On some problems in Extremal Combinatorics

by

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Contents

Abstract

The thesis is divided into 3 parts, namely extremal poset theory, extremal graph theory and extremal hypergraph theory.

Part I: In the first part of the thesis we will study problems in extremal poset theory, where the main question is the following: What is the maximum size, $La(n, P)$, of a family of subsets of $[n] = \{1, 2, \ldots, n\}$ not containing a given poset P as a subposet. The induced version of the problem asks for the maximum size, $La^{\#}(n, P)$, of a family of subsets of $[n]$ not containing P as an induced subposet. These problems are a natural generalization of the well-known Sperner's theorem. In 1945, Erdős obtained the exact value of $La(n, P)$ when P is a *path poset*, generalizing Sperner's theorem. A more formal study of this problem was initiated by Katona and Tarján in 1983. Since then there have been numerous papers in this area, one of which is a result of Bukh which determines the asymptotic value of $La(n, T)$ for every tree poset T.

One of the major open problems in this area is determining $\mathrm{La}(n, \mathcal{Q}_2)$ where $\mathcal{Q}_2$ is the poset on 4 elements with the relations $a < b, c < d$ where b and c are incomparable, called the diamond poset. It is conjectured that $\mathrm{La}(n, \mathcal{Q}_2) = (2 + o(1))\binom{n}{\lfloor n/2 \rfloor}$, and infinitely many significantly different, asymptotically tight constructions are known. In Chapter 3, we use a partitioning of the maximal chains and introduce an induction method to show that $\mathrm{La}(n, \mathcal{Q}_2) \le (2.20711 + o(1))\binom{n}{\lfloor n/2 \rfloor}$, improving on the earlier bound of Kramer, Martin and Young. The results in this chapter are based on the paper "An upper bound on the size of diamond-free families of sets" co-authored with Grósz and Tompkins, published in *Journal of Combinatorial Theory, (Series A)*, 156 (2018), 164–194.

Much less is known about $La^{\#}(n, P)$. Katona, and Lu and Milans conjectured that for every poset P, there is a constant C_P such that the size of any family of subsets of $\{1, 2, \ldots, n\}$ that does not contain P as an induced subposet is at most $C_P\binom{n}{\lfloor \frac{n}{2} \rfloor}$. In Chapter 2, we prove this conjecture by establishing a connection to the theory of forbidden submatrices and then applying a higher dimensional variant of the Marcus-Tardos theorem, proved by Klazar and Marcus. We also give a new proof of their result. These results are based on the paper "Forbidden hypermatrices imply general bounds on induced forbidden subposet problems" co-authored with Pálvölgyi, published in *Combinatorics, Probability and Computing* 26.4 (2017), 593–602.

Part II: In the second part of the thesis, we study some problems in extremal graph theory.

In Chapter 6, we study problems concerning subgraphs of C_{2k}-free graphs. Győri proved that every bipartite, C_6-free graph contains a C_4-free subgraph with at least half as many edges. Later Kühn and Osthus showed that every bipartite C_{2k}-free graph G contains a C_4-free subgraph with at least $1/(k-1)$ fraction of the edges of G. We give a new and very short proof of this result. Moreover, we answer a question of Kühn and Osthus about C_{2k}-free graphs obtained by pasting together $C_{2\ell}$'s (with $k > \ell \ge 3$). Let c denote the largest constant such that every C_6-free graph G contains a bipartite and C_4-free subgraph having c fraction of edges of G. Győri *et al.* showed that $\frac{3}{8} \le c \le \frac{2}{5}$. We prove that $c = \frac{3}{8}$. Our proof uses the following statement, which we prove using probabilistic ideas, generalizing a theorem of Erdős: For any $\varepsilon > 0$, and any integers a,

b, $k \geq 2$, there exists an a-uniform hypergraph H of girth greater than k which does not contain any b-colorable subhypergraph with more than $\left(1 - \frac{1}{b^{a-1}}\right)(1+\varepsilon)$ fraction of the hyperedges of H. We prove further generalizations of this theorem. The results in this chapter are based on the paper "On subgraphs of C_{2k}-free graphs and a problem of Kühn and Osthus" co-authored with Grósz and Tompkins [69].

In Chapter 7, we study the problem of simultaneously forbidding an induced copy of a graph and a (not necessarily induced) copy of another graph, which was introduced by Loh, Tait, Timmons and Zhou. This question is related to the well-studied areas of Ramsey–Turán Theory and the Erdős–Hajnal Conjecture. Our main result of Chapter 7 is the following: Let $k \geq 2$ be an integer. If $s = 2$ and $t \geq 2$, or $s = t = 3$, then the maximum possible number of edges in a C_{2k+1}-free graph containing no induced copy of $K_{s,t}$ is asymptotically equal to $(t-s+1)^{1/s}(n/2)^{2-1/s}$ except when $k = s = t = 2$. This strengthens a result of Allen, Keevash, Sudakov and Verstraëte [1], and answers a question of Loh, Tait, Timmons and Zhou [110]. The results of this chapter are based on the paper "Turán number of an induced complete bipartite graph plus an odd cycle" co-authored with Ergemlidze and Győri, published in *Combinatorics, Probability and Computing* (2018): 1-12.

In Chapter 8, we study problems concerning Turán numbers of ordered graphs: An *ordered graph* is a simple graph with a linear ordering on its vertex set. An ordered graph H is an *ordered subgraph* of G if there is an embedding of H in G that respects the ordering of the vertices. The Turán problem for a set of ordered graphs $\mathcal{H}$ asks for the maximum number $\mathrm{ex}_<(n, \mathcal{H})$ of edges that an ordered graph on n vertices can have without containing any $H \in \mathcal{H}$ as an ordered subgraph. A classical result of Bondy and Simonovits in extremal graph theory states that if a graph on n vertices contains no cycle of length $2k$ then it has at most $O(n^{1+1/k})$ edges. However, matching lower bounds are only known for $k = 2, 3, 5$. In this chapter we study ordered variants of this problem and prove some tight estimates for a certain class of ordered cycles that we call *bordered cycles*. In particular, we show that the maximum number of edges in an ordered graph avoiding bordered cycles of length at most $2k$ is $\Theta(n^{1+1/k})$. Strengthening the result of Bondy and Simonovits in the case of 6-cycles, we also show that it is enough to forbid these bordered orderings of the 6-cycle to guarantee an upper bound of $O(n^{4/3})$ on the number of edges. The results of this chapter are based on the paper "On the Turán number of some ordered even cycles" co-authored with Győri, Korándi, Tomon, Tompkins and Vizer, published in *European Journal of Combinatorics* 73 (2018): 81–88.

In Chapter 9, we study Generalized Turán problems where the main question is the following: Given a graph H and a set of graphs $\mathcal{F}$, what is the maximum possible number $\mathrm{ex}(n, H, \mathcal{F})$ of copies of H in an $\mathcal{F}$-free graph on n vertices? In this chapter, we investigate the function $\mathrm{ex}(n, H, \mathcal{F})$, when H and members of $\mathcal{F}$ are cycles. Let C_k denote the cycle of length k and let $\mathcal{C}_k = \{C_3, C_4, \dots, C_k\}$. We highlight some of the main results below.

(i) We show that $\mathrm{ex}(n, C_{2\ell}, C_{2k}) = \Theta(n^\ell)$ for any $\ell, k \geq 2$. Moreover, in some cases we determine it asymptotically: We show that $\mathrm{ex}(n, C_4, C_{2k}) = (1+o(1))\frac{(k-1)(k-2)}{4}n^2$.

(ii) Erdős's Girth Conjecture states that for any positive integer k, there exist a constant $c > 0$ depending only on k, and a family of graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq cn^{1+1/k}$ with girth more than $2k$.

Solymosi and Wong [138] proved that if this conjecture holds, then for any $\ell \geq 3$ we have $\mathrm{ex}(n, C_{2\ell}, \mathcal{C}_{2\ell-1}) = \Theta(n^{2\ell/(\ell-1)})$. We prove that their result is sharp in the sense that forbidding any other even cycle decreases the number of $C_{2\ell}$'s significantly: For any $k > \ell$, we have $\mathrm{ex}(n, C_{2\ell}, \mathcal{C}_{2\ell-1} \cup \{C_{2k}\}) = \Theta(n^2)$. More generally, we show that for any $k > \ell$ and $m \geq 2$ such that $2k \neq m\ell$, we have $\mathrm{ex}(n, C_{m\ell}, \mathcal{C}_{2\ell-1} \cup \{C_{2k}\}) = \Theta(n^m)$.

(iii) We prove $\mathrm{ex}(n, C_{2\ell+1}, \mathcal{C}_{2\ell}) = \Theta(n^{2+1/\ell})$, provided a stronger version of Erdős's Girth Conjecture holds (which is known to be true when $\ell = 2, 3, 5$). This result is also sharp in the sense that forbidding one more cycle decreases the number of $C_{2\ell+1}$'s significantly: More precisely, we have $\mathrm{ex}(n, C_{2\ell+1}, \mathcal{C}_{2\ell} \cup \{C_{2k}\}) = O(n^{2-\frac{1}{\ell+1}})$, and $\mathrm{ex}(n, C_{2\ell+1}, \mathcal{C}_{2\ell} \cup \{C_{2k+1}\}) = O(n^2)$ for $\ell > k \geq 2$.

These results are based on the paper "Generalized Turán problems for even cycles" co-authored with Gerbner, Győri and Vizer.

Part III: In the third part of the thesis we study some problems in Extremal Hypergraph theory, in particular, Berge hypergraphs.

In Chapter 11, we prove an Erdős-Gallai type theorem for Berge paths and settle a conjecture of Győri, Katona and Lemons: A Berge path of length ℓ in a hypergraph is a set of $\ell + 1$ distinct vertices $v_1, \ldots, v_{\ell+1}$ and ℓ distinct hyperedges $e_1, \ldots, e_\ell$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq \ell$. A well-known theorem of Erdős and Gallai asserts that a graph with no path of length k contains at most $\frac{1}{2}(k-1)n$ edges. Recently, Győri, Katona and Lemons [77] gave an extension of this result to hypergraphs by determining the maximum number of hyperedges in an r-uniform hypergraph containing no Berge path of length k for all values of r and k except for $k = r + 1$. We settle the remaining case by proving that an r-uniform hypergraph with more than n hyperedges must contain a Berge path of length $r + 1$. This result is based on the paper "An Erdős-Gallai type theorem for uniform hypergraphs" which is co-authored with Davoodi, Győri and Tompkins, published in *European Journal of Combinatorics* 69 (2018), 159–162.

A natural generalization of the definitions of Berge cycles and Berge paths to arbitrary Berge graphs is the following: For a graph F, a hypergraph $\mathcal{H}$ is a Berge copy of F (or a Berge-F in short), if there is a bijection $f : E(F) \to E(\mathcal{H})$ such that for each $e \in E(F)$ we have $e \subset f(e)$. A hypergraph is Berge-F-free if it does not contain a Berge copy of F. The *Turán number* of Berge-F, denoted $\mathrm{ex}_r(n, F)$, is the maximum number of hyperedges in an r-uniform hypergraph on n vertices which does not contain a Berge-F. Similarly, for a given family of graphs, $\mathcal{F}$ let $\mathrm{ex}_r(n, \mathcal{F})$ denote the maximum possible number of edges in an r-uniform hypergraph on n vertices containing no Berge-F as a subhypergraph (for every $F \in \mathcal{F}$).

In Chapter 12, we study the general behavior of $\mathrm{ex}_r(n, F)$ for a fixed F as r grows: For small enough r and non-bipartite F, $\mathrm{ex}_r(n, F) = \Omega(n^2)$; we show that for sufficiently large r, $\mathrm{ex}_r(n, F) = o(n^2)$. Let $\mathrm{th}(F) = \min\{r_0 : \mathrm{ex}_r(n, F) = o(n^2) \text{ for all } r \geq r_0\}$. We show lower and upper bounds for $\mathrm{th}(F)$, the uniformity threshold of F. In particular, we obtain that $\mathrm{th}(\triangle) = 5$, improving a result of Győri [76]. We also study the analogous problem for linear hypergraphs. Let $\mathrm{ex}_r^L(n, F)$ denote the maximum number of hyperedges in an r-uniform linear hypergraph on n vertices which does not contain a Berge-F, and let the linear unformity threshold $\mathrm{th}^L(F) = \min\{r_0 : \mathrm{ex}_r^L(n, F) = o(n^2) \text{ for all } r \geq r_0\}$. We show that $\mathrm{th}^L(F)$ is equal to the chromatic number of F. These results are based on the paper "Uniformity thresholds for the asymptotic size of extremal Berge-F-free hypergraphs" co-authored with Grósz and Tompkins.

In Chapter 13, we determine the asymptotics for the Turán number of Berge-$K_{2,t}$ by showing

$$\mathrm{ex}_3(n, K_{2,t}) = (1 + o(1))\frac{1}{6}(t-1)^{3/2} \cdot n^{3/2}$$

for any given $t \geq 7$. We study the analogous question for linear hypergraphs and show

$$\mathrm{ex}_3(n, \{C_2, K_{2,t}\}) = (1 + o_t(1))\frac{1}{6}\sqrt{t-1} \cdot n^{3/2}.$$

In fact, we prove a general theorem that provides bounds on the Turán numbers of a class of graphs including $\mathrm{ex}_r(n, K_{2,t})$, $\mathrm{ex}_r(n, \{C_2, K_{2,t}\})$, and $\mathrm{ex}_r(n, C_{2k})$ for $r \geq 3$. Our bounds improve the results of Gerbner and Palmer, Füredi and Özkahya, Timmons, and provide a new proof of a result of Jiang and Ma. These results are based on the paper "Asymptotics for the Turán number of Berge-$K_{2,t}$" co-authored with Gerbner and Vizer.

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Part I

Extremal poset theory

Chapter 1

Background on Forbidden subposet problems

Let $[n] = \{1, 2, \ldots, n\}$. The Boolean lattice $2^{[n]}$ is defined as the family of all subsets of $[n] = \{1, 2, \ldots, n\}$, and the *ith level* of $2^{[n]}$ refers to the collection of all sets of size i.

In 1928, Sperner [139] proved if $\mathcal{F}$ is a family of subsets of $[n]$ such that no set contains another ($A, B \in \mathcal{F}$ implies $A \not\subset B$), then $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$. Moreover, equality occurs if and only if $\mathcal{F}$ is a level of maximum size in $2^{[n]}$.

Definition 1.1. *Let P be a finite poset, and $\mathcal{F}$ be a family of subsets of $[n]$. We say that P is contained in $\mathcal{F}$ as a* weak *subposet if and only if there is an injection $\alpha : P \to \mathcal{F}$ satisfying $x_1 <_p x_2 \Rightarrow \alpha(x_1) \subset \alpha(x_2)$ for all $x_1, x_2 \in P$. $\mathcal{F}$ is called P-free if P is not contained in $\mathcal{F}$ as a weak subposet. We define the corresponding extremal function as*

$$La(n, P) := \max\{|\mathcal{F}| \;\mid\; \mathcal{F} \text{ is } P\text{-free}\}.$$

We say that P is an induced subposet *of Q if there exists an injection $\alpha : P \to \mathcal{F}$ satisfying $x_1 <_p x_2 \iff \alpha(x_1) \subset \alpha(x_2)$ for all $x_1, x_2 \in P$. $\mathcal{F}$ is called induced P-free if P is not contained in $\mathcal{F}$ as an induced subposet. We define the corresponding extremal function as*

$$La^{\#}(n, P) := \max\{|\mathcal{F}| \;\mid\; \mathcal{F} \text{ is induced } P\text{-free}\}.$$

If we wish to forbid a pair of posets P and Q, we simply write $\mathrm{La}(n, P, Q)$ and $\mathrm{La}^{\#}(n, P, Q)$ respectively. We denote the number of elements of a poset P by $|P|$. The linearly ordered poset on k elements, $a_1 < a_2 < \ldots < a_k$, is called a chain of length k, and is denoted by P_k. Using our notation Sperner's theorem can be stated as follows.

Theorem 1.2 (Sperner [139])**.**

$$La(n, P_2) = \binom{n}{\lfloor n/2 \rfloor}.$$

Proof. Let $\mathcal{A}$ be a P_2-free family. Consider a maximal chain,

$$\mathcal{C} = \{\varnothing, \{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \ldots, [n]\}$$

formed by adding sequentially the elements $x_1, x_2, \ldots, x_n$ in this order. Let us double count the number of pairs $(A, \mathcal{C})$ where $A \in \mathcal{A}$ and $A \in \mathcal{C}$. For a fixed $A \in \mathcal{A}$, the number of maximal chains containing A is $|A|!\,(n - |A|)!$. So,

$$|\{(A, \mathcal{C}) \mid A \in \mathcal{A} \text{ and } A \in \mathcal{C}\}| = \sum_{A \in \mathcal{A}} |A|!\,(n - |A|)! \tag{1.1}$$

On the other hand, a fixed maximal chain $\mathcal{C}$ has at most 1 set from $\mathcal{A}$ for otherwise we will have P_2 as a subposet of $\mathcal{A}$, contradiction. Clearly there are $n!$ maximal chains. So,

$$|\{(A, \mathcal{C}) \mid A \in \mathcal{A} \text{ and } A \in \mathcal{C}\}| \leq n! \tag{1.2}$$

Combining (1.1) and (1.2) proves our theorem. □

Erdős extended Sperner's theorem to P_k-free families for all $k \geq 2$.

Theorem 1.3 (Erdős [33]). *$La(n, P_k)$ is equal to the sum of the $k-1$ largest binomial coefficients of order n. This implies*

$$La(n, P_k) \leq (k-1)\binom{n}{\lfloor n/2 \rfloor}.$$

Proof. The same proof of Theorem 1.2 applies. The only difference is that the right-hand side of (1.2) would now be $(k-1)n!$ because a fixed maximal chain can contain at most $(k-1)$ sets from a P_k-free family. □

Notice that, since any poset P is a weak subposet of a chain of length $|P|$, Theorem 1.3 implies

Corollary 1.4.

$$La(n, P) \leq (|P|-1)\binom{n}{\lfloor n/2 \rfloor} = O\left(\binom{n}{\lfloor n/2 \rfloor}\right).$$

For a variety of posets, P, the value of $\text{La}(n, P)$ has been determined asymptotically. The first forbidden poset result was due to Katona and Tarján [95] in 1983. They considered the V poset defined on $\{x, y, z\}$ with relations $x \leq y, z$. They proved

$$\left(1 + \frac{1}{n} + o\left(\frac{1}{n}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \text{La}(n, V) \leq \left(1 + \frac{2}{n}\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

This result was later generalized by De Bonis and Katona [26] who obtained bounds for the r-fork poset, V_r defined by the relations $x \leq y_1, y_2, \ldots, y_r$. Other posets for which the asymptotic value of $La(n, P)$ has been determined include complete two level posets, batons [142], crowns O_{2k} (cycle of length $2k$ on two levels, asymptotically solved except for $k \in \{3, 5\}$ [68, 113]), butterfly [27], skew-butterfly [124], the N poset [65], harp posets $\mathcal{H}(l_1, l_2, \ldots, l_k)$, defined by k chains of length l_i between two fixed elements [67], and recently the complete 3 level poset $K_{r,s,t}$ [133] among others.

Fewer exact results are known. Already, in their paper introducing the La function, Katona and Tarjan [95] proved that $\text{La}(n, V, \Lambda) = \text{La}^{\#}(n, V, \Lambda) = 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$, where V and Λ are the 2-fork and 2-brush, respectively. Define the butterfly poset, B, by 4 elements a, b, c, d with $a, b \leq c, d$. De Bonis, Katona and Swanepoel [27] showed that $\text{La}(n, B) = \Sigma(n, 2)$. Griggs, Li and Lu [67] determined exact results for the k-diamond, D_k ($w \leq x_1, x_2, \ldots, x_k \leq z$), for an infinite set of values of k. They also obtained exact results for harp posets, $\mathcal{H}(l_1, l_2, \ldots, l_k)$ in the case when the l_i are all distinct. See [66] for an excellent survey by Griggs and Li about all the posets that have been studied so far.

One of the first general results is due to Bukh who determined the asymptotic value of $La(n, P)$ for all posets whose Hasse diagram is a tree. Let $h(P)$ denote the height (maximum length of a chain) of P.

Theorem 1.5 (Bukh [20]). *If T is a finite poset whose Hasse diagram is a tree of height $h(T) \geq 2$, then*

$$La(n,T) = (h(T)-1)\binom{n}{\lfloor n/2 \rfloor}\left(1 + O\left(\frac{1}{n}\right)\right). \tag{1.3}$$

For general posets, it is natural to conjecture[1] that $\lim \frac{La(n,P)}{\binom{n}{\lfloor n/2 \rfloor}}$ exists, and equals the maximum number of complete and consecutive middle levels of the Boolean lattice whose union is P-free. Most notoriously, this conjecture is open for the *diamond* $\mathcal{Q}_2$, the poset on 4 elements with the relations $a < b, c < d$ where b and c are incomparable, for which the best known bound is $(2.20711 + o(1))\binom{n}{\lfloor n/2 \rfloor}$ due to Grósz, Methuku and Tompkins [71]. One of the central questions in this area is to determine $La(n, \mathcal{Q}_2)$. This is the topic of the third chapter of this thesis.

Using a general structure called *double chain* instead of chains for double counting, Burcsi and Nagy obtained a similar but weaker version of this theorem for general posets thereby improving Corollary 1.4. Before we state their theorem we introduce the notion of a double chain.

Definition 1.6 (Double chain). *Let $\emptyset = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = [n]$ be a maximal chain (so $|A_i| = i$). The double chain associated to this chain is given by*

$$\mathcal{D} = \{A_0, A_1, \ldots, A_n, M_1, M_2, \ldots, M_{n-1}\},$$

where $M_i = A_{i-1} \cup \{A_{i+1} \setminus A_i\}$.

Theorem 1.7 (Burcsi, Nagy [21]). *For any poset P, when n is sufficiently large, we have*

$$\mathrm{La}(n,P) \leq \left(\frac{|P| + h(P)}{2} - 1\right)\binom{n}{\lfloor n/2 \rfloor}. \tag{1.4}$$

This result was improved by Chen and Li [23]. The idea of their proof was to generalize the double chain to a more complicated structure.

Theorem 1.8 (Chen, Li [23]). *For any poset P, when n is sufficiently large, the inequality*

$$La(n,P) \leq \frac{1}{m+1}\left(|P| + \frac{1}{2}(m^2+3m-2)(h(P)-1) - 1\right)\binom{n}{\lfloor n/2 \rfloor} \tag{1.5}$$

holds for any fixed $m \geq 1$.

Putting $m = \left\lceil \sqrt{\frac{|P|}{h(P)}} \right\rceil$ in the above formula, they obtained

$$\mathrm{La}(n,P) = \mathcal{O}(|P|^{1/2}\, h(P)^{1/2})\binom{n}{\lfloor n/2 \rfloor}. \tag{1.6}$$

Grósz, Methuku and Tompkins improved Theorem 1.8, by showing that

Theorem 1.9 (Grósz, M. and Tompkins [70]). *For any poset P, when n is sufficiently large, the inequality*

$$\mathrm{La}(n,P) \leq \frac{1}{2^{k-1}}\left(|P| + (3k-5)2^{k-2}(h(P)-1) - 1\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

holds for any fixed $k \geq 2$.

[1] This conjecture motivated much of the early work of Katona and his co-authors, though it has not been explicitly stated. The first appearance seems to be in [20], and a couple of months later in [68].

The main idea behind the above theorem is to use a more general chain structure called a *k-interval chain*, which is a family $\mathcal{H} \subset 2^{[n]}$ of sets defined as follows: Define the interval $[A, B]$ to be the set $\{C : A \subseteq C \subseteq B\}$. Fix a maximal chain $\mathcal{C} = \{A_0 = \emptyset, A_1, \dots, A_{n-1}, A_n = [n]\}$ where $A_i \subset A_{i+1}$ for $0 \le i \le n-1$. From $\mathcal{C}$ we define the k-interval chain $\mathcal{C}_k$ as $\mathcal{C}_k := \bigcup_{i=0}^{n-k}[A_i, A_{i+k}]$.

Notice that putting $k = 2$ in the above theorem, we get Theorem 1.7 and Theorem 1.8 for $m = 1$. Putting $k = 3$, we get Theorem 1.8 for $m = 3$. For $k > 3$, our result strictly improves Theorem 1.8.

By choosing k appropriately in our theorem, we obtain the following improvement of (1.6):

Corollary 1.10 (Grósz, M. and Tompkins [70]). *For every poset P and sufficiently large n,*

$$\mathrm{La}(n, P) = \mathcal{O}\left(h(P)\log_2\left(\frac{|P|}{h(P)} + 2\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The following proposition shows that this bound cannot be improved for general P.

Proposition 1.11 (Grósz, M. and Tompkins [70]). *For $P = K_{a,a,\dots,a}$, we have*

$$\mathrm{La}(n, P) \ge ((h(P) - 2)\log_2 a)\binom{n}{\lfloor \frac{n}{2} \rfloor} = \left((h(P) - 2)\log_2\left(\frac{|P|}{h(P)}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Despite considerable progress made on forbidden weak subposets, little is known about forbidden induced subposets (except for P_k, where the weak and induced containment are equivalent). The first result of this type is due to Carroll and Katona [22] who showed $La^{\#}(n, V_2) = \binom{n}{\lfloor n/2 \rfloor}(1 + o(1))$. Later Katona [93] showed that $La^{\#}(n, V_{r+1}) = \binom{n}{\lfloor n/2 \rfloor}(1 + o(1))$ for any $r \ge 1$. Boehnlein and Jiang [12] generalized this by extending Bukh's result to induced containment, proving $La^{\#}(n, T) = (h(T) - 1)\binom{n}{\lfloor n/2 \rfloor}(1 + o(1))$. Recently, Patkós [133] determined the asymptotic behavior of $La^{\#}(n, P)$ for all complete two level posets and some complete multilevel posets.

However, no nontrivial general upper bound was known for $La^{\#}(n, P)$. A few years ago Katona [94], and recently, Lu and Milans [114], independently conjectured that the analogue of Erdős's bound holds for induced posets as well, namely, $La^{\#}(n, P) = O\left(\binom{n}{\lfloor n/2 \rfloor}\right)$. The main result of Chapter 2 is to prove their conjecture for all posets P.

Theorem 1.12 (M., Pálvölgyi [123]). *For every poset P, $La^{\#}(n, P) \le C_P\binom{n}{\lfloor \frac{n}{2} \rfloor}$ for some constant C_P that depends only on P.*

Our main idea is to make a connection to the theory of forbidden submatrices and use a version of Marcus-Tardos theorem for higher dimensions. Therefore, the order of the constant C_P following from their proof is typically exponential in $|P|$. Very recently, Tomon [145] showed that if the height of the poset is constant, then this can be improved. More precisely, he showed the following:

Theorem 1.13 (Tomon [145]). *For every positive integer h there exists a constant c_h such that if P has height at most h, then*

$$\mathrm{La}^{\#}(n, P) \le |P|^{c_h}\binom{n}{\lfloor n/2 \rfloor}.$$

Chapter 2

Forbidden hypermatrices imply general bounds on induced forbidden subposet problems

The main result of this chapter is the following theorem, settling a conjecture of Katona [94], and Lu and Milans [114].

Theorem 2.1 (M., Pálvölgyi [123]). *For every poset P, $La^{\#}(n, P) \leq C_P\binom{n}{\lfloor \frac{n}{2} \rfloor}$ for some constant C_P that depends on P.*

It is interesting to note that the constant C_P in our upper bound on $La^{\#}(n, P)$ does not depend on $h(P)$, which appears in the upper bounds on $La(n, P)$, but on the *dimension* of P. The *dimension* of a poset P is the least integer d for which there exists t linear orderings, $<_1, \dots, <_d$, of the elements of P such that for every x and y in P, $x <_P y$ if and only if $x <_i y$ for all $1 \leq i \leq d$. Just like in the non-induced case, one might conjecture that $\lim \frac{La^{\#}(n,P)}{\binom{n}{\lfloor n/2 \rfloor}}$ exists, and equals the maximum number of complete and consecutive middle levels of the Boolean lattice whose union is induced P-free.

To establish our theorem, we use a method that is a certain generalization of the so-called *circle method* of Katona [92]. That is, if $\mathcal{F}$ is a family of subsets which is induced P-free, we define some special families $\mathcal{Q}$ of subsets of $[n]$, and double count all pairs $(\mathcal{Q}, F)$ such that $F \in \mathcal{F}$ and $F \in \mathcal{Q}$. What is novel in our approach is that we associate a d-dimensional $0-1$ hypermatrix to $\mathcal{Q}$ and establish a connection to the theory of forbidden submatrices. Then using this connection, we calculate the number of pairs $(\mathcal{Q}, F)$ for a fixed $\mathcal{Q}$. This is made more precise in the proof of Proposition 2.2, for which we first need to introduce some notation.

A d-dimensional *hypermatrix* is an $n_1 \times \cdots \times n_d$ sized ordered array. For short, we refer to such a hypermatrix as a *d-matrix*. So a vector is a 1-matrix and a matrix is a 2-matrix. Moreover, we simply say that a d-matrix is of size n^d if $n_1 = \ldots = n_d = n$. We refer to the entries of a d-matrix M as $M(\underline{i})$ where $\underline{i} = (i_1, \dots, i_d)$ and $1 \leq i_j \leq n_j$ for every $j \in [d]$. In this chapter we only deal with d-matrices whose entries are all 0 and 1. We denote the number of 1's in a d-matrix M by $|M|$. We say that a d-matrix M *contains* a d-matrix A if it has a d-submatrix $M' \subset M$ that is of the same size as A such that $A(\underline{i}) = 1 \Rightarrow M'(\underline{i}) = 1$. If M does not contain A, then we say that M is *A-free*. We define the corresponding extremal function as

$$ex_d(n_1 \times \cdots \times n_d, A) := \max\{|M| \mid M \text{ is an } A\text{-free } d\text{-matrix of size } n_1 \times \cdots \times n_d\}$$

and if $n_1 = \ldots = n_d = n$, we use $ex_d(n, A) := ex_d(n_1 \times \cdots \times n_d, A)$ for convenience. Notice that $ex_1(n, A) = \min\{n, |A|-1\}$ and $ex_2(n, A)$ is the well-studied forbidden submatrix problem: see [50, 141].

We also need to generalize the notion of a permutation matrix to higher dimensions. We say that a d-matrix M of size k^d is a *permutation d-matrix* if $|M| = k$ and it contains exactly one 1 in each axis-parallel hyperplane. In other words, for every $j \in [d]$ and $1 \le i_j \le k$ there is a unique $\underline{i} = (i_1, \ldots, i_d)$ such that $M(\underline{i}) = 1$. From the definition of poset dimension, we get that for every poset P of size k whose dimension is d, there is a unique permutation d-matrix M_P of size k^d that *represents* P in the following sense: The 1-entries of M_P are in bijection with the elements of P such that if the element $M_P(\underline{i})$ is in bijection with $p \in P$ and the element $M_P(\underline{i}')$ is in bijection with $p' \in P$, then $p < p' \Leftrightarrow \forall j \;\; i_j < i'_j$. This M_P can be constructed as follows. Consider the d linear orderings, $<_1, \ldots, <_d$, of the elements of P such that for every x and y in P, $x <_P y$ if and only if $x <_i y$ for all $1 \le j \le d$. For each $p \in P$ the coordinates of the associated 1-entry of M_P are $\underline{i} = (i_1, \ldots, i_d)$ where i_j is the position of p in the linear ordering $<_j$.

The following is our key proposition establishing a connection to the theory of forbidden submatrices.

Proposition 2.2 (M., Pálvölgyi [123])**.** *We have* $La^{\#}(n, P) \le C_d \frac{ex_d(n, M_P)}{n^{d-1}} \binom{n}{\lfloor \frac{n}{2} \rfloor}$ *for every d-dimensional poset P for some constant* C_d *that depends on d.*

We note that in the special case $d = 2$ we get $La^{\#}(n, P) \le (1 + o(1)) \frac{ex_2(n, M_P)}{n} \binom{n}{\lfloor \frac{n}{2} \rfloor}$ as n tends to infinity in the above statement.

We can combine the above proposition with the following theorem, which is a higher dimensional variant of the Marcus-Tardos theorem [121] about forbidden submatrices.

Theorem 2.3 (Klazar-Marcus [98])**.** *If a d-dimensional* $0-1$ *hypermatrix of size* $n \times \cdots \times n$ *does not contain a given d-dimensional permutation hypermatrix of size* $k \times \cdots \times k$*, then it has at most* $C_k n^{d-1}$ *non-zero elements for some constant* C_k *that depends on k.*

Notice that Theorem 2.1 follows from Proposition 2.2 and Theorem 2.3. After uploading the first version of our manuscript [123], we have learned that Theorem 2.3 has been proved earlier by Klazar and Marcus [98]. Surprisingly, our proof of Theorem 2.3 is different from their proof and appears to be a bit shorter, perhaps due to the use of the Loomis-Whitney inequality [111].

The organization of the rest of this chapter is the following. In Section 2.1 we prove our main result, Proposition 2.2, that establishes the connection between the two theories. In Section 2.2 we give a new proof of Theorem 2.3.

2.1 Proof of Proposition 2.2

Define a *permutation d-partition* $\mathcal{Q} := Q_1|Q_2|\ldots|Q_d$ as an ordered partition of a permutation of the elements of $[n]$ into d parts $Q_1, Q_2, \ldots, Q_d$. We denote the i^{th} element of Q_j by $Q_j(i)$. The set of the form $Q_j[i) := \{Q_j(1), \ldots, Q_j(i-1)\}$ is called a *prefix* of Q_j and the set of the form $\mathcal{Q}[\underline{i}) := \bigcup_{j=1}^{d} Q_j[i_j)$ is called a *prefix union* of $\mathcal{Q}$.

An example of a permutation 3-partition is $\mathcal{Q} = 142|5|3$ and $\mathcal{Q}[3, 1, 2) = \{1, 3, 4\}$ is a prefix union of $142|5|3$. Notice that since the order of the parts is respected, we consider, say, $\mathcal{Q}' = 5|142|3$ as a different permutation 3-partition, but of course the prefixes of $\mathcal{Q}$

and $\mathcal{Q}'$ are the same. Some parts might also be empty in a permutation 3-partition, as in 142||53.

The total number of possible permutation d-partitions is $(n+d-1)!\,/(d-1)!$, by taking all permutations of the elements of $[n]$ and the $d-1$ separators.

We are now ready to start the proof of Proposition 2.2. Let $\mathcal{F}$ be an induced-P-free family of subsets of $[n]$. We double count pairs $(\mathcal{Q}, F)$ where $F \in \mathcal{F}$ and F is a prefix union of $\mathcal{Q}$. First, let us fix a set $F \in \mathcal{F}$ and calculate the number of permutation d-partitions $\mathcal{Q}$ such that F is a prefix union of $\mathcal{Q}$.

Lemma 2.4. *Given $F \subset [n]$ and $d \in \mathbb{N}$, there are exactly $\frac{(n+2d-2)!}{((d-1)!)^2}\binom{n+2d-2}{|F|+d-1}^{-1}$ permutation d-partitions $\mathcal{Q}$ of $[n]$ such that F is a prefix union of $\mathcal{Q}$.*

Proof. Permute the elements of F and $d-1$ separators "|" in $(|F|+d-1)!\,/(d-1)!$ ways. Each such permutation is of the form $L_1|L_2|\dots|L_d$. Also permute the elements of $[n]\setminus F$ and $d-1$ separators "|" in $(n-|F|+d-1)!\,/(d-1)!$ ways. Each such permutation is of the form $R_1|R_2|\dots|R_d$.

Now, we concatenate $L_1|L_2|\dots|L_d$ and $R_1|R_2|\dots|R_d$ as $L_1R_1|L_2R_2|\dots|L_dR_d$ to obtain a permutation d-partition for which F is a prefix union. Since

$$\frac{(|F|+d-1)!}{(d-1)!}\cdot\frac{(n-|F|+d-1)!}{(d-1)!} = \frac{(n+2d-2)!}{((d-1)!\,)^2}\binom{n+2d-2}{|F|+d-1}^{-1},$$

the proof is complete. □

The following property about monotonicity for matrices avoiding submatrices will be useful.

Proposition 2.5. *If $\forall i : m_i \le n_i$, then*

$$ex_d(n_1\times\cdots\times n_d, A) \le \frac{n_1}{m_1}\times\cdots\times\frac{n_d}{m_d}ex_d(m_1\times\cdots\times m_d, A).$$

Proof. Let M be an A-free d-matrix of size $n_1\times\cdots\times n_d$. Any M' d-submatrix of M is also A-free. If M' is of size $m_1\times\cdots\times m_d$, then $|M'|\le ex_d(m_1\times\cdots\times m_d, A)$. Averaging over all submatrices of this size the statement follows, as any entry of M has probability $\frac{m_1}{n_1}\times\cdots\times\frac{m_d}{n_d}$ to be in a submatrix. □

Now, let us fix a $\mathcal{Q} = Q_1|Q_2|\dots|Q_d$ and calculate the number of sets $F \in \mathcal{F}$ such that F is a prefix union of $\mathcal{Q}$.

Lemma 2.6. *Let P be a poset of dimension d, and let M_P be the d-dimensional permutation matrix that represents P. Given an induced-P-free family $\mathcal{F}$ of subsets of $[n]$ and a permutation d-partition $\mathcal{Q}$ of $[n]$, there exist at most $\left(\frac{n+d}{nd}\right)^d ex_d(n, M_P)$ sets $F \in \mathcal{F}$ such that F is a prefix union of $\mathcal{Q}$.*

Proof. We first associate a d-matrix $M_\mathcal{Q}$ of size $(|Q_1|+1)\times\cdots\times(|Q_d|+1)$ to $\mathcal{Q}$ where $|Q_j|$ denotes the length of Q_j. This is done by setting $M_\mathcal{Q}(\underline{i}) = 1$ if the prefix union $\mathcal{Q}[\underline{i}] \in \mathcal{F}$ and $M_\mathcal{Q}(\underline{i}) = 0$ otherwise. Now consider the permutation d-matrix M_P of size $|P|^d$ that represents P. Notice that $\mathcal{Q}[\underline{i}'] \subset \mathcal{Q}[\underline{i}]$ if and only if $\forall j : i'_j \le i_j$ and equality can hold only if $\underline{i} = \underline{i}'$. From this it follows that if $M_\mathcal{Q}$ contains M_P, then the same relations hold in $\mathcal{F}$ and thus $\mathcal{F}$ contains an induced copy of P, which is impossible. Therefore $M_\mathcal{Q}$ is

M_P-free. Using Proposition 2.5, we have that the number of sets $F \in \mathcal{F}$ such that F is a prefix union of $\mathcal{Q}$ is

$$|M_{\mathcal{Q}}| \leq \frac{(|Q_1|+1) \times \cdots \times (|Q_d|+1)}{n^d} ex_d(n, M_P) \leq \left(\frac{n+d}{nd}\right)^d ex_d(n, M_P).$$

□

Now combining the lower and upper bounds that we get from Lemmas 2.4 and 2.6 for the number of pairs $(\mathcal{Q}, F)$ such that $\mathcal{Q}$ is a permutation d-partition of $[n]$ and $F \in \mathcal{F}$ is a prefix union of $\mathcal{Q}$, we get

$$\frac{(n+2d-2)!}{((d-1)!\,)^2} \sum_{F \in \mathcal{F}} \binom{n+2d-2}{|F|+d-1}^{-1} \leq \frac{(n+d-1)!}{(d-1)!} \left(\frac{n+d}{nd}\right)^d ex_d(n, M_P).$$

Using $(n+2d-2)! \geq (n+d-1)!\,(n+d)^{d-1}$ and multiplying by $((d-1)!\,)^2$ on both sides, we get,

$$|\mathcal{F}| \binom{n+2d-2}{\lfloor \frac{n}{2} \rfloor + d - 1}^{-1} \leq \sum_{F \in \mathcal{F}} \binom{n+2d-2}{|F|+d-1}^{-1} \leq \frac{(d-1)!}{d^{d-1}} \left(\frac{n+d}{nd}\right) \frac{ex_d(n, M_P)}{n^{d-1}}.$$

Now using that $\binom{n+2d-2}{\lfloor \frac{n}{2} \rfloor + d-1} \leq 4^{d-1} \binom{n}{\lfloor \frac{n}{2} \rfloor}$, we have proved Proposition 2.2. □

2.2 Proof of Theorem 2.3

The proof is similar to the proof of Marcus and Tardos [121], except that we use induction on d and n, just like Klazar and Marcus [98]. However, surprisingly, even though both the Klazar-Marcus proof and our proof are a very natural generalization of the Marcus-Tardos proof, they are still quite different. Below we present our proof.

We prove by induction on d and n that any d-matrix of size n^d not containing some permutation d-matrix of size k^d has at most $C_{k,d} n^{d-1}$ non-zero elements (k is fixed throughout the proof and $C_{k,d}$ is a constant that depends on k and d). As $d \leq k$, our final constant $C_k = C_{k,k}$. The statement trivially holds for $d = 1$, for all $n \geq 1$. We will show that it is true for d and n.

Let M be a d-matrix of size n^d and A a permutation d-matrix of size k^d. If S is a d-matrix, denote by $Proj_i S$ the $(d-1)$-matrix obtained by orthogonally projecting S to the hyperplane orthogonal to the i^{th} axis. Notice that $Proj_i A$ is a permutation $(d-1)$-matrix of size k^{d-1}.

We partition M into smaller d-matrices of size s^d called *blocks* (for convenience, suppose that n is divisible by s) in the following way: we partition $[n]$ into intervals $I_1 < I_2 < \ldots < I_{\frac{n}{s}}$ each of length s. For any $\underline{b} = (b_1, \ldots b_d)$ with $b_i \in \{I_1, I_2, \ldots, I_{\frac{n}{s}}\}$, we define the block $S_{\underline{b}} := \{M(\underline{a}) \mid \underline{a} = (a_1, \ldots a_d)$ and $a_i \in I_i\}$.

An *i-blockcolumn* is a series of blocks parallel to the i^{th}-axis, i.e., $\{S_{\underline{b}} \mid b_i = I_1, \ldots, I_{\frac{n}{s}}\}$ and an *i-column* is simply a series of matrix elements parallel to the i^{th}-axis, i.e., $\{M(\underline{a}) \mid a_i = 1, 2, \ldots, n\}$. A block S is called *i-wide* if $Proj_i S$ contains $Proj_i A$ as a $(d-1)$-submatrix. Using induction on the dimension, if this is not the case, then $|Proj_i S| \leq C_{k,d-1} s^{d-2} = O(s^{d-2})$. If a block is not i-wide for any $i = 1, \ldots, d$, we call it *thin*.

For the induction, we also need to use the following inequality of Loomis and Whitney.

Lemma 2.7 (Loomis-Whitney [111])**.** $|S|^{d-1} \leq \Pi_{i=1}^{d} |Proj_i S|$.

If a block S is thin, then using the above inequality and that $|Proj_i S| \leq C_{k,d-1} s^{d-2}$ (for all $1 \leq i \leq d$), we get $|S| \leq (C_{k,d-1} s^{d-2})^{\frac{d}{d-1}} = O(s^{d-1-\frac{1}{d-1}}) = o(s^{d-1})$. The number of i-wide blocks in an i-blockcolumn is at most $(k-1)\binom{s^{d-1}}{k}$, because if $Proj_i A$ would occur k times, in the same k i-columns, then we could "build" a copy of A from them (here we use that A is a permutation d-matrix).

We define the d-matrix M' of size $(\frac{n}{s})^d$ as $M'_{\underline{b}} = 1$ if and only if the block $S_{\underline{b}}$ is thin. As M' must be also A-free, we get the following bound by induction on d and n, where k is fixed.

$$\begin{aligned}
|M| \leq & \sum_{S \text{ is thin}} |S| + \sum_{i=1}^{d} \sum_{\substack{BC \text{ is an} \\ i\text{-blockcolumn}}} \sum_{\substack{S \in BC \text{ is} \\ i\text{-wide}}} |S| \leq \sum_{S \text{ is thin}} o(s^{d-1}) \\
& + \sum_{i=1}^{d} \sum_{\substack{BC \text{ is an} \\ i\text{-blockcolumn}}} \sum_{\substack{S \in BC \text{ is} \\ i\text{-wide}}} s^d \leq |M'| o(s^{d-1}) + \sum_{i=1}^{d} \sum_{\substack{BC \text{ is an} \\ i\text{-blockcolumn}}} (k-1)\binom{s^{d-1}}{k} s^d \\
& \leq C_{k,d} \left(\frac{n}{s}\right)^{d-1} o(s^{d-1}) + d\left(\frac{n}{s}\right)^{d-1} (k-1)\binom{s^{d-1}}{k} s^d,
\end{aligned}$$

which for a sufficiently large s, is less than $(1-\delta)C_{k,d} n^{d-1} + s^{dk} n^{d-1} \leq C_{k,d} n^{d-1}$ for some $\delta > 0$. With a more precise calculation, we can upper bound $C_{k,d}$ by $k^{k^{d}((d+1)!)^2} = 2^{k^{\Theta(d)}}$. This is much weaker than the bound achieved by Klazar and Marcus [98], which gives $C_k = 2^{O_d(k \log k)}$. Very recently it has been proved by Geneson and Tian [56] that $C_k = 2^{O_d(k)}$.

Chapter 3

Diamond-free families

3.1 Introduction

Let us recall the definitions of the following posets which we use in this chapter.

Definition 3.1 (Posets $\mathcal{Q}_2, V$ and Λ)**.** The diamond poset, denoted $\mathcal{Q}_2$ (or $\mathcal{D}_2$ or $\mathcal{B}_2$), is a poset on four elements $\{x, y, z, w\}$, with the relations $x < y, z$ and $y, z < w$. That is, $\mathcal{Q}_2$ is a subposet of a family of sets $\mathcal{A}$ if there are different sets $A, B, C, D \in \mathcal{A}$ with $A \subset B, C$ and $B, C \subset D$. (Note that B and C are not necessarily unrelated.) The V poset is a poset on $\{x, y, z\}$ with the relations $x \leq y, z$; the Λ poset is defined on $\{x, y, z\}$ with the relations $x, y \leq z$. That is, the Λ is a subposet of a family of sets $\mathcal{A}$ if there are different sets $B, C, D \in \mathcal{A}$ with $B, C \subset D$.

The most investigated poset for which even the asymptotic value of $\mathrm{La}(n, P)$ has yet to be determined is the diamond $\mathcal{Q}_2$ which is the topic of this chapter. The two middle levels of the Boolean lattice do not contain a diamond, so $\mathrm{La}(n, \mathcal{Q}_2) \geq (2 - o(1))\binom{n}{\lfloor n/2 \rfloor}$. Czabarka, Dutle, Johnston and Székely [24] gave infinitely many asymptotically tight constructions by using random set families defined from posets based on Abelian groups. Such constructions suggest that the diamond problem is hard. Using a simple and elegant argument, Griggs, Li and Lu [67] showed that $\mathrm{La}(n, \mathcal{Q}_2) < 2.296\binom{n}{\lfloor n/2 \rfloor}$. Some time after they had announced this bound, Axenovich, Manske and Martin [8] improved the upper bound to $2.283\binom{n}{\lfloor n/2 \rfloor}$. This bound was further improved to $2.273\binom{n}{\lfloor n/2 \rfloor}$ by Griggs, Li and Lu [67]. Kramer, Martin and Young [106] later showed that $\mathrm{La}(n, \mathcal{Q}_2) \leq (2.25 + o(1))\binom{n}{\lfloor n/2 \rfloor}$.

Definition 3.2. A *maximal chain* or, for the rest of this chapter, simply a *chain* of the Boolean lattice is a sequence of sets $\emptyset, \{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \ldots, [n]$ with $x_1, x_2, x_3 \ldots \in [n]$. We refer to $\{x_1, \ldots, x_i\}$ as the ith set on the chain. In particular, we refer to $\{x_1\}$ as the first set on the chain, or just say that the chain starts with the element x_1 (as a singleton). We refer to x_i as the ith element added to form the chain.

Definition 3.3. The *Lubell function* of a family of sets $\mathcal{F} \subseteq 2^{[n]}$ is defined as

$$l(n, \mathcal{F}) = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}.$$

The notation is shortened to just $l(\mathcal{F})$ when there is no ambiguity as to the dimension of the Boolean lattice.

Observation 3.4. The Lubell function of a family $\mathcal{F}$ is the average number of sets from $\mathcal{F}$ on a chain, taken over all $n!$ chains. In particular, the Lubell function of a level is 1, and the Lubell function of an antichain $\mathcal{F}$ is the number of chains containing a set from $\mathcal{F}$ divided by $n!$. The Lubell function is additive across a union of disjoint families of sets. Furthermore, $|\mathcal{F}| \leq l(\mathcal{F})\binom{n}{\lfloor n/2 \rfloor}$ ([116]).

The Lubell function was derived from the celebrated YMBL inequality which was independently discovered by Yamamoto, Meshalkin, Bollobás and Lubell. Using the Lubell function terminology, it states that

YMBL inequality (Yamamoto, Meshalkin, Bollobás, Lubell [149, 122, 13, 116])**.** *If $\mathcal{F} \subseteq 2^{[n]}$ is an antichain, then $l(\mathcal{F}) \leq 1$.*

For a poset P, let $\bar{l}(n, P)$ be the maximum of $l(n, \mathcal{F})$ over all families $\mathcal{F} \subseteq 2^{[n]}$ which are both P-free and contain the empty set. Let $\bar{l}(P) = \limsup_{n\to\infty} \bar{l}(n, P)$. Griggs, Li and Lu proved that

Lemma 3.5 (Griggs, Li and Lu [67])**.**

$$\mathrm{La}(n, \mathcal{Q}_2) \leq \left(\bar{l}(\mathcal{Q}_2) + o(1)\right)\binom{n}{\lfloor n/2 \rfloor}.$$

Kramer, Martin and Young used flag algebras to prove that

Lemma 3.6 (Kramer, Martin and Young [106])**.** $\bar{l}(\mathcal{Q}_2) = 2.25$

thereby proving

Theorem 3.1 (Kramer, Martin and Young [106])**.**

$$\mathrm{La}(n, \mathcal{Q}_2) \leq (2.25 + o(1))\binom{n}{\lfloor n/2 \rfloor}.$$

The following construction shows that $\bar{l}(\mathcal{Q}_2) \geq 2.25$ in Lemma 3.6. There are other constructions known as well.

Example 3.7. Let $\mathcal{F} \subseteq 2^{[n]}$ consist of all the sets of the following forms: $\emptyset, \{e\}, \{e, o\}, \{o_1, o_2\}$ where e denotes any even number in $[n]$, and o, o_1 and o_2 denote any odd numbers in $[n]$. This family is diamond-free, and $l(\mathcal{F}) = 2.25 \pm o(1)$.

Example 3.8. This construction is a generalization of the previous one. Let $A \subseteq [n]$ with $|A| = an$. Let $\mathcal{F} \subseteq 2^{[n]}$ consist of all the sets of the following forms: $\emptyset, \{e\}, \{e, o\}, \{o_1, o_2\}$ where now e denotes any element of A, while o, o_1 and o_2 denote any elements of $[n] \setminus A$. This family is diamond-free, and $l(\mathcal{F}) = 2 + a - a^2 \pm o(1)$. This family contains all size 2 sets that do not form a diamond with $\emptyset$ and the singletons, so all maximal diamond-free families on levels 0, 1 and 2 that contain $\emptyset$ are of this form.

The following restriction of the problem of diamond-free families has been investigated: How big can a diamond-free family be if it can only contain sets from the middle three levels of $2^{[n]}$ (denoted $\mathcal{B}(n, 3)$)? Better bounds are known with this restriction. Axenovich, Manske and Martin showed that

Theorem 3.2 (Axenovich, Manske and Martin [8])**.** *If $\mathcal{F} \subseteq \mathcal{B}(n, 3)$ is diamond-free, then $|\mathcal{F}| \leq (2.20711 + o(1))\binom{n}{\lfloor n/2 \rfloor}$.*

Later, Manske and Shen improved it to $2.1547\binom{n}{\lfloor n/2 \rfloor}$ in [119] and recently, Balogh, Hu, Lidický and Liu gave the best known bound of $2.15121\binom{n}{\lfloor n/2 \rfloor}$ in [9] using flag algebras.

Definition 3.9. We call a chain *maximal–non-maximal (MNM)* with respect to (w.r.t.) $\mathcal{F}$ if it contains a set from $\mathcal{F}$, and the biggest set contained in $\mathcal{F}$ on the chain is not maximal in $\mathcal{F}$ (i.e., there are other sets from $\mathcal{F}$ containing it on some other chains).

It is easy to see that an $\emptyset$-free family is Λ-free if and only if the family we get by adding $\emptyset$ is diamond-free; adding $\emptyset$ increases the Lubell function by 1. In Section 3.2 of this chapter, we prove the following lemma:

Lemma 3.10 (Grósz, M., Tompkins [71])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be a Λ-free family that does* not *contain the empty set, nor any set of size bigger than $n - n'$ for some $n' \in \mathbb{N}$ (that can be chosen independently of n). Assume that there are $cn!$ MNM chains w.r.t. $\mathcal{F}$. Then $l(\mathcal{F}) \leq 1 - \min\left(c + \frac{1}{n'}, \frac{1}{4}\right) + \sqrt{\min\left(c + \frac{1}{n'}, \frac{1}{4}\right)} + \frac{3}{n'}$.*

It is easy to see that in Example 3.8 the number of MNM-chains is approximately $a^2 n!$ (so $a \approx \sqrt{c}$): these are the chains whose second set is $\{e_1, e_2\}$ with $e_1, e_2 \in A$. Thus, this lemma is (asymptotically) sharp, and states that for a given number of MNM chains, Example 3.8 cannot be beaten (with some restriction on the sizes of the sets). Barring the requirement that the topmost n' levels be empty, Lemma 3.10 is a generalization of Lemma 3.6. The proof of Lemma 3.5 in [106] actually works with the restriction of Lemma 3.10 concerning the topmost sets (that there is no set of size bigger than $n - n'$) with $n' = n/2 - n^{2/3}$, immediately giving a new proof of Theorem 3.1. Our proof of Lemma 3.10 includes an intricate induction step and a (non-combinatorial) lemma about functions involving a lot of elementary algebra and calculus, and it does not use details of the structure of $\mathcal{F}$ above the second level (except inside the induction).

Section 3.3 of this chapter uses Lemma 3.10 to prove our main theorem of this chapter:

Theorem 3.3 (Grósz, M., Tompkins [71])**.** $\mathrm{La}(n, \mathcal{Q}_2) \leq \left(\frac{\sqrt{2}+3}{2} + o(1)\right)\binom{n}{\lfloor n/2 \rfloor} < (2.20711 + o(1))\binom{n}{\lfloor n/2 \rfloor}$.

Our proof is inspired by the proof of Theorem 3.2 (the same bound when restricted to 3 levels) as in [8], using the idea of grouping chains by the smallest set contained in $\mathcal{F}$ on a chain (as developed in [67] and [106]).

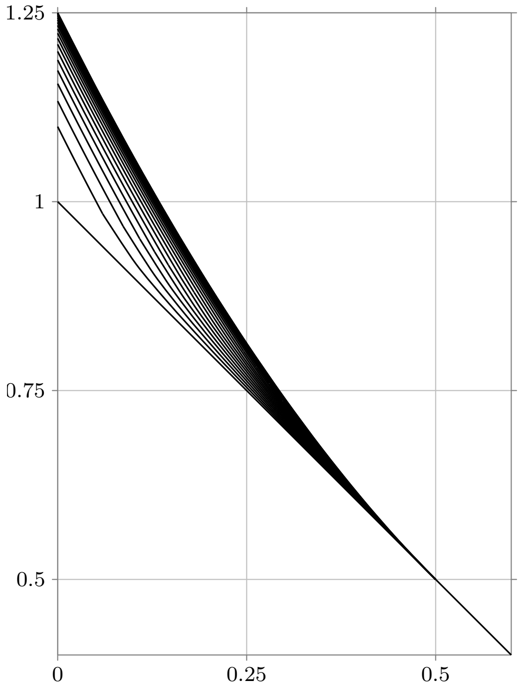
3.2 Λ-free families – Proof of Lemma 3.10

3.2.1 Definitions and main lemma

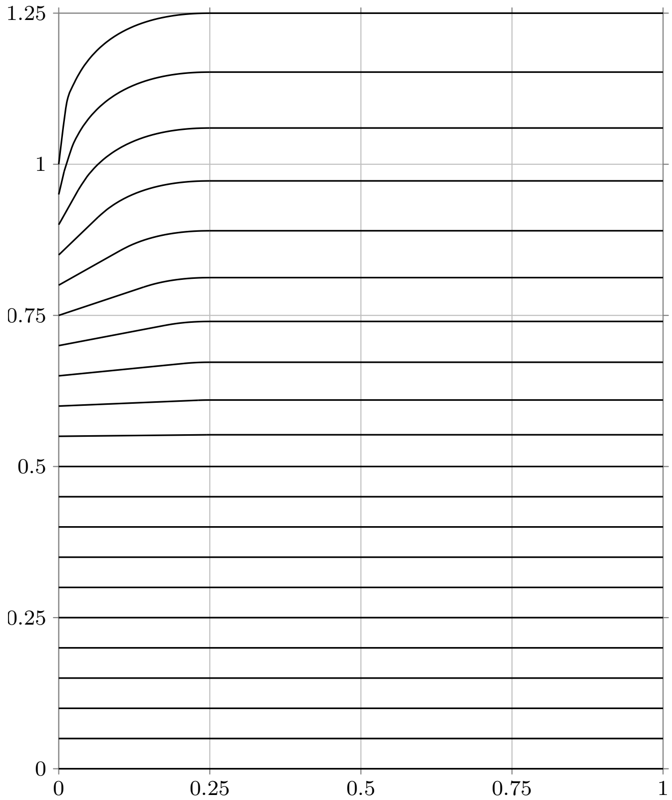
Definition 3.11. We define the following functions:

- For $x \in [0, 1], c \in [0, \infty)$,

$$f(x, c) = \begin{cases} 1 - x + \left(\frac{1}{4(x-x^2)} - 1\right)c & \text{if } x \leq \frac{1}{2} \text{ and } c < 4\left(x - x^2\right)^2 \\ x^2 - 2x + 1 - c + \sqrt{c} & \text{if } x \leq \frac{1}{2} \text{ and } 4\left(x - x^2\right)^2 \leq c \leq \frac{1}{4} \\ x^2 - 2x + 1.25 & \text{if } x \leq \frac{1}{2} \text{ and } \frac{1}{4} \leq c \\ 1 - x & \text{if } \frac{1}{2} \leq x. \end{cases}$$



Values of $f(x,c)$ plotted in x, for $c = 0, 0.0125, 0.025, \ldots, 0.25$ (bottom to top). Note that the $x \geq 0.5$ part of the plots coincide.



Values of $f(x,c)$ plotted in c, for $x = 0, 0.05, 0.01, \ldots, 1$ (top to bottom).

- For $x \in [0,1), c \in [0,\infty), a \in [0,1-x), \tilde{a} \in \big[0, \min\big(a, \frac{c}{x+a}\big)\big]$,

$$g(x,c,a,\tilde{a}) = a + (1-x-a) f\left(x+a, \frac{c-\tilde{a}(x+a)}{1-x-a}\right) + 2\tilde{a}(1-x-a).$$

- For $x \in [0,1), c \in [0,\infty), a \in [0,1-x), \tilde{a} \in \big[0, \min\big(a, \frac{c}{x+a} - x\big)\big]$,

$$h(x,c,a,\tilde{a}) = a+(1-x-a) f\left(x+a, \frac{c-(x+\tilde{a})(x+a)}{1-x-a}\right)+2\tilde{a}(1-x-a)+x-3x(x+a).$$

Lemma 3.12. *The functions above satisfy the following conditions:*

1. *For all $c \in [0,\infty)$, if $\tilde{c} = \min\big(c, \frac{1}{4}\big)$, then $f(0,c) = 1 - \tilde{c} + \sqrt{\tilde{c}}$.*
2. *$f(x,c)$ is concave and monotonously increasing in c, and monotonously decreasing in x.*
3. *For all $x \in [0,1], c \in [0,\infty), a \in [0,1-x), \tilde{a} \in \big[0, \min\big(a, \frac{c}{x+a}\big)\big]$: $g(x,c,a,\tilde{a}) \leq f(x,c)$.*
4. *For all $x \in [0,1], c \in [0,\infty), a \in [0,1-x), \tilde{a} \in \big[0, \min\big(a, \frac{c}{x+a} - x\big)\big]$: $h(x,c,a,\tilde{a}) \leq f(x,c)$.*
5. *For all $c \in [0,\infty) : 1 - x \leq f(x,c)$.*

We prove Lemma 3.12 in Appendix A.

Rather that proving Lemma 3.10 directly, we prove a strengthening of it – Lemma 3.13. This strengthened version involves additional parameters, X and $\mathcal{X}$, and their functions x, α and μ, which we introduce in order to make the inductive proof possible. Lemma 3.10 is a special case of Lemma 3.13 with $X = \mathcal{X} = \emptyset$. In the rest of Section 3.2, we prove Lemma 3.13.

Lemma 3.13. *Let $\mathcal{F} \subseteq 2^{[n]}$ be a Λ-free family which does not contain $\emptyset$, nor any set larger than $n - n'$ for some $n' \in \mathbb{N}$. Let us assume that we are given a "forbidden" set $X \subseteq [n]$, with $x = \frac{|X|}{n}$. Also, let $\mathcal{X} \subset 2^{[n]}$ be a "forbidden" antichain in which each set contains exactly one element of X (and may or may not be a singleton). Let us assume that the sets in $\mathcal{F}$ are disjoint from X, and unrelated to every set in $\mathcal{X}$. Let $\alpha = l(\mathcal{X})$, and let $\mu n!$ be the number of chains which start with an element of X as a singleton, but do not contain any set in $\mathcal{X}$. Assume, furthermore, that there are $cn!$ MNM chains w.r.t. $\mathcal{F}$. Then $l(\mathcal{F}) \leq f\big(x, c + \mu + \frac{1}{n'}\big) - (\alpha - \mu - x) + \frac{3}{n'}$.*

First we verify the base case of the induction.

Proposition 3.14. *Lemma 3.13 holds for $n \leq n'$.*

Proof. $\mathcal{F} = \emptyset$. $\mathcal{X}$ is an antichain, so, by the YMBL inequality, $\alpha \leq 1$. By Lemma 3.12 Point 2 and Point 5, $f\big(x, c + \mu + \frac{1}{n'}\big) - (\alpha - \mu - x) + \frac{3}{n'} \geq f(x, c+\mu) - (\alpha - \mu - x) \geq 1 - x - (\alpha - x) \geq 0 = l(\mathcal{F})$. □

From now on we assume $n' \leq n - 1$.

Notation. Let $A \subseteq [n] \setminus X$ be the set of elements of $[n]$ that appear as singletons in $\mathcal{F}$, and let $a = \frac{|A|}{n}$. Let $\mathcal{B}$ be the family of those sets in $\mathcal{F}$ which contain at least one element of A, but which are not singletons. Let β be the Lubell function of $\mathcal{B}$. Let $\mathcal{C}$ be the family of those sets in $\mathcal{F}$ which only contain elements of $[n] \setminus X \setminus A$.

Table 3.1: Summary of notation

X	"Forbidden" set (sets in $\mathcal{F}$ are disjoint from it – parameter of Lemma 3.13)		$x = \lvert X \rvert / n$
$\mathcal{X}$	"Forbidden" antichain (sets in $\mathcal{F}$ are unrelated to sets in it – parameter of Lemma 3.13)		$\alpha = l(\mathcal{X})$
μ	$\#\{\text{chains containing } \{d\} \text{ for } d \in X \text{ but no set from } \mathcal{X}\}/n!$		
c	$\#\{\text{MNM chains}\}/n!$ (parameter of Lemma 3.10 / Lemma 3.13)		
A	$\{e \in [n] : \{e\} \in \mathcal{F}\}$		$a = \lvert A \rvert / n$
$\mathcal{B}$	$\{B \in \mathcal{F} : (\lvert B \rvert \geq 2, A \cap B \neq \emptyset)\}$	$\{\{e\} : e \in A\} \cup \mathcal{B} \cup \mathcal{C} = \mathcal{F}$	$\beta = l(\mathcal{B})$
$\mathcal{C}$	$\{C \in \mathcal{F} : C \subseteq [n] \setminus X \setminus A\}$		
$\tilde{A}$	$\{e \in A : (\exists B \in \mathcal{B} : e \in B)\}$		$\tilde{a} = \lvert \tilde{A} \rvert / n$
ν	$\#\{\text{chains containing } \{e\} \text{ and } \{e,o\} \text{ for } e \in \tilde{A}, o \in [n] \setminus X \setminus A \text{ but no set from } \mathcal{B}\}/n!$		
c_0	$\#\{\text{chains containing } \{e\} \text{ for } e \in \tilde{A} \text{ but no set from } \mathcal{B}\}/n!$		

Let $\tilde{A} = \{e \in A : (\exists B \in \mathcal{B} : e \in B)\}$, and let $\tilde{a} = \frac{|\tilde{A}|}{n}$. Let $c_0 n!$ be the number of chains that start with $\{e\}$ as a singleton for some $e \in \tilde{A}$, but do not contain any set from $\mathcal{B}$. Let $\nu n!$ be the number of chains that start with $\{e\}$ as a singleton for some $e \in \tilde{A}$, continue with an element of $[n] \setminus X \setminus A$ as the second element added to form the chain, yet do not contain any set from $\mathcal{B}$.

Let $\overline{1} = \frac{n}{n-1} > 1$ and $\underline{1} = \frac{(x+a)n-1}{(x+a)(n-1)} \leq 1$. These correction factors will account for the difference from the asymptotic behavior. (They are both typically close to 1. If $x+a = 0$, let $\underline{1} = 1$; it is irrelevant as it will always be multiplied by $x + a$.)

Outline of the proof: In Subsection 3.2.2, we make some observations on the structure of $\mathcal{X}$ and $\mathcal{B}$. In Subsection 3.2.4, we will finish the proof by applying induction to the Boolean lattices $[\{o_i\}, [n]]$ where $o_i \in [n] \setminus X \setminus A$. When applying Lemma 3.13 by induction, we will use $X \cup A$ in the place of X, while sets from $\mathcal{X}$ and $\mathcal{B}$ will contribute to the family we use in the place of $\mathcal{X}$ (which we will denote by $\mathcal{X}'_i$). We know little about the parameters of each $\mathcal{X}'_i$, but we will be able to bound their sums. The relevant calculations are done in Subsection 3.2.3.

3.2.2 On the structure of $\mathcal{X}$ and $\mathcal{B}$

Proposition 3.15. *Every $D \in \mathcal{X}$ is of the form $\{d, o_1, \ldots, o_k\}$ with $d \in X, o_1, \ldots, o_k \in [n] \setminus X \setminus A$ (where k may be 0).*

Proof. D contains exactly one element of X by definition. Let $e \in A$; then $e \notin D$ for otherwise D and $\{e\} \in \mathcal{F}$ would be related. □

Proposition 3.16. *Sets in $\mathcal{B}$ only contain one element of A. $\mathcal{B}$ is an antichain, and the sets in $\mathcal{B}$ are also unrelated to every set in $\mathcal{C}$.*

Proof. If $e_1 \in B \in \mathcal{B}$ with $e_1 \in A$, and B was related to another set $S \in \mathcal{F}$, then $\{e_1\}$, B and S would form a Λ. This applies to any $S \in \mathcal{B} \cup \mathcal{C}$, as well as $S = \{e_2\}$ for any $e_1 \neq e_2 \in A$. □

Proposition 3.17. *$\tilde{a}(x+a)\underline{1} \leq \tilde{a}(x+a)\underline{1} + \nu = c_0 \leq c$, and thus $\tilde{a} \leq \frac{c}{(x+a)\underline{1}}$.*

Proof. Any chain on which the singleton is $\{e\}$ and the second set is $\{e,d\}$ with $e \in \tilde{A}$ and $d \in X \cup A$ is always an MNM chain: $\{e,d\}$ and any set that contains it is forbidden from being in $\mathcal{B}$ either because it is not disjoint from X (when $d \in X$), or because it would contain two elements of A (when $d \in A$). The number of such chains is $\tilde{a}n \cdot (an + xn - 1) \cdot (n-2)! = \tilde{a}(x+a)n!\,\underline{1}$. And out of the chains which start with $\{e\}$, and whose second set is $\{e,o\}$ with some $o \in [n] \setminus X \setminus A$, $\nu n!$ do not contain any set from $\mathcal{B}$.

We have $c_0 \leq c$ because a chain whose first set is $\{e\}$ for some $e \in \tilde{A}$, but does not contain any set from $\mathcal{B}$, is an MNM chain. □

For a family of sets $\mathcal{A} \subseteq 2^{[n]}$, let $m(\mathcal{A})n!$ be the number of chains which start with an element of X as a singleton and do not contain any set from $\mathcal{A}$. (For example, $m(\mathcal{X}) = \mu$, and therefore $l(\mathcal{X}) - m(\mathcal{X}) = \alpha - \mu$.) For a fixed $d \in X$, let $m_d(\mathcal{A})n!$ be the number of chains on which the singleton is $\{d\}$, and do not contain any element of $\mathcal{A}$.

Proposition 3.18. *For any $d \in X$, let $\mathcal{X}_d = \{D \in \mathcal{X} : d \in D\}$. We can assume without loss of generality that for any $d_1, d_2 \in X$, $\{D \setminus \{d_1\} : D \in \mathcal{X}_{d_1}\} = \{D \setminus \{d_2\} : D \in \mathcal{X}_{d_2}\}$. That is, if $\mathcal{X}$ does not satisfy this condition, we show a family $\hat{\mathcal{X}}$ which does, and also satisfies the conditions of Lemma 3.13's statement (each set contains exactly one element of X, the sets are unrelated to each other and to every set in $\mathcal{F}$), and for which $f\left(x, c + m(\hat{\mathcal{X}}) + \frac{1}{n'}\right) - (l(\hat{\mathcal{X}}) - m(\hat{\mathcal{X}}) - x) \leq f\left(x, c + \mu + \frac{1}{n'}\right) - (\alpha - \mu - x)$.*

Proof. Let $d_0 \in X$ be such that

$$f\left(x, c + |X|\, m_{d_0}(\mathcal{X}_{d_0}) + \frac{1}{n'}\right) - (|X|\, l(\mathcal{X}_{d_0}) - |X|\, m_{d_0}(\mathcal{X}_{d_0}) - x)$$
$$= \min_{d \in X}\left[f\left(x, c + |X|\, m_d(\mathcal{X}_d) + \frac{1}{n'}\right) - (|X|\, l(\mathcal{X}_d) - |X|\, m_d(\mathcal{X}_d) - x)\right].$$

Let $\hat{\mathcal{X}} = \{D \setminus \{d_0\} \cup \{d\} : d \in X, D \in \mathcal{X}_{d_0}\}$.

$\mathcal{X} = \bigsqcup_{d \in X} \mathcal{X}_d$, so $\alpha = \sum_{d \in X} l(\mathcal{X}_d)$. It immediately follows from the definition of $\mathcal{X}_d$ that if a chain has $\{d\}$ as a singleton, and does not contain any set from $\mathcal{X}_d$, then it does not contain any set from $\mathcal{X}$. So $\mu = \sum_{d \in X} m_d(\mathcal{X}_d)$. Similarly, $l(\hat{\mathcal{X}}) = |X|\, l(\mathcal{X}_{d_0})$ and $m(\hat{\mathcal{X}}) = |X|\, m_{d_0}(\mathcal{X}_{d_0})$. Since $f(x,c)$ is monotonously increasing and concave in c, using Jensen's inequality

$$f\left(x, c + |X|\, m_{d_0}(\mathcal{X}_{d_0}) + \frac{1}{n'}\right) - (|X|\, l(\mathcal{X}_{d_0}) - |X|\, m_{d_0}(\mathcal{X}_{d_0}) - x)$$
$$\leq \frac{1}{|X|}\left[\sum_{d \in X} f\left(x, c + |X|\, m_d(\mathcal{X}_d) + \frac{1}{n'}\right) - (|X| \sum_{d \in X} l(\mathcal{X}_d) - |X| \sum_{d \in X} m_d(\mathcal{X}_d) - |X|\, x)\right]$$
$$\leq f\left(x, c + \mu + \frac{1}{n'}\right) - (\alpha - \mu - x).$$

Sets in $\hat{\mathcal{X}}$ contain exactly one element of X, and form an antichain. They are also unrelated to every set $S \in \mathcal{F}$: S cannot contain any element of X, so it could only be related to a set in $\hat{\mathcal{X}}$ by being its subset. But S must also be unrelated to every $D \in \mathcal{X}_{d_0} \subseteq \mathcal{X}$, so it cannot be a subset of $D \setminus \{d_0\} \cup \{d\}$ either. □

In fact we will only use the following simple corollary of Proposition 3.18. In many parts of the rest of this section we will treat the two cases of the corollary below separately.

Corollary 3.19. *With the assumption of Proposition 3.18,*

- *either* $\mathcal{X} = \{\{d\} : d \in X\}$ *(we refer to it as the* ***singletons case****),*
- *or* $\mathcal{X}$ *does not contain any singleton (referred to as the* ***no singleton case****).*

Proof. Let $d_1 \in X$. (If $X = \emptyset$, both statements trivially hold.) If $\{d_1\} \in \mathcal{X}_{d_1} = \{D \in \mathcal{X} : d_1 \in D\}$, then $\mathcal{X}_{d_1} = \{\{d_1\}\}$, because sets in $\mathcal{X}_{d_1}$ are unrelated. So either $\mathcal{X}_{d_1} = \{\{d_1\}\}$ or $\mathcal{X}_{d_1}$ does not contain any singleton, yielding the two cases above by Proposition 3.18. □

Remark. The fact that sets in $\mathcal{X}$ contain an element of X implies that sets in $\mathcal{F}$ do not contain sets in $\mathcal{X}$. Now, let us consider what restrictions are imposed on $\mathcal{F}$ by the fact that sets in $\mathcal{F}$ are not contained in the sets in $\mathcal{X}$, beyond the other conditions of Lemma 3.13 (namely that all the sets in $\mathcal{F}$ are disjoint from X).

In the singletons case, clearly there are no such additional restrictions. However, in the no singleton case, there are two additional restrictions that are not already implied by the set X:

- The union of singletons in $\mathcal{F}$, $A \subseteq [n] \setminus \bigcup \mathcal{X}$.
- Sets in $\mathcal{C}$ must not be contained in sets in $\mathcal{X}$. Clearly this imposes a restriction only if $\mathcal{X}$ contains sets bigger than 2.

Example 3.20. Let $C \subseteq [n] \setminus X$, and let $\mathcal{X} = \{\{d, o\} : d \in X, o \in C\}$. Then $\alpha = l(\mathcal{X}) = \frac{2 \cdot xn \cdot |C| \cdot (n-2)!}{n!} = 2x\frac{|C|}{n}\overline{1}$, and $\mu = \frac{xn \cdot (xn+an-1) \cdot (n-2)!}{n!} = x(x+a)\underline{1}$. The only restriction on $\mathcal{F}$ that this $\mathcal{X}$ creates is that the union of singletons $A \subseteq [n] \setminus X \setminus C$.

In other words, let us assume that $\alpha = l(\mathcal{X}) = 2x\gamma\overline{1}$ for $\gamma \in \mathbb{R}$ (without assuming that $\mathcal{X}$ is of the above form). Then it is possible that $a = \frac{|A|}{n}$ can be as big as $1 - x - \gamma$ with $\mathcal{X}$ not creating any restrictions on $\mathcal{C}$ (depending on the actual structure of $\mathcal{X}$, namely, if it is made up of sets of size 2 as above; then $\alpha = 2x(1 - x - a)\overline{1}$ and $\mu = x(x+a)\underline{1}$). But if $a > 1 - x - \gamma$, then $\alpha = 2x\gamma\overline{1}$ implies that $\mathcal{X}$ contains sets bigger than 2, and thus it creates restrictions on $\mathcal{C}$. So, in the no singleton case, one way to understand the calculations that follow is to check them for $\mathcal{X} = \{\{d, o\} : d \in X, o \in C\}$; then check what happens if x, c and a are fixed, but $\mathcal{X}$ is changed.

3.2.3 Chain calculations

Now we estimate the numbers of certain types of chains, in preparation for applying induction.

Proposition 3.21. *In the no singleton case,* $(\alpha - x + \mu)\, n!$ *chains start with* $\{o\}$ *for some* $o \in [n] \setminus X \setminus A$*, and contain a set from* $\mathcal{X}$*.*

Proof. A total of $\alpha n!$ chains contain a set $D \in \mathcal{X}$. By Proposition 3.15, the singleton on such a chain is either from X or $[n] \setminus X \setminus A$. The number of chains which start with an element of X as their singleton and do not contain a set from $\mathcal{X}$ is $\mu n!$, so the number of chains which contain a set from $\mathcal{X}$, and which start with an element of X, is $(x - \mu)n!$. On the rest, the singleton is from $[n] \setminus X \setminus A$. □

Proposition 3.22. $\left(\beta - \tilde{a}(1 - x - a)\overline{1} + \nu\right) n!$ *chains start with* $\{o\}$ *for some* $o \in [n] \setminus X \setminus A$*, and contain a set from* $\mathcal{B}$*.*

Proof. A total of $\beta n!$ chains contain a set from $\mathcal{B}$. A set in $\mathcal{B}$ is of the form $\{e, o_1, \ldots, o_k\}$ with $e \in \tilde{A}, o_1, \ldots, o_k \in [n] \setminus X \setminus A, k \geq 1$. A chain that contains a $B \in \mathcal{B}$, and does not start with $\{o\}$ for some $o \in [n] \setminus X \setminus A$, must start with an element of $\tilde{A}$, and continue with an element of $[n] \setminus X \setminus A$ as the second element added to form the chain. There are $\tilde{a}n \cdot (1-x-a)n \cdot (n-2)! = \tilde{a}(1-x-a)\overline{1}n!$ such chains, out of which $\nu n!$ do not contain any set from $\mathcal{B}$. So $\left(\tilde{a}(1-x-a)\overline{1} - \nu\right) n!$ chains contain a set from $\mathcal{B}$ and start with an element of $\tilde{A}$. The rest start with $\{o\}$ for some $o \in [n] \setminus X \setminus A$. □

Proposition 3.23. *In the no singleton case,* $\mu \geq x(x+a)\underline{1}$*; and the number of chains of the form* $\emptyset, \{d\}, \{d, o\}, \ldots$ *with* $d \in X, o \in [n] \setminus X \setminus A$*, which do not contain any set from* $\mathcal{X}$*, is* $(\mu - x(x+a)\underline{1})n!$.

Proof. A total of $\mu n!$ chains start with an element of X and do not contain any set from $\mathcal{X}$. The chains of the form $\emptyset, \{d_1\}, \{d_1, d_2\}, \ldots$ with $d_1 \in X, d_2 \in X \cup A$ never contain a set from $\mathcal{X}$ when $\mathcal{X}$ contains no singleton. The number of these chains is $xn \cdot (xn + an - 1) \cdot (n-2)! = (x(x+a)\underline{1})n!$. For the rest, the second element added to form the chain is from $[n] \setminus X \setminus A$. □

Notation. Let $X' = X \cup A$. Let $\mathcal{Y} = \{\{d, o\} : d \in X, o \in [n] \setminus X \setminus A\}$, and let $\mathcal{Z} = \{\{e, o\} : e \in A \setminus \tilde{A}, o \in [n] \setminus X \setminus A\}\}$. In the *singletons case*, let $\mathcal{X}' = \mathcal{Y} \sqcup \mathcal{B} \sqcup \mathcal{Z}$. (Note that here and in the rest of the chapter, $\sqcup$ stands for a union of sets which are pairwise disjoint.) In the *no singleton case*, let $\mathcal{X}' = \mathcal{X} \sqcup \mathcal{B} \sqcup \mathcal{Z}$.

Proposition 3.24. *The three families which make up* $\mathcal{X}'$ *are indeed disjoint in each case, and their union forms an antichain.*

Proof. $\mathcal{B}$ is an antichain by Proposition 3.16; $\mathcal{X}$ is an antichain by definition; and $\mathcal{Y}$ and $\mathcal{Z}$ are antichains because both consist of size 2 sets only. Let $D = \{d, o_1, \ldots, o_k\} \in \mathcal{X}$, $Y = \{d, o\} \in \mathcal{Y}$, $B = \{e_1, p_1, \ldots, p_l\} \in \mathcal{B}$ and $Z = \{e_2, q\} \in \mathcal{Z}$ with $d \in X$, $e_1 \in \tilde{A}$, $e_2 \in A \setminus \tilde{A}$, $o_i, o, p_i, q \in [n] \setminus X \setminus A$, and $l \geq 1$. B is unrelated to D by definition, and to Y because $d \notin B$ and $|B| \geq 2$. Z is unrelated to D and Y because $d \notin Z$ and $e_2 \notin D, Y$; Z is unrelated to B because $e_1 \notin Z$ and $e_2 \notin B$. □

Proposition 3.25. *Sets in* $\mathcal{C}$ *are disjoint from* X'*, and they are unrelated to every set in* $\mathcal{X}'$ *(in both cases).*

Proof. For every $C \in \mathcal{C}$, $C \subseteq [n] \setminus X'$ and it is unrelated to every set in $\mathcal{X}$ by definition. C is unrelated to every set in $\mathcal{B}$ by Proposition 3.16. It also cannot be a superset of a $Y \in \mathcal{Y}$ or a $Z \in \mathcal{Z}$, since those contain an element of X or A; neither a proper subset of Y or Z because $|Y| = |Z| = 2 \leq |C|$. □

Proposition 3.26. *The number of chains that start with an element of* $[n] \setminus X'$ *and contain a set from* $\mathcal{X}'$ *is*

- *at least* $\left[x(1-x-a)\overline{1} + \left(\beta - \tilde{a}(1-x-a)\overline{1} + \nu\right) + (1-x-a)\,(a-\tilde{a})\,\overline{1}\right] n!$ *in the* singletons case*, and*
- *at least* $\left[(\alpha - x + \mu) + \left(\beta - \tilde{a}(1-x-a)\overline{1} + \nu\right) + (1-x-a)\,(a-\tilde{a})\,\overline{1}\right] n!$ *in the* no singleton case*.*

Proof. The number of chains on which the singleton is $\{o\}$ with $o \in [n] \setminus X' = [n] \setminus X \setminus A$, and the second set is $\{o, d\} \in \mathcal{Y}$ with $d \in X$, is $xn \cdot (1-x-a)n \cdot (n-2)! = x(1-x-a)\overline{1}n!$. The number of chains on which the singleton is $\{o\}$, and the second set is $\{o, e\} \in \mathcal{Z}$ with $e \in A \setminus \tilde{A}$, is $(1-x-a)n \cdot (a-\tilde{a})n \cdot (n-2)! = (1-x-a)\,(a-\tilde{a})\,\overline{1}n!$. The rest follows from Proposition 3.21 and Proposition 3.22. □

Proposition 3.27. *The number of chains on which the singleton is $\{o\}$ with $o \in [n] \setminus X'$, the second set is $\{o, d\}$ with $d \in X' = X \cup A$, and which do not contain any set from $\mathcal{X}'$, is*

- $\nu n!$ *in the* singletons case*, and*
- $(\mu - x(x+a)\underline{1} + \nu)n!$ *in the* no singleton case*.*

Proof. Let $\mathcal{A} = \emptyset, A_1, A_2, \ldots, A_{n-1}, [n]$ be a chain with $\emptyset \subset A_1 \subset A_2 \subset \ldots \subset A_{n-1} \subset [n]$. Let $\varphi(\mathcal{A})$ be the chain $\emptyset, A_2 \setminus A_1, A_2, A_3, \ldots, A_{n-1}, [n]$. (In other words, in the order in which elements of $[n]$ are added to form the chain, the first two are swapped.) φ is a bijection.

It is easy to check that $\mathcal{X}'$ does not contain singletons. φ is a bijection between chains of the form $\emptyset, \{o\}, \{o, d\}, \ldots$ containing no set from $\mathcal{X}'$, and chains of the form $\emptyset, \{d\}, \{o, d\}, \ldots$ containing no set from $\mathcal{X}'$, with $o \in [n] \setminus X'$ and $d \in X \cup A$. Below we classify the chains $\emptyset, \{d\}, \{o, d\}, \ldots$ based on what set d belongs to and count them separately.

- For $d \in X$, $\{o, d\} \in \mathcal{Y}$ in the singletons case. In the no singleton case, $(\mu - x(x+a)\underline{1})n!$ chains of the form $\emptyset, \{d\}, \{o, d\}, \ldots$ contain no set from $\mathcal{X}$ by Proposition 3.23; these chains also contain no set from $\mathcal{B}$ or $\mathcal{Z}$, since sets from those do not contain any element of X.
- For $d \in \tilde{A}$, the number of chains of this form which contain no set from $\mathcal{B}$ is $\nu n!$; these chains also contain no set from $\mathcal{X}, \mathcal{Y}$ or $\mathcal{Z}$, since sets from those contain no element of $\tilde{A}$.
- For $d \in A \setminus \tilde{A}$, $\{o, d\} \in \mathcal{Z}$.

Summing these cases, we get the statement of the proposition. □

3.2.4 Inductive step

Notation. Using standard notation for intervals, let $[A, [n]]$ denote the Boolean lattice $\{S \subseteq [n] : A \subseteq S\}$. Let $[n] \setminus X' = [n] \setminus X \setminus A = \{o_1, o_2, \ldots, o_{(1-x-a)n}\}$; and for a family of sets $\mathcal{A}$, let $\mathcal{A} - o_i = \{S \setminus \{o_i\} : S \in \mathcal{A}\}$. Let $\mathcal{C}'_i = (\mathcal{C} \cap [\{o_i\}, [n]]) - o_i$, and $\mathcal{X}'_i = (\mathcal{X}' \cap [\{o_i\}, [n]]) - o_i$. Let $\alpha'_i = l(n-1, \mathcal{X}'_i)$. (Here the Lubell function on the Boolean lattice $2^{[n]\setminus\{o_i\}}$ of order $n-1$ is used.)

$\mathcal{C}'_i \subseteq 2^{[n]\setminus\{o_i\}}$ is a Λ-free family which does not contain $\emptyset$ (since $o_i \notin A$, so $\{o_i\} \notin \mathcal{F}$), nor any set larger than $n - 1 - n'$. Sets in $\mathcal{C}'_i$ are disjoint from X', and are unrelated to sets in $\mathcal{X}'_i$ by Proposition 3.25. Moreover, every set in $\mathcal{X}'_i$ contains exactly one element of X'. Therefore, the conditions of Lemma 3.13 are satisfied for the family $\mathcal{C}'_i \subseteq 2^{[n]\setminus\{o_i\}}$ where the corresponding "forbidden" set is $X' \subseteq [n] \setminus \{o_i\}$, with $\frac{|X'|}{n-1} = (x+a)\overline{1}$ and the corresponding "forbidden" antichain is $\mathcal{X}'_i$.

Since $\mathcal{X}'_i$ is an antichain, $\alpha'_i(n-1)!$ is the number of chains in $2^{[n]\setminus\{o_i\}}$ that contain a set from $\mathcal{X}'_i$. Chains of $2^{[n]\setminus\{o_i\}}$ correspond to chains of $2^{[n]}$ that start with $\{o_i\}$. So by Proposition 3.26, in the *singletons case*

$$\sum_{i=1}^{(1-x-a)n} \alpha'_i \geq \left[x(1-x-a)\overline{1} + \left(\beta - \tilde{a}(1-x-a)\overline{1} + \nu\right) + (1-x-a)\left(a - \tilde{a}\right)\overline{1}\right] n,$$

and in the *no singleton case*

$$\sum_{i=1}^{(1-x-a)n} \alpha_i' \geq \left[(\alpha - x + \mu) + \left(\beta - \tilde{a}(1-x-a)\overline{1} + \nu\right) + (1-x-a)\,(a-\tilde{a})\,\overline{1}\right] n.$$

Let $\mu_i'(n-1)!$ be the number of chains in the Boolean lattice $2^{[n]\setminus\{o_i\}}$ which start with an element of X' as a singleton, but do not contain any set from $\mathcal{X}_i'$. By Proposition 3.27, in the *singletons case*

$$\sum_{i=1}^{(1-x-a)n} \mu_i' = \nu n,$$

and in the *no singleton case*

$$\sum_{i=1}^{(1-x-a)n} \mu_i' = (\mu - x(x+a)\underline{1} + \nu)n.$$

Let $c_i'(n-1)!$ be the number of MNM chains w.r.t. $\mathcal{C}_i'$ in $2^{[n]\setminus\{o_i\}}$. The corresponding $2^{[n]}$-chains, starting with $\{o_i\}$, are MNM chains w.r.t. $\mathcal{F}$. The total number of MNM chains w.r.t. $\mathcal{F}$ is $cn!$, out of which $c_0 n!$ start with an element of A as a singleton. By Proposition 3.17,

$$\sum_{i=1}^{(1-x-a)n} c_i' = (c - c_0)n = (c - \tilde{a}(x+a)\underline{1} - \nu)\,n.$$

The following two examples are typical cases where, in the induction step for the $\mathcal{C}_i'$'s, we will get the singletons case and the no singleton case respectively.

Example 3.28. Let $X = \mathcal{X} = \emptyset$ and $\mathcal{B} = \{\{e,o\} : e \in A, o \in [n] \setminus A\}$. Then $\tilde{A} = A$, $\beta = 2a(1-a)\overline{1}$, and $\nu = 0$. $X' = A$, and $\mathcal{X}_i' = \{\{e\} : e \in A\}$. $\sum_{i=1}^{(1-a)n} \alpha_i' = a(1-a)\overline{1}n$, $\alpha_i' = a\overline{1} = \frac{|X'|}{n-1}$ and $\mu_i' = 0$. $\sum_{i=1}^{(1-x-a)n} c_i' = (c-a^2)\,n$ and the average of the c_i''s is $\frac{c-a^2}{1-a}$.

Example 3.29. Let $X = \mathcal{X} = \emptyset$ and $\mathcal{B} \subseteq \hat{\mathcal{B}} := \{\{e, o_1, o_2\} : e \in A, o_1, o_2 \in [n] \setminus A\}$. Then $X' = A$, and $\mathcal{X}_i' \subseteq \{\{e,o\} : e \in A, o \in [n] \setminus A \setminus \{o_i\}\}$. Chains on $2^{[n]}$ of the form $\emptyset, \{e_1\}, \{e_1, o\}, \{e_1, o, e_2\}, \ldots$ do not intersect $\mathcal{B}$. So $\sum_{i=1}^{(1-a)n} \mu_i' = \nu \geq a^2(1-a)\overline{1}^2 \frac{\overline{1}a(n-1)-1}{\overline{1}a(n-2)} n$ (greater if $\mathcal{B} \subsetneq \hat{\mathcal{B}}$), and the average of the μ_i''s is $\geq a^2\overline{1}^2 \frac{\overline{1}a(n-1)-1}{\overline{1}a(n-2)} = x'^2 \frac{x'(n-1)-1}{x'((n-1)-1)}$ where $x' = \frac{|X'|}{n-1}$. In the case of $\mathcal{B} = \hat{\mathcal{B}}$, the size of the sets in $\mathcal{C}$ is at least 3, and the size of those in $\mathcal{C}_i'$ is at least 2.

Proposition 3.30.

$$l(\mathcal{C}) = \frac{1}{n} \sum_{i=1}^{(1-x-a)n} l(n-1, \mathcal{C}_i') \quad \text{and} \quad l(\mathcal{F}) = a + \beta + l(\mathcal{C}) = a + \beta + \frac{1}{n} \sum_{i=1}^{(1-x-a)n} l(n-1, \mathcal{C}_i').$$

(Still understanding the one parameter version $l(\mathcal{F})$ as $l(n, \mathcal{F})$ for a family $\mathcal{F} \subseteq 2^{[n]}$.)

Proof. Every chain in the Boolean lattice $2^{[n]}$ that intersects $\mathcal{C}$ has an $\{o_i\}$ as a singleton, and thus corresponds to a chain in the Boolean lattice $[\{o_i\}, [n]] - o_i$ that intersects $\mathcal{C}_i'$.

$$l(\mathcal{C}) = \frac{1}{n!} \sum_{\mathcal{H} \text{ is a chain in } 2^{[n]}} |\mathcal{H} \cap \mathcal{C}| = \frac{1}{n!} \sum_{i=1}^{(1-x-a)n} \sum_{\mathcal{H} \text{ is a chain in } [\{o_i\},[n]]-o_i} |\mathcal{H} \cap \mathcal{C}_i'|$$

$$= \frac{1}{n} \sum_{i=1}^{(1-x-a)n} l(n-1, \mathcal{C}'_i).$$

Let $\mathcal{A} = \{\{e\} : e \in A\}$. Then $\mathcal{F} = \mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{C}$. So $l(\mathcal{F}) = l(\mathcal{A}) + l(\mathcal{B}) + l(\mathcal{C})$ with $l(\mathcal{A}) = \frac{|A|}{n} = a$ and $l(\mathcal{B}) = \beta$. $\square$

We now prove Lemma 3.13 (and thus Lemma 3.10) using induction on n. According to Proposition 3.14, Lemma 3.13 holds for $n \leq n'$. By induction and Lemma 3.12 Point 2,

$$\begin{aligned} l(n-1, \mathcal{C}'_i) &\leq f\left((x+a)\overline{1}, c'_i + \mu'_i + \frac{1}{n'}\right) - \left(\alpha'_i - \mu'_i - (x+a)\overline{1}\right) + \frac{3}{n'} \\ &\leq f\left(x+a, c'_i + \mu'_i + \frac{1}{n'}\right) - \left(\alpha'_i - \mu'_i - (x+a)\overline{1}\right) + \frac{3}{n'}. \end{aligned}$$

So, by Proposition 3.30, we have

$$\begin{aligned} l(\mathcal{C}) = \frac{1}{n} \sum_{i=1}^{(1-x-a)n} l(n-1, \mathcal{C}'_i) &\leq \frac{1}{n} \sum_{i=1}^{(1-x-a)n} f\left(x+a, c'_i + \mu'_i + \frac{1}{n'}\right) \\ - \frac{1}{n} &\left(\sum_{i=1}^{(1-x-a)n} \alpha'_i - \sum_{i=1}^{(1-x-a)n} \mu'_i - \sum_{i=1}^{(1-x-a)n} (x+a)\overline{1} \right) + \frac{1}{n} \cdot \frac{3(1-x-a)n}{n'}. \end{aligned}$$

We handle the case of $1-x-a = 0$ separately. If $1-x-a=0$, $A = [n] \setminus X$ and, since any non-singleton $\{e_1, e_2, \ldots\} \in \mathcal{F}$ would form a Λ with the singletons $\{e_1\}, \{e_2\} \in \mathcal{F}$, we have $\mathcal{F} = \mathcal{A}$ and $l(\mathcal{F}) = a = 1-x$. This is only possible in the singletons case, since a non-singleton in $\mathcal{X}$ would have to contain elements of $[n] \setminus X \setminus A$. In the singletons case $\alpha = x$ and $\mu = 0$, so $l(\mathcal{F}) = 1 - x \leq f(x,c) \leq f\big(x, c + \mu + \frac{1}{n'}\big) - (\alpha - \mu - x) + \frac{3}{n'}$ by Lemma 3.12 Point 5. From now on, we assume that $1-x-a > 0$.

Since f is concave in c, by Jensen's inequality, and since f is monotonously decreasing in x,

$$\begin{aligned} l(\mathcal{C}) \leq (1-x-a) f &\left(x+a, \frac{\sum_{i=1}^{(1-x-a)n} c'_i + \sum_{i=1}^{(1-x-a)n} \mu'_i}{(1-x-a)n} + \frac{1}{n'}\right) \\ - &\left(\frac{1}{n} \sum_{i=1}^{(1-x-a)n} \alpha'_i - \frac{1}{n} \sum_{i=1}^{(1-x-a)n} \mu'_i - (1-x-a)(x+a)\overline{1}\right) + \frac{3(1-x-a)}{n'}. \end{aligned}$$

Correction term calculations that we will use later (assuming $n' \leq n-1$):

$$\begin{aligned} (1-\underline{1})(x+\tilde{a})(x+a) + \frac{1-x-a}{n'} &= \frac{(x+\tilde{a})(1-x-a)}{n-1} + \frac{1-x-a}{n'} \\ &\leq \frac{(1+x+\tilde{a})(1-x-a)}{n'} \leq \frac{1-(x+a)^2}{n'} \leq \frac{1}{n'}. \end{aligned} \tag{3.1}$$

$$(1-\underline{1})\tilde{a}(x+a) + \frac{1-x-a}{n'} \leq (1-\underline{1})(x+\tilde{a})(x+a) + \frac{1-x-a}{n'} \leq \frac{1}{n'}. \tag{3.2}$$

$$\begin{aligned} 2\tilde{a}(1-x-a)(\overline{1}-1) + 2(\overline{1}-1)x - \big(2(\overline{1}-1) + (\underline{1}-1)\big)\, x(x+a) + \frac{3(1-x-a)}{n'} \\ \leq \frac{2a+3x}{n-1} + \frac{3(1-x-a)}{n'} \leq \frac{3}{n'}. \end{aligned} \tag{3.3}$$

$$2\tilde{a}(1-x-a)(\overline{1}-1)+\frac{3(1-x-a)}{n'} \leq \frac{2a}{n-1}+\frac{3(1-x-a)}{n'} \leq \frac{3}{n'}. \tag{3.4}$$

In the singletons case:

$$\begin{aligned}
l(\mathcal{F}) &\leq a+\beta+(1-x-a)f\left(x+a, \frac{(c-\tilde{a}(x+a)\underline{1}-\nu)+\nu}{1-x-a}+\frac{1}{n'}\right) \\
&-\left(\left[x(1-x-a)\overline{1}+\left(\beta-\tilde{a}(1-x-a)\overline{1}+\nu\right)+(1-x-a)\left(a-\tilde{a}\right)\overline{1}\right]\right. \\
&\left.-\nu-(1-x-a)(x+a)\overline{1}\right)+\frac{3(1-x-a)}{n'} \\
&= a+(1-x-a)f\left(x+a, \frac{c-\tilde{a}(x+a)\underline{1}}{1-x-a}+\frac{1}{n'}\right)+2\tilde{a}(1-x-a)\overline{1}+\frac{3(1-x-a)}{n'} \\
&= a+(1-x-a)f\left(x+a, \frac{c-\tilde{a}(x+a)+(1-\underline{1})\tilde{a}(x+a)+\frac{1-x-a}{n'}}{1-x-a}\right)+2\tilde{a}(1-x-a) \\
&+2\tilde{a}(1-x-a)(\overline{1}-1)+\frac{3(1-x-a)}{n'}.
\end{aligned}$$

By Lemma 3.12 Point 2 and Point 3, and (3.2) and (3.4) in the Correction term calculations (note that in this case $\alpha = x$ and $\mu = 0$),

$$l(\mathcal{F}) \leq g\left(x, c+\frac{1}{n'}, a, \tilde{a}\right)+\frac{3}{n'} \leq f\left(x, c+\frac{1}{n'}\right)+\frac{3}{n'} = f\left(x, c+\mu+\frac{1}{n'}\right)-(\alpha-\mu-x)+\frac{3}{n'}.$$

(Note that $\tilde{a} \leq \frac{c}{(x+a)\underline{1}}$, so $0 \leq \frac{c-\tilde{a}(x+a)\underline{1}}{1-x-a} \leq \frac{c+\frac{1}{n'}-\tilde{a}(x+a)}{1-x-a}$, and $\tilde{a} \leq \frac{c+\frac{1}{n'}}{x+a}$.)

In the no singleton case:

$$\begin{aligned}
l(\mathcal{F}) &\leq a+\beta+(1-x-a)f\left(x+a, \frac{(c-\tilde{a}(x+a)\underline{1}-\nu)+\mu-x(x+a)\underline{1}+\nu}{1-x-a}+\frac{1}{n'}\right) \\
&-\left(\left[(\alpha-x+\mu)+\left(\beta-\tilde{a}(1-x-a)\overline{1}+\nu\right)+(1-x-a)\left(a-\tilde{a}\right)\overline{1}\right]\right. \\
&\left.-\left[\mu-x(x+a)\underline{1}+\nu\right]-(1-x-a)(x+a)\overline{1}\right)+\frac{3(1-x-a)}{n'} \\
&= a+(1-x-a)f\left(x+a, \frac{c+\mu-(x+\tilde{a})(x+a)\underline{1}}{1-x-a}+\frac{1}{n'}\right)-(\alpha-x(x+a)\underline{1}-x) \\
&+2\tilde{a}(1-x-a)\overline{1}+(2\cdot\overline{1}-1)x-(2\cdot\overline{1}+\underline{1})x(x+a)+\frac{3(1-x-a)}{n'} \\
&= a+(1-x-a)f\left(x+a, \frac{c+\mu-(x+\tilde{a})(x+a)+(1-\underline{1})(x+\tilde{a})(x+a)+\frac{1-x-a}{n'}}{1-x-a}\right) \\
&+2\tilde{a}(1-x-a)+x-3x(x+a)-(\alpha-x(x+a)\underline{1}-x) \\
&+2\tilde{a}(1-x-a)(\overline{1}-1)+2(\overline{1}-1)x-\left(2(\overline{1}-1)+(\underline{1}-1)\right)x(x+a)+\frac{3(1-x-a)}{n'}.
\end{aligned}$$

By Lemma 3.12 Point 2 and Point 4, Proposition 3.23. and (3.1) and (3.3) in the Correction term calculations,

$$l(\mathcal{F}) \leq h\left(x, c+\mu+\frac{1}{n'}, a, \tilde{a}\right)-(\alpha-\mu-x)+\frac{3}{n'} \leq f\left(x, c+\mu+\frac{1}{n'}\right)-(\alpha-\mu-x)+\frac{3}{n'}.$$

(Note that $\tilde{a} \leq \frac{c}{(x+a)\underline{1}}$, so $0 \leq \frac{c-\tilde{a}(x+a)\underline{1}}{1-x-a} \leq \frac{c+\mu+\frac{1}{n'}-(x+\tilde{a})(x+a)}{1-x-a}$, and $\tilde{a} \leq \frac{c+\mu+\frac{1}{n'}}{x+a}-x$.)

3.3 Diamond-free families – Proof of Theorem 3.3

Let $\mathcal{F}$ be a diamond-free family on $2^{[n]}$.

We cite Lemma 1 from [8]:

Lemma 3.31 (Axenovich, Manske, Martin [8])**.**

$$\sum_{\substack{k\in\{0,1,\dots,n\}\\ |k-n/2|\geq n^{2/3}}} \binom{n}{k} \leq 2^{n-\Omega(n^{1/3})} = 2^{-\Omega(n^{1/3})}\binom{n}{\lfloor n/2\rfloor}.$$

By this lemma, the number of sets in $\mathcal{F}$ in the top and bottom $n' := n/2 - n^{\frac{2}{3}}$ levels is $o(1)\binom{n}{\lfloor n/2\rfloor}$, so, since we are bounding the cardinality of $\mathcal{F}$, we may assume that those levels do not contain any set from $\mathcal{F}$.

Notation. For $c \in [0,1]$, let $\tilde{c} = \min\left(c, \frac{1}{4}\right)$, and let $f(c) = 1 - \tilde{c} + \sqrt{\tilde{c}}$. (This is equal to $f(0,c)$ as defined in Definition 3.11.) For $A \in \mathcal{F}$, recall that $[A,[n]]$ denotes the Boolean lattice $\{S \subseteq [n] : A \subseteq S\}$. A chain of this lattice is of the form $A \subset A_{|A|+1} \subset A_{|A|+2} \subset \ldots \subset A_{n-1} \subset [n]$. (When saying just "chain", we continue to mean a maximal chain in the Boolean lattice $2^{[n]}$.) Let

$$c(A) = \frac{1}{(n-|A|)\,!}\#\left\{\begin{matrix}\mathcal{C} \text{ is a chain in } [A,[n]] :\\ \mathcal{C} \text{ is MNM w.r.t. } \mathcal{F}\cap[A,[n]]\end{matrix}\right\}.$$

Further, we can assume without loss of generality that

$$C := \frac{1}{n!}\#\left\{\begin{matrix}\mathcal{C} \text{ is a chain: } \mathcal{C}\cap\mathcal{F}=\emptyset \text{ or}\\ \min(\mathcal{C}\cap\mathcal{F}) \text{ is not minimal in } \mathcal{F}\end{matrix}\right\} \geq \frac{1}{n!}\#\left\{\begin{matrix}\mathcal{C} \text{ is a chain: } \mathcal{C}\cap\mathcal{F}=\emptyset \text{ or}\\ \mathcal{C} \text{ is MNM w.r.t. } \mathcal{F}\end{matrix}\right\}.$$

(If this does not hold, we can replace $\mathcal{F}$ with $\{[n]\setminus A : A \in \mathcal{F}\}$: this family is diamond-free, has the same cardinality, and the opposite inequality holds.)

Clearly

$$C \geq \frac{1}{n!}\#\left\{\begin{matrix}\mathcal{C} \text{ is a chain:}\\ \mathcal{C} \text{ is MNM w.r.t. } \mathcal{F}\end{matrix}\right\}.$$

$$l(\mathcal{F}) = \frac{1}{n!}\sum_{\mathcal{C}\text{ is a chain}} \#(\mathcal{C}\cap\mathcal{F}) = \frac{1}{n!}\sum_{A\in\mathcal{F}}\sum_{\substack{\mathcal{C}\text{ is a chain}\\ A=\min(\mathcal{C}\cap\mathcal{F})}} \#(\mathcal{C}\cap\mathcal{F}),$$

since each chain $\mathcal{C}$ will be counted when $A = \min(\mathcal{C}\cap\mathcal{F})$ – except if $\mathcal{C}\cap\mathcal{F} = \emptyset$, but then $\#(\mathcal{C}\cap\mathcal{F}) = 0$.

Continuing, $l(\mathcal{F})$ is equal to

$$\frac{1}{n!}\sum_{\substack{A\in\mathcal{F}\\ \exists\text{ a chain }\mathcal{C}:\ A=\min(\mathcal{C}\cap\mathcal{F})}} \#\left\{\begin{matrix}\mathcal{C} \text{ is a chain containing } A :\\ A=\min(\mathcal{C}\cap\mathcal{F})\end{matrix}\right\} \frac{\displaystyle\sum_{\substack{\mathcal{C}\text{ is a chain}\\ A=\min(\mathcal{C}\cap\mathcal{F})}} \#(\mathcal{C}\cap\mathcal{F})}{\#\left\{\begin{matrix}\mathcal{C} \text{ is a chain containing } A :\\ A=\min(\mathcal{C}\cap\mathcal{F})\end{matrix}\right\}}.$$

Each chain on $[A,[n]]$ can be extended to a full $2^{[n]}$-chain in $|A|\,!$ ways. Furthermore, the Boolean lattice $[A,[n]]$ can be made equivalent to the Boolean lattice $2^{[n]\setminus A}$ by subtracting

A from each set; for $\mathcal{A} \subseteq [A, [n]]$, we denote $\mathcal{A} - A = \{S \setminus A : S \in \mathcal{A}\}$. If $A = \min(\mathcal{C} \cap \mathcal{F})$, $\#(\mathcal{C} \cap \mathcal{F}) = \#(\mathcal{C} \cap [A, [n]] \cap \mathcal{F})$. If A is minimal in $\mathcal{F}$ (that is, on every chain),

$$\frac{\displaystyle\sum_{\substack{\mathcal{C} \text{ is a chain} \\ A = \min(\mathcal{C} \cap \mathcal{F})}} \#(\mathcal{C} \cap \mathcal{F})}{\#\left\{\begin{matrix} \mathcal{C} \text{ is a chain containing } A : \\ A = \min(\mathcal{C} \cap \mathcal{F}) \end{matrix}\right\}} = \frac{|A|\,! \displaystyle\sum_{\mathcal{C} \text{ is a chain in } [A,[n]]} \#(\mathcal{C} \cap \mathcal{F} \cap [A, [n]])}{|A|\,!\,(n - |A|)\,!}$$

$$= \frac{\displaystyle\sum_{\mathcal{C} \text{ is a chain in } 2^{[n] \setminus A}} \#(\mathcal{C} \cap ((\mathcal{F} \cap [A, [n]]) - A))}{(n - |A|)\,!} = l(n - |A|\,, (\mathcal{F} \cap [A, [n]]) - A).$$

$(\mathcal{F} \cap [A, [n]]) - A$ is diamond-free, so $((\mathcal{F} \cap [A, [n]]) - A) \setminus \emptyset$ is Λ-free; and the top n' levels are assumed to be empty. Using Lemma 3.10 as well as that $\frac{1}{n'} = \frac{1}{\Omega(n)} = o(1)$ and the subadditivity of the square root function,

$$l(n - |A|\,, ((\mathcal{F} \cap [A, [n]]) - A) \setminus \emptyset) \leq 1 - \min\left(c(A) + \frac{1}{n'}, \frac{1}{4}\right) + \sqrt{\min\left(c(A) + \frac{1}{n'}, \frac{1}{4}\right) + \frac{3}{n'}}$$

$$\leq f(c(A)) + \sqrt{\frac{1}{n'}} + \frac{3}{n'} = f(c(A)) + o(1),$$

so $l(n - |A|\,, (\mathcal{F} \cap [A, [n]]) - A) \leq 1 + f(c(A)) + o(1)$. Whereas if A is not minimal in $\mathcal{F}$, i.e. $\exists S \in \mathcal{F}$ such that $A \supsetneq S$, then for any chain $\mathcal{C}$ for which $\min(\mathcal{C} \cap \mathcal{F}) = A$, we have $\#(\mathcal{C} \cap \mathcal{F}) \leq 2$ (otherwise S and three sets in $\mathcal{C} \cap \mathcal{F}$ would form a diamond), so

$$\frac{\sum_{\substack{\mathcal{C} \text{ is a chain} \\ A = \min(\mathcal{C} \cap \mathcal{F})}} \#(\mathcal{C} \cap \mathcal{F})}{\#\left\{\substack{\mathcal{C} \text{ is a chain through } A: \\ A = \min(\mathcal{C} \cap \mathcal{F})}\right\}} \leq 2.$$

$$\begin{aligned} l(\mathcal{F}) \leq \frac{1}{n!} &\sum_{\substack{A \in \mathcal{F} \\ A \text{ is minimal in } \mathcal{F}}} \#\left\{\mathcal{C} \text{ is a chain containing } A\right\}(1 + f(c(A)) + o(1)) \\ &+ \frac{1}{n!} \sum_{\substack{A \in \mathcal{F} \\ A \text{ is not minimal in } \mathcal{F}}} \#\left\{\begin{matrix} \mathcal{C} \text{ is a chain containing } A : \\ A = \min(\mathcal{C} \cap \mathcal{F}) \end{matrix}\right\} \cdot 2 \\ &\leq 2 + \frac{1}{n!} \sum_{\substack{A \in \mathcal{F} \\ A \text{ is minimal in } \mathcal{F}}} \#\left\{\mathcal{C} \text{ is a chain containing } A\right\}(f\left(c\left(A\right)\right) - 1 + o(1))\,. \end{aligned}$$

Since f is concave, we can use Jensen's inequality with the weights $\frac{\#\{\mathcal{C} \text{ is a chain containing } A\}}{(1-C)n!}$ (where A is minimal in $\mathcal{F}$). Notice that the sum of all the weights is 1 because the sum of numerators is the total number of chains $\mathcal{C}$ where $\min(\mathcal{C} \cap \mathcal{F})$ is minimal in $\mathcal{F}$, that is, $(1 - C)n!$.

$$l(\mathcal{F}) \leq 2 + (1 - C)\left(f\left(\frac{\displaystyle\sum_{\substack{A \in \mathcal{F} \\ A \text{ is minimal in } \mathcal{F}}} \#\left\{\mathcal{C} \text{ is a chain containing } A\right\} c(A)}{(1 - C)n!}\right) - 1 + o(1)\right).$$

$c(A)$ is the fraction of the chains containing A which are MNM, so

$$\# \{\mathcal{C} \text{ is a chain containing } A\} \, c(A)$$

is the number of MNM chains through A. In the numerator, each MNM chain in the whole Boolean lattice is counted once, except if the minimal element on it is not a global minimal, then it is not counted. So the numerator is less than or equal to the total number of MNM chains in the Boolean lattice, which is at most $Cn!$. Substituting, we get

$$l(\mathcal{F}) \leq 2 + (1-C)\left(f\left(\frac{n!\,C}{n!\,(1-C)}\right) - 1 + o(1)\right) = 1 + C + (1-C) f\left(\frac{C}{1-C}\right) + o(1).$$

C varies between 0 and 1. $\frac{C}{1-C}$ is increasing in C. Above $\frac{C}{1-C} = \frac{1}{4}$ (corresponding to $C = \frac{1}{5}$), $f(\frac{C}{1-C})$ is constant $\frac{5}{4}$, so $1 + C + (1-C) f\left(\frac{C}{1-C}\right) = \frac{9}{4} - \frac{C}{4}$ is decreasing in C. So it is enough to take the maximum in the interval $\left[0, \frac{1}{5}\right]$:

$$\begin{aligned} \frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}} \leq l(\mathcal{F}) &\leq \max_{C \in \left[0, \frac{1}{5}\right]} \left(1 + C + (1-C)\left(1 - \frac{C}{1-C} + \sqrt{\frac{C}{1-C}}\right)\right) + o(1) \\ &= \frac{\sqrt{2}+3}{2} + o(1) < 2.20711 + o(1). \end{aligned}$$

Open problems and remarks

We conclude Part I of the thesis by presenting some open problems in this area.

As discussed in this chapter, the most investigated (and the simplest) poset for which even the asymptotic value of $La(n, P)$ has yet to be determined is the diamond poset. For the induced version of the diamond problem an upper bound of $2.58\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is known [114].

Besides the diamond, there are still many posets for which the asymptotic value of $La(n, P)$ is not known. One such poset is the crown poset O_{2t} defined by the relations $x_1 < y_1 > x_2 < y_2 \ldots x_t < y_t > x_1$. $La(n, O_{2t})$ has been determined asymptotically for all values of t except when $t = 3, 5$ in [68, 113]. The best known upper bound of roughly $1.70711\binom{n}{\lfloor \frac{n}{2} \rfloor}$ in these two open cases is proved by Griggs and Lu [68]. A better bound is obtained in the case when the family is restricted to 2 levels: Gerbner, Grósz, Martin, Methuku, Walker and Uzzell [57] proved an upper bound of $1.371\binom{n}{\lfloor \frac{n}{2} \rfloor}$ in this case.

Some examples include the crown poset O_{2t} defined by the relations $x_1 < y_1 > x_2 < y_2 \ldots x_t < y_t > x_1$, the harp poset $\mathcal{H}(l_1, \ldots, l_k)$ consisting of paths $P_{l_1}, \ldots, P_{l_k}$ with their top elements identified and their bottom elements identified where $k \geq 1$ and $l_1 \geq \ldots \geq l_k \geq 3$, generalized diamond poset $D_k := \mathcal{H}(3, \ldots, 3)$ (i.e., each $l_i = 3$ for $1 \leq i \leq k$). For any poset P, Griggs and Lu [68] proposed the following conjecture:

Conjecture 3.32 (Griggs, Lu [68])**.** The limit $\pi(P) := \lim_{n\to\infty} \frac{La(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ exists and is an integer.

We now list some well-known posets for which $\pi(P)$ hasn't been determined yet. Note that we only list the best known bound to our knowledge and not the previous bounds and only those cases which haven't been settled yet. Since we do not know if $\pi(P)$ exists, to save space we just write $a \le \pi(P) \le b$ instead of

$$a \le \liminf_{n\to\infty} \frac{La(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \le \limsup_{n\to\infty} \frac{La(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \le b.$$

Poset P	$\pi(P)$
Crowns O_6 and O_{10}	$1 \le \pi \le 1.70711$
Crown O_6 (if the family is restricted to middle 2 levels)	$1 \le \pi \le 1.371$
Diamond D_2	$2 \le \pi \le 2.20711$
Diamond (if the family is restricted to middle 3 levels)	$2 \le \pi \le 2.15121$
Induced Diamond (i.e., with relations $a < b < d$ and $a < c < d$ but b and c must be unrelated)	$2 \le \pi \le 2.58$
Generalized Diamonds D_k with $k \in [2^m - \binom{m}{\frac{m}{2}}, 2^m - 2]$ where $m := \lceil log_2(k+2) \rceil$	$m \le \pi < m+1$
Harp $\mathcal{H}(l_1, \ldots, l_k)$ where the path lengths l_i are not all distinct	Open

Part II

Extremal graph theory

Chapter 4

Background on Extremal graph theory

Basic Notation and Definitions

A *graph* G is a pair of sets $V(G)$ and $E(G)$, where $V(G)$ denotes the *vertices* and $E(G)$ denotes the set of *edges* where the edges are sets of two distinct vertices. Except when stated otherwise, we will only allow a pair of vertices to occur as an edge once. Usually an edge will be written as uv where u and v are vertices. We say that two vertices are *adjacent* if they form an edge and that a vertex and edge are *incident* if the vertex is in the edge. Two edges that share a vertex will also be called *incident*. The total number of edges in a graph G is denoted $e(G) = |E(G)|$. A *subgraph* is defined as follows: H is a subgraph of G if it is possible to obtain the graph H after the removal of some number of edges and vertices from G. We use the notation $H \subset G$ to denote that H is a subgraph of G.

Given a vertex v in a graph G, the *degree* of v is the number of edges incident to v; it is denoted $d(v)$. The *maximum degree* in a graph G is the largest degree among all of the vertices. *Minimum degree* is defined similarly.

A graph is *connected* if we can travel between every pair of vertices along edges of the graph. The *chromatic number* of a graph G is the minimum integer k such that we can assign colors $1, 2, \ldots, k$ to the vertices of G and have no edge with the same color on each vertex; it is denoted $\chi(G)$.

The *complete graph* (or *clique*) on r vertices is denoted K_r; the *complete bipartite graph* with class sizes s and t is denoted $K_{s,t}$; the k-vertex *cycle* is denoted C_k and the k-vertex *path* is denoted P_k. *Length* of a path P_k is $k-1$, the number of edges in it. A connected graph that does not contain cycles is a *tree*.

A hypergraph is a generalization of a graph where an edge may contain any number of vertices. Thus, if we fix a ground set $[n]$ a hypergraph $\mathcal{H}$ is simply a family of subsets of $[n]$. These subsets are called *hyperedges*. A hypergraph is *k-uniform* if all hyperedges are of size k. Thus a graph is a 2-uniform hypergraph. In certain contexts the term *set system* or *family of sets* is used instead of hypergraph.

Throughout the rest of the thesis we use standard order notions. When it is ambiguous, we write the parameter(s) that the constant depends on, as a subscript.

Turán-type problems

Turán-type problems are generally formulated in the following way: one fixes some graph properties and tries to determine the maximum number of edges an n-vertex graph with the prescribed properties can have. These kinds of extremal problems have a rich history in combinatorics, going back to 1907, when Mantel [120] determined the maximum number of edges possible in a triangle-free graph.

Theorem 4.1 (Mantel [120]). *The maximum number of edges in a graph on n vertices with no triangle subgraph is $\lfloor \frac{n^2}{4} \rfloor$.*

Proof. The proof goes by induction on n. If $n = 1, 2$ we are done, so assume $n > 2$ and that the statement of the theorem holds for smaller graphs. Let G be a triangle-free graph on n vertices and let xy be an edge of G. The graph $G - xy$ obtained by removing the vertices x and y from G, is obviously triangle-free and has $n-2$ vertices, so it has at most $\lfloor \frac{(n-2)^2}{4} \rfloor$ edges by induction. The edge xy has at most $n - 2$ edges incident (otherwise there is a triangle). Thus G has at most $1 + (n - 2) + \frac{(n-2)^2}{4} = \frac{n^2}{4}$ edges. □

Observe that the n-vertex complete bipartite graph with class sizes $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ has no triangle subgraph and has exactly $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n^2}{4} \rfloor$ edges. So it remains to show that we cannot have more edges.

The systematic study of these problems began with Turán [146], who generalized Mantel's result to arbitrary complete graphs (see [54, 137] for surveys on this topic).

Theorem 4.2 (Turán [146]). *The maximum number of edges in a graph on n vertices with no K_{r+1} is at most $\left(1 - \frac{1}{r}\right) \frac{n^2}{2}$.*

For simple graphs G and F, we say that G is *F-free* if G does not contain F as a subgraph.

Definition 4.3. *Given a set of graphs $\mathcal{F}$ and a positive integer n, the* Turán number *of $\mathcal{F}$ is*

$$\mathrm{ex}(n, \mathcal{F}) := \max\{|E(G)| : |V(G)| = n \text{ and } G \text{ is } F\text{-free for every } F \in \mathcal{F}\}.$$

For a bipartite graph F, the bipartite Turán number $\mathrm{ex}(m, n, F)$ is the maximum number of edges in an F-free bipartite graph with m and n vertices in its color classes.

Erdős, Stone and Simonovits [38, 39] showed that the behavior of the Turán number of a general graph F is determined by its chromatic number, $\chi(F)$, when $\chi(F) \geq 3$. They proved that if F is a simple graph, then

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right) \frac{n^2}{2} + o(n^2),$$

which is an asymptotically correct result except when F is bipartite.

In the bipartite case, one of the most natural problems is to estimate the Turán number of even cycles. A classical result of Bondy and Simonovits [15] from 1974 is the following.

Theorem 4.4 (Bondy, Simonovits [15]). *For any $k \geq 2$, we have $\mathrm{ex}(n, C_{2k}) = O(n^{1+\frac{1}{k}})$.*[1]

[1]Throughout the thesis, we use the standard asymptotic notations O, o, Θ understood as $n \to \infty$.

A major open question in extremal graph theory is whether this upper bound gives the correct order of magnitude. This was verified for $k = 2, 3$ and 5. For example, the best known bounds for hexagons are due to Füredi, Naor and Verstraëte [52], who proved that

$$0.5338\, n^{4/3} \le \mathrm{ex}(n, C_6) \le 0.6272\, n^{4/3}. \tag{4.1}$$

The corresponding girth problem was studied by Erdős and Simonovits [37], who conjectured that the same lower bound holds even if we also forbid cycles of shorter lengths.

Conjecture 4.5 (Erdős–Simonovits)**.** For any $k \ge 2$, we have

$$\mathrm{ex}(n, \{C_3, \ldots, C_{2k}\}) = \Theta(n^{1+\frac{1}{k}}).$$

This conjecture is only known to hold for $k = 2, 3, 5$. The case when all cycles longer than a given length are forbidden, was considered by Erdős and Gallai [34].

Theorem 4.1 (Erdős, Gallai [34])**.** *If a graph does not contain any cycle of length more than k, then it has at most* $\frac{(k-1)n}{2}$ *edges.*

On the other hand, if all the short cycles are forbidden, Alon, Hoory and Linial [2] proved the following. To state their result let us introduce the following notation: let A be a set of integers, each at least 3. Then let the set of cycles $\mathcal{C}_A = \{C_a : a \in A\}$. If $A = \{3, 4, ..., k\}$ for some integer k, then we denote the corresponding set of cycles by $\mathcal{C}_k$.

Theorem 4.2 (Alon, Hoory, Linial [2])**.** *For any* $k \ge 2$ *we have*

(i) $\mathrm{ex}(n, \mathcal{C}_{2k}) < \frac{1}{2}n^{1+1/k} + \frac{1}{2}n$,

(ii) $\mathrm{ex}(n, \mathcal{C}_{2k+1}) < \frac{1}{2^{1+1/k}}n^{1+1/k} + \frac{1}{2}n$.

For more information on Turán number of cycles one can consult the survey [147].

One of the most important results concerning the Turán number of complete bipartite graphs is due to Kővári, Sós and Turán [105], who showed that $\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s})$, where $s \le t$. Kollár, Rónyai and Szabó [99] provided a lower bound matching the order of magnitude, when $t > s!$. Later Alon, Rónyai and Szabó [4] provided a matching lower bound for $t > (s-1)!$.

For $s = 2$, Füredi proved the following nice result determining the asymptotics for the Turán number of $K_{2,t}$. His construction is inspired by that of Mörs [126] for the Zarankiewicz problem.

Theorem 4.3 (Füredi [48])**.** *For any fixed* $t \ge 2$*, we have*

$$\mathrm{ex}(n, K_{2,t}) = \frac{\sqrt{t-1}}{2} \cdot n^{3/2} + O(n^{4/3})$$

Moreover, he also determined the asymptotics for the balanced case of the bipartite Turán number of $K_{2,t}$.

Theorem 4.4 (Füredi [48])**.** *For any fixed* $t \ge 2$*, we have*

$$\mathrm{ex}(n, n, K_{2,t}) = \sqrt{t-1} \cdot n^{3/2} + O(n^{4/3}).$$

For a survey on the extremal number of bipartite graphs, we refer the reader to [54].

Chapter 5

On subgraphs of C_{2k}-free graphs and a problem of Kühn and Osthus

5.1 Introduction

Recall that $e(H)$ denotes the number of (hyper)edges in a (hyper)graph H and that the girth of a graph is defined as the length of a shortest cycle if it exists, and infinity otherwise. In [75], Győri proved that every bipartite, C_6-free graph contains a C_4-free subgraph with at least half as many edges. Later Kühn and Osthus [107] generalized this result by showing

Theorem 5.1 (Kühn and Osthus [107])**.** *Let $k \geq 3$ be an integer and G a C_{2k}-free bipartite graph. Then G contains a C_4-free subgraph H with $e(H) \geq \frac{e(G)}{k-1}$.*

In Section 5.2 we give a new short proof of their result, using Mirsky's theorem. The complete bipartite graphs $K_{k-1,m}$ (for large enough m) show that the factor $\frac{1}{k-1}$ cannot be replaced by anything larger (see Proposition 5 in [107]).

Füredi, Naor and Verstraëte [52] gave another generalization of Győri's theorem by showing that every C_6-free graph G has a subgraph of girth larger than 4 with at least half as many edges as G. Again, $K_{2,m}$ shows that this factor cannot be improved. It follows that $\mathrm{ex}(n, C_6) \leq 2 \cdot \mathrm{ex}(n, \{C_4, C_6\})$. Since any graph has a bipartite subgraph with at least half as many edges, Theorem 5.1 shows that $\mathrm{ex}(n, C_{2k}) \leq 2(k-1) \cdot \mathrm{ex}(n, \{C_4, C_{2k}\})$. These results confirm special cases of the compactness conjecture of Erdős and Simonovits [37] which states that for every finite family $\mathcal{F}$ of graphs, there exists an $F \in \mathcal{F}$ such that $\mathrm{ex}(n, F) = O(\mathrm{ex}(n, \mathcal{F}))$.

Since any C_6-free graph contains a bipartite subgraph with at least half as many edges, using any of the results above it is easy to show that any C_6-free graph G has a bipartite, C_4-free subgraph with at least $\frac{1}{4}$ of the edges of G. Győri, Kensell and Tompkins [78] improved this factor by showing that

Theorem 5.2 (Győri, Kensell and Tompkins [78])**.** *If c is the largest constant such that every C_6-free graph G contains a C_4-free and bipartite subgraph B with $e(B) \geq c \cdot e(G)$, then $\frac{3}{8} \leq c \leq \frac{2}{5}$.*

The complete graph K_5 (as well as a graph consisting of vertex disjoint K_5's) gives that $c \leq \frac{2}{5}$. To show that $\frac{3}{8} \leq c$ they use a probabilistic deletion procedure where they first randomly two-color the vertices, and then delete some additional edges carefully in order to remove the remaining C_4's. In this chapter we show that $c = \frac{3}{8}$. In fact, we prove the following two general results; putting $k = 3$ in either of the statements below gives that $c = \frac{3}{8}$. To prove these theorems we will construct graphs by replacing the hyperedges of certain (probabilistically constructed) hypergraphs with fixed small graphs.

Theorem 5.3 (Grósz, M., Tompkins [69]). *For any $\varepsilon > 0$, and any integer $k \geq 2$, there is a C_{2k}-free graph G which does not contain a bipartite subgraph of girth greater than $2k$ with more than $\left(1 - \frac{1}{2^{2k-2}}\right) \frac{2}{2k-1} e(G)(1+\varepsilon)$ edges.*

Note that the graph K_{2k-1} is C_{2k}-free, and its only subgraphs with girth greater than $2k$ are forests. This immediately implies that if c_k is the largest constant such that every C_{2k}-free graph G contains a subgraph of girth greater than $2k$ with $c \cdot e(G)$ edges, then $c_k \leq \frac{2}{2k-1}$ (even without requiring the subgraph to be bipartite). Theorem 5.3 improves this trivial upper bound, and in its proof we will get the factor $1 - \frac{1}{2^{2k-2}}$ as the probability that a random two-coloring of K_{2k-1} is not monochromatic.

Theorem 5.4 (Grósz, M., Tompkins [69]). *For any $\varepsilon > 0$, and any integer $k \geq 2$, there is a C_{2k}-free graph G which does not contain a bipartite and C_4-free subgraph with more than $\left(1 - \frac{1}{2^{k-1}}\right) \frac{1}{k-1} e(G)(1+\varepsilon)$ edges.*

Theorem 5.4 improves the upper bound of $\frac{1}{k-1} e(G)(1+\varepsilon)$, which is given by the complete bipartite graphs $K_{k-1,m}$. (We cannot replace the factor $\frac{1}{k-1}$ with anything larger, even if we do not require the C_4-free subgraph to be bipartite.) Take a random bipartition of the vertices of $K_{k-1,m}$ and consider the bipartite subgraph B between the colour classes of this bipartition. In the proof of Theorem 5.4, we get the factor $\left(1 - \frac{1}{2^{k-1}}\right) \frac{1}{k-1}$ as the limit of the expected value of the fraction of edges of $K_{k-1,m}$ in the biggest C_4-free subgraph of B as $m \to \infty$. (Note that because any graph has a bipartite subgraph with at least half of its edges, Theorem 5.1 implies that every C_{2k}-free graph contains a bipartite and C_4-free subgraph with at least $\frac{1}{2(k-1)}$ fraction of its edges.) Interestingly, our proofs use theorems about hypergraphs that are generalizations of the following theorem of Erdős [32].

Every graph G has a bipartite subgraph with at least $\frac{1}{2}$ as many edges as G, and the complete graph K_n shows that the factor $\frac{1}{2}$ cannot be improved. Interestingly, Erdős showed that even if one requires girth to be large, the factor $\frac{1}{2}$ still cannot be improved. More precisely,

Theorem 5.5 (Erdős [32]). *For any $\varepsilon > 0$, and any integer $k \geq 2$, there exists a graph G with girth greater than k which does not contain a bipartite subgraph with more than $\frac{1}{2} e(G)(1+\varepsilon)$ edges.*

In Section 5.3, we prove a series of lemmas about hypergraphs which are broad generalizations of Theorem 5.5, and which may be of independent interest. These lemmas have the theme that for most hypergraphs, every fixed coloring behaves like a random coloring with color classes of the same sizes as in the fixed coloring. Our proof of Theorem 5.4 uses these general lemmas directly. The proof of Theorem 5.3 uses a more direct analogue of the above statement for hypergraphs: Theorem 5.6, which we present below. We will prove Theorem 5.6 from the more general lemmas.

A Berge-cycle of length l in a hypergraph H is a subhypergraph consisting of $l \geq 2$ distinct hyperedges $e_1, \ldots, e_l$ and containing l distinct vertices $v_1, \ldots, v_l$ (called its *defining* vertices), such that $v_i \in e_i \cap e_{i+1}$, $i = 1, \ldots, l$, where addition in the indices is taken modulo l. The girth of a hypergraph H is the length of a shortest Berge-cycle if it exists, and infinity otherwise. (Note that having girth greater than 2 implies that no two hyperedges share more than one vertex.) A hypergraph is b-colorable if there is a coloring of its vertices using b colors so that none of its hyperedges are monochromatic. Erdős and Hajnal [35] showed the existence of hypergraphs of any uniformity, arbitrarily high girth and arbitrarily high chromatic number. Lovász [112] gave a constructive proof for this; several newer proofs exist as well. The following simple proposition is easy to see. We include its proof for completeness.

Proposition 5.1 (Grósz, M., Tompkins [69]). *For any integers $a, b \geq 2$, every a-uniform hypergraph H contains a b-colorable subhypergraph with at least $\left(1 - \frac{1}{b^{a-1}}\right) e(H)$ hyperedges.*

Proof. Color each vertex of H randomly and independently, using b colors with equal probability. For each hyperedge f of H, the probability that f is monochromatic is $\frac{b}{b^a} = \frac{1}{b^{a-1}}$. Therefore, the expected number of monochromatic hyperedges in H is $\frac{e(H)}{b^{a-1}}$. So there exists a coloring of the vertices of H such that there are at most $\frac{e(H)}{b^{a-1}}$ monochromatic hyperedges in that coloring. Thus, the subhypergraph of H consisting of all the non-monochromatic hyperedges of H contains at least $\left(1 - \frac{1}{b^{a-1}}\right) e(H)$ hyperedges and is b-colorable, as desired. □

Again the complete a-uniform hypergraph shows that the factor $\left(1 - \frac{1}{b^{a-1}}\right)$ cannot be improved in the above proposition. We show that (as in case of graphs), this factor cannot be improved even if one requires the girth to be large.

Theorem 5.6 (Grósz, M., Tompkins [69]). *For any $\varepsilon > 0$, and any integers $a, b, k \geq 2$, there exists an a-uniform hypergraph H of girth more than k which does not contain a b-colorable subhypergraph with more than $\left(1 - \frac{1}{b^{a-1}}\right) e(H) (1 + \varepsilon)$ hyperedges.*

Clearly, letting $a = b = 2$ in the above theorem, we get Theorem 5.5. The hypergraph lemmas in Section 5.3 can be used to prove statements analogous to Theorem 5.6 with different notions of colorability. As an example application, we will prove the analogous Proposition 5.10 about rainbow (or strong) colorable subhypergraphs. More generally, a graph G is called H-colorable (where H is a fixed graph) if there is a homomorphism $G \to H$. Our Lemma 5.6 can be said to generalize the notion of H-coloring to hypergraphs, and allow for proving statements similar to Theorem 5.6 for H-colorability or analogous hypergraph conditions.

In Section 5.5, we answer a question of Kühn and Osthus in [107] in the negative. A graph is said to be *pasted together* from $C_{2\ell}$'s if it can be obtained from a $C_{2\ell}$ by successively adding new $C_{2\ell}$'s which have at least one edge in common with the previous ones.

Question 5.2 (Kühn, Osthus [107]). Given integers $k > \ell \geq 2$, does there always exist a number $d = d(k)$ such that every C_{2k}-free graph which is pasted together from $C_{2\ell}$'s has average degree at most d?

Kühn and Osthus show in [107] that an affirmative answer to the above question, even when restricted to bipartite graphs, would imply that any C_{2k}-free graph G contains a $C_{2\ell}$-free subgraph containing a constant fraction of the edges of G. They gave a positive answer to the question when $\ell = 2$ and the graph is bipartite: they showed that if $k \geq 3$

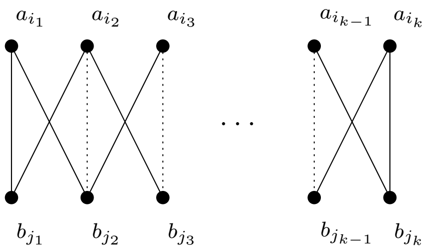


Figure 5.1: The solid edges form a C_{2k}

is an integer and G is a bipartite C_{2k}-free graph which is obtained by pasting together C_4's, then the average degree of G is at most $16k$.

We answer Question 5.2 negatively by showing two different pastings of C_6's to form a C_8-free graph with high average degree. These two examples show (in two very different ways) that many C_6's can be packed into a graph while still keeping it C_8-free. We will show that the first example can be easily generalized to any pair k, ℓ with $k > \ell \geq 3$, showing that $\ell = 2$ is the only case when any C_{2k}-free graph obtained by pasting together $C_{2\ell}$'s has average degree bounded by a constant.

The chapter is organized as follows: In Section 5.2, we give a short proof of Theorem 5.1. In Section 5.3, we prove a series of hypergraph lemmas and Theorem 5.6. Our proofs in Section 5.3 use counting arguments and probabilistic ideas very similar to Erdős's proof. In Section 5.4 we prove Theorem 5.3 and Theorem 5.4. In Section 5.5, we give two examples of pasting together C_6's to form a C_8-free graph with high average degree, answering Question 5.2.

5.2 A simple proof of a theorem of Kühn and Osthus (Theorem 5.1)

The following proof appears in my paper [69] that is co-authored with Grósz and Tompkins.

Proof of Theorem 5.1. Let G be a C_{2k}-free bipartite graph, and let us label its color classes as $A = \{a_1, a_2, \dots, a_l\}$ and $B = \{b_1, b_2, \dots, b_m\}$ for some $l, m \geq 1$. We define a partial order $P = (E(G), \leq_P)$ on the edge set of G as the transitive closure of the following relation: for any two edges $a_i b_j, a_{i'} b_{j'} \in E(G)$, we say that $a_i b_j \leq_P a_{i'} b_{j'}$ if $a_i b_j = a_{i'} b_{j'}$, or if $i < i'$, $j < j'$ and $a_i, b_j, a_{i'}, b_{j'}$ induce a C_4.

It is easy to see that if there is a chain of length k in P, then G contains a cycle of length $2k$, a contradiction (see Figure 5.1). So the length of a longest chain in P is at most $k-1$, which implies that the size of a largest antichain in P is at least $\frac{1}{k-1}|E(G)|$ by Mirsky's theorem [125]. Since G is bipartite, any C_4 in G contains two edges ab, $a'b' \in E(G)$ such that $ab \leq_P a'b'$, so the subgraph H of G consisting of the edges in this largest antichain is C_4-free, completing the proof of the theorem. □

5.3 Hypergraph lemmas and proof of Theorem 5.6

Let $\mathcal{H}(a,n,m)$ denote the family of all a-uniform hypergraphs with n vertices and m hyperedges for some $a \geq 2$. $|\mathcal{H}(a,n,m)| = \binom{\binom{n}{a}}{m}$. Given a coloring $C : [n] \to [b]$ of the vertex set $[n]$ with b colors (with $b \geq 2$), let n_j^C be the number of vertices of color j. The multiset of the colors of the vertices of a hyperedge e (with the multiplicity with which they occur in e) is called the *color multiset of e (with respect to C)*, denoted $c^C(e)$. For an a-element multiset of colors T, let $p^C(T)$ be the probability that the color multiset of a random hyperedge of the complete a-uniform hypergraph on n vertices, with the coloring C, is T. (Note that in this chapter, when we mention a coloring, we mean an arbitrary coloring of the vertex set, not necessarily a proper coloring of a hypergraph, unless indicated.)

The following proposition is a simple consequence of the definition of $p^C(T)$, so we omit the proof.

Proposition 5.3. *For $n \to \infty$, asymptotically*

$$p^C(T) = \frac{\prod_{j=1}^{b} \binom{n_j^C}{I_T(j)}}{\binom{n}{a}} \sim \frac{\prod_{j=1}^{b} \left(n_j^C\right)^{I_T(j)}}{n^a} \cdot \frac{a!}{\prod_{j=1}^{b} I_T(j)!}$$

where $I_T(j)$ denotes the multiplicity of j in the multiset T.

We will also use the following tail bound on the binomial and the hypergeometric distributions. Hoeffding proves this bound in a more general setting, see Section 2 in [90] for the binomial distribution and Section 6 for the hypergeometric distribution. If a random variable X has binomial distribution with m trials and success probability p, we write $X \sim \text{Binomial}(m,p)$. If X has hypergeometric distribution with a population of size N containing pN successes, and with m draws, we write $X \sim \text{Hypergeometric}(pN, N, m)$.

Proposition 5.4 (Hoeffding [90])**.** *Let $m, N \in \mathbb{N}$ and $p, \varepsilon \in [0,1]$, and let X be a random variable with $X \sim \text{Binomial}(m,p)$ or $X \sim \text{Hypergeometric}(pN, N, m)$. Then*

$$P(|X - pm| > \varepsilon m) \leq 2e^{-2\varepsilon^2 m}.$$

Lemma 5.5. *Let $n \to \infty$ and $\frac{m}{n} \to \infty$. For any fixed $\varepsilon > 0$, for every hypergraph H in $\mathcal{H}(a,n,m)$, with the exception of $o\left(\binom{\binom{n}{a}}{m}\right)$ hypergraphs, the following holds:*

> *For any coloring C of the vertex set $[n]$ with b colors, and any a-element multiset of colors T, the number of hyperedges of H whose color multiset is T is $\left(p^C(T) \pm \varepsilon\right) m$.* (5.1)

(Note: In this chapter, whenever we write $X = Y \pm \varepsilon$, we mean $X \in [Y - \varepsilon, Y + \varepsilon]$.)

Proof. Let u be the number of hypergraphs in $\mathcal{H}(a,n,m)$ for which (5.1) does not hold. Corresponding to each such hypergraph H there is at least one b-coloring C of its vertices, and a multiset of colors T, such that (5.1) does not hold for C and T. Therefore

$$u \leq |\{(H,C,T) : H \in \mathcal{H}(a,n,m), \text{(5.1) does not hold for } H, C \text{ and } T\}|.$$

The number of b-colorings of n vertices with b fixed colors is $|\mathcal{C}| = b^n$. The number of multisets of a elements of b colors is $\binom{a+b-1}{a}$. Therefore

$$u \leq b^n \binom{a+b-1}{a} \max_{\substack{\text{coloring } C \\ \text{multiset of colors } T}} \left| \left\{ \begin{array}{c} H \in \mathcal{H}(a,n,m): \\ (5.1) \text{ does not hold for } H, C \text{ and } T \end{array} \right\} \right|.$$

Fix a b-coloring C and a multiset of colors T. A hypergraph $H \in \mathcal{H}(a,n,m)$ consists of m hyperedges, out of $\binom{n}{a}$ possibilities. Out of all possible hyperedges, $p^C(T)\binom{n}{a}$ have T as their color multiset. So $\left|\{e \in H : c^C(e) = T\}\right| \sim \text{Hypergeometric}\big(p^C(T)\binom{n}{a}, \binom{n}{a}, m\big)$. (5.1) fails to hold for H, C and T if

$$\Big| \left|\{e \in H : c^C(e) = T\}\right| - p^C(T)m \Big| > \varepsilon m.$$

By the tail bound for the hypergeometric distribution in Proposition 5.4, the number of hypergraphs $H \in \mathcal{H}(a,n,m)$ for which this holds is at most

$$\binom{\binom{n}{a}}{m} \cdot 2e^{-2\varepsilon^2 m}, \text{ so}$$

$$u \leq 2b^n \binom{a+b-1}{a} e^{-2\varepsilon^2 m} \binom{\binom{n}{a}}{m} = o\left(\binom{\binom{n}{a}}{m}\right)$$

as $\frac{m}{n} \to \infty$. □

The following lemma is a corollary of Lemma 5.5.

Lemma 5.6. *Let $\mathcal{T}$ be a family of multisets of a elements which are in $[b]$. Let $n \to \infty$ and $\frac{m}{n} \to \infty$. For a b-coloring of n vertices C, let $p^C(\mathcal{T}) = \sum_{T \in \mathcal{T}} p^C(T)$ (that is, the probability that the color multiset of a random hyperedge of the complete a-uniform hypergraph is in $\mathcal{T}$); and let C_M be a b-coloring for which $p^C(\mathcal{T})$ takes its maximum. For a hypergraph $H \in \mathcal{H}(a,n,m)$, let $q(H)$ be the number of hyperedges in the biggest subhypergraph of H which is colorable in such a way that the color multiset of every hyperedge of H is in $\mathcal{T}$. For any fixed $\varepsilon > 0$, for every hypergraph H in $\mathcal{H}(a,n,m)$, with the exception of $o\left(\binom{\binom{n}{a}}{m}\right)$ hypergraphs,*

$$q(H) \leq p^{C_M}(\mathcal{T}) m \left(1 + \varepsilon\right). \tag{5.2}$$

Proof. If $\mathcal{T} = \emptyset$, then $q(H) = 0$ for any H. From now we assume that $\mathcal{T} \neq \emptyset$. We show that we may also assume that $p^{C_M}(\mathcal{T}) > \frac{|\mathcal{T}|}{2b^a}$ when n is sufficiently large. Let $T \in \mathcal{T}$, and let C_E be a b-coloring in which every color class has size $\approx \frac{n}{b}$. Then, by Proposition 5.3, asymptotically

$$p^{C_E}(T) = \frac{a!}{b^a \prod_{j=1}^{b} I_T(j)!} \geq \frac{1}{b^a}.$$

Since $p^{C_M}(\mathcal{T}) \geq p^{C_E}(\mathcal{T}) = \sum_{T \in \mathcal{T}} p^{C_E}(T)$, for sufficiently large n, $p^{C_M}(\mathcal{T}) \geq \frac{|\mathcal{T}|}{2b^a}$.

An equivalent definition of the function q is

$$q(H) = \max_{b\text{-coloring } C} \left|\{e \in H : c^C(e) \in \mathcal{T}\}\right|.$$

We use Lemma 5.5 with $\frac{\varepsilon}{2b^a}$ in place of ε. For almost every hypergraph $H \in \mathcal{H}(a,n,m)$, for every coloring C,

$$\left|\left\{e \in H : c^C(e) \in \mathcal{T}\right\}\right| = \sum_{T \in \mathcal{T}} \left|\left\{e \in H : c^C(e) = T\right\}\right| \le \sum_{T \in \mathcal{T}} \left(p^C(T) + \frac{\varepsilon}{2b^a}\right) m$$
$$= \left(p^C(\mathcal{T}) + \frac{\varepsilon\,|\mathcal{T}|}{2b^a}\right) m \le \left(p^{C_M}(\mathcal{T}) + \frac{\varepsilon\,|\mathcal{T}|}{2b^a}\right) m \le p^{C_M}(\mathcal{T}) m\,(1+\varepsilon)$$

using that $p^{C_M}(\mathcal{T}) \ge \frac{|\mathcal{T}|}{2b^a}$. □

We define an *oriented hypergraph* as a set of ordered sequences without repetition (called hyperedges) over a vertex set, such that two hyperedges are not allowed to differ only in their order. (The order of the vertices on different hyperedges is independent of each other.) An oriented hypergraph is thus equivalent to a hypergraph along with a total order on the vertices of each hyperedge. Let $\mathcal{O}(a,n,m)$ denote the family of all a-uniform oriented hypergraphs with n vertices and m hyperedges. (Note that other meanings of the term "oriented hypergraph" exist in the literature.)

Let $C : [n] \to [b]$ be a coloring of the vertex set $[n]$ with b colors ($b \ge 2$). We call the *color sequence (with respect to C)* of an a-tuple of vertices $e = (v_1, \ldots, v_a)$ the sequence $c^C(e) = (C(v_1), \ldots, C(v_a))$. If we choose a random a-tuple of the vertex set V without repetition, the probability that its color sequence is a given sequence of colors $s = (s_1, \ldots, s_a)$ is

$$\frac{1}{\binom{n}{a} a!} \prod_{j=1}^{b} \frac{n_j^C!}{\left(n_j^C - |\{i \in [a] : s_i = j\}|\right)!} \sim \prod_{i=1}^{a} \frac{n_{c_i}^C}{n}$$

if $n \to \infty$.

The following lemma is a variant of Lemma 5.5 for oriented hypergraphs.

Lemma 5.7. *Let $n \to \infty$ and $\frac{m}{n} \to \infty$. For any fixed $\varepsilon > 0$, for every oriented hypergraph O in $\mathcal{O}(a,n,m)$, with the exception of $o(|\mathcal{O}(a,n,m)|)$ hypergraphs, the following holds:*

For any coloring C of the vertex set $[n]$ with b colors, and any a-tuple of colors s, the number of hyperedges of O whose color sequence is s is $\left(\prod_{i=1}^{a} \frac{n_{s_i}^C}{n} \pm \varepsilon\right) \cdot m$. (5.3)

Proof. We use Lemma 5.5 with $\frac{\varepsilon}{4}$ in the place of ε, i.e. that (5.1) holds (with $\frac{\varepsilon}{4}$) for almost every hypergraph $H \in \mathcal{H}(a,n,m)$. In every hypergraph in $\mathcal{H}(a,n,m)$, the hyperedges can be ordered in the same number of ways: $(a!)^m$. So for almost every $O \in \mathcal{O}(a,n,m)$, (5.1) holds for the corresponding hypergraph (obtained by forgetting the orders on the hyperedges).

Let $\tilde{\mathcal{O}}(a,n,m) \subset \mathcal{O}(a,n,m)$ be the family of oriented hypergraphs for which (5.1) holds (forgetting the orders) with $\frac{\varepsilon}{4}$ in the place of ε. Let u be the number of oriented hypergraphs in $\tilde{\mathcal{O}}(a,n,m)$ for which (5.3) does not hold. Corresponding to each such oriented hypergraph $O \in \tilde{\mathcal{O}}(a,n,m)$, there is at least one b-coloring C of its vertices, and an a-tuple of colors s, such that (5.3) does not hold for C and s. Therefore

$$u \le \left|\left\{(O, C, s) : O \in \tilde{\mathcal{O}}(a,n,m),\ \text{(5.3) does not hold for } O,\ C \text{ and } s\right\}\right|.$$

The number of b-colorings of n vertices with b fixed colors is $|\mathcal{C}| = b^n$. The number of a-tuples of b colors is b^a. Therefore

$$u \le b^{n+a} \max_{\substack{\text{coloring } C \\ a\text{-tuple of colors } s}} \left| \left\{ \begin{array}{c} (O,C,s) : O \in \tilde{\mathcal{O}}(a,n,m), \\ (5.3) \text{ does not hold for } O,\ C \text{ and } s \end{array} \right\} \right|.$$

Fix a b-coloring C and an a-tuple of colors s. Let T be the multiset consisting of the elements of s with the multiplicity with which they occur in s (that is, T is s forgetting the order). If (5.1) holds for a $H \in \mathcal{H}(a,n,m)$ with $\frac{\varepsilon}{4}$, the number of hyperedges whose color multiset is T is

$$\begin{aligned} M_H &:= \left(p^C(T) \pm \frac{\varepsilon}{4}\right) m = \left(\frac{\prod_{j=1}^b \left(n_j^C\right)^{I_T(j)}}{n^a} \cdot \frac{a!}{\prod_{j=1}^b I_T(j)!} \pm \frac{\varepsilon}{2} \right) m \\ &= \left(\left(\prod_{i=1}^a \frac{n_{s_i}^C}{n} \right) \cdot \frac{a!}{\prod_{j=1}^b I_T(j)!} \pm \frac{\varepsilon}{2} \right) m \end{aligned}$$

using the Proposition 5.3 for large enough n. (Changing $\frac{\varepsilon}{4}$ to $\frac{\varepsilon}{2}$ accounts for the fact that Proposition 5.3 is asymptotic.) We can obtain an oriented hypergraph from H by ordering its hyperedges in one of the $a!$ possible ways, independently from each other. If we take a hyperedge whose color multiset is T, some of these orders yield the color sequence s. The number of such orders is $\prod_{j=1}^b I_T(j)!$, so if we take a random ordering of a hyperedge whose color multiset is T, the probability that it has color sequence s is

$$\frac{\prod_{j=1}^b I_T(j)!}{a!}.$$

So if we obtain an oriented hypergraph O by randomly ordering every hyperedge of H, then $\left|\left\{e \in O : c^C(e) = s\right\}\right| \sim \text{Binomial}\left(M_H, \frac{\prod_{j=1}^b I_T(j)!}{a!}\right)$, and the expected value of the number of hyperedges whose color sequence is s is

$$E_H := \frac{\prod_{j=1}^b I_T(j)!}{a!} M_H = \left(\prod_{i=1}^a \frac{n_{s_i}^C}{n} \pm \frac{\varepsilon}{2} \right) m.$$

If the number of hyperedges whose color sequence is s is in the range $\left[E_H - \frac{\varepsilon}{2}m, E_H + \frac{\varepsilon}{2}m\right]$, then (5.3) holds for O, C and s, since

$$E_H \pm \frac{\varepsilon}{2} m = \left(\prod_{i=1}^a \frac{n_{s_i}^C}{n} \pm \varepsilon \right) m.$$

We want to bound the probability that in a randomly selected oriented hypergraph obtained from H, the number of hyperedges whose color sequence is s is not in the range $\left[E_H - \frac{\varepsilon}{2}m, E_H + \frac{\varepsilon}{2}m\right] = \left[E_H - \frac{\varepsilon m}{2M_H} M_H, E_H + \frac{\varepsilon m}{2M_H} M_H\right]$. By the tail bound for the binomial distribution in Proposition 5.4, this probability is at most

$$2 \cdot e^{-2((\varepsilon m)/(2M_H))^2 M_H} = 2e^{-\left(\varepsilon^2/2\right)\cdot(m/M_H)\cdot m} \le 2e^{-\left(\varepsilon^2/2\right)\cdot m}, \text{ so}$$

$$u \le b^{n+a} 2e^{-\left(\varepsilon^2/2\right)\cdot m} \left|\tilde{\mathcal{O}}(a,n,m)\right| = o(|\mathcal{O}(a,n,m)|)$$

as $\frac{m}{n} \to \infty$. □

Lemma 5.8. *Let $n \to \infty$, $k \geq 2$ and $m = o\left(n^{1+\frac{1}{k}}\right)$. Every hypergraph H in $\mathcal{H}(a,n,m)$, with the exception of $o\left(\binom{\binom{n}{a}}{m}\right)$ hypergraphs, has at most n Berge-cycles with k or fewer hyperedges.*

Proof. A Berge-cycle of length l has l defining vertices, and each of its l hyperedges contains $a-2$ additional vertices. So the number of Berge-cycles of length l is less than $n^{(a-1)l}$. The number of hypergraphs in $\mathcal{H}(a,n,m)$ which contain a fixed Berge-cycle of length l is $\binom{\binom{n}{a}-l}{m-l}$, since the l hyperedges of the Berge-cycle can be arbitrarily extended to a hypergraph of m hyperedges. Therefore the number of pairs (H,B) where $H \in \mathcal{H}(a,n,m)$ and B is any Berge-cycle of length l in H, is less than

$$n^{(a-1)l}\binom{\binom{n}{a}-l}{m-l} < n^{(a-1)l}\binom{\binom{n}{a}}{m}\left(\frac{m}{\binom{n}{a}}\right)^{l} = O\left(\left(\frac{m}{n}\right)^{l}\binom{\binom{n}{a}}{m}\right).$$

Let $f_k(H)$ denote the number of Berge-cycles of length k or less in H. Using $m = o\left(n^{1+\frac{1}{k}}\right)$, we have

$$\sum_{H\in\mathcal{H}(a,n,m)} f_k(H) = \sum_{l=2}^{k} O\left(\left(\frac{m}{n}\right)^{l}\binom{\binom{n}{a}}{m}\right) = O\left(\left(\frac{m}{n}\right)^{k}\binom{\binom{n}{a}}{m}\right) = o\left(n\binom{\binom{n}{a}}{m}\right).$$

The number of hypergraphs $H \in \mathcal{H}(a,n,m)$ with more than n Berge-cycles of length k or less is clearly

$$\frac{o\left(n\binom{\binom{n}{a}}{m}\right)}{n} = o\left(\binom{\binom{n}{a}}{m}\right),$$

proving Lemma 5.8. □

Proposition 5.9. *For any $\varepsilon > 0$ and $k \geq 2$, there exists an a-uniform hypergraph H of girth more than k for which* (5.1) *in Lemma 5.5 and* (5.2) *in Lemma 5.6 hold. There also exists an a-uniform oriented hypergraph O of girth more than k (using the usual meaning of girth, not taking the orders on the hyperedges into consideration) for which* (5.3) *in Lemma 5.7 holds.*

Proof. Take a sufficiently large n, and $m = o\left(n^{1+\frac{1}{k}}\right)$ but such that $\frac{m}{n} \to \infty$ as $n \to \infty$. Then there is a hypergraph $H \in \mathcal{H}(a,n,m)$ such that (5.1) in Lemma 5.5 holds with $\frac{\varepsilon}{4b^a}$ in place of ε, and H contains at most n Berge-cycles with k or fewer hyperedges (indeed, all but $o\left(\binom{\binom{n}{a}}{m}\right)$ hypergraphs have both properties). Now remove a hyperedge from every Berge-cycle of length k or smaller in H. The resulting hypergraph H' has $m-n$ hyperedges. Fix any coloring C and an a-element multiset of colors T. In H, the number of hyperedges whose color multiset with respect to C is T is $\left(p^C(T) \pm \frac{\varepsilon}{4b^a}\right)m$. The number of such hyperedges in H' is at least $\left(p^C(T) - \frac{\varepsilon}{4b^a}\right)m - n$ and at most $\left(p^C(T) + \frac{\varepsilon}{4b^a}\right)m$, so it is in the range $\left(p^C(T) \pm \frac{\varepsilon}{2b^a}\right)(m-n)$ for big enough n because $\frac{m}{n} \to \infty$. So (5.1) in Lemma 5.5 holds for H', even with $\frac{\varepsilon}{2b^a}$ in the place of ε. From the proof of Lemma 5.6 it is clear that if (5.1) holds with $\frac{\varepsilon}{2b^a}$, then (5.2) holds.

In every hypergraph in $\mathcal{H}(a,n,m)$, the hyperedges can be ordered in the same number of ways, so Lemma 5.8 holds for oriented hypergraphs too. The proof in the previous paragraph works similarly for oriented hypergraphs, proving the existence of O. □

Now we use Proposition 5.9 to prove Theorem 5.6.

Proof of Theorem 5.6. By Proposition 5.9, there is an a-uniform hypergraph H of girth more than k for which (5.2) in Lemma 5.6 holds. We use (5.2) with $\mathcal{T}$ consisting of those multisets which contain at least two different colors, and with $\frac{\varepsilon}{2}$ in the place of ε. With the notation of Lemma 5.6,

$$q(H) < p^{C_M}(\mathcal{T})m\left(1+\frac{\varepsilon}{2}\right) = \left(\sum_{T\in\mathcal{T}} p^{C_M}(T)\right) m\left(1+\frac{\varepsilon}{2}\right)$$
$$= \left(1-\sum_{j=1}^{b} p^{C_M}\left(\{\overbrace{j,j,\dots,j}^{a}\}\right)\right) m\left(1+\frac{\varepsilon}{2}\right) \le \left(1-\sum_{j=1}^{b}\left(\frac{n_j^{C_M}}{n}\right)^a\right) m\left(1+\varepsilon\right)$$

using the asymptotic Proposition 5.3 for sufficiently large n. $\sum_{j=1}^{b} n_j^{C_M} = n$, and using the power mean inequality we get that

$$\left(\frac{1}{b}\sum_{j=1}^{b}\left(\frac{n_j^{C_M}}{n}\right)^a\right)^{\frac{1}{a}} \ge \frac{1}{b}.$$

So $\sum_{j=1}^{b}\left(\frac{n_j^{C_M}}{n}\right)^a \ge \frac{1}{b^{a-1}}$, which implies the statement. □

We show another example application of Lemma 5.6 and Proposition 5.9. A b-coloring of the vertices of a hypergraph is called a *rainbow (or strong) coloring* if all the vertices have different colors in every hyperedge. (For a-uniform hypergraphs, this is only possible if $a \le b$.)

Proposition 5.10. *Let $n \to \infty$ and $\frac{m}{n} \to \infty$. For any fixed $\varepsilon > 0$ and integers $2 \le a \le b$, every hypergraph H in $\mathcal{H}(a, n, m)$, with the exception of $o\left(\binom{\binom{n}{a}}{m}\right)$ hypergraphs, contains no subhypergraph that is rainbow colorable with b colors with more than $\binom{b}{a}\frac{a!}{b^a}e(H)(1+\varepsilon)$ hyperedges. Furthermore, for any $\varepsilon > 0$ and integers $k \ge 2$ and $2 \le a \le b$, there exists an a-uniform hypergraph H of girth more than k which does not contain a subhypergraph that is rainbow colorable with b colors with more than $\binom{b}{a}\frac{a!}{b^a}e(H)(1+\varepsilon)$ hyperedges.*

Proof. A hypergraph coloring is a rainbow coloring if the color multiset of every hyperedge is a conventional set (i.e., every color appears at most once in the multiset). Let $\mathcal{T} = \binom{[b]}{a}$. We will prove that if (5.2) in Lemma 5.6 holds for a hypergraph H with this $\mathcal{T}$ and with $\frac{\varepsilon}{2}$ in the place of ε, then it does not contain a subhypergraph that is rainbow colorable with b colors with more than $\binom{b}{a}\frac{a!}{b^a}e(H)(1+\varepsilon)$ hyperedges. The first statement of the proposition then follows directly from Lemma 5.6, while the second statement follows from Lemma 5.9.

Assume that (5.2) in Lemma 5.6 holds for a hypergraph H. With the notation of Lemma 5.6, and using the asymptotic Proposition 5.3 for large enough n,

$$q(H) < p^{C_M}(\mathcal{T})m\left(1+\frac{\varepsilon}{2}\right) = \left(\sum_{T\in\mathcal{T}} p^{C_M}(T)\right) m\left(1+\frac{\varepsilon}{2}\right)$$
$$\le \left(\sum_{T\in\mathcal{T}}\prod_{j\in T}\frac{n_j^{C_M}}{n}\right) a!\, m\left(1+\varepsilon\right). \tag{5.4}$$

We claim that, under the assumption that $\sum_{j=1}^{b} n_j^{C_M} = n$, (5.4) takes its maximum when $n_1^{C_M} = \ldots = n_b^{C_M} = \frac{n}{b}$. Let us assume that the $n_j^{C_M}$'s are not all equal – then there is a j_1 and j_2 such that $n_{j_1}^{C_M} < \frac{n}{b} < n_{j_2}^{C_M}$. Rewriting the first factor in (5.4), we have

$$\sum_{T\in\mathcal{T}}\prod_{j\in T}\frac{n_j^{C_M}}{n} = \overbrace{\sum_{\tilde{T}\in\binom{[b]\setminus\{j_1,j_2\}}{a}}\prod_{j\in\tilde{T}}\frac{n_j^{C_M}}{n}}^{|T\cap\{j_1,j_2\}|=0} + \overbrace{\left(n_{j_1}^{C_M}+n_{j_2}^{C_M}\right)\sum_{\tilde{T}\in\binom{[b]\setminus\{j_1,j_2\}}{a-1}}\prod_{j\in\tilde{T}}\frac{n_j^{C_M}}{n}}^{|T\cap\{j_1,j_2\}|=1}$$
$$+\overbrace{n_{j_1}^{C_M}n_{j_2}^{C_M}\sum_{\tilde{T}\in\binom{[b]\setminus\{j_1,j_2\}}{a-2}}\prod_{j\in\tilde{T}}\frac{n_j^{C_M}}{n}}^{|T\cap\{j_1,j_2\}|=2}.$$

If we replace $n_{j_1}^{C_M}$ with $\frac{n}{b}$, and $n_{j_2}^{C_M}$ with $n_{j_2}^{C_M} - \frac{n}{b} + n_{j_1}^{C_M}$, (5.4) does not decrease: the first two terms do not change, while in the third term, $n_{j_1}^{C_M}n_{j_2}^{C_M}$ is replaced by $\frac{n}{b}\left(n_{j_2}^{C_M} - \frac{n}{b} + n_{j_1}^{C_M}\right) = n_{j_1}^{C_M}n_{j_2}^{C_M} + \left(n_{j_2}^{C_M} - \frac{n}{b}\right)\left(\frac{n}{b} - n_{j_1}^{C_M}\right) > n_{j_1}^{C_M}n_{j_2}^{C_M}$. Repeating this step, we can increase the number of $n_j^{C_M}$'s which are equal to $\frac{n}{b}$ without decreasing (5.4), until all of them equal $\frac{n}{b}$.

So

$$q(H) \leq \left(\sum_{T\in\mathcal{T}}\prod_{j\in T}\frac{1}{b}\right) a!\, m\,(1+\varepsilon) = \binom{b}{a}\frac{a!}{b^a} m\,(1+\varepsilon)\,. \qquad \square$$

5.4 Subgraphs of C_{2k}-free graphs – Proofs of Theorems 5.3 and 5.4

Proof of Theorem 5.3. Fix $\varepsilon > 0$. By Theorem 5.6, there exists a $2k-1$-uniform hypergraph H with girth more than $2k$ which does not contain a 2-colorable subhypergraph having more than $\left(1 - \frac{1}{2^{2k-2}}\right) e(H)\,(1+\varepsilon)$ hyperedges. We produce a graph G_H from the hypergraph H by replacing each hyperedge of H with a complete graph (i.e. a clique) on $2k-1$ vertices. We refer to these complete graphs as $2k-1$-cliques. It is easy to check that the resulting graph G_H is C_{2k}-free.

Notice that since the girth of H is more than $2k \geq 4$, no two hyperedges of H intersect in more than 1 vertex. Therefore, the $2k-1$-cliques of G_H are edge-disjoint, and by definition every edge of G_H is in some $2k-1$-clique. We show that G_H does not have a bipartite subgraph with girth more than $2k$ which has more than $\left(1 - \frac{1}{2^{2k-2}}\right)\frac{2k-2}{\binom{2k-1}{2}} e(G_H)\,(1+\varepsilon) = \left(1 - \frac{1}{2^{2k-2}}\right)\frac{2}{2k-1} e(G_H)\,(1+\varepsilon)$ edges. Assume that B is a bipartite subgraph of G_H with girth more than $2k$. Notice that any set of more than $2k-2$ edges from a clique on $2k-1$ vertices must contain a cycle of length at most $2k-1$. Therefore B can contain at most $2k-2$ edges from each $2k-1$-clique of G_H. Furthermore, since B is bipartite, there is a 2-coloring of the vertices so that the edges of B are properly colored. If an edge of B is contained in a $2k-1$-clique of G_H, then the corresponding hyperedge of H contains two vertices with different colors in this 2-coloring. By our assumption on H, at most $\left(1 - \frac{1}{2^{2k-2}}\right)(1+\varepsilon)$ fraction of the hyperedges are not monochromatic in this 2-coloring of the vertices. So B has at most $\left(1 - \frac{1}{2^{2k-2}}\right)(2k-2)e(H)\,(1+\varepsilon)$ edges. Since $e(G_H) = \binom{2k-1}{2}e(H)$, B has at most $\left(1 - \frac{1}{2^{2k-2}}\right)\frac{2}{2k-1}e(G_H)\,(1+\varepsilon)$ edges, as desired. $\square$

In the proof of Theorem 5.4, we use the following proposition. For a proof, see the proof of Proposition 5 in [107]. (Note that the bound can be attained when $w \geq \binom{u}{2}$.)

Proposition 5.11. *In the complete bipartite graph $K_{u,w}$, a C_4-free subgraph has at most $w + \binom{u}{2}$ edges.*

Proof of Theorem 5.4. Let l be a large integer. By Proposition 5.9, there exists a $(k-1+l)$-uniform oriented hypergraph O with girth more than $2k$ for which (5.3) in Lemma 5.7 holds with $\frac{\varepsilon}{24 \cdot 2^{k-1+l}}$ in place of ε. Let n be the number of vertices of O. We produce a graph G_O from the oriented hypergraph O by replacing each hyperedge of O with a copy of $K_{k-1,m}$ the following way: in a hyperedge $(v_1, \ldots, v_{k-1+l})$, we connect every vertex in $\{v_1, \ldots, v_{k-1}\}$ with every vertex in $\{v_k, \ldots, v_{k-1+l}\}$ with an edge. The resulting graph G_O is C_{2k}-free.

Since the girth of O is more than $2k \geq 4$, no two hyperedges of O intersect in more than 1 vertex. Therefore the copies of $K_{k-1,l}$ in G_O are edge-disjoint, and by definition every edge of G_O is in one of the copies of $K_{k-1,l}$. We show that G_O does not have a bipartite and C_4-free subgraph which has more than $\left(1 - \frac{1}{2^{k-1}}\right) \frac{1}{k-1} e(G_O)(1+\varepsilon)$ edges. Assume that B is a bipartite and C_4-free subgraph of G_O, its classes being pn red vertices and $(1-p)n$ blue vertices. Now consider a random hyperedge $e = (v_1, \ldots, v_{k-1}, v_k, \ldots, v_{k-1+l})$ of O. How many edges of B can there be between the vertices of e? Each such edge has a red and a blue endpoint; also, each such edge has an endpoint in $\{v_1, \ldots, v_{k-1}\}$ and an endpoint in $\{v_k, \ldots, v_{k-1+l}\}$. Let u and w be the number of red vertices among $\{v_1, \ldots, v_{k-1}\}$ and $\{v_k, \ldots, v_{k-1+l}\}$ respectively. The restriction of B to the vertices of e (which we will denote $B|_e$) is thus a C_4-free subgraph of the union of a $K_{u,l-w}$ and a $K_{k-1-u,w}$ on disjoint vertex sets. We have three possibilities:

- $u \notin \{0, k-1\}$. Then, by Proposition 5.11, $B|_e$ consists of at most $l - w + \binom{u}{2} + w + \binom{k-1-u}{2} < l + \binom{k-1}{2}$ edges.
- $u = k-1$. Then $K_{k-1-u,w}$ is degenerate (as $k-1-u = 0$), and $B|_e$ has at most $l - w + \binom{k-1}{2}$ edges.
- $u = 0$. Then $K_{u,l-w}$ is degenerate, and $B|_e$ has at most $w + \binom{k-1}{2} = l + \binom{k-1}{2} - (l-w)$ edges.

Let $(C_1, \ldots, C_{k-1+l})$ be the color sequence of e (with $C_i \in \{\text{red}, \text{blue}\}$). For any color sequence $(c_1, \ldots, c_{k-1+l})$ (with $c_i \in \{\text{red}, \text{blue}\}$), the probability that $(C_1, \ldots, C_{k-1+l}) = (c_1, \ldots, c_{k-1+l})$ is $p^{|\{i : c_i = \text{red}\}|}(1-p)^{|\{i : c_i = \text{blue}\}|} \pm \frac{\varepsilon}{24 \cdot 2^{k-1+l}}$ since (5.3) in Lemma 5.7 holds for O with $\frac{\varepsilon}{24 \cdot 2^{k-1+l}}$. (Note that e was chosen as a random hyperedge of O.) Let $\tilde{C}_1, \ldots, \tilde{C}_{k-1+l}$ be independent random variables which take the value "red" with probability p and the value "blue" with probability $1-p$. Let $f(C_1, \ldots, C_{k-1+l})$ be a real valued function of a color sequence. We claim that

$$|\mathrm{E}(f(C_1, \ldots, C_{k-1+l})) - \mathrm{E}(f(\tilde{C}_1, \ldots, \tilde{C}_{k-1+l}))| \leq \frac{\varepsilon}{24} \max |f| \,. \tag{5.5}$$

Indeed,

$$\mathrm{E}(f(\tilde{C}_1, \ldots, \tilde{C}_{k-1+l})) = \sum_{\substack{(c_1, \ldots, c_{k-1+l}) \\ \in \{\text{red}, \text{blue}\}^{k-1+l}}} p^{|\{i : c_i = \text{red}\}|}(1-p)^{|\{i : c_i = \text{blue}\}|} f(c_1, \ldots, c_{k-1+l}), \text{ and}$$

$$
\begin{aligned}
\mathrm{E}(f(C_1,\dots,C_{k-1+l})) = &\sum_{\substack{(c_1,\dots,c_{k-1+l})\\ \in\{\text{red},\text{blue}\}^{k-1+l}}} \left(p^{|\{i:c_i=\text{red}\}|}(1-p)^{|\{i:c_i=\text{blue}\}|} \pm \frac{\varepsilon}{24\cdot 2^{k-1+l}}\right)\cdot \\
&\cdot f(c_1,\dots,c_{k-1+l}) = \mathrm{E}(f(\tilde{C}_1,\dots,\tilde{C}_{k-1+l})) \\
&+ \sum_{\substack{(c_1,\dots,c_{k-1+l})\\ \in\{\text{red},\text{blue}\}^{k-1+l}}} \left(\pm\frac{\varepsilon}{24\cdot 2^{k-1+l}}\right) f(c_1,\dots,c_{k-1+l}) \\
&= \mathrm{E}(f(\tilde{C}_1,\dots,\tilde{C}_{k-1+l})) \pm \frac{\varepsilon}{24}\max|f|\,,
\end{aligned}
$$

proving (5.5).

Using (5.5) with

$$
f_1(C_1,\dots,C_{k-1+l}) = \begin{cases} 1 & \text{if } C_1 = \ldots = C_{k-1} = \text{red} \\ 0 & \text{otherwise} \end{cases},
$$

we have $P(u=k-1) = \mathrm{E}(f_1(C_1,\dots,C_{k-1+l})) = p^{k-1} \pm \frac{\varepsilon}{24}$. Using (5.5) with

$$
f_2(C_1,\dots,C_{k-1+l}) = \begin{cases} 1 & \text{if } C_1 = \ldots = C_{k-1} = \text{blue} \\ 0 & \text{otherwise} \end{cases},
$$

we have $P(u=0) = \mathrm{E}(f_2(C_1,\dots,C_{k-1+l})) = (1-p)^{k-1} \pm \frac{\varepsilon}{24}$. Using (5.5) with $f_3(C_1,\dots,C_{k-1+l}) = |\{i \in \{k,\dots,k-1+l\} : C_i = \text{red}\}|$, we have $\mathrm{E}(w) = \mathrm{E}(f_3(C_1,\dots,C_{k-1+l})) = pl \pm \frac{\varepsilon}{24}l$. So

$$
\begin{aligned}
\mathrm{E}(e(B|_e)) \le\ & P(u\notin\{0,k-1\})\left(l+\binom{k-1}{2}\right) + P(u=k-1)\,\mathrm{E}\left(l-w+\binom{k-1}{2}\right) \\
&+ P(u=0)\,\mathrm{E}\left(l+\binom{k-1}{2}-(l-w)\right) \\
&\le l+\binom{k-1}{2} - \left(p^{k-1}\pm\frac{\varepsilon}{24}\right)\left(p\pm\frac{\varepsilon}{24}\right)l - \left((1-p)^{k-1}\pm\frac{\varepsilon}{24}\right)\left(1-p\pm\frac{\varepsilon}{24}\right)l \\
&\le l+\binom{k-1}{2} - p^k l - (1-p)^k l + \frac{\varepsilon}{4}l \le \left(1-\frac{1}{2^{k-1}}\right)l + \binom{k-1}{2} + \frac{\varepsilon}{4}l
\end{aligned}
$$

assuming $\varepsilon \le 1$.

That is, if O has m hyperedges, $e(B) = \left(\left(1-\frac{1}{2^{k-1}}\right)l + \binom{k-1}{2} + \frac{\varepsilon}{4}l\right)m$, while $e(G_O) = m(k-1)l$. Let $l \ge \frac{k(k-2)}{\varepsilon}$, then

$$
\begin{aligned}
e(B) &\le \left(\left(1-\frac{1}{2^{k-1}}\right)\frac{1}{k-1} + \frac{k-2}{2l} + \frac{\varepsilon}{4(k-1)}\right)e(G_O) \\
&\le \left(\left(1-\frac{1}{2^{k-1}}\right)\frac{1}{k-1} + \frac{\varepsilon}{k}\right)e(G_O) \le \left(1-\frac{1}{2^{k-1}}\right)\frac{1}{k-1}e(G_O)(1+\varepsilon). \qquad \square
\end{aligned}
$$

5.5 Pasting C_6's to produce a C_8-free graph

We will make use of the following proposition of Nešetřil and Rödl [128] in the second example, and in the general version of the first example.

Proposition 5.12 (Nešetřil, Rödl [128])**.** *For any positive integers $r \ge 2$ and $s \ge 3$, there exists an $n_0 \in \mathbb{N}$ such that for any integer $n \ge n_0$ there is a r-uniform hypergraph with girth at least s and more than $n^{1+1/s}$ hyperedges.*

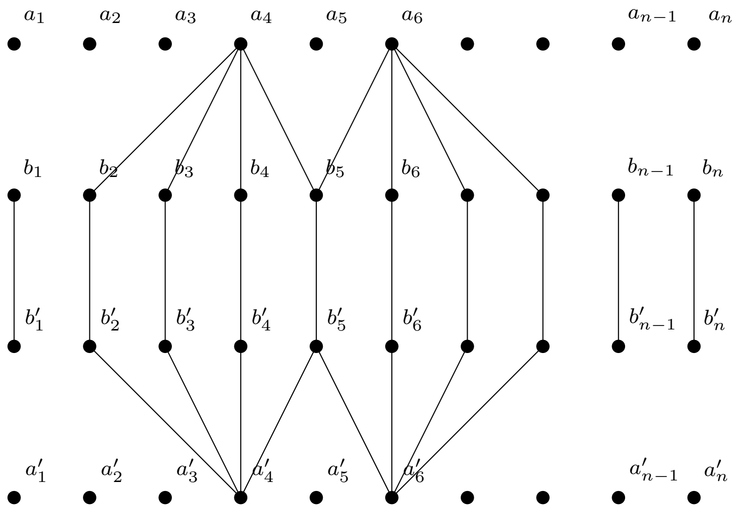


Figure 5.2: First pasting

5.5.1 First example

For our construction here we will need a bipartite graph of girth at least 10 with many edges (and with degree at least 2 in every vertex). We will derive such a graph from the following construction of Benson [10].

Theorem 5.7 (Benson [10])**.** *Let q be an odd prime power. There is a $(q+1)$-regular, bipartite, girth 12 graph Q with $2(q^5+q^4+q^3+q^2+q+1)$ vertices.*

First, let us notice that since Q is a regular bipartite graph, it has color classes of equal size. Moreover, we may assume that Q is connected, for otherwise we may add some edges to make it connected without creating cycles. So we have the following corollary.

Corollary 5.13. *There exists a connected bipartite graph of girth at least 10 with $n/2$ vertices in each color class such that every vertex has degree at least $(n/2)^{1/5}(1-o(1))$. (So it contains at least $(1-o(1))(n/2)^{6/5}$ edges.)*

Theorem 5.8. *There exists a C_8-free graph G on $4n$ vertices with average degree at least $4n^{1/5}$ which is pasted together from C_6's.*

To prove Theorem 5.8, let us take a connected, bipartite graph G_1 of girth at least 10 on $2n$ vertices such that each vertex has degree at least $n^{1/5}(1-o(1))$ (and having n vertices in each color class). The existence of such a graph is guaranteed by Corollary 5.13. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be the two color classes of G_1. Now let G_2 be a copy of G_1 with vertices $a'_1, a'_2, \ldots, a'_n$ and $b'_1, b'_2, \ldots, b'_n$ and edge set $E(G_2) = \{a'_i b'_j \mid a_i b_j \in E(G_1)\}$. Finally, the graph G is defined to have the vertex set $V(G) = V(G_1) \cup V(G_2)$ and the edge set $E(G) = E(G_1) \cup E(G_2) \cup \{b_i b'_i \mid 1 \le i \le n\}$ (see Figure 5.2). So G has $4n$ vertices and $2n^{6/5}(1-o(1)) + n$ edges.

To show that G is pasted together from C_6's, we have to show that every edge is contained in a C_6, and for any two edges $e_1, e_2 \in E(G)$, there is a sequence $O_1, O_2, \ldots, O_m$ of C_6's in G such that for any $1 \le i \le m-1$, O_i and O_{i+1} share at least one edge, and e_1 and e_2 are edges of O_1 and O_m respectively. The graph can then be built starting from an arbitrary fixed edge. It is easy to see that every edge is contained in some C_6 of the form $a_i b_j b'_j a'_i b'_k b_k$, and so we can assume that both e_1 and e_2 are of the form $a_i b_j$. Let

$(a_{i_0})b_{i_1}a_{i_2}b_{i_3}a_{i_4}b_{i_5}\dots a_{i_{t-1}}b_{i_t}(a_{i_{t+1}})$ be a path starting with e_1 and ending with e_2, with $e_1 = a_{i_0}b_{i_1}$ or $e_1 = b_{i_1}a_{i_2}$, and $e_2 = a_{i_{t-1}}b_{i_t}$ or $e_2 = b_{i_t}a_{i_{t+1}}$ (such a path exists since G_1 is connected). Then the path $b'_{i_1}a'_{i_2}b'_{i_3}a'_{i_4}b'_{i_5}\dots a'_{i_{t-1}}b'_{i_t}$ is contained in G_2. These two paths together with the edges $b_{i_1}b'_{i_1}, b_{i_2}b'_{i_2}, \dots, b_{i_t}b'_{i_t}$ give the desired sequence of C_6's (together with an arbitrary C_6 of the form $a_{i_0}b_{i_1}b'_{i_1}a'_{i_0}b'_jb_j$ if $e_1 = a_{i_0}b_{i_1}$, and a C_6 of the form $a_{i_{t+1}}b_{i_t}b'_{i_t}a'_{i_{t+1}}b'_kb_k$ if $e_2 = b_{i_t}a_{i_{t+1}}$).

It remains to show that G is C_8-free. Suppose for a contradiction that it has a C_8. Since the graph G_1 is of girth at least 10, this C_8 cannot be completely in G_1 or G_2. So it has to contain at least one edge from each of the three sets $E(G_1)$, $E(G_2)$ and $\{b_ib'_i \mid 1 \le i \le n\}$. Moreover, it is easy to see that it contains exactly two edges from $\{b_ib'_i \mid 1 \le i \le n\}$, say $b_ib'_i$ and $b_jb'_j$. So there is a path of q edges between b_i and b_j in G_1 and a path of r edges between b'_i and b'_j in G_2 such that $q + r = 6$. Let these paths be $b_ia_{i_1}b_{i_2}\dots a_{i_{q-1}}b_j$ and $b'_ia'_{j_1}b'_{j_2}\dots a'_{j_{r-1}}b'$ respectively. By construction, the second path in G_2 implies that G_1 contains the path $b_ia_{j_1}b_{j_2}\dots a_{j_{r-1}}b_j$, which, together with $b_ia_{i_1}b_{i_2}\dots a_{i_{q-1}}b_j$, would produce a cycle of length 4 or 6 in G_1, a contradiction.

Remark 5.14. We may modify the above construction as described below to find a pasting of C_{2l}'s to produce a C_{2k}-free graph G for any given integers $k > l \ge 3$ and having average degree at least $\Omega\big(n^{1/(2k+2)}\big)$.

Proof. A graph of girth $2k+1$ and having $\Omega\big(n^{1+1/(2k+1)}\big)$ edges exists by applying Proposition 5.12 with $r = 2$. So it has average degree $\Omega\big(n^{1/(2k+1)}\big)$. It is easy to find a bipartite subgraph of such a graph, with equal color classes and having a constant fraction of all the edges. Then we can delete vertices of degree 1 without decreasing its average degree, so we can assume it has minimum degree at least 2, and as usual, we can assume it is connected, because otherwise we can add edges without creating a cycle to make it connected. Let G_1 be this bipartite, connected graph of girth greater than $2k$ on $2n$ vertices with average degree $\Omega\big(n^{1/(2k+1)}\big)$. Then let G_2 be defined in the same way as in the above proof (based on G_1). However, now, for each i we connect the vertices $b_i \in V(G_1)$ and $b'_i \in V(G_2)$ by a path containing $l-2$ edges and let the resulting graph be G. Using the same argument as in the above proof, we can see that this gives a pasting of C_{2l}'s and that G is C_{2k}-free. □

5.5.2 Second example

A hypergraph H is *connected* if for any two vertices $u, v \in V(H)$, there exist hyperedges $h_i \in E(H)$, $1 \le i \le m$, such that $u \in h_1, v \in h_m$ and $h_i \cap h_{i+1} \ne \emptyset$ for all $1 \le i \le m-1$. A minimal collection of such hyperedges is called a path between u and v in H. We may assume that the hypergraph provided by Proposition 5.12 is connected, for otherwise we can simply take a connected component of it containing the biggest number of hyperedges.

Theorem 5.9. *There exists a C_8-free graph G on $2n$ vertices with average degree at least $6 \cdot n^{1/9}$ which is pasted together from C_6's.*

To prove Theorem 5.9, we apply Proposition 5.12 to find a (connected) 3-uniform hypergraph H_1 on n vertices with girth at least 9 and more than $n^{1+1/9}$ hyperedges. Let $V(H_1) = \{u_1, u_2, \dots, u_n\}$. Replace each vertex $u_i \in V(H_1)$ with a pair of vertices u_i, u'_i so that every hyperedge containing u_i now contains both u_i and u'_i. This produces a 6-uniform hypergraph which we denote by H_2.

Now we construct the desired graph G from H_2 in the following fashion. If $\{u_i, u'_i, u_j, u'_j, u_k, u'_k\}$ is a hyperedge in H_2 with $1 \leq i \leq j \leq k \leq n$, then we add the edges $u_iu'_i, u'_iu_j, u_ju'_j, u'_ju_k, u_ku'_k, u'_ku_i$ to $E(G)$. We repeat this procedure for every hyperedge of H_2. Let us call the edges $u_iu'_i \in E(G)$ $(1 \leq i \leq n)$ *fat* edges and the rest of the edges of G *thin* edges.

Note that two fat edges never share a vertex. We claim that a thin edge cannot be added by two different hyperedges of H_2. Suppose by contradiction that u'_iu_j is a thin edge added by two different hyperedges h_1, h_2 of H_2. Then since a hyperedge of H_2 either contains both vertices u_r, u'_r or neither of them for any given $1 \leq r \leq n$, it follows that $\{u_i, u'_i, u_j, u'_j\} \subset h_1$ and $\{u_i, u'_i, u_j, u'_j\} \subset h_2$. Consider the two hyperedges in H_1 which correspond to h_1 and h_2. They both contain u_i and u_j; so they intersect in at least two vertices, which is a contradiction since H_1 is a linear hypergraph. (Notice, on the other hand, that a fat edge may have been added by several hyperedges.) So each hyperedge in H_2 adds precisely 3 new thin edges to $E(G)$. Therefore the number of thin edges in G is three times the number of hyperedges in H_2. Since the number of fat edges is n, we have $|E(G)| = 3 \cdot n^{1+1/9} + n$. Thus it has the desired average degree.

Since H_1 is connected, we can construct it by adding hyperedges one by one, in such a way that each hyperedge intersects one of the previous hyperedges in at least one vertex. We can construct H_2 by adding the C_6's corresponding to the hyperedges of H_1 in the same order; this shows that G is pasted together from C_6's.

It only remains to show that G is C_8-free. We say an edge is between two edges e_1, e_2 if one of its end vertices is in e_1 and the other is in e_2.

Claim 5.15. *There is at most one thin edge between any two fat edges of G.*

Proof. Consider any two fat edges $u_iu'_i$ and $u_ju'_j$ of G. As noted earlier, any thin edge between them is added by a unique hyperedge h of H_2, and h contains all four vertices u_i, u'_i, u_j, u'_j. Because of the linearity of H_1, no hyperedge of H_2 other than h may contain all four vertices u_i, u'_i, u_j, u'_j Now note that in our procedure, any hyperedge of H_2 adds at most one thin edge between any two fat edges contained in it, proving the claim. □

Now suppose for a contradiction that G contains a C_8. Since no two fat edges in G share a vertex, there can be at most four fat edges in this C_8. Contract every pair of vertices u_i, u'_i in G into a single vertex u_i. Then this C_8 would become a closed walk C' of length at most 8 and at least 4 (only thin edges remain after contraction). While this closed walk may have repeated vertices, we show that it cannot have repeated edges (i.e., it is actually a circuit). Suppose that after contracting every pair of vertices u_i, u'_i to u_i, some two thin edges xy and zw coincide. Then, for some i and j, we have $x, z \in \{u_i, u'_i\}$ and $y, w \in \{u_j, u'_j\}$. Between the fat edges $u_iu'_i$ and $u_ju'_j$, there are two thin edges (namely xy and zw), contradicting Claim 5.15.

The 2-shadow of a hypergraph H is the graph which contains an edge uv if and only if there is a hyperedge of H which contains u and v. C' must be contained in the 2-shadow of H_1. Since H_1 has girth at least 9, it is not difficult to see that the only possible length of a circuit in its 2-shadow that is between 4 and 8 is 6 and it must be of the form ab, bc, ce, ed, dc, ca (notice that c is a repeated vertex). Therefore, the original C_8 in G must be contained in the set of edges added by the hyperedges $\{a, a', b, b', c, c'\}$ and $\{c, c', d, d', e, e'\}$ of H_2, but this is impossible as these edges consist of two C_6's sharing exactly one edge.

Chapter 6

Turán number of an induced complete bipartite graph plus an odd cycle

Recall the classical theorem of Kővári, Sós and Turán [105] concerning the Turán number of complete bipartite graphs states that $\mathrm{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{1/s} n^{2-1/s} + O(n)$, where $s \leq t$. Let $\mathrm{ex}_{bip}(n, \mathcal{F})$ denote the maximum number of edges in an n-vertex bipartite $\mathcal{F}$-free graph. Füredi [49] showed that

$$\mathrm{ex}_{bip}(n, K_{s,t}) \leq (1+o(1))(t-s+1)^{1/s} \left(\frac{n}{2}\right)^{2-1/s}. \tag{6.1}$$

In the cases $s = 2$ and $s = t = 3$ asymptotically sharp values are known: Recall that Füredi [48] determined the asymptotics for the Turán number of $K_{2,t}$ by showing that for any fixed $t \geq 2$, we have $\mathrm{ex}(n, K_{2,t}) \leq \frac{1}{2}(t-1)^{1/2} n^{3/2} + O(n^{4/3})$. Using an example of Brown [18] for the lower bound and a theorem of Füredi [49] for the upper bound, it is known that $\mathrm{ex}(n, K_{3,3}) = \frac{1}{2} n^{5/3} + O(n^{5/3-c})$ for some $c > 0$.

Erdős and Simonovits [37] conjectured that given any family $\mathcal{F}$ of graphs, there exists $k \geq 1$ such that $\mathrm{ex}(n, \mathcal{F} \cup \{C_3, C_5, \ldots, C_{2k+1}\})$ is equal to $ex_{bip}(n, \mathcal{F})$ asymptotically. Allen, Keevash, Sudakov and Verstraëte [1] proved this conjecture for complete bipartite graphs $K_{2,t}$ and $K_{3,3}$ in the following stronger form.

Theorem 6.1 (Allen, Keevash, Sudakov and Verstraëte [1])**.** *Let $k \geq 2$ be an integer. If $s = 2$ and $t \geq 2$, or $s = t = 3$, then*

$$\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\}) = (1+o(1))(t-s+1)^{1/s} \left(\frac{n}{2}\right)^{2-1/s}.$$

In fact, they proved a more general result for so-called"smooth" families (for more details, see [1]). We also note that the case $s = t = k = 2$ was solved earlier by Erdős and Simonovits [37].

In the rest of the chapter we use the following asymptotic notation. Given two functions $f(n)$ and $g(n)$, we write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. Otherwise, we write $f(n) \not\sim g(n)$.

6.1 Induced Turán numbers

Loh, Tait, Timmons and Zhou [110] introduced the problem of simultaneously forbidding an induced copy of a graph and a (not necessarily induced) copy of another graph. Let ex$(n, H, F\text{-ind})$ denote the maximum possible number of edges in an n-vertex graph containing no induced copy of F and no copy of H as a subgraph.

The question of determining ex$(n, H, F\text{-ind})$ is related to the well-studied areas of Ramsey–Turán Theory and the Erdős–Hajnal Conjecture. The Ramsey–Turán number $RT(n, H, m)$, is defined as the maximum number of edges that an n-vertex graph with independence number less than m may have without containing H as a subgraph. If we let F be an independent set of order m then ex$(n, H, F\text{-ind}) = RT(n, H, m)$. So the problem of determining ex$(n, H, F\text{-ind})$ is a generalization of the Ramsey-Turán problem. We refer the reader to [110] for a detailed discussion on this general problem and its relation to other areas.

Loh, Tait, Timmons and Zhou showed the following:

Theorem 6.2 (Loh, Tait, Timmons, Zhou [110]). *For any integers $k \geq 2$ and $t \geq 2$ there is a constant β_k, depending only on k, such that*

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-}ind\}) \leq (\alpha(k,t)^{1/2} + 1)^{1/2}\frac{n^{3/2}}{2} + \beta_k n^{1+1/2k}$$

where $\alpha(k,t) = (2k-2)(t-1)((2k-2)(t-1)-1)$.

They asked whether Theorem 6.2 determines the correct growth rate in k, and showed that it determines the correct growth rate in n and t.

In this chapter we answer their question by showing that ex$(n, \{C_{2k+1}, K_{2,t}\text{-ind}\})$ does not depend on k asymptotically, thus improving the upper bound in Theorem 6.2 significantly. Our main result determines ex$(n, \{C_{2k+1}, K_{2,t}\text{-ind}\})$ asymptotically in all the cases except when $k = t = 2$, and is stated below.

Theorem 6.3 (Ergemlidze, Győri, M. [41]). *For any integers $k \geq 2$, $t \geq 2$ where $(k,t) \neq (2,2)$, we have*

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-}ind\}) = (1 + o(1))(t-1)^{1/2}\left(\frac{n}{2}\right)^{3/2}.$$

Note that this shows ex$(n, \{C_{2k+1}, K_{2,t}\text{-ind}\}) \sim$ ex$(n, \{C_{2k+1}, K_{2,t}\})$ for all $k \geq 2$, $t \geq 2$ except in the case $k = t = 2$, which is studied in the next theorem. Therefore, Theorem 6.3 is a strengthening of Theorem 6.1 in the case $s = 2$ (except when $k = t = 2$); thus our proof of Theorem 6.3 provides a new proof of Theorem 6.1 in this case.

For the lower bound in Theorem 6.3, simply consider a bipartite $K_{2,t}$-free graph G with $n/2$ vertices in each color class and containing $\sqrt{t-1} \cdot (n/2)^{3/2} + O(n^{4/3})$ edges. The existence of such a graph is shown by Füredi in [48]. Clearly, G contains no copy of C_{2k+1} and as it contains no copy of $K_{2,t}$, of course, it contains no induced copy of $K_{2,t}$ either.

In the case $k = t = 2$, we improve the upper bound in Theorem 6.2 as follows.

Theorem 6.4 (Ergemlidze, Győri, M. [41]).

$$(1 + o(1))\frac{2}{3\sqrt{3}}n^{3/2} \leq \mathrm{ex}(n, \{C_5, K_{2,2}\text{-}ind\}) \leq (1 + o(1))\frac{n^{3/2}}{2}.$$

For the lower bound, just as in [110], we use the following example of Bollobás and Győri from [14], to construct an induced-$K_{2,2}$-free and C_5-free graph G with $(1+o(1))\frac{2}{3\sqrt{3}}n^{3/2}$ edges.

Bollobás–Győri Construction: Take a $K_{2,2}$-free bipartite graph G_0 with $n/3$ vertices in each part and $(1+o(1))(n/3)^{3/2}$ edges. In one part, replace each vertex u by a pair of two new vertices u_1 and u_2, where u_1 and u_2 are adjacent to each other, and to all the vertices that were adjacent to u. It is easy to check that this new graph G contains no C_5 and no induced copy of $K_{2,2}$. Moreover, G contains approximately twice as many edges as G_0.

We leave open the question of determining the asymptotics of $\mathrm{ex}(n, \{C_5, K_{2,2}\text{-ind}\})$.

Organization of the proofs of Theorem 6.3 and Theorem 6.4: Our proof of Theorem 6.3 is divided into two cases: $k \geq 3$ and $k = 2$ (see Remark 6.9 for an explanation for why the case $k = 2$ is special). In Section 6.3, we prove Theorem 6.3 in the case $k \geq 3$ and in Section 6.4 we prove Theorem 6.3 in the case $k = 2$, along with Theorem 6.4.

6.1.1 $\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\text{-ind}\})$ versus $\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\})$

In Section 6.2 we prove the following proposition which connects $\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\text{-ind}\})$ with $\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\})$.

Proposition 6.5 (Ergemlidze, Győri, M. [41])**.** *For any $k, s, t \geq 2$ we have*

$$\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\text{-}ind\}) \leq \mathrm{ex}(n, \{C_3, C_{2k+1}, K_{s,t}\}) + 3c_k n^{1+1/k},$$

where c_k is a constant depending only on k.

Let $k \geq 2$ be an integer. We will compare the asymptotic values of $\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\text{-ind}\})$ and $\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\})$ in the cases $s = 2$, $t \geq 2$, and $s = t = 3$, and deduce Theorem 6.6, below, from Theorem 6.1, Theorem 6.3 and Proposition 6.5.

First let us consider the case $s = 2$, $t \geq 2$. As already noted, by comparing Theorem 6.1 and Theorem 6.3, it is easy to see that for all $k \geq 2$, $t \geq 2$ except when $k = t = 2$, we have

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\}) \sim \mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\}). \tag{6.2}$$

Now consider the case $s = t = 3$. It follows from Theorem 6.1 that for all $k \geq 2$, we have $\mathrm{ex}(n, \{C_3, C_{2k+1}, K_{3,3}\}) \leq (1+o(1))(n/2)^{5/3}$. Combining this with Proposition 6.5 (substituting $s = t = 3$) we get,

$$\mathrm{ex}(n, \{C_{2k+1}, K_{3,3}\text{-ind}\}) \leq (1+o(1))(n/2)^{5/3} + 3c_k n^{1+1/k}.$$

Now note that $3c_k n^{1+1/k} = o(n^{5/3})$ for all $k \geq 2$. Therefore, $\mathrm{ex}(n, \{C_{2k+1}, K_{3,3}\text{-ind}\}) \leq (1+o(1))(n/2)^{5/3} = \mathrm{ex}(n, \{C_{2k+1}, K_{3,3}\})$. This implies,

$$\mathrm{ex}(n, \{C_{2k+1}, K_{3,3}\text{-ind}\}) \sim \mathrm{ex}(n, \{C_{2k+1}, K_{3,3}\}). \tag{6.3}$$

Therefore, (6.2) and (6.3) imply the following strengthening of Theorem 6.1 in all but one special case.

Theorem 6.6 (Ergemlidze, Győri, M. [41]). *Let $k \geq 2$ be an integer. If $s = 2$ and $t \geq 2$, or $s = t = 3$, then*

$$\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\text{-}ind\}) = (1 + o(1))(t - s + 1)^{1/s} \left(\frac{n}{2}\right)^{2-1/s},$$

except when $k = s = t = 2$.

Surprisingly, in the special case $k = s = t = 2$ the functions $\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\text{-ind}\})$ and $\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\})$ seem to behave quite differently: As we noted before, in this case it is known by a theorem of Erdős and Simonovits [37] (or by Theorem 6.1) that $\mathrm{ex}(n, \{C_5, K_{2,2}\}) = (1+o(1))(n/2)^{3/2}$, whereas $\mathrm{ex}(n, \{C_5, K_{2,2}\text{-ind}\}) \geq (1+o(1))(2n^{3/2}/3\sqrt{3})$ by the lower bound in Theorem 6.4. Therefore, in this very special case

$$\mathrm{ex}(n, \{C_5, K_{2,2}\text{-ind}\}) \not\sim \mathrm{ex}(n, \{C_5, K_{2,2}\}).$$

6.1.2 Notation and the Blakley–Roy inequality

Let G be a graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$ respectively. Given a set $S \subseteq V(G)$, the subgraph of G induced by S is denoted $G[S]$.

Given a graph and a vertex v in it, the first neighborhood of v, $N_1(v)$ is the set of vertices adjacent to v and for $i \geq 2$, let $N_i(v)$ denote the set of vertices at distance exactly i from v. Notice that for $i \neq j$, $N_i(v) \cap N_j(v) = \emptyset$.

A walk of length k, or a *k-walk*, is a sequence $v_0e_0v_1e_1v_2e_2v_3 \dots v_{k-1}e_{k-1}v_k$ of vertices and edges such that $e_i = v_iv_{i+1}$ for $0 \leq i \leq k-1$. In this chapter, we are only interested in walks of length 3. For convenience, we simply denote a 3-walk by $v_0v_1v_2v_3$ instead of $v_0e_0v_1e_1v_2e_2v_3$. The vertices v_0 and v_3 are called the first and last vertices of the 3-walk respectively, and the edges e_0 and e_2 are called the first and last edges respectively. We say the 3-walk starts with the edge e_0 and ends with the edge e_2. We refer to v_0, v_1, v_2, v_3 as first, second, third and fourth vertices of the walk respectively. Note that the 3-walks $v_0v_1v_2v_3$ and $v_3v_2v_1v_0$ are generally considered different. Also note that edges can be repeated in a 3-walk. A *3-path* is a 3-walk with no repeated vertices or edges.

Blakley and Roy [11] showed that the number of k-walks in any graph of average degree d with n vertices, is at least nd^k. In our proofs we will only use this statement with $k = 3$. Note that the Blakley–Roy inequality is in fact, a more general statement concerning inner products, which can be seen as a matrix version of Hölder's inequality.

6.2 Proof of Proposition 6.5

Let G be a C_{2k+1}-free graph containing no induced copy of $K_{s,t}$. Let G_Δ be the subgraph of G consisting of the edges which are contained in the triangles of G. Let $G \setminus G_\Delta$ be the graph obtained after deleting all the edges of G_Δ from G. Of course, $G \setminus G_\Delta$ is triangle free, and the number of edges in G_Δ is at most three times the number of triangles in G.

Győri and Li [85] showed that in a C_{2k+1}-free graph there are at most $c_kn^{1+1/k}$ triangles where c_k is a constant depending only on k. This implies that $|E(G_\Delta)| \leq 3c_kn^{1+1/k}$.

Claim 6.7. *$G \setminus G_\Delta$ is $K_{s,t}$-free.*

Proof. Assume for a contradiction that there is a (not necessarily induced) copy of $K_{s,t}$ in $G \setminus G_\Delta$, and let A, B be its color classes. Then since G contains no induced copy of

$K_{s,t}$, there must be an edge xy of G which is contained in A or B. In either case, it is easy to see that xy and some two edges of the $K_{s,t}$ form a triangle. However, this means that these two edges of $K_{s,t}$ are contained in G_Δ by definition, contradicting the assumption that this $K_{s,t}$ is contained in $G \setminus G_\Delta$. Therefore, the claim follows. □

Using Claim 6.7, and the fact that $G \setminus G_\Delta$ is a triangle-free subgraph of G, we have $|E(G \setminus G_\Delta)| \le \mathrm{ex}(n, \{C_3, C_{2k+1}, K_{s,t}\})$. Therefore,

$$|E(G)| = |E(G_\Delta)| + |E(G \setminus G_\Delta)| \le 3c_k n^{1+1/k} + \mathrm{ex}(n, \{C_3, C_{2k+1}, K_{s,t}\}),$$

proving Proposition 6.5.

6.3 Proof of Theorem 6.3 in the case $k \ge 3$

Using Proposition 6.5 for $s = 2$, we have

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\}) \le \mathrm{ex}(n, \{C_3, C_{2k+1}, K_{2,t}\}) + 3c_k n^{1+1/k}, \tag{6.4}$$

where c_k is a constant depending only on k.

Moreover, we have the following proposition that is proved below.

Proposition 6.8. *For all integers $k \ge 2, t \ge 2$, we have*

$$\mathrm{ex}(n, \{C_3, C_{2k+1}, K_{2,t}\}) \le (1 + o(1))\sqrt{t-1}\left(\frac{n}{2}\right)^{3/2}.$$

Combining Proposition 6.8 and (6.4), we get,

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\}) \le (1 + o(1))\sqrt{t-1}\left(\frac{n}{2}\right)^{3/2} + 3c_k n^{1+1/k}. \tag{6.5}$$

Since $k \ge 3$, clearly $3c_k n^{1+1/k} = o(n^{3/2})$, which completes the proof of Theorem 6.3 in the case $k \ge 3$.

Remark 6.9. Note that when $k = 2$, the number of edges in G_Δ can be as large as $\Theta(n^{3/2})$. For example, observe that this is the case in the Bollobás–Győri construction which is stated after Theorem 6.4 (note that every edge in the construction is contained in a triangle). So using Proposition 6.5, we cannot get an upper bound better than $(1 + o(1))(3c_2 + \sqrt{t-1}\left(\frac{1}{2}\right)^{3/2})n^{3/2}$ where $c_2 > 0$ is a constant. Therefore, the current approach does not work in the case $k = 2$; we will give a different proof for this case in Section 6.4.

Now it only remains to prove Proposition 6.8. Note that Proposition 6.8 follows from Theorem 6.1. However, since our proof of Proposition 6.8 is simpler than the general proof in [1], we present it below. Before we proceed with the details, let us briefly sketch the main ideas of the proof.

Sketch of the proof of Proposition 6.8: First we will show that we may assume the minimum degree in our C_{2k+1}-free, induced-$K_{2,t}$-free graph G is at least $d/2$ and the maximum degree in G is at most $4d$. Using the Blakley–Roy inequality, we find a vertex v from which at least d^3 3-walks start. We will then show that at least $d^3 - O(d^2)$ of these 3-walks are of the form $vv_1v_2v_3$ where $v_i \in N_i(v)$. The final edges of such 3-walks form a bipartite graph with parts $N_2(v)$ and $N_3(v)$. It will turn out that this bipartite graph has at least $(d^3 - O(d^2))/(t-1)$ edges but, of course, it can have at most $ex_{bip}(n, K_{2,t})$ edges. This will give the desired bound on d.

Below we continue with the detailed proof.

Proof of Proposition 6.8. Let d be the average degree of a graph G containing no triangle, C_{2k+1} or $K_{2,t}$. Our aim is to bound d from above. If a vertex has degree smaller than $d/2$, then we can delete the vertex and the edges incident on it without decreasing the average degree. It is easy to see that if the desired upper bound on the average degree holds for the new graph, then it holds for the original graph as well. Therefore, we may assume that G has minimum degree at least $d/2$.

Suppose there is a vertex u in G with degree more than $4d$. That is, $|N_1(u)| > 4d$. Since the minimum degree in G is at least $d/2$, there are at least $(d/2-1)|N_1(u)|$ edges between $N_1(u)$ and $N_2(u)$ as there are no edges contained in $N_1(u)$ (because G is triangle-free). On the other hand, if a vertex $w \in N_2(u)$ is adjacent to t vertices in $N_1(u)$, then u, w and these t vertices form a $K_{2,t}$, a contradiction. So there are at most $|N_2(u)|(t-1)$ edges between $N_1(u)$ and $N_2(u)$. Combining, we get,

$$|N_2(u)|(t-1) \geq \left(\frac{d}{2} - 1\right)|N_1(u)| > \left(\frac{d}{2} - 1\right)4d = 2d^2 - 4d.$$

Now since, $n \geq |N_1(u)| + |N_2(u)|$, we get

$$n > 4d + (2d^2 - 4d)/(t-1) \geq 2d^2/(t-1).$$

Therefore, $d < \sqrt{(t-1)}\sqrt{n/2}$ and the bound in Proposition 6.8 holds because $|E(G)| = dn/2$.

So from now on we can assume that the maximum degree d_{max} in G is at most $4d$. First let us show that $N_2(v)$ doesn't induce many edges.

Claim 6.10. *For any vertex v, the number of edges induced by $N_2(v)$ is at most $(2k-4)16d^2$.*

Proof. For each $q \in N_1(v)$, let S_q be the set of neighbors of q in $N_2(v)$. Of course the sets $\{S_q \mid q \in N_1(v)\}$ cover all the vertices of $N_2(v)$. So we can choose sets $S'_q \subset S_q$ such that $\{S'_q \mid q \in N_1(v)\}$ partition $N_2(v)$. Note that the sets S'_q do not induce any edges as G is triangle-free, so every edge induced by $N_2(v)$ is between some two sets S'_{q_1} and S'_{q_2}, where $q_1, q_2 \in N_1(v)$ with $q_1 \neq q_2$. Color each set S'_q red or blue with probability 1/2. Each vertex in $N_2(v)$ is colored by the color of the set it is contained in. It is easy to see that there exists a coloring of the sets S'_q such that at least half of all the edges induced by $N_2(v)$ are not monochromatic. (Indeed, the probability that an edge of $N_2(v)$ is not monochromatic is 1/2.) If there is a path of length $2k-3$ in the graph B consisting of these non-monochromatic edges, then since $2k-3$ is odd, the end vertices y_1, y_2 of the path are of different colors, so there exist distinct vertices $x_1, x_2 \in N_1(v)$ such that $x_1y_1, x_2y_2 \in E(G)$. However, then $vx_1, vx_2, x_1y_1, x_2y_2$ and this path of length $2k-3$ form a C_{2k+1}-cycle in G, a contradiction. Therefore, using the Erdős–Gallai theorem [34] and the fact that B contains at least half of the edges induced by $N_2(v)$, we get

$$|G[N_2(v)]| \leq \frac{2k-3-1}{2}|N_2(v)| \cdot 2 = (2k-4)|N_2(v)| \leq (2k-4)d_{max}^2 \leq (2k-4)16d^2,$$

proving the claim. □

By the Blakley–Roy inequality, there are at least nd^3 3-walks in G, so there exists a vertex v which is the first vertex of at least d^3 3-walks. A 3-walk of the form $vv_1v_2v_3$ where $v_i \in N_i(v)$ for $1 \le i \le 3$ is called *good*. If a 3-walk is not good but has v as its first vertex, then either $v_2 = v$ or $v_2 \in N_2(v), v_3 \in N_1(v)$ or $v_2 \in N_2(v), v_3 \in N_2(v)$. (Note that here we used that $N_1(v)$ doesn't contain any edges - as G is triangle-free.) Below we show that the number of 3-walks starting from v that are not good are very few.

Number of 3-walks starting from v where $v_2 = v$ is at most $d_{max}^2 \le 16d^2$. Indeed, there are at most d_{max} choices for v_1 and v_3 since they are both adjacent to v. Now we estimate the number of 3-walks starting from v where $v_2 \in N_2(v)$, $v_3 \in N_1(v)$. Any given $v_2 \in N_2(v)$ has at most $t-1$ neighbors in $N_1(v)$ for otherwise we can find a copy of $K_{2,t}$ in G, a contradiction. So the number of such 3-walks is at most $d_{max}^2(t-1) \le 16(t-1)d^2$ because there are at most d_{max}, d_{max} and $t-1$ choices for v_1, v_2 and v_3 respectively. Finally, we estimate the number of 3-walks where $v_2 \in N_2(v)$ and $v_3 \in N_2(v)$. So the edge $v_2v_3 \in G[N_2(v)]$. For a given edge $xy \in G[N_2(v)]$, either $v_2 = x, v_3 = y$ or $v_2 = y, v_3 = x$ and for fixed v_2, v_3, there are at most $t-1$ choices for $v_1 \in N_1(v)$. Therefore, the number of such 3-walks is at most $|G[N_2(v)]|\, 2(t-1)$. Now by Claim 6.10, this is at most $(2k-4)16d^2 \cdot 2(t-1) = 64(k-2)(t-1)d^2$. Therefore, by summing these estimates, we get that the number of 3-walks starting from v that are not good is at most

$$16d^2 + 16(t-1)d^2 + 64(k-2)(t-1)d^2 = 16d^2 + 16(4k-7)(t-1)d^2 \le 32(4k-7)(t-1)d^2.$$

Thus the number of good 3-walks starting at v is at least $d^3 - 32(4k-7)(t-1)d^2$. Let us consider the graph H consisting of the last edges of these good 3-walks. Observe that H is a $K_{2,t}$-free bipartite graph with color classes $N_2(v)$ and $N_3(v)$. It is easy to see that an edge of H belongs to at most $t-1$ good 3-walks, otherwise we can find a $K_{2,t}$. Therefore there are at least $(d^3 - 32(4k-7)(t-1)d^2)/(t-1)$ edges in H. It follows from (6.1) (or by a simple double counting of number of paths of length two) that a bipartite $K_{2,t}$-free graph on n vertices (or less) contains at most $(1+o(1))\sqrt{t-1}(n/2)^{3/2}$ edges, so H contains at most this many edges. Combining the two estimates, we get

$$(d^3 - 32(4k-7)(t-1)d^2)/(t-1) \le (1+o(1))\sqrt{t-1}\left(\frac{n}{2}\right)^{3/2}.$$

Rearranging, we get $(d - c_{k,t})^3 \le (1+o(1))(t-1)^{3/2}(n/2)^{3/2}$ where $c_{k,t} = \frac{32}{3}(4k-7)(t-1)$. Therefore, $d \le (1+o(1))\sqrt{t-1}(n/2)^{1/2}$ which implies that

$$|E(G)| = \frac{nd}{2} \le (1+o(1))\sqrt{t-1}\left(\frac{n}{2}\right)^{3/2},$$

completing the proof of Proposition 6.8, and the proof of Theorem 6.3 in the case $k \ge 3$. □

Remark 6.11. Note that a very useful idea in the proof of Theorem 6.3 was to remove all the edges contained in triangles first (see Proposition 6.5). This helped us avoid technicalities that would otherwise arise in the proof. For example, after destroying all of the triangles, it is straightforward that for any vertex v in the resulting graph, $N_1(v)$ does not induce any edges; moreover, it becomes very easy to argue that $N_2(v)$ does not induce many edges which is an important step of the proof.

6.4 Proof of Theorem 6.3 in the case $k = 2$ and Proof of Theorem 6.4

Before we proceed with the proof, let us quickly explain the differences between the proof in this section (for the case $k = 2$) and in the previous section (for the case $k \geq 3$). Firstly, as explained in Remark 6.9, we cannot apply Proposition 6.5 in this case, so we cannot assume our C_5-free, induced-$K_{2,t}$-free graph G is triangle-free like in the previous section. Moreover, if we pick a vertex v from which at least d^3 3-walks start (where d is the average degree of G), like in the previous section, then we cannot claim that the bipartite graph H with parts $N_2(v)$ and $N_3(v)$ is $K_{2,t}$-free. Therefore, our idea is to use the following approach instead.

Sketch of the proof: As usual, we will show that we can assume the minimum degree in G is $d/2$ and the maximum degree is at most $O(d)$.

We pick an edge xy in G which is the first edge of at least $2d^2 - O(d)$ 3-paths (applying the Blakley–Roy inequality). Then we show that most of these 3-paths (i.e., at least $2d^2 - O(d)$ of them) are such that the second and fourth vertex in them are not adjacent in G. On the other hand, it will turn out that for any vertex w different from x and y, at most $\max(3,t) - 1$ such 3-paths start with the edge xy, and have w as their last vertex. This means that there are at least $(2d^2 - O(d))/(\max(3,t) - 1)$ vertices in G but the number of vertices in G is at most n. This gives the desired upper bound on d.

Now we continue with the detailed proof.

Proof. Let G be a C_{2k+1}-free graph containing no induced copy of $K_{2,t}$. Let d be the average degree of G. It suffices to show that $d \leq (1 + o(1))\sqrt{(\max(3,t) - 1)}\sqrt{n/2}$. As usual we can assume that each vertex of G has degree at least $d/2$, for otherwise we can delete it and the edges incident on it to obtain a new graph with average degree at least d.

Claim 6.12. *Any two non-adjacent vertices u, v in G have at most* $\max(3,t)-1$ *common neighbors.*

Proof. Suppose for a contradiction that u and v have $\max(3,t)$ common neighbours and let S denote the set of these common neighbours. Then since u, v and vertices in S cannot form an induced copy of $K_{2,t}$, there must be an edge xy among the vertices in S. However, then $uxyvw$ is a five cycle in G for some $w \in S \setminus \{x, y\}$, a contradiction. □

Now we will show that we can assume the maximum degree d_{max} of G is at most $6d$. Suppose that there is a vertex v in G with degree more than $6d$. Of course $|N_1(v)| > 6d$. Since G is C_5-free, there is no path on 4 vertices in the subgraph of G induced by $N_1(v)$. Therefore, the number of edges induced by $N_1(v)$ is at most $(4-2)/2 \cdot |N_1(v)| = |N_1(v)|$ by the Erdős–Gallai theorem [34]. Since the minimum degree is at least $d/2$, the sum of degrees of the vertices in $N_1(v)$ is at least $\frac{d}{2}|N_1(v)|$. In this sum, the edges between v and $N_1(v)$ are counted once, the edges induced by $N_1(v)$ are counted twice, so the number of edges between $N_1(v)$ and $N_2(v)$ is at least

$$\frac{d}{2}|N_1(v)| - |N_1(v)| - 2|N_1(v)| = \left(\frac{d}{2} - 3\right)|N_1(v)|.$$

On the other hand, a vertex u in $N_2(v)$ is adjacent to at most $\max(3,t)-1$ vertices in $N_1(v)$ by applying Claim 6.12 to u and v. So there are at most $|N_2(v)|(\max(3,t)-1)$ edges between $N_1(v)$ and $N_2(v)$. So combining, we get

$$|N_2(v)|(\max(3,t)-1) \geq \left(\frac{d}{2}-3\right)|N_1(v)| > \left(\frac{d}{2}-3\right)6d = 3d^2-18d.$$

Since, $n \geq |N_1(v)| + |N_2(v)|$, we have that

$$n > 6d + (3d^2-18d)/(\max(3,t)-1) \geq$$

$$(3d^2-6d)/(\max(3,t)-1) \geq 2d^2/(\max(3,t)-1)$$

whenever $d \geq 6$. Therefore, $d < (1+o(1))\sqrt{(\max(3,t)-1)}\sqrt{n/2}$, so

$$|E(G)| = \frac{nd}{2} < (1+o(1))\sqrt{(\max(3,t)-1)}\left(\frac{n}{2}\right)^{3/2},$$

proving Theorem 6.3 in the case $k=2$ and $t \geq 3$ and Theorem 6.4.

So from now on, we may assume $d_{max} \leq 6d$. A 3-path is called *good* if the second and fourth vertex in it are not adjacent.

Claim 6.13. *There is an edge xy in G such that the number of good 3-paths starting with xy is at least $2d^2-84d$.*

Proof. By the Blakley–Roy inequality, there are at least nd^3 3-walks in G, so there is an edge xy that is the first edge of at least $nd^3/|E(G)| = nd^3/(nd/2) = 2d^2$ 3-walks. First let us show that most of these $2d^2$ 3-walks are 3-paths. Suppose $xyzw$ is a 3-walk that is not a 3-path. Then either $z=x$, or $w=y$, or $w=x$. In each of these cases, there are at most d_{max} 3-walks, so in total there are at most $3d_{max} \leq 18d$ such 3-walks. Similarly there are at most $18d$ 3-walks of the form $yxzw$ that are not 3-paths. Therefore, there are at least $2d^2-36d$ 3-paths starting with the edge xy.

If a 3-path $xyzw$ or $yxzw$ is not good, then the edge $zw \in G[N_1(y)]$ or $zw \in G[N_1(x)]$ respectively. Moreover, given an edge $zw \in G[N_1(y)]$ or $zw \in G[N_1(x)]$ there are at most 4 paths starting with the edge xy and containing zw as its last edge, so the total number of 3-paths starting with the edge xy that are not good is at most $4(|G[N_1(x)]|+|G[N_1(y)]|)$. Since the neighborhood of a vertex doesn't contain a path on 4 vertices,

$$|G[N_1(x)]| + |G[N_1(y)]| \leq \frac{(4-2)}{2}\cdot|N_1(x)| + \frac{(4-2)}{2}\cdot|N_1(y)| \leq 2d_{max} \leq 12d$$

by the Erdős–Gallai theorem [34]. So the number of good 3-paths starting with the edge xy is at least $2d^2-36d-48d = 2d^2-84d$. □

Claim 6.14. *For any vertex w different from x and y, at most* $\max(3,t)-1$ *good 3-paths start with the edge xy and have w as their last vertex.*

Proof. Suppose for a contradiction that $\max(3,t)$ good 3-paths start with the edge xy and have w as their last vertex.

Let us suppose that among these good 3-paths, there is one of the form xyz_1w and another of the form yxz_2w. Now if $z_1 \neq z_2$, then xyz_1wz_2 forms a C_5 in G, a contradiction. Therefore, $z_1 = z_2$. Now since $\max(3,t) \geq 3$, there must be another good 3-path starting with the edge xy and having w as its last vertex - it is either of the form $xyz'w$ or of the form $yxz'w$ for some $z' \neq z_1$. In the first case, $xyz'wz_1$ is a C_5 and in the second case $yxz'wz_1$ is a C_5 in G, a contradiction.

Therefore, all the $\max(3,t)$ good 3-paths starting with the edge xy and having w as their last vertex are all either of the form $xyv_1w, xyv_2w, \ldots, xyv_{\max(3,t)}w$ or of the form $yxv_1w, yxv_2w, \ldots, yxv_{\max(3,t)}w$ where $v_1, v_2, \ldots, v_{\max(3,t)}$ are distinct vertices. However, in the first case, y and w are non-adjacent (as these are good 3-paths) and in the second case x and w are non-adjacent. Moreover, in both of these cases they have $v_1, v_2, \ldots, v_{\max(3,t)}$ as their common neighbors, contradicting Claim 6.12. $\square$

It follows from Claim 6.13 and Claim 6.14 that there are at least $(2d^2-84d)/(\max(3,t)-1)$ vertices in G. Therefore, $(2d^2-84d)/(\max(3,t)-1) \leq n$. Simplifying, we get

$$(d-21)^2 \leq \frac{n}{2}(\max(3,t)-1)+441.$$

So, $d \leq (1+o(1))\sqrt{(\max(3,t)-1)}\sqrt{n/2}$, implying that

$$|E(G)| = \frac{nd}{2} \leq (1+o(1))\sqrt{(\max(3,t)-1)}(n/2)^{3/2},$$

completing the proof of Theorem 6.3 in the cases $k=2$ and $t \geq 3$ and Theorem 6.4 (after substituting $k=t=2$). $\square$

Chapter 7

On the Turán number of some ordered even cycles

An *ordered graph* is a simple graph $G = (V, E)$ with a linear ordering on its vertex set. We say that the ordered graph H is an *ordered subgraph* of G if there is an embedding of H in G that respects the ordering of the vertices. The Turán problem for a set of ordered graphs $\mathcal{H}$ asks the following. What is the maximum number $\mathrm{ex}_<(n, \mathcal{H})$ of edges that an ordered graph on n vertices can have without containing any $H \in \mathcal{H}$ as an ordered subgraph? When $\mathcal{H}$ contains a single ordered graph H, we simply write $\mathrm{ex}_<(n, H)$.

The systematic study of this problem was initiated by Pach and Tardos [131]. They noted that the following analog of the Erdős–Stone–Simonovits result holds (see also [17]):

$$\mathrm{ex}_<(n, H) = \left(1 - \frac{1}{\chi_<(H) - 1}\right)\frac{n^2}{2} + o(n^2),$$

where $\chi_<(H)$, the interval chromatic number of H, is the minimum number of intervals the (linearly ordered) vertex set of H can be partitioned into, so that no two vertices belonging to the same interval are adjacent in H. This formula determines the asymptotics of the ordered Turán number, except when $\chi_<(H) = 2$.

The case $\chi_<(H) = 2$ turns out to be closely related to a well-studied problem of forbidden patterns in 0-1 matrices. To formulate it, let A_H be the bipartite adjacency matrix of H, where rows and columns correspond to the two intervals of H (in the appropriate ordering), and 1-entries correspond to edges in H. Then we are interested in the maximum number of 1-entries in an $n \times m$ matrix that does not contain the pattern A_H in the sense that A_H is not a submatrix, nor can it be obtained from a submatrix by changing some 1-entries to 0-entries.

The problem of finding the extremal number of matrix patterns was introduced by Füredi and Hajnal [50] about 25 years ago, and several results have been obtained since then (see e.g. [100, 121, 131, 141] and the references therein), although most of them concern matrices of *acyclic* graphs. One notable exception is a result of Pach and Tardos [131] that establishes $\mathrm{ex}_<(n, \mathcal{H}) = \Theta(n^{4/3})$ for an infinite set of ordered cycles $\mathcal{H}$ that they call "positive" cycles. The definition of positive cycles is motivated by an incidence geometry problem, where they correspond to a class of forbidden configurations.

In this chapter we estimate $\mathrm{ex}_<(n, \mathcal{H})$ for various finite sets $\mathcal{H}$ of ordered cycles that all come from the class of bordered cycles that we define as follows. In an ordered graph with interval chromatic number 2 and intervals U and V (U preceding V), we call the edge connecting the first vertex of U and the last vertex of V an *outer border*, and the edge connecting the last vertex of U and the first vertex of V an *inner border*. Then a *bordered cycle* is an ordered cycle with interval chromatic number 2 that contains both an inner and an outer border. For example, out of the six ordered bipartite 6-cycles, three are bordered (see Figure 7.1).

Let us emphasize that there is no containment relationship between our bordered cycles and the positive cycles of Pach and Tardos. For example, using the notation of Figure 7.1, the positive 6-cycles are C_6^1, C_6^3, C_6^I and C_6^O, whereas the bordered 6-cycles are C_6^1, C_6^2 and C_6^3.

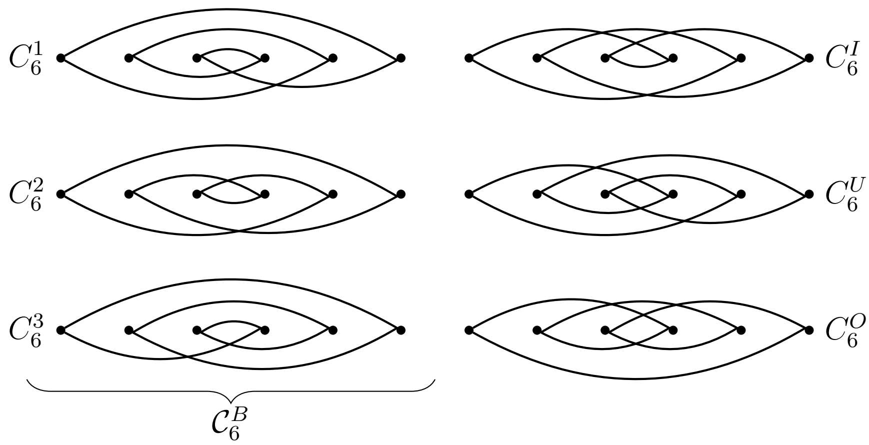


Figure 7.1: Bordered ($\mathcal{C}_6^B = \{C_6^1, C_6^2, C_6^3\}$), outbordered ($C_6^O$), unbordered ($C_6^U$) and inbordered ($C_6^I$) 6-cycles. Orderings shown in the same row can be obtained from each other by reversing the order of vertices in the second interval.

Our results

Let $\mathcal{C}_{2k}^B$ be the set of bordered $2k$-cycles. Our main result determines the asymptotics of the maximum number of edges in an ordered graph with no bordered cycle of length up to $2k$. This can be thought of as an analog of Conjecture 4.5 for bordered cycles.

Theorem 7.1 (Győri, Korándi, M., Tomon, Tompkins, Vizer [79]). *For every fixed integer $k > 1$,*

$$\mathrm{ex}_<(n, \{\mathcal{C}_4^B, \mathcal{C}_6^B, \dots, \mathcal{C}_{2k}^B\}) = \Theta(n^{1+1/k}).$$

We do not know whether the bordered version of Theorem 4.4 is true in general, i.e., if forbidding $\mathcal{C}_{2k}^B$ alone suffices to get the same asymptotic upper bound for the extremal number. However, we can prove this for $k = 3$.

Theorem 7.2 (Győri, Korándi, M., Tomon, Tompkins, Vizer [79]). *For bordered 6-cycles,*

$$\mathrm{ex}_<(n, \mathcal{C}_6^B) = \Theta(n^{4/3}).$$

Actually, Theorem 7.2 is an immediate consequence of Theorem 7.1 and the fact that when $l-1$ divides $k-1$, then any $\mathcal{C}_{2k}^B$-free ordered graph contains a large $\mathcal{C}_{2l}^B$-free subgraph. This is established by the following theorem.

Theorem 7.3 (Győri, Korándi, M., Tomon, Tompkins, Vizer [79]). *Let $k, l \geq 2$ be integers such that $k-1$ is divisible by $l-1$. Then any $\mathcal{C}_{2k}^B$-free ordered graph G contains a $\mathcal{C}_{2l}^B$-free subgraph with at least $\frac{l-1}{k-1}$ fraction of the edges of G.*

Note that for $l = 2$, Theorem 7.3 is a generalization of a theorem of Kühn and Osthus [97] which states that any bipartite C_{2k}-free graph G contains a C_4-free subgraph with at least $1/(k-1)$ fraction of the edges of G. Indeed, if we order the vertices of G so that all of the vertices in one of its color classes appear before the vertices of the other color class, then any C_4 in G is bordered. Then Theorem 7.3 gives a C_4-free subgraph of G that has at least $1/(k-1)$ fraction of the edges of G.

This chapter is organized as follows. In Section 7.1 we prove the lower bound of Theorem 7.1 by constructing a dense ordered graph without short ordered cycles. The matching upper bound is shown in Section 7.2. In Section 7.3 we give a short proof of Theorem 7.3. We conclude the chapter with some remarks and open problems in Section 7.4.

7.1 Lower bound construction

Our construction is based on generalized Sidon sets defined as follows.

Definition 7.4. *Let $k \geq 2$ be an integer. A set of integers S is called a B_k-set if all k-sums of elements in S are different, i.e., if for every integer C, there is at most one solution to*

$$x_1 + x_2 + \cdots + x_k = C$$

in S, up to permuting the elements x_i (the x_i need not be distinct).
We denote the maximum size of a B_k-set $S \subseteq \{1, 2, \ldots, n\}$ by $F_k(n)$.

Note that this definition implies that if $x_1 + x_2 + \cdots + x_l = x'_1 + x'_2 + \cdots + x'_l$ for some $l \leq k$ and $x_i, x'_i \in S$, then $\{x_1, x_2, \ldots, x_l\} = \{x'_1, x'_2, \ldots, x'_l\}$ as multisets.

Bose and Chowla [16] proved that

$$F_k(n) \geq n^{1/k} + o(n^{1/k}).$$

For a fixed $n \geq 1$, let $S \subset \{1, 2, ..., n\}$ be a B_k-set of size $|S| = F_k(n)$. Our construction will be an ordered graph G on $4n$ vertices that we define through its bipartite adjacency matrix $A_G \in \{0,1\}^{2n \times 2n}$ as follows. For $1 \leq i, j \leq 2n$ we put

$$A_G(i,j) = \begin{cases} 1 & \text{if } i - j + n \in S \text{ and } 1 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that G contains no $2l$-cycle with a border edge for any $l \leq k$. Note that the edges of a $2l$-cycle correspond to 1-entries in S at coordinates $(i_1, j_1), \ldots, (i_{2l}, j_{2l})$, where

1. $i_{2s-1} = i_{2s}$ and $j_{2s-1} \neq j_{2s}$ for $s = 1, 2, \ldots, l$, and
2. $j_{2s} = j_{2s+1}$ and $i_{2s} \neq i_{2s+1}$ for $s = 1, 2, \ldots, l$ (taking indices modulo $2l$).

This readily implies $\sum_{s=1}^{l} i_{2s} = \sum_{s=1}^{l} i_{2s-1}$ and $\sum_{s=1}^{l} j_{2s} = \sum_{s=1}^{l} j_{2s-1}$, and in particular

$$\sum_{s=1}^{l} (i_{2s} - j_{2s} + n) = \sum_{s=1}^{l} (i_{2s-1} - j_{2s-1} + n). \tag{7.1}$$

By the definition of A_G, we know that $i_s - j_s + n \in S$ for every $1 \le s \le 2l$, so both sides of (7.1) are l-sums in S. But S was a B_k-set with $l \le k$, so by the observation above, the two sums must have the same terms (possibly in a different order).

However, an outer (resp. inner) border of this cycle uniquely minimizes (resp. maximizes) $i_s - j_s$ over all $s = 1, 2, \dots, 2l$ so if our cycle has a border edge, then there is a unique number among the terms, a contradiction.

Therefore, G does not contain any cycle of length at most $2k$ with a border edge, and it is easy to see that it contains $nF_k(n) \ge n^{1+1/k} + o(n^{1+1/k})$ edges. This proves the lower bound of Theorem 7.1.

7.2 Upper bound

Let $G = (V, E)$ be an ordered graph on the vertex set $V = \{x_1 < x_2 < \dots < x_n\}$ that avoids all cycles in $\mathcal{C}_4^B, \dots, \mathcal{C}_{2k}^B$. We want to show that the number of edges m in G is $O(n^{1+1/k})$. Let us call a path $P = v_0 v_1 \dots v_k$ *k-zigzag*, if $v_k < v_{k-2} < \dots < v_0 < v_1 < v_3 < \dots < v_{k-1}$ for k even, and $v_{k-1} < v_{k-3} < \dots < v_0 < v_1 < v_3 < \dots < v_k$ for k odd.

Claim 7.5. *The graph G contains at most one k-zigzag path between any pair of vertices.*

Proof. Suppose to the contrary that $v_0 \dots v_k$ and $v_0' \dots v_k'$ are two different k-zigzag paths such that $v_0 = v_0'$ and $v_k = v_k'$. Let $s \in \{0, \dots, k\}$ be the largest index such that $v_i = v_i'$ for every $i \in \{0, \dots, s\}$, and let t be the smallest index larger than s such that $v_t = v_t'$. Then $v_s v_{s+1} \dots v_t v_{t-1}' v_{t-2}' \dots v_s'$ are the consecutive vertices of a cycle of length $2(t-s)$, where $2 \le t - s \le k$. Also, this cycle has two border edges, namely $v_s \min\{v_{s+1}, v_{s+1}'\}$ and $\max\{v_{t-1}, v_{t-1}'\} v_t$. But then G contains a cycle in $\mathcal{C}_{2(t-s)}^B$, a contradiction. □

This tells us that the number of k-zigzag paths in G is at most n^2. Now let us bound the number of k-zigzag paths from below.

Claim 7.6. *The graph G contains at least*

$$\frac{m^k}{k^k (3n)^{k-1}}$$

k-zigzag paths.

Proof. We will define a sequence of graphs $G_k, \dots, G_1 \subseteq G$ recursively as follows. We set $G_k = G$, and we will obtain G_{i-1} from G_i by deleting the edges between each vertex and its $u = \lfloor m/2kn \rfloor$ largest and u smallest neighbors.

More precisely, we define the left and right neighborhood of a vertex $x_s \in V$ in G_i as

$$L_i(x_s) = \{x_j : j < s, x_j x_s \in G_i\} \qquad \text{and} \qquad R_i(x_s) = \{x_j : j > s, x_s x_j \in G_i\},$$

respectively. Also, let $L_i^+(x_s)$ be the u smallest elements of $L_i(x_s)$, and let $R_i^+(x_s)$ be the u largest elements of $R_i(x_s)$. (If $|L_i(x_s)| < u$, then we define $L_i^+(x_s) = L_i(x_s)$, and we do the same for $R_i^+(x_s)$.) We then set

$$G_{i-1} = G_i \setminus \left(\bigcup_{s=1}^{n} \{x_j x_s : x_j \in L_i^+(x_s)\} \cup \bigcup_{s=1}^{n} \{x_s x_j : x_j \in R_i^+(x_s)\} \right).$$

Let us collect some properties of the graphs G_i.

1. We delete at most $2nu \leq m/k$ edges from G_i to obtain G_{i-1}, so we have $|E(G_i)| \geq mi/k$ for every i, and in particular, $|E(G_1)| \geq m/k$.
2. For every $x \in V$ and every i, we have $L^+_{2i}(x) < L^+_{2i+2}(x)$ and $R^+_{2i+1}(x) < R^+_{2i-1}(x)$, where we write $A < B$ for some sets $A, B \subset V$ if $\max A < \min B$.
3. For every $x \in V$, if $L_{2i-1}(x)$ is non-empty, then $|L^+_{2i}(x)| = u$. Similarly, if $R_{2i}(x)$ is non-empty, then $|R^+_{2i+1}(x)| = u$.

Now we show that for every edge $f = v_0v_1 \in G_1$, there are at least u^{k-1} k-zigzag paths starting with f. Observe that every sequence of vertices $v_0, v_1, \dots, v_k$ satisfying $v_i \in L^+_i(v_{i-1})$ for i even, and $v_i \in R^+_i(v_{i-1})$ for i odd is a k-zigzag path by property 2. Also, the number of such paths is exactly u^{k-1} by property 3. Hence, using $u \geq m/3kn$, we have at least $|E(G_1)|u^{k-1} > m^k/k^k(3n)^{k-1}$ different k-zigzag paths in G. □

Now comparing our lower and upper bound for the number of k-zigzag paths in G, we arrive at the inequality

$$n^2 \geq \frac{m^k}{k^k(3n)^{k-1}},$$

which yields $m < 3kn^{1+1/k}$. □

7.3 Deleting small cycles

Our proof of Theorem 7.3 is inspired by the proof of Grósz, Methuku and Tompkins in [69] on deleting 4-cycles, which is a simple proof of a theorem of Kühn and Osthus [97]. We make use of the following result of Gallai [55] and Roy [135].

Theorem 7.7 (Gallai–Roy)**.** *If a directed graph G contains no directed path of length h then $\chi(G) \leq h$.*

Proof of Theorem 7.3. Let $G = (V, E)$ be an ordered graph which is $\mathcal{C}^B_{2k}$-free, where the elements of V are $x_1 < \dots < x_n$. Define the directed graph H on E as a vertex set such that for $f, f' \in E$, $\overrightarrow{ff'}$ is a directed edge of H if there exists a bordered $2l$-cycle with outer border f and inner border f'. Note that H is acyclic, because if $\overrightarrow{ff'} \in E(H)$, where $f = ab$, $f' = a'b'$, $a < b$ and $a' < b'$, then $a < a' < b' < b$.

We show that the longest directed path in H has length less than $h = \frac{k-1}{l-1}$. Suppose to the contrary that there is a directed path $f_1 \dots f_{h+1}$ in H. Then for every $i = 1, \dots, h$, there is a bordered cycle C_i with outer border f_i and inner border f_{i+1}. Then it is easy to see that $\left(\bigcup_{i=1}^h C_i\right) \setminus \{f_2, \dots, f_h\}$ is a bordered cycle of length $2lh - 2h + 2 = 2k$, with outer border f_1, and inner border f_{h+1}, contradicting the choice of G.

Hence we can apply Theorem 7.7 to get a proper h-coloring of H. Here the largest color class $E_0 \subseteq E$ is an independent set of size at least $\frac{l-1}{k-1}|E|$, so there is no cycle in $\mathcal{C}^B_{2l}$ that has all its edges in E_0. The edges of E_0 will then form an ordered subgraph of G that satisfies our conditions. □

7.4 Concluding remarks

Note that Theorem 7.2 is stronger than the $k = 3$ case of Theorem 4.4 because it only forbids three out of the six orderings of the hexagon. In fact, it is enough to forbid two orderings of the hexagon to achieve the same asymptotic bound.

Theorem 7.8. *Let* $\mathcal{C}_1 = \{C_6^2, C_6^1\}$, $\mathcal{C}_2 = \{C_6^2, C_6^3\}$, $\mathcal{C}_3 = \{C_6^U, C_6^I\}$, *and* $\mathcal{C}_4 = \{C_6^U, C_6^O\}$. *For any* $i \in \{1, 2, 3, 4\}$, *we have* $\mathrm{ex}_<(n, \mathcal{C}_i) = \Theta(n^{4/3})$.

Sketch of the proof. It is enough to show that every $\mathcal{C}_i$-free ordered graph G on $2n$ vertices has $O(n^{4/3})$ edges between the first n and the last n vertices. Indeed, an inductive argument applied to the two halves of G then yields a $O(n^{4/3})$ upper bound on the total number of edges, as well. We first show this for $\mathcal{C}_1$, so let G be an ordered graph on the vertex set $A \cup B$ with $|A| = |B| = n$ and $A < B$ that has no edges induced by A or B, and avoids C_6^1 and C_6^2.

Note that G cannot contain two bordered 4-cycles such that the inner border of one is the outer border of the other, because they would create a copy of C_6^2. So by the argument of Theorem 7.3, we can assume that G does not contain any bordered 4-cycle. The rest of the proof follows that of Theorem 7.1; we only need that, analogously to Claim 7.5, if for some $x \in A, y \in B$, we have two 3-zigzag paths P_1 and P_2 from x to y, then $P_1 \cup P_2$ is either C_6^1 or C_6^2, or it induces a bordered 4-cycle. So we once again get that the number of 3-zigzag paths in G is at most n^2, and can finish the argument as before.

To obtain an upper bound on $\mathrm{ex}_<(n, \mathcal{C}_i)$ for $i \in \{2, 3, 4\}$, note that we can obtain each $\mathcal{C}_i$ from $\mathcal{C}_1$ by reversing the order of the vertices in one (or both) of the color intervals. This means, for example, that the graph G above is $\mathcal{C}_1$-free if and only if the graph G', obtained from G by reversing the order of vertices in B, is $\mathcal{C}_3$-free. In particular, such a G' has $O(n^{4/3})$ edges, and a similar reduction works for all other i. □

As we mentioned before, Pach and Tardos [131] showed that $\mathrm{ex}_<(n, \mathcal{C}) = \Theta(n^{4/3})$ for a certain set of cycles that they call "positive". They also asked if it would be enough to forbid the positive 6-cycles (i.e., $C_6^1, C_6^3, C_6^O, C_6^I$) to get the same upper bound. More generally, we propose the following conjecture.

Conjecture 7.9. Let C be an ordered 6-cycle of interval chromatic number 2. Then

$$\mathrm{ex}_<(n, C) = \Theta(n^{4/3}).$$

Finally, let us remark that while we are unable to prove that $\mathrm{ex}_<(n, \mathcal{C}_{2k}^B) = O(n^{1+1/k})$, there is certainly no absolute constant $\varepsilon > 0$ such that $\mathrm{ex}_<(n, \mathcal{C}_{2k}^B) \geq n^{1+\varepsilon}$ for every k:

Theorem 7.10. *There exists a sequence of positive real numbers* $(\lambda_k)_{k=2,3,\ldots}$ *such that* $\mathrm{ex}_<(n, \mathcal{C}_{2k}^B) = O(n^{1+\lambda_k})$ *and* $\liminf_{k\to\infty} \lambda_k = 0$.

Proof. We will show that we can choose $\lambda_{(m-1)!+1} = 1/m$. Let $k = (m-1)! + 1$ and let G be a $\mathcal{C}_{2k}^B$-free ordered graph with n vertices. Then for any $2 \leq l \leq m$, $l - 1$ divides $k - 1$. Therefore, applying Theorem 7.3 repeatedly, we obtain a subgraph G' of G such that G' is $\{\mathcal{C}_4^B, \mathcal{C}_6^B, \ldots, \mathcal{C}_{2m}^B\}$-free, and G' has at least $(m-1)!/(k-1)^{m-1}$ proportion of the edges of G. But then, by Theorem 7.1, $|E(G')| = O(n^{1+1/m})$, and thus $|E(G)| = O(n^{1+1/m})$, as well. □

We have recently learned that Timmons [143] also studied the Turán number of ordered cycles, and observed that $\mathrm{ex}_<(n, \mathcal{C}_{2k}) = O(n^{1+1/k})$ for the family $\mathcal{C}_{2k}$ of all ordered $2k$-cycles with interval chromatic number 2. On the other hand, he found the construction we presented in Section 7.1, and asked whether a matching upper bound holds, i.e., if the upper bound $O(n^{1+1/k})$ holds when only the ordered $2k$-cycles with an inner or an outer border are forbidden. Our Theorem 7.2 (or Theorem 7.8) answers this question positively for $k = 3$ (for an even smaller subfamily), and Theorem 7.1 answers a variant for every k where shorter cycles are also forbidden. We kept the construction in this chapter for completeness.

Chapter 8

Generalized Turán problems for even cycles

8.1 Introduction

First we introduce some basic notation and definitions used throughout this chapter.

8.1.1 Notation and definitions

The girth of a graph is the length of a shortest cycle in it. We say a graph has even girth if its girth is of even length, otherwise we say it has odd girth. Now we introduce basic notation that we will use throughout the chapter.

- let A be a set of integers, each at least 3. Then let the set of cycles $\mathcal{C}_A = \{C_a : a \in A\}$. If $A = \{3, 4, ..., k\}$ for some integer k, then we denote the corresponding set of cycles by $\mathcal{C}_k$.
- We will denote by $v_1v_2 \dots v_{k-1}v_kv_1$ a cycle C_k with vertices $v_1, v_2, \dots, v_k$ and edges v_iv_{i+1}
$(i = 1, \dots, k-1)$ and v_kv_1. Similarly $v_1v_2 \dots v_{k-1}v_k$ denotes a path P_k with vertices $v_1, v_2, \dots, v_k$ and edges v_iv_{i+1} $(i = 1, \dots, k-1)$.
- For two graphs H and G, let $\mathcal{N}(H, G)$ denote the number of copies of H in G.
- For a vertex v in G, let $N_i(v)$ denote the set of vertices at distance exactly i from v.
- For any two positive integers n and l, let $(n)_l$ denote the product

$$n(n-1)(n-2)\dots(n-(l-1)).$$

8.1.2 Generalized Turán problems

Given a graph H and a set of graphs $\mathcal{F}$, let

$$\text{ex}(n, H, \mathcal{F}) = \max_G\{\mathcal{N}(H, G) : G \text{ is an } \mathcal{F}\text{-free graph on } n \text{ vertices.}\}$$

If $\mathcal{F} = \{F\}$, we simply denote it by $\text{ex}(n, H, F)$. This problem was initiated by Erdős [30], who determined $\text{ex}(n, K_s, K_t)$ exactly. Concerning cycles, Bollobás and Győri [14] proved that

$$(1 + o(1))\frac{1}{3\sqrt{3}}n^{3/2} \le ex(n, C_3, C_5) \le (1 + o(1))\frac{5}{4}n^{3/2}$$

and this result was extended by Győri and Li [85] showing that

$$\mathrm{ex}(n, C_3, C_{2k+1}) \leq \frac{(2k-1)(16k-2)}{3} \cdot ex(n, C_{2k})$$

for $k \geq 2$. This was later improved by Füredi and Özkahya [53] by a factor of $\Omega(k)$.

The systematic study of the function $\mathrm{ex}(n, H, F)$ was initiated by Alon and Shikhelman in [5], where they improved the result of Bollobás and Győri by showing that $\mathrm{ex}(n, C_3, C_5) \leq (1 + o(1))\frac{\sqrt{3}}{2}n^{3/2}$. This bound was further improved in [44] and then very recently in [46] by Ergemlidze and Methuku who showed that $\mathrm{ex}(n, C_3, C_5) < (1 + o(1))0.231975n^{3/2}$. Another notable result is the exact computation of $\mathrm{ex}(n, C_5, C_3)$ by Hatami, Hladký, Král, Norine, and Razborov [89] and independently by Grzesik [73], where they showed that it is equal to $(\frac{n}{5})^5$. Very recently, the asymptotic value of $\mathrm{ex}(n, C_k, C_{k-2})$ was determined for every odd k by Grzesik and Kielak in [74]. Concerning paths, Győri, Salia, Tompkins and Zamora [87] determined $\mathrm{ex}(n, P_l, P_k)$ asymptotically.

In [5], Alon and Shikhelman characterized the graphs F with $\mathrm{ex}(n, C_3, F) = O(n)$ and more recently, Gerbner and Palmer [62] showed that for every $l \geq 4$ and every graph F we have either $\mathrm{ex}(n, C_l, F) = \Omega(n^2)$ or $\mathrm{ex}(n, C_l, F) = O(n)$, and characterized the graphs F for which the latter bound holds. They also showed

Theorem 8.1 (Gerbner, Palmer [63])**.** *For $t \geq 2$ and $l \geq 4$ we have*

$$\mathrm{ex}(n, C_l, K_{2,t}) = \frac{1}{2l}(t-1)^{l/2}n^{l/2}, \qquad ex(n, P_l, K_{2,t}) = \frac{1}{2}(t-1)^{(l-1)/2}n^{(l+1)/2}.$$

Note that the case $t = 2$ was proved independently by Gishboliner and Shapira [64].

In this chapter, we mainly focus on the case when H is an even cycle of given length and $\mathcal{F}$ is a family of cycles.

8.1.3 Forbidding a set of cycles

The famous Girth Conjecture of Erdős [31] asserts the following.

Conjecture 8.1 (Erdős's Girth Conjecture [31] for k)**.** For any positive integer k, there exist a constant $c > 0$ depending only on k, and a family of graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq cn^{1+1/k}$ and the girth of G_n is more than $2k$.

This conjecture has been verified for $k = 2, 3, 5$, see [10, 18, 134, 148]. For a general k, Sudakov and Verstraëte [140] showed that if such graphs exist, then they contain a C_{2l} for any l with $k < l \leq Cn$, for some constant $C > 0$. More recently, Solymosi and Wong [138] proved that if such graphs exist, then in fact, they contain many C_{2l}'s for any fixed $l > k$. More precisely they proved:

Theorem 8.2 (Solymosi, Wong [138])**.** *If Erdős's Girth Conjecture holds for k, then for every $l > k$ we have*

$$\mathrm{ex}(n, C_{2l}, \mathcal{C}_{2k}) = \Omega(n^{2l/k}).$$

The following remark shows that in many cases this bound is sharp.

Remark 8.2. If $k+1$ divides $2l$, then

$$\mathrm{ex}(n, C_{2l}, \mathcal{C}_{2k}) = O(n^{2l/k}).$$

Indeed, let us associate to each C_{2l}, one fixed ordered list of $2l/(k+1)$ edges $(e_1, e_{k+1}, e_{2k+1}, \ldots)$, where e_1 appears as the first edge (chosen arbitrarily) on the C_{2l}, e_{k+1} as the $(k+1)$-th edge, e_{2k+1} as the $(2k+1)$-th edge and so on. Note that at most one C_{2l} is associated to an ordered tuple $(e_1, e_{k+1}, e_{2k+1}, \ldots)$, because there is at most one path of length $k-1$ connecting the endpoints of any two edges (as all the short cycles are forbidden). Since there are at most $O(n^{1+1/k})$ ways to select each edge, this shows the number of C_{2l}'s is at most $O((n^{1+1/k})^{2l/(k+1)}) = O(n^{2l/k})$, showing that the bound in Theorem 8.2 is sharp when $k+1$ divides $2l$.

It is worth mentioning that Gerbner, Keszegh, Palmer and Patkós [58] considered a similar problem, where a finite list of allowed cycle lengths is given (thus the list of forbidden cycle lengths is infinite). Another main difference is that in [58], all cycles of allowed lengths are counted, as opposed to only counting the number of cycles of a given length like in this chapter.

Constructions

Before mentioning our results in the next section, we present typical constructions of graphs (with many copies of a cycle) that we will refer to, in the rest of the chapter.

- For $l, t \geq 1$ the *(l, t)-theta-graph* with endpoints x and y is the graph obtained by joining two vertices x and y, by t internally disjoint paths of length l.
- For a simple graph F and $n, l \geq 1$ the *theta-*(n, F, l) *graph* is a graph on n vertices obtained by replacing every edge xy of F by an (l, t)-theta-graph with endpoints x and y, where t is chosen as large as possible, with some isolated vertices if needed. More precisely let $t = \lfloor \frac{n-|V(F)|}{|E(F)|(l-1)} \rfloor$, and we add $n-(t|E(F)|(l-1)+|V(F)|)$ isolated vertices.

8.2 Our results

8.2.1 Forbidding a cycle of given length

We determine the order of magnitude of $\mathrm{ex}(n, C_{2l}, C_{2k})$ below.

Theorem 8.3 (Gerbner, Győri, M., Vizer [61])**.** *For any $l \geq 3$ and $k \geq 2$ we have*

$$\mathrm{ex}(n, C_{2l}, C_{2k}) \leq (1+o(1)) \frac{2^{l-2}(k-1)^l}{2l} n^l.$$

For any $k > l \geq 2$ we have

$$\mathrm{ex}(n, C_{2l}, C_{2k}) \geq (1+o(1)) \frac{(k-1)_l}{2l} n^l.$$

For any $l > k \geq 3$ we have

$$\mathrm{ex}(n, C_{2l}, C_{2k}) \geq (1+o(1)) \frac{1}{l^l} n^l.$$

Theorem 8.3 and Theorem 8.4 (stated below) show that $\mathrm{ex}(n, C_{2l}, C_{2k}) = \Theta(n^l)$ for any $k, l \geq 2$, except for the lower bound in the case $k = 2$, which can be easily shown by counting cycles in the well-known C_4-free graph constructed by Erdős and Rényi [36] (See Theorem 8.1 and [64]).

This theorem has recently been proven independently by Gishboliner and Shapira [64]. Our proof is different from theirs, and it gives a better bound if k is fixed (moreover, if l is fixed, then their bound and our bound are both tight). They study odd cycles as well, determining the order of magnitude of $\mathrm{ex}(n, C_l, C_k)$ for every $l > 3$ and k, and also provide interesting applications of these results in the study of the graph removal lemma and graph property testing.

Solymosi and Wong [138] asked whether a similar lower bound (to that of Theorem 8.2) on the number of C_{2l}'s holds, if just C_{2k} is forbidden instead of forbidding $\mathcal{C}_{2k}$. Theorem 8.3 answers this question in the negative.

If we go beyond determining the order of magnitude we can ask the asymptotics of these functions. In many cases it is a much harder question than the order of magnitude question [5, 14, 44, 117].

We determine $\mathrm{ex}(n, C_4, C_{2k})$ asymptotically.

Theorem 8.4 (Gerbner, Győri, M., Vizer [61]). *For $k \geq 2$ we have*

$$\mathrm{ex}(n, C_4, C_{2k}) = (1 + o(1))\frac{(k-1)(k-2)}{4}n^2.$$

8.2.2 Forbidding a set of cycles

Theorem 8.2 implies that if Erdős's Girth Conjecture is true (recall that it is known to be true for $l = 2, 3, 5$), then $\mathrm{ex}(n, C_{2l}, \mathcal{C}_{2l-2}) = \Omega(n^{2l/(l-1)})$ for any $l \geq 3$. On the other hand, by Remark 8.2, this number is at most $O(n^{2l/(l-1)})$. This implies $\mathrm{ex}(n, C_{2l}, \mathcal{C}_{2l-2}) = \Theta(n^{2l/(l-1)})$. By Lemma 8.8 (which is straightforward to prove), we know that when counting copies of an even cycle, forbidding an odd cycle does not change the order of magnitude. Therefore, we have

Corollary 8.3 (Gerbner, Győri, M., Vizer [61]). *Suppose $l \geq 3$ and Erdős's Girth Conjecture is true for $l - 1$. Then we have*

$$\mathrm{ex}(n, C_{2l}, \mathcal{C}_{2l-1}) = \Theta(n^{2l/(l-1)}).$$

In other words, the maximum number of C_{2l}'s in a graph of girth $2l$ is $\Theta(n^{2l/(l-1)})$. We prove that the previous theorem is sharp in the sense that forbidding one more even cycle decreases the order of magnitude significantly: The maximum number of C_{2l}'s in a C_{2k}-free graph with girth $2l$ is $\Theta(n^2)$. That is,

$$\mathrm{ex}(n, C_{2l}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^2).$$

More generally, we show the following.

Theorem 8.5 (Gerbner, Győri, M., Vizer [61]). *For any $k > l$ and $m \geq 2$ such that $2k \neq ml$ we have*

$$\mathrm{ex}(n, C_{ml}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m).$$

Observe that forbidding even more cycles does not decrease the order of magnitude, as long as we do not forbid C_{2l} itself, as shown by $(l, \lfloor n/l \rfloor)$-theta graph and some isolated vertices (i.e. the theta-(n, K_2, l) graph). On the other hand it is easy to see that if we forbid every cycle of length other than $2l+1$, then there are $O(n)$ copies of C_{2l+1}.

Corollary 8.3 determines the order of magnitude of maximum number of C_{2l}'s in a graph of girth $2l$. It is then very natural to consider the analogous question for odd cycles: What is the maximum number of C_{2k+1}'s in a graph of girth $2k+1$? Before answering this question, we state a strong form of Erdős's Girth Conjecture that is known to be true for small values of k.

A graph G on n vertices, with average degree d, is called *almost-regular* if the degree of every vertex of G is $d + O(1)$.

Conjecture 8.4 (Strong form of Erdős's Girth Conjecture)**.** For any positive integer k, there exist a family of almost-regular graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq \frac{n^{1+1/k}}{2}$ and G_n is $\{C_4, C_6, \dots, C_{2k}\}$-free.

Lazebnik, Ustimenko and Woldar [109] showed Conjecture 8.4 is true when $k \in \{2, 3, 5\}$ using the existence of polarities of generalized polygons. We show the following that can be seen as the 'odd cycle analogue' of Theorem 8.2.

Theorem 8.6 (Gerbner, Győri, M., Vizer [61])**.** *Suppose $k \geq 2$ and Strong form of Erdős's Girth Conjecture is true for k. Then we have*

$$\mathrm{ex}(n, C_{2k+1}, \mathcal{C}_{2k}) = (1 + o(1)) \frac{n^{2+\frac{1}{k}}}{4k+2}.$$

To show that Theorem 8.6 is sharp and to give an analogue of Theorem 8.5 (in the case of $m = 2$) for odd cycles, we prove that if we forbid one more even cycle, then the order of magnitude goes down significantly:

Theorem 8.7 (Gerbner, Győri, M., Vizer [61])**.** *For any integers $k > l \geq 2$, we have*

$$\Omega(n^{1+\frac{1}{2k+1}}) = ex(n, C_{2l+1}, \mathcal{C}_{2l} \cup \{C_{2k}\}) = O(n^{1+\frac{l}{l+1}}).$$

However, if the additional forbidden cycle is of odd length, we can only prove a quadratic upper bound. We conjecture that the truth is also sub-quadratic here (see Section 8.8, Theorem 8.20).

Theorem 8.8 (Gerbner, Győri, M., Vizer [61])**.** *For any integers $k > l \geq 2$, we have*

$$\Omega(n^{1+\frac{1}{2k+2}}) = ex(n, C_{2l+1}, \mathcal{C}_{2l} \cup \{C_{2k+1}\}) = O(n^2).$$

Concerning forbidding a set of cycles we also determine the asymptotics of $\mathrm{ex}(n, C_4, \mathcal{C}_A)$ for every possible set A. Let A_e be the set of even numbers in A and A_o be the set of odd numbers in A.

Theorem 8.9 (Gerbner, Győri, M., Vizer [61])**.** *For any $k \geq 3$, we have*

$$\mathrm{ex}(n, C_4, \mathcal{C}_A) = \begin{cases} 0 & \text{if } 4 \in A \\ (1+o(1))\frac{(k-1)(k-2)}{4} n^2 & \text{if } 4 \notin A \text{ and } 2k \text{ is the smallest element of } A_e \\ (1+o(1))\frac{1}{64} n^4 & \text{if } A_e = \emptyset. \end{cases}$$

We also determine the order of magnitude of $\mathrm{ex}(n, C_6, \mathcal{C}_A)$ by proving

Theorem 8.10 (Gerbner, Győri, M., Vizer [61])**.**

$$\mathrm{ex}(n, C_6, \mathcal{C}_A) = \begin{cases} 0 & \textit{if } 6 \in A, \\ \Theta(n^2) & \textit{if } 6 \notin A,\ 4 \in A \textit{ and } |A_e| \geq 2, \\ \Theta(n^3) & \textit{if } 4, 6 \notin A \textit{ and } A_e \neq \emptyset, \textit{ or if } A_e = \{4\}, \\ \Theta(n^6) & \textit{if } A_e = \emptyset. \end{cases}$$

8.2.3 Maximum number of P_l's in a graph avoiding a cycle of given length

We study the maximum possible number of paths in a C_{2k}-free graph and prove the following results.

Theorem 8.11. *For $l, k \geq 2$, we have*

$$ex(n, P_l, C_{2k}) \leq (1 + o(1))\frac{1}{2}(k-1)^{\frac{l-1}{2}} n^{\frac{l+1}{2}}.$$

Theorem 8.12. *If $2 \leq l < 2k$, then*

$$ex(n, P_l, C_{2k}) \geq (1 + o(1))\frac{1}{2}(k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil}.$$

If $l \geq 2k$, then

$$ex(n, P_l, C_{2k}) \geq (1 + o(1)) \max\left\{ \left(\frac{n}{\lfloor l/2 \rfloor}\right)^{\lceil l/2 \rceil}, \left(\frac{(k-1)}{4(k-2)^{k+2}}\right)^{\lceil \frac{l}{2} \rceil} (k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil} \right\}.$$

Note that if $l < 2k$ and l is odd, Theorem 8.11 and Theorem 8.12 show that $ex(n, P_l, C_{2k})$ is equal to $(1 + o(1))\frac{1}{2}(k-1)^{\frac{l-1}{2}} n^{\frac{l+1}{2}}$ as k and n tend to infinity.

Finally, we determine the maximum number of copies of P_l in a C_{2k+1}-free graph, asymptotically.

Theorem 8.13. *For $k \geq 1$ and $l \geq 2$, we have*

$$ex(n, P_l, C_{2k+1}) = (1 + o(1)) \left(\frac{n}{2}\right)^l.$$

Structure of the chapter: In Section 8.3 we prove Theorem 8.3, determining the order of magnitude of $\mathrm{ex}(n, C_{2l}, C_{2k})$

In Section 8.4 we determine the asymptotics of $\mathrm{ex}(n, C_4, C_{2k})$ (Theorem 8.4).

In Section 8.5 we prove some basic lemmas for the case when a set of cycles are forbidden and prove Theorem 8.5 concerning graphs of even girth, along with results about $\mathrm{ex}(n, C_4, \mathcal{C}_A)$ and $\mathrm{ex}(n, C_6, \mathcal{C}_A)$ for every possible set A (i.e., Theorem 8.9 and Theorem 8.10).

In Section 8.6 we prove our results concerning graphs of odd girth: Theorem 8.6, Theorem 8.7 and Theorem 8.8.

In Section 8.7 we prove our results about counting paths in a C_k-free graph: Theorem 8.11, Theorem 8.12 and Theorem 8.13.

Finally in Section 8.8, we make some remarks and pose some questions.

8.3 Maximum number of C_{2l}'s in a C_{2k}-free graph

Below we prove Theorem 8.3. Note that the case $l = 2$ of Theorem 8.5 gives back Theorem 8.3, and the proof here is also a special case of the proof of that more general statement. We decided to include it here separately for two reasons. On the one hand, this is an important special case. On the other hand, it serves as an introduction to the similar, but more complicated proof of Theorem 8.5.

Proof of Theorem 8.3. Let us start with the lower bound and assume first that $2 \le l < k$. Then $K_{k-1,n-k+1}$ is C_{2k}-free and it contains

$$\frac{1}{2l}\binom{k-1}{l}\binom{n-k+1}{l} l!\, l! = (1+o(1))\frac{(k-1)(k-2)\dots(k-l)}{2l}n^l = (1+o(1))\frac{(k-1)_l}{2l}n^l$$

copies of C_{2l}.

Let us now assume $l > k \ge 3$. Consider a copy of C_{2l} and replace every second vertex u by $\lfloor n/l - 1 \rfloor$ or $\lceil n/l - 1 \rceil$ copies of it, each connected to the two neighbors of u in the C_{2l}. The resulting graph only contains cycles of length 4 and $2l$, thus it is C_{2k}-free and it contains

$$(1+o(1))\frac{1}{l^l}n^l$$

copies of C_{2l}.

Let us continue with the upper bound. Consider a C_{2k}-free graph G. First we introduce the following notation. For two distinct vertices $a, b \in V(G)$, let

$$f(a,b) := \text{ number of common neighbors of } a \text{ and } b.$$

Then we have

$$\frac{1}{2}\sum_{a \ne b,\, a,b \in V(G)} \binom{f(a,b)}{2} \le (1+o(1))\frac{(k-1)(k-2)}{4}n^2, \tag{8.1}$$

by Theorem 8.4, since the left-hand-side is equal to the number of C_4's in G.

Claim 8.5. *For every $a \in V(G)$ we have*

$$\sum_{b \in V(G)\setminus\{a\}} f(a,b) \le (2k-2)n.$$

Note that the left-hand side of the above inequality is the number of P_3's starting at a.

Proof. Recall that $N_1(a)$ is the set of vertices adjacent to a and $N_2(a)$ is the set of vertices at distance exactly 2 from a. Let E_1 be the set of edges induced by $N_1(a)$ and E_2 be the set of edges uv with $u \in N_1(a)$ and $v \in N_2(a)$. It is easy to see that $\sum_{b \in V(G)\setminus\{a\}} f(a,b) = 2|E_1| + |E_2|$.

We claim that there is no cycle of length longer that $2k-2$ in $E_1 \cup E_2$.

First suppose by contradiction that there is a cycle C of length $2k-1$ in $E_1 \cup E_2$. Since the cycle is of odd length it must contain an edge $uv \in E_1$. The subpath of length $2k-2$ between the vertices u and v in C together with the edges ua and va forms a C_{2k} in G, a contradiction.

Now suppose that there is a cycle C of length at least $2k$ in $E_1 \cup E_2$. Observe first that a subpath of length $2k-2$ of C starting from a vertex in $N_1(a)$ cannot have its other endpoint in $N_1(a)$, as that would form a C_{2k} together with the vertex a. Thus there has to be an edge v_1v_2 of E_1 in C. Consider the subpath $v_1, v_2, \ldots, v_{2k-1}v_{2k}$ of C. The vertices v_{2k-1} and v_{2k} are both in $N_2(a)$ because they are endpoints of paths of length $2k-2$ starting in v_1 and v_2 respectively. But then, the edge $v_{2k-1}v_{2k} \in E(C)$ is not in $E_1 \cup E_2$, a contradiction again.

Then by Theorem 4.1 we have $|E_1|+|E_2|= |E_1 \cup E_2| \leq (k-1)n$, which implies the claim.

□

The above claim implies that we have

$$\sum_{a \neq b,\, a,b \in V(G)} f(a,b) \leq (k-1)n^2. \tag{8.2}$$

Let us fix vertices $v_1, v_2, \ldots, v_l$ and let $g(v_1, v_2, \ldots, v_l)$ be the number of C_{2l}'s in G where v_i is the $2i$-th vertex ($i \leq l$). Clearly $g(v_1, v_2, \ldots, v_l) \leq \prod_{j=1}^{l} f(v_j, v_{j+1})$ (where $v_{l+1} = v_1$ in the product). If we add up $g(v_1, v_2, \ldots, v_l)$ for all possible l-tuples $(v_1, v_2, \ldots, v_l)$ of l distinct vertices in $V(G)$, we count every C_{2l} exactly $4l$ times. It means the number of C_{2l}'s is at most

$$\frac{1}{4l} \sum_{(v_1,v_2,\ldots,v_l)} \prod_{j=1}^{l} f(v_j v_{j+1}) \leq \frac{1}{4l} \sum_{(v_1,v_2,\ldots,v_l)} \frac{f^2(v_1,v_2) + f^2(v_2,v_3)}{2} \prod_{j=3}^{l} f(v_j v_{j+1}). \tag{8.3}$$

Fix two vertices $u, v \in V(G)$ and let us examine what factor $f^2(u,v)$ is multiplied with in (8.3). It is easy to see that $f^2(u,v)$ appears in (8.3) whenever $u = v_1, v = v_2$ or $u = v_2, v = v_1$ or $u = v_2, v = v_3$ or $u = v_3, v = v_2$. Let us start with the case $u = v_1$ and $v = v_2$. Then $f^2(u,v)$ is multiplied with

$$\frac{1}{8l} \left(\prod_{j=3}^{l-1} f(v_j v_{j+1}) \right) f(v_l, u) = \frac{1}{8l} f(u, v_l) \prod_{j=3}^{l-1} f(v_j v_{j+1})$$

for all the choices of tuples $(v_3, v_4, \ldots, v_l)$ (where these are all different vertices). We claim that

$$\sum_{(v_3,v_4,\ldots,v_l)} \frac{1}{8l} f(u, v_l) \prod_{j=3}^{l-1} f(v_j v_{j+1}) \leq \frac{(2k-2)^{l-2} n^{l-2}}{8l}.$$

Indeed, we can rewrite the left-hand side as

$$\frac{1}{8l} \sum_{v_l \in V(G)} f(u, v_l) \sum_{v_{l-1} \in V(G)} f(v_l, v_{l-1}) \ldots \sum_{v_3 \in V(G)} f(v_4, v_3),$$

and each factor is at most $(2k-2)n$ by Claim 8.14.

Similar calculation in the other three cases gives the same upper bound, so adding up all four cases, we get an additional factor of 4, showing that the number of copies of C_{2l} is at most

$$\frac{1}{2l} \sum_{u \neq v, u,v \in V} f^2(u,v)(2k-2)^{l-2} n^{l-2}.$$

Finally observe that,

$$\sum_{u\neq v, u,v\in V} f^2(u,v) = 2\sum_{u\neq v, u,v\in V}\binom{f(u,v)}{2} + \sum_{u\neq v, u,v\in V} f(u,v) \leq (1+o(1))(k-1)(k-2)n^2+(k-1)n^2,$$

which is at most $(1+o(1))(k-1)^2n^2$. Note that the above inequality follows from (8.1) and (8.2). This finishes the proof. □

Remark 8.6. Note that if G is bipartite, or even just triangle-free, then in Claim 8.14, E_1 is empty. Therefore the same proof gives the better upper bound $\sum_{b\in V(G)\setminus\{a\}} f(a,b) \leq (k-1)n$. So we get

$$\sum_{a\neq b,\, a,b\in V(G)} f(a,b) \leq \frac{k-1}{2}n^2$$

instead of (8.4). Hence we can obtain

$$ex_{bip}(n, C_{2l}, C_{2k}) \leq ex(n, C_{2l}, \{C_3, C_{2k}\}) \leq (1+o(1))\frac{(k-\frac{3}{2})(k-1)^{l-1}}{2l}n^l.$$

Observe that the construction given in Theorem 8.3 is bipartite. Thus the ratio of the upper and lower bounds of $ex_{bip}(n, C_{2l}, C_{2k})$ is

$$\frac{(k-\frac{3}{2})(k-1)^{l-2}}{(k-1)_l},$$

which goes to 1 as k increases.

8.4 Asymptotic results

We will first prove the following simple result and use it in the proof of Theorem 8.9.

Theorem 8.14. *For any k, l, we have*

$$\text{ex}(n, C_{2l}, C_{2k+1}) = (1+o(1))\frac{1}{2l}\left(\frac{n^2}{4}\right)^l.$$

Proof. The lower bound is given by the complete bipartite graph $K_{n/2,n/2}$.

Let G be a graph which is C_{2k+1}-free. By a theorem of Győri and Li [85] there are at most $O(n^{1+1/k})$ triangles in G, so let us delete an edge from each of them and call the resulting triangle-free graph G'. This way we delete at most $O(n^{1+1/k})n^{2l-2} = o(n^{2l})$ copies of C_{2l}. So it suffices to estimate the number of C_{2l}'s in G'.

First we count the number of ordered tuples of l independent edges $M_l = (e_1, e_2, \ldots, e_l)$ in G'. As the maximum number of edges in a triangle-free graph is at most $\lfloor n^2/4\rfloor$ by Mantel's theorem, we can pick an edge $e_1 = uv$ of G in at most $\lfloor n^2/4\rfloor$ ways. Then we can pick the edge e_2 disjoint from e_1 in at most $\lfloor (n-2)^2/4\rfloor$ ways as the subgraph of G induced by $V(G)\setminus\{u,v\}$ is also triangle-free. We can pick e_3 in at most $\lfloor (n-4)^2/4\rfloor$ ways, e_4 at most $\lfloor (n-6)^2/4\rfloor$ ways and so on. Thus G' contains at most

$$\left\lfloor\frac{n^2}{4}\right\rfloor\left\lfloor\frac{(n-2)^2}{4}\right\rfloor\left\lfloor\frac{(n-4)^2}{4}\right\rfloor\cdots\left\lfloor\frac{(n-2l+2)^2}{4}\right\rfloor = (1+o(1))\left(\frac{n^2}{4}\right)^l$$

copies of M_l.

Now we count the number of C_{2l}'s containing a fixed copy of $M_l = (e_1, e_2, \dots, e_l)$, where $e_i = u_i v_i$. To obtain a cycle C_{2l} from M_l, we decide for every i, whether u_i follows v_i or v_i follows u_i in a clock-wise ordering. However, for any given i, after deciding the order for u_i and v_i, we claim that the order for u_{i+1}, v_{i+1} is determined. Indeed, suppose w.l.o.g that v_i follows u_i. Then v_i can be adjacent to at most one of the vertices u_{i+1}, v_{i+1} because G' is triangle-free. Thus the order of u_{i+1}, v_{i+1} is determined. So once the order of u_1 and v_1 is fixed (in two ways) the cycle C_{2l} is determined. Thus the number of C_{2l}'s obtained from a fixed copy of M_l is at most 2. Note that in this way each copy of C_{2l} in G' is obtained exactly $4l$ times. So the total number of C_{2l}'s in G' is at most

$$(1+o(1))\left(\frac{n^2}{4}\right)^l \cdot \frac{2}{4l} = (1+o(1))\frac{1}{2l}\left(\frac{n^2}{4}\right)^l$$

as required. □

8.4.1 Proof of Theorem 8.4: Maximum number of C_4's in a C_{2k}-free graph

Here we prove Theorem 8.4.

Theorem. *For $k \geq 2$ we have:*

$$\mathrm{ex}(n, C_4, C_{2k}) = (1+o(1))\frac{(k-1)(k-2)}{4}n^2.$$

Proof. For the lower bound consider the complete bipartite graph $K_{k-1,n-k+1}$.

For the upper bound consider a C_{2k}-free graph G. We call a pair of vertices *fat* if they have at least k common neighbors, otherwise it is called non-fat. We call a C_4 *fat* if both pairs of opposite vertices in that C_4 are fat. First we claim that the number of non-fat C_4's is at most $\binom{k-1}{2}\binom{n}{2}$. Indeed, there are at most $\binom{n}{2}$ non-fat pairs, and each of them is contained in at most $\binom{k-1}{2}$ C_4's as an opposite pair.

In the remaining part of the proof we will prove that the number of fat C_4's is $O(n^{1+1/k})$, by using an argument inspired by the reduction lemma of Győri and Lemons [83]. We go through the fat C_4's in an arbitrary order, one by one, and pick exactly one edge (from the four edges of the C_4); we always pick the edge which was picked the smallest number of times before (in case there is more than one such edge, then we pick one of them arbitrarily).

After this procedure, every edge e is picked a certain number of times. Let us denote this number by $m(e)$, and we call it the *multiplicity* of e. Note that $\sum_{e \in E(G)} m(e)$ is equal to the number of fat C_4's in G.

If

$$m(e) < 2(k-2)k^2\binom{2k}{k}$$

for each edge e, then the number of fat C_4's in G is at most

$$2(k-2)k^2\binom{2k}{k}|E(G)| = O(n^{1+1/k})$$

by Theorem 4.4, as desired.

Hence we can assume there is an edge e with $m(e) \geq 2(k-2)k^2\binom{2k}{k}$. In this case we will find a C_{2k} in G, which will lead to a contradiction, finishing the proof. More precisely, we are going to prove the following statement:

Claim 8.7. *For every $2 \leq l \leq k$ there is a C_{2l} in G, that contains an edge e_l with*

$$m(e_l) \geq 2(k-l)k^2\binom{2k}{k}.$$

Proof. We prove it by induction on l. For the base case $l = 2$, consider any C_4 containing $e = e_2$. Let us assume now we have found a cycle C of length $2l$ in G and one of its edges $e_l = uv$ has

$$m(e_l) \geq 2(k-l)k^2\binom{2k}{k}.$$

For any $i \leq 2(k-l)k^2\binom{2k}{k}$, when e_l was picked for the ith time, the corresponding fat C_4 contained four edges each of which had been picked earlier at least $i-1$ times, thus they have multiplicity at least $i-1$. Let $\mathcal{F}_l$ be the set of those fat C_4's where e_l was picked for the last $2k^2\binom{2k}{k} - 1$ times. All the three other edges of each of these fat C_4's have multiplicity at least

$$2(k-l)k^2\binom{2k}{k} - 2k^2\binom{2k}{k} = 2(k-l-1)k^2\binom{2k}{k}.$$

At most $\binom{2l-2}{2}$ of the C_4's in $\mathcal{F}_l$ have all four of their vertices in C (note that they all contain the edge e_l).

Observe that G is $K_{k,k}$-free, as C_{2k} is a subgraph of $K_{k,k}$. This means that any k vertices in C have at most $k-1$ common neighbors. We claim that there are at most $(k-1)\binom{2l}{k}$ vertices in $V(G)\setminus V(C)$ that are connected to at least k vertices in C. Indeed, otherwise by pigeon hole principle, there are k vertices in C such that each of them is connected to the same k vertices in $V(G) \setminus V(C)$, a contradiction. Therefore, at most $(2l-2)(k-1)\binom{2l}{k}$ C_4's have a vertex in C and a vertex w outside C such that w is connected to at least k vertices in C.

Thus, there are at least

$$(2k^2\binom{2k}{k} - 1) - \binom{2l-2}{2} - (2l-2)(k-1)\binom{2l}{k} \geq 1$$

four-cycle(s) in $\mathcal{F}_l$ such that one of the following two cases hold. Let $uvxyu$ be one such four-cycle (recall that $e_l = uv$).

Case 1. $x, y \in V(G) \setminus V(C)$.

We replace the edge e_l of C by the path consisting of the edges vx, xy, yu, thus obtaining a cycle of length $2l+2$. The edges vx, xy, yu have multiplicity at least $2(k-l-1)k^2\binom{2k}{k}$, which finishes the proof in this case.

Case 2. $x \in V(C), y \notin V(C)$ and y has less than k neighbors in C.

Note that in this case $\{y, v\}$ is a fat pair, thus y and v have at least k common neighbors. At least one of those, say w, is not in C. Let us replace the edge e_l of C by the path consisting of the edges uy, yw, wv. This way we obtain a cycle of length $2l+2$, and one of its edges uy has $m(uy) \geq 2(k-l-1)k^2\binom{2k}{k}$, which finishes the proof of the claim and the theorem. □

□

8.5 Forbidding a set of cycles

In this section we study the case when multiple cycles are forbidden. Recall that if A is a set of integers, such that each integer is at least 3, then the set of cycles $\mathcal{C}_A = \{C_a : a \in A\}$, A_e is the set of even numbers in A and A_o is the set of odd numbers in A.

8.5.1 Basic Lemmas

The following simple lemma shows that if we count copies of an even cycle of given length, then forbidding odd cycles does not change the order of magnitude.

Lemma 8.8. *If $2k \notin A$, then*

$$\mathrm{ex}(n, C_{2k}, \mathcal{C}_A) = \Theta(ex(n, C_{2k}, \mathcal{C}_{A_e})).$$

Proof. It is obvious that $\mathrm{ex}(n, C_{2k}, \mathcal{C}_A) \leq ex(n, C_{2k}, \mathcal{C}_{A_e})$, as a $\mathcal{C}_A$-free graph is also $\mathcal{C}_{A_e}$-free. Let G be a $\mathcal{C}_{A_e}$-free graph. We are going to show that it has a $\mathcal{C}_A$-free subgraph G' that contains at least $1/2^{2k-1}$ fraction of the $2k$-cycles of G, finishing the proof.

Let us consider a random 2-coloring of the vertices of G, where every vertex becomes blue with probability $1/2$, and red otherwise. Let us delete the edges inside the color classes, and let G' be the resulting graph. As G' is bipartite, it does not contain any cycle in $\mathcal{C}_{A_o}$. The probability that a $2k$-cycle of G is also in G' is $1/2^{2k-1}$, as the first vertex can be of any color, but then the color of all the other vertices is determined. Thus the expected number of $2k$-cycles in G' is $1/2^{2k-1}$ fraction of the $2k$-cycles in G, hence there is a 2-coloring with at least that many $2k$-cycles. □

Next we show that if we count copies of an odd cycle of given length, then forbidding shorter odd cycles does not change the order of magnitude.

Lemma 8.9. *Let O_k be the set of odd integers less than $2k+1$. Then*

$$\mathrm{ex}(n, C_{2k+1}, \mathcal{C}_A) = \Theta(ex(n, C_{2k+1}, \mathcal{C}_{A \setminus O_k})).$$

Proof. The proof goes similarly to that of the previous lemma, one of the directions is again trivial. Let G be a $\mathcal{C}_{A\setminus O_k}$-free graph. We are going to show that it has a $\mathcal{C}_A$-free subgraph G' that contains at least a constant fraction of the $2k$-cycles of G.

Let us consider a random partition of the vertices of G into $2k+1$ classes $V_1, \dots, V_{2k+1}$, where each vertex goes to each class with the same probability $1/(2k+1)$. We keep the edges between V_i and V_{i+1} for $i \leq 2k$, and the edges between V_{2k+1} and V_1. We delete all the other edges and let G' be the resulting graph. It is easy to see that if we delete V_i from G', we obtain a bipartite graph, hence an odd cycle has to contain a vertex from V_i, for every $i \leq 2k+1$. This means every odd cycle has length at least $2k+1$.

It is left to prove that G' contains many $(2k+1)$-cycles. An arbitrary cycle in G is a cycle in G' with probability $1/(2k+1)^{2k}$, finishing the proof. □

Lemma 8.10. *Let m and s be fixed positive integers, and let G be a graph on n vertices. Suppose there is a partition of $V(G)$ into sets $V_1, \dots, V_s$ satisfying the following properties:*

(i) *there is no P_3 with both endpoints in V_i for $i < s$,*

(ii) *there is no P_{m+1} with endpoints in V_i and V_j if $i \neq j$, and*

(iii) *V_s is an independent set.*

Then $|E(G)| = O(n)$.

Proof. Let us first delete every vertex with degree at most m. Then repeat this procedure until we obtain a graph with minimum degree greater than m, or a graph with no vertices. We will show that the resulting graph has no vertices, which would imply that G has at most $mn = O(n)$ edges, since obviously at most mn edges were deleted.

If the resulting graph contains a vertex, it has to contain a vertex $x_1 \in V_j$ with $j \neq s$ (because V_s is an independent set). Starting from x_1, we build a path P_m greedily. First we pick a neighbor x_2 of x_1. For each $2 \leq i \leq m-1$, after picking x_i, we pick a neighbor x_{i+1} of it that has not appeared in the path; it forbids at most $i-1 < m$ neighbors, thus we can pick such a neighbor. After picking x_m, we add one more condition: the neighbor of x_m we pick as x_{m+1} should not be in V_j. As x_m has at most one neighbor in V_j by **(i)**, this is at most one more forbidden neighbor, so we can still find one satisfying all these conditions. This way we obtain a P_{m+1} with endpoints in different parts, a contradiction with **(ii)**. □

8.5.2 Proof of Theorem 8.5: Forbidding an additional cycle in a graph of given even girth

Now we prove Theorem 8.5. We restate it here for convenience.

Theorem. *For any $k > l$ and $m \geq 2$ such that $2k \neq ml$ we have*

$$\mathrm{ex}(n, C_{ml}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m)$$

Proof. By Lemma 8.9 and 8.8 it is enough to prove

$$\mathrm{ex}(n, C_{ml}, \mathcal{C}_{2l-2} \cup \{C_{2k}\}) = \Theta(n^m).$$

The lower bound is given by the theta-(n, C_m, l) graph. It contains $\Omega(n^m)$ cycles of length ml, and additionally contains only cycles of length $2l$.

First we prove the upper bound for $m = 2$. We consider a graph G on n vertices that does not contain any of the forbidden cycles. We can assume it is bipartite by Lemma 8.8. Then G has girth at least $2l$, thus it is easy to see that for any vertices u, v and any length $i \leq l-1$, there is at most one path of length i between u and v.

Claim 8.11. *If C and C' are $2l$-cycles in G sharing a path of length $l-1$, then their intersection is exactly a path of length $l-1$ or l.*

Proof. Observe that if we can find a closed walk of length less than $2l$ which is not contained in a tree, then it implies the existence of a cycle of length less than $2l$, a contradiction.

Let us consider the longest path $Q = u_1 \dots u_i$ shared by C and C'. If Q has length more than l, then there are paths of length less than l in both C and C' with endpoints u_1 and u_i, and these two paths cannot be the same. Thus they form a closed walk of length less than $2l$.

Let $v \in (V(C) \cap V(C')) \setminus V(Q)$ be the vertex that is the closest to u_1 in C. Let x (resp. x') be the distance between v and u_1 in C (resp. C'). Suppose first that $x \neq x'$. Without loss of generality, we may suppose $x < x'$. Then the subpath of length x between u_1 and v in C, the subpath of length $2l - (i-1) - x'$ between v and u_i in C', and the path Q of length $i-1$ between u_i and u_1 form a closed walk of length less than $2l$.

Hence we can assume $x = x'$. If $x < l$, then the paths of length x between u_1 and v in C and C' form a closed walk of length less than $2l$. If $x \geq l$, then either $v \in V(Q)$ or is adjacent to u_l, contradicting either our assumption that $v \in (V(C) \cap V(C')) \setminus V(Q)$ or our assumption that Q was the longest shared path. □

The proof of Theorem 8.5 goes similarly to the proof of Theorem 8.4 from here. We call a pair of vertices *fat* if there are at least $4l^2$ paths, each of length l (i.e., l edges) between them. We call a copy of C_{2l} *fat* if all the l pairs of opposite vertices are fat.

Claim 8.12. *Let $\{u, v\}$ be a fat pair and X be a set of at most $4l$ vertices. Then there is a path P of length l between u and v such that $(V(P) \setminus \{u, v\}) \cap X = \emptyset$.*

Proof. For every $i \leq l-1$, there are at most $4l$ paths $uu_1 \dots u_i \dots u_{l-1}v$ such that u_i is in X. Indeed, there are at most $4l$ ways to choose u_i from X, and after that there is only one choice for the remaining vertices, because there is at most one path of length i from u to u_i, and at most one path of length $l-i$ from u_i to v. Since there are $l-1$ ways to choose i, altogether there are at most $4l(l-1)$ paths intersecting X, finishing the proof. □

Observe now that the number of non-fat C_{2l}'s is at most $\binom{4l^2-1}{2}\binom{n}{2}$, as there are at most $\binom{n}{2}$ non-fat pairs and each of them is contained in at most $\binom{4l^2-1}{2}$ C_{2l}'s as an opposite pair. This way we count every non-fat C_{2l} at least once.

Let us only consider fat C_{2l}'s from now on. We go through them in an arbitrary order, one by one, and pick exactly one path $u_1u_2 \dots u_l$ of length $l-1$ from each of them (from the $2l$ paths of length $l-1$ in the C_{2l}); we always pick the path which was picked the smallest number of times before (in case there is more than one such path, then we pick one of them arbitrarily).

After this procedure, every path Q of length $l-1$ is picked a certain number of times. Let us denote this number by $m(Q)$, and we call it the *multiplicity* of Q. Note that adding up $m(Q)$ for all the paths Q of length $l-1$ gives the number of fat C_{2l}'s in G. Assume first that at the end of this algorithm $m(Q) \leq \frac{2k(k-l)}{l-2}$ for every path Q of length $l-1$. Then the number of fat C_{2l}'s is at most $\frac{2k(k-l)}{l-2}$ times the number of the paths of length $l-1$, which is at most $\frac{2k(k-l)}{l-2}\binom{n}{2}$, as there is at most one path of length $l-1$ between any two vertices.

Hence we can assume there is a path Q of length $l-1$ with multiplicity greater than $\frac{2k(k-l)}{l-2}$. We claim that in this case there is a copy of C_{2k} in G, which leads to a contradiction, finishing the proof. More precisely, we are going to prove the following statement:

Claim 8.13. *For every $l \leq r \leq k$ there is a C_{2r} in G, that contains a path Q_r of length $l-1$ with*

$$m(Q_r) \geq \frac{2k(k-r)}{l-2}.$$

Proof. We prove it by induction on r. More precisely, are going to assume that the statement is true for r and show that it is true for $r+l-1$. Therefore, we need to start with the base cases $l \leq r \leq 2l-2$. For the base case $r = l$, consider any C_{2l} containing $Q = Q_l$. For the other base cases consider $Q = u_1u_2 \dots u_l$ with $m(Q) \geq \frac{2k(k-l)}{(l-2)}$. We have two fat C_{2l}'s, say C and C', containing Q such that every subpath of length $l-1$ in each of them has multiplicity at least $\frac{2k(k-l)}{(l-2)} - 2$. By Claim 8.11 the intersection of C and C'

is a path Q' of length l or $l-1$. It means either $Q = Q'$ or Q' consists of Q plus an additional vertex adjacent to either u_1 or u_l.

Note that for any pair u, v of vertices that are opposite in either C or C', there is a path $P(u,v)$ of length l between u and v such that $V(P) \setminus \{u,v\} \cap (V(C) \cup V(C')) = \emptyset$, by Claim 8.12, since u, v is a fat pair.

Let us assume first that $Q' = Q$. Let $1 \leq i \leq l-2$ and let w be the vertex opposite to u_i in C'. Consider the subpath of Q from u_1 to u_i, the path $P(u_i, w)$ of length l from u_i to w, the subpath of length i from w to u_l in C', and the path of length $l+1$ from u_l to u_1 in C. They form a cycle of length $2l+2i$, that contains a subpath of C of multiplicity at least

$$\frac{2k(k-l)}{l-2} - 1 \geq \frac{2k(k-l-i)}{l-2}.$$

If $Q' \neq Q$, we can assume without loss of generality that $Q' = u_1u_2 \dots u_lu_{l+1}$. Let $1 \leq i \leq l-3$ and let w' be the vertex opposite to u_{i+1} in C'. Consider the subpath of Q' from u_1 to u_{i+1}, the path $P(u_{i+1}, w')$ of length l from u_{i+1} to w', the subpath of length i from w' to u_{l+1} in C', and the path of length l (different from Q') from u_{l+1} to u_1 in C. They form a cycle of length $2l+2i$, that contains a subpath of C of multiplicity at least

$$\frac{2k(k-l)}{l-2} - 1 \geq \frac{2k(k-l-i)}{l-2},$$

finishing the base cases.

Let us continue with the induction step. Assume we are given a cycle C of length $2r$ that contains a path $Q_r = u_1u_2 \dots u_l$ with $m(Q) \geq \frac{2k(k-r)}{l-2}$, and we are going to find a cycle of length $2r+2l-2$ that contains a path of length $l-1$ with multiplicity at least $\frac{2k(k-r-l+2)}{l-2}$. For any $i \leq \frac{2k(k-r)}{l-2}$, when Q_r was picked for the ith time, the corresponding fat C_{2l} contained $2l$ paths of length $l-1$ each of which had been picked earlier at least $i-1$ times, thus they have multiplicity at least $i-1$. Let $\mathcal{F}_r$ be the set of those fat C_{2l}'s where Q_r was picked for the last

$$\frac{2k(k-r)}{l-2} - \frac{2k(k-r-l+2)}{l-2} = 2k$$

times, so $|\mathcal{F}_r| \geq 2k$. All the other paths of length $l-1$ in each of these fat C_{2l}'s have multiplicity at least

$$\frac{2k(k-r)}{l-2} - 2k = \frac{2k(k-r-l+2)}{l-2}.$$

First observe that for every vertex $w \in V(C) \setminus V(Q_r)$ there is at most one cycle in $\mathcal{F}_r$ which contains w such that w is neither next to u_1, nor to u_l in that cycle. Indeed, two such cycles would contradict Claim 8.11. As there are less than $2k$ choices for w, either there exists a cycle $C' = u_1u_2 \dots u_lv_1 \dots v_lu_1 \in \mathcal{F}_r$ such that all of the vertices $v_1, \dots, v_l$ are not in C or there exists a cycle $C' = u_1u_2 \dots u_lv_1 \dots v_lu_1 \in \mathcal{F}_r$ such that only v_l or v_1 is in C among $v_1, \dots, v_l$. Without loss of generality, suppose v_1 is in C.

Consider a path P of length l from u_2 to v_2 that avoids $(V(C) \cup V(C')) \setminus \{u_2, v_2\}$ (such a path exists because u_2, v_2 is an opposite pair in C', thus it is a fat pair and we can apply Claim 8.12). Let P' be a subpath of length $l-1$ in C' from v_2 to u_1. We replace the edge u_1u_2 in C by the union of paths P and P'. Note that we replaced an edge with a path of length $2l-1$, so the resulting cycle has length $2r+2l-2$ and it contains the subpath $v_2 \dots v_lu_1$, which is a subpath of C', thus it has multiplicity at least

$$\frac{2k(k-r-l+2)}{l-2},$$

as required.

□

We are done with the case $m = 2$, now we consider the case m is larger. Let G again be a graph that does not contain $C_3, C_4, \dots, C_{2l-2}, C_{2k}$. From here we follow the proof of Theorem 8.3. First we introduce the following notation. For two distinct vertices $a, b \in V(G)$, let

$$f_l(a,b) := \text{ number of paths of } l \text{ edges between } a \text{ and } b.$$

In particular $f_2(a,b) = f(a,b)$. Then we have

$$\frac{1}{2}\sum_{a\neq b,\, a,b\in V(G)} \binom{f_l(a,b)}{2} = O(n^2), \tag{8.4}$$

by the case $m = 2$, since the left-hand side is equal to the number of C_{2l}'s in G.

Claim 8.14. *For every $a \in V(G)$ we have*

$$\sum_{b\in V(G)\setminus\{a\}} f_l(a,b) = O(n).$$

Proof. Notice that for any $i < l$, the set $N_i(a)$ does not contain any edges. It is easy to see that $\sum_{b\in V(G)\setminus\{a\}} f_l(a,b)$ is equal to the number of edges between $N_{l-1}(a)$ and $N_l(a)$. We will show this number is $O(n)$ by using Lemma 8.10 to the bipartite graph G' with vertex set $N_{l-1}(a) \cup N_l(a)$ and the edge set being the set of edges of G between $N_{l-1}(a)$ and $N_l(a)$.

Let $w_1, \dots, w_{s-1}$ be the neighbors of v, and let $V_i = N_{l-1}(a) \cap N_{l-2}(w_i)$ for $1 \le i \le s-1$. Let $V_s = N_l(a)$. It is easy to see that $V_1, V_2, \dots, V_s$ partition $V(G')$. Observe that a P_3 in G' with both endpoints inside V_i (for $i < s$) would create a cycle of length at most $2l-2$. This implies **(i)** of Lemma 8.10 is satisfied. A path $P_{2k-2l+3}$ in G' with endpoints in V_i and V_j with $i \neq j$ would create a cycle of length $2k$ in G, which shows that **(ii)** of Lemma 8.10 is satisfied. Since G' is bipartite, clearly V_s is independent in G', thus we can apply Lemma 8.10, finishing the proof of the claim. □

Let us fix vertices $v_1, v_2, \dots, v_m$ and let $g(v_1, v_2, \dots, v_m)$ be the number of C_{ml}'s in G where v_i is li'th vertex ($i \le m$). Clearly $g(v_1, v_2, \dots, v_m) \le \prod_{j=1}^{m} f_l(v_j, v_{j+1})$ (where $v_{l+1} = v_1$ in the product).

If we add up $g(v_1, v_2, \dots, v_m)$ for all possible m-tuples $(v_1, v_2, \dots, v_m)$ of l distinct vertices in $V(G)$, we count every C_{ml} exactly $4m$ times. It means the number of C_{ml}'s is at most

$$\frac{1}{4m}\sum_{(v_1,v_2,\dots,v_m)}\prod_{j=1}^{m} f_l(v_jv_{j+1}) \le \frac{1}{4m}\sum_{(v_1,v_2,\dots,v_m)} \frac{f_l^2(v_1,v_2)+f_l^2(v_2,v_3)}{2}\prod_{j=3}^{m} f_l(v_jv_{j+1}). \tag{8.5}$$

Fix two vertices $u, v \in V(G)$ and let us examine what factor $f_l^2(u,v)$ is multiplied with in (8.5). It is easy to see that $f_l^2(u,v)$ appears in (8.5) whenever $u = v_1, v = v_2$ or

$u = v_2, v = v_1$ or $u = v_2, v = v_3$ or $u = v_3, v = v_2$. Let us start with the case $u = v_1$ and $v = v_2$. Then $f_l^2(u,v)$ is multiplied with

$$\frac{1}{8m}\left(\prod_{j=3}^{m-1} f_l(v_j v_{j+1})\right) f(v_m, u) = \frac{1}{8m} f_l(u, v_m) \prod_{j=3}^{m-1} f_l(v_j v_{j+1})$$

for all the choices of tuples $(v_3, v_4, \dots, v_m)$ (where these are all different vertices). We claim that

$$\sum_{(v_3,v_4,\dots,v_m)} \frac{1}{8m} f_l(u, v_m) \prod_{j=3}^{m-1} f_l(v_j v_{j+1}) \leq \frac{(2k-2)^{m-2} n^{m-2}}{8m}.$$

Indeed, we can rewrite the left-hand side as

$$\frac{1}{8m} \sum_{v_l \in V(G)} f_l(u, v_m) \sum_{v_{m-1} \in V(G)} f_l(v_m, v_{m-1}) \dots \sum_{v_3 \in V(G)} f_l(v_4, v_3),$$

and each factor is at most $O(n)$ by Claim 8.14.

Similar calculation in the other three cases gives the same upper bound, so adding up all four cases, we get an additional factor of 4, showing that the number of copies of C_{ml} is at most

$$\frac{1}{ml} \sum_{u \neq v, u, v \in V} f_l^2(u,v) O(n^{m-2}).$$

Finally observe that

$$\sum_{u \neq v, u, v \in V} f_l^2(u,v) = O(n^2),$$

by (8.4) and Claim 8.14. This finishes the proof. □

8.5.3 Proofs of Theorem 8.9 and Theorem 8.10: Counting C_4's or C_6's when a set of cycles is forbidden

Below we determine the asymptotics of $\mathrm{ex}(n, C_4, \mathcal{C}_A)$ and the order of magnitude of $\mathrm{ex}(n, C_6, \mathcal{C}_A)$. For the convenience of the reader, we restate them.

Theorem. *For any $k \geq 3$ we have*

$$\mathrm{ex}(n, C_4, \mathcal{C}_A) = \begin{cases} 0 & \text{if } 4 \in A \\ (1+o(1))\frac{(k-1)(k-2)}{4} n^2 & \text{if } 4 \notin A \text{ and } 2k \text{ is the smallest element of } A_e \\ (1+o(1))\frac{1}{64} n^4 & \text{if } A_e = \emptyset. \end{cases}$$

Proof. The first line is obvious. For the second line, the upper bound follows from Theorem 8.4 as C_{2k} is forbidden, while the lower bound is given by the complete bipartite graph $K_{k-1,n-k+1}$. For the third line, the lower bound is given by the graph $K_{n/2,n/2}$, while the upper bound follows from Theorem 8.14.

□

Theorem.

$$\mathrm{ex}(n, C_6, \mathcal{C}_A) = \begin{cases} 0 & \text{if } 6 \in A, \\ \Theta(n^2) & \text{if } 6 \notin A,\ 4 \in A \text{ and } |A_e| \geq 2, \\ \Theta(n^3) & \text{if } 4, 6 \notin A \text{ and } A_e \neq \emptyset, \text{ or if } A_e = \{4\} \\ \Theta(n^6) & \text{if } A_e = \emptyset. \end{cases}$$

Proof. The first line is obvious. For the second line, Theorem 8.5 with $m = 2$ and $l = 3$ gives the upper bound and the lower bound. For the third line, the upper bound follows from Theorem 8.3 and if $4, 6 \notin A$, then the lower bound is given by the graph $K_{3,n-3}$.

If $A_e = \{4\}$, the upper bound is given by Theorem 8.3. For the lower bound let us consider the C_4-free graph G given by [62, 64], which contains $\Theta(n^3)$ C_6's. G might contain some forbidden odd cycles. However, Lemma 8.8 shows they do not change the order of magnitude.

For the fourth line, the lower bound is given by the graph $K_{n/2,n/2}$, while the upper bound is obvious. □

8.6 Counting cycles in graphs with given odd girth

8.6.1 Proof of Theorem 8.6: Maximum number of C_{2k+1}'s in a graph of girth $2k + 1$

In this subsection we prove Theorem 8.6.

Let G be an almost-regular, $\{C_4, C_6, \ldots, C_{2k}\}$-free graph on n vertices with $n^{1+1/k}/2$ edges given by Conjecture 8.4. It follows that the degree of each vertex of G is $n^{1/k}+O(1)$. We will show that G contains at least

$$(1 - o(1))\frac{1}{2k + 1}\frac{n^{2+1/k}}{2}$$

copies of C_{2k+1}.

Consider an arbitrary vertex $v \in V(G)$. Recall that $N_i(v)$ denotes the set of vertices at distance i from v. (Note that $N_1(v)$ is simply the neighborhood of v.) First we show the following.

Claim 8.15. *Let $2 \le i \le k$. Each vertex $u \in N_{i-1}(v)$ has at least $n^{1/k} + O(1)$ neighbors in $N_i(v)$. Moreover, no two vertices of $N_{i-1}(v)$ have a common neighbor in $N_i(v)$.*

Proof. Consider an arbitrary vertex $u \in N_{i-1}(v)$. If there are two edges ux, uy with $x, y \in N_{i-2}(v)$, let w be the first common ancestor of x and y. Then the length of the cycle formed by the two paths from w to x and from w to y and the two edges ux, uy is at most $2k$ and is even, a contradiction. So there is at most one edge from u to the set $N_{i-2}(v)$. Now if there are two edges ux, uy with $x, y \in N_{i-1}(v)$, then again consider the first common ancestor w, of x and y and we can find an even cycle of length at most $2k$ similarly. So the degree of u in $G[N_{i-2}(v)]$ is at most one. Therefore, each vertex $u \in N_{i-1}(v)$ has at least $n^{1/k} + O(1)$ neighbors in $N_i(v)$ (recall the degree of u is $n^{1/k} + O(1)$), proving the first part of the claim.

Suppose for a contradiction that there are two vertices $u, u' \in N_{i-1}(v)$, which have a common neighbor in $N_i(v)$. Then consider the first common ancestor of u and u', and we can again find an even cycle of length at most $2k$. This completes the proof of the claim. □

The above claim implies the following.

Claim 8.16. *For all $1 \le i \le k$, we have $|N_i(v)| = (1 + o(1))n^{i/k}$.*

Proof. Notice that Claim 8.15 implies that there are at least $|N_{i-1}(v)|\,(n^{1/k}+O(1))$ vertices in $N_i(v)$ for each $2 \le i \le k$. Since $N_1(v) = n^{1/k}+O(1)$, this proves the claim. $\square$

Now we claim that there are $(1-o(1))\frac{n^{1+1/k}}{2}$ edges of G in $N_k(v)$ (i.e., basically all the edges of G are in $N_k(v)$). Indeed, notice that

$$|N_k(v)| = (1+o(1))n^{k/k} = (1+o(1))n$$

by Claim 8.16, so $|V(G)\setminus N_k(v)| = o(n)$. Therefore the sum of degrees of the vertices in $V(G)\setminus N_k(v)$ is

$$o(n)\cdot(n^{1/k}+O(1)) = o(n^{1+1/k}),$$

showing that the number of edges incident to vertices outside $N_k(v)$ are negligible. This shows that there are $(1-o(1))\frac{n^{1+1/k}}{2}$ edges in $G[N_k(v)]$, proving the claim.

Now we color each edge ab of $G[N_k(v)]$ in the following manner: If the first common ancestor of a and b is not v, then ab is colored with the color red, but if the only common ancestor of a and b is v then it is colored blue. We want to show that most of the edges in $G[N_k(v)]$ are of color blue. To this end, let us upper bound the number of edges in $G[N_k(v)]$ of color red.

Consider an arbitrary vertex $w \in N_1(v)$. Applying Claim 8.15 repeatedly, one can obtain that w has

$$(n^{1/k}+O(1))^{k-1} = (1+o(1))n^{(k-1)/k}$$

descendants in $N_k(v)$. By Theorem 4.4, in the subgraph of G induced by this set of descendants, there are at most

$$O(n^{(k-1)/k})^{1+1/k} = (1+o(1))O(n^{(k^2-1)/k^2})$$

edges.

On the other hand, the end vertices of each red edge must have an ancestor $w \in N_1(v)$, so the total number of red edges is at most

$$(1+o(1))\,|N_1(v)|\,O(n^{(k^2-1)/k^2}) = (1+o(1))n^{1/k}O(n^{(k^2-1)/k^2}) = o(n^{1+1/k}).$$

This shows that there are $\frac{n^{1+1/k}}{2}(1-o(1))$ blue edges in $G[N_k(v)]$. Notice that any blue edge ab, together with the two paths joining a and b to v, forms a C_{2k+1} in G containing v. This shows that there are $\frac{n^{1+1/k}}{2}(1-o(1))$ copies of C_{2k+1} in G containing v. As v was arbitrary, summing up for all the vertices of G, we get that there are at least

$$(1-o(1))\frac{1}{2k+1}\frac{n^{2+1/k}}{2}$$

copies of C_{2k+1} in G.

Now it only remains to upper bound the number of C_{2k+1}'s in a graph H of girth $2k+1$. For a pair (v, xy) where $v \in V(H)$, and $xy \in E(H)$, there is at most one C_{2k+1} in H such that xy is the edge in the C_{2k+1} opposite to v. Indeed, there is at most one path of length k in H joining v and x, and at most one path of length k in H joining v and y, as H has no cycles of length at most $2k$. On the other hand, a fixed C_{2k+1} consists of $2k+1$ pairs (v, xy) such that v is opposite to an edge xy of the cycle. Therefore, the number of C_{2k+1}'s in H is at most

$$\frac{n}{2k+1}\,|E(H)| \le (1+o(1))\frac{n}{2k+1}\frac{n^{1+1/k}}{2},$$

by Theorem 4.2. This completes the proof.

8.6.2 Proofs of Theorem 8.7 and Theorem 8.8: Forbidding an additional cycle in a graph of odd girth

In this subsection, we study the maximum number of C_{2l+1}'s in a $\mathcal{C}_{2l} \cup \{C_{2k}\}$-free graph and the maximum number of C_{2l+1}'s in a $\mathcal{C}_{2l} \cup \{C_{2k+1}\}$-free graph, and prove Theorem 8.7 and Theorem 8.8.

If $l = 1$ we count triangles in a C_{2k}-free graph or a C_{2k+1}-free graph. The second question was studied by Győri and Li [85] and Alon and Shikhelman [5]. The first question was studied by Gishboliner and Shapira [64]. They showed the following.

Theorem 8.15 (Győri-Li, Alon-Shikhelman and Gishboliner-Shapira)**.** *For any $k \geq 2$ we have*

(i) $\Omega(ex(n, \mathcal{C}_{2k})) \leq ex(n, C_3, C_{2k}) \leq O_k(ex(n, C_{2k}))$.

(ii) $\Omega(ex(n, \mathcal{C}_{2k})) \leq ex(n, C_3, C_{2k+1}) \leq O(k \cdot ex(n, C_{2k}))$.

The above lower and upper bounds are known to be of the same order of magnitude, $\Theta(n^{1+1/k})$ when $k \in \{2, 3, 5\}$ (see [10, 148]).

In the rest of the section we study the case $l \geq 2$. For the lower bounds, we will use the following result of Nešetřil and Rödl [128]. Girth of a hypergraph H is defined as the length of a shortest Berge cycle in it. More formally, it is the smallest integer k such that H contains a Berge-C_k.

Theorem 8.16 (Nešetřil, Rödl [128])**.** *For any positive integers $r \geq 2$ and $s \geq 3$, there exists an integer n_0 such that for all $n \geq n_0$, there is an r-uniform hypergraph on n vertices with girth at least s and having at least $n^{1+1/s}$ edges.*

Here we restate and prove Theorem 8.7.

Theorem. *For any $k \geq l + 1$, we have*

$$\Omega(n^{1+\frac{1}{2k+1}}) = ex(n, C_{2l+1}, \mathcal{C}_{2l} \cup \{C_{2k}\}) = O(n^{1+\frac{l}{l+1}}).$$

Proof. For the lower bound, consider a $(2l + 1)$-uniform hypergraph of girth $2k + 1$ with $n^{1+1/(2k+1)}$ hyperedges (guaranteed by Theorem 8.16) and then replace each hyperedge by a copy of C_{2l+1}. It is easy to check that the resulting graph does not contain any cycle of length at most $2k$ except $2l + 1$.

Now we prove the upper bound. Consider a $\mathcal{C}_{2l} \cup \{C_{2k}\}$-free graph G. Since all the cycles of length at most $2l$ are forbidden, for any vertex v in G, there are no edges inside $N_i(v)$ for $i < l$, and the number of cycles of length $2l + 1$ containing v is equal to the number of edges in $N_l(v)$. For a neighbor w of v let $Q(v, w) = N_l(v) \cap N_{l-1}(w)$.

Claim 8.17. *For any vertex $v \in V(G)$, there exists a constant $c = c(k, l) \geq 2$ such that there are at most*

(i) *$c|N_l(v)|$ edges inside $N_l(v)$, and*

(ii) *$c(|N_l(v)|+|N_{l+1}(v)|)$ edges between $N_l(v)$ and $N_{l+1}(v)$.*

Proof. For **(i)** let $w_1, \dots, w_{s-1}$ be the neighbors of v, and let $V_i = Q(v, w_i)$ for $1 \leq i \leq s - 1$. This gives a partition of $N_l(v)$. Observe that an edge inside V_i would create a cycle of length at most $2l - 1$, as both its vertices are connected to w_i with a path of length $l - 1$. Similarly a P_3 with both endpoints inside V_i would create a cycle of length at most $2l$. Finally, a $P_{2k-2l+1}$ with endpoints in V_i and V_j would create a cycle of length

$2k$ together with the two (internally disjoint) paths of l connecting its endpoints to v. Thus we can apply Lemma 8.10 to finish the proof.

For **(ii)** we add $V_s = N_{l+1}(v)$ to the family of sets V_i, $1 \le i \le s-1$ defined before, and delete the edges inside V_s, as well as the edges inside $N_l(v)$. Observe that if a $P_{2k-2l+1}$ has endpoints in different parts V_i and V_j, then $i \ne s \ne j$ because of the parity of the length of the path. Thus we can apply Lemma 8.10 to finish the proof. □

Now we delete every vertex which is contained in at most $4c^{l+1}n^{\frac{l}{l+1}}$ copies of C_{2l+1} from G, and we repeat this procedure until we obtain a graph G' where every vertex is contained in more than $4c^{l+1}n^{\frac{l}{l+1}}$ copies of C_{2l+1}. We claim that G' has at most $O(n^{1+\frac{l}{l+1}})$ copies of C_{2l+1}. As we deleted at most $O(n^{\frac{l}{l+1}})$ C_{2l+1}'s with every vertex, this will finish the proof.

Assume G' contains more than $cn^{1+\frac{l}{l+1}}$ copies of C_{2l+1}. First we show that the maximum degree in G' is at least $cn^{\frac{1}{l+1}}$. Indeed, otherwise $N_i(v)$ contains at most $c^i n^{\frac{i}{l+1}}$ vertices for every $1 \le i \le l$ (here we use that G' is $\mathcal{C}_{2l}$-free), thus there are at most $c^l n^{\frac{l}{l+1}}$ vertices in $N_l(v)$, hence there are at most $c^{l+1}n^{\frac{l}{l+1}}$ edges inside $N_l(v)$ by **(i)** of Claim 12.1, which means v is contained in at most $c^{l+1}n^{\frac{l}{l+1}}$ copies of C_{2l+1}, so it should have been deleted, a contradiction.

Thus we can assume there is a vertex v of degree at least $cn^{\frac{1}{l+1}}$. We will show that either v or one of its neighbors is contained in at most $c^l n^{\frac{l}{l+1}}$ copies of C_{2l+1}. For a neighbor w of v let $S_0(w) = N_l(w) \cap N_{l-1}(v)$, $S_1(w) = N_l(w) \cap N_l(v)$ and $S_2(w) = N_l(w) \cap N_{l+1}(v)$. Notice that $N_l(w) = S_0(w) \cup S_1(w) \cup S_2(w)$.

Let us sum up the number of edges pq with $p \in Q(v,w)$ and $q \in N_l(v) \cup N_{l+1}(v)$, over all the neighbors w of v. This way we counted every edge inside $N_l(v)$ or between $N_l(v)$ and $N_{l+1}(v)$ at most twice; moreover the number of such edges is at most $2cn$ by Claim 12.1. Therefore, the total sum is at most $4cn$. As $d(v) \ge cn^{\frac{1}{l+1}}$, v has a neighbor w such that there are at most $4n^{\frac{l}{l+1}}$ edges between vertices in $Q(v,w)$ and vertices in $N_l(v) \cup N_{l+1}(v)$. This also means $|S_1(w) \cup S_2(w)| \le 4n^{\frac{l}{l+1}}$.

We claim that there are at most $(c+1)n^{\frac{l}{l+1}}$ edges inside $N_l(w) = S_0(w) \cup S_1(w) \cup S_2(w)$. There is no edge inside $S_0(w)$ as there is no edge inside $N_{l-1}(v)$. There is no edge between $S_0(w)$ and $S_2(w)$, since otherwise its endpoint in $S_2(w)$ would have to be in $N_l(v)$. A vertex $u \in S_1(w)$ is connected to at most one vertex in $S_0(w)$, otherwise we would obtain two distinct paths of length l between u and v, giving us a cycle of length at most $2l$. Hence the number of edges inside $N_l(w)$ incident to elements of $S_0(w)$ is at most $|S_1(w)| \le 4n^{\frac{l}{l+1}}$.

Let us now partition $N_l(w)$ into sets $Q(w,w') = N_l(w) \cap N_{l-1}(w')$ for each neighbor w' of w. Observe that $Q(w,v) = S_0(w)$. For the remaining $d(w)-1$ parts we want to apply Lemma 8.10 similarly to **(i)** of Claim 12.1. In fact, by deleting $S_0(w)$ we obtain another graph G'' where the same cycles are forbidden and the l–th neighborhood of w is $S_1(w) \cup S_2(w)$. Thus applying Claim 12.1 we obtain that there are at most $c(|S_1(w) \cup S_2(w)|) \le 4cn^{\frac{l}{l+1}}$ edges inside $S_1 \cup S_2$.

Thus altogether there are at most $4(c+1)n^{\frac{l}{l+1}} < 4c^{l+1}n^{\frac{l}{l+1}}$ edges inside $N_l(w)$ (where $c > 0$ is a constant chosen so that the previous inequality is satisfied), hence w should have been deleted, a contradiction.

□

Here we restate Theorem 8.8 and prove it.

Theorem. *For $k > l \geq 2$ we have*

$$\Omega(n^{1+\frac{1}{2k+2}}) = ex(n, C_{2l+1}, \mathcal{C}_{2l} \cup \{C_{2k+1}\}) = O(n^2).$$

Proof. For the lower bound, consider a $(2l+1)$-uniform hypergraph of girth $2k+2$ with $n^{1+1/(2k+2)}$ hyperedges and then replace each hyperedge by a copy of C_{2l+1}.

Now we prove the upper bound. Let v be an arbitrary vertex in a $\mathcal{C}_{2l} \cup \{C_{2k+1}\}$-free graph G. We will upper bound the number of C_{2l+1}'s containing v. There are no edges inside $N_i(v)$ for each $i < l$. Indeed, if there is an edge then we can find a forbidden short odd cycle containing that edge (because the end points of that edge have a common ancestor). This shows that every C_{2l+1} containing v must use an (actually exactly one) edge from $N_l(v)$. So the number of C_{2l+1}'s containing v is upper bounded by the number of edges in $N_l(v)$. We claim the following.

Claim 8.18. *The number of edges in $N_l(v)$ is $O(|N_l(v)|) = O(n)$.*

Proof of Claim. Color each vertex in $N_l(v)$ by its (unique) ancestor in $N_1(v)$. Then the resulting color classes $A_1, A_2, \dots A_t$ partition $N_l(v)$. There are no edges inside the color classes, because such an edge would be contained in a forbidden short odd cycle.

One can partition the color classes into two parts $\{A_i \mid i \in I\}$ and $\{A_i \mid i \in J\}$ (with $I \cup J = \{1, 2, \dots, t\}$ and $I \cap J = \emptyset$), so that at least half of all the edges in $N_l(v)$ are between the vertices of the two parts. Now as C_{2k+1} is forbidden, there is no path of length $2k+1-2l$ between the two parts, as such a path would have its end vertices in different classes. (Note that here we use that the parity of the path length $2k+1-2l$ is odd.) Thus by Erdős-Gallai theorem there are only at most $O(|N_l(v)|) = O(n)$ edges between the two parts. This implies that the total number of edges in $N_l(v)$ is at most twice as many, completing the proof of the claim. □

So using Claim 8.18, the number of C_{2l+1}'s containing any fixed vertex v is $O(n)$. Thus the total number of C_{2l+1}'s in G is at most $O(n^2)$, as desired. This completes the proof. □

We conjecture that even if the additional forbidden cycle has odd length, we can only have a sub-quadratic number of C_{2l+1}'s.

Conjecture 8.19. For any integers $2 \leq k < l$, there is an $\epsilon > 0$ such that

$$\mathrm{ex}(n, C_{2k+1}, \mathcal{C}_{2k} \cup \{C_{2l+1}\}) = O(n^{2-\epsilon}).$$

8.7 Number of copies of P_l in a graph avoiding a cycle of given length

In the first subsection, we will consider the case when an even cycle is forbidden and in the next subsection, we will deal with the case when an odd cycle is forbidden.

8.7.1 Bounds on $\mathrm{ex}(n, P_l, C_{2k})$

For the upper bound, we use a spectral method similar to the one used in [87].

The spectral radius of a finite graph is defined to be the spectral radius of its adjacency matrix. Given a graph G, the spectral radius of G is denoted by $\mu(G)$. Given a matrix A, its spectral radius is denoted by $\mu(A)$. If A is the adjacency matrix of G, then of course, $\mu(G) = \mu(A)$.

Nikiforov [129] showed the following.

Theorem 8.17 (Nikiforov)**.** *Let G be a C_{2k}-free graph on n vertices. Then for any $k \geq 1$, we have*

$$\mu(G) \leq \frac{k-1}{2} + \sqrt{(k-1)n} + o(n).$$

Note that in the case $k = 2$, a sharper bound is known: The maximum spectral radius of a C_4-free graph on n vertices is $\frac{1}{2} + \sqrt{n - 3/4} + O(1/n)$, where for odd n the $O(1/n)$ term is zero. Now we prove Theorem 8.11, restated below.

Theorem. *We have*

$$ex(n, P_l, C_{2k}) \leq (1 + o(1))\frac{1}{2}(k-1)^{\frac{l-1}{2}} n^{\frac{l+1}{2}}.$$

Proof. Let A be the adjacency matrix of a C_{2k}-free graph G. Recall that $\mathcal{N}(P_l, G)$ denotes the number of copies of P_l in G. Let $\mathcal{N}(W_l, G)$ denote the number of walks consisting of l vertices in G. Note that $2\mathcal{N}(P_l, G) \leq \mathcal{N}(W_l, G)$, since every path corresponds to two walks.

Then we have,

$$\frac{2\mathcal{N}(P_l, G)}{n} \leq \frac{\mathcal{N}(W_l, G)}{n} = \frac{\mathbf{1}^t A^{l-1} \mathbf{1}}{\mathbf{1}^t \mathbf{1}} \tag{8.6}$$

Note that $\mathbf{1}$ is the column vector with all entries being 1. The right-hand-side of (8.6) is at most $\mu(A^{l-1})$ because the spectral radius of any Hermitian matrix M is the supremum of the quotient $\frac{x^*Mx}{x^*x}$, where x ranges over $\mathbb{C}^n \setminus \{0\}$. Moreover, using Theorem 8.17, we have

$$\mu(A^{l-1}) = (\mu(A))^{l-1} = (\mu(G))^{l-1} \leq (1 + o(1))((k-1)n)^{\frac{l-1}{2}},$$

completing the proof. □

Now we provide some lower bounds on $\mathrm{ex}(n, P_l, C_{2k})$.

Constructing C_{2k}-free graphs with many copies of P_l

We prove Theorem 8.12. Note that the behavior of the extremal function seems to be very different in the cases $l < 2k$ and $l \geq 2k$.

Theorem. *If $2 \leq l < 2k$, then*

$$\mathrm{ex}(n, P_l, C_{2k}) \geq (1 + o(1))\frac{1}{2}(k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil}.$$

If $l \geq 2k$, then

$$\mathrm{ex}(n, P_l, C_{2k}) \geq (1 + o(1)) \max\left\{ \left(\frac{n}{\lfloor l/2 \rfloor}\right)^{\lceil l/2 \rceil}, \left(\frac{(k-1)}{4(k-2)^{k+2}}\right)^{\lceil \frac{l}{2} \rceil} (k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil} \right\}.$$

Proof. In the case $l < 2k$, we take a complete bipartite graph B with parts of size $k-1$ and $n-(k-1)$. Clearly, B is C_{2k}-free and the number of copies of P_l in B is at least

$$\frac{1}{2}(k-1)_{\lfloor \frac{l}{2} \rfloor}(n-(k-1))_{\lceil \frac{l}{2} \rceil} = (1+o(1))\frac{1}{2}(k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil}.$$

Now we consider the case $l \geq 2k$. First we give a simple construction. Consider a path $v_1 v_2 \dots v_l$ and for each odd i, replace the vertex v_i by b vertices $v_i^1, v_i^2, \dots, v_i^b$ where each of them is adjacent to the same vertices that v_i was adjacent to. Choose $b = \frac{n - \lfloor l/2 \rfloor}{\lfloor l/2 \rfloor} = (1+o(1))\frac{n}{\lfloor l/2 \rfloor}$. The resulting graph only contains cycles of length 4, so it is C_{2k}-free as long as $k \neq 2$. Moreover, it contains at least

$$(1+o(1))b^{\lceil l/2 \rceil} = (1+o(1))\left(\frac{n}{\lfloor l/2 \rfloor}\right)^{\lceil l/2 \rceil}$$

copies of P_l. The case $k = 2$ is dealt with in Theorem 8.1

Now we give a different construction which gives a better lower bound when l is large compared to k. We will use the following theorem of Ellis and Linial [28]. Recall that girth of a hypergraph is the length of a shortest (Berge) cycle in the hypergraph.

Theorem 8.18 (Ellis, Linial [28]). *Let r,d and g be integers with $d \geq 2$ and $r, g \geq 3$. Then there exists an r-uniform, d-regular hypergraph $\mathcal{H}$ with girth at least g, and at most*

$$n_r(g,d) := (r-1)\left(1 + d(r-1)\frac{(d-1)^g(r-1)^g - 1}{(d-1)(r-1)-1}\right) < 4((d-1)(r-1))^{g+1}$$

vertices.

Let $g = k+1$ and $r = k-1$. Consider the hypergraph $\mathcal{H}$ given by Theorem 8.18 with

$$|V(\mathcal{H})| \leq n_r(g,d) = n_{k-1}(k+1,d). \tag{8.7}$$

Notice that the number of hyperedges in $\mathcal{H}$ is

$$|E(\mathcal{H})| \leq \frac{d \cdot n_r(g,d)}{r} = \frac{d \cdot n_{k-1}(k+1,d)}{k-1}. \tag{8.8}$$

Let $E(\mathcal{H}) = \{h_1, h_2, \dots, h_m\}$. To each hyperedge $h_i \in E(\mathcal{H})$, we add a set S_i of new vertices with

$$|S_i| = \frac{(n - |V(\mathcal{H})|)}{|E(\mathcal{H})|}$$

(note for $i \neq j$, we take $S_i \cap S_j = \emptyset$). Now we construct a graph G as follows: For each i with $1 \leq i \leq m$, consider the sets h_i, S_i and add all possible edges between h_i and S_i. That is, $E(G) = \{uv \mid u \in h_i, v \in S_i \text{ for some } 1 \leq i \leq m\}$. It is easy to check that G is C_{2k}-free. Note that G is a bipartite graph with parts $U := V(\mathcal{H})$ and $D := \cup_{i=1}^m S_i$. Moreover, the degree of every vertex of G in U is d times the size of a set S_i, so it is

$$\frac{d(n - |V(\mathcal{H})|)}{|E(\mathcal{H})|}.$$

And the degree of every vertex in D is the size of a set h_i, so it is $k-1$. Therefore, the number of copies of P_l in G is at least

$$\left(\frac{d(n-|V(\mathcal{H})|)}{|E(\mathcal{H})|}\right)_{\lceil \frac{l}{2} \rceil}(k-1)_{\lfloor \frac{l}{2} \rfloor} = \left(\frac{dn}{|E(\mathcal{H})|}\right)^{\lceil \frac{l}{2} \rceil}(k-1)_{\lfloor \frac{l}{2} \rfloor}(1+o(1)).$$

Using (8.8), this is at least

$$(1+o(1))\left(\frac{(k-1)n}{n_{k-1}(k+1,d)}\right)^{\lceil \frac{l}{2} \rceil}(k-1)_{\lfloor \frac{l}{2} \rfloor}.$$

Choosing $d=2$ and using Theorem 8.18, we have

$$n_{k-1}(k+1,d) = (k-2)\left(1+2(k-2)\frac{(k-2)^{k+1}-1}{(k-2)-1}\right) < 4(k-2)^{k+2}.$$

So, $\mathrm{ex}(n,P_l,C_{2k})$ is at least

$$(1+o(1))\left(\frac{(k-1)n}{n_{k-1}(k+1,2)}\right)^{\lceil \frac{l}{2} \rceil}(k-1)_{\lfloor \frac{l}{2} \rfloor} > (1+o(1))\left(\frac{(k-1)}{4(k-2)^{k+2}}\right)^{\lceil \frac{l}{2} \rceil}(k-1)_{\lfloor \frac{l}{2} \rfloor}n^{\lceil \frac{l}{2} \rceil}.$$

□

8.7.2 Bounds on $\mathrm{ex}(n,P_l,C_{2k+1})$

For the upper bound we will again use a spectral bound. We will use the following theorem of Nikiforov [130].

Theorem 8.19 (Nikiforov). *Let G be a C_{2k+1}-free graph on n vertices. Then for any $k \geq 1$ and $n > 320(2k+1)$, we have*

$$\mu(G) \leq \sqrt{n^2/4}.$$

Now we prove Theorem 8.13, restated below.

Theorem. *We have*

$$\mathrm{ex}(n,P_l,C_{2k+1}) = (1+o(1))\left(\frac{n}{2}\right)^l.$$

Proof. For the lower bound, consider a complete bipartite graph B with $n/2$ vertices on each side. Then clearly, B does not contain any odd cycle and it contains at least

$$(1+o(1))\left(\frac{n}{2}\right)^l$$

copies of P_l.

The proof of the upper bound is similar to that of the proof of Theorem 8.11. Let A be the adjacency matrix of a C_{2k+1}-free graph G. Then, for n large enough, using Theorem 8.19 we get,

$$\frac{2\mathcal{N}(P_l,G)}{n} \leq \frac{\mathcal{N}(W_l,G)}{n} = \frac{\mathbf{1}^t A^{l-1}\mathbf{1}}{\mathbf{1}^t\mathbf{1}} \leq \mu(A^{l-1}) = (\mu(A))^{l-1} = (\mu(G))^{l-1} \leq \left(\sqrt{\frac{n^2}{4}}\right)^{l-1} = \left(\frac{n}{2}\right)^{l-1}.$$

Thus,

$$\mathcal{N}(P_1,G) \leq \left(\frac{n}{2}\right)^l,$$

completing the proof of the theorem. □

Remark 8.20. Note that the number of copies of P_{2l} in a graph G is at least $2l$ times the number of copies of C_{2l} in G. Indeed, every copy of C_{2l} contains $2l$ copies of P_{2l}. Moreover, a copy of P_{2l} belongs to at most one copy of C_{2l}. Thus Theorem 8.13 implies Theorem 8.14.

8.8 Concluding remarks and questions

We finish this chapter by posing some questions.

• Naturally, it would be interesting to prove asymptotic or exact results corresponding to our results where we only know the order of magnitude. For example it would be nice to close the gap between the lower and upper bounds in Theorem 8.3.

We also pose some conjectures when a family of cycles are forbidden.

• We proved in Theorem 8.5, that for any $k > l$ and $m \geq 2$ such that $2k \neq ml$ we have

$$\mathrm{ex}(n, C_{ml}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m).$$

We conjecture that it is true for longer cycles as well.

Conjecture 8.21. For any $k > l$, $m \geq 2$ and $1 \leq j < l$ with $ml + j \neq 2k$ we have

$$\mathrm{ex}(n, C_{ml+j}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m).$$

• We prove in Theorem 8.8 that for $l > k \geq 2$ we have

$$\mathrm{ex}(n, C_{2k+1}, \mathcal{C}_{2k} \cup \{C_{2l+1}\}) = O(n^2).$$

However, we conjecture that the truth is smaller.

Conjecture 8.22. For any integers $k < l$, there is an $\epsilon > 0$ such that

$$\mathrm{ex}(n, C_{2k+1}, \mathcal{C}_{2k} \cup \{C_{2l+1}\}) = O(n^{2-\epsilon}).$$

The following theorem supports Conjecture 8.22.

Theorem 8.20 (Gerbner, Győri, M., Vizer [61])**.** *We have*

$$\mathrm{ex}(n, C_5, \mathcal{C}_4 \cup \{C_9\}) = O(n^{11/12}).$$

Proof. Let us consider a $\mathcal{C}_4 \cup \{C_9\} = \{C_3, C_4, C_9\}$-free graph G. First we delete every edge that is contained in less than 17 C_5's, then repeat this until every edge is contained in at least 17 C_5's. We have deleted at most $17|E(G)|= O(n^{3/2})$ C_5's this way (note that $|E(G)|= O(n^{3/2})$ follows from the fact that G is C_4-free). Let G' be the graph obtained this way.

Observe that if a five-cycle $C := v_1v_2v_3v_4v_5v_1$ shares the edge v_1v_2 with another C_5, then they either share also the edge v_2v_3 or v_5v_1 and no other vertices, or they share only the edge v_1v_2. If there are at least six five-cycles sharing only v_1v_2 with C, we say v_1v_2 is an *unfriendly* edge for C, otherwise it is called a *friendly* edge for C. Our plan is to show first that a C_5 cannot contain both friendly and unfriendly edges, then using this we will show that a C_5 cannot contain friendly edges. Thus every edge is unfriendly for every C_5, and this will imply that G' is C_6-free.

Assume C contains both friendly and unfriendly edges. Then it is easy to see that it contains an unfriendly edge, say v_1v_2, and a path P of two edges not containing v_1v_2 such that C shares P with a set $\mathcal{S}$ of at least 6 other C_5's. (Note that the cycles in $\mathcal{S}$ only share P.) Now there is a cycle $v_1v_2w_3w_4w_5v_1$ by the unfriendliness of v_1v_2 that contains three new vertices w_3, w_4, w_5. Then we replace v_1v_2 in C with $v_1w_3w_4w_5v_2$ to obtain a C_8. Afterwards, there is a cycle in $\mathcal{S}$ that does not contain any of w_3, w_4 and w_5 as the elements of $\mathcal{S}$ are vertex disjoint outside C. Thus we can replace P in this C_8 with a path of three edges to obtain a C_9, a contradiction.

Assume now C contains only friendly edges. The edge v_1v_2 is contained in at least 6 other C_5's together with one of its two neighboring edges, say v_2v_3. At least of these 6 C_5's does not contain the vertices v_4 and v_5, let it be $v_1v_2v_3w_1w_2v_1$. Thus replacing the two-edge path $v_1v_2v_3$ with the three-edge path $v_3w_1w_2v_1$ to obtain the six-cycle $v_4v_5v_1w_2w_1v_3v_4$. The edge w_1w_2 is friendly for $v_1v_2v_3w_1w_2v_1$ (because otherwise, it would contain both friendly and unfriendly edges). Thus w_1w_2 is in at least 6 other C_5's together with either v_3w_1 or w_2v_1. In the same way as before, we can replace this two-edge path with a three-edge path to obtain a seven-cycle. Repeating this procedure we can obtain an eight-cycle and then a nine-cycle, a contradiction. Indeed, at each step, we are given a cycle C' of length between 5 and 8, and we add two new vertices to it in place of one of its vertices by replacing a two-edge path P with a three-edge path to increase the length of C'. We have to make sure that the two new vertices are disjoint from the other vertices of C'. Since there are 6 C_5's containing P which are vertex-disjoint outside P, it is easy to find a C_5 that avoids the at most 5 vertices of C' outside P.

Hence every edge is unfriendly to every C_5 in G'. Then we claim that there is no C_6 in G'. Indeed, otherwise we consider an arbitrary edge uv of that C_6, there is a set $\mathcal{S}'$ of at least 17 C_5's that each contain uv. Because of the unfriendliness of uv to each of the cycles in $\mathcal{S}'$, they do not share any other vertices except u and v, so at least one of them is disjoint from the other vertices of the C_6, thus we can exchange e to a 4-edge-path in the C_6, obtaining a C_9, a contradiction.

We obtained that after deleting $O(n^{3/2})$ edges, the resulting graph G' is $\{C_3, C_4, C_6\}$-free, thus it contains at most $O(n^{11/12})$ C_5's by Theorem 8.7.

□

Remarks about $\mathrm{ex}(n, C_l, \mathcal{C}_A)$ for a given set A of cycle lengths

After the investigation carried out in this chapter it is natural to ask to determine $\mathrm{ex}(n, C_l, \mathcal{C}_A)$ for any set A.

Let us note that the behavior of $\mathrm{ex}(n, C_l, \mathcal{C}_A)$ is more complicated if l is not 4 or 6. A simple construction of a $\mathcal{C}_A$-free graph G is the following. Let $2r$ be the shortest length of an even cycle which is allowed (note that if no even cycle is allowed, then the total number of cycles is $O(n)$ by a theorem in [58]). Let $p = \lfloor l/r \rfloor$. If r divides l, then the theta-(n, C_p, r) graph contains $\Omega(n^p)$ copies of C_l, some C_{2r}'s and no other cycles. If r does not divide l, it is easy to see that we can add a path with $l - pr$ new vertices between the two end vertices of a theta-$(n-(l-pr), P_{p+1}, r)$ graph to obtain a graph with $\Omega(n^p)$ many C_l's, some C_{2r}'s and no other cycles.

Observe that in these cases, we still have an integer in the exponent. However, Theorem 8.2 (by Solymosi and Wong) shows that if $l \geq 4$ is even, then $\mathrm{ex}(n, C_{2l}, \mathcal{C}_6) = \Theta(n^{l/3})$

since it is known that Erdős's Girth Conjecture holds for $m = 3$. This shows an example where the exponent is not an integer.

The situation is even more complicated when l is odd. Let us examine the simplest case $l = 5$, i.e. $\mathrm{ex}(n, C_5, \mathcal{C}_A)$. If A contains only one element, Gishboliner and Shapira [64] determined the order of magnitude (it is 0 or n^2 or $n^{5/2}$). If there are at least two elements in A but $4 \notin A$, then the construction described above gives $\mathrm{ex}(n, C_5, \mathcal{C}_A) = \Omega(n^2)$, while the result of Gishboliner and Shapira [64] implies $\mathrm{ex}(n, C_5, \mathcal{C}_A) = O(n^2)$. If $A = \{C_3, C_4\}$, then Lemma 8.9 shows $\mathrm{ex}(n, C_5, \{C_3, C_4\}) = \Theta(ex(n, C_5, C_4))$, which is $\Theta(n^{5/2})$ by Theorem 8.1. What remains is the case A contains 4 and another number. In this case Theorem 8.7 and Theorem 8.8 give some bounds that are not sharp.

Part III

Extremal hypergraph theory

Chapter 9

Background on Berge Hypergraphs

Turán-type extremal problems in graphs and hypergraphs are the central topic of extremal combinatorics and has a vast literature. For a survey of recent results we refer the reader to [54, 96, 127].

The classical definition of a hypergraph cycle is due to Berge.

Definition 9.1. *A Berge cycle of length l in a hypergraph is a set of l distinct vertices $\{v_1, \dots, v_l\}$ and l distinct hyperedges $\{e_1, \dots, e_l\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo l. The vertices $v_1, \dots, v_l$ are called the defining vertices of the Berge cycle.*

A Berge path of length l in a hypergraph is a set of $l+1$ distinct vertices $v_1, \dots, v_{l+1}$ and l distinct hyperedges $e_1, \dots, e_l$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \le i \le l$.

Gerbner and Palmer [63] gave the following natural generalization of the definitions of Berge cycles and Berge paths.

Definition 9.2. *Let $F = (V(F), E(F))$ be a graph and $\mathcal{B} = (V(\mathcal{B}), E(\mathcal{B}))$ be a hypergraph. We say $\mathcal{B}$ is Berge-F if there is a bijection $\phi : E(F) \to E(\mathcal{B})$ such that $e \subseteq \phi(e)$ for all $e \in E(F)$. In other words, given a graph F we can obtain a Berge-F by replacing each edge of F with a hyperedge that contains it.*

Given a family of graphs $\mathcal{F}$, we say that a hypergraph $\mathcal{H}$ is Berge-$\mathcal{F}$-free if for every $F \in \mathcal{F}$, the hypergraph $\mathcal{H}$ does not contain a Berge-F as a subhypergraph.

The maximum possible number of hyperedges in a Berge-$\mathcal{F}$-free hypergraph on n vertices is the Turán number *of Berge-$\mathcal{F}$.*

Notation 9.3. For a set of simple graphs $\mathcal{F}$ and $n \ge r \ge 2$, let

$$\mathrm{ex}_r(n, \mathcal{F}) := \max\{|\mathcal{H}| \colon \mathcal{H} \subset \binom{[n]}{r} \text{ is Berge-}\mathcal{F}\text{-free}\}.$$

If $\mathcal{F} = \{F\}$, then instead of $\mathrm{ex}_r(n, \{F\})$, we simply write $\mathrm{ex}_r(n, F)$.

An important Turán-type extremal result for Berge cycles is due to Lazebnik and Verstraëte [108], who studied the maximum number of hyperedges in an r-uniform hypergraph containing no Berge cycle of length less than five (i.e., girth five). They showed

Theorem 9.4 (Lazebnik, Verstraëte, [108])**.**

$$\mathrm{ex}_3(n, \{C_2 \cup C_3 \cup C_4\}) = \frac{1}{6} n^{3/2} + o(n^{3/2}).$$

Very recently this was strengthened by Ergemlidze, Győri and Methuku [40] showing that $\mathrm{ex}_3(n, \{C_2, C_3, C_4\}) \sim ex_3(n, \{C_2, C_4\})$.

Interestingly, Lazebnik and Verstraëte [108] relate the question of estimating the maximum number of edges in a hypergraph of given girth with the famous question of estimating generalized Turán numbers initiated by Brown, Erdős and Sós [19] and show that the two problems are equivalent in some cases. Since then Turán-type extremal problems for hypergraphs in the Berge sense have attracted considerable attention: see e.g., [14, 25, 53, 63, 62, 77, 83, 81, 91, 144].

The systematic study of the Turán number of Berge cycles started with the study of Berge triangles by Győri [76], and continued with the study of Berge-C_5 by Bollobás and Győri [14] who showed that $n^{3/2}/3\sqrt{3} \leq ex_3(n, C_5) \leq \sqrt{2}n^{3/2} + 4.5n$. Very recently, this estimate was improved by Ergemlidze, Győri, Methuku [42] and they also considered [40] the analogous question for linear hypergraphs and proved that $\mathrm{ex}_3(n, \{C_2, C_5\}) = n^{3/2}/3\sqrt{3} + o(n^{3/2})$. Surprisingly, even though the lower bound here is the same as the lower bound in the Bollobás-Győri theorem, the hypergraph they construct in order to establish their lower bound is very different from the hypergraph used in the Bollobás-Győri theorem. The latter is far from being linear.

Győri and Lemons [80] generalized the Bollobás-Győri theorem to other cycle lengths in a series of papers.

Theorem 9.5 (Győri, Lemons, [83, 84, 82])**.** *If $r \geq 2$, then we have $\mathrm{ex}_r(n, C_{2l}) = O(n^{1+1/l})$.*

If $r \geq 3$, then we have $\mathrm{ex}_r(n, C_{2l+1}) = O(n^{1+1/l})$.

Note that this upper bound has the same order of magnitude as the upper bound on the maximum possible number of edges in a C_{2l}-free graph (see the Even Cycle Theorem of Bondy and Simonovits [15]). This shows the surprising fact that the maximum number of hyperedges in a Berge-C_{2l+1}-free hypergraph is significantly different from the maximum possible number of edges in a C_{2l+1}-free graph. The multiplicative factors depending on l were improved by Jiang and Ma [91], by Füredi and Özkahya [53], and by Gerbner, Methuku and Vizer [60]. This is discussed further in Chapter 12.

Győri, Katona and Lemons [77] generalized the Erdős–Gallai theorem to Berge-paths. Let P_k denote a path of length k.

Theorem 9.1 (Győri, Katona, Lemons [77])**.** *If $k > r + 1 > 3$, then we have*

$$ex_r(n, P_k) \leq \frac{n}{k}\binom{k}{r}.$$

If $r \geq k > 2$, then we have

$$ex_r(n, P_k) \leq \frac{n(k-1)}{r+1}.$$

For the case $k = r + 1$, Győri, Katona and Lemons conjectured that the upper bound should have the same form as the $k > r + 1$ case. This was settled by Davoodi, Győri, Methuku and Tompkins [25] who showed the following theorem, which is the subject of Chapter 10.

Theorem 9.2 (Davoodi, Győri, M., Tompkins [25])**.** *Fix $k = r + 1 > 2$. Then,*

$$ex_r(n, P_k) \leq \frac{n}{k}\binom{k}{r} = n.$$

The bounds in the above two theorems are sharp for each k and r for infinitely many n. Győri, Methuku, Salia, Tompkins and Vizer [86] proved a significantly smaller bound on the maximum number of hyperedges in an n-vertex connected r-graph with no Berge path of length k. Their bound is asymptotically exact when r is fixed and k and n are sufficiently large.

Recently, Füredi, Kostochka and Luo [51] proved similar results for Berge cycles. Before, we can state them, we need to introduce some notation. Let $\mathcal{H}$ be a hypergraph. Then its *2-shadow*, $\partial_2\mathcal{H}$ is the collection of pairs that lie in some hyperedge of $\mathcal{H}$. Given a set $S \subseteq V(\mathcal{H})$, the subhypergraph of $\mathcal{H}$ induced by S is denoted by $\mathcal{H}[S]$. We say $\mathcal{H}$ is not connected if and only if $\partial_2(\mathcal{H})$ is not a connected graph. A hyperedge $h \in E(\mathcal{H})$ is called a "cut-hyperedge" of $\mathcal{H}$ if $\mathcal{H} \setminus \{h\} := (V(\mathcal{H}), E(\mathcal{H}) \setminus \{h\})$ is not connected.

Theorem 9.3 (Füredi, Kostochka, Luo [51]). *Let $r \geq 3$ and $k \geq r+3$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$. Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{k-1}$ and $\mathcal{H}[D] = K^r_{k-1}$.*

Moreover, Kostochka and Luo [101] found exact bounds for $k \leq r-1$ and asymptotic bounds for $k = r$. For the remaining two cases $k = r+2$ and $k = r+1$, Füredi, Kostochka, Luo [51] conjectured that a similar statement as that of Theorem 9.3 holds. Recently, Ergemlidze, Győri, Methuku, Tompkins, Salia and Zamora [45] proved these conjectures. In the case $k = r+2$, they showed the following.

Theorem 9.4 (Ergemlidze, Győri, M., Tompkins, Salia and Zamora [45]). *Let $r \geq 3$ and $n \geq 2$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length $r+2$ or longer. Then $e(\mathcal{H}) \leq \frac{r+1}{r}(n-1)$. Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D] = K^r_{r+1}$.*

In the case $k = r+1$, they proved the following.

Theorem 9.5 (Ergemlidze, Győri, M., Tompkins, Salia and Zamora [45]). *Let $r \geq 3$ and $n \geq 2$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length $r+1$ or longer. Then $e(\mathcal{H}) \leq n-1$. Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D]$ consists of r hyperedges.*

Gerbner and Palmer [63] showed the following simple and interesting result: For any graph F and $r \geq |V(F)|$, we have $\mathrm{ex}_r(n, F) \leq \mathrm{ex}(n, F) = O(n^2)$. This means that as r increases starting from two, $\mathrm{ex}_r(n, F)$ can increase above n^2, but after a while, it stops increasing and goes back to $O(n^2)$. Grósz, Methuku and Tompkins [72] examined this phenomenon and showed the decrease does not stop here. For any F if r is large enough, we have $\mathrm{ex}_r(n, F) = o(n^2)$. They examined the threshold, and showed that for example $\mathrm{ex}_r(n, K_3) = o(n^2)$ if and only if $r \geq 5$, improving a result of Győri [76] on Berge triangles. This is the subject of Chapter 11.

A topic that is closely related to Berge hypergraphs is *expansions of graphs.* Let F be a fixed graph and let $r \geq 3$ be a given integer. The *r-uniform expansion of F* is the r-uniform hypergraph F^+ obtained from F by adding $r-2$ new vertices to each edge of F which are disjoint from $V(F)$ such that distinct vertices are added to distinct edges of F. This notion generalizes the notion of a loose cycle for example. The Turán number of F^+ is the maximum number of edges in an r-uniform hypergraph on n vertices that does not contain F^+ as a subhypergraph. In [102, 103, 104], Kostochka, Mubayi and Verstraëte studied expansions of paths, cycles, trees, bipartite graphs and other graphs. Of particular interest to us is their result showing that the Turán number of $K^+_{2,t}$ is asymptotically equal to $\binom{n}{2}$. Interestingly, as we discuss in Chapter 12, the asymptotic behavior of the Turán number of Berge-$K_{2,t}$ is quite different: $\mathrm{ex}_3(n, K_{2,t}) = (1+o(1))\frac{1}{6}(t-1)^{3/2}n^{3/2}$. Chapter 12 also shows a general theorem that improves many existing results on Berge hypergraphs.

Throughout the rest of the thesis we consider *simple* hypergraphs, which means there are no duplicate hyperedges and we use the term *linear* for a hypergraph if any two different hyperedges contain at most one common vertex (observe that a hypergraph is linear if and only if it is Berge-C_2-free). Note that there is some ambiguity around these words in the theory of hypergraphs. Some authors use the word 'simple' for hypergraphs that we call linear. For ease of notation sometimes we consider a hypergraph as a set of hyperedges. The *degree* $d(v)$ of a vertex v in a hypergraph is the number of hyperedges containing it.

Chapter 10

An Erdős-Gallai type theorem for uniform hypergraphs

Given a hypergraph $\mathcal{H}$, we denote the vertex and edge sets of $\mathcal{H}$ by $V(\mathcal{H})$ and $E(\mathcal{H})$ respectively. Moreover, let $e(\mathcal{H}) = |E(\mathcal{H})|$ and $n(\mathcal{H}) = |V(\mathcal{H})|$.

Recall that Győri, Katona and Lemons determined the largest number of hyperedges possible in an r-uniform hypergraph without a Berge path of length k for both the range $k > r+1$ and the range $k \leq r$.

Theorem 10.1 (Győri, Katona, Lemons, [77])**.** *Let $\mathcal{H}$ be an n-vertex r-uniform hypergraph with no Berge path of length k. If $k > r+1 > 3$, we have*

$$e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}.$$

If $r \geq k > 2$, we have

$$e(\mathcal{H}) \leq \frac{n(k-1)}{r+1}.$$

The case when $k = r+1$ remained unsolved. Győri, Katona and Lemons conjectured that the upper bound in this case should have the same form as the $k > r+1$ case:

Conjecture 10.2 (Győri, Katona, Lemons, [77])**.** Fix $k = r+1 > 2$ and let $\mathcal{H}$ be an n-vertex r-uniform hypergraph containing no Berge path of length k. Then,

$$e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r} = n.$$

In this chapter we settle their conjecture by proving

Theorem 10.3 (Davoodi, Győri, M., Tompkins [25])**.** *Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices. If $e(\mathcal{H}) > n$, then $\mathcal{H}$ contains a Berge path of length at least $r+1$.*

A construction with a matching lower bound when $r+1$ divides n is given by disjoint complete hypergraphs on $r+1$ vertices. Observe that by induction it suffices to prove Theorem 12.8 when the hypergraph is connected. We will prove the following stronger theorem.

Theorem 10.4 (Davoodi, Győri, M., Tompkins [25])**.** *Let $\mathcal{H}$ be a connected r-uniform hypergraph on n vertices. If $e(\mathcal{H}) \geq n$, then for every vertex $v \in V(\mathcal{H})$ either there exists a Berge path of length $r+1$ starting from v or there exists a Berge cycle of length $r+1$ with v as one of its vertices.*

To see that Theorem 10.4 implies Theorem 12.8, suppose $e(\mathcal{H}) > n$ and assume that after applying Theorem 10.4 we find a Berge cycle of length $r+1$. If the Berge cycle is not the complete r-uniform hypergraph on $r+1$ vertices, then there is a vertex in one of its edges that does not belong to the set of vertices that define this Berge cycle. Starting from this vertex and then using all of the edges of the Berge cycle would yield a Berge path of length $r+1$. If the Berge cycle is a complete hypergraph, then by connectivity and the assumption $e(\mathcal{H}) > n$, there must be another hyperedge which intersects it, and we may find a Berge path of length $r+1$ again.

We will need the following Lemma in the proof of Theorem 10.4.

Lemma 10.5. *Let v be a vertex and e be an edge in a hypergraph $\mathcal{H}$ with $v \in e$. Consider a Berge cycle of length r with vertices $\{v_1, \dots, v_r\}$ and edges $\{e_1, \dots, e_r\}$ such that $v \notin \{v_1, \dots, v_r\}$ and $e \notin \{e_1, \dots, e_r\}$ and assume that it spans a set X of vertices such that $X \cap (e \setminus \{v\}) \neq \varnothing$. Then, there is a Berge path of length $r+1$ starting at v or a Berge cycle of length $r+1$ containing v.*

Proof. First, suppose that $X \cap (e \setminus \{v\})$ contains a vertex $u \notin \{v_1, \dots, v_r\}$. Without loss of generality, let $u \in e_1$. Then, we have the Berge path $v, e, u, e_1, v_2, e_2, \dots, v_r, e_r, v_1$ of length $r+1$ starting at v. Now suppose $X \cap (e \setminus \{v\}) \subset \{v_1, \dots, v_r\}$, and assume without loss of generality that $v_1 \in X \cap (e \setminus \{v\})$. Consider the edges e_1 and e_r. If either contains an element not in $\{v_1, \dots, v_r, v\}$, then we will find a Berge path of length $r+1$. Indeed, suppose u is such an element and $u \in e_r$, then we have the Berge path $v, e, v_1, e_1, v_2, e_2, \dots, v_r, e_r, u$. Finally, if neither e_1 nor e_r contains elements outside of $\{v_1, \dots, v_r, v\}$, then since they are distinct sets at least one of them contains v, say e_r. We can then find a Berge cycle of length $r+1$ with v as a vertex, namely $v, e, v_1, e_1, v_2, e_2, \dots, v_r, e_r, v$. □

Proof of Theorem 10.4. We will use induction first on r, and for each r, on n. First, we prove the statement for graph case ($r = 2$). Let G be a graph and fix a vertex $v \in V(G)$. Consider a breadth-first search spanning tree T with root v. If there is no path of length three starting from v, then T has two levels, $N_1(v)$ and $N_2(v)$. By assumption G has at least n edges. Hence, G has at least one more edge than T. If both ends of this edge belong to $N_1(v)$, then we have a triangle containing v. Otherwise, it is easy to see that there is a path of length three starting from v.

Now, let $\mathcal{H} = (V, E)$ be a connected r-uniform hypergraph with $r \geq 3$ and let $v \in V(\mathcal{H})$ be an arbitrary vertex.

First, suppose that there is a cut vertex v_0, that is, the (non-uniform) hypergraph $\mathcal{H}' = (V', E')$ where $V' = V \setminus \{v_0\}$ and $E' = \{e \setminus \{v_0\} : e \in E\}$ is not connected. In this case, let the connected components be $C_1, \dots, C_s$, and for each i, let $\mathcal{H}_i$ be the hypergraph attained by adding back v_0 to the edges in C_i. At least one of these $\mathcal{H}_i$'s, say $\mathcal{H}_1$ satisfies the conditions of the theorem since, if $e(\mathcal{H}_i) \leq n(\mathcal{H}_i) - 1$ for all i, then

$$e(\mathcal{H}) = \sum_{i=1}^{s} e(\mathcal{H}_i) \leq \sum_{i=1}^{s} n(\mathcal{H}_i) - s = n(\mathcal{H}) - 1,$$

a contradiction. If $v \in V(\mathcal{H}_1)$ (this includes the case when $v = v_0$), then we are done by applying induction to $\mathcal{H}_1$. Assume $v \neq v_0$ and let $v \in V(\mathcal{H}_i)$, $i \neq 1$, then by induction, $\mathcal{H}_1$ contains a Berge path of length r starting from v_0 (as a Berge cycle of length $r+1$ with v_0 as a vertex yields a Berge path of length r starting at v_0), and since $\mathcal{H}_i$ contains a Berge path from v to v_0, their union is a Berge path of length at least $r+1$ starting at

v, as desired. Therefore, from now on we may assume there is no cut vertex in $\mathcal{H}$, so in particular v is not a cut vertex.

Let $e \in E(\mathcal{H})$ be an edge containing v and let $\mathcal{H}'$ be the hypergraph defined by removing e from the edge set of $\mathcal{H}$ and deleting v from all remaining edges in $\mathcal{H}$. Let $C_1, \dots, C_s$, $s \geq 1$ be the connected components of $\mathcal{H}'$ and observe that each of them contains a vertex of $e \setminus \{v\}$. By the pigeonhole principle there is some component C_i such that $e(C_i) \geq n(C_i)$. In order to apply the induction hypothesis, we will replace the r-edges in the component C_i by edges of size $r-1$ in such a way that no multiple edges are created and the component remains connected. We proceed by considering one r-edge at a time and attempting to remove an arbitrary vertex from it.

Suppose for some r-edge, say f, this is not possible. If for every vertex u in f, replacing f with $f \setminus \{u\}$ disconnects the hypergraph, then every hyperedge which intersects f intersects it in only one point, and hyperedges which intersect f in different points will be in different components if we delete f. Let $F_1, F_2, \dots, F_r$ be the connected components in C_i obtained from deleting f. Then, by the pigeonhole principle we find a component F_j with $e(F_j) \geq n(F_j)$ and continue the procedure on that component instead.

Thus, we may assume there exists some vertex of f whose removal from f does not disconnect the hypergraph. Now, consider the case when the deletion of any vertex of f would lead to multiple edges in the hypergraph. This means that every $r-1$ subset of f is already an edge of the hypergraph. Clearly, in this case there is a Berge cycle of length r using each vertex of f. In the original hypergraph, if this Berge cycle spans a vertex of $e \setminus \{v\}$, then it can be extended to a Berge path of length $r+1$ starting from v or a Berge cycle of length $r+1$ with v as one of its vertices by Lemma 10.5. If it does not span a vertex of $e \setminus \{v\}$, then there is a Berge path of length at least two from v to the Berge cycle which, in turn, can easily be extended to a Berge path of length $r+1$.

We may now assume that f contains at least one element whose removal does not disconnect the hypergraph and at least one element whose removal does not create a multiple edge. If there is an element w such that removing w from f disconnects the hypergraph, then no element of $f \setminus \{w\}$ will yield a multiple edge if deleted (for then w would not disconnect the hypergraph) and so we can find an element to remove from f. If there is no such element w whose removal disconnects the hypergraph, we are also done since we can simply take any element of f whose removal does not make a multiple edge.

Therefore, we can transform C_i into an $(r-1)$-uniform and connected hypergraph $\mathcal{H}^*$ satisfying $e(\mathcal{H}^*) \geq n(\mathcal{H}^*)$. By the induction hypothesis, for every vertex $z \in V(\mathcal{H}^*)$ there exists a Berge path of length r starting from z or there exists a Berge cycle of length r containing z. Choose z to be in the edge e. The associated Berge path (or cycle) in original hypergraph is a Berge path (or cycle) of the same length. If the result is a Berge path, then we are done trivially by extending it with e and v. If the result is a Berge cycle, then we are done by Lemma 10.5. □

Chapter 11

Uniformity thresholds for the asymptotic size of extremal Berge-F-free hypergraphs

11.1 Introduction and main results

Given a hypergraph $\mathcal{H}$, we denote by $\Gamma(\mathcal{H})$ the 2-shadow of $\mathcal{H}$, that is, the graph on the same vertex set, containing all 2-element subsets of hyperedges from $\mathcal{H}$ as edges. Observe that $\mathcal{H}$ contains a Berge-F as a subgraph if and only if $\Gamma(\mathcal{H})$ contains a copy of F such that $\mathcal{H}$ has a distinct hyperedge containing each edge of this copy of F.

Recall that for a graph F, $\mathrm{ex}_r(n, F)$ denotes the maximum number of hyperedges in an r-uniform hypergraph on n vertices which does not contain a Berge-F as a subgraph. The case when $r = 2$ is the classical Turán function $\mathrm{ex}(n, F)$. We will also consider what happens if we impose the additional assumption that the hypergraph is linear (i.e., any two hyperedges intersect in at most one element). We denote the maximum number of hyperedges in a linear r-uniform hypergraph on n vertices which does not contain a Berge-F by $\mathrm{ex}_r^L(n, F)$.

It follows from Győri's results in [76] that $\mathrm{ex}_r(n, \triangle) \leq \frac{n^2}{8(r-2)}$ if n is large enough. For $r = 3, 4$ this result is asymptotically sharp. We studied this problem in higher uniformities, and determined that, in fact, $\mathrm{ex}_r(n, \triangle) = o(n^2)$ when $r \geq 5$, improving Győri's result. This will be obtained as a special case of more general theorems presented later.

The following result can be proved easily.

Proposition 11.1 (Gerbner and Palmer [63])**.** *For any graph F and $r \geq |V(F)|$, we have*

$$\mathrm{ex}_r(n, F) \leq \mathrm{ex}(n, F) = O(n^2).$$

We include its proof in Section 11.3. By the Erdős–Stone theorem, for any bipartite graph F, we have $\mathrm{ex}(n, F) = o(n^2)$. Moreover, in the graph case, if F is not bipartite, then we have $\mathrm{ex}(n, F) = \Omega(n^2)$. We will show that for any graph F and for any sufficiently large r, we have $\mathrm{ex}_r(n, F) = o(n^2)$. We introduce the following threshold functions.

Definition 11.2. Let F be a graph. We define the uniformity threshold of F as

$$\mathrm{th}(F) = \min\{r_0 \geq 2 : \mathrm{ex}_r(n, F) = o(n^2) \text{ for all } r \geq r_0\}.$$

We define the linear uniformity threshold of F as

$$\mathrm{th}^L(F) = \min\{r_0 \geq 2 : \mathrm{ex}_r^L(n, F) = o(n^2) \text{ for all } r \geq r_0\}.$$

Our first theorem gives an upper bound for the value of $\mathrm{th}(F)$ for any graph F. (Note that if F is bipartite, then $\mathrm{th}(F) \leq |V(F)|$ by Proposition 11.1.) The Ramsey number of two graphs $R(G, H)$ is defined to be the smallest n such that every 2-coloring of the edges of the complete graph K_n contains a copy of G in the first color or H in the second color. For every G and H this number is known to be finite by Ramsey's theorem.

For a graph F containing an edge e, let $F \setminus e$ denote the graph formed by deleting e from F.

Theorem 11.1 (Grósz, M., Tompkins [72]). *For any graph F (with at least two edges), and any of its edges e, we have*

$$\mathrm{th}(F) \leq R(F, F \setminus e).$$

Our next result is a construction giving a lower bound on $\mathrm{th}(F)$.

Theorem 11.2 (Grósz, M., Tompkins [72]). *Let F be a graph with clique number $\omega(F) \geq 2$. For any $2 \leq r \leq (\omega(F) - 1)^2$, there exists an r-uniform, Berge-F-free hypergraph on n vertices with $\Omega(n^2)$ hyperedges. Therefore we have*

$$\mathrm{th}(F) \geq (\omega(F) - 1)^2 + 1.$$

The above two theorems imply $\mathrm{th}(\triangle) = 5$.

Erdős, Frankl and Rödl [33] constructed a linear r-uniform Berge-triangle-free hypergraph with more than $n^{2-\varepsilon}$ hyperedges for any $r \geq 3$ and $\varepsilon > 0$. This implies that when $F = \triangle$, in our definition of the functions th and th^L, $o(n^2)$ cannot be replaced by a function of n with smaller exponent.

Finally, we consider linear hypergraphs. (In Section 11.5 we prove Theorem 11.2 by blowing up a linear, Berge-F-free hypergraph.) It is easy to see that a linear hypergraph on n vertices has at most $\binom{n}{2}$ hyperedges: fix a pair of vertices in each hyperedge; by the definition of a linear hypergraph, all these pairs must be distinct. Timmons [144] showed that (with our notation) $\mathrm{th}^L(F) \leq |V(F)|$. We prove the following exact result.

Theorem 11.3 (Grósz, M., Tompkins [72]). *For any (non-empty) graph F, we have*

$$\mathrm{th}^L(F) = \chi(F),$$

and for any $2 \leq r < \chi(F)$, there exists an r-uniform, linear, Berge-F-free hypergraph on n vertices with $\Omega(n^2)$ hyperedges.

Note that $\chi(F)$ may be bigger than the lower bound in Theorem 11.2, and it obviously also bounds $\mathrm{th}(F)$ from below. Generalizing the proof of Theorem 11.2, we prove the following common generalization of Theorem 11.2 and the lower bound on $\mathrm{th}(F)$ coming from Theorem 11.3.

For a graph F, we define a t-admissible partition of F as a partition of $V(F)$ into sets of size at most t, such that between any two sets there is at most one edge in F. 'Contracting' a set S of vertices in a graph produces a new graph in which all the vertices of S are replaced with a single vertex s such that s is adjacent to all the vertices to which any of the vertices of S was originally adjacent.

Theorem 11.4 (Grósz, M., Tompkins [72]). *Let F be a graph, and let $1 \leq t \leq |V(F)|-1$. Consider all the graphs obtained by contracting each set in some t-admissible partition of F to a point, and let c be the minimum of the chromatic numbers of all such graphs. If $c \geq 3$, then* $\mathrm{th}(F) \geq (c-1)t+1$.

For $t = 1$, the only t-admissible partition of a graph F is putting every vertex into a different set, so $c = \chi(F)$, and we just get back the lower bound in Theorem 11.3. We also get Theorem 11.2 as a special case of Theorem 11.4 when $t = \omega(F) - 1$: In any $(\omega(F) - 1)$-admissible partition of F, every vertex in a maximal clique of F must belong to a different set of the partition. Indeed, no set of the partition may contain all vertices of an $\omega(F)$-clique. But if a set A from the partition contained two or more vertices of an $\omega(F)$-clique, and another set B contained another vertex of that clique, then there would be two or more edges between A and B, contradicting the definition of a t-admissible partition. This means that the graph we get after contracting all the sets of an $(\omega(F) - 1)$-admissible partition contains an $\omega(F)$-clique, so its chromatic number is at least $\omega(F)$. Therefore $c \geq \omega(F)$.

As an example where Theorem 11.4 gives an improvement, consider $F = K_{2,1,1}$. Putting $t = 3$ and $c = 3$, we get $\mathrm{th}(K_{2,1,1}) \geq 7$. Indeed, the only 3-admissible partition of $K_{2,1,1}$ is to put every vertex of $K_{2,1,1}$ into a different set. Theorem 11.2 gives a lower bound of just 5, while Theorem 11.3 gives 3. We give further corollaries of Theorem 11.4 about blowups of graphs in Section 11.5.

Until now we focused on uniformities r for which $\mathrm{ex}_r(n, F)$ is subquadratic. In Section 11.2, we discuss the behavior of $\mathrm{ex}_r(n, F)$ as r grows, more generally. In particular, we discuss uniformities r for which $\mathrm{ex}_r(n, F)$ is superquadratic using the relationship between $\mathrm{ex}_r(n, F)$ and the maximum number of K_r's in an F-free graph.

11.2 Behavior of $\mathrm{ex}_r(n, F)$ as r increases

Recall that the maximum number of copies of a graph T in an F-free graph on n vertices, is denoted by $\mathrm{ex}(n, T, F)$. In the following proposition, we paraphrase Propositions 2.1 and 2.2 from the paper of Alon and Shikhelman [5].

Proposition 11.3 (Alon and Shikhelman [5]). *Let $r \geq 2$. Then $\mathrm{ex}(n, K_r, F) = \Omega(n^r)$ if and only if $r < \chi(F)$. Moreover, if $r < \chi(F)$, then $\mathrm{ex}(n, K_r, F) = (1+o(1))\binom{\chi(F)-1}{r}\left(\frac{n}{\chi(F)-1}\right)^r$, otherwise $\mathrm{ex}(n, K_r, F) \leq n^{r-\epsilon(r,F)}$ for some $\epsilon(r, F) > 0$.*

For the $r < \chi(F)$ case, a construction showing $\mathrm{ex}(n, K_r, F) = \Omega(n^r)$ is a complete r-partite graph on n vertices with roughly $\lfloor \frac{n}{r} \rfloor$ vertices in each part. It has chromatic number r, so it does not contain F, and it contains $\Omega(n^r)$ copies of K_r.

Clearly $\mathrm{ex}(n, K_r, F) \leq \mathrm{ex}_r(n, F)$: Take an F-free graph with $\mathrm{ex}(n, K_r, F)$ r-cliques, and replace each r-clique with a hyperedge containing the vertices of the clique. The resulting hypergraph cannot contain a Berge-F, as its 2-shadow does not even contain a copy of F. The converse is not true. If we take a Berge-F-free hypergraph with $\mathrm{ex}_r(n, F)$ hyperedges, and we replace its hyperedges with r-cliques (i.e., we take its 2-shadow), it might contain a copy of F. The upper bound in the following proposition, which relates $\mathrm{ex}_r(n, F)$ to $\mathrm{ex}(n, K_r, F)$, was discovered by Gerbner and Palmer [62]. As the proof is very simple, we include it for completeness.

Proposition 11.4 (Gerbner and Palmer [62])**.** *For any $r \geq 3$,*

$$\mathrm{ex}(n, K_r, F) \leq \mathrm{ex}_r(n, F) \leq \mathrm{ex}(n, K_r, F) + \mathrm{ex}(n, F).$$

Proof. We have already seen $\mathrm{ex}(n, K_r, F) \leq \mathrm{ex}_r(n, F)$. To prove $\mathrm{ex}_r(n, F) \leq \mathrm{ex}(n, K_r, F) + \mathrm{ex}(n, F)$, let $\mathcal{H}$ be an r-uniform, Berge-F-free hypergraph on a vertex set V of n elements. We consider the hyperedges of $\mathcal{H}$ one-by-one, and we will mark elements of $\binom{V}{2} \cup \binom{V}{r}$. For each hyperedge, we mark a pair of its vertices that we have not marked yet; if all those pairs are already marked, then we mark the hyperedge itself.

Let $\tilde{\mathcal{H}}$ be the set of the marked pairs and hyperedges. $\tilde{\mathcal{H}} \cap \binom{V}{2}$ is a graph with no copy of F. Indeed, since we only marked one edge for each hyperedge, if the graph contained a copy of F, its edges would be contained by distinct hyperedges of $\mathcal{H}$, which would form a Berge-F. So $|\tilde{\mathcal{H}} \cap \binom{V}{2}| \leq \mathrm{ex}(n, F)$. Meanwhile each hyperedge in $\tilde{\mathcal{H}} \cap \binom{V}{r}$ was marked because each pair of vertices in it had already been marked, so they form an r-clique in $\tilde{\mathcal{H}} \cap \binom{V}{2}$. But $\tilde{\mathcal{H}} \cap \binom{V}{2}$ is F-free, so the number of r-cliques in it is at most $\mathrm{ex}(n, K_r, F)$. Thus, $|\tilde{\mathcal{H}} \cap \binom{V}{r}| \leq \mathrm{ex}(n, K_r, F)$. Since $|\mathcal{H}| = |\tilde{\mathcal{H}}|$, the proof is complete. □

Proposition 11.4 implies that the two functions $\mathrm{ex}(n, K_r, F)$ and $\mathrm{ex}_r(n, F)$ differ by only $O(n^2)$. So $\mathrm{ex}_r(n, F) = O(n^2)$ if and only if $\mathrm{ex}(n, K_r, F) = O(n^2)$, and we have $\mathrm{ex}_r(n, F) = \omega(n^2)$ if and only if $\mathrm{ex}(n, K_r, F) = \omega(n^2)$. Moreover, if $\mathrm{ex}(n, K_r, F) = \omega(n^2)$, then $\mathrm{ex}_r(n, F) = (1 + o(1))\,\mathrm{ex}(n, K_r, F)$. If F is bipartite, since $\mathrm{ex}(n, F) = o(n^2)$, the difference is even smaller — only $o(n^2)$. So for bipartite F, $\mathrm{ex}_r(n, F) = o(n^2)$ if and only if $\mathrm{ex}(n, K_r, F) = o(n^2)$, and if $\mathrm{ex}(n, K_r, F) = \Omega(n^2)$, then $\mathrm{ex}_r(n, F) = (1 + o(1))\,\mathrm{ex}(n, K_r, F)$.

On the other hand, note that for any non-bipartite F, even if we know $\mathrm{ex}(n, K_r, F) = o(n^2)$, Proposition 11.4 does not imply $\mathrm{ex}_r(n, F) = o(n^2)$; so $\mathrm{ex}(n, K_r, F)$ does not tell us much about $\mathrm{th}(F)$.

Combining Proposition 11.3 and Proposition 11.4, we can obtain the following nice proposition discovered by Palmer et al. [132]. We note, however, that the proof given in [132] is different from the simple proof mentioned here.

Proposition 11.5 (Palmer, Tait, Timmons, Wagner [132])**.** *Let $r \geq 2$. If $r < \chi(F)$, then $\mathrm{ex}_r(n, F) = \Theta(n^r)$ and if $r \geq \chi(F)$, then $\mathrm{ex}_r(n, F) = o(n^r)$.*

More precisely, if $r < \chi(F)$, then $\mathrm{ex}_r(n, F) = (1 + o(1))\binom{\chi(F)-1}{r}\left(\frac{n}{\chi(F)-1}\right)^r$, and if $r \geq \chi(F)$, then $\mathrm{ex}_r(n, F) \leq n^{r-\epsilon(r,F)}$ for some $\epsilon(r, F) > 0$.

Below we outline some interesting facts about the behavior of $\mathrm{ex}_r(n, F)$ as r grows.

Proposition 11.5 shows that as r increases from 2 until $\chi(F)-1$, the function $\mathrm{ex}_r(n, F)$ increases, and from $r = \chi(F)$, it is $o(n^r)$. From $r \geq |F|$ (at the latest), it becomes $O(n^2)$ again (by Proposition 11.1). However, the decrease does not stop there. As shown by Theorem 11.1, from some point, it becomes $o(n^2)$.

In general, we do not know much about the behavior of $\mathrm{ex}_r(n, F)$ when r is between $\chi(F)$ and $|F|-1$. In the special case of $F = K_s$, we know more. As r increases from $\chi(F)-1$ to $\chi(F)$, $\mathrm{ex}_r(n, F)$ immediately drops from $\Theta(n^{\chi(F)-1})$ to $O(n^2)$ (by Proposition 11.1 since $|V(F)| = \chi(F) = s$), and it is at most $O(n^2)$ for all $r \geq \chi(F)$. It would be very interesting to determine the precise threshold $\mathrm{th}(K_s)$ for when it becomes sub-quadratic.

It is also notable that $\mathrm{ex}_r(n, F)$ may increase with r in the range $\chi(F) \leq r \leq |F| - 1$ (as will be shown by the proposition below). Theorem 1.2 in Alon and Shikhelman's paper [5] shows that if $2 \leq r \leq \frac{s}{2} + 1$ and $t \geq (s-1)!+1$, then $\mathrm{ex}(n, K_r, K_{s,t}) = \Theta(n^{r-\binom{r}{2}/s})$.

More recently, Ma, Yuan and Zhang [118] showed that given any two integers s and r with $2 \le r \le s+1$, there is a constant $f(s,r) > 0$ depending only on s and r such that for any integer $t \ge f(s,r)$, we have $\mathrm{ex}(n, K_r, K_{s,t}) = \Theta(n^{r-\binom{r}{2}/s})$.

It is easy to check that $n^{r-\binom{r}{2}/s}$ is non-decreasing in r between 2 and $s+1$ (and monotonously increasing until s), and $n^{r-\binom{r}{2}/s} = \Omega(n^2)$ when $3 \le r \le s+1$ if $3 \le s$. So combining Ma, Yuan and Zhang's result with Proposition 11.4, we get

Proposition 11.6. *Given two integers $s \ge 3$ and r such that $2 \le r \le s+1$, there is a constant $f(s,r) > 0$ such that for any $t \ge f(s,r)$, we have $\mathrm{ex}_r(n, K_{s,t}) = \Theta(n^{r-\binom{r}{2}/s})$.*

Of course $\mathrm{ex}_2(n, K_{s,t}) = o(n^2)$, while $\mathrm{ex}_3(n, K_{s,t}) = \Omega(n^{2.25})$ if $s \ge 4$, which implies that $\mathrm{th}(F)$ is not necessarily the smallest $r \ge 2$ for which $\mathrm{ex}_r(n, F) = o(n^2)$. (Then from some point later on — $r \ge s+t$ at the latest — it becomes sub-quadratic again and remains so.)

For a non-bipartite graph F, Theorem 11.1 implies that for any $r \ge R(F, F \setminus e)$, we have $\mathrm{ex}_r(n, F) = o(\mathrm{ex}(n, F))$. However, it is unclear if the same holds for some bipartite graphs. If F is a forest, then it is known that $\mathrm{ex}_r(n, F) = \Theta(n) = \Theta(\mathrm{ex}(n, F))$.

Question 11.7. Is there a bipartite graph F containing a cycle for which the following statement holds: There exists an integer $r_0(F)$ such that $\mathrm{ex}_r(n, F) = o(\mathrm{ex}(n, F))$ for all $r \ge r_0(F)$? If yes, is the same statement true for every bipartite F containing a cycle?

The analogous question for linear hypergraphs was asked by Verstraëte [144]. For $F = C_4$, we ask the following, more precise question about the threshold.

Question 11.8. Is it true that $\mathrm{ex}_r(n, C_4) = o(n^{1.5})$ for all $r \ge 7$?

One can show that for $2 \le r \le 6$, we have $\mathrm{ex}_r(n, C_4) = \Omega(n^{1.5})$: Consider a bipartite C_4-free graph G having $\Omega(n^{1.5})$ edges with parts A and B. Let $1 \le i \le 3$ and $1 \le j \le 3$. Now replace each vertex $a \in A$ with i vertices $a_1, \dots, a_i$, and each vertex $b \in B$ with j vertices $b_1, \dots, b_j$, so that each edge $ab \in E(G)$ is replaced by the hyperedge $\{a_1, \dots, a_i, b_1, \dots, b_j\}$, Let A' and B' be the sets replacing A and B respectively. Clearly, the resulting hypergraph H is $(i+j)$-uniform.

We claim that H is Berge-C_4-free. Indeed, suppose for a contradiction that $abcd$ is a C_4 in the 2-shadow of H such that ab, bc, cd, da are contained in distinct hyperedges. Since $i, j \le 3$, it is impossible that the vertices of the C_4 are all contained in A' or B'. Notice that two of the vertices a, b, c, d must correspond to the same vertex of G, because otherwise $abcd$ would be a C_4 in G as well. Furthermore, if two adjacent vertices of $abcd$ are in A' (or in B'), then they correspond to the same vertex of G. We have the following cases.

- If two opposite vertices of $abcd$, say a and c, are in A', and b and d are in B', then suppose w.l.o.g. that a and c correspond to the same vertex of G. Then ab and bc are not contained in distinct hyperedges of H, a contradiction.
- If two adjacent vertices, say a and b are in A', and c and d are in B', then a and b correspond to the same vertex of G, and so do c and d. Then ad and bc are not contained in distinct hyperedges of H.
- If three vertices, say a, b, c, are in A', and d is in B' (or vice-versa), then a, b, c correspond to the same vertex of G, so ad and cd are not contained in distinct hyperedges of H.

As $2 \le i+j \le 6$, this shows that $\mathrm{ex}_r(n, C_4) = \Omega(\mathrm{ex}(n, C_4)) = \Omega(n^{1.5})$ for $2 \le r \le 6$.

11.3 Upper bound — Proof of Theorem 11.1

First we prove Proposition 11.1 by showing that $\mathrm{ex}(n, F)$ is an upper bound for $\mathrm{ex}_r(n, F)$, whenever $r \geq |V(F)|$.

Proof of Proposition 11.1. Assume $\mathcal{H} = (V, \mathcal{E})$ is an r-uniform hypergraph which contains no Berge-F. One by one, for every hyperedge $h \in \mathcal{E}$ we take an edge $e \subset h$ which has not yet been taken. By our assumption $r \geq |V(F)|$ we can always do this, for otherwise we would have a complete K_r, and the corresponding hyperedges would form a Berge-F. After completing this procedure, we obtain a graph G in which the number of edges is equal to the number of hyperedges in $\mathcal{H}$. Clearly G is F-free and thus has at most $\mathrm{ex}(n, F)$ edges, completing the proof. □

Another essential tool in some of our proofs is the graph removal lemma. We recall it here without proof.

Lemma 11.9 (Graph removal lemma)**.** *Let F be a fixed graph. For any $\varepsilon > 0$, there is a $\delta > 0$ such that for every graph G which has at most $\delta\,|V(G)|^{|V(F)|}$ copies of F, there exists a set $S \subseteq E(G)$ of εn^2 edges such that every copy of F in G contains at least one edge from S.*

We wish to apply the graph removal lemma to the 2-shadow of a Berge-F-free hypergraph $\mathcal{H}$, denoted $\Gamma(\mathcal{H})$. To this end, we prove the following claim.

Claim 11.10. *Let F be a fixed graph and $r \geq |V(F)|$. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices with no Berge-F. Then the number of copies of F in $\Gamma(\mathcal{H})$ is $o(n^{|V(F)|})$.*

Proof. Any copy of F in $\Gamma(\mathcal{H})$ has at least two edges (and therefore at least three vertices) in some hyperedge of $\mathcal{H}$, otherwise the hyperedges containing the edges of F would form a Berge-F. Thus we have the following upper bound:

$$\#\{F\text{-copies in } \Gamma(\mathcal{H})\} \leq |E(\mathcal{H})| \binom{r}{3} n^{|V(F)|-3}.$$

$\binom{r}{3}$ is a constant, and by Proposition 11.1, we have $|E(\mathcal{H})| = O(n^2)$. So the number of copies of F is $O(n^{|V(F)|-1})$ and so $o(n^{|V(F)|})$. □

From now on, we consider F to be a fixed graph, $e \in E(F)$, and $r \geq R(F, F \setminus e)$. We consider an r-uniform hypergraph $\mathcal{H}$ with no Berge-F.

By Claim 11.10 and the graph removal lemma, there are $o(n^2)$ edges such that every copy of F in the 2-shadow of $\mathcal{H}$ contains one of these edges. Call the set of these edges $\mathcal{R}$.

Claim 11.11. *Every hyperedge of $\mathcal{H}$ contains an edge from $\mathcal{R}$ which is contained in at most $|E(F)| - 1$ hyperedges.*

Proof. By contradiction, assume that there is a hyperedge h such that every edge from $\mathcal{R}$ contained in h is in at least $|E(F)|$ hyperedges. By the definition of $\mathcal{R}$, $\Gamma(\{h\}) \setminus \mathcal{R}$ cannot contain a copy of F. Applying Ramsey's theorem with the edges of $\Gamma(\{h\}) \setminus \mathcal{R}$ colored with the first color and those in $\Gamma(\{h\}) \cap \mathcal{R}$ colored with the second, we obtain that $\Gamma(\{h\}) \cap \mathcal{R}$ must contain a copy of $F \setminus e$. Let $\hat{e}$ be an edge in h whose addition would complete this copy of F. By our assumption we can select $|E(F)|$ different hyperedges to represent every edge in this copy of F: h itself for $\hat{e}$, and other hyperedges containing the rest of the edges. These hyperedges form a Berge-F in $\mathcal{H}$, a contradiction. □

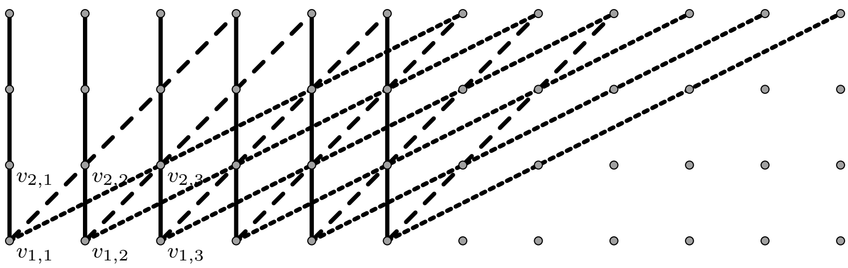


Figure 11.1: The construction in Section 11.4.1 (with $r = 4$, $n = 48$). The solid, dashed and dotted lines represent $e_{x,0}$'s, $e_{x,1}$'s and $e_{x,2}$'s respectively.

We are now ready to complete the proof of Theorem 11.1. For every hyperedge $h \in \mathcal{H}$ we apply Claim 11.11 to find an edge $e \in \mathcal{R}$, $e \subset h$ which is contained in at most $|E(F)|-1$ hyperedges. It follows that the number of edges in $\mathcal{H}$ is bounded by $|E(F)|-1$ times the number of edges found in this way, and thus

$$|E(\mathcal{H})| \leq (|E(F)| - 1)\,|\mathcal{R}| = o(n^2).$$

11.4 Linear hypergraphs — Proof of Theorem 11.3

11.4.1 Construction showing $\mathrm{th}^L(F) \geq \chi(F)$

First, we show that for any F we have $\mathrm{th}^L(F) \geq \chi(F)$. Let $2 \leq r \leq \chi(F) - 1$. We construct an r-uniform linear hypergraph on n vertices with $\Omega(n^2)$ edges and no Berge-F. Take r sets $V_1, V_2, \dots, V_r$ of $\lfloor \frac{n}{r} \rfloor$ vertices each. For each i, $1 \leq i \leq r$, let $V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,\lfloor n/r \rfloor}\}$. The hyperedges are the sets of $v_{i,j}$'s of the form $e_{x,m} = \{v_{1,x}, v_{2,x+m}, \dots, v_{r,x+(r-1)m}\}$ where $x \in \{1, 2, \dots, \lfloor \frac{n}{2r} \rfloor\}$ and $m \in \{0, 1, \dots, \lfloor \frac{n}{2r(r-1)} \rfloor\}$.

The number of hyperedges in this hypergraph is $\lfloor \frac{n}{2r} \rfloor (\lfloor \frac{n}{2r(r-1)} \rfloor + 1)$. The hypergraph is linear: if two different vertices v_{i_1,j_1} and v_{i_2,j_2} are contained in a hyperedge, then $i_1 \neq i_2$; and the two vertices uniquely determine the parameters of the hyperedge as $m = \frac{j_2 - j_1}{i_2 - i_1}$ and $x = j_1 - (i_1 - 1)m$, so they cannot be contained in two hyperedges. Moreover, the hypergraph contains no Berge-F since the 2-shadow contains no copy of F: the 2-shadow is r-partite but $\chi(F) > r$.

11.4.2 Proof of sharpness: $\mathrm{th}^L(F) \leq \chi(F)$

Now we show that $\mathrm{th}^L(F) \leq \chi(F)$. Let $\mathcal{H}$ be an r-uniform ($r \geq \chi(F)$) linear hypergraph on n vertices. A w-blowup of $K_{\chi(F)}$ is a complete $\chi(F)$-partite graph with w vertices in each class.

Lemma 11.12. *For large enough w (which depends on F and r, but not on n), if the 2-shadow of $\mathcal{H}$, denoted $\Gamma(\mathcal{H})$, contains a w-blowup of $K_{\chi(F)}$, then $\mathcal{H}$ contains a Berge-F.*

Proof. Let U be the vertices of a w-blowup of $K_{\chi(F)}$ in $\Gamma(\mathcal{H})$, and let $U_1, \dots, U_{\chi(F)}$ be its vertex classes. Let $v_1, \dots, v_{|V(F)|}$ be the vertices of F, and fix a proper coloring $c : V(F) \to \{1, \dots, \chi(F)\}$. Consider a map $\psi : V(F) \to U$ such that $\forall i : \psi(v_i) \in U_{c(v_i)}$. For every edge $v_iv_j \in E(F)$, $c(v_i) \neq c(v_j)$, so $\psi(v_i)\psi(v_j)$ is an edge of $\Gamma(\mathcal{H})$. If the hyperedges of $\mathcal{H}$ containing the edges $\psi(v_i)\psi(v_j)$ are distinct, then they form a Berge-F. We prove that $\mathcal{H}$ contains a Berge-F by estimating the number of maps $\psi : V(F) \to U$

such that $\forall i : \psi(v_i) \in U_{c(v_i)}$, and upper bounding the number of such maps that do not yield a Berge-F.

There are more than $(w - |V(F)|)^{|V(F)|} = \Omega(w^{|V(F)|})$ such maps in total. Indeed, we can choose the image of $v_1, v_2, \ldots \in V(F)$ one after the other. For each vertex v_i, $\left|U_{c(v_i)}\right| = w$, out of which at most $i-1$ vertices may already be taken, so we have more than $(w - |V(F)|$ choices.

Now fix two edges $e_1, e_2 \in E(F)$. We upper bound the number of maps such that the images of e_1 and e_2 are contained in the same hyperedge. There are w^2 ways to choose the images of the endpoints of $e_1 = v_i v_j$ in $\Gamma(\mathcal{H})$, since they have to be in $U_{c(v_i)}$ and $U_{c(v_j)}$ respectively. Because $\mathcal{H}$ is linear, there is only one hyperedge containing the image of e_1, so there are less than $\binom{r}{2}$ ways to choose the image of the endpoints of e_2 in $\Gamma(\mathcal{H})$ so that it is contained in the same hyperedge as e_1. For the image of the remaining vertices $V(F) \setminus (e_1 \cup e_2)$ we have less than $w^{|V(F)|-3}$ or $w^{|V(F)|-4}$ choices ($e_1 \cup e_2$ contains 3 or 4 vertices depending on whether e_1 and e_2 share a vertex).

In total, considering all pairs $e_1, e_2 \in F$, we have less than $\binom{|E(F)|}{2}\binom{r}{2} w^2 w^{|V(F)|-3} = O(w^{|V(F)|-1})$ maps ψ which do not yield a Berge-F. So for large enough w, $\mathcal{H}$ must contain a Berge-F. □

By Proposition 11.3, if a graph contains $\Omega(n^{\chi(F)})$ copies of $K_{\chi(F)}$, then (for large enough n) it contains an arbitrarily large (constant) blowup of $K_{\chi(F)}$ (because this graph has chromatic number $\chi(F)$). Therefore, by Lemma 11.12, assuming that $\mathcal{H}$ does not contain a Berge-F, $\Gamma(\mathcal{H})$ contains only $o(n^{\chi(F)})$ copies of $K_{\chi(F)}$.

By the graph removal lemma, there is a set S of $o(n^2)$ edges in $\Gamma(\mathcal{H})$ such that every copy of $K_{\chi(F)}$ in $\Gamma(\mathcal{H})$ contains at least one edge from S. Since $r \geq \chi(F)$, the 2-shadow of every hyperedge contains a $K_{\chi(F)}$, and therefore it contains an edge from S. Since $\mathcal{H}$ is a linear hypergraph, every edge in $\Gamma(\mathcal{H})$ is contained in only one hyperedge, so $|E(\mathcal{H})| \leq |S| = o(n^2)$.

11.5 Lower bound — Proof of Theorems 11.2 and 11.4

Definition 11.13. Given a k-uniform hypergraph $\mathcal{H}$ on a vertex set V, a blowup of $\mathcal{H}$ by a factor of w is a kw-uniform hypergraph $\mathcal{H}'$ obtained by replacing each vertex u_i of $\mathcal{H}$ by w vertices $v_{i,1}, \ldots, v_{i,w}$; the hyperedges of the new hypergraph are

$$\{\{v_{i,j} : u_i \in e, j = 1, \ldots, w\} : e \in \mathcal{H}\}.$$

We say that the vertices $v_{i,1}, \ldots, v_{i,w}$ of $\mathcal{H}'$ *originate from* the vertex u_i of $\mathcal{H}$, and the hyperedge $\{v_{i,j} : u_i \in e, j = 1, \ldots, w\}$ originates from the hyperedge e of $\mathcal{H}$. We may also blow up the vertices of $\mathcal{H}$ by different factors, replacing a vertex $u_i \in V$ with $w(u_i)$ vertices ($w(u_i) \geq 1$). If $\sum_{u \in e} w(u) = r$ for every $e \in \mathcal{H}$, then the new hypergraph is r-uniform.

Note that this blowup definition is not analogous with the graph blowup definition used in Section 11.4.2.

To motivate the reader, we first show the (simpler) proof of Theorem 11.2, and then generalize it to prove Theorem 11.4.

Proof of Theorem 11.2. Let $\omega(F) = s$. Assume for simplicity that $s-1$ divides n (if it does not, then take the construction below on $(s-1)\lfloor \frac{n}{s-1} \rfloor$ vertices, and supplement it with a few isolated vertices). We construct an $(s-1)^2$-uniform hypergraph on n vertices which does not contain a Berge-K_s. Since K_s is a subgraph of F, it is easy to see that it does not contain a Berge-F either. By Theorem 11.3, we have a linear $(s-1)$-uniform hypergraph $\mathcal{L}$ on $\lfloor \frac{n}{s-1} \rfloor$ vertices with $\Omega(n^2)$ hyperedges that does not contain a Berge-K_s. Let $\mathcal{H}$ be the $(s-1)^2$-uniform hypergraph obtained by blowing up $\mathcal{L}$ by a factor of $s-1$. $\mathcal{H}$ has the same number of hyperedges as $\mathcal{L}$.

Assume by contradiction that $\mathcal{H}$ has a Berge-K_s. Then there is an s-clique in the 2-shadow graph $\Gamma(\mathcal{H})$. Let $v_1, \dots, v_s$ be the vertices of an s-clique in $\Gamma(\mathcal{H})$ which corresponds to a Berge-K_s in $\mathcal{H}$. Let u_i be the vertex of $\mathcal{L}$ that v_i originates from. Because the blow-up factor is $s-1$, it is impossible for all u_i's to be the same vertex. It is also impossible for all u_i's to be different, since then the Berge-K_s in $\mathcal{H}$ would correspond to a Berge-K_s in $\mathcal{L}$. Thus we have $u_i = u_j \neq u_k$ for some $i \neq j \neq k$. But since $\mathcal{L}$ is linear, there is at most one hyperedge in $\mathcal{L}$ containing $u_i = u_j$ and u_k, so there are no distinct hyperedges in $\mathcal{H}$ containing the edges $v_i v_k$ and $v_j v_k$, contradicting that those vertices are part of a Berge-K_s in $\mathcal{H}$.

Using the construction in Section 11.4.1, we can construct r-uniform hypergraphs for $r < (s-1)^2$ similarly. Let the sets V_i be defined as in Section 11.4.1. If $r > s-1$, then blow up each vertex in V_i by the same factor w_i, where $1 \leq w_i \leq s-1$, in such a way that $\sum_i w_i = r$. If $r \leq s-1$, then just take an r-uniform linear hypergraph with no Berge-K_s. □

Proof of Theorem 11.4. For any r between 2 and $(c-1)t$, we construct an r-uniform hypergraph on n vertices with $\Omega(n^2)$ hyperedges and no Berge-F. Since putting each vertex of F in a separate set is a t-admissible partition, $c \leq \chi(F)$. If $r < c$, just take an r-uniform linear hypergraph with no Berge-F (given by Section 11.4.1). Otherwise let $\mathcal{L}$ be the linear $(c-1)$-uniform hypergraph with $\Omega(n^2)$ hyperedges given by Section 11.4.1 with c in the place of r. $\mathcal{L}$ does not contain any Berge-G with $\chi(G) \geq c$. Now fix blow-up factors $w_1, \dots, w_{c-1}$ between 1 and t such that $\sum w_i = r$, and let $\mathcal{H}$ be a blow-up of $\mathcal{L}$ obtained by blowing up every vertex in V_i by w_i, for all i. $\mathcal{H}$ is r-uniform, and it has $\Omega(n^2)$ hyperedges.

Let $|V(F)| = s$. Assume that $\mathcal{H}$ contains a Berge-F. Let $v_1, \dots, v_s$ be the vertices of F, and let ψ be the bijection that maps the vertices of F to the vertices of the Berge-F in $\mathcal{H}$ (as in Definition 9.2). Let $\tilde{\psi}(v_i)$ be the vertex of $\mathcal{L}$ from which $\psi(v_i)$ originates. Now partition the vertices of F with v_i and v_j belonging to the same set if $\tilde{\psi}(v_i) = \tilde{\psi}(v_j)$. We claim that this is a t-admissible partition. Indeed, first notice that at most t vertices of $\mathcal{H}$ originate from any vertex of $\mathcal{L}$ because the blow-up factors were taken between 1 and t. So the size of each set in the partition is at most t. Now assume for a contradiction that there are two different edges of F, $v_i v_j$ and $v_k v_l$, between some two sets of the partition. In other words, let $\tilde{\psi}(v_i) = \tilde{\psi}(v_k)$, and $\tilde{\psi}(v_j) = \tilde{\psi}(v_l)$. But $\mathcal{L}$ is linear, so there is at most one hyperedge containing both $\tilde{\psi}(v_i)$ and $\tilde{\psi}(v_j)$, so there are no distinct hyperedges in $\mathcal{H}$ containing the edges $\psi(v_i)\psi(v_j)$ and $\psi(v_k)\psi(v_l)$ of $\Gamma(\mathcal{H})$, contradicting the assumption that ψ maps to a Berge-F in $\mathcal{H}$.

So we have a t-admissible partition of F. Let G be the graph obtained by contracting each set in the partition. G is isomorphic to the graph $\tilde{G}$ with vertex set $\{\tilde{\psi}(v) : v \in V(F)\}$ and edge set $\{\tilde{\psi}(v_i)\tilde{\psi}(v_j) : v_i v_j \in E(F), \tilde{\psi}(v_i) \neq \tilde{\psi}(v_j)\}$. All the edges $\psi(v_i)\psi(v_j)$ of $\Gamma(\mathcal{H})$ corresponding to edges $v_i v_j$ of F are contained in distinct hyperedges of $\mathcal{H}$, since ψ maps to a Berge-F. Since the hyperedges of $\mathcal{H}$ originate from distinct hyperedges of $\mathcal{L}$, all the edges $\tilde{\psi}(v_i)\tilde{\psi}(v_j)$ of $\tilde{G}$ are contained in distinct hyperedges of $\mathcal{L}$. Thus, $\mathcal{L}$ contains a Berge-G. But $\chi(G) \geq c$, contradicting the assumption that $\mathcal{L}$ does not contain any Berge-G with $\chi(G) \geq c$. □

We show two corollaries of Theorem 11.4. The following is a simple observation that helps in reasoning about admissible partitions (already alluded to after the statement of Theorem 11.4).

Observation 11.14. If two vertices v and w are in the same set A of a t-admissible partition of F, and a third vertex u is connected to both of them, then $u \in A$: if it was in a different set B, then uv and uw would be two edges between A and B, which is not allowed in a t-admissible partition.

In the following corollaries, a blowup F of a graph G is the graph obtained from G by blowing up each vertex v_i in G by a factor w_i (in the usual graph sense).

Corollary 11.15. *Let $s \geq 3$, and let F be an arbitrary blowup of K_s. Then* $\mathrm{th}(F) \geq (s-1)(|V(F)|-1)+1$.

Proof. Let $t = |V(F)| - 1$. Let $V(F) = V_1 \cup \ldots \cup V_s$, and let $V_i = \{v_{i,1}, \ldots, v_{i,w_i}\}$ where $v_{i,j}$ and $v_{k,l}$ are adjacent in F if $i \neq k$.

We claim that the only t-admissible partition of F is into singletons. Assuming this claim, there are no contractions to be made, so we have $c = \chi(F) = s$, proving the corollary.

Assume that $v_{i,j}$ and $v_{k,l}$ are in the same set A in a t-admissible partition (where i and k may be different or equal). For every $p \notin \{i,k\}$ and $q \in 1, \ldots, w_p$, $v_{p,q}$ must be in A by Observation 11.14, since $v_{p,q}$ is connected to both $v_{i,j}$ and $v_{k,l}$. Now A contains at least one vertex from every V_i, and by choosing appropriate pairs of vertices, it is easy to see that the remaining vertices must be in A too. But $t < |V(F)|$, so putting all vertices in the same set is not a t-admissible partition. □

Corollary 11.16. *Let G be a connected, non-bipartite graph on the vertex set $u_1, \ldots, u_s$, and let F be a blowup of G such that every vertex is blown up by a factor of at least 2: Let $V_i = \{v_{i,1}, \ldots, v_{i,w_i}\}$ with $w_i \geq 2$, and let $V(F) = V_1 \cup \ldots \cup V_s$, where $v_{i,j}$ and $v_{k,l}$ are connected if $u_iu_k \in E(G)$. Then* $\mathrm{th}(F) \geq (\chi(G)-1)(|V(F)|-1)+1$.

Proof. Let $t = |V(F)| - 1$. Take a t-admissible partition of F. We claim that no two vertices in the same V_i belong to the same set of the partition. Assume that $v_{i,j}$ and $v_{i,l}$ are in the same set A in a t-admissible partition. Let u_p be a neighbor of u_i in G. Then every $v_{p,q}$ is connected to both $v_{i,j}$ and $v_{i,l}$, so $V_p \subset A$ by Observation 11.14. Since G is connected, and $|V_i| = w_i \geq 2$ for every i, we can traverse G starting from v_i, and inductively show that every V_i is a subset of A. But then $|A| = |V(F)| > t$, contradicting that the partition is t-admissible.

We also claim that for any u_i, at most one of the vertices $v_{i,1}, \ldots, v_{i,w_i}$ belongs to a set of the partition that also contains a vertex $v_{k,l}$ such that u_i and u_k are connected in G. Assume that $v_{i,j}, v_{k,l} \in A$ and $v_{i,m}, v_{p,q} \in B$, where A and B are sets in our t-admissible partition of F, and $u_iu_k, u_iu_p \in E(G)$; then $v_{i,j}v_{p,q}$ and $v_{i,m}v_{k,l}$ are two edges between A and B in F.

Now we prove the corollary. Let F' be the graph obtained by contracting each set in a t-admissible partition of F. Let $f(v_{i,j})$ be the vertex of F' corresponding to the set $v_{i,j}$ belongs to. We are going to prove that $\chi(G) \leq \chi(F')$; since the t-admissible partition we took is arbitrary, this implies that c in Theorem 11.4 is equal to $\chi(G)$ (and at least 3 since G was assumed to be bipartite), proving the corollary. Take a proper coloring of F' with $\chi(F')$ colors. We properly color the vertices of G with the same colors. For every vertex u_i, we pick a vertex $v_{i,g(i)}$ such that the set of the partition that contains $v_{i,g(i)}$ does not contain any vertex $v_{k,l}$ such that u_i and u_k are connected in G; assign u_i the color of $f(v_{i,g(i)})$ in F'. Now let $u_i u_k$ an arbitrary edge of G. Then $v_{i,g(i)}$ and $v_{k,g(k)}$ must be in different sets, so $f(v_{i,g(i)}) \neq f(v_{k,g(k)})$; and $v_{i,g(i)} v_{k,g(k)} \in E(F)$, so $f(v_{i,g(i)}) f(v_{i,g(i)}) \in E(F')$. So $f(v_{i,g(i)})$ and $f(v_{k,g(k)})$ have different colors in F', thus we assigned u_i and u_k different colors. So we have obtained a proper coloring of G with $\chi(F')$ colors. $\square$

Chapter 12

Asymptotics for the Turán number of Berge-$K_{2,t}$

Only a handful of results are known about the asymptotic behaviour of Turán numbers for hypergraphs. Our main goal in this chapter is to determine sharp asymptotics for the Turán number of Berge-$K_{2,t}$ and Berge-$\{C_2, K_{2,t}\}$ in 3-uniform hypergraphs. In fact, we prove a general theorem which also provides bounds in the r-uniform case.

Structure of the chapter and notation

In Section 12.1 we highlight the results that we improve. In Section 12.2 we state our results and prove some of our corollaries. In Section 12.3 we provide proofs of our theorems about r-uniform, Berge-F-free hypergraphs, while in Section 12.4 we provide proofs of our theorems about linear, r-uniform, Berge-$K_{2,t}$-free hypergraphs. Finally we provide some remarks and connections with other topics.

Given graphs H and F, recall that $ex(n, H, F)$ denotes the maximum number of copies of H in an F-free graph on n vertices.

In most cases we use F to denote the forbidden graph, G for the base graph and $\mathcal{H}$ for the hypergraph.

12.1 History, known results

We list a couple of useful results that are needed later in this chapter. Alon and Shikhelman determined the asymptotics of $\mathrm{ex}(n, K_3, K_{2,t})$.

Theorem 12.1 (Alon and Shikhelman [5]). *For $t \geq 2$, we have*

$$\mathrm{ex}(n, K_3, K_{2,t}) = \frac{1}{6}(t-1)^{3/2}n^{3/2}(1 + o(1)).$$

Recently Luo determined $\mathrm{ex}(n, K_r, P_k)$ exactly.

Theorem 12.2 (Luo [115]). *For $n \geq k \geq 2$ and $r \geq 1$ we have*

$$\mathrm{ex}(n, K_r, P_k) = \frac{n}{k-1}\binom{k-1}{r}.$$

12.1.1 Turán-type results for Berge-F-free hypergraphs

In this section we briefly present the results that we improve in this chapter.

For Berge cycles of even length, Füredi and Özkahya proved the following bound.

Theorem 12.3 (Füredi and Özkahya [53]). *For $k \geq 2$, we have*

$$\mathrm{ex}_3(n, C_{2k}) \leq \frac{2k}{3}\,\mathrm{ex}(n, C_{2k}),$$

and

$$\mathrm{ex}(n, K_3, C_{2k}) \leq \frac{2k-3}{3}\,\mathrm{ex}(n, C_{2k}).$$

We improve the first inequality in Corollary 12.5 by showing $\frac{2k}{3}$ can be replaced by $\frac{2k-3}{3}$ provided $k \geq 5$.

Győri and Lemons [82] also showed that for general r-uniform hypergraphs with $r \geq 4$, $\mathrm{ex}_r(n, C_{2k+1}) \leq O(k^{r-2}) \cdot ex_3(n, C_{2k+1})$ and $\mathrm{ex}_r(n, C_{2k}) \leq O(k^{r-1}) \cdot \mathrm{ex}(n, C_{2k})$. Jiang and Ma [91] improved these results by an $\Omega(k)$ factor. In particular, for the even cycle case they showed the following.

Theorem 12.4 (Jiang and Ma [91]). *If $n, k \geq 1$ and $r \geq 4$, then we have*

$$\mathrm{ex}_r(n, C_{2k}) \leq O_r(k^{r-2}) \cdot \mathrm{ex}(n, C_{2k}).$$

We give a new proof of the above result in Corollary 12.6 (with a better constant factor).

Gerbner and Palmer proved the following about r-uniform Berge $K_{2,t}$-free hypergraphs.

Theorem 12.5 (Gerbner and Palmer [63]). *If $t \leq r-2$, then*

$$\mathrm{ex}_r(n, K_{2,t}) = O(n^{3/2}),$$

and if $t = r-2$, then

$$\mathrm{ex}_r(n, K_{2,t}) = \Theta(n^{3/2}).$$

We extend this result to other ranges of t and r, and prove more precise bounds in Corollary 12.3.

Timmons studied the same problem for linear hypergraphs and proved the following nice result.

Theorem 12.6 (Timmons [144]). *For all $r \geq 3$ and $t \geq 1$, we have*

$$\mathrm{ex}_r(n, \{C_2, K_{2,t}\}) \leq \frac{\sqrt{2(t+1)}}{r} n^{3/2} + \frac{n}{r}.$$

Let $r \geq 3$ be an integer and l be any integer with $2l+1 \geq r$. If $q \geq 2lr^3$ is a power of an odd prime and $n = rq^2$, then

$$\mathrm{ex}_r(n, \{C_2, K_{2,t}\}) \geq \frac{l}{r^{3/2}} n^{3/2} + O(n),$$

where $t - 1 = (r-1)(2l^2 - l)$.

Note that Timmons mentioned that the upper bound was pointed out by Palmer using methods similar to the ones used in [144].
We improve Theorem 12.6 in Theorem 12.9, Theorem 12.10 and Theorem 12.11.

Finally let us recall the simple but useful result of Gerbner and Palmer that connects $\mathrm{ex}(n, K_r, F)$ and $\mathrm{ex}_r(n, F)$.

Theorem 12.7 (Gerbner and Palmer [62]). *For $r \geq 2$ and any graph F, we have*

$$\mathrm{ex}(n, K_r, F) \leq \mathrm{ex}_r(n, F) \leq \mathrm{ex}(n, K_r, F) + \mathrm{ex}(n, F).$$

One of our main results shown in the next section, determines the asymptotics for the Turán number of Berge-$K_{2,t}$ for $t \geq 7$ in the case that $r = 3$. Note that Theorem 12.7 combined with Theorem 12.1 gives

$$\frac{1}{6}(t-1)^{3/2}n^{3/2}(1+o(1)) \leq ex_3(n, K_{2,t}) \leq \frac{1}{6}(t-1)^{3/2}n^{3/2}(1+o(1)) + \frac{\sqrt{t-1}}{2}n^{3/2}(1+o(1)).$$

Thus we have an upper bound which differs from the lower bound by

$$\mathrm{ex}(n, K_{2,t}) = \frac{\sqrt{t-1}}{2}n^{3/2}(1 + o(1)).$$

This has the same order of magnitude in n and a lower order of magnitude in t compared to the lower bound. However, the simple idea used in the proof of Theorem 12.7 is not useful to reduce this gap, we will introduce new ideas. Our main focus in this chapter is to determine sharp asymptotics.

12.2 Our results

12.2.1 A general theorem

First we state a general theorem that applies to many graphs and not just $K_{2,t}$. For convenience of notation in the rest of the chapter, let us define

$$f(r) = \left(\binom{r}{2} - 2\right)\left(1 + \left(\binom{r}{2} - 1\right)\left(\binom{r}{2} - 3\right)\right)$$

and

$$g(r) = f(r) + \binom{r}{2} - 2.$$

Theorem 12.8 (Gerbner, M., Vizer [60]). *Let F be a K_r-free graph. Let F' be a graph we get by deleting a vertex from F (where this vertex may be chosen arbitrarily). Moreover, suppose that there are constants c and i with $0 \leq i \leq r-1$ with $\mathrm{ex}(n, K_{r-1}, F') \leq cn^i$ for every n.*

(a) If $cn^{i-1} \geq rg(r)/2$, then we have

$$\mathrm{ex}_r(n, F) \leq 2c\frac{\mathrm{ex}(n, F)n^{i-1}}{r}.$$

(b) If $cn^{i-1} \leq rg(r)/2$, then we have

$$\mathrm{ex}_r(n, F) \leq g(r) \cdot \mathrm{ex}(n, F).$$

(c) If $i > 1$ and n is large enough, then we have

$$\mathrm{ex}_r(n, F) \leq c(r-1)\,\mathrm{ex}(n, F)^i \left(\frac{2}{n}\right)^{i-1}.$$

Remark 12.1. The proof of the above theorem can be modified to show that if F contains K_r, similar upper bounds hold with slightly different multiplicative constant factors. Theorem 12.8 together with these inequalities show that if every cycle in F contains the same vertex v for some $v \in V(F)$, then we have

$$\mathrm{ex}_r(n, F) = O(\mathrm{ex}(n, F))$$

for every $r \geq 3$.

Also note that $g(3) = 2$. Moreover, if $F = K_{2,t}$ and $F' = K_{1,t}$, or if $F = C_{t+2}$ and $F' = P_{t+1}$, then we have

$$\mathrm{ex}(n, K_2, F') = \mathrm{ex}(n, F') \leq (t-1)\frac{n}{2}.$$

Therefore, using Theorem 12.8 part (a) with $c = \frac{t-1}{2}$ and $i = 1$ implies the upper bounds in Corollary 12.2 and Corollary 12.5 which are given below.

Asymptotics for Berge-$K_{2,t}$

Corollary 12.2 (Gerbner, M., Vizer [60])**.** *Let $t \geq 7$. Then*

$$\mathrm{ex}_3(n, K_{2,t}) = \frac{1}{6}(t-1)^{3/2} n^{3/2}(1 + o(1)).$$

Our lower bound in the above result follows from Theorem 12.7 and Theorem 12.1. The latter theorem considers the $K_{2,t}$-free graph G constructed by Füredi in Theorem 4.3 and shows that the number of triangles in it is at least $\frac{1}{6}(t-1)^{3/2}n^{3/2}(1+o(1))$. Replacing each triangle in G by a hyperedge on the same vertex set, we get a Berge-$K_{2,t}$-free hypergraph containing the desired number of hyperedges.

Below we show the analogous result for general r-uniform hypergraphs that is sharp in the order of magnitude of n.

Corollary 12.3 (Gerbner, M., Vizer [60])**.** *(a) If $t > \lceil \frac{r}{2} \rceil - 2 \geq 0$, then we have*

$$(1 + o(1))\frac{\sqrt{t-1}}{r^{3/2}} n^{3/2} \leq ex_r(n, K_{2,t}).$$

(b) If $\frac{\binom{t}{r-1}}{t} \geq rg(r)/2$, then we have

$$\mathrm{ex}_r(n, K_{2,t}) \leq (1 + o(1))\frac{\sqrt{(t-1)\binom{t}{r-1}}}{r \cdot t} n^{3/2}.$$

If $\frac{\binom{t}{r-1}}{t} \leq rg(r)/2$, then we have

$$\mathrm{ex}_r(n, K_{2,t}) \leq (1 + o(1))g(r)\sqrt{t-1}\,n^{3/2}.$$

This result improves Theorem 12.5.

Remark 12.4. If $t \leq 6$, then Theorem 12.8 gives

$$\mathrm{ex}_3(n, K_{2,t}) \leq \sqrt{t-1} \cdot n^{3/2}$$

and the lower bound in Corollary 12.2 still holds. On the other hand putting $r = 3$ and $t = 2$ into Corollary 12.3 (a), we get a lower bound of $n^{3/2}/3\sqrt{3}$, which is larger. For this particular case, the best upper bound known is $\mathrm{ex}_3(n, K_{2,2}) \leq (1 + o(1))n^{3/2}/\sqrt{10}$ due to Ergemlidze, Győri, Methuku, Tompkins and Salia [43].

Improved bounds for Berge-C_{2k}

Corollary 12.5 (Gerbner, M., Vizer [60]). *Let $k \geq 5$. Then*

$$\mathrm{ex}_3(n, C_{2k}) \leq \frac{2k-3}{3} \cdot \mathrm{ex}(n, C_{2k}).$$

Note that $\mathrm{ex}(n, K_3, C_{2k}) \leq \frac{2k-3}{3}\,\mathrm{ex}(n, C_{2k})$ (the second inequality from Theorem 12.3) and Theorem 12.7 implies

$$\mathrm{ex}_3(n, C_{2k}) \leq \frac{2k}{3}\,\mathrm{ex}(n, C_{2k}),$$

which is the first inequality from Theorem 12.3. Here we remove the difference between $\mathrm{ex}_3(n, C_{2k})$ and $\mathrm{ex}(n, K_3, C_{2k})$, as we do in the case of $K_{2,t}$ in Corollary 12.2.

For the r-uniform case, we have the following corollary giving a new proof of Theorem 12.4. We note that the multiplicative factor given in Corollary 12.6 is better than the one obtained in the proof of Theorem 12.4 by Jiang and Ma [91], whenever $r \geq 8$ or $k \geq 3$.

Corollary 12.6 (Gerbner, M., Vizer [60]). *If $n, k \geq 2$ and $r \geq 4$, then we have*

$$\mathrm{ex}_r(n, C_{2k}) \leq \max\left\{\frac{1}{r(k-1)}\binom{2k-2}{r-1}, g(r)\right\} \cdot \mathrm{ex}(n, C_{2k}) =$$
$$= O_r(k^{r-2}) \cdot \mathrm{ex}(n, C_{2k}).$$

Proof. Using Theorem 12.2, we get

$$\mathrm{ex}(n, K_{r-1}, P_{2k-1}) = \frac{n}{2k-2}\binom{2k-2}{r-1}.$$

So in Theorem 12.8, we can choose $i = 1$, $F' = P_{2k-1}$ and $c = \frac{1}{2k-2}\binom{2k-2}{r-1}$. To decide whether part (a) or part (b) of Theorem 12.8 applies, we need to compare

$$c = \frac{1}{2k-2}\binom{2k-2}{r-1} \quad \text{and} \quad \frac{rg(r)}{2}.$$

If the first one is larger, then we get

$$\mathrm{ex}_r(n, C_{2k}) \leq \frac{2}{r(2k-2)}\binom{2k-2}{r-1} \cdot \mathrm{ex}(n, C_{2k}).$$

If the second one is larger, we get

$$\mathrm{ex}_r(n, C_{2k}) \leq g(r) \cdot \mathrm{ex}(n, C_{2k}).$$

□

Remark 12.7. Let us assume $2k > \lceil r/2 \rceil$, take a bipartite graph G of girth more than $2k$ containing $\lfloor n/r \rfloor$ vertices in both color classes and replace each vertex in one part by $\lfloor r/2 \rfloor$ copies of it, and each vertex in the other part by $\lceil r/2 \rceil$ copies of it. We claim that if the resulting r-uniform hypergraph $\mathcal{H}$ contains a Berge-C_{2k}, then G contains a cycle of length at most $2k$, which is impossible. Indeed, consider a Berge cycle of length $2k$, v_1,h_1,v_2,$h_2, \ldots, v_{2k}$,h_{2k},v_1 in $\mathcal{H}$. Then each v_i is a copy of a vertex u_i of G. So this Berge cycle corresponds to a closed walk $u_1 u_2, \ldots, u_{2k} u_1$ in G of length $2k$. We claim that this closed walk contains a cycle of length at most $2k$. Indeed, otherwise either an edge is repeated in the walk or it consists of only one vertex; we will show both of these cases are impossible: As $2k > \lceil r/2 \rceil$, there are two vertices v_i and v_j that are copies of two different vertices u_i and u_j (respectively) of G, which means the walk contains at least two different vertices of G. Also observe that if an edge is repeated in the closed walk $u_1 u_2, \ldots, u_{2k} u_1$, say $u_i u_{i+1} = u_j u_{j+1}$ (addition in the subscripts is modulo $2k$), then we must have $h_i = h_j$ contradicting the definition of a Berge cycle. This gives a lower bound of

$$\mathrm{ex}_r(n, C_{2k}) \geq ex\left(\left\lfloor \frac{n}{r} \right\rfloor, \left\lfloor \frac{n}{r} \right\rfloor, \{C_3, C_4, \ldots, C_{2k}\}\right).$$

12.2.2 Asymptotics for Berge-$K_{2,t}$ - the linear case

First we prove the following upper bound.

Theorem 12.9 (Gerbner, M., Vizer [60]). *For all $r, t \geq 2$, we have*

$$\mathrm{ex}_r(n, \{C_2, K_{2,t}\}) \leq \frac{\sqrt{t-1}}{r(r-1)} n^{3/2} + O(n).$$

Note that putting $r = 2$ in Theorem 12.9, we can recover the upper bound in Füredi's theorem - Theorem 4.3.

Our main focus in the rest of this section is to prove lower bounds and determine the asymptotics of the Turán number of Berge-$K_{2,t}$ in 3-uniform linear hypergraphs. Putting $r = 3$ in Theorem 12.9 we get an upper bound of

$$(1 + o(1)) \frac{\sqrt{t-1}}{6} n^{3/2},$$

and putting $r = 3$ in Theorem 12.6 we get a lower bound of

$$\frac{\sqrt{t-1}}{6\sqrt{3}} n^{3/2} + O(n)$$

for some special n and t. First we present a slightly weaker lower bound but its proof is much simpler than that of Theorem 12.6 and it also works for every n and t:

Theorem 12.10 (Gerbner, M., Vizer [60]). *If $n \geq t \geq 2$, we have*

$$\mathrm{ex}_3(n, \{C_2, K_{2,t}\}) \geq \frac{\sqrt{t-1}}{12} n^{3/2} + O(n).$$

However, when t is large enough, we can do much better. In Theorem 12.11 below, we prove a lower bound of

$$(1 + o_t(1)) \frac{\sqrt{t-1}}{6} n^{3/2},$$

where o_t depends on t. Thus, it shows that for large enough t our upper bound in Theorem 12.9 is close to being asymptotically correct for $r = 3$.

Theorem 12.11 (Gerbner, M., Vizer [60]). *There is an absolute constant c such that for any $t \geq 2$, we have,*

$$\mathrm{ex}_3(n, \{C_2, K_{2,t}\}) \geq \left(1 - \frac{c}{\sqrt{t-1}} \ln^{3/2}(t-1)\right) \frac{\sqrt{t-1}}{6} n^{3/2}.$$

Note that for $t = 2$, this gives a lower bound matching the upper bound in Theorem 12.9 for $r = 3$. Therefore, we can recover a sharp result of Ergemlidze, Győri and Methuku in [40], showing $\mathrm{ex}_3(n, \{C_2, C_4\}) = (1 + o(1))n^{3/2}/6$.

12.3 Proofs of the results about r-uniform Berge-F-free hypergraphs

12.3.1 Proof of Theorem 12.8

Let us introduce some notation. Let $\mathcal{H}$ be an r-uniform Berge-F-free hypergraph. We call a pair of vertices u, v a *blue edge* if it is contained in at least one and at most $\binom{r}{2} - 2$ hyperedges in $\mathcal{H}$ and a *red edge* if it is contained in more than $\binom{r}{2} - 2$ hyperedges.

We call a hyperedge *blue* if it contains a blue edge, and *red* otherwise. Let us denote the set of blue edges by S and the number of blue hyperedges by s. We choose a largest subset $S' \subset S$ with the property that every blue hyperedge contains at most one edge of S'.

Claim 12.8. $|S'| \geq s/f(r)$.

Proof. We build an auxiliary bipartite graph A with parts P and Q where P consists of all the blue edges and Q consists of all the blue hyperedges of $\mathcal{H}$, and we connect a vertex of P with a vertex of Q if the corresponding blue edge is contained in the corresponding blue hyperedge.

By definition, S' is the largest subset of P such that any two vertices of S' are at distance more than two in the graph A. We claim that every vertex of Q is at distance at most three from a vertex of S'. Indeed, otherwise any of its neighbors can be added to S'.

Now we show that the number of vertices in Q that are at distance at most three from a vertex of S' is at most $|S'|f(r)$. Indeed, a blue edge is contained in at most $(\binom{r}{2} - 2)$ blue hyperedges, and they each contain at most $\binom{r}{2} - 1$ other blue edges; those blue edges are in turn contained in at most $\binom{r}{2} - 3$ other blue hyperedges. So a vertex of S' is at distance at most three from at most $(\binom{r}{2} - 2)(1 + (\binom{r}{2} - 1)(\binom{r}{2} - 3)) = f(r)$ vertices in Q. Since we must have $|Q| = s \leq |S'|f(r)$, the proof is complete.

□

Now consider an auxiliary graph G, consisting only of the blue edges of S' and all the red edges. Let us assume there is a copy of F in G. We build an auxiliary bipartite graph B. One of its classes B_1 consists of the edges of that copy of F, and the other class B_2 consists of the hyperedges of $\mathcal{H}$ that contain them. We connect a vertex of B_1 with a vertex of B_2 if the corresponding hyperedge of $\mathcal{H}$ contains the corresponding edge of F. Note that every hyperedge can contain at most $\binom{r}{2} - 1$ edges from a copy of F (since F is K_r-free), thus vertices of B_2 have degree at most $\binom{r}{2} - 1$.

Notice that a matching in B covering B_1 would give a Berge-F in $\mathcal{H}$. Thus by Hall's theorem there is a subset $X \subset B_1 = E(F)$ with $|N(X)| < |X|$ for any copy of F in G. We call such a subset X *bad*.

Claim 12.9. *There is a blue edge in every copy of F in G.*

Proof. Otherwise we can find a bad set $X \subset B_1$ such that every element in it is a red edge of G, thus the corresponding vertices have degree at least $\binom{r}{2} - 1$ in B. On the other hand since every vertex of $N(X)$ is in B_2, they have degree at most $\binom{r}{2} - 1$ in B. This implies $|N(X)| \geq |X|$, contradicting the assumption that X is a bad set. □

The following claim shows that every bad subset of edges in a copy of F in G contains a red edge which is contained in few red hyperedges. Our plan will be to recolor such a red edge of a bad set in each copy of F in G to green, to make sure that every copy of F in G contains a green edge.

Claim 12.10. *Every bad set X (in any copy of F in G) contains a red edge that is contained in at most $\binom{r}{2} - 2$ red hyperedges.*

Proof. Let us assume indirectly that every red edge of G is contained in at least $\binom{r}{2} - 1$ red hyperedges. Let x be the number of red hyperedges in $N(X)$ and y be the number of blue hyperedges. Then the number of blue edges in X is at most y (since S' has the property that every blue hyperedge contains at most one pair of S'). Since $|N(X)| < |X|$, this implies the number of red edges in X is more than x, hence the number of edges in B between the red edges in X and red hyperedges is at least $(x+1)(\binom{r}{2} - 1)$. However, a hyperedge in B_2 has at most $\binom{r}{2} - 1$ neighbors in B_1 and so there can be at most $x(\binom{r}{2} - 1)$ edges in B between red hyperedges and red edges in X, a contradiction. □

We will call a red edge (in each copy of F in G) guaranteed by the above claim, *special red edge*. As every copy of F in G contains a bad set, it contains a special red edge too.

We consider each copy of F, one by one, and recolor a special red edge in it to *green*, and also recolor all the red hyperedges containing it to *green*. Note that it is possible that some of the red hyperedges containing a special red edge of G may have already turned green. However, after this procedure, the total number of green hyperedges of $\mathcal{H}$ is obviously at most $\binom{r}{2} - 2$ times the number of green edges of G. Notice that each remaining red hyperedge still contains $\binom{r}{2}$ red edges of G.

Let us now recolor the remaining red edges of G and red hyperedges of $\mathcal{H}$ to *purple* to avoid confusion. Thus G now contains blue, green and purple edges, while $\mathcal{H}$ contains blue, green and purple hyperedges. Every blue hyperedge of $\mathcal{H}$ contains at most one blue edge of G, and every green hyperedge of $\mathcal{H}$ contains a green edge of G (possibly more than one), while a purple hyperedge contains $\binom{r}{2}$ purple edges of G (i.e., every pair contained in a purple hyperedge is a purple edge of G).

Furthermore, let G_1 be the subgraph of G consisting of blue and purple edges, and let G_2 be the subgraph of G consisting of green and purple edges. Clearly G_1 is F-free because every copy of F in G contains a green edge. We claim G_2 is also F-free – indeed, notice that we recolored only red edges to green or purple, so the edges in G_2 were all originally red. Therefore, by Claim 12.9, G_2 cannot contain a copy of F.

Claim 12.11. *If an F-free graph G contains x edges, then it contains at most*

$$\min\left\{\frac{2cxn^{i-1}}{r}, cx(r-1)\left(\frac{2\,\mathrm{ex}(n,F)}{n}\right)^{i-1}\right\}$$

copies of K_r.

Proof. Obviously the neighborhood of every vertex is F'-free. An F'-free graph on $d(v)$ vertices contains at most

$$\mathrm{ex}(d(v), K_{r-1}, F') \le cd(v)^i$$

copies of K_{r-1} by the definition of c. It means v is in at most that many copies of K_r, so summing up for every vertex v, every K_r is counted r times. On the other hand as $\sum_{v\in V(G)} d(v) = 2x$ and $d(v) \le n$ we have

$$\sum_{v\in V(G)} cd(v)^i \le \sum_{v\in V(G)} cn^{i-1}d(v) = 2cxn^{i-1},$$

showing that the number of copies of K_r in G is at most

$$\frac{2cxn^{i-1}}{r}.$$

Now we show that the number of copies of K_r is also at most

$$cx(r-1)\left(\frac{2\,\mathrm{ex}(n,F)}{n}\right)^{i-1}.$$

Let a be the number of the copies of K_r in G. Let us consider an edge that is contained in less than a/x copies of K_r, and delete it. We repeat this as long as there exists such an edge. Altogether we deleted at most x edges, hence we deleted less than a copies of K_r. So the resulting graph G_1 is non-empty. Let us delete the isolated vertices of G_1. The resulting graph G_2 is F-free on, say, $n' < n$ vertices, hence it contains at most $\mathrm{ex}(n', F)$ edges. This implies G_2 contains a vertex v with degree

$$d(v) \le \frac{2\,\mathrm{ex}(n',F)}{n'} \le \frac{2\,\mathrm{ex}(n,F)}{n}.$$

Let us consider the number of copies of K_{r-1} in the neighborhood of v in G_2. On one hand it is at most

$$\mathrm{ex}(d(v), K_{r-1}, F') \le cd(v)^i.$$

On the other hand, it is equal to the number of copies of K_r that contain v, which is at least

$$\frac{ad(v)}{x(r-1)}.$$

Indeed, the $d(v)$ edges incident to v are all contained in at least a/x copies of K_r, and such copies are counted $r-1$ times. So combining, we get

$$a \le cx(r-1)d(v)^{i-1} \le cx(r-1)\left(\frac{2\,\mathrm{ex}(n,F)}{n}\right)^{i-1},$$

completing the proof of the claim. □

Now we continue the proof of Theorem 12.8. Let x be the number of purple edges in G. Then, by Claim 12.11, the number of copies of K_r consisting of purple edges in G is at most

$$\frac{2cxn^{i-1}}{r},$$

but then the number of purple hyperedges in $\mathcal{H}$ is also at most this number because any pair contained in a purple hyperedge forms a purple edge in G.

Let $y := \mathrm{ex}(n, F)$. Then the number of blue edges in G is at most $y - x$ as G_1 is F-free, and similarly, the number of green edges in G is at most $y - x$.

By Claim 12.8 the total number of blue hyperedges in $\mathcal{H}$ is at most $f(r)$ times the number of blue edges, i.e., at most $f(r)(y - x)$.

Moreover, we claim that the number of green hyperedges in $\mathcal{H}$ is at most $\binom{r}{2} - 2$ times the number of green edges of G – indeed, by Claim 12.10, any (special) red edge of G that was recolored green, was originally contained in at most $\binom{r}{2} - 2$ red hyperedges (which were all recolored to become green hyperedges). Thus a green edge of G is in at most $\binom{r}{2} - 2$ green hyperedges of $\mathcal{H}$ and every green hyperedge contains a green edge. Therefore, the number of green hyperedges in $\mathcal{H}$ is at most $\left(\binom{r}{2} - 2\right)(y - x)$.

Therefore the total number of hyperedges in $\mathcal{H}$ is at most

$$\frac{2cxn^{i-1}}{r} + \left(f(r) + \binom{r}{2} - 2\right)(y - x) \leq \max\left\{\frac{2cn^{i-1}}{r}, g(r)\right\}(x + y - x),$$

and we are done with (a) and (b) by the assumption on c.

For (c) we use the other upper bound on the number of K_r's from Claim 12.11, namely:

$$cx(r-1)\left(\frac{2\,\mathrm{ex}(n,F)}{n}\right)^{i-1}.$$

Observe that it goes to infinity as n grows, since $i > 1$. The same calculation gives that for large enough n the number of hyperedges is at most

$$\max\left\{c(r-1)\left(\frac{2\,\mathrm{ex}(n,F)}{n}\right)^{i-1}, g(r)\right\}(x + y - x) = c(r-1)\left(\frac{2\,\mathrm{ex}(n,F)}{n}\right)^{i-1} y$$

$$\leq c(r-1)\,\mathrm{ex}(n,F)^i\left(\frac{2}{n}\right)^{i-1}.$$

□

12.3.2 Proof of Corollary 12.3

First we prove (a):

Let G be a bipartite $K_{2,t}$-free graph with $\frac{n}{r}$ vertices in both color classes and containing

$$\sqrt{t-1}\left(\frac{n}{r}\right)^{3/2}(1 + o(1))$$

edges. The existence of such a graph is guaranteed by Theorem 4.4. Let

$$A = \{a_1, a_2, \dots, a_{\frac{n}{r}}\}, \text{ and } B = \{b_1, b_2, \dots, b_{\frac{n}{r}}\}$$

be the color classes of G.

Let us replace each vertex $a_i \in A$ with a set A_i of $\left\lfloor \frac{r}{2} \right\rfloor$ copies of a_i, and each vertex $b_i \in B$ with a set B_i of $\left\lceil \frac{r}{2} \right\rceil$ copies of b_i to get an r-uniform hypergraph $\mathcal{H}$. Let

$$A_{new} := \cup_i A_i, \text{ and}$$

$$B_{new} := \cup_i B_i.$$

It is easy to see that the number of hyperedges in $\mathcal{H}$ is equal the number of edges in G, as required. It remains to show that $\mathcal{H}$ is Berge-$K_{2,t}$-free.

Suppose for a contradiction that $\mathcal{H}$ contains a Berge-$K_{2,t}$. Then there is a bijective map from the hyperedges of the Berge-$K_{2,t}$ to the edges of the graph $K_{2,t}$ such that each edge is contained in the hyperedge that was mapped to it. Let $\{p,q\}$ and T be the color classes of $K_{2,t}$. If $\{p,q\} \subset A_{new}$ and $T \subset B_{new}$ (or vice versa), then p and q cannot be in the same A_i as each A_i and each vertex of B_{new} are contained in at most one hyperedge of $\mathcal{H}$, however the hyperedges of the Berge-$K_{2,t}$ containing the edges pr, qr for some $r \in T$ must be different, a contradiction. Therefore, p and q belong to distinct A_i and similarly, the vertices of T must belong to distinct B_i, but this implies that G contains a $K_{2,t}$, a contradiction. So there are two vertices $x \in \{p,q\}$ and $y \in T$ such that $x, y \in A_{new}$ or $x, y \in B_{new}$.

Suppose first that $x, y \in B_{new}$. There must be a hyperedge in $\mathcal{H}$ containing both x and y. However, there is no hyperedge in $\mathcal{H}$ containing a vertex of B_i and a vertex of B_j with $i \neq j$, so x and y are both contained in some B_i. Every vertex of $\{p,q\} \cup T$ must be contained in a hyperedge with x or y, thus each vertex of $\{p,q\} \cup T$ must be in B_i or A_{new}. As the size of B_i is

$$\left\lceil \frac{r}{2} \right\rceil < |\{p,q\} \cup T| = t + 2$$

by assumption, there must be at least one vertex $z \in \{p,q\} \cup T$ in A_{new}. There is exactly one hyperedge of $\mathcal{H}$ that contains z and any other vertex of $B_i \cup A_{new}$. However, the degree of z in the Berge-$K_{2,t}$ is at least 2, a contradiction.

If $x, y \in A_{new}$ then we can again get a contradiction by the same reasoning as above. Therefore, $\mathcal{H}$ is Berge-$K_{2,t}$-free.

Now we prove (b):

In a $K_{1,t}$-free graph, since the degree of any vertex is at most $t-1$, there are at most

$$\binom{t-1}{r-2}$$

cliques of size $r-1$ containing any vertex. Thus we get the following.

$$\mathrm{ex}(n, K_{r-1}, K_{1,t}) \leq \frac{n}{r-1}\binom{t-1}{r-2} = \frac{n}{t}\binom{t}{r-1}.$$

Therefore in Theorem 12.8, we can choose $i = 1$, $F' = K_{1,t}$ and $c = \frac{1}{t}\binom{t}{r-1}$. To apply Theorem 12.8 part (a) or (b), we need to compare

$$c = \frac{1}{t}\binom{t}{r-1} \quad \text{and} \quad \frac{rg(r)}{2}.$$

If the first one is larger, then we get

$$\mathrm{ex}_r(n, K_{2,t}) \leq \frac{2}{r \cdot t}\binom{t}{r-1} \cdot \mathrm{ex}(n, K_{2,t}),$$

and by Theorem 4.3 we are done. If the second one is larger, we get

$$\mathrm{ex}_r(n, K_{2,t}) \leq g(r) \cdot \mathrm{ex}(n, K_{2,t}),$$

and we are again done by Theorem 4.3. □

12.4 Proofs of the results about r-uniform linear Berge-$K_{2,t}$-free hypergraphs

12.4.1 Proof of Theorem 12.9

Let $\mathcal{H}$ be an r-uniform linear hypergraph containing no Berge-$K_{2,t}$.

First let us fix $v \in V(\mathcal{H})$. Let the first neighborhood and second neighborhood of v in $\mathcal{H}$ be defined as

$$N_1^{\mathcal{H}}(v) := \{x \in V(\mathcal{H}) \setminus \{v\} \mid \exists h \in E(\mathcal{H}) \text{ such that } v, x \in h\}, \text{ and}$$

$$N_2^{\mathcal{H}}(v) := \{x \in V(\mathcal{H}) \setminus (N_1^{\mathcal{H}}(v) \cup \{v\}) \mid \exists h \in E(\mathcal{H}) \text{ such that } x \in h \text{ and } h \cap N_1^{\mathcal{H}}(v) \neq \emptyset\},$$

respectively.

Claim 12.12. *For any $u \in N_1^{\mathcal{H}}(v)$, the number of hyperedges $h \in E(\mathcal{H})$ containing u such that*

$$\left|h \cap N_1^{\mathcal{H}}(v)\right| \geq 2$$

is at most $(r-1)(t-1)+1$ ($\leq (r-1)t$).

Proof. Suppose for a contradiction that there is a vertex $u \in N_1^{\mathcal{H}}(v)$ which is contained in $(r-1)(t-1)+2$ hyperedges h such that $\left|h \cap N_1^{\mathcal{H}}(v)\right| \geq 2$. At most one of them contains v because $\mathcal{H}$ is linear.

From each of the $(r-1)(t-1)+1$ other hyperedges h_i ($1 \leq i \leq (r-1)(t-1)+1$), that do not contain v, we select exactly one pair $uy_i \subset h_i$ arbitrarily. These pairs are distinct since H is linear. Then (by pigeonhole principle) it is easy to see that there exist t distinct vertices

$$p_1, p_2, \dots, p_t \in \{y_1, y_2, \dots, y_{(r-1)(t-1)+1}\}$$

and t distinct hyperedges $f_1, f_2, \dots, f_t$ containing v such that $p_i \in f_i$. The t hyperedges containing the pairs up_i and the t hyperedges f_i ($1 \leq i \leq t$), form a Berge-$K_{2,t}$ in $\mathcal{H}$, a contradiction. □

For each $u \in N_1^{\mathcal{H}}(v)$, let

$$E_u := \{h \in E(\mathcal{H}) \mid h \cap N_1^{\mathcal{H}}(v) = \{u\}\}, \text{ and}$$

$$V_u := \{w \in N_2^{\mathcal{H}}(v) \mid \exists h \in E_u \text{ with } w \in h\}.$$

Notice that by Claim 12.12 we have

$$|E_u| \geq d(u) - (r-1)t \quad \text{and} \quad |V_u| = (r-1)|E_u|$$

(except for $r = 2$, in which case $|E_u| \geq d(u) - t + 1$ and $|V_u| = |E_u| - 1$, because the edge $vu \in E_u$. However, inequality (12.1) will still hold).

Therefore,

$$|V_u| \geq (r-1)d(u) - (r-1)^2 t. \tag{12.1}$$

Let the hyperedges incident to v be $e_1^v, e_2^v, \dots, e_{d(v)}^v$, where

$$e_i^v =: \{v, u_{1,i}, u_{2,i}, \dots, u_{r-1,i}\}$$

for $1 \leq i \leq d(v)$, and let us define the sets

$$V_i := \cup_{j=1}^{r-1} V_{u_{j,i}}$$

for each $1 \leq i \leq d(v)$.

Claim 12.13. *For each $1 \leq i \leq d(v)$, we have*

$$|V_i| \geq \sum_{j=1}^{r-1} \left|V_{u_{j,i}}\right| - \binom{r-1}{2}(2rt).$$

Proof. Note that

$$|V_i| = \left|\cup_{j=1}^{r-1} V_{u_{j,i}}\right| \geq \sum_{j=1}^{r-1} \left|V_{u_{j,i}}\right| - \sum_{1 \leq p < q \leq r-1} \left|V_{u_{p,i}} \cap V_{u_{q,i}}\right|. \tag{12.2}$$

First we will show that

$$\left|V_{u_{p,i}} \cap V_{u_{q,i}}\right| \leq (2r-3)t - 1.$$

Suppose by contradiction that $\left|V_{u_{p,i}} \cap V_{u_{q,i}}\right| \geq (2r-3)t$. We will construct an auxiliary graph G whose vertex set is $V_{u_{p,i}} \cap V_{u_{q,i}}$ and whose edge set is the union of the two sets

$$\{xy \mid xy \subset h \text{ for some } h \in E_{u_{p,i}} \text{ and } x, y \in V_{u_{p,i}} \cap V_{u_{q,i}}\}, \text{ and}$$

$$\{xy \mid xy \subset h \text{ for some } h \in E_{u_{q,i}} \text{ and } x, y \in V_{u_{p,i}} \cap V_{u_{q,i}}\}.$$

It is easy to see that each set consists of pairwise vertex disjoint cliques of size at most $r-1$. Therefore, the maximum degree in G is at most $2(r-2)$, so it has chromatic number at most $2r-3$, which implies that it has an independent set I of size at least

$$\frac{|V(G)|}{2r-3} = \frac{\left|V_{u_{p,i}} \cap V_{u_{q,i}}\right|}{2r-3} \geq t.$$

Let $x_1, x_2, \ldots, x_t \in I$ be distinct vertices. Then consider the set of hyperedges containing the pairs

$$u_{p,i}x_1, u_{p,i}x_2, \ldots, u_{p,i}x_t$$

and the set of hyperedges containing the pairs

$$u_{q,i}x_1, u_{q,i}x_2, \ldots, u_{q,i}x_t.$$

The hyperedges in the first set are different from each other since $x_1, x_2, \ldots, x_t$ is an independent set. Similarly, the hyperedges in the second set are different from each other. A hyperedge of the first set and a hyperedge of the second set can not be same since that would imply that there is a hyperedge in $E_{u_{p,i}}$ containing the pair $u_{p,i}u_{q,i}$. However, this is impossible since such a hyperedge contains exactly one vertex from $N_1^{\mathcal{H}}(v)$ by definition. So all the hyperedges are different and they form a Berge-$K_{2,t}$, a contradiction. Therefore, we have

$$\left|V_{u_{p,i}} \cap V_{u_{q,i}}\right| \leq (2r-3)t - 1 < 2rt.$$

Using this upper bound in (12.2), we get

$$|V_i| \geq \sum_{j=1}^{r-1} \left|V_{u_{j,i}}\right| - \sum_{1 \leq p < q \leq r-1} \left|V_{u_{p,i}} \cap V_{u_{q,i}}\right| \geq \sum_{j=1}^{r-1} \left|V_{u_{j,i}}\right| - \binom{r-1}{2}(2rt),$$

completing the proof of the claim. □

Claim 12.14. *We have*

$$\sum_{i=1}^{d(v)} |V_i| \leq (t-1)n.$$

Proof. It suffices to show that a vertex $x \in V(\mathcal{H})$ belongs to at most $t-1$ of the sets V_i for any $1 \leq i \leq d(v)$. Suppose for a contradiction that there is a vertex x that is contained in t sets $V_{i_1}, V_{i_2}, \dots, V_{i_t}$ for some distinct $i_1, i_2, \dots, i_t \in \{1, 2, \dots, d(v)\}$. For notational simplicity, we may assume that $i_1 = 1$, $i_2 = 2$, . . . , and $i_t = t$. This means that there are t hyperedges $h_1, h_2, \dots, h_t$ containing the pairs $xz_1, xz_2, \dots, xz_t$, respectively, where

$$z_j \in e_j^v \setminus \{v\} = \{u_{1,j}, u_{2,j}, \dots, u_{r-1,j}\}$$

for $1 \leq j \leq t$. The hyperedges $h_1, h_2, \dots, h_t$ are distinct since they contain exactly one vertex from $N_1^{\mathcal{H}}(v)$. Moreover, the t hyperedges e_j^v for $1 \leq j \leq t$ are distinct from $h_1, h_2, \dots, h_t$ as a hyperedge in the former set contains v but not x and a hyperedge in the latter set contains x but not v. Therefore, these $2t$ hyperedges form a Berge-$K_{2,t}$, a contradiction. □

Lemma 12.15. *For any $v \in V(\mathcal{H})$, we have*

$$\sum_{u \in N_1^{\mathcal{H}}(v)} (r-1)d(u) \leq (t-1)n + (r-1)(2r^2 - 4r + 1)td(v).$$

Proof. By (12.1),

$$\begin{aligned} \sum_{u \in N_1^{\mathcal{H}}(v)} (r-1)d(u) \leq \sum_{u \in N_1^{\mathcal{H}}(v)} (|V_u| + (r-1)^2 t) = \\ (r-1)^2 t \cdot (r-1)d(v) + \sum_{u \in N_1^{\mathcal{H}}(v)} |V_u|. \end{aligned} \tag{12.3}$$

Moreover, by Claim 12.13,

$$\sum_{u \in N_1^{\mathcal{H}}(v)} |V_u| = \sum_{i=1}^{d(v)} \sum_{j=1}^{r-1} \left|V_{u_{j,i}}\right| \leq \sum_{i=1}^{d(v)} (|V_i| + \binom{r-1}{2}(2rt)),$$

and so using Claim 12.14 we have

$$\sum_{u \in N_1^{\mathcal{H}}(v)} |V_u| \leq (t-1)n + \binom{r-1}{2}(2rt)d(v). \tag{12.4}$$

Combining (12.3) and (12.4), we get

$$\sum_{u \in N_1^{\mathcal{H}}(v)} (r-1)d(u) \leq (t-1)n + \left(\binom{r-1}{2}(2rt) + (r-1)^2 t \cdot (r-1)\right) d(v).$$

Simplifying, we get

$$\sum_{u\in N_1^{\mathcal{H}}(v)} (r-1)d(u) \le (t-1)n + (r-1)(2r^2-4r+1)td(v),$$

as desired. □

On the one hand, if d denotes the average degree in $\mathcal{H}$, by Lemma 12.15 we have

$$\sum_{v\in V(H)} \sum_{u\in N_1^{\mathcal{H}}(v)} (r-1)d(u) \le (t-1)n^2 + (r-1)(2r^2-4r+1)tnd.$$

On the other hand,

$$\sum_{v\in V(\mathcal{H})} \sum_{u\in N_1^{\mathcal{H}}(v)} (r-1)d(u) = \sum_{u\in V(\mathcal{H})} (r-1)d(u)\cdot(r-1)d(u) =$$

$$\sum_{u\in V(\mathcal{H})} (r-1)^2 d(u)^2 \ge (r-1)^2 nd^2.$$

Here the first equality follows by taking an arbitrary vertex u and counting how many times it appears in the first neighborhood of a vertex v, while the last inequality follows from the Cauchy-Schwarz inequality. So combining, we get

$$(r-1)^2 nd^2 \le (t-1)n^2 + (r-1)(2r^2-4r+1)tnd.$$

That is,

$$(r-1)^2 d^2 \le (t-1)n + (r-1)(2r^2-4r+1)td.$$

Rearranging, we get

$$((r-1)d - c_1(r,t))^2 \le (t-1)n + c_2(r,t),$$

where

$$c_1(r,t) := \frac{(2r^2-4r+1)t}{2} \quad \text{and} \quad c_2(r,t) := (c_1(r,t))^2.$$

This gives that

$$d \le \frac{\sqrt{n(t-1)}}{r-1}\left(1 + O\left(\frac{1}{n}\right)\right) + c_3(r,t)$$

for some constant $c_3(r,t)$ depending only on r and t. So the number of hyperedges in $\mathcal{H}$ is

$$\frac{nd}{r} \le \frac{n^{3/2}\sqrt{t-1}}{r(r-1)} + O(n),$$

completing the proof of our theorem. □

12.4.2 Proofs of Theorem 12.10 and 12.11

In order to prove both theorems, we take the $K_{2,t}$-free graph G constructed by Füredi [48] (which is used to prove the lower bound in Theorem 4.3), and replace its triangles by hyperedges as usual. However, the resulting hypergraph is far from linear, so our main idea is to delete some hyperedges in it to get a linear hypergraph. The graph G contains many triangles and this number is calculated by Alon and Shikhelman to prove their lower bound in Theorem 12.1. In our proofs of both theorems (Theorem 12.10 and 12.11) we do not need many specific properties of G. In the proof of Theorem 12.10 we use that it is $K_{2,t}$-free and contains

$$(1+o(1))\frac{1}{6}(t-1)^{3/2}n^{3/2}$$

triangles. In the proof of Theorem 12.11 we also use that it contains

$$(1+o(1))\frac{1}{2}(t-1)^{1/2}n^{3/2}$$

edges and all but $o(n^{3/2})$ edges are contained in $t-1$ triangles, while the remaining edges are contained in $t-2$ triangles. One can easily check these well-known properties of Füredi's construction [48], so we omit the proofs of these properties.

To conclude the proof of Theorem 12.10, we construct an auxiliary graph G'. Its vertices are the triangles of G, and two vertices of G' are connected by an edge if the corresponding triangles in G share an edge. Obviously, we want to find a large independent set in G'. A theorem of Fajtlowicz states the following.

Theorem 12.12 ([47]). *Any graph F contains an independent set of size at least*

$$\frac{2|V(F)|}{\Delta(F)+\omega(F)+1},$$

where $\Delta(F)$ and $\omega(F)$ denotes the maximal degree and the size of the maximal clique of F, respectively.

Clearly we have $\Delta(G') \leq 3(t-2) = 3t-6$ since each of the three edges of a triangle in G is contained in at most $t-2$ other triangles. Now notice that if a set of triangles of G pairwise intersect in two vertices then they either share a common edge or they are all contained in a K_4. In both cases, it is easy to see that $\omega(G') \leq t+1$. Substituting these bounds in Theorem 12.12 and using that $|V(G')| = (1+o(1))\frac{1}{6}(t-1)^{3/2}n^{3/2}$ completes the proof of Theorem 12.10.

□

To prove Theorem 12.11, we define an auxiliary hypergraph $\mathcal{H}$ to be the 3-uniform hypergraph whose vertices are the edges of G, and three vertices e_1, e_2 and e_3 form a hyperedge in $\mathcal{H}$ if there is a triangle in G whose edges are e_1, e_2 and e_3. Then $\mathcal{H}$ is linear since given any two edges of G, there is at most one triangle in G that contains both of them. Further, $\mathcal{H}$ is 3-uniform and all but $o(n^{3/2})$ vertices in $\mathcal{H}$ have degree $t-1$, while the rest have degree $t-2$. It is easy to see that we can construct another hypergraph $\mathcal{H}'$ by adding a set X of $o(n^{3/2})$ vertices to the vertex set of $\mathcal{H}$, such that $\mathcal{H}'$ is linear, 3-uniform and $(t-1)$-regular.

We will use the following special case of a theorem of Alon, Kim and Spencer [3].

Theorem 12.13 ([3])**.** *Let $\mathcal{H}'$ be a linear, 3-uniform, $(t-1)$-regular hypergraph on N vertices. Then there exists a matching M in $\mathcal{H}'$ covering at least*

$$N - \frac{c_0 N \ln^{3/2}(t-1)}{\sqrt{t-1}}$$

vertices, where c_0 is an absolute constant.

Note that $\mathcal{H}$ has

$$(1+o(1))\frac{1}{2}(t-1)^{1/2}n^{3/2}$$

vertices, thus the number of vertices in $\mathcal{H}'$ is

$$N = (1+o(1))\frac{1}{2}(t-1)^{1/2}n^{3/2} + o(n^{3/2}).$$

Applying Theorem 12.13 we get a matching M in $\mathcal{H}'$. We delete at most $o(n^{3/2})$ hyperedges of M that contain a vertex from X. This way we get a matching M' in $\mathcal{H}$ that covers all but

$$\frac{c_0 N \ln^{3/2}(t-1)}{\sqrt{t-1}} + o(n^{3/2})$$

vertices of $\mathcal{H}$. This implies,

$$|M'| \geq \left(1 - \frac{c_0}{\sqrt{t-1}} \ln^{3/2}(t-1)\right) \frac{\sqrt{t-1}}{6} n^{3/2} + o(n^{3/2}),$$

Finally, $\mathrm{ex}_3(n, \{C_2, K_{2,t}\}) \geq |M'|$ – indeed, by definition, M' corresponds to a set of triangles in G such that no two of them share an edge. So replacing them by hyperedges we get a 3-uniform Berge-$K_{2,t}$-free linear hypergraph with $|M'|$ hyperedges, as desired. Note that the lower bound in Theorem 12.11 does not have the additive term $o(n^{3/2})$ because we can choose c in Theorem 12.11 to be large enough (compared to c_0) so that the right hand side of the above inequality is at least the bound mentioned in our theorem. □

12.5 Remarks

We finish this chapter with some questions and remarks concerning our results.

• In Corollary 12.2 we provided an asymptotics for $\mathrm{ex}_3(n, K_{2,t})$ for $t \geq 7$. It would be interesting to determine the asymptotics in the remaining cases. We conjecture the following.

Conjecture 12.16. For $t = 3, 4, 5, 6$, we have

$$\mathrm{ex}_3(n, K_{2,t}) = (1+o(1))\frac{1}{6}(t-1)^{3/2}n^{3/2}.$$

• In Theorem 12.9 and Theorem 12.11 we showed that the asymptotics of $\mathrm{ex}_3(n, \{C_2, K_{2,t}\})$ is close to being sharp for large enough t. However, it would be interesting to determine the asymptotics for all $t \geq 3$.

• In Theorem 12.8, we studied a class of r-uniform Berge-F-free hypergraphs. It would be interesting to extend these results to a larger class of hypergraphs. Similarly, it would be interesting to see if our results in the linear case (in Section 12.2.2) can be extended.

• Finally we note that there is a correspondence between Turán-type questions for Berge hypergraphs and forbidden submatrix problems (for an updated survey of the latter topic see [6]). For more information about this correspondence, see [7], where they prove results about forbidding small hypergraphs in the Berge sense and they are mostly interested in the order of magnitude. Very recently, similar research was carried out in [136] and also see the references therein. We note that our results provide improvements of some special cases of Theorem 5.8. in [136].

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Appendix A

Proof of Lemma 3.12

Points 1 and 5 are easy to see.

Proof of Point 2. It is also easy to check that f is continuous at the points $x = \frac{1}{2}$, $c = 4\left(x - x^2\right)^2$, $c = \frac{1}{4}$, and that the function is monotonously decreasing in x and increasing in c in each range.

$f(x,0) = 1 - x$; $x^2 - 2x + 1 - c + \sqrt{c}$ is a concave and monotonously increasing expression in c. When $0 < x \le \frac{1}{2}$, $c \mapsto 1 - x + \left(\frac{1}{4(x-x^2)} - 1\right) c$ is the tangential line of the graph of $c \mapsto x^2 - 2x + 1 - c + \sqrt{c}$ at the point $c = 4\left(x - x^2\right)^2$, since both their values, and their derivatives at this point coincide. So f is concave in c. $\square$

Since the graph of a concave function is below the tangent line at any point, we also have that for $x \in \left[0, \frac{1}{2}\right], c \in \left[0, \frac{1}{4}\right]$,

$$f(x,c) \ge x^2 - 2x + 1 - c + \sqrt{c} =: \tilde{f}(x,c). \tag{A.1}$$

We will use this inequality in the proof of Point 3 and 4.

Proof of Point 3. If $x + a = 0$, $g(x,c,a,\tilde{a}) = f(x,c)$. From now on, we assume that $x + a > 0$.

We first show that g is monotonously increasing in $\tilde{a}$.

$$\begin{aligned}\left(\frac{\partial}{\partial c} f\right)(x,c) &= \left(\begin{cases} \frac{1}{4(x-x^2)} - 1 & \text{if } x \le \frac{1}{2} \text{ and } c < 4\left(x-x^2\right)^2 \\ -1 + \frac{1}{2\sqrt{c}} & \text{if } x \le \frac{1}{2} \text{ and } 4\left(x-x^2\right)^2 \le c \le \frac{1}{4} \\ 0 & \text{if } \frac{1}{4} \le c \text{ or } \frac{1}{2} \le x \end{cases}\right) \\ &\le \begin{cases} \frac{1}{4(x-x^2)} - 1 & \text{if } x \le \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le x \end{cases}\end{aligned} \tag{A.2}$$

So,

$$\frac{\partial}{\partial \tilde{a}} g(x,c,a,\tilde{a}) = 2(1-x-a)$$
$$+(1-x-a)\cdot\left(\frac{\partial}{\partial c} f\right)\left(x+a, \frac{c-\tilde{a}(x+a)}{1-x-a}\right)\cdot\frac{\partial}{\partial \tilde{a}}\left(\frac{c-\tilde{a}(x+a)}{1-x-a}\right)$$
$$\geq 2(1-x-a)-(x+a)\left(\begin{cases}\frac{1}{4((x+a)-(x+a)^2)}-1 & \text{if } x+a\leq\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}\leq x+a\end{cases}\right)$$
$$\geq 2(1-x-a)-(x+a)\left(\begin{cases}\frac{1}{4\cdot\frac{1}{2}(x+a)}-1 & \text{if } x+a\leq\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}\leq x+a\end{cases}\right)$$
$$\geq 2(1-x-a)-\left(\begin{cases}\frac{1}{2} & \text{if } x+a\leq\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}\leq x+a\end{cases}\right)\geq 0.$$

Therefore, from now on we assume $\tilde{a}=\min\left(a,\frac{c}{x+a}\right)$ since if $g(x,c,a,\tilde{a})\leq f(x,c)$ holds for $\tilde{a}=\min\left(a,\frac{c}{x+a}\right)$ then it also holds for any $\tilde{a}\in\left[0,\min\left(a,\frac{c}{x+a}\right)\right]$.

Case 1. First assume $a\leq\frac{c}{x+a}$ (so $\tilde{a}=a$), which is equivalent to $a(x+a)\leq c$ or $a\leq\frac{-x+\sqrt{x^2+4c}}{2}$. Let $x'=x+a$ and $c'=\frac{c-a(x+a)}{1-x-a}$.

Case 1.1. When $x'\leq\frac{1}{2}$ and $4\left(x'-x'^2\right)^2\leq c'$, we bound g from above:

$$g(x,c,a,a)=a+(1-x-a)f\left(x+a,\frac{c-a(x+a)}{1-x-a}\right)+2a(1-x-a)$$
$$\leq a+(1-x-a)f\left(x+a,\frac{c}{1-x-a}\right)+2a(1-x-a)=:\tilde{g}(x,c,a,a).$$

We now consider subcases based on the values of c and $\frac{c}{1-x-a}$ compared to $\frac{1}{4}$. Note that $4\left(x'-x'^2\right)^2\leq c'\leq\frac{c}{1-x-a}$.

Case 1.1.1. When $c\leq\frac{c}{1-x-a}\leq\frac{1}{4}$, using (A.1),

$$g(x,c,a,a)-f(x,c)\leq\tilde{g}(x,c,a,a)-\tilde{f}(x,c)=a+(1-x-a)\Bigg((x+a)^2-2(x+a)+1$$
$$-\frac{c}{1-x-a}+\frac{\sqrt{c}}{\sqrt{1-x-a}}\Bigg)+2a(1-x-a)-\left(x^2-2x+1-c+\sqrt{c}\right)$$
$$=-x(1-x-2a)+(1-x-a)(x+a)^2+\left(\sqrt{1-x'}-1\right)\sqrt{c}. \tag{A.3}$$

Thus,

$$\frac{\partial}{\partial c}\left(\tilde{g}(x,c,a,a)-\tilde{f}(x,c)\right)=\frac{\sqrt{1-x-a}-1}{2\sqrt{c}}\leq 0.$$

So it is enough to check that $\tilde{g}(x,c,a,a)-\tilde{f}(x,c)\leq 0$ when $c'=4\left(x'-x'^2\right)^2$ or, equivalently, when $c=4\left(x'-x'^2\right)^2(1-x')+a(x+a)$; then it is also ≤ 0 for bigger c. First some auxiliary calculations:

$$\sqrt{1-x'}-1\leq\sqrt{1-x'+\frac{x'^2}{4}}-1=-\frac{x'}{2}\leq 0. \tag{A.4}$$

$$4(x'-x'^2)^2 \le 4x'^2 \quad \text{and} \quad 4(x'-x'^2)^2 \le 4\cdot\left(\frac{1}{4}\right)^2 = \frac{1}{4}, \quad \text{so}$$

$$4(x'-x'^2)^2 \le \min\left(4x'^2, \frac{1}{4}\right) \le x'. \tag{A.5}$$

So,

$$\begin{aligned} 4(x'-x'^2)^2(1-x') + a(x+a) &\ge 4(x'-x'^2)^2(1-x') + 4(x'-x'^2)^2 a \\ &= \Big(2(x'-x'^2)\Big)^2(1-x) \ge \Big(2(x'-x'^2)\Big)^2(1-x)^2. \end{aligned} \tag{A.6}$$

Putting $c = 4(x'-x'^2)^2(1-x') + a(x+a)$ in (A.3) and then using (A.4) and (A.6),

$$\begin{aligned} \tilde{g}(x,c,a,a) - \tilde{f}(x,c) &= -x(1-x-2a) + (1-x-a)(x+a)^2 \\ &\quad + \left(\sqrt{1-x'}-1\right)\sqrt{4(x'-x'^2)^2(1-x') + a(x+a)} \\ &\le -x(1-x-2a) + (1-x-a)(x+a)^2 - x'(x'-x'^2)(1-x) \\ &= -x\left[(1-x-2a) - (1-x-a)(x+a)^2\right] \\ &\le -x\left[(1-x-2a) - (1-x-a)\frac{1}{4}\right] \\ &= -x\left(\frac{3-3x-7a}{4}\right), \end{aligned}$$

which is ≤ 0 when $a \le \frac{3-3x-3a}{4} = \frac{3-3x'}{4}$. Assume $a > \frac{3-3x'}{4}$. Since $x' \ge a$, $x' > \frac{3}{7}$ (and we have also assumed $\frac{1}{2} \ge x'$); and

$$\begin{aligned} c = 4(x'-x'^2)^2(1-x') + a(x+a) &> 4(x'-x'^2)^2(1-x') + \frac{(3-3x')x'}{4} \\ > 4\cdot\left(\frac{3}{7}-\left(\frac{3}{7}\right)^2\right)^2\left(1-\frac{1}{2}\right) + \frac{\left(3-3\cdot\frac{1}{2}\right)\cdot\frac{3}{7}}{4} &= \frac{5391}{19208} > \frac{1}{4} \end{aligned}$$

contrary to our assumption that $c \le \frac{c}{1-x-a} \le \frac{1}{4}$.

Case 1.1.2. When $c \le \frac{1}{4} < \frac{c}{1-x-a}$ (and recall $4(x'-x'^2)^2(1-x') + a(x+a) \le c$), using (A.1),

$$\begin{aligned} g(x,c,a,a) - f(x,c) &\le \tilde{g}(x,c,a,a) - \tilde{f}(x,c) \\ &= a + (1-x-a)\big((x+a)^2 - 2(x+a) + 1.25\big) \\ &\quad + 2a(1-x-a) - \big(x^2 - 2x + 1 - c + \sqrt{c}\big) \\ &= -x(1-x-2a) + (1-x-a)\big((x+a)^2 + 0.25\big) + c - \sqrt{c}. \end{aligned} \tag{A.7}$$

Case 1.1.2.1. If $4(x'-x'^2)^2(1-x') + a(x+a) \le \frac{1}{4}(1-x-a)$ (which is $< c$), since $t-\sqrt{t}$ is decreasing in $0 \le t \le \frac{1}{4}$, replacing c by $\frac{1}{4}(1-x-a)$ in (A.7), we get

$$\begin{aligned} g(x,c,a,a) - f(x,c) &\le -x(1-x-2a) + (1-x-a)\big((x+a)^2+0.25\big) + \frac{1}{4}(1-x-a) \\ - \sqrt{\frac{1}{4}(1-x-a)} &= \tilde{g}\left(x, \frac{1}{4}(1-x-a), a, a\right) - \tilde{f}\left(x, \frac{1}{4}(1-x-a)\right) \le 0, \end{aligned}$$

as it falls in Case 1.1.1. above.

Case 1.1.2.2. If $\frac{1}{4}(1-x-a) \leq 4(x'-x'^2)^2(1-x') + a(x+a)$ (which is $\leq c$), again, by (A.7) we have,

$$\begin{aligned} g(x,c,a,a) - f(x,c) \leq & -x(1-x-2a) + (1-x-a)\left((x+a)^2+0.25\right) \\ & + 4(x'-x'^2)^2(1-x') + a(x+a) - \sqrt{4(x'-x'^2)^2(1-x') + a(x+a)}. \end{aligned} \tag{A.8}$$

Let $b = \max\left(4(x'-x'^2)^2(1-x') + x'^2 - \frac{1}{4}, 0\right)$. Now some auxiliary calculations follow. Since $\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{t} = \frac{1}{2\sqrt{t}}$, and by (A.6),

$$\begin{aligned} & \sqrt{4(x'-x'^2)^2(1-x') + x'^2 - b} - \sqrt{4(x'-x'^2)^2(1-x') + a(x+a)} \\ & \leq \frac{1}{2\sqrt{4(x'-x'^2)^2(1-x') + a(x+a)}}(x(x+a)-b) \\ & \leq \frac{1}{4\left(x'-x'^2\right)(1-x)}(x(x+a)-b). \end{aligned} \tag{A.9}$$

$$(1-a)\cdot 4\left(x'-x'^2\right)(1-x) \geq 4\left(x'-x'^2\right)(1-x') = 4(1-x')^2x' \geq 4\cdot\left(\frac{1}{2}\right)^2 x' = x+a.$$

So,

$$\begin{aligned} & -x(1-x-2a) + \left(\frac{1}{4\left(x'-x'^2\right)(1-x)} - 1\right)(x(x+a)-b) \\ & \leq -x(1-x-2a) + \left(\frac{1-a}{x+a} - 1\right)x(x+a) = 0. \end{aligned} \tag{A.10}$$

By (A.5), $4(x'-x'^2)^2 \leq x'$, so

$$4(x'-x'^2)^2(1-x') + x'^2 \geq 4(x'-x'^2)^2(1-x') + 4(x'-x'^2)^2x' = \left(2(x'-x'^2)\right)^2. \tag{A.11}$$

$$4(x'-x'^2)^2(1-x') + x'^2 - b \geq \left(2(x'-x'^2)\right)^2, \tag{A.12}$$

since if $b > 0$, $4(x'-x'^2)^2(1-x') + x'^2 - b = \frac{1}{4} \geq \left(2(\frac{1}{4} - (\frac{1}{2}-x')^2)\right)^2 = \left(2(x'-x'^2)\right)^2$ (if b = 0, then it holds by (A.11)). Now using (A.9) in (A.8), and then using (A.10) and (A.12) and that $t-\sqrt{t}$ is decreasing in $0 \leq t \leq 1/4$, we get

$$\begin{aligned} g(x,c,a,a) - f(x,c) \leq & (1-x-a)\left((x+a)^2+0.25\right) + 4(x'-x'^2)^2(1-x') + x'^2 - b \\ & - \sqrt{4(x'-x'^2)^2(1-x') + x'^2 - b} - x(1-x-2a) \\ & + \left(\frac{1}{4\left(x'-x'^2\right)(1-x)} - 1\right)(x(x+a)-b) \\ \leq & (1-x-a)\left((x+a)^2+0.25\right) + \left(2(x'-x'^2)\right)^2 - 2(x'-x'^2) \\ = & 4\left(x'-\frac{1}{4}\right)\left(x'-\frac{1}{2}\right)^2(x'-1), \end{aligned}$$

which is ≤ 0 when $\frac{1}{4} \leq x' \leq 1$. When $x' < \frac{1}{4}$, we show that this subcase cannot hold:

$$\begin{aligned} &4(x' - x'^2)^2 (1-x') + a(x+a) - \frac{1}{4}(1-x-a) \\ &< \left(4 \cdot \left(\frac{1}{4} - \left(\frac{1}{4}\right)^2\right)^2 - \frac{1}{4}\right)(1-x') + x'^2 \qquad \text{(A.13)} \\ = -\frac{7}{64}(1-x') + x'^2 &< -\frac{1}{12}(1-x') + x'^2 = \left(x' - \frac{1}{4}\right)\left(x' + \frac{1}{3}\right) < 0. \end{aligned}$$

Case 1.1.3. When $\frac{1}{4} \leq c$ (which is $< \frac{c}{1-x-a}$),

$$\begin{aligned} g(x,c,a,a) - f(x,c) &\leq \tilde{g}(x,c,a,a) - f(x,c) \\ &= a + (1-x-a)\big((x+a)^2 - 2(x+a) + 1.25\big) \\ &+ 2a(1-x-a) - \big(x^2 - 2x + 1.25\big) = \tilde{g}\left(x, \frac{1}{4}, a, a\right) - \tilde{f}\left(x, \frac{1}{4}\right) \leq 0, \end{aligned}$$

as it falls in Case 1.1.1. or 1.1.2. above.
Case 1.2. When $x' \leq \frac{1}{2}$ and $c' \leq 4(x' - x'^2)^2$,

$$g(x,c,a,a) = a + (1-x-a)\left(1 - x' + \left(\frac{1}{4\left(x' - x'^2\right)} - 1\right)c'\right) + 2a(1-x-a),$$

which is linear in c', so also in c. $f(x,c)$ is concave in c, so it is enough to check that g is smaller than f in the ends of the interval $c \in \Big[a(a+x), 4(x'-x'^2)^2(1-x') + a(x+a)\Big]$:

$$\begin{aligned} g(x, a(a+x), a, a) &= a + (1-x-a)^2 + 2a(1-x-a) \\ &\leq a + (1-x-a)^2 + 2a(1-x-a) + \frac{2ax\left(\frac{1}{2} - x - a + \left(1 - \frac{a}{2}\right)x\right)}{\sqrt{a(x+a)} + ax + a} \\ &= x^2 - 2x + 1 - a(x+a) + \sqrt{a(x+a)} \leq f(x, a(x+a)), \end{aligned}$$

since $\frac{1}{2} - x - a \geq 0$ and $a(x+a) \leq x'^2 \leq \frac{1}{4}$, and using (A.1). Whereas the higher end of the interval was handled above in Case 1.1. since f is continuous.
Case 1.3. Finally, when $\frac{1}{2} \leq x'$,

$$g(x,c,a,a) = a + (1-x-a)^2 + 2a(1-x-a) = x^2 - 2x + 1 - a^2 + a.$$

If $x \leq \frac{1}{2}$, let $\tilde{c} = \min\big(c, \frac{1}{4}\big)$. Since $c \geq a(x+a) \geq a^2$, and by (A.1)

$$g(x,c,a,a) = x^2 - 2x + 1 - a^2 + a \leq x^2 - 2x + 1 - \tilde{c} + \sqrt{\tilde{c}} \leq f(x,c).$$

If $\frac{1}{2} \leq x$, then $a \leq 1 - x \leq \frac{1}{2}$. $-a^2 + a$ is monotonously increasing in $a \in \big[0, \frac{1}{2}\big]$, so

$$g(x,c,a,a) = x^2 - 2x + 1 - a^2 + a \leq x^2 - 2x + 1 - (1-x)^2 + 1 - x = 1 - x \leq f(x,c),$$

by Point 5.

Case 2. Now consider $a \geq \frac{c}{x+a}$, that is, $a \geq \frac{-x+\sqrt{x^2+4c}}{2}$. Then $\tilde{a} = \frac{c}{x+a} \leq \frac{c}{x+\frac{-x+\sqrt{x^2+4c}}{2}} = \frac{-x+\sqrt{x^2+4c}}{2}$, and $\frac{c-\tilde{a}(x+a)}{1-x-a} = 0$, so $f\left(x+a, \frac{c-\tilde{a}(x+a)}{1-x-a}\right) = 1-x-a$.

$$g(x,c,a,\tilde{a}) = a+(1-x-a)^2+2\frac{c}{x+a}(1-x-a) \leq a+(1-x-a)^2+\left(-x+\sqrt{x^2+4c}\right)(1-x-a),$$

which is quadratic in a with a positive leading coefficient, so its maximum is at one end of the interval $\left[\frac{-x+\sqrt{x^2+4c}}{2}, 1-x\right]$. $a = \frac{-x+\sqrt{x^2+4c}}{2}$ (i.e., $a = \frac{c}{x+a} = \tilde{a}$) was handled above in Case 1. If $a = 1-x$, the right side of the inequality equals $1-x$ which is $\leq f(x,c)$ by Point 5. □

Proof of Point 4. If $x+a = 0$, $h(x,c,a,\tilde{a}) = f(x,c)$. From now on, we assume that $x+a > 0$.

We first show that h is monotonously increasing in $\tilde{a}$. Using (A.2) and a calculation similar to the one in the proof of Point 3, we have

$$\begin{aligned}\frac{\partial}{\partial \tilde{a}} h(x,c,a,\tilde{a}) &= 2(1-x-a) + (1-x-a)\cdot\left(\frac{\partial}{\partial c}f\right)\left(x+a, \frac{c-(x+\tilde{a})(x+a)}{1-x-a}\right) \\ &\qquad \cdot \frac{\partial}{\partial \tilde{a}}\left(\frac{c-(x+\tilde{a})(x+a)}{1-x-a}\right) \\ &\geq 2(1-x-a) - (x+a)\left(\begin{cases}\frac{1}{4((x+a)-(x+a)^2)} - 1 & \text{if } x+a \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x+a\end{cases}\right) \geq 0.\end{aligned}$$

Therefore, from now on we assume $\tilde{a} = \min\left(a, \frac{c}{x+a} - x\right)$.

Case 1. First assume $a \leq \frac{c}{x+a} - x$ (so $\tilde{a} = a$), which is equivalent to $(x+a)^2 \leq c$ or $a \leq \sqrt{c} - x$. Let $x' = x+a$ and $c' = \frac{c-(x+a)^2}{1-x-a}$.

Case 1.1. When $x' \leq \frac{1}{2}$ and $4\left(x'-x'^2\right)^2 \leq c'$, we bound h from above:

$$\begin{aligned}h(x,c,a,a) &= a + (1-x-a)f\left(x+a, \frac{c-(x+a)^2}{1-x-a}\right) + 2a(1-x-a) + x - 3x(x+a) \\ &\leq a + (1-x-a)f\left(x+a, \frac{c}{1-x-a}\right) + 2a(1-x-a) + x - 3x(x+a) =: \tilde{h}(x,c,a,a).\end{aligned}$$

We now consider subcases based on the values of c and $\frac{c}{1-x-a}$ compared to $\frac{1}{4}$. Note that $4\left(x'-x'^2\right)^2 \leq c' \leq \frac{c}{1-x-a}$.

Case 1.1.1. When $c \leq \frac{c}{1-x-a} \leq \frac{1}{4}$, using (A.1),

$$\begin{aligned}h(x,c,a,a) - f(x,c) \leq \tilde{h}(x,c,a,a) - \tilde{f}(x,c) &= a + (1-x-a)\Bigg((x+a)^2 - 2(x+a) + 1 \\ &- \frac{c}{1-x-a} + \frac{\sqrt{c}}{\sqrt{1-x-a}}\Bigg) + 2a(1-x-a) + x - 3x(x+a) - \left(x^2 - 2x + 1 - c + \sqrt{c}\right) \\ &= -2x^2 - xa + (1-x-a)(x+a)^2 + \left(\sqrt{1-x'} - 1\right)\sqrt{c}.\end{aligned} \tag{A.14}$$

$$\frac{\partial}{\partial c}\Big(\tilde{h}(x,c,a,a)-\tilde{f}(x,c)\Big)=\frac{\sqrt{1-x-a}-1}{2\sqrt{c}}\leq 0.$$

So it is enough to check that $\tilde{h}(x,c,a,a)-\tilde{f}(x,c)\leq 0$ when $c'=4(x'-x'^2)^2$ or, equivalently, when $c=4(x'-x'^2)^2(1-x')+x'^2$; then it is also ≤ 0 for bigger c. As seen in (A.11) and (A.4) in the proof of Point 3, $4(x'-x'^2)^2(1-x')+x'^2\geq \big(2(x'-x'^2)\big)^2$, and $\sqrt{1-x'}-1\leq -\frac{x'}{2}\leq 0$. Putting $c=4(x'-x'^2)^2(1-x')+x'^2$ in (A.14) and using these inequalities, we get

$$\begin{aligned}\tilde{h}(x,c,a,a)-\tilde{f}(x,c)&=-2x^2-xa+(1-x-a)(x+a)^2\\&\quad+\left(\sqrt{1-x'}-1\right)\sqrt{4(x'-x'^2)^2(1-x')+x'^2}\\&\leq -2x^2-xa+(1-x-a)(x+a)^2-x'(x'-x'^2)=-2x^2-xa\leq 0.\end{aligned}$$

Case 1.1.2. When $c\leq\frac{1}{4}<\frac{c}{1-x-a}$ (and recall $\big(2(x'-x'^2)\big)^2\leq 4(x'-x'^2)^2(1-x')+x'^2\leq c$), using (A.1),

$$\begin{aligned}h(x,c,a,a)-f(x,c)&\leq\tilde{h}(x,c,a,a)-\tilde{f}(x,c)\\&=a+(1-x-a)\big((x+a)^2-2(x+a)+1.25\big)\\&\quad+2a(1-x-a)+x-3x(x+a)-\big(x^2-2x+1-c+\sqrt{c}\big)\\&=-2x^2-xa+(1-x-a)\big((x+a)^2+0.25\big)+c-\sqrt{c}.\end{aligned}\tag{A.15}$$

Case 1.1.2.1. If $4(x'-x'^2)^2(1-x')+x'^2\leq\frac{1}{4}(1-x-a)$ (which is $<c$), since $t-\sqrt{t}$ is decreasing in $0\leq t\leq\frac{1}{4}$, replacing c by $\frac{1}{4}(1-x-a)$ in (A.15), we get

$$\begin{aligned}h(x,c,a,a)-f(x,c)&\leq -2x^2-xa+(1-x-a)\big((x+a)^2+0.25\big)+\frac{1}{4}(1-x-a)\\&\quad-\sqrt{\frac{1}{4}(1-x-a)}\\&=\tilde{h}\left(x,\frac{1}{4}(1-x-a),a,a\right)-\tilde{f}\left(x,\frac{1}{4}(1-x-a)\right)\leq 0,\end{aligned}$$

as it falls in Case 1.1.1. above.

Case 1.1.2.2. If $\frac{1}{4}(1-x-a)\leq 4(x'-x'^2)^2(1-x')+x'^2$ (which is $\leq c$), again, by (A.15) (and since $-2x^2-xa\leq 0$), we get

$$\begin{aligned}h(x,c,a,a)-f(x,c)&\leq(1-x-a)\big((x+a)^2+0.25\big)+\Big(2(x'-x'^2)\Big)^2-2(x'-x'^2)\\&=4\left(x'-\frac{1}{4}\right)\left(x'-\frac{1}{2}\right)^2(x'-1),\end{aligned}$$

which is ≤ 0 when $\frac{1}{4}\leq x'\leq 1$. When $x'<\frac{1}{4}$, we show that this subcase cannot hold:

$$4(x'-x'^2)^2(1-x')+x'^2-\frac{1}{4}(1-x-a)<\left(4\cdot\left(\frac{1}{4}-\left(\frac{1}{4}\right)^2\right)^2-\frac{1}{4}\right)(1-x')+x'^2<0,$$

like in (A.13) in the proof of Point 3.

Case 1.1.3. When $\frac{1}{4} \leq c$ (which is $< \frac{c}{1-x-a}$),

$$h(x,c,a,a) - f(x,c) \leq \tilde{h}(x,c,a,a) - f(x,c)$$
$$= a + (1-x-a)\left((x+a)^2 - 2(x+a) + 1.25\right)$$
$$+ 2a(1-x-a) + x - 3x(x+a) - \left(x^2 - 2x + 1.25\right) = \tilde{h}\left(x, \frac{1}{4}, a, a\right) - \tilde{f}\left(x, \frac{1}{4}\right) \leq 0,$$

as it falls in Case 1.1.1. or 1.1.2. above.

Case 1.2. When $x' \leq \frac{1}{2}$ and $c' \leq 4(x' - x'^2)^2$,

$$h(x,c,a,a) = a + (1-x-a)\left(1 - x' + \left(\frac{1}{4\left(x' - x'^2\right)} - 1\right) c'\right) + 2a(1-x-a) + x - 3x(x+a),$$

which is linear in c', so also in c. $f(x,c)$ is concave in c, so it is enough to check that h is smaller than f in the ends of the interval $c \in \left[(x+a)^2, 4(x' - x'^2)^2 (1 - x') + x'^2\right]$:

$$\begin{aligned} h\left(x, (x+a)^2, a, a\right) &= a + (1-x-a)^2 + 2a(1-x-a) + x - 3x(x+a) \\ &\leq a + (1-x-a)^2 + 2a(1-x-a) + x - 3x(x+a) + x(2x+a) \\ &= x^2 - 2x + 1 - (x+a)^2 + (x+a) \leq f\left(x, (x+a)^2\right) \end{aligned}$$

since $(x+a)^2 = x'^2 \leq \frac{1}{4}$, and using (A.1). Whereas the higher end of the interval was handled above in Case 1.1. since f is continuous.

Case 1.3. Finally, when $\frac{1}{2} \leq x'$, $c \geq (x+a)^2 \geq \frac{1}{4}$, so

$$h(x,c,a,a) = a + (1-x-a)^2 + 2a(1-x-a) + x - 3x(x+a) \leq x^2 - 2x + 1 - (x+a)^2 + (x+a).$$

If $x \leq \frac{1}{2}$, then

$$h(x,c,a,a) \leq x^2 - 2x + 1 - (x+a)^2 + (x+a) \leq x^2 - 2x + 1.25 = f(x,c).$$

If $\frac{1}{2} \leq x$, then $-(x+a)^2 + (x+a)$ is monotonously decreasing in a, so

$$h(x,c,a,a) \leq x^2 - 2x + 1 - (x+a)^2 + (x+a) \leq x^2 - 2x + 1 - x^2 + x = 1 - x = f(x,c).$$

Case 2. Now we consider $a \geq \frac{c}{x+a} - x$, that is, $a \geq \sqrt{c} - x$. Then $\tilde{a} = \min\left(a, \frac{c}{x+a} - x\right) = \frac{c}{x+a} - x \leq \sqrt{c} - x$, and $\frac{c-(x+\tilde{a})(x+a)}{1-x-a} = 0$, so $f\left(x+a, \frac{c-(x+\tilde{a})(x+a)}{1-x-a}\right) = 1 - x - a$.

$$\begin{aligned} h(x,c,a,\tilde{a}) &= a + (1-x-a)^2 + 2\left(\frac{c}{x+a} - x\right)(1-x-a) + x - 3x(x+a) \\ &\leq a + (1-x-a)^2 + 2\left(\sqrt{c} - x\right)(1-x-a) + x - 3x(x+a), \end{aligned}$$

which is quadratic in a with a positive leading coefficient, so its maximum is at one end of the interval $[\sqrt{c} - x, 1 - x]$. $a = \sqrt{c} - x$ (i.e., $a = \frac{c}{x+a} - x = \tilde{a}$) was handled above in Case 1. If $a = 1 - x$, the right side of the inequality equals $1 - 3x$ which is $\leq f(x,c)$ by Point 5. □