Borel-Haefliger type theorems

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Declaration

I hereby declare that

- the dissertation contains no materials accepted for any other degrees in any other institution, and
- that the dissertation contains no materials previously written and/or published by another person except where appropriate acknowledgement is made in the form of bibliographic reference etc.

Budapest, 1 April, 2019 Ákos Matszangosz

Γιαγιά emlékére.

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Contents

Introduction

The focus of this thesis are certain U(1)-manifolds, whose fixed point manifold is half-dimensional. We show that under certain additional conditions, the rational cohomology ring of the space is isomorphic to the cohomology ring of its fixed point set with the degrees halved.

Such a phenomenon was first observed by Borel and Haefliger [BH61], for the $\mathbb{Z}_2$-action of complex conjugation on complex algebraic varieties for mod 2 coefficient cohomology. The Borel-Haefliger theorem (Theorem 3.1.4) states that if $X_{\mathbb{C}}$ is a smooth variety, which is the complexification of the real variety $X_{\mathbb{R}}$, and $H^*(X_{\mathbb{R}};\mathbb{F}_2)$ and $H^*(X_{\mathbb{C}};\mathbb{F}_2)$ are additively generated by real algebraic cycles and their complexifications respectively, then the complexification map is a multiplicative degree doubling isomorphism.

More recently, Hausmann, Holm and Puppe [HHP05] introduced a class of (topological) $\mathbb{Z}_2$-spaces called conjugation spaces with such a degree-halving ring isomorphism

$$\kappa : H^{2*}(X;\mathbb{F}_2) \to H^*(X^\Gamma;\mathbb{F}_2),$$

using equivariant cohomology. The relationship of conjugation spaces to the Borel-Haefliger theorem in terms of geometrically defined cycles was examined and clarified in a paper of van Hamel [VH07].

The other main theorem in the paper of Borel and Haefliger (Theorem 3.2.12) relates equivariant fundamental classes of real and complex singularity loci, also known as Thom polynomials. The theorem states that the Thom polynomial of a complexified singularity locus $\eta^{\mathbb{C}}$ expressed in terms of Chern classes is the same as the Thom polynomial of the real singularity locus $\eta^{\mathbb{R}}$ expressed in terms of Stiefel-Whitney classes - mod 2:

$$tp(\eta^{\mathbb{C}}; \bmod 2) = p(c_*, c'_*; \bmod 2) \qquad \Longleftrightarrow \qquad tp(\eta^{\mathbb{R}}) = p(w_*, w'_*) \tag{1}$$

Our primary objective in this thesis is to obtain analogues of the first Borel-Haefliger theorem and Hausmann, Holm and Puppe's theory of conjugation spaces for U(1)-actions and rational coefficient cohomology. We also prove a theorem similar to (1) but relating (rational) Thom polynomials of real singularities in Pontryagin classes to Thom polynomials of complex singularities in Chern classes.

In this thesis we extend the definition of conjugation spaces to U(1)-spaces and rational coefficient cohomology and we call the resulting spaces *circle spaces* (**Definition 2.1.3**). Hausmann, Holm and Puppe's 'topological' definition of conjugation spaces generalizes without any modifications. However, van Hamel's 'geometric' proof of the Borel-Haefliger theorem requires a slightly different approach. Namely, it requires an interpretation of the main coefficient α_d of the restriction of an equivariant fundamental class $[Z \subseteq X]_\Gamma$ to the fixed point set X^Γ:

$$[Z \subseteq X]_\Gamma|_{X^\Gamma} = \alpha_d u^d + \ldots + \alpha_1 u + \alpha_0,$$

where $H^*_\Gamma(X^\Gamma) \cong H^*(X^\Gamma)[u]$. Our approach uses the equivariant excess intersection formula. By using circle spaces, we obtain the following *generalized Borel-Haefliger theorem* (**Theorem 3.1.1,** see also Remark 3.1.2 iii))

Theorem. *Let $\Gamma = \mathrm{U}(1)$ and let X be a compact oriented Γ-manifold, whose rational cohomology groups are additively generated by good $\mathrm{U}(1)$-invariant cycles $[Z_i]$, such that $\operatorname{codim}_{\mathbb{R}} Z_i = 2\operatorname{codim}_{\mathbb{R}} Z_i^\Gamma$. Assume that the $\mathrm{U}(1)$-equivariant normal bundle $\nu(X^\Gamma \hookrightarrow X)$ has only one weight $\lambda \in \mathbb{Z}$. Then the assignment sending $[Z_i]$ to $[Z_i^\Gamma]$ determines a degree-halving multiplicative isomorphism between $H^{2*}(X;\mathbb{Q})$ and $H^*(X^\Gamma;\mathbb{Q})$.*

Our main examples of circle spaces are real partial flag manifolds $\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$, parametrizing flags of real, even dimensional subspaces (**Theorem 4.2.2**). This allows us to complete the geometric part of the Casian-Kodama conjecture [CK13] about the cohomology ring structure of real Grassmannians. Namely, we obtain a description of the cohomology ring of Grassmannians in terms of fundamental classes of Schubert cycles (**Propositions 4.2.7, 4.2.8** and **4.2.9**). Other examples of circle spaces include quaternionic flag manifolds (**Theorem 4.2.13**).

The definition and properties of conjugation spaces also extend to Sp(1)-actions, we are aware of two such spaces: the octonionic flag manifolds $\mathbb{O}P^2$ and $\mathrm{Fl}(\mathbb{O})$ (**Theorem 4.3.4**). We call conjugation spaces, circle spaces and their Sp(1)-analogues collectively *halving spaces*.

Returning to the context of conjugation spaces, a result of Franz and Puppe [FP06] (Theorem 2.2.10) relates certain equivariant cohomology classes to Steenrod squares. Applying the excess intersection formula yields an alternative proof of this result in the algebraic case (**Proposition 2.2.11**).

One of our motivations for determining the cohomology rings of real flag manifolds in terms of Schubert cycles was to obtain lower bounds in real enumerative geometry. Schubert calculus over the real numbers is still not completely understood, and it is a subject of investigation, see [Sot97], [Vak06], [MTV09], [BL].

In the real case, the number of solutions to a real Schubert problem (Section 4.4) depends on the configuration. An upper bound is given by the number of complex solutions, which is independent of the configuration. A natural goal is to obtain a lower bound. The cohomological product of the Schubert cycles is a signed sum of the solutions, so it gives such a lower bound. This was our motivation to determine the cohomology ring structure of flag manifolds in terms of Schubert cycles. In particular, we get that the number of solutions of a 'double' real Schubert problem is bounded below by the number of the half sized complex one (**Proposition 4.4.1**).

The cohomology ring structure of general real flag manifolds $\mathrm{Fl}_{\mathcal{D}}^{\mathbb{R}}$ in terms of characteristic classes is well-understood by the theory of Cartan (see **Proposition C.2.3**). However, translating this description to Schubert cycles, appears to be not well understood, and nontrivial combinatorially. We attempt to illustrate this in Appendix F. We hope that these computations could be a step towards understanding the cohomology rings of real flag manifolds in terms of Schubert cycles.

We also obtain an extension of the other Borel-Haefliger theorem (1) mentioned above, which we call the *equivariant Borel-Haefliger theorem* (**Theorem 3.2.10**). A simple version is the following:

Theorem. *Let* $\Gamma = \mathrm{U}(1)$ *act on* $G = \times_i \mathrm{GL}(2n_i, \mathbb{R})$ *by inner automorphisms where* $\mathrm{U}(1) \hookrightarrow G$ *is identified with the diagonal circle subgroup, with fixed point set* $G^\Gamma = \times_i \mathrm{GL}(n_i, \mathbb{C})$*. Let* G *and* Γ *act on a real vector space* V *compatibly and let* $Z \subseteq V$ *be a* G*-cycle that is also* Γ*-invariant and* $\dim_{\mathbb{R}} Z = 2\dim_{\mathbb{R}} Z^\Gamma$*. Then the same polynomial expresses* $[Z]_G$ *and* $[Z^\Gamma]_{G^\Gamma}$*, i.e.*

$$[Z]_G = q(p_*) \qquad \Longleftrightarrow \qquad [Z^\Gamma]_{G^\Gamma} = q(c_*)$$

where $H^*_G \cong \mathbb{Q}[p_*]$ *and* $H^*_{G^\Gamma} \cong \mathbb{Q}[c_*]$*, and by* p_* *and* c_* *we abbreviate the set of universal characteristic classes* c^i_j, p^i_j*,* $j = 1, \dots, n_i$*.*

More generally, the equivariant Borel-Haefliger theorem holds for $\Gamma = \mathbb{Z}_2, \mathrm{U}(1), \mathrm{Sp}(1)$ acting on groups G, such that BG is a halving space in a well-defined sense. We call such groups G *halving groups*; for their precise definition, see Definition 3.2.5.

The equivariant Borel-Haefliger theorem can be used to deduce the equivariant fundamental class of matrix Schubert varieties (**Theorem 5.4.3**), the real Thom polynomials of the Thom-Boardman singularities $\Sigma^{2i}_{\mathbb{R}}(2\ell)$ originally computed by Ronga (**Theorem 5.2.1**), and Thom polynomials in equioriented A_n quivers (**Theorem 5.3.6**).

A subtle point which reappears throughout the thesis is that not all real algebraic varieties have fundamental classes over $\mathbb{Z}$. Over $\mathbb{F}_2$, real algebraic varieties are cycles, as was shown by Borel and Haefliger. However, since we will be interested in fundamental classes over $\mathbb{Z}$, cycleness requires careful verification in every case.

The structure of the thesis is the following.

In **Chapter 1**, we review some preliminaries we will use, namely equivariant formality, equivariant localization and the excess intersection formula. We state the excess weight lemma (Lemma 1.2.14), which is the main ingredient in the proof of the generalized Borel-Haefliger theorem.

In **Chapter 2**, we introduce halving spaces, which are a collective term we use for conjugation spaces, circle spaces and their quaternionic analogue. We also prove their main properties (Cor. 2.1.5, 2.1.6, Prop. 2.2.5) and give a sufficient condition for a space to be a halving space (Lemma 2.2.8).

In **Chapter 3**, we state and prove the generalized and equivariant Borel-Haefliger theorems (Thm. 3.1.1 and 3.2.10). We also introduce halving groups (Def. 3.2.5), which are the central objects in the equivariant Borel-Haefliger theorem. We prove the halving bundle lemma (Lemma 3.3.2), which informally states that if $F \to E \to B$ is a fiber bundle whose fiber and base are halving spaces, then E is also a halving space. This is the main ingredient in the proof of the equivariant Borel-Haefliger theorem.

In **Chapter 4**, we apply the generalized Borel-Haefliger theorem to real, quaternionic and octonionic flag manifolds (Thm. 4.2.2, 4.2.13 and 4.3.4) and determine their cohomology ring structure in terms of geometrically defined cycles. We deduce some consequences of these descriptions in enumerative geometry (Prop. 4.4.1, 4.4.3).

In **Chapter 5**, we give examples for halving groups (Ex. 5.1.1, 5.1.2). We apply the equivariant Borel-Haefliger theorem to compute real matrix Schubert varieties, real quiver Thom polynomials and real and quaternionic equivariant Schubert classes (Thm. 5.2.1, 5.3.6, 5.4.3, 5.4.4, Prop. 5.5.1, 5.5.3).

This concludes the main part of the thesis. In the Appendix we include some computations and some more classical material, for the convenience of the reader and to make this work more self-contained.

In **Appendix A** we discuss the notions of fundamental classes of real varieties and stratified spaces we use, and give sufficient conditions for their existence.

In **Appendix B**, we discuss the definition and classification of R-spaces also known as generalized real flag manifolds, since all our examples of halving spaces are instances of R-spaces. We also discuss some of the geometry of R-spaces, for instance their Schubert cell decomposition.

In **Appendix C**, we discuss some aspects of the additive and multiplicative structure of the cohomology of classical real flag manifolds $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$. In particular, we compute the incidence coefficients in the Vassiliev complex (Prop. C.1.8, C.1.11) and the Borel-Cartan type description of $H^*(\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N);\mathbb{Q})$ (Prop. C.2.3). We also determine the rational cycles for $\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$ (Prop. C.1.9)

In **Appendix D** we carry out some elementary representation theoretic computations, that are used in Theorems 4.2.2 and 4.2.13.

In **Appendix E** we discuss equivariant principal bundles in the sense of tom Dieck, as these are central objects in the equivariant Borel-Haefliger theorem.

In **Appendix F** we include some tables listing Schubert generators for the rational cohomology rings of real flag manifolds.

Chapter 1

Preliminaries

There is a class of Γ-spaces called equivariantly formal spaces [GKM97] with the nice property that $H^*_\Gamma(X)$ is a free H^*_Γ-module. One of the necessary conditions for a Γ-space X to be a halving space is equivariant formality. In Section 1.1 we discuss some of the properties of equivariantly formal spaces.

The main ingredient in our proof of the generalized Borel-Haefliger theorem is a lemma we call the excess weight lemma (Lemma 1.2.14), which is a corollary of the (equivariant) Excess Intersection Formula. It gives a geometric description of the main coefficient in the restriction of an equivariant fundamental class to the fixed point set. In Section 1.2, we state the Excess Intersection Formula, deduce the lemma and refer to Quillen's paper [Qui71a] for further details. In this thesis, equivariant cohomology signifies Borel-equivariant singular cohomology $H^*_\Gamma(X;R) = H^*(B_\Gamma X;R)$. We will deal with manifolds, and by manifold and submanifold we always mean *smooth* manifolds and submanifolds. We expect that with enough care, some of the discussion generalizes to topological, even cohomological manifolds, however we will not need this generality.

1.1 Equivariant cohomology

The Leray-Hirsch theorem states that for a fiber bundle $F \to E \to X$, if the restriction to the fiber $H^*(E) \to H^*(F)$ has a section, then it induces an isomorphism $H^*(E) \cong H^*(F) \otimes_R H^*(X)$ of $H^*(X)$-modules. In slightly more technical terms, the degeneration of the Leray-Serre spectral sequence follows from the existence of a section of the edge homomorphism. A Γ-space X is said to be equivariantly formal, if this condition is satisfied for the bundle $X \to B_\Gamma X \to B\Gamma$, in particular $H^*_\Gamma(X) \cong H^*(X) \otimes H^*_\Gamma$ as H^*_Γ-modules.

In general the Leray-Hirsch theorem only gives information about the H^*_Γ-module structure of $H^*_\Gamma(X)$, but not about the ring structure. Halving spaces are a special class of equivariantly formal Γ-spaces X, with the property that even though the action on X is nontrivial, the cohomology ring structure $H^*_\Gamma(X)$ is isomorphic to a tensor product $H^*(X) \otimes_R H^*_\Gamma$. In this section we discuss some sufficient conditions for equivariant formality. By fiber bundles – unless explicitly mentioned– we mean continuous fiber bundles over an arbitrary topological space.

1.1.1 Equivariant formality

Let R be a principal ideal domain. We need the following form of the Leray-Hirsch theorem (e.g. [Hat02, Theorem 4D.1]):

Theorem 1.1.1 (Leray-Hirsch)**.** *Let $p : E \to X$ be a fiber bundle with fiber F over a connected base X. If $H^*(F; R)$ is a finitely generated free R-module in every degree, and $\rho : H^*(E; R) \to H^*(F; R)$ has a graded R-module section σ, then the induced map*

$$\hat{\sigma} : H^*(F; R) \otimes_R H^*(X; R) \to H^*(E; R)$$

is a $H^*(X; R)$-module isomorphism.

A bundle satisfying the conditions of the theorem is called *Leray-Hirsch*, and we will call a section σ as in the theorem a *Leray-Hirsch section*. Such a section σ is by no means unique, and all choices of a section determine an isomorphism as above. The section σ is sometimes also called a *cohomology extension of the fiber* [tD87, III.1.12].

Recall the following well-known sufficient condition, see e.g. [Tu17]:

Proposition 1.1.2. *Let $F \to E \to X$ be a fiber bundle where X is simply connected and the cohomology of F and X are concentrated in even degrees. Then E is Leray-Hirsch and its cohomology is concentrated in even degrees.*

Proof. By simply connectedness of X, the Leray-Serre spectral sequence of E has E_2-page $E_2^{p,q} = H^p(F; H^q(X))$. By evenness, all differentials are zero, the spectral sequence degenerates on the E_2-page, so E is Leray-Hirsch, and therefore has nonzero cohomology only in even degrees. $\square$

Let Γ be a Lie group and let X be a Γ-space. We want to apply the Leray-Hirsch theorem to the Borel construction $B_\Gamma X \to B\Gamma$ which motivates the following definition: A Γ-space X is said to be *equivariantly formal* if $B_\Gamma X \to B\Gamma$ satisfies the condition of the Leray-Hirsch theorem.

A surjective map onto a free R-module always has a section. In case R is a field, every R-module is free. So with field coefficients equivariant formality can also be defined as ρ being surjective [GKM97]. From now on, we will usually make the simplifying assumption that R is a field. The following simple Proposition can be useful:

Proposition 1.1.3. *Let F be a Γ-equivariantly formal space. Then any fiber bundle $E = P \times_\Gamma F \to X$ with fiber F and structure group Γ over a paracompact base X is Leray-Hirsch.*

Proof. Let $\mathcal{K} : X \to B\Gamma$ be the classifying map of P and let $\tilde{\mathcal{K}} : E \to B_\Gamma F$ be the covering map. The restriction $\rho : H^*_\Gamma(F) \to H^*(F)$ factors through $H^*(E)$ via $\tilde{\mathcal{K}}^*$. If σ is a section of ρ, then $\tilde{\mathcal{K}}^* \circ \sigma$ is a section of $H^*(E) \to H^*(F)$. $\square$

A more geometric sufficient condition is given by

Proposition 1.1.4. *Let Γ be a connected Lie group or let the coefficients of cohomology be $R = \mathbb{F}_2$. If the cohomology of the smooth Γ-manifold X is freely generated as an R-module by Γ-invariant cycles, finitely many in each degree, then X is Γ-equivariantly formal.*

Proof. Existence of equivariant classes $[Z]_\Gamma$ is the content of Proposition A.3.2, which provides a Leray-Hirsch section $\sigma : [Z_i] \mapsto [Z_i]_\Gamma$ defined on a system of additive generators $[Z_i]$. $\square$

If Γ is not connected and $R = \mathbb{Q}$, the Proposition generalizes under an extra condition, see Proposition A.3.2 c).

1.1.2 Equivariant localization

If X is a finite dimensional Γ-manifold, then the cohomological restriction map to the fixed point set X^Γ becomes an isomorphism after inverting Euler classes, if

a) $(\Gamma, R) = (\mathrm{U}(1)^k, \mathbb{Q})$, by Borel [Bor60], Quillen [Qui71b],

b) $(\Gamma, R) = (\mathbb{Z}_2^k, \mathbb{F}_2)$ see e.g. [AP93, Corollary 3.1.8.].

This statement is often called *equivariant localization theorem.* There is also a localization theorem for noncommutative compact Lie groups, however one has to be more careful, in particular certain orbit types must be forbidden, see [tD87, p. 192]. For example, if $\Gamma = \mathrm{SU}(2)$, then $\mathrm{SU}(2)/\mathrm{U}(1)$ should not appear as an orbit, which is a rather stringent condition, so the localization is rarely an isomorphism in the nonabelian case.

Proposition 1.1.5. *Let (Γ, R) be one of a) or b) and let X be an equivariantly formal Γ-manifold. Then the restriction $r : H^*_\Gamma(X) \to H^*_\Gamma(X^\Gamma)$ is injective.*

Proof. Localization induces a commutative diagram:

$$\begin{tikzcd}
H^*_\Gamma(X) \arrow[r, "r"] \arrow[d, hook, "l_1"] & H^*_\Gamma(X^\Gamma) \arrow[d, hook, "l_2"] \\
S^{-1}H^*_\Gamma(X) \arrow[r, "S^{-1}r", "\cong"'] & S^{-1}H^*_\Gamma(X^\Gamma)
\end{tikzcd}$$

Since $H^*_\Gamma(X)$ is a free H^*_Γ-module by equivariant formality, l_1 is injective and l_2 is always injective. Commutativity implies that r is also injective. $\square$

1.2 Excess intersection formula

The main tool in our proof of the generalized Borel-Haefliger theorem is the (equivariant) excess intersection formula. Namely, it has a Corollary which we call excess weight lemma, describing the restriction of an equivariant fundamental class $[Z \subseteq X]_\Gamma$ to the fixed point set $[Z \subseteq X]_\Gamma|_{X^\Gamma}$.

In this section we introduce the excess intersection formula and state it in the form that is needed for us. Although we only need these notions in the special case of ordinary singular Borel-equivariant cohomology, Quillen's treatment [Qui71a] being so elegant, we didn't see any disadvantage to state it in its general form.

1.2.1 Excess intersection formula

Whereas cohomology theories are contravariant functors on TOP, oriented cohomology theories h^* have an additional functorial property: for oriented proper maps between smooth manifolds $f : X \to Y$, there is an induced morphism $f_! : h^*(X) \to h^*(Y)$ called *Gysin map*, see Appendix A.4. For cartesian diagrams (transversal intersections), Gysin maps commute with pullbacks. For diagrams that are no longer cartesian, there is a correction term involved which is described by the *excess intersection formula* which holds for the more general case of clean intersections.

Clean intersection, excess bundle

Smooth submanifolds $Y, Z \hookrightarrow X$ are said to *intersect cleanly*, if their intersection $W := Y \cap Z$ is a submanifold and $TY|_W \cap TZ|_W = TW$. The *excess bundle* of a clean intersection is $\eta(Y,Z) := TX|_W/(TY|_W + TZ|_W)$. Denoting the inclusion maps

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, hook, "g"'] & Z \arrow[d, hook, "f"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd} \tag{EIF}$$

the relations $\nu_i|_W \cong \nu_j \oplus \eta$ and $\nu_f|_W \cong \nu_g \oplus \eta$ hold: for clean intersections the defining short exact sequence of η induces

$$0 \longrightarrow \underbrace{(TY|_W + TZ|_W)/TZ|_W}_{\nu_g} \longrightarrow \underbrace{TX|_W/TZ|_W}_{\nu_f|_W} \longrightarrow \eta \longrightarrow 0$$

where the first term is isomorphic to ν_g by the isomorphism theorems. If f, g are cooriented, then there is a unique compatible orientation on η such that $\nu_f|_W = \nu_g \oplus \eta$ as oriented bundles, see Remark A.1.5.

Remark 1.2.1.
- The direct sum orientation depends on the order of $\nu_g \oplus \eta$, so let us adopt this convention.
- If f, i, j are cooriented, then ν_g is orientable, with a unique compatible orientation satisfying

$$\nu_f|_W \oplus \nu_j = \nu_i|_W \oplus \nu_g$$

as oriented bundles. This can be useful for clean intersections $Z^\Gamma = X^\Gamma \cap Z$, as the normal bundles of fixed point sets are often oriented.

The equivariant excess intersection formula

The excess intersection formula describes how Gysin maps commute with pullbacks in the case of clean intersections $W = Y \cap Z$. The nonequivariant proof of the excess intersection formula adapts to oriented cohomology theories (see also Appendix A.4):

Theorem 1.2.2 ((Equivariant) Excess Intersection Formula)**.** *Let $Y, Z \hookrightarrow X$ be Γ-invariant smooth submanifolds which intersect cleanly. Using the notations of* (EIF)*, assume that f, g are cooriented. Then for all $z \in h^*_\Gamma(Z)$,*

$$i^* f_! z = g_!(j^* z \cdot e(\eta))$$

in $h^*_\Gamma(Y, Y \backslash W)$ where η is compatibly oriented with f and g.

Proof. The main observation is that $j^*\nu_f$ splits as Γ-equivariant bundles $\nu_g \oplus \eta$; let $j_0 : \nu_g \to j^*\nu_f$ be the inclusion. The theorem follows from the following diagram

$$\begin{tikzcd}
h^*_\Gamma(W) \arrow[d, "\tau_g(e(\eta)\cdot)"] \arrow[dr, "\tau_{j^*\nu_f}"] \arrow[dd, bend right=60, "g_!(e(\eta)\cdot)"'] & & h^*_\Gamma(Z) \arrow[ll, "j^*"'] \arrow[d, "\tau_f"] \arrow[dd, bend left=60, "f_!"] \\
\tilde{h}^{*+c_f}_\Gamma(\mathrm{Th}(\nu_g)) \arrow[d] & \tilde{h}^{*+c_f}_\Gamma(\mathrm{Th}(j^*\nu_f)) \arrow[l, "j_0^*"'] & \tilde{h}^{*+c_f}_\Gamma(\mathrm{Th}(\nu_f)) \arrow[l, "\mathrm{Th}(j)^*"'] \arrow[d] \\
h^{*+c_f}_\Gamma(Y, Y\backslash W) & & h^{*+c_f}_\Gamma(X, X\backslash Z) \arrow[ll, "i^*"']
\end{tikzcd}$$

whose commutativity follows from naturality of Thom classes and excision, the definitions and the following lemma, which is a simple corollary of the definitions:

Lemma 1.2.3. *Let E^m, F^n be Γ-equivariant complex oriented bundles over X. Then*

$$\begin{tikzcd}
\tilde{h}^{*+m+n}_\Gamma(\mathrm{Th}(E\oplus F)) \arrow[r, "\mathrm{Th}(i)^*"] & \tilde{h}^{*+m+n}_\Gamma(\mathrm{Th}(E)) \\
h^*_\Gamma(X) \arrow[u, "\tau_{E\oplus F}"] \arrow[ur, "\tau_E(e(F)\cdot)"']
\end{tikzcd}$$

commutes where $i : E \to E \oplus F$.

∎

Remark 1.2.4. One can change the orientation of g, then $g_!$ also changes. However then the induced orientation on η also changes, changing $e(\eta)$, so that the right hand side remains invariant.

Applications

Next we apply the excess intersection formula to the fixed point set $Z^\Gamma \hookrightarrow X^\Gamma$ of a Γ-invariant submanifold $Z \hookrightarrow X$:

Corollary 1.2.5. *Let $\Gamma = U(1)$. Let $Z \hookrightarrow X$ be a Γ-invariant oriented smooth submanifold. Then $Z \cap X^\Gamma$ is a clean intersection and all maps in*

$$\begin{tikzcd} Z^\Gamma \arrow[r, hook, "j"] \arrow[d, hook, "g"'] & Z \arrow[d, hook, "f"] \\ X^\Gamma \arrow[r, hook, "i"] & X \end{tikzcd}$$

can be compatibly oriented, and with these orientations

$$i^* f_! z = g_!(j^* z \cdot e(\eta))$$

in $h^*_\Gamma(X^\Gamma, X^\Gamma \backslash Z^\Gamma)$.

Proof. $X^\Gamma \cap Z = Z^\Gamma$ is a clean intersection by the slice theorem [GGK02, Theorem B.24]. Also, $j : Z^\Gamma \hookrightarrow Z$ and $i : X^\Gamma \hookrightarrow X$ have Γ-equivariant normal bundles with no trivial Γ-representations (no zero weights), again by the slice theorem. This induces orientations on j and i. Together with the orientation of f, this induces a compatible orientation on g by Remark 1.2.1. So one can apply the Excess Intersection Formula. □

Remark 1.2.6. It can be shown as in [Qui71a], that the EIF gives localization formulas in generalized equivariant cohomology theories of Atiyah-Bott–Berline-Vergne and Lefschetz-Riemann-Roch type.

Equivariant Euler classes

Let Γ be a compact connected Lie group and from now on we restrict to $h^*_\Gamma := H^*_\Gamma(;R)$. In this section we describe the Γ-equivariant Euler class of an equivariant vector bundle $E \to X$, where Γ acts on X trivially. For such an X, $H^*_\Gamma(X) \cong H^*(X) \otimes_R H^*_\Gamma$ as graded R-algebras. If Γ is connected and E is oriented then $B_\Gamma E \to B_\Gamma X$ is orientable, since by the Leray-Serre spectral sequence

$$H^k_\Gamma(E, E\backslash 0) = H^k(E, E\backslash 0)^C = H^k(E, E\backslash 0),$$

where $C = \Gamma/\Gamma^0 = \{e\}$ is the connected components - for further details see Proposition A.3.2. By this identification $B_\Gamma E \to B_\Gamma X$ has an induced orientation and therefore $e_\Gamma(E)$ exists for $R = \mathbb{Z}$.

Proposition 1.2.7. *Let $E \to X$ be a Γ-equivariant real oriented bundle over a connected smooth manifold X with trivial action. Let V be a fiber of E as a Γ-representation (regarded as a vector bundle over a point $y \in X$) and let $|E|$ denote the real rank of E. Then*

$$e_\Gamma(E) = e_\Gamma(V) \cdot 1 + \deg_\Gamma^{<|E|}, \tag{1.1}$$

in $H^*_\Gamma(X) \cong H^*(X) \otimes_R H^*_\Gamma$. Here $1 \in H^0(X)$, $e_\Gamma(V) \in H^{|E|}_\Gamma \cong H^{|E|}_\Gamma(y)$ and $\deg_\Gamma^{<|E|} \in H^*_\Gamma(X)$ denotes a sum of elements whose H^*_Γ-degree is less than $|E|$.

Proof. Restriction to a point $i^* : H^0(X) \otimes H^{|E|}_\Gamma \to H^0(y) \otimes H^{|E|}_\Gamma$ is injective since X is connected. □

Remark 1.2.8. In the real case, the Euler classes of orientable (but not oriented) bundles inherently contain a sign ambiguity depending on the chosen orientation. In most of our applications this sign does not contain relevant information, so we will not complicate matters by keeping track of the sign. However, see Section D.1 for an orientation convention which makes the signs positive in our applications, see also Remark 2.2.2.

1.2.2 Restricting cycles to the fixed point set

As we have indicated earlier, the main ingredient to the generalized Borel-Haefliger theorem is a formula (excess weight lemma) for the restriction of equivariant fundamental classes to the fixed point set $[Z \subseteq X]_\Gamma|_{X^\Gamma}$. For the definition of topological varieties and invariant cycles, see Appendixes A.1 and A.3.

Excess weights

Let Γ be a compact connected Lie group. Let $\pi : E \to X$ be a Γ-equivariant real oriented vector bundle with trivial Γ-action on the base X. Then the *(multi)set of weights* $W_x(E)$ of $E \to X$ at

$x \in X$ is defined as the multiset $\{e_\Gamma(V_i) \in H^*_\Gamma\}$, where V_i are the irreducible summands of the Γ-representation $\pi^{-1}(x)$. Note that we also allow Γ to be noncommutative, in which case these weights may be of different degrees and could also be zero; see Appendix D.3.3 for $\Gamma = \mathrm{Sp}(1)$. If X is connected, then $W_x(E)$ is independent of x and we simply write $W(E)$. In the following definition, we refer to topological subvarieties and their regular points defined in Definition A.1.7, and the discussion following it.

Definition 1.2.9. Let X be a Γ-manifold and $Z \subseteq X$ be a Γ-invariant topological subvariety. The *excess weight* $w_z \in H^*_\Gamma$ at a regular point fixed by Γ, $z \in Z^\Gamma_R$ is the product of the weights $w_z := \prod_{w \in W_z(\eta)} w$, where η denotes the excess bundle of the (clean) intersection $Z^\Gamma = Z \cap X^\Gamma$.

Remark 1.2.10. i) The set of excess weights $W_z(\eta)$ at a regular fixed point $z \in Z^\Gamma_R$ is the difference of multisets

$$W_z(\eta) = W_z(\nu_X) \backslash W_z(\nu_Z)$$

where $\nu_X = \nu(X^\Gamma \hookrightarrow X)$ and $\nu_Z = \nu(Z^\Gamma_R \hookrightarrow Z_R)$.

ii) If the set of weights is the same $W_x(\nu_X) = \{\lambda\}$ for all $x \in X^\Gamma$, then the excess weight of Z at $z \in Z^\Gamma_R$ is $w_z = \lambda^{r_z}$, where $r_z \cdot \deg(\lambda)$ is the (real) rank of the excess bundle at z.

iii) For $\Gamma = \mathbb{Z}_2$, $R = \mathbb{F}_2$ the same definitions can be given. In this case $w_z = u^k \in \mathbb{F}_2[u]$ always, since the normal bundles of $Z^\Gamma_R \hookrightarrow Z_R$ and $X^\Gamma \hookrightarrow X$ have no zero weights by the slice theorem [GGK02, Theorem B.24].

For $\Gamma = \mathrm{U}(1)$, $R = \mathbb{Q}$, $w_z \neq 0$ always holds, since $e_\Gamma(V) = 0$ iff V contains a trivial subrepresentation which cannot happen by the slice theorem. As a consequence, the real codimension of $X^\Gamma \hookrightarrow X$ is always even.

In the case of Γ nonabelian (e.g. $\Gamma = \mathrm{Sp}(1)$), it might happen that $e_\Gamma(\nu|_x) = 0$ (see Section D.3.3), even though there are no trivial subrepresentations in $\nu := \nu(X^\Gamma \hookrightarrow X)$ by the slice theorem. Nonvanishing of $e_\Gamma(\nu|_x)$ is a necessary condition for the localization theorem to hold. Indeed, using the adjunction formula, it can be shown that the localized pushforward $S^{-1}i_!$ is injective iff $e_\Gamma(\nu|_x)$ is not a zero-divisor for x in each connected component. See also the discussion in Section 1.1.2.

We mention the Atiyah-Bott lemma [AB83, Proposition 13.4] which can be used in the nonabelian situation.

Lemma 1.2.11 (Atiyah-Bott)**.** *Let Γ be a compact Lie group and let $E \to X$ be a Γ-equivariant real vector bundle over connected X. If there exists a subtorus $T \leq \Gamma$ acting trivially on X, with no trivial T-subrepresentations in E_x, then $\mathrm{e}_\Gamma(E)$ is not a zero-divisor in $H^*_\Gamma(X;\mathbb{Q})$.*

Excess weight lemma

For the definition of topological subvarieties and fat nonsingular subsets, we refer to Definition A.1.7. Let Γ be a compact connected Lie group. Let Z be a Γ-invariant topological subvariety with a Γ-invariant fat nonsingular U and singular set Σ. This does not ensure that Z^Γ is a topological subvariety, since it might happen that Σ^Γ has too large dimension. This motivates the following definition, introduced for $\Gamma = \mathbb{Z}_2$ in [VH07]:

Definition 1.2.12. A Γ-invariant closed subset $Z \subseteq X$ is a *good Γ-invariant subvariety of codimension type* (k, l) if

- $Z \subseteq X$ is a topological subvariety of codimension k with Γ-invariant fat nonsingular set U
- $Z^\Gamma \subseteq X^\Gamma$ is a (nonempty) topological subvariety of codimension l with fat nonsingular set U^Γ.

We call such a set U a *Γ-invariant fat nonsingular set*. If in addition $Z \subseteq X$ is a cycle and its excess weight $w = w_z$ is independent of $z \in Z^\Gamma_R$, then we say that Z is a *good Γ-invariant cycle of codimension type* (k, l).

Example 1.2.13. Let $Z \subseteq X$ be a k-codimensional Γ-invariant stratified submanifold with Γ-invariant top stratum Z_k. If Z^Γ has a stratification whose *unique top stratum is* $(Z_k)^\Gamma$, then Z is a good Γ-invariant subvariety of codimension type (k, l) where $l = \mathrm{codim}(Z_k)^\Gamma$. If additionally Z is a Γ-invariant cycle and $(Z_k)^\Gamma$ is connected, then Z is a good Γ-invariant cycle. This will be relevant in the case of real double Schubert varieties, see Theorem 4.2.2. ♣

By Proposition A.3.2, a good Γ-invariant cycle Z represents an equivariant cohomology class $[Z]_\Gamma$.

Lemma 1.2.14 (Excess weight lemma)**.** *Let $\Gamma = \mathrm{U}(1)$ and $R = \mathbb{Q}$ be the coefficients of cohomology, and let $u \in H^2(B\Gamma;\mathbb{Q})$ be a generator. Let X be a Γ-manifold and let $Z \subseteq X$ be a good*

Γ-invariant cycle of codimension type (k,l). Then $Z^\Gamma \subseteq X^\Gamma$ is a cycle and it has a fundamental class $[Z^\Gamma \subseteq X^\Gamma]$ satisfying

$$\begin{aligned} H_\Gamma^*(X) &\xrightarrow{r} H_\Gamma^*(X^\Gamma) \cong H^*(X^\Gamma) \otimes H_\Gamma^* \\ [Z \subseteq X]_\Gamma &\longmapsto w \cdot [Z^\Gamma \subseteq X^\Gamma] + \deg_\Gamma^{<k-l} \end{aligned}$$

where $w \in H_\Gamma^{k-l}$ is the excess weight of Z (Definition 1.2.9) and $\deg_\Gamma^{<k-l}$ denotes a sum of elements whose H_Γ^*-degree is less than $k-l$.

Proof. First, consider the case when $Z \subseteq X$ is a smooth Γ-invariant cycle. Then there is a commutative diagram of equivariant inclusions:

$$\begin{tikzcd} Z^\Gamma \arrow[r, hook, "j"] \arrow[d, hook, "g"'] & Z \arrow[d, hook, "f"] \\ X^\Gamma \arrow[r, hook, "i"] & X \end{tikzcd}$$

By Corollary 1.2.5, $Z^\Gamma = Z \cap X^\Gamma$ is a clean intersection and therefore has an excess bundle η. Since $X^\Gamma \hookrightarrow X$ and $Z^\Gamma \hookrightarrow Z$ are cooriented (their normal bundle is complex via the Γ-action), and by Remark 1.2.1, this induces a compatible orientation of g and therefore on η. By Proposition 1.2.7 the equivariant Euler class can be written as

$$e_\Gamma(\eta) = w + \deg_\Gamma^{<k-l}$$

and by the equivariant EIF and using that $g_!$ is a H_Γ^*-module homomorphism

$$i^* f_! 1 = g_!(e_\Gamma(\eta)) = w \cdot g_! 1 + \deg_\Gamma^{<k-l},$$

proving the claim.

Now let $Z = U \amalg \Sigma$ be a good Γ-invariant cycle of X of codimension (k,l) with Γ-invariant fat nonsingular U. Consider the following commutative diagram:

$$\begin{tikzcd} H_\Gamma^*(X, X \setminus Z) \arrow[r, "s"] \arrow[d, "r"'] & H_\Gamma^*(X \setminus \Sigma, X \setminus Z) \arrow[d, "r_U"] \\ H^*(X^\Gamma, X^\Gamma \setminus Z^\Gamma)[u] \arrow[r, "s_U"] & H^*(X^\Gamma \setminus \Sigma^\Gamma, X^\Gamma \setminus Z^\Gamma)[u] \end{tikzcd}$$

Since $Z^\Gamma \subseteq X^\Gamma$ is an l-codimensional singular subvariety, by definition, $H^i(X^\Gamma, X^\Gamma\backslash Z^\Gamma) = 0$ for $i < l$. So one has

$$r[Z]_\Gamma = \sum_{i=0}^{(k-l)/2} \xi_{k-2i} u^i$$

for $\xi_i \in H^{2i}(X^\Gamma, X^\Gamma\backslash Z^\Gamma)$; by Remark 1.2.10 iii), $k-l$ is even. By commutativity of the diagram,

$$\sum_{i=0}^{(k-l)/2} s_U(\xi_{k-2i})u^i = s_U r[Z]_\Gamma = r_U s[Z]_\Gamma = r_U[U]_\Gamma = w \cdot [U^\Gamma] + \deg_\Gamma^{<k-l} \tag{1.2}$$

for a fundamental class $[U^\Gamma]$, by the first case, since $U \hookrightarrow X\backslash\Sigma$ is smooth. Therefore $s_U(\xi_l)u^{(k-l)/2} = w \cdot [U^\Gamma]$. Write $w = \mu u^{(k-l)/2}$, then $[Z^\Gamma] := \xi_l/\mu$ is a fundamental class with the required property ($\mu \neq 0$ by Remark 1.2.10,iii)). □

Remark 1.2.15. i) If $R = \mathbb{Z}$, then the proof also shows that equation (1.2) holds, but then ξ_l might not be divisible by μ, and therefore $Z^\Gamma \subseteq X^\Gamma$ is not necessarily a cycle.

ii) Let $Z \subseteq X$ be a good Γ-invariant subvariety and a cycle. If $W(\nu(X^\Gamma \subseteq X))$ consists of a single weight λ (with multiplicity), then the excess weight is $\lambda^{d/2}$, independently of $z \in Z_R^\Gamma$. Therefore Z is a *good* Γ-invariant cycle.

iii) In the proof we used that $0 \notin W(\nu(X^\Gamma \hookrightarrow X))$, which always holds for $\Gamma = \mathrm{U}(1)$ as indicated in Remark 1.2.10. For $\Gamma = \mathrm{Sp}(1)$, this is not necessarily true; for example $\mathrm{Gr}_k(\mathbb{H}^n) \hookrightarrow \mathrm{Gr}_{4k}(\mathbb{R}^{4n})$ does not satisfy this. If one assumes $w \neq 0$, and that $Z^\Gamma \subseteq X^\Gamma$ is a cycle, the Lemma also holds for $\Gamma = \mathrm{Sp}(1)$, $D = 4$.

iv) For $\Gamma = \mathbb{Z}_2$, $R = \mathbb{F}_2$ and $D = 1$, this proves Corollary 1.3 in [VH07]. In this case $w = u^{k-l}$ always as we remarked earlier. The proof in [VH07] uses the localization theorem. Since the localization theorem also holds for $\Gamma = \mathrm{U}(1)$, that proof could be adapted to our situation, however that would only prove that the main coefficient is $\mu[Z^\Gamma]$ for some nonzero μ (which in the $\mathbb{Z}_2$-case amounts to the same). This description also gives a geometric interpretation of μ. We used the EIF, which is in fact related to the localization theorem [Qui71a, PT07].

Chapter 2

Halving spaces

In this section we extend the definition of conjugation spaces [HHP05] to $\Gamma = \mathrm{U}(1)$ and $\mathrm{Sp}(1)$-actions and cohomology with $\mathbb{Z}$ or $\mathbb{Q}$-coefficients. We separate the discussion into Topology and Geometry: the first concerns general properties of halving spaces and the second one concerns how the halving space structure respects geometrically defined cycles.

For the topology part, most of the proofs of Hausmann, Holm and Puppe [HHP05] generalize word by word to halving spaces, we give a discussion for the sake of completeness. Van Hamel [VH07] gave a proof of the Borel-Haefliger theorem using conjugation spaces (see Theorem 3.1.4). Adapting the proof of van Hamel to the context of circle spaces shows that $\kappa[Z] = \mu[Z^\Gamma]$, where μ is an undetermined constant. This constant is explicitly described by the excess intersection formula as described in Chapter 1, and allows us to generalize the Borel-Haefliger theorem to U(1) and Sp(1)-actions.

2.1 Topology

2.1.1 Definition

Definition 2.1.1. Let Γ be a topological group, and R be a ring. We say that (Γ, R) is a *halving pair*, if $H^*(B\Gamma; R) \cong R[u]$ with $u \in H^D(B\Gamma; R)$, some $D \in \mathbb{N}_+$.

Example 2.1.2. [See also Steenrod's polynomial realization problem [Ste61]]

- $(\mathbb{Z}_2, \mathbb{F}_2)$ is a halving pair with $D = 1$, since $H^*(B\mathbb{Z}_2; \mathbb{F}_2) \cong \mathbb{F}_2[u]$ for $u = w_1(S)$ where $S \to \mathbb{R}P^\infty$ is the tautological bundle.
- $(\mathrm{U}(1), \mathbb{Z})$ is a halving pair with $D = 2$, since $H^*(B\,\mathrm{U}(1); \mathbb{Z}) \cong \mathbb{Z}[u]$ for $u = c_1(S)$ where $S \to \mathbb{C}P^\infty$ is the tautological bundle.
- $(\mathrm{Sp}(1), \mathbb{Z})$ is a halving pair with $D = 4$, since $H^*(B\,\mathrm{Sp}(1); \mathbb{Z}) \cong \mathbb{Z}[u]$, for $u = q_1(S)$ where $S \to \mathbb{H}P^\infty$ is the tautological bundle and q_i denotes the quaternionic Pontryagin class.

♣

From now on, throughout this section fix a halving pair

$$(\Gamma, R) \in \{(\mathbb{Z}_2, \mathbb{F}_2), (\mathrm{U}(1), \mathbb{Q}), (\mathrm{Sp}(1), \mathbb{Q})\}$$

and the corresponding D, and take all cohomologies with R-coefficients. To simplify the discussion we work with field coefficients, although some of the discussion generalizes to $R = \mathbb{Z}$, especially in this section.

Halving spaces

Let (Γ, R) be a halving pair, $u \in H^D(B\Gamma; R)$ the generator. By adapting Hausmann, Holm and Puppe's definition of conjugation spaces to halving pairs, we obtain the central definition of this thesis:

Definition 2.1.3. [Halving space] Let X be a Γ-space, and denote by X^Γ its fixed point set. Assume X has nonzero cohomology only in $2Di$ degrees, and that there exists a Leray-Hirsch section $\sigma : H^*(X) \to H^*_\Gamma(X)$ satisfying the following *degree condition*:

(DC) : for all $x \in H^{2Di}(X)$, $r(\sigma(x))$ is a polynomial in u of degree *exactly* i where $r : H^*_\Gamma(X) \to H^*(X^\Gamma)[u]$ is the restriction map, $u \in H^D_\Gamma$.

A Γ-space X satisfying these conditions is called a *halving space*.

Let us unravel this definition. The halving space structure involves the following maps, which will be used in the following:

$$\begin{tikzcd} H^*_\Gamma(X) \arrow[r, "r"] \arrow[d, "\rho"] & H^*(X^\Gamma)[u] \arrow[d] \\ H^*(X) \arrow[r] \arrow[u, bend left, "\sigma"] & H^*(X^\Gamma) \end{tikzcd}$$

Let

$$\kappa : H^{2*}(X) \to H^*(X^\Gamma) \tag{2.1}$$

be the degree halving R-module homomorphism $\kappa(x) := \deg^i_u(r\sigma(x))$ for $x \in H^{2Di}(X)$, where $u \in H^D_\Gamma$. The pair (κ, σ) is called a *cohomology frame*.

With this notation, the degree condition (DC) means that κ is injective and that for $x \in H^{2Di}(X)$

$$r\sigma(x) = \kappa(x)u^i + \lambda_1 u^{i-1} + \ldots + \lambda_{i-1}u + \lambda_i, \tag{2.2}$$

where $\lambda_j \in H^{D(i+j)}(X^\Gamma)$. We call equation (2.2) *restriction equation*, it is called *conjugation equation* in [HHP05]. For $(\Gamma, R) = (\mathbb{Z}_2, \mathbb{F}_2)$, the definition of halving spaces is the same as the definition of conjugation spaces in [HHP05], except that we don't require κ to be surjective.

It would be more precise to call a halving space a (Γ, R)-halving space, however when (Γ, R) is fixed, we simply say X is a halving space. We consider halving spaces for the following halving pairs (Γ, R):

- Hausmann-Holm-Puppe's conjugation spaces [HHP05] for the halving pair $(\mathbb{Z}_2, \mathbb{F}_2)$.
- *Circle spaces* for the halving pair $(\mathrm{U}(1), \mathbb{Q})$. This is the main case we will consider.
- *Quaternionic halving spaces* for the halving pair $(\mathrm{Sp}(1), \mathbb{Q})$.

Since the Borel-Haefliger theorem only involves spaces and not pairs of spaces, we do not discuss the relative version of conjugation/halving spaces, which is discussed in [HHP05].

2.1.2 Main properties

To motivate the following discussion, we list some of the nice properties of halving spaces:

- κ is a degree-halving ring homomorphism,
- σ is a ring homomorphism, therefore the Leray-Hirsch isomorphism induced by σ is a H^*_Γ-algebra isomorphism,
- the cohomology frame (κ, σ) is unique.

The proof of these properties relies on the following lemma, which is implicitly used in [HHP05], its proof is the same, we repeat it for the sake of completeness.

Lemma 2.1.4 (Degree Lemma)**.** *Let X be a halving space with cohomology frame (κ, σ). Let D denote the degree of the generator $u \in H^*_\Gamma$. Then for $x \in H^{2Dk}_\Gamma(X; R)$*

$$x \in \operatorname{Im} \sigma \iff \deg_u(rx) = k$$

Proof. The direction $\Rightarrow$ holds by definition. For the other direction, let $x \in H^{2Dk}_\Gamma(X; R)$, and assume $x \notin \operatorname{Im} \sigma$. By the Leray-Hirsch theorem

$$x = \sum_{i=0}^{k} \sigma(\xi_i) u^{2(k-i)}$$

for some $\xi_i \in H^{2Di}(X)$. Since r is an $R[u]$-module morphism,

$$rx = \sum_{i=0}^{k} (r\sigma\xi_i) u^{2(k-i)} = \sum_{i=0}^{k} p_i(u) u^{2(k-i)}$$

where $r\sigma\xi_i = p_i(u) \in H^*(X^\Gamma)[u]$ is a polynomial in u of degree $\leq i$. Then $p_i(u)u^{2(k-i)}$ has degree $\leq 2k - i$ for each i. Take the smallest $0 \leq i < k$, such that $\xi_i \neq 0$. Then $r\sigma\xi_i = \kappa(\xi_i)u^i + ...$ and since κ is injective, $r\sigma\xi_i$ has degree i. It follows that rx is a polynomial in u of degree $2k - i$. Since $i < k$, this is a contradiction, since rx is a polynomial of degree k by assumption. Therefore $\xi_i = 0$ for $i < k$ and $x = \sigma(\xi_k)$.

□

In particular, by using the Leray-Hirsch theorem, this implies that if $x \in H^{2Dk}_\Gamma(X; R)$, and $\deg_u(rx) < k$, then $x = 0$.

Corollary 2.1.5. *Let X be a halving space. Then κ and σ are multiplicative.*

Proof. Let $a \in H^{2Dk}(X)$, $b \in H^{2Dl}(X)$. Set $x := \sigma(a)\sigma(b)$, note that $\rho(x) = ab$. Then

$$rx = r(\sigma(a))r(\sigma(b)) = (\kappa(a)u^k + ...)(\kappa(b)u^l + ...)$$

so $x = \sigma(y)$ for some $y \in H^{2D(k+l)}(X)$ by the degree lemma. Since

$$ab = \rho(x) = \rho(\sigma(y)) = y,$$

so $x = \sigma(y) = \sigma(ab)$ and by definition $x = \sigma(a)\sigma(b)$ proving multiplicativity of σ. Using

$$r\sigma(ab) = r(\sigma(a))r(\sigma(b)),$$

the degree $k+l$ part of the left hand side is $\kappa(ab)$ and on the right hand side $\kappa(a)\kappa(b)$. □

In particular, multiplicativity of σ implies that the Leray-Hirsch isomorphism induced by σ

$$H^*_\Gamma(X) \cong H^*(X) \otimes_R H^*_\Gamma$$

is a ring isomorphism.

Corollary 2.1.6 (Naturality)**.** *Let X, Y be halving spaces with some cohomology frames (κ_X, σ_X) and (κ_Y, σ_Y) for X and Y respectively. If $f : X \to Y$ is a Γ-equivariant map, then*

$$\sigma_X \circ H^* f = H^*_\Gamma f \circ \sigma_Y : H^*(Y) \to H^*_\Gamma(X)$$

and

$$\kappa_X \circ H^* f = H^* f^\Gamma \circ \kappa_Y : H^{2*}(Y) \to H^*(X^\Gamma)$$

The proof proceeds similarly as the proof of multiplicativity using the degree lemma. Note that naturality implies uniqueness of the cohomology frame (κ, σ).

We will also need the following proposition. Its proof in the case of conjugation spaces can be found in [HHP05, Proposition 4.6] and it generalizes verbatim to halving spaces.

Proposition 2.1.7. *Let (X_i, f_{ij}) be a direct system of halving spaces which are T_1 and f_{ij} are Γ-equivariant inclusions. Then $X = \varinjlim_i X_i$ is a halving space with cohomology frame $(\varprojlim \kappa_i, \varprojlim \sigma_i)$.*

2.1.3 Olbermann's definition

Let $\Gamma = \mathbb{Z}_2$. Olbermann [Olb07, Proposition 2.1.1.] gave the following equivalent definition of conjugation spaces: A Γ-space X is a conjugation space iff

$$q \circ r : H^*_\Gamma(X) \to H^*(X^\Gamma)[u] \Big/ \bigoplus_{j>k} H^j(X^\Gamma) \cdot u^k$$

is an additive isomorphism, where q denotes the quotient map.

Whereas the original definition involves the existence of a section σ satisfying the degree condition (DC), this definition has the advantage that it is intrinsic, in the sense that the condition does not involve the existence of an additional structure. This definition can also be adapted to halving spaces. However, since for us σ and κ have geometric meaning, we use the original definition.

2.2 Geometry

The motto of this section is 'halving spaces respect geometry'. In this section we restrict our attention to *halving manifolds*, smooth manifolds which are halving spaces. In the context of conjugation manifolds, many of their properties can be found in [HH11, Section 2.7]. In this section let (Γ, R) be $(\mathbb{Z}_2, \mathbb{F}_2)$ or $(\mathrm{U}(1), \mathbb{Q})$ and fix the corresponding degree $D = 1, 2$. Most of the results also hold for $(\mathrm{Sp}(1), \mathbb{Q})$ under additional assumptions, see Remarks 2.2.6 and 2.2.9 iii).

2.2.1 Halving cycles

The original Borel-Haefliger theorem (Theorem 3.1.4) states that for a smooth complexified projective algebraic variety $X_\mathbb{C}$, under certain conditions, the complexification map $[Z_\mathbb{R}] \mapsto [Z_\mathbb{C}]$ from $H^*(X_\mathbb{R}; \mathbb{F}_2) \to H^{2*}(X_\mathbb{C}; \mathbb{F}_2)$ is a multiplicative isomorphism. Van Hamel [VH07] showed that under the conjugation action, $X_\mathbb{C}$ is a conjugation space and that κ is the inverse of the complexification map (see Theorem 3.1.4). In particular, $\kappa[Z] = [Z^\Gamma]$ if Z is the complexification of a real subvariety. Let us consider the question whether such a statement holds for circle spaces, and simultaenously introduce our first example of a circle space:

Example 2.2.1. Let $X = \mathrm{Gr}_2(\mathbb{R}^{2n})$. Consider the $\Gamma = \mathrm{U}(1)$-action on $\mathbb{R}^{2n}$ by identifying it with $\mathbb{C}^n$ and acting by complex multiplication. This induces an action on X. With this action, X is a circle space with fixed point set $X^\Gamma = \mathbb{C}P^{n-1}$. ♣

Proof. Since the U(1)-invariant subspaces are exactly the complex subspaces, X^Γ can be identified with $\mathrm{Gr}_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$ (for more details, see Section 4.1.2). In terms of characteristic classes, the ring structure can be written as

$$H^*(\mathrm{Gr}_2(\mathbb{R}^{2n});\mathbb{Q}) = \mathbb{Q}[x]/x^n, \qquad H^*(\mathbb{C}P^{n-1};\mathbb{Q}) = \mathbb{Q}[y]/y^n$$

where $x = p_1(S_\mathbb{R})$, $y = c_1(S_\mathbb{C})$, and $S_\mathbb{R} \to X$, $S_\mathbb{C} \to \mathbb{C}P^{n-1}$ are the tautological quotient bundles (see e.g. Proposition C.2.3). Let σ be defined on the additive generators by $\sigma(x^i) := (p_1^\Gamma(S_\mathbb{R}) - u^2)^i$, where $p_*^\Gamma(S_\mathbb{R}) = p_*(B_\Gamma S_\mathbb{R} \to B_\Gamma X)$ is the equivariant Pontryagin class. This σ is a Leray-Hirsch section, and it satisfies the degree condition (DC), which can be shown by the following computation. First,

$$B_\Gamma(S_\mathbb{R})|_{X^\Gamma} = B_\Gamma(S_\mathbb{R}|_{X^\Gamma}) = B_\Gamma S_\mathbb{C}.$$

Since Γ acts on $S_\mathbb{C}$ by complex multiplication, we can rewrite it as the tensor product of equivariant bundles $S_\mathbb{C} = S_\mathbb{C}^0 \otimes_\mathbb{C} \mathbb{C}^{tw}$, where $S_\mathbb{C}^0$ denotes $S_\mathbb{C}$ with the trivial action and $\mathbb{C}^{tw}$ denotes the trivial bundle, with the nontrivial Γ-action given by complex multiplication. Then as bundles over $B_\Gamma X^\Gamma = B\Gamma \times X^\Gamma$,

$$B_\Gamma S_\mathbb{C} = S_\mathbb{C}^0 \otimes_\mathbb{C} \tau,$$

where $B_\Gamma \mathbb{C}^{tw} = \tau \to \mathbb{C}P^\infty$ is the tautological bundle and we omit the notation for the pullbacks to the product. Therefore

$$rp_i^\Gamma(S_\mathbb{R}) = p_i(B_\Gamma S_\mathbb{C}) = (-1)^i c_{2i}((S_\mathbb{C}^0 \otimes_\mathbb{C} \tau) \otimes_\mathbb{R} \mathbb{C})$$

and

$$c_*((S_\mathbb{C}^0 \otimes_\mathbb{C} \tau) \otimes_\mathbb{R} \mathbb{C}) = c_*(S_\mathbb{C}^0 \otimes_\mathbb{C} \tau) c_*(\overline{S_\mathbb{C}^0 \otimes_\mathbb{C} \tau}) = (1 + y + u)(1 - y - u),$$

so

$$r\sigma(x^i) = r(p_1^\Gamma(S_\mathbb{R}) - u^2)^i = ((y+u)^2 - u^2)^i = (2yu + y^2)^i,$$

hence σ satisfies the degree condition, and (κ, σ) is a cohomology frame with $\kappa(x^i) = 2^i y^i$.

What happens to cycles? Fix a complete flag $F_\bullet$ in $\mathbb{R}^{2n}$ such that F_{2i} is U(1)-invariant for all i; then $F_\bullet^{\mathbb{C}} = (F_0 \leq F_2 \leq \ldots \leq F_{2n})$ is a complex flag. Let $Z = \sigma_{\boxplus}(F_\bullet)$. Its Γ-fixed point set is

$$Z^\Gamma = \{L \in \mathbb{C}P^1 : L \leq F_{n-1}^{\mathbb{C}}\} = \sigma_{\square}^{\mathbb{C}}(F_\bullet^{\mathbb{C}})$$

Similarly, for $Z = \sigma_{(2i,2i)}^{\mathbb{R}}(F_\bullet)$, Z^Γ is the set of complex lines contained in $F_{n-i}^{\mathbb{C}}$, so $Z^\Gamma = \sigma_i^{\mathbb{C}}(F_\bullet^{\mathbb{C}})$. If $Q_{\mathbb{R}} \to X$ and $Q_{\mathbb{C}} \to X^\Gamma$ denote the tautological quotient bundles,

$$[\sigma_{(2i,2i)}^{\mathbb{R}}] = p_i(Q_{\mathbb{R}}) = (-x)^i, \qquad [\sigma_i^{\mathbb{C}}] = c_i(Q_{\mathbb{C}}) = (-y)^i, \tag{2.3}$$

Summarizing, $\kappa[Z] = 2^i[Z^\Gamma]$.

□

Remark 2.2.2. i) In equation (2.3), there is a sign ambiguity depending on the orientation of $X = \mathrm{Gr}_{2k}(\mathbb{R}^{2n})$ chosen. Under appropriate orientation conventions, e.g. the lexicographical ordering of the tangent bundle of X the equation holds, see Section D.1 for some further details on orientations.

ii) This proof generalizes to $\mathrm{Gr}_{2k}(\mathbb{R}^{2n})$ by a similar computation. We will prove this independently by using the generalized Borel-Haefliger theorem in Theorem 4.2.2.

This example illustrates that in the case of circle spaces it is no longer true that $\kappa[Z] = [Z^\Gamma]$, but there is a constant which appears. This constant is in fact an excess weight, as we will explain below, but let us first examine more generally how the cohomology frame behaves with respect to Γ-invariant cycles. Let X be a halving manifold with cohomology frame (κ, σ). Let $Z \subseteq X$ be a Γ-invariant cycle. We consider the following question:

When is $\kappa[Z] = \mu[Z^\Gamma]$ for some $0 \neq \mu \in R$ and $\sigma[Z] = [Z]_\Gamma$?

In case $[Z] \neq 0$, an immediate necessary condition for $\kappa[Z] = \mu[Z^\Gamma]$ is that Z^Γ has half the codimension of Z, as was the case in the example above. We introduce a definition:

Definition 2.2.3. Let X be a Γ-manifold. A good Γ-invariant cycle $Z \subseteq X$ (Definition 1.2.12) of codimension type $(2k, k)$ is called a *halving cycle.*

Remark 2.2.4. For $\Gamma = \mathrm{U}(1)$, a halving cycle Z has codimension divisible by 4. Indeed, the codimension of $Z \subseteq X$ has the same parity as the codimension of $Z^\Gamma \subseteq X^\Gamma$, by Remark 1.2.10, iii).

To answer our question, it turns out that being a halving cycle is sufficient:

Proposition 2.2.5. *Let (Γ, R) be $(\mathbb{Z}_2, \mathbb{F}_2)$ or $(U(1), \mathbb{Q})$. Let X be a halving manifold, and $Z \subseteq X$ be a halving cycle. Then*

$$\sigma[Z] = [Z]_\Gamma, \qquad \kappa[Z] = \mu[Z^\Gamma],$$

where $w = \mu u^k$ is the excess weight of Z (Definition 1.2.9) and Z has codimension $2Dk$, $u \in H^D_\Gamma$.

Proof. By the excess weight lemma (Lemma 1.2.14)

$$r[Z]_\Gamma = w \cdot [Z^\Gamma] + \deg_\Gamma^{<Dk}, \tag{2.4}$$

where $\deg_\Gamma^{<Dk}$ denotes a polynomial in u of degree less than k.

By the degree lemma (Lemma 2.1.4) $[Z]_\Gamma = \sigma(x)$ for some $x \in H^{2Dk}(X)$. Since

$$[Z] = \rho[Z]_\Gamma = \rho\sigma(x) = x,$$

$[Z]_\Gamma = \sigma(x) = \sigma[Z]$. Restricting, using the definition of κ (2.1) and (2.4),

$$\kappa[Z]u^k + \deg_\Gamma^{<Dk} = r\sigma[Z] = r[Z]_\Gamma = w \cdot [Z^\Gamma] + \deg_\Gamma^{<Dk}$$

therefore $\kappa[Z] = \mu[Z^\Gamma]$, where $w = \mu u^k$ is the excess weight of Z. □

Remark 2.2.6. The lemma also holds for the halving pair $(\Gamma, R) = (\mathrm{Sp}(1), \mathbb{Q})$, if one assumes that $Z^\Gamma \subseteq X^\Gamma$ is a cycle and that $w \neq 0$, see Remark 1.2.15.

We conclude by emphasizing that $\sigma[Z] = [Z]_\Gamma$ and $\kappa[Z] = \mu[Z^\Gamma]$, $\mu \neq 0$ does not hold if Z is not of codimension type $(2k, k)$. As we have already remarked, if $[Z] \neq 0$ then $\kappa[Z] = \mu[Z^\Gamma]$ forces Z to have codimension type $(2k, k)$. However, even $[Z] = 0$ does not imply $[Z]_\Gamma = 0$ nor $[Z^\Gamma] = 0$. This can already be seen for the class of the fixed point set $Z = X^\Gamma = \mathbb{R}P^1$ in the conjugation space $X = \mathbb{C}P^1$.

In the algebraic case, complexified cycles $Z_\mathbb{C}$, are $\mathbb{Z}_2$-halving cycles over $R = \mathbb{F}_2$, see Lemma 3.1.3.

2.2.2 Poincaré duality

In this section we give some sufficient conditions for a Γ-space X to be a halving space.

Definition 2.2.7. We say that a Γ-space X is *almost a halving space*, if X has nonzero cohomology only in degrees $2Di$ and X is equivariantly formal with a Leray-Hirsch section $\sigma : H^*(X) \to H^*_\Gamma(X)$ satisfying a weaker form of the degree condition:

(DC-) for all $x \in H^{2Di}(X)$, $r\sigma(x)$ is a polynomial of degree *at most* i where $r : H^*_\Gamma(X) \to H^*(X^\Gamma)[u]$ is the restriction map, $u \in H^D_\Gamma$.

To put it simply, (DC-) allows the u-degree of $r\sigma(x)$ to be smaller than i. The following lemma can be found (implicitly) in van Hamel [VH07] for the case of conjugation spaces and its proof is the same. For $(\Gamma, R) = (U(1), \mathbb{Q})$ we also have to assume Poincaré duality/orientability:

Lemma 2.2.8 (Injectivity lemma)**.** *Let* $(\Gamma, R) = (\mathrm{U}(1), \mathbb{Q})$*,* $H^*_\Gamma \cong \mathbb{Q}[u]$*,* $u \in H^D_\Gamma$*,* $D = 2$*. Let* X *be a smooth* Γ*-manifold which is almost a halving space with* σ*. If* X *is compact, orientable and* $\dim X \geq 2 \dim X^\Gamma$*, then* X *satisfies the degree condition (DC). In particular,* X *is a halving space with the same* σ*, and* κ *defined by* (2.1) *is an isomorphism.*

Proof. Since X has nonzero cohomology only in degrees $2Di$, $\dim X = 2Dn$. Let $\kappa : H^{2*}(X) \to H^*(X^\Gamma)$ be the degree halving R-module homomorphism $\kappa(x) := \deg_u^k(r\sigma x)$ for $x \in H^{2Dk}(X)$. To show (DC), it is enough to show that κ is injective.

Assume $0 \neq \beta \in \ker \kappa \cap H^{2Dk}(X)$. By Poincaré duality, there exists $\gamma \in H^{2D(n-k)}(X)$, such that $\beta\gamma \neq 0$. Since

$$\rho(\sigma(\beta)\sigma(\gamma)) = (\rho\sigma(\beta))(\rho\sigma(\gamma)) = \beta\gamma \neq 0,$$

so $\sigma(\beta)\sigma(\gamma) \neq 0$. On the other hand, by (DC-) and the definition of κ:

$$r\sigma(\beta) = \kappa(\beta)u^k + \eta_{k+1}u^{k-1} + \ldots + \eta_{2k}$$

for some $\eta_i \in H^{Di}(X^\Gamma)$, where $\kappa(\beta) = 0$ by assumption, and

$$r\sigma(\gamma) = \xi_{n-k}u^{n-k} + \xi_{n-k+1}u^{n-k-1} + \ldots + \xi_{2(n-k)}$$

for some $\xi_i \in H^{Di}(X^\Gamma)$. Then

$$r(\sigma(\beta)\sigma(\gamma)) = r\sigma(\beta)r\sigma(\gamma) = 0$$

since multiplying the two equations, all u-coefficients are in $H^{>Dn}(X^\Gamma) = (0)$, since $\dim(X^\Gamma) \leq Dn$ by assumption. Since X satisfies the localization theorem (Section 1.1.2), r is injective by Proposition 1.1.5, so $\sigma(\beta)\sigma(\gamma) = 0$ which is a contradiction. So κ is injective.

Surjectivity of κ follows from its injectivity, which can be proved similarly to [VH07, p. 1563].

$\square$

Remark 2.2.9. i) The lemma can be generalized to non-orientable *Poincaré duality spaces* [AP93, Definition 5.1.1.] by replacing $\dim(X)$ with formal dimension $\mathrm{fd}(X)$. Indeed, compactness and orientability was only used for Poincaré duality, which is satisfied by a larger class of spaces, which need not be orientable nor compact. Note that the formal dimension of a manifold X can be smaller than the dimension of X as a manifold. For example, $\mathbb{R}P^{2n}$ is a $\mathbb{Q}$-Poincaré duality space with formal dimension 0. More generally, all real partial flag manifolds $\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$ are $\mathbb{Q}$-Poincaré duality spaces, see Appendix C.2. This allows us to extend the applications of the generalized Borel-Haefliger theorem only slightly (see Proposition 4.2.8), so we didn't add the extra conditions.

ii) For $(\Gamma, R) = (\mathbb{Z}_2, \mathbb{F}_2)$, the same lemma holds, without having to assume orientability. Every manifold satisfies $\mathbb{F}_2$-Poincaré duality – indeed, van Hamel's original proof [VH07] does not assume orientability.

iii) The lemma can also be generalized to $\Gamma = \mathrm{Sp}(1)$, if one makes the additional assumption that the localization theorem holds for the manifold X, see Section 1.1.2.

2.2.3 Coefficients and Steenrod squares

Franz and Puppe [FP06] completely determined the coefficients in the restriction equation (2.2):

Theorem 2.2.10 (Franz-Puppe [FP06])**.** *Let $(\Gamma, R) := (\mathbb{Z}_2, \mathbb{F}_2)$ and X be a conjugation space with cohomology frame (κ, σ). Then for all $x \in H^*(X)$:*

$$r\sigma x = \mathrm{Sq}(\kappa(x)),$$

where $\mathrm{Sq} : H^*(X^\Gamma) \to H^*(X^\Gamma)[u]$ denotes the (homogenized, total) Steenrod square:

$$\mathrm{Sq}(y) = \sum_{i=0}^{d} \mathrm{Sq}^i(y) u^{d-i}$$

for $y \in H^d(X^\Gamma; \mathbb{F}_2)$. In particular, this implies

$$\kappa \circ \mathrm{Sq}^{2i} = \mathrm{Sq}^i \circ \kappa \tag{2.5}$$

for all i.

Equation (2.5) was conjectured by Borel and Haefliger [BH61, 5.17].

Together with the theorem of Van Hamel [VH07], this proves the topological version of a classical theorem of Chow [Cho63], namely that $[Z]|_{X^\Gamma} = [Z^\Gamma]^2$. Using the Excess Intersection Formula we can give a simple proof of a weaker version of Theorem 2.2.10, namely in the algebraic case:

Proposition 2.2.11. *Let X be the complexification of a real algebraic variety which is smooth, and let $Z \subseteq X$ be a complexified subvariety which is a smooth cycle. Then*

$$r\sigma[Z] = \mathrm{Sq}(\kappa[Z])$$

Proof. By the Excess Intersection Formula

$$[Z]_\Gamma|_{X^\Gamma} = g_!(e_\Gamma(\eta)) = g_!(w_*^u(\nu)) = \mathrm{Sq}[Z^\Gamma],$$

where $g : Z^\Gamma \hookrightarrow X^\Gamma$ and w_*^u denotes the total homogenized Stiefel-Whitney class. Indeed, since Z is a complexification, the excess bundle equivariantly is $\eta = \nu(Z^\Gamma \subseteq X^\Gamma) \otimes_\mathbb{R} i\mathbb{R}$ where $i\mathbb{R}$ denotes the trivial line bundle with nontrivial $\mathbb{Z}_2$-action, since

$$\nu(Z \subseteq X)|_{Z^\Gamma} = \nu(Z^\Gamma \subseteq X^\Gamma) \otimes_\mathbb{R} \mathbb{C}.$$

Then $e_\Gamma(E \otimes_\mathbb{R} i\mathbb{R}) = w_*^u(E)$, and we conclude by Thom's theorem [Tho50] saying that for $i : Z \hookrightarrow X$: $i_!(w_*(\nu)) = \mathrm{Sq}[Z]$. ∎

It would be nice to have a similar description of the coefficients for the case of circle spaces, however there are no nontrivial stable rational cohomology operations. Preliminary computations suggest that the coefficients are related to Landweber-Novikov operations in complex cobordism; more precisely to the integral squaring operations discussed by Wood [Woo97], however this relationship remains to be explored.

Chapter 3

Borel-Haefliger type theorems

The goal of this section is to state and prove generalizations of the Borel-Haefliger theorems. The first, generalized Borel-Haefliger theorem relates the cohomology ring of Γ-spaces X to the cohomology ring of their fixed point sets X^Γ in terms of geometric cycles. The second, equivariant Borel-Haefliger theorem states a similar relation, but involving an extra G-action on X compatible with the Γ-action. This is already interesting if X is a vector space; indeed, this was the content of the original Borel-Haefliger theorem.

In Section 3.1 we state and prove the generalized Borel-Haefliger theorem. In Section 3.2 we state the equivariant Borel-Haefliger theorem and prove it using the "halving bundle lemma", whose proof can be found in the last part of this chapter.

3.1 Generalized Borel-Haefliger theorem

For the definitions of good invariant cycles and excess weight see Definitions 1.2.12 and 1.2.9. Recall that for good invariant cycles Z there exists $[Z]_\Gamma$ and $[Z^\Gamma]$ by Proposition A.3.2 and Lemma 1.2.14.

Theorem 3.1.1 (Generalized Borel-Haefliger theorem)**.** *Let* (Γ, R) *be* $(\mathbb{Z}_2, \mathbb{F}_2)$ *or* $(\mathrm{U}(1), \mathbb{Q})$ *and* $H^*(B\Gamma; R) \cong R[u]$, $u \in H^D_\Gamma$*. Let* X *be a smooth compact* Γ*-manifold, orientable if* $R = \mathbb{Q}$*. Assume that* $H^*(X)$ *is nonzero only in degrees divisible by* D *and has a basis of halving cycles:*

good Γ-invariant cycles $Z_i \subseteq X$ of codimension type $(2Dk_i, Dk_i)$. Then X is a halving space with cohomology frame

$$\sigma[Z_i] = [Z_i]_\Gamma, \qquad \kappa[Z_i] = \mu_i[Z_i^\Gamma] \tag{3.1}$$

where $\mu_i u^{k_i} \neq 0$ is the excess weight of $Z_i \subseteq X$. Furthermore, (3.1) holds for any halving cycle $Z \subseteq X$.

Proof. Since $H^*(X)$ is generated by halving cycles, it is equivariantly formal by Proposition 1.1.4. By the excess weight lemma (Lemma 1.2.14),

$$r\sigma[Z_i] = \kappa[Z_i]u^{k_i} + \deg_\Gamma^{<Dk_i}, \qquad \text{for all } i,$$

where $\deg_\Gamma^{<Dk_i}$ denotes a polynomial in u of degree less than k_i. Since $[Z_i]$ form a basis of $H^*(X)$, X is almost a halving space, cf. Definition 2.2.7. Since X is a compact orientable manifold, the class of a point is a cycle which is represented by a halving cycle by assumption, so $\dim X = 2\dim X^\Gamma$. Therefore X is a halving space with cohomology frame (κ, σ) by the injectivity lemma (Lemma 2.2.8). Finally, (3.1) holds for any halving cycle $Z \subseteq X$ by Proposition 2.2.5. □

Remark 3.1.2. i) The orientability assumption is not essential, see Remark 2.2.9. However, then one has to add conditions on the formal dimensions: $\mathrm{fd}(X) \geq 2\,\mathrm{fd}(X^\Gamma)$, for the injectivity lemma to hold.

ii) For $(\Gamma, R) = (\mathrm{Sp}(1), \mathbb{Q})$ there is an analogous theorem, if one makes additional assumptions: the Z_i^Γ have to be assumed to be cycles and the localization theorem must be satisfied (Section 1.1.2) for the injectivity lemma to hold.

iii) To translate this to the Theorem stated in the Introduction of the thesis, recall that halving $\mathrm{U}(1)$-cycles have codimensions divisible by 4 by Remark 2.2.4. Also, the excess weight is multiplicative, since $\mu_i = \lambda^{k_i}$ if the normal bundle has only one weight λ. Therefore $[Z_i] \mapsto [Z_i^\Gamma]$ is multiplicative.

In order to make a connection with the original form of the Borel-Haefliger theorem we need the following lemma (see Section A.5.1 for the definitions):

Lemma 3.1.3. *Let $X_\mathbb{C}$ be a nonsingular quasiprojective variety which is the complexification of $X_\mathbb{R}$. A complexified subvariety $Z_\mathbb{C} \subseteq X_\mathbb{C}$ is a halving cycle.*

Proof. This follows from Whitney's lemma (Lemma A.5.1) stating that the real dimension of a real (nonempty) algebraic variety $Z_{\mathbb{R}}$ –in the sense specified above Lemma A.5.1– is the same as the complex dimension of its complexification $Z_{\mathbb{C}}$. Cycleness follows from Borel and Haefliger's theory. It is also a good cycle: it is clearly conjugation invariant, so it remains to show that it has an invariant fat nonsingular set. Since the singular subset $\Sigma_{\mathbb{C}}$ of the complexification $Z_{\mathbb{C}}$ is defined over $\mathbb{R}$ and has at least one complex codimension in $Z_{\mathbb{C}}$, its real part has at least one real codimension in $Z_{\mathbb{R}}$. So the nonsingular subset $U_{\mathbb{C}} \subseteq Z_{\mathbb{C}}$ is a conjugation invariant fat open, whose real part is a fat open in $Z_{\mathbb{R}}$.

For further details, see Appendix A.5. □

Theorem 3.1.4 (Borel-Haefliger, Proposition 5.15 in [BH61]). *Let X be a nonsingular projective variety over $\mathbb{C}$, which is the complexification of its real part, let $\Gamma = \mathbb{Z}_2$ act on X by complex conjugation. Assume that all $x \in H^*(X;\mathbb{F}_2)$ and $y \in H^*(X^\Gamma;\mathbb{F}_2)$ can be represented by algebraic cycles defined over $\mathbb{R}$ and real algebraic cycles respectively. Then the complexification map (of real algebraic cycles) induces a degree doubling ring isomorphism:*

$$\lambda : H^*(X^\Gamma;\mathbb{F}_2) \to H^{2*}(X;\mathbb{F}_2).$$

Proof. By the previous Lemma 3.1.3, every complexified subvariety Z is a halving cycle. Since X is compact, it is a halving space by Theorem 3.1.1, satisfying $\kappa[Z] = [Z^\Gamma]$ $(0 \neq \mu \in \mathbb{F}_2 \Rightarrow \mu = 1)$, which is the inverse of the complexification map λ. □

Remark 3.1.5.

i) Strictly speaking Van Hamel's theorem [VH07] is slightly weaker than the original Borel-Haefliger theorem in the following sense. Van Hamel's theorem proves that for a basis of complexifications $\kappa[Z_i] = [Z_i^\Gamma]$ holds, whereas the Borel-Haefliger theorem proves this for all complexifications. The additional ingredient we used to obtain the full theorem is Proposition 2.2.5 which involved the degree lemma (Lemma 2.1.4), in particular to show that $\sigma[Z] = [Z]_\Gamma$.

ii) Borel-Haefliger only assumed that X is quasi-projective, however we will not need this generality.

iii) Analogously to spherical conjugation complexes, if a Γ-manifold has a cell decomposition, such that all cells and attaching maps are Γ-equivariant and all cells are $4i$-dimensional with half-dimensional fixed point set, then X is a circle space, this follows from the generalized Borel-Haefliger theorem. However not all circle spaces are of this form, as we will see in the case of even real flag manifolds $\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$; they have torsion cohomology in odd degrees, so they cannot have such a decomposition. Nevertheless, the generalized Borel-Haefliger theorem still applies to even real flag manifolds with rational coefficients. For this reason we didn't develop the analogue of spherical conjugation complexes, besides the fact that it would also require developing the analogue of conjugation pairs.

3.2 Equivariant Borel-Haefliger theorem

In this section we discuss the equivariant Borel-Haefliger theorem, regarding halving spaces equipped with an additional G-action. It states that if G is a halving group (Definition 3.2.5) and X is a halving G-manifold (Definition 3.2.8), then $B_G X$ is a halving space with a cohomology frame (κ, σ) and fixed point set $B_{G^\Gamma} X^\Gamma$. This also implies that the G-equivariant cohomology of X and the G^Γ-equivariant cohomology of X^Γ are isomorphic by a degree halving isomorphism. For a similar theorem in the context of conjugation spaces, see [HHP05, Corollary 7.6].

Furthermore, if $Z \subseteq X$ is a halving G-cycle (Definition 3.2.9), then κ maps $[Z]_G$ to $\mu[Z^\Gamma]_{G^\Gamma}$ where μ is its excess weight.

In Section 3.2.1, we define halving groups and Γ-approximations of BG. In Section 3.2.2 we state the equivariant Borel-Haefliger theorem and prove it using the halving bundle lemma, which we prove in Section 3.3. This section involves (generalized) equivariant principal bundles as in tom Dieck [tD87, Chapter I.8] and Lashof-May [LM86], we collected some of their properties in Appendix E.

3.2.1 Halving groups

In a first approximation, a halving group is a Lie group G, such that $EG \to BG$ has the structure of a Γ-equivariant principal G-bundle, such that BG is a halving space. However, to prove statements about the G-equivariant fundamental classes, the actual definition we will use involves Γ-approximations of $EG \to BG$, which are based on approximations of Edidin-Graham–Totaro [EG98], [Tot99].

Let Γ, G be topological groups and let $\alpha : \Gamma \to \mathrm{Aut}(G)$ be a group homomorphism, such that the map $\Gamma \times G \to G$ sending $(\gamma, g) \mapsto \alpha(\gamma)(g)$ is continuous and in the following always assume this about group homomorphisms $\alpha : \Gamma \to \mathrm{Aut}(G)$. Set $S := G \rtimes_\alpha \Gamma$. An S-space P is a Γ-*equivariant principal* G-*bundle* (or (Γ, G)-*bundle* in short) if $P \to P/G$ is a principal G-bundle. In other words, the principal G-bundle is also equipped with a compatible Γ-action.

Example 3.2.1. Consider $\mathbb{Z}_2$ acting on $\mathrm{GL}(n, \mathbb{C})$ by complex conjugation. The conjugation action on $BG = \mathrm{Gr}_n(\mathbb{C}^\infty)$ lifts to the tautological principal $\mathrm{GL}(n, \mathbb{C})$-bundle $\mathrm{Inj}(\mathbb{C}^n, \mathbb{C}^\infty) \to BG$, giving it the structure of a (Γ, G)-bundle; this bundle is in fact a universal (Γ, G)-bundle, see Proposition E.2.15. For more details, see Appendix E. ♣

Definition 3.2.2. Let $EG \to BG$ have the structure of a (Γ, G)-bundle. We say that $(E_k \to B_k, \iota_k, \mathcal{K}_k)$ is a Γ-*approximation* of $EG \to BG$, if

- $p_k : E_k \to B_k$ are smooth (Γ, G)-bundles,
- $\mathcal{K}_k : B_k \to BG$ are Γ-equivariant classifying maps, i.e. there are (Γ, G)-bundle maps $\tilde{\mathcal{K}}_k : E_k \to EG$ covering $\mathcal{K}_k$ with the property that

$$\pi_j^\Delta(\mathcal{K}_k) : \pi_j^\Delta(B_k) \xrightarrow{\cong} \pi_j^\Delta(BG),$$

  for all $j < k$ and $\Delta \leq \Gamma$,
- $\iota_k : B_k \to B_{k+1}$ are Γ-equivariant maps such that $\mathcal{K}_k = \mathcal{K}_{k+1} \circ \iota_k$.

We will often omit ι_k and $\mathcal{K}_k$ from the notation of a Γ-approximation.

Example 3.2.3. In Example 3.2.1 above, the tautological principal $\mathrm{GL}(n, \mathbb{C})$-bundles over finite Grassmannians $E_N = \mathrm{Inj}(\mathbb{C}^n, \mathbb{C}^N) \to \mathrm{Gr}_n(\mathbb{C}^N) = B_N$ with the complex conjugation action are a Γ-approximation of the (Γ, G)-bundle $\mathrm{Inj}(\mathbb{C}^n, \mathbb{C}^\infty) \to \mathrm{Gr}_n(\mathbb{C}^\infty)$. ♣

Remark 3.2.4. i) In general, there exists a *universal* (Γ, G)-*bundle* $EG \to BG$, where BG is uniquely defined up to Γ-homotopy, see [tD87], [LM86], see also Theorem E.2.4. We do *not* require that $EG \to BG$ be universal as a (Γ, G)-bundle, even though in some of our examples this is the case, as in the case of $\mathbb{Z}_2$ acting on $\mathrm{GL}(n, \mathbb{C})$, see Proposition E.2.15. In the other cases, universality holds in a weaker sense, see Proposition E.2.16.

ii) Note that by definition, $\pi_j^\Delta(Y) = [S^n, Y]_\Delta = [S^n, Y^\Delta] = \pi_j(Y^\Delta)$.

If Y is an S-space, denote by

$$Y(k) := E_k \times_G Y \tag{3.2}$$

which is a Γ-equivariant fiber bundle over B_k. Approximations allow us to study $B_G X$ using arguments from the smooth (or algebraic) category. The following proposition illustrates this principle; for its proof, we refer to the Appendix, Proposition E.2.19:

Proposition. *Let $EG \to BG$ have the structure of a (Γ, G)-bundle and let $E_k \to B_k$ be a Γ-approximation of $EG \to BG$. Let X be an S-space and let $\tilde{\mathcal{K}}_k : X(k) \to B_G X$ be the map covering $\mathcal{K}_k : B_k \to BG$ where $X(k)$ is defined as in (3.2). Then*

$$\tilde{\mathcal{K}}_k^* : H^j_\Delta(B_G X) \xrightarrow{\cong} H^j_\Delta(X(k))$$

is an isomorphism for all $j < k-1$, $\Delta \le \Gamma$. (As always, $H^*_\Delta(Y) = H^*(B_\Delta Y)$.)

We define halving groups via Γ-approximations:

Definition 3.2.5. Let $\Gamma = \mathbb{Z}_2$ or $\mathrm{U}(1)$ and $\alpha : \Gamma \to \mathrm{Aut}(G)$. We say that a triple (Γ, α, G) is a *halving group* if $EG \to BG$ has the structure of a (Γ, G)-bundle such that

- $(EG)^\Gamma$ is contractible and
- $EG \to BG$ has a Γ-approximation $E_k \to B_k$, where B_k are halving spaces with cohomology frame (κ_k, σ_k).

Example 3.2.6. $\mathbb{Z}_2$ acting on $\mathrm{GL}(n, \mathbb{C})$ by complex conjugation is a halving group, if $\mathbb{Z}_2$ acts by complex conjugation on the universal principal bundle $\mathrm{Inj}(\mathbb{C}^n, \mathbb{C}^\infty) \to BG$ which has a Γ-approximation by the finite Grassmannians which are conjugation spaces. For further details, see Example 5.1.1. We will give further examples of halving groups in Section 5.1. ♣

When the Γ-action is clear from the context, we will simply say that G is a halving group. See Section 5.1 for explicit descriptions of examples.

Remark 3.2.7. i) If G is a halving group, with a (Γ, G)-structure on $EG \to BG$, then $(EG \to BG)^\Gamma$ is a model of the universal G^Γ-bundle since EG^Γ is contractible and it is a G^Γ-principal bundle by Lemma E.1.6. In other words, $(EG \to BG)^\Gamma = E(G^\Gamma) \to B(G^\Gamma)$.

ii) If G is a halving group, then BG is a halving space with cohomology frame $(\varprojlim \kappa_k, \varprojlim \sigma_k)$ by Proposition 2.1.7.

iii) We could alternatively *define* a halving group by the condition that BG is a halving space, and this would already imply that $B_G X$ is a halving space by the halving bundle lemma (Lemma 3.3.2). However, we will use the Γ-approximation to prove the statement about cycles: $\kappa[Z]_G = [Z^\Gamma]_{G^\Gamma}$.

iv) We expect that if Γ acts on G and $EG \to BG$ has the structure of a (Γ, G)-bundle, then it always has a Γ-approximation by an Edidin-Graham–Totaro type construction [EG98], [Tot99]. In particular, we expect that if BG is a halving space, then it has a Γ-approximation by halving spaces. However, we don't require this as in our examples we will only use the explicit descriptions given in Section 5.1.

3.2.2 The equivariant Borel-Haefliger theorem

In this section, we state and prove the equivariant Borel-Haefliger theorem using halving groups (Definition 3.2.5) and halving G-manifolds, which are halving manifolds with an additional G-action:

Definition 3.2.8. Let $(\Gamma, R) = (\mathbb{Z}_2, \mathbb{F}_2)$ or $(\mathrm{U}(1), \mathbb{Q})$. Let Γ act on G by automorphisms. If Γ and G act on a manifold X compatibly $(\gamma.(g.x) = (\gamma.g).(\gamma.x))$, such that for the Γ-action it is a halving space, we call X a *halving G-manifold*.

Definition 3.2.9. Let $(\Gamma, R) = (\mathbb{Z}_2, \mathbb{F}_2)$ or $(\mathrm{U}(1), \mathbb{Q})$. Let Γ act on G by automorphisms and let X be a halving G-manifold. If $Z \subseteq X$ is a G-invariant halving cycle (Definition 2.2.3) whose Γ-invariant fat nonsingular subset $Y \subseteq Z$ is also G-invariant, and if Z is a G-cycle (Definition A.3.1), then we say that Z is a *halving G-cycle*.

Before stating the equivariant Borel-Haefliger theorem, let us make some preliminary remarks. If G is connected, and Z is a halving G-cycle, then $[Z]_G$ exists, see Proposition A.3.2. For G not connected, a sufficient condition for the existence of $[Z]_G$ is given if G acts on the normal bundle $\nu(U \hookrightarrow X)$ of a G-invariant nonsingular fat subset $U \subseteq Z$ in an orientation preserving way, see Proposition A.3.3. If $Z = \overline{G.x}$ is the closure of a G-orbit which is Γ-invariant, then the orbit $G.x$ is a Γ and G-invariant fat nonsingular subset; this will be the case in our applications.

If $P \to B$ is a (Γ, G)-bundle and Γ and G act on X compatibly, then $P \times_G X \to B$ is a Γ-equivariant fiber bundle, and

$$(P \times_G X)^\Gamma = P^\Gamma \times_{G^\Gamma} X^\Gamma,$$

see Proposition E.1.7. In particular, if G is a halving group, $(B_G X)^\Gamma = B_{G^\Gamma} X^\Gamma$.

Theorem 3.2.10 (Equivariant Borel-Haefliger)**.** *Let (Γ, R) be either $(\mathbb{Z}_2, \mathbb{F}_2)$ or $(\mathrm{U}(1), \mathbb{Q})$. Let G be a halving group and X be a halving G-manifold, which is G-equivariantly formal. If $\Gamma = \mathbb{Z}_2$, assume in addition that $B_G X$ is Γ-equivariantly formal.*

Then $B_G X$ is a halving space for the Γ-action induced on $B_G X = EG \times_G X$ with fixed point set $(B_G X)^\Gamma = B_{G^\Gamma} X^\Gamma$. Denote the cohomology frame (κ, σ), where $\kappa : H_G^{2*}(X) \to H^*_{G^\Gamma}(X^\Gamma)$. Furthermore, if $Z \subseteq X$ is a halving G-cycle with excess weight μu^i, then

$$\kappa[Z]_G = \mu[Z^\Gamma]_{G^\Gamma}.$$

The proof uses the following halving bundle lemma, whose proof we defer to the next section.

Lemma (Halving bundle lemma)**.** *Let (Γ, R) be either $(\mathbb{Z}_2, \mathbb{F}_2)$ or $(\mathrm{U}(1), \mathbb{Q})$. Let Γ act on G by automorphisms and let $S := G \rtimes \Gamma$. Let $P \to X$ be a Γ-equivariant principal G-bundle over a halving space X. Let F be an S-space, such that for the $\Gamma \leq S$-action it is a halving space. Assume $E := P \times_G F \to X$ is a Leray-Hirsch bundle and E is Γ-equivariantly formal. Then E is a halving space.*

Proof of theorem. Let $E_k \to B_k$ be a Γ-approximation of $EG \to BG$ given by the definition of halving groups. For an S-space Y, recall the notation $Y(k) := E_k \times_G Y$.

The Γ-manifold $X(k)$ is a halving manifold by the halving bundle lemma, whose conditions are satisfied: G-formality of X implies that $X(k) \to B_k$ is Leray-Hirsch by Proposition 1.1.3 and $X(k)$ is Γ-equivariantly formal by Proposition 1.1.2 or by assumption if $\Gamma = \mathbb{Z}_2$. So by the halving bundle lemma $X(k)$ is a halving space for all k, denote by (κ_k, σ_k) the corresponding cohomology frames. By naturality of halving spaces (Corollary 2.1.6), κ_k and σ_k are compatible as k varies. Since $H^j_\Gamma(B_G X) \xrightarrow{\cong} H^j_\Gamma(X(k))$ for k large enough (Proposition E.2.19), one can define

$$(\kappa, \sigma) := (\varprojlim_k \kappa_k, \varprojlim_k \sigma_k) \tag{3.3}$$

as in Proposition 2.1.7 and $B_G X$ is a halving space.

To show the second part of the theorem let $Z \subseteq X$ be a halving G-cycle with excess weight μ. Then $Z(k) \subseteq X(k)$ is also a halving cycle with excess weight μ by Proposition E.1.11. Therefore

$$\kappa[Z]_G = \varprojlim_k \kappa_k[Z(k)] = \varprojlim_k \mu[Z(k)^\Gamma] = \mu[Z^\Gamma]_{G^\Gamma}$$

by definition of κ, Proposition 2.2.5 and since $E_k^\Gamma \to B_k^\Gamma$ is an approximation of $EG^\Gamma \to BG^\Gamma$ (Proposition E.2.20).

□

Remark 3.2.11. i) We conjecture that the condition that $B_G X$ is equivariantly formal is not needed in the $\Gamma = \mathbb{Z}_2$ case, see also the discussion following Proposition 3.3.3. In the Leray-Serre spectral sequence of $B_\Gamma B_G X \to B\Gamma$, $B\Gamma$ is not simply connected, so local coefficients have to be examined. It also has $\mathbb{F}_2$-cohomology in odd degrees, so that Proposition 1.1.2 can no longer be applied.

ii) In the proof we did not use that π_j^Δ are isomorphisms for all $\Delta \leq \Gamma$, only that π_j and π_j^Γ are. Indeed, this is all that is needed from Proposition E.2.19. However, we found it more natural to impose a Γ-equivariant universality property on $EG \to BG$, especially since this distinction does not make a difference in any of our examples. See Appendix E.2 for further discussion.

Corollaries

We deduce some corollaries of the equivariant Borel-Haefliger theorem. First, in the case of conjugation spaces we get back the classical Borel-Haefliger theorem about Thom polynomials. For the definitions of Thom polynomials and complexification, see Sections 5.2 and A.5.1.

Theorem 3.2.12 (Borel-Haefliger, Theorem 6.2 in [BH61])**.** *Let $\eta^{\mathbb{C}} \subseteq J^k(\mathbb{C}^n, \mathbb{C}^p)$ be the complexification of a real singularity type $\eta^{\mathbb{R}} \subseteq J^k(\mathbb{R}^n, \mathbb{R}^p)$ where J^k denotes the k-jet space. Then their Thom polynomials are given by the same polynomial mod 2:*

$$tp(\eta^{\mathbb{C}}; \mathrm{mod}\, 2) = p(c_*, c'_*; \mathrm{mod}\, 2) \qquad \iff \qquad tp(\eta^{\mathbb{R}}) = p(w_*, w'_*)$$

Proof. We check the conditions of the equivariant Borel-Haefliger theorem. First, $\mathrm{Diff}_n \times \mathrm{Diff}_p$ is homotopy equivalent to $G = \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(p, \mathbb{C})$ with the conjugation action, which is a halving group by Example 5.1.1. The jet space $V = J^k(\mathbb{C}^n, \mathbb{C}^p)$ is trivially a halving G-manifold and G-equivariantly formal since its cohomology is trivial, and similarly, B_GV is Γ-equivariantly formal (BG is since G is a halving group).

The Thom polynomials are the G and G^Γ-equivariant fundamental classes of $\eta^{\mathbb{C}}$ and $\eta^{\mathbb{R}}$. Since $\eta^{\mathbb{C}}$ is a complexification, it is a G-equivariant halving cycle. So by the previous theorem, $\kappa[\eta^{\mathbb{C}}]_G = [\eta^{\mathbb{R}}]_{G^\Gamma}$ (excess weight is 1). By Example 5.1.7, $\kappa(c_i) = w_i$ (c_i is taken $\bmod 2$). So applying κ to $[\eta^{\mathbb{C}}]_G = p(c_*, c'_*)$, we get

$$[\eta^{\mathbb{R}}]_{G^\Gamma} = \kappa p(c_*, c'_*) = p(\kappa c_*, \kappa c'_*) = p(w_*, w'_*).$$

□

In the case $\Gamma = \mathrm{U}(1)$, we get the following analogue. We do not need to assume that the ambient space is a vector space, only that it is a halving G-manifold (this is also true for the case $\Gamma = \mathbb{Z}_2$). Recall, that if X is a G-equivariantly formal space with a Leray-Hirsch section σ and $x_i \in H^*(X)$ is a basis, then any $\alpha \in H^*_G(X)$ can be uniquely written as a linear combination

$$\alpha = q(\sigma(x_i), p_*) = \sum \alpha_i(p_*)\sigma(x_i),$$

where the coefficients $\alpha_i(p_*) \in H^*_G \cong \mathbb{Q}[p_*]$ are polynomials in characteristic classes.

Theorem 3.2.13. *Let $\Gamma = \mathrm{U}(1)$ act on $G = \times\, \mathrm{GL}(2n_i, \mathbb{R})$ by conjugation, where $\mathrm{U}(1) \hookrightarrow G$ via the diagonal circle subgroup, with fixed point set $G^\Gamma = \times\, \mathrm{GL}(n_i, \mathbb{C})$ (Section 4.1.2). Let X be a halving G-manifold.*

a) If $H^*(X)$ has a basis $[Z_i]$ of halving G-cycles Z_i, then B_GX is a halving space with a cohomology frame (κ, σ) and fixed point set $B_{G^\Gamma}X^\Gamma$.

b) Furthermore, if all normal weights of $X^\Gamma \subseteq X$ are $2u$, then for any halving G-cycle Z,

$$[Z]_G = \sum q_i(p_*)[Z_i]_G \qquad \Longleftrightarrow \qquad [Z^\Gamma]_{G^\Gamma} = \sum q_i(c_*)[Z_i^\Gamma]_{G^\Gamma}$$

In words, the coefficients $q_i(p_*) \in H^*_G$ of $[Z]_G$ in the basis $[Z_i]_G$ are described by the same polynomials as the coefficients $q_i(c_*) \in H^*_{G^\Gamma}$ of $[Z^\Gamma]_{G^\Gamma}$ in the basis $[Z_i^\Gamma]_{G^\Gamma}$. Here p_* abbreviates the set of Pontryagin classes p_j^l in H^*_G and c_* abbreviates the Chern classes c_j^l in $H^*_{G^\Gamma}$ and $\kappa(p_j^l) = 2^j c_j^l$ for each $\mathrm{GL}(2n_l, \mathbb{R})$.

Proof. The conditions of the equivariant Borel-Haefliger theorem are satisfied: X is G-equivariantly formal by $[Z_i] \mapsto [Z_i]_G$, so $B_G X$ is a halving space, proving a). Then

$$\kappa[Z_i]_G = 2^{k_i}[Z_i^\Gamma],$$

where $[Z_i]_G \in H_G^{4k_i}(X)$. If Z is an arbitrary halving G-cycle with excess weight $(\mu u)^k$, write $[Z]_G = q([Z_i]_G, p_*)$, then

$$\kappa[Z]_G = \kappa q([Z_i]_G, p_j^l) = q(2^{k_i}[Z_i^\Gamma]_{G^\Gamma}, 2^j c_j^l) = 2^k [Z^\Gamma]_{G^\Gamma}.$$

□

In particular, if X is a vector space, then q is a polynomial in solely Chern/Pontryagin classes. The theorem holds for other halving groups, such as $\mathbb{Z}_2$ acting on products of $\mathrm{GL}(n_i, \mathbb{C})$, $\mathrm{U}(1)$ acting on products of $\mathrm{GL}(n_i, \mathbb{H})$ - the proof is the same. We will give several applications of these theorems in the last chapter.

3.3 Halving bundles

In this section we prove the halving bundle lemma, which states that if $F \to E \to B$ is a fiber bundle, such that F and B are halving spaces, then E is a halving space. We use this lemma in the equivariant Borel-Haefliger theorem, to show that for halving groups G and halving spaces X, (Definition 3.2.5) $B_G X$ is a halving space.

In the case of conjugation spaces, a similar theorem can be found in [HHP05, Proposition 5.3], involving spherical conjugation complexes – informally, these are $\mathbb{Z}_2$-equivariant cell complexes built out of even dimensional cells C_i with half-dimensional fixed point set C_i^Γ. In order to adapt the proof in [HHP05] to circle spaces, one would have to develop an analogue of spherical conjugation complexes. However one of our more interesting examples, real flag manifolds cannot be built up from cells of dimension $4i$, so we needed a different approach. Our proof of the halving bundle lemma is more algebraic in nature and is fairly technical. One of the trade-offs is that it does not require developing the theory of pairs of halving spaces, their main properties and the analogue of spherical conjugation complexes [HHP05, pp. 934–942], although they should adapt readily. We work with cohomology with field coefficients.

3.3.1 Halving bundle lemma

We are going to need the following Proposition in the proof of the halving bundle lemma.

Proposition 3.3.1. *Let*

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi_E"] & E' \arrow[d, "\pi_{E'}"] \\ X \arrow[r, "\varphi"] & X' \end{tikzcd}$$

be a pullback of Leray-Hirsch fiber bundles with fiber F. Assume that $\varphi^* : H^*(X') \to H^*(X)$ has a section σ_X (is surjective). Let σ be a Leray-Hirsch section of E. Then there exists a Leray-Hirsch section σ' making the following diagram commute

$$\begin{tikzcd} & H^*(E') \arrow[d, "\varphi^*"] \\ H^*(F) \arrow[ur, "\sigma'"] \arrow[r, "\sigma"] & H^*(E) \end{tikzcd}$$

Proof. We have a surjective map of graded vector spaces

$$H^*(E') \to H^*(E) \to H^*(F).$$

Therefore the section $\sigma : H^*(F) \to H^*(E)$ can be lifted to $\sigma' : H^*(F) \to H^*(E')$. □

Lemma 3.3.2 (Halving bundle lemma)**.** *Let $P \to X$ be a Γ-equivariant principal G-bundle over a halving space X. Let $S := G \rtimes \Gamma$ act on F, such that for the $\Gamma \leq S$-action it is a halving space. Assume $E := P \times_G F \to X$ is a Leray-Hirsch bundle and E is Γ-equivariantly formal. Then E is a halving space.*

Proof. The idea of the proof is very simple: we build a cohomology frame $(\hat{\kappa}, \hat{\sigma})$ from the cohomology frames (κ_X, σ_X) and (κ_F, σ_F) using that E is Leray-Hirsch. However, the fair amount of Leray-Hirsch sections involved requires some careful bookkeeping and renders the proof a bit technical. To alleviate notation we assume that $D = 1$, this can be achieved by dividing the grading by D.

Since X is a halving space, it has a fixed point, so there is a commutative diagram as follows:

$$\begin{tikzcd}
H^*_\Gamma(X) \arrow[r, "\pi^*_\Gamma"] \arrow[d, "\rho_X"] & H^*_\Gamma(E) \arrow[r, "i^*_\Gamma"] \arrow[d, "\rho_E"] & H^*_\Gamma(F) \arrow[d, "\rho_F"'] \\
H^*(X) \arrow[r, "\pi^*"] \arrow[u, bend left, "\sigma_X"] & H^*(E) \arrow[r, "i^*"] & H^*(F) \arrow[u, bend right, "\sigma_F"'] \arrow[ul, dashed, "\sigma_\Gamma"] \arrow[l, dashed, bend left, "\sigma"]
\end{tikzcd}$$

where σ_X and σ_F are the halving space sections. We will construct σ_Γ, such that it is a lift of σ_F with an additional property (P) (see below) and we will define σ to be $\rho_E \circ \sigma_\Gamma$. Commutativity of $i^*_\Gamma \circ \pi^*_\Gamma \circ \sigma_X = \sigma_F \circ i^* \circ \pi^*$ follows from naturality of the halving space structure (Corollary 2.1.6), since $\pi \circ i : F \to X$ is an equivariant map between halving spaces.

We also use the notation $\sigma_0 : H^*(F^\Gamma) \to H^*(E^\Gamma)$ for a Leray-Hirsch section and $\pi_0 : E^\Gamma \to X^\Gamma$ (we will show in a moment that $E^\Gamma \to X^\Gamma$ is Leray-Hirsch).

To prove the proposition, we want to show that E is a halving space, i.e. there exist $(\hat{\kappa}, \hat{\sigma})$, $\hat{\sigma} : H^*(E) \to H^*_\Gamma(E)$ Leray-Hirsch section, $\hat{\kappa} : H^{2*}(E) \to H^*(E^\Gamma)$ injective satisfying the degree condition

$$\hat{r}\hat{\sigma}\hat{x} = \hat{\kappa}(\hat{x})u^d + l.d.t. \tag{DC}$$

for all $\hat{x} \in H^{2d}(E)$, where $l.d.t.$ denotes lower degree terms in u.

We claim that if $(\sigma_\Gamma, \sigma_0)$ are Leray-Hirsch sections of $B_\Gamma E \to B_\Gamma X$ and $E^\Gamma \to X^\Gamma$ respectively satisfying property

$$\hat{r}\sigma_\Gamma(f) = \sigma_0 \kappa_F(f) u^d + l.o.t. \tag{P}$$

then $\sigma := \rho_E \circ \sigma_\Gamma$ is a Leray-Hirsch section of $E \to X$ (this is trivial) and that $(\hat{\kappa}, \hat{\sigma})$ defined by

$$\hat{\sigma}(\pi^* x \cdot \sigma(f)) := \pi^*_\Gamma \sigma_X(x) \cdot \sigma_\Gamma(f)$$

and

$$\hat{\kappa}(\pi^* x \cdot \sigma(f)) := \pi^*_0 \kappa_X(x) \cdot \sigma_0 \kappa_F(f)$$

form a cohomology frame for E (these definitions extend to $H^*(E)$ uniquely using that σ is a Leray-Hirsch section). First, $\hat{\sigma}$ is a section:

$$\rho_E \hat{\sigma}(\pi^* x \cdot \sigma(f)) = \rho_E \pi^*_\Gamma \sigma_X(x) \cdot \rho_E \sigma_\Gamma(f) = \pi^* \rho_X \sigma_X(x) \cdot \sigma(f) = \pi^* x \cdot \sigma(f)$$

satisfying the degree condition:

$$\hat{r}\hat{\sigma}(\pi^* x \cdot \sigma(f)) = \hat{r}\pi_\Gamma^* \sigma_X(x) \cdot \hat{r}\sigma_\Gamma(f) = (\pi_0^* \kappa_X(x) u^i + l.d.t.)(\sigma_0 \kappa_F(f) u^d + l.d.t.) \tag{DC}$$

using (P) and that $\hat{r}\pi_\Gamma^* = \pi_0^* r_X$. Finally $\hat{\kappa}$ is injective since $H^*(E^\Gamma)$ is a free π_0^*-module and κ_X and κ_F are injective.

Now we turn to constructing a pair $(\sigma_\Gamma, \sigma_0)$ satisfying (P).

Using Proposition 3.3.1 (here we use that E is Γ-equivariantly formal), let σ_Γ be an arbitrary section which restricts to σ_F:

$$\begin{tikzcd}
H_\Gamma^*(E) \arrow[r, "\hat{r}"] \arrow[d, "\hat{\rho}"] & H_\Gamma^*(E^\Gamma) \arrow[d, "\rho=\rho_0\otimes\mathrm{id}"] \\
H_\Gamma^*(F) \arrow[r, "r_F"] \arrow[d] & H_\Gamma^*(F^\Gamma) \arrow[d] \\
H^*(F) \arrow[r] \arrow[uu, bend left=60, "\sigma_\Gamma"] \arrow[u, bend left, "\sigma_F"] & H^*(F^\Gamma)
\end{tikzcd}$$

We construct the correct $(\sigma_\Gamma, \sigma_0)$ by induction on the degree. The induction step consists of two phases. In the *first phase*, we modify σ_Γ, so that it satisfies $\deg_u \hat{r}\sigma_\Gamma(f) \leq d$ for all $f \in H^{2d}(F)$ (and it still satisfies $\hat{\rho} \circ \sigma_\Gamma = \sigma_F$). In the *second phase* we define $\sigma_0(y) := \deg_u^d(\hat{r}\sigma_\Gamma \kappa_F^{-1}(y))$, and show that this is indeed a section, in particular $E^\Gamma \to X^\Gamma$ is Leray-Hirsch. Then $(\sigma_\Gamma, \sigma_0)$ satisfy property (P) in degree $2d$.

As the first step of the induction, σ_Γ satisfies $\hat{r}\sigma_\Gamma(1) = 1$ on $H^0(X)$, therefore property (P). For the induction step, assume that $(\sigma_\Gamma, \sigma_0)$ satisfy property

$$\hat{r}\sigma_\Gamma(f) = \sigma_0 \kappa_F(f) u^i + l.d.t. \tag{P}$$

for $f \in H^{2i}(F)$, $i \leq d-1$ (σ_0 is defined up to degree $d-1$). Now we describe the induction step.

First phase. Let $f \in H^{2d}(F)$. Then

$$\hat{r}\sigma_\Gamma(f) = \sum_{i=k}^{2d} \xi_i u^{2d-i}$$

with $\xi_i \in H^i(E^\Gamma)$. If $k < d$, then $\xi_k \in \ker(\rho_0 : H^*(E^\Gamma) \to H^*(F^\Gamma))$ by the upper square in the diagram above (since $r_F\sigma_F(f)$ satisfies degree condition).

Then $\xi_k \in \ker\rho_0 = \pi_0^* H^{>0}(X^\Gamma) \cdot H^*(E^\Gamma)$, so we can write

$$\xi_k = \sum_{i=1}^{r} \pi_0^*\beta_i \cdot \sigma_0(y_i).$$

for some $\beta_i \in H^{>0}(X^\Gamma)$, and $y_i \in H^{<k}(F^\Gamma)$.

Let

$$\sigma'_\Gamma(f) := \sigma_\Gamma(f) - \sum_{i=1}^{r} \pi_\Gamma^*\sigma_X\kappa_X^{-1}(\beta_i) \cdot \sigma_\Gamma(\kappa_F^{-1}(y_i))u^{2d-2k}.$$

This is also a section satisfying $\hat{\rho} \circ \sigma'_\Gamma = \sigma_F$, as $\hat{\rho}$ maps $H^{>0}(X)$ to zero.

Since X is a halving space,

$$\hat{r}\pi_\Gamma^*\sigma_X\kappa_X^{-1}(\beta_i) = \pi_0^* r_X\sigma_X(\kappa_X^{-1}(\beta_i)) = \beta_i u^{\deg\beta_i} + l.d.t.$$

and by the induction step,

$$\hat{r}(\sigma_\Gamma(\kappa_F^{-1}(y_i))) = \sigma_0(y_i)u^{\deg y_i} + l.d.t.$$

so

$$\deg_u^{2d-k} \hat{r}\sigma'_\Gamma(f) = \sum_{i=1}^{r} \beta_i \cdot \sigma_0(y_i) - \sum_{i=1}^{r} \beta_i \cdot \sigma_0(y_i) = 0$$

and the degree decreases. Let $\sigma_\Gamma := \sigma'_\Gamma$, and repeat this, until $k = d$ for all $f \in H^{2d}(F)$. This finishes the first step.

Second phase. Define $\sigma_0(y) := \deg_u^d(\hat{r}\sigma_\Gamma\kappa_F^{-1}(y))$. We claim that $\rho_0\sigma_0(y) = y$. The following diagram commutes by naturality of Künneth formula:

$$\begin{array}{ccc}
H^*_\Gamma(E^\Gamma) & \xrightarrow{\cong} & H^*(E^\Gamma) \otimes H^*_\Gamma \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho_0 \otimes \mathrm{id}} \\
H^*_\Gamma(F^\Gamma) & \xrightarrow{\cong} & H^*(F^\Gamma) \otimes H^*_\Gamma
\end{array}$$

We need the following relations:

$$\deg_u^d \rho = \rho_0 \deg_u^d \qquad \text{which follows from } \rho = \rho_0 \otimes \mathrm{id},$$

$$\rho\hat{r}\sigma_\Gamma = r_F\sigma_F \qquad \text{which follows from } \rho\hat{r} = r_F\hat{\rho} \text{ and } \hat{\rho}\sigma_\Gamma = \sigma_F$$

Using these, we get:

$$\rho_0\sigma_0(y) = \rho_0 \deg_u^d(\hat{r}\sigma_\Gamma\kappa_F^{-1}(y)) = \deg_u^d \rho\hat{r}\sigma_\Gamma\kappa_F^{-1}(y)$$

as claimed. This finishes the induction argument, and we get $(\sigma_\Gamma, \sigma_0)$ satisfying (P), which finishes the proof. □

Hausmann, Holm and Puppe proved a similar Theorem ([HHP05, Proposition 5.3]) over *spherical conjugation complexes*. These are $\mathbb{Z}_2$-equivariant cell complexes built from *conjugation cells* (closed unit disks $D^{2k} \subseteq \mathbb{C}^k$ with complex conjugation $\mathbb{Z}_2$-action). Let us recall this Theorem:

Proposition 3.3.3 (Proposition 5.3 in [HHP05])**.** *Let* $(\Gamma, R) = (\mathbb{Z}_2, \mathbb{F}_2)$*. Let* $P \to X$ *be a* Γ*-equivariant principal* G*-bundle over a spherical conjugation complex* X*. Let* F *be an* S*-space, such that for the* $\Gamma \leq S$ *action it is a conjugation space,* $S := G \rtimes_\alpha \Gamma$*. Then* $E = P \times_G F$ *is a conjugation space.*

In the context of conjugation spaces, the halving bundle lemma (Lemma 3.3.2) is *formally* stronger than Proposition 3.3.3: the conditions of Proposition 3.3.3 guarantee the conditions of the halving bundle lemma. We need to check two conditions: Leray-Hirschness and $\mathbb{Z}_2$-equivariant formality of E. X is simply connected since it is a spherical conjugation complex (it has no one-cells) and therefore E is Leray-Hirsch by Proposition 1.1.2. Although we did not manage to prove $\mathbb{Z}_2$-equivariant formality of E directly, we know it holds, since E is a conjugation space by Proposition 3.3.3. It would be interesting to find an independent direct proof of $\mathbb{Z}_2$-equivariant formality of E under the conditions of Proposition 3.3.3.

Chapter 4

Applications - Borel-Haefliger theorem

In this chapter we discuss applications of the generalized Borel-Haefliger theorem. Since conjugation spaces are discussed in [HHP05], we concentrate on the case of circle spaces. Our main type of application is the following: we deduce the cohomology ring structure of X from that of X^Γ, in particular we determine the structure constants of Schubert cycles in X from the structure constants of the Schubert cycles in X^Γ.

The discussion of our examples follows the following pattern: for a candidate halving space X, we describe its geometry in Appendix B, generators for its cohomology (additively) in Appendix C. In this chapter, we define the Γ-actions on X and for each example we show that the additive generators $[Z_i]$ can be chosen such that they are halving cycles with respect to the Γ-action. We decided to put these topics in the Appendix, since they are somewhat lengthy and would distract from our main focus, namely the generalized Borel-Haefliger theorems.

In Section 4.1 we define the group actions for our main examples of halving spaces. Then, in Section 4.2 we apply the generalized Borel-Haefliger theorem to show that these spaces are circle spaces. In Section 4.3 we give further examples for circle spaces, which are quaternionic halving spaces as well. Computing intersection products of cycles has consequences in enumerative geometry; we conclude this chapter by discussing these in Section 4.4.

4.1 The group actions

In this short section we introduce the group actions that provide most of our examples of halving spaces. We will refer to them as Galois type and pseudo Galois type actions.

4.1.1 Galois type actions

There are four normed division $\mathbb{R}$-algebras $\mathbb{F}_i$, $i = 1, 2, 3, 4$:

$$\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} \subseteq \mathbb{O}.$$

In each case, there is a subgroup $\Gamma \cong \mathrm{O}(\mathbb{F}_{i-1})$ of the algebra automorphisms $\mathrm{Aut}(\mathbb{F}_i)$, such that the fixed point set $\mathbb{F}_i^\Gamma$ is the previous division algebra $\mathbb{F}_{i-1}$, $i = 2, 3, 4$ – hence the name "Galois type". Here $\mathrm{O}(\mathbb{F}_i)$ denotes the subgroup of elements of norm one in $\mathbb{F}_i$. Explicitly, these actions can be described as follows.

Example 4.1.1. The group of (continuous) automorphisms is $\Gamma = \mathrm{Aut}(\mathbb{C}) = \mathbb{Z}_2$, acting on $\mathbb{F} = \mathbb{C}$ by complex conjugation with fixed point set $\mathbb{R}$. ♣

Example 4.1.2. $\Gamma = \mathrm{U}(1) \leq \mathrm{Aut}(\mathbb{H}) = \mathrm{SO}(3)$ acts on $\mathbb{F} = \mathbb{H}$ with fixed point set $\mathbb{C}$. ♣

Proof. By the Skolem-Noether theorem, all ring automorphisms of $\mathbb{H}$ are inner, and are in fact $\mathbb{R}$-algebra automorphisms. Inner automorphisms leave pure quaternions invariant, which shows that the automorphism group is $\mathrm{Aut}(\mathbb{H}) = \mathrm{Sp}(1)/\mathbb{Z}_2 \cong \mathrm{SO}(3)$. Then $\Gamma = \mathrm{U}(1) \leq \mathrm{Sp}(1)$ acts on $\mathbb{H}$ by weight 2 by left-right conjugation, where the inclusion $\Gamma \leq \mathbb{H}$ is the $\mathrm{U}(1)$ spanned by $(1, i)$. The fixed point set $\mathbb{H}^\Gamma$ is the centralizer of $\mathrm{U}(1)$ in $\mathbb{H}$, which is $\mathbb{C} = \langle 1, i \rangle$. □

Example 4.1.3. $\Gamma = \mathrm{Sp}(1) \leq \mathrm{Aut}(\mathbb{O}) = \mathrm{G}_2$ and $\Gamma' = \mathrm{U}(1) \leq \Gamma$ both act on $\mathbb{F} = \mathbb{O}$ with fixed point set $\mathbb{F}^\Gamma = \mathbb{F}^{\Gamma'} = \mathbb{H}$. Here G_2 denotes the compact real form. ♣

For the proof, see Proposition B.4.1.

We deduce some facts from these actions. First of all, these actions induce an action $\mathrm{GL}(n, \mathbb{F}_i)$, except on "$\mathrm{GL}(n, \mathbb{O})$", which is not defined:

Proposition 4.1.4. $\Gamma = \mathrm{O}(\mathbb{F}_{i-1})$ *acts on* $\mathrm{GL}(n, \mathbb{F}_i)$ *via group automorphisms with fixed point set* $\mathrm{GL}(n, \mathbb{F}_{i-1})$*,* $i = 2, 3$.

Proof. $\mathrm{GL}(n, \mathbb{F}_i)$ can be represented by $n \times n$ matrices with entries in $\mathbb{F}_i$, and the induced action acting entry-wise is compatible with matrix multiplication, so the fixed point set is as claimed. $\square$

The Galois type actions Γ acting on $\mathbb{F}_i$ induce an action on linear subspaces of $\mathbb{F}_i^N$, and therefore on their flag manifolds: $\mathbb{Z}_2$ acts on $\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^N)$, $\mathrm{U}(1)$ acts on $\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N)$:

Proposition 4.1.5. $\Gamma = \mathrm{O}(\mathbb{F}_{i-1})$ *acts on* $\mathrm{Fl}_{\mathcal{D}}(\mathbb{F}_i^N)$ *with fixed point set* $\mathrm{Fl}_{\mathcal{D}}(\mathbb{F}_{i-1}^N)$*,* $i = 2, 3$*.*

For the proof, see Appendix D.2.

For $i = 4$, the octonionic case, vector space structure and subspaces are not defined. Flag manifolds $\mathrm{Fl}_{\mathcal{D}}$ also only make sense for $\mathcal{D} \in \{(1,1), (1,2), (1,1,1)\}$ and are usually defined via Jordan algebras. There is an induced $\mathrm{Sp}(1)$-action on octonionic flag manifolds, but an alternative proof is required, see Proposition B.4.2. We will refer to all the actions defined in this section collectively as *Galois type actions.*

4.1.2 Pseudo Galois type actions

There is another type of action we will be interested in, for lack of a standard name, we call them "pseudo Galois type actions" (no relation to pseudo-Galois extensions). Namely each $\mathbb{F}_i$-module V is an $\mathbb{R}$-vector space ($i = 2, 3$). Furthermore, $\mathbb{F}_i$-submodules $W \leq V$ are those $\mathbb{R}$-subspaces which are $\Gamma := \mathrm{O}(\mathbb{F}_i)$-invariant, essentially by definition. Therefore

Proposition 4.1.6. $\Gamma = \mathrm{O}(\mathbb{F}_i)$ *acts on* $\mathrm{Fl}_{\delta\mathcal{D}}(\mathbb{R}^{\delta N})$ *with fixed point set* $\mathrm{Fl}_{\mathcal{D}}(\mathbb{F}_i^N)$*, where* $\delta = \dim_{\mathbb{R}} \mathbb{F}_i$*,* $i = 2, 3$*.*

Since $O(\mathbb{F}_i)$ acting on $\mathbb{F}_i^n$ is $\mathbb{R}$-linear, it acts on $\mathrm{GL}(\delta n, \mathbb{R})$ via inner conjugation. The following Proposition follows from the definition of complex/quaternionic linear transformations:

Proposition 4.1.7. $\Gamma = \mathrm{O}(\mathbb{F}_i)$ *acts on* $\mathrm{GL}(\delta n, \mathbb{R})$ *via group automorphisms with fixed point set* $\mathrm{GL}(n, \mathbb{F}_i)$*, where* $\delta = \dim_{\mathbb{R}} \mathbb{F}_i$*,* $i = 2, 3$*. Similarly,* Γ *acts on* $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{\delta k}, \mathbb{R}^{\delta l})$ *with fixed point set* $\mathrm{Hom}_{\mathbb{F}_i}(\mathbb{F}_i^k, \mathbb{F}_i^l)$*.*

For us, the most important action is $\mathrm{U}(1)$ acting on $\mathrm{Fl}_{\delta\mathcal{D}}(\mathbb{R}^N)$. We will refer to these actions collectively as *pseudo Galois actions.*

4.2 Examples: circle spaces

All our examples of circle spaces are homogeneous spaces, in particular they are all *R-spaces*, more commonly known as (generalized) real flag manifolds. In this thesis, we reserve the terminology real flag manifolds to $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$, which are also R-spaces.

Out of the R-spaces, we have four main classes of examples of circle spaces: spheres S^{4n}, even real flag manifolds, quaternionic and octonionic flag manifolds. We describe them in this order.

4.2.1 Warm-up: spheres S^{4n}

Let $\mathrm{U}(1)$ act on $\mathbb{R}^{4n+1}$ as the linear orthogonal representation, which splits into n weight one and $2n+1$ trivial representations; $S^{4n} \subseteq \mathbb{R}^{4n+1}$ is $\mathrm{U}(1)$-invariant.

Proposition 4.2.1. *With this Γ-action S^{4n} is a circle space with fixed point set S^{2n}.*

Proof. The fixed point set of S^{4n} is $S^{4n} \cap \mathbb{R}^{2n+1} = S^{2n}$. Since $H^*(S^{4n}) = \mathbb{Z}[x]/(x^2)$ is generated by a Γ-invariant halving cycle, namely the class of a fixed point, by the generalized Borel-Haefliger theorem, S^{4n} is a circle space. $\square$

This example illustrates the general method of proof we will apply in the following.

4.2.2 Even real flag manifolds

The next example is given by *even real flag manifolds*, i.e. flag manifolds $\mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^{2n})$, where all dimensions are even. In the first part of this section we give a brief overview of what is known about the cohomology of real flag manifolds, and in the second part we show that $\mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^{2n})$ is a circle space and deduce consequences. For the notation and geometry of real flag manifolds, see Appendix B.2.

Overview

Complex flag manifolds have a cell decomposition given by the Bruhat cells, which are the complex B^+-orbits. Since all cells are even dimensional, all boundary maps are trivial, and all cohomology is concentrated in even degrees. The closures of the Bruhat cells are the Schubert varieties, which are all cycles and their fundamental cohomology classes generate additively the cohomology ring of complex flag manifolds. Their multiplicative structure constants are given by Schubert calculus which is a classical and well-developed theory [KL72].

Real flag manifolds also have a cell decomposition, however the boundary maps are no longer trivial. The boundary maps have been first examined by Ehresmann [Ehr37]: he computed them completely for the case of real Grassmannians $\mathrm{Gr}_p(\mathbb{R}^{p+q}) = \mathrm{Fl}_{p,q}$ [Ehr37, p. 80], up to sign for flag manifolds of type $\mathrm{Fl}_{p,q,r}$ [Ehr37, p. 85] and he determined the cycles in the case of $\mathrm{Fl}_{1,q,r}$ [Ehr37, p. 87]. There is a summary of Ehresmann's computations for the case of the Grassmannian in Chern [Che51, p. 73].

General R-spaces also have a Bruhat cell decomposition [DKV83], see Appendix B.1.2. If all multiplicities of the restricted roots are greater than 1, then there are no cells of neighboring dimensions [DKV83]. Therefore, the boundary relations of this cell decomposition are trivial, so additively its cohomology is freely generated by the closures of the Bruhat cells. If the multiplicities are not such (as in the case of real flag manifolds), the boundary relations are no longer trivial. Kocherlakota [Koc95] computed the boundary relations for general R-spaces up to sign by Morse theoretic methods. As he remarks, the open cells determined by the Morse function coincide with the Bruhat cells, so his computations determine the incidence coefficients, up to sign. There is a modern treatment in Casian-Stanton [CS99] connecting the question to the infinite dimensional representation theory of real reductive groups. In the case of Grassmannians the incidence coefficients are described combinatorially (up to sign) in Casian-Kodama [CK13]. The latest development is [RM18], who complete Kocherlakota's computation for R-spaces by determining the signs of the incidence coefficients via a CW homology approach. In Appendix C.1 we also compute the incidence coefficients via a slightly different approach, namely using the geometry of the Schubert cell description.

Once the incidence coefficients are known, it is a nontrivial combinatorial problem to determine what the homology groups are and which Schubert varieties are cycles. The only infinite families where we can determine which Schubert varieties are nonzero rational cycles is the case of even flag manifolds $\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$ and Grassmannians, see Propositions C.1.9 and 4.2.7. We compute some further small examples in Appendix F.

Once the cycles have been determined, the next step is to determine the structure constants of the cycles. We will carry this out for the cycles of $\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$ with rational coefficients, see Corollary 4.2.3.

Returning to the complex case, another kind of description of the cohomology ring of the complex flag manifolds is given in terms of characteristic classes of their tautological bundles. Namely, $H^*(\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^N);\mathbb{Z})$ is generated as an algebra by the Chern classes $c_i(D_j)$ of the tautological quotient bundles $D_j = S_j/S_{j-1}$. In modern language, this can be formulated as surjectivity of the Kirwan map [Kir84]. The relations between these generators are given by the graded parts of $\prod_{j=1}^m c_*(D_j)$, see e.g. [BT82, Chapter 23]. In the case of the Grassmannians, the relationship between these two descriptions is given by the *Giambelli formula*

$$[\sigma_\lambda] = \det(c_{\lambda_i+j-i}(Q)).$$

In the real case, Pontryagin classes do not always generate the cohomology ring $H^*(G/P;\mathbb{Q})$; this is only the case if G and P have the same rank, i.e. even real flag manifolds $\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$. In other words, the "rational real Kirwan map" is no longer surjective in general. Whenever it is surjective, we express $[\sigma_\lambda]$ in terms of Pontryagin classes, see Corollary 4.2.4. The intersection theory of real Grassmannians, more precisely their Chow-Witt rings have been recently considered in [Wen].

We can summarize the previous discussion in the following questions:

- Q1a) Which Schubert varieties are cycles? Which ones are $\mathbb{Q}$-cycles?
- Q1b) What are their structure constants?
- Q2) What are the relations between Pontryagin classes? What are the additional generators and what are the relations?
- Q3) How to express one set of generators from the other? $\sigma_\lambda(p_i) = ?$ $p_i(\sigma_\lambda) = ?$

Casian-Kodama [CK13] examined the incidence relations in the case of Grassmannians ($\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$, $\mathcal{D} = (k,l)$), and made a conjecture about the ring structure of $H^*(\mathrm{Gr}_k(\mathbb{R}^n))$. This has been answered in the form of Q2) for the case of Grassmannians using different approaches by Takeuchi

[Tak62], equivariantly by He [He16] using a generalized GKM theory, Sadykov [Sad17] using Borel's theorem and Carlson [Car16] following Cartan's methods.

In Section 4.2.3 we settle the geometric part of Casian-Kodama's conjecture for Grassmannians, namely we answer Q1a) Q1b) and Q3) rationally.

In Appendix C.1.7 we answer Q1a) for $\mathcal{D}$ even and $R = \mathbb{Q}$ using two approaches: one by resolutions (Remark 5.4.8) and another by using the Vassiliev complex. For $\mathcal{D}$ not even, the answer is more complicated, as we illustrate on some small examples in Appendix F. In Section C.2 we answer Q2) for general $\mathcal{D}$ by Cartan's model. In this section, using the generalized Borel-Haefliger theorem and the computations of Appendix C.1.7 which answer Q1a), we deduce Q1b), Q3) for $\mathcal{D}$ even, see Corollaries 4.2.3 and 4.2.4.

Note, that Q1a), Q2) and Q3) imply Q1b) rationally, at least in theory; in practice giving combinatorial rules to compute the structure constants is not immediate and has been extensively studied in the complex case for different kind of cohomology theories by Littlewood-Richardson rules, checkers, puzzles [Ful97], [Buc02], [Vak06], [KT03].

$\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$ are circle spaces

Recall the pseudo Galois type $\Gamma = \mathrm{U}(1)$-action on real flag manifolds: the identification of $\mathbb{R}^{2n} \leftrightarrow \mathbb{C}^n$ as real Γ-representations induces an action on $\mathrm{Fl}_{\mathcal{E}}(\mathbb{R}^{2n})$, $\mathcal{E} = (e_1, \dots, e_r)$. The flag manifold $\mathrm{Fl}_{\mathcal{E}}(\mathbb{R}^N)$ has a Schubert cell decomposition

$$\Omega_I(A_\bullet) = \{F_\bullet \in \mathrm{Fl}_{\mathcal{E}}(\mathbb{R}^N) : \dim F_i \cap A_k = r_I(i,k)\}, \tag{4.1}$$

where

$$I \in \mathrm{OSP}(\mathcal{E}) = S_N/(S_{e_1} \times \ldots \times S_{e_r})$$

is an *ordered set partition*, and $r_I(i,k) = \#\{l \in I_1 \cup \ldots \cup I_i : l \leq k\}$, see also Section B.2.1. If $2\mathcal{D} = (2d_1, 2d_2, \dots, 2d_r)$ and $I \in \mathrm{OSP}(\mathcal{D})$, then *the doubled ordered set partition* $DI \in \mathrm{OSP}(2\mathcal{D})$ is obtained by replacing each $i \in I_j$ by $(2i-1, 2i) \in DI_j$. A *double Schubert variety* $\sigma^{\mathbb{R}}_{DI} \subseteq \mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$ is a Schubert variety corresponding to $DI \in \mathrm{OSP}(2\mathcal{D})$. In the case of the Grassmannian $\mathcal{D} = (k,l)$, $DI \in \binom{2(k+l)}{2k}$ corresponds to the Young diagram obtained by subdividing each square into 2×2 squares in the Young diagram corresponding to $I \in \binom{k+l}{k}$, see Figure 4.1.

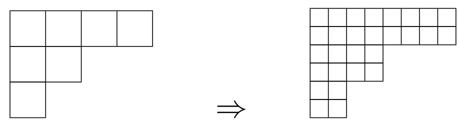


Figure 4.1: The double of a Young diagram

By Proposition C.1.9, the double Schubert varieties $\sigma^{\mathbb{R}}_{DI}$ are cycles and their classes form a basis of $H^*(\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}};\mathbb{Q})$. Using circle spaces we can deduce their structure constants.

Theorem 4.2.2. *With the pseudo Galois type action,* Γ *acting on* $\mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^{2n})$ *is a circle space, with fixed point set* $\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^n)$*. Furthermore*

$$\kappa[\sigma^{\mathbb{R}}_{DI}] = 2^{|I|}[\sigma^{\mathbb{C}}_{I}],$$

where $[\sigma^{\mathbb{C}}_{I}] \in H^{2|I|}(\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^N))$.

Proof. By the generalized Borel-Haefliger theorem, it is enough to show that a) for an appropriate complete real flag $F_\bullet$, the Schubert varieties $\sigma^{\mathbb{R}}_{DI}(F_\bullet)$ are halving cycles, b) form a basis of rational cohomology and c) have fixed point set $\sigma^{\mathbb{C}}_{I}(F^{\mathbb{C}}_\bullet)$. The $[\sigma^{\mathbb{R}}_{DI}]$ form a basis by Proposition C.1.9, so b) holds. It remains to choose a flag $F_\bullet$ satisfying a) and c).

a) Let $F_\bullet$ be a complete flag in $\mathbb{R}^{2n}$, such that F_{2i} are Γ-invariant, and let $F^{\mathbb{C}}_\bullet$ denote the corresponding complex flag $(F_0, F_2, \dots, F_{2n})$ in $\mathbb{C}^n$ by the identification $\mathbb{R}^{2n} \leftrightarrow \mathbb{C}^n$. Then, $\sigma^{\mathbb{R}}_{DI}(F_\bullet)$ are halving cycles. Indeed, by the rank conditions (4.1), the complex points of $\sigma^{\mathbb{R}}_{DI}(F_\bullet)$ are the points of $\sigma^{\mathbb{C}}_{I}(F^{\mathbb{C}}_\bullet)$ since

$$\dim_{\mathbb{R}}(W \cap W') = 2k \iff \dim_{\mathbb{C}}(W \cap W') = k$$

for any Γ-invariant subspaces $W, W' \leq \mathbb{R}^{2n}$. Therefore for this choice of $F_\bullet$, the $\sigma^{\mathbb{R}}_{DI}(F_\bullet)$ are Γ-invariant and $(\sigma^{\mathbb{R}}_{DI}(F_\bullet))^\Gamma = \sigma^{\mathbb{C}}_{I}(F^{\mathbb{C}}_\bullet)$, so c) holds. A dimension count shows

$$\operatorname{codim}_{\mathbb{R}} \sigma^{\mathbb{R}}_{DI}(F_\bullet) = 2 \operatorname{codim}_{\mathbb{R}} \sigma^{\mathbb{C}}_{I}(F^{\mathbb{C}}_\bullet)$$

and since $\Omega^{\mathbb{C}}_I$ is the unique top stratum of $\sigma^{\mathbb{C}}_I$, the $\sigma^{\mathbb{R}}_{DI}$ are good and therefore halving cycles by Example 1.2.13. Finally, the computation in Section D.3.1 shows that the normal weights are all 2.

□

Corollary 4.2.3 (Littlewood-Richardson coefficients)**.**

$$[\sigma^{\mathbb{R}}_{DI}] \cdot [\sigma^{\mathbb{R}}_{DJ}] = \sum_K c^K_{IJ} [\sigma^{\mathbb{R}}_{DK}]$$

where c^K_{IJ} are the same Littlewood-Richardson coefficients as in

$$[\sigma^{\mathbb{C}}_{I}] \cdot [\sigma^{\mathbb{C}}_{J}] = \sum_K c^K_{IJ} [\sigma^{\mathbb{C}}_{K}].$$

Proof. By the Theorem and multiplicativity of κ,

$$\kappa([\sigma^{\mathbb{R}}_{DI}] \cdot [\sigma^{\mathbb{R}}_{DJ}]) = \kappa([\sigma^{\mathbb{R}}_{DI}]) \cdot \kappa([\sigma^{\mathbb{R}}_{DJ}]) = 2^{|I|+|J|}[\sigma^{\mathbb{C}}_{I}] \cdot [\sigma^{\mathbb{C}}_{J}]$$

Applying κ to the basis expansion, one gets

$$\kappa\left(\sum_K c^K_{IJ}[\sigma^{\mathbb{R}}_{DK}]\right) = 2^{|I|+|J|} \sum_K c^K_{IJ}[\sigma^{\mathbb{C}}_{K}]$$

□

Corollary 4.2.4 (Giambelli formula type description)**.**

$$[\sigma^{\mathbb{R}}_{DI}] = q(p_*(S^{\mathbb{R}}_i)) \qquad \Longleftrightarrow \qquad [\sigma^{\mathbb{C}}_{I}] = q(c_*(S^{\mathbb{C}}_i)),$$

that is the same polynomial describes the double real Schubert classes and complex Schubert classes in terms of Pontryagin and Chern classes.

Proof. Since κ is a ring isomorphism and $S^{\mathbb{R}}_i$ is associated to a Γ-equivariant principal $\mathrm{GL}(2s_i,\mathbb{R})$-bundle (Example E.1.9), which has a Γ-equivariant classifyng map $\mathcal{K} : X \to BG$, the claim follows from Proposition 5.1.5 and Example 5.1.9. □

Corollary 4.2.5.

$$H^*(\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}) = \mathbb{Q}[p_*(S^{\mathbb{R}}_i)]/\mathcal{R}(p_*(S^{\mathbb{R}}_i)) \qquad \Longleftrightarrow \qquad H^*(\mathrm{Fl}^{\mathbb{C}}_{\mathcal{D}}) = \mathbb{Q}[c_*(S^{\mathbb{C}}_i)]/\mathcal{R}(c_*(S^{\mathbb{C}}_i)),$$

where $\mathcal{R}(x^i_*)$ denotes an ideal in the variables x^i_j, that is the same polynomial relations hold in the two cohomology rings in terms of Pontryagin and Chern classes of the respective tautological bundles.

Proof. Since $p_j(S_i^{\mathbb{R}})$ are algebra generators, and $\kappa(p_j(S_i^{\mathbb{R}})) = 2^j c_j(S_i^{\mathbb{C}})$, this follows from multiplicativity of κ. $\square$

Corollary 4.2.6 (Equivariant Giambelli formula)**.** *For the case of Grassmannians* $\mathrm{Fl}_{2\mathcal{D}}^{\mathbb{R}} = \mathrm{Gr}_{2k}(\mathbb{R}^{2(k+l)})$*,* $\mathcal{D} = (k, l)$*, the (doubled) Giambelli formula holds, even* Γ*-equivariantly*

$$[\sigma_{D\lambda}]_\Gamma = \det \begin{vmatrix} [\sigma_{D\lambda_1}]_\Gamma & [\sigma_{D(\lambda_1+1)}]_\Gamma & \cdots & [\sigma_{D(\lambda_1+k)}]_\Gamma \\ [\sigma_{D(\lambda_2-1)}]_\Gamma & [\sigma_{D\lambda_2}]_\Gamma & \cdots & [\sigma_{D(\lambda_2+k-1)}]_\Gamma \\ \vdots & \vdots & \ddots & \vdots \\ [\sigma_{D(\lambda_k-k)}]_\Gamma & \cdots & \cdots & [\sigma_{D\lambda_k}]_\Gamma \end{vmatrix}$$

where $D\lambda$ *denotes the double of the partition* $\lambda \subseteq k \times l$ *and* $Da = (2a, 2a)$ *for* $a \in \mathbb{Z}$*.*

Proof. Nonequivariantly, this follows from the complex Giambelli formula and from X being a circle space with $\kappa[\sigma_{D\lambda}^{\mathbb{R}}] = 2^{|\lambda|}[\sigma_\lambda^{\mathbb{C}}]$. Equivariantly, this follows from σ being multiplicative (Corollary 2.1.5) and from the generalized Borel-Haefliger theorem, $\sigma[\sigma_{D\lambda}] = [\sigma_{D\lambda}]_\Gamma$. $\square$

4.2.3 Real Grassmannians

The Casian-Kodama conjecture [CK13] concerns the cohomology ring structure of real Grassmannians. The characteristic class description was completely settled even equivariantly by [He16], [Sad17], [Car16]. However, it seems that the question of the cohomology ring structure in terms of Schubert cycles (fundamental classes of Schubert varieties) has not yet been addressed. We complete this description by using the generalized Borel-Haefliger theorem, and via the computation of incidence coefficients in Appendix C.

First, we describe the additive structure, see Proposition 4.2.7. To deduce the multiplicative structure, we show that $\mathrm{Gr}_K(\mathbb{R}^N)$ are circle spaces, except when K is odd, N is even, see Proposition 4.2.8. This gives some examples for nonorientable circle spaces. For K odd, N even, we use an additional geometric argument to deduce the multiplicative structure, see Proposition 4.2.9.

Additive structure

A convenient way to parametrize the Schubert varieties in Grassmannians is by Young diagrams $\lambda \subseteq K \times (N-K)$. One has the following conversion formulas between $\lambda \subseteq K \times (N-K)$ and

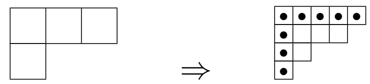


Figure 4.2: The L-operation on $\lambda = (3, 1, 0) \subseteq 3 \times 4$

$I \in \binom{N}{K}$:

$$\lambda_j = N - K + j - I_j, \qquad I_j = N - K + j - \lambda_j. \tag{4.2}$$

Before stating the Schubert cycle description of $H^*(\mathrm{Gr}_K(\mathbb{R}^N); \mathbb{Q})$ it is convenient to introduce the *L-operation* on Young diagrams: given $\lambda \subseteq K \times (N - K)$, let $L\lambda \subseteq (K + 1) \times (N - K + 1)$ be the partition

$$L\lambda := (N - K + 1, \lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_K + 1).$$

In terms of Young diagrams, the diagram contains the first row and column, and the complement of this L-shape is the Young diagram λ, see Figure 4.2 (the added L-shape is marked with bullet points). We call the corresponding Schubert varieties $\sigma_{L\lambda}$ *L-Schubert varieties*. Recall the definition of the double $D\lambda \subseteq 2k \times 2(n - k)$ of a Young diagram $\lambda \subseteq k \times (n - k)$ (Figure 4.1).

Proposition 4.2.7.

$$H^*(\mathrm{Gr}_K(\mathbb{R}^N); \mathbb{Q}) = \begin{cases} \mathbb{Q}\Big\langle [\sigma_{D\lambda}], [\sigma_{L(D\lambda)}] : \lambda \subseteq k \times (n - k) \Big\rangle & N \textit{ even } K \textit{ odd} \\ \mathbb{Q}\Big\langle [\sigma_{D\lambda}] : \lambda \subseteq k \times (n - k) \Big\rangle & \textit{else.} \end{cases}$$

where $k = \lfloor K/2 \rfloor$, $n = \lfloor N/2 \rfloor$.

Proof. The computation of the incidence coefficients in Appendix C is summarized in equation (C.3) (modulo sign). For Grassmannians, it can be rewritten in terms of Young diagrams as follows (see also [CK13]). If μ is a partition obtained from λ by increasing λ_j by 1, then

$$[\sigma_\lambda, \sigma_\mu] = \begin{cases} 0 & \lambda_j - j \text{ odd} \\ \pm 2 & \lambda_j - j \text{ even} \end{cases}$$

In particular, it follows that σ_λ is a cycle if and only if for all partitions μ obtained by increasing λ_j by 1, $\lambda_j - j$ is odd. Pictorially this means that the 'inner' corners on the Young diagram λ only lie on even antidiagonals. This implies that double Schubert varieties $\sigma_{D\lambda}$ and double L-Schubert varieties $\sigma_{L(D\lambda)}$ are cycles – possibly torsion.

However, the boundary relations imply that the coefficient of $\sigma_{D\lambda}$ in every incidence relation vanishes:

$$[\sigma_{D\lambda}, \sigma_\mu] = 0$$

for all $\dim \sigma_\mu = \dim \sigma_{D\lambda} + 1$, so each $[\sigma_{D\lambda}]$ do not appear in any relations and therefore are linearly independent. A similar computation shows that if K odd and N even, $[\sigma_{L(D\lambda)}]$ appear with zero coefficient in every incidence relation, so each of them generates a free $\mathbb{Q}$-submodule.

The Cartan description of Section C.2 implies that

$$\dim_{\mathbb{Q}} H^*(\mathrm{Gr}_K(\mathbb{R}^N); \mathbb{Q}) = \begin{cases} 2\binom{n}{k} & N \text{ even } K \text{ odd} \\ \binom{n}{k} & \text{else.} \end{cases}$$

which implies that these Schubert classes form a basis. □

Multiplicative structure

The $\mathrm{Gr}_K(\mathbb{R}^N)$ are circle spaces unless K is odd and N is even. If K and N is even, this is contained in Theorem 4.2.2. The remaining cases: K odd N even and K even N odd are both nonorientable, see Corollary B.2.9. This gives examples of nonorientable circle spaces.

Identify $\mathbb{R}^{2n+1} = \mathbb{R} \oplus \mathbb{C}^n$ as $\Gamma = \mathrm{U}(1)$-representations; let the trivial representation be the first coordinate $\mathbb{R} = \langle e_1 \rangle$. This induces actions on $\mathrm{Gr}_{2k+1}(\mathbb{R}^{2n+1})$ and $\mathrm{Gr}_{2k}(\mathbb{R}^{2n+1})$, whose fixed point set can be identified with $\mathrm{Gr}_k(\mathbb{C}^n)$.

Proposition 4.2.8. *The natural inclusions induce isomorphisms*

$$H^*(\mathrm{Gr}_{2k}(\mathbb{R}^{2n})) \cong H^*(\mathrm{Gr}_{2k}(\mathbb{R}^{2n+1})) \cong H^*(\mathrm{Gr}_{2k+1}(\mathbb{R}^{2n+1}))$$

compatibly with the descriptions given in Corollaries 4.2.3, 4.2.4, 4.2.5.

Proof. Let $F_\bullet^{\mathbb{R}} \in \mathrm{Fl}(\mathbb{R}^{2n+1})$ be the standard complete flag. Then by our definition of the U(1)-action, $F_\bullet^{\mathbb{C}} = (\langle e_2, e_3 \rangle, \langle e_2, e_3, e_4, e_5 \rangle, \ldots, \langle e_2, \ldots, e_{2n+1} \rangle)$ is a complete complex flag $F_\bullet^{\mathbb{C}} \in \mathrm{Fl}(\mathbb{C}^n)$. The rank description (B.3) implies that double Schubert varieties $\sigma_{D\lambda}(F_\bullet^{\mathbb{R}})$ are halving cycles, with fixed point set $\sigma_\lambda^{\mathbb{C}}(F_\bullet^{\mathbb{C}})$.

Since we are no longer in the orientable case, in order to use the generalized Borel-Haefliger theorem, we need that the Grassmannians are Poincaré duality spaces and that $\mathrm{fd}(X) = 2\,\mathrm{fd}(X^\Gamma)$ according to Remark 3.1.2 i). Both of these follow from the Cartan description (Section C.2): any flag manifold is a Poincaré duality space, and $\mathrm{fd}(X) = 4k(n-k)$ and $\mathrm{fd}(X^\Gamma) = 2k(n-k)$. Then one can apply the generalized Borel-Haefliger theorem as in the proof of Theorem 4.2.2.

Finally, that the natural inclusions induce the isomorphisms can be shown as follows: the characteristic classes are mapped into each other via the natural inclusions, so there is a system of generators mapped into a system of generators with the same relations. For example,

$$i : \mathrm{Gr}_{2k}(\mathbb{R}^{2n}) \hookrightarrow \mathrm{Gr}_{2k}(\mathbb{R}^{2n+1})$$

pulls back $i^*p_j(S^{2n+1}) = p_j(S^{2n})$ and $i^*p_j(Q^{2n+1}) = p_j(Q^{2n})$ (even though the pullback $i^*Q^{2n+1} = Q^{2n} \oplus \varepsilon$.) $\square$

Proposition 4.2.9. *The structure constants of $[\sigma_{D\lambda}]$ and $[\sigma_{L(D\lambda)}]$ in $H^*(\mathrm{Gr}_{2k+1}(\mathbb{R}^{2n}))$ are completely determined by the Littlewood-Richardson structure constants of $[\sigma_{D\lambda}]$ (Corollary 4.2.3) and*

$$[\sigma_{D\lambda}] \cdot [\sigma_{L0}] = [\sigma_{L(D\lambda)}], \qquad [\sigma_{L0}]^2 = 0.$$

Proof. Since $[\sigma_{L0}] \in H^{2n-1}$ lives in odd degree, and multiplication is graded commutative, $[\sigma_{L0}]^2$ is 2-torsion, therefore zero rationally.

To show $[\sigma_{D\lambda}] \cdot [\sigma_{L0}] = [\sigma_{L(D\lambda)}]$, we use the following lemma. $\square$

Lemma 4.2.10. *In $\mathrm{Gr}_k(\mathbb{R}^n)$, for appropriate transverse flags $E_\bullet, F_\bullet$ there exists a flag $G_\bullet$, such that*

$$\sigma_{L0}(E_\bullet) \cap \sigma_\lambda(F_\bullet) = \sigma_{L\lambda}(G_\bullet) \tag{4.3}$$

Proof. Let $E_\bullet$ be the standard flag, $F_\bullet$ the opposite flag

$$F_\bullet = \langle e_n \rangle \le \langle e_n, e_{n-1} \rangle \le \ldots \le \langle e_n, e_{n-1}, \ldots e_2 \rangle \le \langle e_n, e_{n-1}, \ldots, e_1 \rangle$$

and $G_\bullet$ be obtained from $F_\bullet$ by exchanging e_1 and e_n:

$$G_\bullet = \langle e_1 \rangle \le \langle e_1, e_{n-1} \rangle \le \ldots \le \langle e_1, e_{n-1}, \ldots e_2 \rangle \le \langle e_1, e_{n-1}, \ldots, e_3, e_2, e_n \rangle.$$

$E_\bullet$ and $F_\bullet$ are transverse flags, which imply that the Schubert varieties $\sigma_{L0}(E_\bullet) \cap \sigma_\lambda(F_\bullet)$ intersect transversely. The rank conditions defining $\sigma_{L0}(E_\bullet)$ translate to

$$U \in \sigma_{L0} \qquad \Longleftrightarrow \qquad E_1 \leq U \leq E_{n-1}$$

in particular, $\sigma_{L0}(E_\bullet)$ is a subGrassmannian $\mathrm{Gr}_k(\mathbb{R}^{n-2})$. The following observation allows us to conclude: if $E_1 \leq U \leq E_{n-1}$, then

$$\dim(U \cap F_j) = k \qquad \Longleftrightarrow \qquad \dim(U \cap G_j) = k+1$$

By comparing the rank conditions defining σ_λ and $\sigma_{L\lambda}$ the two sides of (4.3) are equal. $\square$

Remark 4.2.11. i) The previous lemma was stated for arbitrary k and n; a simple verification shows that the subGrassmannian σ_{L0} is always coorientable and therefore a cycle. However, the computation which shows that it appears in every incidence relation with zero coefficient only holds if k is odd and n is even, otherwise $[\sigma_{L0}]$ is a 2-torsion element.

ii) If one defines the usual U(1)-action on $\mathbb{R}^{2n}$ by identifying it with $\mathbb{C}^n$, the induced action on $\mathrm{Gr}_{2k+1}(\mathbb{R}^{2n})$ has no fixed points; indeed, $\mathbb{C}^n$ has no real odd dimensional invariant subspaces. $\mathrm{Gr}_{2k}(\mathbb{R}^{2n+1})$ has zero Euler characteristic so it cannot be a circle space, see also Section 4.2.5.

Richardson varieties

The original Borel-Haefliger theorem states that for *any* complexified subvariety $Z \subseteq \mathrm{Gr}_k(\mathbb{C}^n)$, $\kappa[Z] = [Z^\Gamma]$ holds, and not only for the basis of Schubert varieties. For example, one could consider *Richardson varieties*

$$\sigma^{\mathbb{C}}_{\lambda,\mu} := \sigma^{\mathbb{C}}_\lambda(F_\bullet) \cap \sigma^{\mathbb{C}}_\mu(F'_\bullet)$$

for transversal real flags $F_\bullet \pitchfork F'_\bullet$, $\lambda, \mu \in \binom{n}{k}$. Applying the mod 2 Borel-Haefliger theorem yields that expressing the fundamental class of a complex Richardson variety $[\sigma^{\mathbb{C}}_{\lambda,\mu}]$ in tautological Chern classes agrees with the real $[\sigma^{\mathbb{R}}_{\lambda,\mu}]$ in tautological Stiefel-Whitney classes, as in the proof of Corollary 4.2.4.

By the generalized Borel-Haefliger theorem, the same type of relation holds in the case of circle spaces. In particular, for real double Richardson varieties in $\mathrm{Gr}_{2k}(\mathbb{R}^{2n})$:

Proposition 4.2.12.
$$\kappa[\sigma^{\mathbb{R}}_{D\lambda,D\mu}] = 2^{|\lambda|+|\mu|}[\sigma^{\mathbb{C}}_{\lambda,\mu}].$$

4.2.4 Quaternionic flag manifolds

Recall the Galois type $\Gamma = \mathrm{U}(1)$-action on quaternionic flag manifolds. Let $\mathbb{H}^n$ be a (right) quaternionic vector space. Then $\Gamma = \mathrm{U}(1) \subseteq \mathbb{C} = \langle 1, i \rangle \leq \mathbb{H}$ acts on $\mathbb{H}$ via inner (left-right) conjugation, which induces an action on $\mathbb{H}^n$ and $\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^n)$, with fixed point set $\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^n)$. The flag manifold $\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^n)$ has a Schubert cell decomposition $\sigma_I^{\mathbb{H}}$ where $I \in \mathrm{OSP}(\mathcal{D})$, see Section B.3.

Theorem 4.2.13. *With the Galois type* $\Gamma = \mathrm{U}(1)$*-action (from the left),* $\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^n)$ *is a circle space, with fixed point set* $\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^n)$*. Furthermore*

$$\kappa[\sigma_I^{\mathbb{H}}] = 2^{|I|}[\sigma_I^{\mathbb{C}}],$$

where $[\sigma_I^{\mathbb{C}}] \in H^{2|I|}(\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^n))$*.*

Proof. Similarly to the case of real even flag manifolds, by the generalized Borel-Haefliger theorem it is enough to show that a) the Schubert varieties $\sigma_I^{\mathbb{H}}(F_\bullet)$ are halving cycles with respect to an appropriate complete flag $F_\bullet \in \mathrm{Fl}(\mathbb{H}^n)$, b) $[\sigma_I^{\mathbb{H}}]$ form a basis of rational cohomology and c) they have fixed points $\sigma_I^{\mathbb{C}}(F_\bullet^{\mathbb{C}})$.

a) If $F_\bullet^{\mathbb{C}}$ is a complex flag, then $F_\bullet := F_\bullet^{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{H}$ is a quaternionic flag, which is Γ-invariant, therefore the $\sigma_I^{\mathbb{H}}(F_\bullet)$ are also Γ-invariant. Here being a good cycle is more straightforward by Example 1.2.13, as all strata are even dimensional and Γ-invariant, whose fixed point sets give a stratification of $\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^n)$. This also implies b).

c) The fixed point sets of $\sigma_I^{\mathbb{H}}(F_\bullet)$ are $\sigma_I^{\mathbb{C}}(F_\bullet^{\mathbb{C}})$ by the observation that for any complex subspaces $W, W' \leq \mathbb{C}^n$,

$$\dim_{\mathbb{H}}(W_{\mathbb{H}} \cap W'_{\mathbb{H}}) = k \iff \dim_{\mathbb{C}}(W \cap W') = k.$$

Then the rank conditions describing Schubert varieties imply c). To conclude part a), a dimension count shows that the $\sigma_I^{\mathbb{H}}(F_\bullet)$ are halving cycles. The normal weights are all $2u$ as the computation in Section D.3.2 shows. $\square$

Corollary 4.2.14 (Littlewood-Richardson coefficients)**.**

$$[\sigma_I^{\mathbb{H}}] \cdot [\sigma_J^{\mathbb{H}}] = \sum_K c_{IJ}^K [\sigma_K^{\mathbb{H}}]$$

where c_{IJ}^K are the same Littlewood-Richardson coefficients as in

$$[\sigma_I^{\mathbb{C}}] \cdot [\sigma_J^{\mathbb{C}}] = \sum_K c_{IJ}^K [\sigma_K^{\mathbb{C}}].$$

Proof. Exactly the same as Corollary 4.2.3. □

Corollary 4.2.15 (Giambelli formula type description)**.**

$$[\sigma_I^{\mathbb{H}}] = q(p_*(S_i^{\mathbb{H}})) \qquad \Longleftrightarrow \qquad [\sigma_I^{\mathbb{C}}] = q(c_*(S_i^{\mathbb{C}})),$$

where p_* denotes quaternionic Pontryagin classes. In words, the same polynomial describes the quaternionic and complex Schubert varieties in terms of characteristic classes.

In the case of Grassmannians, this was already noticed by Pragacz-Ratajski [PR97]; as they remark, the proof of the Pieri formula in Griffiths-Harris [GH78] can be replicated in the quaternionic case implying the same description of the cohomology rings (complex and quaternionic) with degrees doubled.

4.2.5 Further examples - remarks

We conclude this section by some remarks on finding further examples.

First, no complex projective variety X can be a circle space; $H^2(X)$ contains the non-zero hyperplane section, which violates the condition of having nonzero cohomology groups only in degrees $4i$.

Second, if Γ acting on a homogeneous space $X = G/H$ is a circle space, then $\mathrm{rk}(G) = \mathrm{rk}(H)$. Indeed, it is classical (e.g. [GHZ06]), that the Euler-characteristic of a homogeneous space is zero if $\mathrm{rk}(G) > \mathrm{rk}(H)$, which means that X has nonzero cohomology in some odd degree, again violating the condition on even degrees. Indeed, all of the examples above satisfied this condition.

In the case of $\Gamma = \mathbb{Z}_2$, smooth toric manifolds are conjugation spaces [HHP05, Example 8.7]. There is a quaternionic analogue of toric varieties introduced by Scott [Sco95], which come naturally equipped with an SO(3)-action. The cohomology of such a space is degree-doubling isomorphic to its complex counterpart [Sco93, Theorems 3.3.2. and 5.5.1], which suggests that these are also circle spaces (via the $\mathrm{SO}(2) \leq \mathrm{SO}(3)$-action). This still needs to be verified. I am indebted to Matthias Franz for recommending this example.

A remark on the weights

In the main examples of circle spaces encountered the normal weights were equal to 2, one might wonder whether this has some deeper geometric meaning. The answer is no: the actions above can be modified in such a way to obtain circle spaces of arbitrary normal weights.

This follows from the following trivial remark. Any action $\alpha : \Gamma \to \mathrm{Aut}(X)$ factors through $\Gamma/\ker(\alpha)$. This applies to $\Gamma = \mathrm{U}(1)$ acting on $X = \mathrm{Gr}_{2k}(\mathbb{R}^{2n})$; it is not hard to see that there are two types of orbits: $\mathrm{U}(1)/\mathbb{Z}_2$ and fixed points, so $\ker(\alpha) = \mathbb{Z}_2$. In particular, with this action $\mathrm{U}(1)/\mathbb{Z}_2 \cong \mathrm{U}(1)$ acts semi-freely on X.

For a semi-free action, all normal weights are one, which can be shown using the slice theorem. Conversely, given a semi-free action, by precomposing with the n-fold cover $\pi_n : \mathrm{U}(1) \to \mathrm{U}(1)$, one can obtain any normal weight n. Explicitly, the whole Borel construction factors through this lift: $B\pi_2 : E\Gamma \times_{\pi_2} \Gamma \to E\Gamma$ induces a commutative diagram

$$\begin{tikzcd} H^*_\Gamma \arrow[r] \arrow[d, "B\pi_2^*"] & H^*_\Gamma(\tilde{X}) \arrow[r, "\tilde{r}"] \arrow[d, "\widetilde{B\pi}_2^*"] & H^*_\Gamma(X^\Gamma) \arrow[d] \\ H^*_\Gamma \arrow[r] & H^*_\Gamma(X) \arrow[r, "r"] & H^*_\Gamma(X^\Gamma) \end{tikzcd}$$

where $\tilde{X}$ denotes X as a Γ-space with the modified action $\tilde{\alpha} : \Gamma \to \mathrm{Aut}(X)$, and

$$\widetilde{B\pi_2} : B_\Gamma X = (E\Gamma \times_{\pi_2} \Gamma) \times_{\tilde{\alpha}} X \to E\Gamma \times_{\tilde{\alpha}} X.$$

However, notice that the $\mathrm{U}(1)$-action on $\mathrm{Gr}_{2k}(\mathbb{R}^{2n})$ only lifts to the tautological bundle $S_\mathbb{R}$ when this weight is even. This also indicates the natural appearance of the powers of 2 in the proof of Example 2.2.1. When an extra G-action is involved as in the equivariant Borel-Haefliger theorem, the Γ and G-actions can only be modified together so that the actions remain compatible.

4.3 Quaternionic halving spaces

In this section we give some examples for quaternionic halving spaces, i.e. halving spaces for $\mathrm{Sp}(1)$: the octonionic flag manifolds. We will also show that they are circle spaces by restricting to $\mathrm{U}(1) \leq \mathrm{Sp}(1)$. These are the only two examples of quaternionic halving spaces that we are aware of.

4.3.1 Octonionic flag manifolds

By octonionic flag manifolds we mean the following three examples: $\mathbb{O}P^1, \mathbb{O}P^2, \mathrm{Fl}(\mathbb{O})(= \mathrm{Fl}(\mathbb{O}^3))$. Nonassociativity of octonions leads to the fact that there are no analogues of higher dimensional octonionic flag manifolds. We briefly summarize what we need about these spaces in Appendix B.4 and refer to [Bae02], [Fre85], [Esc], [MW13] for further details. Since $\mathbb{O}P^1 \cong S^8$ which is easily seen to be both a circle space and a quaternionic halving space, we start with $\mathbb{O}P^2$.

Octonionic projective plane: $\mathbb{O}P^2$

In the case of $\mathbb{OP}^2$ a purely topological proof can be given using Hopf fibrations.

Proposition 4.3.1. *The Hopf fibrations are Γ-equivariant principal G-bundles (in the sense of Appendix E) each with fixed point set the previous Hopf fibration:*

- $\pi_2 : S^1 \to S^3 \to S^2$ *is* $\Gamma = \mathbb{Z}_2$*-equivariant with fixed point set* $\pi_1 : S^0 \to S^1 \to S^1$
- $\pi_3 : S^3 \to S^7 \to S^4$ *is* $\Gamma = \mathrm{U}(1)$*-equivariant with fixed point set* $\pi_2 : S^1 \to S^3 \to S^2$
- $\pi_4 : S^7 \to S^{15} \to S^8$ *is* $\Gamma = \mathrm{Sp}(1)$*-equivariant with fixed point set* $\pi_3 : S^3 \to S^7 \to S^4$

Proof. The definition of these bundles involves the division algebra structure of $\mathbb{F}$, so they are naturally $\mathrm{Aut}(\mathbb{F})$-equivariant. The fixed point sets follow from the Galois type actions: Examples 4.1.1, 4.1.2, 4.1.3. $\square$

Corollary 4.3.2. $\mathbb{O}P^2$ *is a halving space with the Galois type* $\mathrm{U}(1)$*-action, with fixed point set* $\mathbb{H}P^2$*.*

Proof sketch. The projective planes $\mathbb{FP}^2$ can be obtained by gluing along the Hopf fibrations:

$$\mathbb{R}P^2 = D^2 \coprod_{\pi_1} S^1, \quad \mathbb{C}P^2 = D^4 \coprod_{\pi_2} S^2, \quad \mathbb{H}P^2 = D^8 \coprod_{\pi_3} S^4, \quad \mathbb{O}P^2 = D^{16} \coprod_{\pi_4} S^8$$

and the Γ-action descends to the projective planes, with fixed point set the the previous one. From the naturally occurring cell decompositions we get that each space is a halving space with the fixed point set the previous one, in particular we get that $\mathbb{O}P^2$ is a quaternionic halving/circle space. Alternatively, one can adapt the proof of the next section. $\square$

Octonionic flag manifold: $\mathrm{Fl}(\mathbb{O})$

$\mathbb{O}P^2$ has a description by (restricted) homogeneous coordinates as follows. The *points of* $\mathbb{O}P^2$ are triples $(a,b,c) \in \mathbb{O}^3$, such that at least one of them is real, modulo the relation that two such elements are equal if they differ by left-multiplication by an element of $\mathbb{O}$. The *lines* $(\mathbb{O}P^2)^*$ *of* $\mathbb{O}P^2$ are defined similarly, but now the equivalence relation is right multiplication. A point $x = (x_1, x_2, x_3) \in \mathbb{O}P^2$ *is incident to the line* $l = (l_1, l_2, l_3) \in (\mathbb{O}P^2)^*$ denoted $x \in l$, if $x_1 l_1 + x_2 l_2 + x_3 l_3 = 0$ for representatives chosen such that at least two of the sets $\{x_i, l_i\}$ contain a real number. The *flag manifold* $\mathrm{Fl}(\mathbb{O})$ can be defined as the set of incident point-lines:

$$\mathrm{Fl}(\mathbb{O}) := \{(x,l) : x \in l\} \subseteq \mathbb{O}P^2 \times (\mathbb{O}P^2)^*.$$

Remark 4.3.3. The description in terms of coordinates is in fact isomorphic to the usual model of $\mathbb{O}P^2$ by the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$, see [Fre64], [GOMV94], [MW13]. These identifications are due to [Asl91], [All97], see also [Ros97, Theorem 7.2], [CK11].

There is a $\Gamma = \mathrm{Sp}(1)$-action on $\mathbb{O}$, with fixed point set $\mathbb{O}^\Gamma = \mathbb{H}$, see Proposition B.4.1. This induces a coordinate-wise action on $Y = \mathbb{O}P^2$ with fixed point set $Y^\Gamma = \mathbb{H}P^2$. Since the action is compatible with the incidence relation, it also induces an action on $X = \mathrm{Fl}(\mathbb{O})$ with fixed point set $X^\Gamma = \mathrm{Fl}(\mathbb{H}^3)$. For further details, see Proposition B.4.2.

Theorem 4.3.4. *With the Galois type* $\Gamma = \mathrm{Sp}(1)$*-action,* $\mathrm{Fl}(\mathbb{O})$ *is a quaternionic halving space, with fixed point set* $\mathrm{Fl}(\mathbb{H}^3)$*. Furthermore*

$$\kappa[\sigma_w^{\mathbb{O}}] = [\sigma_w^{\mathbb{H}}]$$

where $w \in S_3$ *and* $[\sigma_w^{\mathbb{H}}] \in H^{2|w|}(\mathrm{Fl}(\mathbb{H}^3))$*.*

Proof. The theorem follows from Proposition B.4.3 stating that for $d_\bullet \in \mathrm{Fl}(\mathbb{O})^\Gamma = \mathrm{Fl}(\mathbb{H}^3)$, the flag manifold $\mathrm{Fl}(\mathbb{O})$ has a decomposition into Γ-invariant Schubert $8i$-cells $\Omega_w^{\mathbb{O}}(d_\bullet)$, defined by incidence relations, whose fixed point sets are $\Omega_w^{\mathbb{H}}(d_\bullet)$. In particular, the closures of the Schubert cells are Γ-invariant halving cycles $\sigma_w^{\mathbb{O}}(d_\bullet)$ by a dimension count.

The conditions of the generalized Borel-Haefliger theorem for $\Gamma = \mathrm{Sp}(1)$ have to be checked according to Remark 3.1.2 ii). First, the Schubert cycles are $\mathrm{Sp}(1)$-invariant halving cycles and their fixed point sets are cycles (this is straightforward). Second, $\mathrm{Fl}(\mathbb{O})$ satisfies the localization theorem for $\mathrm{Sp}(1)$ by [tD87, Theorem III.3.8.], see also [MW13, Theorem 1.3].

□

Remark 4.3.5. i) By the general theory, $\mathrm{Fl}(\mathbb{O})$ has a Bruhat cell decomposition as N-orbits ([DKV83], [MW13], see also Appendix B.1.2). This agrees with the Schubert cell decomposition - this can be verified through the Jordan algebra model of $\mathbb{O}P^2$.

ii) These examples are also examples of circle spaces. Indeed, one can restrict the action of $\Gamma = \mathrm{Sp}(1)$ to a $\Delta = \mathrm{U}(1)$-action, with the same fixed point set.

4.4 Enumerative geometry: Schubert problems

The cohomology ring structure in terms of Schubert classes gives information about enumerative geometric *Schubert problems*:

Given generic complete flags $F_\bullet^1, \dots, F_\bullet^r$ in $\mathbb{F}^N$, what is the cardinality of

$$\left| \bigcap_{j=1}^{r} \sigma_{I_j}(F_\bullet^j) \right| = ?$$

for $\sigma_j := \sigma_{I_j}$ of total dimension $\mathrm{Fl}_{\mathcal{D}}(\mathbb{F}^N)$ $(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$? $I_j \in \mathrm{OSP}(\mathcal{D})$.

The word generic is a subtle point here: we will say that the flags are *generic*, if the corresponding Schubert varieties are transversal. This is an open condition in the configuration space by the Kleiman-Bertini theorem. The main property of generic configurations $\mathcal{G}$ relevant to us is that the number of solutions is locally constant on $\mathcal{G}$.

In the case of two flags, if the flags are transversal, then they are generic. This is no longer true if multiple Schubert varieties are involved, as we will illustrate below in Section 4.4.4.

4.4.1 Complex Schubert problems

In the complex case $\mathbb{F} = \mathbb{C}$, a Schubert problem can be solved by multiplying Schubert cycles: since everything is complex, at a smooth transversal intersection all tangent spaces have canonical orientations, therefore all intersections come with the same sign. Therefore the cohomology product of $[\sigma_j]$ is an element $n[*]$ of $H^{top}(\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^N)) \cong \mathbb{Z}\langle[*]\rangle$, and this number n is the answer to the Schubert problem.

4.4.2 Real Schubert problems - lower bounds

In the real case $\mathbb{F} = \mathbb{R}$, there are no canonical orientations, therefore each transversal intersection p comes with a sign, depending on whether the orientation of the tangent spaces $T_p\sigma_j$ agrees with the orientation of the tangent space $T_p\,\mathrm{Fl}$. Therefore the cohomological calculation only gives a signed sum of the points, hence a lower bound to the Schubert problem. The actual number of solutions depends on the configuration (the choice of the flags $F^i_\bullet$), and there is a range of numbers that might appear as the number of solutions. This range is not known in general, a special class of examples has been computed via elementary methods in the case of Grassmannians [FM16]. Recently [Wen] considered similar problems for Grassmannians using Chow-Witt rings. For example, in $\mathrm{Gr}_8(\mathbb{R}^{16})$ the number of solutions to the Schubert problem σ_λ^4 for $\lambda = (4,4,4,4)$ can be $\{6, 14, 30, 70\}$, see [FM16].

The dependence of the number of solutions on the given configuration has the following explanation. In the complex case, the singular configurations form an at least one complex codimensional subvariety of the configuration space, so the space of nonsingular configurations is connected. In the real case, the singular configurations can be one *real* codimensional, in which case the configuration space falls apart into connected components (*chambers*).

An upper bound for the range is given by the number of solutions for the corresponding generic complex Schubert problem. Here some caution is required when discussing genericity: one has to show that there exist real generic flags which are complex generic when regarded as complex flags. Indeed, this is the case: the subset of complex nongeneric configurations can be defined by real equations, so there exist real flags $F^j_\bullet$ which are complex generic. For such flags, all intersections of $\sigma_j^{\mathbb{C}}(F^j_\bullet)$ are transverse, therefore so are those of $\sigma_j^{\mathbb{R}}(F^j_\bullet)$, so such configurations are also real generic.

It is a natural question, whether a real enumerative problem is *maximal/fully real* [Sot97], i.e. whether there exists a configuration for which the number of solutions agrees with the number of solutions for the same complex problem. This is true for real Schubert problems in Grassmannians as shown in [Vak06].

Another natural question in real enumerative geometry, is to find a lower bound for the range of solutions [Wel05], [HHS13], and as we have already mentioned, the cohomology calculation gives such a lower bound. By the description of the real Littlewood-Richardson coefficients (Corollary 4.2.3) we have:

Proposition 4.4.1. *The number of solutions of a double real Schubert problem* (DI_j) *is bounded below by the number of solutions to the half sized complex one* (I_j)*.*

The cohomology of real Grassmannians (and flag manifolds) with integer coefficients contains $\mathbb{Z}_2$-torsion [He17]. If in a Schubert problem all Schubert varieties are cycles, but one of them is $\mathbb{Z}_2$-torsion, then the corresponding cycles multiply to zero in cohomology (at least if the flag manifold is orientable), which is uninteresting as a lower bound. Note however that the corresponding Schubert problem can be, and usually is nontrivial. Summarizing: for the purpose of obtaining enumerative lower bounds, we don't lose anything by working with rational coefficient cohomology.

Alternatively, using the original Borel-Haefliger theorem over $R = \mathbb{F}_2$, one can obtain mod 2 information about a Schubert problem:

Proposition 4.4.2. *The number of solutions of a real Schubert problem has the same parity as the number of solutions of the corresponding complex Schubert problem.*

We conclude with a question:

Question 1. *Is the lower bound of Proposition 4.4.1 sharp?*

We make some remarks without giving a complete answer. Take real complete flags $F^j_\bullet \in \mathrm{Fl}(\mathbb{R}^n)$, such that F^j_{2i} are U(1)-invariant. Then the set of solutions

$$S := \bigcap_j \sigma^{\mathbb{R}}_{DI_j}(F^j_\bullet)$$

is a U(1)-invariant subset. If S is finite, then each point $W \in S$ is a U(1)-fixed point, i.e. complex. Therefore, it is a solution to the corresponding half sized complex Schubert problem

$$W \in S_{\mathbb{C}} = \bigcap_j \sigma_{I_j}^{\mathbb{C}}(F_{\bullet,\mathbb{C}}^j).$$

This reduces the question to one about genericity: Does there exist a complex generic configuration of flags $F_{\bullet,\mathbb{C}}^j$, which as real flags $F_{\bullet,\mathbb{R}}^j$ are real generic (for the double sized real problem)?

4.4.3 Quaternionic Schubert problems

In the quaternionic case $\mathbb{F} = \mathbb{H}$ the situation can be reduced to the complex case:

Proposition 4.4.3. *The number of solutions of a quaternionic Schubert problem is the same as the corresponding complex Schubert problem.*

Proof. If the Schubert varieties $\sigma_j(F_\bullet^{\mathbb{H}})$ are transverse, then since the tangent spaces are canonically oriented (they are complex via $\mathrm{U}(1) \leq \mathbb{H}$), the cohomology computation gives the exact number of solutions. We can conclude by the Littlewood-Richardson coefficients of Corollary 4.2.14. □

Modulo a question on genericity left unanswered, we also give an elementary proof attempt by a direct reduction to the complex case:

Proof sketch. Consider the complex problem and its solutions $W_1, \dots, W_r \in \mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^N)$. Tensoring everything by $\otimes_{\mathbb{C}}\mathbb{H}$, we get a (right) quaternionic Schubert problem, w.r.t. flags $F_\bullet^j \otimes_{\mathbb{C}} \mathbb{H}$ whose solutions are $W_j \otimes_{\mathbb{C}} \mathbb{H}$. Indeed, for any complex subspace $W \leq \mathbb{C}^n$, $\dim_{\mathbb{C}} W = \dim_{\mathbb{H}}(W \otimes_{\mathbb{C}} \mathbb{H})$. To show that there are no more solutions, notice that the set of solutions S to such a problem is U(1)-invariant (the flags $F_\bullet^j$ are). If there are finitely many solutions, then each quaternionic subspace $U \in S$ is U(1)-fixed. Linear algebra shows that then it is of the form $W \otimes_{\mathbb{C}} \mathbb{H}$, so every solution must come from a complex solution (see Proposition D.2.3). It remains to answer the following question: □

Question 2. *Given a generic complex Schubert problem, is its quaternionification generic?*

Although the complex intersections will remain transverse, it could be the case that S has further components in $\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N) \setminus \mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^N)$. One can modify the definition of complex genericity as those complex problems whose quaternionification is generic; hopefully this is an open condition.

4.4.4 An example

We illustrate the subtleties concerning genericity on the following problem:

Problem 1. *How many lines intersect four given lines in $\mathbb{H}P^3$?*

By Proposition 4.4.3, the answer is the same as in the complex case, which is 2. Forgetting the quaternionic structure and remembering only the complex structure, we can ask the double of this problem:

Problem 2. *How many $W \in \mathrm{Gr}_4(\mathbb{C}^8)$ intersect four given $U_i \in \mathrm{Gr}_4(\mathbb{C}^8)$ in 2 dimensions?*

If the configuration is complex generic, this has more solutions, namely 6. 2 of these solutions are going to be $\mathrm{Sp}(1)$-invariant as subspaces, however note that the $\mathrm{Sp}(1)$-action does not descend to $\mathrm{Gr}_4(\mathbb{C}^8)$, since the $\mathrm{Sp}(1)$-action does not commute with the complex structure. The $\mathrm{Sp}(1)$-action does descend to $\mathrm{Gr}_8(\mathbb{R}^{16})$, so we can forget the complex structure, and consider the double sized real problem:

Problem 3. *How many $W \in \mathrm{Gr}_8(\mathbb{R}^{16})$ intersect four given $U_i \in \mathrm{Gr}_8(\mathbb{R}^{16})$ in 4 dimensions?*

Since the U_i obtained by forgetting the complex structure are $\mathrm{U}(1)$-invariant, the set of real solutions S is $\mathrm{U}(1)$-invariant. If the configuration is real generic, then there are finitely many solutions S, which are $U(1)$-fixed points, therefore solutions to Problem 2. It follows that the number of real solutions is six.

On the other hand, the problem was obtained from a quaternionic problem by forgetting the quaternionic structure, so the U_i are $\mathrm{Sp}(1)$-invariant, and so is the set of solutions S. As before, S is $\mathrm{Sp}(1)$-invariant ergo quaternionic, therefore by this argument there should only be 2 real solutions, which is a contradiction. The resolution of this seeming contradiction is that complex and real problems obtained by forgetting the quaternionic structure are usually *not real generic.*

A notable exception is if we intersect *two* Schubert varieties, as this is equivalent to the flags being transverse. For example, two lines in $\mathbb{H}P^2$ always intersect in a point.

On this concrete example, nongenericity can be seen even more explicitly as follows. To a given configuration $U_1, \dots, U_4$, one can associate a real linear map $\varphi : U_1 \to U_1$ with the property that the problem is real generic iff all four eigenvalues of this map are distinct, and different from 0 and 1, see [FM16, Remark 4.14]. If the U_i are complex (U(1)-invariant), the corresponding map $\varphi : U_1 \to U_1$ is complex linear. In case the complex eigenvalues of φ contain no complex conjugate pairs, then the problem is also real generic. However, if the U_i are quaternionic (Sp(1)-invariant) the map is quaternionic linear, and the real eigenvalues of $\varphi \in \mathrm{Sp}(1) = \mathrm{SU}(2)$ are of the form $\{\lambda, \lambda, \overline{\lambda}, \overline{\lambda}\}$, so the problem is *not* real generic.

Chapter 5

Applications - Equivariant Borel-Haefliger

In this chapter we discuss applications of the equivariant Borel-Haefliger theorem (Theorem 3.2.10). The equivariant Borel-Haefliger theorem provides a degree-halving ring ismorphism between $H_G^*(X)$ and $H_{G^\Gamma}^*(X^\Gamma)$, when X is a halving G-manifold and G is a halving group. We will give examples of halving groups in Section 5.1. We relate the characteristic classes of G and G^Γ in the formalism of halving spaces. In the rest of the chapter, we give applications of the equivariant Borel-Haefliger theorem for $\Gamma = \mathrm{U}(1)$. These include (real, rational) Thom polynomials (Section 5.2), Quiver Thom polynomials (Section 5.3), Matrix Schubert varieties (Section 5.4) and some applications when X is not a vector space (Section 5.5).

5.1 Halving groups

Recall that Γ acting on G by automorphisms is a *halving group*, if $EG \to BG$ has a Γ-approximation $E_k \to B_k$ with B_k halving spaces and $(EG)^\Gamma$ is contractible. Our main application of halving groups is the equivariant Borel-Haefliger theorem, which relates the G-equivariant cohomology of a halving space X with the G^Γ-equivariant cohomology of its fixed point set X^Γ.

5.1.1 Examples

In all examples below, it is well-known from characteristic class theory that there is a degree halving ring isomorphism between H^*_G and $H^*_{G^\Gamma}$. However, being a halving group is more, as this also involves the Γ-equivariant cohomology structure of BG. According to our definition of halving groups, we give Γ-approximations of BG which we already know are halving spaces and using Proposition 2.1.7 we can pass to the limit.

Conjugation-halving groups

Cohomology is taken with $\mathbb{F}_2$ coefficients.

Example 5.1.1. Let $\Gamma = \mathbb{Z}_2$ act on

a) $G = \mathrm{U}(1)$ by complex conjugation with fixed point set $G^\Gamma = \mathbb{Z}_2$.

b) $G = \mathrm{U}(1)^n$ by complex conjugation with fixed point set $G^\Gamma = \mathbb{Z}_2^n$.

c) $G = \mathrm{GL}(n,\mathbb{C})$ by complex conjugation with fixed point set $G^\Gamma = \mathrm{GL}(n,\mathbb{R})$.

Then G is a halving group. ♣

Proof. Let Γ act on

a) $EG = \mathrm{Inj}_\mathbb{C}(\mathbb{C},\mathbb{C}^\infty) \to \mathbb{C}P^\infty = BG$

b) $EG = \mathrm{Inj}_\mathbb{C}(\mathbb{C},\mathbb{C}^\infty)^{\times n} \to (\mathbb{C}P^\infty)^{\times n} = BG$

c) $EG = \mathrm{Inj}_\mathbb{C}(\mathbb{C}^n,\mathbb{C}^\infty) \to \mathrm{Gr}_n(\mathbb{C}^\infty) = BG$

by complex conjugaton. Then $EG \to BG$ is a (Γ, G)-bundle with fixed point set the universal bundle over BG^Γ which is a) $\mathbb{R}P^\infty$, b) $(\mathbb{R}P^\infty)^{\times n}$ c) $\mathrm{Gr}_n(\mathbb{R}^\infty)$. The (Γ, G)-bundle $EG \to BG$ has a Γ-approxmation by conjugation spaces via the natural inclusions, a) $\varinjlim_N \mathbb{C}P^N$, b) $\varinjlim_N (\mathbb{C}P^N)^{\times n}$ c) $\varinjlim_N \mathrm{Gr}_n(\mathbb{C}^{N+n})$. Clearly $(EG)^\Gamma$ is contractible, so G is a halving group. (The Γ-fixed point sets are the real Grassmannians, which approximate BG^Γ.) These examples already appear in [HHP05, Section 7]. □

In [Esh09, Section 2.4.5] the following question was considered. For which compact simply connected simple Lie groups G can BG be a conjugation space? Example 5.1.1 c) corresponds to $\mathrm{SU}(n)$, which is the type A simple Lie group, and $\mathbb{Z}_2$ acting on $\mathrm{Sp}(n)$ is type C (note that this action extends to the $\mathrm{U}(1)$-action of Example 5.1.2 b) giving us further information about its rational cohomology). Since for all other types $H^*(BG;\mathbb{F}_2)$ contains odd degree elements (except G_2!), they cannot be halving groups, with the possible exception of G_2.

Circle-halving groups

We describe two halving groups with fixed point set $\mathrm{GL}(n,\mathbb{C})$: they are $\mathrm{GL}(2n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{H})$. H_G^* is generated by Pontryagin classes and quaternionic Pontryagin classes respectively. Cohomology is taken with $\mathbb{Q}$ coefficients.

Example 5.1.2. Let $\Gamma = \mathrm{U}(1)$ act on

a) $G = \mathrm{GL}(2n,\mathbb{R})$ by the pseudo Galois type action, with fixed point set $\mathrm{GL}(n,\mathbb{C})$.

b) $G = \mathrm{GL}(n,\mathbb{H})$ by the Galois type action, with fixed point set $\mathrm{GL}(n,\mathbb{C})$.

Then G is a halving group. ♣

Proof. Let Γ act on

a) $EG = \mathrm{Inj}_{\mathbb{R}}(\mathbb{R}^{2n},\mathbb{R}^\infty) \to \mathrm{Gr}_{2n}(\mathbb{R}^\infty) = BG$

b) $EG = \mathrm{Inj}_{\mathbb{H}}(\mathbb{H}^{n},\mathbb{H}^\infty) \to \mathrm{Gr}_{n}(\mathbb{H}^\infty) = BG$

by the left-right conjugation action. Then $EG \to BG$ is a (Γ, G)-bundle with fixed point set the universal bundle over $BG^\Gamma = \mathrm{Gr}_n(\mathbb{C}^\infty)$ in both cases. The (Γ, G)-bundle $EG \to BG$ has a Γ-approximation by a) $\varinjlim_N \mathrm{Gr}_{2n}(\mathbb{R}^{2n+N})$, b) $\varinjlim_N \mathrm{Gr}_n(\mathbb{H}^{n+N})$ which are circle spaces by Theorems 4.2.2 and 4.2.13. Since $(EG)^\Gamma$ is contractible, therefore G is a halving group. (The Γ-fixed point sets are the complex Grassmannians, which approximate BG^Γ.) □

Remark 5.1.3. i) We conjecture that $G = \mathrm{Sp}(2n,\mathbb{C})$ is a $\mathrm{U}(1)$-halving group, with fixed point set $G^\Gamma = \mathrm{GL}(n,\mathbb{C})$. It is not hard to give a $\mathrm{U}(1)$-action on G satisfying $G^\Gamma = \mathrm{GL}(n,\mathbb{C})$, and the characteristic classes of G are the symplectic Pontryagin classes. What is missing is the Γ-approximation of $EG \to BG$.

The maximal compact subgroup of both $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{H})$ is $\mathrm{Sp}(n)$, which is $\mathrm{U}(1)$-invariant. It is interesting to note that the $\mathrm{U}(1)$-action extends in both cases to the whole group, even though there is no containment between the groups [BtD85, Diagram 1.14].

ii) It would be also interesting to find a Lie theoretical explanation of a group G being a halving group, in terms of Weyl groups and roots.

Let us conclude with the following easy Proposition, which allows us to get further examples of halving groups. Its proof relies on the fact that the product $X \times Y$ of halving spaces X, Y is a halving space with the product action (e.g. by the halving bundle lemma), we omit the details of the proof.

Proposition 5.1.4. *Let G, H be halving groups for Γ. Then $G \times H$ is a halving group for the product action.*

5.1.2 Characteristic classes

Let X be a paracompact halving space, Γ acting on G a halving group and $P \to X$ a Γ-equivariant principal G-bundle, assume it is classified via a Γ-equivariant $\mathcal{K} : X \to BG$. Then the characteristic classes $c \in H^*_G$ of P relate to the characteristic classes of the fixed, principal G^Γ bundle $P^\Gamma \to X^\Gamma$ as follows:

Proposition 5.1.5.

$$\kappa_X(c(P)) = \kappa_G(c)(P^\Gamma)$$

where $\kappa_G := \kappa_{BG}$.

Proof. The claim follows from naturality of κ applied to the Γ-equivariant classifying map of P between the halving spaces $\mathcal{K} : X \to BG$:

$$\kappa_X(c(P)) = \kappa_X(\mathcal{K}^* c) = (\mathcal{K}^\Gamma)^* \kappa_G(c) = \kappa_G(c)(P^\Gamma)$$

using also $(\mathcal{K}^\Gamma)^*(EG)^\Gamma = (\mathcal{K}^* EG)^\Gamma = P^\Gamma$, see Appendix E. □

In the following examples we determine explicitly the value of κ_G on the usual generators of H^*_G: Chern classes, Pontryagin classes.

Conjugation halving groups

Example 5.1.6. Let $\Gamma = \mathbb{Z}_2$ act on $G = \mathrm{GL}(1, \mathbb{C})$ by the Galois type action, with coefficients of cohomology $R = \mathbb{F}_2$, then

$$\kappa_G(e_{\mathbb{C}}) = e_{\mathbb{R}}.$$

♣

Proof. Indeed, by Example 5.1.1, BG has a Γ-approximation by halving spaces $\mathbb{C}P^n$ and κ maps $[\mathbb{C}P^{n-1}]$ to $[\mathbb{R}P^{n-1}]$. Since $e_{\mathbb{C}}(S^*_{\mathbb{C}}) = \varprojlim_n [\mathbb{C}P^{n-1}]$ and $e_{\mathbb{R}}(S^*_{\mathbb{R}}) = \varprojlim_n [\mathbb{R}P^{n-1}]$, the claim follows by Proposition 2.1.7. ☐

Example 5.1.7. Let $\Gamma = \mathbb{Z}_2$ act on $G = \mathrm{GL}(n, \mathbb{C})$ by the Galois type action, with coefficients of cohomology $R = \mathbb{F}_2$, then

$$\kappa_G(c_i) = w_i.$$

♣

Proof. By Example 5.1.1, BG is a conjugation space. Take a Γ-invariant maximal torus $\iota : T \hookrightarrow G$. Then $B\iota : BT \to BG$ is a Γ-map between conjugation spaces, so by naturality and multiplicativity of κ_T,

$$(B\iota^\Gamma)^* \kappa_G(c_i) = \kappa_T(B\iota^* c_i) = \kappa_T(s_i(t_*)) = s_i(u_*)$$

where s_j is the jth elementary symmetric polynomial, t_* and u_* are the Chern and Stiefel-Whitney roots, $\kappa_T(t_*) = u_*$ by the previous example. Using injectivity of $(B\iota^\Gamma)^*$ ($\mathbb{F}_2$-splitting principle), $\kappa_G(c_i) = w_i$. Alternatively, as in the previous example one can identify Schubert classes in the approximations with Chern and Stiefel-Whitney classes. We will do this explicitly in the case of circle halving groups. ☐

Remark 5.1.8. [Equivariant Chern classes] In the previous example Γ-equivariant Chern classes as Chern classes of $B_\Gamma S_{\mathbb{C}} \to B_\Gamma \mathrm{Gr}_n \mathbb{C}^\infty$ do not make sense, since the $\mathbb{Z}_2$-action on the fibers of $S_{\mathbb{C}}$ are not complex linear, see Appendix E. However, it is possible to construct such objects, by using equivariant cohomology with twisted coefficients. They have been first considered in [Kah87] and from a conjugation space perspective in [PS13].

Circle halving groups

Example 5.1.9. Let $\Gamma = \mathrm{U}(1)$ act on $G = \mathrm{GL}(2k, \mathbb{R})$ by the pseudo Galois type action with coefficients of cohomology $R = \mathbb{Q}$, then

$$\kappa_G(p_r) = 2^r c_r.$$

♣

Proof. This can be seen using the degeneracy locus description of Pontryagin and Chern classes as follows. Let $k > r$ and let $S_\mathbb{R} \to \mathrm{Gr}_{2k}(\mathbb{R}^{2n})$ denote the tautological bundle. Fix a linear form $\varphi \in \mathrm{Hom}_\mathbb{R}(\mathbb{R}^{2(k-r+1)}, (\mathbb{R}^{2n})^*)$. It defines a section of the bundle $E := \mathrm{Hom}_\mathbb{R}(\mathbb{R}^{2(k-r+1)}, S^*)$ by restriction $|_W : (\mathbb{R}^{2n})^* \to W^*$:

$$s_\varphi(W) := \varphi|_W, \qquad s_\varphi \in \Gamma(E).$$

By the degeneracy locus description of Pontryagin classes, for a generic section s_φ:

$$p_r(S^*_\mathbb{R}) = [\Sigma^2_\mathbb{R}(s_\varphi)] = [\sigma^\mathbb{R}_{2^{2r}}]$$

where the last equality can be seen explicitly by taking $\varphi : \mathbb{R}^{2(k-r+1)} \to (\mathbb{R}^{2n})^*$ to be the last $2(k-r+1)$ coordinates $x_{2(n-k-r)-1}, \ldots, x_{2n}$ on $\mathbb{R}^{2n}$ and taking $\sigma^\mathbb{R}_{2^{2r}}$ with respect to the standard flag. Then κ maps $[\sigma^\mathbb{R}_{2^{2r}}]$ to $2^r[\sigma^\mathbb{C}_{1^r}] = 2^r c_r(S^*_\mathbb{C})$ by the generalized Borel-Haefliger theorem. By Example 5.1.2 a), BG has a Γ-approximation by halving spaces $\varinjlim_n \mathrm{Gr}_{2k}(\mathbb{R}^{2n})$ and the claim follows by Proposition 2.1.7 using $p_r = \varprojlim_n p_r(S^*_\mathbb{R})$ and $c_r = \varprojlim_n c_r(S^*_\mathbb{C})$. □

Example 5.1.10. Let $\Gamma = \mathrm{U}(1)$ act on $G = \mathrm{GL}(n, \mathbb{H})$ by the Galois type action, with coefficients of cohomology $R = \mathbb{Z}$, then $\kappa_G(q_i) = 2^i c_i$. ♣

Proof. A similar computation as in the previous example. □

5.2 Thom polynomials

The Thom polynomial of a singularity type η in a jet space $J^k(n, p)$ is its equivariant fundamental cohomology class $[\eta]_\mathcal{A}$. Let us briefly recall these notions.

The *jet space of order k* is the space of polynomials maps of degree at most k from V to W sending 0 to 0:

$$J^k(V,W) := \mathrm{Pol}^{\leq k}(V,W) = \mathrm{Hom}_{\mathbb{C}}\left(\bigoplus_{j\leq k} \mathrm{Sym}^j V, W\right)$$

We denote $J^k(m,n) := J^k(\mathbb{C}^m,\mathbb{C}^n)$. Jets $J^k(n,n)$ form a semigroup with identity under composition, so they have a group of units denoted $\mathrm{Diff}^k(n)$. Let $\mathcal{A}^k(m,n) := \mathrm{Diff}^k(m)\times \mathrm{Diff}^k(n)$, which comes with a natural representation on $J^k(m,n)$. We call a *singularity type* η an $\mathcal{A}^k$-invariant subset in $J^k(m,n)$. If (the closure of) η is a cycle, it defines a polynomial in Chern classes

$$[\eta \subseteq J^k]_{\mathcal{A}^k} \in H^*_{\mathcal{A}^k}(J^k) \cong H^*_{\mathrm{GL}(m,n)} = \mathbb{Z}[a_1,\dots,a_m,b_1,\dots,b_n],$$

since $\mathcal{A}^k(m,n)$ is homotopy equivalent to $\mathrm{GL}(m,n) = \mathrm{GL}(m,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})$. The same definitions make sense over $\mathbb{R}$, however being a cycle over $\mathbb{Z}$ is a more restrictive condition. Using Borel-Haefliger's theory (cf. Appendix A.5), we get that if a singularity type is real algebraic, then it is a cycle over $\mathbb{F}_2$. Not all singularity types are real algebraic, for example $\Sigma^{i,j}_{\mathbb{R}} \subseteq J_{\mathbb{R}}(2,1)$ is not in general, see [Lev71].

To apply the U(1)-equivariant Borel-Haefliger theorem (Theorem 3.2.10) to the case of Thom polynomials, one needs to find singularity types $\eta\subseteq J^k$ which are halving cycles. Such examples are given by $\Sigma^{2i}_{\mathbb{R}} \subseteq J^1_{\mathbb{R}}(2m,2n)$ with fixed point set $\Sigma^i_{\mathbb{C}} \subseteq J^1_{\mathbb{C}}(m,n)$. Applying the equivariant Borel-Haefliger theorem, we recover a theorem of Ronga [Ron71] on first order real Thom-Boardman singularities of even degree with rational coefficients:

Theorem 5.2.1 (Ronga, [Ron71])**.** *The rational Thom polynomial of $\Sigma^{2i}_{\mathbb{R}} \subseteq \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{2m},\mathbb{R}^{2n})$ is given by*

$$[\Sigma^{2i}_{\mathbb{R}}]_G = q(p_1,\dots,p_m,p'_1,\dots,p'_n) \qquad \iff \qquad [\Sigma^i_{\mathbb{C}}]_{G^\Gamma} = q(c_1,\dots,c_m,c'_1,\dots,c'_n)$$

where $G = \mathrm{GL}(2m,\mathbb{R})\times\mathrm{GL}(2n,\mathbb{R})$ and $G^\Gamma = \mathrm{GL}(m,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})$. In particular, in quotient Pontryagin classes

$$[\Sigma^{2i}_{\mathbb{R}}(2\ell)]_G = \det[p_{\ell+i+k-l}]_{k,l}$$

where $\ell = n-m$ and $k,l = 1,\dots,i$.

Proof. Take the pseudo Galois type action of $\Gamma = \mathrm{U}(1)$ on $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{2m}, \mathbb{R}^{2n})$ as described in Section 4.1.2. Namely, let Γ act by left-right conjugation via a diagonal circle subgroup $\mathrm{U}(1) \leq \mathrm{GL}(2m, \mathbb{R}) \times \mathrm{GL}(2n, \mathbb{R})$, with fixed point set $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^m, \mathbb{C}^n)$. Since for complex maps φ, $\mathrm{rk}_{\mathbb{R}}\, \varphi = 2\, \mathrm{rk}_{\mathbb{C}}\, \varphi$, the Γ-fixed point set of $Z := \overline{\Sigma^{2i}_{\mathbb{R}}}$ is $Z^\Gamma = \overline{\Sigma^{i}_{\mathbb{C}}}$. To use the equivariant Borel-Haefliger theorem, we have to show that Z is a halving G-cycle.

The proof of G-cycleness of Z is the same as in Ronga: the stratum neighboring the top stratum in the stratification of $\overline{\Sigma^{2i}_{\mathbb{R}}}$ has codimension at least 3, therefore it is enough to show that $\Sigma^{2i}_{\mathbb{R}}$ is G-coorientable.

The maximal compact subgroup of the stabilizer subgroup of $\Sigma^{2i}_{\mathbb{R}}$ in G can be shown to be $\mathrm{O}(2i) \times \mathrm{O}(2i + 2\ell) \times \mathrm{O}(2m - 2i)$ and the normal isotropy representation is the left-right representation on the normal space $N = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{2i}, \mathbb{R}^{2(i+\ell)})$, the factor $\mathrm{O}(2m - 2i)$ acting trivially. These statements follow from Proposition 5.3.7. Since the normal space N is an even-even dimensional $\mathrm{Hom}_{\mathbb{R}}$-space, the normal isotropy representation is oriented, so $\Sigma^{2i}_{\mathbb{R}}$ is a G-cycle by Proposition A.3.3.

Since $\mathrm{codim}_{\mathbb{R}}\, \Sigma^{2i}_{\mathbb{R}}(2\ell) = (2i)(2i + 2\ell)$ and $\mathrm{codim}_{\mathbb{C}}\, \Sigma^{i}_{\mathbb{C}}(\ell) = i(i + \ell)$, Z is a halving G-cycle, so the equivariant Borel-Haefliger theorem 3.2.13 allows us to conclude. □

The codimension of higher order real Thom-Boardman singularities $\Sigma^I(\ell)$, $I = (i_1 \geq i_2 \geq \ldots \geq i_r)$ is given by the following formula due to Boardman (cf. [Mat73, Proposition 3]):

$$\mathrm{codim}\, \Sigma^I(\ell) = (\ell + i_1)\mu(i_1, \ldots, i_r) - (i_1 - i_2)\mu(i_2, \ldots, i_r) - \ldots - (i_{r-1} - i_r)\mu(i_r)$$

where $\mu(i_s, \ldots, i_r)$ is the number of integer sequences $j_s \geq \ldots \geq j_r$ such that $i_u \geq j_u \geq 0$ for all u, and $j_s > 0$. For example:

$$\mathrm{codim}\, \Sigma^{(i,j)} = (ij + i - j^2/2 + j/2)\ell + (i^2 j + i^2 - ij^2/2 - ij/2 + j^2).$$

These computations indicate that among the higher order real Thom-Boardman singularities there is no evident example of halving cycles. However a natural generalization of the Ronga theorem arises in the context of quiver Thom polynomials.

5.3 Quiver Thom polynomials

Quiver Thom polynomials are equivariant fundamental classes of orbit closures $[\overline{G.x}]_G \in H^*_G(V)$ in a quiver representation V, which have been investigated in the complex case see [BF99], [FR02a], [KMS06]. Our aim is to use the equivariant Borel-Haefliger theorem to determine such classes in the real case, see Theorem 5.3.6. As usual, in the real case one has to verify that the orbit closures are cycles. In the case of equioriented A_n quivers for $n > 2$, we are no longer in the simple situation where cycleness follows from the fact that there are no strata of neighboring dimension. Instead, to prove cycleness, we use resolutions, as in Proposition A.4.3. In particular, we will use the resolution given by Reineke [Rei03]. We briefly recall some notions and notation about quiver representations; for further details we refer to [Rei03], [Buc02], [Rim14].

5.3.1 Quiver representations

A *quiver* is a oriented graph $Q = (Q_0, Q_1)$ where Q_0 denotes the set of its vertices and Q_1 the set of its edges. An oriented edge $a \in Q_1$ connects $a : t(a) \to h(a)$, its tail $t(a) \in Q_0$ and head $h(a) \in Q_0$. The *dimension vector* of a Q_0-graded vector space $V = \oplus_{i\in Q_0} V_i$ is the function $\gamma \in \mathbb{N}^{Q_0}$ such that $\gamma(i) = \dim V_i$. The (left-right) conjugation action of $\mathrm{GL}(|\gamma|)$ on $\mathrm{End}(\mathbb{F}^{|\gamma|})$ restricts to an action

$$\mathrm{GL}_\gamma := \bigtimes_{i\in Q_0} \mathrm{GL}(\gamma(i)) \curvearrowright \mathrm{Rep}_\gamma := \bigoplus_{a\in Q_1} \mathrm{Hom}(\mathbb{F}^{\gamma(t(a))}, \mathbb{F}^{\gamma(h(a))}) \tag{5.1}$$

A *quiver representation of* Q is a pair $(\gamma \in \mathbb{N}^{Q_0}, \varphi \in \mathrm{Rep}_\gamma)$. Less formally, it is an assignment to each vertex $i \in Q_0$ a vector space V_i and to each edge a linear map between the vector spaces on the vertices. Given two quiver representations of Q, one can define their direct sum in an obvious manner. A *morphism of quiver representations* $f : V = (\gamma, \varphi) \to V' = (\gamma', \varphi')$ is a linear map $f = \oplus_i f_i$, where $f_i : V_i \to V'_i$ such that $\varphi'_a \circ f_{t(a)} = f_{h(a)} \circ \varphi_a$ for all $a \in Q_1$. Denote the resulting abelian category by $\mathrm{Rep}(Q)$.

It follows from the definitions that the orbits of the action (5.1) are in one-to-one correspondence with the isoclasses of quiver representations with dimension vector γ. Given such an isomorphism class $M \in \mathrm{Rep}(Q)$, it decomposes uniquely (up to isomorphism) as a direct sum of indecomposables $M = \oplus D_i^{m_i}$. So isomorphism classes $M \in \mathrm{Rep}(Q)$ correspond to an $\mathbb{N}$-valued function on the indecomposables $D_i \in \mathrm{Rep}(Q)$, so it is enough to understand these.

By Gabriel's theorem, for a Dynkin quiver Q, isomorphism classes M_β of indecomposable quiver representations correspond to positive roots $\beta \in \Phi^+$ of the Dynkin diagram; the positive roots are the dimension vector of such M_β. In particular, the representation has finitely many orbits. Gabriel's theorem also states the converse: if the representation has finitely many orbits, its underlying graph is Dynkin. Summarizing, the orbits of a fixed dimension vector γ can be parametrized by *Kostant partitions of* γ; functions $m \in \mathbb{N}^{\Phi^+}$ such that $\sum m_\beta \beta = \gamma$. We will denote the GL_γ-orbits by Ω_m, where $m \in \mathbb{N}^{\Phi^+}$.

Quiver representations can also be described as representations of the *path algebra* P_Q of a quiver, however for us this point of view is not important.

5.3.2 Reineke's resolution

We briefly describe Reineke's [Rei03] resolution of quiver orbit closures $Z_m := \overline{\Omega_m}$ of an orbit Ω_m, $m \in \mathbb{N}^{\Phi^+}$. The variant we present follows [Rim18].

Let $\underline{\gamma} = (\gamma_1, \dots, \gamma_r) \in (\mathbb{N}^{Q_0})^{\times r}$ be a list of dimension vectors and let $\gamma = \sum_{j=1}^r \gamma_j$, componentwise. Let

$$\mathrm{Fl}_{\underline{\gamma}} := \times_{i \in Q_0} \mathrm{Fl}_{\underline{\gamma}(i)}, \qquad \mathrm{Fl}_{\underline{\gamma}(i)} = \{F_\bullet = (F_{s_0} \leq F_{s_1} \leq \ldots \leq F_{s_r} = \mathbb{F}^{\gamma(i)})\},$$

for $i \in Q_0$, where $s_j = \sum_{k \leq j} \gamma_k(i)$; this is consistent with the usual notation $\mathrm{Fl}_{\underline{\gamma}(i)} = \mathrm{Fl}_{\mathcal{D}}$ for $\mathcal{D} = \underline{\gamma}(i)$ (cf. Section B.2). Let

$$\Sigma_{\underline{\gamma}} := \{(F^i_\bullet, \varphi_a) \in \mathrm{Fl}_{\underline{\gamma}} \times \mathrm{Rep}_\gamma : \varphi_a(F_j^{t(a)}) \subseteq F_j^{h(a)}\} \subseteq \mathrm{Fl}_{\underline{\gamma}} \times \mathrm{Rep}_\gamma,$$

where $i \in Q_0$, $j = 1, \dots, r$ and $a \in Q_1$. Note that $\Sigma_{\underline{\gamma}}$ is smooth, as it is a vector bundle over $\mathrm{Fl}_{\underline{\gamma}}$. It also has a projection $\pi_{\underline{\gamma}}$ to the second factor Rep_γ. Reineke's resolution of an orbit closure $Z_m \subseteq \mathrm{Rep}_\gamma$, consists of an appropriate choice of dimension vectors $\underline{\gamma}$, such that $\pi_{\underline{\gamma}}$ is a resolution of Z_m. Given a GL_γ-orbit closure Z_m, this choice is constructed as follows.

Definition 5.3.1. An ordering $>$ on the set of positive roots $\Phi^+ = \{\beta_i\}$ is a *Reineke order*, if for all $i > j$: $\mathrm{Hom}(M_{\beta_j}, M_{\beta_i}) = 0$ and $\mathrm{Ext}(M_{\beta_i}, M_{\beta_j}) = 0$, where $M_{\beta_i} \in \mathrm{Rep}(Q)$ denotes the indecomposable isomorphism class corresponding to β_i.

More generally, Reineke defines *directed partitions* of the positive roots. For any orientation of a Dynkin quiver, it has a Reineke order [Rei03]; it is not unique in general. Reineke's resolution [Rei03] can be stated as follows, see also [Rim18, Proposition 3.6]:

Proposition 5.3.2. *Let Q be a Dynkin quiver and let $\beta_1 < \ldots < \beta_N$ be a Reineke order on the positive roots. Let $\Omega_m \subseteq \mathrm{Rep}_\gamma$ be an orbit, where $m = \sum m_i\beta_i$ is a Kostant partition of γ. Let*

$$\delta_m = (m_1\beta_1, \ldots, m_N\beta_N).$$

Let $\Sigma_m := \Sigma_{\delta_m}$ and $\pi_m := \pi_{\delta_m}$. Then

$$\pi_m : \Sigma_m \to Z_m \subseteq \mathrm{Rep}_\gamma$$

is a resolution of the orbit closure $Z_m := \overline{\Omega_m}$.

Remark 5.3.3. The Reineke resolution is a resolution defined over $\mathbb{R}$.

Example 5.3.4. Consider the equioriented quiver $(Q, \gamma) = 6 \to 6 \to 8$ and the orbit Ω_m corresponding to

$$m = 4(1,0,0) + 2(1,1,1) + 4(0,1,1) + 2(0,0,1).$$

A Reineke order is given by

$$(1,0,0) < (1,1,0) < (0,1,0) < (1,1,1) < (0,1,1) < (0,0,1),$$

so

$$\delta_m = ((4,0,0),(2,2,2),(0,4,4),(0,0,2)).$$

Then

$$\Sigma_{\delta_m} = \{(U_\bullet, V_\bullet, W_\bullet, \varphi_1, \varphi_2) : \text{compatible}\} \subseteq \mathrm{Fl}_{\delta_m} \times \mathrm{Rep}_\gamma$$

i.e. satisfying

$$\begin{cases} \varphi_1(U_4) = 0, \\ \varphi_1(U_6) \subseteq V_2, \\ \varphi_2(V_2) \subseteq W_2, \\ \varphi_2(V_6) \subseteq W_6. \end{cases}$$

where $U_\bullet \in \mathrm{Fl}_{4,2}$, $V_\bullet \in \mathrm{Fl}_{0,2,4,0}$, $W_\bullet \in \mathrm{Fl}_{0,2,4,2}$ and the subscripts in $U_\bullet$ denote the dimensions. ♣

5.3.3 A_n quivers

We will be concerned with equioriented A_n quivers $\bullet \to \bullet \to \ldots \to \bullet$. To translate the discussion to the real case, the following Proposition can be used, which is a special case of the Noether-Deuring theorem (e.g. [CR06, Theorem 29.7]):

Proposition 5.3.5. *Let k be an arbitrary field and $K := \overline{k}$ be its algebraic closure. Let Q be a quiver. If $x, y \in \mathrm{Rep}_\gamma(k) \subseteq \mathrm{Rep}_\gamma(K)$ are in the same $\mathrm{GL}_\gamma(K)$-orbit, then they are in the same $\mathrm{GL}_\gamma(k)$-orbit.*

Therefore the real $\mathrm{GL}_\gamma^{\mathbb{R}}$-orbits $\Omega_m^{\mathbb{R}}$ in $\mathrm{Rep}_\gamma^{\mathbb{R}}$ are also parametrized by $m = \sum \mu_i \beta_i$, if β_i denote the positive roots. The next application of the equivariant Borel-Haefliger theorem is the following:

Theorem 5.3.6. *Let Q be the equioriented A_n quiver. Let $\gamma \in \mathbb{N}^n$ be a dimension vector and $Z_m \subseteq \mathrm{Rep}_\gamma^{\mathbb{C}}$ be a $\mathrm{GL}_\gamma^{\mathbb{C}}$-orbit closure, $m = \sum \mu_{ij} l_{ij}$, where $l_{ij} \in \Phi^+$ are the positive roots. Then*

$$[Z_m^{\mathbb{C}} \subseteq \mathrm{Rep}_\gamma^{\mathbb{C}}]_{\mathrm{GL}_\gamma^{\mathbb{C}}} = q(c_*) \qquad \Longleftrightarrow \qquad [Z_{2m}^{\mathbb{R}} \subseteq \mathrm{Rep}_{2\gamma}^{\mathbb{R}}]_{\mathrm{GL}_{2\gamma}^{\mathbb{R}}} = q(p_*)$$

in $H^*(BGL_\gamma^{\mathbb{C}};\mathbb{Q})$ and $H^*(BGL_{2\gamma}^{\mathbb{R}};\mathbb{Q})$ respectively where $2\gamma = (2\gamma_1, \ldots, 2\gamma_r)$ and $2m = \sum 2\mu_{ij} l_{ij}$.

Proof. Using the equivariant Borel-Haefliger theorem in the form Theorem 3.2.13, it is enough to show the following:

i) There is a $\Gamma = \mathrm{U}(1)$-action on $\mathrm{Rep}_{2\gamma}^{\mathbb{R}}$ with fixed point set $\mathrm{Rep}_\gamma^{\mathbb{C}}$, compatible with $\mathrm{GL}_{2\gamma}^{\mathbb{R}}$ acting on $\mathrm{Rep}_{2\gamma}^{\mathbb{R}}$ in (5.1),

ii) $Z_{2m}^{\mathbb{R}}$ are $\mathrm{GL}_{2\gamma}^{\mathbb{R}}$-halving cycles: $\mathrm{GL}_{2\gamma}^{\mathbb{R}}$-cycles, which are also good Γ-invariant cycles, with half-dimensional Γ-fixed point set $Z_m^{\mathbb{C}}$.

The pseudo-Galois action as defined in Section 4.1.2 satisfies i). The rest of this section is devoted to proving ii). To break the proof down, we need to show the following points, where C stands for cycleness and H for halving cycleness:

C1) The Reineke resolution $\Sigma_{2m}^{\mathbb{R}}$ is oriented $\Rightarrow$ $Z_{2m}^{\mathbb{R}}$ is a cycle, representing a class $[Z_{2m}^{\mathbb{R}}]$

C2) The normal isotropy representation $G_{2m} \to \mathrm{GL}(N_{2m}^{\mathbb{R}})$ of $\Omega_{2m}^{\mathbb{R}}$ is oriented $\Rightarrow$ $Z_{2m}^{\mathbb{R}}$ is a $\mathrm{GL}_{2\gamma}^{\mathbb{R}}$-cycle, representing a class $[Z_{2m}^{\mathbb{R}}]_{\mathrm{GL}_{2\gamma}^{\mathbb{R}}}$

H1) $Z_{2m}^{\mathbb{R}}$ is Γ-invariant and its Γ-fixed point set is $Z_m^{\mathbb{C}}$ and half dimensional.

H2) $Z_{2m}^{\mathbb{R}}$ is a *good* Γ-invariant cycle: the orbit $\Omega_{2m}^{\mathbb{R}}$ is connected.

In C1), the conclusion $\Rightarrow$ holds, since an oriented resolution of $Z_{2m}^{\mathbb{R}}$ defined over $\mathbb{R}$ proves cycleness, see Propositions A.4.3, A.5.4 and Remark 5.3.3. The conclusion $\Rightarrow$ in C2) follows from Propositions A.3.2 and A.3.3.

First, we sketch why C1) holds. The key observation is that the Reineke resolution $\Sigma_{2m}^{\mathbb{R}}$ is a vector bundle over a product of flag manifolds, which splits as a direct sum of even-even rank Hom-bundles, which are orientable by Proposition B.2.6.

C2) follows from the following Proposition, which is the real version of Propositions 3.6, 3.11, 3.12 and Lemma 4.4. from [FR02a], its proof is similar:

Proposition 5.3.7. *Let $\Omega_m \subseteq \mathrm{Rep}_\gamma^{\mathbb{R}}$ be an A_n-quiver orbit, $m = \sum \mu_{ij} l_{ij}$. The maximal compact subgroup of its stabilizer subgroup is*

$$G_m \cong \bigtimes_{i \leq j} \mathrm{O}(\mu_{ij}).$$

The normal isotropy G_m-representation N_m of Ω_m is isomorphic to

$$N_m \cong \bigoplus_{i+1 \leq k \leq j+1 \leq l} \mathrm{Hom}(\mathbb{R}^{\mu_{ij}}, \mathbb{R}^{\mu_{kl}}),$$

via the natural G_m-action ($\mathrm{O}(\mu_{ij})$ acts on $\mathbb{R}^{\mu_{ij}}$).

Since every μ_{ij} is even, N_m is an even-even Hom-space, therefore the source-target representation has positive determinant by Proposition D.1.2, implying C2).

To show H1), the GL_γ-orbit closures Z_m are clearly Γ-invariant, since $\Gamma \leq \mathrm{GL}_\gamma$. The codimensions are halved by Proposition 5.3.7:

$$\dim_{\mathbb{R}} N_m^{\mathbb{C}} = 2 \sum_{r_{ij,kl} \neq 0} \mu_{ij}\mu_{kl}, \qquad \dim_{\mathbb{R}} N_{2m}^{\mathbb{R}} = \sum_{r_{ij,kl} \neq 0} (2\mu_{ij})(2\mu_{kl})$$

so $\dim_{\mathbb{R}} N_{2m}^{\mathbb{R}} = 2 \dim_{\mathbb{R}} N_m^{\mathbb{C}}$, where $N_{2m}^{\mathbb{R}}$ denotes the real normal space of $\Omega_{2m}^{\mathbb{R}} \subseteq \mathrm{Rep}_\gamma^{\mathbb{R}}$.

H2) is also satisfied, since the stabilizer subgroup G_{2m} contains an element from each connected component of $\mathrm{GL}_{2\gamma}$, so $\mathrm{GL}_{2\gamma}/G_{2m}$ is connected. We can conclude by the equivariant Borel-Haefliger theorem. $\square$

5.4 Matrix Schubert classes

In the complex case, a *matrix Schubert variety* $M_w \subseteq V := \mathrm{End}(\mathbb{C}^n)$ is the closure of a $B^- \times B^+$-orbit $\Omega_w = B^-.w.B^+$, $w \in S_N$ in V, see [Ful92]. Their classes are related to Schubert varieties $\sigma_w \subseteq \mathrm{Fl}(\mathbb{C}^n)$ as follows. Since $\mathrm{Fl}(\mathbb{C}^n) = \mathrm{End}(\mathbb{C}^n)/\!\!/B^+$, it has a classifying map $\mathcal{K} : \mathrm{Fl}(\mathbb{C}^n) \to BB^+$, in particular, the classifying map of the principal bundle $B^+ \to \mathrm{GL}(n,\mathbb{C}) \to \mathrm{Fl}(\mathbb{C}^n)$. Then $\mathcal{K}^*[M_w]_{B^+} = [\sigma_w]$ via the *Kirwan map* $\mathcal{K}^*$ [Kir84].

We will discuss a similar example in the real case, matrix Schubert varieties over real Grassmannians $\mathrm{Gr}_K(\mathbb{R}^N)$. Let $\Sigma^0 := \mathrm{Inj}(\mathbb{R}^K, \mathbb{R}^N)$, the injective linear maps. Consider the principal $\mathrm{GL}(K,\mathbb{R})$-bundle $\pi : \Sigma^0 \to \mathrm{Gr}_K(\mathbb{R}^N)$, where $i : \Sigma^0 \hookrightarrow \mathrm{Hom}(\mathbb{R}^K, \mathbb{R}^N) =: V$. The *matrix Schubert varieties* are defined as

$$Z_\lambda := \overline{\pi^{-1}\sigma_\lambda}^V.$$

They are orbit closures of the natural $G_0 := \mathrm{GL}(K,\mathbb{R}) \times B^+(N,\mathbb{R})$-action on V. Fixing the standard flag $F_\bullet$ in $\mathbb{R}^N$,

$$\pi^{-1}\sigma_\lambda(F_\bullet) = \{\varphi \in \Sigma^0 : \dim(\mathrm{Im}\,\varphi \cap F_{I(j)}) \geq j,\ j = 1, \ldots, K\}, \tag{5.2}$$

where $I(j) = N - K + j - \lambda_j$.

Not all matrix Schubert varieties are cycles, similarly to the case of Grassmannians. However, $Z_{D\lambda}$ turn out to be cycles; our aim is to compute their equivariant fundamental class. Let $DB^+(N) := \mathrm{GL}(\mathcal{D}_N) \leq \mathrm{GL}(N,\mathbb{R})$ be the parabolic subgroup corresponding to $\mathcal{D}_N = (2, 2, \ldots, 2)$ or $\mathcal{D}_N = (2, 2, \ldots, 2, 1)$ depending on the parity of N. $Z_{D\lambda}$ are orbit closures of a larger group action, namely the natural representation of $G = \mathrm{GL}(K,\mathbb{R}) \times DB^+(N)$ on $\mathrm{Hom}(\mathbb{R}^K, \mathbb{R}^N)$.

Our aim is to compute the class $[Z_{D\lambda}]_G$. We will use the equivariant Borel-Haefliger theorem – as usual, we have to show that $Z_{D\lambda}$ is a G-cycle. We will use fibered resolutions.

5.4.1 Fibered resolutions

Definition 5.4.1. Let V be a real G-representation and $Y \subseteq V$ a G-invariant stratified submanifold. A G-equivariant resolution $\varphi : \tilde{Y} \to V$ of Y is called a *fibered resolution of* Y if $E = \tilde{Y}$ is the total space of a G-equivariant sub-vector bundle $p: E \to M$ of $\pi_M : M \times V \to M$, such that

$$\begin{tikzcd} E \arrow[r, "i"'] \arrow[rr, bend left, "\varphi"] \arrow[dr, "p"'] & M \times V \arrow[r, "\pi_V"'] \arrow[d, "\pi_M"] & V \\ & M & \end{tikzcd}$$

We describe a fibered resolution of the double matrix Schubert varieties $Z_{D\lambda} \subseteq V$, see also [Kőm10]. Let $D\lambda \subseteq K \times (N-K)$ be a double Young diagram, where $\lambda = (\lambda_1, \dots, \lambda_k)$, $k := \lfloor K/2 \rfloor$, $n = \lfloor N/2 \rfloor$, let $F_\bullet$ be the standard flag in $\mathbb{R}^N$.

In the notation of Definition 5.4.1, let $M := \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^K)$ for $\mathcal{D} = (2, 2, \dots, 2)$ or $\mathcal{D} = (2, 2, \dots, 2, 1)$ depending on the parity of K. Let $Y := Z_{D\lambda} \subseteq V := \mathrm{Hom}(\mathbb{R}^K, \mathbb{R}^N)$, and

$$E := \{(A_\bullet, \varphi) : \varphi(A_j) \subseteq F_{I(j)}, j = 1, \dots, k\} \subseteq M \times V,$$

where $I(j) = N - K - 2\lambda_j + 2j$. Define $p : E \to M$ by restricting $p := \pi_M|_E$. Then E is a subbundle of $M \times V$, namely

$$E = \bigoplus_{j=1}^{l} \mathrm{Hom}(D_j, \mathbb{R}^{I(j)})$$

where $l := \lceil K/2 \rceil$ with the convention that $I(l) = N$ if K odd.

Orientability

The subbundle E is G-invariant, so $\varphi := \pi_V|_E : E \to V$ is a G-equivariant fibered resolution of $Z_{D\lambda}$.

Proposition 5.4.2. *$\varphi : E \to Y = Z_{D\lambda}$ is an orientable resolution, i.e. E is orientable as a manifold.*

Proof. Note that TE is orientable iff $w_1(TM) = w_1(E)$, since $TE \cong p_E^*(E \oplus TM)$. Let us compute both. Let $l = \lceil \frac{K}{2} \rceil$. By Proposition B.2.8:

$$w_1(TM) = \sum_{i=1}^{l} (K - d_i) w_1(D_i).$$

By the decomposition of E above,

$$w_1(E) = \sum_{j=1}^{l} I(j) w_1(D_j). \tag{5.3}$$

We distinguish two cases: K even or odd. If K even, then $w_1(TM) = 0$, since K and d_i are all even. On the other hand, $I(j) \equiv N \bmod 2$, so by the relation $\sum w_1(D_j) = 0$, $w_1(E) = 0$ as well.

Now assume that K is odd. Then $d_i = 2$ for all i, except $d_l = 1$. So

$$w_1(TM) = w_1(D_l) + \sum_{i=1}^{l} w_1(D_i) = w_1(D_l),$$

again by the relation $\sum w_1(D_j) = 0$. On the other hand, by (5.3), since $I(j) \equiv N - 1 \bmod 2$ for all j, except $I(l) \equiv N$, $w_1(E) = w_1(D_l) = w_1(TM)$, proving the claim.

$\square$

Given an oriented fibered resolution of $Z \subseteq V$, it follows that Z is a nonequivariant cycle, see Proposition A.4.3. To show that $Z \subseteq V$ is a G-cycle in case G is not connected, some further work is needed. One method is to compute $[Z]_{G^0}$ for the connected component G^0 of G and show that it is the restriction of a nonzero element in $H_G(V)$ - this implies that Z is a G-cycle by Proposition A.3.4. Alternatively –as we did in the case of quivers – if $Z = \overline{G.X}$ is a G-orbit closure, one can show that the normal isotropy representation at x is oriented, see Proposition A.3.3. We will use the second approach in the even case, and the first approach in the odd case.

5.4.2 Double matrix Schubert classes - even case

In case when K and N are even, the equivariant Borel-Haefliger theorem can be applied to compute $[Z_{D\lambda}]_G$ - once one knows that they are G-cycles.

Theorem 5.4.3 (Matrix Schubert classes). *Let K, N be even, and let $G := \mathrm{GL}(K, \mathbb{R}) \times DB^+(N)$ act on $\mathrm{Hom}(\mathbb{R}^K, \mathbb{R}^N)$ via the natural left-right representation. Then $Z_{D\lambda} \subseteq V$ are halving G-cycles. Then*

$$[Z^{\mathbb{R}}_{D\lambda}]_G = q(p_*, x_*) \qquad \Longleftrightarrow \qquad [Z^{\mathbb{C}}_{\lambda}]_{G^{\Gamma}} = q(c_*, y_*)$$

where $H^*_G \cong \mathbb{Q}[p_*, x_*]$ *and* $H^*_{G^\Gamma} \cong \mathbb{Q}[c_*, y_*]$, $G^\Gamma = \mathrm{GL}(k,\mathbb{C}) \times B^+_{\mathbb{C}}$. *In particular,*

$$[Z_{D\lambda}]_{\mathrm{GL}(K,\mathbb{R})} = \det(p_{\lambda'_i+j-i})$$

where λ' *denotes the conjugate partition (reflection on the diagonal).*

Proof. Using the equivariant Borel-Haefliger theorem in the form Theorem 3.2.13, it is enough to show that $Z^{\mathbb{R}}_{D\lambda}$ are G-halving cycles: G-cycles, which are also good Γ-invariant cycles, with half-dimensional Γ-fixed point set $Z^{\mathbb{C}}_\lambda$.

To break the proof down, we have to show the following points, as in the proof of quiver Thom polynomials. Again, C stands for cycleness and H for halving cycleness:

C1) The fibered resolution $E \to Z_{D\lambda}$ is oriented $\Rightarrow Z^{\mathbb{R}}_{D\lambda}$ is a cycle, representing a class $[Z^{\mathbb{R}}_{D\lambda}]$

C2) The normal isotropy representation $G_{D\lambda} \to \mathrm{GL}(N^{\mathbb{R}}_{D\lambda})$ of the G-orbit $\Omega^{\mathbb{R}}_{D\lambda}$ is oriented $\Rightarrow Z^{\mathbb{R}}_{D\lambda}$ is a G-cycle, representing a class $[Z^{\mathbb{R}}_{D\lambda}]_G$

H1) $Z^{\mathbb{R}}_{D\lambda}$ is Γ-invariant with half dimensional Γ-fixed point set $Z^{\mathbb{C}}_\lambda$.

H2) $Z^{\mathbb{R}}_{D\lambda}$ is a *good* Γ-invariant cycle: the orbit $\Omega^{\mathbb{R}}_{D\lambda}$ is connected.

C1) is the content of Proposition 5.4.2. A computation similar to [FR03, p. 426] shows that the maximal compact subgroup of the stabilizer $G_{D\lambda}$ is $\mathrm{O}(2)^n$ $(n = N/2)$, acting on a direct sum of even-even Hom-spaces, acting by conjugation for 2×2 identity blocks in $\mathrm{Hom}(\mathbb{R}^K, \mathbb{R}^N)$, and otherwise acting only on the target. Since a source-target action on even-even Hom-spaces has positive determinant (Proposition D.1.2), C2) follows.

H1) holds since Γ is a subgroup of G, and $Z^{\Gamma}_{D\lambda} = Z^{\mathbb{C}}_\lambda$ follows from the rank condition description of matrix Schubert varieties as in (5.2). Finally, H2) is satisfied, since $G_{D\lambda}$ contains an element from each connected component of G. Therefore $Z_{D\lambda}$ are halving G-cycles, and one can conclude from the equivariant Borel-Haefliger theorem and the complex description of $[Z^{\mathbb{C}}_\lambda]_{G^\Gamma}$, see [FR03]. $\square$

5.4.3 Double matrix Schubert classes - odd case

In fact, given a fibered resolution, one can use Atiyah-Bott–Berline-Vergne (ABBV) localization to compute double matrix Schubert classes in the other cases as well.

Theorem 5.4.4. *$Z_{D\lambda} \subseteq \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^K, \mathbb{R}^N)$ is a G-cycle for $G = \mathrm{GL}(K, \mathbb{R})$ and*

$$[Z_{D\lambda}]_G(p_1, \dots, p_k) = \det(p_{\lambda'_i + j - i})$$

where λ' denotes the conjugate partition (reflection on the diagonal).

We devote the rest of this section to proving this theorem via ABBV localization. First, given a fibered resolution, one has (see [BS12] or [FR12, Section 5]):

Proposition 5.4.5. *If $\varphi : E \to V$ is a fibered resolution of a G-cycle Y, then*

$$[Y]_G = co_V^*(co_M)_! e_G(\underline{V}/E),$$

where $\underline{V}$ denotes the trivial V-bundle over M. Here $co_X : X \to pt$ denotes the collapse map.

First, we will compute $[Y]_T$ for the maximal torus $T = T(k+n) \leq G$ whose action has finitely many fixed points on M by applying the ABBV formula:

$$[Y]_T = (co_M)_! e_T(V/E) = \sum_{\pi \in M^T} \frac{e_T(V/E_\pi)}{e_T(T_\pi M)}$$

Fixed points Let $k := \lfloor \frac{K}{2} \rfloor$ and $l := \lceil \frac{N}{2} \rceil$. Write $\mathbb{R}^K = \oplus_{j=1}^l W_i$, where W_i are T-invariant subspaces with all $\dim W_i = 2$ except if K odd, $\dim W_l = 1$. Let $F_\bullet \in M$ be the standard double flag in $\mathbb{R}^K$ where $F_i := \bigoplus_{j=1}^i W_j$. M has finitely many $T(k)$-fixed points, indexed by permutations $\pi \in S_k$, namely $F_\bullet^\pi$ where $F_i^\pi = \bigoplus_{j=1}^i W_{\pi(j)}$. We are going to use the abuse of notation and denote both the permutation and the flag by π.

Denominator Recall (Corollary B.2.2)

$$TM = \bigoplus_{i=1}^{l-1} \bigoplus_{j=i+1}^{l} \mathrm{Hom}(D_i, D_j)$$

Restricted to the fixed point corresponding to a permutation $\pi \in S_k$, if K even,

$$T_\pi M = \bigoplus_{i=1}^{k-1} \bigoplus_{j=i+1}^{k} \mathrm{Hom}(W_{\pi(i)}, W_{\pi(j)})$$

and if K odd,

$$T_\pi M = \operatorname{Hom}(\mathbb{R}^{K-1}, W_l) \oplus \bigoplus_{i=1}^{k-1} \bigoplus_{j=i+1}^{k} \operatorname{Hom}(W_{\pi(i)}, W_{\pi(j)})$$

The weight on $\operatorname{Hom}_{\mathbb{R}}(W_i, W_j)$ is $(\alpha_j + \alpha_i + 2u)(\alpha_j - \alpha_i)$ by Proposition D.3.1. If K is odd, the weight on $\operatorname{Hom}(\mathbb{R}^{K-1}, W_l)$ is $\Delta := \prod_{i=1}^{k}(-\alpha_i)$.

Numerator We have to compute for a T-fixed flag $\pi \in M^T$

$$e_T(V/E_\pi) = [E_\pi \subseteq V]_T.$$

By definition,

$$V/E = \operatorname{Hom}(\mathbb{R}^K, \mathbb{R}^N) \Big/ \bigoplus_{j=1}^{l} \operatorname{Hom}(D_j, \mathbb{R}^{I(j)}) = \bigoplus_{j=1}^{l} \operatorname{Hom}(D_j, \mathbb{R}^N/\mathbb{R}^{I(j)})$$

and for $\pi \in M^T$ and using that $I(l) = N$,

$$V/E_\pi = \bigoplus_{j=1}^{k} \operatorname{Hom}(W_{\pi(j)}, \mathbb{R}^N/\mathbb{R}^{I(j)}).$$

The weight on $\operatorname{Hom}(W_i, \mathbb{R}^N/\mathbb{R}^{I(j)})$ is $(-\alpha_j)^{N-I(j)}$ by the same argument as above, except here β acts trivially.

Summarizing, if K even

$$[Z_{D\lambda}]_T = \sum_{\pi \in S_k} \frac{\prod_{j=1}^{k}(-\alpha_{\pi(j)})^{N-I(j)}}{\prod_{i=1}^{k-1}\prod_{j=i+1}^{k} \alpha_{\pi(j)}^2 - \alpha_{\pi(i)}^2} = \frac{\sum_{\pi\in S_k}(-1)^\pi \prod_{j=1}^{k} \alpha_{\pi(j)}^{2(k-j+\lambda_j)}}{V(\alpha_1^2, \dots, \alpha_k^2)}$$

where $V(x_1, \dots, x_k)$ denotes the Vandermonde polynomial and if $K = 2k+1$,

$$[Z_{D\lambda}]_T = \frac{\sum_{\pi\in S_k}(-1)^\pi \prod_{j=1}^{k}(-\alpha_{\pi(j)})^{N-I(j)}}{\Delta \cdot V(\alpha_1^2, \dots, \alpha_k^2)} = \frac{\sum_{\pi\in S_k}(-1)^\pi \prod_{j=1}^{k} \alpha_{\pi(j)}^{2(k-j+\lambda_j)}}{V(\alpha_1^2, \dots, \alpha_k^2)}.$$

In both cases, this is by definition the Schur polynomial

$$[Z_{D\lambda}]_T = \frac{\det(\alpha_i^{2(k-j+\lambda_j)})}{V(\alpha_1^2, \dots, \alpha_k^2)} = s_\lambda(\alpha_1^2, \dots, \alpha_k^2)$$

Using that the Pontryagin classes are elementary symmetric polynomials in α_j^2, the (second) Jacobi-Trudy formula gives

$$[Z_{D\lambda}]_G(p_1,\dots,p_k) = \det(p_{\lambda'_i+j-i})$$

as claimed. Since this expression is a nonzero element in H_G^*, we can conclude the proof by noting that $[Z_{D\lambda}]$ is a G-cycle by Proposition A.3.4. (This must be verified, since G is not connected.)

Remark 5.4.6. If one proves that $\mathrm{GL}(2k+1,\mathbb{R})$ are halving groups, one could also use the equivariant Borel-Haefliger theorem in the odd case as well.

Relation to Schubert classes

As in the case of complex matrix Schubert classes, if $\mathcal{K} : \mathrm{Gr}_K(\mathbb{R}^N) \to BGL(K,\mathbb{R})$ is the classifying map of the tautological bundle, then

$$\mathcal{K}^*[Z_{D\lambda}]_{\mathrm{GL}(K,\mathbb{R})} = [\sigma_{D\lambda}].$$

Thus we obtain an alternative proof for

Theorem 5.4.7 (Real Giambelli formula). *The $\sigma_{D\lambda}$ are cycles and*

$$[\sigma_{D\lambda}] = \det(p_{\lambda'_i+j-i}(S_{\mathbb{R}}))$$

Remark 5.4.8. This gives an alternative proof of cycleness of $\sigma_{D\lambda}$. However, from this proof it is not immediately clear that $[\sigma_{D\lambda}] \neq 0$; the relations between Pontryagin classes must be known from another approach (e.g. the Cartan model, Section C.2). This is one advantage of the Vassiliev complex approach, a disadvantage being the lengthy computations of Section C.1.

5.5 Equivariant intersection theory

We use the term equivariant intersection theory in a fairly liberal sense: we compute equivariant fundamental cohomology classes in the case when the ambient space is no longer a vector space. We give two applications of Theorem 3.2.13.

Real flag manifolds

The torus equivariant cohomology of real Grassmannians $\mathrm{Gr}_k(\mathbb{R}^{k+l}) = \mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$, $\mathcal{D} = (k, l)$ has been considered by GKM methods in [He16]. Let us first describe the compatible G and Γ-action we consider on $\mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^{2n})$. Let $G := \bigtimes_n \mathrm{GL}(2, \mathbb{R})$ act on X by restricting the $\mathrm{GL}(2n, \mathbb{R})$-action, where $G \leq \mathrm{GL}(2n, \mathbb{R})$ as the subgroup consisting of 2×2 blocks on the diagonal. The diagonal subgroup $\Gamma := \mathrm{U}(1) \leq G$ acts on G via inner conjugation with centralizer $G^\Gamma = \bigtimes_n \mathrm{GL}(1, \mathbb{C})$ and Γ acts on X compatibly with G by restriction. The Γ-action is the pseudo Galois type action described in Section 4.1.2.

Proposition 5.5.1. *Let $\Gamma = \mathrm{U}(1)$ act on $X = \mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^{2n})$ by the pseudo Galois type action and let $G = \bigtimes_n \mathrm{GL}(2, \mathbb{R})$ act on X compatibly with Γ in the way described above. The Schubert varieties $\sigma^{\mathbb{R}}_{DI}(F_\bullet)$ form a basis of halving G-cycles of $H^*(\mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^{2n}); \mathbb{Q})$ for an appropriate complete flag $F_\bullet \in \mathrm{Fl}(\mathbb{R}^{2n})$. Then $B_G X$ is a circle space and there is a degree-halving ring isomorphism*

$$\kappa : H^{2*}_G(\mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^{2n})) \xrightarrow{\cong} H^*_{G^\Gamma}(\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^n)).$$

Furthermore, if $Z \subseteq X$ is a halving G-cycle, then

$$[Z \subseteq \mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^N)]_G = q(y_i, [\sigma^{\mathbb{R}}_{DI}]_G) \qquad \Longleftrightarrow \qquad [Z^\Gamma \subseteq \mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^n)]_{G^\Gamma} = q(x_i, [\sigma^{\mathbb{C}}_I]_{G^\Gamma})$$

where $H^*_G \cong \mathbb{Q}[y_1, \dots, y_n]$ and $H^*_{G^\Gamma} \cong \mathbb{Q}[x_1, \dots, x_n]$.

Proof. We can use the equivariant Borel-Haefliger theorem, in the form of Theorem 3.2.13. First, we have to prove that $\sigma^{\mathbb{R}}_{DI}(F_\bullet)$ are G-cycles for an appropriate $F_\bullet$. Fix $F_\bullet \in \mathrm{Fl}(\mathbb{R}^{2n})$ such that every F_{2i} is G-invariant. The $\sigma^{\mathbb{R}}_{DI}(F_\bullet)$ are G-invariant, since G stabilizes the partial flag $DF_\bullet = (F_2 \leq F_4 \leq \ldots \leq F_{2n})$ and the rank conditions describing $\sigma^{\mathbb{R}}_{DI}$ only depend on $DF_\bullet$, cf. (B.3). The $\sigma^{\mathbb{R}}_{DI}$ are halving cycles by Theorem 4.2.2 and form a basis of $H^*(X; \mathbb{Q})$ by Proposition C.1.9. The $\sigma^{\mathbb{R}}_{DI}(F_\bullet)$ are also G-cycles by Proposition A.3.2: the normal space of the orbit Ω_{DI} is a direct sum of even-even rank Hom-spaces, see Proposition B.2.4.

We can conclude by Theorem 3.2.13, whose conditions are satisfied: $H^*(X)$ is generated by the halving G-cycles $[\sigma^{\mathbb{R}}_{DI}]$ and the normal weights are all $2u$ as we have verified in Theorem 4.2.2. □

Remark 5.5.2. By homotopy equivalence, the conclusion holds for the larger group DB^+ the "real double Borel subgroup" of $\mathrm{GL}(2n, \mathbb{R})$, i.e. the parabolic subgroup $DB^+ = \mathrm{GL}(\mathcal{D})$, for $\mathcal{D} = (2, 2, \dots, 2)$

using the notation of Section B.2.1. Its fixed point set is $B^+_{\mathbb{C}} \leq \mathrm{GL}(n,\mathbb{C})$ the usual complex Borel subgroup. However, the action of DB^+ has finitely many orbits on X and therefore there are finitely many DB^+-cycles, so the second part of the theorem is uninteresting for DB^+.

One could also take the even larger halving group $G = \mathrm{GL}(2n,\mathbb{R})$ and get a degree halving isomorphism between the G-equivariant cohomology of $\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$ to the $G^\Gamma = \mathrm{GL}(n,\mathbb{C})$-equivariant cohomology of $\mathrm{Fl}^{\mathbb{C}}_{\mathcal{D}}$ - but again, from a geometric point of view this is uninteresting, as there are no nontrivial G-invariant cycles.

Quaternionic manifolds

Our next example concerns quaternionic flag manifolds. The $T\mathbb{H} = \mathrm{Sp}(1)^N$-equivariant cohomology of complete quaternionic flag manifolds $\mathrm{Fl}(\mathbb{H}^N)$ has been first described in [Mar08] by GKM methods, and the existence of an abstract degree halving isomorphism between $H^*(\mathrm{Fl}(\mathbb{H}^N))$ and $H^*(\mathrm{Fl}(\mathbb{C}^N))$ has also been observed. From the point of view of the Borel-Haefliger theorem, this isomorphism becomes concrete. We can apply the equivariant Borel-Haefliger theorem to obtain a geometric description of all partial quaternionic flag manifolds $\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N)$, and relate it to the $T\mathbb{C} = \mathrm{U}(1)^N$-torus equivariant cohomology of the complex flag manifold $\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^N)$.

Let us describe the compatible G and Γ-actions we consider on $X = \mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N)$. Let $G = T\mathbb{H} = \times_N \mathrm{Sp}(1)$ act on X by restricting the $\mathrm{GL}(N,\mathbb{H})$-action, where $G \leq \mathrm{GL}(N,\mathbb{H})$ is the "quaternionic maximal torus" consisting of diagonal quaternionic matrices with unit quaternions on the diagonal ($\mathrm{Sp}(1)^N$). The diagonal subgroup $\Gamma = \mathrm{U}(1) \leq G$ acts on G via inner conjugation with $G^\Gamma = \times_N \mathrm{U}(1)$ and Γ acts on X by the Galois type action (Section 4.1.1) compatibly with G.

Proposition 5.5.3. *Let $\Gamma = \mathrm{U}(1)$ act on $X = \mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N)$ by the Galois type action and let $G = T\mathbb{H} = \mathrm{Sp}(1)^N$ act on X compatibly with Γ in the way described above. The Schubert varieties $\sigma_I^{\mathbb{H}}(F_\bullet)$ form a basis of halving G-cycles of $H^*(\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N);\mathbb{Q})$ for an appropriate complete flag $F_\bullet \in \mathrm{Fl}(\mathbb{H}^N)$. Then B_GX is a circle space and there is a degree halving isomorphism*

$$\kappa : H^{2*}_{T\mathbb{H}}(\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N)) \xrightarrow{\cong} H^*_{T\mathbb{C}}(\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^N)).$$

Furthermore, if $Z \subseteq X$ is a halving $T\mathbb{H}$-cycle, then the same polynomial describes the classes

$$[Z \subseteq \mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N)]_{T\mathbb{H}} = q(y_i, [\sigma_I^{\mathbb{H}}]_{T\mathbb{H}}) \qquad \Longleftrightarrow \qquad [Z^\Gamma \subseteq \mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^n)]_{T\mathbb{C}} = q(x_i, [\sigma_I^{\mathbb{C}}]_{T\mathbb{C}})$$

where $H^*_{T\mathbb{H}} \cong \mathbb{Z}[y_1, \ldots, y_N]$ *and* $H^*_{T\mathbb{C}} \cong \mathbb{Z}[x_1, \ldots, x_N]$.

Proof. The proof repeats the proof of Proposition 5.5.1 above; the choice of the complete flag $F_\bullet \in \mathrm{Fl}(\mathbb{H}^N)$ must be taken to be $\mathrm{U}(1)$-invariant. $\square$

Future directions

We conclude the core part of the thesis with a list of problems.

- Give an interpretation of the coefficients of the restriction equation $r\sigma(x)$ of circle spaces, similar to Theorem 2.2.10.
- Determine for general $\mathcal{D}$, which sums of Schubert varieties are cycles rationally for real flag manifolds $\mathrm{Fl}_{\mathcal{D}}$, as in Section F.
- Find a Lie theoretical interpretation of a group G being a halving group in terms of Weyl groups and roots.
- Give an Edidin-Graham–Totaro [EG98], [Tot99] type construction of approximations of Γ-equivariant principal bundles.
- Motivic characteristic classes: examine whether $\kappa c^{SM}(Z) = w^S(Z^\Gamma)$ holds, where c^{SM} denotes the Chern-Schwartz-MacPherson class and w^S denotes Sullivan's mod 2 motivic characteristic class.
- Generalized cohomology theories: Can the methods generalize to complex vs. real oriented theories and quaternionic vs. complex oriented theories?

Appendix A

Fundamental classes of real varieties

The Borel-Haefliger theorems involve cohomology classes of real and complex algebraic varieties, and there are slightly different approaches in defining these. We use the definitions of *singular topological varieties* of the original Borel-Haefliger paper [BH61] and van Hamel [VH07]. More precisely, we use a variation, in the sense that we work in cohomology instead of homology; in particular we use coorientability instead of orientability.

Another approach relevant to us, is fundamental cohomology classes of *stratified submanifolds* as developed by [Whi65], [Tho69], [Mat73], [Gor81], [Vas88]. We discuss stratified submanifolds and their relationship to the Borel-Haefliger definition in Section A.2.

Finally we discuss extensions of fundamental classes to equivariant cohomology.

This section is standard and well-known to experts, however because of the slight variations of these notions, we found it reasonable to include some details at least to fix terminology.

We work in the smooth category so by manifold and submanifold we always mean *smooth* manifolds and submanifolds. We expect that with enough care, most of the discussion generalizes to topological, even cohomological manifolds, however we will not need this generality.

A.1 Borel-Haefliger's fundamental class

Borel-Haefliger define fundamental classes for a class of topological spaces they call *singular topological varieties* (or in their notation VS_n spaces). This is a class of topological spaces which includes analytic manifolds and algebraic varieties; both real and complex. They are defined as spaces X which have a *fat open subset* U which is an n-dimensional manifold. Fat open subset means that its complement has small *cohomological dimension*.

In this section we attempt to make this introduction precise. Throughout the discussion, fix a smooth ambient manifold X and a principal ideal domain K (typically $K = \mathbb{Z}$ or a field $\mathbb{F}_p, \mathbb{Q}, \mathbb{R}$), and all cohomology is taken with K-coefficients.

A.1.1 Cohomology class of a submanifold

A submanifold $Z \hookrightarrow X$ is *cooriented*, if its normal bundle is oriented. Let $Z \hookrightarrow X$ be a cooriented submanifold with normal bundle ν. By the tubular neighbourhood theorem there exists an open neighbourhood $U \subseteq X$, such that $t : (\nu, \nu \backslash Z) \cong (U, U \backslash Z)$. This gives rise to a quotient map $q : X \to (\operatorname{Th} \nu, pt)$ where Th is the Thom space of ν.

Definition A.1.1. The *fundamental cohomology class* $[Z] \in H^k(X, X\backslash Z)$ is the unique class which restricts via the excisive map $e : (U, U\backslash Z) \hookrightarrow (X, X\backslash Z)$ to the Thom class of ν using t.

We remark that this object is sometimes called *refined fundamental class* [Ful97, B.3.] and denoted $[\![Z]\!] \in H^k(X, X\backslash Z)$; we will abuse terminology and call both $[\![Z]\!]$ and its restriction $[Z] \in H^k(X)$ fundamental classes. We hope that this does not cause confusion.

Alternatively, we could define $[Z] \in H^k(X)$ as the pullback of the Thom class τ_ν via the quotient map $q : X \to (\operatorname{Th} \nu, pt)$:

Proposition A.1.2.

$$[Z] = q^* \tau_\nu,$$

where $\tau_\nu \in \tilde{H}^k(\operatorname{Th} \nu)$ is the Thom class of ν, and $q : X \to \operatorname{Th} \nu$ is the quotient map.

Proof. Omitting the details, a diagram-chase on the following commutative diagram:

$$\begin{tikzcd}
(U, \partial U) \arrow[r, hook, "htop."] \arrow[d] \arrow[dr, hook] & (U, U\backslash Z) \arrow[r, hook, "exc"] & (X, X\backslash Z) \\
(U/\partial U, pt) & (X, \partial U \cup U^c) \arrow[l, two heads] \arrow[ur, hook] & X \arrow[l, hook'] \arrow[u, hook, "i"'] \arrow[ll, two heads, bend left, "q"]
\end{tikzcd}$$

$\square$

Remark A.1.3. It can be shown that if $Z \hookrightarrow X$ is a k-codimensional (closed, not necessarily orientable) submanifold, then

$$H^i(X, X \backslash Z) = 0, \qquad i < k,$$

using either Thom isomorphism in the orientable case or Thom spaces in the unorientable case.

Thom isomorphism implies the following trivial, but useful remark:

Remark A.1.4. Let $Z \subseteq X$ be a connected, cooriented k-codimensional submanifold, with a tubular neighborhood U. Let $D_y \subseteq U$ be a k-dimensional disk centered around y, intersecting Z transversally in the single point y. Then the restriction

$$H^k(X, X \backslash Z) \to H^k(D_y, D_y \backslash y)$$

is an isomorphism.

Remark A.1.5. Let

$$0 \longrightarrow A \longrightarrow B \xrightarrow{q} C \longrightarrow 0$$

be a short exact sequence of bundles over some X, with A and B oriented. Then there is a unique *compatible orientation* of C, under which all splittings $j : C \to B$ of q, the orientation of B agrees with the orientation of $A \oplus j(C)$. Notice that the orientation of $A \oplus j(C)$ and $j(C) \oplus A$ can be different we fix once and for all this ordering convention.

Although we will not need this generality, we remark that the arguments in this section can be generalized to any complex oriented cohomology theory, when $Z \hookrightarrow X$ has a complex coorientation, compare with Section A.4.

A.1.2 Singular topological varieties

Let X be a smooth, connected manifold, not necessarily orientable.

Definition A.1.6. $\Sigma \subseteq X$ has *cohomological codimension* $\operatorname{codim}_K \Sigma \geq k$ if $H^i(X, X \backslash \Sigma) = 0$ for all $i \leq k-1$. It has *cohomological codimension* $\operatorname{codim}_K \Sigma = k$, if $\operatorname{codim}_K \Sigma \geq k$ and $\operatorname{codim}_K \Sigma \ngeq k+1$.

We use the convention $\operatorname{codim}_K \emptyset = \infty$. If $Z \hookrightarrow X$ is a smooth, connected k-codimensional submanifold which is not coorientable, then $\operatorname{codim}_{\mathbb{Z}} Z \geq k+1$.

Definition A.1.7. A connected, closed subset $Z \subseteq X$ is a *topological subvariety of codimension k* if there exists an open subset $Y \subseteq Z$ which is a k-codimensional submanifold in X and its complement $\Sigma := Z \backslash Y$ has $\operatorname{codim}_K \Sigma \geq k+1$. Such a set $Y \subseteq Z$ is called a *fat nonsingular open*. More generally, a closed subset $Z \subseteq X$ is a *topological subvariety of codimension k* if each of its connected components is.

Topological subvarieties behave similarly to algebraic ones: if $Z \subseteq X$ is a topological subvariety of codimension k, then it has a set of points *regular points* Z_R, which are the points having neighbourhoods that are locally submanifolds. Then the *singular points* $Z_S = Z \backslash Z_R$ are contained in the complement Σ of any fat nonsingular open, and the long exact sequence of $(X, X\backslash Z_S, X\backslash\Sigma)$ shows that Z_R is also a fat nonsingular open, using also Remark A.1.3.

Although the definition may seem inconvenient at first, we will see that algebraic subvarieties (real and complex) and stratified submanifolds are all examples of topological subvarieties over $K = \mathbb{Z}$, see Example A.2.2, Propositions A.2.5.

Definition A.1.8. Let $Z \subseteq X$ be a topological subvariety of codimension k with fat open subset Y, let $y \in Y$. A *normal disk $D_y \subseteq X$ to $y \in Y$* is a k-dimensional disk $D_y \subseteq X$ centered at y, intersecting Y in the single point y transversally.

As is standard, we will later extend this notion to the singular set under some regularity assumptions and define normal slices, see Definition A.2.9.

Remark A.1.9. [Remarks on the definitions] To define cohomological dimension, [BH61] uses the notion of *Φ-supported sheaf (co)homology* $H_*^{\Phi}(X;\mathcal{S})$, where Φ is *a family of supports*, with particular emphasis on compactly supported cohomology. Also, they consider fundamental homology classes as opposed to cohomology classes. Since our final aim is to compute intersection products, we translated everything to cohomology.

A.1.3 Fundamental class

The following construction is an extension of the fundamental cohomology class of Definition A.1.1 to topological subvarieties.

Definition A.1.10. Let $Z \subseteq X$ be a k-codimensional topological subvariety over K and D_x a normal disk of Z_R at x. A *local coorientation at* $x \in Z_R$ is a generator of $H^k(D_x, D_x \backslash x)$.

Although this definition depends on the normal disk, the following property does not: let $U \subseteq X$ be an open set containing two normal disks D_x, D'_x and $\alpha \in H^k(U, U \backslash Z)$; then $\alpha|_{(D_x, D_x \backslash x)}$ is a local coorientation iff $\alpha|_{(D'_x, D'_x \backslash x)}$ is one. We are led to the following definition:

Definition A.1.11. Let $Z \subseteq X$ be a k-codimensional topological subvariety over K. A *fundamental cohomology class* (over K) is an element $[\![X]\!] \in H^k(X, X \backslash Z; K)$ whose restriction to $H^k(D_x, D_x \backslash x; K)$ is a generator for all regular points $x \in Z_R$, where D_x is a normal disk over x. If such a fundamental class exists, we say that Z is a *cycle*.

Notice that the definition extends the notion of Thom class: if $Z \hookrightarrow X$ is a submanifold, then it is a cycle iff it is coorientable. Even if a topological subvariety $Z \subseteq X$ is a cycle, the (non-refined class) $[Z] \in H^k(X)$ can be zero (although this cannot happen to the refined class $[\![Z]\!] \in H^k(X, X \backslash Z)$). In this section we distinguish the notation of refined class $[\![Z]\!]$ and $[Z]$, but later we will not.

The following characterization can be useful:

Proposition A.1.12. *Let $Z \subseteq X$ be a k-codimensional topological subvariety. Let $A \subseteq Z_R$ be a subset intersecting each component of Z_R, and for each $a \in Z_R$ let D_a be a normal disk to Z_R at a. A cohomology class $\alpha \in H^k(X, X \backslash Z)$ is a fundamental class if and only if it restricts to a generator of $H^k(D_a, D_a \backslash a)$ for all $a \in A$, and α is uniquely determined by these restrictions.*

Proof. Let C be a component $C \subseteq Z_R$, assume it is not coorientable. Then $H^k(U, U \backslash C) = 0$ for a tubular neighborhood U of C, so $\alpha|_{D_a, D_a \backslash a}$ cannot be a generator. Therefore all components are coorientable, hence the restrictions of α determine it uniquely by the Thom isomorphism

$$H^k(X \backslash \Sigma, X \backslash Z) \cong \bigoplus_{C \subseteq Z_R} H^0(C)$$

for connected components $C \subseteq Z_R$. □

This implies in particular by a Mayer-Vietoris argument, that given open sets $U_i \subseteq X$ covering Z and a compatible system of fundamental classes α_i on U_i, Z has a fundamental class restricting to the α_i. We state two propositions that will be used in later characterizations, see Propositions A.2.8 and A.4.3.

Proposition A.1.13. *Let $Z \subseteq X$ be a k-codimensional topological subvariety, with fat nonsingular open Y and $\Sigma := Z \backslash Y$. Then Z is a cycle if and only if $Y \subseteq X \backslash \Sigma$ is a cycle and has a fundamental class $[\![Y]\!]$ such that $\delta[\![Y]\!] = 0$, where*

$$\delta : H^k(X\backslash\Sigma, X\backslash Z) \to H^{k+1}(X, X\backslash\Sigma)$$

is the connecting homomorphism of the triple $(X, X\backslash\Sigma, X\backslash Z)$.

Proof. This follows directly from the previous proposition and the LES of $(X, X\backslash\Sigma, X, \backslash Z)$. □

Proposition A.1.14. *Let $Z \subseteq X$ be a k-codimensional topological subvariety, with fat nonsingular open Y and $\Sigma := Z\backslash Y$. Then Z is a cycle if and only if*

- *$Y \subseteq X\backslash\Sigma$ is a cycle with a refined class $[\![Y]\!] \in H^k(X\backslash\Sigma, X\backslash Z)$, and*
- *there exists $[Z] \in H^k(X)$, such that $[Z]|_{X\backslash\Sigma} = [Y] \in H^k(X\backslash\Sigma)$.*

In this case, $[\![Y]\!]$ has a unique lift to a refined class $[\![Z]\!] \in H^k(X, X\backslash Z)$, and this class restricts to $[Z]$.

Proof. A diagram chase on the LES of the pairs $(X, X\backslash Z)$ and $(X\backslash\Sigma, X\backslash Z)$. □

The following proposition summarizes some existence and uniqueness properties of fundamental classes, which follow from the previous discussion.

Proposition A.1.15. *Let $Z \subseteq X$ be a k-codimensional topological subvariety with a fat open subset U, $\Sigma = Z\backslash U$.*

- *Uniqueness: Given a fundamental class $[\![U]\!] \in H^k(X\backslash\Sigma, X\backslash Z)$, Z has at most one fundamental class $[\![Z]\!]$ restricting to $[\![U]\!]$.*
- *Uniqueness for $K = \mathbb{F}_2$: If $K = \mathbb{F}_2$, then Z has at most one fundamental class $[\![Z]\!]$.*
- *Existence: If U has a fundamental class $[\![U]\!]$, and if $\operatorname{codim}_K \Sigma \geq k+2$, then X has a fundamental class restricting to $[\![U]\!]$.*

Remark A.1.16. If Z is a complex subvariety, then there is a canonical choice for the fundamental class - the complex structure induces an orientation on the normal bundle of the smooth part. If $K = \mathbb{F}_2$, then by the previous proposition, the fundamental class is unique. However, for $K = \mathbb{Z}$, if Z is a connected topological subvariety, there are no longer canonical orientations and there is a sign ambiguity in the choice of the fundamental class. In this case, any result involving fundamental classes inherently contains such a sign ambiguity.

The last important property that we will prove after introducing Gysin maps, is that if $Z \subseteq X$ has an oriented resolution $f : \tilde{Z} \to X$, then Z is a cycle and $f_! 1 \in H^*(X)$ is a fundamental class (cf. Proposition A.4.3).

Unless we have some extra knowledge as in Propositions A.1.15, A.4.3 or A.5.3 it is not an easy task to decide whether a given topological subvariety is a cycle. In the next section we describe a general method that in some cases helps [Ohm94], [FR02b].

A.2 Stratified submanifolds

Whenever a Lie group G acts nicely on X (e.g. algebraically, with finitely many orbits), a natural stratification of X arises given by the orbit structure. In particular, orbit closures $Z = \overline{G.x}$ are topological subvarieties, with $G.x \subseteq Z$ a fat open subset. Vassiliev [Vas88] introduced a chain complex associated to the group action which is a free $\mathbb{Z}$-module generated by the orbits, with the property that the closure of the orbit is a cycle in the sense of Definition A.1.11 if and only if it is a cycle in this chain complex.

In this section we give a very brief introduction to stratified spaces and their cycles. For further details on stratified spaces see [Mat73], [Wal75], [GWdPL76] or [GM88], and for their cycles we refer to [Gor81] and [Vas88].

A.2.1 Definition, main examples

Definition A.2.1. Let X be an n-dimensional manifold. A *stratification* $(X_\alpha)_{\alpha \in A}$ *of* X is a locally finite partition of X into disjoint submanifolds X_α called *strata* (not necessarily closed or connected), which satisfy the *frontier condition*:

$$X_\alpha \cap \overline{X_\beta} \neq \emptyset \Rightarrow X_\alpha \subseteq \overline{X_\beta} \backslash X_\beta \text{ and } k_\alpha > k_\beta$$

where $k_\alpha := \operatorname{codim} X_\alpha$.

A closed subset $Z \subseteq X$ is a *k-codimensional stratified submanifold of* X if there exists a stratification $(X_\alpha)_{\alpha \in A}$ of X, such that $Z = \overline{X_\alpha}$, $k_\alpha = k$.

In particular, a stratified submanifold is a disjoint union of strata. Our main examples are the following.

Example A.2.2. [*The singular stratification*] Let $Z \subseteq X$ be a complex k-codimensional closed algebraic subset of a smooth n-dimensional variety X. Then Z has a stratification obtained as follows. Let $\mathcal{S}$ be the system of constructible subsets containing Z and closed under the following operations:

- taking singular subset of an irreducible component ($C \subseteq A \in \mathcal{S}$ component $\Rightarrow \operatorname{Sing}(C) \in \mathcal{S}$),
- taking intersections and differences, ($A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}, A \backslash B \in \mathcal{S}$),
- taking closures ($A, B \in \mathcal{S} \Rightarrow \overline{A} \in \mathcal{S}$)

Using the observation that the singular subset of a variety has smaller dimension, it is not hard to see that $\mathcal{S}$ consists of finitely many subsets of Z and that the minimal subsets of $\mathcal{S}$ are nonsingular and form a stratification of Z, (in particular they satisfy the frontier condition).

The analogous theorem in the real case is more complicated and uses Whitney's condition, see e.g. [GWdPL76, Section 2.7]. ♣

Example A.2.3. [*Orbit stratification*] Let G be a Lie group acting smoothly on a smooth manifold X semialgebraically and with finitely many orbits. Then the orbit structure gives a stratification of X: the frontier condition is satisfied, since the frontier of an orbit consists of smaller dimensional orbits by [Dim87, Remark 4.9] and local finiteness clearly holds. The stratification is also Whitney regular, see e.g. [DK00] for a more general theorem. ♣

To relate stratifications to topological varieties, first we need a lemma:

Lemma A.2.4. *Let Y be a manifold with a codimension stratification (Y^i). Then for all $i < k$*

$$H^i(Y, Y \backslash \overline{Y^k}) = 0.$$

Proof. Prove this by downward induction on k. For $k > \dim Y$, this is clearly true. Assume that it holds for $k+1$, let $i < k$. Consider the following segment of the long exact sequence of the triple $(Y, Y\backslash\overline{Y^{k+1}}, Y\backslash\overline{Y^k})$:

$$H^i(Y, Y\backslash\overline{Y^{k+1}}) \to H^i(Y, Y\backslash\overline{Y^k}) \to H^i(Y\backslash\overline{Y^{k+1}}, Y\backslash\overline{Y^k})$$

The first term is zero by assumption, and since $Y^k \subseteq Y\backslash\overline{Y^{k+1}}$ is a closed k-codimensional submanifold, the last term is zero by Remark A.1.3. □

A direct consequence is

Proposition A.2.5. *Let $Z = \overline{X^k}$ be a k-codimensional stratified submanifold of X. Then it is a k-codimensional topological variety (over $\mathbb{Z}$) with a fat open subset X^k.*

We conclude this section by commenting on the assumptions used in the definition of stratified submanifolds, in particular on Whitney stratifications.

Remark A.2.6. [Remarks on the definitions] The definition of stratification used here is closest to [Vas88, Definition 7.1].

The frontier condition used here is stronger than the usual one, $k_\alpha > k_\beta$ is often not included in the definition. However, condition $k_\alpha > k_\beta$ follows from Whitney condition b) [Mat12, Proposition 4.5]. For building the spectral sequence in the next section we only require this property from our stratification and not the whole Whitney condition. This is not an important distinction, as a large class of spaces have been proven to have Whitney stratifications, we mention real/complex analytic/algebraic varieties and semialgebraic varieties, see [Łoj65], [GWdPL76].

For our purposes the main application of Whitney stratifications is that it enables a computation of the incidence coefficients using local topological structure of the strata. For a precise formulation, see Section A.2.3.

The locally finite condition and the stronger version of the frontier condition implies that the union X^k of all k-codimensional strata is a k-codimensional submanifold; this induces another stratification, which we call *codimension stratification* $(X^k)_k$; the stratum X^k is a k-codimensional submanifold, closed in $X \backslash \overline{X^{k+1}}$.

It is also customary to call the *connected components* of a stratification strata, and require the frontier condition for these connected components - note that this is not automatic. Since we also want to deal with examples when a stratum is not connected we do not assume connectedness.

Being submanifolds, the strata are *locally closed* (e.g. [Mat73] assumes only this and does not assume that they are submanifolds).

As a final side remark, local finiteness and Whitney condition b) imply the frontier condition, [Mat12, Corollary 10.5].

A.2.2 Spectral sequence of a filtration

In nice cases, the spectral sequence of the open filtration associated to a codimension stratification $(X^i)_i$ reduces to the bottom row of the E_1-page, also called the Vassiliev complex. We describe the spectral sequence and the degeneration conditions.

Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a finite filtration of X:

$$F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = X.$$

Then there is a cohomological spectral sequence

$$E_\infty^{p,q} \Rightarrow H^{p+q}(X)$$

with E_1-term

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}).$$

In particular,

$$E_\infty^{p,q} = F_{p-1}H^{p+q}(X)/F_pH^{p+q}(X),$$

where

$$F_iH^r(X) = \ker(H^r(X) \to H^r(F_i))$$

is the filtration induced on $H^r(X)$.

Let X be an n-dimensional manifold with a codimension stratification (X^i). Take the *open filtration* $\mathcal{F} = (F_i)$ given by

$$F_i := X \backslash \overline{X^{i+1}} = \coprod_{j \leq i} X^j$$

which for $i > 0$ are open submanifolds of X. Note that $(F_j)_{j \leq i}$ is a filtration of the n-dimensional manifold F_i and that $F_i \backslash F_{i-1} = X^i \subseteq F_i$ is a closed i-codimensional submanifold of F_i. Applying Lemma A.2.4 to $Y = F_p$ implies that

$$E_1^{p,q} = H^{p+q}(F_p, F_p \backslash X^p) = 0, \qquad \text{for } q < 0$$

that is $E_\infty^{p,q}$ is a first quadrant spectral sequence.

Example A.2.7. [Vassiliev complex] We say that *the stratification* $(X_\alpha)_{\alpha \in A}$ *satisfies Vassiliev's condition*, if all strata are connected and contractible. Let $(X^i)_i$ be the associated codimension stratification and $\mathcal{F} = (F_i)_i$ be the associated open filtration. Then all strata are clearly coorientable, so by Thom isomorphism

$$E_1^{p,q} = H^{p+q}(F_p, F_p \backslash X^p) \cong H^q(X^p) \cong \begin{cases} \mathbb{Z}^{c_p}, & q = 0 \\ 0, & q \neq 0 \end{cases}$$

where c_p denotes the cardinality of strata X_α of codimension p. So the E_1-page consists of a single row, which is a chain complex. In singularity theory this chain complex is also often called the *Vassiliev complex*. Therefore this spectral sequence degenerates on the E_2-page, $E_2 = E_\infty$, hence computing the cohomology of the Vassiliev complex completely computes the additive structure of $H^*(X; \mathbb{Z})$.

Denote by d_1 the differential on E_1; it is the connecting homomorphism of the long exact sequence of the triple (F_{p+1}, F_p, F_{p-1})

$$d_1 : H^p(F_p, F_p \backslash X^p) \to H^{p+1}(F_{p+1}, F_{p+1} \backslash X^{p+1})$$

It is completely determined by the incidence coefficients $[X_\alpha, X_\beta]$, defined by

$$d_1 X_\alpha = \sum_{k_\beta = k_\alpha + 1} [X_\alpha, X_\beta] X_\beta$$

where $k_\alpha = \operatorname{codim} X_\alpha$. In the next section we describe a geometric way of computing these incidence coefficients. ♣

Finally, to show that the Vassiliev complex actually computes when a stratified submanifold Z represents a cohomology class $[Z] \in H^k(X)$, we have

Proposition A.2.8. *Let $(X_\alpha)_{\alpha\in A}$ be a stratification of X satisfying Vassiliev's condition and let Y be a union of equidimensional cooriented strata. Then the stratified submanifold $Z = \overline{Y}$ is a cycle in the sense of Definition A.1.11 if and only if Y is a cycle in the Vassiliev complex.*

Proof. As before, let $\Sigma := Z\backslash Y$ and (X^k) be the codimension stratification. Using Proposition A.1.13, both directions follow from commutativity of

$$\begin{tikzcd}
H^k(X\backslash\Sigma, X\backslash Z) \arrow[r, "\delta"] \arrow[d, "\rho"] & H^{k+1}(X, X\backslash\Sigma) \arrow[d, hook, "\rho'"] \\
H^k(X\backslash\overline{X^{k+1}}, X\backslash\overline{X^k}) \arrow[r, "d_1"] & H^{k+1}(X\backslash\overline{X^{k+2}}, X\backslash\overline{X^{k+1}})
\end{tikzcd}$$

and $\rho[Y] = [Y]$. The injectivity of ρ' follows from the LES's of $(X, X\backslash\Sigma, X\backslash\overline{X^{k+1}})$ and $(X, X\backslash\overline{X^{k+2}}, X\backslash\overline{X^{k+1}})$. □

We remark that in case a stratification does not satisfy Vassiliev's condition, the corresponding filtration gives a spectral sequence which does not degenerate.

A.2.3 Vassiliev complex - incidence coefficients

Throughout this section let $(X_\alpha)_{\alpha\in A}$ be a Whitney stratification of X satisfying Vassiliev's condition. We recall a geometric description of the incidence coefficients $d_{\alpha\beta}$ of neighboring strata $X_\beta \subseteq \partial X_\alpha$ in the Vassiliev complex (Example A.2.7). The procedure is the following. Take a normal slice N_b of $\overline{X_\alpha}$ at some $b \in X_\beta$, which is a finite union of cooriented curves L_i, whose closure contains b. Then comparing each coorientation of L_i to that of b gives a signed sum $n_{\alpha\beta} \in \mathbb{Z}$. By Proposition A.2.10 below, these two coefficients agree: $d_{\alpha\beta} = n_{\alpha\beta}$. To make this introduction precise, we have to elaborate on some technical points: independence of the normal slice on the choice of b, how to compare the two coorientations, and finally $d_{\alpha\beta} = n_{\alpha\beta}$.

Let $Z := \overline{X_\alpha}$ and $B := X_\beta \subseteq \partial X_\alpha$ be a connected stratum. Using Whitney regularity, take a small enough tubular neighborhood $p : U \to B$ intersecting all strata of Z transversely. Then

$E := U \cap Z$ is a stratified submanifold, and by Thom's first isotopy lemma, $E \to B$ is a locally trivial fibration (see [GM88, 1.11].) This allows us to extend normal slices (Definition A.1.8) to lower dimensional strata as follows.

Definition A.2.9. A *normal disk* to the stratum $X_\beta \subseteq \partial X_\alpha$ of a Whitney stratified submanifold $Z = \overline{X_\alpha}$ at $b \in X_\beta$, is a k_β-dimensional open disk D_b centered at b, intersecting all strata transversely, in particular $D_b \cap X_\beta = \{b\}$. A *normal slice* of Z to X_β at b is $N_b := D_b \cap Z$.

The previous discussion shows that the the topological type of the normal slice $N_b \cap Z$ is independent of the choice of b. For the computation of incidence numbers we will need normal slices in the special case when the strata are of neighboring dimensions.

Next, we define $n_{\alpha\beta}$ in more detail. Let $A := X_\alpha$ be a k-codimensional and $B := X_\beta \subseteq \overline{X_\alpha}$ be a $k+1$-codimensional cooriented stratum, with normal bundles ν_α, ν_β respectively. Let $D := D^{k+1}$ be a normal disk to B at $b \in B$, let $L := D \cap A$ which by transversality is a disjoint union of connected curves L_i, whose closure contains b. For each L_i, choose a splitting of the quotient map $TX \to \nu_\alpha|_{L_i}$, such that

$$TD|_{L_i} = \nu_\alpha|_{L_i} \oplus TL_i.$$

Then TL_i is oriented by taking the orientation pointing towards b and ν_α is oriented by the coorientation of A, so they determine an orientation of $TD|_{L_i}$ which determines an orientation of TD at b as well. Since D is a normal disk, the orientation of $\nu_\beta|_b$ determines an orientation of $TD|_b$. If these two orientations agree for L_i, then set $n^i_{\alpha\beta} := +1$, otherwise -1. Set $n_{\alpha\beta} := \sum_i n^i_{\alpha\beta}$. The following Proposition states that the two coefficients agree:

Proposition A.2.10.

$$d_{\alpha\beta} = n_{\alpha\beta}$$

The proof uses the transversal pullback property; we omit the details.

A.3 Equivariant fundamental classes

In this section, we examine equivariant lifts $[Z]_\Gamma$ of fundamental classes $[Z]$.

Definition A.3.1. Let X be a Γ-space. We say that a Γ-invariant cycle $Z \subseteq X$ *is a* Γ-*cycle*, if there exists $[Z]_\Gamma \in H^k_\Gamma(X, X \backslash Z)$ restricting to $[Z] \in H^k(X, X\backslash Z)$.

If Γ is a connected Lie group, this definition is redundant, by the following Proposition. We introduce this definition, since we will be concerned with nonconnected Lie groups as well. In this case a Γ-invariant cycle is not always a Γ-cycle as the example of the $\Gamma = \mathbb{Z}_2$-invariant cycle $0 \hookrightarrow \mathbb{R}$ shows for the reflection action.

Proposition A.3.2. *Let X be a Γ-manifold and let $Z \subseteq X$ be a Γ-invariant cycle with fundamental class $[Z] \in H^k(X, X\backslash Z)$. If (Γ, R) is*

a) $(\Gamma, \mathbb{F}_2)$, Γ *arbitrary Lie group,*

b) $(\Gamma, \mathbb{Q})$, Γ *connected Lie group,*

c) $(\Gamma, \mathbb{Q})$, Γ *arbitrary Lie group and if* Γ *acts on* $\nu := \nu(Z_R \hookrightarrow X)$ *in an orientation preserving way, where* Z_R *denotes the regular points of* Z*,*

then Z is a Γ-cycle, i.e. there exists a unique $[Z]_\Gamma \in H^k_\Gamma(X, X\backslash Z)$ restricting to $[Z]$.

Proof. Consider the spectral sequence of the (relative) fibration $(X, X\backslash Z) \to B_\Gamma(X, X\backslash Z) \to B\Gamma$. Its E_2-page is

$$E_2^{p,q} = H^p(B\Gamma; \mathcal{H}^q(X, X\backslash Z)) \Rightarrow H^{p+q}_\Gamma(X, X\backslash Z)$$

where $\mathcal{H}^*(X, X\backslash Z)$ denotes the local coefficient system over $B\Gamma$. Since Z is a topological subvariety, $E_2^{p,q} = 0$ for $q < k$, therefore $E_2^{0,k}$ consists entirely of permanent cycles and $E_\infty^{0,k} = E_2^{0,k}$. So by convergence of the spectral sequence,

$$H^k_\Gamma(X, X\backslash Z) = H^0(B\Gamma; \mathcal{H}^k(X, X\backslash Z))$$

This can be identified as follows. The connected components $C := \Gamma/\Gamma^0$ act on $H^*(X, X\backslash Z)$. Then

$$H^0(B\Gamma; \mathcal{H}^k(X, X\backslash Z)) = H^k(X, X\backslash Z)^C$$

by an identification via group cohomology, see e.g. [Whi78, Theorem VI.3.2.].

a) Over $R = \mathbb{F}_2$, every cycle Z has a *unique* fundamental class $[Z]$. Since $\gamma^*[Z]$ for $\gamma \in C$ acting on $H^k(X, X\backslash Z)$ are also fundamental classes, $[Z]$ is Γ-invariant and lifts to a fundamental class $[Z]_\Gamma$.

b) If Γ is connected, then C is trivial and every fundamental class $[Z]$ is C-invariant, lifting to a unique $[Z]_\Gamma$.

c) We identify the C-action on $H^k(X, X\backslash Z)$. First, consider the case when Z is smooth with some fundamental class $[Z] \in H^k(X, X\backslash Z)$. If Γ acts on ν by orientation preserving bundle maps, then $B_\Gamma \nu \to B_\Gamma Z$ is also orientable and the Thom class of ν lifts to a unique equivariant Thom class of $B_\Gamma \nu \to B_\Gamma Z$. This maps to an equivariant fundamental class $[Z]_\Gamma$ via excision

$$H^k_\Gamma(\nu, \nu\backslash 0) \cong H^k_\Gamma(X, X\backslash Z).$$

More generally, let $Z \subseteq X$ be a cycle, with Γ-invariant fat nonsingular U such that Γ acts on $\nu(U \hookrightarrow X)$ by oriented bundle maps. Then consider the commutative diagram:

$$\begin{tikzcd}
H^k_\Gamma(X, X\backslash Z) \arrow[r, two heads, "e_Z"] \arrow[d] & H^k(X, X\backslash Z)^C \arrow[d, hook] \\
H^k_\Gamma(X\backslash\Sigma, X\backslash Z) \arrow[r, two heads, "e_U"] & H^k(X\backslash\Sigma, X\backslash Z)^C
\end{tikzcd}$$

where e_Z, e_U denote the edge homomorphisms. Since $[U] \in H^k(X\backslash\Sigma, X\backslash Z)$ lifts to $[Z] \in H^k(X, X\backslash Z)$, and $[U]$ is C-invariant, $[Z]$ is also C-invariant. Therefore it lifts to a class $[Z]_\Gamma$. $\square$

More generally, if $R = \mathbb{Z}$ and H^*_Γ is free of finite type, $E_2 = H^*(X) \otimes H^*_\Gamma$ as well.

Proposition A.3.3. *Let X be a Γ-manifold and $\Omega = \Gamma.x \subseteq X$ an orbit of the form $\Omega = \Gamma/\Delta$.*

a) If the normal isotropy representation $\Delta \to \mathrm{GL}_+(N_x)$ has positive determinant, then Ω is coorientable.

b) If $Z = \overline{\Omega}$ is a cycle, then Z is a Γ-cycle iff the normal isotropy representation $\Delta \to \mathrm{GL}_+(N_x)$ has positive determinant.

Proof. a) If $\Delta \to \mathrm{GL}_+ := \mathrm{GL}_+(N_x)$, then the normal bundle of Ω in X is

$$\nu = \Gamma \times_\Delta N_x = (\Gamma \times_\Delta \mathrm{GL}_+) \times_{\mathrm{GL}_+} N_x$$

b) By the previous Proposition, it is enough to prove that if $\Delta \to \mathrm{GL}_+$, then Γ acts by oriented bundle maps, and indeed,

$$B_\Gamma \nu = B_\Gamma(\Gamma \times_\Delta N_x) = B_\Delta N_x = (E\Delta \times_\Delta \mathrm{GL}_+) \times_{\mathrm{GL}_+} N_x$$

is orientable iff $\Delta \to GL_+$. □

The converse of a) does not hold, as the nontrivial $\mathbb{Z}_2$-representation on $0 \in \mathbb{R}$ shows.

Finally, we give a sufficient condition, which allows to determine when a Γ^0-equivariant class $[Z]_{\Gamma^0}$ lifts to a Γ-equivariant class $[Z]_\Gamma$, where Γ^0 is the connected component of Γ.

Proposition A.3.4. *Let $Z \subseteq X$ be a Γ-invariant cycle, such that $0 \neq [Z]_{\Gamma^0} \in (H^k_{\Gamma^0}(X))^C$, where $C := \Gamma/\Gamma^0$. Then Z is a Γ-cycle, and given a refined class $[\![Z]\!]_\Gamma$ restricting to $[Z]_{\Gamma^0}$, there is a unique class $[\![Z]\!]_\Gamma$ restricting to $[\![Z]\!]_{\Gamma^0}$.*

Proof. By the fibrations $C \to B_{\Gamma^0}X \to B_\Gamma X$, one has a commutative diagram

$$\begin{tikzcd} H^k_\Gamma(X, X\backslash Z) \arrow[r] \arrow[d, hook] & H^k_\Gamma(X) \arrow[r] \arrow[d, hook] & H^k_\Gamma(X\backslash Z) \arrow[d, hook] \\ H^k_{\Gamma^0}(X, X\backslash Z) \arrow[r] & H^k_{\Gamma^0}(X) \arrow[r] & H^k_{\Gamma^0}(X\backslash Z) \end{tikzcd}$$

where the top row maps to the C-invariant part of the bottom row. Since $[Z]_{\Gamma^0}$ is C-invariant, it lifts to $0 \neq [Z]_\Gamma \in H^k_\Gamma(Z)$. By exactness, it lifts to a nonzero class $[\![Z]\!]_\Gamma \in H^k_\Gamma(X, X\backslash Z)$. Finally, a diagram chase shows that $[\![Z]\!]_{\Gamma^0}$ has a unique lift $[\![Z]\!]_\Gamma$. □

The example of the $\mathbb{Z}_2$-invariant cycle $0 \hookrightarrow \mathbb{R}$ with the reflection action does not satisfy the condition $[Z]_{\Gamma^0} = [Z] \neq 0$.

A.4 Gysin maps

Given a smooth map $f : X \to Y$ of codimension k between smooth oriented manifolds, there is a Gysin map $f_! : H^*(X) \to H^{*+k}(Y)$. If $f : X \hookrightarrow Y$ is an embedding, then $f_! 1 = [X]$. More generally, if $Z \subseteq Y$ is a subvariety and $f : X \to Z \subseteq Y$ is a resolution, then $f_! 1 = [Z]$. We give less details in this section, but good references for these notions are [Dye69], [BJ] and [Swi02].

A.4.1 Gysin maps

Let $f : X \to Y$ be a cooriented map of codimension k with the orientation $\tilde{\nu} := \nu_{\tilde{f}}$ fixed, where $i : X \hookrightarrow V$ embedding, V a vector space and $\tilde{f} := (f, i) : X \hookrightarrow Y \times V$. There exists a tubular neighborhood $\tilde{f}(X) \subseteq U \subseteq BV$, such that $j : \tilde{\nu} \to U$ is an isomorphism, such that $j \circ s = \tilde{f}$, where $s : X \to \tilde{\nu}$ is the zero section.

Definition A.4.1. Let $f : X \to Y$ be a cooriented proper map. The *Gysin homomorphism* $f_! : H^*(X) \to H^{*+k}(Y)$ is given by the composition

$$\begin{tikzcd} \tilde{H}^{*+k+|V|}(\mathrm{Th}(\tilde{\nu})) \arrow[r, "p^*"] & \tilde{H}^{*+k+|V|}(Y_+ \wedge S^V) \arrow[d, "\cong"', "(\Sigma^V)^{-1}"] \\ H^*(X) \arrow[u, "\tilde{\tau}", "\cong"'] \arrow[r, "f_!"] & H^{*+k}(Y) \end{tikzcd}$$

where

$$p : (\mathrm{Th}(\tilde{\nu}), pt) \cong (U/\partial U, pt) \to (Y \times BV/Y \times SV, pt)$$

is the Pontryagin-Thom collapse map.

Independence of the Gysin homomorphism on the representative of the orientation follows from differential topological lemmas, see e.g. [Dye69].

Properties

In any generalized equivariant cohomology theory h^*_Γ, Gysin maps have the following axiomatic properties (see [Qui71a]):

- $f_!$ is a h^*_Γ-module homomorphism
- If $f : X \hookrightarrow Y$ is a cooriented Γ-equivariant embedding, then $f_!$ factors through

$$\begin{tikzcd} & h^{*+k}_\Gamma(Y, Y \backslash X) \arrow[d] \\ h^*_\Gamma(X) \arrow[ur, "f_!", "\cong"'] \arrow[r, "f_!"] & h^{*+k}_\Gamma(Y) \end{tikzcd}$$

- $$f_!(f^*y \cdot x) = y \cdot f_!x \qquad \text{(Adjunction formula)}$$

For proper cooriented Γ-maps $f : X \to Y$ we denote $e_f := f^*f_!1$. The following lemma is an easy corollary of the adjunction formula:

Lemma A.4.2. *Let $f : X \hookrightarrow Y$ be a cooriented Γ-equivariant embedding. Then*

a) $f^*f_!x = x \cdot e_f$

b) $f_!x \cdot f_!y = f_!(x \cdot y \cdot e_f)$

c) if e_f is not a zero-divisor, then $f_!1$ is not a zero-divisor in $h^*_\Gamma(Y, Y\backslash X)$, and after localizing

$$\frac{f_!x}{f_!1} = f_!\frac{x}{e_f}$$

Proof. a) Using the adjunction formula for $f_!1 \cdot f_!x = f_!x \cdot f_!1$ in both ways:

$$f_!(f^*f_!x \cdot 1) = f_!(f^*f_!1 \cdot x).$$

Since $f_!$ is an isomorphism,

$$f^*f_!x = f^*f_!1 \cdot x.$$

b) Using a) and the adjunction formula

$$f_!(x \cdot y \cdot f^*f_!1) = f_!(f^*f_!x \cdot y) = f_!x \cdot f_!y$$

c) Assume that $f_!1 \cdot y = 0$. Since $f_!$ is an isomorphism, $y = f_!x$ for some x. Then by part b)

$$f_!1 \cdot f_!x = f_!(x \cdot e_f) = 0$$

and since $f_!$ is an isomorphism, $x \cdot e_f = 0$. Since e_f is not a zero-divisor, $x = 0$, and $y = f_!0 = 0$. We can conclude by applying part b) to x and $y = 1$. □

Gysin maps are related to fundamental classes via the following Proposition:

Proposition A.4.3. *Let $Z \subseteq X$ be a k-codimensional topological subvariety, and let $f : \tilde{Z} \to X$ be a smooth, proper, cooriented map, such that $f(\tilde{Z}) = Z$. If there exists a fat nonsingular open subset $U \subseteq Z_R$ intersecting each component of Z_R, and such that $g := f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a diffeomorphism, then*

- *Z is a cycle,*
- *there exists a unique fundamental class $[\![Z]\!] \in H^k(X, X\backslash Z)$ restricting to $g_! 1 = [\![U]\!] \in H^k(X\backslash\Sigma, X\backslash Z)$ where $\Sigma := Z\backslash U$, and*
- *$f_! 1 = [\![Z]\!]|_X \in H^k(X)$ is a fundamental class.*

Definition A.4.4. We call a map f with the properties as in the Proposition a *resolution of Z*.

Proof. It is enough to show that

$$f_! 1|_{X\backslash\Sigma} = [\![U]\!]|_{X\backslash\Sigma} \in H^k(X\backslash\Sigma),$$

by Proposition A.1.14. This follows from the definition of Gysin maps, the naturality of suspensions and naturality of Thom isomorphism.

□

For the relationship of this Proposition to resolutions in algebraic geometry, see the next section.

A.5 Real algebraic varieties

In this section we discuss real algebraic varieties, complexifications and Borel and Haefliger's proof that real algebraic varieties possess a fundamental class mod 2. This section is based on [BH61] and [Whi65].

A.5.1 Complexification

A *real algebraic variety* $X \subseteq \mathbb{R}P^n$, is the zero locus $X = V(f_1, \dots, f_r) \subseteq \mathbb{R}P^n$ of homogeneous polynomials $f_i \in \mathbb{R}[x_0, \dots, x_n]$. An *algebraic variety* $X \subseteq \mathbb{C}P^n$ *defined over* $\mathbb{R}$, is the zero locus

$X = V(f_1, \ldots, f_r) \subseteq \mathbb{C}P^n$ of homogeneous polynomials $f_i \in \mathbb{R}[x_0, \ldots, x_n]$, X irreducible in the real Zariski topology.

A complex variety $X \subseteq \mathbb{C}P^n$ is a *complexification* if it is equal to the Zariski closure of its real part: $X = \overline{X_{\mathbb{R}}}$, where $X_{\mathbb{R}} = X \cap \mathbb{R}P^n$. The Zariski closure $\overline{X_{\mathbb{R}}}$ is not necessarily the complex locus of its defining equations, for example $V(x^2 + y^2) \subseteq \mathbb{R}P^2$. In fact, the Zariski closure $\overline{X_{\mathbb{R}}}$ corresponds to $V(I(X_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C})$.

If $X \subseteq \mathbb{C}P^n$ is an algebraic variety defined over $\mathbb{R}$, then its real part $X_{\mathbb{R}} \subseteq \mathbb{R}P^n$ is a real algebraic variety. The Zariski topology on $\mathbb{R}P^n$ coincides with the subspace topology induced from the Zariski topology of $\mathbb{C}P^n$.

The *(algebraic) dimension of a real/complex algebraic variety* $Z \subseteq \mathbb{F}P^n$ is $\dim^{\text{alg}}_{\mathbb{F}} Z = n - r$, where r is the maximum over $p \in Z$ of the rank of the Jacobian $J(f_1, \ldots, f_s)(p)$ of a generating system $(f_1, \ldots, f_s)$ of the ideal $I(Z)$.

The following lemma is due to Whitney [Whi57, Lemma 9]:

Lemma A.5.1 (Whitney's lemma)**.** *Let $X_{\mathbb{R}}$ be a nonempty real algebraic variety. Then*

$$\dim^{\text{alg}}_{\mathbb{R}} X_{\mathbb{R}} = \dim^{\text{alg}}_{\mathbb{C}} X_{\mathbb{C}}.$$

A.5.2 Real algebraic cycles

Definition A.5.2. A topological variety X of dimension n is *locally real algebraic*, if every $x \in X$ has a neighborhood homeomorphic to a real algebraic subset of some open $U \subseteq \mathbb{R}^q$.

A locally real algebraic variety has a fundamental class mod 2 by a theorem of Borel and Haefliger [BH61, Section 3.8]. We briefly sketch the proof.

Proposition A.5.3. *Let X be a topological variety of dimension n. If it is locally real algebraic, then it has a fundamental class mod 2.*

Proof. It is enough to show locally that real algebraic varieties $X_{\mathbb{R}}$ have fundamental classes.

Recall the following: complex varieties have normalisations, and normal varieties have the property that the singular subset is at least 2 (complex) codimensional. If X is the complexification of a real variety $X_{\mathbb{R}}$, then the normalisation $F : Y \to X$ is defined over $\mathbb{R}$ (which is a surjective proper map, which is a bijection on $F^{-1}X_{reg}$ where X_{reg} are the regular points of X).

We can apply Proposition A.4.3 to the resolution $F : Y \to X$. We show that the conditions are satisfied: Let $Y_{\mathbb{R}}$ be the real part of Y. Using that F is defined over $\mathbb{R}$, i.e. $F(Y_{\mathbb{R}}) \subseteq X_{\mathbb{R}}$, $F|_{Y_{\mathbb{R}}}$ is also proper. F is a homeomorphism restricted to $F^{-1}(X_{reg} \cap X_{\mathbb{R}})$. Therefore it is a homeomorphism on $F^{-1}(X_{reg} \cap U)$, where $U \subseteq X_{\mathbb{R}}$ is the fat open subset.

It remains to show that $Y_{\mathbb{R}}$ has a fundamental class. Indeed, the singular points of $Y_{\mathbb{R}}$ are contained in the real part of singular points of Y:

$$\mathrm{Sing}(Y_{\mathbb{R}}) \subseteq \mathrm{Sing}(Y)_{\mathbb{R}},$$

therefore

$$\dim_{\mathbb{Z}}(\mathrm{Sing}(Y_{\mathbb{R}})) \leq \dim_{\mathbb{Z}}(\mathrm{Sing}(Y)_{\mathbb{R}})/2 \leq n-2.$$

□

The mod 2 condition was used in that the nonsingular part of $Y_{\mathbb{R}}$ is always coorientable mod 2. If it is coorientable, then X has a fundamental class.

Since we are concerned with a generalization of the Borel-Haefliger theorem to U(1)-actions, it is relevant that normalisation is a non-topological operation. In the purely topological case we either have to include the existence of a fundamental class $[Z^{\Gamma} \subseteq X^{\Gamma}]$ in the definition, or find a sufficient condition for their existence (and include that in the definition); van Hamel chooses the latter approach when defining good equivariant topological varieties.

Proposition A.5.4. *Let $X^{\mathbb{C}} \subseteq \mathbb{C}P^n$ be a nonsingular projective variety, which is the complexification of $X^{\mathbb{R}} \subseteq \mathbb{R}P^n$, assume that $X^{\mathbb{R}}$ is orientable. Let $Z^{\mathbb{C}} \subseteq X^{\mathbb{C}}$ be the complexification of a real algebraic variety $Z^{\mathbb{R}} \subseteq X^{\mathbb{R}}$. If $\pi : Y^{\mathbb{C}} \to Z^{\mathbb{C}}$ is a resolution of the complex algebraic variety $Z^{\mathbb{C}}$ defined over $\mathbb{R}$, and if the real part $Y^{\mathbb{R}}$ of $Y^{\mathbb{C}}$ is orientable, then $\pi^{\mathbb{R}} : Y^{\mathbb{R}} \to Z^{\mathbb{R}} \subseteq X^{\mathbb{R}}$ is a resolution in the sense of Definition A.4.4.*

The proof is analogous to the proof of the previous Proposition A.5.3, we omit the details.

Appendix B

R-spaces: flag manifolds for $\mathbb{R}, \mathbb{H}, \mathbb{O}$

Our main examples satisfying the generalized Borel-Haefliger theorem are real, quaternionic and octonionic flag manifolds. These are all examples of R-spaces, also known as generalized real flag manifolds. In Section B.1, we collect some generalities on R-spaces, namely their definition, classification and Bruhat cell decomposition is described. In Section B.2, we describe some geometry of the real flag manifolds $\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$ that will be used in the computation of its Vassiliev complex in Chapter C. In Section B.3, we describe the Schubert cell decomposition of the quaternionic flag manifold. In Section B.4, we give the definition of $\mathbb{O}P^2$ and $\mathrm{Fl}(\mathbb{O})$ via Jordan algebras and describe its Schubert cell decomposition.

B.1 R-spaces

First, we briefly recall the case of complex flag manifolds, since there are many similarities with the real case. Then we summarize some facts about semisimple Lie groups and Lie algebras and their relation to R-spaces. For the material about semisimple Lie groups and algebras we followed [Oni04] and [GOMV94], see also [Kna86] and [Hel78]. For further details about R-spaces see [DKV83], [Koc95], [BCO16].

B.1.1 Complex flag manifolds

A (real or complex) Lie algebra $\mathfrak{g}$ is *semisimple* if it is a direct sum of simple Lie algebras. A (not necessarily connected) Lie group G is *semisimple* if its Lie algebra $\mathfrak{g}$ is semisimple.

Let G be a complex semisimple Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{h} \leq \mathfrak{g}$ be a Cartan subalgebra, $\Delta = \Delta_+ \cup \Delta_-$ be an ordering of the set of roots and let $\Delta^s \subseteq \Delta_+$ be the simple roots. Then

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$$

is called the corresponding *Borel subalgebra*. The connected Lie group $B \leq G$ with Lie algebra $\mathfrak{b}$ is called a *Borel subgroup*. *Parabolic subgroups* P are subgroups of G containing B. The parabolic subgroups P_Θ are in one-one correspondence with subsets $\Theta \subseteq \Delta^s$ which can be represented by subsets of vertices of the Dynkin diagram. To a parabolic subgroup there corresponds a (complex) flag manifold $\mathrm{Fl}^\Theta = G/P_\Theta$. The B-orbits on the complete flag manifold $\mathrm{Fl} = G/B$ give a cell decomposition of Fl called the *Bruhat decomposition*: $G/B = BW$, where W is the Weyl group of G. More generally, the B-orbits give a cell decomposition of the flag manifolds $G/P = BW_\Theta$ where $W_\Theta = W/\langle s_\vartheta : \vartheta \in \Theta \rangle$, s_ϑ denotes the reflection to the hyperplane orthogonal to ϑ.

B.1.2 R-spaces

Now we turn to the real case. First, we recall some of the structure theory of real semisimple Lie algebras.

Cartan decomposition

A bilinear form b on a real Lie algebra $\mathfrak{g}$ is *invariant* if it is $\operatorname{Int}\mathfrak{g}$-invariant. A real Lie algebra $\mathfrak{g}$ is *compact* if there exists an invariant scalar product on $\mathfrak{g}$.

A real Lie subalgebra $\mathfrak{g} \leq \mathfrak{g}_\mathbb{C}$ is a *real form* of the complex Lie algebra $\mathfrak{g}_\mathbb{C}$, if $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$. The complexification operation gives a correspondence between real and complex semisimple Lie algebras by the *Cartan criterion*: the Killing form of a complex/real Lie algebra $\mathfrak{g}$ is nondegenerate iff $\mathfrak{g}$ is semisimple. Thus classifying real semisimple Lie algebras consists of two steps: classifying complex ones, and classifying real forms of complex semisimple Lie algebras (up to inner automorphisms of $\mathfrak{g}_{\mathbb{C}}$).

Real forms $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ can be classified by *real structures/antiinvolutions*: complex antilinear involutions. Every complex semisimple Lie algebra has a compact real form $\mathfrak{u} = (\mathfrak{g}_{\mathbb{C}})^{\tau}$, corresponding to a real structure τ [Oni04, Theorem 3.1]. É. Cartan showed that any real structure σ in $\mathfrak{g}_{\mathbb{C}}$ is conjugate to a real structure commuting with τ. Then $\theta = \sigma\tau$ is an involutive (linear) automorphism of $\mathfrak{g}_{\mathbb{C}}$. The real form $\mathfrak{g} = (\mathfrak{g}_{\mathbb{C}})^{\sigma}$ is then invariant under τ and θ.

The involution θ determines a decomposition $\mathfrak{g}_{\mathbb{C}} = (\mathfrak{g}_{\mathbb{C}})_+ \oplus (\mathfrak{g}_{\mathbb{C}})_-$. By θ-invariance of $\mathfrak{g}$ and $\mathfrak{u}$,

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \qquad \mathfrak{u} = \mathfrak{u}_+ \oplus \mathfrak{u}_-.$$

Let $\mathfrak{k} = \mathfrak{g}_+$ and $\mathfrak{s} = \mathfrak{g}_-$ ($\mathfrak{p}$ is standard here instead of $\mathfrak{s}$, but we reserve it for parabolic subalgebras). Since the three involutions commute, $\mathfrak{k} = \mathfrak{u}_+$ and $\mathfrak{s} = i\mathfrak{u}_-$. Therefore

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}, \qquad \mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{s}. \tag{B.1}$$

Such a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is called a *Cartan decomposition* and it satisfies $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{s}] \subseteq \mathfrak{s}$, $[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{k}$ with the Killing form b is negative definite on $\mathfrak{k}$, positive definite on $\mathfrak{s}$. In fact, any decomposition with these properties can be obtained this way (from a compact real form $\mathfrak{u}$), and it is unique up to inner automorphisms of $\mathfrak{g}$ [Oni04, Theorem 5.1]. The involution θ induces a positive definite bilinear form $b_\theta(x, y) := -b(x, \theta y)$.

Example B.1.1. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$, with compact real form $\mathfrak{u} = \mathfrak{su}_n = (\mathfrak{g}_{\mathbb{C}})^{\tau}$ for $\tau(X) = -\overline{X}^T$. Consider the (split) real form $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ corresponding to the real structure $\tau(X) = \overline{X}$. Then $\theta(X) = \sigma\tau(X) = -X^T$. Then $\mathfrak{k} = \mathfrak{sl}(n, \mathbb{R})^{\theta} = \mathfrak{so}(n)$ and $\mathfrak{s} = S(n, \mathbb{R})$, the symmetric trace 0 matrices. The Cartan decomposition (B.1) is

$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus S(n, \mathbb{R}), \qquad \mathfrak{su}_n = \mathfrak{so}(n) \oplus iS(n, \mathbb{R}).$$

♣

Let G be a real semisimple Lie group and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition of its Lie algebra. Let K be the subgroup of G consisting of orthogonal transformations of $\mathfrak{g}$ with respect to b_θ:

$$K := \{g \in G : \operatorname{Ad} g \in O(\mathfrak{g})\}.$$

Let $S := \exp(\mathfrak{s})$, note that it is not a subgroup of G in general. Then $G = KS$ and every element $g \in G$ can be uniquely written as a product ks for $k \in K$, $s \in S$ [GOMV94, Theorem IV.3.2]. This is called a *Cartan decomposition* of G. If $\mathfrak{k}$ is semisimple and G has finitely many connected components, then K is compact.

Example B.1.2. Let $G = \mathrm{SL}(n, \mathbb{R})$ with $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. The Cartan decomposition of G is the *polar decomposition*: $G = \mathrm{SO}(n)S$, where S denotes the symmetric positive definite matrices. ♣

R-spaces: definition

The previous description implies that K acts on $\mathfrak{s}$ via the adjoint representation. These representations are sometimes also called *s-representations* and can also be thought of as the isotropy representation of the homogeneous space G/K. The orbits of this action are called *R-spaces* or generalized real flag manifolds. The reason is that they are classified by subsystems of Roots (Racines), as we will describe below.

Example B.1.3. For $G = \mathrm{SL}(n, \mathbb{R})$, the s-representation is left-right conjugation by $\mathrm{SO}(n)$ on symmetric matrices. The orbits are parametrized by the spectrum $\Sigma = \{\lambda^{\mathcal{D}} = \lambda_1^{d_1}, \ldots, \lambda_r^{d_r}\}$ of an $n \times n$ symmetric matrix (as a multiset). Indeed, by the spectral theorem a symmetric matrix is diagonalizable by an orthogonal one, and two diagonal matrices are in the same $\mathrm{SO}(n)$-orbit if they are in the same $S_n \leq \mathrm{O}(n)$-orbit.

By definition, the R-spaces are the orbits of the $\mathrm{SO}(n)$-action; they are the real flag manifolds $\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$. The points of the orbit $\lambda^{\mathcal{D}}$, correspond to points of the real flag manifold $\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$ by taking the successive direct sum of the eigenspaces.

Notice that a priori $\mathcal{D}$ in $\lambda^{\mathcal{D}}$ is not ordered, whereas in $\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$ it is: the reorderings $\mathcal{D}'$ of $\mathcal{D}$ correspond to isomorphisms between the flag manifolds $\mathrm{Fl}_{\mathcal{D}}$ and $\mathrm{Fl}_{\mathcal{D}'}$.

♣

Real root systems

A subalgebra $\mathfrak{a}$ of a real semisimple Lie algebra $\mathfrak{g}$ is $\mathbb{R}$-*diagonalizable* if the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ is simultaneously diagonalizable. Then $\mathfrak{g}$ has a *root decomposition w.r.t.* $\mathfrak{a}$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$$

where $\Sigma \subseteq \mathfrak{a}^*$ is called the *root system of* $\mathfrak{g}$ *w.r.t.* $\mathfrak{a}$ and $\mathfrak{g}_\lambda$ are the *weight subspaces*.

If $\mathfrak{a}$ is maximal $\mathbb{R}$-diagonalizable, then the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ can be taken so that $\mathfrak{a} \subseteq \mathfrak{s}$ and the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ is of the form

$$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a},$$

where $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}$. Any two maximal $\mathbb{R}$-diagonalizable subalgebras are conjugate and their dimension is called the *real rank of* $\mathfrak{g}$. Let $\mathfrak{a}$ be maximal $\mathbb{R}$-diagonalizable. Then $\Sigma \neq \emptyset$ iff $\mathfrak{a} \neq 0$ iff $\mathfrak{g}$ is noncompact. The root system $\Sigma \subseteq \mathfrak{a}^*$ of a real semisimple $\mathfrak{g}$ w.r.t. $\mathfrak{a}$ is a root system (in the usual sense). The $\lambda \in \Sigma$ are called *restricted roots* ($\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$). However, contrary to the complex case, the root system is in general neither reduced (α root$\Rightarrow$ only $\pm\alpha$ root) nor is it true that $\dim g_\lambda = 1$. The dimension of g_λ is called the *mulitplicity of the root*.

Satake diagrams

Any complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ has at least two real forms; the compact form (encountered above) and the split form. The rest are described by their *Satake diagrams*, which encode the relationship between the root system of $\mathfrak{g}_{\mathbb{C}}$ and the real root system of $\mathfrak{g}$. We briefly recall their definition.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be its Cartan decomposition with respect to an involution θ. Let $\mathfrak{a} \subseteq \mathfrak{s}$ be a maximal $\mathbb{R}$-diagonalizable subalgebra and take the corresponding root system Σ.

Take a Cartan subalgebra $\mathfrak{a} \leq \mathfrak{h} \leq \mathfrak{g}$ (a maximal commutative subalgebra consisting of semisimple elements), Then $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra in $\mathfrak{g}_{\mathbb{C}}$ and one can consider its root system $\Delta \leq \mathfrak{h}_{\mathbb{C}}^*$. The restricted roots $\Sigma \subseteq \mathfrak{a}^*$ defined previously are the nonzero restrictions of the roots to $\mathfrak{a} \leq \mathfrak{h}_{\mathbb{C}}$.

There are two types of roots in Δ: those restricting to zero are called *compact roots* Δ_0 and the rest are the *noncompact roots* Δ_1.

Next, one can prove ([Oni04, § 9]) that the positive roots Δ^+ and Σ^+ can be chosen in Δ and Σ compatibly in the following sense: the corresponding simple roots Δ^s and Σ^s satisfy $r(\Delta_1^s) \supseteq \Sigma^s$, where $\Delta_1^s = \Delta^s \cap \Delta_1$ and $r : \mathfrak{h}_{\mathbb{C}}^* \to \mathfrak{a}^*$ is the restriction. In fact, $r(\Delta_1^s) = \Sigma^s$.

Definition B.1.4. The *Satake diagram of a real semisimple Lie algebra* $\mathfrak{g}$ is an annotated Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ defined as follows:

- the simple roots corresponding to Δ_0^s are colored black,
- the simple roots corresponding to Δ_1^s are colored white,
- simple roots corresponding to $\alpha, \beta \in \Delta_1^s$ are connected by an arrow if $r(\alpha) = r(\beta)$.

By a lemma of Satake, an element can be connected to at most another element. The Satake diagram of a split form $\mathfrak{g}$ is the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ with all nodes colored white, and for the compact form all nodes are colored black. We will give some examples for Satake diagrams below.

Real parabolic subgroups

A subalgebra $\mathfrak{p}$ of a real semisimple Lie algebra is *parabolic* if $\mathfrak{p}_{\mathbb{C}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. A *parabolic subgroup* $P \leq G$ is the normalizer of a parabolic subalgebra $\mathfrak{p}$ in G. Parabolic subalgebras (up to inner conjugacies) can be described by real root systems. They correspond to subsets of the simple roots, or subsets of the white nodes (glued by the arrows) on their Satake diagram.

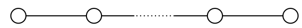
Let $\Theta \subseteq \Sigma^s$ be a subset of the simple real roots. Then the *standard parabolic subalgebra* is defined as

$$\mathfrak{p}(\Theta) := \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in [\Theta]} \mathfrak{g}_\lambda \oplus \bigoplus_{\lambda \in \Sigma^+ \setminus [\Theta]} \mathfrak{g}_\lambda$$

where $[\Theta] = \langle \Theta \rangle \cap \Sigma$.

By a theorem of Borel-Tits [BT65, Proposition 5.14], any parabolic subgroup/subalgebra is conjugate to a (unique) standard one. We describe the R-spaces in the cases relevant to us.

Example B.1.5. Let $G = \mathrm{SL}(n, \mathbb{R})$ with $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, the split real form of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$ as in Example B.1.3. The Satake diagram is



The parabolic subgroups/R-spaces correspond to subsets of the white nodes of the Satake diagram, which agrees with the Dynkin diagram of $\mathfrak{sl}(n, \mathbb{C})$ so there are the same number of them as complex flag manifolds. In particular, if d_i denotes the distance between the $i-1$th and ith chosen node, then the corresponding R-space is $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^n)$. This corresponds to the description as in Example B.1.3. ♣

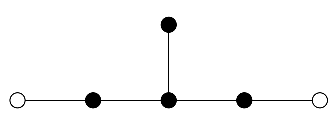
Example B.1.6. Let $G = \mathrm{SL}(n, \mathbb{H})$ with $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{H})$, whose complexification is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2n, \mathbb{C})$. The Satake diagram is [Oni04]

The R-spaces correspond to subsets of the white nodes of the Satake diagram, which again agrees with the Dynkin diagram of $\mathfrak{sl}(n, \mathbb{C})$ so there are the same number of them as complex flag manifolds, and they have the same description as manifolds of partial flags in $\mathbb{H}^n$: $\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^n)$. ♣

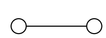
Example B.1.7. Let $G = E_{6(-26)}$ with $\mathfrak{g} = \mathfrak{e}_{6(-26)}$. The Cartan decomposition of $\mathfrak{g}$ is [Fre85, Section 8.1.1], [MW13, Proposition C.1]

$$\mathfrak{e}_{6(-26)} = \mathfrak{f}_4 \oplus \mathfrak{h}_3^0(\mathbb{O})$$

where $\mathfrak{f}_4$ is the compact real form of $\mathfrak{f}_4^{\mathbb{C}}$ and $\mathfrak{h}_3^0(\mathbb{O})$ denotes the trace 0 elements in the Jordan algebra of octonionic matrices. The real rank is 2, and $\mathfrak{a} \leq \mathfrak{h}_3^0(\mathbb{O})$ can be chosen to be the 3×3 diagonal trace zero real matrices. The Satake diagram of $\mathfrak{g}$ is [Oni04]



so the Dynkin diagram corresponding to the root system of $\mathfrak{g}$ w.r.t. $\mathfrak{a}$ is



The parabolic subgroups correspond to subsets of the nodes (up to symmetry!) so there are two R-spaces: $\mathbb{O}P^2$ and $\mathrm{Fl}(\mathbb{O})$.

♣

Bruhat decomposition

Finally, by results of Borel-Tits [BT65] and [DKV83], R-spaces have a cell decomposition consisting of the N-orbits.

A real connected semisimple Lie group G has an *Iwasawa decomposition* $G = KAN$, $\mathrm{Lie}(K) = \mathfrak{k}$, $\mathrm{Lie}(A) = \mathfrak{a}$ and $\mathrm{Lie}(N) = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$. Let C^+ denote the fundamental Weyl chamber corresponding to the simple real roots Σ^s. Let $\Theta \subseteq \Sigma^s$ be a subset and let H be be an element in the closure of C^+ such that Θ is precisely the subset of Σ^s vanishing at H. The K-orbit of H (via Ad) is an R-space $K.H = K/K_H$. In fact, it can be identified with G/P_Θ, where P_Θ is the parabolic subgroup corresponding to Θ, i.e. the normalizer of $\mathfrak{p}_\Theta$. The identification is via a decomposition $P_\Theta = K_H AN$, which induces a natural map

$$i : K/K_H \to KAN/K_H AN = G/P_\Theta$$

which is a K-equivariant isomorphism.

The Weyl group is defined as $W = N_K\mathfrak{a}/Z_K\mathfrak{a}$, let W_H be the subgroup stabilizing $H \in C^+$. Since $K_H \supseteq Z_K\mathfrak{a}$, there is an embedding $W.K_H \hookrightarrow K/K_H$, and $W.K_H \cong W/W_H$.

Theorem B.1.8 (Bruhat decomposition, [DKV83])**.** *The R-spaces G/P_Θ have a cell-decomposition as a disjoint union of N-orbits through the points of $W/W_H \hookrightarrow G/P_\Theta$.*

In all the examples we are aware of, where the R-spaces parametrize flags (as in the real, quaternionic and octonionic case), the Bruhat cells coincide with Schubert cells, where by Schubert cells we mean subsets described by geometric incidence conditions, such as (B.3) below.

B.2 Real flag manifolds

As described in Examples B.1.3 and B.1.5, real flag manifolds are R-spaces for $G = \mathrm{SL}(N,\mathbb{R})$. Since $\mathrm{SL}(N,\mathbb{R})$ is a split form of $\mathrm{SL}(N,\mathbb{C})$, there are the same number of isomorphism classes of R-spaces for G, as complex flag manifolds of $\mathrm{SL}(N,\mathbb{C})$. Furthermore, they have the same geometric description as a sequence of flags in $\mathbb{R}^N$. In this section we describe their Schubert cell decomposition and describe some of its geometric structure relevant when computing the incidence coefficients of these cells, carried out in Appendix C.

Remark on notation: In this section, since we are interested in the real case, $\mathrm{GL}(K)$ denotes $\mathrm{GL}(K,\mathbb{R})$, $B^+ := B^+(N,\mathbb{R})$ denotes the upper triangular matrices.

B.2.1 Geometry

This section is standard, see [Bri05], [Ful97] for the complex case. We include it to fix some notation and properties we will use in the computations of Section C.

Schubert varieties, orbit structure

Denote the standard basis in $\mathbb{R}^N$ by $e_1,\dots,e_N$, their one-dimensional spans by $\varepsilon_i = \langle e_i\rangle$, and the standard flag $E_j = \bigoplus_{i=1}^{j}\varepsilon_i$. The stabilizer of $E_\bullet$ in $G := \mathrm{GL}(N)$ is B^+. Choose a parabolic subgroup, i.e. $B^+ \le P \le G$. Similarly to the complex case,

$$P = \mathrm{GL}(\mathcal{D}), \qquad \text{for some } \mathcal{D} = (d_1,\dots,d_m)$$

which is the subgroup of block upper-triangular matrices with elements of $\mathrm{GL}(d_i)$ on the diagonal and arbitrary entries above the blocks.

The corresponding homogeneous space $X = G/P$ is the partial flag manifold $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$. In general $\mathrm{GL}(\mathcal{D})$ does not contain a maximal torus of $\mathrm{GL}(N)$, only a maximal 2-torus, in which case G/P is not GKM, and has zero Euler characteristic (see e.g. [GHZ06]). In this notation d_i denotes the difference in the dimensions, introduce $\mathcal{S} = (s_1,\dots,s_m)$, $s_i - s_{i-1} = d_i$ for the dimensions.

The B^+-orbits on X are called *Bruhat cells* $\Omega_I(E_\bullet)$, each of which contains a unique $\mathbb{Z}_2$-torus fixed point $E_\bullet^I \in \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$, which are indexed by *ordered set partitions* $I \in \mathrm{OSP}(\mathcal{D})$, where

$$\mathrm{OSP}(\mathcal{D}) := \binom{N}{\mathcal{D}} = S_N/S_{d_1}\times\dots\times S_{d_m},$$

in particular $I_j \in \binom{N}{d_j}$, $j = 1,\dots,m$ and $\amalg_j I_j = [N]$. (Caution: $E_\bullet$ denotes a complete flag, $E_\bullet^I$ a partial one.) It is sometimes convenient to write $I \in \binom{N}{\mathcal{D}}$ as a function: $I : \{1,\dots,N\} \to \{1,\dots,m\}$ satisfying $|I^{-1}(j)| = d_j$ for all j.

For the dimension of Ω_I, $I \in \mathrm{OSP}(\mathcal{D})$, introduce $\ell(I)$ be the number of elements in reverse order:

$$\ell(I) := |\{(a,b) : a > b, a \in I_\alpha, b \in I_\beta, \alpha < \beta\}|. \tag{B.2}$$

Then $\dim \Omega_I = \ell(I)$.

Given a general complete flag $A_\bullet$, the Bruhat cells coincide with the following *Schubert cell* description:

$$\Omega_I(A_\bullet) = \{F_\bullet \in \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N) : \dim F_i \cap A_k = r_I(i,k)\}, \tag{B.3}$$

where $r_I(i,k) = \#\{l \in I_1 \cup \ldots \cup I_i : l \leq k\}$. When we omit the flag from the notation Ω_I, that means that we take the standard flag $E_\bullet$. The closure of the orbit Ω_I is called a *Schubert variety* and is denoted σ_I. If a Schubert variety σ_I is a cycle (in the sense of Definition A.1.11), we call it a *Schubert cycle* and its class $[\sigma_I]$ a *Schubert class*. The orbit structure is described by the *Bruhat order* (cf. [Koc95, Theorem 2.3.2] for the real case):

$$\sigma_I = \bigcup_{J \leq I} \Omega_J \tag{B.4}$$

where $J \leq I$ iff $(J_1 \cup \ldots \cup J_i)_{rtiv} \leq (I_1 \cup \ldots \cup I_i)_{rtiv}$ for all i, where *rtiv* means "reordered to increasing value" and the partial order $(a_1, \ldots, a_j) \leq (b_1, \ldots, b_j)$ is the lexicographic one. This is also equivalent to $r_I(i,k) \geq r_J(i,k)$ for all i, k.

Tangent bundle of $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$

We recall a well-known decomposition of the tangent bundle of X in terms of tautological bundles.

Let $G := \mathrm{GL}(N)$ and $P \leq G$ be a parabolic subgroup; $P = \mathrm{GL}(\mathcal{D})$. P has projections to subgroups $p_i : P \to \mathrm{GL}(s_i)$ which are homomorphisms, whose defining representations induce the tautological bundles. For example, the defining representation of $\mathrm{GL}(s_i)$ on $\mathbb{R}^{s_i}$ induces the ith tautological bundle over G/P:

$$S_i \cong \mathrm{GL}(N) \times_P \mathbb{R}^{s_i}.$$

The quotient and difference bundles are defined by the following exact sequences of bundles over X:

$$0 \longrightarrow S_i \longrightarrow \mathbb{R}^N \longrightarrow Q_i \longrightarrow 0$$

$$0 \longrightarrow S_{i-1} \longrightarrow S_i \longrightarrow D_i \longrightarrow 0$$

with the convention $S_0 = 0$. Notice that $s_i = \dim S_i$, $d_i = \dim D_i$, and let $q_i = \dim Q_i$. Recall the following general fact about the tangent bundle of homogeneous spaces:

Proposition B.2.1. *Let $X = G/H$ be a homogeneous space and let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively. Then the G-equivariant vector bundle $TX \to X$ fits into the short exact sequence of G-equivariant bundles*

$$G \times_H \mathfrak{h} \to G \times_H \mathfrak{g} \to G \times_H (\mathfrak{g}/\mathfrak{h}) \cong TX$$

where H acts on $\mathfrak{g}, \mathfrak{h}$ via the adjoint representation.

Corollary B.2.2.

$$TX \cong \bigoplus_{i=1}^{m-1} \operatorname{Hom}(D_i, Q_i) \cong \bigoplus_{1 \leq i < j \leq m} \operatorname{Hom}(D_i, D_j)$$

Proof. Apply the Proposition to the homogeneous space $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N) = \mathrm{GL}(N)/\,\mathrm{GL}(\mathcal{D})$. □

Corollary B.2.3. *The partial flag manifold $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$ has even dimension iff the number of odd d_i's is 0 or 1 mod 4.*

Proof. The dimension of $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$ is given by $\sum_{1 \leq i < j \leq m} d_i d_j$, which mod 2 is equal to $\binom{o}{2}$, where o is the number of odd d_i's. □

A choice of the basis $e_i \in \mathbb{R}^N$ induces splittings $Q_i \to \mathbb{R}^N$, $D_i \to \mathbb{R}^N$, which is not essential, but facilitates computations, in particular it realizes TX as a subbundle of $\operatorname{End}(\mathbb{R}^N)$. In particular, using this identification and Corollary B.2.2, the tangent and normal spaces of the B^+-orbits can be described as follows:

Proposition B.2.4. *The tangent and normal spaces of Ω_I at $E^I_\bullet$, $I \in \mathrm{OSP}(\mathcal{D})$ are given by*

$$T_I \Omega_I = \bigoplus_{(c,d) \in T_I} \varepsilon_{cd}, \qquad N_I \Omega_I = \bigoplus_{(c,d) \in N_I} \varepsilon_{cd}$$

where $\varepsilon_{cd} = \operatorname{Hom}(\varepsilon_c, \varepsilon_d)$ and

$$T_I := \{(c,d) \in [N]^2 : c > d,\ I(c) < I(d)\}, \qquad N_I := \{(c,d) \in [N]^2 : c < d,\ I(c) < I(d)\}.$$

In particular, the dimension of $\Omega_I \subseteq \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$ is given by $|T_I| = \ell(I)$.

Orientability of real flag manifolds

The main aim of this section is to state Proposition B.2.6 and Corollary B.2.9. Let $X = \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$ be the flag manifold.

Proposition B.2.5.

$$H^*(X; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_j(D_i)] \Big/ \prod_{i=1}^{m} w_*(D_i)$$

Proof. The analogous result for complex flag varieties is well-known, and complex flag varieties are conjugation spaces, implying the Proposition. □

Recall that for real vector bundles E, F of ranks a, b respectively

$$w_1(E^*) = w_1(E), \qquad w_1(E \oplus F) = w_1(E) + w_1(F), \qquad w_1(E \otimes F) = bw_1(E) + aw_1(F). \tag{B.5}$$

Using that E orientable iff $w_1(E) = 0$, we get the following Proposition:

Proposition B.2.6. *Let A, B be real vector bundles of even rank over X. Then* $\mathrm{Hom}_{\mathbb{R}}(A, B)$ *is orientable.*

Remark B.2.7. Even more is true; by fixing a convention once and for all, there is a *canonical* orientation for all such Hom-bundles. For example, the lexicographic ordering is such a convention; the antilexicographic is another one; for further details see D.1.

In the case of real flag manifolds we can give a more precise statement.

Proposition B.2.8. *For $X = \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$, $\mathcal{D} = (d_1, \ldots, d_m)$:*

$$w_1(TX) = \sum_{i=1}^{m} (N - d_i) w_1(D_i),$$

Proof. Using (B.5),

$$w_1(TX) = \sum_{i=1}^{m-1} w_1 \left(\mathrm{Hom}(D_i, Q_i) \right) = \sum_{i=1}^{m-1} q_i w_1(D_i) + d_i w_1(Q_i)$$

and using that $Q_i = \oplus_{j=i+1}^m D_j$ and $q_i = \sum_{j=i+1}^m d_j$,

$$\mathrm{coeff}(w_1(D_i) \in w_1(TX)) = \sum_{j=i+1}^m d_j + \sum_{j<i} d_j = N - d_i.$$

□

Corollary B.2.9.

$$\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N) \text{ is orientable} \qquad \Longleftrightarrow \qquad d_1 \equiv d_2 \equiv \ldots \equiv d_m \bmod 2$$

where $d_i = s_i - s_{i-1}$ according to the notation above.

Proof. X is orientable iff $w_1(TX) = 0$. By Proposition B.2.5, the only $\mathbb{Z}_2$-linear combination of $w_1(D_i)$'s that is zero, is if all the coefficients are 1 or all 0, therefore TX is orientable iff all $N - d_i$ have the same parity, iff all d_i have the same parity. □

Direct sum maps

We are going to make use of the following natural maps between flag manifolds. Let $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{N}^m$ be two ordered sets of natural numbers, where we allow zero and let $\mathcal{D}_1 + \mathcal{D}_2$ be the element-wise sum. Direct sum induces the following *direct sum map* of flag manifolds:

$$F_{\mathcal{D}_1,\mathcal{D}_2} : \mathrm{Fl}_{\mathcal{D}_1}(A_1) \times \mathrm{Fl}_{\mathcal{D}_2}(A_2) \hookrightarrow \mathrm{Fl}_{\mathcal{D}_1+\mathcal{D}_2}(A_1 \oplus A_2)$$

defined by

$$F_{\mathcal{D}_1,\mathcal{D}_2}(F^1_\bullet, F^2_\bullet)_\kappa := F^1_\kappa \oplus F^2_\kappa.$$

This map is a $\mathrm{GL}(A_1) \times \mathrm{GL}(A_2) \leq \mathrm{GL}(A_1 \oplus A_2)$-equivariant embedding. If $B^+ \leq \mathrm{GL}(A_1 \oplus A_2)$ is the stabilizer of a complete flag $E_\bullet \leq A_1 \oplus A_2$, such that $E_\bullet = \pi_1 E_\bullet \oplus \pi_2 E_\bullet$ for $\pi_i : A_1 \oplus A_2 \to A_i$, then

$$B_i^+ := B^+ \cap \mathrm{GL}(A_i) \leq \mathrm{GL}(A_i)$$

is a Borel subgroup and a $B_1^+ \times B_2^+$-orbit embeds to a B^+-orbit.

A direct sum decomposition $V = \bigoplus_i A_i$ induces a direct sum decomposition

$$\mathrm{End}(V) = \bigoplus_{i,j} \mathrm{Hom}(A_i, A_j),$$

in particular we obtain inclusions $\iota_j : \mathrm{End}(A_j) \hookrightarrow \mathrm{End}(V)$.

Proposition B.2.10. *Let $\mathrm{Fl}_{\mathcal{D}_i}(A_i)$ be two flag manifolds with $\mathcal{D}_i \in \mathbb{N}^m$, and let $E_\bullet \in \mathrm{Fl}_{\mathcal{D}_2}(A_2)$ be a fixed flag. Then*

$$f : \mathrm{Fl}_{\mathcal{D}_1}(A_1) \to \mathrm{Fl}_{\mathcal{D}_1+\mathcal{D}_2}(A_1 \oplus A_2)$$

defined by $f := F_{\mathcal{D}_1,\mathcal{D}_2}(\cdot, E_\bullet)$ is an isomorphism onto its image, and

$$df = \iota_1|_{T\,\mathrm{Fl}_{\mathcal{D}_1}} : T\,\mathrm{Fl}_{\mathcal{D}_1} \to T\,\mathrm{Fl}_{\mathcal{D}_1+\mathcal{D}_2}$$

where $T\,\mathrm{Fl}_{\mathcal{D}_1} \leq \mathrm{End}(A_1)$ and $T\,\mathrm{Fl}_{\mathcal{D}_1+\mathcal{D}_2} \leq \mathrm{End}(A_1 \oplus A_2)$, using the identification as a subbundle determined by bases $(a_i^1 \in A_1)$, $(a_i^2 \in A_2)$.

B.3 Quaternionic flag manifolds

As described in Example B.1.6, quaternionic flag manifolds are R-spaces for $G = \mathrm{SL}(n, \mathbb{H})$. Since the Dynkin diagram of the real root system is the same as the Dynkin diagram of $\mathrm{SL}(n, \mathbb{C})$, there are the same number of R-spaces for G, and have the same geometric description as a sequence of flags in $\mathbb{H}^n$. As in the case of real flag manifolds, $\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N)$ denotes the space parametrizing partial flags where $\mathcal{D}$ denotes the differences in the dimensions.

The subgroup $N \leq \mathrm{SL}(n, \mathbb{H})$ in the Iwasawa decomposition can be identified with the subgroup of strictly upper triangular quaternionic matrices. Since the G-action leaves incidences $W \leq U$ invariant, and the standard flag $E_\bullet \in \mathrm{Fl}(\mathbb{H}^N)$ is fixed by N, the dimension function $d_{ij}(F_\bullet) = \dim(F_i \cap E_j)$ is an invariant of the N-orbits $(F_\bullet \in \mathrm{Fl}_{\mathcal{D}}(\mathbb{H}))$; it turns out to be a complete invariant.

Proposition B.3.1. $\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N)$ *has a cell decomposition given by the Schubert cells*

$$\Omega_I(A_\bullet) = \{F_\bullet \in \mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N) : \dim F_i \cap A_k = r_I(i,k)\},$$

where $r_I(i,k) = \#\{l \in I_1 \cup \ldots \cup I_i : l \leq k\}$, $I \in \mathrm{OSP}(\mathcal{D})$.

The standard complex proof as in [Bri05], [Ful97] generalizes without difficulty to the quaternionic case, we omit the details.

B.4 Octonionic flag manifolds

In this section we describe the Schubert cell decomposition of octonionic flag manifolds and the group actions that make them circle spaces.

The classical construction of octonionic flag manifolds is via Jordan algebras and due to Freudenthal [Fre85] and J. Tits; since we will not need this description in this thesis, we did not include it. That description is better suited to analyze them Lie theoretically as homogeneous spaces $\mathbb{O}P^2 = F_4/\operatorname{Spin}(9)$ and $\mathrm{Fl}(\mathbb{O}) = F_4/\operatorname{Spin}(8)$. To the interested reader, we recommend [GOMV94, Chapter 5], [MW13] and [Bae02] where these descriptions, and much more can be found. The two descriptions of $\mathbb{O}P^2$ and $\mathrm{Fl}(\mathbb{O})$ via Jordan algebras and restricted homogeneous coordinates are compatible, see [All97].

B.4.1 Galois type action on $\mathrm{Fl}(\mathbb{O})$

Proposition B.4.1. *$\Gamma = \mathrm{Sp}(1) \leq \mathrm{Aut}(\mathbb{O}) = \mathrm{G}_2$ and $\Gamma' = \mathrm{U}(1) \leq \Gamma$ both act on $\mathbb{F} = \mathbb{O}$ with fixed point set $\mathbb{F}^{\Gamma} = \mathbb{F}^{\Gamma'} = \mathbb{H}$. Here G_2 denotes the compact real form.*

Proof. For the proof of $\mathrm{Aut}(\mathbb{O}) = \mathrm{G}_2$ see [Bae02, 4.1] or [SV00].

We identify $\Gamma \leq \mathrm{Aut}(\mathbb{O})$. Any subalgebra of $\mathbb{O}$ is isomorphic either to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. A triple of octonions (e_1, e_2, e_3) is called a *basic triple* if $e_i^2 = -1$, they anticommute and $(e_1e_2)e_3 = -e_3(e_1e_2)$. In particular, the subalgebra generated by a basic triple equals $\mathbb{O}$. The importance for us is that $\mathrm{Aut}(\mathbb{O})$ acts freely and transitively on basic triples. Indeed, a short computation using the Moufang and alternative identities shows that for a basic triple,

$$(e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, (e_1e_2)e_3)$$

together with 1, forms a standard basis of $\mathbb{O}$ (pairwise anticommuting, square roots of -1) and that the linear map associated to the change of basis is in fact an algebra automorphism - the multiplication table is the same.

The subgroup Γ of $\mathrm{Aut}(\mathbb{O})$ fixing the subalgebra $\mathbb{H}$ generated by e_1, e_2 is therefore the same as the subgroup fixing the pair (e_1, e_2). Any choice of $e_3 \in S^3 \subseteq \mathbb{H}^{\perp}$ determines a basic triple (e_1, e_2, e_3) as a short computation shows. Therefore the subgroup Γ acts freely and transitively on S^3, so Γ is S^3 as a manifold, and is therefore $\mathrm{Sp}(1)$.

Restricting the action of $\mathrm{Sp}(1)$ to $\Delta := \mathrm{U}(1) \leq \mathrm{Sp}(1)$ also acts freely on S^3, therefore $\mathbb{O}^\Delta = \mathbb{H}$ as well. This is the action we will consider, and it will simplify our discussion, as we can stay in the more well-behaved realm of circle spaces.

□

Proposition B.4.2. $\Gamma = \mathrm{Sp}(1)$ *and* $\Delta = \mathrm{U}(1)$ *act on*

a) $X = \mathbb{O}P^2$ *with fixed point set* $X^\Gamma = X^\Delta = \mathbb{HP}^2$ *and*

b) $Y = \mathrm{Fl}(\mathbb{O}) \subseteq \mathbb{O}P^2 \times \mathbb{O}P^2$ *is invariant for the product action with fixed point set* $Y^\Gamma = Y^\Delta = \mathrm{Fl}(\mathbb{H}^3)$.

Proof. a) By the previous Proposition, Γ and Δ act on $\mathbb{O}$ by automorphisms with fixed point set $\mathbb{H}$. This action induces an action on the restricted homogeneous coordinates $\mathbb{O}^3$ of $\mathbb{O}P^2$ and $(\mathbb{O}P^2)^*$; if

$$(1, \lambda, \mu) \sim (\lambda^{-1}, 1, \lambda^{-1}\mu) \qquad \Rightarrow \qquad (1, \gamma(\lambda), \gamma(\mu)) \sim (\gamma(\lambda^{-1}), 1, \gamma(\lambda^{-1})\gamma(\mu))$$

for $\gamma \in \Gamma$ and $\lambda, \mu \in \mathbb{O}$ and similarly when the real entries are in different coordinates. The fixed point set is the of restricted homogeneous coordinates in $\mathbb{H}^3$, i.e. $\mathbb{H}P^2$. The proof is the same for $(\mathbb{O}P^2)^*$.

b) It is enough to show that the incidence relation is invariant for the Γ-action, and this holds since Γ acts by algebra automorphisms. □

To show that these actions make $\mathbb{O}P^2$ and $\mathrm{Fl}(\mathbb{O})$ circle spaces, we want to use the generalized Borel-Haefliger theorem; we have to show that their cohomology is generated by halving cycles, see the next Section.

B.4.2 Schubert cell decomposition of $\mathrm{Fl}(\mathbb{O})$

Let $(d_1, d_2) \in \mathrm{Fl}(\mathbb{O})$ be the standard flag given in homogeneous coordinates by $d_1 = [1 : 0 : 0]$ and $d_2 = [0 : 1 : 0]$.

Proposition B.4.3. *The Schubert cells* $\Omega_w^{\mathbb{O}}(d_\bullet)$ *defined by the usual incidence relations (see Section C.1.5) give a* Γ*-invariant* $8i$*-cell decomposition of* $\mathrm{Fl}(\mathbb{O})$*, with fixed point set* $\Omega_w^{\mathbb{H}}(d_\bullet)$ *for* $d_\bullet \in \mathrm{Fl}(\mathbb{H}^3) = \mathrm{Fl}(\mathbb{O})^\Gamma$.

Proof. Recall that the incidence relation for $(x, L) \in \mathbb{O}P^2 \times (\mathbb{O}P^2)^*$ is defined by $x_1 l_1 + x_2 l_2 + x_3 l_3 = 0$. Then the Schubert cells can be written in terms of homogeneous coordinates $(x, L) \in \mathrm{Fl}(\mathbb{O})$ (for the defining relations see Section C.1.5):

$$\Omega_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Omega_{213} = \begin{pmatrix} \alpha & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\Omega_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1 \end{pmatrix}, \qquad \Omega_{231} = \begin{pmatrix} \alpha & 0 & 1 \\ 1 & \beta & -\alpha \end{pmatrix},$$

$$\Omega_{312} = \begin{pmatrix} \alpha & 1 & -\beta \\ 0 & \beta & 1 \end{pmatrix}, \qquad \Omega_{321} = \begin{pmatrix} \alpha & 1 & \beta \\ 1 & \beta\gamma - \alpha & \gamma \end{pmatrix},$$

where $\alpha, \beta, \gamma \in \mathbb{O}$ are free parameters. This gives a partition of $\mathrm{Fl}(\mathbb{O})$ into Schubert $8i$-cells, so the cohomology of $\mathrm{Fl}(\mathbb{O})$ is additively generated by the classes of the closures. Since (d_1, d_2) is $\mathrm{Sp}(1)$-fixed, the Schubert cells $\Omega_w(d_\bullet)$ are $\mathrm{Sp}(1)$-invariant, and this incidence description implies that the fixed point sets are the quaternionic Schubert cells: $\Omega_w(d_\bullet)^\Gamma = \Omega_w^{\mathbb{H}}(d_\bullet)$. □

Appendix C

Cohomology of real flag manifolds

Analogously to complex flag varieties, there are two approaches towards computing the cohomology of an R-space (generalized real flag manifold) X: a) via the Vassiliev stratification given by Bruhat decomposition which gives additive generators, or b) in terms of characteristic classes which (together with $\hat{P}$, the Samelson subspace) are algebra generators. Compared to the complex case, the added complication in the case of R-spaces is that there can be cells in neighboring dimensions, which allows for torsion in the $\mathbb{Z}$-coefficient cohomology. If multiplicities of the restricted roots of G/P (cf. Section B.1) are larger than one, then this issue does not arise.

Approach a) computes the cohomology additively, with $\mathbb{Z}$-coefficients. However, computing the incidence coefficients is a nontrivial task. Particular cases were studied by Ehresmann [Ehr37] for certain real flag varieties and Casian-Kodama [CK13] gave a combinatorial description for real Grassmannians. The general case of R-spaces was first studied by Kocherlakota [Koc95] using Morse theory. He computed the Morse-Bott complex whose cohomology computes the cohomology of X. The complex is generated by Weyl-group elements/Schubert cells, and the coefficients turn out to be either 0 or ± 2. His computation was not entirely complete, as he did not determine the sign of ± 2. The sign has recently been computed by Rabelo-San Martin [RM18] using a CW decomposition. In this section we give an alternative computation of these coefficients using the Vassiliev complex for R-spaces of type A, i.e. the real flag manifolds.

Once the coefficients are known, it is a nontrivial combinatorial task to determine which Schubert varieties represent nonzero cohomology classes rationally. We are able to completely determine this in the case of even real flag manifolds (Proposition C.1.9) and Grassmannians (Proposition 4.2.7). In the other cases we do not have a description besides some particular cases, see Appendix F. Once this description is understood, the final step in the complete picture would be to understand the multiplicative structure constants of the cycles, and giving combinatorial rules for computing them. For the even real flag manifolds and Grassmannians, we do this, see Corollary 4.2.3, Propositions 4.2.8 and 4.2.9.

Approach b) computes $H^*(G/P)$ only with rational coefficients, but has the advantage that it also yields the multiplicative structure. A general method has been devised by Cartan [Car51] to compute the cohomology of homogeneous spaces G/P in terms of characteristic classes, when (G, P) is a Cartan pair (see Section C.2). We carry out the computation for R-spaces of type A. For the equivariant cohomology of R-spaces in the case of uniform multiplicities ≥ 2, cf. [Mar06].

Note that the second approach would also partially answer the first question, if one knows how to relate the Schubert classes $[\sigma_\lambda]$ to characteristic classes. For a 'complete' answer, one would also like to derive combinatorial rules for multiplying these Schubert classes. However as small examples show, the combinatorics is non-trivial already for determining the additive structure; not all rational cohomology classes are represented by a Schubert variety, but in general for flag manifolds some are represented by *signed sums* of Schubert cells. In all the examples we have computed (for some of which see Appendix F), all coefficients of Schubert cells appearing are ± 1, so the closure of the union of these cells is a cycle.

On a final note, having a nonzero Steenrod square $\mathrm{Sq}^1[\sigma_\lambda] \neq 0$ is an obstruction for a Schubert variety to be a cycle as observed by Lenart [Len98], whose results suggest that the relationship is much stronger: in the case of Grassmannians $\mathrm{Sq}^1[\sigma_\lambda]$ is the sum of exactly those $[\sigma_\mu]$ whose incidence coefficient $[\sigma_\lambda, \sigma_\mu]$ is nonzero. It would be interesting to see if the combinatorial description of Sq in the case of real flag manifolds given by Duan, Zhao [DZ07] yields a simple combinatorial description of which Schubert cells have to be glued to obtain a cycle.

C.1 The Vassiliev complex of $\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$

In this section we compute the Vassiliev complex (A.2.7) associated to the group action B^+ acting on $\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$ whose orbits are the Schubert cells Ω_I. Explicitly, we determine the incidence coefficients $[\Omega_I, \Omega_J]$. To compute the coefficients $[\Omega_I, \Omega_J]$, one has to coorient Ω_I, and compare these coorientations by extending them to the neighboring orbits Ω_J along normal slices, see Section A.2.3. The proof we give has many similarities to the one given by Kocherlakota, but instead of Morse theory we emphasize the geometry of the Schubert cells. Let us give a brief outline.

In Section C.1.1 we define the coorientation of the orbits. In Section C.1.2 we define normal disks to the orbits and the normal slices whose closures are the Richardson curves σ_I^J (defined below). In Section C.1.3 we relate the incidence coefficients $[\Omega_I, \Omega_J]$ (up to sign) to a bundle $\nu(W)$ over $\sigma_I^J \cong \mathbb{R}P^1$. In Section C.1.4 we describe a splitting of $T\,\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}|_{\sigma_I^J}$ into line bundles. In Section C.1.5 we compute $\nu(W)$ in the special case $\mathrm{Fl}(\mathbb{R}^3)$. In Section C.1.6 we deduce $[\Omega_I, \Omega_J]$ for $\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$ up to sign. In Section C.1.7 we deduce which Schubert varieties σ_I are nonzero rational cycles in the even case $\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$. In Section C.1.8 we relate our computations to the theorem of Kocherlakota. Finally we compute the signs of the incidence coefficients.

We remark that in the case of real (and complex) flag manifolds, each B^+-orbit is homeomorphic to an affine space, so the orbit stratification yields a CW decomposition. Therefore computing the incidence coefficients agrees with the incidence coefficients of the CW complex, which have been examined by Ehresmann [Ehr37] for the Grassmannians, later by Kocherlakota using the Morse complex [Koc95] and most recently by Rabelo and San Martin [RM18]. We will use the notation of Section B.2.1.

C.1.1 Coorientation of the orbits

We describe the Vassiliev complex of $X = \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$, $\mathcal{D} = (d_1, \dots, d_m)$. First, coorient all orbits by coorienting Ω_I at the $\mathbb{Z}_2^N$-torus fixed points $E^I_\bullet$ (see Section B.2.1 for the notation). Using the decomposition of the tangent and normal spaces given in Proposition B.2.4, orient both the tangent and normal spaces by the lexicographic ordering of $e_{kl} = (e_k \mapsto e_l)$. Since Ω_I is contractible, this determines a coorientation on the whole of it.

C.1.2 Normal slices: Richardson curves

Our aim is to determine the incidence numbers $[\Omega_I, \Omega_J]$, for $\ell(J) = \ell(I) - 1$ and $J \leq I$ (recall the notations (B.2) and (B.4)). The Bruhat order implies that J is obtained from I by interchanging $a \in I_\alpha$ with some $b \in I_\beta$, $a > b$, $\alpha < \beta$. Fix this data in the upcoming discussion.

According to Proposition A.2.10, we will fix a normal disk D to Ω_J at $E_\bullet^J$; natural candidates are the B^--orbits. Indeed, the B^--orbits $B^- E_\bullet^J$ have the following characterization:

$$B^- E_\bullet^J = \Omega_{J^D}(E_\bullet^\vee) = \{F_\bullet \in \mathrm{Fl}_\mathcal{D}(\mathbb{R}^N) : \dim F_i \cap E_k^\vee = r_{J^D}(i,k)\}$$

where $J_i^D := N + 1 - J_i$ and $E_\bullet^\vee$ is the standard dual flag:

$$E_i^\vee = \langle e_N, \dots, e_{N-i+1} \rangle.$$

Since the flags $E_\bullet, E_\bullet^\vee$ are transverse, the intersection $B^- E_\bullet^J \cap \Omega_J = \{E_\bullet^J\}$ is transverse and locally $B^- E_\bullet^J$ is a normal disk to Ω_J at $E_\bullet^J$.

For general I, J, the intersections $\sigma_I^J = \sigma_I(E_\bullet) \cap \sigma_{J^D}(E_\bullet^\vee)$ are called *Richardson varieties*. To determine the incidence numbers $[\Omega_I, \Omega_J]$, we will be interested in the *Richardson curves* σ_I^J when $\ell(J) = \ell(I) - 1$ and $J \leq I$. Intuitively, the Richardson curve is the curve between the coordinate flags $E_\bullet^I$ and $E_\bullet^J$ obtained by continuously exchanging the coordinates ε_a and ε_b in $E_\bullet^I$.

More precisely, in terms of the direct sum maps of Proposition B.2.10, the Richardson curve σ_I^J is the isomorphic image of $f : \mathbb{P}(A_1) \hookrightarrow \mathrm{Fl}_\mathcal{D}(\mathbb{R}^N)$ for $A_1 := \varepsilon_a \oplus \varepsilon_b$, $A_2 := A_1^\vee = \bigoplus_{i \neq a,b} \varepsilon_i$ and $E_\bullet := E_\bullet^I \cap E_\bullet^J$,

$$f(\cdot) = F_{\mathcal{D}_1, \mathcal{D}_2}(\cdot, E_\bullet), \qquad \mathcal{D}_2 = (d_1, \dots, d_\alpha - 1, \dots, d_\beta - 1, \dots, d_m),$$

for $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$ and $\mathcal{D}_1 = (1,1)$ in positions α, β. The Richardson curves have tautological bundles $\rho \to R$ defined as follows. Let $\tilde{f} : A_1 \to A_1$ be the trivial bundle map covering f and let $\tau \to \mathbb{P}(A_1)$ denote the tautological subbundle of A_1. Define $\rho := \tilde{f}(\tau)$ which is a subbundle of the trivial bundle $A_1 \leq \mathbb{R}^N$ over R.

Note that the intersection $\Omega_I \cap B^- E_\bullet^J$ is the Richardson curve minus two points $\sigma_I^J \backslash \{E_\bullet^I, E_\bullet^J\}$, i.e. $\mathbb{R}P^1$ minus two points. We remark that the branches of the Richardson curve are the *pairs of flows* in the terminology of Kocherlakota.

C.1.3 Incidence coefficients

Let $W := \Omega_I \cup \Omega_J$ and let $D := B^- E_\bullet^J \cup B^- E_\bullet^I$ which is locally a normal disk to Ω_J at $E_\bullet^J$. The Richardson curve $R := \sigma_I^J$ is the transversal intersection of W and D. Sometimes to alleviate notation we will use an abuse of notation and denote $I := E_\bullet^I$ and $J := E_\bullet^J$. Let $R_+ \cup R_- = R \backslash \{I, J\}$ denote the two branches of the Richardson curve.

To compute $[\Omega_I, \Omega_J]$, we are going to use the fact that W and D are smooth; indeed by normality of Schubert varieties, the singularities of $\overline{\Omega_I}$ are of codimension 2 and are unions of Schubert cells, so Ω_I union neighboring cells is always smooth.

Proposition C.1.1.

$$[\Omega_I, \Omega_J] = \begin{cases} 0 & \nu(W \hookrightarrow X)|_R \ \textit{trivial} \\ \pm 2 & \nu(W \hookrightarrow X)|_R \ \textit{nontrivial} \end{cases}$$

Proof. Since $R = W \pitchfork D$, there is a short exact sequence

$$0 \longrightarrow TR \longrightarrow TD|_R \longrightarrow \nu(W)|_R \longrightarrow 0$$

where $\nu(W) = \nu(W \hookrightarrow X)$. Take a splitting of this short exact sequence

$$TD|_R = N_W \oplus TR.$$

According to Proposition A.2.10, one has to compare the following two orientations for each branch $R_\pm$:

- $TD|_J$ oriented by the coorientation of Ω_J
- $N_W|_J$ oriented by extending the coorientation of $\Omega_I|_{R_\pm}$ to J and TR oriented towards J on both branches $R_\pm$.

To compute $[\Omega_I, \Omega_J]$ modulo sign, it is enough to compare how the coorientations $\Omega_I|_{R_\pm}$ extend to $N_W|_J$. This amounts to deciding orientability of $N_W \cong \nu(W)|_R$. If N_W is orientable, then since its orientation on both branches agrees with its orientation at I, the orientations of $N_W|_{R_\pm}$ extend to J identically. Since the orientations of TR induced by the orientations of $TR_\pm$ differ at J, in this case $[\Omega_I, \Omega_J] = 0$.

If N_W is not orientable, then the orientations of $N_W|_{\overline{R}_+}$ and $N_W|_{\overline{R}_-}$ agree at I, so they are different at J. In this case $[\Omega_I, \Omega_J] = \pm 2$. $\square$

We determine triviality of $\nu(W)|_R$ by giving linearly independent line subbundles $\lambda_{cd} \leq \nu(W)|_R$ spanning it (Corollary C.1.7), and counting the nontrivial ones (since $R \cong S^1$, each λ_{cd} is either a Möbius bundle or a trivial one). Kocherlakota computes the incidence coefficients (up to sign) using a very similar idea: he computes the relative orientations of pairs of flows from I to J, which are in our terminology the branches of the Richardson curves.

Remark C.1.2. As we have mentioned before, since Schubert varieties are normal, the singularities have codimension 2 and the singular part is a union of Schubert cells. Therefore Ω_I union the neighboring cells is smooth. Let's denote this union by $\mathring{\sigma}_I$. This gives a new stratification of σ_I, with empty one codimensional stratum, but now the stratification no longer satisfies Vassiliev's condition (Example A.2.7). Now σ_I is a cycle if and only if $\mathring{\sigma}_I$ is coorientable. Indeed, by the previous Proposition, this is the information encoded in $[\Omega_I, \Omega_J]$: the normal bundle of $\mathring{\sigma}_I$ restricted to the Richardson curve σ_I^J is orientable iff this coefficient vanishes. Then $\mathring{\sigma}_I$ is coorientable iff $\nu(\mathring{\sigma}_I)$ restricted to the Richardson curve σ_I^J is orientable (trivial) for all neighboring J.

In general, we cannot get rid of the one codimensional stratum, but when we can, this method is sufficient to decide cycleness. However to compute the cohomology groups we need more, namely to determine the incidence coefficients, which cannot be deduced only from orientability.

C.1.4 Splitting $TX|_R$

To determine triviality of $\nu(W)|_R$, we split $TX|_R$ into line subbundles λ_{cd}, parametrized by $T_I \amalg N_I$ (for the notation T_I, N_I, see Proposition B.2.4). We will show that $TW|_R = \bigoplus_{(c,d)\in T_I} \lambda_{cd}$, so $\bigoplus_{(c,d)\in N_I} \lambda_{cd}$ is isomorphic to $\nu(W)|_R$, see Proposition C.1.5.

In (C.2) we specify $\lambda_{cd} \to R$ as subbundles of $TX|_R \leq \mathrm{End}(\mathbb{R}^N)$, in particular each λ_{cd} is of the form $\mathrm{Hom}(\mu_1, \mu_2)$: $\mu_i \in \{\rho, \rho^\vee, \varepsilon_k : k \in [N]\}$, where $\rho \to R$ is the tautological bundle and $\rho^\vee$ denotes its orthogonal complement.

A choice of a basis $e_i \in \varepsilon_i$ induces a scalar product on $\mathbb{R}^N$; this realizes the quotient bundles Q_i, D_i as subbundles of $\mathbb{R}^N$. This realizes the following splittings over $X = \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$:

$$\mathbb{R}^N = S_i \oplus Q_i = \bigoplus_{j=1}^{m} D_j = \bigoplus_{k=1}^{N} \varepsilon_k, \qquad S_i = \bigoplus_{j=1}^{i} D_j, \qquad Q_i = \bigoplus_{j=i+1}^{m} D_j$$

for all $i = 1, \dots, m$. By restricting to the Richardson curve $R = \sigma_I^J$,

$$D_i|_R = \begin{cases} \bigoplus_{j \in I_i} \varepsilon_j, & i \neq \alpha, \beta, \\ \rho \oplus \bigoplus_{a \neq j \in I_\alpha} \varepsilon_j, & i = \alpha \\ \rho^\vee \oplus \bigoplus_{b \neq j \in I_\beta} \varepsilon_j, & i = \beta \end{cases} \tag{C.1}$$

where $\rho \to R$ is the tautological bundle defined previously. Then via

$$TX \cong \bigoplus_{i<j} \mathrm{Hom}(D_i, D_j) \leq \mathrm{End}(\mathbb{R}^N)$$

the decomposition (C.1) induces a splitting of $TX|_R$ into line bundles $\lambda_{cd} \to R$ parametrized by $(c, d) \in T_I \amalg N_I$, defined as follows

$$\lambda_{cd} := \begin{cases} \mathrm{Hom}(\rho, \rho^\vee), & \text{if } c = a, d = b \\ \mathrm{Hom}(\rho, \varepsilon_d), & \text{if } c = a,\ I(a) < I(d) \leq I(b), d \neq b \\ \mathrm{Hom}(\varepsilon_c, \rho^\vee), & \text{if } d = b,\ I(a) \leq I(c) < I(b), c \neq a \\ \varepsilon_{cd}, & \text{else.} \end{cases} \tag{C.2}$$

where in the else line we use that $\mathrm{Hom}(\rho \oplus \rho^\vee, \varepsilon_k) = \mathrm{Hom}(\varepsilon_a \oplus \varepsilon_b, \varepsilon_k)$. In the next two sections, we show that $\{\lambda_{cd} : (c, d) \in T_I\}$ span $TW|_R$. We show this by reducing the general case to the flag manifolds $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^3)$, which we describe in the next section.

C.1.5 A special case

Let $\mathcal{D} = (1, 1, 1)$, $\varepsilon_i = \langle e_i \rangle$ and $E_\bullet$ be the standard flag, $E_i = \oplus_{j=1}^i \varepsilon_j$. Then

- $\sigma_{321} = X$
- $\sigma_{231} = \{F_\bullet : F_1 \leq E_2\}$
- $\sigma_{312} = \{F_\bullet : E_1 \leq F_2\}$
- $\sigma_{213} = \{F_\bullet : F_2 = E_2\}$
- $\sigma_{132} = \{F_\bullet : F_1 = E_1\}$

- $\sigma_{123} = \{F_\bullet : F_1 = E_1, F_2 = E_2\}$

All of these Schubert varieties are smooth. The tangent bundles of the $\geq$ 2-dimensional Schubert varieties are therefore:

$$T\sigma_{231} = \mathrm{Hom}(S_1, E_2/S_1) \oplus \mathrm{Hom}(D_2, D_3)$$

$$T\sigma_{312} = \mathrm{Hom}(S_1, D_2) \oplus \mathrm{Hom}(S_2/E_1, D_3)$$

$$T\sigma_{321} = TX.$$

Let $I, J \in \mathrm{OSP}(\mathcal{D})$, $\ell(J) = \ell(I) - 1$ and J be obtained by $a \in I_\alpha \leftrightarrow b \in I_\beta$, $a > b$, $\alpha < \beta$. Then by restricting to the Richardson curves $R = R_{ab} = \sigma_I^J$, we get the following table for $T\sigma_I|_R/TR$:

(a,b)	(2,1)	(3,1)	(3,2)
σ_{321}	$\varepsilon_{32} \oplus \varepsilon_{31}$	$\varepsilon_{32} \oplus \varepsilon_{21}$	$\varepsilon_{31} \oplus \varepsilon_{21}$
σ_{231}	$\mathrm{Hom}(\varepsilon_3, \rho^\vee)$	ε_{21}	–
σ_{312}	–	ε_{32}	$\mathrm{Hom}(\rho, \varepsilon_1)$

where $\varepsilon_{ij} = \mathrm{Hom}_{\mathbb{R}}(\varepsilon_i, \varepsilon_j) \leq TX|_R$ and $\rho \to R$ is the tautological bundle as described earlier. These bundles are the λ_{cd} spanning $TW|_R$ for $R = \sigma_I^J$ and $(c,d) \in T_I$.

In case $\mathcal{D} = (1,2)$, $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^3) = \mathbb{P}^2$, the only $\geq$ 2-dimensional orbit is $I = (3)$, $J = (2)$ and

$$T\sigma_I|_R/TR = \mathrm{Hom}(\rho, \varepsilon_1)$$

In case $\mathcal{D} = (2,1)$, $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^3) = \mathrm{Gr}_2(\mathbb{R}^3)$, the only $\geq$ 2-dimensional orbit is $I = (2,3)$, $J = (1,3)$ and

$$T\sigma_I|_R/TR = \mathrm{Hom}(\varepsilon_3, \rho^\vee)$$

C.1.6 Decomposing TW

Let us return to the general case $X = \mathrm{Fl}_{\mathcal{D}}^{\mathbb{R}}$, and fix neighboring I, J, $a \in I_\alpha, b \in I_\beta$, $R = \sigma_I^J$, $W = \Omega_I \cup \Omega_J$ as before.

In this section we show that $TW|_R = \bigoplus_{(c,d)\in T_I} \lambda_{cd}$. We show this by embedding smooth submanifolds $f : W' \hookrightarrow W$, such that $\lambda_{cd} \leq df(TW')|_R \leq TW|_R$ for all $(c,d) \in T_I$. The W' are submanifolds of smaller flag manifolds $\mathrm{Fl}_{\mathcal{D}_1}$ which are embedded in $\mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$ via the direct sum maps of Section B.2.1.

For each $(c,d) \in T_I$ we specify a direct sum map. Set $\vartheta := \{a,b,c,d\}$, and let

$$(\mathcal{D}_1)_\kappa := |\{k \in \vartheta : I(k) = \kappa\}|, \qquad \kappa = 1,\ldots,m$$

the number of distinct elements in ϑ which are in I_κ (this is either 0, 1 or 2). Let $\mathcal{D}_2 := \mathcal{D} - \mathcal{D}_1$ and $\Theta = \langle \varepsilon_k : k \in \vartheta \rangle$. The decomposition $\mathbb{R}^N = \Theta \oplus \Theta^\vee$ induces the direct sum map

$$F : \mathrm{Fl}_{\mathcal{D}_1}(\Theta) \times \mathrm{Fl}_{\mathcal{D}_2}(\Theta^\vee) \to \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N).$$

Let $E_\bullet^{IJ} := E_\bullet^I \cap E_\bullet^J \in \mathrm{Fl}_{\mathcal{D}_2}(\Theta^\vee)$. Define the embedding $f : \mathrm{Fl}_{\mathcal{D}_1}(\Theta) \hookrightarrow \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N)$ by $F(\cdot, E_\bullet^{IJ})$.

Proposition C.1.3. *Given I, J as above, there exist (unique) $I', J' \in \binom{|\vartheta|}{\mathcal{D}_1}$, such that the following diagram commutes:*

$$\begin{tikzcd} R' \arrow[r, hook] \arrow[d, "\cong"] & W' \arrow[r, hook] \arrow[d, hook] & \mathrm{Fl}_{\mathcal{D}_1}(\Theta) \arrow[d, hook, "f"] \\ R \arrow[r, hook] & W \arrow[r, hook] & \mathrm{Fl}_{\mathcal{D}}(\mathbb{R}^N) \end{tikzcd}$$

where $R' = \sigma_{I'}^{J'} \subseteq \mathrm{Fl}_{\mathcal{D}_1}(\Theta)$, $W' = \Omega_{I'} \cup \Omega_{J'} \subseteq \mathrm{Fl}_{\mathcal{D}_1}(\Theta)$ are smooth submanifolds.

Proof. Since f is an embedding, $E_\bullet^I$ and $E_\bullet^J$ have at most one preimage each. There are unique order preserving maps

$$m : \{1,\ldots,|\vartheta|\} \to \vartheta, \qquad n : \{1,\ldots,|I(\vartheta)|\} \to I(\vartheta).$$

Let the maps

$$I', J' : \{1,\ldots,|\vartheta|\} \to \{1,\ldots,|I(\vartheta)|\}$$

be defined by $I'(i) := n^{-1}(I(m(i)))$, $J'(i) := n^{-1}(J(m(i)))$. Then $f(E_\bullet^{I'}) = E_\bullet^I$ and $f(E_\bullet^{J'}) = E_\bullet^J$. Since f is $\mathrm{GL}(\Theta)$-equivariant, $f(R') = R$, and $f(W') \subseteq W$. $\square$

Corollary C.1.4. $df(TW') \leq TW$.

Proposition C.1.5. $\{\lambda_{cd} : (c,d) \in T_I\}$ *are subbundles of* $TW|_R$.

Proof. Let $(c,d) \in T_I$ and set $\vartheta = \{a,b,c,d\}$, $\vartheta_1 = \{a,b\}$ and $\vartheta_2 = \vartheta \backslash \vartheta_1$.

If $|\vartheta| = 4$, then by Proposition B.2.4 $(c,d) \in T_J$. In this case we can further decompose $\mathcal{D}_1$ as

$$(\mathcal{D}_{1i})_\kappa = |\{k \in \vartheta_i : I(k) = \kappa\}|, \qquad \kappa = 1, \dots, m$$

$\mathcal{D}_1 = \mathcal{D}_{11} + \mathcal{D}_{12}$ and $\Theta = \Theta_1 \oplus \Theta_2$ for $\Theta_i = \langle \varepsilon_k : k \in \vartheta_i \rangle$. Note that since $\mathcal{D}_{1i} = (1,1)$ in the appropriate positions. This decomposition induces another direct sum map

$$g : \mathbb{P}\Theta_1 \times \mathbb{P}\Theta_2 \to \mathrm{Fl}_{\mathcal{D}_1}(\Theta).$$

Let $W'' := \mathbb{P}\Theta_1 \times (\mathbb{P}\Theta_2 \backslash \mathbb{P}\varepsilon_d)$ and $R'' := \mathbb{P}\Theta_1 \times \mathbb{P}\varepsilon_c$. As in the proof of Proposition C.1.3, equivariance shows that $W' := \Omega_{I'} \cup \Omega_{J'} = g(W'')$ and $R' = g(R'')$. Then $\varepsilon_{cd} \le TW''|_{R''}$. By Proposition B.2.10 for g and f,

$$\lambda_{cd} = d(f \circ g)\varepsilon_{cd} \le TW|_R.$$

If $|\vartheta| = 3$, we are in the case of $\mathcal{D}_1 = (1,1,1)$, $\mathcal{D}_1 = (2,1)$ or $\mathcal{D}_1 = (1,2)$. The table of the previous section shows that the Proposition holds for $I', J' \in \binom{3}{\mathcal{D}_1}$ and $(c',d') \in T_{I'}$. By Proposition B.2.10, $\lambda_{cd} \le TW|_R$. $\square$

Corollary C.1.6. $\{\lambda_{cd} : (c,d) \in T_I\}$ *are linearly independent and therefore span* $TW|_R$.

Proof. Set $\vartheta_1 = \{a,b\}$ and $\Theta_1 = \varepsilon_a \oplus \varepsilon_b$. It is enough to show that λ_{cd} are linearly independent in each summand

$$\mathrm{End}(\mathbb{R}^N) = \mathrm{End}(\Theta_1^\vee) \oplus \mathrm{End}(\Theta_1) \oplus \bigoplus_{k \ne a,b} (\mathrm{Hom}(\Theta_1, \varepsilon_k) \oplus \mathrm{Hom}(\varepsilon_k, \Theta_1))$$

Given c, d, set $\vartheta = \{a,b,c,d\}$, $\vartheta_2 = \vartheta \backslash \vartheta_1$, and $\Theta_2 = \langle \varepsilon_k : k \in \vartheta_2 \rangle$.

If $|\vartheta| = 4$, then $\lambda_{cd} = \varepsilon_{cd}$ which are linearly independent in $\mathrm{End}(\Theta_1^\vee)$.

If $|\vartheta| = 3$, then $|\vartheta_2| = 1$, denote its single element by k. Then

$$\lambda_{cd} \le \mathrm{Hom}(\Theta_1, \varepsilon_k) \oplus \mathrm{Hom}(\varepsilon_k, \Theta_1)$$

For fixed k there are at most 2 such (c,d) pairs, since $c > d$ for $(c,d) \in T_I$. So it is enough to check linear independence of such pairs.

If $a > k > b$, then $\lambda_{ak} \leq \mathrm{Hom}(\Theta_1, \varepsilon_k)$, $\lambda_{kb} \leq \mathrm{Hom}(\varepsilon_k, \Theta_1)$, so they are independent.

If $a > b > k$ then in order for $(a, k), (b, k) \in T_I$ to hold, $I(a) < I(b) < I(k)$ must hold, then $\lambda_{ak} = \varepsilon_{ak}, \lambda_{bk} = \varepsilon_{bk}$ are independent in $\mathrm{Hom}(\Theta_1, \varepsilon_k)$. The case $k > a > b$ is similar.

Finally, $\lambda_{ab} = \mathrm{Hom}(\rho, \rho^\vee) \leq \mathrm{End}(\Theta_1)$. □

Corollary C.1.7. *$\nu(W)|_R$ is isomorphic to $\bigoplus_{(c,d)\in N_I} \lambda_{cd}$.*

Introduce the notations

$$G_I(c, \gamma, \delta) := |\{d > c : \gamma < I(d) \leq \delta\}|, \qquad L_I(c, \gamma, \delta) := |\{d < c : \gamma < I(d) \leq \delta\}|$$

$$T_I(a, b) := L_I(a, \alpha, \beta) + G_I(b, \alpha, \beta), \qquad N_I(a, b) := G_I(a, \alpha, \beta) + L_I(b, \alpha, \beta)$$

$$G_I(a) := G_I(a, I(a), m), \qquad G_I(a, \delta) := G_I(a, \delta, m)$$

Note that by Definition (C.2) of λ_{cd}, $N_I(a, b)$ is the number of nontrivial λ_{cd} for $(c, d) \in N_I$. Summarizing, by Proposition C.1.1 and Corollary C.1.7, we have:

Proposition C.1.8. *If $I, J \in \mathrm{OSP}(\mathcal{D})$, $\ell(J) = \ell(I) - 1$ and J is obtained from I by interchanging $a \in I_\alpha \leftrightarrow b \in I_\beta$, $a > b$, $\alpha < \beta$, then*

$$[\Omega_I, \Omega_J] = \begin{cases} 0, & N_I(a, b) \text{ even} \\ \pm 2, & N_I(a, b) \text{ odd} \end{cases} \tag{C.3}$$

C.1.7 Determining the cycles

In the even case $\mathrm{Fl}_{2\mathcal{D}}^{\mathbb{R}}$, equation (C.3) is actually sufficient to determine the rational coefficient cohomology $H^*(\mathrm{Fl}_{2\mathcal{D}}^{\mathbb{R}}; \mathbb{Q})$ additively in terms of Schubert cycles. For the $\mathbb{Z}$-coefficient cohomology, the signs are required as well, see the next section.

Recall that if $2\mathcal{D} = (2d_1, 2d_2, \ldots, 2d_r)$ and $I \in \mathrm{OSP}(\mathcal{D})$, then *the doubled ordered set partition $DI \in \mathrm{OSP}(2\mathcal{D})$* is obtained by replacing each $i \in I_j$ by $(2i - 1, 2i) \in DI_j$; each element $k \in DI_j$ has a unique pair $k' \in DI_j$. We call Schubert varieties σ_{DI} *double Schubert varieties.*

Proposition C.1.9. *In $\mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^N)$ the double Schubert varieties σ_{DI} are non-torsion cycles and their classes $[\sigma_{DI}]$ generate a free $\mathbb{Z}$-submodule of $H^*(\mathrm{Fl}_{2\mathcal{D}}^{\mathbb{R}}; \mathbb{Z})$. Rationally, $[\sigma_{DI}]$ form a basis of $H^*(\mathrm{Fl}_{2\mathcal{D}}^{\mathbb{R}}; \mathbb{Q})$.*

Proof. Let $I = DI' \in \mathrm{OSP}(2\mathcal{D})$ be a doubled ordered set partition. As above, if $\Omega_J \subseteq \sigma_I$ and $\ell(I) - \ell(J) = 1$, then J is obtained by $a \in I_\alpha \leftrightarrow b \in I_\beta$, $\alpha < \beta$, $a > b$. Since I is doubled, both terms in the sum

$$N_I(a, b) = G_I(a, \alpha, \beta) + L_I(b, \alpha, \beta) \tag{C.4}$$

are even; e.g. if $k > a$ and $I(k) > \alpha$, then its pair k' also satisfies $k' > a$ and $I(k') = I(k) > \alpha$. So all coefficients $[\Omega_I, \Omega_J]$ vanish and σ_I is a cycle.

Now assume that $\Omega_I \subseteq \sigma_J$ and $\ell(J) - \ell(I) = 1$, and I be obtained by $a \in J_\alpha \leftrightarrow b \in J_\beta$, $\alpha < \beta$, $a > b$. We again have to determine the parity of (C.4), but now for $N_J(a, b)$. Let a' and b' denote the pairs of a and b respectively. Since I is a doubled ordered set partition, $a' \in I_\beta$ and $b' \in I_\alpha$. $\ell(J) - \ell(I) = 1$ implies that $a < a'$ and $b' < b$. As before, everything in J is in pairs, except $a < a'$ and $b < b'$ which shows that $G_J(a, \alpha, \beta)$ and $L_J(a, \alpha, \beta)$ are both odd. So $N_J(a, b)$ is even and Ω_I appears in all incidence relations with zero coefficient $[\Omega_J, \Omega_I] = 0$, so $[\sigma_I]$ generates a free $\mathbb{Z}$-submodule.

Finally, $\dim_{\mathbb{Q}} H^*(\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}; \mathbb{Q}) = |\,\mathrm{OSP}(\mathcal{D})|$, which follows e.g. from Proposition C.2.3. This agrees with the number of doubled ordered set partitions of $2\mathcal{D}$. □

C.1.8 Kocherlakota's theorem

In the special case of the classical real flag manifolds $\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$ Proposition C.1.8 gives an alternate proof of [Koc95, Theorem A]. Before we state it we have to introduce some further notation (see also Section B.1.2).

Let $\mathfrak{g}$ be a real split semisimple Lie algebra. Let $\mathfrak{a}$ be a maximal $\mathbb{R}$-diagonalizable subalgebra and let $\Sigma \subseteq \mathfrak{a}^*$ be the restricted root system. Since $\mathfrak{g}$ is split, all root multiplicities are one. Choose a regular element $\xi \in \mathfrak{a}$, which determines a positive Weyl chamber C^+ and $\Sigma = \Sigma^+ \coprod \Sigma^-$. The reflections r_φ in the root planes $\ker \varphi$, $\varphi \in \Sigma^+$ generate the Weyl group W of the root system. The Weyl group acts freely and transitively on the Weyl chambers, and the Weyl chambers C_w are labeled by $w \in W$, $C^+ = C_1$. Given $H \in \overline{C^+}$, let $\Theta \subseteq \Sigma^s$ be the simple roots vanishing at H. Then the Weyl orbit of H is W/W_H, which parametrizes the Bruhat cells of G/P_Θ (cf. B.1.2). Now we state Kocherlakota's theorem. Given $x \in \mathfrak{a}$, let

$$N(x) = \{\varphi \in \Sigma^+ : \varphi(x) < 0\},$$

$$\sigma(x) := \sum_{\varphi \in N(x)} \varphi \in \mathfrak{a}^*,$$

and $\ell(x) = |N(x)|$. This is consistent with notation (B.2), as we will show below, and in general it is the dimension of $\Omega_x \subseteq G/P_\Theta$ for $x \in W.H$. We will give another interpretation of $N(x)$, see (C.5).

Theorem C.1.10 (Kocherlakota)**.** *Let $x, y \in W.H = W/W_H$ and $\ell(y) = \ell(x) - 1$. If $r_\varphi(x) = y$ for a reflection r_φ, $\varphi \in \Sigma^+$, then $\sigma(x) - \sigma(y) = m\varphi$ for some $m \in \mathbb{Z}$. The incidence coefficients are given by*

$$[\Omega_x, \Omega_y] = \begin{cases} 0, & m \text{ odd} \\ \pm 2, & m \text{ even} \end{cases}$$

Before giving the proof for $\mathfrak{g} = \mathfrak{sl}(N, \mathbb{R})$, let us recall some specifics about the root system of type A_{N-1}. The roots in an appropriate basis are $\pm e_{ij}$ where $e_{ij} = e_i - e_j$, $i < j$. The simple roots are $\delta_i = e_{i,i+1}$, and in terms of the simple roots $e_{ij} = \sum_{k=i}^{j} \delta_i$. Its Weyl group is $W \cong S_N$ and the reflections r_{ij} in $\ker e_{ij}$ correspond to the transpositions $(ij) \in S_N$.

The Weyl-orbit of a regular element $H \in C^+$ can be parametrized by $W \cong S_N$. If $H \in \overline{C^+}$ is not regular, list the simple roots δ_i not vanishing on H: $\delta_{s_1}, \delta_{s_2}, \ldots, \delta_{s_r}$, such that $s_1 < s_2 < \ldots s_r$, and set $s_{r+1} := N$. Then the Weyl-orbit $W.H = W/W_H = \mathrm{OSP}(\mathcal{D})$, where $\mathcal{D} = (d_1, \ldots, d_r)$, for $d_i = s_{i+1} - s_i$.

This implies that positive roots $e_{ij} \in \Sigma^+$ have the following property: given $I \in W/W_H$, $e_{ij}(I) < 0$ iff (i, j) is an inversion of $I \in W/W_H$. Thus $N(I)$ is the set of inversions of $I \in W/W_H$:

$$N(I) = \{e_{ij} : (i, j) \text{ is an inversion of } I\} \tag{C.5}$$

In particular, for $I \in \mathrm{OSP}(\mathcal{D})$, $|N(I)| = \ell(I) = \dim_{\mathbb{R}} \Omega_I$ as we have stated above (e.g. by Proposition B.2.4).

Proof of Theorem C.1.10 for $\mathfrak{g} = \mathfrak{sl}(N, \mathbb{R})$. Let $I, J \in W/W_H = \mathrm{OSP}(\mathcal{D})$, such that $r_{ab}(I) = J$, and $\ell(J) = \ell(I) - 1$, $a > b$, $a \in I_\alpha, b \in I_\beta$. By (C.5), the set theoretic difference of $N(J) \setminus N(I)$ consists of those e_{ij}, for which i, j is an inversion in J, but not in I.

Clearly all such i, j pairs must contain a or b. A simple verification shows that there are three types of elements in $N(I) \setminus N(J)$ (the other cases can be excluded using $\ell(J) = \ell(I) - 1$):

- If $e_{bj} \in N(I)\backslash N(J)$ and $a < j$, then $e_{aj} \in N(J)\backslash N(I)$,
- if $e_{ja} \in N(I)\backslash N(J)$ and $j < b$, then $e_{jb} \in N(J)\backslash N(I)$, and
- $e_{ba} \in N(I)\backslash N(J)$.

Since $e_{bj} - e_{aj} = e_{ba}$ for $b < a < j$ and $e_{ja} - e_{jb} = e_{ba}$ for $j < b < a$

$$\sigma(I)-\sigma(J) = \sum_{e_{ij} \in N(I)\backslash N(J)} e_{ij} - \sum_{e_{ij} \in N(J)\backslash N(I)} e_{ij} = (G_I(a,\alpha,\beta)+L_I(b,\alpha,\beta)+1)e_{ba} = (N_I(a,b)+1)e_{ba}$$

by the definitions preceding Proposition C.1.8. We can conclude by Proposition C.1.8.

□

C.1.9 Signs

For cooriented Ω_I and Ω_J, determining the actual signs of $[\Omega_I, \Omega_J]$ requires some further work. We coorient all Ω_I lexicographically as described in Section C.1.1 and compute the signs of $[\Omega_I, \Omega_J]$ relative to these orientations.

Let $I \to J$ be obtained by $a \in I_\alpha \leftrightarrow b \in I_\beta$. Using the decomposition of the bundles $TW|_R$ and $\nu(W \hookrightarrow X)|_R$ (Corollaries C.1.6 and C.1.7), we can compare the coorientations of Ω_I and Ω_J at J. We do this by taking locally defined nowhere vanishing sections s_{cd} of $\lambda_{cd} \leq \nu(W \hookrightarrow X)|_R$, which are defined on a connected open set $U \subseteq R$ containing one of the branches $\overline{R_\pm}$, such that the lexicographical ordering of $(s_{cd}(I) : (c,d) \in N_I)$ agrees with the coorientation of Ω_I at I. Then the lexicographic ordering of $s_{cd}|_{R_+}$ agrees with the coorientation of Ω_I all along R_+. Extending the orientation determined by the sections s_{cd} to J, we can compare the two orientations $TR \oplus \nu(W)|_J = N_J$.

Proposition C.1.11. *If $[\Omega_I, \Omega_J] \neq 0$, then its sign is given by $[\Omega_I, \Omega_J] = (-1)^{s(I,J)} \cdot 2$, where*

$$s(I,J) = c_1 + c_2 + c_3 + c_4$$

and

$$c_1 = \sum_{c<b} G_I(c) + |\{b < c < a : J(c) > J(b)\}|$$

$$c_2 = \sum_{\substack{a<d \\ \alpha<I(d)\leq\beta}} \sum_{b<c<a} G_I(c) + G_J(a, J(a))$$

$$c_3 = \sum_{\substack{c<b \\ \alpha\leq I(c)<\beta}} G_I(b, I(c)) - G_I(a, I(c))$$

$$c_4 = G_I(a, \alpha, \beta)$$

Proof. Let R_+ be the branch of R, on which the vector in $TR|_{R_+}$ pointing towards J converges towards $\mathbb{R}_+(e_b \mapsto e_a)$. Let $U \subsetneq R$ be a connected open set containing $\overline{R_+}$. Let $r \in \Gamma(\rho|_U)$ and $r^\vee \in \Gamma(\rho^\vee|_U)$ be nowhere vanishing sections, such that $r(I) = r^\vee(J) = e_a$ and $r(J) = r^\vee(I) = e_b$ (they exist since U is contractible). Define sections of $\lambda_{cd}|_U$ for $(c,d) \in N_I$ as

$$s_{cd} := \begin{cases} (e_c \mapsto e_d), & \text{if } (c,d) \in N_J \\ (r \mapsto e_d), & \text{if } c = a,\ I(a) < I(d) \leq I(b) \\ (e_c \mapsto r^\vee), & \text{if } d = b,\ I(a) \leq I(c) < I(b) \end{cases}$$

We call s_{cd} trivial if $s_{cd} = (e_c \mapsto e_d)$ and nontrivial otherwise. Let us compare the following two orientations of N_J:

- the lexicographical orientation $N_J^1 := \langle (e_c \mapsto e_d) : (c,d) \in N_J \rangle$ and
- $N_J^2 := \langle (e_b \mapsto e_a), s_{cd}(J) : (c,d) \in N_I \rangle$, where $(c,d) \in N_I$ are ordered by the lexicographical orientation.

The upper index N_J^i, $i = 1, 2$ signifies the respective orientation. Therefore the combinatorial task is to determine the (relative) sign of two permutations. The first permutation is simply the elements of N_J^1 listed lexicographically:

$$\ldots, (b-1, d^{b-1}_{n_{b-1}}), (b, d^b_1), \ldots, (b, a), \ldots, (b, d^b_{n_b}), , \ldots, (c, d^c_1), \ldots, (c, d^c_{n_c}), \ldots, (a, d^a_1), \ldots, (a, d^a_{n_a}), \ldots$$

The second permutation is obtained by listing (b, a) and then the elements of N_I lexicographically

$$(b, a), \ldots, (b-1, f^{b-1}_{m_{b-1}}), (b, f^b_1), \ldots, (b, f^b_{m_b}), , \ldots, (c, f^c_1), \ldots, (c, f^c_{m_c}), \ldots, (a, f^a_1), \ldots, (a, f^a_{m_a}), \ldots$$

and making the following substitutions, which compared to N_J^1 contribute a certain number of transpositions that we will determine below:

- listing (b, a) first: this contributes c_1 many transpositions,
- replacing (a, d) with (b, d) for all nontrivial sections s_{ad}: c_2 many transpositions,
- replacing (c, b) with (c, a) for all nontrivial sections s_{cd}: c_3 many transpositions,

One must also count the sign differences coming from the values of $s_{cd}(J) = \pm(e_c \mapsto e_d)$. These sign differences are represented by the term c_4. Now we determine c_1, c_2, c_3, c_4.

The first difference is that $(e_b \mapsto e_a)$ is the first element in N_J^2. This contributes

$$c_1 = \sum_{\substack{c \leq b \\ I(c) \leq \alpha}} G_I(c) + \{b < c < a : J(b) < J(c)\}$$

many transpositions.

For nontrivial sections s_{ad}, $(a, d) \in N_I$: $s_{ad}(J) = \pm(e_b \mapsto e_d)$, which has the following distance from its final position at (b, d) in N_J^1: $(a > b)$

$$\#\{a < c < d : b \to c\} + \#\{d < c : b \to c\} + \sum_{b<c<a} G_I(c) =$$

$$= G_I(a, \beta) - G_I(d, \beta) + G_I(d, \beta) + \sum_{b<c<a} G_I(c)$$

as $(e_a \mapsto e_d)$ swaps place with every $(e_f \mapsto e_g)$ pair whose position doesn't change and precedes it. The sum of these for nontrivial s_{ad} pairs is the term c_2.

Similarly, for $(c, b) \in N_I$, $s_{cb}(J) = \pm(e_c \mapsto e_a)$ has the following distance to its final position:

$$\#\{b < c < a : d \to c\} = G_I(b, \delta, m) - G_I(a, \delta, m)$$

the sum of which for nontrivial s_{cb} pairs is c_3.

Finally, the sections s_{cd} induce $G_I(a, \alpha, \beta)$ many sign changes: the trivial bundles have trivial sections, some of the nontrivial bundles λ_{cd} introduce no sign change, and by the initial choice of the branch R_+ $G_I(a, \alpha, \beta)$ many of them do. $\square$

C.2 The Cartan model

A pair (G, K) of compact connected Lie groups is a *Cartan pair*, if $K \leq G$ and H_G^* has $\operatorname{rk} G$ many polynomial generators, $\operatorname{rk} G - \operatorname{rk} K$ many of which restrict to zero via $\rho^* : H_G^* \to H_K^*$.

For Cartan pairs, there is a simple description of $H^*(G/K; \mathbb{Q})$, due to Cartan and Borel [Bor53], see also [Ter11] for a summary.

Theorem C.2.1 (Borel, Cartan)**.** *For a Cartan pair (G, K)*

$$H^*(G/K) \cong H_K^*/(\operatorname{Im} \rho^*) \bigotimes \bigwedge [x_{r_i - 1}]_{i=p+1}^n$$

where r_i are the degrees of the polynomial generators restricting to zero via $\rho^* : H_G^* \to H_K^*$, $n = \operatorname{rk} G$, $p = \operatorname{rk} K$.

Let $\mathrm{SO}(\mathcal{D}) := \prod_{i=1}^m \mathrm{SO}(d_i)$, $\mathrm{O}(\mathcal{D}) := \prod_{i=1}^m \mathrm{O}(d_i)$ $N := \sum d_i$.

Proposition C.2.2. *$(G, K_0) = (\mathrm{SO}(N), \mathrm{SO}(\mathcal{D}))$ is a Cartan pair for all $\mathcal{D}$.*

Proof. One has

$$\rho^*(p_*(S)) = \prod_{i=1}^m p_*(S_i),$$

where $S \to BSO(N)$ and $S_i \to BSO(d_i)$ denote the tautological bundles. Let $n = \lfloor \frac{N}{2} \rfloor$ be the rank of $\mathrm{SO}(N)$ and $q = \sum_{i=1}^m \lfloor \frac{d_i}{2} \rfloor$ be the rank of $\mathrm{SO}(\mathcal{D})$. Since $p_{\text{top}}(S_i) = p_{\lfloor \frac{d_i}{2} \rfloor}(S_i)$, by examining degrees, one sees that $\rho^*(p_j(S)) = 0$ for $j > q$.

□

For $\mathcal{D}$, denote by $\mathcal{D}^-$ the unique sequence of $d_i - 1 \leq d_i^- \leq d_i$ such that all d_i^- are even. Let $\mathcal{D}_0$ be the sequence of $d_i^-/2$. We can extend the Borel-Cartan description of $H^*(G/K; \mathbb{Q})$ to $K = S(\mathrm{O}(\mathcal{D}))$ which is no longer connected as follows.

Proposition C.2.3. *Let $n = \lfloor \frac{N}{2} \rfloor$ be the rank of $\mathrm{SO}(N)$ and $q = \sum_{i=1}^m \lfloor \frac{d_i}{2} \rfloor$ be the rank of $\mathrm{SO}(\mathcal{D})$, then*

$$H^*(\mathrm{Fl}_{\mathcal{D}}^{\mathbb{R}}) \cong H^{2*}(\mathrm{Fl}_{\mathcal{D}_0}^{\mathbb{C}}) \otimes \bigwedge [y_i]_{i=q+1}^n$$

where $y_i = x_{4i-1}$ except if N even, $y_n = x_{N-1}$. H^{2*} means that the degrees are doubled and $\deg x_j = j$.

Proof. Let $G = \mathrm{SO}(N)$, $K = \mathrm{S}(\mathrm{O}(\mathcal{D}))$ and $K_0 = \mathrm{SO}(\mathcal{D})$. The Cartan description can be obtained by analyzing the Leray-Serre spectral sequence $K_0 \to B_{K_0}G \to BK_0$. Indeed, since $EK_0 \to B_{K_0}G \to G/K_0$ is a fibration with contractible fiber, $B_{K_0}G \sim G/K_0$.

In the Cartan description, the $\Gamma := \mathbb{Z}_2^{m-1}$-action induced by the fibration $\Gamma \to G/K \to \mathrm{Fl}_{\mathcal{D}}^{\mathbb{R}}$ acts by multiplying the Euler classes of H_K^* by -1. This follows from the fact that $B_{K_0}G \to B_KG$ and $BK_0 \to BK$ are principal Γ-bundles and $B_{K_0}G \to B_KG$ is a principal G-bundle map. It follows that

$$H^*(\mathrm{Fl}_{\mathcal{D}}^{\mathbb{R}}) \cong H^*(G/K_0)^{\Gamma} = (H_{K_0}^*/\operatorname{Im}\rho^*)^{\Gamma} \bigotimes \bigwedge [x_{r_i-1}]_{i=q+1}^{n}$$

where r_i are the degrees of the polynomial generators restricting to zero. The invariant part is therefore generated by solely Pontryagin classes (the Euler class generators are squared) and

$$(H_{K_0}^*/\operatorname{Im}\rho^*)^{\Gamma} \cong H^{2*}(\mathrm{Fl}_{\mathcal{D}_0}^{\mathbb{C}}).$$

For the antisymmetric part, as above, $p_i \in H_{\mathrm{SO}(N)}^*$ restricts to zero for $i > q$, which proves the odd case. In the even case, the generators are $p_1, \ldots, p_{n-1}, e$. There are two cases. If all d_i's are even, then the ranks $n = q$ are equal, and there is no antisymmetric part. If there is an odd d_i, then the Euler class restricts to zero, so there is a generator $y_n = x_{N-1}$. □

Notice that this implies that $\mathrm{Fl}_{\mathcal{D}}^{\mathbb{R}}$ is a $\mathbb{Q}$-Poincaré dual space for any $\mathcal{D}$.

Example C.2.4. $\mathcal{D} = (1, 3, 3, 3)$: $D^- = (0, 2, 2, 2)$, $\mathcal{D}_0 = (1, 1, 1)$, $N = 10$, $n = 5$, $q = 3$,

$$\rho^*(p_*^N) = \rho^*(1 + p_1 + p_2 + p_3 + p_4 + p_5) = (1 + p_1)(1 + p_1')(1 + p_1'')$$

$p_1, \ldots, p_4, e$ are the generators of $H^*(BSO(10))$, so $p_4 \in H^{16}, e \in H^{10}$ restrict to zero. Therefore

$$H^*(\mathrm{Fl}_{\mathcal{D}}^{\mathbb{R}}; \mathbb{Q}) = H^{2*}(\mathrm{Fl}_{\mathcal{D}_0}^{\mathbb{C}}; \mathbb{Q}) \bigotimes \bigwedge [x_9, x_{15}],$$

whose Poincaré polynomial is

$$(1 + t^4)(1 + t^4 + t^8)(1 + t^9)(1 + t^{15}).$$

♣

Example C.2.5. $\mathcal{D} = (1,1,3,5)$: $D^- = (0,0,2,4)$, $\mathcal{D}_0 = (1,2)$, $N = 10$, $n = 5$, $q = 3$,

$$\rho^*(p_*^N) = \rho^*(1 + p_1 + p_2 + p_3 + p_4 + p_5) = (1 + p_1)(1 + p_1' + p_2')$$

$p_1, \ldots, p_4, e$ are the generators of $H^*(BSO(10))$, so $p_4 \in H^{16}, e \in H^{10}$ restrict to zero. Therefore

$$H^*(\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}; \mathbb{Q}) = H^{2*}(\mathrm{Fl}^{\mathbb{C}}_{\mathcal{D}_0}; \mathbb{Q}) \bigotimes \bigwedge [x_9, x_{15}]$$

whose Poincaré polynomial is

$$(1 + t^4 + t^8)(1 + t^9)(1 + t^{15}).$$

♣

Appendix D

Algebra

D.1 Canonical orientations

The aim of this section is to give a universal choice of orientations of a) complex vector spaces and b) real $\mathrm{Hom}_{\mathbb{R}}$-spaces, compatible in a well-defined sense. Fix the following universal conventions:

a) If A is a complex vector space, its *complex orientation* is defined by taking a complex basis $a_1, \ldots, a_n$ and orienting A by
$$\langle a_1, ia_1, \ldots, a_n, ia_n \rangle.$$

b) If A, B are even dimensional real vector spaces, then its *lexicographical orientation* is obtained by taking real bases $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ and orienting $\mathrm{Hom}_{\mathbb{R}}(A, B)$ by the lexicographical ordering of the basis vectors $\varphi_{ij} = (a_i \mapsto b_j)$.

c) If A, B are complex vector spaces, then $\mathrm{Hom}_{\mathbb{R}}(A, B)$ is a complex vector space through the complex structure of the image B: $(\lambda \cdot \varphi)(x) := \lambda \cdot (\varphi(x))$.

Both choices a) and b) are independent of the chosen basis. In particular, complex vector bundles and real $\mathrm{Hom}_{\mathbb{R}}$-bundles have canonical orientations using these conventions. These choices are compatible with each other in the following sense:

Proposition D.1.1. *If A, B are complex vector spaces, then the complex orientation of* $\mathrm{Hom}_{\mathbb{R}}(A,B)$ *as a complex vector space (via convention c)) coincides with its lexicographical orientation as a* $\mathrm{Hom}_{\mathbb{R}}$*-space.*

Proof. Let $a_i \in A$, $b_j \in B$ be complex bases. Then

$$\mathrm{Hom}_{\mathbb{R}}(A,B) = \bigoplus_{i,j} \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}a_i, \mathbb{C}b_j),$$

so it is enough to show that the two orientations coincide on $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$. Take the real standard basis $\varphi_{ab} \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$, $a, b \in \{1, i\}$. The complex orientation a) is given by

$$\varphi_{11} + \varphi_{ii}, \quad \varphi_{1i} - \varphi_{i1}, \quad \varphi_{11} - \varphi_{ii}, \quad \varphi_{1i} + \varphi_{i1}$$

The lexicographic orientation b) is $\varphi_{11}, \varphi_{1i}, \varphi_{i1}, \varphi_{ii}$. A computation shows that the change of basis matrix has positive determinant. □

For example, this implies that in Example 2.2.1 the two orientation conventions on $TX|_{X^\Gamma} = \mathrm{Hom}_{\mathbb{R}}(S_{\mathbb{C}}, Q_{\mathbb{C}})$ coincide. With this orientation convention, the excess weights appear with positive signs in all of our examples, see also Section C.1.1. We used the following Proposition together with Proposition A.3.3.

Proposition D.1.2. *Let A, B be real even dimensional vector spaces. Then the natural representation*

$$\rho : \mathrm{GL}(A) \times \mathrm{GL}(B) \to \mathrm{GL}(\mathrm{Hom}_{\mathbb{R}}(A,B))$$

has positive determinant, i.e. $\mathrm{Im}(\rho) \subseteq \mathrm{GL}_+(\mathrm{Hom}_{\mathbb{R}}(A,B))$.

Proof. By a connectedness argument, it is enough to check this for some $g \in \mathrm{GL}(A)$ of negative determinant. This is easy to check for g reflection to a hyperplane; it multiplies the determinant by $(-1)^{|B|} = 1$. □

D.2 Galois type actions - details

We prove the following – intuitively clear – statements. Conjugation-invariant complex subspaces in $\mathbb{C}^n$ are complexifications of real subspaces. Quaternionic subspaces $W \leq \mathbb{H}^n$ invariant under inner automorphisms by $\mathrm{U}(1)$ are quaternionifications of complex subspaces.

D.2.1 $\Gamma = \mathbb{Z}_2$ acts on $\mathbb{C}^n$

Let $\Gamma = \mathbb{Z}_2$ act on $\mathbb{C}^n$ by complex conjugation.

Proposition D.2.1. *There is a natural one-to-one correspondence between k-dimensional Γ-invariant complex subspaces $W \leq \mathbb{C}^n$ and k-dimensional real subspaces $W_{\mathbb{R}} \leq \mathbb{R}^n \leq \mathbb{C}^n$ via $W \mapsto W_{\mathbb{R}} = W \cap \mathbb{R}^n$.*

Proof. The following trivial lemma coupled with Maschke's theorem implies the Proposition: one can decompose W successively into one dimensional complex Γ-invariant subspaces. $\square$

Lemma D.2.2. *Let $W \leq \mathbb{C}^n$ be a complex, Γ-invariant nontrivial subspace. Then $W \cap \mathbb{R}^n \neq (0)$.*

Proof. Let $0 \neq w \in W$. Then $w + \overline{w} \in W^\Gamma \leq \mathbb{R}^n$. If $w + \overline{w} \neq 0$, then we are done. Otherwise, write $w = -\overline{w}$, so $w = iv$ for some $0 \neq v \in \mathbb{R}^n$, which is in W, since W is a complex subspace. $\square$

D.2.2 $\Gamma = \mathrm{U}(1)$ acts on $\mathbb{H}^n$

Decompose the right quaternionic vector space $\mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n$ and let $\Gamma = \mathrm{U}(1) \leq \mathbb{H}^*$ act on $\mathbb{H}^n$ by left multiplication. This gives $\mathbb{H}^n$ the structure of a (right)-complex $\mathrm{U}(1)$-representation, splitting into n weight 1 ($V_+ := \mathbb{C}^n$) and n weight -1 ($V_- := j\mathbb{C}^n$) representations. (Summarizing, $\mathrm{U}(1)$ acts from the left, complex multiplication $\mathbb{C} \leq \mathbb{H}$ from the right.)

Proposition D.2.3. *There is a natural one to one correspondence between k-dimensional $\mathrm{U}(1)$-invariant quaternionic subspaces $W \leq \mathbb{H}^n$ and k-dimensional complex subspaces $W_{\mathbb{C}} \leq \mathbb{C}^n \leq \mathbb{H}^n$ via $W \mapsto W_{\mathbb{C}} = W \cap \mathbb{C}^n$.*

Proof. Again, by Maschke's theorem, the following lemma implies the Proposition.

$\square$

Lemma D.2.4. *Let $W \leq \mathbb{H}^n$ be a $\mathrm{U}(1)$-invariant nontrivial quaternionic subspace. Then $W \cap \mathbb{C}^n \neq (0)$.*

Proof. Let $W \leq \mathbb{H}^n$ be a $\mathrm{U}(1)$-invariant nontrivial quaternionic subspace; then as complex subspaces it splits as $W = W_+ \oplus W_-$, where $W_+ := W \cap \mathbb{C}^n$ and $W_- := W \cap j\mathbb{C}^n$. Since $W_- = jW_+$, W_+ must contain a nonzero element, otherwise $W = W_+ \oplus jW_+ = (0)$. $\square$

D.3 Representation theoretic computations

The three main actions in this thesis are $\mathrm{U}(1)$ acting on $\mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^N)$, $\mathrm{U}(1)$ acting on $\mathrm{Fl}_{\mathcal{D}}(\mathbb{H}^N)$ and $\mathrm{Sp}(1)$ acting on $\mathrm{Fl}_{\mathcal{D}}(\mathbb{O})$. In Borel-Haefliger I, we have to understand the weights of the normal representation. In this section we carry out these elementary representation theoretic computations.

D.3.1 $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^k, \mathbb{C}^n)$ as a $\mathrm{U}(1)$-representation

Recall that if A, B are complex vector spaces, then $\mathrm{Hom}_{\mathbb{R}}(A, B)$ is also a complex vector space via $\mathrm{U}(1)$ acting on the target, see Section D.1.

Proposition D.3.1. *Let A, B be complex a and b-dimensional $\mathrm{U}(1)$-representations, let $\alpha \in \mathbb{Z}^a$ and $\beta \in \mathbb{Z}^b$ be their characters. Then the character of the complex $\mathrm{U}(1)$-representation $\mathrm{Hom}_{\mathbb{R}}(A, B)$ is*

$$(\beta \oplus (-\alpha), \alpha \oplus \beta) \in \mathbb{Z}^{2ab},$$

where $\oplus$ denotes the Minkowski sum of α and β.

Corollary D.3.2. *If A, B are complex vector spaces regarded as weight one complex $\mathrm{U}(1)$-representations, then the character of $\mathrm{Hom}_{\mathbb{R}}(A, B)$ is $(0^{ab}, 2^{ab})$.*

The character 0 signifies the *complex* trivial representation (of real dimension 2). If $\mathrm{Hom}_{\mathbb{R}}(A, B)$ is regarded as a complex vector space, then the sign of the weight is $+2$ – this follows from Proposition D.1.1.

D.3.2 $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}^k, \mathbb{H}^n)$ as a $\mathrm{U}(1)$-representation

All quaternionic vector spaces V are considered to be right quaternionic. If one identifies $V \cong \mathbb{H}^n$, then it also has a left $\mathbb{H}$-module structure - this structure depends on the identification. However, any two structures are isomorphic as bimodules, so fix an identification with $\mathbb{H}^n$. Thus $\mathbb{H}^n$ inherits a left $\mathrm{U}(1) \leq \mathbb{H}^*$-action by restricting the left $\mathbb{H}$-module structure. This action is clearly right $\mathbb{H}$-linear.

Proposition D.3.3. *Let* $\mathrm{U}(1) \leq \mathbb{H}^*$ *act on (the right* $\mathbb{H}$*-linear homomorphisms)* $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}^k, \mathbb{H}^n)$ *via conjugation*

$$(\lambda.\varphi)(x) = \lambda\varphi(\lambda^{-1}x).$$

Then as complex $\mathrm{U}(1)$*-representations,* $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}^k, \mathbb{H}^n)$ *splits as* kn *weight 0 and* kn *weight 2 representations.*

Proof. Since $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}^k, \mathbb{H}^n)$ splits as the direct sum of U(1)-representations $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}, \mathbb{H})^{\oplus kn}$, it is enough to understand $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}, \mathbb{H})$ as a U(1)-representation. But this is simply $\mathbb{H}$ acted on by U(1) via $\lambda.x = \lambda x \lambda^{-1}$, which as a (right complex) U(1)-representation splits as $\mathbb{C} \oplus j\mathbb{C}$, which is the direct sum of a weight 0 and a weight 2 representation. □

D.3.3 SU(2)-equivariant Euler classes

In this section we compute the equivariant Euler classes of the irreducible (real and complex) representations of $\mathrm{Sp}(1) = \mathrm{SU}(2)$, in order to illustrate that the set of weights $W(E)$ for a non-commutative Lie group can be zero or can be of higher degrees.

Complex irreducible representations

A complete list of irreducible complex $\Gamma = \mathrm{SU}(2)$ representations is given by the action on $V_d := \mathrm{Pol}^d(\mathbb{C}^2)$ induced by the defining representation; $\dim_{\mathbb{C}} V_d = d + 1$. In other words $V_d = \mathrm{Sym}^d_{\mathbb{C}}(V_1)$.

The classifying space of SU(2) is $B\Gamma = \mathbb{H}P^\infty$. Write

$$H^*(\mathbb{H}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[u] \text{ for } u = e(S_{\mathbb{H}}), \qquad \text{and} \qquad H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[v] \text{ for } v = e(S_{\mathbb{C}}).$$

Essentially by definition, $B_\Gamma V_1 \cong S_{\mathbb{H}}$ as (right-) quaternionic bundles over $B\Gamma$, therefore $e_\Gamma(V_1) = u$. We want to compute $e_\Gamma(V_d)$ in general - notice that they exist, since SU(2) is connected. It is enough to compute $e_\Gamma(V_d)|_{\mathbb{C}P^\infty}$, since $|_{\mathbb{C}P^\infty}$ is injective. Since

$$S_{\mathbb{H}}|_{\mathbb{C}P^\infty} \cong S_{\mathbb{C}} \oplus \overline{S_{\mathbb{C}}},$$

$u|_{\mathbb{C}P^\infty} = -v^2$. Write

$$B_\Gamma V_d|_{\mathbb{C}P^\infty} = \mathrm{Sym}^d_{\mathbb{C}}(B_\Gamma V_1|_{\mathbb{C}P^\infty}) = \mathrm{Sym}^d_{\mathbb{C}}(S_{\mathbb{C}} \oplus \overline{S_{\mathbb{C}}})$$

It is a standard computation to get

$$e(\mathrm{Sym}^d_{\mathbb{C}}(S_{\mathbb{C}} \oplus \overline{S_{\mathbb{C}}})) = \prod_{i=0}^{d}(d-2i)v = \begin{cases} 0 & d = 2k \\ -(d!!)^2 v^{d+1} & d = 4k+1 \\ (d!!)^2 v^{d+1} & d = 4k+3 \end{cases}.$$

Therefore

$$e_\Gamma(V_d) = \begin{cases} 0 & d = 2k \\ (d!!)^2 u^{k+1} & d = 2k+1 \end{cases}$$

Real irreducible representations

A complete list of real irreducible representations of Γ is given by the following two types:

- $V_d^{\mathbb{R}}$ obtained by forgetting the complex structure on V_d for d odd,
- it turns out that V_d is the complexification of a real irreducible representation W_d for d even.

For example, V_2 is the complexification of the SU(2)-lift of the defining representation of SO(3).

For the first case, with $d = 2k+1$, the same reasoning as in the case of U(1)-representations gives that

$$e_\Gamma(V_d^{\mathbb{R}}) = e_\Gamma(V_d) = (d!!)^2 u^{k+1}.$$

For $d = 2k$, since all odd cohomology groups are zero,

$$e_\Gamma(W_d) = 0 \in H^{2k+1}(\mathbb{H}P^\infty; \mathbb{Z}).$$

Appendix E

Equivariant principal bundles

We include some discussion on equivariant principal bundles, since they appear in the equivariant Borel-Haefliger theorem. Namely, let Γ act on G by automorphisms and let $EG \to BG$ have the structure of a (Γ, G)-bundle. If the semidirect product $S = G \rtimes \Gamma$ acts on X, then Γ acts on $B_G X \to BG$. The equivariant Borel-Haefliger theorem states that if BG and X are halving spaces, then so is $B_G X$. We review these constructions. The definitions and some basic properties can be found in [tD69], [tD87, Chapter I.8], [LM86] and [May96, Chapter VII].

E.1 Definition and properties

Throughout this Appendix, let Γ be a compact Lie group, G a topological group, and $\alpha : \Gamma \to \mathrm{Aut}(G)$ be a group homomorphism, such that the map $\Gamma \times G \to G$ sending $(\gamma, g) \mapsto \alpha(\gamma)(g)$ is continuous. Denote by S the semidirect product $S := G \rtimes_\alpha \Gamma$. An S-space X is naturally a Γ and G-space via the fixed inclusions $\Gamma, G \leq S$.

A Γ-*equivariant principal* G-*bundle* is an S-space P, such that the G-action restricted from S makes it a principal G-bundle. This is equivalent to saying that Γ *acts compatibly with* G on the principal G-bundle P, i.e.

$$\gamma.(p.g) = (\gamma.p).(\gamma.g),$$

where $\gamma.g = \alpha(\gamma)(g)$ by definition. This induces a unique Γ-action on the base, such that projection is Γ-equivariant. We will also use the shorter names *equivariant principal bundle*, or (Γ, G)-*bundle*.

Remark E.1.1. In the literature, equivariant principal bundles often correspond to the special case when α acts trivially on G, this was also tom Dieck's terminology, see also [LU14]. What we call equivariant principal bundles are called (Γ, α, G)-bundles by tom Dieck. There is a more general notion of principal bundles associated to group extensions of G by Γ, see Lashof-May [LM86], however we will not be concerned with this case. What we denote (Γ, G)-bundles are called $(G; G \rtimes \Gamma)$-bundles in [LM86].

Example E.1.2. The motivating example is Atiyah's KR theory of Real bundles [Ati66]; the underlying Γ-equivariant principal G-bundles correspond to the Galois action of $\Gamma = \mathbb{Z}_2$ on $G = \mathrm{GL}(n, \mathbb{C})$, with fixed point set $\mathrm{GL}(n, \mathbb{R})$. The semidirect product $S = \mathrm{GL}(n, \mathbb{C}) \rtimes \mathbb{Z}_2$ is also called the *semilinear group*, denoted $\Gamma L(n, \mathbb{C})$. The Real bundles are obtained by associating the vector bundle via the defining (Γ-equivariant) representation of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. ♣

Example E.1.3. For us, another example of interest is $\Gamma = \mathrm{U}(1)$ acting on $G = \mathrm{GL}(2n, \mathbb{R})$ with fixed point set $\mathrm{GL}(n, \mathbb{C})$ (via inner conjugation as described in Section 4.1.2). This also leads to an equivariant K-theory KC as one can take direct sums of such bundles. There are forgetful group homomorphisms $KC(X) \to KO(X)$ and $KC(X) \to K(X^\Gamma)$. However contrary to KR, there is no product structure on KC compatible with these forgetful maps, as a computation with characteristic classes shows. ♣

E.1.1 Compatible actions

As we have already remarked, Γ and G act on X compatibly is equivalent to X being an S-space. We discuss some generalities on such compatible actions.

Faithful actions

Assume $G \leq \mathrm{Aut}(F)$ (G-action faithful), let F be a Γ-space. Then there is at most one $\alpha : \Gamma \to \mathrm{Aut}(G)$, which makes the actions compatible on F. Indeed, the compatibility condition and faithfulness implies that the action can only be the restriction of the conjugation action of Γ on $\mathrm{Aut}(F)$:

$$(\gamma.\varphi)(x) := \gamma.(\varphi(\gamma^{-1}.x)).$$

In case $G \leq \mathrm{Aut}(F)$ is Γ-invariant, then this action restricts to $G \leq \mathrm{Aut}(F)$. In this case it is also easy to see that $G^\Gamma \leq G$ is a subgroup, furthermore $F^\Gamma \subseteq F$ is G^Γ-invariant, so there is a homomorphism $G^\Gamma \to \mathrm{Aut}(F^\Gamma)$, possibly with nontrivial kernel. Indeed, for example $\mathbb{Z}_2$ acts on $[-1, 1]$ by reflection; there are many $g \in \mathrm{Aut}(F)$ fixed by $\mathbb{Z}_2$, such that $g.0 = 0$.

Successive quotients

We mention the following triviality for future reference.

Proposition E.1.4. *If X is an S-space, then there is a unique induced Γ-action on the orbit space X/G making the projection $\pi : X \to X/G$ Γ-equivariant. Also, $X/S \cong (X/G)/\Gamma$ as topological spaces.*

Proof. This follows from the splitting of the short exact sequence $(1) \to G \to S \to \Gamma \to (1)$ and that Γ, G generate S (and therefore the equivalence relation). □

E.1.2 Bundle constructions

The usual bundle constructions also work for equivariant principal bundles, we briefly review them.

Associated fiber bundles

Let F be an S-space and P be a (Γ, G)-bundle. Then $E := P \times_G F \to X$ is the Γ-*equivariant fiber bundle associated to* $P \to X$; it has fiber F, and Γ acts on E by Proposition E.1.4, explicitly: $\gamma.[p, f] := [\gamma.p, \gamma.f]$. Association commutes with pullbacks:

Proposition E.1.5. *Let $P \to Y$ be a (Γ, G)-bundle and F be an S-space. If $f : X \to Y$ is a Γ-equivariant map, then*

$$f^*(P) \times_G F \cong f^*(P \times_G F)$$

as Γ-equivariant fiber bundles over X.

Fixed bundle

Lemma E.1.6. *Let $p : P \to X$ be a (Γ, G)-bundle. Then the Γ-fixed point set*

$$\pi := p|_{P^\Gamma} : P^\Gamma \to X^\Gamma$$

is a G^Γ-principal bundle. Furthermore, the structure group of $P|_{X^\Gamma}$ reduces to G^Γ:

$$P|_{X^\Gamma} \cong P^\Gamma \times_{G^\Gamma} G$$

as (Γ, G)-bundles where G^Γ acts on G by left multiplication.

Proof. The free G-action on P restricts to a free G^Γ-action under which P^Γ is invariant. The G^Γ-action on P^Γ is transitive on fibers: assume that $x, y \in P^\Gamma$, such that $\pi(x) = \pi(y)$, then $x = y.g$ for some $g \in G$. By using the defining properties, one can see that $g \in G^\Gamma$. For local triviality of $P^\Gamma \to X^\Gamma$, we use [LM86, Proposition 4], stating that if the total space is completely regular, then local triviality follows.

To see reduction of the structure group of $P|_X^\Gamma$, notice that $P^\Gamma \times_{G^\Gamma} G$ is a principal (Γ, G)-bundle. Let $P^\Gamma \times_{G^\Gamma} G \to P|_{X^\Gamma}$ be the map $(p, g) \mapsto p.g$. This map is Γ and G-equivariant, therefore an isomorphism of principal (Γ, G)-bundles. □

Proposition E.1.7. *Let $P \to X$ be a (Γ, G)-bundle and F be an S-space. Then*

$$(P \times_G F)^\Gamma \cong P^\Gamma \times_{G^\Gamma} F^\Gamma$$

as fiber bundles over X^Γ with fiber F^Γ. This isomorphism is functorial: if $F' \subseteq F$ is G and Γ-invariant, then there is a commutative diagram:

$$\begin{tikzcd}
(P \times_G F)^\Gamma \arrow[r, "\cong"] & P^\Gamma \times_{G^\Gamma} F^\Gamma \\
(P \times_G F')^\Gamma \arrow[r, "\cong"] \arrow[u, hook] & P^\Gamma \times_{G^\Gamma} (F')^\Gamma \arrow[u, hook]
\end{tikzcd}$$

Proof. First, $(P \times_G F)^\Gamma = (P \times_G F|_{X^\Gamma})^\Gamma$, since projection is Γ-equivariant. By Proposition E.1.5 and Lemma E.1.6,

$$P \times_G F|_{X^\Gamma} \cong P|_{X^\Gamma} \times_G F \cong (P^\Gamma \times_{G^\Gamma} G) \times_G F \cong P^\Gamma \times_{G^\Gamma} F$$

so

$$(P \times_G F)^\Gamma \cong (P^\Gamma \times_{G^\Gamma} F)^\Gamma \cong P^\Gamma \times_{G^\Gamma} F^\Gamma$$

Each of these isomorphisms is functorial, proving the claim. □

Remark E.1.8. [Reduction of structure group] More generally, instead of taking fixed points, if $F' \subseteq F$ is a G^Γ-invariant subset, then $E' := P^\Gamma \times_{G^\Gamma} F' \subseteq E|_{X^\Gamma}$ is a subbundle with structure group G^Γ. In particular, if $F' = F$, then the structure group of $E|_{X^\Gamma}$ can be reduced to G^Γ.

We demonstrate these notions on an example, in particular we show that tautological bundles on flag manifolds are associated to equivariant principal bundles.

Example E.1.9. [Equivariance of tautological bundles on flag manifolds] Let $\Gamma = \mathrm{U}(1)$ act on $G = \mathrm{GL}(2n, \mathbb{R})$ by the pseudo Galois action. By definition of the Γ-action on G, they act compatibly on the defining G-representation of $\mathbb{R}^{2n}$. Fix the parabolic subgroup

$$P = \mathrm{GL}(2k, 2(n-k), \mathbb{R}) \leq \mathrm{GL}(2n, \mathbb{R}),$$

and consider the tautological Γ-equivariant bundle $\gamma \to X = G/P = \mathrm{Gr}_{2k}(\mathbb{R}^{2n})$. First, $G_1 := \mathrm{GL}(2k, \mathbb{R}) \leq G$ is a Γ-invariant subgroup and the projection $\pi : P \to G_1$ is a Γ-equivariant group homomorphism. The tautological principal bundle over $X = G/P$ is

$$Q = G \times_P G_1 \to X,$$

where P acts on G_1 on the left via $\pi : P \to G_1$. Since Γ acts compatibly with G_1 on Q, Q is a (Γ, G_1)-bundle. Then the tautological Γ-equivariant bundle γ is obtained as $\gamma = Q \times_{G_1} \mathbb{R}^{2k}$, where G_1 acts on $\mathbb{R}^{2k}$ via its defining representation, compatibly with Γ.

By restricting $\gamma|_{X^\Gamma}$, one gets $Q^\Gamma \times_{G_1^\Gamma} \mathbb{C}^k$ which is the tautological vector bundle over $\mathrm{Gr}_k(\mathbb{C}^n)$ with structure group $\mathrm{GL}(k, \mathbb{C})$, as described in Remark E.1.8. Note that this is not the fixed bundle, but just the restricted bundle. The examples for P arbitrary parabolic subgroup are similar. ♣

Example E.1.10. The example of $(\Gamma, G) = (\mathbb{Z}_2, \mathrm{GL}(n, \mathbb{C}))$ is analogous. In the end however, the tautological bundle $\gamma_{\mathbb{R}} \to \mathrm{Gr}_n(\mathbb{R}^N)$ is the fixed bundle $\gamma_{\mathbb{C}}^\Gamma$ of $\gamma_{\mathbb{C}} \to \mathrm{Gr}_n(\mathbb{C}^N)$. ♣

Associating cycles

Proposition E.1.11. *Let $P \to B$ be a smooth (Γ, G)-principal bundle and let X be an $S = G \rtimes \Gamma$-manifold. Let $Z \subseteq X$ be a good Γ-invariant G-cycle of codimension type (k, l) and assume it has a fat nonsingular subset $Y \subseteq Z$, both Γ and G-invariant. Then $P \times_G Z \subseteq P \times_G X$ is a good Γ-invariant cycle of codimension type (k, l).*

Proof. Γ-invariance of $P \times_G Z \subseteq P \times_G X$ follows from the definitions of associated fiber bundles (Section E.1.2). Since Z is a G-cycle, G acts in an orientation preserving way on $\nu := \nu(Y \hookrightarrow X)$, and therefore

$$\nu(P \times_G Y \hookrightarrow P \times_G X) = P \times_G \nu \to P \times_G Y$$

is oriented. Since the restriction $H^k_G(X, X \backslash Z) \to H^k(X, X \backslash Z)$ factors through

$$H^k(P \times_G X, P \times_G (X \backslash Z)),$$

$P \times_G Z \subseteq P \times_G X$ is a cycle, which is Γ-invariant, with Γ-invariant fat nonsingular open $P \times_G Y$. Independence of the excess weight follows from Proposition E.1.7. □

E.2 Universal bundles

We proceed to discuss universal principal (Γ, G)-bundles. The (Γ, G)-bundles $EG \to BG$ naturally occurring in the equivariant Borel-Haefliger theorem are not necessarily universal as (Γ, G)-bundles, so we introduce a more general concept, universal $\mathcal{F}$-spaces, where $\mathcal{F}$ is a family of subgroups of S. The identification of EG as a universal $\mathcal{F}$-space is not logically required, but we feel that it gives some insight into what kind of an object $EG \to BG$ with the additional (Γ, G)-structure is.

Most of this section is based on [LM86], [tD87, Chapters I.6-8] and [Ser02].

E.2.1 Existence

Definition E.2.1. Let S be a Lie group. A *family $\mathcal{F}$ of subgroups of S* is a set of closed subgroups $K \leq S$, such that $\mathcal{F}$ is closed under taking subgroups and conjugacy ($H \sim K \in \mathcal{F} \Rightarrow H \in \mathcal{F}$).

Definition E.2.2. Let $\mathcal{F}$ be a family of subgroups of S. Then an S-space X is called an $\mathcal{F}$*-space*, if all isotropy subgroups $S_x \in \mathcal{F}$, for all $x \in X$.

Example E.2.3.

- If $\mathcal{F} = \{(1)\}$, then an $\mathcal{F}$-space X is a free S-space, $X \to X/S$ is a principal S-bundle.
- If $\mathcal{F}$ is the family of all subgroups, then an $\mathcal{F}$-space is an S-space.
- If $S = G \rtimes \Gamma$, and $\mathcal{F}$ is a family of subgroups $H \leq S$, all satisfying $H \cap G = (1)$, then an $\mathcal{F}$-space P is a (Γ, G)-bundle via $P \to P/G$.

♣

The category of $\mathcal{F}$-spaces have universal spaces $E\mathcal{F}$, generalizing $ES \to BS$. For the following theorem, see [tD87, Theorem 6.6] or [May96, Chapter V].

Theorem E.2.4. *Let $\mathcal{F}$ be a family of subgroups of S. There exists a universal numerable $\mathcal{F}$-space $E\mathcal{F}$, unique up to S-homotopy equivalence:*

i) Every numerable $\mathcal{F}$-space P admits an S-equivariant classifying map $\mathcal{K} : P \to E\mathcal{F}$.

ii) The classifying map $\mathcal{K}$ is S-homotopically unique.

Example E.2.5.

- For $\mathcal{F} = \{(e)\}$, $E\mathcal{F} = ES$, the universal principal S-bundle $ES \to BS$.
- For $\mathcal{F}$ the family of all subgroups, $E\mathcal{F}$ is a point.
- If $S = G \rtimes \Gamma$ and $\mathcal{F}$ is the family of *all* subgroups $H \leq S$ satisfying $H \cap G = (1)$, then unraveling the definitions shows that $E\mathcal{F}$ is a universal (Γ, G)-bundle. For further details about universal (Γ, G)-bundles, see [LM86], [May96], [GMM17], [tD87, Theorem 8.12] and [MS95].
- If $S = G \rtimes \Gamma$ and $\mathcal{F}$ is the family of subgroups $H \leq \Gamma$ and their conjugates, then $E\mathcal{F}$ is still a (Γ, G)-bundle $E\mathcal{F} \to E\mathcal{F}/G$, but universal for a smaller class of (Γ, G)-bundles. This is the class of examples naturally appearing in the equivariant Borel-Haefliger theorem.

♣

E.2.2 Characterization

A numerable principal G-bundle $E \to X$ is universal iff E is contractible. A similar characterization can be given for $\mathcal{F}$-spaces, see [LM86], [May96], [tD87, Chapter I.6].

Theorem E.2.6 (Characterization). *A numerable $\mathcal{F}$-space is universal iff E^H is contractible for all $H \in \mathcal{F}$.*

In particular, an S-space E is a universal (Γ, G)-bundle, if E^H is contractible for all $H \leq S$ satisfying $H \cap G = (1)$, see [LM86, Theorem 9]. This condition is not necessarily easy to verify in practice; even if Γ is finite, $S = G \rtimes \Gamma$ can have many subgroups H satisfying $H \cap G = (1)$. For example, every reflection in $\mathrm{O}(2) = \mathrm{SO}(2) \rtimes \mathbb{Z}_2$ generates such a subgroup. However, this condition can be reduced to the study of the nonabelian cohomology $H^1(\Gamma, G)$, as we will describe in the next section.

E.2.3 Nonabelian cohomology

To characterize when a (Γ, G)-space is universal, we are interested in the family $\mathcal{F}$ of subgroups $H \leq S = G \rtimes \Gamma$ satisfying $G \cap H = (1)$. This family can be described as the family generated by subgroups $H_t \leq S$, parametrized by $[t] \in H^1(\Delta, G)$, $\Delta \leq \Gamma$ see Proposition E.2.11. In this section we describe the details of this description, it is partially based on [Ser02, Chapter I.5] and [tD87].

Local objects

Fix a Γ-action $\alpha : \Gamma \to \mathrm{Aut}(G)$. We proceed to define $H^1(\Gamma, G)$ from the topological point of view. The most basic (Γ, G)-bundles are those over orbits - all (Γ, G)-bundles are glued together from such bundles.

Definition E.2.7. A *local object* is a (Γ, G)-bundle $P \to X$ over a Γ-orbit $X = \Gamma/\Delta$.

Our aim is to classify local objects up to equivalence. To a local object $q : P \to X = \Gamma/\Delta$, one can associate a map $t : \Delta \to G$ as follows: Pick $x \in q^{-1}(\Delta)$. Then for all $\lambda \in \Delta$:

$$\lambda.x = x \cdot t(\lambda)$$

for a unique $t(\lambda) \in G$. A short computation shows that this map t satisfies

$$t(\lambda\mu) = (\lambda.t(\mu)) \cdot t(\lambda)$$

for all $\lambda, \mu \in \Delta$. For local objects over points (i.e. $\Gamma = \Delta$), we obtain the definition of the cocycles of $H^1(\Gamma, G)$:

Definition E.2.8. A *cocycle* is a continuous map $t : \Gamma \to G$ satisfying

$$t(\lambda\mu) = (\lambda.t(\mu)) \cdot t(\lambda).$$

Conversely, given such a map $t : \Delta \to G$, there corresponds to it a local object $P_t \to \Gamma/\Delta$ as follows. The graph

$$H_t := \mathrm{graph}(t) \leq S = G \rtimes \Gamma$$

of t is a subgroup; $\tilde{t} := (t, \mathrm{id}_\Delta) : \Delta \to G \rtimes \Gamma$ is a group homomorphism. Then S acts on $H_t \backslash S$ from the right; since $H_t \cap G = (1)$, the G-action is free, furthermore $H_t \backslash S$ is a local object by [tD87, Lemma I.8.9].

When do such maps yield isomorphic local objects? Clearly, by choosing another $x' = g \cdot x \in q^{-1}(\Delta)$ leads to the same bundle, but a different map:

$$t'(\lambda) = (\lambda.g) \cdot t(\lambda) \cdot g^{-1}$$

as a simple computation shows. Equivalently, conjugate subgroups $H_t \leq S$ define isomorphic local objects. Serre [Ser02] calls local objects over a point principal homogeneous spaces over G.

This leads to the definition of the first nonabelian cohomology:

Definition E.2.9. Two cocycles $t, t' : \Gamma \to G$ are *cohomologous* if there exists $g \in G$, such that

$$t'(\lambda) = (\lambda.g) \cdot t(\lambda) \cdot g^{-1}$$

for all $\lambda \in \Gamma$. The *first nonabelian cohomology set* $H^1(\Gamma, G)$ is the set of equivalence classes of cocycles modulo the equivalence relation of cohomology.

Remark E.2.10. i) Unless further assumptions are made, $H^1(\Gamma, G)$ only has the structure of a pointed set.

ii) The notation is ambiguous in the sense that it depends on the action $\alpha : \Gamma \to \mathrm{Aut}(G)$, however this notation is standard.

iii) Every subgroup $H \leq S$ satisfying $H \cap G = (1)$ is of the form H_t.

iv) In case $\Gamma = \mathrm{Gal}(K|k)$, and G is a linear algebraic group defined over k, we get the definition of the Galois cohomology set $H^1(\Gamma, G)$. We will also be interested in cases, when Γ is not a Galois group, but Γ is a subgroup of G acting via inner automorphisms, as in Section 4.1.2. For $\Gamma = \mathbb{Z}_n$ finite cyclic, G connected, $H^1(\Gamma, G)$ has been considered in [AW08].

Summarizing the previous discussion, elements $[t] \in H^1(\Gamma, G)$ correspond bijectively to isomorphism classes of (Γ, G)-bundles P_t over a point.

Second characterization of universal bundles

Let $S = G \rtimes \Gamma$ and let

$$C \subseteq \bigcup_{\Delta \leq \Gamma} H^1(\Delta, G)$$

be a set of cohomology classes. Denote by $\mathcal{F}(C)$ the family of subgroups of S, generated by $(H_t : [t] \in C)$ via conjugation and taking subgroups.

Proposition E.2.11. *The S-space E is a model of $E\mathcal{F}(C)$ iff E^K is contractible for all $K \leq H_t$, for some representatitve $t \in [t]$ in each $[t] \in C$.*

Proof. Using the characterization of universal $\mathcal{F}$-spaces given by Theorem E.2.6, it is enough to show that if E^K is contractible for $K \leq H_t$, then E^L is contractible for its conjugates $L = sKs^{-1}$ for $s \in S$. But this is immediate, as $E^L = s.E^K$. $\square$

Corollary E.2.12. *The (Γ, G)-bundle $EG \to BG$ is universal iff EG^{H_t} is contractible for some representative t in each $[t] \in H^1(\Delta, G)$, for all closed subgroups $\Delta \leq \Gamma$.*

Proof. $EG \to BG$ is universal as a (Γ, G)-bundle iff EG is a universal $\mathcal{F}$-space for the family $\mathcal{F}$ of all subgroups $H \leq S$ satisfying $H \cap G = (1)$. But by Remark E.2.10 iii), this family equals $\mathcal{F}(C)$ for

$$C = \bigcup_{\Delta \leq \Gamma} H^1(\Delta, G).$$

$\square$

Corollary E.2.13. *Assume $H^1(\Delta, G) = 0$ for all $\Delta \leq \Gamma$. Then the (Γ, G)-bundle $EG \to BG$ is universal iff EG^Δ is contractible for all closed subgroups $\Delta \leq \Gamma$.*

Remark E.2.14. Corollary E.2.12 is a reformulation of a characterization given by tom Dieck [tD87]: a numerable (Γ, G)-bundle $EG \to BG$ is a universal (Γ, G)-bundle if and only if for every local object $P \to X = \Gamma/\Delta$, $\Delta \leq \Gamma$, the Γ-equivariant bundle

$$E = P \times_G EG \to X$$

is Γ-homotopy equivalent to its base.

Proposition E.2.15. *Let $\Gamma = \mathbb{Z}_2$ act on $G = \mathrm{GL}(n, \mathbb{C})$ by complex conjugation. Then Γ acting on*

$$EG = \mathrm{Inj}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^\infty) \to \mathrm{Gr}_n(\mathbb{C}^\infty) = BG$$

by complex conjugation is a model of the universal (Γ, G)-bundle.

Proof. By Hilbert 90, $H^1(\mathbb{Z}_2, \mathrm{GL}(n, \mathbb{C})) = 0$. Since $EG^\Gamma = \mathrm{Inj}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^\infty)$ is contractible, so by the previous Corollary, $EG \to BG$ is a model of the universal (Γ, G)-bundle. □

Proposition E.2.16. *Let $C = \{e\} \subseteq H^1(\Gamma, G)$ be the trivial class. Then for*

a) $\Gamma = \mathrm{U}(1)$ *acting on* $\mathrm{GL}(2n, \mathbb{R})$ *by the pseudo Galois action,*

$$EG = \mathrm{Inj}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^\infty) \to \mathrm{Gr}_{2n}(\mathbb{R}^\infty) = BG$$

with the left-right conjugation action is a model of $E\mathcal{F}(C)$ and

b) $\Gamma = \mathrm{U}(1)$ *acting on* $\mathrm{GL}(n, \mathbb{H})$ *by the Galois type action,*

$$EG = \mathrm{Inj}_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H}^\infty) \to \mathrm{Gr}_n(\mathbb{H}^\infty) = BG$$

with the left-right conjugation action is a model of $E\mathcal{F}(C)$.

Proof. By Corollary E.2.12, it is enough to show that EG^Δ is contractible for all $\Delta \leq \Gamma$. There are two types of fixed point sets: the whole space EG if $\Delta \leq \mathbb{Z}_2 \leq \mathrm{U}(1)$ and $EG^\Delta = \mathrm{Inj}_{\mathbb{H}}(\mathbb{C}^n, \mathbb{C}^\infty)$ otherwise. □

Remark E.2.17. In fact, in these examples $H^1(\Gamma, G) \neq 0$, so these bundles are not universal as (Γ, G)-bundles.

E.2.4 Approximations

In this section we prove some propositions regarding Γ-approximations used in the proof of the equivariant Borel-Haefliger theorem. First, let us recall the definition of Γ-approximations:

Definition E.2.18. Let $EG \to BG$ have the structure of a (Γ, G)-bundle. We say that $(E_k \to B_k, \iota_k, \mathcal{K}_k)$ is a Γ-*approximation of* $EG \to BG$, if

- $p_k : E_k \to B_k$ are smooth (Γ, G)-bundles,
- the Γ-equivariant classifying maps $\mathcal{K}_k : B_k \to BG$ induce isomorphisms:
$$\pi_j^{\Delta}(\mathcal{K}_k) : \pi_j^{\Delta}(B_k) \xrightarrow{\cong} \pi_j^{\Delta}(BG),$$
for all $j < k$ and $\Delta \leq \Gamma$,
- $\iota_k : B_k \to B_{k+1}$ are Γ-equivariant maps such that $\mathcal{K}_{k+1} = \mathcal{K}_{k+1} \circ \iota_k$.

Proposition E.2.19. *Let $EG \to BG$ have the structure of a (Γ, G)-bundle, and let $E_k \to B_k$ be a Γ-approximation. Let X be an S-space and let $\tilde{\mathcal{K}}_k : X(k) \to B_G X$ be the covering map using the notation of* (3.2). *Then*
$$\tilde{\mathcal{K}}_k^* : H^j_{\Delta}(B_G X) \xrightarrow{\cong} H^j_{\Delta}(X(k))$$
is an isomorphism for all $j < k - 1$, $\Delta \leq \Gamma$.

Proof. We give the proof for $\Delta = \Gamma$, as it is the same. By the long exact sequence of the fibrations $B_k \to B_\Gamma B_k \to B\Gamma$, $BG \to B_\Gamma BG \to B\Gamma$ and naturality,

$$\begin{tikzcd}
\dots \arrow[r] & \pi_{j+1}(B\Gamma) \arrow[r] \arrow[d, "\cong"] & \pi_j(B_k) \arrow[r] \arrow[d, "\cong"] & \pi_j(B_\Gamma B_k) \arrow[r] \arrow[d] & \pi_j(B\Gamma) \arrow[r] \arrow[d, "\cong"] & \pi_{j-1}(B_k) \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & \pi_{j+1}(B\Gamma) \arrow[r] & \pi_j(BG) \arrow[r] & \pi_j(B_\Gamma BG) \arrow[r] & \pi_j(B\Gamma) \arrow[r] & \pi_{j-1}(BG) \arrow[r] & \dots
\end{tikzcd}$$

for $j < k$, so by the five-lemma the middle arrow is also an isomorphism. Applying again the long exact sequence of the fibrations $X \to B_\Gamma X(k) \to B_\Gamma B_k$, $X \to B_\Gamma B_G X \to B_\Gamma BG$ and naturality,

$$\begin{tikzcd}
\dots \arrow[r] & \pi_{j+1}(B_\Gamma B_k) \arrow[r] \arrow[d, "\cong"] & \pi_j(X) \arrow[r] \arrow[d, "\cong"] & \pi_j(B_\Gamma X(k)) \arrow[r] \arrow[d] & \pi_j(B_\Gamma B_k) \arrow[r] \arrow[d, "\cong"] & \pi_{j-1}(X) \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & \pi_{j+1}(B_\Gamma BG) \arrow[r] & \pi_j(X) \arrow[r] & \pi_j(B_\Gamma B_G X) \arrow[r] & \pi_j(B_\Gamma BG) \arrow[r] & \pi_{j-1}(X) \arrow[r] & \dots
\end{tikzcd}$$

so by the five-lemma the middle arrow is an isomorphism if $j < k - 1$, in other words $\tilde{\mathcal{K}}_k$ is $(k - 2)$-connected. The Hurewicz theorem and the universal coefficient theorem allows us to conclude:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{j-1}(B_\Gamma B_G X), \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H^j_\Gamma(B_G X) \arrow[r] \arrow[d] & \mathrm{Hom}(H_j(B_\Gamma B_G X), \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{j-1}(B_\Gamma X(k)), \mathbb{Z}) \arrow[r] & H^j_\Gamma(X(k)) \arrow[r] & \mathrm{Hom}(H_j(B_\Gamma X(k)), \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

□

Proposition E.2.20. *Let $EG \to BG$ have the structure of a (Γ, G)-bundle, such that EG^Γ is contractible and let $E_k \to B_k$ be a Γ-approximation. Then $E_k^\Gamma \to B_k^\Gamma$ is an approximation of the universal G^Γ-bundle $EG^\Gamma \to BG^\Gamma$.*

Proof. By definition, if X is a Γ-space,

$$\pi_j^\Gamma(X) = [S^n, X]_\Gamma = [S^n, X^\Gamma] = \pi_j(X^\Gamma),$$

so by definition of Γ-approximations, $\pi_j(\mathcal{K}_k^\Gamma)$ is an isomorphism $\pi_j(B_k^\Gamma) \xrightarrow{\cong} \pi_j(BG^\Gamma)$ for $j < k$. By Lemma E.1.6, $E_k^\Gamma \to B_k^\Gamma$ are principal G^Γ-bundles, and are therefore an approximation of $EG^\Gamma \to BG^\Gamma$, which is a universal G^Γ-bundle, since EG^Γ is contractible. □

Appendix F

Schubert generators of $H^*(\mathrm{Fl}_{\mathcal{D}}; \mathbb{Q})$

Using the coefficients of the Vassiliev complex described in Section C, we computed some of the Schubert cell generators of $H^*(\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}; \mathbb{Q})$. For the computations we used SageMath's homology package. We illustrate some of the small examples, but to compute larger examples, there are computational limits.

We include below cases which are not covered by Theorem 4.2.2, in particular, complete, odd and other examples. The Schubert cells $\Omega_I \subseteq \mathrm{Fl}_{\mathcal{D}}$ are parametrized by ordered set partitions $I \in \mathrm{OSP}(\mathcal{D})$. As we have already mentioned, in the non-even case it is no longer true that the cohomology classes can be represented by Schubert varieties, but a signed sum of Schubert cells. In all the examples we have computed, the coefficients of these Schubert cells are ± 1. In the tables, we use the following conventions.

We do not keep track of the sign of the cells, as this can vary according to convention (even though relative to each other, the signs do make sense). For ordered set partitions $I \in \mathrm{OSP}(\mathcal{D})$, we use one-line notation. We use the convention that we list elements of I_j in increasing order, and I_j and I_{j+1} is separated by a comma. The $+$ sign separates the Schubert cells whose sums are generators. In the last table, the ordered set partitions are elements of $1, 2, \ldots, 11$; for typographical reasons 10 and 11 are preceded by a space.

The complete case

The two extreme cases of real flag manifolds are Grassmannians and complete flag manifolds. We understand the Schubert calculus of Grassmannians by Propositions 4.2.8 and 4.2.9, see also [He16]. For the case of complete flag manifolds, the answer appears to be less simple, as we illustrate in Tables F.1 and F.3.

For the case of $\mathrm{Fl}(\mathbb{R}^3)$ see also [Koc95, p. 5] and for $\mathrm{Fl}(\mathbb{R}^4)$, see also [CS99, p. 529].

The odd case

There are two cases when flag manifolds $\mathrm{Fl}^{\mathbb{R}}_{\mathcal{D}}$ are orientable (see Corollary B.2.9). If all d_i are even, we understand Schubert calculus by Theorem 4.2.2. If all d_i are odd, the answer again appears to be less simple, see Tables F.2 F.5 (and also the complete cases). Due to the computational limitations, we only have a limited number of examples.

The other cases

See Table F.4 for a nonorientable case.

These examples hopefully illustrate that although there is a simple description of the cohomology of real flag manifolds in terms of characteristic classes (cf. Cartan model, Section C.2), in general there is some nontrivial combinatorics involved in translating that description to the Schubert calculus setting.

deg	$\mathrm{Fl}(\mathbb{R}^3)$	$\mathrm{Fl}(\mathbb{R}^4)$	$\mathrm{Fl}(\mathbb{R}^5)$	$\mathrm{Fl}(\mathbb{R}^6)$
	$\Lambda[x_3]$	$\Lambda[x_3, y_3]$	$\Lambda[x_3, x_7]$	$\Lambda[x_3, x_7, y_5]$
0	321	4321	54321	654321
3	123	2341+4123	34521+52341+54123	456321+634521+652341+654123
3		3214		
5				365214
6		1234		
7			14325	432561+632145
8				345216+523416+541236
10			1234	234561+236145+412563+612345
12				125436
15				123456

Table F.1: Sums of Schubert cells generating $H^*(\mathrm{Fl}(\mathbb{R}^n); \mathbb{Q})$, labeled by permutations S_n

deg	Fl_{333}
	$H^*(\mathrm{Fl}_{222}; \mathbb{Q}) \otimes \Lambda[x_{15}]$
0	123,456,789
4	789,236,145
4	569,478,123
8	349,678,125+369,458,127+389,256,147+589,234,167
8	569,238,147+589,234,167
12	349,258,167
15	167,258,349
19	167,234,589
19	145,278,369+147,256,389
23	123,478,569
23	145,236,789
27	789,456,123

Table F.2: Sums of Schubert cells generating $H^*(\mathrm{Fl}_{333}; \mathbb{Q})$, labeled by $\mathrm{OSP}(3, 3, 3)$

deg	$\mathrm{Fl}(\mathbb{R}^7)$
	$\Lambda[x_3, x_7, x_{11}]$
0	7654321
3	5674321+7456321+7634521+7652341+7654123
7	5436721+7432561+7632145
10	3456721+3472561+3672145+5236741+5416723+5436127+5632147+7234561+7236145+7412563+7612345
11	1476325
14	1456327+3416527+5216347+5412367
18	1236547
21	1234567

Table F.3: Sums of Schubert cells generating $H^*(\mathrm{Fl}(\mathbb{R}^7);\mathbb{Q})$, labeled by permutations S_7

deg	Fl_{234}
	$H^*(\mathrm{Fl}_{224};\mathbb{Q})$
0	89,567,1234
4	89,347,1256
4	67,589,1234
8	89,127,3456
8	45,789,1236
8	47,569,1238+67,349,1258
12	45,369,1278
12	23,789,1456
12	27,369,1458+67,129,3458
16	23,569,1478
16	25,349,1678+45,129,3678
20	23,149,5678

Table F.4: Sums of Schubert cells generating $H^*(\mathrm{Fl}_{234};\mathbb{Q})$, labeled by $\mathrm{OSP}(2,3,4)$

deg	Fl_{335}
	$H^*(\mathrm{Fl}_{224};\mathbb{Q})\otimes\Lambda[x_{19}]$
0	9 10 11,678,12345
4	9 10 11,458,12367
4	78 11, 69 10,12345
8	9 10 11, 238, 14567
8	56 11,89 10,12347
8	58 11,67 10,12349+78 11,45 10,12369
12	56 11,47 10,12389
12	34 11,89 10,12567
12	38 11,47 10,12569+78 11,23 10,14569
16	34 11,67 10,12589
16	36 11,45 10,12789+56 11,23 10,14789
19	189,27 10,3456 11
20	34 11,25 10,16789
23	127,89 10,3456 11+167,29 10,3458 11
23	169,25 10,3478 11+189,256,347 10 11
27	167,258,349 10 11
27	125,69 10,3478 11+145,29 10,3678 11
27	149,23 10,5678 11+189,234,567 10 11
31	125,678,349 10 11+145,278,369 10 11
31	123,49 10,5678 11
31	147,238,569 10 11+167,234,589 10 11
35	145,236,789 10 11
35	123,478,569 10 11
39	123,456,789 10 11

Table F.5: Sums of Schubert cells generating $H^*(\mathrm{Fl}_{335};\mathbb{Q})$, labeled by $\mathrm{OSP}(3,3,5)$

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Glossary

- **Γ-manifold**: a smooth manifold X with a smooth Γ-action.
- **Γ-space**: a topological space X with a continuous Γ-action.
- (Γ, G)**-bundle**: If Γ acts on G by automorphisms, and $S = G \rtimes \Gamma$, then a (Γ, G)-bundle is an S-space P such that $P \to P/G$ is a principal G-bundle, p. 159.
- **Bruhat cell**: an N-orbit in the R-space X, see p. 122 and 123.
- **Circle space** X is a halving space for $(\Gamma, R) = (U(1), \mathbb{Z})$ or $(\mathrm{U}(1), \mathbb{Q})$, p. 15.
- **Cohomology frame** (κ, σ) of a Γ-space X: $\kappa : H^{2*}(X) \to H^*(X^\Gamma)$ degree-halving additive isomorphism, $\sigma : H^*(X) \to H^*_\Gamma(X)$ a Leray-Hirsch section satisfying the degree condition (DC), p. 14, 15.
- **Compatible action**: If Γ acts on G by automorphisms, then Γ, G *act on* X *compatibly* if $\gamma.(g.x) = (\gamma.g).(\gamma.x)$ for all $\gamma \in \Gamma, g \in G, x \in X$, p. 159.
- **Conjugation space**: X is a halving space for $(\Gamma, R) = (\mathbb{Z}_2, \mathbb{F}_2)$, p. 15.
- **Doubled ordered set partition**: $DI \in \mathrm{OSP}(2\mathcal{D})$ which is obtained from $I \in \mathrm{OSP}(\mathcal{D})$ by replacing $i \in I_j$ by $(2i-1, 2i) \in DI_j$, p. 47.
- **Double Schubert variety**: a Schubert variety $\sigma^{\mathbb{R}}_{DI}$, where $DI \in \mathrm{OSP}(2\mathcal{D})$ is a doubled set partition, p. 47.
- **Even real flag manifold**: a real flag manifold $\mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$ where all the dimensions are even, p. 44.
- **Excess weight**: If Z is a Γ-invariant subvariety of X, its excess weight at a regular fixed point $z \in Z^\Gamma_R$ is the product of the weights of the excess bundle of $Z \cap X^\Gamma$ at z, p. 9.

- **Flag manifold** $\mathrm{Fl}_{\mathcal{D}}(\mathbb{F}^N)$: manifold of flags $F_\bullet = (V_1 \leq \ldots \leq V_r = \mathbb{F}^N)$, such that $\dim_{\mathbb{F}} V_i = s_i$, $d_i = s_i - s_{i-1}$, $\mathcal{D} = (d_1, \ldots, d_r)$, $\sum d_i = N$, p. 123.
- **Galois type action**: A Γ-action on $\mathrm{Fl}_{\mathcal{D}}(\mathbb{F}_i^N)$ whose fixed point set is $\mathrm{Fl}_{\mathcal{D}}(\mathbb{F}_{i-1}^N)$, p. 42.
- **pseudo Galois type action**: A Γ-action on $\mathrm{Fl}_{2\mathcal{D}}(\mathbb{R}^{2N})$ whose fixed point set is $\mathrm{Fl}_{\mathcal{D}}(\mathbb{C}^N)$, p. 43.
- **Halving pair**: A pair (Γ, R) consisting of a topological group Γ and a ring R such that $H^*(B\Gamma; R) \cong R[u]$, $u \in H_\Gamma^D$, p. 13.
- **Halving space** X: a Γ-space with a cohomology frame (κ, σ) where (Γ, R) is a halving pair, p. 14.
- **Leray-Hirsch section** $\sigma : H^*(F) \to H^*(E)$: If $F \to E \to X$ is a fiber bundle, a section of the restriction $\rho : H^*(E) \to H^*(F)$, p. 2.
- **Normal disk at** p: For a Whitney stratified submanifold $Z \subseteq X$, a disk $p \in D \subseteq X$ intersecting all strata transversely, p. 105.
- **Normal slice at** p: For a Whitney stratified submanifold $Z \subseteq X$, the intersection of Z with a normal disk D, p. 105.
- **Ordered set partition**: $I \in \mathrm{OSP}(\mathcal{D}) = S_N / S_{d_1} \times \ldots \times S_{d_r}$, where $\mathcal{D} = (d_1, \ldots, d_r)$ and $N = \sum d_i$. p. 47.
- **Quaternionic halving space** X is a halving space for $(\Gamma, R) = (\mathrm{Sp}(1), \mathbb{Q})$, p. 15.
- **Schubert cell** Ω_I: a cell in the flag manifold $X = \mathrm{Fl}_{\mathcal{D}}(\mathbb{F}^n)$ defined by rank conditions, p. 124.
- **Schubert class** $[\sigma_\lambda]$: cohomology class represented by a Schubert cycle 124.
- **Schubert cycle**: a Schubert variety which is a cycle/represents a cohomology class, p. 124.
- **Schubert problem**: "Given fixed transversal Schubert varieties σ_j of complementary dimension what is the cardinality $\left|\bigcap_{j=1}^r \sigma_j\right|$ =?", p. 60
- **Schubert variety** σ_I: closure of a Schubert cell, p. 124
- $\mathcal{D} \in \mathbb{N}^r$, $\mathcal{D} = (d_1, \ldots, d_r)$ is *even* if all d_i are even.

Notation

- $\deg_\Gamma^{<d}$: For a trivial Γ-space X, $H^*_\Gamma(X) \cong H^*(X) \otimes H^*_\Gamma$; $\deg_\Gamma^{<d}$ denotes a sum of elements whose H^*_Γ-degree is less than d p. 8.
- $\deg_u(x)$: For a trivial $\Gamma = \mathrm{U}(1)$-space X, any element $x \in H^*_\Gamma(X) \cong H^*(X)[u]$, can be written as a polynomial in $u \in H^2_\Gamma$. Then $\deg_u(x)$ denotes the u-degree of this element, p. 15.
- X^Γ: the Γ-fixed point set of a Γ-space X.
- $\nu(Z \hookrightarrow X)$: the normal bundle of a smooth submanifold $Z \hookrightarrow X$.
- $\eta = \eta(Y, Z)$: the excess bundle of a clean intersection $Y \cap Z$, p. 5.
- $l.d.t.$: lower degree terms.
- $DF_\bullet$: an even real flag $DF_\bullet \in \mathrm{Fl}^{\mathbb{R}}_{2\mathcal{D}}$, where $\mathcal{D} = (1, 1, \ldots, 1)$.
- $\mathrm{Inj}_{\mathbb{F}}(\mathbb{F}^k, \mathbb{F}^n)$: injective $\mathbb{F}$-linear maps from $\mathbb{F}^k$ to $\mathbb{F}^n$.
- $\mathrm{Fl}(\mathbb{F}^n)$: the complete flag manifold in $\mathbb{F}^n$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.
- $\mathrm{Fl}^{\mathbb{F}}_{\mathcal{D}}$ or $\mathrm{Fl}_{\mathcal{D}}(\mathbb{F}^n)$: the manifold of partial flags, where $\mathcal{D} = (d_1, \ldots, d_r) \in \mathbb{N}^r$ denotes the differences in the dimensions, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.
- $\mathrm{Fl}(\mathbb{O})$: the complete octonionic flag manifold in $\mathbb{O}^3$.
- $\mathrm{OSP}(\mathcal{D})$: the set of ordered set partitions $S_N / S_{d_1} \times \ldots \times S_{d_r}$, p. 47.
- $W_x(E)$ is the multiset of weights of the Γ-representation E_x for a Γ-equivariant bundle $E \to X$ over a Γ-fixed point x, 8.
- Z_R, Z_S: the regular/singular points of a topological subvariety $Z \subseteq X$.

Index