# On some Applications of Convexity and Differential Equations

by

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# ABSTRACT

We investigate some applications of convexity and differential equations to study on the planar $L_p$-Minkowski problem for $0 < p < 1$ and the minimum time function, in particular. We first establish necessary and sufficient conditions for the existence of solutions to the asymmetric $L_p$- Minkowski problem in $\mathbb{R}^2$ for $0 < p < 1$, which amounts to solve a Monge-Ampère type differential equation on $\mathbb{S}^1$ in the regular case. In addition, we investigate the $\varphi$-convexity of the epigraph of the minimum time function $T$ associated with a nonlinear control system with a general closed target under the condition that the sublevel sets of $T$ are $\varphi_0$-convex for some appropriate nonnegative constant $\varphi_0$, where $\varphi$ is a continuous function which can be computed explicitly. This property of $T$ is proved based on some suitable sensitivity relation results. We also provide some sufficient conditions for convexity of sublevel sets of $T$. Furthermore, we provide an invariant result for the set of non-Lipschitz points of the minimum time function.

# CONTENTS

It is fundamental that differential equations and convexity are widely studied for their applications in pure and applied mathematics, physics, engineering, and in many other fields. In this thesis, we are interested in using them to study the planar $L_p$-Minkowski problem for $0 < p < 1$ and the minimum time function for a nonlinear control system, in particular.

The classical Minkowski problem is one of the cornerstones of the Brunn-Minkowski theory. The problem asks for necessary and sufficient conditions on a Borel measure $\mu$ on $\mathbb{S}^{n-1}$ that guarantee the existence of a convex body such that its surface area measure is $\mu$ (see Gardner [Gar06], Gruber [Gru07] or Schneider [Sch14] for reference). Let $K$ be a convex body in $\mathbb{R}^n$, that is, a compact convex set with nonempty interior. The surface area measure $S_K$ on $\mathbb{S}^{n-1}$ is defined for a Borel set $\omega \subset \mathbb{S}^{n-1}$ by

$$S_K(\omega) = \int_{x \in \nu_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x)$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure normalized in a way such that it coincides with the Lebesgue measure on $\mathbb{R}^{n-1}$ and $\nu_K(x)$ stands for exterior unit normal to the boundary, $\mathrm{bd}K$, of $K$ at the boundary point $x$, which is unique for $\mathcal{H}^{n-1}$ almost all $x \in \mathrm{bd}K$. The classical Minkowski existence theorem, due to Minkowski himself in the case of polytopes or discrete measures and to Alexandrov for the general case, states that a Borel measure $\mu$ on $\mathbb{S}^{n-1}$ is the surface area measure of a convex body if and only if the measure of any open hemisphere is positive and

$$\int_{S^{n-1}} u d\mu(u) = 0.$$

The solution is unique up to translation. If the measure $\mu$ has a density function $f$ with respect to $\mathcal{H}^{n-1}$ on $\mathbb{S}^{n-1}$, then the solution amounts to solve a Monge-Ampère type differential equation

$$\det(\nabla^2 h + hI) = nf$$

on $\mathbb{S}^{n-1}$ where $h$ is the unknown non-negative function on $\mathbb{S}^{n-1}$ to be found (the support function), $\nabla^2 h$ denotes the Hessian matrix of $h$ with respect to an orthonormal frame on $\mathbb{S}^{n-1}$, and $I$ is the identity matrix. In this case, even the regularity of the solution is well understood, see Lewy [Lew38], Nirenberg [Nir53], Cheng and Yau [CY76], Pogorelov [Pog78], and Caffarelli [Caf90].

The $L_p$- Minkowski problem is a central problem within the $L_p$- Brunn-Minkowski theory. The study of the so called $L_p$- surface area measure for any $p \in \mathbb{R}$ as initiated by Lutwak [Lut93]. For a convex compact set $K$ in $\mathbb{R}^n$, let $h_K$ be its support function, which is $h_K(u) := \max\{\langle x, u\rangle : x \in K\}$ for $u \in \mathbb{R}^n$ where $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product. Let $\mathcal{K}_o^n$ denote the family of convex bodies in $\mathbb{R}^n$ containing the origin $o$. For $p \leq 1$ and $K \in \mathcal{K}_o^n$, the $L_p$-surface area measure of $K$ is defined by

$$dS_{K,p} = h_K^{1-p}\, dS_K.$$

In particular, if $\omega \subset \mathbb{S}^{n-1}$ is Borel, then

$$S_{K,p}(\omega) = \int_{x \in \nu_K^{-1}(\omega)} \langle x, \nu_K(x)\rangle^{1-p} d\mathcal{H}^{n-1}(x).$$

For $p > 1$, the same formula $dS_{K,p} = h_K^{1-p}\, dS_K$ defines the $L_p$-surface area measure, only one needs to assume that either $o \in \operatorname{int} K$, or $o \in \operatorname{bd}K$ and $\int_{\mathbb{S}^{n-1}} h_K^{1-p}\, dS_K < \infty$. The case $p = 1$ corresponds to the surface area measure $S_K$, and $p = 0$ is corresponding to the so called cone volume measure.

The $L_p$- surface area measure has been intensively investigated in the recent decades, see, for example, [Ale42, BGMN05, CG02, GM77, Hab12, HP14b, HP14a, HL14, Lud03, Lud10, LR10, LYZ00a, LYZ00b, LYZ02a, LYZ04b, LZ97, Nao07, NR03, Pao06, PW12]. In [Lut93], Lutwak posed the associated $L_p$- Minkowski problem for $p \geq 1$ which extends the classical Minkowski problem. If $p > 1$ and $p \neq n$, then the $L_p$- Minkowski problem is solved by Chou, Wang [CW06], Guan, Lin [GL] and Hug, Lutwak, Yang, Zhang [HLYZ05]. In addition, the $L_p$- Minkowski problem for $p < 1$ was publicized by a series of talks by Lutwak in the 1990's. The $L_p$- Minkowski problem is the classical Minkowski problem when $p = 1$, while the $L_p$- Minkowski problem is the so called logarithmic Minkowski problem when $p = 0$, see, for example, [BH16, BLYZ13, BLYZ12, BLYZ15, Lud03, Lud10, LR10, Nao07, NR03, Pao06, Sta02, Sta03, Zhu14]. The $L_p$- Minkowski problem is interesting for all real $p$, and has been studied by Lut-wak [Lut93], Lut-wak and Oliker [LO95], Chou and Wang [CW06], Guan and Lin [GL], Hug, etal. [HLYZ05], Böröczky, et al. [BLYZ13]. Additional references regarding the $L_p$- Minkowski problem and Minkowski-type problems can be found, for example, in [Che06, GG02, GM77, Hab12, HL05, HLYZ10, HMS04, Jia10, Kla04, LW13, Lut93, LO95, LYZ04a, Min97, Sta02, Sta03, Zhu15a, Zhu15b]. Applications of the solutions to the $L_p$- Minkowski problem can be found in, e.g., [And99, And03, Cho85, Zha99, GH86, LYZ02b, CLYZ09, HS09b, Hui84, Iva13, HS09a, HSX12, Wan12].

For a given real number $p$, $L_p$**-Minkowski problem asks for necessary and sufficient conditions on a finite Borel measure** $\mu$ **on** $\mathbb{S}^{n-1}$ **to ensure that it is the** $L_p$**-surface area measure of a convex body in** $\mathbb{R}^n$. Besides discrete measures corresponding to polytopes, an important special case is when

$$d\mu = f\, d\mathcal{H}^{n-1}$$

for some nonnegative measurable function $f$ on $\mathbb{S}^{n-1}$. If $p < 1$ and this equation holds, then the $L_p$-Minkowski problem amounts to solve the Monge-Ampère type equation

$$h^{1-p} \det(\nabla^2 h + hI) = nf$$

where $h$ is the unknown non-negative function on $\mathbb{S}^{n-1}$ to be found (the support function), $\nabla^2 h$ again denotes the Hessian matrix of $h$ with respect to an orthonormal frame on $\mathbb{S}^{n-1}$, and $I$ is again the identity matrix. If $n = 2$, then we may assume that both $h$ and $f$ are nonnegative periodic functions on $\mathbb{R}$ with period $2\pi$. In this case the corresponding differential equation is

$$h^{1-p}(h'' + h) = 2f.$$

After earlier work by V. Umanskiy [Uma03] and W. Chen [Che06], the previous equation in the $\pi$-periodic case that corresponds to planar origin symmetric convex bodies has been thoroughly investigated by M.Y. Jiang [Jia10] if $p > -2$, and by M.N. Ivaki [Iva13] if $p = -2$ (the "critical case").

For $p \in (1,\infty) \setminus \{n\}$, it has been handled for the general case by Chou, Wang [CW06], Guan, Lin [GL], Hug, Lutwak, Yang, and Zhang [HLYZ05]. They established that a Borel measure $\mu$ on $\mathbb{S}^{n-1}$ is the $L_p$- surface area of a convex body in $\mathbb{R}^n$ if and only if $\mu$ is not concentrated on a closed hemisphere.

The $L_p$-Minkowski problem in full generality is still in question for $p \in (-\infty, 1] \cup \{n\}$. For $p \in (0,1)$, some particular cases have been taken care of. Zhu [Zhu15b] solved the $L_p$-Minkowski problem for *polytopes*, stating that for $p \in (0,1)$ and $n \geq 2$, a non-trivial discrete Borel measure $\mu$ on $\mathbb{S}^{n-1}$ is the $L_p$-surface area measure of a polytope $P$ in $\mathbb{R}^n$ containing the origin in its interior if and only if $\mu$ is not concentrated on any closed hemisphere. Another result was given by Haberl, Lutwak, Yang, and Zhang [HLYZ10] for even measures, or equivalently, for origin symmetric convex bodies. They confirmed that for $p \in (0,1)$ and $n \geq 2$, a non-trivial bounded even Borel measure $\mu$ on $\mathbb{S}^{n-1}$ is the $L_p$-surface area measure of an origin symmetric $K \in \mathcal{K}_o^n$ with $o \in \text{int}K$ if and only if $\mu$ is not concentrated on any great subsphere. In addition, the case when $\mu$ has a positive density function is handled by Chou, Wang [CW06]. They proved that if $p \in (-n,1)$, $n \geq 2$, and $\mu$ is a Borel measure on $\mathbb{S}^{n-1}$ satisfying $d\mu = f\, d\mathcal{H}^{n-1}$ where $f$ is bounded and $\inf_{u \in S^{n-1}} f(u) > 0$, then $\mu$ is the $L_p$-surface area measure of a convex body $K \in \mathcal{K}_o^n$. Here, we note that if $p \in (2-n, 1)$, then there exists $K \in \mathcal{K}_o^n$ with $o \in \text{bd}K$ such that $dS_{K,p} = f\, d\mathcal{H}^{n-1}$ for a positive continuous $f : \mathbb{S}^{n-1} \to \mathbb{R}$ (see Example 1.3.5).

In Chapter 2 of this thesis, we concentrate on the case $p \in (0,1)$. It is our main goal to solve the planar $L_p$- Minkowski problem in full generality if $p \in (0,1)$. More precisely, denoting by $\text{supp}\,\mu$ the support of the measure $\mu$ on $\mathbb{S}^1$, we aim to prove the following.

**Theorem 0.0.1.** *For $p \in (0,1)$ and a non-trivial finite Borel measure $\mu$ on $\mathbb{S}^1$, $\mu$ is the $L_p$-surface area measure of a convex body $K \in \mathcal{K}_o^2$ if and only if $\text{supp}\,\mu$ does not consist of a pair of antipodal vectors.*

It is worth mentioning that our method of proving this theorem fails to apply to higher dimensions (see Example 2.1.2), unfortunately. For a general $n$ dimensional Euclidean space, recently, it was proved by Chen, Li, Zhu [CLZ] that for $p \in (0,1)$ and any $n \geq 2$, every non-trivial bounded Borel measure $\mu$ on $\mathbb{S}^{n-1}$ not concentrated on any great subsphere is the $L_p$-surface area measure of a convex body in $\mathbb{R}^n$.

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a Lipschitz continuous sublinear multifunction and $\mathcal{K}$ be a closed subset in $\mathbb{R}^n$. We consider the minimum time function with the target $\mathcal{K}$ for the differential inclusion

$$(1) \qquad \begin{cases} y'(t) \in F(y(t)) & \text{a.e. } t > 0, \\ y(0) = x \in \mathbb{R}^n. \end{cases}$$

A trajectory (starting from $x$) of $F$ is an absolutely continuous arc $y(\cdot)$ that satisfies (1). By $y'(t)$, we mean the derivative of $y(\cdot)$ at the time $t$ and it is the right derivative if $t = 0$.

The time optimal control problem for the differential inclusion (1) is a problem in which the goal is to steer an initial point $x \in \mathbb{R}^n$ to the target $\mathcal{K}$ in minimum time, denoted by $T(x)$, along trajectories of $F$. $T(x)$ could be $+\infty$ if there is no trajectory starting from $x$ can reach $\mathcal{K}$. The function $x \mapsto T(x)$ is called the *minimum time function*, i.e.,

$$T(x) := \inf\{t > 0 : \exists y(\cdot) \text{ satisfying (1) with } y(0) = x \text{ and } y(t) \in \mathcal{K}\},$$

with $\inf \emptyset = +\infty$.

The regularity of the minumum time function is a classical and widely studied topic in control theory (see, e.g., [HL69, CS95, CS04, CFS00, CMW06, CN10, Ngu10, CN11, CN13, CMW12, CNN14, FN15, CN15, Ngu16] and references therein), for linear control systems, i.e., $F$ is of the form $F(x) = \{Ax + u : u \in \mathcal{U}\}$ where $A$ is an $n \times n$ matrix and $\mathcal{U}$ is a compact convex subset of $\mathbb{R}^n$, in particular. It is well known that the locally Lipschitz continuity of $T$ is established if *Petrov's controllability condition* is satisfied (see [CS95]). However, in general, $T$ is not everywhere differentiable. Therefore, it is natural to identify a new type of regularity of $T$ in situations where the locally Lipschitz continuity of $T$ can be relaxed.

Colombo, Marigonda, and Wolenski [CMW06], proved that for a linear control system with a convex target, the epigraph of $T$ satisfies an external sphere condition with locally uniform radius, provided that $T$ is continuous; this property, for general sets, is referred as *positive reach*, *proximal smoothness* or *$\varphi$-convex*. In particular, convex sets and sets with $C^{1,1}$-boundary are $\varphi$-convex. It is known that functions with $\varphi$-convex epigraphs are semiconvex if and only if they are locally Lipschitz, and they have several fine properties (see, e.g., Colombo and Marigonda [CM06]). We note that under the appropriate assumptions in [CMW06], the convexity of sublevel sets of $T$ is obtained, (see Proposition 3.1 in [CMW06]), and this property of sublevel sets is essential to ensure that the $\varphi$-convexity of the epigraph of $T$ is established, apparently. Colombo and Nguyen [CN13] proved that for *two dimensional nonlinear affine control systems*: $F(x) = \{f(x) + g(x)u : u \in \mathcal{U}\}$ with $f : \mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to M_{2\times m}(\mathbb{R})$, $\mathcal{U} = [-1, 1]^m$, $m = 1, 2$, and $\mathcal{K} = \{o\}$, the epigraph of $T$ is $\varphi$-convex in a small neighborhood of the origin. As a matter of fact, they used the convexity of sublevel sets of $T$ (in small times) and the fact that every sufficiency close to the origin point is *optimal*. Motivated by the results in [CMW06, CN13], Nguyen [Ngu16] proved, under suitable assumptions, that if sublevel sets of $T$ are $\varphi_0$-convex for some suitable nonnegative number $\varphi_0$, then there exists a continuous function $\varphi$ such that the epigraph of $T$ is $\varphi$-convex. Nguyen [Ngu16] studied the case of differential inclusions (1) where $F$ may not admit a smooth parameterization. It is assumed in Nguyen [Ngu16] that the *maximized Hamiltionian*,

$$H(x, p) := \max_{v\in F(x)}\langle v, p\rangle,$$

satisfies the following assumption

(H) $\nabla_p H(x, p)$ exists and is Lipschitz in $x$ on $B(o, r)$, uniformly for $p \in \mathbb{R}^n \setminus \{o\}$, for every $r > 0$.

This assumption, however, is not fulfilled in [CMW06] and [CN13]. Indeed, we point out, in the following example, that the assumptions in [CMW06] are satisfied but assumption (H). Therefore, the $\varphi$- convexity result in [Ngu16] does not cover the coressponding results in [CMW06] and [CN13].

We consider the minimum time function to reach the origin for the following linear control system

$$y' = Ay + Bu,$$

where $A$ is an $n \times n$ matrix, $B$ is an $n \times m$ matrix and $u \in \mathcal{U} := [-1, 1]^m$ with $1 \leq m \leq n$. Assume $B = [b_1, \cdots, b_m]$ and $u = (u_1, \cdots, u_m)^\top$ where $b_1, \cdots, b_m$ are columns of the matrix $B$. Assume further that the *normality condition* is satisfied, i.e.,

$$\text{rank}[b_i, Ab_i, \cdots, A^{n-1}b_i] = n, \qquad \forall i = 1, \cdots, m.$$

Then all assumptions in [CMW06] are fulfilled. In this case, the maximized Hamiltonian is computed as follows: for $x, p \in \mathbb{R}^n$

$$\begin{aligned} H(x,p) &= \max_{u \in \mathcal{U}} \langle Ax + Bu, p \rangle \\ &= \langle Ax, p \rangle + \max_{u \in \mathcal{U}} \sum_{i=1}^{m} \langle b_i u_i, p \rangle \\ &= \langle Ax, p \rangle + \sum_{i=1}^{m} |\langle b_i, p \rangle|. \end{aligned}$$

It is obvious that if $p$ is such that $\langle b_i, p \rangle = 0$ for all $i = 1, \cdots, m$, then $H(x, \cdot)$ is not differentiable at $p$. In other words, (H) is not satisfied if $\text{rank} B < n$.

In Chapter 3, it is our purpose to prove a similar $\varphi$-convexity result for the epigraph of $T$ for nonlinear control systems under assumptions in which (H) is not necessarily satisfied. More precisely, considering the minimum time function $T$ for the nonlinear control system

$$\begin{cases} y'(t) = f(y(t), u(t)) & \text{a.e. } t > 0, \\ u(t) \in \mathcal{U} & \text{a.e. } t \geq 0, \\ y(0) = x, \end{cases}$$

we show that if sublevel sets of $T$ are $\varphi_0$-convex for some constant $\varphi_0 \geq 0$, then there exists a continuous function $\varphi$ such that the epigraph of $T$ is $\varphi$-convex. Unlike the proofs in [CMW06, CN13, Ngu16] where only the existence of the function $\varphi$ is accomplished, we compute $\varphi$ explicitly. Our proof relies on suitable sensitivity relation results.

Sensitivity relations are also widely studied in control theory for their various application such as to optimality conditions and regularity of the value functions. The dual arc satisfying an inclusion of an appropriate generalized gradient of the value function is included. For the minimal time problem, Cannarsa, Frankowska, and Sinestrari [CFS00] initiated investigating the sensitivity relations for smooth parameterized systems with the target having an interior sphere condition. Later, sensitivity relations have been widely studied for differential inclusions (see, e.g., [CMN15], [CS15], [CNN14], [FN15], [Ngu16], and references therein). It is shown by Frankowska and Nguyen [FN15] that the proximal subdifferential of $T$ propagates along optimal trajectories except at the terminal points. Similar results confirming that the proximal subdifferential of $T$ propagates wholly along optimal trajectories was given by Nguyen [Ngu16]. The tool used in the proofs is the relationship between normals to the epigraph and to sublevel sets of $T$ via the value at relevant points of the minimized Hamiltonian $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ associated with the previous differential inclusion defined by

$$h(x, \zeta) = \min_{u \in F(x)} \langle u, \zeta \rangle, \qquad \forall x, \zeta \in \mathbb{R}^n.$$

Using the tool as in [Ngu16], we prove the similar sensitivity relation results for nonlinear control system but using different approach from [CNN14] or [Ngu14] under the condition that (H) is not necessarily satisfied.

This thesis consists of three chapters and one appendix. In Chapter 1, we fix the notation and recall some definitions as well as preliminary statements needed in the sequel. Some basic concepts related to convexity and nonsmooth analysis, the $L_p$-Minkowski problem, nonlinear control systems, and the minimum time function are recollected. We also prove some important properties of measures on a sphere and investigate some properties of the solution to the system (1.8), in order to support our main results in the following chapters.

Chapter 2 deals with the planar $L_p$- Minkowski problem for $0 < p < 1$. More precisely, necessary and sufficient conditions for the existence of solutions to the asymmetric $L_p$- Minkowski problem in $\mathbb{R}^2$ is established for $0 < p < 1$. We prove, for $0 < p < 1$, that whenever a non-trivial bounded Borel measure on the unit circle has its support consisting no pair of antipodal vectors, there always exists a convex body containing the origin for which that measure is its $L_p$-surface area measure. The first part of this chapter shows how the problem is handled in the case when the measure of any open semicircle is positive while the last part presents the method to deal with the case when the support of the measure is concentrated on a closed semicircle. The results in this chapter can also be found in [BT17].

In Chapter 3, we study the relationship between sublevel sets and the epigraph of the minimum time function $T$ for a nonlinear control system with a general closed target, see also in [NT]. The main purpose of this chapter is presented in Section 3.2. We establish that if the sublevel sets of $T$ are $\varphi_0$-convex for some appropriate nonnegative constant $\varphi_0$, then the epigraph of $T$ is $\varphi$-convex where $\varphi$ is a continuous function which can be computed explicitly. In order to do that, we provide some suitable sensitivity relations as in Section 3.1, including inclusions for normal cones to the epigraph and to the sublevel sets of the minimum time function. We note that the minimum time function may not be (locally) Lipschitz when $\text{epi}(T)$ is $\varphi$-convex. In this case, we can characterize the set $\mathcal{S}$ of non-Lipschitz points of $T$. Moreover, we prove that the set $\mathcal{S}$ is invariant for optimal trajectories, i.e., if $y(\cdot)$ is an optimal trajectory starting at a point $x$ in $\mathcal{S}$ then $y(t) \in \mathcal{S}$ for all $0 \leq t < T(x)$. This extends the corresponding in [CNN14] with much shorter proof.

The appendix gives a clear explanation for our remark given below Theorem 1.3.2 accomplished by Zhu [Zhu15b].

CHAPTER 1 PRELIMINARIES

## 1.1 Notation

This section is devoted to fix our notation and collect some basic definitions used throughout this thesis. We shall work in $n$-dimensional real Euclidean vector space, $\mathbb{R}^n$, with origin $o$, standard scalar product $\langle\cdot,\cdot\rangle$, and induced norm $\|\cdot\|$. We also use $\langle\cdot,\cdot\rangle$ to denote the scalar product on $\mathbb{R}^n\times\mathbb{R}$, which is given by $\langle(x,\eta),(y,\beta)\rangle=\langle x,y\rangle+\eta\beta$, and $\|\cdot\|$ to denote the associate norm, accordingly. By $\mathcal{H}^m$, $m\le n$, we mean the $m$-dimensional Hausdorff measure normalized in a way such that it coincides with the Lebesgue measure on $\mathbb{R}^m$.

For any $A\subset\mathbb{R}^n$, $\operatorname{lin}A$ and $\operatorname{aff}A$ stand for the *linear hull* and *affine hull* of $A$, respectively. For $x,y\in\mathbb{R}^n$, we denote by $[x,y]$ the closed segment with end points $x$ and $y$, i.e., $[x,y]:=\{\lambda x+(1-\lambda)y:0\le\lambda\le1\}$. Given $A,B\subset\mathbb{R}^n$ and $\lambda\in\mathbb{R}$, we define $A+B:=\{a+b:a\in A,b\in B\}$ and $\lambda A:=\{\lambda a:a\in A\}$. We denote by $\operatorname{cl}A,\operatorname{int}A$, and $\operatorname{bd}A$, respectively, the closure, interior, and boundary of the subset $A$ in $\mathbb{R}^n$.

For a real matrix $M\in\mathbb{R}^{n\times m}, m\in\mathbb{Z}^+$, we write $M^\top$ for its transpose and $\|M\|$ for its norm, as a linear operator. For a function $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^k$ associating to each $x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$ an element in $\mathbb{R}^k$, we denote by $\mathrm{D}f$ its Jacobian matrix and by $\mathrm{D_x}f$, $\mathrm{D_y}f$ its associated partial Jacobians.

We shall also use the following metric notions. For any two points $x,y\in\mathbb{R}^n$ and a nonempty subset $A$ in $\mathbb{R}^n$, $\|x-y\|$ is the *distance* between $x$ and $y$ and $\mathrm{d_A}(x):=\inf\{\|x-y\|:y\in A\}$ is the distance of $x$ from $A$. For a nonempty bounded subset $A$ in $\mathbb{R}$, the *diameter* of $A$ is defined by $\operatorname{diam}A:=\sup\{\|x-y\|:x,y\in A\}$. The set $B^n:=\{x\in\mathbb{R}^n:\|x\|\le1\}$ is the *unit ball* and $\mathbb{S}^{n-1}:=\{x\in\mathbb{R}^n:\|x\|=1\}$ is the *unit sphere* of $\mathbb{R}^n$. By $B(x,r)$, we denote the open ball, $\{u\in\mathbb{R}^n:\|u-x\|<r\}$, centered at $x$ with radius $r>0$.

## 1.2 Basic convexity and nonsmooth analysis

We recall some basic concepts of convex analysis and nonsmooth analysis which can be found in, e.g., [CLSW98] and [Roc72]. We first recall that a subset $A$ in $\mathbb{R}^n$ is *convex* if for any two points $x, y \in A$, it also contains the segment $[x, y]$. We denote by $\mathrm{conv} A$ the *convex hull* of $A$. A nonempty, compact (bounded), convex subset of $\mathbb{R}^n$ is called a *convex body*, as a central notion of Chapter 2. By $\mathcal{K}_o^n$ we denote the family of convex bodies in $\mathbb{R}^n$ containing the origin $o$, and by $\mathcal{K}_{(o)}^n$ we mean the family of convex bodies in $\mathcal{K}_o^n$ containing the origin in interiors.

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ is said to be *convex* if it is *proper*, which means that $\{f = -\infty\} = \emptyset$ and $\{f = \infty\} \neq \mathbb{R}^n$, and if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in \mathbb{R}^n$ and for $\lambda \in [0, 1]$.

For a nonempty closed convex subset $K$ in $\mathbb{R}^n$, the *support function* of $K$, $h_K$, is defined by $h_K(u) := \max\{\langle x, u\rangle : x \in K\}$ for $u \in \mathbb{R}^n$. Clearly, $h_K(\lambda u) = h_{\lambda K}(u)$ for any nonnegative number $\lambda$ and if $K \neq \mathbb{R}^n$, then $h_K(\cdot)$ is convex.

The *Hausdorff distance* of the nonempty compact subsets $K$ and $L$ in $\mathbb{R}^n$ is defined by

$$\delta(K, L) := \min\{\lambda \geq 0 : K \subset L + \lambda B^n, L \subset K + \lambda B^n\},$$

which turns out to be equal to $\max\{h_K(u) - h_L(u) : u \in \mathbb{S}^{n-1}\}$ and be a metric on $\mathcal{C}^n$, the set of nonempty compact subsets of $\mathbb{R}^n$. It is well known that the metric space $(\mathcal{C}^n, \delta)$ is complete (e.g., see Schneider [Sch14]). Consequently, the following fundamental result is stated.

**Theorem 1.2.1** (Blaschke selection theorem)**.** *Every bounded sequence of convex bodies has a subsequence that converges to a convex body.*

Let $K$ be a closed subset in $\mathbb{R}^n$. Given $x \in K$ and $v \in \mathbb{R}^n$, we say that $v$ is a *proximal normal* to $K$ at $x$ if there exists a nonnegative constant $\sigma$ depending on $x$ and $v$ such that $\langle v, y-x\rangle \leq \sigma\|y-x\|^2$ for all $y \in K$. We denote the set of all proximal normals to $K$ at $x$ by $N_K^P(x)$ and call it the *proximal normal cone* to $K$ at $x$. Equivalently, $v$ is a proximal normal to $K$ at $x$ if there exist positive constants $C$ and $\eta$ such that $\langle v, y-x\rangle \leq C\|y-x\|^2$ for all $y \in B(x,\eta)\cap K$. We note that if $K$ is convex, then the proximal normal cone to $K$ at $x$ coincides with the normal cone in the sense of convex analysis.

Let $f : \Omega \to \mathbb{R}\cup\{+\infty\}$, where $\Omega$ is an open subset in $\mathbb{R}^n$, be an extended real-valued function, the *affective domain* of $f$ is the set $\mathrm{dom}(f) := \{x \in \Omega : f(x) < +\infty\}$ and the *epigraph* of $f$ is the set $\mathrm{epi}(f) := \{(x,\beta) \in \Omega\times\mathbb{R} : x \in \mathrm{dom}(f), \beta \geq f(x)\}$. We say that $f$ is *lower semicontinuous* at $x_0 \in \mathbb{R}^n$ if for every $\varepsilon > 0$, there is a neighborhood $U$ of $x_0$ such that $f(x) \geq f(x_0) - \varepsilon$ for all $x$ in $U$ when $f(x_0) < +\infty$ and $f(x)$ tends to $+\infty$ as $x$ tends to $x_0$ when $f(x_0) = +\infty$. In other words, $\liminf_{x\to x_0} f(x) \geq f(x_0)$. We say that $f$ is lower semicontinuous if it is so at every $x_0$ in $\Omega$. We observe that if $f$ is lower semicontinuous then its sublevel sets are closed.

Given a lower semicontinuous function $f$ and let $x \in \mathrm{dom}(f)$, the *proximal subdifferential* of $f$ at $x$ is defined by

$$\partial^P f(x) := \{v \in \mathbb{R}^n : (v,-1) \in N^P_{\mathrm{epi}(f)}(x, f(x))\}.$$

An element of $\partial^P f(x)$ is called a *proximal subgradient* of $f$ at $x$. Equivalently, by saying that $v$ belongs to $\partial^P f(x)$ we mean there exist positive constants $c$ and $\delta$ such that $f(y) - f(x) - \langle v, y-x\rangle \geq -c\|y-x\|^2$ for all $y \in B(x,\delta)$. The *horizontal proximal subdifferential* of $f$ at $x$ is defined by

$$\partial^\infty f(x) := \{v \in \mathbb{R}^n : (v,0) \in N^P_{\mathrm{epi}(f)}(x, f(x))\}.$$

which consists of all *proximal horizontal subgradients* of $f$ at $x$.

**Definition 1.2.2.** *Suppose $K \subset \mathbb{R}^n$ is closed and $\varphi : K \to [0,+\infty)$ is continuous. We say that $K$ is $\varphi$-convex if for all $x \in \mathrm{bd}K$ we have*

$$\langle v, y-x\rangle \leq \varphi(x)\|v\|\|y-x\|^2$$

*for all* $y \in K$ *and all* $v \in N_K^P(x)$*. By* $\varphi_0$*-convexity, we mean* $\varphi$*-convexity with* $\varphi \equiv \varphi_0$*.*

Clearly, we can see that for the case when $\varphi$ is trivial, Definition 1.2.2 deduces to the definition of the convexity of $K$. Equivalently rephrasing, $\varphi$-convexity is a generalization of convexity. Moreover, if the boundary of $K$ is the graph of a $C^{1,1}$ function then $K$ is $\varphi$-convex with $\varphi$ being a suitable constant function.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. It is convenient to state the $\varphi$-convexity of the epigraph of $f$ in view of Definition 1.2.2. The epigraph of $f$ is $\varphi$-convex if there exists a continuous function $\varphi$ such that for all $x \in \mathrm{dom}(f)$, we have

$$\langle (\zeta, \eta), (y, \beta) - (x, f(x)) \rangle \leq \varphi(x) \|(\zeta, \eta)\| (\|y - x\|^2 + |\beta - f(x)|^2)$$

for all $y \in \mathrm{dom}(f)$, $\beta \geq f(y)$ and $(\zeta, \eta) \in N^P_{\mathrm{epi}(f)}(x, f(x))$.

It is worth mentioning that functions whose epigraphs are $\varphi$-convex enjoy good regularity properties that are similar to properties of convex functions (see [CM06]).

## 1.3 The $L_p$- Minkowski problem

For a given convex body $K$ in $\mathbb{R}^n$, we define the *surface area measure*, $S_K$, of $K$ to be a Borel measure on the unit sphere, $\mathbb{S}^{n-1}$, such that for a Borel $\omega \subset \mathbb{S}^{n-1}$ (see, e.g., Schneider [Sch14]), we have

$$S_K(\omega) = \int_{x \in \nu_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \mathrm{db}'K \to \mathbb{S}^{n-1}$ is the Gauss map of $K$, defined on $\mathrm{db}'K$, the set of boundary points in $\mathrm{db}K$ that have a unique exterior unit normal. For a given number $p \leq 1$ and a given convex body $K \in \mathcal{K}_o^n$, the $L_p$*-surface area measure* is defined by

$$dS_{K,p} = h_K^{1-p}\, dS_K.$$

In particular, if $\omega \subset \mathbb{S}^{n-1}$ is Borel, then

$$S_{K,p}(\omega) = \int_{x \in \nu_K^{-1}(\omega)} \langle x, \nu_K(x) \rangle^{1-p} d\mathcal{H}^{n-1}(x). \tag{1.1}$$

In the case when $p = 1$, it is corresponding to the surface area measure $S_K$ while $p = 0$ corresponds to the so called *cone volume measure*. For $p > 1$, the $L_p$-surface area measure is defined by the same formula, $dS_{K,p} = h_K^{1-p}\, dS_K$, only one needs to assume that either $o \in \text{int}\, K$, or $o \in \text{bd}K$ and $\int_{\mathbb{S}^{n-1}} h_K^{1-p}\, dS_K < \infty$.

The $L_p$-Minkowski problem is posed as follows: ***Given a real number $p$, what are the necessary and sufficient conditions on a finite Borel measure $\mu$ on $\mathbb{S}^{n-1}$ to ensure that $\mu$ is the $L_p$- surface area measure of a convex body in $\mathbb{R}^n$?***

For the case when $p$ being different from $n$ ranges over $(1, \infty)$, the $L_p$-Minkowski problem has been solved by Chou, Wang [CW06], Guan, Lin [GL], Hug, Lutwak, Yang, and Zhang [HLYZ05].

**Theorem 1.3.1** (Chou, Wang, Guan, Lin, Hug, Lutwak, Yang, and Zhang)**.** *If $p > 1$ and $p \neq n$, then a Borel measure $\mu$ on $\mathbb{S}^{n-1}$ is the $L_p$- surface area of a convex body in $\mathbb{R}^n$ if and only if $\mu$ is not concentrated on a closed hemisphere.*

Naturally, it has been calling attention to the question about whether or not the $L_p$-Minkowski problem has a solution for the case when $p \in (-\infty, 1] \cup \{n\}$. We first notice that for $p \in (0, 1)$, the $L_p$-Minkowski problem for *polytopes* has been solved by Zhu [Zhu15b]. Here, polytope is the notion for a convex hull of a finite set having positive $n$-dimensional volume.

**Theorem 1.3.2** (Zhu)**.** *For $p \in (0, 1)$ and $n \geq 2$, a non-trivial discrete Borel measure $\mu$ on $\mathbb{S}^{n-1}$ is the $L_p$-surface area measure of a polytope $P \in \mathcal{K}^n_{(o)}$ if and only if $\mu$ is not concentrated on any closed hemisphere.*

It is worth remarking, for the measure $\mu$ and the polytope $P$ as in Theorem 1.3.2, that if $G$ is a subgroup in $O(n)$ such that $\mu(\{Au\}) = \mu(\{u\})$ for any $u \in \mathbb{S}^{n-1}$ and $A \in G$, then one may assume that $AP = P$ for any $A \in G$, as we explain in the Appendix.

In addition, the $L_p$-Minkowski problem for even measures, or equivalently, for origin symmetric convex bodies, was also answered for $p \in (0, 1)$. Haberl, Lutwak, Yang, and Zhang [HLYZ10] stated the following result.

**Theorem 1.3.3** (Haberl, Lutwak, Yang, and Zhang)**.** *For $p \in (0,1)$ and $n \geq 2$, a non-trivial bounded even Borel measure $\mu$ on $\mathbb{S}^{n-1}$ is the $L_p$-surface area measure of an origin symmetric $K \in \mathcal{K}^n_{(o)}$ if and only if $\mu$ is not concentrated on any great subsphere.*

Regrading the $L_p$-Minkowski problem, besides discrete measures corresponding to polytopes, an important special case is when

$$d\mu = f\, d\mathcal{H}^{n-1} \tag{1.2}$$

for some nonnegative measurable function $f$ on $\mathbb{S}^{n-1}$. If $p < 1$ and (1.2) is satisfied, then the $L_p$-Minkowski problem amounts to solving the Monge-Amper type equation

$$h^{1-p} \det(\nabla^2 h + hI) = nf \tag{1.3}$$

where $h$ stands for the unknown nonnegative function on $\mathbb{S}^{n-1}$ to be found (the support function), $\nabla^2 h$ denotes the Hessian matrix of $h$ with respect to an orthonormal frame on $\mathbb{S}^{n-1}$, and $I$ is the identity matrix.

For the particular case when $n = 2$, we may assume that both $h$ and $f$ are nonnegative periodic functions on $\mathbb{R}$ with period $2\pi$. In this case the corresponding differential equation is

$$h^{1-p}(h'' + h) = 2f. \tag{1.4}$$

For $p \in (-n, 1)$, Chou and Wang [CW06] handled the $L_p$-Minkowski problem in $\mathbb{R}^n$, $n \geq 2$, under the condition that $\mu$ has a positive density function.

**Theorem 1.3.4** (Chou, Wang)**.** *If $p \in (-n, 1)$, $n \geq 2$, and $\mu$ is a Borel measure on $\mathbb{S}^{n-1}$ satisfying (1.2) where $f$ is bounded and $\inf_{u \in \mathbb{S}^{n-1}} f(u) > 0$, then $\mu$ is the $L_p$-surface area measure of a convex body $K \in \mathcal{K}^n_o$.*

We end this section by giving an example of a Borel measure $\mu$ on $\mathbb{S}^{n-1}$ being not concentrated on a closed hemisphere such that the origin is a boundary point of a convex body $K$ for which $dS_{K,p} = f\, d\mathcal{H}^{n-1}$ for a positive continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$. This example is based on Example 4.1 of Hug, Lutwak, Yang, and

Zhang [HLYZ05], examples in the preprint of Guan, Lin [GL], and in Chou, Wang [CW06].

**Example 1.3.5.** *If $p \in (2-n, 1)$, then there exists $K \in \mathcal{K}_o^n$ with $C^2$ boundary having $o \in \mathrm{bd}K$ such that $dS_{K,p} = f\, d\mathcal{H}^{n-1}$ for a positive continuous $f : \mathbb{S}^{n-1} \to \mathbb{R}$.*

*We fix a vector $v \in \mathbb{S}^{n-1}$ and set $B^{n-1} := v^\perp \cap B^n$. For any $x \in v^\perp$ and any $t \in \mathbb{R}$, we write the point $(x,t) = x + tv$. For*

$$q = \frac{2(n-1)}{n+p-2} > 2,$$

*we consider the $C^2$ function $g(x) := \|x\|^q$ on $B^{n-1}$. We define the convex body $K$ in $\mathbb{R}^n$ with $C^2$ boundary in a way such that $o \in \mathrm{bd}K$ and the graph $\{(x, g(x)) : x \in B^{n-1}\}$ of $g$ above $B^{n-1}$ is a subset of $\mathrm{bd}K$. We may assume that $\mathrm{bd}K$ has positive Gauß curvature at each $z \in \mathrm{bd}K\backslash\{o\}$.*

*We observe that $K$ is strictly convex and $-v$ is the exterior unit normal at $o$, and hence $S_K(\{-v\}) = 0$. If $z \in \mathrm{bd}K$, then we write $\nu(z)$ for the exterior unit normal at $z$, and $\kappa(\nu(z))$ for the Gauß curvature at $z$, therefore even if $\kappa(-v) = 0$, we have*

$$dS_K = \kappa^{-1}\, d\mathcal{H}^{n-1}.$$

*In turn, we deduce that*

$$dS_{K,p} = h_K^{1-p}\kappa^{-1}\, d\mathcal{H}^{n-1}. \tag{1.5}$$

*Let $x \in B^{n-1}$ satisfy $0 < \|x\| < 1$ and let $z = (x, g(x))$. Hence, $\kappa(\nu(z)) > 0$. We observe that $\nabla g(x) = q\|x\|^{q-2}x$ and $\nu(z) = a(x)^{-1}(\nabla g(x), -1)$ where $a(x)$ is defined to be $a(x) := (1 + \|\nabla g(x)\|^2)^{1/2}$. In particular, writing $u = \nu(z)$, we have*

$$h_K(u) = \langle u, z\rangle = a(x)^{-1}\left(\langle \nabla g(x), x\rangle - g(x)\right) = a(x)^{-1}(q-1)\|x\|^q.$$

*In addition,*

$$\kappa(u) = a(x)^{-(n+1)}\det(\nabla^2 g(x)) = (q-1)q^{n-1}a(x)^{-(n+1)}\|x\|^{(q-2)(n-1)}.$$

*Therefore, the Radon-Nikodym derivative in (1.5) is*

$$h_K(u)^{1-p}\kappa(u)^{-1} = (q-1)^{-p}q^{1-n}a(x)^{n+p}\|x\|^{q(1-p)-(q-2)(n-1)} = (q-1)^{-p}q^{1-n}a(x)^{n+p}.$$

*Since $a(\cdot)$ is a continuous and positive function on $B^{n-1}$, we deduce that $S_{K,p}$ has a positive and continous Radon-Nikodym derivative $f$ with respect to $\mathcal{H}^{n-1}$ on $\mathbb{S}^{n-1}$.*

## 1.4 Some properties of measures on a sphere

In this section, we present some basic properties of measures on $\mathbb{S}^{n-1}$ which will be taken into account for technical use in Chapter 2. For any unit vector $v$ in $\mathbb{S}^{n-1}$ and any number $t$ in $[0,1)$, we define $\Omega(v,t)$ to be the subset $\{u \in \mathbb{S}^{n-1} : \langle u, v\rangle > t\}$ of $\mathbb{S}^{n-1}$. In particular, $\Omega(v,0)$ is the open hemisphere centered at $v$.

**Lemma 1.4.1.** *If $\mu$ is a finite Borel measure on $\mathbb{S}^{n-1}$ such that the measure of any open hemisphere is positive, then there exists $\delta \in (0, \frac{1}{2})$ such that for any $v \in \mathbb{S}^{n-1}$,*

$$\mu\left(\Omega(v,\delta)\right) > \delta.$$

It is remarkable that $\delta$ can possibly be chosen to be small enough to ensure also $\mu(\mathbb{S}^{n-1}) < 1/\delta$.

*Proof.* Suppose, to the contrary, that for any $k \in \mathbb{N}$, $k > 1$, there exists a vector $u_k$ in $\mathbb{S}^{n-1}$ for which the $\mu$ measure of $\Omega(u_k, 1/k)$ is at most $1/k$. It follows from the compactness of $\mathbb{S}^{n-1}$ that there is a subsequence of $\{u_k\}$, denoted by $\{u_{k_j}\}$, converging to some unit vector $u$.

Since the $\mu$ measure of the open hemisphere centered at $u$ is positive, there exists $\tau := \cos\alpha$ for $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that the $\mu$ measure of $\Omega(u,\tau)$ is positive. Obviously, there is a sufficiently large $k_j \in \mathbb{N}$ such that the $\mu$ measure of $\Omega(u,\tau)$ is greater than $1/k_j$, $\frac{1}{k_j} < \cos\frac{\pi+2\alpha}{4}$, and the angle $\theta$ between $u_{k_j}$ and $u$ is at most $\frac{\pi-2\alpha}{4}$. Since

$$\cos(\alpha+\theta) \geq \cos\left(\alpha + \frac{\pi-2\alpha}{4}\right) = \cos\frac{\pi+2\alpha}{4} > \frac{1}{k_j},$$

the spherical triangle inequality yields that $\Omega(u,\tau)$ is contained in $\Omega(u_{k_j}, 1/k_j)$. In other words, we obtain

$$\mu\left(\Omega\left(u_{k_j}, \frac{1}{k_j}\right)\right) \geq \mu\left(\Omega(u,\tau)\right) > \frac{1}{k_j},$$

which is a contradiction to the definition of $u_k$. □

We recall that the sequence of convex compact sets $K_m$ is said to converge to a convex compact set $K$ in $\mathbb{R}^n$ if

$$\lim_{m\to\infty} \max\{u \in \mathbb{S}^{n-1} : \|h_{K_m}(u) - h_K(u)\|\} = 0.$$

We also note that the surface area measure can be extended to compact convex sets (see Schneider [Sch14]). Let $K$ be a compact convex set in $\mathbb{R}^n$. If $\dim K \leq n-2$, then $S_K$ is the constant zero measure. In addition, if $\dim K = n-1$ and $v \in \mathbb{S}^{n-1}$ is normal to aff $K$, then $S_K$ is concentrated on $\{\pm v\}$ and $S_K(\{v\}) = S_K(\{-v\}) = \mathcal{H}^{n-1}(K)$.

**Lemma 1.4.2.** *If $\varphi : [0,\infty) \to [0,\infty)$ is continuous and the sequence of compact convex sets $K_m$ with $o \in K_m$ tends to the convex compact set $K$ in $\mathbb{R}^n$, then the measures $\varphi \circ h_{K_m}\, dS_{K_m}$ tend weakly to $\varphi \circ h_K\, dS_K$.*

*Proof.* Since $o \in K_m$ for all $m$, we have $o \in K$. As $h_{K_m}$ tends uniformly to $h_K$ on $\mathbb{S}^{n-1}$, $\varphi \circ h_{K_m}$ converges uniformly to $\varphi \circ h_K$ for any continuous function $\varphi : [0,\infty) \mapsto [0,\infty)$ and it follows that $f\varphi \circ h_{K_m}$ tends uniformly to $f\varphi \circ h_K$ as well for any $f \in C(\mathbb{S}^{n-1})$. The continuity and boundedness of $\varphi \circ h_{K_m} : \mathbb{S}^{n-1} \to [0,\infty)$ and $\varphi \circ h_K : \mathbb{S}^{n-1} \to [0,\infty)$ follow from the continuity of $\varphi$ and $h_{K_m}$, the compactness of $\mathbb{S}^{n-1}$, and the uniform convergence of $\{\varphi \circ h_{K_m}\}$, respectively.

These imply that for any $m$ and any $f \in C(\mathbb{S}^{n-1})$, we have

$$\lim_{m\to\infty} \int_{\mathbb{S}^{n-1}} f(u)\varphi \circ h_{K_m}(u) S_{K_m}(du) = \int_{\mathbb{S}^{n-1}} f(u)\varphi \circ h_K(u) S_{K_m}(du).$$

Moreover, since $S_{K_m}$ tends weakly to $S_K$ according to Theorem 4.2.1 in Schneider [Sch14], we conclude that for all $f \in C(\mathbb{S}^{n-1})$,

$$\lim_{m\to\infty} \int_{\mathbb{S}^{n-1}} f(u)\varphi \circ h_K(u) S_{K_m}(du) = \int_{\mathbb{S}^{n-1}} f(u)\varphi \circ h_K(u) S_K(du).$$

□

Paying attention to the case when $p \le 1$, we can apply Lemma 1.4.2 to obtain an essential statement which will be used in Section 2.1.

**Corollary 1.4.3.** *If $p \le 1$ and a sequence of compact convex sets $K_m$ with $o \in K_m$ tends to the compact convex set $K$ in $\mathbb{R}^n$, then $S_{K_m,p}$ tends weakly to $S_{K,p}$.*

We recall that the *positive hull* of the vectors $u_1, \dots, u_k$ in $\mathbb{R}^n$ is the set of all *positive combinations* of $u_1, \dots, u_k$, namely,

$$\text{pos}\{u_1, \dots, u_k\} := \{\lambda_1 u_1 + \dots + \lambda_k u_k : \lambda_1, \dots, \lambda_k \ge 0\}.$$

**Lemma 1.4.4.** *If $x \in \mathbb{R}^n$, $u_1, \dots, u_k \in \mathbb{S}^{n-1}$, and $u \in \mathbb{S}^{n-1} \cap \text{pos}\{u_1, \dots, u_k\}$ satisfy that $\langle u_i, x \rangle \ge 0$ for $i = 1, \dots, k$, then*

$$\langle u, x \rangle \ge \min\{\langle u_1, x \rangle, \dots, \langle u_k, x \rangle\}.$$

*Proof.* Since the unit sphere is convex, there exist nonnegative constants $\lambda_1, ..., \lambda_k$ with $\lambda_1 + ... + \lambda_k \ge 1$ such that $u$ is a positive combination of $u_1, \dots, u_k$ represented by $u = \lambda_1 u_1 + ... + \lambda_k u_k$.

Hence,

$$\langle u, x \rangle = \sum_{i=1}^{k} \lambda_i \langle u_i, x \rangle \ge \min\{\langle u_1, x \rangle, \dots, \langle u_k, x \rangle\} \sum_{i=1}^{k} \lambda_i \ge \min\{\langle u_1, x \rangle, \dots, \langle u_k, x \rangle\}.$$

□

For a planar convex body $K$ in $\mathbb{R}^2$, we say that the two boundary points of $K$, $x_1$ and $x_2$, are *opposite* if there exists an exterior normal $u \in \mathbb{S}^1$ at $x_1$ such that $-u$ is an exterior unit normal at $x_2$. If the boundary points of $K$, $x_1$ and $x_2$, are not opposite, then we denote by $\sigma(K, x_1, x_2)$ the arc of $\text{bd}K$ connecting $x_1$ and $x_2$ not containing any opposite points. It is possible that $x_1 = x_2$. Obviously, it is observable that if $x \in \sigma(K, x_1, x_2) \backslash \{x_1, x_2\}$, then

$$\nu_K(x) \in \text{pos}\{\nu_K(x_1), \nu_K(x_2)\}. \tag{1.6}$$

The following estimate is for technical use later in the proof of Proposition 2.1.1. The statement is given by applying observation (1.6) above together with Lemma 1.4.4.

**Claim 1.4.5.** *For $p<1$, a planar convex body $K$ in $\mathbb{R}^2$, and non-opposite $x_1, x_2 \in \operatorname{bd}K$, if $\langle x_1, \nu_K(x_2)\rangle > 0$ and $\langle x_2 - x_1, u\rangle > 0$ for $u \in \mathbb{S}^1$, then*

$$\min\{h_K(\nu_K(x_1)), \langle x_1, \nu_K(x_2)\rangle\}^{1-p}\langle x_2 - x_1, u\rangle \le \int_{\mathbb{S}^1} h_K^{1-p}\, dS_K.$$

*Proof.* As $\langle x_1, \nu_K(x_1)\rangle = h_K(\nu_K(x_1))$, if $x \in \sigma(K, x_1, x_2)$ is a smooth point, then (1.6) and Lemma 1.4.4 yield

$$\langle x, \nu_K(x)\rangle \ge \langle x_1, \nu_K(x)\rangle \ge \min\{h_K(\nu_K(x_1)), \langle x_1, \nu_K(x_2)\rangle\}.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{S}^1} h_K^{1-p}\, dS_K &= \int_{\operatorname{bd}K} \langle x, \nu_K(x)\rangle^{1-p}\, d\mathcal{H}^1(x) > \int_{\sigma(K,x_1,x_2)} \langle x, \nu_K(x)\rangle^{1-p}\, d\mathcal{H}^1(x)\\ &\ge \min\{h_K(\nu_K(x_1)), \langle x_1, \nu_K(x_2)\rangle\}^{1-p}\mathcal{H}^1(\sigma(K, x_1, x_2)),\end{aligned}$$

and finally Claim 1.4.5 follows from the fact that $\mathcal{H}^1(\sigma(K, x_1, x_2)) \ge \langle x_2 - x_1, u\rangle$. □

## 1.5 Nonlinear control systems

We begin this section by introducing the control system in order to define the minimum time function minimizing a functional depending only on the final endpoint of the trajectory. Standards references are in [CS04]. The definition of a nonlinear control system is given as the following.

**Definition 1.5.1.** *A control system is a pair $(f; \mathcal{U})$ where $\mathcal{U}$ is a nonempty closed subset in $\mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ is a continuous function. The set $\mathcal{U}$ is called the control set, while $f$ is called the dynamics of the system. The state equation associated with the system is*

$$(1.7)\qquad \begin{cases} y'(t) = f(y(t), u(t)) & a.e.\ t > 0,\\ u(t) \in \mathcal{U} & a.e.\ t \ge 0,\\ y(0) = x, & \end{cases}$$

*where $u : [0, \infty) \to \mathbb{R}^m$ is a measure function. The function $u$ is called a control strategy or simply a control. The solution of (1.7) is called the trajectory of the system corresponding to the initial condition $y(0) = x$ and to the control $u$.*

In Chapter 3, we require the following assumptions on the function $f$ and the control set $\mathcal{U}$:

(A1) $\mathcal{U}$ is compact and $f(x,\mathcal{U})$ is convex for every $x \in \mathbb{R}^n$.

(A2) $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ is continuous and satisfies

$$\|f(x,u) - f(y,u)\| \leq L\|x - y\| \qquad \text{for all } x, y \in \mathbb{R}^n,\ u \in \mathcal{U},$$

and for a positive constant $L$.

(A3) The differential of $f$ with respect to the first variable $D_x f$ exists everywhere, is continuous with respect to both $x$ and $u$, and satisfies

$$\|D_x f(x,u) - D_x f(y,u)\| \leq L_1\|x - y\| \qquad \text{for all } x, y \in \mathbb{R}^n,\ u \in \mathcal{U},$$

and for a positive constant $L_1$.

We denote by $\mathcal{U}_{ad}$ the set of admissible controls, i.e., the set of all measure functions $u : [0,\infty) \to \mathbb{R}^m$ such that $u(t) \in \mathcal{U}$ a.e. $t \geq 0$. According to the theory of ordinary differential equations, it is well known that the existence of a unique global solution to the state equation (1.7), denoted by $y^{x,u}(\cdot)$, corresponding to any $u(\cdot) \in \mathcal{U}_{ad}$ and any $x \in \mathbb{R}^n$, is sufficiently ensured whenever the assumption (A2) is satisfied. We also observe that under assumptions (A1) and (A2), the continuity of the function $f$ leads to

$$\|f(x,u)\| \leq C + L\|x\|$$

for every $x \in \mathbb{R}^n$ and $u \in \mathcal{U}$, where $C = \max\{f(o,u) : u \in \mathcal{U}\}$. As a consequence, the *attainable set* $\mathcal{A}^T(x)$ from $x$ in time $T$, $\mathcal{A}^T(x) := \{y^{x,u}(t) : t \leq T, u(\cdot) \in \mathcal{U}_{ad}\}$ is bounded for every $x \in \mathbb{R}^n$ and finite $T$. For studying the properties of the minimum time function, assumption (A3) plays an essential role as we can see in Section 1.6 and Chapter 3.

We end this section by recalling some basic properties of the solution of (1.7) and presenting some estimates for the solution of (1.8) below. These results will be used in Chapter 3. We denote $K_1 := \max_{u\in\mathcal{U}}\|f(o,u)\|$ and $K_2 := \max_{u\in\mathcal{U}}\|D_x f(o,u)\|$. The following elementary estimates are established by Colombo and Nguyen [CN10].

**Lemma 1.5.2.** *[Colombo and Nguyen] Assume (A1) - (A3). Let $y(t) := y^{x,u}(t)$ be the solution of (1.7). The following estimates hold for all $t > 0$:*

*(i)* $\|y(t) - x\| \le \frac{1}{L}(L\|x\| + K_1)(e^{Lt} - 1) \le (L\|x\| + K_1)e^{Lt}t.$

*(ii)* $\|y(t)\| \le e^{Lt}\|x\| + \frac{K_1}{L}(e^{Lt} - 1).$

*(iii)* $\|f(y(t), u(t))\| \le (L\|x\| + K_1)e^{Lt}.$

*(iv)* $||D_x f(y(t), u(t))|| \le L_1 e^{Lt}\|x\| + \frac{L_1 K_1}{L}(e^{Lt} - 1) + K_2.$

**Lemma 1.5.3.** *Assume (A1) - (A3). Let $y(t) := y^{x,u}(t)$ be the solution of (1.7). Let $p(\cdot)$ be the solution of*

$$\begin{cases} p'(t) &= -D_x f(y(t), u(t))^\top p(t) \quad a.e.\ t > 0, \\ p(0) &= p_0 \in \mathbb{R}^n. \end{cases} \tag{1.8}$$

*Then for $t > 0$, we have*

*(i)* $\|p(t)\| \le e^{l(x,t)t}\|p_0\|$ *and*

*(ii)* $\|p(t) - p_0\| \le l(x,t)e^{l(x,t)t}t\|p_0\|$

*where*

$$l(x,t) = L_1 e^{Lt}\|x\| + \frac{L_1 K_1}{L}(e^{Lt} - 1) + K_2.$$

*Proof.* One can prove easily by using Lemma 1.5.2 and Theorem 2.2.1, p. 23 in [BP07]. □

## 1.6 Minimum time function

It is the purpose of this section to the introduction to the center notions of Chapter 3, the minimum time function and related notation. Further studies on the minimum time function can be found in, e.g., [CS04]. Together with the control system (1.7) as in Section 1.5, we now consider a nonempty closed set $\mathcal{K} \subset \mathbb{R}^n$, which we shall call the *target*. For a given point $x \in \mathbb{R}^n \setminus \mathcal{K}$ and $u(\cdot) \in \mathcal{U}_{ad}$, we define

$$\theta(x,u) := \min\{t \geq 0 : y^{x,u}(t) \in \mathcal{K}\}.$$

Obviously, $\theta(x,u) \in [0,+\infty]$ and $\theta(x,u)$ is the time at which the trajectory $y^{x,u}(\cdot)$ reaches the target for the first time provided $\theta(x,u) < +\infty$. The *minimum time function* $T : \mathbb{R}^n \to \mathbb{R}$ determining the minimum time $T(x)$ to reach $\mathcal{K}$ from $x$ is defined by

$$T(x) := \inf\{\theta(x,u) : u(\cdot) \in \mathcal{U}_{ad}\}. \tag{1.9}$$

The infimum in (1.9) is not established in general. However, it is proven as in Theorem 8.1.2 in [CS04] that the infimum is obtained when the dynamic of the control system and the control set satisfy the conditions (A1) and (A2).

**Theorem 1.6.1.** *Assume that the control system satisfies (A1) and (A2). Then*

$$T(x) = \min\{\theta(x,u) : u(\cdot) \in \mathcal{U}_{ad}\}.$$

When the infimum in (1.9) is attained, a minimizing control, say $\bar{u}(\cdot)$, is called an *optimal control* for $x$ and the corresponding trajectory $y^{x,\bar{u}}(\cdot)$ is called an *optimal trajectory* for $x$, or we simply call $(y(\cdot), u(\cdot))$ an *optimal pair* for $x$.

The minimum time function $T$ satisfies the so-called *Dynamic Programming Principle*. This important property of $T$ is demonstrated in [CS04] and will be used in Section 3.1.

**Theorem 1.6.2.** *Assume that $\mathcal{U}$ is compact and (A2) holds. Then*

$$T(x) = t + \inf\{T(y) : y \in \mathcal{A}^t(x)\}$$

*for every $x \in \mathbb{R}^n \setminus \mathcal{K}$ and $t \in [0, T(x)]$. Equivalently, for all $u(\cdot) \in \mathcal{U}_{ad}$, the function $t \mapsto t + T(y^{x,u}(t))$ is increasing on $[0, T(x)]$.*

*Moreover, if $y^{x,u}(\cdot)$ is an optimal trajectory then $t \mapsto t + T(y^{x,u}(t))$ is constant on $[0, T(x)]$, i.e.,*

$$T(y^{x,u}(t)) = t - s + T(y^{x,u}(s)) \quad \textit{for } 0 \leq s \leq t \leq T(x).$$

For any $t > 0$, we denote by $\mathcal{R}(t)$ the *$t$-sublevel set* of the function $T$, that is, $\mathcal{R}(t) := \{x \in \mathbb{R}^n : T(x) \leq t\}$, and by $\mathcal{R}$ the set of points which can be steered to the target in finite time, i.e., $\mathcal{R} := \{x \in \mathbb{R}^n : T(x) < +\infty\}$. $\mathcal{R}$ is called the *reachable set* and, obviously, $\mathcal{R} = \cup_{t>0}\mathcal{R}(t)$.

# CHAPTER 2 ON THE PLANAR $L_p$- MINKOWSKI PROBLEM

In this chapter, we aim to give necessary and sufficient conditions for the existence of solutions to the asymmetric $L_p$- Minkowski problem in $\mathbb{R}^2$ for $0 < p < 1$. We establish, as in Theorem 2.0.1, that the planar $L_p$- Minkowski problem, $p \in (0,1)$, in full generality has a solution whose support does not contain a pair of antipodal vectors. Given a non-zero finite Borel measure $\mu$ on $\mathbb{S}^1$, the idea behind the planar $L_p$- Minkowski problem, $p \in (0,1)$, is to distinguish the possibilities of whether or not the support of $\mu$ is concentrated on a closed semicircle. Section 2.1 deals with the case when the measure is not concentrated on a closed semicircle, or in other words, the measure of any open semicircle is positive, based on the important statement in Proposition 2.1.1. Section 2.2 deals with the other case when the measure is concentrated on a closed semicircle based on the fact stated in Lemma 2.2.1. Our main goal is to establish:

**Theorem 2.0.1.** *For $p \in (0,1)$ and a non-trivial finite Borel measure $\mu$ on $\mathbb{S}^1$, $\mu$ is the $L_p$-surface area measure of a convex body $K \in \mathcal{K}_o^2$ if and only if $\operatorname{supp}\mu$ does not consist of a pair of antipodal vectors.*

**Remark 2.0.2.** *For the $\mu$ and $K$ as in Theorem 2.0.1, if $G$ is a finite subgroup in $O(2)$ such that $\mu(A\omega) = \mu(\omega)$ for every Borel $\omega \subset \mathbb{S}^1$ and $A \in G$, then one may assume that $AK = K$ for any $A \in G$.*

**Corollary 2.0.3.** *For $p \in (0,1)$ and every nonnegative $2\pi$-periodic function $f \in L_1([0,2\pi])$, the differential equation (1.4) has a nonnegative $2\pi$-periodic weak solution.*

**Remark** If the $f$ in (1.4) is even, or is periodic with respect to $2\pi/k$ for an integer $k \geq 2$, then the solution $h$ can be also chosen even, or periodic with respect to $2\pi/k$, respectively.

## 2.1 The measure of any open semicircle is positive

Let $p \in (0,1)$ and let $\mu$ be a finite Borel measure on $\mathbb{S}^1$ such that the measure of any open semicircle is positive. Following from Lemma 1.4.1, there is a constant $\delta \in (0, \frac{1}{2})$ depending on $\mu$ for which the measure of $\Omega(v,\delta)$ is greater than $\delta$ where we may assume that $\mu(\mathbb{S}^1) < 1/\delta$. We construct a sequence $\{\mu_m\}$ of discrete Borel measures on $\mathbb{S}^1$ converging weakly to $\mu$ such that the $\mu_m$ measure of any open semicircle is positive for each $m$. It is the easiest way to construct the sequence by identifying $\mathbb{R}^2$ with $\mathbb{C}$. For $m \geq 3$, we write $u_{jm} = e^{2ij\pi/m}$ , $i = \sqrt{-1}$, for $j = 1, \ldots, m$, and we define $\mu_m$ to be the measure having the support $\{u_{1m}, \ldots, u_{mm}\}$ with

$$\mu_m(\{u_{jm}\}) = \frac{1}{m^2} + \mu\left(\{e^{it} : (j-1)2\pi/m < t \leq j2\pi/m\}\right) \quad \text{for } j = 1, \ldots, m.$$

According to Zhu [Zhu15b], there exists a polygon $P_m$ containing the origin as its interior point satisfying $d\mu_m = h_{P_m}^{1-p}\, dS_{P_m}$ for each $m$. Lemma 1.4.2 then allows us to assume that for each $m$,

$$\int_{\mathbb{S}^1} h_{P_m}^{1-p}\, dS_{P_m} < 1/\delta. \tag{2.1}$$

In order to prove our statement given in Theorem 2.0.1 for the case when the measure of any open semicircle is positive, the boundedness of $\{P_m\}$ is essentially required. The following proposition is devoted to verify this significant property of the sequence $\{P_m\}$.

**Proposition 2.1.1.** *$\{P_m\}$ is bounded.*

*Proof.* We assume that the diameters of $P_m$, $d_m := \operatorname{diam} P_m$, tend to infinity as $m$ tends to $\infty$, and seek a contradiction. Choose $y_m, z_m \in P_m$ such that $\|z_m - y_m\| = d_m$ and $\|z_m\| \geq \|y_m\|$. We denote by $v_m$ the unit vector $(z_m - y_m)/\|z_m - y_m\|$ and let $w_m \in \mathbb{S}^1$ be orthogonal to $v_m$. We observe that $v_m$ and $-v_m$ are exterior normals of $P_m$ at $z_m$ and $y_m$, respectively, as $[y_m, z_m]$ is a diameter of $P_m$. It follows that $\langle z_m, v_m\rangle \geq d_m/2$. By possibly taking subsequences, we may assume that $v_m$ tends to some $\tilde{v} \in \mathbb{S}^1$. According to Lemma 1.4.1 and Lemma 1.4.2, if $m$ is large, then

$$\int_{\Omega(-v_m,\delta/2)} h_{P_m}^{1-p}\, dS_{P_m} > \delta/2. \tag{2.2}$$

More precisely, as $\mu\left(\Omega(-\tilde{v},\delta)\right) > \delta$ by Lemma 1.4.1, applying Lemma 1.4.2 with a continuous function $g : \mathbb{S}^1 \to [0,1]$ defined by

$$g(x) = \begin{cases} 1, & \text{if } x \in \Omega(-\tilde{v},\delta), \\ 0, & \text{if } x \notin \Omega(-\tilde{v},3\delta/4), \end{cases}$$

we obtain

$$\mu_m\left(\Omega\left(-\tilde{v},3\delta/4\right)\right) \geq \int_{\mathbb{S}^1} g d\mu_m \xrightarrow{m\to\infty} \int_{\mathbb{S}^1} g d\mu \geq \mu(\Omega(-\tilde{v},\delta)) > \delta.$$

That is, $\mu_m\left(\Omega\left(-\tilde{v},3\delta/4\right)\right) > \delta/2$ for sufficiently large $m$. Here, the number $3\delta/4$ only plays a role as any other positive constant in $(1/2,1)$ so that the similar inequality to the previous one turns out to have $\delta/2$ as the smaller value. In addition to the previous inequality, we observe that if $m$ is large enough so that the angle between $-\tilde{v}$ and $-v_m$ is at most $\arccos\frac{\delta}{2} - \arccos\frac{3\delta}{4}$, then $\Omega\left(-\tilde{v},3\delta/4\right) \subset \Omega\left(-v_m,\delta/2\right)$. Hence, we obtain $\mu_m\left(\Omega\left(-v_m,\delta/2\right)\right) > \frac{\delta}{2}$ as desired.

Let $a_m$ and $b_m$ be boundary points of $P_m$ such that $\langle a_m - b_m, w_m\rangle$ is positive and $\langle a_m, v_m\rangle$ and $\langle b_m, v_m\rangle$ are both exactly equal to $d_m/4$. Consequently, we also deduce that $[a_m,b_m] \cap \operatorname{int} P_m \neq \emptyset$ for the segment $[a_m,b_m]$. Our observation about the values of the support function at the exterior unit normals at $a_m$ and $b_m$ is a positive one. Indeed, because $\langle z_m - a_m, v_m\rangle \geq d_m/4$ and $\langle z_m - b_m, v_m\rangle \geq d_m/4$ by the definitions of $z_m$, $a_m$, and $b_m$, it follows from (2.1) and Claim 1.4.5 with $x_1 = a_m$, $x_1 = b_m$, alternately, $x_2 = z_m$, and $u = v_m$ that *there exists a positive constant $c_1$ depending on $\mu$ and $p$ such that if $m$ is large, then*

$$h_{P_m}(\nu_{P_m}(a_m)) \leq c_1 d_m^{\frac{-1}{1-p}} \ \textit{and}\ \ h_{P_m}(\nu_{P_m}(b_m)) \leq c_1 d_m^{\frac{-1}{1-p}}. \tag{2.3}$$

Our intermediate goal is to indicate that $\nu_{P_m}(a_m)$ and $\nu_{P_m}(b_m)$ point essentially to the same direction as $w_m$ and $-w_m$, respectively, or in other words,

$$\lim_{m\to\infty}\langle \nu_{P_m}(a_m), v_m\rangle = \lim_{m\to\infty}\langle \nu_{P_m}(b_m), v_m\rangle = 0.$$

We shall frequently use the fact that $\langle \nu_{P_m}(x_0), x_0 - x\rangle$ is nonnegative for all boundary points $x_0$ and for all points $x$ in $P_m$. Particularly, $\langle \nu_{P_m}(a_m), w_m\rangle$ and $\langle \nu_{P_m}(b_m), -w_m\rangle$ are both positive for $\langle \nu_{P_m}(a_m), a_m - b_m\rangle$ and $\langle \nu_{P_m}(b_m), b_m - a_m\rangle$ are positive as well, respectively, by the fact above and $[a_m, b_m] \cap \operatorname{int} P_m \neq \emptyset$. We also keep in mind that as $\{v_m, w_m\}$ is an orthogonal system by definition, any point $x$ in $\mathbb{R}^2$ can be represented as $x = \langle x, v_m\rangle v_m + \langle x, w_m\rangle w_m$.

First of all, we establish some basic properties of $\nu_{P_m}(a_m)$ and $\nu_{P_m}(b_m)$ with respect to the orthogonal systems $\{v_m, w_m\}$ and $\{v_m, -w_m\}$, respectively. We achieve the initial statement that *for any* $P_m$,

$$\frac{|\langle \nu_{P_m}(a_m), v_m\rangle|}{\langle \nu_{P_m}(a_m), w_m\rangle} \le \frac{\langle a_m - z_m, w_m\rangle}{d_m/4} \quad\text{and}\quad \frac{|\langle \nu_{P_m}(b_m), v_m\rangle|}{\langle \nu_{P_m}(b_m), -w_m\rangle} \le \frac{\langle b_m - z_m, -w_m\rangle}{d_m/4}. \tag{2.4}$$

More specifically, since the roles of $a_m$ together with $w_m$ and $b_m$ together with $-w_m$ are "symmetric", it suffices that the statement about $\nu_{P_m}(a_m)$ is verified. We notice that $\langle a_m - z_m, v_m\rangle \le -d_m/4$, as a remark of the definition of $a_m$. In the case when $\langle \nu_{P_m}(a_m), v_m\rangle$ is nonnegative, we obtain

$$\begin{aligned} 0 &\le \langle \nu_{P_m}(a_m), a_m - z_m\rangle \\ &= \langle \nu_{P_m}(a_m), v_m\rangle\langle a_m - z_m, v_m\rangle + \langle \nu_{P_m}(a_m), w_m\rangle\langle a_m - z_m, w_m\rangle \\ &\le -\langle \nu_{P_m}(a_m), v_m\rangle(d_m/4) + \langle \nu_{P_m}(a_m), w_m\rangle\langle a_m - z_m, w_m\rangle, \end{aligned}$$

which yields our desired inequality. Otherwise, in the case when $\langle \nu_{P_m}(a_m), -v_m\rangle$ is nonnegative, using $\langle a_m - y_m, -v_m\rangle \le -d_m/4$, $\langle a_m - y_m, w_m\rangle = \langle a_m - z_m, w_m\rangle$, we deduce

$$\begin{aligned} 0 &\le \langle \nu_{P_m}(a_m), a_m - y_m\rangle \\ &= \langle \nu_{P_m}(a_m), -v_m\rangle\langle a_m - y_m, -v_m\rangle + \langle \nu_{P_m}(a_m), w_m\rangle\langle a_m - y_m, w_m\rangle \\ &\le -\langle \nu_{P_m}(a_m), -v_m\rangle(d_m/4) + \langle \nu_{P_m}(a_m), w_m\rangle\langle a_m - y_m, w_m\rangle, \end{aligned}$$

and in turn completely verify the first inequality in (2.4).

Consequently, our previous observation (2.4) gives a clear version of the "position" of $\nu_{P_m}(a_m)$ and $\nu_{P_m}(b_m)$ with respect to $w_m$ and $-w_m$, respectively. More precisely, *for any $P_m$, we have*

$$\langle \nu_{P_m}(a_m), w_m\rangle > \frac{1}{5} \text{ and } \langle \nu_{P_m}(b_m), -w_m\rangle > \frac{1}{5}. \tag{2.5}$$

Indeed, let $\gamma_m$ stand for the angle formed by the vectors $\nu_{P_m}(a_m)$ and $w_m$. Seeing that $\langle a_m - z_m, w_m\rangle \leq d_m$ follows from $\|a_m - z_m\| \leq d_m$, we evidently conclude from the first inequality in (2.4) that $\tan\gamma_m \leq 4$. Therefore,

$$\langle \nu_{P_m}(a_m), w_m\rangle = \cos\gamma_m = (1+\tan^2\gamma_m)^{-1/2} \geq \frac{1}{\sqrt{17}} > \frac{1}{5}.$$

The statement about $\nu_{P_m}(b_m)$ in (2.5) can be similarly verified as a consequence of the second inequality in (2.4).

Secondly, our first observation about the "position" of $\nu_{P_m}(a_m)$ and $\nu_{P_m}(b_m)$ with respect to $v_m$ is revealed. For the sake of simplicity, we denote $t_m := \langle z_m, v_m\rangle$, $s_m := -\langle y_m, v_m\rangle$, and $r_m := \langle z_m, w_m\rangle = \langle y_m, w_m\rangle$ for each $m$. Then we deduce, by definition, that $t_m$ is at least $d_m/2$ and $s_m$ is positive, and also observe that $z_m = t_m v_m + r_m w_m$ and $y_m = -s_m v_m + r_m w_m$ for each $m$. Possibly interchanging $w_m$ with $-w_m$ and the role of $a_m$ and $b_m$, without loss of generality, we may assume that $r_m$ is nonnegative. Based on the fact that $\langle \nu_{P_m}(a_m), z_m\rangle$ is at most $\langle \nu_{P_m}(a_m), a_m\rangle$, we can easily see that if $\langle \nu_{P_m}(a_m), v_m\rangle$ is nonnegative, then the first inequalities in (2.3) and (2.5) imply

$$\frac{r_m}{5} < t_m\langle \nu_{P_m}(a_m), v_m\rangle + r_m\langle \nu_{P_m}(a_m), w_m\rangle = \langle \nu_{P_m}(a_m), z_m\rangle \leq c_1 d_m^{\frac{-1}{1-p}},$$

which in turn gives a upper bound for $r_m$ as well as a upper bound for $\langle \nu_{P_m}(a_m), v_m\rangle$ for $t_m \geq d_m/2$, with respect to $d_m$. Similarly, if $\langle \nu_{P_m}(a_m), -v_m\rangle$ is positive, then again, according to (2.3), (2.5), and the fact that $\langle \nu_{P_m}(a_m), y_m\rangle$ is less than or equal to $\langle \nu_{P_m}(a_m), a_m\rangle$, we have

$$\frac{r_m}{5} < s_m\langle \nu_{P_m}(a_m), -v_m\rangle + r_m\langle \nu_{P_m}(a_m), w_m\rangle = \langle \nu_{P_m}(a_m), y_m\rangle \leq c_1 d_m^{\frac{-1}{1-p}},$$

and conclude that $\{r_m\}$ is bounded from above, with respect to $d_m$. In other words, we say that *there exists a positive constant $c_2$ depending on $\mu$ and $p$ such that if $m$ is large, then*

$$r_m \leq c_2 d_m^{\frac{-1}{1-p}}. \tag{2.6}$$

We consider the case when $\langle \nu_{P_m}(b_m), v_m\rangle$ is nonnegative. As $\langle \nu_{P_m}(b_m), -w_m\rangle$ is positive and less than or equal to $1$ and $\langle \nu_{P_m}(b_m), z_m\rangle$ is at most $\langle \nu_{P_m}(b_m), b_m\rangle$, the second inequality in (2.3) implies

$$t_m\langle \nu_{P_m}(b_m), v_m\rangle - r_m\langle \nu_{P_m}(b_m), -w_m\rangle = \langle \nu_{P_m}(b_m), z_m\rangle \leq c_1 d_m^{\frac{-1}{1-p}},$$

which leads to $\langle \nu_{P_m}(b_m), v_m\rangle \leq 2(c_1 + c_2) d_m^{\frac{p-2}{1-p}}$ because of (2.6) and $t_m \geq d_m/2$. In summary, our current statement is established that *there exist positive constants $c_3$ and $c_4$ depending on $\mu$ and $p$ such that if $m$ is large, then*

$$\langle \nu_{P_m}(a_m), v_m\rangle \leq c_3 d_m^{\frac{p-2}{1-p}} \text{and } \langle \nu_{P_m}(b_m), v_m\rangle \leq c_4 d_m^{\frac{p-2}{1-p}}. \tag{2.7}$$

Finally, we are at the stage to reach our intermediate goal where our observation about the lower bounds of $\langle \nu_{P_m}(a_m), v_m\rangle$ and $\langle \nu_{P_m}(b_m), v_m\rangle$ is achieved. We claim that *there exist positive constants $c_5$ and $c_6$ depending on $\mu$ and $p$ such that if $m$ is large, then*

$$\langle \nu_{P_m}(a_m), v_m\rangle \geq -c_5 d_m^{\frac{p-1}{3-3p+p^2}-1} \text{ and } \langle \nu_{P_m}(b_m), v_m\rangle \geq -c_6 d_m^{\frac{p-1}{3-3p+p^2}-1}. \tag{2.8}$$

As a matter of fact, according to our observation (2.4) above, our claim is equivalent to saying that there exist positive constants $c_7$ and $c_8$ depending on $\mu$ and $p$ such that

$$\begin{aligned} \alpha_m := \langle a_m - z_m, w_m\rangle &\leq c_7 d_m^{\frac{p-1}{3-3p+p^2}} \text{ provided } \langle \nu_{P_m}(a_m), v_m\rangle < 0, \text{ and} \\ \beta_m := \langle b_m - z_m, -w_m\rangle &\leq c_8 d_m^{\frac{p-1}{3-3p+p^2}} \text{ provided } \langle \nu_{P_m}(b_m), v_m\rangle < 0. \end{aligned}$$

We begin with estimating $\alpha_m$ with note that $\alpha_m \leq \frac{\sqrt{15}}{4} d_m$ follows from $\|a_m - z_m\| \leq d_m$ and $|\langle a_m - z_m, v_m\rangle| \geq d_m/4$. We denote $\eta_m := \left(\frac{\alpha_m}{d_m}\right)^{\frac{1-p}{2-p}}$, which turns out to be at most $\left(\frac{\sqrt{15}}{4}\right)^{\frac{1-p}{2-p}}$. The constant $\eta_m$ here is chosen in a way such that the following calculations will lead to the same estimate up to a constant factor. We consider the unit vector $e_m$

such that $\langle e_m, v_m \rangle = \eta_m$ and $\langle e_m, w_m \rangle$ is positive. It is obvious that

$$\langle e_m, w_m \rangle \geq c_9 := \left(1 - \left(\sqrt{15}/4\right)^{\frac{2(1-p)}{2-p}}\right)^{\frac{1}{2}}.$$

There exist a boundary point $a'_m$ in $\sigma(P_m, a_m, z_m)$ such that $w_m$ is an exterior unit normal of $P_m$ at $a'_m$ and a boundary point $\tilde{a}_m$ in $\sigma(P_m, a'_m, z_m)$ such that $e_m$ is an exterior unit normal of $P_m$ at $\tilde{a}_m$. In particular, we may assume that $\nu_{P_m}(a'_m) = w_m$ and $\nu_{P_m}(\tilde{a}_m) = e_m$. Thus, as $\langle z_m, w_m \rangle$ is nonnegative, we have

$$\langle a'_m, w_m \rangle \geq \langle a'_m - z_m, w_m \rangle = h_{P_m}(w_m) - \langle z_m, w_m \rangle \geq \langle a_m - z_m, w_m \rangle = \alpha_m.$$

We distinguish two cases. The first case is when $\langle \tilde{a}_m - z_m, w_m \rangle < \alpha_m/2$. Since both of $\langle a'_m, w_m \rangle$ and $\langle e_m, w_m \rangle$ are positive, $\langle a'_m, v_m \rangle \geq d_m/4$, $\langle e_m, v_m \rangle = \eta_m$, and $d_m \geq \alpha_m$, we deduce that $\langle a'_m, e_m \rangle \geq \alpha_m/4$ from

$$\langle a'_m, e_m \rangle = \langle a'_m, v_m \rangle \langle e_m, v_m \rangle + \langle a'_m, w_m \rangle \langle e_m, w_m \rangle \geq (d_m/4)\eta_m = \tfrac{1}{4}\, \alpha_m^{\frac{1-p}{2-p}} d_m^{\frac{1}{2-p}}.$$

Moreover, $h_{P_m}(w_m) \geq \alpha_m$, then we observe that $\min\{h_{P_m}(w_m), \langle a'_m, e_m \rangle\} \geq \alpha_m/4$. Seeing that $\langle \tilde{a}_m - a'_m, w_m \rangle < -\alpha_m/2$ by our assumption, we obtain

$$\begin{aligned} 0 &\leq \langle \tilde{a}_m - a'_m, e_m \rangle = \langle \tilde{a}_m - a'_m, v_m \rangle \langle e_m, v_m \rangle + \langle \tilde{a}_m - a'_m, w_m \rangle \langle e_m, w_m \rangle \\ &\leq \langle \tilde{a}_m - a'_m, v_m \rangle \eta_m - \frac{c_9 \alpha_m}{2}, \end{aligned}$$

and consequently,

$$\langle \tilde{a}_m - a'_m, v_m \rangle \geq \frac{c_9 \alpha_m}{2\eta_m} = \frac{c_9}{2}\, \alpha_m^{\frac{1}{2-p}} d_m^{\frac{1-p}{2-p}}.$$

Thus, an appropriate positive constant $c_7$ and hence $c_5$ can be achieved from

$$\left(\frac{\alpha_m}{4}\right)^{1-p} \frac{c_9}{2}\, \alpha_m^{\frac{1}{2-p}} d_m^{\frac{1-p}{2-p}} < \frac{1}{\delta},$$

by taking into account (2.1) and Claim 1.4.5 with $x_1 = a'_m$, $x_2 = \tilde{a}_m$, and $u = v_m$.

The other case is when $\langle \tilde{a}_m - z_m, w_m \rangle \geq \alpha_m/2$. Now as $\langle z_m, e_m \rangle \geq (d_m/4)\eta_m$ by $\langle z_m, w_m \rangle \geq 0$, $h_{P_m}(v_m) \geq d_m/2$ yields

$$\min\{h_{P_m}(v_m), \langle z_m, e_m \rangle\} \geq (d_m/4)\eta_m = \frac{1}{4}\, \alpha_m^{\frac{1-p}{2-p}} d_m^{\frac{1}{2-p}}.$$

Therefore, we can apply Claim 1.4.5 with $x_1 = z_m$, $x_2 = \tilde{a}_m$, and $u = w_m$, and use (2.1) in order to obtain

$$\left(\frac{1}{4}\,\alpha_m^{\frac{1-p}{2-p}} d_m^{\frac{1}{2-p}}\right)^{1-p} \frac{\alpha_m}{2} < \frac{1}{\delta},$$

which gives the desired lower bound for $\{\langle \nu_{P_m}(a_m), v_m\rangle\}$.

Next, we turn to the statement about $\{\langle \nu_{P_m}(b_m), v_m\rangle\}$ where the argument is similar to the argument for $\{\langle \nu_{P_m}(a_m), v_m\rangle\}$ with an important remark that $\langle z_m, -w_m\rangle$ is negative. For this, we will keep in mind that $\langle z_m, -w_m\rangle = -r_m > -c_2 d_m^{\frac{-1}{1-p}}$ according to (2.7). If $\beta_m$ is less than $d_m^{\frac{p-1}{3-3p+p^2}}$, then we are done proving the second statement. Therefore, we assume otherwise that $\beta_m \geq d_m^{\frac{p-1}{3-3p+p^2}}$. Since $\frac{-1}{1-p} < \frac{p-1}{3-3p+p^2}$, we may suppose that $m$ is sufficiently large to ensure

$$\beta_m \geq d_m^{\frac{p-1}{3-3p+p^2}} > 4c_2 d_m^{\frac{-1}{1-p}} \geq 4r_m. \tag{2.9}$$

In particular, if $m$ is sufficiently large, then

$$\langle b_m, -w_m\rangle \geq \frac{3\beta_m}{4}. \tag{2.10}$$

Since $\|b_m - z_m\| \leq d_m$ and $|\langle b_m - z_m, v_m\rangle| \geq d_m/4$ yield $\beta_m \leq \frac{\sqrt{15}}{4}\, d_m$, we have

$$\theta_m := \left(\frac{\beta_m}{d_m}\right)^{\frac{1-p}{2-p}} \leq \left(\frac{\sqrt{15}}{4}\right)^{\frac{1-p}{2-p}} < 1.$$

As above, we choose the constant $\theta_m$ in a way such that the calculations below will lead to the same estimate up to a constant factor. We consider the vector $f_m \in \mathbb{S}^1$ such that $\langle f_m, v_m\rangle = \theta_m$ and $\langle f_m, -w_m\rangle$ is positive. The choice of $f_m$ leads to $\langle f_m, -w_m\rangle \geq c_9$ where $c_9$ being positive and depending on $p$ is given as above. The existence of a boundary point $b'_m$ in $\sigma(P_m, b_m, z_m)$ such that $-w_m$ is an exterior unit normal of $P_m$ at $b'_m$ and a boundary point $\tilde{b}_m$ in $\sigma(P_m, b'_m, z_m)$ for which $f_m$ is an exterior unit normal of $P_m$ at $\tilde{b}_m$ is guaranteed. In particular, we may assume that $\nu_{P_m}(b'_m) = -w_m$ and $\nu_{P_m}(\tilde{b}_m) = f_m$. It is worth mentioning that

$$\langle b'_m - z_m, -w_m\rangle = h_{P_m}(-w_m) - \langle z_m, -w_m\rangle \geq \langle b_m - z_m, -w_m\rangle \geq \beta_m.$$

Again, we shall consider two cases. We first assume that $\langle \tilde{b}_m - z_m, -w_m\rangle < \beta_m/2$. Since both of $\langle b'_m, -w_m\rangle$ and $\langle f_m, -w_m\rangle$ are positive, $\langle b'_m, v_m\rangle \geq d_m/4$, $\langle f_m, v_m\rangle = \theta_m$, and $d_m \geq \beta_m$, we deduce that $\langle b'_m, f_m\rangle \geq \beta_m/4$ from

$$\langle b'_m, f_m\rangle = \langle b'_m, v_m\rangle\langle f_m, v_m\rangle + \langle b'_m, -w_m\rangle\langle f_m, -w_m\rangle \geq (d_m/4)\theta_m = \tfrac{1}{4}\,\beta_m^{\frac{1-p}{2-p}}\,d_m^{\frac{1}{2-p}}.$$

In addition, $h_{P_m}(-w_m)$ is at least $3\beta_m/4$ by (2.10), as a consequence of it, we observe that $\min\{h_{P_m}(-w_m), \langle b'_m, f_m\rangle\} \geq \beta_m/4$. Because $\langle \tilde{b}_m - b'_m, -w_m\rangle < -\beta_m/2$ by our assumption, it holds

$$\begin{aligned} 0 &\leq \langle \tilde{b}_m - b'_m, f_m\rangle = \langle \tilde{b}_m - b'_m, v_m\rangle\langle f_m, v_m\rangle + \langle \tilde{b}_m - b'_m, -w_m\rangle\langle f_m, -w_m\rangle \\ &\leq \langle \tilde{b}_m - b'_m, v_m\rangle\theta_m - \frac{c_9\beta_m}{2}, \end{aligned}$$

and hence,

$$\langle \tilde{b}_m - b'_m, v_m\rangle \geq \frac{c_9\beta_m}{2\theta_m} = \frac{c_9}{2}\,\beta_m^{\frac{1}{2-p}}\,d_m^{\frac{1-p}{2-p}}.$$

Therefore, applying (2.1) and Claim 1.4.5 with $x_1 = b'_m$, $x_2 = \tilde{b}_m$, and $u = v_m$, we get

$$\left(\frac{\beta_m}{4}\right)^{1-p}\frac{c_9}{2}\,\beta_m^{\frac{1}{2-p}}\,d_m^{\frac{1-p}{2-p}} < \frac{1}{\delta},$$

and in turn conclude the desired inequality. Now we assume $\langle \tilde{b}_m - z_m, -w_m\rangle \geq \beta_m/2$. In this case, (2.7) implies

$$\langle z_m, f_m\rangle = \langle z_m, v_m\rangle\langle f_m, v_m\rangle + \langle z_m, -w_m\rangle\langle f_m, -w_m\rangle \geq (d_m/4)\theta_m - c_2 d_m^{\frac{-1}{1-p}}.$$

Here, by the definition of $\theta_m$ and (2.9), we note that for sufficiently large $m$,

$$d_m\theta_m = d_m\left(\frac{\beta_m}{d_m}\right)^{\frac{1-p}{2-p}} = \beta_m^{\frac{1-p}{2-p}}\,d_m^{\frac{1}{2-p}} \geq \left(4c_2 d_m^{\frac{-1}{1-p}}\right)^{\frac{1-p}{2-p}} d_m^{\frac{1}{2-p}} = (4c_2)^{\frac{1-p}{2-p}} > 8c_2 d_m^{\frac{-1}{1-p}}.$$

Thus, $\langle z_m, f_m\rangle \geq (d_m/8)\theta_m$. This observation together with $h_{P_m}(v_m) \geq d_m/2$ yield

$$\min\{h_{P_m}(v_m), \langle z_m, f_m\rangle\} \geq (d_m/8)\theta_m = \frac{1}{8}\,\beta_m^{\frac{1-p}{2-p}}\,d_m^{\frac{1}{2-p}}.$$

Therefore, we complete verifying our claim (2.8) by obtaining an appropriate lower bound for $\{\langle \nu_{P_m}(b_m), v_m\rangle\}$ from

$$\left(\frac{1}{8}\beta_m^{\frac{1-p}{2-p}}d_m^{\frac{1}{2-p}}\right)^{1-p}\frac{\beta_m}{2} < \frac{1}{\delta},$$

when (2.1) is sued and Claim 1.4.5 is applied with $x_1 = z_m$, $x_2 = \tilde{b}_m$, and $u = -w_m$.

Eventually, based on our previous observations, we are in the position to estimate the $\mu_m$ measures of $\Omega(-v_m, \delta/2)$ in view of its definition. In order to do that, we denote by $a_m^*$ the boundary point of $P_m$ maximizing $\langle a_m^*, w_m\rangle$ under the condition that the $\gamma_m \in \mathbb{S}^1$ with $\langle \gamma_m, -v_m\rangle = \delta/2$ and $\langle \gamma_m, w_m\rangle > 0$ is an exterior unit normal at $a_m^*$, and by $b_m^*$ the boundary point of $P_m$ maximizing $\langle b_m^*, -w_m\rangle$ under the condition that the $\xi_m \in \mathbb{S}^1$ with $\langle \xi_m, -v_m\rangle = \delta/2$ and $\langle \xi_m, -w_m\rangle > 0$ is an exterior unit normal at $b_m^*$. More precisely, we aim to investigate

$$\int_{\Omega(-v_m,\delta/2)} h_{P_m}^{1-p}\, dS_{P_m} = \int_{\sigma(P_m,a_m^*,b_m^*)} \langle x, \nu_{P_m}(x)\rangle^{1-p}\, d\mathcal{H}^1(x). \tag{2.11}$$

For our purpose, it is desirable to evaluate $\mathcal{H}^1(\sigma(P_m, a_m^*, b_m^*))$, which can conveniently be considered by comparing with $\langle a_m^* - b_m^*, w_m\rangle$ according to the definition of $a_m^*$ and $b_m^*$ together the fact that $v_m$ and $w_m$ are orthogonal. We are going on the right track with the following auxiliary observation: *there exist positive constants $c_{10}$ and $c_{11}$ depending on $\mu$ and $p$ such that if $m$ is large, then*

$$\langle a_m^* - y_m, w_m\rangle \le c_{10} d_m^{\frac{-1}{1-p}} \text{ and } \langle b_m^* - y_m, -w_m\rangle \le c_{11} d_m^{\frac{-1}{1-p}}. \tag{2.12}$$

To verify this claim, we first remark that $\langle y_m, w_m\rangle = r_m$ being nonnegative yields

$$\langle a_m^* - y_m, w_m\rangle \le \langle a_m^*, w_m\rangle.$$

This means that in order to establish the first inequality in (2.12), it is sufficient to have an appropriate upper bound for $\langle a_m^*, w_m\rangle$. Owing to the fact that $\langle a_m^* - y_m, v_m\rangle$ and $\langle a_m^* - y_m, w_m\rangle$ are both nonnegative, we have

$$\begin{aligned} 0 &\le \langle a_m^* - y_m, \gamma_m\rangle = \langle a_m^* - y_m, v_m\rangle\langle \gamma_m, v_m\rangle + \langle a_m^* - y_m, w_m\rangle\langle \gamma_m, w_m\rangle \\ &\le -\frac{\delta}{2}\langle a_m^* - y_m, v_m\rangle + \langle a_m^* - y_m, w_m\rangle. \end{aligned}$$

Consequently,

$$\langle a_m^* - y_m, v_m\rangle \le \frac{2}{\delta}\langle a_m^* - y_m, w_m\rangle \le \frac{2}{\delta}\langle a_m^*, w_m\rangle. \tag{2.13}$$

Moreover, for the fact that

$$h_{P_m}(\nu_{P_m}(a_m)) \ge \langle a_m^*, \nu_{P_m}(a_m)\rangle = \langle a_m^*, v_m\rangle\langle \nu_{P_m}(a_m), v_m\rangle + \langle a_m^*, w_m\rangle\langle \nu_{P_m}(a_m), w_m\rangle,$$

it follows from (2.3) and (2.5) that

$$c_1 d_m^{\frac{-1}{1-p}} \ge h_{P_m}(\nu_{P_m}(a_m)) \ge \langle a_m^*, v_m\rangle\langle \nu_{P_m}(a_m), v_m\rangle + \langle a_m^*, w_m\rangle/5. \tag{2.14}$$

We shall consider the following three cases according to the signs of $\langle a_m^*, v_m\rangle$ and $\langle \nu_{P_m}(a_m), v_m\rangle$. The first case is when $\langle a_m^*, v_m\rangle$ and $\langle \nu_{P_m}(a_m), v_m\rangle$ share the same signs, i.e., $\langle a_m^*, v_m\rangle\langle \nu_{P_m}(a_m), v_m\rangle$ is nonnegative. In this case, our argument is directly verified by (2.14). Another case is when $\langle a_m^*, v_m\rangle$ is positive while $\langle \nu_{P_m}(a_m), v_m\rangle$ is negative. For this case, we notice that the inequality (2.13) implies $\langle a_m^*, v_m\rangle \le \frac{2}{\delta}\langle a_m^*, w_m\rangle$ for $\langle y_m, v_m\rangle$ is non-positive. Furthermore, as $|\langle \nu_{P_m}(a_m), v_m\rangle| < \frac{\delta}{20}$ for sufficiently large $m$ according to (2.8), we conclude from (2.14) that

$$c_1 d_m^{\frac{-1}{1-p}} \ge -\frac{2}{\delta}\langle a_m^*, w_m\rangle\frac{\delta}{20} + \frac{\langle a_m^*, w_m\rangle}{5} = \frac{\langle a_m^*, w_m\rangle}{10},$$

which gives an appropriate $c_{10}$. We come to the last case when $\langle a_m^*, v_m\rangle$ is negative and $\langle \nu_{P_m}(a_m), v_m\rangle$ is positive. In this case, it is worth recalling that $\langle a_m^*, v_m\rangle \ge -d_m$ on the one hand and $\langle \nu_{P_m}(a_m), v_m\rangle < c_3 d_m^{\frac{p-2}{1-p}}$ by (2.7) on the other hand. Therefore, (2.14) yields

$$c_1 d_m^{\frac{-1}{1-p}} \ge -d_m c_3 d_m^{\frac{p-2}{1-p}} + \frac{\langle a_m^*, w_m\rangle}{5} = -c_3 d_m^{\frac{-1}{1-p}} + \frac{\langle a_m^*, w_m\rangle}{5},$$

and then completely verifies the first inequality in (2.12).

Now we turn to the second argument in (2.12) where we may assume that

$$\langle b_m^* - y_m, -w_m\rangle \ge 2c_2 d_m^{\frac{-1}{1-p}},$$

since otherwise we are readily done with $c_{11} = 2c_2$. Inasmuch $\langle b_m^* - y_m, -w_m\rangle$ is less than or equal to $\langle b_m^*, -w_m\rangle + c_2 d_m^{\frac{-1}{1-p}}$ according to (2.7), we have

$$\langle b^* - y_m, -w_m\rangle \le 2\langle b_m^*, -w_m\rangle.$$

By this observation, we can confirm the statement about $b_m^*$ by using similar argument to the previous one.

Based on the fact that $\mathcal{H}^1(\sigma(P_m, a_m^*, b_m^*)) \le \frac{2}{\delta}\langle a_m^* - b_m^*, w_m\rangle$, our estimate (2.12) leads to

$$\mathcal{H}^1(\sigma(P_m, a_m^*, b_m^*)) \le \frac{2(c_{10}+c_{11})}{\delta} d_m^{\frac{-1}{1-p}}$$

for sufficiently large $m$. As a consequence, we use (2.2), (2.11), and the fact that $\langle x, \nu_{P_m}(x)\rangle \le d_m$ for any boundary point $x$ of $P_m$, to conclude

$$\frac{\delta}{2} \le d_m^{1-p}\mathcal{H}^1(\sigma(P_m, a_m^*, b_m^*)) \le \frac{2(c_{10}+c_{11})}{\delta} d_m^{1-p} d_m^{\frac{-1}{1-p}} = \frac{2(c_{10}+c_{11})}{\delta} d_m^{\frac{p(p-2)}{1-p}}$$

for sufficiently large $m$, which turns out to be absurd as $\frac{p(p-2)}{1-p}$ is negative and $d_m$ tends to infinity. This contradiction ends the proof of Proposition 2.1.1. □

Now we turn to the **proof of Theorem 2.0.1 if the measure of any open semicircle is positive**. Since the sequence $\{P_m\}$ is bounded and each $P_m$ contains the origin according to Proposition 2.1.1, the Blaschke selection theorem 1.2.1 provides a subsequence $\{P_{m'}\}$ converging to a compact convex set $K$ containing the origin. It follows from Corollary 1.4.3 that $S_{P_{m'},p}$ tends weakly to $S_{K,p}$. However, $\mu_{m'} = S_{P_{m'},p}$ tends weakly to $\mu$ by construction. Therefore, $\mu = S_{K,p}$. Since any open semicircle of $\mathbb{S}^1$ has positive $\mu$ measure, we conclude that $K$ has nonempty interior.

Finally, we pay our attention to Remark 2.0.2. Given $G$ to be a finite subgroup of $O(2)$ such that $\mu(A\omega) = \mu(\omega)$ for any Borel $\omega \subset \mathbb{S}^1$ and $A \in G$. The idea is that for large $m$, we subdivide $\mathbb{S}^1$ into arcs of length less than $2\pi/m$ in a way such that the subdivision is symmetric with respect to $G$ and each endpoint has $\mu$ measure $0$.

We fix a regular $l$-gon $Q$, $l \ge 3$, whose vertices lie on $\mathbb{S}^1$ such that $G$ is a subgroup of the symmetry group of $Q$. In addition, we consider the set $\Sigma$ of atoms of $\mu$, namely, the set of all unit vectors $u \in \mathbb{S}^1$ such that $\mu(\{u\})$ is positive. In particular, $\Sigma$ is countable.

For $m \geq 2$, we denote by $Q_m$ the regular polygon with $lm$ vertices such that all vertices of $Q$ are vertices of $Q_m$, and denote by $G_m$ the symmetry group of $Q_m$. We observe that $G_m$ contains rotations by angle $\frac{2\pi}{lm}$. We write $\Sigma_m$ for the set obtained from repeated applications of the elements of $G_m$ to the elements of $\Sigma$. We note that $\Sigma_m$ is countable, as well. For a fixed $x_0 \in \mathbb{S}^1 \backslash \Sigma_m$, we consider the orbit $G_m x_0 = \{Ax_0 : A \in G_m\}$ and let $\mathcal{I}_m$ be the set of open arcs of $\mathbb{S}^1$ that are the components of $\mathbb{S}^1 \backslash G_m x_0$. We observe that $G_m x_0$ is disjoint from $\Sigma_m$ and, consequently, $\mu(\sigma) = \mu(\mathrm{cl}\,\sigma)$ for $\sigma \in \mathcal{I}_m$.

Now we define $\mu_m$. It is concentrated on the set of midpoints of all $\sigma \in \mathcal{I}_m$ and the $\mu_m$ measure of the midpoints of a $\sigma \in \mathcal{I}_m$ is $\mu(\sigma)$. In particular, $\mu_m$ is invariant under $G_m$ and hence, $\mu_m$ is invariant under $G$. Since the length of each arc in $\mathcal{I}_m$ is at most $\frac{2\pi}{lm}$, we deduce that $\mu_u$ tends weakly to $\mu$.

According to the remark after Theorem 1.3.2 due to Zhu [Zhu15b], we may assume that each $P_m$ is invariant under $G$. The argument above shows that some subsequence of $\{P_m\}$ tends to a convex body $K$ satisfying $S_{K,p} = \mu$ and readily $K$ is invariant under $G$.

Unfortunately, the proof of Theorem 2.0.1 we present does not extend to higher dimensions. Apparently, regarding the planar $L_p$-Minkowski problem when $p$ ranges over the interval $(0,1)$, the most important key is the following statement: If $0 < p < 1$, $\mu$ is a bounded Borel measures on $\mathbb{S}^1$ such that the $\mu$ measure of any open semicircle is positive, and $\{P_m\} \subset \mathcal{K}^2_{(o)}$ is a sequence of convex bodies such that $S_{P_m,p}$ tends weakly to $\mu$, then it is bounded. However, this statement fails to hold for higher dimensions. The following Example 2.1.2 is evidence of its failure in $n = 3$ dimension where the solution to the $L_p$-Minkowski problem exists without requiring the boundedness of the sequence $\{P_m\}$.

**Example 2.1.2.** *For $p \in (0,1)$, there exist a measure $\mu$ on $\mathbb{S}^2$ ensuring that any open hemisphere has positive measure and an unbounded sequence of polytopes $\{P_m\} \subset \mathcal{K}^3_{(o)}$ in $\mathbb{R}^3$*

*such that $S_{P_{m},p}$ converges weakly to $\mu$.*

More specific details are given as in the following. For $x_1, \dots, x_k \in \mathbb{R}^3$, we write $[x_1, \dots, x_k]$ for their convex hull. We denote by $u_0, u_1, u_2, u^+$, and $u^-$ the vectors $(1,0,0)$, $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$, $(0,0,1)$, and $(0,0,-1)$, respectively, and we define the discrete measure $\mu$ on $\mathbb{S}^2$ to have $\operatorname{supp}\mu = \{u_0, u_1, u_2, u^+, u^-\}$ with

$$\mu(\{u_0\}) = 8,\ \mu(\{u_1\}) = \mu(\{u_2\}) = 2^{\frac{p}{2}},\ \mu(\{u^+\}) = \mu(\{u^-\}) = 3.$$

Obviously, any open hemisphere has positive measure.

For $m \geq 2$ and $a_m := m^{-(2-p)}$, we set

$$v_{1,m} = (0,m,0),\ v_{1,m}^+ = (m, 2m, a_m),\ v_{1,m}^- = (m, 2m, -a_m),$$

$$v_{2,m} = (0,-m,0),\ v_{2,m}^+ = (m, -2m, a_m),\ v_{2,m}^- = (m, -2m, -a_m),$$

and denote by $\widetilde{P}_m$ their convex hull. The exterior unit normals of the facets of $\widetilde{P}_m$, $F_{0,m} := [v_{i,m}^+, v_{i,m}^-]_{i=1,2}$, $F_{1,m} := [v_{1,m}, v_{1,m}^+, v_{1,m}^-]$, and $F_{2,m} := [v_{2,m}, v_{2,m}^+, v_{2,m}^-]$ are, respectively, the unit vectors $u_0, u_1$, and $u_2$, which are independent of $m$. In addition, $\widetilde{P}_m$ has two more facets, $F_m^+ = [v_{1,m}, v_{2,m}, v_{1,m}^+, v_{2,m}^+]$ and $F_m^- = [v_{1,m}, v_{2,m}, v_{1,m}^-, v_{2,m}^-]$, whose exterior unit normals are, respectively,

$$u_m^+ = \left(\frac{-a_m}{\sqrt{a_m^2+m^2}}, 0, \frac{m}{\sqrt{a_m^2+m^2}}\right) \text{ and } u_m^- = \left(\frac{-a_m}{\sqrt{a_m^2+m^2}}, 0, \frac{-m}{\sqrt{a_m^2+m^2}}\right),$$

which satisfy $\lim_{m\to\infty} u_m^+ = u^+$ and $\lim_{m\to\infty} u_m^- = u^-$.

For $i = 1,2$, we have $h_{\widetilde{P}_m}(u_0) = m$, $h_{\widetilde{P}_m}(u_1) = h_{\widetilde{P}_m}(u_2) = \frac{m}{\sqrt{2}}$, and $h_{\widetilde{P}_m}(u_m^+) = h_{\widetilde{P}_m}(u_m^-) = 0$. It implies that

$$\begin{aligned} S_{\widetilde{P}_m,p}(\{u_0\}) &= h_{\widetilde{P}_m}(u_0)^{1-p}\mathcal{H}^2(F_{0,m}) = m^{1-p} 8 m a_m = 8 \text{ and} \\ S_{\widetilde{P}_m,p}(\{u_i\}) &= h_{\widetilde{P}_m}(u_i)^{1-p}\mathcal{H}^2(F_{i,m}) = \left(\frac{m}{\sqrt{2}}\right)^{1-p} \sqrt{2} m a_m = 2^{\frac{p}{2}} \text{ for } i = 1,2. \end{aligned}$$

Now we translate $\widetilde{P}_m$ in order to alter $S_{\widetilde{P}_m,p}(\{u_m^+\})$. We define positive constants $t_m$ in a way such that $P_m = \widetilde{P}_m - t_m u_0$ satisfy

$$h_{P_m}(u_m^+) = m^{\frac{-2}{1-p}}.$$

It follows that

$$m^{\frac{-2}{1-p}} = h_{P_m}(u_m^+) = t_m\langle u_m^+, u_0\rangle = \frac{t_m a_m}{\sqrt{m^2+a_m^2}} > \frac{t_m}{2m^{3-p}}.$$

We observe that $r = 3 - p - \frac{2}{1-p} < 3 - 2 = 1$ if $p \in (0,1)$ and, as its consequence, $\lim_{m\to\infty} t_m/m = 0$. We deduce that

$$\begin{aligned}
\lim_{m\to\infty} S_{P_m,p}(\{u_0\}) &= 8,\\
\lim_{m\to\infty} S_{P_m,p}(\{u_i\}) &= 2^{\frac{p}{2}} \text{ for } i = 1,2, \text{ and}\\
\lim_{m\to\infty} S_{P_m,p}(\{u_m^+\}) &= \lim_{m\to\infty} h_{P_m}(u_m^+)^{1-p}\mathcal{H}^2(F_m^+) = \lim_{m\to\infty} m^{-2}3m\sqrt{m^2+a_m^2} = 3.
\end{aligned}$$

Therefore, $S_{P_m,p}$ tends weakly to $\mu$.

## 2.2 The measure is concentrated on a closed semicircle

Dealing with the case when there is a possibility that $L_p$- surface area measure of a convex body $K$ containing the origin of an open semicircle can be equal to zero, we first show that it cannot be supported on two antipodal points.

**Lemma 2.2.1.** *If $K \in \mathcal{K}_o^2$, then $\operatorname{supp} S_{K,p}$ is not a pair of antipodal points.*

*Proof.* We suppose that $\operatorname{supp} S_{K,p} = \{v, -v\}$ for some unit vector $v \in \mathbb{S}^1$ and seek a contradiction. Let $w \in \mathbb{S}^1$ be orthogonal to $v$.

If $o \in \operatorname{int} K$, then $\operatorname{supp} S_{K,p} = \operatorname{supp} S_K$, which is not contained in any closed semicircle. Therefore, $o \in \operatorname{bd} K$. Let $C$ be the exterior normal cone at $o$, namely, $C \cap \mathbb{S}^1 = \{u \in \mathbb{S}^1 : h_K(u) = 0\}$. Since $\operatorname{supp} S_{K,p} = \{v, -v\}$, $h_K(v)$ and $h_K(-v)$ are both positive and it follows that neither $v$ nor $-v$ belong to $C$. Thus, we may assume possibly after replacing $w$ with $-w$ that $C \cap \mathbb{S}^1$ is contained in $\Omega(-w, 0)$. It leads to the observation that $h_K(u)$ is positive for any $u$ in $\Omega(w, 0)$, and since $S_K(\Omega(w,0))$ is positive, it also follows that

$$S_{K,p}(\Omega(w,0)) = \int_{\Omega(w,0)} h_K^{1-p}\, dS_K > 0.$$

This contradicts to $\operatorname{supp} S_{K,p} = \{v, -v\}$ and then completes the proof of Lemma 2.2.1.

□

Let $\mu$ be a non-trivial measure on $\mathbb{S}^1$ that is concentrated on a closed semicircle $\sigma$ of $\mathbb{S}^1$ connecting the unit vectors $v$ and $-v$ in $\mathbb{S}^1$ such that $\operatorname{supp} \mu$ is not a pair of antipodal points. We may assume that for the $w \in \sigma$ orthogonal to $v$, we have either $\operatorname{supp} \mu = \{w\}$ or

$$w \in \operatorname{int} \operatorname{pos}(\operatorname{supp} \mu). \tag{2.15}$$

We consider the case when $\operatorname{supp} \mu = \{w\}$. Let the unit vectors $w_1$ and $w_2$ in $\mathbb{S}^1$ be such that $w_1 + w_2 = -w$ and let $K_0$ be the regular triangle

$$K_0 = \{x \in \mathbb{R}^2 : \langle x, w_1 \rangle \leq 0,\ \langle x, w_2 \rangle \leq 0,\ \langle x, w \rangle \leq 1\}.$$

For $\lambda = \mu(\{w\})/S_{K_0,p}(\{w\})$ and $\lambda_0 = \lambda^{\frac{1}{2-p}}$, we have $S_{\lambda_0 K_0, p} = \mu$.

Now we consider the other case when $w \in \operatorname{int} \operatorname{pos}(\operatorname{supp} \mu)$. We denote by $A$ the reflection through the line $\operatorname{lin} v$. We define a measure $\tilde{\mu}$ on $\mathbb{S}^1$ by

$$\tilde{\mu}(\omega) = \mu(\omega) + \mu(A\omega) \ \text{ for Borel sets } \omega \subset \mathbb{S}^1.$$

We observe that $\tilde{\mu}$ is invariant under $A$,

$$\begin{aligned} \tilde{\mu}(\omega) &= \mu(\omega) \ \text{ if } \omega \subset \Omega(w, 0), \\ \tilde{\mu}(\{v\}) &= 2\mu(\{v\}), \text{ and} \\ \tilde{\mu}(\{-v\}) &= 2\mu(\{-v\}). \end{aligned}$$

It follows from $w \in \operatorname{int} \operatorname{pos}(\operatorname{supp} \mu)$ that there is not any closed semicircle containing $\operatorname{supp} \tilde{\mu}$. We deduce from the previous section, where the case when the measure of any open semicircle is positive has been already proved, that there exists a convex body $\widetilde{K} \in \mathcal{K}_o^2$ invariant under $A$ for which $S_{\widetilde{K},p} = \tilde{\mu}$.

We claim that

$$S_{K,p} = \mu \text{ for } K = \{x \in \widetilde{K} : \langle x, w\rangle \geq 0\}. \tag{2.16}$$

For any convex body $M$ and any unit vector $u \in \mathbb{S}^1$, we write $F(M,u) := \{x \in M : \langle x, u\rangle = h_M(u)\}$ for the face of $M$ with exterior unit normal $u$ and for any $x, y \in \mathbb{R}^2$, we write $[x, y]$ for the convex hull of $x$ and $y$, which is a segment if $x \neq y$. Since $\widetilde{K}$ is invariant under $A$, there exist nonnegative numbers $t$ and $s$ such that $tv$ and $-sv$ are boundary points of $\widetilde{K}$, and the exterior normals at $tv$ and $-sv$ are $v$ and $-v$, respectively. In addition, $\mathcal{H}^1(F(\widetilde{K}, v)) = 2\,\mathcal{H}^1(F(K, v))$, $\mathcal{H}^1(F(\widetilde{K}, -v)) = 2\,\mathcal{H}^1(F(K, -v))$, and $F(K, -w) = [tv, -sv]$.

To prove (2.16), first we observe that by definition and have

$$\mu(\{v\}) = \frac{\tilde{\mu}(\{v\})}{2} = \frac{h_{\widetilde{K}}(v)^{1-p}\mathcal{H}^1(F(\widetilde{K}, v))}{2} = h_K(v)^{1-p}\mathcal{H}^1(F(K, v)) = S_{K,p}(\{v\}),$$

and, similarly, $\mu(\{-v\}) = S_{K,p}(\{-v\})$. Next (1.1) yields that

$$S_{K,p}(\Omega(-w, 0)) = \int_{[tv,-sv]} \langle x, w\rangle^{1-p}\, d\mathcal{H}^1(x) = 0 = \mu(\Omega(-w, 0)).$$

Finally, if $\omega \subset \Omega(w, 0)$, then $\nu_{\widetilde{K}}^{-1}(\omega) = \nu_K^{-1}(\omega)$, which yields

$$\mu(\omega) = \tilde{\mu}(\omega) = S_{\widetilde{K},p}(\omega) = S_{K,p}(\omega),$$

and in turn (2.16).

Therefore, all we are left to do is to check the symmetries of $\mu$. Actually, the only possible symmetry is the reflection $B$ through $\operatorname{lin} w$. In this case, $\tilde{\mu}$ is also invariant under $B$ and hence, we may assume that $\widetilde{K}$ is also invariant under $B$. We conclude that $K$ is invariant under $B$, completing the proof of Theorem 2.0.1.

# CHAPTER 3 ON THE MINIMUM TIME FUNCTION

The purpose of the present chapter is to study the $\varphi$-convexity of the epigraph of the minimum function $T$ for a nonlinear control system with a general closed target, provided that the sublevel sets of $T$ are $\varphi_0$-convex for some nonnegative constant $\varphi_0$. This property of the minimum time function $T$ will be demonstrated in Section 3.2 via the relationship between the sublevel sets and the epigraph of $T$. The most important key is the appropriate sensitivity relation results stated in Section 3.1.

For given function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ and control set $\mathcal{U} \subset \mathbb{R}^m$, we consider the nonlinear control system

$$\begin{cases} y'(t) = f(y(t), u(t)) & \text{a.e. } t > 0, \\ u(t) \in \mathcal{U} & \text{a.e. } t > 0, \\ y(0) = x, \end{cases} \tag{3.1}$$

The global existence of a unique solution to (3.1) is ensured by the following essential assumptions on the function $f$ and the control set $\mathcal{U}$:

(A1) $\mathcal{U}$ is compact and $f(x, \mathcal{U})$ is convex for every $x \in \mathbb{R}^n$.

(A2) $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ is continuous and satisfies

$$\|f(x, u) - f(y, u)\| \le L\|x - y\| \qquad \text{for all } x, y \in \mathbb{R}^n,\ u \in \mathcal{U},$$

and for a positive constant $L$.

(A3) The differential of $f$ with respect to the $x$ variable $D_x f$ exists everywhere, is continuous with respect to both $x$ and $u$, and satisfies

$$\|D_x f(x, u) - D_x f(y, u)\| \le L_1\|x - y\| \qquad \text{for all } x, y \in \mathbb{R}^n,\ u \in \mathcal{U},$$

and for a positive constant $L_1$.

For a given nonempty closed target $\mathcal{K} \subset \mathbb{R}^n$, we define the minimum time function $T$, which is well-defined under the assumptions (A1) - (A3) (see ([CS04])), as in Section 1.6. Throughout this chapter, the notations are also referred to those in Section 1.5 and Section 1.6.

## 3.1 Sensitivity relations

Sensitivity relations, which consist of the *dual arc* satisfying an inclusion of an appropriate generalized gradient of the value function, are widely studied on the minimal time problem (see, e.g., [CFS00], [CMN15], [CS15], [CNN14], [FN15], [Ngu16], and references therein). In this section, dealing with the minimum time function $T$ associated with the nonlinear control system (3.1), we present similar propagation results concerning with both the proximal subdifferential and the proximal horizontal subdifferential of $T$, which play an important role in Section 3.2. We prove inclusions for normal cones to the epigraph and to the sublevel sets of the minimum time function. As a consequence, we come to conclusion that the proximal subdifferential and the proximal horizontal subdifferential of $T$ propagate wholly along optimal trajectories. Although the result is similar to the result given in [Ngu16] where the author deals with the minimum time function for differential inclusions, we work under different assumptions and use different techniques. With regard to the nonlinear control system (3.1), under assumptions (A1)-(A3), the sensitivity relations are given in Theorem 3.1.4 based on the characterization of the proximal subdifferential and the horizontal proximal subdifferential of $T$ as well as the relationship between normals to the epigraph and to sublevel sets of $T$ via the value at relevant points of the minimized Hamiltonian $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ associated with the control system (3.1) defined by

$$h(x, \zeta) = \min_{u \in \mathcal{U}} \langle \zeta, f(x, u) \rangle \quad \forall x, \zeta \in \mathbb{R}^n.$$

We first recall the characterization of the proximal subdifferential and the horizontal proximal subdifferential of $T$ at a point outside the target given by Wolenski and

Yu [WY98] and Nguyen [Ngu16].

**Theorem 3.1.1** (Wolenski, Yu, and Nguyen). *Assume (A1) - (A3). Let $x \in \mathcal{R} \setminus \mathcal{K}$. We have*

*(i)* $\partial^P T(x) = N^P_{\mathcal{R}(T(x))}(x) \cap \{\zeta \in \mathbb{R}^n : h(x,\zeta) = -1\}$.

*(ii)* $\partial^\infty T(x) = N^P_{\mathcal{R}(T(x))}(x) \cap \{\zeta \in \mathbb{R}^n : h(x,\zeta) = 0\}$.

Another useful variational result, established by Nguyen [Ngu16], is a connection between normal cones to sublevel sets and to the epigraph of the minimum time function $T$.

**Theorem 3.1.2** (Nguyen). *Assume (A1) - (A3). Let $x \in \mathcal{R} \setminus \mathcal{K}$.*

*(i) If* $\zeta \in N^P_{\mathcal{R}(T(x))}(x)$, *then* $(\zeta, h(x,\zeta)) \in N^P_{\mathrm{epi}(T)}(x, T(x))$.

*(ii) If* $\zeta \in \mathbb{R}^n$ *and* $\eta \in \mathbb{R}$ *satisfy* $(\zeta, \eta) \in N^P_{\mathrm{epi}(T)}(x, T(x))$, *then* $\eta \le 0$, $\zeta \in N^P_{\mathcal{R}(T(x))}(x)$, *and* $h(x,\zeta) = \eta$.

We also recall the Maximum Principle in the following form, see, e.g., [CFS00].

**Theorem 3.1.3.** *Assume (A1) - (A4). Let $x \in \mathcal{R} \setminus \mathcal{K}$ and let $u(\cdot)$ be an optimal control for $x$ and $y(\cdot) := y^{x,u}(\cdot)$ be the corresponding optimal trajectory. Let $\zeta \in N^P_{\mathcal{K}}(y(T(x)))$. Then the solution of the system*

$$\begin{cases} p'(t) &= -D_x f(y(t), u(t))^\top p(t) \qquad a.e.\, t \in [0, T(x)] \\ p(T(x)) &= \zeta \end{cases}$$

*satisfies*

$$\langle f(y(t), u(t)), p(t) \rangle = h(y(t), p(t)) \qquad a.e.\, t \in [0, T(x)].$$

Here, a non-trivial absolutely continuous function $p(\cdot)$ satisfying the system above in Theorem 3.1.3 is called a *dual arc* associated to the optimal trajectory $y^{x,u}(\cdot)$.

The following sensitivity relations are the main result of this section. To demonstrate these relations, we need to take the results given in Theorem 3.1.1 and Theorem 3.1.2 into account.

**Theorem 3.1.4.** *Assume (A1) - (A3). Let $x \in \mathcal{R} \setminus \mathcal{K}$ and let $(y(\cdot), w(\cdot))$ be an optimal pair for $x$. Let $p : [0, T(x)] \to \mathbb{R}^n$ be a solution of the equation*

$$p'(t) = -D_x f(y(t), w(t))^\top p(t) \qquad a.e.\, t \in [0, T(x)]. \tag{3.2}$$

*We have the following.*

*(i) If $(p(0), h(x, p(0))) \in N^P_{\text{epi}(T)}(x, T(x))$, then*

$$(p(t), h(y(t), p(t))) \in N^P_{\text{epi}(T)}(y(t), T(y(t))) \text{ for all } t \in [0, T(x))$$

*and*

$$h(y(t), p(t)) = h(x, p(0)) \text{ for all } t \in [0, T(x)].$$

*(ii) If $p(0) \in N^P_{\mathcal{R}(T(x))}(x)$, then for all $t \in [0, T(x)]$,*

$$p(t) \in N^P_{\mathcal{R}(T(y(t)))}(y(t)) \text{ and } h(y(t), p(t)) = h(x, p(0)).$$

*(iii) If $p(0) \in \partial^P T(x)$, then for all $t \in [0, T(x)]$,*

$$p(t) \in \partial^P T(y(t)) \text{ and } h(y(t), p(t)) = -1.$$

*(iv) If $p(0) \in \partial^\infty T(x)$, then for all $t \in [0, T(x)]$,*

$$p(t) \in \partial^\infty T(y(t)) \text{ and } h(y(t), p(t)) = 0.$$



*Proof.* We first remark that if $p(\cdot)$ is a solution of the equation (3.2) then either $p(t) = o$ for all $t \in [0, T(x)]$ or $p(t) \neq o$ for all $t \in [0, T(x)]$. Moreover, there is a positive constant $K$ such that $\|p(t)\| \leq K$ for all $t \in [0, T(x)]$.

(i) In the case when $p(\cdot)$ is trivial, our conclusion is obvious. We now consider the case that $p(\cdot)$ is non-trivial. For the sake of simplicity, we shall denote $\alpha := h(x, p(0))$. Since $(p(0), \alpha)$ is an element of $N^P_{\text{epi}(T)}(x, T(x))$ according to our assumption, by definition of the proximal normal cone, there exist positive constants $c$ and $\eta$ such that

$$\langle p(0), y - x \rangle + \alpha(\beta - T(x)) \leq c(\|y - x\|^2 + |\beta - T(x)|^2) \tag{3.3}$$

for all $y \in B(x, \eta)$ and $\beta \geq T(y)$. We now fix $t \in [0, T(x))$. Let $u \in B(o, \eta)$ and let $y_u(\cdot)$ denote the solution of the system

$$\begin{cases} y_u'(s) &= f(y_u(s), w(s)) \\ y_u(t) &= y(t) + u \end{cases} \qquad \text{a.e. } s \in [0, T(x)].$$

The Lipschitz continuity of $f$ implies that for all $s \in [0, T(x)]$,

$$\begin{aligned} \|y_u(s) - y(s)\| &= \left\| \int_s^t (y_u'(\tau) - y'(\tau)) d\tau + y(t) - y_u(t) \right\| \\ &\leq \|u\| + \int_s^t \|f(y_u(\tau), w(\tau)) - f(y(\tau), w(\tau))\| d\tau \\ &\leq \|u\| + L \int_s^t \|y_u(\tau) - y(\tau)\| d\tau. \end{aligned}$$

Consequently, by Gronwall's lemma, there exists a constant $\kappa > 1$ independent of $u$ such that

$$\|y_u(s) - y(s)\| \leq \kappa \|u\| \qquad \text{for all } s \in [0, T(x)]. \tag{3.4}$$

Moreover, we have

$$\langle p(t), y_u(t) - y(t) \rangle - \langle p(0), y_u(0) - y(0) \rangle = \int_0^t \frac{d}{ds} \langle p(s), y_h(s) - y(s) \rangle ds.$$

We first decompose the formula under the integral in the right hand side of this equality by the differentiation rule for a combined function, then use (3.2), and finally apply conjugate symmetry property of the inner product in order to obtain that the left hand side of the previous equality can be represented to be

$$\int_0^t \langle -p(s), D_x f(y(s), w(s))(y_u(s) - y(s)) + f(y_u(s), w(s)) - f(y(s), w(s)) \rangle ds. \tag{3.5}$$

We consider the inner product under the integral (3.5) when keeping in mind that the term $f(y_u(s), w(s)) - f(y(s), w(s))$ can be represented in a form of a integral as

$$f(y_u(s), w(s)) - f(y(s), w(s)) = (y_u(s) - y(s)) \int_0^1 -D_x f(y(s) + \tau(y_u(s) - y(s)), w(s)) d\tau$$

In addition to applying Cauchy–Schwarz inequality for the inner product in (3.5) where $D_x f(y(s), w(s))$ will be necessarily put under the integral as in the revised form of $f(y_u(s), w(s)) - f(y(s), w(s))$ mentioned above and recalling that $\|p(s)\|$ is bounded above by $K$ on $[0, T(x)]$, we take assumption (A3) into account to finally deduce that

$$\langle p(t), y_u(t) - y(t)\rangle - \langle p(0), y_u(0) - y(0)\rangle \le KL_1 \int_0^t \|y_u(s) - y(s)\|^2 ds \int_0^1 \tau d\tau.$$

According to (3.4), there exists a positive constant $C_1$ independent of $u$ and $t$ for which

$$\langle p(t), y_u(t) - y(t)\rangle - \langle p(0), y_u(0) - y(0)\rangle \le C_1 \|u\|^2. \tag{3.6}$$

We can choose $\eta > 0$ sufficiently small such that $y_u([0,t]) \cap \mathcal{K} = \emptyset$ for all $u \in B(o, \eta)$. The dynamic programming principle 1.6.2 leads to

$$T(x) = T(y(t)) + t \text{ and } T(y_u(0)) \le T(y_u(t)) + t. \tag{3.7}$$

Let $\bar{\beta} \ge T(y_u(t))$. It follows from (3.7) that

$$T(y_u(0)) \le \bar{\beta} + T(x) - T(y(t)).$$

We substitute $y$ and $\beta$ by $y_u(0)$ and $\bar{\beta} + T(x) - T(y(t))$ in (3.3), respectively, while keeping in mind that $x = y(0)$, to obtain

$$\langle p(0), y_u(0) - y(0)\rangle + \alpha(\bar{\beta} - T(y(t)) \le c(\|y_u(0) - y(0)\|^2 + |\bar{\beta} - T(y(t))|^2). \tag{3.8}$$

By simply adding and subtracting $\langle p(0), y_u(0) - y(0)\rangle$, according to our observations (3.4), (3.6), and (3.8), we have

$$\langle p(t), y_u(t) - y(t)\rangle \le -\alpha(\bar{\beta} - T(y(t))) + (C_1 + c\kappa)\left(\|u\|^2 + |\bar{\beta} - T(y(t))|^2\right),$$

and as a consequence,

$$\langle p(t), y_u(t) - y(t)\rangle + \alpha(\bar{\beta} - T(y(t))) \le (C_1 + c\kappa)(\|y_u(t) - y(t)\|^2 + |\bar{\beta} - T(y(t)|^2).$$

This means that for any $u \in B(o, \eta)$ and any $\bar{\beta} \ge T(y(t) + u)$, it holds

$$\langle p(t), u\rangle + \alpha(\bar{\beta} - T(y(t))) \le (C_1 + c\kappa)(\|u\|^2 + |\bar{\beta} - T(y(t)|^2).$$

In other words, $(p(t), \alpha)$ is an element in $N^P_{\text{epi}(T)}(y(t), T(y(t)))$. Thus, we deduce from

Theorem 3.1.2 that $h(y(t), p(t)) = \alpha$. Our desired argument follows from the fact that $t \in [0, T(x))$ is chosen arbitrarily and from the continuity of $h$.

(ii) Since $p(0)$ is an element of $N^P_{\mathcal{R}(T(x))}(x)$, it follows from Theorem 3.1.2 that $(p(0), h(x, p(0)))$ belongs to $N^P_{\text{epi}(T)}(x, T(x))$. Thus, (i) guarantees that $(p(t), h(y(t), p(t)))$ is an element of $N^P_{\text{epi}(T)}(y(t), T(y(t)))$ for all $t \in [0, T(x))$ and $h(y(t), p(t)) = h(x, p(0))$ for all $t \in [0, T(x)]$. Again, Theorem 3.1.2 points out that $p(t)$ belongs to $N^P_{\mathcal{R}(T(y(t)))}(y(t))$ for any $t \in [0, T(x))$. It is left to show that $p(T(x))$ is actually an element of $N^P_{\mathcal{K}}(y(T(x)))$. As $p(0)$ is contained in $N^P_{\mathcal{R}(T(x))}(x)$ according to our assumption, by the definition of the proximal normal cone, there are positive constants $C_0$ and $\eta_0$ such that

$$\langle p(0), y - x\rangle \leq C_0 \|y - x\|^2 \qquad \forall y \in \mathcal{R}(T(x)) \cap B(x, \eta_0). \tag{3.9}$$

Let $z \in \mathcal{K} \cap B(y(T(x)), \eta_0)$ and set $v := z - y(T(x)) \in B(o, \eta_0)$. We denote by $x_v(\cdot)$ the solution of the system

$$\begin{cases} x_v'(s) &= f(x_v(s), w(s)) \\ x_v(T(x)) &= y(T(x)) + v \end{cases} \qquad a.e.\ s \in [0, T(x)].$$

We observe that $T(x_v(0)) \leq T(x)$, which means $x_v(0)$ is contained in $\mathcal{R}(T(x))$.

Having similar argument as in the proof of (i), we can see the existence of positive constants $C_2$ and $C_3$ such that

$$\|x_v(s) - y(s)\| \leq C_2 \|v\| \qquad \forall\, s \in [0, T(x)] \tag{3.10}$$

and

$$\langle p(T(x)), x_v(T(x)) - y(T(x))\rangle - \langle p(0), x_v(0) - y(0)\rangle \leq C_3 \|v\|^2. \tag{3.11}$$

By simply adding and subtracting $\langle p(0), x_v(0) - y(0)\rangle$ with note that $z = x_v(T(x))$, we take (3.9), (3.10), and (3.11) into account to conclude that

$$\langle p(T(x)), z - y(T(x))\rangle \leq (C_3 + C_0 C_2^2)\|v\|^2 = (C_3 + C_0 C_2^2)\|z - y(T(x))\|^2,$$

or in other words, $p(T(x))$ is contained in $N^P_{\mathcal{K}}(y(T(x)))$.

(iii) and (iv) follow from Theorem 3.1.1 and (ii). □

We end this section by stating the following corollary, as a consequence of Theorem 3.1.4 and the Maximum Principle given in Theorem 3.1.3. This corollary will be needed in Section 3.2.

**Corollary 3.1.5.** *Assume (A1) - (A3). Let* $x \in \mathcal{R} \setminus \mathcal{K}$ *and let* $(y(\cdot), w(\cdot))$ *be an optimal pair for* $x$*. Assume* $N^P_{\mathcal{R}(T(x))}(x) \neq \{o\}$*. Let* $o \neq \zeta \in N^P_{\mathcal{R}(T(x))}(x)$ *and* $p : [0, T(x)] \to \mathbb{R}^n$ *be the solution of the system*

$$\begin{cases} p'(t) &= -D_x f(y(t), w(t))^\top p(t) \qquad a.e.\, t \in [0, T(x)]. \\ p(0) &= \zeta \end{cases}$$

*Then we have*

$$\langle f(y(t), w(t)), p(t) \rangle = h(y(t), p(t)) \qquad a.e.\, t \in [0, T(x)].$$

*Proof.* Since $p(0) = \zeta$ is an element of $N^P_{\mathcal{R}(T(x))}(x)$ according to Theorem 3.1.4 (ii), we observe that $\zeta_1 := p(T(x))$ belongs to $N^P_{\mathcal{K}}(y(T(x)))$. Thus, $p(\cdot)$ is the unique solution of the equation

$$\begin{cases} p'(t) &= -D_x f(y(t), w(t))^\top p(t) \qquad a.e.\, t \in [0, T(x)] \\ p(T(x)) &= \zeta_1 \in N^P_{\mathcal{K}}(y(T(x))). \end{cases}$$

Therefore, by Maximum Principle 3.1.3, we obtain

$$h(y(t), p(t)) = \langle p(t), f(y(t), w(t)) \rangle \qquad a.e.\, t \in [0, T(x)].$$

□

## 3.2 Relationship between sublevel sets and the epigraph

In this section, we present the main result of this chapter, namely, the relationship between sublevel sets and the epigraph of the minimum time function associating with the nonlinear control system (3.1). More precisely, we show that the epigraph of $T$ is $\varphi$-convex for some appropriate continuous function $\varphi$, provided that the sublevel sets of $T$ are $\varphi_0$-convex for some nonnegative constant $\varphi_0$. Our main goal is to construct a continuous function $\varphi$ in view of Definition 1.2.2, based on the suitable sensitivity relation results given in Section 3.1 and the $\varphi_0$-convexity of the sublevel sets of $T$. The function $\varphi$ is explicitly computed as in Theorem 3.2.1.

Given a positive number $\sigma$, we denote by $\mathcal{S}(\sigma)$ the set $\mathcal{R}(\sigma) \setminus \mathcal{K}$. For a subset $\mathcal{O}$ of $\mathbb{R}^n$, let $T_{|\mathcal{O}} : \mathcal{O} \to \mathbb{R}^n$ stand for the restriction of $T$ on $\mathcal{O}$, i.e., $T_{|\mathcal{O}}(x) = T(x)$ for all $x \in \mathcal{O}$. The following assumptions will be needed.

(Q1) There is a nonnegative constant $\varphi_0$ such that $\mathcal{R}(t)$ is $\varphi_0$-convex for all $t \in [0, \sigma]$.

(Q2) $T$ is continuous on $\mathcal{S}(\sigma)$.

**Theorem 3.2.1.** *Assume (A1) - (A3) and (Q1) - (Q2). Then there exists a continuous function $\varphi$, which can be computed explicitly, such that the epigraph of $T_{|\mathcal{S}(\sigma)}$ is $\varphi$-convex.*

*Proof.* We shall construct a continuous function $\varphi$ ensuring that for any $x$ and $y$ in $\mathcal{S}(\sigma)$, any $\beta \geq T(y)$, and any element $(\zeta, \alpha)$ of $N^P_{\text{epi}(T)}(x, T(x))$, it holds

$$\langle(\zeta, \alpha), (y - x, \beta - T(x))\rangle \leq \varphi(x)\|(\zeta, \alpha)\|(\|y - x\|^2 + |\beta - T(x)|^2). \tag{3.12}$$

We first note that if $(\zeta, \alpha)$ belongs to $N^P_{\text{epi}(T)}(x, T(x))$, then $\zeta$ is an element of $N^P_{\mathcal{R}(T(x))}(x)$ and $h(x, \zeta) = \alpha$, according to Theorem 3.1.2.

For the sake of simplicity, we denote by $\varphi_i : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$, $i = 1, ..., 7$, the following functions.

$$\varphi_1(x, t) := 2\left(\varphi_0[1 + (L\|x\| + K_1)^2 e^{2Lt}] + L[(L\|x\| + K_1)e^{Lt} + 1]\right),$$

$$\varphi_2(x, t) := 2\left(\varphi_0[1 + (L\|x\| + L + K_1)^2 e^{2L}] + L[(L\|x\| + L + K_1)e^{L} + 1]\right),$$

$$\varphi_3(x, t) := \left(L_1 e^{Lt}\|x\| + \frac{L_1 K_1(e^{Lt} - 1)}{L} + K_2\right) e^{l(x,t)t},$$

$$\varphi_4(x, t) := \varphi_3(x, t)[1 + (L\|x\| + K_1)e^{Lt}],$$

$$\varphi_5(x, t) := \varphi_0 e^{l(x,t)t}[1 + (L\|x\| + K_1)e^{Lt}],$$

$$\varphi_6(x, t) := l(x, t)e^{l(x,t)t}(L\|x\| + K_1)e^{Lt}, \text{ and}$$

$$\varphi_7(x, t) := \max\{\varphi_0, \varphi_4(x, t) + \varphi_5(x, t) + \varphi_6(x, t)\}$$

where $K_1 := \max_{u\in\mathcal{U}}\|f(o,u)\|$ and $K_2 := \max_{u\in\mathcal{U}}\|D_x f(o,u)\|$ and $l(x,t)$ is defined as in Lemma 1.5.3.

Now, we define $\varphi : \mathcal{S}(\sigma) \to [0,\infty)$ by $\varphi(x) := \max\{\varphi_2(x,T(x)), 2, \varphi_7(x,T(x))\}$. Then $\varphi(\cdot)$ is continuous on $\mathcal{S}(\sigma)$ as $T$ is continuous.

It is sufficient to verify (3.12). We begin with the special case when $\zeta = o$. In the case, we observe that $\alpha = 0$. Obviously, our inequality holds true. We pay attention to the case when $\zeta \neq o$. For any $x$ in $\mathcal{S}(\sigma)$, we denote by $r$ the value $T(x)$ and by $(y(\cdot), w(\cdot))$ an optimal pair for $x$. Let $p : [0, T(x)] \to \mathbb{R}^n$ stand for the solution of the system

$$\text{(3.13)} \qquad \begin{cases} p'(t) &= -D_x f(y(t), w(t))^\top p(t) \\ p(0) &= \zeta \end{cases} \qquad \text{a.e. } t \in [0,r].$$

We shall continue to verity (3.12) based on the fact that for any pair $x$ and $y$ in $\mathcal{S}(\sigma)$, there are only two possibilities in the comparison between $T(x)$ and $T(y)$, namely, $T(y) \geq T(x)$ and $T(y) < T(x)$.

We first assume that $T(y) \geq T(x)$. We denote by $(\bar{y}(\cdot), \bar{w}(\cdot))$ an optimal pair for $y$ and set $y_1 := \bar{y}(r_1)$ where $r_1 := T(y) - T(x)$. According to Lemma 1.5.2 (i), for all $s \in [0, r_1]$,

$$\text{(3.14)} \qquad \|y - \bar{y}(s)\| = \|\bar{y}(s) - \bar{y}(0)\| \leq (L\|y\| + K_1)e^{Ls}s \leq (L\|y\| + K_1)e^{Lr_1}r_1.$$

As $\langle \zeta, y - x\rangle = \langle \zeta, y - y_1\rangle + \langle \zeta, y_1 - x\rangle$, we shall estimate $\langle \zeta, y - x\rangle$ via $\langle \zeta, y - y_1\rangle$ and $\langle \zeta, y_1 - x\rangle$. Since $y_1 \in \mathcal{R}(r)$ and $\mathcal{R}(r)$ is $\varphi_0$-convex, using the triangle inequality for norm and (3.14), we have

$$\begin{aligned} \langle \zeta, y_1 - x\rangle &\leq 2\varphi_0\|\zeta\|(\|y - x\|^2 + \|y - y_1\|^2) \\ &\leq 2\varphi_0\|\zeta\|\left(\|y - x\|^2 + (L\|y\| + K_1)^2 e^{2Lr_1} r_1^2\right) \\ \text{(3.15)} \qquad &\leq 2\varphi_0\|\zeta\|\left(1 + (L\|y\| + K_1)^2 e^{2Lr_1}\right)\left(\|y - x\|^2 + r_1^2\right). \end{aligned}$$

In addition, observing that $\langle \zeta, y - y_1\rangle = \langle \zeta, \bar{y}(0) - \bar{y}(r_1)\rangle$, using the definition of $\bar{y}(\cdot)$, the definition of $h(\cdot)$, Cauchy-Schwarz inequality, the triangle inequality for norm, and the assumption (A2), we deduce from (3.14) that

$$\begin{aligned}
\langle \zeta, y - y_1 \rangle &= -\int_0^{r_1} \langle \zeta, \bar{y}'(s) \rangle ds \\
&= -\int_0^{r_1} \langle \zeta, f(\bar{y}(s), \bar{w}(s)) \rangle ds \\
&= -\int_0^{r_1} \langle \zeta, f(x, \bar{w}(s)) \rangle ds + \int_0^{r_1} \langle \zeta, f(x, \bar{w}(s)) - f(\bar{y}(s), \bar{w}(s)) \rangle ds \\
&\leq -h(x, \zeta) r_1 + L\|\zeta\| \int_0^{r_1} \|\bar{y}(s) - x\| ds \\
&\leq -h(x, \zeta) r_1 + L\|\zeta\| \int_0^{r_1} (\|\bar{y}(s) - y\| + \|y - x\|) ds \\
&\leq -\alpha r_1 + L\|\zeta\| \|y - x\| r_1 + L\|\zeta\| \int_0^{r_1} (L\|y\| + K_1) e^{L r_1} r_1 ds \\
&\leq -\alpha r_1 + L\|\zeta\| \|y - x\| r_1 + L\|\zeta\| (L\|y\| + K_1) e^{L r_1} r_1^2 \\
&\leq -\alpha r_1 + 2L\|\zeta\| \left[ (L\|y\| + K_1) e^{L r_1} + 1 \right] \left( \|y - x\|^2 + r_1^2 \right). \qquad (3.16)
\end{aligned}$$

By some simple computation, our observations (3.15) and (3.16) lead to

$$\begin{aligned}
\langle \zeta, y - x \rangle &= \langle \zeta, y - y_1 \rangle + \langle \zeta, y_1 - x \rangle \\
&= -\alpha(T(y) - T(x)) + \varphi_1(y, r_1) \|\zeta\| (\|y - x\|^2 + |T(y) - T(x)|^2). \qquad (3.17)
\end{aligned}$$

We notice that if $\|y - x\| + |T(y) - T(x)| \leq 1$, then $\|y\| \leq \|x\| + 1$ and $0 \leq r_1 = T(y) - T(x) \leq 1$. It follows that

$$\varphi_1(y, r_1) \leq \varphi_2(x, r). \qquad (3.18)$$

Otherwise, if $\|y - x\| + |T(y) - T(x)| > 1$, then

$$\begin{aligned}
\langle \zeta, y - x \rangle + \alpha(T(y) - T(x)) &\leq \|(\zeta, \alpha)\| (\|y - x\| + |T(y) - T(x)|) \\
&\leq 2\|(\zeta, \alpha)\| (\|y - x\|^2 + |T(y) - T(x)|^2). \qquad (3.19)
\end{aligned}$$

According to (3.17) - (3.19) and the definition of $\varphi$, we obtain

$$\langle \zeta, y - x \rangle + \alpha(T(y) - T(x)) \leq \varphi(x) \|(\zeta, \alpha)\| (\|y - x\|^2 + |T(y) - T(x)|^2) \qquad (3.20)$$

for $T(y) \geq T(x)$.

Recalling that $\alpha = h(x, \zeta)$ is non-positive by Theorem 3.1.2, we deduce from (3.20) that

$$\langle(\zeta,\alpha),(y-x,\beta-T(x))\rangle \le \varphi(x)\|(\zeta,\alpha)\|(\|y-x\|^2+|\beta-T(x)|^2) \tag{3.21}$$

for $\beta \ge T(y) \ge T(x)$.

Finally, we assume that $T(y) < T(x)$. Since $y \in \mathcal{R}(r)$ and $\mathcal{R}(r)$ is $\varphi_0$-convex, it holds

$$\langle\zeta, y-x\rangle \le \varphi_0\|\zeta\|\|y-x\|^2. \tag{3.22}$$

It follows from $\alpha \le 0$ that for all $\beta \ge T(x) > T(y)$,

$$\langle\zeta, y-x\rangle + \alpha(\beta-T(x)) \le \varphi_0\|(\zeta,\alpha)\|(\|y-x\|^2+|\beta-T(x)|^2). \tag{3.23}$$

We suppose that $T(y) \le \beta \le T(x)$ and set $x_1 := y(r_2)$ where $r_2 := T(x)-\beta$. Since

$$\langle\zeta, y-x\rangle = \langle\zeta-p(r_2), y-x_1\rangle + \langle p(r_2), y-x_1\rangle + \langle\zeta, x_1-x\rangle,$$

we shall estimate $\langle\zeta, y-x\rangle$ via $\langle\zeta-p(r_2), y-x_1\rangle$, $\langle p(r_2), y-x_1\rangle$, and $\langle\zeta, x_1-x\rangle$ with note that

$$\|y-x_1\| \le \|y-x\| + \|x-x_1\| \le \|y-x\| + (L\|x\|+K_1)e^{Lr_2}r_2. \tag{3.24}$$

by Lemma 1.5.2. Recalling that $\zeta = p(0)$, we can revise $\langle\zeta-p(r_2), y-x_1\rangle$ to be of the form $-\int_0^{r_2}\langle p'(s), y-x_1\rangle ds$. Using the definition of $p(\cdot)$, Cauchy-Schwarz inequalities, applying Lemma 1.5.2 and 1.5.3, and taking (3.24) into account, by simple calculation, we obtain

$$\begin{aligned}
\langle\zeta-p(r_2), y-x_1\rangle &= \int_0^{r_2}\langle D_x f(y(s),w(s))^\top p(s), y-x_1\rangle ds \\
&\le \int_0^{r_2}\|D_x f(y(s),w(s))\|\|p(s)\|\|y-x_1\|ds \\
&\le \left(L_1 e^{Lr_2}\|x\| + \frac{L_1K_1(e^{Lr_2}-1)}{L} + K_2\right) e^{l(x,r_2)r_2}\|p(0)\|\|y-x_1\|r_2 \\
&\le \varphi_3(x,r_2)\|\zeta\|\left(\|y-x\|r_2 + (L\|x\|+K_1)e^{Lr_2}r_2^2\right) \\
&\le \varphi_4(x,r)\|\zeta\|(\|y-x\|^2+r_2^2). \qquad (3.25)
\end{aligned}$$

Regarding $\langle p(r_2), y-x_1\rangle$ , we remark that $p(r_2)$ belongs to $N^P_{\mathcal{R}(T(x_1))}(x_1)$ following from Theorem 3.1.4 (ii). Thus, by $\varphi_0$-convexity, using Lemma 1.5.3 and (3.24), we deduce

$$\begin{aligned}\langle p(r_2), y - x_1\rangle &\leq \varphi_0\|p(r_2)\||y - x_1|^2\\ &\leq \varphi_0 e^{l(x,r_2)r_2}\|p(0)\|[\|y-x\| + (L\|x\| + K_1)e^{Lr_2}r_2]^2\\ &\leq \varphi_5(x,r)\|\zeta\|(\|y-x\|^2 + r_2^2). \qquad (3.26)\end{aligned}$$

To estimate $\langle \zeta, x_1 - x\rangle$, using the definition of $y(\cdot)$, the definition of $h(\cdot)$, Cauchy-Schwarz inequality, and Lemma 1.5.3, we achieve

$$\begin{aligned}\langle \zeta, x_1 - x\rangle &= \int_0^{r_2}\langle \zeta, y'(s)\rangle ds\\ &= \int_0^{r_2}\langle p(s), f(y(s), w(s))\rangle ds + \int_0^{r_2}\langle p(0) - p(s), f(y(s), w(s))\rangle ds\\ &\leq \int_0^{r_2} h(y(s), p(s))ds + \int_0^{r_2}\|p(s) - p(0)\|\|f(y(s), w(s))\|ds\\ &\leq h(x,\zeta)r_2 + \int_0^{r_2} l(x,r_2)e^{l(x,r_2)r_2}r_2\|p(0)\|(L\|x\| + K_1)e^{Lr_2}ds\\ &= h(x,\zeta)r_2 + l(x,r_2)e^{l(x,r_2)r_2}(L\|x\| + K_1)e^{Lr_2}\|\zeta\|r_2^2\\ &\leq h(x,\zeta)r_2 + \varphi_6(x,r)\|\zeta\|r_2^2. \qquad (3.27)\end{aligned}$$

Combining our observations (3.25)- (3.27), we get

$$\langle \zeta, y - x\rangle \leq h(x,\zeta)r_2 + (\varphi_4(x,r) + \varphi_5(x,r) + \varphi_6(x,r))\|\zeta\|(\|y-x\|^2 + r_2^2),$$

which leads to

$$\langle(\zeta,\alpha), (y - x, \beta - T(x))\rangle \leq \varphi_7(x,r)\|(\zeta,\alpha)\|(\|y-x\|^2 + |\beta - T(x)|^2) \qquad (3.28)$$

for all $T(x) \geq \beta \geq T(y)$. Therefore, it follows from (3.23) and (3.28) that

$$\langle(\zeta,\alpha), (y - x, \beta - \mathcal{T}(x))\rangle \leq \varphi_7(x,r)\|(\zeta,\alpha)\|(\|y-x\|^2 + |\beta - T(x)|^2) \qquad (3.29)$$

for $\beta \geq T(y)$ and $T(y) < T(x)$.

By combining (3.21) and (3.29), we are completely done verifying (3.12). □

We continue this section by presenting some examples in which assumption (Q1) is satisfied for $\varphi_0 = 0$ and for some positive number $\sigma$, i.e., there is some positive number $\sigma$ such that $\mathcal{R}(t)$ is convex for any $t \in [0, \sigma]$. These examples are based on Proposition 3.1 of Colombo, Marigonda, and Wolenski [CMW06], and Theorem 5.1 of Colombo and Nguyen [CN13].

**Example 3.2.2.** *Let $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ be defined by $f(x, u) = Ax + u$ where $U$ is a nonempty compact convex subset of $\mathbb{R}^n$ and $A \in \mathbb{M}_{n\times n}(\mathbb{R})$. If the target $\mathcal{K}$ is a closed convex subset in $\mathbb{R}^n$, then $\mathcal{R}(t)$ is convex for any positive $t$.*

**Example 3.2.3.** *Let $\mathcal{K} = \{0\}$ and let $f : \mathbb{R}^2 \times [-1, 1]^m \to \mathbb{R}^2$, $m = 1, 2$, be defined by $f(x, u) = \ell(x) + g(x)u$ where $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{M}_{2\times m}(\mathbb{R})$ are of class $C^{1,1}$ (with Lipschitz constant $L$) and satisfy*

*(i)* $\ell(o) = o$,

*(ii)* $\operatorname{rank}[g_i(o), D\ell(o)g_i(o)] = 2$, *for* $i = 1, m$ *where* $g = (g_1, g_m)$,

*(iii)* $Dg(o) = o$.

*Then there exists a positive constant $\tau$ depending only on $L$, $f(o)$, and $g(o)$ such that $\mathcal{R}(t)$ is strictly convex for any $t \in [0, \tau]$.*

**Definition 3.2.4.** *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map. We say that $F$ is convex if and only if for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, we have*

$$\lambda F(x) + (1-\lambda)F(y) \subset F(\lambda x + (1-\lambda)y).$$

Here, we give a simple example of a convex multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Define $F(x) := \{Ax + g(x)u : u \in U\}$ where $A$ is an $n \times n$ matrix, $g : \mathbb{R}^n \to (0, \infty)$ is concave, and $U \subset \mathbb{R}^n$ satisfies $tU \subset sU$ if $0 < t \leq s$. For all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned}
\lambda F(x) + (1-\lambda)F(y) &= \lambda(Ax + g(x)U) + (1-\lambda)(Ay + g(y)U) \\
&= A(\lambda x + (1-\lambda)y) + (\lambda g(x) + (1-\lambda)g(y))U \\
&\subset A(\lambda x + (1-\lambda)y) + g(\lambda x + (1-\lambda)y)U = F(\lambda x + (1-\lambda)y).
\end{aligned}$$

Thus $F$ is convex.

The following statement points out that the convexity of the sublevel sets of $T$ for the control system (3.1) is ensured when $F(\cdot) := \{f(\cdot, u) : u \in \mathcal{U}\}$ is convex with $f$ and $\mathcal{U}$ satisfy the assumptions (A1)-(A3).

**Proposition 3.2.5.** *Consider the control system (3.1) with a nonempty closed convex target $\mathcal{K}$, and $f$ and $\mathcal{U}$ satisfy (A1)- (A3). Define $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by $F(x) = \{f(x, u) : u \in \mathcal{U}\}$ for all $x \in \mathbb{R}^n$. If $F$ is convex, then $\mathcal{R}(t)$ is convex for any $t > 0$.*

*Proof.* We fix $T > 0$ and consider the control system

$$\begin{cases} y'(t) & \in & -F(y(t)) \qquad \text{a.e. } t > 0 \\ y(0) & = & x. \end{cases} \tag{3.30}$$

Set

$$\mathcal{A}(T) := \{y(T) : y(\cdot) \text{ solves } (3.30) \text{ with } x \in \mathcal{K}\}.$$

We first show that $\mathcal{A}(T)$ is convex. For this, we note that the set of all trajectories of (3.30) is convex, i.e., if $y_1(\cdot)$ and $y_2(\cdot)$ are trajectories of (3.30) with $y_1(0) = x_1 \in \mathcal{K}$ and $y_2(0) = x_2 \in \mathcal{K}$ then for $\lambda \in [0, 1]$, the curve $y(\cdot) := \lambda y_1(\cdot) + (1 - \lambda)y_2(\cdot)$ is a trajectory of (3.30) with $y(0) = \lambda x_1 + (1 - \lambda)x_2 \in \mathcal{K}$. Indeed, by the convexity of $F$, for a.e. $t > 0$

$$\begin{aligned} y'(t) = \lambda y_1'(t) + (1 - \lambda)y_2'(t) &\in -[\lambda F(y_1(t)) + (1 - \lambda)F(y_2(t))] \\ &\subset -F(\lambda y_1(t) + (1 - \lambda)y_2(t)) = -F(y(t)). \end{aligned}$$

Observe that

(i) $\mathcal{A}(T) \subset \mathcal{R}(T)$,

(ii) $\text{bd}\mathcal{R}(T) \subset \mathcal{A}(T)$.

Since $\mathcal{A}(T)$ is convex, we conclude that $\mathcal{R}(T)$ is convex. Here we use the fact that if $A, B \subset \mathbb{R}^n$, $A \subset B$, $\text{bd}B \subset A$ and $A$ is convex, then $B$ is convex (see [Ngu16]). □

**Corollary 3.2.6.** *Consider the control system (3.1) with a nonempty closed convex target $\mathcal{K}$, and $f$ and $\mathcal{U}$ satisfy (A1)- (A3). Assume that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by $F(x) = \{f(x, u) : u \in \mathcal{U}\}$ for all $x \in \mathbb{R}^n$ is convex. For any $\sigma > 0$, if $T$ is continuous in $\mathcal{S}(\sigma)$, then there exists a continuous function $\varphi$ such that the epigraph of $T_{|\mathcal{S}(\sigma)}$ is $\varphi$-convex.*

The minimum time function $T$ may not be Lipschitz in the case the epigraph of $T$ is $\varphi$-convex. However, as in [CNN14], we can characterize the set of points where the (locally) Lipschitz continuity of $T$ is not guaranteed. We end this section by extending the corresponding results for linear and two dimensional affine control systems and singleton targets given in [CNN14] to more general setting.

**Proposition 3.2.7.** *Assume (A1) - (A3) and (Q1) - (Q2). $T$ is not Lipschitz at $x \in \mathcal{S}(\sigma)$ if and only if there exists $o \neq \zeta \in \mathbb{R}^n$ such that $h(x, \zeta) = 0$ and $\zeta \in N^P_{\mathcal{R}(T(x))}(x)$.*

*Proof.* We note that under our assumptions, according to Theorem 3.2.1, the epigraph of $T|_{S(\sigma)}$ is $\varphi$ - convex. In this case, the proximal normal cone and the Fréchet normal cone to the epigraph of $T|_{S(\sigma)}$ at $(x, T(x))$ with $x \in S(\sigma)$ coincide. By Theorem 9.13 in [RW98], $T$ is not Lipschitz at $x$ if and only if $\partial^\infty T(x) \neq \{o\}$. It is equivalent to, by Theorem 3.1.1,

$$N^P_{\mathcal{R}(T(x))}(x) \cap \{\zeta \in \mathbb{R}^n : h(x, \zeta) = 0\} \neq \{o\}.$$

The proof is complete. ∎

Set

$$\mathcal{S} := \{x \in \mathcal{S}(\sigma) : \exists \zeta \in \mathbb{R}^n, \zeta \neq o \text{ such that } h(x, \zeta) = 0\}. \tag{3.31}$$

We observe that $\mathcal{S}$ is the set of all non-Lipschitz points of $T$ in $\mathcal{S}(\sigma)$. In the next result, we show that $\mathcal{S}$ is invariant for optimal trajectories. This extends Proposition 5.1 in [CNN14] to more general setting with a much shorter proof.

**Proposition 3.2.8.** *Assume (A1) - (A3) and (Q1) - (Q2) and let $\mathcal{S}$ be defined according to (3.31). Then $\mathcal{S}$ is invariant for optimal trajectories.*

*Proof.* We are going to prove that if $x \in \mathcal{S}$ and $y(\cdot)$ is an optimal trajectory for $x$ then $y(t) \in \mathcal{S}$ for all $0 \leq t < T(x)$. Since $x \in \mathcal{S}$, by Proposition 3.2.7, there exists $o \neq \zeta \in \mathbb{R}^n$ such that $h(x, \zeta) = 0$ and $\zeta \in N^P_{\mathcal{R}(T(x))}(x)$. Let $p : [0, T(x)] \to \mathbb{R}^n$ be the solution of the system

$$\begin{cases} p'(t) &= -D_x f(y(t), w(t))^\top p(t) \qquad a.e.\, t \in [0, T(x)] \\ p(T(x)) &= \zeta. \end{cases}$$

Then $p(t) \neq o$ for all $t \in [0, T(x)]$ and by Theorem 3.1.4 we have $p(t) \in N^P_{\mathcal{R}(T(y(t)))}(y(t))$ and $h(y(t), p(t)) = h(x, \zeta) = 0$ for all $t \in [0, T(x)]$. This emplies $y(t) \in \mathcal{S}$ for all $0 \leq t < T(x)$. □

# APPENDIX

Let $p \in (0,1)$, let $\mu$ be a discrete measure on $\mathbb{S}^{n-1}$ such that any open hemisphere has positive measure, and let $G$ be a subgroup in $O(n)$ such that $\mu(\{Au\}) = \mu(\{u\})$ for any unit vector $u$ in $\mathbb{S}^{n-1}$ and any $A \in G$. We review the proof of Theorem 1.3.2 due to Zhu [Zhu15b] to show that for the polytope $P$ with $o \in \operatorname{int} P$ and $S_{P,p} = \mu$, one may even assume that $AP = P$ for every $A \in G$.

We set $\operatorname{supp} \mu = \{u_1, \ldots, u_N\}$ and $\alpha_i = \mu(\{u_i\}) > 0$ for $i = 1, \ldots, N$, and we denote by

$$\mathcal{P}^G(u_1, \ldots, u_N)$$

the family of $n$-dimensional polytopes whose exterior unit normals are among $u_1, \ldots, u_N$ and are $G$ invariant. In particular, if $P \in \mathcal{P}^G(u_1, \ldots, u_N)$ and $A \in G$, then $h_P(Au_i) = h_P(u_i)$ for $i = 1, \ldots, N$.

In order to find the a polytope $P_0 \in \mathcal{P}^G(u_1, \ldots, u_N)$ with $S_{P_0,p} = \mu$, following Zhu [Zhu15b], we consider

$$\Phi_P(\xi) = \int_{\mathbb{S}^{n-1}} h_{P-\xi}^p \, d\mu = \sum_{i=1}^{N} \alpha_i (h_P(u_i) - \langle \xi, u_i \rangle)^p$$

for $P \in \mathcal{P}^G(u_1, \ldots, u_N)$ and $\xi \in P$, and show that the extremal problem

$$\inf \left\{ \sup_{\xi \in P} \Phi_P(\xi) : \ P \in \mathcal{P}^G(u_1, \ldots, u_N) \text{ and } V(P) = 1 \right\}$$

has a solution that is a dilated copy of $P_0$.

According to Lemma 3.1 and Lemma 3.2 in [Zhu15b], if $P \in \mathcal{P}^G(u_1, \ldots, u_N)$, then there exists a unique $\xi(P) \in \operatorname{int} P$ such that

$$\sup_{\xi \in P} \Phi_P(\xi) = \Phi_P(\xi(P)).$$

The uniqueness of $\xi(P)$ yields that

$$A\xi(P) = \xi(P) \ \text{ for } A \in G.$$

We deduce from Lemma 3.3 in [Zhu15b] that $\xi(P)$ is a continuous function of $P$.

Let $\mathcal{P}_N^G(u_1,\dots,u_N)$ be the family of all $P \in \mathcal{P}^G(u_1,\dots,u_N)$ with $N$ facets. Based on Lemma 3.4 and Lemma 3.5 in [Zhu15b], slightly modifying the argument for Lemma 3.6 in [Zhu15b], we deduce the existence of $\widetilde{P} \in \mathcal{P}_N^G(u_1,\dots,u_N)$ with $V(\widetilde{P}) = 1$ such that

$$\Phi_{\widetilde{P}}(\xi(\widetilde{P})) = \inf\left\{\Phi_P(\xi(P)) : P \in \mathcal{P}^G(u_1,\dots,u_N) \text{ and } V(P) = 1\right\}.$$

The only change in the argument in the argument for Lemma 3.6 in [Zhu15b] is making the definition of $P_\delta$ $G$ invariant. So supposing that $\dim F(\widetilde{P}, u_{i_0}) \le n-2$, let $I \subset \{1,\dots,N\}$ be defined by

$$\{Au_{i_0} : A \in G\} = \{u_i : i \in I\}.$$

Therefore, for small $\delta > 0$, we set

$$P_\delta = \{x \in P : \langle x, u_i\rangle \le h_{\widetilde{P}}(u_i) - \delta \text{ for } i \in I\}.$$

The rest of the argument for Lemma 3.6 in [Zhu15b] carries over.

Finally, in the proof of Theorem 4.1 in [Zhu15b], the only necessary change is that for the numbers $\delta_1,\dots,\delta_N$, we assume that for any $A \in G$ and $i \in \{1,\dots,N\}$, if $u_j = Au_i$, then $\delta_j = \delta_i$.

# REFERENCES


[Ale42] A.D. Alexandrov. Existence and uniqueness of a convex surface with a given integral curvature. *(Doklady) Acad. Sci. USSR (N.S.)*, 35:131–134, 1942.

[And99] B. Andrews. Gauss curvature flow: the fate of the rolling stones. *Invent. Math.*, 138:151–161, 1999.

[And03] B. Andrews. Classification of limiting shapes for isotropic curve flows. *J. Amer. Math. Soc.*, 16:443–459, 2003.

[BGMN05] F. Barthe, O. Guédon, S. Mendelson, and A. Naor. A probabilistic approach to the geometry of the $l_p^n$-ball. *Ann. Probab.*, 33:480–513, 2005.

[BH16] K.J. Böröczky and M. Henk. Cone-volume measure of general centered convex bodies. *Adv. Math.*, 286:703–721, 2016.

[BLYZ12] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The log-Brunn-Minkowski inequality. *Adv. Math.*, 231:1974–1997, 2012.

[BLYZ13] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The logarithmic Minkowski problem. *J. Amer. Math. Soc.*, 26:831–852, 2013.

[BLYZ15] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. Affine images of isotropic measures. *J. Differential. Geom.*, 99:407–442, 2015.

[BP07] A. Bressan and B. Piccoli. *Introduction to the Mathematical Theory of Control.* Appl. Math. 2, Amer. Inst. Math. Sci., Springfield, MO, 2007.

[BT17] K. J. Böröczky and H. T. Trinh. The planar $L_p$- Minkowski problem for $0 < p < 1$. *Adv. Appl. Math.*, 87:58–81, 2017.

[Caf90] L. Caffarelli. Interior $W^{2,p}$-estimates for solutions of the Monge-Ampère equation. *Ann. of Math.*, 131(2):135–150, 1990.

[CFS00] P. Cannarsa, H. Frankowska, and C. Sinestrari. Optimality conditions and synthesis for the minimum time problem. *Set-Valued Anal.*, 8:127–148, 2000.

[CG02] S. Campi and P. Gronchi. $L^p$-Busemann-Petty centroid inequality. *Adv. Math.*, 167:128–141, 2002.

[Che06] W. Chen. $L_p$ Minkowski problem with not necessarily positive data. *Adv. Math.*, 201:77–89, 2006.

[Cho85] K. -S. Chou. Deforming a hypersurface by its Gauss-Kronecker curvature. *Comm. Pure Appl. Math.*, 38:867–882, 1985.

[CLSW98] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth Analysis and Control Theory*. Springer, New York, 1998.

[CLYZ09] A. Cianchi, E. Lutwak, D. Yang, and G. Zhang. Affine Moser-Trudinger and Morrey-Sobolev inequalities. *Calc. Var. Partial Differential Equations*, 36:419–436, 2009.

[CLZ] S. Chen, Q.-R. Li, and G. Zhu. The logarithmic Minkowski problem for non-symmetric measures. *Submitted for publication*.

[CM06] G. Colombo and A. Marigonda. Differentiability properties for a class of nonconvex functions. *Calc. Var. Partial Differential Equations*, 25:1–31, 2006.

[CMN15] P. Cannarsa, A. Marigonda, and K.T. Nguyen. Optimality conditions and regularity results for time optimal control problems with differential inclusion. *J. Math. Anal. Appl.*, 427:202–228, 2015.

[CMW06] G. Colombo, A. Marigonda, and P. R. Wolenski. Some new regularity properties for the minimal time function. *J. Control Optim.*, 44:2285–2299, 2006.

[CMW12] P. Cannarsa, F. Marino, and P. R. Wolenski. Semiconcavity of the minimum time function for differential inclusions. *Discrete Contin. Dyn. Syst. Ser. B*, 19:187–206, 2012.

[CN10] G. Colombo and K. T. Nguyen. On the structure of the minimum time function. *SIAM J. Control Optim.*, 48:4776–4814, 2010.

[CN11] P. Cannarsa and K. T. Nguyen. Exterior sphere condition and time optimal control for differential inclusions. *SIAM J. Control Optim.*, 46:2558–2576, 2011.

[CN13] G. Colombo and K. T. Nguyen. On the minimum time function around the origin. *Math. Control Relat. Fields*, 3:51–82, 2013.

[CN15] G. Colombo and L. V. Nguyen. Differentiability properties of the minimum time function for normal linear systems. *J. Math. Anal. Appl.*, 429(1):143–174, 2015.

[CNN14] G. Colombo, K.T. Nguyen, and L.V. Nguyen. Non-Lipschitz points and the SBV regularity of the minimum time function. *Calc. Var. Partial Differential Equations*, 51:439–463, 2014.

[CS95] P. Cannarsa and C. Sinestrari. Convexity properties of the minimum time function. *Calc. Var. Partial Differential Equations*, 3:273–298, 1995.

[CS04] P. Cannarsa and C. Sinestrari. *Semiconcave Functions, Hamilton-Jacobi Equations and Optimal Control*. Birkhauser, Boston, 2004.

[CS15] P. Cannarsa and T. Scarinci. Conjugate times and regularity of the minimum time function with differential inclusions,. *Analysis and Geometry in Control Theory and its Applications, Springer INdAM Ser.*, 11:85–110, 2015.

[CW06] K. -S. Chou and X. J. Wang. The $L_p$-Minkowski problem and the Minkowski problem in centroaffine geometry. *Adv. Math.*, 205:33–83, 2006.

[CY76] S. -Y. Cheng and S. - T. Yau. On the regularity of the solution of the $n$-dimensional Minkowski problem. *Comm. Pure Appl. Math.*, 29:495–561, 1976.

[FN15] H. Frankowska and L.V. Nguyen. Local regularity of the minimum time function. *J. Optim. Theory Appl.*, 164:68–91, 2015.

[Gar06] R. J. Gardner. *Geometric Tomography*. Encyclopedia Math. Appl. (Second Edition). Cambridge University Press, Cambridge, 2006.

[GG02] B. Guan and P. Guan. Convex hypersurfaces of prescribed curvatures. *Ann. of Math.*, 156(2):655–673, 2002.

[GH86] M. Gage and R. Hamilton. The heat equation shrinking convex plane curves. *J. Differential Geom.*, 23:69–96, 1986.

[GL] P. Guan and C.-S. Lin. On equation $\det(u_{ij} + \delta_{ij} u) = u^p f$ on $S^n$. *Preprint*.

[GM77] P. Guan and X. Ma. The Christoffel-Minkowski problem I: Convexity of solutions of a Hessian equation. *Invent. Math.*, 151:2003, 553-577.

[Gru07] P.M. Gruber. *Convex and Discrete Geometry,*, volume 336 of *Grundlehren Math. Wiss.* Springer, Berlin, 2007.

[Hab12] C. Haberl. Minkowski valuations intertwining with the special linear group. *J. Eur. Math. Soc.*, 14:1565–1597, 2012.

[HL69] H. Hermes and J. P. LaSalle. *Functional analysis and time optimal control*. Academic Press, New York-London, 1969.

[HL05] Y. Huang and Q. Lu. On the regularity of the $L_p$-Minkowski problem. *Adv. in Appl. Math.*, 50:268–280, 2005.

[HL14] M. Henk and E. Linke. Cone-volume measures of polytopes. *Adv. Math.*, 253:50–62, 2014.

[HLYZ05] D. Hug, E. Lutwak, D. Yang, and G. Zhang. On the $L_p$ Minkowski problem for polytopes. *Discrete Comput. Geom.*, 33:699–715, 2005.

[HLYZ10] C. Haberl, E. Lutwak, D. Yang, and G. Zhang. The even Orlicz Minkowski problem. *Adv. Math.*, 224:2485–2510, 2010.

[HMS04] C. Hu, X. Ma, and C. Shen. On the Christoffel-Minkowski problem of Firey's p-sum. *Calc. Var. Partial Differential Equations*, 21:137–155, 2004.

[HP14a] C. Haberl and L. Parapatits. The centro-affine Hadwiger theorem. *J. Amer. Math. Soc.*, 27:685–705, 2014.

[HP14b] C. Haberl and L. Parapatits. Valuations and surface area measures. *J. Reine Angew. Math.*, 687:225–245, 2014.

[HS09a] C. Haberl and F. Schuster. Asymmetric affine $L_p$ Sobolev inequalities. *J. Funct. Anal.*, 257:641–658, 2009.

[HS09b] C. Haberl and F. Schuster. General $L_p$ affine isoperimetric inequalities. *J. Differential Geom.*, 83:1–26, 2009.

[HSX12] C. Haberl, F. Schuster, and J. Xiao. An asymmetric affine Pólya-Szegö principle. *Math. Ann.*, 352:517–542, 2012.

[Hui84] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20:237–266, 1984.

[Iva13] M.N. Ivaki. A flow approach to the $L_{-2}$ Minkowski problem. *Adv. in Appl. Math.*, 50:445–464, 2013.

[Jia10] M.Y. Jiang. Remarks on the 2-dimensional $L_p$-Minkowski problem. *Adv. Nonlinear Stud.*, 10:297–313, 2010.

[Kla04] D. Klain. The Minkowski problem for polytopes. *Adv. Math.*, 185:270–288, 2004.

[Lew38] H. Lewy. On differential geometry in the large. I. Minkowski problem. *Trans. Amer. Math. Soc.*, 43:258–270, 1938.

[LO95] E. Lutwak and V. Oliker. On the regularity of solutions to a generalization of the Minkowski problem. *J. Differential Geom.*, 41:227–246, 1995.

[LR10] M. Ludwig and M. Reitzner. A classification of SL(n) invariant valuations. *Ann. of Math.*, 172(2):1219–1267, 2010.

[Lud03] M. Ludwig. Ellipsoids and matrix-valued valuations. *Duke Math. J.*, 119:159–188, 2003.

[Lud10] M. Ludwig. General affine surface areas. *Adv. Math.*, 224:2346–2360, 2010.

[Lut93] E. Lutwak. The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geom.*, 38:131–150, 1993.

[LW13] J. Lu and X.-J. Wang. Rotationally symmetric solution to the $L_p$-Minkowski problem. *J. Differential Equations*, 254:983–1005, 2013.

[LYZ00a] E. Lutwak, D. Yang, and G. Zhang. $L_p$ affine isoperimetric inequalities. *J. Differential Geom.*, 56:111–132, 2000.

[LYZ00b] E. Lutwak, D. Yang, and G. Zhang. A new ellipsoid associated with convex bodies. *Duke Math. J.*, 104:375–390, 2000.

[LYZ02a] E. Lutwak, D. Yang, and G. Zhang. The Cramer-Rao inequality for star bodies. *Duke Math. J.*, 112:59–81, 2002.

[LYZ02b] E. Lutwak, D. Yang, and G. Zhang. Sharp affine $L_p$ Sobolev inequalities. *J. Differential Geom.*, 62:17–38, 2002.

[LYZ04a] E. Lutwak, D. Yang, and G. Zhang. On the $L_p$-Minkowski problem. *Trans. Amer. Math. Soc.*, 356:4359–4370, 2004.

[LYZ04b] E. Lutwak, D. Yang, and G. Zhang. Volume inequalities for subspaces of $L_p$. *J. Differential Geom.*, 68:159–184, 2004.

[LZ97] E. Lutwak and G. Zhang. Blaschke-Santaló inequalities. *J. Differential Geom.*, 47:1–16, 1997.

[Min97] H. Minkowski. Allgemeine Lehrsätze über die konvexen Polyeder. *Gött. Nachr.*, 1897:198–219, 1897.

[Nao07] A. Naor. The surface measure and cone measure on the sphere of $l_p^n$. *Trans. Amer. Math. Soc.*, 359:1045–1079, 2007.

[Ngu10] K. T. Nguyen. Hypographs satisfying an external sphere condition and the regularity of the minimum time function. *J. Math. Anal. Appl.*, 372:611–628, 2010.

[Ngu14] L.V. Nguyen. *On Regular and Singular Points of the Minimum Time Function*. Ph.D. Thesis, Universit'a di Padova, Padua, Italy, 2014.

[Ngu16] L. V. Nguyen. Variational analysis and regularity of the minimum time function for differential inclusions. *SIAM J. Control Optim.*, 54(5):2235–2258, 2016.

[Nir53] L. Nirenberg. The weyl and Minkowski problems in differential geometry in the large. *Comm. Pure Appl. Math.*, 6:337–394, 1953.

[NR03] A. Naor and D. Romik. Projecting the surface measure of the sphere of $l_p^n$. *Ann. Inst. H. Poincaré Probab. Stat.*, 39:241–261, 2003.

[NT] L. V. Nguyen and H. T. Trinh. On the relationship between sublevel sets and the epigraph of the minimum time function. *Submitted for publication*.

[Pao06] G. Paouris. Concentration of mass on convex bodies. *Geom. Funct. Anal.*, 16:1021–1049, 2006.

[Pog78] A.V. Pogorelov. *The Minkowski Multidimensional Problem*. V.H. Winston & Sons, Washington, D.C, 1978.

[PW12] G. Paouris and E. Werner. Relative entropy of cone measures and $L_p$ centroid bodies. *Proc. Lond. Math. Soc.*, 104:253–286, 2012.

[Roc72] R. T. Rockafellar. *Convex Analysis*. Princeton Univesity Press, Princeton, 1972.

[RW98] R. T. Rockafellar and R. J-B. Wets. *Variational Analysis*. Springer, Berlin, 1998.

[Sch14] R. Schneider. *Convex bodies: the Brunn-Minkowski Theory*. Encyclopedia of Mathematics and its Applications (Second Edition). Cambridge University Press, Cambridge, 2014.

[Sta02] A. Stancu. The discrete planar $L_0$-Minkowski problem. *Adv. Math.*, 167:160–174, 2002.

[Sta03] A. Stancu. On the number of solutions to the discrete two-dimensional $L_0$-Minkowski problem. *Adv. Math.*, 180:290–323, 2003.

[Uma03] V. Umanskiy. On solvability of two-dimensional $L_p$-Minkowski problem. *Adv. Math.*, 180:176–186, 2003.

[Wan12] T. Wang. The affine Sobolev-Zhang inequality on $BV(\mathbb{R}^n)$. *Adv. Math.*, 230:2457–2473, 2012.

[WY98] P. R. Wolenski and Z. Yu. Proximal analysis and the minimal time function. *SIAM J. Control Optim.*, 36:1048–1072, 1998.

[Zha99] G. Zhang. The affine Sobolev inequality. *J. Differential Geom.*, 53:183–202, 1999.

[Zhu14] G. Zhu. The logarithmic Minkowski problem for polytopes. *Adv. Math.*, 262:909–931, 2014.

[Zhu15a] G. Zhu. The centro-affine Minkowski problem for polytopes. *J. Differential Geom.*, 101:159–174, 2015.

[Zhu15b] G. Zhu. The $L_p$ Minkowski problem for polytopes for $0 < p < 1$. *J. Func Anal.*, 269:1070–1094, 2015.