this m is chain map.

Theorem 1.46. $m : [|B|] \to [|C|][1]$ *is chain map . In other words,* $d^r \circ m = m \circ d^r$.

Proof. Since d^r consists sum of m and Δ, it is enough to show $m \circ m = m \circ m$ and $\Delta \circ m = m \circ \Delta$. $m \circ m = m \circ m$ is obvious . For $\Delta \circ m = m \circ \Delta$ look at case 6 . □

Assume that we have $C^{i,*}(A)$ and we know that it is direct sum of V_α for different α where different α have the same number 1 smoothing. Now suppose that we have $\alpha_1 = 0k_1k_2...k_n$, it is easy to see that we can get V_{α_1} if we apply resolution $\alpha_2 = k_1k_2...k_n$ to the diagram B. Similarly, if we have $\alpha_3 = 1k_1k_2...k_n$, we can get V_{α_3} from diagram C with applying resolution $\alpha_4 = k_1k_2...k_n$. As a consequence, we can say that if α starts with 0 than V_α belongs to $C^{k,*}(B)$ and if α starts with 1 then it belongs to $C^{l,*}(C)$. This shows that $C^{i,*}(A) = C^{k,*}(B) \bigoplus C^{l,*}(C)$. Diagram A and B have the same number negative crossing. Hence $r_{\alpha_1} = r_{\alpha_2}$ and $k = i$. Regarding l, since $r_{\alpha_3} = r_{\alpha_4} + 1$, we have $l = i - 1$ Furthermore, we have chain map

$m: [|B|] \xrightarrow{m} [|C|][1]$. Therefore, we can see $[|A|]$ as a $cone(m)$. This will make our proof easy at this section.
We know that for every circle we assign vector space $V =< v_-, v_+ >$. Now let us assign $V' =< v_+ >$ instead of $V =< v_-, v_+ >$ to the special circle $'o'$ which can be seen in the figure  and denote this chain complex by $[|B|]_{v_+}$. It is obviously subcomplex of $[|B|]$. Now let us define $[|A'|]$ a subcomplex of $[|A|]$. It is a mapping cone of m where we have $m: [|B|]_{v_+} \xrightarrow{m} [|C|][1]$.

Theorem 1.47. *$[|A'|]$ is acyclic (i.e $[|A'|]$ has no homology)*

Before starting prove this claim, we need to prove that the map

$$m: [|B|]_{v_+} \to [|C|][1]$$

is an isomorphism.

Claim. m is an isomorphism.

Proof. To begin with , $m(\cdots - \otimes - \otimes \cdots - \otimes v_+) = \cdots - \otimes - \otimes \cdots - \otimes -$. In short, m takes elements which has v_+ at the end and sends it to elements which have the same components without v_+ at the end. It is obviously one-to-one .
It is onto. Because if we take an element $a \in [|C|][1]$ then $m(a \otimes v_+) = a$.
$m(a_1 \otimes v_+ + a_2 \otimes v_+) = a_1 + a_2 = m(a_1 \otimes v_+) + m(a_2 \otimes v_+)$.
so m is linear, one-to-one and onto. Therefore, it is isomorphism between $[|B|]_{v_+}$ and $[|C|][1]$. □

Proof. of 1.47
As we know $[|A'|]$ is a mapping cone. By the lemma 1.26 we have exact sequence below

$$\cdots \to H^d([|B|]_{v_+}) \xrightarrow{\text{H(m)}} H^d([|C|]) \to H^d(Cone(m)) \to H^{d+1}([|B|]_{v_+}) \to H^{d+1}([|C|]) \cdots$$

Since m is an isomorphism, $H(m)$ is also isomorphism. Hence we have exact sequence as below

$$H^d([|B|]_{v_+}) \xrightarrow{\text{H(m)}} H^d([|C|]) \to H^d(Cone(m)) \to H^{d+1}([|B|]_{v_+}) \xrightarrow{\text{H(m)}} H^{d+1}([|C|])$$

Since $H(m)$ is an isomorphism, we have exact sequence as below

$$H^d([|C|]) \xrightarrow{0} H^d(Cone(m)) \xrightarrow{0} H^{d+1}([|B|]_{v_+})$$

and by exactness $H^d(cone(m)) = 0$ for all d.

□

This means $[|A'|]$ is acyclic. Again with using 1.26 we can prove the following lemma.

Lemma 1.48.

$$H^d([|A|]) \cong H^d([|A|])/[|A'|])$$

Proof. We have short exact sequence below

$$0 \to [|A'|] \xrightarrow{\text{i}} [|A|] \xrightarrow{\text{p}} [|A|]/[|A'|] \to 0$$

and again we get long exact sequence as below by 1.26

$$\to H^d([|A'|]) \to H^d([|A|]) \to H^d([|A|]/[|A'|] \to H^{d+1}([|A'|])$$

We just showed that A' is acyclic i.e $H^d(A') = 0$ for all d. We have exact sequence like below

$$0 \to H^d([|A|]) \to H^d([|A|]/[|A'|] \to 0$$

By exactness of sequence

$$H^d([|A|]) \cong H^d([|A|])/[|A'|])$$

for all d. □

Claim. $[|A|]/[|A'|] \cong [|C|][1]$.

Proof. We know $[|A|] = [|B|] \oplus [|C|][1]$ and $[|A'|] = [|B|]_{v_+} \oplus [|C|][1]$ we can define map

$$\phi : [|A|] \to [|B|]_{v_+}$$

$$(v_+ - \otimes \cdots) \otimes [|C|][1] \to 0$$

$$(v_- - \otimes \cdots -) \otimes [|C|][1] \to (v_- - \otimes \cdots -)$$

where $(v_+ - \otimes \cdots -)$ and $(v_- - \otimes \cdots -) \in [|B|]$.
Here ϕ is obviously onto and we have $[|A'|]$ as a kernel of ϕ. Therefore we have $[|A|]/[|A'|] \cong [|B|]/(v_+ = 0)$. By the Khovanov bracket $[|B|]/(v_+ = 0) = < v_- > \otimes [|C|]$. We know that $< v_- > \otimes [|C|][1] \cong [|C|][1]$. □

Now we have this $0 \to [|A'|] \xrightarrow{\text{i}} [|A|] \xrightarrow{\text{p}} [|A|]/[|A'|] \to 0$
if we apply H functor to this sequence, we get a long exact sequence again. Since $H^d([|A'|]) = 0$ for all d, we have

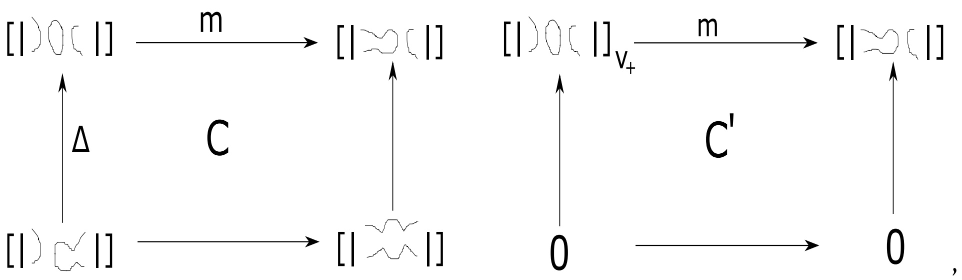
$$H^d([|A|]) \cong H^d([|C|]) \tag{1.1}$$

At the beginnig, we had the diagrams A and C which we can pass from diagram A to diagram C with Reidemeister 1 move and we showed $H([|A|]) \cong H([|C|])$. This means that if we apply $(R1)$ to any diagrams, we get diagrams which has the same Khovanov homology . As a consequence, this shows that the Khovanov homology is an invariant under R1.

1.4.2.2 Invariance of Khovanov homology under R2

At this case we also assign letters to figures so say $D_1 =$  and $D_2 =$ . Now we can prove invariance of Khovanov homology under R2.

Proof. For computing $H([|D_1|])$ we define complex C and C' as below

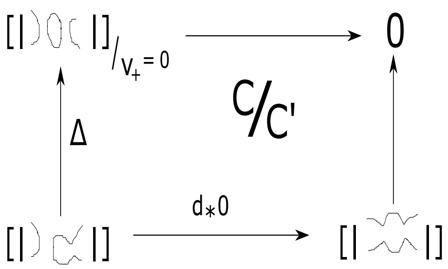


Here we can observe that C' is really subcomplex of C because left upper corner and right upper corner of C' are subcomplex of left upper and right upper corners of C respectively. Furthermore, since $Im(m) \in [|$ $|]$ in C', we say that C' is really subcomplex of C. We have the basic following claim.

Claim. C' is acyclic

Proof. Actually, proof is the same as in $(R1)$. See 1.4.2.1 □

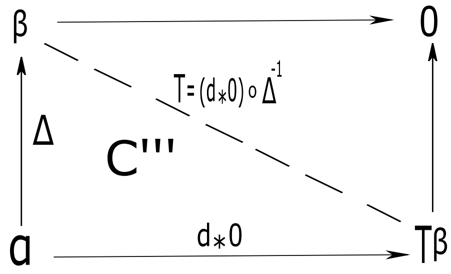
Now we have C/C' complex



Claim. Δ is bijective in C/C'.

Proof. Δ has one circle in its domain and two circles in codomain. Let us say that one of the circle in codomain (say that first circle) is circle which can be seen in . We also know that we kill v_+ in V which assigned for first circle. We have $v_- \otimes v_-$ and $v_- \otimes v_+$ as a base elements of codomain of Δ because $v_+ = 0$ in the first circle. Image of Δ is a vector space which generated by $v_- \otimes v_-$ and $v_- \otimes v_+ + v_+ \otimes v_-$. Since $v_+ = 0$, we have $v_- \otimes v_-$ and $v_- \otimes v_+$ as generators of $Im(\Delta)$, hence Δ is onto. In addition to that, since Δ is linear, onto and its domain and codomain have the same dimension, it should be one-to-one. As a result, Δ is bijective. □

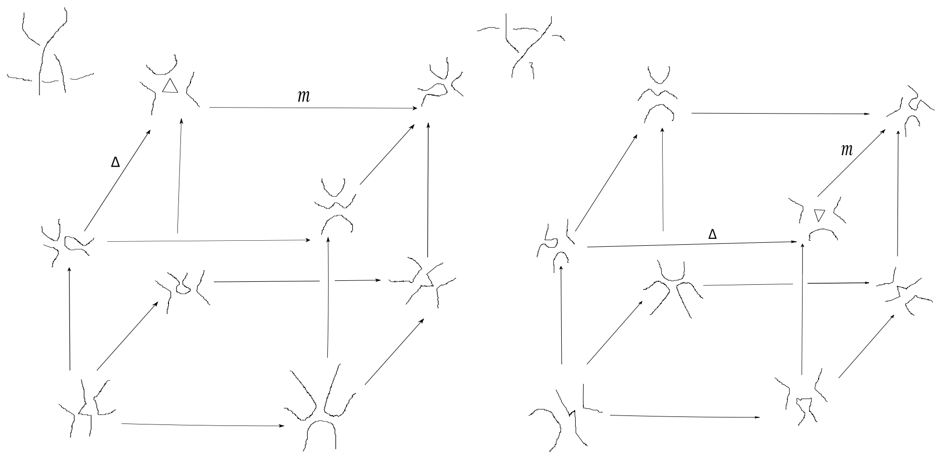
Now let us define C''' subcomplex of C/C' . It consists all $\alpha \in [|D_2|]$ and all pairs of the form $(\beta, d_{*0} o \Delta^{-1}(\beta))$ where $\beta = \Delta(\alpha)$. We have it as below

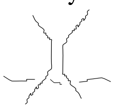
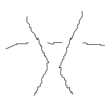
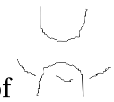
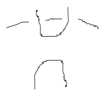


We have $\delta : \alpha \to (\beta, \tau(\beta))$. Since Δ is one-to-one and onto, δ is also one-to-one and onto . Now we can see C''' as mapping cone of α and $(\beta, \tau(\beta))$. We have map δ between them which is an isomorphism so C''' is acyclic.
Now let us look at $C/C'/C'''$. Since we take regular $\alpha \in [|D_2|]$, lower left corner of $C/C'/C'''$ is zero. In addition to that, we are killing $(\beta, \tau(\beta)$ in $[|\ |]_{/v_+=0} \bigoplus [|\ |]$. We know that for any vector spaces V_1 and V_2 and for $f : V_1 \to V_2$ linear map f, we have $V_1 \oplus V_2/\{x, f(x)\} \cong V_2$.
We can generalize this from vector spaces to the chain complex. Because in the Khovanov homology we have vector spaces as modules of our chains so we can apply this vector argument to all our chain members. As a result, when we kill $(\beta, \tau(\beta))$ in $[|\ |]_{/v_+=0} \bigoplus [|\ |]$, we have $H([|C/C'/C'''|]) \cong H([|\ |])$. Finally, we say that $H([|D_1|]) = H(C)$ and since C' is acyclic, we can say $H(C) \cong H(C/C')$. Furthermore, We showed C''' is acyclic.
$H(C/C') \cong H(C/C'/C''') \cong H([|\ |])$. Hence $H([|D_1|]) \cong H([|\ |])$.
Finally, we can say that if we have diagrams and if we apply R2 to this diagram the new diagram will have the same Khovanov homology as the old one. In short, the Khovanov homology is invariant under R2.

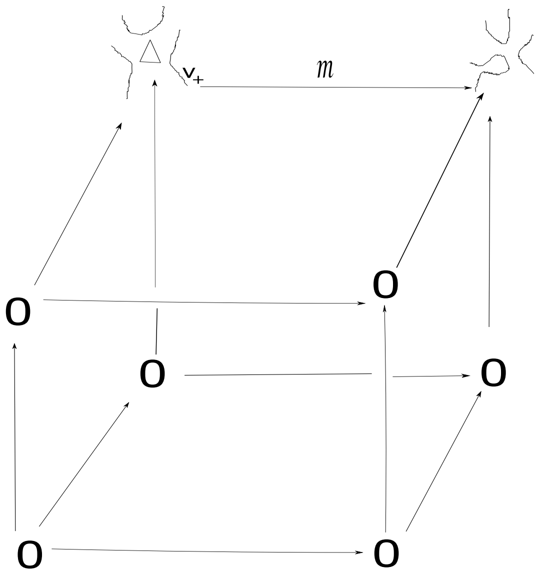
1.4.2.3 Invariance of Khovanov homology under R3

Let us begin with smoothing two sides of (R3) :

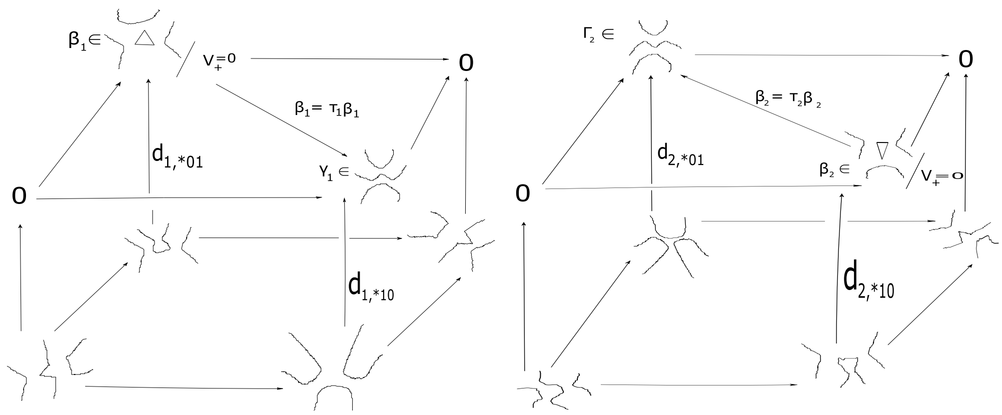


It is easy to observe that the bottom layers of these two cubes correspond to smoothings of  and  since these two diagrams are planar isotopic these two layers are isomorphic. Furthermore, top layers corresponds to smoothings of  and  and top layers are also isomorphic .

Now proof is the similar to the proof of invariance of (R2). Define C' and C''' similarly as we did it in proof of (R2) within the top layers of both cubes. For example, for the left cube above we have C' as below



When we take out quotient by its C' and C''' of both cubes we get two cubes as below



These two cubes are isomorphic where we have map ϕ between these two cubes. It acts on as id on bottom layer and sends pair (β_1, γ_1) to (β_2, γ_2) at upper layer. It is obviously isomorphism at bottom level because it is just id at bottom level. As we mentioned earlier, upper layers of the two cubes which we have in the beginning of the proof are isomorphic so when we take quotient by their C' and C''' of both cubes, this isomorphism will be protected. In other words, ϕ is isomorphism for upper layer also.

We check if ϕ is isomorphism between two complexes or not for it, we need to show $\phi \circ \partial = \partial \circ \phi$ where d is a boundary map. In our cases since we have zero for left and right corner at top level, to show $\phi \circ \partial = \partial \circ \phi$ we need to show $\tau_1 \circ d_{1,*01} = d_{2,*01}$ and $d_{1,*10} = \tau_2 \circ d_{2,*01}$. We just prove $\tau_1 \circ d_{1,*01} = d_{2,*01}$. The other one can be prove similarly. Now we can write $\tau_1 = d_1 \circ \Delta^{-1}$ where d_1 is a map between  and . We know $d_{1,*01}$ is equal to Δ and additionally it is one-to-one. Therefore we have $\tau_1 \circ d_{1,*01} = d_1 \circ \Delta^{-1} \circ d_{1,*01}$. Since  and  are planar isotopic, we say that their chain complexes are isomorphic also, hence we deduce that $\Delta^{-1} \circ d_{1,*01} = id$. Furthermore, since [||] and [||] are isomorphic, [||] and [||] are also isomorphic. Therefore $d_1 = d_{2,*01}$. As a result, we showed $\tau_1 \circ d_{1,*01} = d_1 \circ \Delta^{-1} \circ d_{1,*01}$. $\square$

Chapter 2

Khovanov Homology And Slice Genus

In this chapter, we will use Khovanov homology to define the knot invariant $s(K)$ and we will prove two main theorems. Before theorems we need some definitions.

Definition 2.1. Slice genus of smooth knot K in S^3 is the least integer g such that K is the boundary of a connected, orientable 2-manifold S of genus g embedded in the 4-ball D^4 bounded by S^3. It is Denoted by $g_*(K)$.

Theorem 2.2. $|s(K)| \leq 2g_*(K)$.

Definition 2.3. Given two links $L_0 \subset S^n$ and $L_1 \subset S^n$. We say L_0 and L_1 are concordant if there exist an embedding $f : L_0 \times [0,1] \to S^n \times [0,1]$ such that $f(L_0 \times \{0\}) = L_0 \times \{0\}$ and $f(L_0 \times \{1\}) = L_1 \times \{1\}$.
Knot Concordance is an equivalence relation on the set of knots. Furthermore, knot concordance class form an abelian group under the connected sum operation. We denote this group by $Conc(S^3)$. For more information see [5]

Theorem 2.4. *The map s induces a homomorphism from $Conc(S^3)$ to $\mathbb{Z}$. .*

2.1 Lee homology

Lee homology uses similar contruction for modules but we have m' and Δ' instead of m and Δ. Actually the underlying modules are the same for both Lee homology and Khovanov homology but the maps $m' : V \otimes V \to V$ and $\Delta' : V \to V \otimes V$ are different. They are given by

$m'(v_+ \otimes v_+) = m'(v_- \otimes v_-) = v_+$ $\qquad$ $\Delta'(v_+) = v_+ \otimes v_- + v_- \otimes v_+$
$m'(v_+ \otimes v_-) = m'(v_- \otimes v_+) = v_-$ $\qquad$ $\Delta'(v_-) = v_- \otimes v_- + v_+ \otimes v_+$
we have below equalities.

$$m' = m + \Phi_m$$

$$\Delta' = \Delta + \Phi_\Delta$$

See page 3 of [6]. We denote the resulting chain complex by $CKh'(L)$ and homology by $Kh'(L)$. As we remember, we have $\{v_+, v_-\}$ as basis for V. Let's have $a = v_- + v_+$ and $b = v_- - v_+$,

$\{a, b\}$ is also base for V. With respect to this basis, we have

$m'(a \otimes a) = 2a$ $\qquad \Delta'(a) = a \otimes a$

$m'(a \otimes b) = m'(b \otimes a) = 0 \qquad \Delta'(b) = b \otimes b$

$m'(b \otimes b) = -2b$

We have basis $\{a, b\}$ with respect to this basis we can prove the following theorem

Theorem 2.5. *$Kh'(L)$ has rank 2^n, where n is the components of link L.*

Proof. [6] Theorem 5.1 □

Actually, this theorem tell us that we have a bijection between the possible orientations on a link L and generators of $Kh'(L)$, which we refer to as canonical generators. We can see this bijection as follows. Given an orientation o of L, let D_o be the corresponding oriented resolution, which means resolution will respect orientation. We label the circles in D_o with a and b according to the following rule. To each circle C we assign a mod 2 invariant. We are doing it as follows. Draw a ray in the plane from C to infinity, it can be in any direction, and take the number of it intersect the other circles mod 2. To this number, we add 1 if C has counterclockwise orientation, and 0 if it does not. Label C by a if the resulting invariant is 0, by b if it is 1. We denote the resulting state by s_0.

We have an important observation about orientation as follows.

Lemma 2.6. *Suppose there is a region in the state diagram for s_o containing exactly two segments, as shown in the figure below. Then either the orientations of the two are the same and the labels are different (like part a of the figure) or the orientations are different and the labels are the same (like part b of the figure).*

a) b)

Proof. We have three cases, the first one is that we have just one circle. Two segments belong to the same circle so we have just one label with different orientations like part b of the figure. The Second case is the case where segments belong to two circles, one of which is contained inside the other one. Firstly, assume that we have different orientations for the two circle and suppose that the circle which is inside the other one has counterclockwise orientation. Since it intersects other circles one time and since it has counterclockwise orientation, we have a for this circle. Similarly, the outer circle have a also. And by the same logic, if we chance orientations of circle, we have b for each circle. As a result, we have different orientations but same label for circles. Next, assume that we have same orientation on two circles and let's say that we have counterclockwise orientation on them since inner circle intersects outer one once and since it has counterclockwise orientation, it has label a whereas outer circle has b. If we change orientations on circles, we will have b for inner and a for outer circle. Third case is the case where segments belong to two circle, neither of which is contained inside the other. Let's say that we have same orientation on both of them, then they will intersect each other with the same mod 2 number and since they have same orientation, segment labels will be the same. What if we have different orientation on circles ? Their intersection number of each other will be the same as we did previously one but since they have different orientation, one of them will have label a and other b. □

Corollary 2.7. *If two circles in the state diagram for s_0 share a crossing, they have different labels.*

Proof. Since we resolve crossing point with the orientation preserving way, resulted two circles should have same orientation. □

2.2 Definition and basic properties of the Invariant

Before making formal definition of the invariant, we need some information about filtrations. Suppose C is chain complex. A finite length filtration of C is a sequence of subcomplexes

$$0 = C_n \subset C_{n-1} \subset C_{n-2} \subset \cdots \subset C_m = C$$

We associate grading to this kind of filtration as follows: $x \in C$ has grading i if and only if $x \in C_i$ but $x \notin C_{i-1}$. If $f : C \to C'$ is a map between two filtered chain complexes , we say that f respects the filtration if $f(C_i) \subset C'_i$. More generally, we say that f is filtered map of degree k if $f(C_i) \subset C'_{i+k}$. A filtration $\{C_i\}$ induces filtration $\{S_i\}$ on $H_*(C)$ defined as follows : a class $[x]$ in $H_*(C)$ is in S_i if and only if it has representative which is in $\{C_i\}$. If $f : C \to C'$ is a filtered chain map of degree k, it is easy to see that induced map $f_* : H_*(C) \to H_*(C')$ is also filtered of degree k. Now let us denote by s grading on $Kh'(K)$ induced by the q grading on $CKh'(K)$. We can see this induced q grading as follows.

We have q grading on $CKh'(K)$. This means we have filtration on $C^i Kh'(K)$ (i th module of our chain complex). Now let's define

$$F^j(C^i Kh'(K)) = span\{v \in C^i Kh'(K) | qdeg(v) \geq j\}$$

and we have filtration

$$\{0\} \cdots \subset F^{j+1}(C^i Kh'(K)) \subset F^j(C^i Kh'(K)) \subset \cdots \subset C^i Kh'(K)$$

Now we see how we can reduce this filtration to $H^i_{Lee}(K)$. Since d_{Lee} keeps q grading same or increase it by four, we have

$d_{Lee} : C^i Kh'(K) \to C^{i+1} Kh'(K)$

$d_{Lee} : F^j(C^i Kh'(K)) \to F^j(C^{i+1} Kh'(K)) \oplus F^{j+4}(C^{i+1} Kh'(K)) \subset F^j(C^{i+1} Kh'(K))$

Let $\pi : Kerd_{Lee} \to H^i_{Lee}(K)$ be the projection.
Note that $Kerd_{Lee} \subset C^i Kh'(K)$. Now let's define

$$F^j(H^i_{Lee}) = \pi(Kerd_{Lee} \cap F^j(C^i Kh'(K)))$$

As a result, we have filtration

$$\{0\} \cdots F^{j+1}(H^i_{Lee}) \subset F^j(H^i_{Lee}) \subset \cdots \subset H^i_{Lee}$$

Using this q grading we have

Definition 2.8.

$$s_{min}(K) = min\{s(x) | x \in Kh'(K), x \neq 0\}$$

$$s_{max}(K) = max\{s(x) | x \in Kh'(K), x \neq 0\}$$

For unknot U, in $CKh'(U)$ we have just V in our complex. Note that $j(v_+) = 1$ and $j(v_-) = -1$, so we have $s_{max}(U) = 1$ and $s_{min}(U) = -1$.

2.2.0.1 The invariant s

In this section we define invariant s.

Proposition 2.9.

$$s_{max}(K) = s_{min}(K) + 2$$

which justifies

Definition 2.10.

$$s(K) = s_{max}(K) - 1 = s_{min}(K) + 1$$

Before proving the proposition, we need some lemmas and results.

Lemma 2.11. *Let n be the number of components of L. We have orientation o and $\bar{o}$ where they are opposite to each other. There is a direct sum decomposition $Kh'(L) \cong Kh'_o(L) \oplus Kh'_e(L)$, where $Kh'_o(L)$ is generated by all states with q grading congruent to $2 + n \ mod\ 4$, $Kh'_e(L)$ is generated by all states with q grading congruent to $n \ mod\ 4$. If o is an orientation on L then $s_o + s_{\bar{o}}$ is contained in one of two summands, and $s_o - s_{\bar{o}}$ is contained in other.*

We will use the following lemma to prove Lemma 2.11

Lemma 2.12. *Denote number of components of the link by $|L|$. All elements of the chain complexes have degree equal to $|L| \ mod\ 2$.*

Proof. Let $s \in S(D)$ be a state and $m = v_\pm \otimes \cdots \otimes v_\pm \in C_s$. Let m have k_+ times v_+ and k_- times v_- factors. As we know $j(m) = i(m) + deg(m) + n_+ - n_- = r_\alpha - n_- + k_+ - k_- + n_+ - n_-$. (Note that r_α is number of 1 in smoothing α and α is smoothing which has diagram s after resolving link diagram. Since $k_+ - k_- \equiv k_+ + k_- \equiv \#k(s)\ mod\ 2$ where $\#k(s)$ denote the number of circle in the state s and $r_\alpha - n_- + n_+ - n_- \equiv r_\alpha + n_-\ mod\ 2$, we have $j(m) = \mathrm{r}_\alpha + n_+ + \#k(s)$. We know that we need to resolve each crossing point to go from the link diagram to the Seifert state s_o and we know that each oriented resolution changes number of link components by one. In other words, we have link diagram with $|L|$ components and we get $k(s_o)$ after resolving n crossing points. As a result, we have $|L| + n = k(s_o)$. When we go from any state s to s_o, it is still true that at each crossing number of components(circles) change by one so we have $r_\alpha + k(s) = r_{\alpha'} + k(s_o)$ where α' is smoothing for s_o. Now we have $j(m) = \mathrm{r}_\alpha + n_+ + \#k(s) = j(m) = r_{\alpha'} + n_+ + \#k(s_o)$. We know that $r_{\alpha'} = n_-$ so $j(m) = \mathrm{r}_\alpha + n_+ + \#k(s) = j(m) = n_- + n_+ + \#k(s_o) = n + \#k(s_o) = |L|\ mod\ 2$. □

Corollary 2.13. *For any knot K homogeneous elements in $CK'h(K)$ have odd q degree.*

Proof. of Lemma 2.11
As we know we can write

$$m' = m + \Phi_m$$

$$\Delta' = \Delta + \Phi_\Delta$$

We know that m and Δ do not change q grading but Φ_m and Φ_Δ increase q grading by 4. We showed that for any homogeneous element m, $j(m) = |L|\ mod\ 2$. This means that any homogeneous element has q grading $|L|\ mod\ 4$ or $|L| + 2\ mod\ 4$ so we can write

$$C(D) = C_o(D) \oplus C_e(D)$$

Where $C_o(D)$ contains all elements with q degree with q grading congruent to $2 + |L|\ mod\ 4$ and $C_e(D)$ contains all elements with q degree with q grading congruent to $|L|\ mod\ 4$. Since we know that boundary map will act on these two chain complexes $C_o(D)$ and $C_e(D)$, we can say

$$Kh'(L) \cong Kh'_o(L) \oplus Kh'_e(L)$$

This proves the first statement of lemma 2.11. Now we will show that $s_o + s_{\bar{o}}$ and $s_0 - s_{\bar{o}}$ have different q degree in mod 4 and difference is 2 mod 4.

Lemma 2.14. *$s_o + s_{\bar{o}}$ is the sum of monomials where we have even number of v_+ in these monomials. Where as $s_o - s_{\bar{o}}$ is the sum of monomials where we have odd number of v_+ in these monomials.*

Proof. We use induction on number of circles in the diagram of s_o and $s_{\bar{o}}$. Assume that we have just 1 circle, then $s_o + s_{\bar{o}} = 2v_-$ and $s_o - s_{\bar{o}} = 2v_+$. The lemma is true for 1 circle. Now assume that we have $k+1$ circles and we know for k circles the lemma is true. Let's say for k circles $m = a \otimes b \otimes \cdots \otimes b$ and $n = b \otimes a \otimes \cdots a$. (Note that m and n are tensor product of a and b k times, so what we wrote as m and n is just example). Now take $s_o = a \otimes m$ and $s_{\bar{o}} = b \otimes n$. We have

$$s_o + s_{\bar{o}} = v_+ \otimes (m - n) + v_- \otimes (m + n)$$

By induction, we know that $m + n$ is the sum of monomials where we have even number of v_+ in these monomials and $m - n$ is the sum of monomials where we have even number of

v_+ in these monomials so $s_o + s_{\bar{o}} = v_+ \otimes (m-n) + v_- \otimes (m+n)$ is the sum of monomials where we have even number of v_+ in these monomials. Proof is the same when we take $s_o = b \otimes m$ and $s_{\bar{o}} = a \otimes n$. Also proof for $s_o - s_{\bar{o}}$ is similar. □

Now we can say that monomials with even number of v_+ are the base for one of the two chain complexes and monomials with odd number of v_+ are the base for other chain complex. As a result, $s_o + s_{\bar{o}}$ and $s_o - s_{\bar{o}}$ are in the different chain complexes. □

Corollary 2.15. $s(s_o) = s(s_{\bar{o}}) = s_{min}(K)$

Proof. We know that $[s_o]$ and $[s_{\bar{o}}]$ are basis for Lee homology of knot K. Now assume that we have $[x] \in Kh'(K)$ with the property $s([x]) < s(s_o)$ Since $dimKh'(K) = 2$ $\{[s_o - s_{\bar{o}}], [s_o + s_{\bar{o}}]\}$ is also base for $Kh'(K)$ so we can write $[x]$ as a linear sum of $[s_o + s_{\bar{o}}]$ and $[s_o - s_{\bar{o}}]$. We have

$$s([x]) = s([s_o + s_{\bar{o}}] + [s_o - s_{\bar{o}}]) = s(2[s_o]) = s([s_o])$$

which contradicts with our assumption $s([x]) < s(s_o)$ so $s(s_o) = s(s_{\bar{o}}) = s_{min}(K)$ □

Corollary 2.16. $s_{max}(K) > s_{min}(K)$

Proof. We showed that $[s_o + s_{\bar{o}}]$ and $[s_o - s_{\bar{o}}]$ have different q degree this means at least one of them greater than $s_{min}(K)$. □

Lemma 2.17. *For knots K_1 and K_2, there is short exact sequence*

$$0 \to Kh'(K_1 \# K_2) \xrightarrow{p_*} Kh'(K_1) \otimes Kh'(K_2) \xrightarrow{\partial} Kh'(K_1 \# K_2) \to 0$$

The maps p_* and ∂ are filtered of degree -1.

Proof. See [7] Lemma 3.8. □

Now we can prove proposition 2.9

Proof. Consider the exact sequence of the previous lemma and let $K_1 = K$ and $K_2 = U$, . Denote the canonical generators by s_a and s_b we label these generators by label of circle near the connected sum and denote the canonical generators of U by a and b. We can assume that $S_{max} = s(s_a - s_b)$. From figure A below, we can see that $\partial((s_a - s_b) \otimes a) = s_a$. (We should remember that $m'(b \otimes a) = 0$). We know by the Lemma 2.17 that ∂ is of degree -1 and $s(a) = -1$. We can say that

$$s(s_a - s_b) \otimes a) \leq s_{min} + 1$$

$$s_{max} - 1 \leq s_{min} + 1$$

we know $s_{min} \leq s_{max} - 1 \leq s_{min} + 1$. Since s_{min} and s_{max} are odd, we have

$$s_{max} - 1 = s_{min} + 1.$$

□

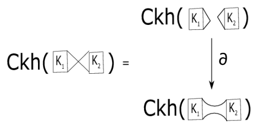


Figure A

2.2.0.2 Properties of s

we check how s behave with respect to the mirror image and direct sum.

Proposition 2.18. *Let $\bar{K}$ be the mirror image of K. Then we have*

$$s_{max}(\bar{K}) = -s_{min}(K)$$

$$s_{min}(\bar{K}) = -s_{max}(K)$$

$$s(\bar{K}) = -s(K)$$

Proof. Suppose that C is a filtered chain complex with filtration $C = C_0 \supset C_1 \supset C_2 \cdots \supset C_n = \{0\}$. Then the dual complex C^* has filtration $\{0\} = C_0^* \subset C_{-1}^* \subset \cdots \subset C_{-n}^* = C^*$, where
$C_{-i}^* = \{x \in C^* | \langle x, y\rangle = 0, \forall y \in C_i\}$. Firstly, we see that $C_i(K) \simeq C_{-i}(\bar{K})$. To see this, let D be a diagram of K and $\bar{D}$ be a diagram of $\bar{K}$. Assume that α is a resolution for D and α' is a resolution for $\bar{D}$ where α' is acquired by changing every digit of α. In other words, change digit if it is 1 in α to 0 to get α' or 0 to 1. We observe $n - r_\alpha = r_{\alpha'}$. Furthermore, we know that positive crossing point in D is negative crossing point at $\bar{D}$. Then we have

$$i = r_\alpha - n_- = n - r_{\alpha'} - n_- = n_+ - r_{\alpha'}$$

But for α' , $n_- = n_+$ of α. We have

$$n_+ - r_{\alpha'} = n_- - r_{\alpha'} = -i.$$

As a result $C_i(K) = C_{-i}(\bar{K})$. Secondly, we have isomorphism

$$V \to V^*$$

$$v_\pm \to v_\mp^*$$

It extends to an isomorphism $V \otimes \cdots \otimes V \to V^* \otimes \cdots \otimes V^*$, so we have an ismorphism $C_{-i}(K) \simeq C_i(K)^*$. Let's denote this isomorphism by Φ. Finally, we need to check if the isomorphism is compatible with boundary map. Assume that we have s and s' two adjacent states and α and α' for their smoothing respectively. Assume that we have merging map between states s and s' so we have $C_\alpha(\bar{K}) \xrightarrow{m} C_{\alpha'}(\bar{K})$, $C_{\alpha'}(\bar{K}) \xrightarrow{\Delta} C_\alpha(\bar{K})$ and the isomorphism Φ between $C_{-i}(K)$ and $C_i(K)$ sends m to $\Delta*$, $C_{\alpha'}(\bar{K})^* \xrightarrow{\Delta^*} C_\alpha(\bar{K})$. Now we have m and Δ^* as boundary maps, so we have $\Delta^* \circ \Phi = \Phi \circ m$. This implies $CKh(\bar{K}) \simeq (CKh(K))^*$. Now we have isomorphism, since we have filtration on $\{0\} = C_0^* \subset C_{-1}^* \subset \cdots \subset C_{-n}^* = C^*$. (Note that we have reverse filtration on C^*) so $s_{max}(K) = -s_{min}(\bar{K})$ and $s_{min}(K) = -s_{max}\bar{K}$. □

Proposition 2.19. $s(K_1 \# K_2) = s(K_1) + s(K_2)$.

Proof. We will use the short exact sequence of Lemma 2.17. Denote the canonical generators of K_i by s_a^i and s_b^i, according to label of circle near the connected sum. We can see that p_* will send the canonical generator of $Kh'(K_1\#K_2)$ to $s_a^1 \otimes s_b^2$. Since we know that p_* has degree -1, we have

$$s(s_o) - 1 \leq s(s_a^1 \otimes s_b^2)$$

Furthermore, because of grading on tensor product of chain complexes we have $s(s_a^1 \otimes s_b^2) \leq s_{min}(K_1) + s_{min}(K_2)$. As a result, we have

$$s_{min}(K_1\#K_2) - 1 \leq s_{min}(K_1) + s_{min}(K_2).$$

Now we apply same argument to $\bar{K_1}$ and $\bar{K_2}$ and we know that $s_{min}(K) = -s_{max}(\bar{K})$, so we get

$$s_{max}(K_1\#K_2) + 1 \geq s_{max}(K_1) + s_{max}(K_2).$$

By converting max to min it is easy to see that

$$s_{min}(K_1\#K_2) - 1 \geq s_{min}(K_1) + s_{min}(K_2).$$

Thus we have

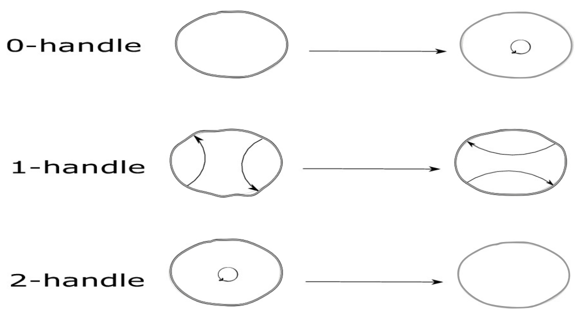
$$s_{min}(K_1\#K_2) = s_{min}(K_1) + s_{min}(K_2) + 1$$

$$s_{max}(K_1\#K_2) = s_{max}(K_1) + s_{max}(K_2) - 1$$

which proves the proposition. □

2.2.0.3 Behaviour under cobordisms

Let L_0 and L_1 be two links in R^3. An oriented cobordism from L_0 to L_1 is a smooth, oriented, compact, properly embedded surface $S \subset R^3 \times [0,1]$ with $S \cap R^3 \times \{i\} = L_i$. In this section, we define and study a map $\phi_S : Kh'(L_o) \to Kh'(L_1)$ induced by such a cobordism. We can write the cobordism as a composition of elementary cobordism where elementary cobordism can be subdivided by two moves : Reidemeister moves which we already know the map and Morse moves which can be seen on the figure below.



Now we will assign function $\phi_S : Kh'(L_o) \to Kh'(L_1)$. We want ϕ_S to be functorial. In other words, given a cobordism $S = S_1 \cup S_2 \cdots \cup S_k$, then we want $\phi_S = \phi_{S_k} \circ \ldots \phi_{S_1}$. Suppose that S is a given cobordism corresponding to the i-th Reidemeister move or its inverse then we define ϕ_S to be p'_{i*} or its inverse. For details of definition of p'_{i*} look at [6] page 5. We

can see that p'_{i*} is degree of 0. For proof of statement of p'_{i*} is degree of 0 see [7] section 6. If S is an elementary cobordism corresponding to a Morse move then we take ϕ_S to be the map induced by $\gamma : CKh'(L_0) \to CKh'(L_1)$. In other words, if the move corresponds to the addition of a 0-handle or a 2-handle, we apply function $\iota' : \mathbb{Q} \to V$, $\iota(1) = (a-b)/2$ or $\epsilon : V \to \mathbb{Q}$, $\epsilon'(a) = \epsilon'(b) = 1$ respectively. If the move correspondens to the addition of 1-handle, the induced map is m' or Δ' depending on whether the move results in a merge or a split at the vertex in question. We can also say ϕ_S is a filtered map of degree 1 for a 0-handle and 2-handle addition and degree -1 for a 1-handle.
The map ϕ_S behaves nicely with respect to canonical generators.

Proposition 2.20. *Suppose S is an oriented cobordism from L_0 to L_1 which is weakly connected, in the sense that every component of S has boundary component in L_0. Then $\phi_S([S_{o_0}])$ is nonzero multiple $\phi_S([S_{o_1}])$.*

Proof. See page 11 proposition 4.1 of [7] □

Corollary 2.21. *if S is connected cobordism between knots K_0 and K_1 then Φ_S is an isomorphism.*

Proof. Fix orientation o on S. Then we have $\{S_{o_0}, S_{\bar{o}_0}\}$ as basis for $Kh'(K_0)$. We show that Φ_S sends basis to

$$\{k_1 S_{o_1}, k_2 S_{\bar{o}_1}\}$$

where k_1,$k_2 \in \mathbb{Q}$ which is basis for $Kh'(K_1)$ so Φ_S is isomorphism. □

Now we can prove one main theorem.

Proof. of Theorem 2.2 Assume that we have a knot $K \subset S^3$ which bounds an oriented surface of genus g. Then there is an orientable connected cobordism of Euler characteristic $-2g$ between K and unknot U in $R^3 \times [0,1]$. Let $x \in Kh'(K)_0$ be a class which has maximal degree. Since Φ_S is an isomorphism, $\Phi_S(x)$ is not zero and since Φ_S has degree $-2g$, we can say that $s(\Phi_S(x)) \geq s(x) - 2g$. We know $s_{max}(U) = 1$, so we have $s(\Phi_S(x)) \leq 1$. It follows that $s(x) \leq 1 + 2g$, so $s_{max}(K) \leq 1 + 2g$ and $s(K) \leq 2g$. To show that $s(K) \geq -2g$, we need to apply the same argument to $\bar{K}$ and we need to use the fact that $s(K) = -s(\bar{K})$. □

Chapter 3

Khovanov homology and topological quantum field theory

In this chapter we talk about the connection between Khovanov homology and topological quantum field theory. We will see some definitions about the cobordism category and Frobenius algebras.

Definition 3.1. Given two closed (n-1)-dimensional manifolds Σ_0 and Σ_1, an (oriented) cobordism from Σ_0 and Σ_1 is an (oriented) n-dimensional manifold M together with the maps

$$\Sigma_0 \to M \leftarrow \Sigma_1$$

such that Σ_0 maps diffeomorphically onto the in-boundary of M and Σ_1 maps diffeomorphically onto the out-boundary of M.

Of couse we have many cobordism between Σ_0 and Σ_1 so we need equivalence between cobordisms. Two cobordisms M and M' from Σ_0 to Σ_1 are called equivalent if there exists a diffeomorphism from M to M' making the diagram below commutative

$$\begin{tikzcd}
 & M' & \\
\Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[ul] \arrow[dl] \\
 & M \arrow[uu, "\simeq"] &
\end{tikzcd}$$

Now we need composition of cobordism. Given one cobordism M_0 from Σ_0 to Σ_1 and M_1 from Σ_1 to Σ_2, then the composition M_0M_1 can be obtained by gluing M_0 to M_1 along Σ_1. It is a cobordism from Σ_0 to Σ_2. Now we can define the cobordism category.

Definition 3.2. The category of $nCob$. The objects of $nCob$ are (n-1)- dimensional closed oriented manifolds. Given two objects Σ_0 and Σ_1, the morphisms from Σ_0 to Σ_1 are equivalence classes of cobordisms from Σ_0 to Σ_1.

Now we can talk about Topological quantum field theory. Roughly, TQFT takes as input spaces and space-times (i.e cobordisms) and associates to them state spaces and time evolution operators. The space is modelled as a closed (n-1)- dimensional manifold, while space-time is an n-dimensional manifold whose boundary represents time 0 and time 1. The state space is a vector space, and a time evolution operator is simply a linear map from the state space of time 0 to the state space of time 1.

Definition 3.3. A topological quantum field theory is a functor F from $nCob$ to $Vect_{\mathbb{k}}$ where $Vect_{\mathbb{k}}$ is the category of vector spaces over $\mathbb{k}$. For an (n-1)-dimensional manifold Σ, $F(\Sigma)$ is a vector space and for a cobordism M between Σ_0 and Σ_1, $F(M)$ is a linear map between $F(\Sigma_0)$ and $F(\Sigma_1)$. The functor F should satisfy the following properies:

1. if $\Sigma = \Sigma_0 \cup \Sigma_1$ then $F(\Sigma) = F(\Sigma_0) \otimes F(\Sigma_1)$
2. $F(\Sigma^*) = F(\Sigma)^*$ where Σ^* is manifold Σ with opposite orientation and $F(\Sigma)^*$ denotes the dual vector space.
3. For the empty (n-1)-dimensional manifolds Σ, $F(\Sigma) = \mathbb{k}$.

Next, we can talk about Frobenius algebra.

Definition 3.4. A unital associative $\mathbb{k}$-algebra is a $\mathbb{k}$-vector space with two $\mathbb{k}$-linear maps

$$\mu : A \otimes A \to A \quad \eta : \mathbb{k} \to A$$

(multiplication and unit map) satisfying the associativity law and the unit law:

$$\mu(\mu \otimes \eta) = \mu(\eta \otimes \mu) \qquad \mu(\eta(1) \otimes Id_A) = Id_A = \mu(Id_a \otimes \eta(1))$$

Definition 3.5. A finite-dimensional, unital, associative algebra A defined over a field $\mathbb{k}$ is said to be a Frobenius algebra if A is equipped with a linear functional

$$\delta : A \to \mathbb{k}$$

such that the kernel of δ contains no nonzero ideal of A.
Equivalently, if A can be equipped with a nondegenrate bilinear form $\sigma : A \times A \to \mathbb{k}$ that satisfy the following equation $\sigma(a.b, c) = \sigma(a, b.c)$.
With this definition we can define the comultiplication $\Delta : A \to A \otimes A$ by $\Delta(v) = \sum_i v'_i \otimes v''_i$ being the unique element such that for all $w \in A$, $\mu(v, w) = \sum_i v'_i < v''_i, w >$.

Some examples for Frobenius algebra.

1. Let A be a finite division ring over k. Since a divison ring has no nontrivial left ideal, any nonzero linear map $A \to \mathbb{k}$ will make A a Frobenius algebra.
2. Any matrix ring $Mat_n(\mathbb{k})$ is a Frobenius algebra with the usual trace map $\delta : Mat_n(\mathbb{k}) \to \mathbb{k}$.

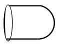
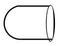
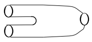
3. $A = \mathbb{k}[x]/(x^2)$ is a Frobenius algebra with linear functional
   $\delta : A \to \mathbb{k}$
   $1 \to 0$
   $x \to 1$

3.0.0.1 TQFT and Frobenius algebra

There is a deep relation between topological quantum field theories and Frobenius algebras: They determine each other. If we have a topological quantum field theory then we can write a Frobenius algebra and it is true for vice versa. Assume that we have a TQFT from $2cob$ to $Vect_{\mathbb{k}}$, then we can write a Frobenius algebra from scratch.

- take a circle and send it to the vector space A via TQFT.
   $\longrightarrow A$

  and we can define multiplication, comultiplication, unit map and linear map via TQFT as below.

-  $\longrightarrow Id : A \to A$
-  $\longrightarrow \delta : A \to \mathbb{k}$
-  $\longrightarrow \eta : \mathbb{k} \to A$
-  $\longrightarrow \mu : A \otimes A \to A$
-  $\longrightarrow \Delta : A \to A \otimes A$

If we know a Frobenius algebra, we can find a TQFT also,hence

Theorem 3.6. *The category of Frobenius algebras and the category of topological quantum field theories are equivalent.*

Proof. [8] □

3.0.0.2 TQFT and Khovanov homology

In Khovanov homology, we have circles after we resolve our knot diagram, we can see these circles as 1-dimensional closed manifolds and we have cobordism between any 1-dimensional manifold. A TQFT takes a circle and sends it to a vector space V and it takes cobordisms and sends to our maps m and Δ. Furthermore, we have
$\epsilon : V \to k$
$v_+ \to 0$
$v_- \to 1.$

It can be seen that a vector space V in Khovanov homology is a Frobenius algebra with multiplication map m, comultiplication map Δ and map ϵ. It is obvious that V is a $\Bbbk$ algebra and it is easy to see that ker of ϵ has no nonzero left ideal. Kernel of ϵ is the one dimensional vector space which is generated by v_+. We have v_+ in any non-trivial ideal in kernel of ϵ. We know $m(v_- \otimes v_+) = v_-$, so v_- is also in any non-trivial ideal. This means that any non-trivial ideal is equal to V. This is a contradiction because V is not in the kernel of ϵ. As a result, the kernel of ϵ has no non-trivial left ideal so V is Frobenius algebra.

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