Trees without Models
Truth-Valuational Semantics and the Tableau Method

by

Péter Susánszky

Submitted to

Central European University

Department of Philosophy

In partial fulfilment of the requirements for the degree of

Master of Arts

Supervisor: Professor Hanoch Ben-Yami

Budapest, Hungary

2018

I, the undersigned, Péter Susánszky, candidate for the degree of Master of Arts at the Central European University Department of Philosophy, declare herewith that the present thesis is exclusively my own work, based on my research and only such external information as properly credited in notes and bibliography. I declare that no unidentified and illegitimate use was made of works of others, and no part the thesis infringes on any person's or intstitution's copyright. I also declare that no part of the thesis has been submitted in this form to any other institution of higher education for an academic degree.

Budapest, 4 June 2018

Signature

Abstract

In this thesis, we will introduce truth-valuational semantics for modal logic and compare it to standard, Kripke-style model-theoretic semantics. We will show the detailed proofs for the soundness and semi-strong (roughly, restricted to finite premise sets) completeness of the truth-valuational propositional modal semantics for the logic K relative to the prefixed propositional modal tableau system K, and will sketch extensions to the logics/systems T, B, $K4$, $S4$ and $S5$. Afterwards, we will sketch three different ways of how one can strengthen the semi-strong completeness theorems for these logics to strong completeness, and will evaluate the pros and cons of each approach. We conclude that a proper solution remains forthcoming.

Acknowledgments

First and foremost, I would like to thank my supervisor, Hanoch Ben-Yami for the $\aleph_0$ times he helped me with problems related to this thesis. I would also like to thank Edi Pavlovic for reading and commenting on a very early draft of it, my family for support, and my wise friends for the great conversations, especially Tamás Antal, Gábor Kőhidai, and Gergő Tóth.

Table of Contents

List of Tables

What is dispensable simply is not of the essence.

Hugues Leblanc

1 Introduction

Practically every introductory or philosophy of logic book starts out by characterizing logic in a rough and ready way as the discipline that concerns itself with arguments and their correctness. So logic is about 'tell[ing] good arguments from bad arguments' (Tomassi 1999, 2), 'it is the study of what constitutes correct reasoning' (Klenk 2002, 2), it is 'the systematic study of principles of correct reasoning' (Jacquette 2002, 2), it is about 'logical consequence' (Sider 2010, 1), it is 'the science of deduction' (Jeffrey 2006, 1), it is 'a handbook (of a highly sophisticated kind, to be sure) for drawing inferences' (Leblanc 1973a, 241), it is 'the study of valid arguments' (Newton-Smith 1985, 1) (cf. (Hodges 1991, 13)), it is 'concerned with the principles of valid inference' (Kneale and Kneale 1962, 1) and its aim is 'to discriminate valid from invalid arguments' (Haack 1978, 1). With validity, the last three definitions already imply that logic has to do with truth, at least as long as it concerns arguments (i.e., that *if* the premises are true, *then* the conclusion must be true as well). According to Dummet (1973, 432), Frege went so far (and Russell followed him) as to claim that the object of study in logic is truth *itself*, at least logical truth (cf. Sider (2010, 2)). But, as Dummett (1973, 432-3) also writes:

> The traditional answer to the question what is the subject-matter of logic is, however, that it is, not truth, but inference, or, more properly, the relation of logical consequence. This was the received opinion all through the doldrums of

logic, until the subject was revitalized by Frege; and it is, surely, the correct view.

Note that Dummett's main point, as he goes on to explain, is not that one should not have an account of validity which captures the semantic aspect of logical consequence, or an account of logical truth (which is really just a special case of the former). Rather, the problem, according to him, is that Frege's axiomatic approach to deduction took *logical truths* as basic in proof theory, which then had to be corrected by Gentzen's sequent calculi and natural deduction, both being deductive systems with only *inference rules*, as is demanded by what logic's proper subject-matter is. As Gentzen (1964, 288) himself writes:

> My starting point was this: The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return. I intended, first of all, to set up a formal system which came as close as possible to actual reasoning. The result was a "calculus of natural deduction".

Insofar as the tableau method, which will be used in this thesis, does away with axioms and makes use only of inference rules, Dummett's criticism is eluded (in contrast to, e.g., the Frege-Hilbert style axiomatic approach Leblanc uses in his publications on truth-valuational semantics). The tableau method is also an intuitive modelling of a natural way of reasoning, different from the kind of reasoning natural deduction models. Specifically, it models the way we reason when we think of validity as the fact that if the premises are true, then the conclusion cannot be false, so if we suppose that the premises are true and the conclusion is false, we will run into a contradiction, i.e., no such example will be forthcoming.[1]

[1]Beth (1969, 40), the inventor of tableaux writes that together with him, Hintikka too 'stresses the interpretation of all proofs of logical truth as proofs of impossibility of counter-examples'. More generally, in his paper, similar to what has been said here, he argues that there are two different but equivalent methods of proving logical consequence, one through straightforward derivation (natural deduction), the other through trying systematically to find a counter-example (tableaux).

However, this is not the main point. What needs to be emphasized is that in all of the above definitions, logic, even in its semantic (as opposed to its syntactic) conception, is *at most* concerned with logical truths (whatever one takes *those* to be) and truth-value relations between sentences. Let us call this the *canonical conception*, or what Leblanc (1973a, 241) called 'logic straight'.

Propositional logic is closest in its semantics to the canonical conception. In checking for the validity of arguments, we look at *truth-value assignments*, or what we have called *truth-valuations*, and check whether the conclusion is *true* under all valuations that make the premises true. If not, the argument is invalid. Logical truth is then simply a formula which is true under all assignments, or more closely to the canonical conception, they are conclusions which are true no matter the premises.

With model-theoretic semantics for different logics, we seem to leave behind propositional purity. The most straightforward example of this is first-order logic. According to the traditional Tarskian conception, in giving an account of truth-value relations between sentences in a first-order language, one needs to posit a model, which consists of a non-empty set of objects (the domain) and an interpretation function, which interprets the expressions of the language in relation to the domain. Of course, the most straightforward aspect of this is when an interpretation function takes a name and outputs its 'referent' in the domain.

Such a model can be thought of as supplying the *necessary* mathematical machinery to give an account of what is *needed*, i.e., a *logical* semantics, an account of truth-value relations between sentences of the language.[2] But insofar as that machinery posits something over and above what is needed, it is not essential for a semantics of logic, and if it is dispensable,

[2]More radically, it can be viewed as the universal form of semantics proper, of any account of meaning, where meaning is construed as something 'whereby symbols are associated with aspects of the world', as Lewis held in his General Semantics (1970, 19). This is a much more robust claim, where models are not viewed as a necessary evil for giving an account of truth-value relations in *logical semantics*, but an essential part to any semantics. Cf. Quine's different but equally radical view about Tarskian semantics in his (1948) and (1969).

we should dispense with it. And indeed, this can be done, as Leblanc showed in his (1976) and (2001).

Given that propositional modal logic is a close cousin of first-order logic[3] it is no surprise that we have a similar problem in the former as in the latter. The importance of Kripke's seminal papers on modal logics was precisely that they successfully adapted the Tarskian model-theoretic framework for providing a semantics for different modal logics, which were then also easily extendible to first-order modal logics. At the same time, by giving an account of truth-value relations between sentences with model-theoretic machinery, it again involved something over and above what was *needed*, i.e., models, and as we will show, something over and above what was *necessary*.[4]

In this thesis, I will follow the research programme initiated by Leblanc around 50 years ago, which aimed to reformulate any model-theoretic semantics for a given logic into a non-model-theoretic, truth-valuational one. I will prove the *weak* completeness of truth-valuational semantics for the propositional modal logic K (sketching the extensions to T, $K4$, B, $S4$, $S5$) relative to the Fitting-style prefixed modal propositional tableau system K (and the corresponding extensions), and will show three ways to extend such results to *strong* completeness. Completeness proofs for different *axiomatic* systems can be found in Dunn's (1973), Leblanc's (1976) and Ben-Yami's (MS). We will return to these in due course, but a full account of similarities and differences will not be possible in the confines of these pages.

The thesis will also provide a more in-depth analysis of truth-valuational semantics for propositional modal logics than can be found in the literature. It will identify a problem for truth-valuational semantics related to the tableaux method (or a problem for the tableaux method related to truth-valuational semantics, depending on how one may construe it),

[3]We think here mainly of the equisatifiability theorem that a modal formula is satisfiable in a Kripke model *iff* its 'standard translation' into the 'first-order correspondence language' is satisfiable in the same model now thought of as a first-order one. For details, see, e.g., Blackburn and van Benthem's (2007).

[4]Interestingly, though not outright hostile, Quine also had reservations with Kripke's modal-theoretic semantics given in terms of possible worlds, which he voiced in his less well-known (1972) book review.

one that is surprisingly hard, if even possible, to overcome. The problem will underline a non-trivial difference between model-theoretic and truth-valuational semantics: the lack of equivalents to distinct but indistinguishable 'possible worlds' in the latter.

Remark. Besides truth-valuational semantics, there are several other 'non-standard' or 'alternative' approaches to the semantics of modal logics. For a short overview of these, see Section 7 of Blackburn and Van Benthem's (2007). An alternative to the usual algebraic approach to modal logics can be found in Agudelo-Agudelo and Carnielli's (2017). Unfortunately, due to length constraints we cannot discuss here the formal and philosophical similarities and differences of these.

2 Syntax

We first introduce the language we will be working with. The language $\mathcal{L}$ of propositional modal logic is an enrichment of standard propositional logic. It is based on the following alphabet.

Definition 1 (Alphabet of $\mathcal{L}$)**.** The alphabet $\mathcal{A}$ of $\mathcal{L}$ is the union of all members of the ordered quadruple $< Var, Log, Mod, Pun >$.

1. The set $Var = \{p_1, p_2, p_3, \dots, p_n, \dots\}$ contains $\aleph_0$ propositional variables.
2. The set $Log = \{\neg, \vee, \wedge, \rightarrow\}$ contains the logical connectives 'not', 'or', 'and' and 'if then', respectively.
3. The set $Mod = \{\Box, \Diamond\}$ contains the two modal operators 'necessarily' and 'possibly', respectively.
4. The set $Pun = \{(,)\}$ consists of the 'left parenthesis' and the 'right parenthesis'.

In the following, the metavariables $P, Q, R, S, \dots$ will range over propositional variables, while the metavariables $X, Y, Z, \dots$ will range over arbitrary formulas. The set of our formulas is given by the following definition.

Definition 2 (Formulas of $\mathcal{L}$)**.** The set of formulas $\mathcal{F}$ of $\mathcal{L}$ is a subset of all finite strings over the alphabet of $\mathcal{L}$, characterized as follows.

1. All propositional variables in Var are in $\mathcal{F}$.

2. If X is in $\mathcal{F}$, so is $\neg X$.
3. If X and Y are in $\mathcal{F}$, so are $(X \vee Y)$, $(X \wedge Y)$ and $(X \rightarrow Y)$.
4. If X is in $\mathcal{F}$, then so are $\Box X$ and $\Diamond X$.
5. Nothing else is in $\mathcal{F}$.

We will at times call members of Var *atomic formulas* or, when it is clear from the context, simply *variables*, and in general, we will call a *literal* every propositional variable P or its negation $\neg P$. Later on, it will be convenient to consider the connectives and operators not only as signs occurring in complex formulas but as *syntactic operations* which output these complex formulas from their constituent formula(s). Thus, if $\circ^2$ is any two-place connective, whenever we apply the function $\circ^2_f$ on two formulas X and Y, which maps them to $X \circ^2 Y$, we write $\circ^2_f(X, Y) = X \circ^2 Y$, and similarly for one-place connectives and operators. Given any set S of formulas and the syntactic operations defined on some members of S, $< S, \circ^1_{f_1}, \circ^2_{f_1}, ..., \circ^1_{f_n}, \circ^2_{f_m} >$ will be denoted by $\mathbf{S}$, and we will call $\mathbf{S}$ a *syntax*.

We also define the *complexity* of our formulas.

Definition 3 (Complexity of Formulas)**.** The complexity of formulas of $\mathcal{F}$ is given by the function $c : \mathcal{F} \rightarrow \mathbb{N}$, defined recursively for any P, X, Y as follows:

1. For any propositional variable P in $\mathcal{F}$, $c(P) = 0$.
2. If $c(X) = n$, then $c(\neg X)$, $c(\Box X)$ and $c(\Diamond X)$ are all $n + 1$.
3. If $c(X) = n_1$ and $c(Y) = n_2$, $c(X \vee Y)$, $c(X \wedge Y)$ and $c(X \rightarrow Y)$ are all $n_1 + n_2 + 1$.

Later on, it will be important to know what cardinality of propositional variables *occur* in a given set of formulas, and what cardinality of variables are *omitted*. Given that the set of all variables of the language is denumerably infinite (i.e., $\aleph_0$), we will use the following terminology:

Definition 4 (Omitting Variables)**.** Given any set S of formulas of the language $\mathcal{L}$, we say S

omits $\aleph_0$ variables *iff* there are $\aleph_0$ propositional variables of $\mathcal{L}$ not occurring in any formula in S.

Note that since any formula is finite, it is trivial that $\aleph_0$ propositional variables do not occur in any one of them, hence the definition's restriction to sets only.

Related to this, a propositional modal language as defined above can be extended to another propositional modal language just by adding new propositional variables to it. We will call this a *variable extension*.

Definition 5 (Variable Extension)**.** If $\mathcal{L}_1$ is a propositional modal language as in Definitions 1 and 2 with $\aleph_0$ propositional variables in Var_1, then by the *variable extension* $\mathcal{L}_2$ *of* $\mathcal{L}_1$, we mean the language $\mathcal{L}_2$ which results from adding $\aleph_0$ new variables $p'_1, ..., p'_n, ...$ to Var_1.

Then, we have the following trivial proposition.

Proposition 6. *Given any set S of formulas of $\mathcal{L}_1$, in every variable extension $\mathcal{L}_2$ of $\mathcal{L}_1$, the same set S of formulas omits $\aleph_0$ variables.*

We may arrive at a similar proposition through a different technique, which is lengthier but does not require extending to new languages. What we want to prove is that if S is any set of formulas, there is a set of formulas syntactically isomorphic to it that omits $\aleph_0$ variables.

Take a function $u : \mathbb{N}^+ \to \mathbb{N}^+$ defined by $u(n) = 2n - 1$.

Proposition 7. *The function u is injective from $\mathbb{N}^+$ to $\mathbb{N}^+$, and bijective between $\mathbb{N}^+$ and its proper subset of all uneven natural numbers $\mathbb{N}^u$.*

Proof. We do not prove this here as it is lengthy and tedious. □

We can then define what we will call a 'variable rewrite'.

Definition 8 (Variable Rewrite)**.** By a variable rewrite, we mean the function $R : \mathcal{F} \to \mathcal{F}$ defined recursively on the complexity of formulas in $\mathcal{F}$ as follows. As the base case, for any propositional variable p_n, set $R(p_n) = p_{u(n)}$. Then, suppose $R(X)$ is defined for formulas up to complexity $n-1$. We define n as follows. If $\circ^1$ is any one-place connective or modal operator and X is a formula of form $(\circ^1 Y)$, set $R(\neg Y) = \neg R(Y)$. If $\circ^2$ is any two-place propositional connective and X is of form $(Y \circ^2 Z)$, then set $R(Y \circ^2 Z) = R(Y) \circ^2 R(Z)$. Finally, by the variable rewrite of a set of formulas S, we mean the variable rewrite $R(X)$ of all $X \in S$. We will sometimes call a variable rewrite $R(X)$ itself, i.e., the output formula of R given an input formula X.

We then have the following proposition.

Proposition 9. *If X is any formula, then any variable rewrite $R(X)$ of X is of the same complexity as X.*

Proof. The proof is by induction on the complexity of X. We won't show it here. □

Proposition 10. *The variable rewrite R is a bijective function from any set of formulas S to $R(S)$.*

Proof. We have to show that R is both injective and surjective. Trivially, since $R(S)$ is the variable rewrite of S, i.e., for all X, $X \in R(S)$ *iff* there is a $Y \in S$ such that $R(Y) = X$, $R : S \to R(S)$ is surjective. That the function is injective is harder to show.

The proof is by double induction on the complexity of formulas X and Y.[1] What we will show is that whenever $R(X) = R(Y)$, $X = Y$, or equivalently, whenever $X \neq Y$, $R(X) \neq R(Y)$, by showing it first for $c(X) = c(Y) = 0$, then showing that if the hypothesis holds for $c(X) = n$ and $c(Y) = m$, then it holds for $c(X) = n$ and $c(Y) = m+1$, and it holds for $c(X) = n+1$ and $c(Y) = m$.

[1] More can be found on double induction in Gunderson's (2014, sect. 3.5).

Take the base case. Then, X is p_n and Y is p_k. Suppose $p_n \neq p_k$. Then, $n \neq k$. By definition of R, $R(p_n) \neq R(p_k)$.

Suppose the hypothesis holds for $c(X) = n$ and $c(Y) = m$. We show it for the pairs $< n, m+1 >$ and $< n+1, m >$.

1. Take $c(X) = n, c(Y) = m + 1$ first. Suppose $X \neq Y$. Suppose $c(X) \neq c(Y)$, i.e., $n \neq m + 1$. By Proposition 9, for any Z, $c[Z] = c[R(Z)]$. Since $c(X) \neq c(Y)$, $c[R(X)] \neq c[R(Y)]$, so obviously, $R(X) \neq R(Y)$.

   While still supposing $X \neq Y$, suppose $c(X) = c(Y)$, i.e., $n = m+1$. Then, $c[R(X)] = c[R(Y)]$. By the definition of formulas, $R(X) = R(Y)$ *iff* both $R(X)$ and $R(Y)$ are governed by the same connective or operator and their immediate constituent formula(s) are identical, i.e., $R(X) = \circ^1 R(Z_1)$ and $R(Y) = \circ^1 R(Z_2)$, where $R(Z_1) = R(Z_2)$, or $R(X) = R(Z_1) \circ^2 R(Z_3)$ and $R(Y) = R(Z_2) \circ^2 R(Z_4)$, where $R(Z_1) = R(Z_2)$ and $R(Z_3) = R(Z_4)$.

   (a) Suppose $R(X) = \circ^1 R(Z_1)$ and $R(Y) = \circ^1 R(Z_2)$. Then, since Z_1 and Z_2 are of smaller complexity than X and Y, the hypothesis holds, so $Z_1 = Z_2$. But then since $X = \circ^1 Z_1$ and $Y = \circ^1 Z_2$, $X = Y$ contra hypothesis.

   (b) Suppose $R(X) = R(Z_1) \circ^2 R(Z_3)$ and $R(Y) = R(Z_2) \circ^2 R(Z_4)$, where $R(Z_1) = R(Z_2)$ and $R(Z_3) = R(Z_4)$. Then again, since Z_1, Z_2, Z_3, Z_4 are of smaller complexity than X and Y, the hypothesis holds, so $Z_1 = Z_2$ and $Z_3 = Z_4$. But then since $X = Z_1 \circ^2 Z_3$ and $Y = Z_2 \circ^2 Z_4$, $X = Y$ contra hypothesis.

2. The reasoning is similar the other way around.

This concludes the proof. $\square$

We may now prove the isomorphism we wanted.

Proposition 11. *Given any set S of formulas and its variable rewrite $R(S)$, call it S^*, the syntax $\mathbf{S}$, built on S with the syntactic operations defined on some elements of S, is isomorphic to $\mathbf{S}^*$ built with the same operations on S^*. In fact, it is R that induces an isomorphism $R : \mathbf{S} \rightarrow \mathbf{S}^*$.*

Proof. The proof is simple. By Proposition 10, we already know that $R : S \rightarrow S^*$ is bijective. Since the syntactic operations of $\mathbf{S}^*$ are given by the identity function from those of $\mathbf{S}$, there is a one-to-one correspondence between the two. Thus, we only need to show that $R[\circ^1_f(X)] = \circ^1_f(R[X])$ and $R[\circ^2_f(X,Y)] = \circ^2_f(R[X], R[Y])$, whenever $\circ^1_f$ is defined on some $X \in S$ or $\circ^2_f$ on some $X, Y \in S$.

The first part is by induction on the complexity of X, where X is in the set of outputs for $\circ^1_f$. Then, X is of form $\circ^1 Y$, so $\circ^1_f$ is defined on Y and its output for it is X.

1. Take the base case, where $\circ^1_f(P) = \circ^1 P$. We want to show that $R[\circ^1_f(P)] = \circ^1_f(R[P])$. By Definition 8, $R[\circ^1_f(P)] = \circ^1_f(R[P])$.

2. Suppose the hypothesis holds for any formula of complexity n. We show it for $n+1$. We want to show that $R[\circ^1_f(X)] = \circ^1_f(R[X])$. By Definition 8, $R[\circ^1_f(X)] = \circ^1_f(R[X])$.

The second part is also by induction on the complexity of X, but this time X being in the set of outputs for $\circ^2_f$. Then, X is of form $Y \circ^2 Z$, so $\circ^2_f$ is defined on Y and Z and its output for the pair is X.

1. Take the base case, where $\circ^2_f(P,Q) = P \circ^2 Q$. We want to show that $R[\circ^2_f(P,Q)] = \circ^2_f(R[P], R[Q])$. By Definition 8, $R[\circ^2_f(P,Q)] = \circ^2_f(R[P], R[Q])$.

2. Suppose the hypothesis holds for any formula of complexity n. We show it for $n+1$. We want to show that $R[\circ^2_f(Y,Z)] = \circ^2_f(R[Y], R[Z])$. By Definition 8, $R[\circ^2_f(Y,Z)] = \circ^2_f(R[Y], R[Z])$.

This concludes the proof. $\square$

Finally, we have the following trivial consequence of the propositions above.

Proposition 12. *If S is any set of formulas of $\mathcal{L}$ that does not omit $\aleph_0$ variables of $\mathcal{L}$, the variable rewrite $R(S)$ of S (which is syntactically isomorphic to S) does omit $\aleph_0$ variables of $\mathcal{L}$, namely, every variable indexed by an even number.*

Note that S may be $\mathcal{F}$, in which case we have that the set of all formulas can be rewritten into a set which omits $\aleph_0$ propositional variables.

3 Semantics

In this chapter, we will introduce two semantics. The first is a variant of the standard, modal-theoretic or Kripke one (which can be found in its classic formulation in his (1963)). The second takes inspiration from Leblanc's truth-valuational semantics from his (1973b) (see also his (1973a) and (1976)), Dunn's (1973), and Ben-Yami's (MS).

Note that even though we do not part ways with the word 'semantics', the semantics presented here does not give an account of the meaning of sentences of the language as a semantics for any natural language would (purport to) do, nor it should aspire to do so. It merely gives an account of truth-value relations between sentences of the language.

3.1 Model-Theoretic Semantics

In the following, we will introduce the standard, model-theoretic or Kripke semantics for propositional modal logic, which uses 'possible worlds' in its models. All definitions, terminology and notation are from Fitting and Mendelsohn's (1998), unless otherwise noted.

Definition 13 (Frame)**.** A modal frame is an ordered double (pair, for short) $< \mathcal{G}, \mathcal{R} >$, where $\mathcal{G}$ is a non-empty set whose members are traditionally called 'possible worlds', and $\mathcal{R}$ is a binary relation defined on $\mathcal{G}$, generally called the 'accessibility relation'. If Γ and Δ are possible worlds and $\Gamma\mathcal{R}\Delta$ holds, we say that Δ is accessible to Γ, or that Δ is possible relative to Γ.

In the following, the metavariables $\Gamma, \Delta, \Theta, \ldots$ will range over members of $\mathcal{G}$.

Definition 14 (Model)**.** A *(modal propositional) model* $\mathcal{M}$ is an ordered triple $< \mathcal{G}, \mathcal{R}, \Vdash >$, where $\mathcal{G}$ and $\mathcal{R}$ are as before and $\Vdash$ is a binary relation between members of $\mathcal{G}$ and members of Var. If Γ is a possible world, P is any propositional variable and $\Gamma \Vdash P$ holds, we say that P is true at the possible world Γ. If, instead, $\Gamma \nVdash P$ holds, we say that P is false at Γ.

Definition 15 (Truth in a Model)**.** Take any model $\mathcal{M}$. We extend the relation $\Vdash$ to arbitrary formulas of $\mathcal{F}$ as follows. For each Γ in $\mathcal{G}$:

1. $\Gamma \Vdash \neg X$ *iff* $\Gamma \nVdash X$.
2. $\Gamma \Vdash (X \vee Y)$ *iff* $\Gamma \Vdash X$ or $\Gamma \Vdash Y$.
3. $\Gamma \Vdash (X \wedge Y)$ *iff* $\Gamma \Vdash X$ and $\Gamma \Vdash Y$.
4. $\Gamma \Vdash (X \rightarrow Y)$ *iff* $\Gamma \nVdash X$ or $\Gamma \Vdash Y$.
5. $\Gamma \Vdash \Box X$ *iff* for every $\Delta \in \mathcal{G}$, if $\Gamma \mathcal{R} \Delta$, then $\Delta \Vdash X$.
6. $\Gamma \Vdash \Diamond X$ *iff* there is a $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$ and $\Delta \Vdash X$.

We will also introduce a definition of our own which will prove to be helpful later on when we discuss the parallels between propositional logic and modal propositional logic.

Definition 16 (U-model)**.** A U-model is a propositional modal model such that for all $\Gamma, \Delta \in \mathcal{G}$, if for all P, $\Gamma \Vdash P$ if and only if $\Delta \Vdash P$, then $\Gamma = \Delta$.

How should one introduce the modal equivalent of truth-functional tautology and first-order validity? As expected, by the above definition of truth in a model, we know that all truth-functional tautologies will be true at all worlds in all models. The modal case, however, is less obvious. Importantly, by relativising the validity of modal formulas to specific classes of frames, we can consider a variety of different valid formulas. Thus, given a class of frames

L defined by certain frame conditions on the accessibility relation $\mathcal{R}$, we say that a formula is L-valid if it is true at all worlds in all models 'based on' such frames. Compare this with first-order logic. There, a valid formula is defined as one which is true in all models. This can be adapted to modal propositional logic as is if by 'truth in a model', we mean a formula being true at every world of every model. Since this disregards frame conditions altogether, it is equivalent to saying that such formulas are L-valid, where L is the class of all frames, i.e., we do not specify any conditions to be met by frames. The class of all frames and the logic determined by the class are both called K. We say that formulas which are true in all models based on any frame are K-valid, and the logic K is just the set of all K-valid formulas. In essence, K allows us to regard any truth-functional tautology as necessary, i.e., for any truth-functional tautology X, $\Box X$ is K-valid. Note that by definition, K-valid formulas are also L-valid for any class L of frames since $L \subseteq K$. For any classes L_1 and L_2 of frames, if $L_1 \subseteq L_2$, L_2 is said to be a sublogic of L_1.

Definition 17 (L-validity)**.** A model $\mathcal{M} =< \mathcal{G}, \mathcal{R}, \Vdash >$ is *based on* the frame $< \mathcal{G}, \mathcal{R} >$. A formula X is *valid in a model* $\mathcal{M}$ if it is true at every world $\Gamma \in \mathcal{G}$ of $\mathcal{M}$. A formula X is *valid in a frame* if it is valid in every model based on that frame. Then, if L is a class of frames, X is L-valid *iff* X is valid in every frame in L. If X is L-valid, we write $\vDash_{L_M} X$.

Parallel to first-order logic, we can also define L_M-satisfiability. We want our definition to follow the first-order theorems that X is valid *iff* $\neg X$ is not satisfiable. The following is not in Fitting and Mendelsohn's (1998), but standard in the literature.

Definition 18 (L_M-Satisfiability for Formulas)**.** A formula X is L_M-satisfiable *iff* there is a model $\mathcal{M}$ based on a member of a class of frames L where there is a $\Gamma \in \mathcal{G}$ such that $\Gamma \Vdash X$. Then, a set S of formulas is L_M-satisfiable *iff* all $X \in S$ are L_M-satisfiable at the same world of the same model.

In general, as exemplified above, we use the subscript M to differentiate model theoretic definitions and abbreviations from their truth-valuational equivalents (with subscript v). However, we will omit such subscripts at will if context disambiguates which is intended. We may also omit the explicit specification of the logic under consideration if no ambiguity results.

The above mentioned theorem follows immediately from the definitions. The proof is mine.

Proposition 19. *A formula X is L-valid iff $\neg X$ is not L_M-satisfiable.*

Proof. From left to right. Suppose that a formula X is L-valid. Suppose also that $\neg X$ is L_M-satisfiable. Then, by the first, X is true at all worlds in all models based on an L-frame. By the second, there is a model based on an L-frame in which there is a world at which $\neg X$ is true. By Definition 15, we have a contradiction. From right to left. Suppose that $\neg X$ is not L_M-satisfiable. Then, there is no world in any model based on an L-frame where $\neg X$ is true. Equivalently, X is true at all worlds in all models based on an L-frame. By Definition 17 then, X is L-valid. □

The most well-known classes of frames and their logics are based on some simple conditions.

Definition 20 (Properties of Frames)**.** A frame $< \mathcal{G}, \mathcal{R} >$ is:

1. *reflexive* if $\Gamma\mathcal{R}\Gamma$, for all $\Gamma \in \mathcal{G}$;
2. *symmetric* if $\Gamma\mathcal{R}\Delta$ then $\Delta\mathcal{R}\Gamma$, for every $\Gamma, \Delta \in \mathcal{G}$;
3. *transitive* if $\Gamma\mathcal{R}\Delta$ and $\Delta\mathcal{R}\Omega$ then $\Gamma\mathcal{R}\Omega$, for every $\Gamma, \Delta, \Omega \in \mathcal{G}$;

These properties of frames then combine to give the frame conditions presented in Table 1 that each determine a class and a logic.[1]

[1] Of course, these are not all. For a fairly long list of modal logics, their corresponding axioms and frame conditions, see Hughes and Cresswell's (1996). As they note, 'any normal modal system may be defined by

Logic	Frame Conditions
K	no conditions
T	reflexive
B	reflexive, symmetric
K4	transitive
S4	reflexive, transitive
S5	reflexive, symmetric, transitive

Table 1: Modal Logics and Frame Conditions

3.2 Truth-Valuational Semantics

In the truth-valuational approach to the semantics of modal propositional logic, we do away with models altogether. Instead, we revert to the truth-value assignments of the semantics of standard propositional logic and extend the approach to modal cases. Take first the propositional case. We rely on the definitions, terminology and notation of Smullyan's (1995) with some inessential tweaks. We first introduce a set $\mathcal{P} = \{t, f\}$ with elements *True* and *False*, respectively. We then define a valuation for a set of formulas as follows.

Definition 21 (Valuation)**.** For any set S of formulas, by a valuation v we mean a function $v : S \to \mathcal{P}$. For any $X \in S$, if $v(X) = t$, we say that X is *true under* v and if $v(X) = f$, we say that X is *false under* v. The value $v(X)$ under v (i.e., t or f) is called the truth value of X under v.

Two valuations of a set S can agree and disagree on formulas and are identical or distinct relative to that.

Definition 22 (Agreement and Distinctness)**.** Two valuations v_1 and v_2 *agree* on a formula X *iff* $v_1(X) = v_2(X)$, and *disagree* otherwise. Two valuations agree on a set S of formulas *iff*

a list of axioms' in addition to K (and the same is true for non-normal modal systems where K need not be included). In turn, with regards to the semantics, we may always ask what logic the class L of frames determines.

they agree on all members of S, and disagree otherwise. Then, two valuations are identical relative to S if they both agree on S, and are distinct relative to S otherwise. If two valuations agree on the set $\mathcal{F}$ of all formulas of the language, then they are identical *simpliciter*.

In classical propositional logic and modal propositional logic, we are interested only in *Boolean* valuations, which are defined as follows.

Definition 23 (Boolean Valuation)**.** By a *Boolean* valuation v_B we mean a valuation v of $\mathcal{F}$ of $\mathcal{L}$ for which the following conditions hold for any $X, Y \in \mathcal{F}$.

1. $v(\neg X) = t$ *iff* $v(X) = f$.
2. $v(X \wedge Y) = t$ *iff* $v(X) = t$ and $v(Y) = t$.
3. $v(X \vee Y) = f$ *iff* $v(X) = f$ and $v(Y) = f$.
4. $v(X \rightarrow Y) = f$ *iff* $v(X) = t$ and $v(Y) = f$.

From now on, we only concern ourselves with Boolean valuations, thus any valuation v mentioned will be understood to be a Boolean one. Note that the non-modal cases in Definition 15 parallel those in Definition 23. In particular, to any Boolean valuation v of the non-modal subset $\mathcal{F}_{\mathcal{P}}$ of the set of all formulas $\mathcal{F}$ of our language, there corresponds a (in fact, several) propositional modal model such that for some $\Gamma \in \mathcal{G}$, if $X \in \mathcal{F}_{\mathcal{P}}$ is assigned t by v, then X is true at Γ. Take a special model of this kind, $\mathcal{M}_{\text{prop}}$. The model $\mathcal{M}_{\text{prop}}$ is a U-model and, again, such that corresponding to every distinct valuation v of $\mathcal{F}_{\mathcal{P}}$, there is a member Γ of $\mathcal{G}$ such that for any X, if $v(X) = t$ then for the corresponding $\Gamma \in \mathcal{G}$, $\Gamma \Vdash X$, and $\Gamma \nVdash X$ otherwise. By the definition of U-model, in $\mathcal{M}_{\text{prop}}$, for any distinct Boolean valuation v there corresponds exactly one possible world, i.e., the cardinality of all distinct valuations of $\mathcal{F}_{\mathcal{P}}$ equals the cardinality of $\mathcal{G}$ in $\mathcal{M}_{\text{prop}}$. From propositional logic, we know that if the cardinality of the set of all propositional variables that figure in any formula of $\mathcal{F}_{\mathcal{P}}$ is n, then there are 2^n distinct valuations for *both* the propositional variable subset of $\mathcal{F}_{\mathcal{P}}$ and $\mathcal{F}_{\mathcal{P}}$ itself. Now note that $\mathcal{F}_{\mathcal{P}}$ is the set of all non-modal formulas of our language. Accordingly, it has $\aleph_0$ many atomic variables and there are $2^{\aleph_0}$ distinct Boolean valuations of $\mathcal{F}_{\mathcal{P}}$. Since $2^{\aleph_0} = \mathfrak{c}$, there are continuum many distinct Boolean valuations $\mathcal{F}_{\mathcal{P}}$ and by our definition of $\mathcal{M}_{\text{prop}}$, $|\mathcal{G}| = \mathfrak{c}$ as well. The model $\mathcal{M}_{\text{prop}}$ here has *all* the possible worlds that can be qualitatively distinguished in the propositional setting with $\aleph_0$ many atomic propositions. In other words, any additional possible world will be *atomically* indistinguishable from one that is already in the model. Naturally, this puts an upper bound on the cardinality of U-models.

What we see from the above considerations is that with any model $\mathcal{M}$, the elements in $\mathcal{G}$ are, restricted to non-modal cases, nothing over and above different Boolean valuations for the set $\mathcal{F}_{\mathcal{P}}$. In other words, whenever we consider a possible world (in the propositional setting), we consider a world in which some atomic propositions *may* have different truth values from those in the actual world (and consequently, the relevant complex propositions do too). We write 'may' because there is no limit on how many atomically indistinguishable possible worlds we can have in a model, i.e., there can be possible worlds $\Gamma, \Delta \in \mathcal{G}$ such that $\Gamma \Vdash X$ iff $\Delta \Vdash X$, from which it clearly follows that $\Gamma \Vdash P$ iff $\Delta \Vdash P$. As we will see below, in general, even though the two semantics are strongly equivalent, this possibility of arbitrarily many atomically indistinguishable yet distinct possible worlds does not have a parallel that holds *unconditionally* for the truth-valuational account.

Ending the digression, if we want to do away with models altogether, we still have to give a truth-valuational account of modal formulas as well, i.e., the whole of $\mathcal{F}$. We already know that in model-theoretic semantics, possible worlds function as different (but not necessarily distinct) Boolean valuations. The account of modal formulas is more elaborate however, since in the model-theoretic account, whether or not a modal formula is true at a given world depends not only on other worlds but whether those worlds are accessible or not. Thus, we have to give a truth-valuational account of the accessibility relation $\mathcal{R}$ as well. The most straightforward way of doing this is by taking a set $\mathcal{V}$ instead of $\mathcal{G}$ that has valuations as its elements. Then, we lose the relation $\Vdash$ but retain $\mathcal{R}$ as a relation defined on $\mathcal{V}$. We can now specify what a *valuational framework*, the truth-valuational equivalent of a Kripke model is.

Definition 24 (Valuational Framework)**.** A valuational framework $\mathcal{F}_v$ is a pair $< \mathcal{V}, \mathcal{R}_v >$, where $\mathcal{V}$ is a non-empty set of modal valuation functions and $\mathcal{R}_v$ is a binary relation defined on $\mathcal{V}$. If v_{M_1} and v_{M_2} are both in $\mathcal{V}$ and $v_{M_1}\mathcal{R}_v v_{M_2}$, we say that v_{M_2} is an alternative valuation to v_{M_1}.[2]

Definition 25 (Modal Valuation)**.** A modal valuation v_M of the set of formulas $\mathcal{F}$ of $\mathcal{L}$ relative to a modal valuational framework $\mathcal{F}_v$ is a Boolean valuation $v_1 \in \mathcal{V}$ for $\mathcal{F}$ for which the following additional conditions hold for any X and $v_2 \in \mathcal{V}$.

1. $v_1(\Box X) = t$ iff for all modal valuations $v_2 \in \mathcal{V}$, if $v_1\mathcal{R}_v v_2$, then $v_2(X) = t$ and f otherwise.
2. $v_1(\Diamond X) = t$ iff there is a modal valuation $v_2 \in \mathcal{V}$ such that $v_1\mathcal{R}_v v_2$ and $v_2(X) = t$ and f otherwise.

We then define how truth-values are assigned to formulas. The method comes from Ben-Yami's (MS).

Definition 26 (Value Assignment)**.** Truth-values for formulas relative to a valuational framework $\mathcal{F}_v$ are assigned recursively as follows.

1. As the base case, we assign truth-values to all propositional variables in S under all modal valuations $v_M \in \mathcal{V}$.
2. Then, we assign truth-values to all complex formulas $(\neg X)$, $(X \wedge Y)$, $(X \vee Y)$ and $(X \rightarrow Y)$ per our rules in Definition 25, if we have not assigned a truth-value to them before, but have already assigned truth-values to both X and Y at an earlier stage.

[2] The terminology 'alternative to', with which we designate the relation $\mathcal{R}_v$, comes from Hintikka's (1969).

3. Then, we assign truth-values to all complex formulas $\Box X$ and $\Diamond X$ if we have not assigned a truth-value to them before, but have already assigned a truth-value to X at an earlier stage.
4. We repeat the process until we have assigned a truth-value to every formula in S under any $v_M \in \mathcal{V}$.

We can now define the truth-valuational equivalent of L-validity. As an extension of the standard propositional case, we will call it an L-modal tautology.

Definition 27 (L-Modal Tautology)**.** Suppose L is a class of valuational frameworks. Then, we call any $\mathcal{F}_v$ in L an L-valuational framework. We say that a formula X is an L-modal tautology *iff* it is true under all valuations $v_M \in \mathcal{V}$ in all L-valuational frameworks. If X is an L-modal tautology, we write $\vDash_{L_v} X$.

Note that this definition is somewhat different from the model-theoretic version in Definition 17 since there, L is specified to be a class of *frames*, not a class of *models*, which are the model-theoretic equivalents of valuational frameworks. The relevant classes of valuational frameworks for the logics considered in this thesis can be given by taking the appropriate classes of all valuational frameworks where $\mathcal{R}_v$ satisfies some specific properties (e.g., transitivity, reflexivity, etc.). We could also define the exact truth-valuational equivalents of the definitions of 17 involving frames, but it would take more elaborate constructions.[3]

Take, for example, T. Per T, the relation $\mathcal{R}_v$ is reflexive. Accordingly, the class T has as members all valuational frameworks where $\mathcal{R}_v$ is reflexive. Take an arbitrary valuational framework where $\mathcal{R}_v$ is reflexive and $\mathcal{V}$ is any non-empty set of modal valuations. If we prove that a given formula X is true under every modal valuation $v_M \in \mathcal{V}$ for such a valuational

[3]We can do the following. Take any Kripke frame $F = < \mathcal{G}, \mathcal{R} >$ as before. Introduce a labelling function $\mathscr{L} : \mathcal{G} \to \mathcal{V}^{\infty}$, where $\mathcal{V}^{\infty}$ is the set of all modal valuations. Then, given a frame F, take the set $F_{\mathcal{F}_v}$ to contain all admissible valuational frameworks $\mathcal{F}_v = < \mathcal{V}, \mathcal{R}_v >$, where for any $\mathcal{F}_v$, there is an $\mathscr{L}$ such that $\mathcal{V} = \{v_M : \exists \Gamma \in \mathcal{G} \text{ such that } \mathscr{L}(\Gamma) = v_M\}$ and if $\Gamma \mathcal{R} \Delta$, then $\mathscr{L}(\Gamma) \mathcal{R}_v \mathscr{L}(\Delta)$.

frame, we prove that it is true under any modal valuation of any valuational frame where $\mathcal{R}_v$ is as specified by T. Thus, we prove that X is a T-modal tautology.

Proposition 28. *Any formula of the form $X \to \Diamond X$ is a T-modal tautology.*

Proof. Take an arbitrary valuational frame $\mathcal{F}_v$ with $\mathcal{R}_v$ being reflexive as specified by T. Then, for all $v_M \in \mathcal{V}$, $v_M \mathcal{R}_v v_M$. Take an arbitrary valuation $v_{M_1} \in \mathcal{V}$ and suppose that $v_{M_1}(X) = t$. For $\Diamond X$ to be true under v_{M_1}, there has to be at least one modal valuation v_{M_2} such that $v_{M_1} \mathcal{R}_v v_{M_2}$ and $v_{M_2}(X) = t$. Suppose $v_{M_1} = v_{M_2}$. By T, we know that $v_{M_1} \mathcal{R}_v v_{M_2}$ holds. We also know that $v_{M_1}(X) = t$, thus $v_{M_2}(X) = t$ also. Therefore, $v_{M_1}(X \to \Diamond X) = t$. $\square$

We also have the sublogic relation in the truth-valuational approach.

Definition 29 (Sublogic)**.** For any L_1 and L_2, if $L_1 \subseteq L_2$, then L_2 is a sublogic of L_1.

Note that by Definition 27, X is an L-modal tautology *iff* it is true under all v_M in all L-valuational frameworks. Then, if X is an L_2-modal tautology and the sublogic relation is as before, then X is also an L_1-modal tautology, hence the terminology.

We can also define L-satisfiability for our truth-valuational account. Later on, we will consider another definition for a different type of expressions.

Definition 30 (Simple L_v-Satisfiability)**.** A formula X is L_v-satisfiable *iff* there is an L-valuational framework $\mathcal{F}_v$ and a $v_M \in \mathcal{V}$ such that $v_M(X) = t$. Then, a set S of formulas is L_v-satisfiable *iff* all members of S are L_v-satisfiable under the same modal valuation v_M.

With definitions of L-modal tautology and L_v-satisfiability at hand, we can go on to prove the truth-valuational equivalent of Proposition 19.

Proposition 31. *X is an L-modal tautology iff $\neg X$ is not L_v-satisfiable.*

Proof. Suppose first that X is an L-modal tautology and $\neg X$ is satisfiable. Then, by Definition 27, in all valuational frameworks of L, for all modal valuations $v_M \in \mathcal{V}$, $v_M(X) = t$, and thus $v_M(\neg X) = f$ also. By Definition 30, if $\neg X$ is satisfiable, there is a valuational framework of L where there is a $v_M \in \mathcal{V}$ such that $v_M(\neg X) = t$. We arrive at a contradiction.

Suppose second that $\neg X$ is not satisfiable and X is *not* an L-modal tautology. First, by Definition 30, there is no valuational framework $\mathcal{F}_v$ of L with a modal valuation $v_M \in \mathcal{V}$ such that $v_M(\neg X) = t$. Thus, for any modal valuation $v_M \in \mathcal{V}$, $v_M(\neg X) = f$. But, we also know that X is not an L-modal tautology, so by Definition 27, there is at least one valuational framework $\mathcal{F}_v$ of L with a $v_M \in \mathcal{V}$ such that $v_M(X) = f$, i.e., $v_M(\neg X) = t$. We have our contradiction again. $\square$

We go on to define semantic entailment. Since modal logic departs significantly from the concept of entailment in propositional logic, take the latter first. In propositional logic, semantic entailment is given as follows (see, e.g., Smullyan's (1995, p. 12)):

Definition 32 (Propositional Semantic Entailment)**.** A set S of formulas semantically entail a formula X *iff* for every valuation v, if $v(Y) = t$ for every $Y \in S$, then $v(X) = t$ also.

In the modal case, things are more complicated: we essentially have two notions of semantic entailment which can also combine. Fitting and Mendelsohn (1998), following the convention in the literature, call these 'local' and 'global' assumptions or premises. The definition, adapted to our truth-valuational account is as follows:

Definition 33 (Modal Semantic Entailment)**.** If S and U are sets of formulas and X is a formula, X is a consequence in L of S as global assumptions and U as local assumptions *iff* for every L-valuational framework where members of S are a modal tautology and for every modal valuation $v_M \in \mathcal{V}$ under which all members of U evaluate to t, $v_M(X) = t$. In such cases, we write $S \vDash_{L_v} U \Rightarrow X$.

More explicitly, when looking at modal semantic entailment, we first take all L-valuational frameworks. Then, we take a subset of all L-valuational frameworks where under all $v_M \in \mathcal{V}$, all members of S evaluate to t. Then, for each of these frameworks, we take all those $v_M \in \mathcal{V}$ where all members of U evaluate to t. Finally, if X also evaluates to t under those v_M where all members of U do, then X is semantically L-entailed by S as global and U as local assumptions. Note that we can model several possible circumstances with this definition. Most importantly, whether S or U is non-empty is not specified. Thus, we can have either S, U or both empty, the last one corresponding to the definition of an L-modal tautology. We show it, and a few other examples.

Proposition 34. $\vDash_L X$ *iff* $\emptyset \vDash_L \emptyset \Rightarrow X$*.*

Proof. The proposition is true by definition. The left hand side says that in every L-valuational framework, X evaluates to t under all modal valuations in $\mathcal{V}$. According to the right hand side, we take every L-valuational framework where all members of $\emptyset$ are a modal tautology, i.e., true under all modal valuations. Since $\emptyset$ has no members, it is vacuously true that all of its members are true under all modal valuations in any L-valuational framework. We proceed to local assumptions. Again, for each framework, it is vacuously true that all members of $\emptyset$ evaluate to t under any $v_M \in \mathcal{V}$. Then, $\emptyset \vDash_L \emptyset \Rightarrow X$ if X is true under all modal valuations in any L-valuational framework. $\square$

Then, whenever we have $\emptyset \vDash_L \emptyset \Rightarrow X$, we can just write $\vDash_L X$. Now the other two cases. Suppose $U = \emptyset$. Again, we can omit $\emptyset$ and write $S \vDash_L X$. Then in all L-valuational frameworks where S is a set of modal tautologies, X is a modal tautology also. Take the following example.

Proposition 35. $S \vDash_K \Box(X \vee Y)$*, where S is the set of all formulas of form $X \vee Y$.*

Proof. First, we take the set of all K-valuational frameworks. Then, we take that subset Q of such a set where for any $\mathcal{F}_v$, for all $v_M \in \mathcal{V}$, $v_M(X \vee Y) = t$. By Definition 29, K is a sublogic of the logic defined by Q. By Definition 25, $v_{M_1}[\Box(X \vee Y)] = t$ *iff* for all v_{M_2}, if $v_{M_1}\mathcal{R}_v v_{M_2}$, then $v_{M_2}(X \vee Y) = t$. Now take an arbitrary $v_{M_1} \in \mathcal{V}$ from an arbitrary $\mathcal{F}_v$ of Q. If there is no $v_{M_2} \in \mathcal{V}$ such that $v_{M_1}\mathcal{R}_v v_{M_2}$, then $v_{M_1}[\Box(X \vee Y)] = t$ is trivially true. Alternatively, given that for all $v_M \in \mathcal{V}$ of any $\mathcal{F}_v$ of Q, $v_M(X \vee Y) = t$, if for any v_{M_2}, $v_{M_1}\mathcal{R}_v v_{M_2}$, then $v_{M_2}(X \vee Y) = t$. Thus, $v_{M_1}[\Box(X \vee Y)] = t$. □

We now move on to $S \vDash_L U \Rightarrow X$, where $S = \emptyset$. We can then write $\vDash_L U \Rightarrow X$. We give an example where the relation does not hold.

Proposition 36. $\nvDash_K X \Rightarrow \Diamond X$

Proof. Suppose that $\vDash_K X \Rightarrow \Diamond X$. Then for all K-valuational frameworks, for any $v_M \in \mathcal{V}$, if $v_M(X) = t$, then $v_M(\Diamond X) = t$ also. Now take the K-valuational framework with exactly one $v_M \in \mathcal{V}$. Set $v_M(X) = t$ and take $v_M \mathcal{R}_v v_M$ to be false. Then, $v_M(\Diamond X) = f$, contra supposition. □

We can establish two results that carry over straightforwardly from the propositional case.

Proposition 37 (Local Semantic Deduction Theorem)**.** *$S \vDash_L U \cup \{X\} \Rightarrow Y$ iff $S \vDash_L U \Rightarrow (X \rightarrow Y)$.*

Proof. From left to right. Suppose that $S \vDash_L U \cup \{X\} \Rightarrow Y$ is true. Then, at all L-valuational frameworks where all members of S are modal tautologies, under all v_M where the members of $U \cup \{X\}$ evaluate to t, Y also does. Then, we take the same set of frameworks. Take an arbitrary $v_M \in \mathcal{V}$ where all elements of U evaluate to t. To establish that $v_M(X \rightarrow Y) = t$, we have to show that if $v_M(X) = t$, then $v_M(Y) = t$ also does. By our supposition, we know that if all elements of U evaluate to t and X also does, then Y does also. Thus, $S \vDash_L U \Rightarrow (X \rightarrow Y)$.

From right to left. Suppose that $S \vDash_L U \Rightarrow (X \to Y)$. Then, at all L-valuational frameworks where all members of S are a modal tautology, under all v_M where all members of U evaluate to t, $(X \to Y)$ also does. Equivalently, whenever $v_M(X) = t$, $v_M(Y) = t$. Now we take the same set of L-valuational frameworks where all members of S are a modal tautology and form the union $U \cup \{X\}$. By our supposition, we already know that for any v_M, if all members of U evaluate to t and X does too, then Y does too. Thus, $S \vDash_L U \cup \{X\} \Rightarrow Y$. □

The global semantic deduction theorem is not so straightforward, but can be proved syntactically through the completeness theorem.

Proposition 38 (Monotonicity). *If $S_1 \subseteq S_2$, $U_1 \subseteq U_2$ and L_1 is a sublogic of L_2, then if $S_1 \vDash_{L_1} U_1 \Rightarrow X$, then $S_2 \vDash_{L_2} U_2 \Rightarrow X$.*

Proof. Suppose that $S_1 \subseteq S_2$, $U_1 \subseteq U_2$ and L_1 is a sublogic of L_2. Suppose also that $S_1 \vDash_{L_1} U_1 \Rightarrow X$. Then, for all L_1 valuational frameworks $\mathcal{F}_v$ where every member of S_1 is a modal tautology, for all $v_M \in \mathcal{V}$ of $\mathcal{F}_v$, if all members of U_1 evaluate to t, then X also. We extend S_1 to S_2 and U_1 to U_2. Since $S_1 \subseteq S_2$, $U_1 \subseteq U_2$ and L_1 is a sublogic of L_2, we first consider the subset of all L_1-valuational frameworks which determine the logic L_2. Then, we consider a subset of the set Q_1 of all L_1-valuational frameworks where all members of S_1 are modal tautologies, precisely that one where all members of $S_2 \setminus S_1$ are modal tautologies also, call it Q_2. Similarly, we take that subset R_2 of all $v_M \in \mathcal{V}$ where all members of U_1 evaluate to t (call it R_1), where all members of $U_2 \setminus U_1$ also evaluate to t. Since we know that $S_1 \vDash_{L_1} U_1 \Rightarrow X$, we know that at *any* member of Q_1, under *any* member of R_1, X evaluates to t. Note that we want to look at L_2-valuational frameworks, but both Q_1 and Q_2 are defined as subsets of all L_1-valuational frameworks. We know that Q_2 is a subset of Q_1. Then, we take the intersection of Q_2 and all L_2-valuational frameworks, Q_3. Then, we still have a subset of Q_1. Then, we take the subset R_2 of R_1. Since at *any* member of Q_1, under *any* member of R_1, X evaluates to t and any members of Q_3 and R_2 are also members of Q_1

and R_1, respectively, then at *any* member of Q_3, under *any* member of R_2, X evaluates to t also. Thus, $S_2 \vDash_{L_2} U_2 \Rightarrow X$. $\square$

Finally, we need to extend the notion of satisfiability to include global and local premise sets.

Definition 39 (L_v-Satisfiability for Sets of Global and Local Premises)**.** A formula X is L_v-satisfiable together with a set S of global and a set U of local premises *iff* there is an L-valuational framework $\mathcal{F}_v$ where under any v_M, for any $Z \in S$, $v_M(Z) = t$, and a $v_M \in \mathcal{V}$ such that for any $Y \in U$, $v_M(Y) = t$ and $v_M(X) = t$ also. Then, the simple L_v-satisfiability for a set of formulas S is L_v satisfiability for the set S construed as the set of local premises.

We then have the following:

Proposition 40. *X is L-entailed by the global premise set S and local premise set U iff the global premise set S, the local premise set U and $\neg X$ are not L_v-satisfiable.*

Proof. Suppose first that X is L-entailed by the global premise set S and local premise set U and that the global premise set S, the local premise set U and $\neg X$ are L_v-satisfiable. Then, by Definition 33, in all valuational frameworks in L where all members of S are true under $v_M \in \mathcal{V}$, for all modal valuations $v_M \in \mathcal{V}$, if $v_M(Y) = t$ for every $Y \in U$, then $v_M(X) = t$ also. Thus, $v_M(\neg X) = f$ also. By Definition 39, if the global premise set S, the local premise set U and $\neg X$ are satisfiable, there is a valuational framework in L where all members of S are true under every $v_M \in \mathcal{V}$, and where there is a $v_M \in \mathcal{V}$ such that for every $Y \in U$, $v_M(Y) = t$ and $v_M(\neg X) = t$ also. We arrive at a contradiction.

Suppose second that the global premise set S, the local premise set U and $\neg X$ are not L_v-satisfiable and X is *not* L-entailed by the global premise set S and local premise set U. First, by Definition 39, there is no valuational framework $\mathcal{F}_v$ in L where under every $v_M \in \mathcal{V}$, $v_M(Z) = t$ for every $Z \in S$ with a modal valuation $v_M \in \mathcal{V}$ such that $v_M(Y) = t$ for every

$Y \in U$ and $v_M(\neg X) = t$ also. Thus, for any modal valuation $v_M \in \mathcal{V}$, $v_M(\neg X) = f$. But, we also know that X is not L-entailed by the global premise set S and local premise set U, so by Definition 33, there is at least one valuational framework $\mathcal{F}_v$ in L where every $Z \in U$ is true under every $v_M \in \mathcal{V}$ and with a $v_M \in \mathcal{V}$ such that for every $Y \in U$, $v_M(Y) = t$, but $v_M(X) = f$, i.e., $v_M(\neg X) = t$. We have our contradiction again. $\square$

3.3 Leblanc, Dunn, and Ben-Yami

As mentioned above, the truth-valuational account presented here is based on Leblanc's (1976), Dunn's (1973), and Ben-Yami's (MS) approach. It differs from Leblanc's (cf. Kripke's (1959)) and *one of* Dunn's semantics in that modal valuations assign truth-values directly to *all* formulas of the language, not just to the atomic ones. In his *other* semantics from the same paper, Dunn presents an account where modal valuations assign truth-values directly to all formulas of the language, similar to the one here and that of Ben-Yami, though in the end, his semantics is rather different than ours and Ben-Yami's.[4]

Leblanc, Dunn, and Ben-Yami also give different accounts of how valuations relate to each other. Leblanc's is essentially the same as we have presented above, and which parallels the Kripkean account. We will continue using it going forward, but will remain agnostic as to what the best conception is philosophically. Dunn shows that his definition (which, again, can be found in his (1973), cf. Goble's (1973)) is equivalent to the Leblanc-Kripke one. And it can also be shown that moving back and forth between the account presented above and that of Ben-Yami is also (at least formally) trivial.

We do it as follows. Instead of defining the $\mathcal{R}_v$ relation between valuations, for any modal

[4]This formulation may be somewhat misleading. Naturally, it is not the case that in the end, there are formulas of the language which do not receive a truth-value under a valuation. Rather, the question is whether we take the set of valuations of a valuational framework as a set of *atomic* valuations, which are then extended according to the rules, or a set of valuations *simpliciter*, which then need not be extended, but the set of valuations and the relation on the set in the framework already have to conform to each other.

valuation v_M, Ben-Yami defines an assignment set S_{v_M}. This assignment set gives us the valuations alternative to a given one. Take the approach with the relation $\mathcal{R}_v$. Then we have expressions of the form $\mathcal{R}_v(v_{M_1}, v_{M_2})$. We can then define the assignment set $S_{v_{M_1}}$ for a modal valuation v_{M_1} by noting that any modal valuation v_{M_2} is in $S_{v_{M_1}}$ if $\mathcal{R}_v(v_{M_1}, v_{M_2})$ holds. Or we can say that given the assignment set $S_{v_{M_1}}$ for a modal valuation v_{M_1}, $\mathcal{R}_v(v_{M_1}, v_{M_2})$ holds if every v_{M_2} is in $S_{v_{M_1}}$.

4 The Tableau Proof Method

The proof systems described in the following come from Fitting's seminal work in the field encompassing several decades. For details, see his (1972), (1983), (1993), (1998) and (2007). The main idea for prefixed modal tableaux came from Fitch's (1966), with significant material coming from Smullyan's classic work on first-order tableaux (1995, originally published in 1968), which in turn already relied on first-order Hintikka sets for completeness (see Hintikka's (1955)).[1]

Before proceeding, we give some background on trees. The terminology and definitions are adapted to prefixed tableaux from Smullyan's (1995). Tableau proofs are trees of the mathematical kind. A tree $\mathcal{T}$ is defined by three things. First, we need a set P of points of our tree. Second, we have a function $l : P \to \mathbb{N}^+$. If $l(x) = n$, we say that the *level of* x is n. There is a unique point p_1 such that $l(p_1) = 1$ which is called the *origin* of the tree. Finally, we have a binary relation xRy defined on P, which gives us the predecessor/successor relation if the following conditions are met. Except p_1, every point has a unique predecessor. Finally, for all $x, y \in P$, if xRy then $l(y) = l(x) + 1$.

Now for some terminology. If a point has no successor, it is called an *end point*. If it has only one successor, it is a *simple point*, if it has more than one successor, a *junction*

[1] There has been intense collaboration between the authors mentioned in this thesis. Dunn wrote his PhD under the supervision of Belnap, with whom he wrote their famous (1968) paper criticizing substitutional quantification. In his (1976), Leblanc came up with a way of countenancing that criticism with help from Hintikka. Both Dunn and Fitting reviewed Leblanc's (1976). Finally, Fitting was a PhD student of Smullyan.

point. By the *descendents* of a point p, we mean the immediate successors of p, their immediate successors, and so on. Naturally, trees have branches. First, a *path* is any finite or denumerably infinite sequence of points, where the first point in the sequence is the origin point and any point is a predecessor of the next (if any) in the sequence. By P_x we mean a finite path whose last point is x. A path is *maximal* if it has a last point which is an end point of $\mathcal{T}$ or if it is infinite. A branch is just a *maximal path*. If we want to speak of the first, second, third, etc. successors of a junction point (where the point at most has denumerably many successors), we need an *ordered tree*. Thus, we need to specify a function s which maps to every junction point x a sequence $s(x)$ of all the successors of x without repetition. Finally, by a *dyadic tree*, we mean a tree with junction points that have exactly 2 successors. With ordered dyadic trees, we can speak of the *left successor* and the *right successor* of any junction point. Any dyadic tree is said to be *finitely generated*, since any point has only finitely many successors (namely, 0, 1 or 2). However, not every dyadic tree is *finite* in the sense that it has finitely many points. We will return to this.

Next, we look at how to actually construct trees from the origin with repeated applications of syntactic rules. In our system, we only concern ourselves with dyadic trees. Any deduction rule either adds to an end point x a sole successor in the conjunctive case or a left successor and a right successor in the disjunctive case. This is called *adjunction*. In the conjunctive and modal cases, by an adjunction of a point y that is outside of our tree $\mathcal{T}$ as the sole successor of point x we mean adding y to the set P, specifying that xRy holds and extending our function l by defining $l(y) = l(x) + 1$. In the disjunctive cases, by an adjunction of points y_1 and y_2 that are outside of our tree $\mathcal{T}$ as the successors of point x, we mean adding both y_1 and y_2 to the set P, specifying that the relation R holds between both x and y_1 and x and y_2 and extending the function l by $l(y_1) = l(y_2) = l(x) + 1$. Finally, we also extend our ordering function s such that $s(x) =< y_1, y_2 >$, i.e., y_1 becomes the left successor of x and y_2 the right.

The points of our proof trees are prefixed formulas. Intuitively, a prefix tells us what 'possible world' or modal valuation we are considering. Prefixed tableaux, i.e., tableaux with prefixed formulas for different modal logics were developed by Fitting in his (1972) and (1983).

Definition 41 (Prefixed Formula)**.** A *prefixed formula* is an expression of the form ωX, where X is a formula and ω is a finite sequence of integers (separated by points for unambiguous human parsing) called a *prefix*.

4.1 K-Tableau Rules

To shorten our proofs, we introduce Smullyan and Fitting's *unified notation*. We will sort our rules into four categories. C-rules will include the conjunctive cases, D-rules will include the disjunctive cases, P-rules will include the K-possibility cases and N-rules will include the K-necessity cases. As mentioned above, modal cases are similar to conjunctive cases in that they do not involve 'branching'. In all four categories, our rules will involve specifying for a prefixed formula of form $\omega\alpha$, $\omega\beta$, $\omega\gamma$ or $\omega\delta$, respectively, what pairs of formulas of form $\omega\alpha_1, \omega\alpha_2$ we can adjoin to $\omega\alpha$ either by themselves or successively, what pairs of formulas of form $\omega\beta_1$ and $\omega\beta_2$ we can adjoin to $\omega\beta$ at once and what formulas of form $\omega.n\gamma_s$ and $\omega.n\delta_s$ we can adjoin to $\omega\gamma$ and $\omega\delta$ as sole successor, respectively. From now on, we will continue to use this notation to refer to such cases throughout.

Definition 42 (Modal Conjunctive Rules)**.** A conjunctive rule is called a *Type-C* rule. For any prefix ω, if some $\omega\alpha$ occurs on some branch P_y, where α is of one of the four forms specified in Table 2 below, one may adjoin either $\omega\alpha_1$ or $\omega\alpha_2$ as sole successor to y (usually, we do both successively).

$\omega\alpha$	$\omega\alpha_1$	$\omega\alpha_2$
$\omega X \wedge Y$	ωX	ωY
$\omega\neg(X \vee Y)$	$\omega\neg X$	$\omega\neg Y$
$\omega\neg(X \rightarrow Y)$	ωX	$\omega\neg Y$
$\omega\neg\neg X$	ωX	ωX

Table 2: Type-C Deduction Rules

Proposition 43. *Given any valuational framework $\mathcal{F}_v$, for any $\alpha, \alpha_1, \alpha_2$, $v_M(\alpha) = t$ iff $v_M(\alpha_1) = t$ and $v_M(\alpha_2) = t$.*

Proof. We show one. We need to prove that $v_M(X \wedge Y) = t$ *iff* $v_M(X) = t$ and $v_M(Y) = t$. Since modal valuations are Boolean, this is true by definition. The rest is shown similarly. □

Definition 44 (Modal Disjunctive Rules)**.** A disjunctive rule is called a *Type-D* rule. For any prefix ω, if some $\omega\beta$ occurs on some branch P_y, where β is of one of the four forms specified in Table 3 below, one may simultaneously adjoin $\omega\beta_1$ as left successor and $\omega\beta_2$ as right successor to y.

$\omega\beta$	$\omega\beta_1$	$\omega\beta_2$
$\omega X \vee Y$	ωX	ωY
$\omega\neg(X \wedge Y)$	$\omega\neg X$	$\omega\neg Y$
$\omega X \rightarrow Y$	$\omega\neg X$	ωY

Table 3: Type-D Deduction Rules

Proposition 45. *Given any valuational framework $\mathcal{F}_v$, for any β, β_1, β_2, $v_M(\beta) = t$ iff $v_M(\beta_1) = t$ or $v_M(\beta_2) = t$.*

Proof. Again, we show one. We need to prove that $v_M(X \vee Y) = t$ *iff* $v_M(X) = t$ or $v_M(Y) = t$. Again, since modal valuations are Boolean, this is true by definition. The rest is shown similarly. □

Definition 46 (K-Possibility Rules)**.** A K-possibility rule is called a *Type-P* rule. If some $\omega\gamma$ occurs on some branch P_y, where γ is of one of the two forms specified in Table 4 below, one may adjoin $\omega.n\gamma_s$ as sole successor to y, if the prefix $\omega.n$ is *new* to the branch.

$\omega\gamma$	$\omega.n\gamma_s$
$\omega\Diamond X$	$\omega.nX$
$\omega\neg\Box X$	$\omega.n\neg X$

Table 4: Type-P Deduction Rules

Proposition 47. *Given any valuational framework $\mathcal{F}_v$, for any $\gamma, \gamma_s, v_{M_1}$, $v_{M_1}(\gamma) = t$ iff there is a v_{M_2} such that $v_{M_1}\mathcal{R}_v v_{M_2}$ and $v_{M_2}(\gamma_s) = t$.*

Proof. Again, we show only one. We need to show that $v_{M_1}(\Diamond X) = t$ *iff* there is a v_{M_2} such that $v_{M_1}\mathcal{R}_v v_{M_2}$ and $v_{M_2}(X) = t$. By Definition 25, this holds. The other case is similar □

Definition 48 (K-Necessity Rule)**.** A K-necessity rule is called a *Type-N* rule. If some $\omega\delta$ occurs on some branch P_y, where δ is of one of the two forms specified in Table 5 below, one may adjoin $\omega.n\delta_s$ as sole successor to y, if the prefix $\omega.n$ *already occurs* on the branch.

$\omega\delta$	$\omega.n\delta_s$
$\omega\Box X$	$\omega.nX$
$\omega\neg\Diamond X$	$\omega.n\neg X$

Table 5: Type-N Deduction Rules

Proposition 49. *Given any valuational framework $\mathcal{F}_v$, for any $\delta, \delta_s, v_{M_1}$, $v_{M_1}(\delta) = t$ iff for all v_{M_2}, if $v_{M_1}\mathcal{R}_v v_{M_2}$, then $v_{M_2}(\delta_s) = t$.*

Proof. Again, we show only one. We need to show that $v_{M_1}(\delta) = t$ *iff* for all v_{M_2}, if $v_{M_1}\mathcal{R}_v v_{M_2}$, then $v_{M_2}(\delta_s) = t$. By Definition 25, this holds. The other case is similar. □

Finally, we present two rules for adding premises to a tableau, one for *global* premises and one for *local* ones.

Definition 50 (Global Rule)**.** A global rule is a Type-G rule. If the set S contains all global assumptions, then we can adjoin ωX to any branch ψ of our tableau, where $X \in S$ and ω is not new to the branch.

Definition 51 (Local Rule)**.** A local rule is a Type-L rule. If the set U contains all local assumptions, then we can adjoin $1\ X$ to any branch ψ of our tableau, where $X \in U$.

Importantly, every formula of our language is either a literal, or of form α, β, γ or δ.

Proposition 52. *Every formula X is either of form P, $\neg P$, α, β, γ or δ.*

Proof. The proof is by induction on the complexity of X. Take the base case. Then, X is of complexity 0, i.e., X is P. Thus, our hypothesis holds. Take the case of formulas of complexity 1. Then, X is either $\neg P$, $P \wedge Q$ (α), $P \vee Q$ (β), $P \to Q$ (β), $\Diamond P$ (γ) or $\Box P$ (δ). Then, again, our hypothesis holds.

Suppose the hypothesis holds for all formulas up to complexity n. We then show it for formulas of complexity $n+1$. We take each case one by one.

1. Any negation of a formula $\neg X$ ($c(\neg X) > 1$) is either of form $\neg\neg Y$ (α), $\neg(Y \wedge Z)$ (β), $\neg(Y \vee Z)$ (α), $\neg(Y \to Z)$ (α), $\neg(\Box Y)$ (γ) or $\neg(\Diamond Y)$ (δ). Then, our hypothesis holds.
2. Any conjunction $X \wedge Y$ of two formulas is of form α, so the hypothesis holds.
3. Any disjunction $X \vee Y$ of two formulas is of form β, so the hypothesis holds.
4. Any conditional $X \to Y$ of two formulas is of form β, so the hypothesis holds.
5. Any formula of form $\Diamond X$ is of form γ, so the hypothesis holds.
6. Any formula of form $\Box X$ is of form δ, so the hypothesis holds. $\Box$

4.2 Extensions to Other Systems

We sketch how to extend our system to stronger modal logics. After introducing these extensions, we will return back to K for our further proofs, but will note extensions to stronger systems throughout.

There are several different ways to accomodate different logics in the modal tableau system. I will use a simple one from Fitting and Mendelsohn's (1998) (which is, in turn, based on Massacci's (1994) and Goré's (1999)), where one adds additional rules to the already presented K-ones. Different *combinations* of rules will result in different systems, parallel to the axiomatic treatment of modal propositional logic. In fact, the introduced rules will show important parallels to the appropriate axioms of the same name. The additional rules only concern formulas of form δ.

Definition 53 (Additional Necessity Rules)**.** Depending on the system used, if some $\omega\delta$ ($\mathfrak{T}$, 4) or $\omega.n\delta$ ($\mathfrak{B}$, $4\mathfrak{r}$) occurs on some branch P_y, where they are as specified by the rules in Table 6 below, then one may adjoin the corresponding $\omega\delta_s$ ($\mathfrak{T}$, $\mathfrak{B}$) or $\omega.n\delta_s^4$ (4) or $\omega\delta_s^4$ ($4\mathfrak{r}$) as sole successor to y, if the prefix ω ($\mathfrak{T}$, $\mathfrak{B}$, $4\mathfrak{r}$) or $\omega.n$ (4) *already occurs* on the branch.

	$\omega\delta$	$\omega\delta_s$
$\mathfrak{T}$	$\omega\Box X$	ωX
	$\omega\neg\Diamond X$	$\omega\neg X$

	$\omega.n\delta$	$\omega\delta_s$
$\mathfrak{B}$	$\omega.n\Box X$	ωX
	$\omega.n\neg\Diamond X$	$\omega\neg X$

	$\omega\delta$	$\omega.n\delta_s^4$
4	$\omega\Box X$	$\omega.n\Box X$
	$\omega\neg\Diamond X$	$\omega.n\neg\Diamond X$

	$\omega.n\delta$	$\omega\delta_s^4$
$4\mathfrak{r}$	$\omega.n\Box X$	$\omega\Box X$
	$\omega.n\neg\Diamond X$	$\omega\neg\Diamond X$

Table 6: Additional Deduction Rules

The system T results from adding the $\mathfrak{T}$-rule to the *system* K (i.e., to the K-rules), *K4* by adding rule 4 to the system K, B by adding the $\mathfrak{B}$-rule and the 4-rule to the system K, *S4* by adding the 4-rule to the system T, and *S5* by adding the $4\mathfrak{r}$ rule to the system *S4*.

4.3 Central Notions

We adapt some additional terminology and definitions from Smullyan's (1995). Every deduction step in our tableau proofs is a *direct extension* of a tree. By the direct L-extension of the ordered dyadic tree $\mathcal{T}_1$ to the ordered dyadic tree $\mathcal{T}_2$, where the points of both are occurences of prefixed formulas, we mean that $\mathcal{T}_2$ was obtained by exactly one application of the rules appropriate for L. We can now define what a modal tableau for a formula is.

Definition 54 (L-Modal Tableau)**.** By a finite L-modal tableau $\mathcal{T}$ for a formula X from the set of global premises S and the set of local premises U we mean an ordered dyadic tree with occurences of prefixed formulas as points that is defined inductively as follows. Every 1-point tree with occurences of prefixed formulas as points with $1\ X$ at its origin is a finite modal tableau for X. If $\mathcal{T}_1$ is a finite modal tableau for X and $\mathcal{T}_2$ is a direct L-extension of it, then $\mathcal{T}_2$ is also a finite modal tableau for X. Nothing else is a finite modal tableau for X. In other words, $\mathcal{T}$ is a finite modal tableau for a formula X from the set of global premises S and the set of local premises U *iff* there is a finite sequence $< \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n = \mathcal{T} >$, where $\mathcal{T}_1$ is a 1-point tree whose sole point is $1\ X$ and for each $i < n$, $\mathcal{T}_{i+1}$ is a direct L-extension of $\mathcal{T}_i$. By an *infinite* modal tableau $\mathcal{T}$, we mean the union tree of the infinite sequence $< \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n, \ldots >$, where for each n, $\mathcal{T}_{n+1}$ is a direct L-extension of $\mathcal{T}_n$.

Naturally, not every modal tableau is a proof of X from S as global and U as local premises and we need one additional definition to specify what makes a modal tableau for X from S as global and U as local premises a tableau proof of Y from S as global and U as local premises.

Definition 55 (Closure)**.** A branch of a modal tableau is *closed* if it contains both ωX and $\omega\neg X$ for some formula X. If a branch is not closed, it is *open*. A tableau is *closed* if every branch of it is closed.

Definition 56 (L-Tableau Proof)**.** A closed L-modal tableau for $1\ \neg X$ from S as global and U as local premises is an *L-tableau proof* of X from S as global and U as local premises. If X has an L-tableau proof from S as global and U as local premises, it is *L-deducible from S and U*. If X is L-deducible from S as global and U as local premises, we write $S \vdash_L U \Rightarrow X$, and if either S or U is empty, we write $\vdash_L U \Rightarrow X$ or $S \vdash_L X$. If X is L-deducible with S and U empty, we say that X is a theorem of L and write $\vdash_L X$.

Before continuing, we provide two examples (Example 1 [p. 38], and Example 2 [p. 39]) of modal tableau proofs. Example 1 shows the necessitation of a theorem of propositional logic (one half of a DeMorgan Law) using just K-rules, Example 2 shows a possible interplay between the axiomatic approach and tableau, giving a proof of $S \vdash_K U \Rightarrow Z$, where $S = \{\Box X \to X : X \in \mathcal{F}\}$, i.e., all instances of the axiom T, and $U = \{\Box Y, \Box(Y \to Z)\}$.

We next need to prove Kőnig's Lemma, a result from the Hungarian mathematician Dénes Kőnig. The proof is Kőnig's original, as presented by Smullyan in his (1995).

Proposition 57 (Kőnig's Lemma)**.** *Any finitely generated infinite tableau $\mathcal{T}$ has at least one infinite branch.*

$\vdash_K \Box((\neg X \vee \neg Y) \to \neg(X \wedge Y))$

$$
\begin{array}{lcl}
1. & 1\ \neg\Box((\neg X \vee \neg Y) \to \neg(X \wedge Y)) & \text{Pr.} \\
2. & 1.1\ \neg((\neg X \vee \neg Y) \to \neg(X \wedge Y)) & 1\ P\text{-type rule} \\
3. & 1.1\ (\neg X \vee \neg Y) & 2\ C\text{-type rule} \\
4. & 1.1\ \neg\neg(X \wedge Y) & 2\ C\text{-type rule} \\
5. & 1.1\ X \wedge Y & 4\ C\text{-type rule} \\
6. & 1.1\ X & 5\ C\text{-type rule} \\
7. & 1.1\ Y & 5\ C\text{-type rule} \\
8. & 1.1\ \neg X \qquad 1.1\ \neg Y & 3\ D\text{-type rule} \\
 & \underset{6,8}{\otimes} \qquad\quad \underset{7,8}{\otimes} &
\end{array}
$$

Example 1: Tableau Proof of a Proposition

$$\{\Box X \to X : X \in \mathcal{F}\} \vdash_K \{\Box Y, \Box(Y \to Z)\} \Rightarrow Z$$

1.		$1\ \neg Z$			Pr.
2.		$1\ \Box Y$			L-type rule
3.		$1\ \Box(Y \to Z)$			L-type rule
4.		$1\ \Box Y \to Y$			G-type rule
5.		$1\ \Box(Y \to Z) \to (Y \to Z)$			G-type rule
6.	$1\ \neg\Box(Y \to Z)$	$1\ Y \to Z$			5 D-type rule
7.		$1\ \neg Y$		$1\ Z$ $\otimes$ 1, 7	6 D-type rule
8.		$1\ \neg\Box Y$	$1\ Y$ $\otimes$ 7,8		4 D-type rule
9.		$1.1\ \neg Y$			8 P-type rule
10.		$1.1\ Y$ $\otimes$ 9, 10			2 N-type rule
11.	$1.1\ \neg(Y \to Z)$				6 P-type rule
12.	$1.1\ Y \to Z$ $\otimes$ 11, 12				3 N-type rule

Example 2: Tableau Proof of an Argument

Proof. We introduce some new terminology. A point of a tree is *good* if it has infinitely many descendents. It is called *bad* if it has only finitely many descendents. We know by hypothesis that $\mathcal{T}$ has infintely many points. Thus, the origin must be good.

Then, we go on to note that every good point p must have at least one good descendent, since for any tree that is finitely generated, the immediate successors of any point p are also finite. The descendents of p are the finite number of immediate successors of p and their successors. Thus, there must be at least one immediate successor to p which has infinitely many descendants and thus is itself good.

Keeping this in mind, our origin point p_1 has a good descendent p_2, which in turn has a good descendent p_3 and so on. We thus generate an infinite branch. $\square$

Then, we have the following proposition:

Proposition 58. *If X has an L-tableau proof from S as global and U as local premises, X has a finite L-tableau proof from S as global and U as local premises.*

Proof. Suppose X has a tableau proof $\mathcal{T}$ from S as global and U as local premises. $\mathcal{T}$ is either finite or infinite. If it is finite, then X has a finite tableau proof. If it is infinite, then we have an infinite tableau $\mathcal{T}$ where for every branch ψ, we have a pair of points $< x, y >$ at levels $l(x)$ and $l(y)$ where $l(x) < l(y)$ such that we have ωX at x and $\omega \neg X$ at y or $\omega \neg X$ at x and ωX at y. By Proposition 57, $\mathcal{T}$ has at least one infinite branch. Again, since $\mathcal{T}$ is closed, for each infinite branch ψ, we have our pair of points $< x, y >$ at some finite levels $l(x)$ and $l(y)$ which close the branch. Then, we can stop constructing any infinite ψ after $l(y)$ many levels, still resulting in a closed tableau. Thus, X has a tableau proof $\mathcal{T}_1$ from S as global and U as local assumptions with only finite long branches. Then, by the converse of Proposition 57, $\mathcal{T}_1$ is not infinite or not finitely generated. Since any modal tableau is finitely generated, $\mathcal{T}_1$ is not infinite. Thus, X has a finite tableau proof from S as global and U as local premises. □

5 Soundness

The soundness proof is based on a straightforward induction on tableau construction steps. However, due to the large number of branch extension rules, and the fact that tableux are trees with junction points, the actual proof is somewhat lengthy and elaborate. The general strategy of it comes from Fitting's (1983).

First, we define satisfiability for a set of prefixed formulas.

Definition 59 (K-Satisfiability for Prefixed Formulas)**.** A set of prefixed formulas S is K-satisfiable in the valuational framework $\mathcal{F}_v =< \mathcal{V}, \mathcal{R}_v >$ *iff* there is a way, Θ, of assigning to each prefix ω that occurs in some element of S some modal valuation $\Theta(\omega) \in \mathcal{V}$ such that:

1. If ω and $\omega.n$ both occur as prefixes in our set S, then $\Theta(\omega)\mathcal{R}_v\Theta(\omega.n)$ holds, i.e., $\Theta(\omega.n)$ is an alternative possible valuation relative to $\Theta(\omega)$.
2. If ωX is in S, then $\Theta(\omega)(X) = t$, i.e., X is true under the modal valuation $\Theta(\omega)$.

A tableau branch is satisfiable if the set of prefixed formulas on it is satisfiable in some valuational framework. A tableau itself is satisfiable if some branch of it is satisfiable.

Remark. In extending to the general, L-satisfiability of a set of prefixed formulas following the systems we introduced above, we only need to introduce some *additional* conditions for prefixes as specified by the logic L under consideration. If the logic under consideration is reflexive, then for any ω occuring as a prefix in any formula in S, $\theta(\omega)\mathcal{R}_v\theta(\omega)$, if symmetric and $\omega.n$ also occurs in S, then $\theta(\omega.n)\mathcal{R}_v\theta(\omega)$, and if transitive and $\omega.\omega_1$ also occurs in S, then $\theta(\omega)\mathcal{R}_v\theta(\omega.\omega_1)$.

To make our proofs less complicated, we define premise or *P-satisfiable* branches and tableaux:

Definition 60 (P-Satisfiability)**.** Suppose S is a set of local and U is a set of global premises. A branch ψ of an L-modal tableau $\mathcal{T}$ is said to be *P-satisfiable iff* it is L-satisfiable together with every $1\ Y$ for each $Y \in S$ and every ωZ for each $Z \in U$ and ω that occurs on ψ. A tableau is P-satisfiable if one of its branches is.

We need one more lemma to prove the soundness theorem. Again, both this and the soundness proof itself are truth-valuational variants of those found in Fitting's (1983).

Proposition 61. *A closed tableau is not satisfiable.*

Proof. Suppose we have a modal tableau that is both closed and satisfiable. By Definition 59, there is a branch of the tableau that is itself satisfiable. Form a set S of the prefixed formulas that are on the satisfiable branch. Then, there is valuational framework in which that set is satisfiable. Suppose it is satisfiable in the valuational framework $\mathcal{F}_v = < \mathcal{V}, \mathcal{R}_v >$, using the mapping Θ of prefixes to modal valuations in $\mathcal{V}$. Since the tableau is closed, by Definition 55, every branch of it is, and in particular, our satisfiable one as well. Thus, S also has as members for some prefix ω and some formula X both ωX and $\omega\neg X$. But then, by Definition 59, there is a modal valuation $v_M \in \mathcal{V}$ such that $\Theta(\omega) = v_M$ and $v_M(X) = v_M(\neg X) = t$, which is impossible. $\square$

Proposition 62 (Soundness)**.** *If S and U are sets of formulas and X is any formula, then if $S \vdash_K U \Rightarrow X$, then $S \vDash_{K_v} U \Rightarrow X$.*

Proof. The proof is by contradiction. Suppose that $S \vdash_K U \Rightarrow X$, but not $S \vDash_K U \Rightarrow X$. Then, we have a *finite* closed K-modal tableau $\mathcal{T}$ with S as global and U as local assumptions with $1\ \neg X$ at its origin. We also know by our supposition that $S \nvDash_K U \Rightarrow X$.

Thus, there is a K-valuational framework $\mathcal{F}_v$ where any member of S evaluates to t under all modal valuations v_M, with a specific modal valuation v_{M_1} such that for any member Y of U, $v_{M_1}(Y) = t$, but not X, i.e., $v_{M_1}(\neg X) = t$. We set $\theta(1) = v_{M_1}$. Thus, we know that $1\ \neg X$ and all prefixed formulas of form $1\ Y$ for all $Y \in U$ are K-satisfiable in $\mathcal{F}_v$ with θ as defined and additionally, for any $Z \in S$ and $v_M \in \mathcal{V}$, $v_M(Z) = t$ (and thus, given any θ, ω and $Z \in S$, $\theta(\omega) = v_M$ such that $v_M(Z) = t$).

We show that any finite tableau $\mathcal{T}^*$ for $1\ \neg X$ with S as global and U as local premises is satisfiable together with all prefixed formulas of form $1\ Y$ for all $Y \in U$ and of form ωZ for all ω occuring on the relevant open branch and all $Z \in S$ (i.e., it is P-satisfiable) in a valuational framework $\mathcal{F}_v$ where for all $v_M \in \mathcal{V}$, $v_M(Z) = t$ for all $Z \in S$. The proof is by induction on the number of tableaux in the tableaux formation sequence $< \mathcal{T}_1, ..., \mathcal{T}_n = \mathcal{T}^* >$, where given $i > 1$, each $\mathcal{T}_i$ is a direct extension of $\mathcal{T}_{i-1}$ and $\mathcal{T}_1$ is the tableau with the sole point $1\ \neg X$ at the origin.

Take the base case. Then, we have a tableau with the sole point $1\ \neg X$ at the origin. Then, by the above, it is P-satisfiable in a $\mathcal{F}_v$ as specified above.

Suppose the inductive hypothesis holds up to any sequence with n-many tableaux. Then, we show it for the sequence with $n+1$ many tableaux, i.e., when $\mathcal{T}_{n+1} = \mathcal{T}^*$. By the induction hypothesis, $\mathcal{T}_n$ is P-satisfiable in $\mathcal{F}_v$, and thus, at least one of its branches is P-satisfiable in a $\mathcal{F}_v$ where under every v_M, any $Z \in S$ receives t and under v_{M_1}, any $Y \in U$ receives t also, along with $\neg X$. Suppose that the P-satisfiable branch is ψ. Then, we look at each possible direct extension of $\mathcal{T}_n$ to $\mathcal{T}_{n+1}$ one by one.

1. Suppose a formula of form $\omega\alpha$ occurs on ψ and we apply a rule of type C to ψ. Then, either $\omega\alpha_1$ or $\omega\alpha_2$ is adjoined to the end of ψ. Since ψ is P-satisfiable in the appropriate $\mathcal{F}_v$ where under any v_M, for any $Z \in S$, $v_M(Z) = t$, and $\omega\alpha$ occurs on it, we know that there are modal valuations $v_M, v_{M_1} \in \mathcal{V}$ such that $\Theta(1) = v_{M_1}$, $\Theta(\omega) = v_M$,

$v_M(\alpha) = v_M(Z) = t$ for all $Z \in S$ and $v_{M_1}(Y) = v_{M_1}(Z) = t$ for all $Y \in U$ and $Z \in S$. By Proposition 43, $v_M(\alpha) = t$ *iff* $v_M(\alpha_1) = t$ and $v_M(\alpha_2) = t$. Thus, either way, we have a P-satisfiable branch. Then, we have a P-satisfiable tableau.

2. Suppose a formula of form $\omega\beta$ occurs on ψ and we apply a rule of type D to ψ. Then, we adjoin $\omega\beta_1$ and $\omega\beta_2$ simultaneously as left and right successors. We get two new branches, ψ_1 with end point $\omega\beta_1$ and ψ_2 with end point $\omega\beta_2$. Again, since ψ is P-satisfiable in the appropriate $\mathcal{F}_v$ as before, and $\omega\beta$ occurs on it, we know that there is a modal valuation $v_M \in \mathcal{V}$ such that $\Theta(\omega) = v_M$ and $v_M(\beta) = t$. By Proposition 45, $v_M(\beta) = t$ *iff* $v_M(\beta_1) = t$ or $v_M(\beta_2) = t$. In the first case, ψ_1 is P-satisfiable, in the second case, ψ_2 is P-satisfiable. Either way, we still have a branch that is P-satisfiable and therefore, our tableau is still P-satisfiable.

3. Suppose a formula of form $\omega\gamma$ occurs on ψ and we apply a rule of type P to ψ. Then, we adjoin $\omega.n\gamma_s$, where $\omega.n$ did not occur previously on the branch. Accordingly, $\Theta(\omega.n)$ is not defined. As before, since ψ is P-satisfiable in the appropriate $\mathcal{F}_v$ as before, and $\omega\gamma$ occurs on it, we know that there is a modal valuation $v_M \in \mathcal{V}$ such that $\Theta(\omega) = v_M$ and $v_M(\gamma) = t$. Then, by Proposition 47, there is a modal valuation $v_{M_2} \in \mathcal{V}$, such that $v_M \mathcal{R}_v v_{M_2}$ and $v_{M_2}(\gamma_s) = t$.

   We go on to define a new mapping, Θ', the following way. For all prefixes occuring in ψ, let Θ' be the same as Θ and set $\Theta'(\omega.n)$ to v_{M_2} (since it was previously undefined). Since Θ and Θ' agree on all the prefixes occurring on ψ, $\Theta'(\omega)$ is also v_M and $\Theta'(1)$ is also v_{M_1}. We already defined $\Theta'(\omega.n) = v_{M_2}$ and know that $v_M \mathcal{R}_v v_{M_2}$ and $v_{M_2}(\gamma_s) = t$. Thus, $\Theta'(\omega)\mathcal{R}_v\Theta'(\omega.n)$ and γ_s is true under $\Theta'(\omega.n)$, which is v_{M_2}. It follows that extending the branch ψ with any instance of a rule of Type C will result in a branch that is P-satisfiable in the desired $\mathcal{F}_v$, using the mapping Θ' instead of the original Θ. Then, we still have a P-satisfiable tableau.

4. Suppose a formula of form $\omega\delta$ occurs on ψ and we apply a rule of type N to ψ. Then, we adjoin $\omega.n\delta_s$, where $\omega.n$ already occured previously on the branch and $\Theta(\omega.n)$ is thus already defined. By Definition 59, we also know that $\omega.n$ is such that $\Theta(\omega)\mathcal{R}_v\Theta(\omega.n)$. Again, since ψ is P-satisfiable in the appropriate $\mathcal{F}_v$ as before, and $\omega\delta$ occurs on it, we know that there is a modal valuation $v_M \in \mathcal{V}$ such that $\Theta(\omega) = v_M$ and $v_M(\delta) = t$. Then, by Proposition 47, for all modal valuations $v_{M_2} \in \mathcal{V}$, if $v_M\mathcal{R}_v v_{M_2}$, then $v_{M_2}(\delta_s) = t$. Since $\Theta(\omega.n)$ is just such a v_{M_2}, δ_s is true under $\Theta(\omega.n)$. Accordingly, it follows that extending the branch ψ with any instance of a rule of Type D will result in a branch that is P-satisfiable. Then, we still have a P-satisfiable tableau.

5. Suppose we extend a P-satisfiable branch ψ of our tableau with our type L rule. Then, we add a formula of form $1\ Y$, where $Y \in U$. As before, we know that ψ is P-satisfiable in the appropriate $\mathcal{F}_v$. But then the resulting branch is still P-satisfiable, and thus the tableau also.

6. Suppose we extend a P-satisfiable branch ψ of our tableau with our type G rule. Then, we add a formula of form ωZ, where ω already occurs as a prefix at a point of the branch ψ and $Z \in S$. As before, we know that ψ is P-satisfiable in the appropriate $\mathcal{F}_v$. But then the resulting branch is still P-satisfiable, and thus the tableau also.

7. Finally, if we apply any rule to a branch ψ' other than the P-satisfiable ψ, we still have a P-satisfiable tableau.

This concludes the inductive part of the proof. By $S \vdash_K U \Rightarrow X$ and Proposition 58, we know we have a *finite* closed K-modal tableau $\mathcal{T}$ with S as global and U as local assumptions with $1\ \neg X$ at its origin. But by the above, $\mathcal{T}$ is also P-satisfiable in an appropriate $\mathcal{F}_v$. We have a contradiction. $\square$

Remark. Extensions to different systems are done by adding the appropriate rules as additional cases to the proof. We show one. Let us suppose the system is *K4*. Then, we add the case for the application of rule 4. Thus, if a formula of form $\omega\delta$ occurs on ψ, we may adjoin $\omega.n\delta_s^4$ as well, where $\omega.n$ already occured on the branch. By Definition 59, we know that $\omega.n$ is such that $\Theta(\omega)\mathcal{R}_v\Theta(\omega.n)$. Since ψ is P-satisfiable and $\omega\delta$ occurs on it, we know that there is a modal valuation $v_M \in \mathcal{V}$ such that $\Theta(\omega) = v_M$ and $v_M(\delta) = t$. Since our valuational framework $\mathcal{F}_v$ is transitive, we have that for any v_{M_2} such that $v_M\mathcal{R}_v v_{M_2}$, $v_{M_2}(\delta_S^4) = t$. For suppose $v_{M_2}(\delta_S^4) = f$. Then, there is a valuation v_{M_3} such that $v_{M_2}\mathcal{R}_v v_{M_3}$ and $v_{M_3}(\delta_s) = f$. But note that by transitivity, we have that $v_M\mathcal{R}_v v_{M_3}$, and by Proposition 47, for all modal valuations $v_{M_1} \in \mathcal{V}$, if $v_M\mathcal{R}_v v_{M_1}$, then $v_{M_1}(\delta_s) = t$, which is a contradiction. Thus, $v_{M_2}(\delta_S^4) = t$ for all v_{M_2}. Since $\Theta(\omega.n)$ is just such a v_{M_2}, δ_s^4 is true under $\Theta(\omega.n)$. The rest is as before but with $K4$ instead of K.

This concludes the proof of soundness.

6 Semi-Strong Completeness

In the following, we will prove the completeness of our tableau system K relative to the truth-valuational semantics we introduced earlier. The definitions and proofs are, again, the truth-valuational variants of those found in Fitting's (1983), with some important changes. The completeness presented here is semi-strong in that it only applies to sets of premises (more precisely, any union of local and global premise sets) which omit $\aleph_0$ propositional variables.

In essence, the strategy for the proof is the usual for completeness theorems in general. What we want to show is that whenever a set of formulas is syntactically consistent (here, this just means that the tableau does not close), then it is satisfiable. But there is also an important difference between the method found here and those used in Henkin-style constructions.

With tableaux, we specify an algorithm, and the tableau itself builds (or rather, we show that it *would be able to* build) the syntactic equivalent of a counter-valuational framework for *each* invalid argument through the systematic application of the tableau rules. This is a natural way to go about proving the appropriate satisfiability result, since tableaux are already syntactic counter-framework building 'frameworks', though naturally, when a person uses them, she is the one building the appropriate syntactic counter-framework in the given 'framework' of rules, not the algorithm. In contrast, Henkin's extension lemma specifies an algorithm that saturates *any* consistent set upwards to larger and larger consistent sets. By referring to consistency, at each step, we ensure that no contradiction can be derived from the extended set, and this way, we refer to actual proofs *inside* the system, or more precisely, the *lack of* specific proofs. But this is far more abstract than with tableaux, where the construction of the appropriate set by the algorithm takes place *entirely* inside the proof system, specifying for each case how to construct the sole modal tableau branch constituting the set.

More precisely, given the specific algorithm, the tableau can at most build the syntactic equivalent of a *fragment* of a counter-framework, but a fragment of which we can be sure, following Hintikka's work (see his (1969), which builds on his (1955)), that it encodes enough information to be extendible to a full counter-framework. Accordingly, let us start with the definition of modal Hintikka sets.

Definition 63 (K-Modal Hintikka Set)**.** By a K-modal Hintikka set or downward saturated set, we mean a set $S^{\downarrow}$ of prefixed formulas such that the following conditions hold for any $\omega\alpha$, $\omega\beta$, $\omega\gamma$, $\omega\delta$ in $S^{\downarrow}$:

1. No prefixed propositional variable ωP and its negation $\omega\neg P$ are both in $S^{\downarrow}$.
2. If $\omega\alpha$ is in $S^{\downarrow}$, then so are $\omega\alpha_1$ and $\omega\alpha_2$.
3. If $\omega\beta$ is in $S^{\downarrow}$, then so is $\omega\beta_1$ or $\omega\beta_2$.
4. If $\omega\gamma$ is in $S^{\downarrow}$, so is $\omega.n\gamma_s$ for some n.
5. If $\omega\delta$ is in $S^{\downarrow}$, so is $\omega.n\delta_s$ for every $\omega.n$ that occurs in an element of $S^{\downarrow}$.

Remark. Extensions to different systems is straightforward. For every rule that comes with the given system, we add the relevant additional condition. Continuing our example of $K4$, given the rule 4, we add that if $\omega\delta$ is in $S^{\downarrow}$, then so is $\omega.n\delta_s^4$ for every $\omega.n$ that occurs in an element of $S^{\downarrow}$. Note how this relates to Condition 5.

Since in this chapter, we only talk about K-modal Hintikka sets in the main body, we will just write 'Hintikka set' whenever we refer to them. Such sets may differ from logic to logic, as noted above. We go on to introduce a *systematic procedure* to create modal tableaux. Our systematic procedure ensures that every open branch (if any) of the constructed modal tableau will constitute a Hintikka set. In the following, we will introduce a 'bookkeeping device' by designating certain occurences of prefixed formulas as 'used'. In practice, this can be thought of as putting a little mark next to occurences of prefixed formulas in our proofs after we have applied the appropriate rules to them so that we do not get confused and know what is left to do. Mathematically, they are used to define a systematic construction algorithm. The specifics are given by the definition of systematic tableau below. The method was first introduced by Smullyan and adapted to modal logic by Fitting. In fact, the systematic construction of first-order tableau introduced by Smullyan in his (1995) is very close to the modal construction. This is no coincidence, since the modal operators play a role similar to quantifiers in first-order logic. The following algorithm comes from Fitting's (1972) and (1983) without modification.

Definition 64 (Systematic Tableau)**.** A modal tableau $\mathcal{T}$ with global and local assumptions for a formula X is called *systematic iff* the tableau is constructed as follows. First, take S to be the set of all global assumptions whose members are arranged in a sequence $A_1, A_2, A_3, \ldots$, and U to be the set of local assumptions whose members are arranged in a sequence $B_1, B_2, B_3, \ldots$. We construct our tableau as follows:

1. As a base case, put our formula at the origin as $1\ X$.

2. Suppose we have completed the nth stage. Then, we adjoin to the end of each open branch ψ ωA_i for each $i \leq n$ and all prefixes ω occuring on ψ, and $1\ B_n$ also, if they are not on ψ already. If the tableau we have constructed is closed, then stop. Also, if all $X \in U$ appear as $1\ X$ on every open branch and all $Y \in S$ appear on every open

branch as ωY for every ω on that branch and every occurrence of each formula of form $\omega\alpha$, $\omega\beta$, $\omega\gamma$, $\omega\delta$ has been used, then stop.

3. If none of the above, we take the leftmost *unused* point x of minimal level (i.e., the smallest n, such that $l(x) = n$) which appears on at least one open branch. Then, for *every* open branch ψ that passes through the point x:

   (a) If x is an $\omega\alpha$, we extend the branch ψ by adjoining α_1 and α_2 successively.

   (b) If x is an $\omega\beta$, we extend the branch ψ simultaneously to two branches ψ_1 and ψ_2, adjoining β_1 in the first case and adjoining β_2 in the second case.

   (c) If x is an $\omega\gamma$, we extend the branch ψ by adjoining $\omega.n\gamma_s$, where n is the smallest integer such that $\omega.n\gamma_s$ does not occur on the branch.

   (d) If x is an $\omega\delta$, we extend the branch ψ by adjoining $\omega.n\delta_s$ for *each* $\omega.n$ that already occurs on the branch and we repeat $\omega\delta$ once more.

If we have done the above for each open branch through x, we declare the occurrence of the formula at x to be used. This concludes the $n+1$st stage.

Remark. Again, we do not have to make significant changes to our systematic procedure to accomodate extensions to different logics. The only thing we have to change is condition 3. (d) of the above definition according to the additional necessity rules of the system L. Let us continue with *K4*. Then, we change condition 3. (d) to the following: If x is an $\omega\delta$, we extend the branch ψ by adjoining $\omega.n\delta_s$ one after the other for *each* $\omega.n$ that already occurs on the branch, *and* we also adjoin $\omega.n\delta_s^4$ for *each* $\omega.n$ that already occurs on the branch, and *then* we repeat $\omega\delta$ once more.

Note that this results in an extension of our tableau rules since we repeat formulas to enable 'bookkeeping'. This makes no difference. In fact, if we want to prove the decidability of K-systems, we lose it (see Fitting's (1983, 410-416)). If the systematic tableau $\mathcal{T}$ did not close but stopped constructing because all points of it were used up, $\mathcal{T}$ is a *finished* systematic tableau. If $\mathcal{T}$ did not close but ran on infinitely, it is called an *infinite* systematic tableau. Otherwise, $\mathcal{T}$ is a *closed* systematic tableau.

From Kőnig's Lemma and the definition of a systematic modal tableau, we immediately have the following.

Proposition 65. *Any infinite systematic modal tableau is open.*

Proof. Suppose $\mathcal{T}$ is an infinite systematic modal tableau. Then, by Kőnig's Lemma, $\mathcal{T}$ has at least one infinite branch. However, if $\mathcal{T}$ is also closed, then every branch of it is of finite length, since we stop constructing our systematic tableau when all of its branches close at some stage n. We have a contradiction. $\square$

We need to prove that the systematic procedure of Fitting actually produces the Hintikka sets we want. The proof is mine.

Proposition 66. *For any finished or infinite systematic K-modal tableau $\mathcal{T}$ for X, every open branch ψ of $\mathcal{T}$ constitutes a Hintikka set $S^{\downarrow}$ which has as members all $Y \in U$ as $1\ Y$ and all $Z \in S$ as ωZ for any ω that occurs in $S^{\downarrow}$.*

Proof. By contraposition, if there is an open branch ψ of $\mathcal{T}$ which does not constitute a Hintikka set $S^{\downarrow}$ which has as members all $Y \in U$ as $1\ Y$ and all $Z \in S$ as ωZ for any ω that occurs in $S^{\downarrow}$, then $\mathcal{T}$ is not a finished or infinite systematic K-modal tableau. We show that our systematic construction does not stop as long as $\mathcal{T}$ has an open branch that does not constitute a Hintikka set $S^{\downarrow}$ which has as members all $Y \in U$ as $1\ Y$ and all $Z \in S$ as ωZ for every ω that occurs in $S^{\downarrow}$.

Take the base case. We put $1\ X$ at the origin. Since our tableau is not closed, we do not stop. If both S and U are empty and X is of form P or $\neg P$ (i.e., a literal), we stop and the set constitutes a Hintikka set. If X is not a literal, then $1\ X$ has not yet been used and if S

or U or both are non-empty, then we do not have all $X \in U$ as $1\ X$ and all $Y \in S$ as ωY for every ω on the open branch, thus in either or both cases, we do not stop.

Now take the nth stage. We either stop if our tableau closes or if all points have been used *and* all $X \in U$ appear as $1\ X$ on every open branch and all $Y \in S$ appear on every open branch as ωY for every ω on that branch. If our tableau closes, we do not have any open branch left. If all points have been used *and* all $X \in U$ appear as $1\ X$ on every open branch and all $Y \in S$ appear on every open branch as ωY for every ω on that branch, then for any open branch ψ, we have the following for any ω and occurrence of formulas of form α, β, γ, δ and their successors as by our rules (including all $X \in U$ that appear as $1\ X$ on the branch and all $Y \in S$ that appear on the branch as ωY for every ω on that branch):

1. If $\omega\alpha$ is on ψ, then so are α_1 and α_2, otherwise that point has not been used.
2. If $\omega\beta$ is on ψ, then so is either β_1 or β_2, otherwise that point has not been used.
3. If $\omega\gamma$ is on ψ, then $\omega.n\gamma_s$ is also on ψ, for some n, otherwise that point has not been used.
4. Finally, if $\omega\delta$ is on ψ, then $\omega.n\delta_s$ is also on ψ, for every $\omega.n$ that occurs on ψ, otherwise that point has not been used.

Clearly, ψ constitutes a Hintikka set in this case which is such that all $X \in U$ appear as $1\ X$ on ψ and all $Y \in S$ appear on ψ as ωY for every ω on ψ. If the construction of our tableau did not stop at any n by either closing or using up all points (including all $X \in U$ as they appear as $1\ X$ on every open branch and all $Y \in S$ as they appear on every open branch as ωY for every ω on that branch), then it is constructed infinitely. By infinite construction, we ensure that any open branch constitutes a Hintikka set which is such that all $X \in U$ appear as $1\ X$ on any open branch and all $Y \in S$ appear on any open branch as ωY for every ω on that open branch (and any such open branch is, naturally, itself infinite).

Finally, note that since ψ is open, no pair of formulas ωX and $\omega\neg X$ are in the Hintikka set $S^{\downarrow}$, and thus no pair of formulas ωP and $\omega\neg P$ are in $S^{\downarrow}$. □

Remark. Extending to different systems is trivial. We merely have to add the relevant additions to clause 4 paralleling the relevant additions in Definition 64.

Finally, we move to the essential part of our completeness proof, a proposition that is the modal equivalent of Hintikka's Lemma for first-order logic (see Smullyan's (1995)). As in first-order logic, the lemma shows that any Hintikka set is satisfiable. Smullyan uses a kind of term-model to construct a model for first-order logic while proving Hintikka's Lemma, while Fitting uses a 'prefix' model for this. Note that while prefixes may *be* (or more precisely, *stand for*) possible worlds in some abstract sense, modal valuations are functions and most definitely not prefixes. Our construction will reflect this. The construction will also reflect an additional difference between model-theoretic and truth-valuational semantics, which also explains our initial restriction to *semi*-strong completeness.

As mentioned, in valuational frameworks, every modal valuation needs to be distinguishable if distinct in at least one value of a formula. In Fitting's proof, in essence, we build a model for the Hintikka set $S^{\downarrow}$ by taking every propositional variable in the set to be true at the world ω (each prefix taken as a 'possible world' of the model) by which the formula is prefixed in the set, and otherwise take all other propositional variables to be false at the relevant ω (whether they occur in $S^{\downarrow}$ or not). However, there is a seeming problem with adapting the equivalent construction for truth-valuation semantics, for so far, we have found no proof that makes sure each constructed valutional framework would be an admissible one, where every valuation is distinguishable if distinct.[1]

Accordingly, in the following, we will *ensure* that each constructed valuation comes out distinguishable from any other. Incidentally, the technique requires that each open branch omit $\aleph_0$ variables, hence our restriction. Note that since tableaux are *analytic*, we do not have

[1]We will discuss this problem more in the next two chapters.

any Cut rule. Thus, any branch of a tableau $\mathcal{T}$ may contain at most as many propositional variables as those in the union of the premise sets and the singleton set of the conclusion itself. Of course, since the conclusion is always a finite formula, if the union of the premise sets omits $\aleph_0$ propositional variables, the union of the premise sets and the singleton set of the conclusion also omits $\aleph_0$ variables. This is why when talking of semi-strong completeness, we talk of a restriction to just those premise sets whose union omits $\aleph_0$ variables, which then entails that any Hintikka set $S^{\downarrow}$ constituted by an open branch ψ of a systematic tableaux $\mathcal{T}$ omits $\aleph_0$ variables too.

Keeping the above in mind, let us present the proof.

Proposition 67 (Hintikka's Lemma)**.** *If $S^{\downarrow}$ is a Hintikka set that omits $\aleph_0$ propositional variables, then $S^{\downarrow}$ is K-satisfiable.*

Proof. To prove the above proposition, we need to construct a valuational framework $\mathcal{F}_v$ for $S^{\downarrow}$ in which it is K-satisfiable. We do it as follows.

First, take the (denumerably infinite) set Q of the variables that do not occur in $S^{\downarrow}$ and suppose they are ordered $p_1, p_2, ...$ Suppose the (at most denumerably infinite) set O of all ω occurring in $S^{\downarrow}$ is ordered $\omega_1, \omega_2, ...$

For every prefix ω that occurs in any formula in $S^{\downarrow}$, we introduce a function $v_\omega : \mathcal{F} \to \{t, f\}$. For any prefix ω and propositional variable P *occurring in* $S^{\downarrow}$, if $\omega P \in S^{\downarrow}$, we set $v_\omega(P) = t$, and if $\omega P \notin S^{\downarrow}$, we set $v_\omega(P) = f$. Then, for any $\omega_n \in O$ and $p_k \in Q$, set $v_{\omega_n}(p_k) = t$ if $k \leq n$, and set $v_{\omega_n}(p_k) = f$ if $n < k$. Then, every $v_\omega \in \mathcal{V}$ is atomically distinct.

Finally, if ω_1 and ω_2 occur in some prefixed formulas in $S^{\downarrow}$, we set $v_{\omega_1}\mathcal{R}_v v_{\omega_2}$ *iff* ω_1 is of form ω and ω_2 is of form $\omega.n$. We thus construct a valuational framework $\mathcal{F}_v =< \mathcal{V}, \mathcal{R}_v >$, where $\mathcal{V} = \{v_\omega : \omega \text{ occurs in } S^{\downarrow}\}$ and $\mathcal{R}_v$ is as we defined.

Next, we want to show that for each formula X and prefix ω, if $\omega X \in S^{\downarrow}$, then $v_\omega(X) = t$, i.e., X is true under v_ω. The proof is by induction on the complexity of the formula X.

Take the base case. Then, X is a propositional variable, i.e., P. By definition, if $\omega P \in S^{\downarrow}$, $v_\omega(P) = t$.

Take one form of formulas of complexity 1, that of $\neg P$, separately. Since $S^{\downarrow}$ is a Hintikka set, if $\omega\neg P \in S^{\downarrow}$, $\omega P \notin S^{\downarrow}$, thus $v_\omega(P) = f$. Then, by Definition 25, $v_\omega(\neg P) = t$.

Suppose that our hypothesis holds for formulas up to complexity n. Then, we show it for all formulas of complexity $n+1$. We take each case in turn.

1. Suppose X is a formula of form α. Then, since $S^{\downarrow}$ is a Hintikka set, if $\omega\alpha$ is in $S^{\downarrow}$, so are $\omega\alpha_1$ and $\omega\alpha_2$. Both are of complexity less than α, thus, by our induction hypothesis, both are true under v_ω. But then so is α, by Definition 25.
2. Suppose X is a formula of form β. Then, again, if $\omega\beta$ is in $S^{\downarrow}$, so is one of $\omega\beta_1$ or $\omega\beta_2$. Again, both are of complexity less than β, thus, by the induction hypothesis, either is true under v_ω if it is in $S^{\downarrow}$. But then so is β, by Definition 25.
3. Suppose X is a formula of form γ. Then, if $\omega\gamma$ is in $S^{\downarrow}$, so is $\omega.n\gamma_s$ for some n. Since γ_s is of smaller complexity than γ, $v_{\omega.n}(\gamma_s) = t$, and by the above construction, $v_\omega \mathcal{R}_n v_{\omega.n}$. But then $v_\omega(\gamma) = t$, by Definition 25.
4. Suppose X is a formula of form δ. Again, if $\omega\delta$ is in $S^{\downarrow}$, so is $\omega.n\delta_s$ for all $\omega.n$ that occurs in $S^{\downarrow}$. Since δ_s is of smaller complexity than δ, $v_{\omega.n}(\delta_s) = t$ for every $v_{\omega.n}$, and by the above construction, $v_\omega \mathcal{R}_v v_{\omega.n}$ and for no other prefix ω_1 we have $v_\omega \mathcal{R}_v v_{\omega_1}$. But then $v_\omega(\delta) = t$, again, by Definition 25.

This concludes the inductive part of the proof. Finally, if we take $\mathcal{F}_v$ as specified above and set $\Theta(\omega) = v_\omega$, $S^{\downarrow}$ is clearly K-satisfiable. □

Remark. In extending to a different system L, we have to construct the relevant L-valuational framework in which our set is L-satisfiable. Let us continue with *K4*, which is determined by the set of K-valuational frameworks with transitivity. First, we change the definition of the relation $\mathcal{R}_v$ used in the proof as follows: if ω and $\omega.n$ occur in some prefixed formulas in

S, we set $v_\omega \mathcal{R}_v v_{\omega.n}$ *and* if $\omega.\omega_1$ *also* occurs in some prefixed formula in $S^{\downarrow}$, we set $v_\omega \mathcal{R}_v v_{\omega.\omega_1}$ (in this case, such a definition is redundant, but not for all systems, so we give it like this). This ensures that the constructed valuational framework is a *K4* one.

Now for its members. As before, we only have to change clause 4. We first want to show that if $\omega\delta$ occurs in $S^{\downarrow}$, we have $\omega.\omega_1\delta_s$ and $\omega.\omega_1\delta_s^4$ for *any* $\omega.\omega_1$ that occurs in $S^{\downarrow}$. We use induction on the number of integers occurring in ω_1. Take the base case. Then, since $S^{\downarrow}$ is a *K4*-modal Hintikka set, we have that if $\omega\delta$ is in $S^{\downarrow}$, so is $\omega.n\delta_s$ and $\omega.n\delta_s^4$ for all $\omega.n$ that occurs in $S^{\downarrow}$. Now suppose our induction hypothesis holds for any ω_1 occuring in $S^{\downarrow}$ with at most k many occurrences of integers. Then, we prove it for $k+1$. Since we have $\omega.\omega_1\delta_s^4$ with ω_1 having k many occurrences of integers, and it is a formula of form $\omega\delta$, we also have $\omega.\omega_1.n\delta_s$ and $\omega.\omega_1.n\delta_s^4$ for every $\omega.\omega_1.n$ that occurs in $S^{\downarrow}$. This concludes the induction.

Thus, we have $\omega.\omega_1\delta_s$ for *any* $\omega.\omega_1$ that occurs in $S^{\downarrow}$. Then, since δ_s is of smaller complexity than δ, for any $\omega.\omega_1$ occuring in $S^{\downarrow}$, we have that $v_{\omega.\omega_1}(\delta_s) = t$, and by the definition of $\mathcal{R}_v$, we know that $\omega \mathcal{R}_v \omega.\omega_1$ and for no other prefixes ω_2 we have $v_\omega \mathcal{R}_v v_{\omega_2}$. Then, by Definition 25, $v_\omega(\delta) = t$. The proof is similar for other systems.

We now have to put together our results, but the hard part is over. Given our sketches above for extending to different systems, the L-equivalents of the following propositions and proofs are trivial variants of what is presented for K. Indeed, there is nothing more to do than replace every occurrence of K with L to arrive at them.

Proposition 68. *In any finished systematic K-modal tableau, every open branch that omits $\aleph_0$ propositional variables is K-satisfiable.*

Proof. Follows immediately from Propositions 66 and 67. □

The next proof is entirely mine.

Proposition 69. *X has a K-tableau proof from S as global and U as local premises iff it has a systematic K-tableau proof with S as global and U as local premises.*

Proof. From left to right. Suppose X has a K-tableau proof from S as global and U as local premises but does not have a *systematic* K-tableau proof from S as global and U as local premises. Then, there is a tableau $\mathcal{T}$ from S as global and U as local premises with $1\ \neg X$ at the origin whose branches all close. However, there is a *systematic* tableau $\mathcal{T}_S$ from S as global and U as local premises with $1\ \neg X$ at the origin which has an open branch ψ such that all members Y of U occur as $1\ Y$ and all members Z of S occur as ωZ on ψ for every ω that occurs on ψ. We know that ψ is K-satisfiable and importantly, since $1\ \neg X$ occurs on ψ and all members Y of U occur as $1\ Y$ and all members Z of S occur as ωZ on ψ for every ω that occurs on ψ, the sets of local and global assumptions together with the negation of X is also K-satisfiable. By assumption, X has a K-tableau proof from S as global and U as local premises, and by the Soundness Theorem, if X has a K-tableau proof from S as global and U as local premises, X is K-entailed by the global premises S and local premises U. But we have just shown that the sets S and U together with $\neg X$ are satisfiable. We have a contradiction. From right to left, it is trivial. If X has a systematic K-tableau proof from S as global and U as local premises, then it has a K-tableau proof from S as global and U as local premises, namely, the systematic one. □

Then, we can finally prove completeness.

Proposition 70 (Semi-Strong Tableau Completeness for K)**.** *For any formula X, global premises S and local premises U whose union omits $\aleph_0$ propositional variables, if $S \vDash_{K_v} U \Rightarrow X$, then $S \vdash_K U \Rightarrow X$.*

Proof. In (other) words, we want to show that if X is K-entailed by S as global and U as local premises, then X has a tableau proof from S as global and U as local premises using the K rules. Suppose X is K-entailed by S as global and U as local premises. By Proposition 69, X has a K-tableau proof from S as global and U as local premises *iff* it has a *systematic* K-tableau proof from S as global and U as local premises. Let $\mathcal{T}$ be a finished or infinite *systematic* modal tableau for $\neg X$ with S as global and U as local premises. If $\mathcal{T}$ contained an open branch ψ, then by Proposition 68, the set that constitutes ψ would be satisfiable. Then, $1\ \neg X$ being at the origin and thus on every branch, and all members Y of U occurring on any open branch as $1\ Y$ and all members Z of S occurring on any open branch as ωZ for any ω that occurs on the branch means that the global and local premises together with the negation of the conclusion are satisfiable. Thus, there is a valuational framework $\mathcal{F}_v$ with $\mathcal{R}_v$ as specified by K where all formulas in the global premise set S are true under all $v_M \in \mathcal{V}$ and a $v_M \in \mathcal{V}$ (specifically, v_1) such that $v_M(\neg X) = t$ and additionally, $v_M(Y) = t$ for any Y in the local premise set U. We have a contradiction. Therefore, $\mathcal{T}$ with S as global and U as local premises must close. $\square$

This concludes our semi-strong completeness proof. Then, we immediately have the following:

Proposition 71 (Semi-Strong Correctness for Truth-Valuational Semantics)**.** *For any formula X, global premise set S and local premise set U whose union omits $\aleph_0$ propositional variables, $S \vDash_{K_v} U \Rightarrow X$ iff $S \vdash_K U \Rightarrow X$.*

As corollaries, we also have compactness and the Löwenheim-Skolem property for our systems.

Proposition 72 (Compactness)**.** *For any formula X, global premise set S_1 and local premise set U_1 whose union omits $\aleph_0$ variables, there are finite subsets $S_2 \subseteq S_1$ and $U_2 \subseteq U_1$ such that if $S_1 \vDash_{K_v} U_1 \Rightarrow X$, then $S_2 \vDash_{K_v} U_2 \Rightarrow X$.*

Proof. We trivially have *syntactic* compactness, since by Proposition 58, if X has a tableau proof from S as global and U as local premises, X has a *finite* tableau proof from S as global and U as local premises, where only some finite subsets of S and U occur in the proof, and the tableau closes (by definition of a tableau proof). Thus, we have that for any formula

X, global premise set S_1 and local premise set U_1 whose union omits $\aleph_0$ variables, there are *finite* subsets $S_2 \subseteq S_1$ and $U_2 \subseteq U_1$ such that if $S_1 \vdash_K U_1 \Rightarrow X$, then $S_2 \vdash_K U_2 \Rightarrow X$. By the Correctness Theorem, we immediately have that if $S_1 \vDash_{K_v} U_1 \Rightarrow X$, then $S_2 \vDash_{K_v} U_2 \Rightarrow X$. □

Proposition 73 (Löwenheim-Skolem Property). *If a formula X is K-satisfiable at all, it is K-satisfiable in a valuational framework $\mathcal{F}_v$ with at most $\aleph_0$ modal valuations.*

Proof. We know that if X is K-satisfiable, $\neg X$ is not a K-modal tautology. Thus, by the Correctness Theorem, $\neg\neg X$ does not have a closed K-tableau. Then, $\neg\neg X$ (and thus, X too) has a systematic K-tableau that is open. As shown above, any open branch of a K-tableau constitutes a K-Hintikka set $S^{\downarrow}$, and since any branch has at most $\aleph_0$ points, and at each point exactly one prefixed formula, there are at most $\aleph_0$ prefixes occurring in any K-Hintikka set. By Hintikka's lemma, any such set is K-satisfiable in a valuational framework which has the same cardinality of modal valuations as prefixes occurring in $S^{\downarrow}$. □

Through Fitting, we already have strong, and *a fortiori* semi-strong correctness for model theoretic semantics, i.e, the following:

Proposition 74 (Strong Correctness for Model-Theoretic Semantics). *For any formula X, global premise set S and local premise set U, $S \vDash_{K_M} U \Rightarrow X$ iff $S \vdash_K U \Rightarrow X$.*

Then, we have semi-strong semantic equivalence between our two semantics.

Proposition 75 (Semi-Strong Semantic Equivalence). *For any formula X, global premise set S and local premise set U whose union omits $\aleph_0$ propositional variables, $S \vDash_{K_v} U \Rightarrow X$ iff $S \vDash_{K_M} U \Rightarrow X$*

7 Three Ways to Strong Completeness

In the following, we will lift our restriction on premise sets and will accept any sets of local and global premises in our proofs. The resultant problems can be countenanced by taking at least three diverging approaches. However, taking any of these three roads results in non-trivial consequences for our semantics.

Before moving on, note the following very important fact. Using the valuational equivalents of canonical models, i.e., canonical frameworks (see Fitting's (2007) for the exact model theoretic definitions and proofs), where for each maximal L-consistent set of an axiomatic (or natural deduction) L-system, there corresponds a modal valuation in $\mathcal{V}$ of $\mathcal{F}_v$, and where $\mathcal{F}_v$ is in L, we can prove the *strong completeness* of any one of the logics we have considered above relative to their corresponding axiomatic (or natural deduction) systems through the fact that every L-consistent set can be extended to a complete and L-consistent set, whose corresponding valuational framework is in $\mathcal{V}$ of the canonical framework $\mathcal{F}_v$.[1]

Such a completeness proof works because every valuation in $\mathcal{V}$ of the canonical framework $\mathcal{F}_v$ is provably pairwise distinguishable from any other valuation in $\mathcal{V}$. The model-theoretic version of this kind of a completeness proof then immediately gives us an expressibility result for model-theoretic semantics that is not well-advertised in the literature, but an

[1]It is not clear whether the construction of Makinson's (1966), which produces somewhat smaller models than the full canonical one, can be made to work with truth-valuational semantics. Again, one would have to show that during the construction, we do not suppose that some indistinguishable valuations are distinct.

expressibility result that is crucial for the truth-valuational approach.

Proposition 76 (An Expressibility Result)**.** *If L is K, T, $K4$, B, $S4$ or $S5$, no satisfiable set S of propositional modal formulas can express a difference between an L-model which has distinct but indistinguishable worlds and an L-model which does not have any.*

Proof. Given any set S of formulas, if S is L-satisfiable at all, then S is L-satisfiable in the canonical L-model $\mathcal{M}$. For suppose this is not the case. Then, S is L-satisfiable but it cannot be satisfied in $\mathcal{M}$. We know that every L-consistent set of sentences is satisfiable in $\mathcal{M}$. Then, by our supposition, S is L-inconsistent. Then, both X and $\neg X$ can be derived from it. Trivially, no L-inconsistent set is satisfiable, contra supposition. Thus, S is L-satisfiable in the canonical L-model $\mathcal{M}$, which by definition only has distinguishable worlds as distinct. $\square$

The problem with prefixed modal tableaux is that its completeness proof, as we have seen, does not use complete consistent sets or canonical valuational frameworks. Instead, for any Hintikka set $S^{\downarrow}$, the construction found in the proof builds a valuational framework in which $S^{\downarrow}$ can be satisfied. That the constructed framework is admissible was, in the previous section, ensured by carefully setting the values of propositional variables so that each valuation of any valuational framework is atomically distinct from any other. This required that the Hintikka set omit $\aleph_0$ propositional variables, hence the restriction on premise sets. However, now that we lifted the restriction, the problem presents itself anew. If we want strong completeness, we need to ensure that the constructed frameworks for our Hintikka sets are admissible. But if the set does not omit $\aleph_0$ variables, this cannot be done the way we have done above.[2]

[2]Note that we have no proof (i.e., a counterexample) that the truth-valuational equivalent of Fitting's original construction produces inadmissible valuational frameworks. However, we do not have a proof to the contrary either, that *every* framework constructed *is* admissible.

Let us see how we can answer this challenge three different ways, and what problems they each introduce.

7.1 Extending the Language

The most straightforward way of making sure Hintikka's lemma goes through is by freeing up $\aleph_0$ propositional variables. An obvious way to do this is simply by taking the initial language $\mathcal{L}_1$ we are working with and taking its variable extension $\mathcal{L}_2$.[3]

The proof is exactly as before up to Proposition 67, i.e., Hintikka's lemma. However, instead of moving straight to our construction, we first extend the language $\mathcal{L}_1$ in which our L-modal Hintikka set $S^{\downarrow}$ is formulated to the variable extension $\mathcal{L}_2$ of $\mathcal{L}_1$. Clearly, since $|Var_2 \setminus Var_1| = \aleph_0$, and only members of Var_1 occur in $S^{\downarrow}$, whether or not $S^{\downarrow}$ omits deumerably infinite many variables *relative to* $\mathcal{L}_1$, it omits $\aleph_0$ variables *relative to* $\mathcal{L}_2$.

Moving on to Hintikka's lemma, the proof is exactly as before. Skipping to the relevant part, we again can assign values to our propositional variables relative to the relevant v_ω in a way that they each come out atomically distinct from any other in $\mathcal{V}$, since in $\mathcal{L}_2$, we again have enough variables to do this for each (at most $\aleph_0$) v_ω.

However, this comes at a price, a price that proponents of truth-valuational semantics know all too well. For note that even though it is not mentioned, it is implicit in the definition of validity and satisfiability that we have the initial language $\mathcal{L}_1$ in mind. Thus, we have to *redefine* both. We give it for the most general definitions, which subsume weaker ones.

Definition 77 (Modal Semantic Entailment for Variable Extensions)**.** If S and U are sets of formulas and X is a formula *formulated in* $\mathcal{L}_1$, X is a consequence in L of S as global assumptions and U as local assumptions *iff* in *every* variable extension $\mathcal{L}_2$ of $\mathcal{L}_1$, for every

[3] The basic idea comes from Dunn and Belnap's (1968), who propose it for substitutional first-order semantics (cf. Dunn's (1973)).

L-valuational framework where all members of S are a modal tautology and for every modal valuation $v_M \in \mathcal{V}$ under which all members of U evaluate to t, $v_M(X) = t$.

Definition 78 (L_v-Satisfiability for Variable Extensions)**.** A formula X is L_v-satisfiable together with a set S of global and a set U of local premises *formulated in* $\mathcal{L}_1$ *iff* there is a variable extension $\mathcal{L}_2$ of $\mathcal{L}_1$, an L-valuational framework $\mathcal{F}_v$ where under any v_M, for any $Z \in S$, $v_M(Z) = t$, and a $v_M \in \mathcal{V}$ such that for any $Y \in U$, $v_M(Y) = t$ and $v_M(X) = t$ also.

In first-order logic, philosophers and logicians have provided, or at least tried to provide, reason to accept such extensions (there, extensions with terms, not propositional variables) and the redefined validity and satisfiability notions along with them. An important part of this defence was the fact that in first-order logic, due to ω-inconsistent theories, the semantics gives the wrong results in general relative to *all* standard systems.

Similarly, if variable extensions were generally required for any strong completeness proof with the truth-valuational approach relative to the standard systems, they could be taken as a non-trivial consequence of it, as in first-order logic. The question, then, would be whether the truth-valuational approach provides sufficiently strong reasons against model-theoretic semantics even in conjunction with this consequence, similarly as in the debate concerning first-order logic.

Now if together with *one of* Dunn's (1973) semantics, we want to hold that there are no irreducible modal facts, then variable extensions *are* required even for an axiomatic proof system with a canonical framework type completeness proof, which provides a *prima facie* strong case that variable extensions *are* a general consequence of a correct account of that semantics, and are perhaps philosophically defensible on some ground along with the whole approach.

However, given that even strong completeness goes through if a canonical framework-type proof is available for a deductive system on the truth-valuational approach, such a general consequence cannot be established, for such a completeness result demonstrates that the semantics itself is fine – the problem is with some specific systems and their completeness proofs. Then, redefining entailment and satisfiability, the central notions of logic, just for the sake of one specific type of completeness proofs, which are required for one specific type of deductive systems, seems completely indefensible. But seeing that we would rather not part with the tableaux method either, as it is a very intuitive, easy-to-use and powerful system, we are at an impasse.

7.2 Rewriting the Hintikka Set

One can also free up $\aleph_0$ variables by rewriting the Hintikka set in a specific way.[4] In fact, we have already specified how to do this. Given any modal Hintikka set $S^{\downarrow}$, take its variable rewrite $R(S^{\downarrow})$.[5] We already know that the resultant set $R(S^{\downarrow})$ is syntactically isomorphic to $S^{\downarrow}$, and it omits $\aleph_0$ variables, namely, all the variables indexed by an even number.

Again, such a move enables us to take the proof of Proposition 67 exactly as it occurs in the section on semi-strong completeness, given the fact that for each Hintikka set $S^{\downarrow}$, there corresponds a Hintikka set $R(S^{\downarrow})$ which omits $\aleph_0$ variables, regardless of whether $S^{\downarrow}$ omits $\aleph_0$ variables or not.

However, as before, we again have to redefine validity and satisfiability for our proof to show something relevant relative to the semantics. To make the following definitions more universal, we will generalize the definition of a *rewrite function* so that it applies to any function $\mathscr{R} : \mathcal{F} \to \mathcal{F}$ (where $\mathcal{F}$ is the set of all formulas of the language) which uniformly substitutes all occurences of propositional variables in a formula for some specific other

[4]The basic idea comes from Leblanc's (1976), who introduces it as an alternative in substitutional first-order semantics to extending the language as specified by Dunn and Belnap in their (1968). As can be seen, such a dilemma for substitutional (and therefore, truth-valuational) first-order semantics reproduces itself for truth-valuational modal logic.

[5]If we want to be really precise, the variable rewrite $R(\omega X)$ of a prefixed formula ωX is just $\omega R(X)$.

propositional variables, and which, when extended, as before, to $\mathscr{R} : \mathcal{P}(\mathcal{F}) \to \mathcal{P}(\mathcal{F})$, is such that if $\mathscr{R}(S) = S^*$, then $\mathscr{R}$ induces an isomorphism between the syntaxes $\mathbf{S}$ and $\mathbf{S}^*$, and moreover, for any set S, $\mathscr{R}(S)$ omits $\aleph_0$ variables of the language. Clearly, the rewrite function R we have been working with is such a function.

Definition 79 (Modal Semantic Entailment for Rewrites)**.** If S and U are sets of formulas and X is a formula, X is a *rewrite* consequence in L of S as global assumptions and U as local assumptions *iff* $\mathscr{R}(X)$ is a standard consequence in L of $\mathscr{R}(S)$ as global assumptions and $\mathscr{R}(U)$ as local assumptions.

Definition 80 (L_v-Satisfiability for Rewrites)**.** A formula X is *rewrite* L_v-satisfiable together with a set S of global and a set U of local premises *iff* the formula $\mathscr{R}(X)$ is standard L_v-satisfiable together with the set $\mathscr{R}(S)$ of global and the set $\mathscr{R}(U)$ of local premises.

As can be seen, such a definition is 'parasitic' on the standard definitions of validity and satisfiability given in Section 3.2. The substantial content of the definitions is not changed, but whenever we are evaluating a given argument, we need to take its variable rewrite and evaluate *that* argument, which definitely omits $\aleph_0$ variables.

As before, such a move works formally, but the price to pay is high for again, given the fact of strong completeness for axiomatic systems, it is not a general consequence of the semantics that we require these modifications, while the redefined central semantic notions of logic are as general as it gets.

7.3 Indexing Valuations

The third and final way to strong completeness is probably the most natural and straightforward. We simply introduce more structure into our frameworks, thereby countenancing the seemingly coincidental fact that valuations are functions and no two functions assigning the same values to all their inputs can be distinct.[6]

We give the following definition of *indexed* valuational frameworks.

Definition 81 (Indexed Valuational Frameworks)**.** An indexed valuational framework $\mathcal{F}_v^i$ is a pair $<\mathscr{V}, \mathcal{R}_v>$, where $\mathscr{V}$ is any subset of the Cartesian product $\mathcal{V}^i \times S$, where $\mathcal{V}^i$ is the set of all indexed modal valuations and S is a non-empty set of any cardinality κ, provided that if $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are in $\mathscr{V}$, then if $\mathfrak{v}_1 =< v_1, \omega >$ and $\mathfrak{v}_2 =< v_2, \omega >$, then $v_1 = v_2$, and thus $\mathfrak{v}_1 = \mathfrak{v}_2$ (i.e., the structure models an injection). As before, $\mathcal{R}_v$ is a binary relation defined on $\mathscr{V}$. If $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are both in $\mathscr{V}$ and $\mathfrak{v}_1\mathcal{R}_v\mathfrak{v}_2$, we say that $\mathfrak{v}_2$ is a (possible) alternative valuation pair to $\mathfrak{v}_2$.

The above construction ensures that every modal valuation gets its own prefix, distinct from the prefix of any other valuation. Let us go forward.

Definition 82 (Indexed Modal Valuation)**.** An indexed modal valuation v_1 in $\mathfrak{v}_1 =< v_1, \omega_1 >$ relative to an indexed valuational framework $\mathcal{F}_v^i$ is a Boolean valuation, for which the following additional conditions hold for any X and v_2 of any $\mathfrak{v}_2 =< v_2, \omega_2 >$ in $\mathscr{V}$.

1. $v_1(\Box X) = t$ *iff* for any indexed modal valuation v_2 in any $\mathfrak{v}_2 =< v_2, \omega_2 >$ in $\mathscr{V}$, if $\mathfrak{v}_1\mathcal{R}_v\mathfrak{v}_2$, then $v_2(X) = t$ and f otherwise.
2. $v_1(\Diamond X) = t$ *iff* there is an indexed modal valuation v_2 in a $\mathfrak{v}_2 =< v_2, \omega_2 >$ in $\mathscr{V}$ such that $\mathfrak{v}_1\mathcal{R}_v\mathfrak{v}_2$ and $v_2(X) = t$ and f otherwise.

Everything else is a relatively straightforward extension of the above. We define indexed L_v-satisfiability.

[6] Again, the basic idea comes from Leblanc's (1976), who refers to Montague as its originator. Leblanc uses it to prove completeness for first-order modal logics other than *S5* where each valuation needs to be *atomically* distinct.

Definition 83 (Indexed L_v-Satisfiability)**.** A formula X is indexed L_v-satisfiable together with a set S of global and a set U of local premises *iff* there is an indexed L-valuational framework $\mathcal{F}_v^i$ where under any v_M of any $\mathfrak{v}$ of $\mathscr{V}$, for any $Z \in S$, $v_M(Z) = t$, and a $\mathfrak{v} \in \mathscr{V}$ such that $\mathfrak{v} =< v_M, \omega >$, and for any $Y \in U$, $v_M(Y) = t$ and $v_M(X) = t$ also.

Moving on to Hintikka's lemma, this time we need to change some things in the proof. As one might have suspected all along, we have been using the ω symbol in our definitions above since in the following proof, prefixes will stand in as indexes of the indexed modal valuations (i.e., the second members of the ordered pairs).

Proposition 84 (Hintikka's Lemma)**.** *If $S^{\downarrow}$ is a Hintikka set, then $S^{\downarrow}$ is indexed K-satisfiable in an indexed valuational framework.*

Proof. To prove the above proposition, we need to construct an indexed valuational framework $\mathcal{F}_v$ for $S^{\downarrow}$ in which it is K-satisfiable. We do it as follows.

For every prefix ω that occurs in any formula in $S^{\downarrow}$, we introduce an indexed valuation-prefix pair $\mathfrak{v}_\omega =< v_\omega, \omega >$, where $v_\omega : \mathcal{F} \to \{t, f\}$. For any prefix ω and propositional variable P, if $\omega P \in S^{\downarrow}$, we set the output of v_ω of $\mathfrak{v}_\omega$ to t for the input P, and if $\omega P \notin S^{\downarrow}$, we set the output of v_ω of $\mathfrak{v}_\omega$ to f for the input P.

Finally, if ω_1 and ω_2 occur in some prefixed formulas in $S^{\downarrow}$, we set $\mathfrak{v}_{\omega_1} \mathcal{R}_v \mathfrak{v}_{\omega_2}$ *iff* ω_1 is of form ω and ω_2 is of form $\omega.n$. We thus construct an indexed valuational framework $\mathcal{F}_v^i =< \mathscr{V}, \mathcal{R}_v >$, where $\mathscr{V} = \{\mathfrak{v}_\omega : \omega \text{ occurs in } S^{\downarrow}\}$, $\mathfrak{v}_\omega =< v_\omega, \omega >$, and $\mathcal{R}_v$ is as we defined.

Again, we want to show that for each formula X and prefix ω, if $\omega X \in S^{\downarrow}$, then $v_\omega(X) = t$, i.e., X is true under v_ω. The proof is by induction on the complexity of the formula X as before. We do not write it out again. □

With some modifications, the above construction returns to Fitting's original. Such a move is feasible since indexed valuational framework ensure that there is one-to-one correspondence between the set of all Kripke models and the set of all indexed valuational frameworks. This may be construed as either a positive or a negative result. If we think of the restriction on unindexed valuational frameworks regarding distinct and indistinguishable valuations as an inessential consequence of truth-valuational semantics that obtains merely because of the accidental mathematical fact that we model formally the different truth-values formulas may take by functions, then indexed modal valuations are the obvious solution.

Yet, one may reason in the opposite direction as well. If we think that it is an essential consequence of truth-valuational semantics for modal logics that there can be no distinct but indistinguishable valuations, e.g., because we construe valuations as alternative assignments of truth-values to sentences, and it makes no sense to consider an assignment of truth-values that is identical to another, distinct one as an alternative to it, then the use of indexed valuational frameworks gives us the wrong results. If we follow this line of thought, employing pairs in place of valuations seem more like dishonest formal trickery than a substantial result.

Accordingly, we conclude that if one wants to retain this non-trivial difference between truth-valuational and model-theoretic semantics, the method presented in this section is not the right one.

8 Conclusion and Further Research

As we have argued in the previous section, we seem to have no straightforward way to establish strong completeness for our semantics relative to tableaux systems. We will now specify more abstractly what properties we require from our constructions. The discussion will be in model-theoretic terms, for the required properties lead to set-theoretic contradictions in truth-valuational semantics if not satisfied by a framework, while their model-theoretic equivalents only restrict the set of all models to a subset of them.

As discussed above, truth-valuational semantics does not admit *distinct* valuations which assign the same truth-values to all formulas, i.e., which are *indistinguishable* relative to a valuational framework. Accordingly, we first define the corresponding (though a bit more general) notion of modal indistinguishability for modal-theoretic semantics.

Definition 85 (Modal Indistinguishability)**.** Two points Γ and Δ in $\mathcal{G}$ of $\mathcal{M}$ and $\mathcal{G}'$ of $\mathcal{M}'$ are modally indistinguishable *iff* $\mathcal{M}, \Gamma \Vdash X$ *iff* $\mathcal{M}', \Delta \Vdash X$. If Γ and Δ are modally indistinguishable, we write $\mathcal{M}, \Gamma \leftrightsquigarrow \mathcal{M}', \Delta$. If we have $\mathcal{M}, \Gamma \leftrightsquigarrow \mathcal{M}, \Delta$, we say that Γ and Δ are modally indistinguishable restricted to the model $\mathcal{M}$.

Then, we give the definition of the admissibility property $\mathscr{A}$.

Definition 86 (Admissibility Property)**.** A model $\mathcal{M} =< \mathcal{G}, \mathcal{R}, \Vdash>$ has the admissibility property $\mathscr{A}$ *iff* given the partitioning of $\mathcal{G}$ under $\leftrightsquigarrow$ ($\leftrightsquigarrow$ being restricted to $\mathcal{M}$), for any

Γ in an equivalence class E^1, we have that if $\Gamma\mathcal{R}\Delta$, where $\Delta \in E^2$, then for each $\Pi \in E^1$, there is an Υ in E^2 such that $\Pi\mathcal{R}\Upsilon$.

We can show that if a valuational framework does not have the equivalent of $\mathscr{A}$, it leads to a contradiction. Equivalently, only models with the property $\mathscr{A}$ have 'corresponding' valuational frameworks.

Proposition 87. *Any valuational framework $\mathcal{F}_v$ has the truth-valuational equivalent of $\mathscr{A}$.*

Proof. Suppose $\mathcal{F}_v$ is a valuational framework, where there are two valutions v_{M_1} and v_{M_2} such that $v_{M_1}(X) = t$ *iff* $v_{M_2}(X) = t$, there is a v_{M_3} such that $v_{M_1}\mathcal{R}_v v_{M_3}$, and there is no v_{M_4} such that $v_{M_4}(X) = t$ *iff* $v_{M_3}(X) = t$ and $v_{M_2}\mathcal{R}_v v_{M_4}$. Then, we know that $v_{M_1} = v_{M_2}$, and since $v_{M_1}\mathcal{R}_v v_{M_3}$, $v_{M_2}\mathcal{R}_v v_{M_3}$. But we also know there is no valuation v_{M_4} such that $v_{M_4} = v_{M_3}$ and $v_{M_2}\mathcal{R}v_{M_4}$. But $v_{M_3} = v_{M_3}$, so it is both alternative and not alternative to v_{M_2}. $\square$

For each model $\mathcal{M}$ with admissibility property $\mathscr{A}$, we can establish a model contraction theorem, which contracts any such model into one which does not have modally indistinguishable but distinct worlds, but where any point and its contraction in the contracted model are modally indistinguishable. Then, one can construct for each contracted model a corresponding valuational framework.

Thus, what we need to establish is *either* that there is a construction which builds models with the property $\mathscr{A}$ (that also retain the properties of $\mathcal{R}$ through contraction) *or* that there can be no such construction. If the latter holds, then we may reason two ways. We may either try and alter the tableaux systems themselves while retaining their desired metalogical properties, and if this is not possible, just accept the limited results, *or* we may argue that given such a result, the truth-valuational approach is not worth pursuing further. Since any *S5* model has the property $\mathscr{A}$ by definition, we know that Fitting's original construction is sufficient to establish the desired result. So far, we have no parallel results for other systems.

References

Agudelo-Agudelo, JC, and Walter Carnielli. 2017. “Polynomial ring calculus for modalities.” *Journal of Logic and Computation* 27 (6): 1853–1870.

Ben-Yami, Hanoch. n.d. Truth and Proof without Models. Unpublished manuscript.

Beth, E. W. 1969. “Semantic Entailment and Formal Derivability.” In *Philosophy of Mathematics (Readings in Philosophy)*, edited by Jaakko Hintikka, 9–41. Oxford: Oxford University Press.

Blackburn, Patrick, and Johan van Benthem. 2007. “Modal Logic: A Semantic Perspective.” In *Handbook of Modal Logic*, edited by Patrick Blackburn, Johan van Benthem, and Frank Wolter, Volume 3 of *Studies in Logic and Practical Reasoning*, 1–84. Amsterdam: Elsevier.

Dummett, Michael. 1973. *Frege: Philosophy of Language.* New York: Harper & Row.

Dunn, J. Michael. 1973. “A Truth Value Semantics for Modal Logic.” In *Truth, Syntax and Modality: Proceedings of the Temple University Conference on Alternative Semantics*, edited by Hugues Leblanc, 87–100. Amsterdam: North-Holland.

Dunn, J. Michael, and Nuel D. Belnap. 1968. “The Substitution Interpretation of the Quantifiers.” *Noûs* 2 (2): 177–185.

Fitch, Frederic B. 1966. "Tree proofs in modal logic (Abstract)." *The Journal of Symbolic Logic* 31 (1): 152.

Fitting, Melvin. 1972. "Tableau Methods of Proof for Modal Logics." *Notre Dame Journal of Formal Logic* 13 (2): 237–247.

———. 1983. *Proof Methods for Modal and Intuitionistic Logics.* Dordrecht: Springer.

———. 1993. "Basic Modal Logic." In *Handbook of Logic in Artificial Intelligence and Logic Programming*, Volume 1, 368–448. Oxford: Oxford University Press.

———. 2007. "Modal Proof Theory." In *Handbook of Modal Logic*, edited by Frank Wolter Patrick Blackburn, Johan van Benthem, Volume 3 of *Studies in Logic and Practical Reasoning*, 85–138. Amsterdam: Elsevier.

Fitting, Melvin, and Richard L. Mendelsohn. 1998. *First-Order Modal Logic.* Kluwer Academic Publishers.

Gentzen, Gerhard. 1964. "Investigations into Logical Deduction." *American Philosophical Quarterly* 1 (4): 288–306.

Goble, L. F. 1973. "A Simplified Semantics for Modal Logic." *Notre Dame Journal of Formal Logic* 14 (2): 151–174.

Goré, Rajeev. 1999. "Tableau Methods for Modal and Temporal Logics." In *Handbook of Tableau Methods*, 297–396. Dordrecht: Springer Netherlands.

Gunderson, David S. 2014. *Handbook of Mathematical Induction: Theory and Applications.* Discrete Mathematics and Its Applications. Boca Raton, FL: Chapman and Hall/CRC.

Haack, Susan. 1978. *Philosophy of Logics.* Cambridge: Cambridge University Press.

Hintikka, Jaakko. 1955. "Form and content in quantification theory." *Acta Philosophica Fennica* 8:11–55.

———. 1969. “Modality and Quantification.” In *Models for Modalities: Selected Essays*, 57–70. Dordrecht: D. Reidel.

Hodges, Wilfrid. 1991. *Logic: An Introduction to Elementary Logic*. London: Penguin Books.

Hughes, G. E., and M. J. Cresswell. 1996. *A New Introduction to Modal Logic*. Oxon: Routledge.

Jacquette, Dale. 2002. “Introduction: Logic, Philosophy, and Philosophical Logic.” In *A Companion to Philosophical Logic*, Blackwell Companions to Philosophy, 1–8. Malden/Oxford/Carlton: Blackwell.

Jeffrey, Richard. 2006. *Formal Logic: Its Scope and Limits*. 4th ed. Indianapolis/Cambridge: Hackett.

Klenk, Virginia. 2002. *Understanding Symbolic Logic*. 4th ed. New Jersey: Prentice Hall.

Kneale, William, and Martha Kneale. 1962. *The Development of Logic*. Oxford: Clarendon Press.

Kripke, Saul. 1959. “A Completeness Theorem In Modal Logic.” *The Journal of Symbolic Logic* 24 (1): 1–14.

———. 1963. “Semantical Analysis of Modal Logic I: Normal Modal Propositional Calculi.” *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 9:67–96.

Leblanc, Hugues. 1973a. “On Dispensing with Things and Worlds.” In *Logic and Ontology*, 241–260. New York: New York University Press.

———. 1973b. “Semantic Deviations.” In *Truth, Syntax and Modality: Proceedings of the Temple University Conference on Alternative Semantics*, edited by Hugues Leblanc, 1–16. Amsterdam: North-Holland.

———. 1976. *Truth-Value Semantics*. Amsterdam: North-Holland.

———. 2001. "Alternatives to Standard First-Order Semantics." In *Handbook of Philosophical Logic*, edited by D. M. Gabbay and F. Guenthner, Volume 2, 2nd ed., 53–132. Dordrecht: Kluwer. Originally published in 1983 as "Alternatives to Standard First-Order Semantics." In *Handbook of Philosophical Logic*, edited by D. M. Gabbay and F. Guenthner, Volume 1, 1st ed., 189–274. Dordrecht: Reidel.

Lewis, David. 1970. "General Semantics." *Synthese* 22 (1/2): 18–67.

Makinson, D. 1966. "On Some Completeness Theorems in Modal Logic." *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 12 (1): 379–384.

Massacci, Fabio. 1994. "Strongly analytic tableaux for normal modal logics." In *Automated Deduction — CADE-12*, 723–737. Berlin/Heidelberg: Springer.

Newton-Smith, W. H. 1985. *Logic: An Introductory Course*. London: Routledge.

Quine, W. V. 1948. "On What There Is." *The Review of Metaphysics* 2 (5): 21–38.

———. 1969. "Existence and Quantification." In *Ontological Relativity & Other Essays*, 91–113. New York: Columbia University Press.

———. 1972. "Reviewed Work: Identity and Individuation. by Milton K. Munitz." *The Journal of Philosophy* 69 (16): 488–497.

Sider, Theodore. 2010. *Logic for Philosophy*. Oxford: Oxford University Press.

Smullyan, Raymond M. 1995. *First-Order Logic*. New York: Dover. Originally published in 1968 as *First-Order Logic*. Heidelberg: Springer-Verlag.

Tomassi, Paul. 1999. *Logic*. London/New York: Routledge.