Topological classification of links associated with plane curve singularities

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Abstract

Consider an irreducible branch $\mathcal{B}$ of a curve germ $(\mathcal{C}, 0)$. We show that the knot $K_{\mathcal{B}}$ is completely determined by the set of Puiseux pairs. We describe the full topological classification of links in terms of the set of Puiseux pairs and linking numbers of knot components forming the link. The second aims of the thesis is to recover the Puiseux pairs by mean of resolution of the singularity. We show that for any curve $\mathcal{C}$ there exist a resolution such that the strict transform of $\mathcal{C}$ and exceptional curves E_i, which come with Euler numbers e_i, form a normal crossing divisor. We show that the set of Puiseux pairs determine the shape of the resolution dual graph together with the Euler numbers as decoration, and vice-versa.

Acknowledgements

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A sincere gratitude to my supervisor Prof. András Némethi for he has shared me his experience and knowledge.

I thank my family, and my best friend for their infinite love.

Ity boky ity dia atolotra ho an'i Dada.

Dia ho feheziko amin'ny teny roa:

Mankasitraka aho...

Contents

1. Introduction and Preliminaries

Suppose we have many ropes (could be of infinite length) and we drop them from the roof top of a tall building to a flat ground. Looking from above, we draw a picture of the ropes on a piece of paper and we call by *plane curve* the picture thus obtained. We could get many configuration of a plane curve. For example, we could see in Figure 1.2 the graph of the ropes when two sections of a rope overlap, or in Figure 1.1 we could see the graph of a rope where a section is folded. In the graph, these particular points, called *singular points*, are represented either by a self-intersection or by non-immersed points of the curve.

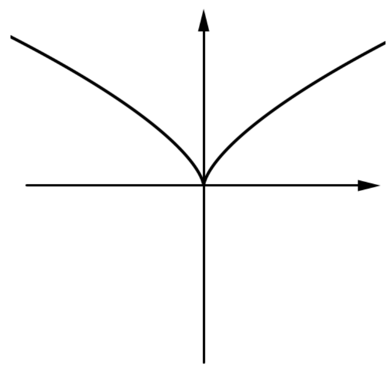


Figure 1.1: An ordinary cusp at the origin.

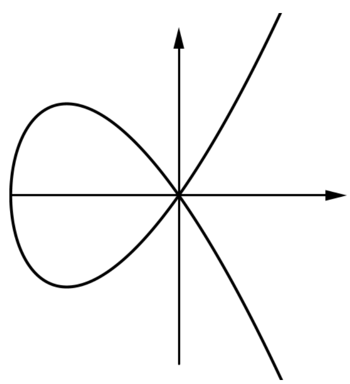


Figure 1.2: An ordinary node at the origin.

In the first part of this chapter, we introduce more precisely the notion of plane curve and singular points. We will restrict our study to *algebraic plane curves*, that is, curve, which could be represented in xy-coordinate system by a polynomial over $\mathbb{C}$. And we state some theorems about the local parametrization of an algebraic curve. In the second part we will discuss about a global visualization of a plane curve in $\mathbb{C}^2$.

In Chapter 2, we introduce a topological invariant associated with a plane curve singularity, called *link*. A link associated with a singularity of a local irreducible branch is completely determined by a set of sequence of pairs $(m_1, n_1), \ldots, (m_g, n_g)$ called *Puiseux pairs* which we could get from a particular parametrization of the curve called *Puiseux parametrization*. We show that we get a complete topological classification of the links associated with the irreducible singularities.

Now, consider the initial ropes before we dropped them as a curve in higher dimension, and the process of dropping the ropes by a projection to a plane. We get every plane algebraic curve over $\mathbb{C}$ by a projection $\pi : T \to \mathbb{C}^2$ of a curve in higher dimension. The reverse process of the projection is called *resolution of a singularity*. In Chapter 3, we show that for a curve $\mathcal{C} \subset \mathbb{C}^2$, there is a such projection such that the pre-image π^{-1} is the union of a curve which has no singular points, and a finite number of curve E_i, called *exceptional curve*, which come with an integer e_i called *Euler number*. Furthermore, all these curves form a normal crossing divisor. We show that the set of Puiseux pairs and the dual resolution graph decorated by the Euler numbers determine each other.

1.1 Affine algebraic plane curves

Consider a polynomial $f \in \mathbb{C}[X, Y]$, we denote the set of solutions of the equation $f(X, Y) = 0$ as $(f = 0)$.

1.1.1 Definition. A subset $\mathcal{C} \subset \mathbb{C}^2$ is called an *affine algebraic curve* if there exist a non-constant polynomial $f \in \mathbb{C}[X, Y]$ such that $\mathcal{C} = (f = 0)$.

Notice that we do not consider the empty set as a curve so it is important that the polynomial f is non-constant.

1.1.2 Example.

- A line is a curve given by linear polynomial.
- The ordinary cusp at the origin in Figure 1.1 is realized by $X^2 - Y^3$.
- The node at the origin in Figure 1.2 is realized by $Y^2 - X^3 - 2X^2$.
- The curve described by the sinus function is not algebraic (non-example).

Now, consider two polynomials $f, g \in \mathbb{C}[X, Y]$ such that g divides f. If $g(x, y) = 0$ at a point $(x, y) \in \mathbb{C}^2$, then $f(x, y) = 0$. This mean that $(f = 0) \subseteq (g = 0)$. The reverse implication is not true in general. A partial converse is stated in the following theorem.

1.1.3 Theorem (Study's lemma). *If g is non-constant and irreducible, and $(g = 0) \subseteq (f = 0)$ then g divides f.*

Proof. See (Fischer, 2001). □

Since a polynomial ring is a unique factorization domain, for $f \in \mathbb{C}[X, Y]$, f splits into a product

$$f = f_1^{k_1} \cdots f_r^{k_r},$$

where the f_i are irreducible and no redundant. Hence we have

$$(f = 0) = \bigcup_i (f_i^{k^i} = 0) = \bigcup_i (f_i = 0).$$

1.1.4 Definition. An algebraic curve $\mathcal{C} \subset \mathbb{C}^2$ is called *reducible* if there exist two algebraic curves $\mathcal{C}_1, \mathcal{C}_2$ such that $\mathcal{C}_1 \neq \mathcal{C}_2$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. In the other case, the curve $\mathcal{C}$ is called *irreducible.*

This leads us to the following theorem.

1.1.5 Theorem. *An algebraic curve $\mathcal{C} \subset \mathbb{C}^2$ splits as*

$$\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$$

where all $\mathcal{C}_i$ are irreducible. $\{\mathcal{C}_i\}$ are called irreducible components.

Proof. See (Fischer, 2001). □

Consider a polynomial $f \in \mathbb{C}[X, Y]$, $\mathcal{C} = (f = 0)$ and $f = f_1^{k_1} \cdots f_r^{k_r}$ a prime factorization of f. Set $\hat{f} = f_1 \cdots f_r$. The polynomial $\hat{f}$ satisfies $(\hat{f} = 0) = (f = 0)$ and $\hat{f}$ is called *minimal polynomial* of $\mathcal{C}$. It leads us to the definition of the degree of a curve.

1.1.6 Definition. If $\mathcal{C} = (f = 0)$ such that f is minimal then the *degree of the curve* $\mathcal{C}$ is

$$\deg \mathcal{C} := \deg f.$$

Our study focuses on the local behaviour of a plane curve. Assume that we are working in a neighborhood of the origin. Algebraically, we can express a polynomial f by its Taylor expansion. Hence it is convenient to work in the local ring $\mathbb{C}[[X, Y]]$. Therefore we say that $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are equivalent if there is some φ in $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f = g \circ \varphi$. The equivalence class of curve is called a *curve germ.*

Consider the curve $\mathcal{C}$ defined by $f = Y^2 - X^3 - X^2$. According to the above definition, though $\mathcal{C}$ is an irreducible curve in $\mathbb{C}^2$, by writing $Y^2 - X^3 - X^2 = (Y - X\sqrt{X+1})(Y + X\sqrt{X+1})$, we can see that, locally, the germ of $\mathcal{C}$ is the union of two germs. In general, consider a curve germ $\mathcal{C} = V(f)$ such that $f \in \mathbb{C}[[X, Y]]$. Assume that f admits an irreducible decomposition $f = f_1^{k_1} \cdots f_r^{k_r}$, $f_i \in \mathbb{C}[[X, Y]]$. A locus $\mathcal{B} = V(f_i^{k_i})$ is called *local branch.*

Write $f \in \mathbb{C}[[X, Y]]$ as $f = \sum a_i(X)Y^i$ where $a_i(X) \in \mathbb{C}[[X]]$. We say that f is *general* in Y if $a_i(X) \in \mathbb{C}$ for some i.

1.2 Tangent lines and singularities

We will pursue the study in this section by introducing the notion of singularity.

1.2.1 Definition. An algebraic curve $\mathcal{C} = (f = 0)$ where f is minimal is said to be *smooth at a point* $P \in \mathcal{C}$ if

$$\nabla_P(f) = \left(\frac{\partial f}{\partial X}(P), \frac{\partial f}{\partial Y}(P) \right) \neq (0,0).$$

Otherwise, it is called *singular* at P.

A curve $\mathcal{C}$ is called *smooth curve* if it is smooth at each point $P \in \mathcal{C}$.

It might happen that higher order derivatives also vanish at a singular point $P = (p_1, p_2)$. We can have a brief picture of behaviour of $\mathcal{C}$ in a neighborhood of P by analyzing the number of these higher derivatives which vanish at P. Define the *order of* $\mathcal{C}$ at P as

$$\mathrm{ord}_P(\mathcal{C}) = \min\{k \ : \ f_k \neq 0\},$$

where $f_k = \sum_{i+j=k} a_{ij}(X - p_1)^i (Y - p_2)^j$ is the k-th term of its Taylor expansion at P.

It is easy to see that $\mathrm{ord}_P(\mathcal{C}) = 1$ if and only if $\mathcal{C}$ is smooth at P.

A first result in the study of plane curve is the following theorem.

1.2.2 Theorem (Implicit function theorem)**.** *Consider a polynomial* $f \in \mathbb{C}[X, Y]$ *such that* $\frac{\partial f}{\partial Y}(0,0) \neq 0$. *Then there exists a series* $\varphi \in \mathbb{C}[[X]]$ *such that* $\varphi(0) = 0$ *and*

$$f(X, \varphi(X)) \equiv 0.$$

Geometrically, the condition $\frac{\partial f}{\partial Y}(0,0) \neq 0$ means that a curve $\mathcal{C}$ defined by f is smooth at the origin. Furthermore, the existence of φ mean that we have a local parametrization $X \mapsto (X, \varphi(X))$ of $\mathcal{C}$, i.e, the curve $\mathcal{C}$ behaves like a complex line in a neighborhood of the origin. Particularly, we have a homeomorphism between a local neighborhood of the origin and a line $T_0(\mathcal{C})$, called *tangent line*, given by

$$T_0(\mathcal{C}) := \left(X \frac{\partial f}{\partial X}(0,0) + Y \frac{\partial f}{\partial Y}(0,0) = 0 \right).$$

1.3 Intersection of two curves

Consider two polynomials $f, g \in \mathbb{C}[X, Y]$. Assume that the curves defined by $(f = 0)$ and $(g = 0)$ do not have any common irreducible component. To describe how the curves intersect, we study one by one the intersection point of the irreducible components. For an easy computation, consider the point at the origin as a point of intersection of the curves. We define the intersection multiplicity as follow:

1.3.1 Definition. The *intersection muptiplicity* $i_P(f, g)$ of two curves $(f = 0)$ and $(g = 0)$, where P is the origin, $P \in (f = 0) \cap (g = 0)$, is given by

$$i_P(f, g) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[X, Y]]}{(f, g)},$$

where (f, g) denotes the ideal generated by f, g.

1.3.2 Example. Consider $f = X^2 - Y^3$ and $g = Y$. The intersection multiplicity at the origin is

$$i_O(f,g) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[X,Y]]}{(f,g)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[X,Y]]}{(X^2 - Y^3, Y)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[X,Y]]}{(X^2, Y)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[X]]}{(X^2)} = 2.$$

1.3.3 Theorem. *Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{C}^2$ be two algebraic curves with no common component. Then we have*

$$\sum_{P \in \mathcal{C}_1 \cap \mathcal{C}_2} i_P(\mathcal{C}_1, \mathcal{C}_2) \leq \deg \mathcal{C}_1 \cdot \deg \mathcal{C}_2.$$

Proof. See (Fischer, 2001). □

1.4 The Riemann surface of an algebraic curve

We defined first a *projective curve* in $\mathbb{CP}^2$ to be a solution set of a homogeneous polynomial over $\mathbb{C}$. Indeed, recall that $\mathbb{CP}^2$ is the set of line through the origin in $\mathbb{C}^3$. Algebraically, we say that two points $(a_1, a_2, a_3), (b_1, b_2, b_3)$ are equivalent if there is $\lambda \in \mathbb{C}^*$ such that $a_i = \lambda b_i$ for $i = 1, \ldots 3$. Therefore, a line through the origin represents an equivalence class, and we denote the equivalence class of (a,b,c) by $[a:b:c]$. So if a polynomial F vanishes at a point $[a:b:c] \in \mathbb{CP}^2$, then not only $F(a,b,c) = 0$ but also $F(\lambda a, \lambda b, \lambda c) = 0$ for $\lambda \in \mathbb{C}^*$. Therefore, F has to be homogeneous.

We know that $[a_0 : a_1 : a_2] \mapsto \left(\frac{a_1}{a_0}, \frac{a_2}{a_0}\right)$ is an isomorphism between $\mathbb{CP}^2 - \{a_0 = 0\}$ and $\mathbb{C}^2$. We describe a projective curve in terms of its affine points. For a given polynomial $f(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$ of degree d, we associate to f the homogeneous polynomial $F \in k[X_0, X_1, \ldots, X_n]$:

$$F(X_0, X_1, \ldots, X_n) = X_0^d f\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right),$$

so that $F(1, X_1, \ldots, X_n) = f(X_1, \ldots, X_n)$. Conversely, to a given homogeneous polynomial $F(X_0, X_1, \ldots, X_n) \in k[X_0, X_1, \ldots, X_n]$ of degree d, we associate to F the polynomial $f \in k[X_1, \ldots, X_n]$:

$$f(X_1, \ldots, X_n) = F(1, X_1, \ldots, X_n).$$

Under the isomorphism stated previously, the restriction $(F = 0) - \{a_0 = 0\} \to (f = 0)$ is also an isomorphism.

Suppose we have two lines in $\mathbb{CP}^2$. In one of chart $\mathbb{C}^2 \simeq \mathbb{CP}^2 - \{a_2 = 0\}$, we have two cases: either they intersect at one point, or they are parallel. Suppose we have two parallel lines given by $f = aX + bY + c_1$ and $g = aX + bY + c_2$. Their projective closure in $\mathbb{CP}^2$, which are given respectively by $aX + bY + c_1 Z$ and $aX + bY + c_2 Z$, intersect at $[b : -a : 0]$. Hence two lines in $\mathbb{CP}^2$ always intersect. In general, we have the following theorem.

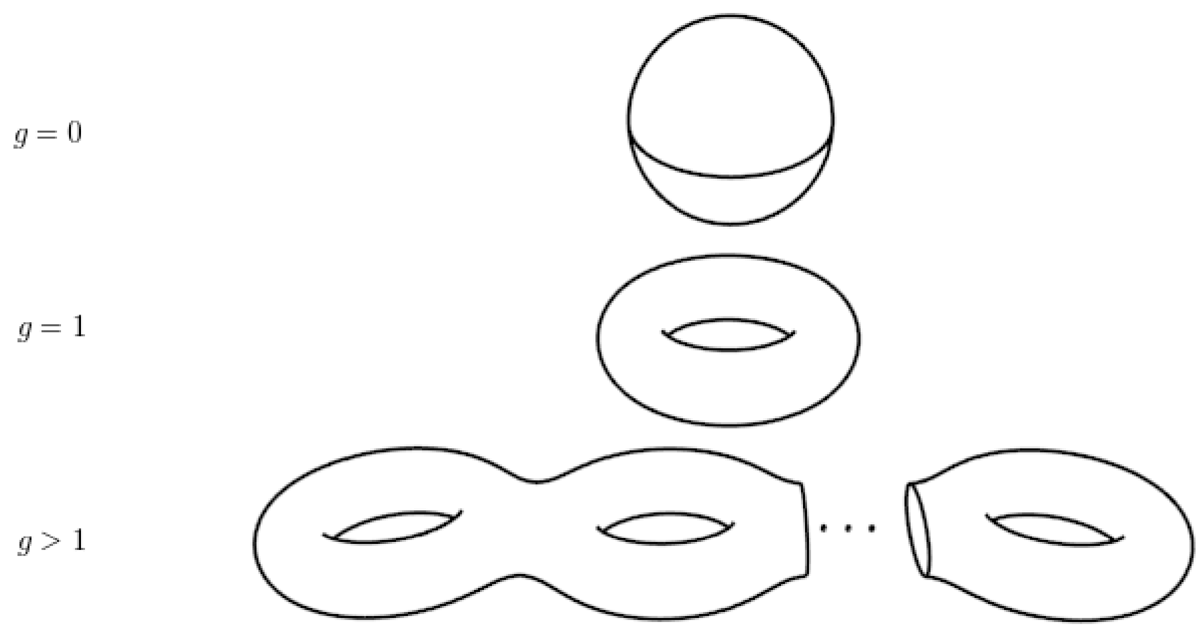
1.4.1 Theorem (Bezout)**.** *Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{CP}^2$ are two curves with no common component. Then we have*

$$\sum_{P \in \mathcal{C}_1 \cap \mathcal{C}_2} i_P(\mathcal{C}_1, \mathcal{C}_2) = \deg \mathcal{C}_1 \cdot \deg \mathcal{C}_2.$$

We come to the visualization of projective curves over $\mathbb{C}$. Over the complex field, we can not illustrate or draw anymore since a curve lies in the real 4−dimensional space. In order to study the topology of an algebraic curve, we define a *Riemann surface* S as a complex manifold of dimension one, so of real dimension 2. Furthermore, it is orientable. Indeed, as complex manifold, consider two charts f, g and a transition function $h = f \circ g^{-1}$. We can consider h as a map from an open set of $\mathbb{R}^2$ to $\mathbb{R}^2$ and orientation is determined by the sign of the determinant of the Jacobian of h, which is positive since h is complex holomorphic.

Consider a complex projective line as $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$ such that $[a : b] \mapsto [0 : a : b]$. Notice that it is algebraic as it is the solution set $(X = 0)$. We have an isomorphism $\mathbb{CP}^1 - \{[0 : 1]\} \to \mathbb{C}$ such that $[a : b] \mapsto \frac{b}{a}$. Therefore, $\mathbb{CP}^1$ is a one point compactification of $\mathbb{C}$, i.e. topologically, we can consider $\mathbb{CP}^1$ as S^2.

In general, a compact orientable real surface (in another words, compact complex curve) is homeomorphic to a sphere with g handles. The number g is called *genus* of S.



For a smooth curve, we have a nice formula to compute g.

1.4.2 Theorem (Genus formula)**.** *A smooth irreducible curve $\mathcal{C} \subset \mathbb{CP}^2$ of degree n has genus*

$$g(\mathcal{C}) = \frac{1}{2}(n-1)(n-2).$$

For the computation of the genus of singular curves, we define the *Milnor number* μ associated with a singular point. Consider a curve $\mathcal{C}$ defined by a polynomial f. Assume that the origin is a singular point of $\mathcal{C}$, then the corresponding Milnor number is given by

$$\mu = i_0\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right) = \dim \frac{\mathbb{C}[[X,Y]]}{\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right)}.$$

Suppose $\mathcal{C}$ has r local branches in a neighborhood of 0. We define a *delta invariant* δ_0 associated with the origin by

$$2\delta_0 = \mu + r - 1.$$

We have a general formula to compute the genus of a curve $\mathcal{C}$ of degree n.

1.4.3 Theorem (Max Noether's Genus formula)**.** *An irreducible curve $\mathcal{C} \subset \mathbb{CP}^2$ of degree n has genus*

$$g(\mathcal{C}) = \frac{1}{2}(n-1)(n-2) - \sum_{P \in Sing(\mathcal{C})} \delta_P,$$

where $Sing(\mathcal{C})$ is the set of all singular points of $\mathcal{C}$.

Consider a curve $\mathcal{C}$ of degree 2. Then $\mathcal{C}$ has at most one singular point. Otherwise, take a line L passing through two singular points $P, Q \in \mathcal{C}$. Then the intersection number of L and $\mathcal{C}$ counted with multiplicities is $i_P(\mathcal{C}, L) + i_Q(\mathcal{C}, L) > 2 = \deg(\mathcal{C})\deg(L)$ which contradicts Theorem 1.3.3. In general, we could show that $Sing(\mathcal{C})$ is a finite set for an algebraic curve $\mathcal{C}$.

1.4.4 Example. Consider the curve $\mathcal{C}$ given by $X^2 - Y^3$. The origin is the only singular point of $\mathcal{C}$. It has only one branch, so $r = 1$. Let $\overline{\mathcal{C}}$ be its projective closure, i.e. $\overline{\mathcal{C}}$ is defined by $ZX^2 - Y^3$. We have:

$$\begin{aligned}
\mu_0 &= \dim \frac{\mathbb{C}[[X,Y]]}{(X, Y^2)} = 2,\\
2\delta_0 &= 2,\\
g(\overline{\mathcal{C}}) &= \frac{1}{2}(3-1)(3-2) - \delta_0 = 0.
\end{aligned}$$

2. Topology of the link

In the previous chapter, we have introduced the notion of plane curve, studied some of its properties and show the importance of the study of singularities. In this chapter, we are going to describe some approach to describe the singularities. In the first section, we introduce the Puiseux parametrization which is a nice way to parametrize a branch. Then in the second section, we describe the link associated with a singularity in term of the obtained parametrization.

2.1 Puiseux parametrization

Consider a branch $\mathcal{C}$ defined by a minimal polynomial $f \in \mathbb{C}[X,Y]$ such that $(0,0) \in \mathcal{C}$. From the implicit function theorem, it follow that if $\frac{\partial f}{\partial Y}(0,0) \neq 0$, then we have a local parametrization in a neighborhood of $(0,0)$:

$$x \mapsto (x, \varphi(x)),$$

such that $f(x, \varphi(x)) \equiv 0$.

Now consider the polynomial $g = X^2 - Y^3$. The implicit function theorem does not apply since $\frac{\partial g}{\partial X}(0,0) = \frac{\partial g}{\partial Y}(0,0) = 0$. However, we have a parametrization

$$t \mapsto (t^3, \varphi(t)) \text{ where } \varphi(t) = t^2.$$

2.1.1 Definition. We define a Puiseux series as an expression of the form

$$f = \sum_{k=k_0}^{\infty} c_k t^{\frac{k}{n}},$$

where $c_k \in \mathbb{C}$, $k_0, n \in \mathbb{Z}$ such that $n > 0$ and $c_{k_0} \neq 0$.

We can express the parametrization of the cusp as:

$$t \mapsto (t, t^{\frac{2}{3}}).$$

This is an application of the following theorem:

2.1.2 Theorem (Puiseux problem)**.** *Let $f \in \mathbb{C}[\![X,Y]\!]$ be general in Y of order $k \geq 1$. Then there exist a natural number $n \geq 1$ and $\varphi \in \mathbb{C}[\![t]\!]$ such that $\varphi(0) = 0$ and*

$$f(t^n, \varphi(t)) = 0 \text{ in } \mathbb{C}[\![t]\!].$$

If f is convergent, then so is φ.

To prove the theorem, we will first introduce some definitions and state some lemmas.

2.1.3 Definition. Let $f \in \mathbb{C}[\![X,Y]\!]$ such that $f = \sum a_{ij}X^iY^j$. We define the *carrier* of f as

$$\operatorname{carr}(f) = \{(i,j) \in \mathbb{N}^2 \ : \ a_{ij} \neq 0\}.$$

2.1.4 Example. A *homogenius* polynomial f of degree $\deg f = d$ is a polynomial of the form

$$f = \sum_{i+j=d} a_{ij}X^iY^j.$$

So the carrier of a homogeneous polynomial of a degree d lies on a line of slope -1.

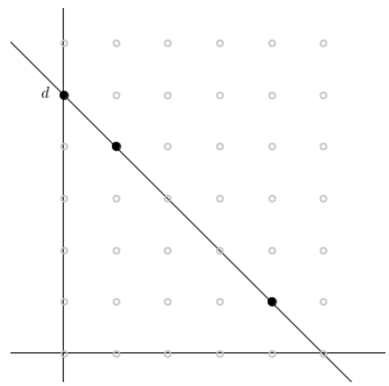


Figure 2.1: Carrier of a homogeneous polynomial.

2.1.5 Example. A polynomial f is said to be a *quasi-homogenius* polynomial if there exist some $p, q, l \in \mathbb{N}^*$ such that

$$f = \sum_{pi+qj=l} a_{ij}X^iY^j.$$

We call p, q the *weights*. Particularly, if $p = q = 1$ then f is homogeneous.

The carrier of a quasi-homogeneous polynomial lies on a line of equation $pi + qj = l$.

Moreover, if f is general in Y of order k, i.e. $f = a_{0k}Y^k + \sum_{pi+qj=l,\ j\neq k} a_{ij}X^iY^j$ for some $k \in \mathbb{N}^*$, then $l = kq$.

2.1.6 Lemma. If f is quasi-homogeneous of weights p, q, and general in Y of order $k \geq 1$, then there exists $\lambda \in \mathbb{C}$ such that

$$f(t^p, \lambda t^q) = 0.$$

Proof. Consider $f = \sum_{pi+qj=l} a_{ij}X^iY^j$. Then we have:

$$\begin{aligned} f(t^p, \lambda t^q) &= \sum_{pi+qj=l} a_{ij}t^{ip+jq}\lambda^j \\ &= t^l \sum_{pi+qj=l} a_{ij}\lambda^j \\ &= t^l g(\lambda). \end{aligned}$$

As we are working over $\mathbb{C}$, $g(\lambda) = 0$ has a solution. $\square$

2.1.7 Example. Consider $f = X^6 - 2X^3X^2 + Y^4$. By solving the system of equation:

$$\begin{cases} 6p = l \\ 3p + 2q = l \\ 4q = l. \end{cases}$$

We deduce that f has weights $p = 2, q = 3$.

Let us now find $\lambda \in \mathbb{C}$ such that $f(t^2, \lambda t^3) = 0$. We have:

$$\begin{aligned} \tilde{f}(t^2, \lambda t^3) &= \left(t^2\right)^6 - 2\left(t^2\right)^3\left(t^3\right)^2 + \left(t^3\right)^4 \\ &= t^{12}(1 - 2\lambda^2 + \lambda^4) \\ &= t^{12}(\lambda^2 - 1)^2 \end{aligned}$$

Let us choose $\lambda = 1$.

Notice that the existence of $\lambda \in \mathbb{C}$ such that $f(t^p, \lambda t^q) = 0$ does not mean that we have a parametrization of curves defined by quasi-homogeneous polynomial. If a quasi-homogeneous polynomial f is not locally irreducible, then $(X = t^p, Y = \lambda t^q)$ is a parametrization of an irreducible branch of the curve defined by f. Consider $f = X^6 + X^2Y^2 - 2Y^3 = (X^2 - Y)(X^4 + X^2Y + 2Y^2)$. The weights of f is $(1, 2)$. Then we have

$$f(t^1, \lambda t^2) = t^6(1 + \lambda^2 - 2\lambda) = t^6(\lambda - 1)^2.$$

Hence we can take $\lambda = 1$. However, $(X = t, Y = t^2)$ parametrizes only $(X^2 - Y = 0)$.

We proved the existence of a solution of the Puiseux problem for a quasi-homogeneous polynomial. In the next step, we show that a polynomial f can be written as $f = \tilde{f} + h$ where $\tilde{f}, h \in \mathbb{C}[X, Y]$ such that the polynomial $\tilde{f}$ is called *quasi-homogeneous initial polynomial* of f. To compute $\tilde{f}$, we use a method elaborated by Newton as follow:

Assuming that f is general in Y of order k, $(0, k) \in \operatorname{carr}(f)$. Consider a line passing through $(0, k)$ and $(0, 0)$. Then rotate this line anti-clockwise around the point $(0, k)$ until it hit another point in $\operatorname{carr}(f)$. The points in $\operatorname{carr}(f)$ that lie on the line obtained determine $\tilde{f}$.

2.1.8 Example. Consider $f = X^6 + X^5Y^3 + X^4Y^3 + X^3Y^2 + X^2Y^4 + X^2Y^2 + Y^3$. So the quasi-homogeneous initial polynomial $\tilde{f} = X^6 + X^2Y^2 + Y^3$, and $f = \tilde{f} + X^5Y^3 + X^4Y^3 + X^3Y^2 + X^2Y^4$.

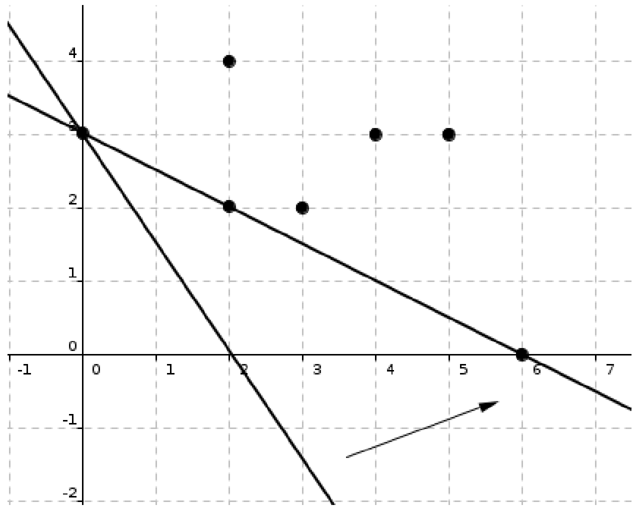


Figure 2.2: Finding the quasi-homogeneous initial polynomial.

From this discussion and Lemma 2.1.6, we can do the following iteration. Suppose $f = \tilde{f} + h$ and that $X = t^p, Y = \lambda t^q$ is a solution of the Puiseux problem for $\tilde{f}$. Set $\tilde{Y} = \lambda X^{\frac{q}{p}}$ such that $\tilde{f}(X, \tilde{Y}) \equiv 0$. We could approximate the solution for f as follow:

$$X = X_1^p,\ Y = \tilde{Y} + X_1^q Y_1 = \lambda X^{\frac{q}{p}} + X_1^q Y_1 = \lambda \left(X^{\frac{1}{q}}\right)^q + X_1^q Y_1 = X_1^q(\lambda + Y_1). \tag{2.1.1}$$

Substituting Equation 2.1.1 into $\tilde{f}$ gives us

$$\begin{aligned}
\tilde{f}(X,Y) &= \sum_{pi+qj=qk} a_{ij} X^i Y^j \\
&= \sum_{pi+qj=qk} a_{ij} \left(X_1^p\right)^i \left(X_1^q(\lambda + Y_1)\right)^j \\
&= \sum_{pi+qj=qk} a_{ij} \left(X_1^p\right)^i \left(X_1^q\right)^j (\lambda + Y_1)^j \\
&= X_1^{qk} \sum_{pi+qj=qk} a_{ij} (\lambda + Y_1)^j \\
&= X_1^{qk} g(\lambda + Y_1).
\end{aligned}$$

Substituting Equation 2.1.1 into h gives us

$$\begin{aligned}
h(X,Y) &= \sum_{pi+qj>qk+1} a_{ij} X^i Y^j \\
&= \sum_{pi+qj>qk+1} a_{ij} \left(X_1^p\right)^i \left(X_1^q(\lambda + Y_1)\right)^j \\
&= \sum_{pi+qj>qk+1} a_{ij} \left(X_1^p\right)^i \left(X_1^q\right)^j (\lambda + Y_1)^j \\
&= X_1^{qk} h^*(X_1, Y_1).
\end{aligned}$$

Therefore, we have

$$\begin{aligned} f(X,Y) &= \tilde{f}(X,Y) + h(X,Y) \\ &= X_1^{qk} g(\lambda + Y_1) + X_1^{qk} h^*(X_1, Y_1) \\ &= X_1^{qk} f_1(X_1, Y_1). \end{aligned}$$

We have the following lemma.

2.1.9 Lemma. Let f and f_1 as above:

- f_1 is general in Y of order k_1, such that $1 \leq k_1 \leq k$,
- if $k_1 = k$, then $q = 1$.

Proof. See (Fischer, 2001). □

Now, we are ready to complete the proof of Theorem 2.1.2.

Let $f(X,Y)$ be a polynomial which is general in Y of order k. For the initialization, set

$$f_0 = f,\ X_0 = X,\ Y_0 = Y,\ k_0 = k.$$

We proceed as in the discussion preceding Lemma 2.1.9 for the first step. To get from i to $i+1$, we proceed as follow: if $Y_i^{k_i}$ divides f_i, then $Y_i = 0$ is solution of $f_i(X_i, Y_i) = 0$. Otherwise, write f_i as $f_i = \tilde{f}_i + h_i$ where $\tilde{f}_i$ is the quasi-homogeneous initial polynomial of f_i with weights p_i, q_i. Set the equation of the carrier of $\tilde{f}_i$ as $p_i\mu + q_i\nu = q_i k_i$. Since Y_i does not divide f_i, then there is some $\lambda_i \in \mathbb{C}^*$ such that $\tilde{Y}_i = \lambda_i X_i^{\frac{q_i}{p_i}}$ is a solution of $f_i(X_i, \tilde{Y}_i) = 0$. Then set

$$X_i = X_{i+1}^{p_i}, \tilde{Y}_i = X_{i+1}^{q_i}(\lambda_i + Y_{i+1}).$$

Then we have:

$$f_i(X_i, Y_i) = X_{i+1}^{q_i k_i} f_{i+1}(X_{i+1}, Y_{i+1}),$$

where f_{i+1} is general in Y_{i+1}.

Finally, we have

$$X = X_1^{p_0} = X_2^{p_0 p_1} = \cdots = X_{N+1}^{p_0 \cdots p_N} = X_{N+1}^n = t^n,$$

for some $N \in \mathbb{N}$, $n = p_0 \cdots p_N$. Moreover, we have

$$\begin{aligned} Y &= \tilde{Y}_0 + X_1^{q_0} Y_1 \\ &= \tilde{Y}_0 + X_1^{q_0}(\tilde{Y}_1 + X_2^{q_1} Y_2) \\ &= \tilde{Y}_0 + X_1^{q_0}\tilde{Y}_1 + X_1^{q_0} X_2^{q_1} Y_2 \\ &= \cdots \\ &= \lambda_0 X_0^{\frac{q_0}{p_0}} + \sum_{i=1}^{\infty} \lambda_i X_1^{q_0} \cdots X_i^{q_{i-1} + \frac{q_i}{p_i}} \\ &= \sum_{i=0}^{\infty} \lambda_i t^{m_i} \end{aligned}$$

where $m_0 = q_0 p_1 \cdots q_N$ and $m_{i+1} = m_i + q_{i+1} \prod_{j>i+1} p_j$.

We are only interested on the construction of the Puiseux series. For the remaining proof that this is a solution and about the convergence of $Y = \sum_{i=0}^{\infty} \lambda_i t^{m_i}$, see (Fischer, 2001).

2.1.10 Example. Consider $f = -X^7 + X^6 - 4X^5Y - 2X^3Y^2 + Y^4$. The carrier of f is

$$\mathrm{carr}(f) = \{(7,0),(6,0),(5,1),(3,2),(0,4)\}.$$

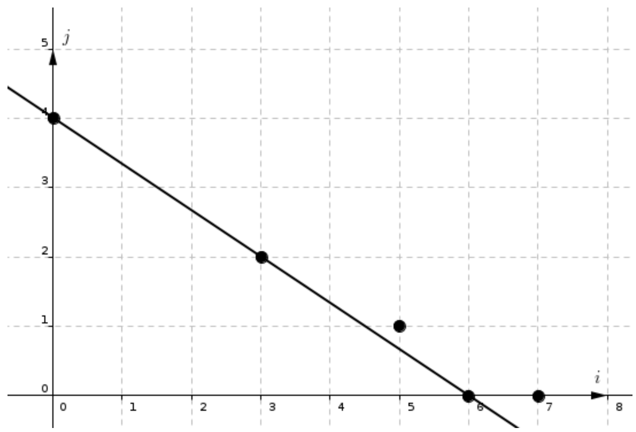


Figure 2.3: The carrier of f.

According to the method of Newton, the quasi-homogeneous initial polynomial of f is

$$\tilde{f} = X^6 - 2X^3X^2 + Y^4.$$

From Example 2.1.7, $(X = t^2, Y = t^3)$ is solution of $\tilde{f} = 0$. For the iteration: set $X = X_1^2, Y = X_1^3(1+Y_1)$. Then we have:

$$\begin{aligned} f(X,Y) &= -\left(X_1^2\right)^7 + \left(X_1^2\right)^6 - 4\left(X_1^2\right)^5 X_1^3(1+Y_1) - 2\left(X_1^2\right)^3 \left(X_1^3(1+Y_1)\right)^2 + \left(X_1^3(1+Y_1)\right)^4 \\ &= -X_1^{14} + X_1^{12} - 4X_1^{13}(1+Y_1) - 2X_1^{12}(1+Y_1)^2 + X_1^{12}(1+Y_1)^4 \\ &= X_1^{12}\left(-X_1^2 + 1 - 4X_1 - 4X_1Y_1 - 2Y_1^2 - 4Y_1 - 2 + Y_1^4 + 4Y_1^3 + 6Y_1^2 + 4Y_1 + 1\right) \\ &= X_1^{12}\left(-X_1^2 - 4X_1Y_1 + Y_1^4 + 4Y_1^3 + 4Y_1^2 - 4X_1\right) \\ &= X_1^{12} f_1(X_1, Y_1). \end{aligned}$$

Now, the carrier of f_1 is

$$\mathrm{carr}(f_1) = \{(2,0),(1,0),(1,1),(0,4),(0,3),(0,2)\}.$$

The quasi-homogeneous initial polynomial of f_1 is $\tilde{f}_1 = 4Y_1^2 - 4X_1$. Let us find $\lambda \in \mathbb{C}^2$ such that $\tilde{f}_1(t^2, \lambda t) = 0$. We have

$$\begin{aligned} \tilde{f}_1(t^2, \lambda t) &= 4\lambda^2 t^2 - 4t^2 \\ &= 4t^2(\lambda - 1). \end{aligned}$$

The solution of $\tilde{f}_1 = 0$ is (t^2, t). Then set $X_1 = X_2^2, Y_1 = X_2(1+Y_2)$. Substituting this into f_1 we have:

$$\begin{aligned}
f_1(X_2^2, X^2(1+Y_2)) &= -\left(X_2^2\right)^2 - 4\left(X_2^2\right)X^2(1+Y_2) - 4X_2^2 + \left(X^2(1+Y_2)\right)^4 \\
&+ 4\left(X^2(1+Y_2)\right)^3 + 4\left(X^2(1+Y_2)\right)^2 \\
&= X_2^4Y_2^4 + 4X_2^4Y_2^3 + 6X_2^4Y_2^2 + 4X_2^3Y_2^3 + 4X_2^4Y_2 \\
&+ 12X_2^3Y_2^2 + 8X_2^3Y_2 + 4X_2^2Y_2^2 + 8X_2^2Y_2 \\
&= X_2^2(X_2^2Y_2^4 + 4X_2^2Y_2^3 + 6X_2^2Y_2^2 + 4X_2Y_2^3 + 4X_2^2Y_2 \\
&+ 12X_2Y_2^2 + 8X_2Y_2 + 4Y_2^2 + 8Y_2 \\
&= X_2^2 f_2(X_2, Y_2).
\end{aligned}$$

We see that Y_2 divides $f_2(X_2, Y_2)$ so $Y_2 = 0$ is a solution of $f_2(X_2, Y_2) = 0$.

We can now finalize $X = X_1^2 = (X_2^2)^2 = X_2^4$ and

$$\begin{aligned}
Y &= X_1^3(1+Y_1) \\
&= \left(X_2^2\right)^3(1 + X_2(1+Y_2)) \\
&= X_2^6(1+X_2) \\
&= X_2^6 + X_2^7.
\end{aligned}$$

Therefore, the solution for the Puiseux problem is $X = t^4, Y = t^6 + t^7$. We can also write $Y = X^{\frac{3}{2}} + X^{\frac{7}{4}}$.

Extraction of the Puiseux pairs. Consider a solution $X = t^m, Y = \sum a_i t^i$, $a_i \in \mathbb{C}$ of $f(X,Y) = 0$. Write $Y = \sum a_k X^k$, $k \in \mathbb{Q}$. We are going to extract a finite sequence of pairs from the exponent which we will use later to describe the link associated to a singularity.

- If $k \in \mathbb{N}$ for every exponent k, then f is regular and there is no pair. Otherwise, take the smallest exponent $k_1 = \frac{n_1}{m_1}$ such that $(n_1 > m_1)$, and $\gcd(n_1, m_1) = 1$. The pair (m_1, n_1) is called the *first Puiseux pair.*
- Next, we choose the next smallest exponent k_2 which is not of the form $\frac{q}{m_1}$, $q \in \mathbb{N}$. We write $k_2 = \frac{n_2}{m_1 \cdot m_2}$ such that $\gcd(n_2, m_2) = 1$.
- Now, assume that we have the first pairs $(m_1, n_1), \ldots, (m_j, n_j)$. Take the next exponent which is not of the form $\frac{q}{m_1 \cdots m_j}$. Write $k_{j+1} = \frac{n_{j+1}}{m_1 \cdots m_j m_{j+1}}$ such that $\gcd(n_{j+1}, m_{j+1}) = 1$.

We could multiply the numerator and denominator by a divisor of $m_1 \cdots m_j$ to get the desired form (see Example 2.1.12). Moreover, this process terminates eventually. So there is some $g \in \mathbb{N}$ such that $m_1 \cdots m_g = m$.

The pairs $(m_1, n_1), \ldots, (m_g, n_g)$ are called *Puiseux pairs*. The series $(m; \beta_1, \ldots, \beta_g)$ where $\beta_i = n_i \cdot m_{i+1} \cdots m_g$ is called *Puiseux characteristic*.

2.1.11 Example. Given the results in the Example 2.1.10. We have $Y = X^{\frac{3}{2}} + X^{\frac{7}{4}}$. Therefore, the Puiseux pairs are $(3,2), (7,2)$ and the Puiseux characteristic is $(4; 6, 7)$.

2.1.12 Example. Consider $Y = X^{\frac{3}{2}} + X^{\frac{5}{3}}$. The first pair is again $(3, 2)$. But to determine the second pair, we write $\frac{5}{3} = \frac{10}{6}$. Therefore, the next pair is $(10, 3)$.

2.2 Description of the topology of the link

So far, we have only describe plane curves singularities algebraically. In this section, we will describe geometrically the local behaviour of a curve near a singular point.

2.2.1 Definition. A *knot* $K_{\mathcal{B}}$ associated to a branch $\mathcal{B}$ is an embedded copy of S^1 into S^3. More formally, assume $0 \in \mathcal{B}$ is an isolated singularity, then $K_{\mathcal{B}}$ is given by the intersection $K_{\mathcal{B}} = \mathcal{B} \cap S^3_\epsilon$ for a small enough $\epsilon \in \mathbb{R}_{>0}$.

A *link* $L_{\mathcal{C}}$ associated to a curve $\mathcal{C} = \cup_{i=1}^s \mathcal{B}_i$ is a disjoint union $L_{\mathcal{C}} = \cup_{i=1}^s K_{\mathcal{B}_i}$.

In this definition, it might appear that the knot is depending on the size of the sphere S^3_ϵ. However, if we take δ small enough, then $\mathcal{B} \cap S^3_\epsilon$ and $\mathcal{B} \cap S^3_\delta$ are isotopic in S^3. This is a consequence of the following lemma:

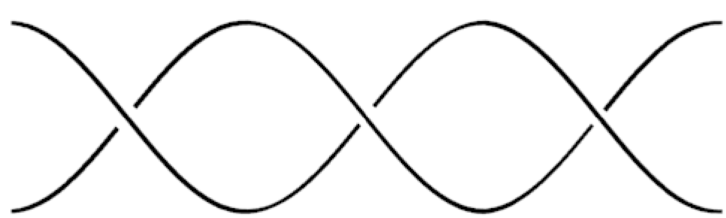
2.2.2 Lemma. For ϵ sufficiently small, $K_{\mathcal{B}}$ is a 1-manifold smoothly embedded in S_ϵ, and there is a homeomorphism of $\mathcal{B} \cap D_\epsilon$ to the cone of $\mathcal{B} \cap S_\epsilon$. Furthermore, the embedded topological type of all these objects are independent of the choice of ϵ.

Now, let us investigate the construction of the knot for an irreducible branch $\mathcal{B}$. First, let us assume that $\mathcal{B}$ has the easiest form of Puiseux parametrization $Y = X^{\frac{m}{n}}$. Write $X = re^{i\theta}$ where $|X| = r$. Therefore, $Y = r^{\frac{m}{n}} e^{i\theta \frac{m}{n}}$. So the curve described by a point in $\mathcal{B} \cap S_\epsilon$ turns once around the circle $|X| = r$, and $\frac{m}{n}$ times around $|Y| = r^{\frac{m}{n}}$. Notice that if we fix X, the equation $X^m = Y^n$ has as solution $\omega Y, \dots, \omega^n Y$ where ω is an n-th root of unity. Therefore, at the initial point $X = r$, we have n points $Y_k = r^{\frac{m}{n}} e^{2\pi i \frac{m}{n}} k$, for $k = 1, \dots, n$.

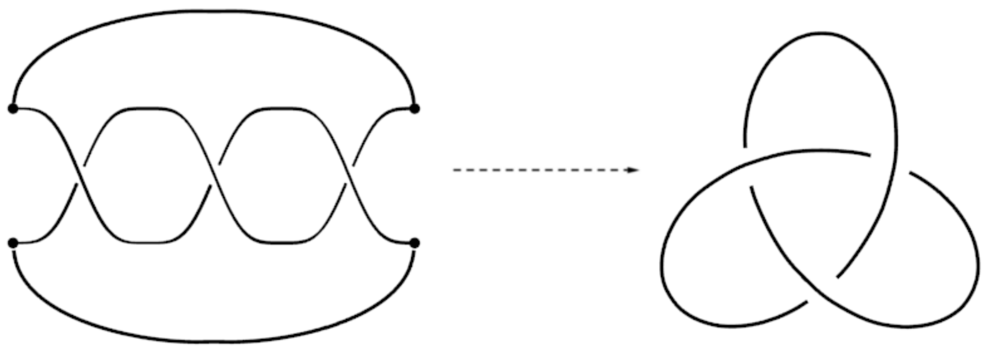
2.2.3 Definition. A *braid* on m strings (and initial points $\{P_1, \dots, P_n\}$) is a homotopy class of closed path with initial and final points $\{P_1, \dots, P_n\}$.

From the previous discussion, a braid results from letting θ run in $[0, 2\pi]$.

2.2.4 Example. Consider the cusp $V(X^2 - Y^3)$. Write $Y = X^{\frac{3}{2}}$. So we have 2 points moving with velocity $\frac{3}{2}$.



By the definition of a braid, notice that a braid has the same initial and final points. We get the associated link (or knot) by gluing the extreme points. Therefore, the knot associated to a cusp is the following.



Now, consider a parametrization $Y = X^{\frac{m_1}{n_1}} + X^{\frac{m_2}{n_1 n_2}} + \cdots + X^{\frac{m_g}{n_1 \cdots n_g}}$. So the Puiseux pairs are $(m_1, n_1), \ldots, (m_g, n_g)$. Then we construct a link as follow.

- From the first pair, at the initial position $X = r$, we have n_1 points $Y_1, \ldots, Y_{n_1}$ moving with velocity $\frac{m_1}{n_1}$ on a circle of center 0 with radius $r^{\frac{m_1}{n_1}}$.
- For each point Y_i, the second pair gives n_2 points moving with velocity $\frac{m_2}{n_2}$ on a circle centered at the points Y_i and of radius $r^{\frac{m_2}{n_2}}$.
- In general, the pair (m_i, n_i) gives n_i points moving with velocity $\frac{m_i}{n_i}$ on a circle centered at the previous $m_1 \cdots m_{i-1}$ points created and of radius $r^{\frac{m_i}{n_i}}$.

For a small enough ϵ, these circles do not interfere with one another. The points on the last circle (of radius $r^{\frac{m_g}{n_g}}$ describe the braid when X run through the circle of radius r.

2.2.5 Example. Consider the parametrization $Y = X^{\frac{3}{2}} + X^{\frac{5}{4}} + X^{\frac{7}{3}}$.

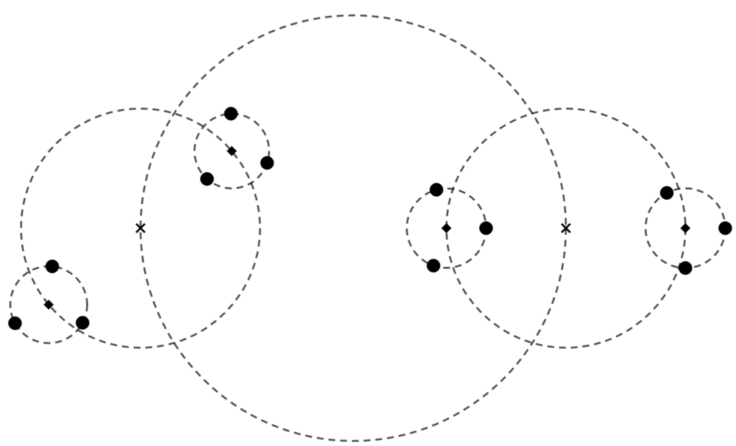
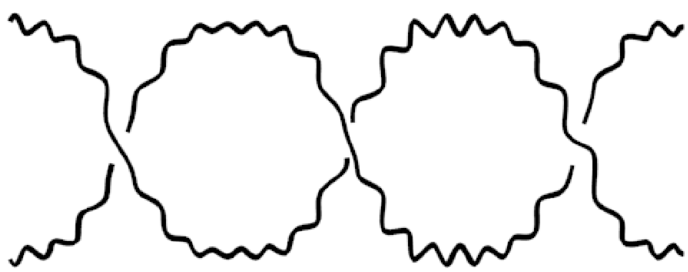


Figure 2.4: Initial and final points of a braid (full circles).

So far, we have described the braid constructed from the Puiseux pairs. Now, consider a parametrization $Y = a_k X^k, k \in \mathbb{Q}$. Let us see the effect of the other exponents which are represented in the pairs. Suppose we have the first Puiseux pair (m_1, n_1), then we constructed the next pair from an exponent which is not of the for $\frac{q}{m_1}, q > n_1$. Let us describe the effect of the exponent of this form. Notice that we get the second pair (m_2, n_2) as the first exponent which is not of the form $\frac{q}{m_1}$ and write it as $\frac{n_2}{m_1 m_2}$. If we write $\frac{q}{m_1}$ of this given form, then the pair we get is $(1, n_2)$. We can write the Puiseux expansion from this pair as $Y = X^{n_2}$. Therefore, similarly as in the discussion preceding Definition 2.2.3, we see that it is just a curve described by one oscillating point. Going back to the original problem, the exponents of the form $\frac{q}{m_1}$ does not alter the number of the points in the braid, it create oscillation around the first approximation, which may be smoothed to get the original braid.

2.2.6 Example. Suppose we have a parametrization $X = t^2, Y = t^3 + t^{33}$. Therefore, we have only one Puiseux pair $(2, 3)$. The first pair gives us a braid as in Example 2.2.4. Thus the final braid we get from the parametrization $Y = X^{\frac{3}{2}} + X^{\frac{33}{2}}$ is the following.



This may be smoothed to get the braid in Example 2.2.4.

This illustrates to the following theorem.

2.2.7 Theorem. *Puiseux parametrizations with the same Puiseux pairs yield equivalent braids.*

Now, we study the configuration of the disjoint knots $K_{\mathcal{B}_i}$ generating a link $L_{\mathcal{C}}$. Consider two embeddings $f_1, f_2 : S^1 \to S^3$ such that the images are disjoint. One can extend f_1 to $\tilde{f}_1 : D^2 \to S^3$, and that the images of $\tilde{f}_1$ and f_2 intersect transversely (see (Bredon, 1993)). Therefore $\tilde{f}_1(D^2) \cap f_2(S^1)$ is a finite set of points. The pre-image $\tilde{f}_1^{-1}(\tilde{f}_1(D^2) \cap f_2(S^1))$ is then also a finite set of points. Consider a point x in this pre-image. The differential of $\tilde{f}_1$ and f_2 induce a 3-frame at $\tilde{f}_1(x) = f_2(x)$. Assign a $+$ if this frame is consistent with the standard 3-frame in $\mathbb{R}^3$, and assign $-$ otherwise.

2.2.8 Definition. Given the above construction. The *linking number* $\mathcal{L}(f, g)$ is the sum of the sign over all such a point x.

2.2.9 Lemma. Consider two branches $\mathcal{B}_1, \mathcal{B}_2$ at 0, their intersection multiplicity at 0 is equal to the linking number of $K_{\mathcal{B}_1}$ and $K_{\mathcal{B}_2}$. Moreover, we have:

$$\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) = \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1).$$

2.2.10 Theorem. *Consider two links $L_{\mathcal{C}}, L_{\mathcal{C}'}$ associated respectively with two curves $\mathcal{C}, \mathcal{C}'$. Two links are equivalent if there is a one-to-one correspondence between the knots forming $L_{\mathcal{C}}, L_{\mathcal{C}'}$ such that:*

- *the corresponding knot is described by the same Puiseux pairs,*
- *the linking number of corresponding knot components is the same.*

2.2.11 Definition. We say that two curves germs $\mathcal{C}, \mathcal{C}'$ are *equisingular* if the corresponding links are equivalent.

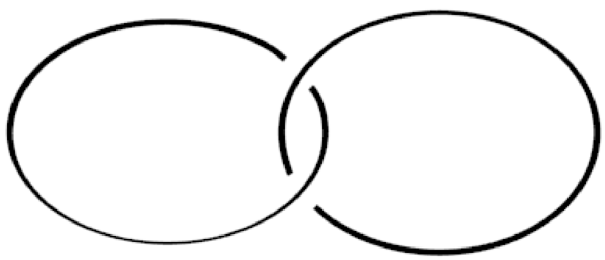
Lemma 2.2.2 show that the topology of curve germs is the same as the topology of the link associated. Therefore, we can have the following classification of plane curve singularities.

2.2.12 Theorem. *Two curve germs $\mathcal{C}, \mathcal{C}'$ are topologically equivalent if and only if they are equisingular.*

2.2.13 Example. Consider the curve defined by XY and the curve defined by $f = Y^2 - X^3 - 2X^2$. The curve defined by f has two smooth branches $\mathcal{B}_a$ and $\mathcal{B}_b$ defined respectively by the $x-$axis and the $y-$axis. In addition, notice that in a neighborhood of the origin, $g = Y^2 - X^3 - 2X^2 = (Y - X\sqrt{X-2})(Y + X\sqrt{X-2})$. As we see in Figure 1.2, the curve defined by g has two smooth branches $\mathcal{B}_\mu$ and $\mathcal{B}_\nu$ defined respectively by $(Y - X\sqrt{X-2})$ and $(Y + X\sqrt{X-2})$. Moreover, we have:

$$\frac{\mathbb{C}[[X,Y]]}{(Y - X\sqrt{X-2}, Y + X\sqrt{X-2})} \simeq \frac{\mathbb{C}[[X,Y]]}{(Y, X\sqrt{X-2})} \simeq \frac{\mathbb{C}[[X,Y]]}{(Y,X)}.$$

Therefore, we have $\mathcal{L}(\mathcal{B}_a, \mathcal{B}_b) = \mathcal{L}(\mathcal{B}_\mu, \mathcal{B}_\nu)$. Then the link associated with the origin of both germs is the following



3. Resolution of singularity

In this chapter, we describe a method to resolve a singularity. There are many ways to resolve a singularity, in this thesis we will discuss only about one method called *blowing up*.

3.1 Blow up of $\mathbb{C}^2$ at the origin.

There are many ways to define a blow up. In this thesis, we describe it geometrically and later we present a more algebraic definition.

Basically, the principle of a blow up is to construct a new surface T and a projection $\phi : T \to \mathbb{C}^2$ such that $\phi^{-1}(0)$ is a curve E isomorphic to $\mathbb{CP}^1$. The curve E is called *exceptional curve*. The map ϕ induces an isomorphism away from E. Each point in E represents a direction in $\mathbb{C}^2$.

More formally, we have the following definition.

3.1.1 Definition. The incidence correspondence of a blow up is a set $T \in \mathbb{C}^2 \times \mathbb{CP}^1$ such that

$$T = \{(p, L)\ :\ p \in L\} \subset \mathbb{C}^2 \times \mathbb{CP}^1.$$

Consider a point $p = (x, y) \in \mathbb{C}^2$ and a line L through the origin which is determined by the ration $[a : b], a, b \in \mathbb{C}$, such that $(a, b) \neq (0, 0)$. Therefore, the relation $p \in L$ in the definition means that $(a, b) = \lambda(x, y)$ for some $\lambda \in \mathbb{C}^*$. Thus we have a relation $ay = bx$. This lead us to another definition of the incidence correspondence:

$$T = \{((x, y), [a : b])\ :\ ay = bx\} \subset \mathbb{C}^2 \times \mathbb{CP}^1.$$

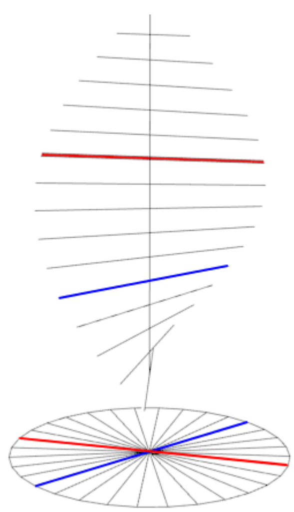


Figure 3.1: Blow up of $\mathbb{C}^2$ at the origin.

Take a point $(x, y) \in \mathbb{C}^2$ such that $(x, y) \neq 0$. Assume $x \neq 0$. Then $b = \frac{ay}{x}$. This implies that $[a : b] = [a : \frac{ay}{x}]$. Therefore, we have $a \neq 0$ and $[a : b] = [1 : \frac{y}{x}]$. The pre-image of (x, y) is given by $\left\{(x, y), [1 : \frac{y}{x}]\right\}$. This lead us to the conclude that there is an isomorphism between $\mathcal{C} - 0$ and $\pi^{-1}(\mathcal{C}) - E$.

3.1.2 Definition. The pre-image $\pi^{-1}(\mathcal{C})$ is called the *total transform*, and the closure of $\pi^{-1} - E$ is called the *strict transform* of $\mathcal{C}$.

We know that $\mathbb{CP}^1 = \{[a : b] \ : \ a \neq 0\} \sqcup \{[a : b] \ : \ b \neq 0\}$. If $a \neq 0$, then we can write $y = \frac{b}{a}x$. Similarly, if $b \neq 0$, then we write $x = \frac{a}{b}y$. Therefore, we could resume the blow up of $\mathbb{C}^2$ to two coordinate charts $(u, v) \mapsto (u, uv)$ and $(u, v) \mapsto (uv, v)$.

3.1.3 Example. Let $\mathcal{C} = V(X^2 - Y^3)$. Applying the blow up at 0, we have :

$$(X = ab, Y = b) \Rightarrow X^2 - Y^3 = b^2(a^2 - b).$$

So the exceptional curve is $V(b)$, and the strict transform is $V(a^2 - b)$.

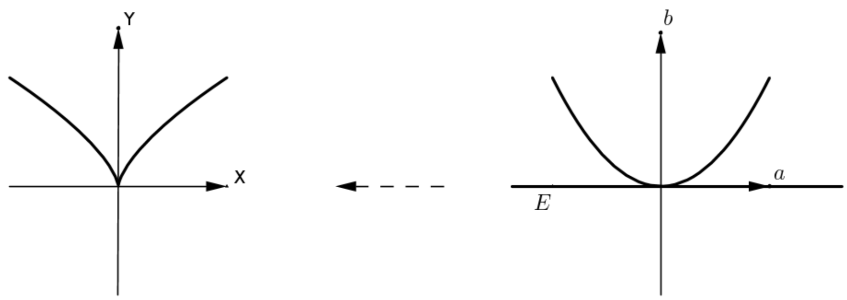


Figure 3.2: Blow up of a cusp $V(X^2 - Y^3)$ at the origin.

Notice that in the other change of coordinate, we have

$$(X = a, Y = ab) \Rightarrow X^2 - Y^3 = a^2(1 - ab^3).$$

The strict transform $V(1 - ab^3)$ is already smooth.

3.1.4 Remark. If the strict transform has a singular point in each coordinate chart, then we have to consider results from both charts.

Now suppose $\mathcal{C} \in T_0 = \mathbb{C}^2$ is a single branch passing through $0_0 = 0$. Blowing up T_0 at 0_0 gives us a new surface T_1 and a projection $\pi_0 : T_1 \to T_0$. Let $E_0 = \pi_0^{-1}(0)$ be the exceptional curve, and $\mathcal{C}^{(1)}$ the strict transform such that E_0 and $\mathcal{C}^{(1)}$ at a unique point $\{0_1\}$.

Now, suppose we have a surface T_i, curves E_j for $0 \leq j \leq i - 1$, and a strict transform $\mathcal{C}^{(i)}$ meeting E_{i-1} at a point 0_i. Blowing up T_i at 0_i gives us a new surface T_{i+1} and a projection $\pi_{i+1} : T_{i+1} \to T_i$. Let $E_i = \pi_{i+1}^{-1}(0)$ be the exceptional curve, $\mathcal{C}^{(i+1)}$ the strict transform of $\mathcal{C}^{(i)}$ such that E_i intersects $\mathcal{C}^{(i+1)}$ uniquely at a point 0_{i+1}.

3.1.5 Definition. From the above construction, the projection $\pi : T_N \to T_0$ is a called a *resolution* of $\mathcal{C}$ if $\mathcal{C}^N$ is smooth for some N.

The previous construction leads us to the following theorem:

3.1.6 Theorem. *A resolution $\pi : T_N \to T_0$ exists for some N, that is, $\mathcal{C}^{(N)}$ is smooth.*

Proof. See (Wall, 2004). □

3.1.7 Example. Consider the curve $\mathcal{C}$ defined by $V(X^2 - Y^5)$. Since we want to see the tangent cone of $\mathcal{C}$ at 0 which is given by $V(X)$, we apply the chart $(a, b) \to (ab, b)$. Then blowing up $\mathbb{C}^2$ at 0 gives us:

$$X^2 - Y^5 = (ab)^2 - b^5 = b^2(a^2 - b^3).$$

So the exceptional curve in $T_1 = \mathbb{C}^2$ is $E_0 = V(b)$, and $\mathcal{C}^{(1)} = V(a^2 - b^3)$. Observe that the other chart $(a, b) \to (a, ab)$ gives us:

$$X^2 - Y^5 = a^2 - (ab)^5 = a^2(1 - a^3b^5).$$

This gives us no information since $\mathcal{C}^{(1)} = V(1-a^3b^5)$ is already smooth and does not intersect the exceptional curve $E_0 = V(a)$.

Now, according to the result we get from the first chart, $E_0 \cap \mathcal{C}^{(1)} = \{0\}$, and $\mathcal{C}^{(1)}$ is still singular at 0. Applying the chart $(s, t) \to (st, t)$ gives us:

$$b^2(a^2 - b^3) = t^2(s^2t^2 - t^3) = t^4(s^2 - t).$$

The exceptional curve is $E_1 = V(t)$ and the strict transform is a smooth curve $\mathcal{C}^{(2)} = V(s^2 - t)$.

Let us also do the computation in the chart $(s, t) \to (s, st)$ to see where is the exceptional curve E_0 sent to. We have:

$$b^2(a^2 - b^3) = s^2t^2(s^2 - s^3t^3) = s^4t^2(1 - st^3).$$

We see that E_0 and E_1 intersect transversely in a point at infinity. See Figure 3.3.

3.2 Geometry of the resolution

In the previous construction, we obtained a resolution such that the strict transform $\mathcal{C}^{(N)}$ is smooth. However, in the above example, the exceptional curve is tangential to the strict transform. We say that a collection of curves in a smooth surface has a *normal crossing* if each curve is smooth, no three meet in one point, and any intersection of two is transverse. In this section, we will show that if we apply repeatedly the blowing up at $0 \in \mathcal{C}$, we will end with a resolution $\pi : T \to \mathbb{C}^2$ such that the collection $\pi^{-1}(\mathcal{C})$ has a normal crossing, and if $E = \pi^{-1}(0)$, then $T - E \simeq \mathbb{C}^2 - 0$. In that case, a such resolution is called a *good resolution*.

According to Theorem 3.1.6, we have a resolution such that the strict transform $\mathcal{C}^{(N)}$ is smooth. In order to keep applying the blow up, the first step is to check that the strict transform of a smooth curve is a smooth curve after applying a blow up.

3.2.1 Lemma. Let $\mathcal{C} \subset \mathbb{C}^2$ be a smooth curve such that $0 \in \mathcal{C}$. Blow up $\mathbb{C}^2$ at 0 gives us an exceptional curve E and a strict transform $\mathcal{C}'$ of $\mathcal{C}$. Then $\mathcal{C}'$ is smooth, it meets E at a single point, and the intersection is transverse. Moreover, $\mathcal{C}'$ is isomorphic to $\mathcal{C}$.

Proof. See (Wall, 2004). □

In the next step, let us investigate on the geometry of the intersection of the exceptional curves. We have the following lemma.

3.2.2 Proposition. The exceptional curve $E_i \subset T_{i+1}$ intersects transversely E_{i-1} and at most one E_j with $j < i-1$. And at most two E_i pass through a common point.

Proof. See C.T.C Wall. □

Then we get the following theorem.

3.2.3 Theorem. *Any plane curve singularity has a good resolution.*

Proof. See (Wall, 2004). □

3.2.4 Definition. Let $\pi : T \to \mathbb{C}^2$ be a good resolution. The strict transform $\tilde{\mathcal{C}}$ is called the *normalisation* of $\mathcal{C}$.

3.2.5 Example. Assume the result from Example 3.1.7 of the resolution of the cusp $V(X^2 - Y^5)$. We ended up having a strict transform $V(a^2 - b)$, a curve $E_1 = V(b^2)$ and another curve E_0 that we do not see in this coordinate chart but intersect E_1 at a far away point. Notice that E_1 is tangent $V(a^2 - b)$, that is, the intersection is not transverse. Then we continue applying a blow-up at 0. Since we want to see where the tangent cone $V(b)$ is sent to in the next picture, we apply the chart $(a, b) \to (u, uv)$. We have:

$$b^4(a^2 - b) = u^4v^4(u^2 - uv) = u^5v^4(u - v).$$

The exceptional curve E_1 correspond to $V(v^2)$. Therefore, we have a new exceptional curve $E_2 = V(u^3)$ and a strict transform $V(u-v)$. Notice that the curve E_1, E_2 and the exceptional curve intersect at 0 so we have to apply a blow up at 0.

Here, we want to see where both of E_1 and E_2, which are the axis, are sent to. Therefore, we have to apply both coordinate charts. In the chart $(u, v) \to (\mu, \mu\nu)$, we have:

$$u^5v^4(u - v) = \mu^5(\mu\nu)^4(\mu - \mu\nu) = \mu^{10}\nu^4(1 - \nu).$$

So we have a new curve $E_3 = V(\mu^4)$. The strict transform is $V(1-\nu)$, and E_1 is represented by $V(\nu^4)$.

In the other coordinate chart $(u, v) \to (\mu\nu, \nu)$, we have:

$$u^5v^4(u - v) = (\mu\nu)^5\nu^4(\mu\nu - \nu) = \mu^5\nu^{10}(\mu - 1).$$

We see that E_2 intersects transversely E_3 at a far away point in the first coordinate chart.

In the end, we have a collection $E_0, E_1, E_2, E_3, \tilde{\mathcal{C}}$ such that every intersection is transverse.

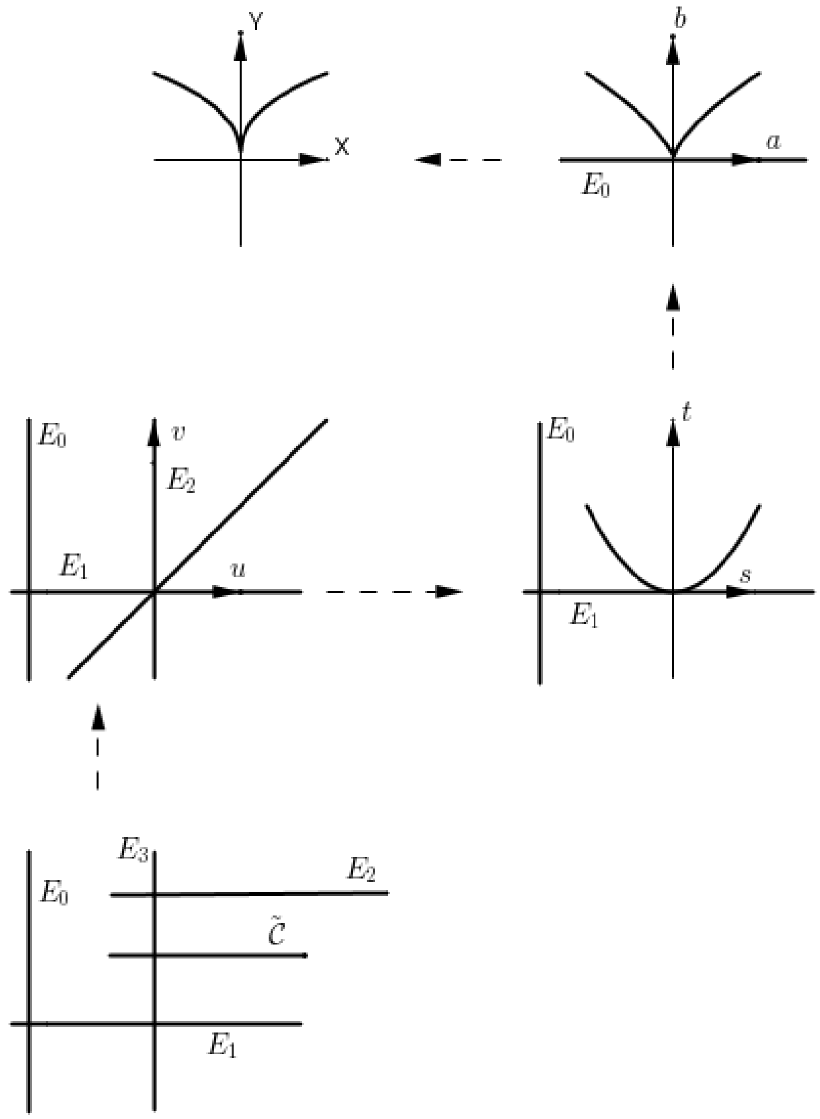


Figure 3.3: Successive blow up of the cusp $V(X^2 - Y^5)$ at 0.

3.3 Resolution and Puiseux parametrization

In this section, we describe the change of the Puiseux parametrization under the blowing up. Let $\mathcal{B}$ be a single branch.

3.3.1 Definition.

- A point in $E_{r-1} \subset T_r$ is said to be an *infinitely near point* of the rth order to 0.
- A point 0_i is said to be *proximate* to 0_j, $j < i$, if for $\varphi : T_{i+1} \to T_{j+1}$, 0_i lies in the strict transform of E_j. And we denote $0_i \to 0_j$.

An infinitely near point comes with a multiplicity $m_i(\mathcal{B})$ at 0_i of the strict transform $\mathcal{B}^{(i)}$.

We present some basic properties of the proximity relation.

3.3.2 Proposition. We have the following proximity relations.

1. $0_i \to 0_{i-1}$,
2. there is at most one $j < i-1$ such that $0_i \to 0_j$,
3. if $0_i \to 0_j$ and $j < k < i$, then $0_k \to 0_j$,
4. $m_j(\mathcal{B}) = \sum_{0_i \to 0_j} m_i(\mathcal{B})$.

In order to relate the sequence of multiplicity of the infinitely near points and the Puiseux parametrization, let us first state a theorem expressing the effect of a blow up to the parametrization.

3.3.3 Theorem. *Suppose given an irreducible curve such that the Puiseux characteristic is $(m; \beta_1, \ldots, \beta_g)$. Then the Puiseux parametrization of the strict transform obtained is*

$$\begin{array}{ll} (m; \beta_1 - m, \ldots, \beta_g - m), & \textit{if } \beta_1 \geq 2m, \\ (\beta_1 - m; m, \beta_2 - \beta_1 + m, \ldots, \beta_g - \beta_1 + m), & \textit{if } \beta_1 < 2m, \textit{ and } (\beta_1 - m) \nmid m, \\ (\beta_1 - m; \beta_2 - \beta_1 + m, \ldots, \beta_g - \beta_1 + m), & \textit{if } \beta_1 < 2m, \textit{ and } (\beta_1 - m) \mid m. \end{array}$$

Proof. See (Wall, 2004). □

Notice that applying this algorithm in this theorem successively yields the sequence of multiplicity $m_i(\mathcal{C})$.

Now, consider a sequence of multiplicity $m_i(\mathcal{C})$.

If $m_i(\mathcal{C}) = 1$ for all i, then the curve is smooth.

If there is only one $m_i(\mathcal{C}) \neq 1$, then, according to the previous theorem, we have $\beta_1 - m = 1$ and the Puiseux characteristic is $(m; m+1)$, where $m = m_0(\mathcal{C})$.

Now suppose that we blow up $\mathcal{C}$ once, and the Puiseux parametrization of the strict transform is $(m_1; \beta_1, \ldots, \beta_g)$. So the Puiseux parametrization of $\mathcal{C}$ is as follow:

$$\begin{array}{ll} (m_1; \beta_1 + m_1, \ldots, \beta_g + m_1), & \text{if } m_0(\mathcal{C}) = m_1(\mathcal{C}) \\ (m_0; \beta_1 + m_1, \ldots, \beta_g + m_1), & \text{if } m_1(\mathcal{C}) \nmid m_0(\mathcal{C}), \\ (m_0; m_0 + m_1, \beta_1 + m_1, \ldots, \beta_g + m_1), & \text{if } m_1(\mathcal{C}) \mid m_0(\mathcal{C}). \end{array}$$

From this construction and the previous theorem, we have the following:

3.3.4 Theorem. *The Puiseux characteristic of a branch $\mathcal{C}$ and the multiplicity sequence $m_i(\mathcal{C})$ determine each other.*

From the Theorem 3.3.3, we could establish some proximity relations as follow:

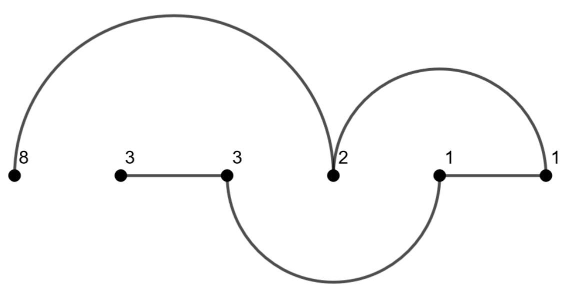
- if $m_0(\mathcal{B}) = m_1(\mathcal{B})$, then $0_1 \to 0_0$,
- otherwise, write $m_0(\mathcal{B}) = m_1(\mathcal{B})q + r$. If $r \neq 0$, then $0_{q+1} \to 0_0$.

3.3.5 Example. Consider the parametrization of the branch $\mathcal{B}$, $X = t^8, Y = t^{11}$. Therefore, the Puiseux parametrization is $(8; 11)$.

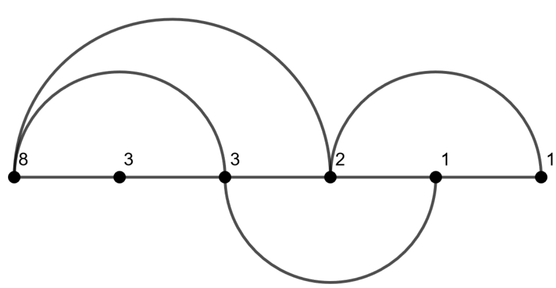
Let us first compute in details the multiplicity sequence of $\mathcal{B}$. By blowing up repeatedly, we get the following sequence of the Puiseux parametrization:

$$(8; 11) \to (3; 8) \to (3; 5) \to (2; 3) \to (1; 2) \to (1; 1).$$

From the discussion following Theorem 3.3.4, we have the following proximity relation.



And by applying the properties of the proximity relation in Proposition 3.3.2, we complete the graph as follow:



We show an efficient way to get the proximity relations from the Puiseux characteristic for a single branch $\mathcal{B} = V(Y^{ad} - X^{ad+bd})$ with $\gcd(a, b) = 1$. Consider the steps in the Euclidian algorithm for finding $\gcd(a, b)$.

$$\begin{aligned} a =& \quad bq_1 + r_1, \\ b =& \quad r_1 q_2 + r_2, \\ \cdots & \quad \cdots \\ r_{f-1} =& \quad r_f q_{f+1}. \end{aligned} \tag{3.3.1}$$

We write $s_k = \sum_{i=1}^{k} q_i$. Then the proximity relations are the following:

- $O_1, \dots, O_{s_1}$ are proximate to O_0,
- $O_{s_1+1}, \dots O_{s_2}$ are proximate to O_{s_1},
- in general, $O_{s_k+1}, \dots, O_{s_{k+1}}$ are proximate to O_{s_k}.

3.4 The resolution dual graph

In this section, we will represent geometrically the data we get from the resolution.

3.4.1 Definition. A *graph* Γ consists of a set $\mathcal{V}(\Gamma$ of vertices V_i and a set $\mathcal{E}(\Gamma)$ of edges joining the vertices.

Let $\mathcal{C} \subset \mathbb{C}^2$ be a curve and $\pi : T \to \mathbb{C}^2$ be a minimal good resolution. Here, we mean by *minimal good resolution* the first good resolution that we have when applying the blow up repeatedly. Consider the exceptional curves E_i for $0 \leq N$ and the strict transform $\mathcal{C}^{(N)}$.

3.4.2 Definition. Consider the graph whose vertex V_i corresponds to the curve E_i, and two vertices V_i and V_j are joined by an edge if an only if E_i intersects E_j. We say that this graph is the *dual graph* of the resolution, denoted by $\Gamma(\mathcal{C})$.

The *augmented dual graph* is the dual graph such that we add arrowhead vertices which represent the strict transform $\mathcal{C}^{(N)}$.

3.4.3 Example. Consider the result from the previous example.

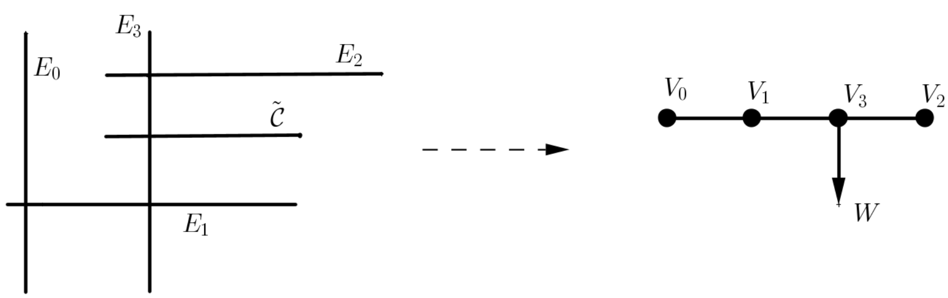
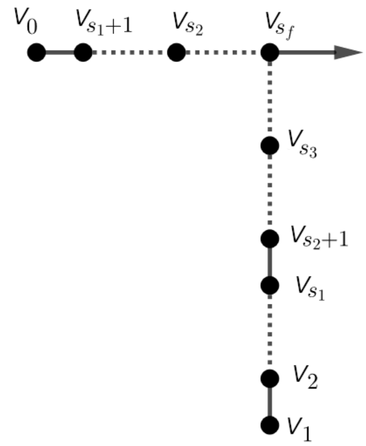


Figure 3.4: Resolution dual graph of the cusp $V(X^2 - Y^5)$.

The blow up of a plane curve consists of two coordinate charts $(x, y) = (uv, v)$ or (u, uv). We have a sequence of transformation of one type, followed by a sequence of transformation of the other type. The change-over corresponds to the proximity relation. Thus the sequence of vertices in the dual resolution graph is determined by the proximity relation which is induced by the Euler algorithm 3.3.1. From the Euler algorithm 3.3.1, we have a dual graph of the following form.



3.4.4 Lemma. If $\mathcal{C}$ is an irreducible curve with Puiseux characteristic $(m; \beta_1, \ldots, \beta_g)$, then the resolution dual graph is a single chain of edges from the initial vertex V_0 to the vertex W, with g side branches, each a single chain, attached to a distinct vertices of the original chain.

Proof. We use induction on g. The above construction describe the sequence of blow up for $g = 1$. Only the first two conditions in Theorem 3.3.3 are satisfied, except at the last blow up. So the above construction describe the sequence of blow up until the length of the Puiseux characteristics decreases. It also follow from this construction that any later vertices of the graph will be attached at v_{s_f}. □

3.5 Combinatorics on the resolution graph

In order to be able to do operations and study the property of the dual graph, we will associate to a vertex V_i of the dual graph three integers m_i, M_i and e_i which are respectively the multiplicity of an infinitely near point, the multiplicity and the Euler number of the exceptional curve E_i. The multiplicity of an infinitely near point is defined in Section 3.3. We define the multiplicity and the Euler number of an exceptional curve. In this section, we also study the relation between this integers.

Multiplicity of the exceptional curves. Consider a curve $\mathcal{C}$ and a minimal good resolution $\pi : T \to \mathbb{C}^2$. Notice that π is the composition of $\pi_i : T_{i+1} \to T_i$ where $T_0 = \mathbb{C}^2$. Now, suppose that in one chart of T_i, $\pi_{i-1}^{-1}(\mathcal{C}) = V(X^a Y^b f)$ such that the Puiseux parametrization of $V(f)$ is $(m; \beta_1, \ldots, \beta_g)$. Therefore, the equation of $\pi_i^{-1}(\mathcal{C})$ is either $u^{a+b+m} v^b f_1$ or $u^a v^{a+b+m} f_1$. In each case, the multiplicity of the new exceptional curve is $a + b + m$. From the properties of infinitely near points in Section 3.3, $m = m_i(\mathcal{C})$, and O_i is proximate to points in $V(X^a Y^b)$. We deduce the following relation between m_i and M_i.

$$M_i(\mathcal{C}) = m_i(\mathcal{C}) + \sum_{O_i \to O_j} M_j(\mathcal{C}).$$

Euler number. First, we introduce the notion of Euler number. Let us recall some results from topology. Consider two Hausdorff spaces T, B and a projection $\pi : T \to B$. We say that π is a *fiber bundle* over the base space B with total space T, fiber F and structure group K if we have a collection Φ of local trivialization $\varphi : U \times F \to \pi^{-1}(U)$, called *charts* over U, such that

- each point $P \in B$ has an open neighborhood U over which there is a chart in Φ,
- if $\varphi : U \times F \to \pi^{-1}(U)$ is in Φ and $V \subset U$ then $\varphi|_{V\times F} : V \times F \to \pi^{-1}(V)$ is in Φ,
- over a non-trivial overlap $U_\alpha \cap U_\beta$, the charts $\varphi_\alpha, \varphi_\beta$ yield a map $\varphi_\beta^{-1}\varphi_\alpha : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ such that $\varphi_\beta^{-1}\varphi_\alpha(u, x) = (u, g_{\alpha\beta}(u)x)$. A such map $g_{\alpha\beta}$ is called *transition function*.

A *section* of the fiber bundle is a continuous map $s : B \to T$ such that $\pi \circ s = \mathrm{id}_B$ and $s|_{U_\alpha} = s_\alpha : U_\alpha \to T$ verifying:

$$s_\alpha(u) = g_{\alpha\beta}(u)s_\beta(u).$$

3.5.1 Example. A blow up realizes a line bundle $\pi : T \to \mathbb{CP}^1$ over $\mathbb{CP}^1$ with fiber $\mathbb{C}$. Take the usual covering of $\mathbb{CP}^1$,

$$U_0 = \{[x : y] \;:\; y \neq 0\} = \{[\frac{x}{y} : 1], x, y \in \mathbb{C}, y \neq 0\},$$
$$U_1 = \{[x : y] \;:\; x \neq 0\} = \{[1 : \frac{y}{x}], x, y \in \mathbb{C}, x \neq 0\}.$$

Therefore, $g_{01}[z : 1] = \frac{1}{z}$. We have

$$s_0([z : 1]) = \frac{1}{z}s_1([z : 1]).$$

The condition $\pi \circ s = \mathrm{id}_B$ implies that $s_i(U_i) \subset \pi^{-1}(U_i) \simeq U_i \times \mathbb{C}$. Therefore, we could consider s_i as polynomial in z.

We define the *first Chern class* of a line bundle $\pi : T \to B$ as the difference

$$c_1(T) = \sharp\{\text{zero of } s\} - \sharp\{\text{pole of } s\}, \text{ where } s \text{ is a non-zero section.}$$

If we write $s_1([1 : z]) = a_0 + \cdots + a_n z^n$, then $s_0([z : 1]) = \frac{1}{z}s_1([z : 1]) = \frac{1}{z}s_1([1 : \frac{1}{z}])$. Notice that the zeros of s_1 are represented as poles of s_0, then counting the number of zeros and poles of s_0 suffices. Therefore, $c_1(T) = -1$.

The previous constructed dual resolution graph was decorated only with the multiplicity of the exceptional curves. Now, we will add a new decoration which are the Euler number of the exceptional curves as follow:

- the Euler number of a new exceptional curve is -1,
- the Euler number of an exceptional curve decrease by 1 if the blowing up point lies in that curve, otherwise, the Euler number remain unchanged.

Let us denote e_i the Euler number associated with an exceptional curve E_i.

3.5.2 Example. Consider the successive blow up at 0 of the cusp $V(X^2 - Y^5)$ as represented in Figure 3.3.

- From the first blow up, we have $e_0 = -1$.
- (2nd blow up) Since $0 \in E_0$, $e_0 = -2, e_1 = -1$.
- (3rd blow up) Since $0 \notin E_0, 0 \in E_1$, $e_0 = -2, e_1 = -2, e_2 = -1$.
- (4th blow up) Since $0 \notin E_0, 0 \in E_1 \cap E_2$, $e_0 = -2, e_1 = -3, e_2 = -2, e_3 = -1$.

3.5.3 Theorem. *Two curves $\mathcal{C}, \mathcal{C}'$ are equisingular if and only if there is an isomorphism of the dual resolution graphs $\Gamma(\mathcal{C}) \to \Gamma(\mathcal{C}')$ of minimal good resolution which preserves the number e_i.*

Remark that we can also define e_i as $e_i = [E_i] \cdot [E_i]$ in the homology group $H_2(T)$. Then for $[X] = \sum M_i(\mathcal{C})[E_i]$, we have

$$\begin{aligned} [X] \cdot [E_i] &= M_i(\mathcal{C})[E_i] \cdot [E_i] + \sum_{i \neq j} M_j(\mathcal{C})[E_j] \cdot [E_i] \\ &= M_i(\mathcal{C})e_i + \sum \{M_j(\mathcal{C}) \ : \ V_j \text{ is adjacent to } V_i\}. \end{aligned}$$

Denote by $\mathcal{E}_i$ the set of all j such that V_j is adjacent to V_i. Then we have the following relation between the multiplicity system and the Euler numbers:

$$M_i(\mathcal{C})e_i + \sum_{j \in \mathcal{E}_i} M_j(\mathcal{C}) = 0. \tag{3.5.1}$$

Define the *intersection matrix* $\mathcal{I} = (a_{ij})$ of a decorated graph Γ such that

$$a_{ij} = \begin{cases} e_i & \text{if } i = j, \\ 1 & \text{if } V_i, V_j \text{ is connected by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

3.5.4 Lemma. The intersection matrix $\mathcal{I}$ is negative definite. And the determinant $\det(-\mathcal{I})$ is invariant under blowing up and its inverse. Particularly, $\det(-\mathcal{I}) = 1$ for resolution of plane curve singularities.

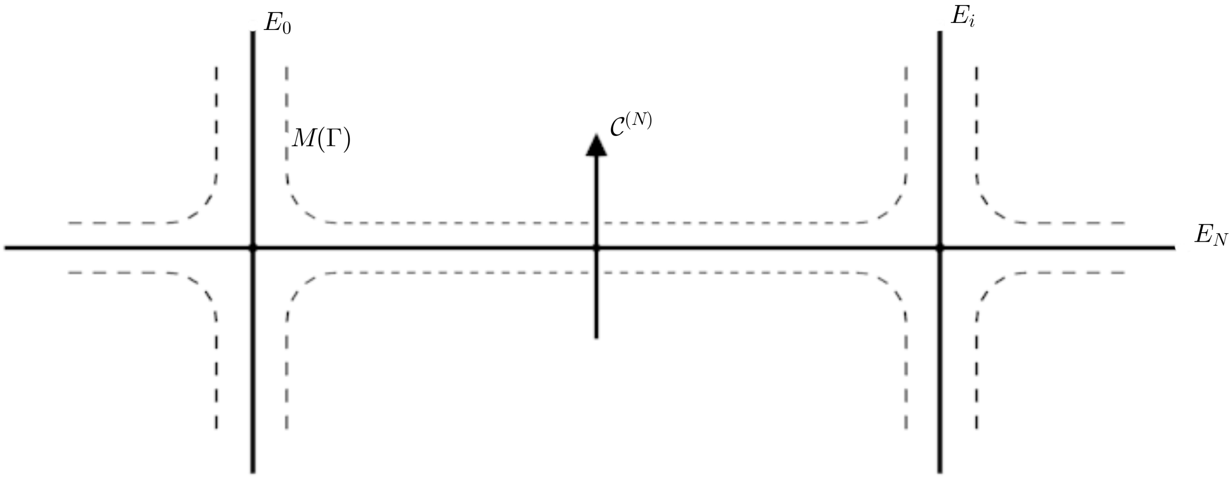
From the above construction, we define *a plumbing graph* as a graph such that any vertex V_i has two integers as decorations: the Euler number e_i and the *genus* $[g_i]$. For the case of the dual resolution graph of plane curve singularities, we omit the genus as they are all zero.

Plumbing construction. As the study is focused on the topology of the link associated with a singularity, we introduce the notion of oriented plumbed 3-manifolds $M(\Gamma)$ associated with a plumbing graph Γ.

We associate to a vertex $V \in \Gamma$ an S^1-bundle $\pi_V : B_V \to S_V$ such that S_V is a closed orientable real surface of genus g_V, and the Euler number (in here, first Chern class) of the bundle is the Euler number e_V. We choose the orientation of S_V and the fibers to be compatible with the orientation of B_V. Now consider two vertices $V_i, i = 1, 2$ joined by an edge. We fix for $i = 1, 2$, a point $P_i \in S_i$, an orientation preserving local trivialization

$D_i \times S^1 \to \pi_i^{-1}(D_i)$ above a small disc $D_i \ni p_i$. So the edge (V_1, V_2) of Γ determines $\partial D_1 \times S^1 \subset B_{V_1}$ and $\partial D_2 \times S^1 \subset B_{V_2}$ (both diffeomorphic to $S^1 \times S^1$) which are glued by an identification map $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We consider the plumber 3-manifold $M(\emptyset)$ associated with the empty graph as S^3. As stated in (Neumann, 1981), the plumbed 3-manifold $M(\Gamma)$ is invariant under action of blowing up and its inverse called blowing down.

For an embedded resolution graph, we can blow down and get an empty graph. Hence, one can see that the arrowheads represent the link associated with the singularities.



Since an algebraic link is completely determined by the set of Puiseux pairs, for an irreducible germ, one sees the relations between the Puiseux pairs and the embedded resolution graph. In order to establish this relation, we define first the *Hirzebruch continued fraction* representing $x \in \mathbb{R}$ as follow:

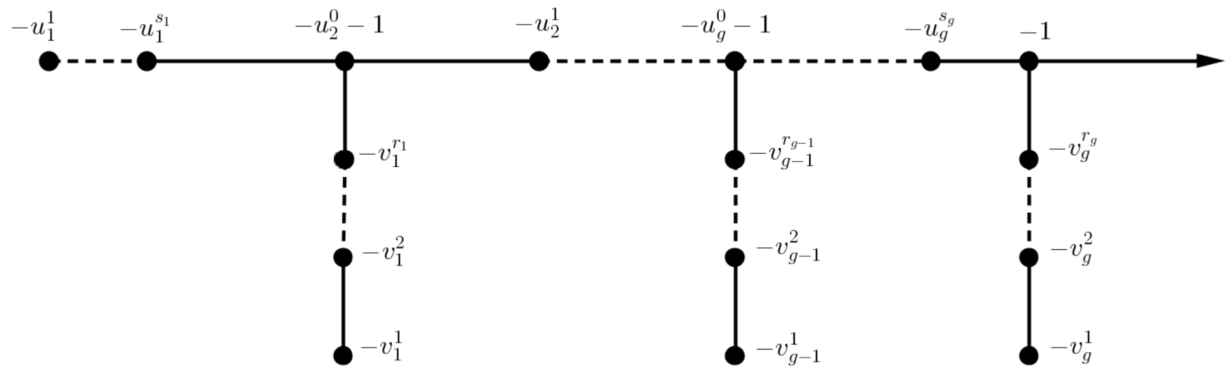
$$x = a_0 - \cfrac{1}{a_1 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}$$

We denote $x = [a_0, a_1, \ldots, a_n]$.

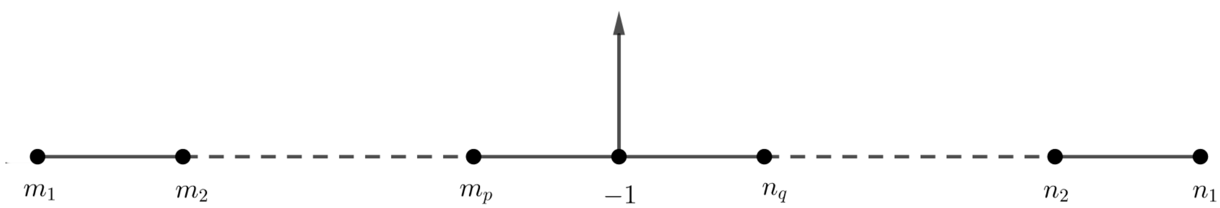
Now, assume we have a single branch curve $\mathcal{B} = V(f)$ and the Puiseux pairs associated $(m_1, n_1), \ldots, (m_g, n_g)$. Consider the associated continued fractions

$$\frac{m_i}{n_i} = [u_i^0, u_i^1, \ldots, u_i^{s_i}],\ \frac{n_i}{m_i} = [v_i^0, v_i^1, \ldots, v_i^{r_i}].$$

Then the embedded resolution graph decorated with Euler numbers is as follow:



Conversely, assume that the dual graph associated with $\mathcal{B}$ is of the following type:



Consider Γ_1 as

and Γ_2 as

Set $m = det(-I_1), n = det(-I_2)$ such that I_1 is the intersection matrix of Γ_1 and I_2 is the intersection matrix of Γ_2. Then (m, n) is the First Puiseux pair of $\mathcal{B}$.

3.5.5 Example. Consider the curve defined by $X^2 - Y^5$, and the results in Example 3.5.2, then $I_1 = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix}$ and $I_2 = -2$. Therefore, the first Puiseux pair is $(5, 2)$.

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