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THREE ESSAYS ON GROWTH, DEMOGRAPHY AND MACROECONOMICS

by

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Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Central European University

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CENTRAL EUROPEAN UNIVERSITY
DEPARTMENT OF ECONOMICS AND BUSINESS

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DISCLOSURE OF CO-AUTHORS CONTRIBUTION

More Gray, More Volatile?

Co-author: Zsuzsa Munkácsi

The nature of the cooperation and the roles of the individual-co-authors and approximate share of each co-author in the joint work: The paper was developed in close cooperation with Zsuzsa Munkácsi throughout all stages. We designed both the main question and the model of the paper together. We share the tasks during the literature review, coding, scenario and regression analysis and writing as well.

Convergence Stories of Post-Socialist Central-Eastern European Countries

Co-author: István Kónya

The nature of the cooperation and the roles of the individual-co-authors and approximate share of each co-author in the joint work: The paper was developed in close cooperation with István Kónya throughout all stages. We designed both the main question and the model of the paper together. We share the tasks during the literature review, coding, scenario and regression analysis and writing as well.

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Abstract

Two of the three essays explore the possible consequences of demographic aging, while the third one estimates the main shocks of the convergence period of the Central-Eastern European countries. The first chapter shows that the secular stagnation hypothesis is valid only for those countries where the economic agents' expectation is consistent with the theory of rational expectation. The second chapter (joint work with Zsuzsa Munkácsi) demonstrates that population aging contributes to a higher volatility of nominal variables, and the monetary policy becomes less efficient in influencing the output gap. The third chapter (joint work with István Kónya) applies a neoclassical growth model and shows that, behind of the post-socialist CEE countries, productivity and financial shocks were the key determinants of the convergence.

Chapter 1 - Secular Stagnation and the Role of Expectations

This paper reconsiders the secular stagnation hypothesis through the lens of bounded rationality. The consequences of population aging on medium- and long-term equilibria are at the core of current macroeconomic discourse. According to the secular stagnation hypothesis, in aging societies, the GDP growth decelerates and the natural rate of interest decreases when households accumulate more savings for a longer lifespan. However, the negative relationship between the old-age dependency ratio and the real interest rate can be rejected or weakly explained by historical data from OECD countries. This paper presents a multi-period, Gertler-type OLG model that incorporates bounded rationality and empirically shows that a declining real interest rate is valid only for those countries where the agents' behavior is consistent with the rational expectation equilibrium, or where the agents have a relatively long planning horizon.

Chapter 2 - More Gray, More Volatile?

Co-author: Zsuzsa Munkácsi

The empirical and theoretical evidence on the impact of population aging on inflation is mixed, and there is no evidence regarding the volatility of inflation. Using advanced economies' data and a DSGE-OLG model - a multi-period general equilibrium framework with overlapping generations - we find that aging leads to downward pressure on inflation and higher inflation volatility. Our paper shows how aging affects the short-term cyclical behavior of the economy and the transmission channels of monetary policy. We also examine the interplay between aging and optimal central bank policies. As aging redistributes wealth among generations, generations behave differently, and the labor force becomes more scarce. Our model suggests that aging makes monetary policy less effective, and aggregate demand less elastic to changes in the interest rate. Moreover, in grayer societies, central banks should react more strongly to nominal variables to compensate for higher inflation volatility.

Chapter 3 - Convergence Stories of Post-Socialist Central-Eastern European Countries

Co-author: István Kónya

This paper views the growth and convergence process of five Central-Eastern European economies - the Czech Republic, Hungary, Poland, Slovenia and Slovakia - through the lens of an open economy, stochastic neoclassical growth model. We estimate for these countries a version of the model augmented by simple financial frictions. Our main question is whether shocks to the growth rate of productivity ("trend"), or shocks to the external interest premium are more important to understand the volatility of GDP growth and its components. We find that while GDP growth fluctuations can be traced back to productivity shocks, the composition of GDP - and

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consumption in particular - was driven particularly by premium shocks. Investment-specific and labor-market shocks are also important. Our panel estimation allows us to separate global and local components for productivity-trend and interest-premium shocks. The results indicate that the global trend component is well approximated by the growth rate of the advanced European Union economies, and we also find tentative evidence that recent investment behavior is largely driven to a large extent by European Union funds. When looking at the global component of the implicit interest rate recovered from the estimation, we find that it tracks the observed real interest rate in the EU 15 countries until 2008, but sharply diverges thereafter. This final finding is consistent with the hypothesis that various capital market wedges and non-price restrictions to lending became important during and after the global financial crisis.

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This thesis is dedicated to the memory of my father who died unjustly early.

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Introduction

This thesis contributes to the growth and demography strands of the macroeconomic literature. In the past few years, my main research efforts have focused on the nature of growth and the effect of demographics on economic behavior. Due to the protracted post-crisis economic recovery, these topics have become highly relevant for both developed and emerging economies. Secular stagnation theory assumes substantially subdued economic growth and lower interest rates for the coming decades. The present work shows that changes in demographic structure impact not only the long-run equilibrium but also the short-run, cyclical behavior of economies, and thus affect the future optimal policies of central banks. In chapters 1 and 2, I revisit the secular stagnation hypothesis from two different perspectives. Chapter 1 shows what happens if we deviate from the rational expectation theory, while chapter 2 presents the consequences of aging on cyclical behavior and monetary policy decisions. Population aging is a challenge for emerging economies, and for Central-Eastern European countries as well, in addition to that the financial crisis challenged the sustainability of their previous solid convergence process and, compared to the pre-crisis period, the structure of growth has changed, too. During the pre-crisis, Great Moderation period, robust productivity-based economic growth was coupled with easy monetary conditions; after the crisis, the investment climate changed, and productivity improvement decelerated. To understand the importance of productivity and financial shocks in this case, chapter 3 estimates the core macroeconomic shocks of post-socialist Central-Eastern European countries in a small, open economy neoclassical model.

Chapter 1 reconsiders the secular stagnation hypothesis through the lens of bounded rationality. The standard overlapping generation models have very similar predictions for aging economies: slower growth accompanies a lower interest rate, and the central banks should keep the interest rate low to follow the natural rate of interest (Carvalho, Ferrero, and Nechio (2016)). However, these models are anchored by the rational expectation. The standard theory assumes that agents are fully aware of all future information, and that their decision in the present is consistent with their future behavior. During periods of population aging, rational economic agents will increase their savings, which exerts negative pressure on domestic interest rates. In chapter 1, I show how the assumption of non-rational behavior can change agents' current and future consumption and savings decision. The introduction of bounded rationality generates a non-decreasing interest rate and, indeed, it weakens the secular stagnation hypothesis. This theoretical contribution sheds light on a possible selection bias problem as well, namely, if one estimates the relationship between the demographic variables and real interest rate and does not control for the countries being fully or bounded rational. Similar to Ferrero, Gross, and Neri (2017) and Aksoy, Basso, Smith, and Grasl (2019), I demonstrate that the negative relationship is not robust in simple panel estimations, and that international spillover effects better explain the dynamic of the real interest rate than underlying demographic processes. However, the estimations with additional interaction terms for the rational behavior, are more robust, and I later show a negative relationship between demography and financial variables for rational economies (exclusively). This finding is in line with the prediction of the overlapping generation model with bounded rationality.

Beyond the bounded rationality extension, chapter 1 indicates that pension system size has an impact on the future equilibrium position of the real interest rate, and that this result is independent of expectation channels. In those economies where the intergenerational fiscal redistribution is large enough. Based on Istenič, Hammer, Šeme, Lotrič Dolinar, and Sambt (2016), the model is calibrated to an average European pay-as-you-go pension system; due to population aging, the natural rate of interest will be higher in the new steady-state equilibrium. This redistribution has a significant impact on the other steady-state variables, and it also changes the short-run behavior of the economies. In chapter 2 (co-authored with Zsuzsa Munkácsi), we compare the cyclical behavior of aging and young economies. In the former, monetary policy is less efficient in influencing the output gap; however, due to the scarcity of labor force, the volatility of nominal variables is higher than in young societies. To provide evidence for the latter phenomenon, we apply a panel estimation for the OECD countries, and we show that there is a significant positive relationship between the old-age-dependency ratio and the volatility of core inflation. These findings also have an impact on the optimal monetary policy decision. In the last part of the paper, we calculate such optimal simple monetary policy rules, showing how the optimal reaction should follow the aging process.

Finally, chapter 3 (co-authored with István Kónya) views the growth and convergence process of five Central-Eastern European economies - the Czech Republic, Hungary, Poland, Slovenia and Slovakia - through the lens of an open economy, stochastic neoclassical growth model. For these countries, we estimate a version of the neoclassical growth model augmented by simple financial frictions in a panel setup that allows us to separate global and local components. Our main question is whether shocks to the growth rate of productivity or shocks to the external interest premium are more important to understand the volatility of GDP growth and its components. We find that while GDP growth can be explained by productivity shocks, and that global productivity growth is strongly linked to the growth of the EU 15 growth rate. The composition of GDP - and consumption in particular - was driven mostly by premium shocks, and, post-crisis, investment-specific shocks, which were triggered by the European Union funds, have become more important.

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Chapter 1

Secular Stagnation and the Role of Expectations

"By the time you're eighty years old you've learned everything. You only have to remember it."

George Burns (1896-1996)

1.1 Introduction

In the past few decades, the aging has become a central topic in both developed and emerging economies. Accordingly, the economic consequences of demographic changes has garnered a great deal of research attention. According to the *secular stagnation* literature, the slower economic growth is coupled with the fall of the natural rate of interest (Ferrero, Gross, and Neri (2017))[1]. The latter phenomenon is a strong and robust prediction of standard DSGE-OLG models. However, the empirical findings for the long-term interest rate do not necessarily align with the theoretical papers. In a global VAR model, Aksoy, Basso, Smith, and Grasl (2019) show that demographic trends have a negative impact on long-term investment, growth and the long-term interest rate, but their results are not robust for two-way estimation. If one controls for the time fixed effect (the global common component of the long-term interest rate), the effect of demographic variables becomes insignificant. Other researchers have come to the same conclusion for the aging European population, and claim that the demographic trends in Europe do not support the secular stagnation hypothesis. In their view, the expected age-structure of the population will generate positive real rates in the future (Favero and Galasso (2015)). Population

[1] In New Keynesian terminology, the interest rate of the flexible price equilibrium is called the "natural rate of interest".

aging is a prominent problem in developed economies, and the common time fixed effect could capture the common demographic trends as well. However, contrary to the prediction of DSGE-OLG models, the non-significant coefficient of demography also means that there is no difference in the real interest rate based on the different demographic structures of different countries. In this paper, I try to provide an explanation for this puzzle and offer two contributions to the macroeconomic literature: (1) I show, in a model-based analysis, that the secular stagnation hypothesis and prediction of falling interest rate during the demographic transition period is valid for those countries where the agents' expectation is close to the rational expectation case - in economies with bounded rational agents, interest rates are less likely to be decreasing; (2) within a panel estimation on OECD countries, I demonstrate that the negative relationship between demographic variables and the interest rate remains robust to two-way estimation or common international spillover effects, if one controls for the behavioral differences between rational and bounded rational economies. My findings challenge the prediction of the standard DSGE-OLG models, although the empirical results are consistent with the bounded rational version of the OLG-model.

In this paper, which is based on a simplified version of the OGRE model (Overlapping Generations for Retirement, Baksa and Munkacsi (2016); Baksa and Munkacsi (2019)), I compare the traditional and bounded rational prediction of OLG models for the natural rate of interest at the time of population aging. The current version of the model assumes a simple, frictionless, one-sector economy. The households' behavior can be described by a Gertler-type OLG framework. The young generation supplies labor, pays lump-sum taxes to the government, and owns the firm of the economy. The old, retired households receive pensions from a pay-as-you-go (PAYG) pension system. Both cohorts are able to save or take loans and finance the public debt. During population aging, the lifespan of the retired cohort increases and the fertility rate of young households decreases. According to the standard theory, based on rational expectations, at the time of aging, agents change their saving or credit position. Households with an increasing lifespan decide to accumulate more savings and decrease their consumption. Young households, for their part, anticipate the increasing future financing needs of the public pension system, and the increasing level of private savings exerts negative pressure on the natural rate of interest. Additionally, this paper demonstrates that, independently from bounded rationality, the long-run position of the real interest rate depends on the size of the public pension system. In those countries where the redistribution from the worker cohort to pensioners is relatively high, and that aging can generate higher steady-state interest rates. Bounded rationality changes not only the long-run position of the real interest rate; it also has an impact on the short-run accommodation. Thus, it changes young households' savings attitude and makes them relatively more indebted, which results in a higher interest rate in the long run than in full rational equilibrium. According to the theory of bounded rationality, agents have cognitive limits, their future expectations are distorted, and households' consumption and savings decisions significantly differ from the rational expectation equilibrium. In behavioral macroeconomics, level-k thinking is the common way to model bounded rationality[2]. One can show that level-k thinking is a special case of myopia, and the size of k can be interpreted as the length of the planning horizon[3] (Lovo (2000)). In these settings, households take into account only the first k periods of future information; after period k, they expect that the economy will revert back to the initial steady state or to the initial balanced growth path equilibrium. The biased expectation channel could be crucial if the economy is affected by permanent demographic shocks. Despite the fact that population aging generates a continuously increasing financing problem in the pension system, agents with bounded rationality are less careful about their own future wealth and more serious fiscal issues. With relatively low k, young households consider increasing taxes or debt financing as a temporary economic event; hence, they do not adjust their permanent income expectation and consumption. In addition, they borrow more to avoid welfare loss - but the higher credit demand elevates the natural rate of interest, which increases financing costs and prompts the retired cohort to accumulate even more savings. In this paper, I compare the demographic transition with different k-s and the rational expectation equilibrium under differently sized PAYG pension systems.

The current paper also tests the empirical relationship between the demographic component and the long-term real interest rate in the OECD countries. In the first *naive* estimation, I show that the negative relationship is not necessarily robust for all countries. If one uses two-way estimation and involves the common time fixed effect or observes the common interest rate, the secular stagnation hypothesis does not hold for all OECD countries and the estimated parameters become weaker or insignificant. This result is consistent with Aksoy, Basso, Smith, and Grasl (2019). Nevertheless, the theory of bounded rationality sheds light on the weakness of the naive

[2]There are other types of non-rational models. According to Sims (2003), there are three categories for non-rational models: (1) behavioral economics literature; (2) learning literature; and (3) robust control literature. Sims (2003) suggested another direction, wherein people have a limited capacity for processing information.

[3]In this framework, the rational expectation can be interpreted as a special case of bounded rationality in which the agents consider all available information about the future.

empirical identification strategy. According to the naive two-way estimation, there is no control on any selection bias, and it is implicitly assumed that all OECD countries follow the same (rational) consumption and savings behavior. However, the model with bounded rationality implies that the negative relationship between demographic factors and interest rate is only true for those economies where the agents' expectation is close to the rational expectation. Therefore, the further empirical estimations are adjusted by interaction terms and the countries are separated into two categories: rational and non-rational. I used two different proxies: (1) the financial literacy indicator (Klapper, Lusardi, and van Oudheusden (2014)); and (2) the time preference from the Global Preference Survey (Falk, Becker, Dohmen, Enke, Huffman, and Sunde (2018)). In both specifications, the estimated coefficients of the demographic factor are significant and have a reasonable, negative value only for the countries considered financially literate or which have a higher index for time preferences. These results confirm the prediction of bounded rational OLG models; namely, the natural rate of interest is expected to be decreasing only in those countries where the agents' expectation is close to the rational case.

In the rest of the paper, Section 1.2 provides a review of the related literature and defines my contribution, and Section 1.3 describes the benchmark model, discusses the phenomenon of secular stagnation and shows the results of naive estimations. Section 1.4 compares the rational and bounded rational equilibria outcomes and long-term properties. Finally, in Section 1.5, by controlling for rational behavior with the interaction terms, I re-estimate the panel model and check the robustness of the results. The appendixes contain the list of the simplified model equations and the derivation of the core behavioral equation of bounded rational equilibrium. In a separate chapter, I provide a detailed derivation of Baksa and Munkacsi (2016) original model.

1.2 Literature review

Since the Great Recession and the prolonged economic recovery that followed it, secular stagnation has become a prominent topic in macroeconomics. The post-crisis US recovery, with decelerated productivity growth and population aging, resulted in a slower potential growth and a historically low interest rate. Nonetheless, according to Summers (2014), the effect of demography is negligible. In the past several decades, a great deal of research has examined the theory of secular stagnation, reconsidering the effect of population aging on gloomy recovery and period of low interest rates.

In the literature, overlapping-generation (OLG) models are the most common tool used to understand and predict the future consequences of population aging. However, the most recent empirical studies are not fully consistent with the theory-based models and lead to different conclusions. OLG models have the same long history as real-business cycle models. In a seminal paper that used a consumption life-cycle model, Auerbach and Kotlikoff (1987) examined the medium-term effect of different tax policies and demographic changes. Their model belongs to the Diamond-style OLG framework, where the households are distributed in well-defined cohorts and are assumed to have a fixed life-time horizon. In other types of OLG models, instead of using an explicit age-cohort assumption, the average life-time of households is expressed by survival rates (Blanchard (1985); Yaari (1965)). These models also dissolve the Ricardian equivalence[4], and through intergenerational redistribution, the fiscal policy can directly influence the agents' behavior. It is also possible to adjust the Blanchard-Yaari specification with an additional cohort to separate workers and retired households[5] according to their labor market participation and eligibility for pension benefit (Gertler (1999)). The Blanchard-Yaari-type and Gertler-type models have a structure that is quite similar to dynamic stochastic general equilibrium (DSGE) models, and they can be easily combined with New-Keynesian features. However, until the Great Recession, these models were not treated widely in macroeconomic discussions. The slow recovery, gloomy demographic outlook, and later the emerging fiscal imbalances, favored the alternative, non-Ricardian interpretation of fiscal policy. The Global Integrated Monetary and Fiscal Model (GIMF) was among the first to have implemented the Blanchard-Yaari version of OLG models (Kumhof, Laxton, Muir, and Mursula (2010)), and later many other papers included these types of non-Ricardian features. The fundamental difference between the OLG and the representative models lies in the description of households' behavior. While in the DSGE-models the infinitely lived representative households' consumption can be described by the Euler equation, in the OLG models, each individual optimizes on an finite horizon and his own initial wealth position; the aggregate behavior cannot be described by a cohort-level Euler equation, thus one should explicitly solve the individual consumption function, and these functions should be aggregated to express cohort-level consumption and savings.

One of the main focal points of today's OLG literature is the current and expected position

[4]Ricardian equivalence is a common property of DSGE or RBC models that is caused by forward-looking rational behavior, an infinite planning horizon, and a lack of liquidity constraints. According to the equivalence theorem, households are neutral regarding fiscal redistribution; thus, in terms of welfare, there is no difference between the timing of a tax increase or a domestically financed debt issuance.

[5]The workers with a given probability become retired, and the retired households survive each period with a given probability.

of the natural rate of interest. The demographic transition has a significant impact on the real economic variable and the long-term interest rate (Carvalho, Ferrero, and Nechio (2016)). The demographic trend explains 1.5 percentage points of the interest rate decline between 1980 and 2030, and it has also been shown that the decrease of the natural rate of interest may contribute to deflationary pressure, if the central bank is not able to follow the flexible price consistent interest rate (Bielecki, Brzoza-Brzezina, and Kolasa (2018)). Additionally, it has been reported that, in the US, demographic factors contributed to the sluggish recovery and depressed monetary conditions (Gagnon, Johannsen, and Lopez-Salido (2016); Eggertsson, Mehrotra, and Robbins (2017); Jones (2018)).

Compared to the theoretical findings, the empirical results are less clear. Ferrero, Gross, and Neri (2017) found a relationship between demographic factors and the interest rate. According to Arslanalp, Lee, and Rawat (2018), in Asian economies, the interest rate and demography is well connected. In Europe, however, the real interest rate will recover, and, because the secular stagnation hypothesis is not valid, the long-term interest will not decrease (Favero and Galasso (2015)). Aksoy, Basso, Smith, and Grasl (2019) showed in a panel VAR model that the demographic structure has an effect on medium-term growth and long-term yields, although their results for the natural interest rate is not robust for the two-way estimation. I will demonstrate later, in a simple *naive* panel estimation, that the relationship between the demographic factor and the long-term real interest rate disappears or becomes weak, as one involves a time-fixed effect to control for international spillover effects. In the rest of the paper, I show possible explanations for the mis-specification, and, based on the extension of bounded rationality, I provide insight into how to adjust the empirical estimation.

As the Great Recession continued, a new flow of economic theories, as well as behavioral macroeconomics, gained popularity. Conlisk (1996) summarized the main advantages of models with bounded rationality. The concept of level-k thinking was introduced by Fair and Taylor (1983), Evans, Honkapohja, and Mitra (2017), Evans and Ramey (1998) and Evans, Honkapohja, and Mitra (2010). These papers showed how different equilibria can be calculated from the iteration process, and compared rational and bounded-rational behaviors. It can be shown that bounded rationality is equivalent to myopic behavior and can be linked to the length of the planning horizon (Lovo (2000)). Despite non-rational behavior in these models, the Ricardian equivalence proposition can continue to hold, and the existence of equivalence depends on the government's transversality conditions (Evans, Honkapohja, and Mitra (2010)). Farhi and Werning (2017) derived analytically how interest rate elasticity can change if agents have bounded rationality with level-k thinking and occasionally binding borrowing constraints. Gabaix (2017) described the properties of a behavioral New-Keynesian model and compared the impulse responses of typical macroeconomic shocks; however, his paper did not focus on fiscal policies. Gabaix (2017) show that a life-cycle model with bounded rationality can generate a hump-shaped consumption profile that matches the US data.

My paper makes several contributions to the macroeconomic literature. To the best of my knowledge, this is the first paper that merges Gertler-type OLG models with bounded rationality and level-k thinking to examine population aging with non-rational expectations. My paper complements that of Farhi and Werning (2017) and Gabaix (2017) with a focus on fiscal policy, demographic shock, and an overlapping generation framework. I describe the general equilibrium effect on the natural interest rate, and do not assume a time-invariant, long-term interest rate, as in Park and Feigenbaumz (2018). Finally, this extension provides economic evidence and insight into the empirical identification strategy, if one seeks to estimate the relationship between demography and the long-term interest rate. I show that the negative coefficient between demographic factors and the interest rate is robust and significant for those economies only where households' behavior is consistent with the rational expectation equilibrium.

1.3 Secular stagnation hypothesis and Overlapping Generations

This section describes a benchmark Gertler-type overlapping generation model and the secular stagnation hypothesis. The model is a simplified version of Baksa and Munkacsi (2019)[6].

1.3.1 An OLG-model a la Gertler

In the Gertler-type OLG models, there are two cohorts (workers and retired). The workers arrive to the cohort with the rate n, and become retired in the next period, with probability ω^Y. The retirees pass away with probability ω^O. The households in each cohort are able to save or consume from their disposable income. The workers earn wage income (wL) from the firms of the economy, and, as the owners of the firms, they also receive profits from the firms. The government is responsible for the pay-as-you-go (PAYG) pension system, where the benefit is the function of the wage income from the pre-retirement period and the exogenous replacement rate (ν). At the time of retirement, based on previous wage income flow, the government calculates the

[6] Detailed derivations of the model can be found in the Appendix.

just-retired pension and supplies this benefit until the death of the retiree. The government also issues risk-free bonds that are accumulated by both households. In the overlapping generation models, the interest rate is the explicit function of the bonds market equilibrium, which also implies, in the steady state, that the interest rate can not simply be calculated by the inverse of time-preference from the households' utility function. Below, I summarize the behavioral equation of the model.

The old-age dependency ratio s, the number of the retired population divided by the working-age population, can be given by survival rates, fertility rates, and the previous value of the old-age-dependency ratio. s^Y, the relative size of the worker cohort can be given by the function of the old-age dependency ratio:

$$s_t = \frac{(1-\omega^O_{t-1})}{(1-\omega^Y_{t-1}+n_t)} s_{t-1} + \frac{\omega^Y_{t-1}}{(1-\omega^Y_{t-1}+n_t)} \tag{1.1}$$

$$s^Y_t = \frac{1}{1+s_t} \tag{1.2}$$

The cohort level ($g^{N,Y}$ is the growth rate of the worker cohort, $g^{N,O}$ is the growth rate of retired cohort), and total population growth (g^N) can be defined as the function of survival rates and the old-age dependency ratio:

$$1+g^{N,Y}_t = 1-\omega^Y_{t-1}+n_t \tag{1.3}$$

$$1+g^{N,O}_t = (1-\omega^O_{t-1}) + \frac{\omega^Y_{t-1}}{s_{t-1}} \tag{1.4}$$

$$1+g^N_t = (1+g^{N,Y}_t)\frac{1+s_t}{1+s_{t-1}} \tag{1.5}$$

The dynamic optimization of individuals can be described by the Bellman equation, where the maximizing utility function is the combination of individual consumption and leisure (leisure matters for the workers only). Due to the overlapping generation framework, the first-order conditions do not describe the representative behavior of the households, because the agents are born in different periods and can have different wealth positions. Therefore, based on the first-order conditions, we need to express the individuals' explicit consumption function from the Euler equation and the intertemporal budget constraint. In the final step of the derivations, with the sum of all individuals' consumption and bonds, one can express the cohort-level aggregate consumption and aggregate bond accumulation. In growth models, it is also common that non-price-related variables are expressed in terms of the balanced growth trend[7]. The aggregate per capita variables are functions of the demography, and thus, the aging shock directly affects the balanced growth path and the short-run dynamics of the normalized variables too.

Following the logic above, first one can derive the consumption function of retirees, and later that of workers. The retirees' consumption (C^O) is the function of the expected permanent income, the initial bonds of the survived retired population (B^O), the inherited bond of just-retired (B^Y), and the marginal propensity to consume (MPC^O). The permanent income is the function of the actual pension (TR) and the discount factor of retired cohort (Ω^O), which takes into account the current and expected real interest rate and the probability of death. The retired cohort consumption function is the following:

$$\tilde{C_t^O} = MPC_t^O \tilde{TR}_t \Omega_t^O + MPC_t^O \frac{(1+r_{t-1})}{1+g_t^N}\left(\omega_{t-1}^Y \tilde{B_{t-1}^Y} + \tilde{B_{t-1}^O}\right) \tag{1.6}$$

where

$$\Omega_t^O = 1 + E_t \frac{1-\omega_t^O}{1+r_t}\Omega_{t+1}^O \tag{1.7}$$

$$\frac{1}{MPC_t^O} = 1 + E_t(1-\omega_t^O)(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\frac{1}{MPC_{t+1}^O} \tag{1.8}$$

γ is the inverse of intertemporal elasticity from the households' utility function, β is the time-preference, and r is the real interest rate.

The pay-as-you-go (PAYG) pension system and pension expenditures are functions of two components: (1) the just-retired initial pension that is linked to the pre-retirement labor income (wL) and the exogenous replacement rate (ν); and (2) the old-retireds' pension:

$$\tilde{TR}_t = \nu \frac{\omega_{t-1}^Y}{1+g_t^N} w_{t-1}\tilde{L}_{t-1} + \frac{(1-\omega_{t-1}^O)}{1+g_t^N}\tilde{TR}_{t-1} \tag{1.9}$$

The pension expenditures are connected to biological factors via the survival rate; thus, the increasing longevity (decreasing ω^O) ceteris paribus generates higher fiscal expenditures. Any intervention to the pension system, i.e.: increase of retirement age or decrease of replacement ratio, can go through only the just-retired benefit; thus, pension reforms can slowly stabilize the fiscal balance.

Workers' behavior can be also described by a dynamic optimization problem. However, their

[7]The $\tilde{x}_t$ denotes the value of x_t normalized by the balanced growth path at period t. In this version of the model, the balanced-growth path is a function of the population growth, and I do not assume productivity growth.

utility function contains leisure and σ shows the relative importance of consumption in the utility function. The individual decision takes into account the probability of the next period of retirement (ω^Y) and the expected value of future labor income. Households do not know the exact time of their own retirement, but, due to the state-contingency assumption at the time of retirement, all of their previously accumulated wealth is transferred into their retired-self balance sheet; thus, individuals are able to smooth out their own consumption between two lifetime periods. After the aggregation and normalization, the workers' cohort-level consumption function (C^Y) can be written as

$$\tilde{C_t^Y} = MPC_t^Y \tilde{Inc_t} + MPC_t^Y \frac{(1+r_{t-1})(1-\omega_{t-1}^Y)}{1+g_t^N} \tilde{B_{t-1}^Y} \tag{1.10}$$

where the expected income is the function of the current disposable income, and the ω^Y weighted next period retired income or $1-\omega^Y$ weighted future net labor income:

$$\tilde{Inc_t} = w_t s_t^Y + \tilde{Profit_t} - \tilde{Tax_t} + E_t \frac{\omega_t^Y \nu w_t \tilde{L_t} \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \tilde{Inc}_{t+1} \tag{1.11}$$

The labor supply curve can be derived from the first-order condition:

$$\frac{\tilde{C_t^Y}}{s_t^Y - \tilde{L_t}} = \frac{\sigma}{1-\sigma} w_t \tag{1.12}$$

The marginal propensity to consume (MPC^Y) is the function of the real interest rate, the weighted average next period MPC-s:

$$\frac{1}{MPC_t^Y} = \frac{1}{\sigma} + E_t(1+r_t)^{\frac{1}{\gamma}-1} \left[(1-\omega_t^Y)\Lambda_t^Y \frac{1}{MPC_{t+1}^Y} + \omega_t^Y \Lambda_t^{YO} \frac{1}{MPC_{t+1}^O} \right] \tag{1.13}$$

For simplification, we assigned an additional two variables from the equation of MPC^Y:

$$\Lambda_t^Y = \beta^{\frac{1}{\gamma}} \left(E_t \frac{w_{t+1}}{w_t} \right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \tag{1.14}$$

$$\Lambda_t^{YO} = \left\{ \frac{\beta}{\sigma} \right\}^{\frac{1}{\gamma}} \left(\frac{1}{\frac{\sigma}{1-\sigma} w_t} \right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \tag{1.15}$$

Due to the two distinct cohorts and two bonds, from the workers' period-t budget constraint one can express the law of motion for risk-free bonds:

$$\tilde{B_t^Y} = w_t\tilde{L_t} + Pr\tilde{o}fit_t - T\tilde{a}x_t - \tilde{C_t^Y} + \frac{(1+r_{t-1})}{1+g_t^N}(1-\omega_{t-1}^Y)\tilde{B_{t-1}^Y} \tag{1.16}$$

The profit-maximizing firms have the usual Cobb-Douglas production function, and for simplicity I assume that these firms are price-takers. Production function can be given by:

$$\tilde{Y_t} = A_t\left(\frac{\tilde{K}_{t-1}}{1+g_t^N}\right)^{\alpha}\tilde{L}_t^{1-\alpha} \tag{1.17}$$

The firms are also responsible for capital accumulation, the law of motion for capital is the following:

$$\tilde{K_t} = I\tilde{n}v_t + (1-\delta)\frac{\tilde{K}_{t-1}}{1+g_t^N} \tag{1.18}$$

where δ is the depreciation of capital and Inv is the level of private investment. Labor demand and implicit capital demand functions can be given by the following equations:

$$w_t = (1-\alpha)\frac{\tilde{Y_t}}{\tilde{L_t}} \tag{1.19}$$

$$1+r_t = E_t\alpha(1+g_{t+1}^N)\frac{\tilde{Y}_{t+1}}{\tilde{K_t}} + (1-\delta) \tag{1.20}$$

The profit can be given as

$$Pr\tilde{o}fit_t = \tilde{Y_t} - w_t\tilde{L_t} - I\tilde{n}v_t \tag{1.21}$$

Following the New-Keynesian terminology, the flexible price assumption implies that the total output and the real interest rate can be interpreted as the potential output and the natural rate of interest.

The government budget constraint describes the law of motion for public debt, which is ceteris paribus increasing if government expenditures exceed tax revenues or the government has to pay a higher interest-cost:

$$D\tilde{e}bt_t = T\tilde{R}_t + G\tilde{o}v_t - T\tilde{a}x_t + \frac{(1+r_{t-1})}{1+g_t}D\tilde{e}bt_{t-1} \tag{1.22}$$

where the government consumption (Gov), taxes (Tax) or the debt level ($Debt$) can be exogenously given. In this setup, I assume that the fiscal policy follows a simple lump-sum tax rule, and in every period the government adjusts the taxes in order to meet its debt target:

$$\tilde{Tax}_t = \tilde{TR}_t + \tilde{Gov}_t + \frac{(1+r_{t-1})}{1+g_t}\tilde{Debt}_{t-1} - \left\{\frac{Debt}{Y}\right\}^{Target} Y_t \tag{1.23}$$

It is also possible to implement a fiscal rule that anchors the variables (see Baksa and Munkacsi (2016) and Baksa, Munkacsi, and Nerlich (2019)).

In OLG models, the bond market equilibrium should be satisfied explicitly. The bond market equilibrium is an essential part of the equilibrium conditions because it determines the equilibrium interest rate:

$$\tilde{Debt}_t = \tilde{B}_t^Y + \tilde{B}_t^O \tag{1.24}$$

Based on the agents' budget constraints, one can derive the usual market clearing conditions that characterize a good market equilibrium:

$$\tilde{Y}_t = \tilde{C}_t^Y + \tilde{C}_t^O + \tilde{Inv}_t + \tilde{Gov}_t \tag{1.25}$$

The model can be simplified into two main equations that characterize the transitional dynamics and steady-state equilibrium. Based on both cohorts' consumption function, labor supply curve and λ-s, one can derive a dynamic IS-curve that explicitly describes the workers' bonds supply curve:

$$\begin{aligned}\left(\frac{1}{MPC_t^Y} - \frac{1}{\sigma}\right)\tilde{C}_t^Y = & \underbrace{\tilde{B}_t^Y\left(1 - \frac{(1-\omega_t^Y)^2}{1+g_{t+1}^{N,Y}}\right)}_{\text{Wealth effect}} + \underbrace{E_t\frac{\omega_t^Y}{1+r_t}\nu\alpha\tilde{Y}_t\Omega_{t+1}^O}_{\text{Expected pension}} + \\ & + \underbrace{E_t\frac{1-\omega_t^Y}{1+r_t}\frac{1+s_{t+1}}{1+s_t}\frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y}}_{\text{Workers' expectation}}\end{aligned} \tag{1.26}$$

where the 'Wealth effect' assigns the direct effect of the accumulated bonds on workers' consumption. Due to overlapping generations, 'Expected pension' also influences the non-retired behavior; however, its effect on expectation is relatively small. 'Workers' expectation' denotes the effect of expected future consumption on contemporaneous consumption. It can be shown that, if $\omega_t^Y = 0$ for all t and there are no retirees in the economy, this equation collapses into the standard Euler equation of representative real business cycle models.

The demand for the workers' bond can be expressed as the following:

$$\begin{aligned} \tilde{B}_t^Y &= \underbrace{\tilde{Debt}_t}_{\text{Public debt}} - \underbrace{(1 - MPC_t^O \Omega_t^O)\tilde{TR}_t}_{\text{Savings from pension}} - \\ & -\frac{(1 + r_{t-1})}{1 + g_t^N}(1 - MPC_t^O) \underbrace{\left[\tilde{Debt}_{t-1} - (1 - \omega_{t-1}^Y)\tilde{B}_{t-1}^Y\right]}_{\text{Non-consumed pensioner savings}} \end{aligned} \tag{1.27}$$

where 'Public debt' is the actual level of government debt (total demand for all households' savings); 'Savings from pension' is the non-consumed part of the life-time income of pensioners; and 'Non-consumed pensioner savings' denotes the reinvested part of the period $t-1$ retirees' bonds. The latter two components denote the current savings of the retired cohort, and the rest of the debt should be covered from the workers' savings. The above two equations determine the optimal consumption-savings position of the workers' cohort. With the labor-supply curve, production function, and good market equilibrium, one can derive the rest of the other variables.

1.3.2 Parametrization of the model

The calibration of the model follows Baksa and Munkacsi (2019) without monopolistic competition and price stickiness (see Table 1.1). The survival probabilities and fertility rates are calculated from the German, Portuguese, Spanish, and Slovak population database of the Eurostat between 1975 and 1990 in such a way as to capture the old-age-dependency ratio in the early 1990s. The worker cohort covers the generation between the 20- and 64-year-old population, and pensioners are the 65+ year-old population. The fertility rate can be calculated by the ratio of the 19-year-old population and the worker cohort.

The parameters of households' behavior are taken from the Pessoa model, the Portuguese version of the GIMF model (Almeida, Castro, Félix, J'ulio, and Maria (2013)). The discount rate is annualized, and the weight of consumption utility and intertemporal elasticity are taken from the households with access to the financial market. The values for α and δ are typical values in the macroeconomic literature, and the same as in the New Area-Wide model (Christoffel, Coenen, and Warne (2008)).

The replacement ratio was calibrated in such a way that the received transfer to the GDP ratio expresses the total pension spending (retirement benefits and other old-age-related entitlements from the public sector). Istenič, Hammer, Šeme, Lotrič Dolinar, and Sambt (2016) calculated the generational redistribution for 25 member states of the European Union. This ratio expresses the overall income and other non-cash payments that governments reallocate from the working-age population to the 65+ age population. The government consumption-to-GDP ratio is calibrated to the average value of government individual consumption, while the debt-to-GDP ratio is the Maastricht criteria, which can be considered as a long-term debt target of European countries.

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Table 1.1: Parametrization of the model

Name	Notation	Value	Source, Comments
Demography			
Probability of retirement	ω^Y	0.02	Eurostat Population data
Probability of death	ω^O	0.09	Eurostat Population data
Fertility rate	n	0.03	Eurostat Population data
Old-age dep. ratio	s	0.2	Implied ratio, $(65+/20-64$ population)
Households			
Discount rate	β	0.9841	Almeida, Castro, Félix, J'ulio, and Maria (2013); annualized
Weight of consumption utility	σ	0.73	Almeida, Castro, Félix, J'ulio, and Maria (2013)
Inverse of intertemporal elasticity	γ	2	Almeida, Castro, Félix, J'ulio, and Maria (2013)
Firms			
Capital share	α	0.36	Christoffel, Coenen, and Warne (2008)
Capital depreciation	δ	0.1	Christoffel, Coenen, and Warne (2008); annualized
Fiscal policy			
Replacement rate	ν	0.9219	Istenič, Hammer, Šeme, Lotrič Dolinar, and Sambt (2016); $\frac{TR}{Y} = 0.118$
Gov. cons. in % of GDP	$\frac{Gov}{Y}$	0.1	Eurostat SNA
Debt in % of GDP	$\frac{Debt}{Y}$	0.6	Maastricht criteria

1.3.3 Population aging and secular stagnation

This part describes the conventional prediction of OLG models and the secular stagnation hypothesis at the time of population aging under differently sized PAYG pension system (see Figure 1.1). This stylized scenario, based on the population projection of selected European countries[8], assumes that, in 15-20 years, the fertility rate and the probability of death gradually decrease from their steady-state level to 1.7% and 3.65%, respectively. The permanent changes of these two shocks increase the old-age dependency ratio from 20% to 60%, which is similar to the past and the expected population changes between 1980 and 2060. The response functions describe the short- and medium-term accommodation, and later I compare the initial and terminal steady state of the equilibrium interest rate and workers' consumption-to-GDP ratio. Additional decomposition charts are available in the Appendix that show separately the contribution of the shock to fertility and shock to the probability of death under different pension systems (see Figure B.1-B.3).

Aging distorts the demographic distribution, and the increasing old-age dependency ratio indicates that the relative size of the retired cohort exceeds its initial level; the changes in fertility and longevity equally increase the old-age dependency ratio. Due to the lower share of the young population and the decreasing labor supply, regardless of the size of the pension system, the GDP per capita starts decreasing. Due to the shrinking GDP, the government has to in-

[8]The values are calculated for Germany, Slovakia, Portugal, Spain; taken from Baksa, Munkacsi, and Nerlich (2019)

crease lump-sum taxes to sustain the targeted public debt-to-GDP ratio[9]. The lower economic performance is strongly connected to the decreasing level of labor supply; hence, the need for higher lump-sum tax revenues is more driven by the shock to fertility (see Figure B.1). The medium and long-term macroeconomic consequences of population aging can be significantly different under differently sized pension system. Namely, the increasing longevity in the PAYG pension system automatically results in a higher pension expenditure and forces the government to issue even more public debt or raise taxes further[10].

The workers realize the longer future lifespan and are also aware that they need to pay more taxes to the government later; therefore, from the first moment of the transition, young households adjust their behavior: they consume less and also give up some savings to minimize the sacrifice of lower consumption, after which they smooth out their future consumption path. If there is no PAYG pension system, old households are forced to accumulate more private savings for the longer future lifespan. The decreasing demand exerts negative pressure on the GDP, the workers' income, and their future consumption; hence, the shock to the probability of death negatively contributes to both cohorts' consumption. Due to the quick accommodation of the private sector, the higher level of private savings puts permanent negative pressure on the real interest rate. These findings are consistent with Carvalho, Ferrero, and Nechio (2016) and the OLG literature.

In the PAYG system, pensioners are able to increases their consumption as they receive relatively more benefits from the government. However, compared to the initial level, they still save more to secure their individual consumption on an elongated life-time horizon. These savings are translated into credit for the young generation and indirectly finance the government budget. In the short run, regardless of the size of the public pension system, due to the quick accommodation of the private sector, the real interest rate falls. Nevertheless, the size of the initial pension system significantly determines the new steady-state position of the interest rate. By relatively large generational redistribution, the demographic changes result in a higher-than-initial interest rate in the steady state, because under an oversized PAYG system the private sector has to pay more taxes and the young generation becomes more indebted, or the government increases the public debt issuance and demands more private savings. Both scenarios elevate long-term interest rates.

[9] In this paper, I always assume that the fiscal policy increases the taxes to offset any debt increase. However, in the closed economy model, most of the government debt is financed by the workers' cohort, in which case the increase of lump-sum taxes or the increasing public debt generates a similar outcome.

[10] This paper does not examine the effect of other pension reforms eg.: increase of retirement age, increase of contribution rate or decrease of replacement rate. In Baksa and Munkacsi (2016) and Baksa, Munkacsi, and Nerlich (2019) paper, we exam the macroeconomic effect of various pension reforms in different European economies.

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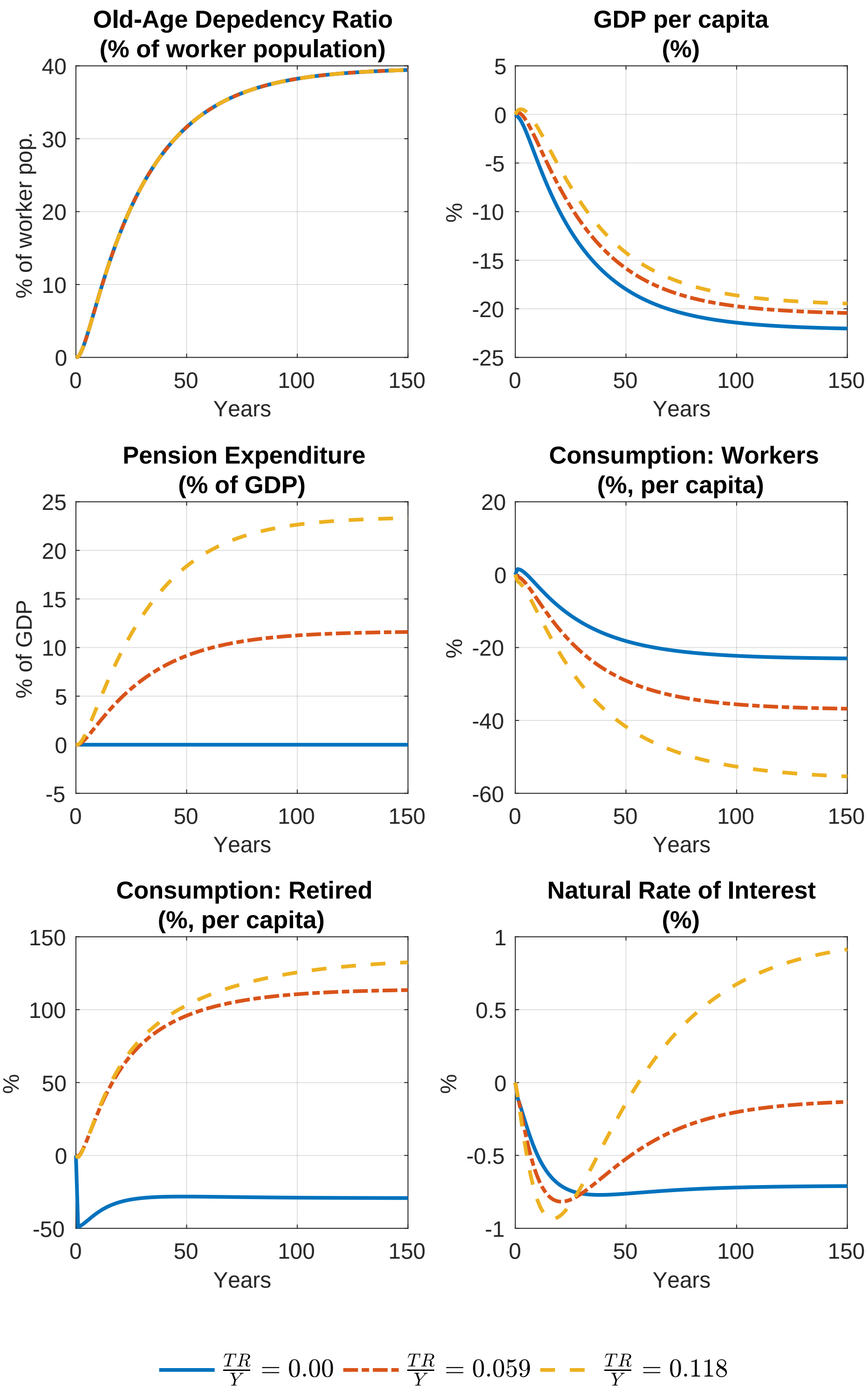

Figure 1.1: Population aging and transitional dynamics in rational expectation equilibrium

Based on the dynamic IS-curve equation and the demand for the young bonds equation, the workers' consumption-to-GDP ratio can be expressed as the function of the real interest rate and other variables. The intersection of solid lines is the initial equilibrium while the dashed lines give the terminal steady state equilibrium under differently sized PAYG systems (see Figure 1.2-1.4). Due to the aging shock, the government increases lump-sum taxes and workers decrease their consumption, which shifts down the dynamic IS-curve. The slope of the curve is the function of initial size of the PAYG pension system. If there is no PAYG system, the IS-curve is flat around the initial steady-state, meaning that the consumption ratio is quite elastic to any changes of the real interest rate. A larger pension system generates less elastic consumption and makes steady-state real interest rate more responsive. In this case, workers try to smooth out their consumption, and they are willing to borrow more for the future expected benefit in order to minimize consumption loss in the present.

The reaction on the bonds market also depends on the size of the pension system. If there is no generational redistribution, the workers' bond curve is relatively stable and the consumption-to-GDP ratio is almost unchanged. In the environment of initial steady-state, it slightly shifts right as old households start accumulating risk-free bonds and young households pay the elevated lump-sum taxes from their own wealth. If the pension system is even larger, under the same demographic shock the government has to collect more taxes or demand more private savings, both of which result in a lower consumption-to-GDP ratio by young households, which ceteris paribus, shifts left the demand curve and increases the real interest rate.

The initial and terminal equilibria can be significantly different under different PAYG systems. In the case of $\frac{TR}{Y} = 0$, the demand for the workers' bond curve is nearly unchanged, but workers decrease their consumption, which implies a lower real interest rate. In other cases, the aging shock generates a larger fiscal redistribution, which moves the bonds' demand curve further leftwards; however, this cannot be compensated for by lower young consumption as these movements can imply larger than initial real interest rate in the new equilibrium.

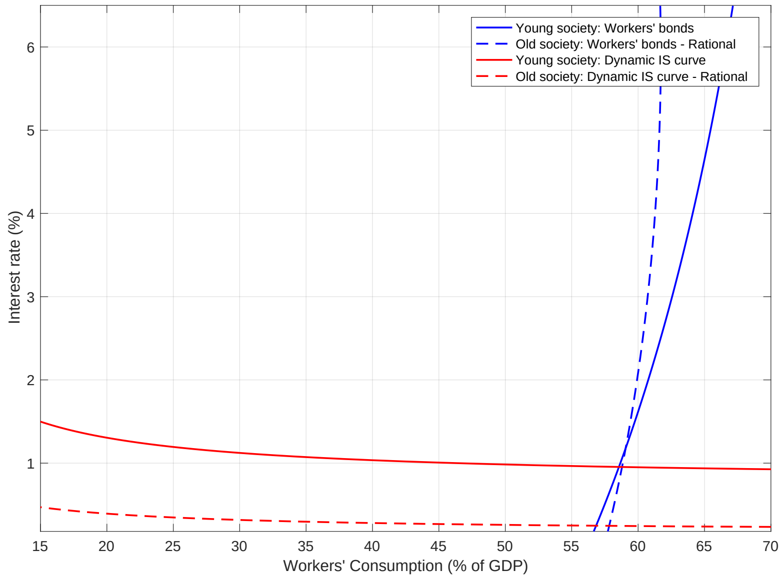

Figure 1.2: Initial and new steady-state equilibrium with $\frac{TR}{Y} = 0$

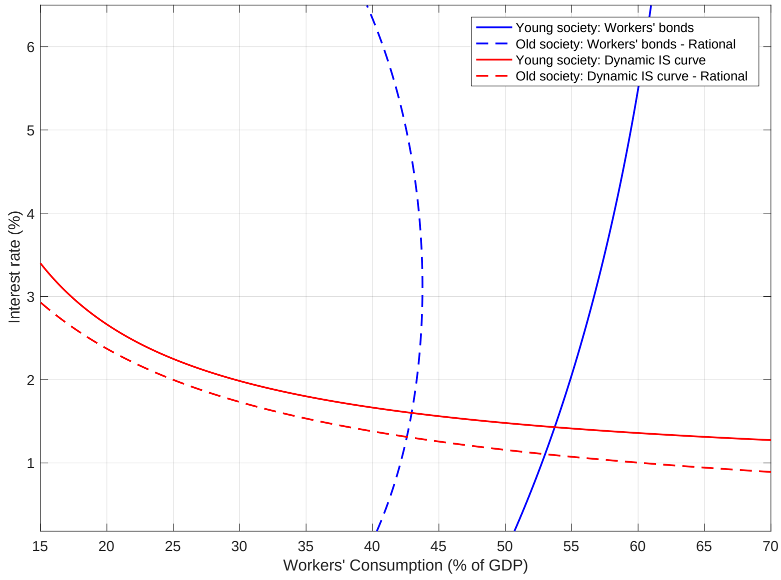

Figure 1.3: Initial and new steady-state equilibrium with $\frac{TR}{Y} = 0.06$

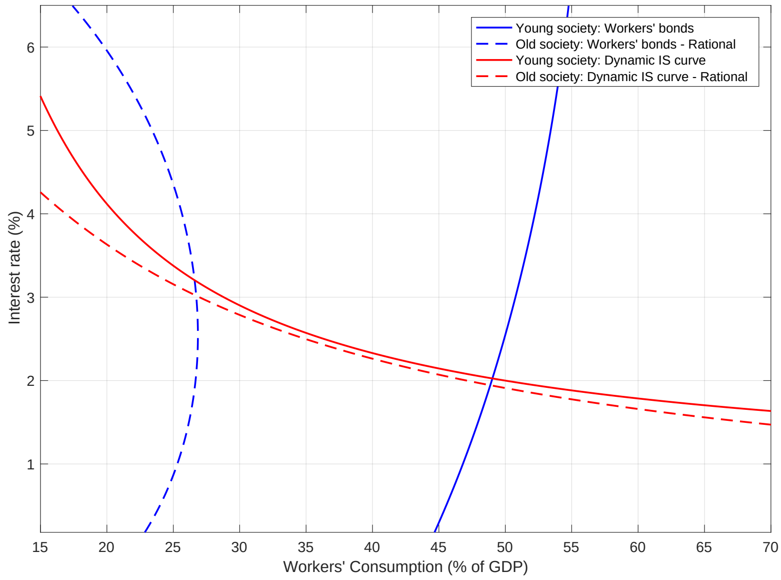

Figure 1.4: Initial and new steady-state equilibrium with $\frac{TR}{Y} = 0.118$

1.3.4 Naive estimation: testing the secular stagnation hypothesis

In the previous section, I showed the effect of population aging in benchmark OLG-models. According these models, during the first 20-30 years of the accommodation, the real interest rate should fall - regardless of the size of the public pension system. Thus, it is essential to check whether the last 20 years of macroeconomic data supports the idea of secular stagnation in the OECD economies. Figure 1.5 and Table 2.3 describe the historical development of the average old-age dependency ratio and average real interest rate. They suggest that there is a negative relationship between the demographic component and the real interest rate. Since the middle of the 1990s the real interest rate decreased by more than 3%, while the old-age dependency ratio increased by 5%. However, there are differences among the countries as the standard deviation of the real interest rate and dependency ratio have increased in the past 20 years.

Table 1.2: Descriptive statistics: OECD countries between 1993 and 2017

		Selected intervals			
	All obs.	1993-1999	2000-2007	2008-2013	2014-2018
$mean(r)$	2.63	4.63	2.37	2.60	1.02
$std(r)$	2.53	1.93	1.85	2.91	2.18
$mean(OADR)$	25.77	24.27	24.17	26.27	29.60
$std(OADR)$	5.98	3.03	5.24	6.41	7.08
$mean\,(TBal)$	0.96	1.06	0.16	0.84	2.54
$std\,(TBal)$	4.71	4.08	5.34	4.47	4.04
$mean\,(\Delta \ln TFP)$	1.54	1.81	2.18	0.93	1.00
$std\,(\Delta \ln TFP)$	1.99	1.21	1.78	2.53	1.68
$mean\left(r^{US}\right)$	2.03	3.63	1.87	1.25	0.72
$std\left(r^{US}\right)$	1.52	0.85	1.04	1.46	1.24
$mean\left(r^{DE}\right)$	2.24	4.35	2.52	1.29	-0.55
$std\left(r^{DE}\right)$	1.94	0.82	0.88	1.36	1.04
Number of countries	24	16	24	24	24

To check this co-movement, I estimate a simple unbalanced panel model on the OECD countries between 1992 and 2017[11]:

$$r_{it} = \rho \cdot r_{i,t-1} + \alpha_i + \delta_t + \gamma \cdot OADR_{it} + \kappa \cdot TBAL_{it} + \iota \cdot \Delta \ln TFP_{it} + u_{it} \tag{1.28}$$

where r_{it} 10Y nominal yields minus next period inflation from OECD database, $OADR$ old-age dependency ratio (65+ over 20-64 years old population) from the UN database, $TBAL$ is the net-export-to-GDP ratio from the OECD database that proxies the foreign indebtedness and risk premium of a given member state. $\Delta \ln TFP_{it}$ is the percentage changes of total factor productivity from the OECD database; this variable controls on the other supply side movements behind real interest rates. α_i country fixed effect, δ_t time fixed effect. The static and dynamic model were estimated in R with a plm package (Croissant and Millo (2008)) [12], and I calculated robust standard errors.

[11]The starting year was arbitrary, although most of the OECD countries published data from the beginning of 1990s, and population aging become more evident in the final decade of the 20th century.

[12]The dynamic panel model was estimated with a within estimator and also with GMM (Arellano and Bond (1991)), but I have not found significant differences between the two results.

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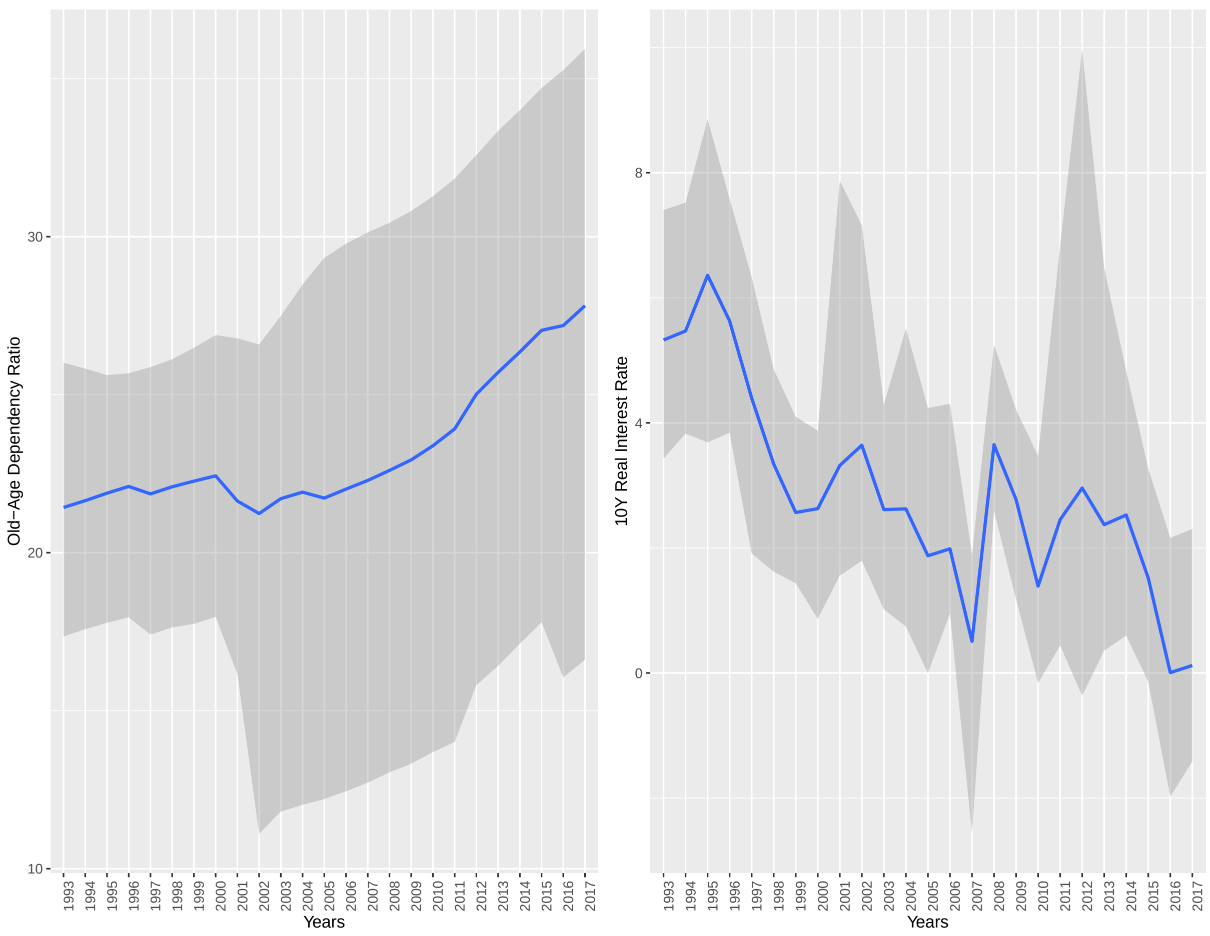

Figure 1.5: Old-Age Dependency Ratio and Real Interest Rate in OECD countries between 1993 and 2016

Table 1.3: Naive estimation of unbalanced panel: Long-term real interest rate and demography

	Dependent variable:								
	r_{it}								
	FE	FE	FE	FE	FE	FE	FE	FE	FE
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
r_{it-1}	0.47	0.51	0.48	0.48	0.44	0.47	0.42	0.57	0.53
	(0.08)***	(0.08)***	(0.07)***	(0.09)***	(0.08)***	(0.1)***	(0.1)***	(0.08)***	(0.08)***
$OADR_{it}$	−0.24	−0.25	−0.27	−0.08	−0.11	0.01	0.02	−0.01	−0.00
	(0.05)***	(0.05)***	(0.05)***	(0.03)***	(0.03)***	(0.06)	(0.06)	(0.04)	(0.04)
$\Delta \ln(TFP_{it})$		−0.16	−0.15	−0.13	−0.12	−0.13	−0.12	−0.14	−0.13
		(0.11)	(0.10)	(0.11)	(0.11)	(0.11)	(0.11)	(0.09)	(0.09)
$TBAL_{it}$			0.07		0.09		0.12		0.09
			(0.04)*		(0.04)**		(0.05)***		(0.04)**
r_t^{US}				0.56	0.58	0.36	0.29		
				(0.1)***	(0.09)***	(0.11)***	(0.12)**		
r_t^{DE}						0.27	0.40		
						(0.21)	(0.22)*		
Country FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	No	No	No	No	No	No	No	Yes	Yes
Observations	506	489	489	464	464	439	439	489	489
Num. of countries	24	24	24	23	23	22	22	24	24
R^2	0.375	0.425	0.431	0.492	0.501	0.483	0.497	0.643	0.650

Note: *p<0.1; **p<0.05; ***p<0.01
Robust standard errors in parentheses

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The fixed-effect estimations are reported in Table 1.3[13]. I have estimated different specifications to test the robustness of the negative relationship between demography and the long-term interest rate. The FE (1)-(3) specifications are the common *naive* estimations where I have found a significant, negative relationship between the old-age dependency ratio and the real interest rate. The lag-term of the interest rate captures the autocorrelation of the data, but it does not influence the sign of the estimated coefficients. It is not possible to involve more lag for the old-age dependency ratio, because the demographic variables are highly smoothed and strongly autocorrelated. Consequently, any additional lag in the estimation would generate a strong multicollinearity. The net-export-to-GDP ratio has a positive and robust coefficient in all estimations. In this reduced form estimation, this result suggests that, before the crisis, these economies accumulated foreign debt and were financed relatively cheaply from the world financial market. After the crisis, however, these countries were forced to engage in strong deleveraging, and the increasing risk premium contributed positively to the long-term real interest rate. The percentage changes of total factor productivity has reasonable negative coefficients, but the estimated values are not significant in any specification.

According to the first three estimations, the demographic components are very good predictors of the long-term real interest rate, and these results appear to be consistent with the prediction of OLG-models. However, the results for the demography are not robust for the other estimations. The FE (8)-(9) assume a time fixed effect component, where the explanatory power of

[13]In this paper I have listed the relevant specifications; other specifications are available upon request.

demographic factors on real interest rate completely disappears. This result is consistent with Aksoy, Basso, Smith, and Grasl (2019). Based on these specifications, the changes in the demographic trend do not generate an automatic decline in the interest rate. This suggests that, in many countries, the fall of the real interest rate is sourced from a common component eg: international spillover effects (which can be captured by time fixed effect). Checking the source of the common time component, to the FE (4)-(5) specifications I have added the US interest rate, to the the FE (6)-(7) specifications I have added the US interest rate and the German interest rate as a common observation, and estimated the following equation:

$$\begin{aligned} r_{it} &= \rho \cdot r_{i,t-1} + \alpha_i + \delta_t + \gamma \cdot OADR_{it} + \kappa \cdot TBAL_{it} + \iota \cdot \Delta \ln TFP_{it} + \\ &+ \xi \cdot r_t^{US} + \lambda \cdot r_t^{DE} + u_{it} \end{aligned} \tag{1.29}$$

where the r^{US} is the US 10Y real interest rate for all countries, r^{DE} is the German 10Y real interest rate for all countries; in those specifications for which I used the US and German interest rate as common components, I dropped the US and German country-level observations from the sample. Since, unlike the European countries, the US is not experiencing a serious aging problem[14], removing the US from the sample does not significantly distort the estimation. Removing the German data is less trivial, since Germany is among the countries that are most affected by the population aging. Nevertheless, the German interest rate can be a more valid benchmark interest rate for the European OECD countries. In the FE (4)-(5), once I involve the US long-term real interest rate rather than the time fixed effect, the size of the coefficients decreases in absolute terms - with or without the net export I get similar results. Adding the German interest rate to FE (6)-(7), the coefficients of the old-age dependency ratio become insignificant and have the wrong sign. The coefficient of US real interest rate is positive and significant in all specifications.

According to the results from FE (4)-(7), the US and German real interest rate could be potential common components as well. Moreover, contrary to the benchmark OLG-model, the negative relationship between demographic factors and long-term interest rate is not necessarily robust for all OECD countries. These results suggest that a significant part of the decline in interest rates is sourced from international spillover effects, and that demography contributes less to the overall dynamic. Population aging is a common problem in developed economies,

[14]Most of the decline in the US rates resulted from *Great Moderation*, the period of low inflation and low interest rate from 1990s until the financial crisis, and the unconventional monetary policy of the post-crisis period.

and the common time fixed effect could also capture common demographic trends. However, the estimated non-significant coefficient of demography means that, beyond the international spillover effects, the country-specific demographic structure and country-level savings decision have no, or only limited, effect on the country-level real interest rate positions.

In the following chapters, I adjust the standard OLG model and show a potential extension with bounded rationality. This modification of the model sheds light on potential identification issues; namely, in the estimation above, we implicitly assumed that all countries follow the same rational behavior. Later, we will see that if we deviate from the rational expectation equilibrium, population aging generates a different natural rate of interest. This also means that if we do not select the rational and non-rational countries within the sample, the selection bias distorts the estimation of the demographic coefficients in the two-way estimations.

1.4 Bounded rationality and OLG-models

A great deal of anecdotal and psychological evidence indicates that economic agents have cognitive limits and are not able to process all the available information about their economic decisions. According to the theory of bounded rationality, agents' expectations are distorted and this bias generates different outcomes compared to the rational expectation equilibrium. Conlisk (1996) provides a summary of the advantages and empirical relevance of bounded rationality. In our simplified Gertler-type OLG-model, the dynamic IS-curve and the bond demand curve anchor the equilibrium where the expectation terms can be distorted under the assumption of bounded rationality. In the following, I give a formal definition for the distortion and implement level-k thinking as a potential interpretation of bounded rationality, after which I combine it with the Gertler-type OLG model. I will show later that the k refers to the set of information that agents take into account when they form their expectations, and the rational expectation is the special case of level-k thinking, where the agents take into account an infinite amount of information in making their decision.

1.4.1 Bounded rationality and level-k thinking

Level-k thinking has a growing presence in recent macroeconomic literature. The main concept is taken from Fair and Taylor (1983) and Evans and Ramey (1992). These papers compute the rational expectation equilibrium from iterative steps, and as one chooses different lengths for the iteration, different equilibrium outcomes can be calculated. Farhi and Werning (2017) and Gabaix (2017) implemented level-k thinking into dynamic stochastic general equilibrium models; however, these papers concentrate on only the short-run fluctuations and the effect of monetary policy. The main advantages of level-k thinking are its tractability and compatibility with standard DSGE-models: it only changes the expectation operator, and assumes the agents have the same behavioral equations as in rational expectation models. In the following section, I show how the forward-looking equation can be written into level-k in a consistent way.

The iterative process can be described by the following forward-looking difference equation where the x is a forward-looking variable and y is another contemporaneous variable:

$$x_t^k = \alpha x_{t+1}^{e,k-1} + \beta f(y_t) \tag{1.30}$$

where $f(\cdot)$ is a well-defined function of y_t fundamental variables, $x_{t+1}^{e,k-1}$ is the net period value from the iteration $k-1$ and $|\alpha| < 1$. For the case of $k = 1$, we assume:

$$x_t^1 = \alpha x_{t+1}^{e,0} + \beta f(y_t) \tag{1.31}$$

where $x_{t+1}^{e,0}$ equal to the initial steady-state value for all t. For larger k, we can write the following equations:

$$\begin{aligned} x_t^2 &= \alpha x_{t+1}^{e,1} + \beta f(y_t) \\ x_t^3 &= \alpha x_{t+1}^{e,2} + \beta f(y_t) \\ &\vdots \\ x_t^k &= \alpha x_{t+1}^{e,k-1} + \beta f(y_t) \end{aligned} \tag{1.32}$$

If one substitutes out the expectation terms, the x_t^k can be expressed as the sum of next period $f(y)$-s and initial value:

$$x_t^k = \alpha^k x_{t+1}^{e,0} + \beta \sum_{n=0}^{k-1} \alpha^n f(y_{t+n}^e) \tag{1.33}$$

The size of α and k are crucial to see how the initial conditions affect the dynamics of x. Based on the formula above, the rational expectation equilibrium (REE) can be interpreted as the special case of bounded rationality. According to the rational expectation assumption, the agents take into account all available information. This formally means $\lim_{k\to\infty} \alpha^k = 0$ and x_t^∞ is independent

of its initial steady-state value. The case $k < \infty$ means that the agents consider only period k information; from the future and beyond period k, they expect that the economy will revert back to the initial equilibrium path. A low value for k makes the agents biased toward the initial equilibrium. In our case, the low k means that, regardless of population aging, the agents do not pay too much attention to future financing issues of the public pension system; thus, they do not adjust their permanent income expectation as they do in the full rational case.
In the next subsection, I present the modified equations of the OLG-model. Based on the formula above, I show level-k consistent expectations and compare the dynamic and steady-state properties of the demographic aging shock.

1.4.2 Gertler-type OLG with level-k thinking

The main contribution of this paper is the combination of a Gertler-type OLG model with level-k thinking and a description of population aging in non-rational economies. Previously, I showed that the model can be simplified into two main equations. The first, namely, the dynamic IS-curve from worker cohort with level-k thinking, can be written as:

$$\begin{aligned} \frac{\tilde{C}_t^{Y,k}}{MPC_t^{Y,k}} &= \frac{\tilde{C}_t^{Y,k}}{\sigma} + \tilde{B}_t^{Y,k}\left(1 - \frac{(1-\omega_t^Y)^2}{1+g_{t+1}^{N,Y}}\right) + \frac{\omega_t^Y}{1+r_t}\nu\tilde{Y}_t\Omega_{t+1}^{O,e,k} + \\ &+ \frac{1-\omega_t^Y}{1+r_t}\frac{1+s_{t+1}}{1+s_t}\frac{\tilde{C}_{t+1}^{Y,e,k-1}}{MPC_{t+1}^{Y,e,k-1}} \end{aligned} \tag{1.34}$$

where the additional variables are

$$\begin{aligned} \frac{1}{MPC_t^{Y,k}} &= \frac{1}{\sigma} + (1+r_t)^{\frac{1}{\gamma}-1}\left[(1-\omega_t^Y)\Lambda_t^Y\frac{1}{MPC_{t+1}^{Y,e,k-1}} + \omega_t^Y\Lambda_t^{YO}\frac{1}{MPC_{t+1}^{O,e,k-1}}\right] \\ \frac{1}{MPC_t^{O,k}} &= 1 + (1-\omega_t^O)(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\frac{1}{MPC_{t+1}^{O,e,k-1}} \\ \Omega_t^{O,k} &= 1 + \frac{1-\omega_t^O}{1+r_t}\Omega_{t+1}^{O,e,k-1} \end{aligned} \tag{1.35}$$

Level-k thinking changes the role of expectation, and, as we have seen above in the case of bounded rationality, the expectation operator is biased towards the initial point. In the Appendix, I also demonstrate that the initial steady state of the bounded and full rational equilibrium can be identical, but the permanent changes generate differences between the terminal steady states of rational and bounded-rational equilibriums. Additionally, in the Appendix, I show the analytical form of the steady state and compare the two equilibria.

Since the retiree's discount factor also contains expectation terms, the retired cohort consumption can be also distorted and the demands for the worker bond differs from the rational equilibrium level:

$$\begin{aligned}\tilde{B}_t^{Y,k} &= \tilde{Debt}_t - (1 - MPC_t^{O,k}\Omega_t^{O,k})\tilde{TR}_t - \\ &- \frac{(1+r_{t-1})}{1+g_t^N}(1 - MPC_t^{O,k})\left[\tilde{Debt}_{t-1} - (1-\omega_{t-1}^Y)\tilde{B}_{t-1}^{Y,k}\right]\end{aligned} \tag{1.36}$$

the rest of the equations and variables are identical to those in the original OLG-model with rational expectations. The two equations above determine the level-k consistent real interest rate. The model assumes capital accumulation. However, the real interest rate via the capital demand function also anchors the expected marginal product of capital, and thus the dynamic of capital accumulation is also consistent with the level-k thinking property of the model.

1.4.3 Aging shock and level-*k* thinking

I ran the same population-aging scenario from the previous section with the bounded rationality model and different k-s. I compared the rational expectation equilibrium (REE); in the bounded rationality (BRE) model, the workers' consumption-savings decision is different (see Figure 1.6-1.8). In the bounded rationality model, the chosen k is the length of the information set of the given cohort. The workers' cohort consists of the 20- and 64-year-old population. All agents - regardless of their age - in the cohort have the same retirement probability, which means that old workers and young workers expect the same chance of the next period retirement, and a 64-year-old worker has the same planning horizon as a 20-year-old worker. This means that the worker cohort incorporates a very long planning horizon, and to approach the rational expectation equilibrium one should choose a large k to cover the whole information set of the worker cohort.

Under rational expectations, from the first moment of the population aging shock, the workers take into account that the government will automatically increase lump-sum taxes to stabilize the public debt and the pension system - especially in the medium-term, when the aging becomes more advanced. Hence, workers immediately adjust their permanent income expectation and decrease their own consumption. In the case of bounded rationality, this pattern is different: young households with lower k are biased toward the initial steady-state consumption, and beyond k period they expect that the economy will revert back to its initial state. Therefore,

contrary to the conventional prediction of OLG models with rational expectations, they are not likely to give up as much consumption as the society starts aging because they do not think that aging permanently changes the long-run position of the economy. Moreover, to offset the increasing taxes, workers start accumulating relatively more domestic credit to smooth out their consumption. The higher level of indebtedness exerts a positive pressure on the real interest rate during the medium-term accommodation. The increasing natural rate of interest becomes a strong incentive for retired households to save even more, and to finance the public debt.
The increasing interest rate redistributes the wealth among generations. Initially, retired households are willing to consume less than in the rational expectation equilibrium. Later, as the economy is converged to the new steady-state equilibrium, retired households with higher levels of savings are able to consume more, young households with higher indebtedness and interest cost consume less, and their consumption subdues the rational expectation equilibrium. The deviations between the rational and bounded rational equilibrium are the same, regardless of the size of the public pension system. However, in those economies where the generational redistribution is larger, the interest rate increases more as a consequence of population aging if the agents have bounded rationality.

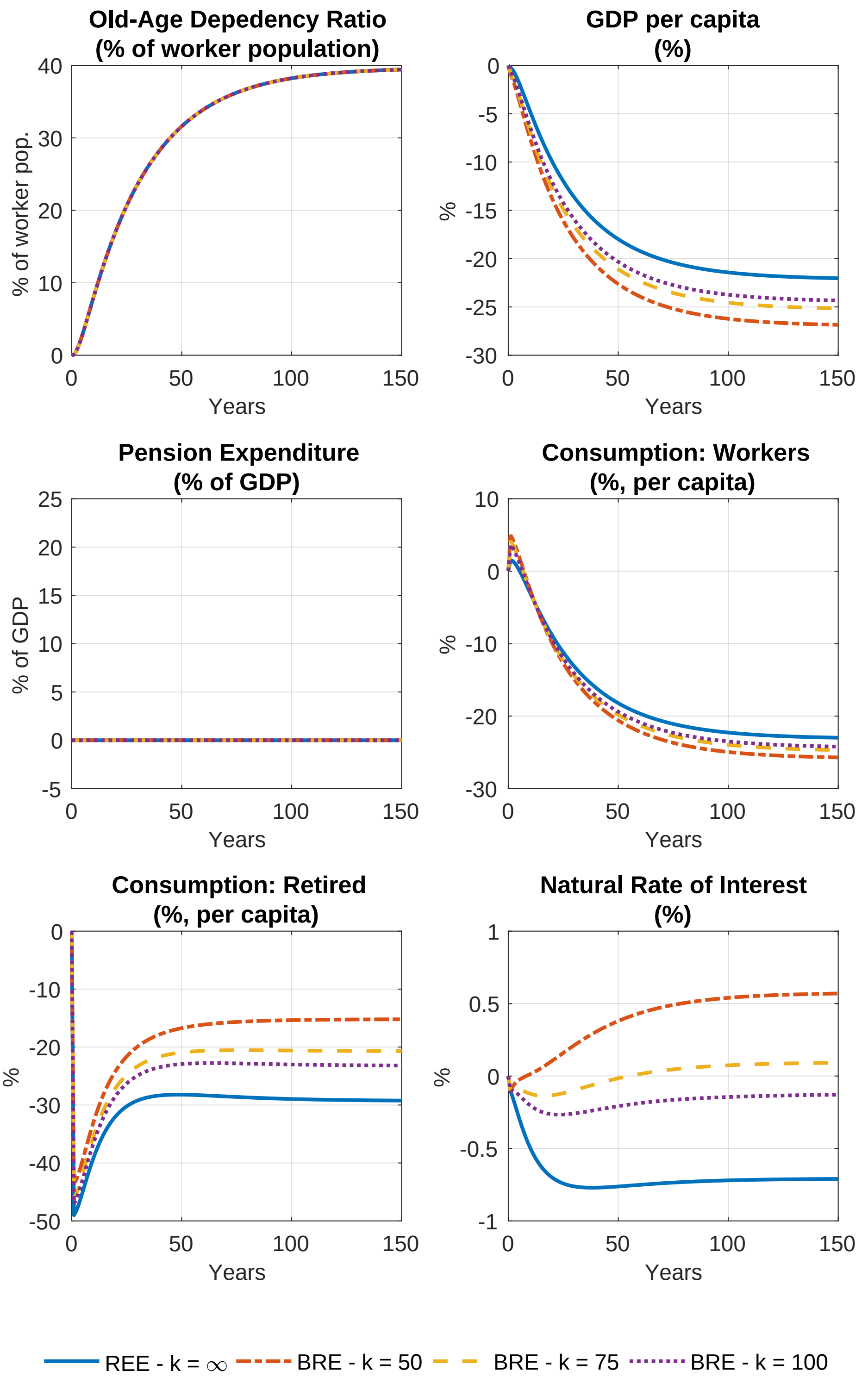

Figure 1.6: Population aging and transitional dynamics: Bounded rationality (BRE) versus full rationality (REE) with $\frac{TR}{Y} = 0$

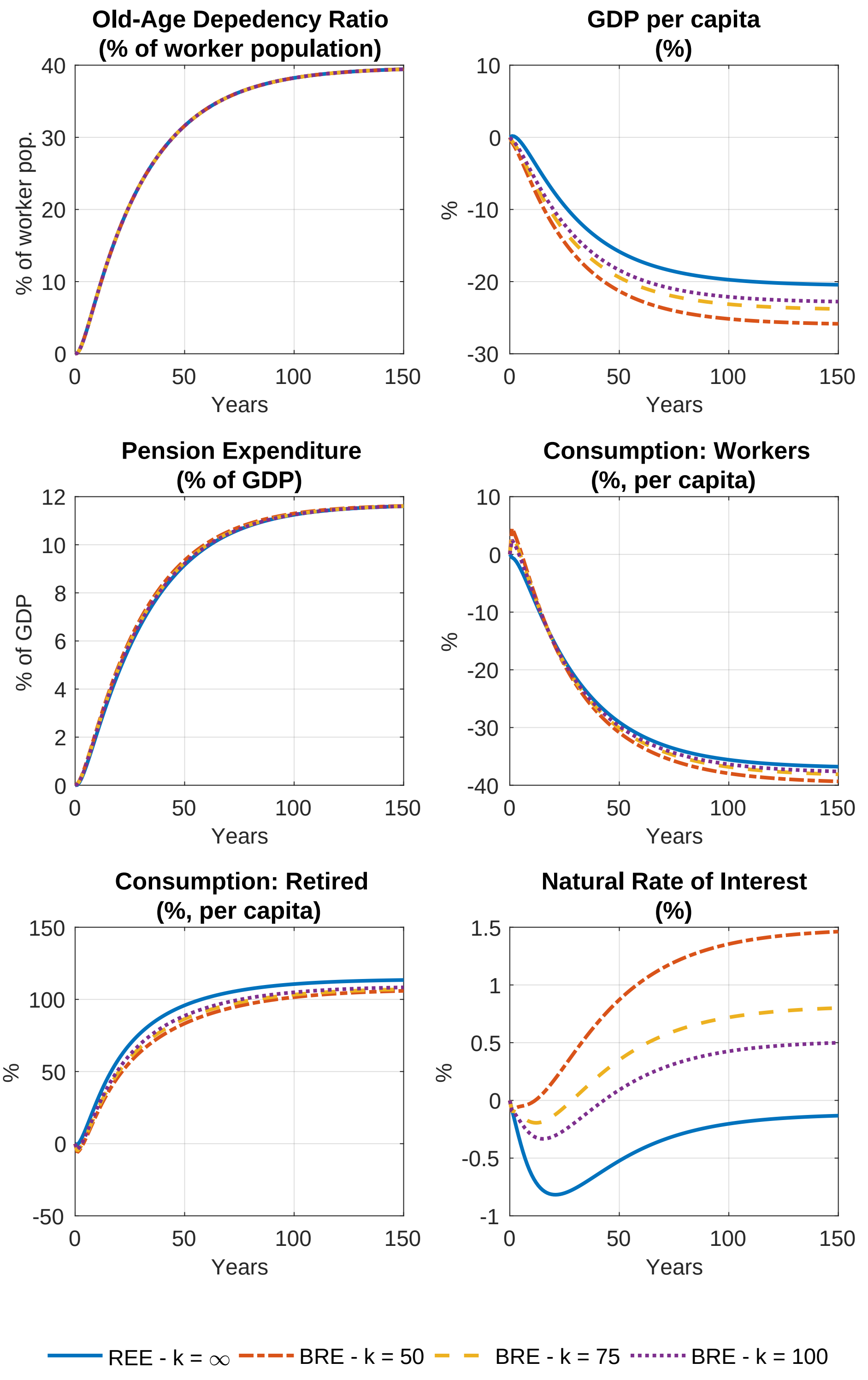

Figure 1.7: Population aging and transitional dynamics: Bounded rationality (BRE) versus full rationality (REE) with $\frac{TR}{Y} = 0.059$

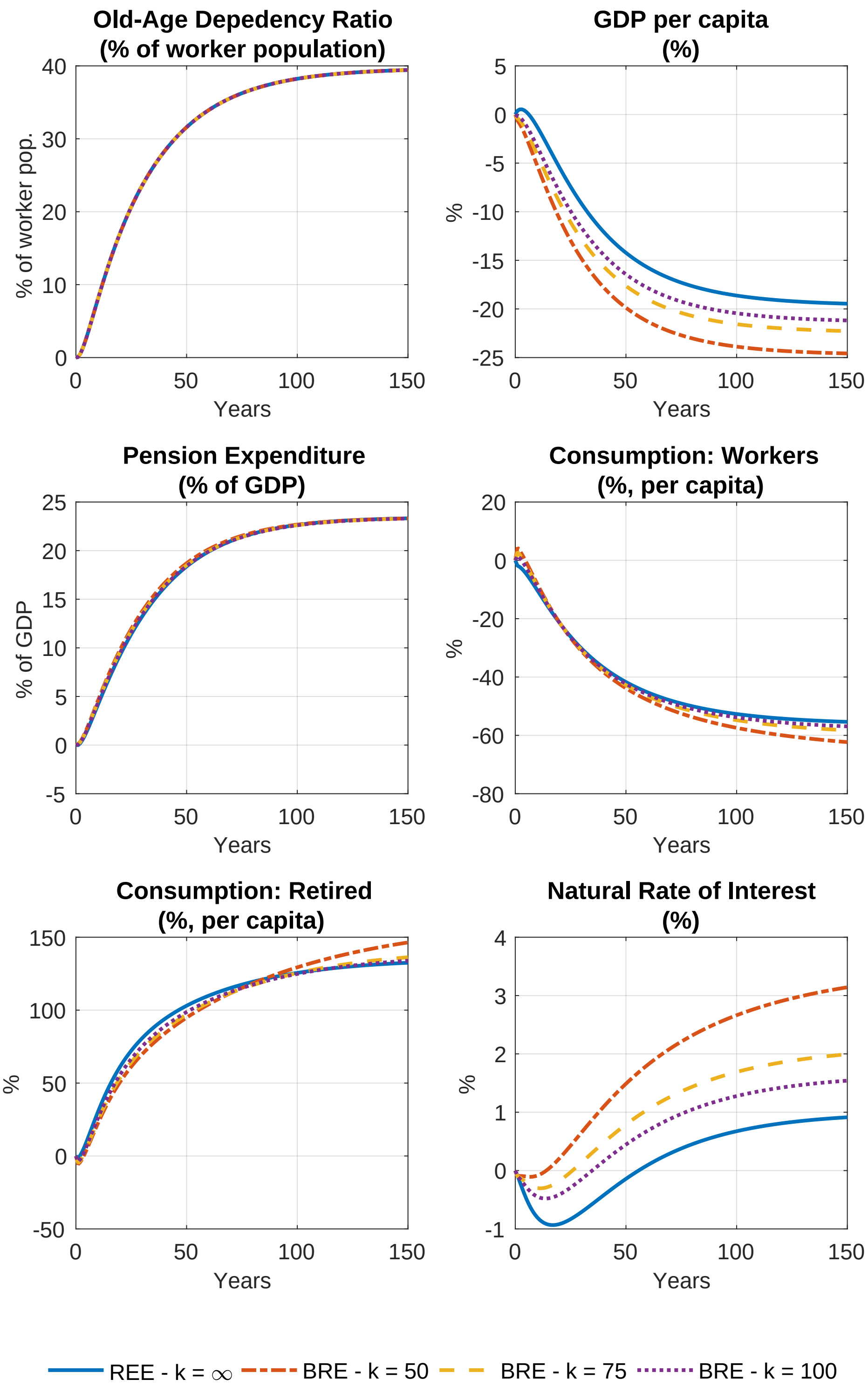

Figure 1.8: Population aging and transitional dynamics: Bounded rationality (BRE) versus full rationality (REE) with $\frac{TR}{Y} = 0.118$

Bounded rationality generates significant differences in the new steady-state equilibrium after population aging (see Figure 1.10-1.11). The two models have the same initial equilibrium point (solid lines), but the level-k thinking (with finite k) shifts and changes the slope of the dynamic IS-curve (see red dotted line). Due to the biased expectation of the worker cohort, the curve is steeper. This means that, under bounded rationality, the young generation is not willing to give up as much consumption as their rational selves, and the interest rate reacts more on the increasing credit demand.

The demand curve for young bonds is also shifted. However, the level-k thinking does not significantly change the slope of this curve. It is mostly affected by the retired households behavior, but old agents have a shorter life-time horizon (relatively large ω^O), less distorted by the bounded rationality.

Due to population aging, the bonds demand curves are shifted almost to the same position as in rational equilibrium. The position of workers' IS curve in under bounded rationality significantly differs from the rational expectation case, which generates higher interest rates in the new equilibrium.

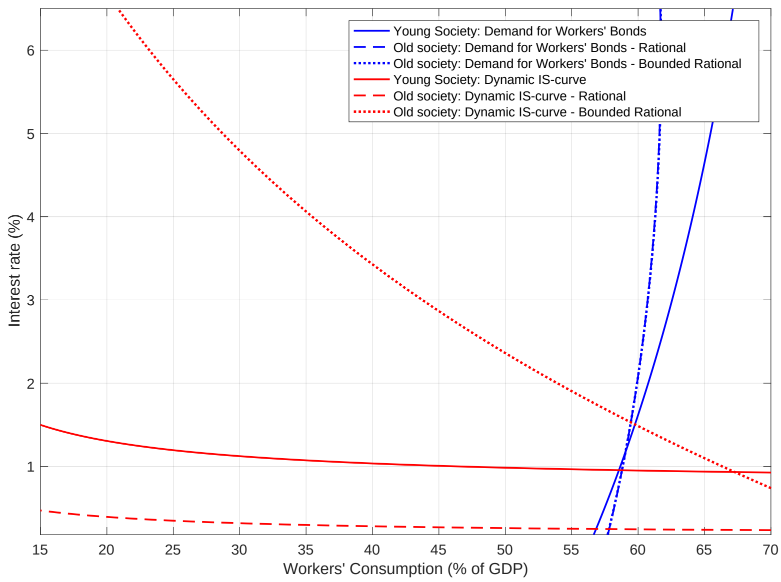

Figure 1.9: Initial and new steady-state equilibrium: bounded rational expectation equilibrium (k=50) versus rational expectation equilibrium with $\frac{TR}{Y} = 0$

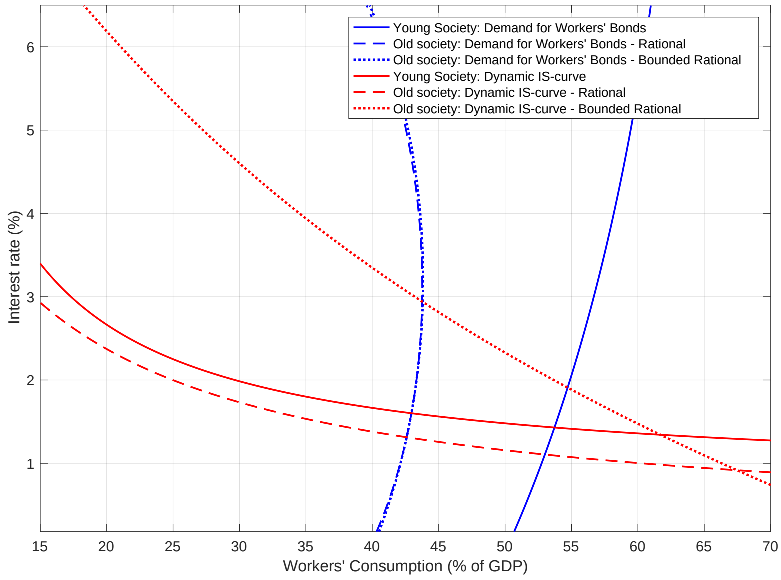

Figure 1.10: Initial and new steady-state equilibrium: bounded rational expectation equilibrium (k=50) versus rational expectation equilibrium with $\frac{TR}{Y} = 0.059$

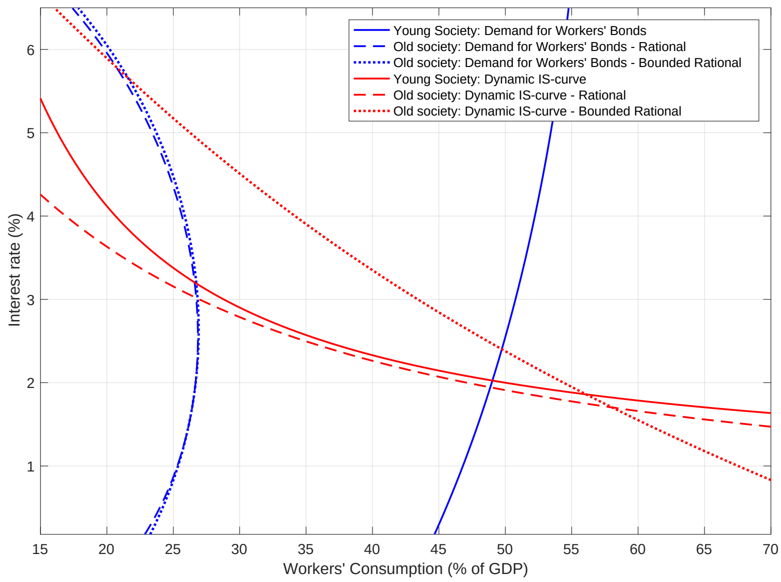

Figure 1.11: Initial and new steady-state equilibrium: bounded rational expectation equilibrium (k=50) versus rational expectation equilibrium with $\frac{TR}{Y} = 0.118$

Bounded rationality sheds light on a possible identification problem of the secular stagnation hypothesis. While standard OLG-models have a similar prediction about the decreasing natural rate of interest at the time of population aging, under bounded rationality the strength and sign of this relationship is the function of the k. Moreover, the negative relationship can be true for those country only where the k is large, which means that the agents have a long enough planning horizon; otherwise, in non-rational economies, aging could generate different dynamic properties and steady state. In the following section, I return to and adjust the initial, *naive*, estimation, show two examples of how to control for rational behavior among countries, and check whether secular stagnation is valid only for rational economies.

1.5 Controlling for rational expectations

The bounded rationality model shows that, in aging societies, the natural rate of interest is not necessarily decreasing. In those countries, according to level-k thinking, where agents have bounded rational expectations, they concentrate on the short period of their expected life-time, and the interest rate could even increase during the demographic aging. In this section, I return to the empirical test, and, based on the intuition of the bounded rational model, I adjust the previous estimation with controlling for those countries where agents' behavior is consistent with rational expectations.

In the final phase of the paper, using the same dataset of OECD countries between 1992 and 2017, I estimate the following unbalanced panel with an additional interaction term:

$$\begin{aligned} r_{it} &= \rho \cdot r_{i,t-1} + \alpha_i + \delta_t + (\gamma + \nu \cdot D_i) \cdot OADR_{it} + \kappa \cdot TBAL_{it} + \iota \cdot \Delta \ln TFP_{it} + \\ &\quad + \xi \cdot r_t^{US} + \lambda \cdot r_t^{DE} + u_{it} \end{aligned} \tag{1.37}$$

where $D_i = 1$ if the country i is considered rational, $D_i = 0$ for the rest of the countries. The essential question is how to separate the rational and non-rational countries within the sample. I found two alternative proxies for the selection, the first being the financial literacy indicator 2014 S&P FinLit Survey (Klapper, Lusardi, and van Oudheusden (2014)). This survey, which was sponsored by the S&P and involved countries all over world, measured participants' understanding of risk diversification, inflation, interest rates, and compound interest rate calculations. The second proxy is the time preference index from the Global Preference Survey (Falk, Becker, Dohmen, Enke, Huffman, and Sunde (2018)), which is a global representative dataset

Table 1.4: Financial literacy and Patience from Global Preference Survey in OECD countries

Both index above the median	Only GPS above the median	Only Financial Literacy above the median	Both index below the median
Australia	Austria	Czechia	Chile
Canada	Switzerland	Israel	Spain
Germany	USA		France
Finland			Greece
United Kingdom			Hungary
Netherlands			Italy
Sweden			Japan
			Korea
			Lithuania
			Mexico
			Poland
			Portugal

from 76 countries around the world. The time preference was measured from the combination of a quantitative and qualitative surveys, a series of binary choices and self-assessments about their willingness to wait for immediate or delayed financial rewards. Here, I have the implicit assumption that the average planning horizon is longer (and the expectations are less bounded) in those countries where agents are financially literate or more patient about future economic pay-offs. The estimated correlation between the two indicators is 0.8, with a [0.77;0.83] 95% confidence interval that refers to a relatively strong relationship. I have calculated the median for both indices and separated the countries as above or below the median on the table 1.4. This exercise is useful to check the overlap among the two data sources, and to determine whether the rational economies are affected by population aging[15].

Before the panel estimation, I also tested whether the negative relationship between the old-age dependency ratio and the interest rate is stronger if the economy has a higher financial literacy index or a larger time preference. For this exercise, I calculate simple correlations between the country-specific real interest rate and the old-age dependency ratio, and create a cross-plot figure between the estimated correlation and financial literacy or time preference indicator. Based on Figure 1.12, there is negative relationship among the indicators and the estimated correlation. The negative co-movement is stronger if I use a time preference as a proxy: the correlation between the time preference and estimated correlations is -0.74 with [-0.89;-0.45] 95% confidence interval, while in the other case it is -0.55 with a [-0.79;-0.15] 95%

[15] I also checked the correlation between the indices above and the GDP per capita or development of the financial system. Among the OECD member states, the most developed countries in the world, I did not find any strong correlations. There could be a positive correlation among the indicators if we extend the sample with low- and middle-income - non-OECD - countries. However, most of these economies are not affected by population aging.

confidence interval. These findings support the idea that in those countries where households have a better understanding of the nature of the financial markets or have a longer planning horizon, households might have more savings - especially at the time of aging, which puts even more negative pressure on the domestic long-term real interest rate. These two proxies could be consistent with the logic of the model and seem to be a good control for the selection as well.

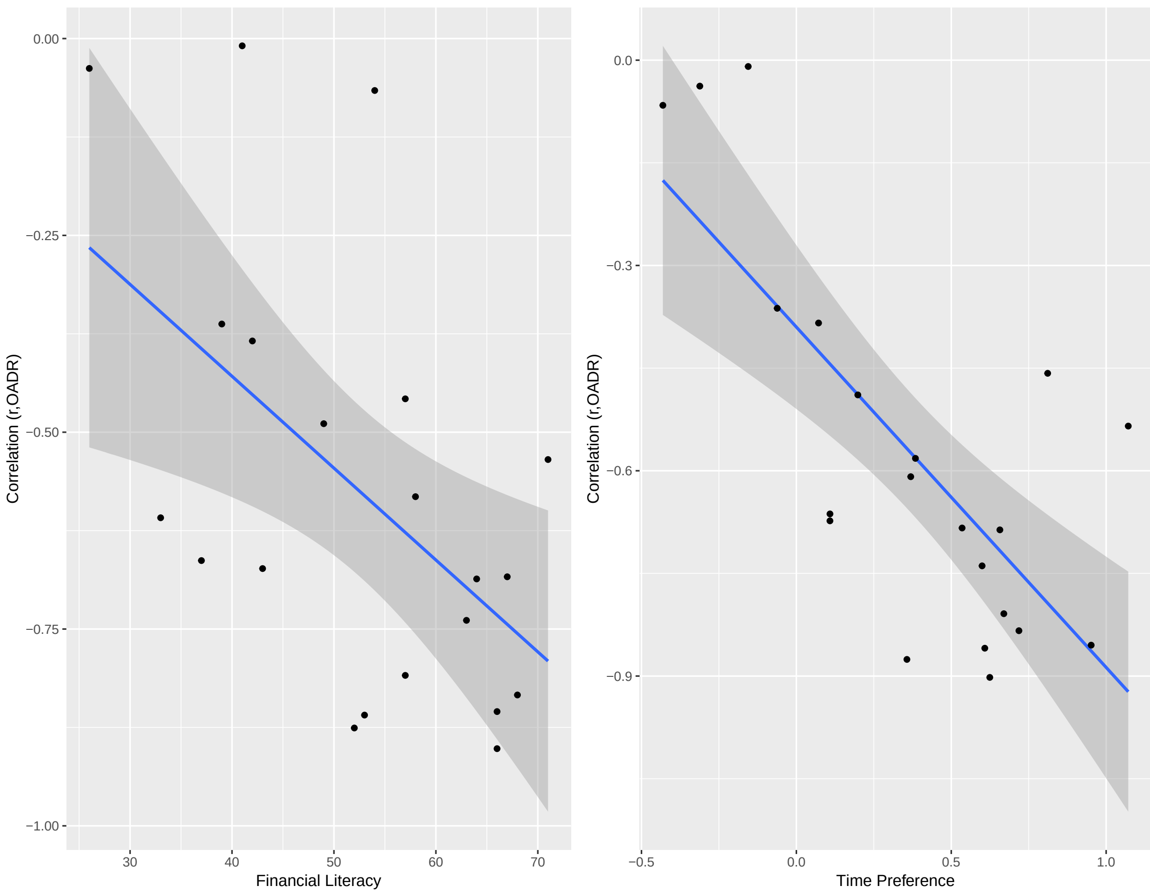

Figure 1.12: Country-level coefficients with financial literacy or time preference

The results of the panel estimations are reported in Table 1.5 and 1.6. FE (10)-(13) are the dynamic panel estimations with the common US real interest rate and without the time fixed effect; FE (14)-(17) are the dynamic panel estimations with the common US and German real interest rate and without the time fixed effect; and FE (18)-(21) are dynamic panel estimations with the time fixed effect. In all estimations, I observe the percentage change of total factor productivity: the estimated coefficients are the same - not significant, as in the previous estimation. FE (11), FE (13), FE (15), FE (17), FE (19) and FE (21) estimations consist of the trade balance-to-GDP ratio; in almost all estimations, the trade balance-to-GDP ratio has the same, robust coefficient as in the previous specifications. FE (10)-(11), FE (14)-(15) and FE (18)-(19) specifications use the financial literacy indicator; and FE (12)-(13), FE (16)-(17) and FE (20)-(21) involve the GPS time preference to control for rational economies. The interaction terms for most of the specifications have a significant negative relationship despite the two-way estimation and the common international interest rate. These results suggest that a one percentage point permanent increase in the old-age dependency ratio ceteris paribus generates a 12-25 basis points cumulative decline in the long-term real interest rate, if the country received a higher than median score for financial literacy index or households have higher than median time preferences. These estimations also mean that, during the past 20 years, the nearly 5 percentage point increase of the old-age dependency ratio contributes 60-130 basis points to the real interest rate on average. This result is consistent with the prediction of the OLG model with rational expectations. For the case of $D_i = 0$, none of the specifications results in significant coefficients. These findings are consistent with the prediction of the model with bounded rationality, namely, that the negative relationship does not necessarily hold for all countries, but only for those where agents' expectations are consistent with rational expectations.

Adding the trade balance ratio to the estimation somewhat weakens the coefficients of the demographic variable, and from FE (16) to the FE (17), the estimated coefficient become insignificant and has a higher value. The changes in the external debt position explain the fluctuation of the real interest rate better. However, this can still be consistent with the theory of bounded rationality. Thus, in those economies where the agents have strong myopia, they are willing to take more foreign or domestic loans to smooth out their consumption, put more weight on present information, and later they react more drastically to the new shocks. To check the relevance of this hypothesis, I calculated the pre-crisis (2001-2009) and post-crisis (2009-2018) average of the trade balance-to-GDP ratios, and the improvement of the post-crisis trade balance-to-GDP ratio, based on the cross-plots (see Figure 1.13). It seems that there is a weak but negative co-movement between the reversion of trade-balance and financial literacy or time preference[16]. This negative relationship supports the idea that in those countries where the agents have stronger myopia, the post-crisis adjustment in the trade balance was stronger than in other, more rational, economies. The stronger deleveraging is the consequence of relatively higher risk premium and higher real interest rate. Thus, the trade-balance can also control for the expectation channels of observed economies, and this could be the reason why it weakens

[16]The estimated correlation is -0.36, with a [-0.69; 0.08] 95% confidence interval for financial literacy, and -0.44, with a [-0.73;-0.01] 95% confidence interval for GPS time preference

the coefficients of the interaction terms that are also used to control for the rational behavior of the observed countries.

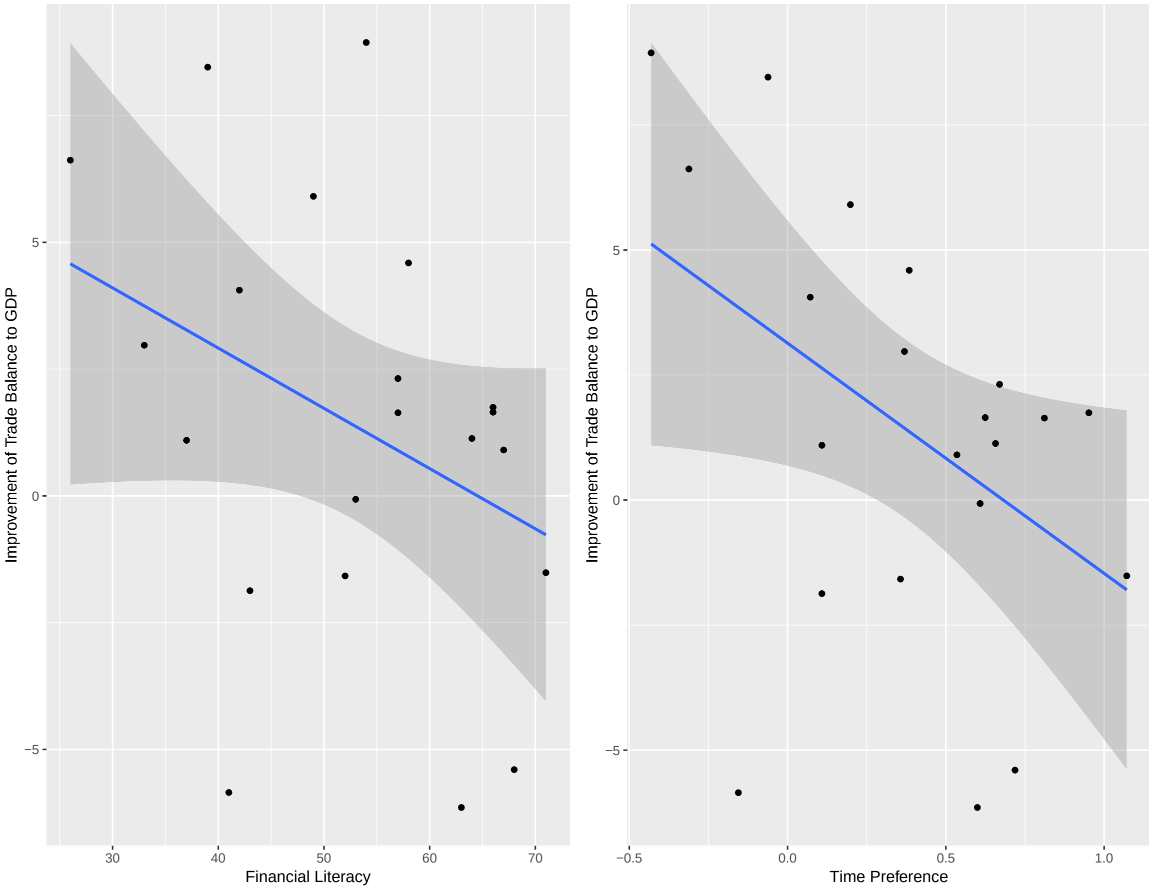

Figure 1.13: Financial literacy, time preference and the improvement in trade-balance ratios

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Table 1.5: Estimation with financial literacy and time preference 1

	Dependent variable:							
	r_{it}							
	FE	FE	FE	FE	FE	FE	FE	FE
	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)
r_{it-1}	0.47	0.43	0.47	0.43	0.46	0.41	0.46	0.42
	$(0.09)^{***}$	$(0.08)^{***}$	$(0.09)^{***}$	$(0.08)^{***}$	$(0.10)^{***}$	$(0.10)^{***}$	$(0.10)^{***}$	$(0.10)^{***}$
$OADR_{it}$	−0.02	−0.05	−0.03	−0.06	0.06	0.06	0.04	0.04
	(0.05)	(0.05)	(0.04)	(0.05)	(0.07)	(0.06)	(0.07)	(0.06)
$D_i(\text{FinLit}) \cdot OADR_{it}$	−0.13	−0.11			−0.14	−0.10		
	$(0.04)^{***}$	$(0.04)^{***}$			$(0.05)^{***}$	$(0.05)^{**}$		
$D_i(\text{GPS}) \cdot OADR_{it}$			−0.13	−0.11			−0.12	−0.07
			$(0.04)^{***}$	$(0.04)^{***}$			$(0.05)^{**}$	(0.05)
$\Delta \ln(TFP_{it})$	−0.13	−0.12	−0.13	−0.12	−0.14	−0.12	−0.13	−0.12
	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.10)	(0.11)	(0.11)
$TBAL_{it}$		0.09		0.08		0.11		0.11
		$(0.04)^{**}$		$(0.04)^{**}$		$(0.05)^{**}$		$(0.05)^{**}$
r_t^{US}	0.55	0.57	0.54	0.56	0.37	0.30	0.37	0.30
	$(0.10)^{***}$	$(0.09)^{***}$	$(0.10)^{***}$	$(0.09)^{***}$	$(0.11)^{***}$	$(0.12)^{**}$	$(0.11)^{***}$	$(0.12)^{**}$
r_t^{DE}					0.25	0.38	0.24	0.37
					(0.21)	(0.23)	(0.21)	(0.24)
Country FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	No	No	No	No	No	No	No	No
Observations	464	464	464	464	439	439	439	439
Number of countries	23	23	23	23	22	22	22	22
R^2	0.497	0.504	0.497	0.504	0.488	0.500	0.487	0.498

Note: *p<0.1; **p<0.05; ***p<0.01
Robust standard errors in parentheses

Table 1.6: Estimation with financial literacy and time preference 2

	Dependent variable:			
	r_{it}			
	FE	FE	FE	FE
	(18)	(19)	(20)	(21)
r_{it-1}	0.56	0.52	0.56	0.52
	(0.08)***	(0.08)***	(0.08)***	(0.08)***
$OADR_{it}$	0.05	0.04	0.04	0.03
	(0.05)	(0.04)	(0.05)	(0.04)
$D_i(\text{FinLit}) \cdot OADR_{it}$	−0.11	−0.09		
	(0.03)***	(0.03)***		
$D_i(\text{GPS}) \cdot OADR_{it}$			−0.11	−0.09
			(0.03)***	(0.03)***
$\Delta \ln(TFP_{it})$	−0.14	−0.13	−0.14	−0.12
	(0.09)	(0.09)	(0.09)	(0.09)
$TBAL_{it}$		0.08		0.08
		(0.04)**		(0.04)**
Country FE	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes
Observations	489	489	489	489
Number of countries	24	24	24	24
R^2	0.646	0.653	0.646	0.653
Note:			*p<0.1; **p<0.05; ***p<0.01	
			Robust standard errors in parentheses	

1.6 Summary and conclusion

This paper reconsiders the secular stagnation hypothesis through the lens of bounded rationality with level-k thinking. To the best of my knowledge, this is the first Gertler-type OLG-model with bounded rationality that examines the effect of population aging in non-rational economies. It can be shown that if agents' expectations are not consistent with rational expectations, demographic changes will not occur with a decreasing real interest rate because, in non-rational cases, young households are less likely to save enough for the longer life-time horizon. This theoretical contribution sheds light on the identification strategy, and provides a possible explanation for why estimations for the relationship between demography and the long-term interest rate are not robust for two-way methods, or the estimated coefficients become weaker if one controls for international spillover effects. In the last section of the chapter, I adjusted my panel estimation with an additional interaction term that differentiates between rational and less rational countries. The rational countries were selected by the S&P Financial Literacy survey, and in another specification by the time preference from the Global Preference Survey. In both adjusted estimations, I showed that the secular stagnation hypothesis only holds for those countries where agents' behavior is more consistent with rational expectations; for the rest of the countries, it is not possible to estimate a significant coefficient. This result is also in line with the findings of the OLG-model with bounded rationality.

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Chapter 2

More Gray, More Volatile? Aging and (Optimal) Monetary Policy

"Old age is like a plane flying through a storm. Once you're aboard there's nothing you can do."

Golda Meir (1898-1978)

"But there is something central banks can do."

The authors

2.1 Introduction

In 2012,[1], William C. Dudley, then president of the Federal Reserve Bank of New York, remarked (Dudley 2012) that "the weaker than expected recovery [since the recent crisis] likely lies in the interplay between secular and cyclical factors", and "demographic factors have played a role in [it]." Thus, in addition to stressing the more straightforward fiscal consequences of aging, Dudley drew attention to the interaction between demographics, nominal variables, and central bank policies. Specifically, he noted that the spending decisions of older-age cohorts are less likely to be easily stimulated by monetary policy, as such age groups tend to spend less of their income on consumer durables and housing.

Dudley's comment signaled a longstanding interest of central bankers in population aging. Numerous empirical and theoretical papers discuss the longer-term monetary (inflation rate, but not inflation volatility) and interest-rate implications of aging. According to the secular stagnation literature, slower economic growth is coupled with a fall in the natural rate of interest (examples include Summers (2014), Favero and Galasso (2015), Eggertsson, Mehrotra, and Rob-

[1] `https://www.newyorkfed.org/newsevents/speeches/2012/dud121015`

bins (2017), Ferrero, Gross, and Neri (2017) and Jones (2018)).[2] Many studies have treated the shorter-term behavior of the macroeconomy and monetary policy transmission (but not based on a multi-period general equilibrium framework with overlapping generations), which we will discuss next. To be best of our knowledge, though, the interplay between aging and optimal monetary policies has been completely overlooked in the literature.

Wong (2018) and Miyamoto and Yoshino (2017) are the closest to our framework, and arrive at similar conclusions. Nevertheless, Wong (2018) presents a partial equilibrium life-cycle model, that is, a household model of mortgages and housing.[3] Hence, she does not take all the general equilibrium channels and effects into account. For their part, Miyamoto and Yoshino (2017) do not incorporate overlapping generations into their framework: the retirees simply are rule-of-thumb agents who consume all of their transfers (pensions). To the best of our knowledge, our model is the first multi-period dynamic general equilibrium model with overlapping generations to explore the short-term cyclical behavior of the macroeconomy and the monetary policy transmission mechanism in the presence of aging.

The model is a dynamic general equilibrium model with demographics and overlapping generations, following Baksa and Munkacsi (2016). Population changes over time, and there are two cohorts: the young (workers) and the old (retired). We assume a simple production sector (one sector with a Cobb-Douglas utility function), a simple labor market block (only labor supply, no unemployment), a simple fiscal block (only lump-sum taxes), and consider a closed economy[4]. The model is parametrized for an advanced economy, and the standard deviations are calibrated to the moments of the euro area time series.

Aging is a challenge: it changes the relative and absolute sizes of each cohort. An increase in longevity increases the number and share of the elderly. If the fertility rate (birth rate) also declines, in addition to the higher old-age dependency ratio, the number of the young declines as well. The most important channel is that as agents live longer and their planning horizon becomes longer, their savings position changes: the young are willing to borrow relatively more and finance the increasing pay-as-you-go pension system, while the retired accumulate more

[2] There is also a related stream on the zero lower bound; however, this topic is beyond the scope of the present paper. Additionally, regarding the cyclical volatility of real variables, Jaimovich and Siu (2009) report that the workforce age composition has a large and significant effect on the cyclical volatility of hours worked.

[3] It is beyond the scope of this paper to review the literature on the role of household balance sheets (e.g., debt and housing) in detail.

[4] As stressed by Batini, Callen, and McKibbin (2006), Boersch-Supan, Ludwig, and Winter (2006), and Krueger and Ludwig (2007), open-economy channels matter in multi-country or global settings. In particular, closed-economy predictions for a decline in the interest rate tend to be overstated, that is, capital mobility tends to moderate the pressure on factor price adjustments.

savings to guarantee their consumption over a longer time horizon. The young and the old also make different consumption-savings decisions. In particular, there are age-dependent elasticities of intertemporal substitution in the model, as the old are less sensitive to changes in monetary conditions than the young.[5] Hence, when the interest rate changes, it has different implications for the young and the old: higher interest rates imply an extra cost for the young, who are indebted, while the old generate higher income. Finally, aging has labor market implications: when the labor force shrinks, the labor market becomes tighter, that is, the labor supply is more inelastic (the Frisch-elasticity decreases as the old-age dependency ratio increases). This also affects real wages: the more scarce the labor supply, the more profound the real wage reaction to shocks.[6] Our model suggests that aging contributes positively to inflation volatility, while monetary policy becomes less effective and the aggregate demand less elastic to changes in the interest rate. To compensate for higher inflation volatility, the central bank should react more strongly to nominal variables.

The rest of the paper is structured as follows: in Section 2.2, we summarize the relevant literature on this topic, while in Section 2.3 we describe the equations of the model, which is a simplified version of Baksa and Munkacsi (2016). In Section 2.4, we discuss how to parametrize the framework. Section 2.5 demonstrates the steady states of young and old societies. Then, in Section 2.6, we present the main impulse-response functions and, in particular, explore the transmission mechanism of monetary policy and show empirical evidence for higher inflation volatility in aging societies. Next, Section 2.7 is devoted to studying the welfare consequences of aging and optimal monetary policy rules. Finally, in Section 2.8, we summarize the main policy conclusions.

2.2 Literature review

The related literature consists of several empirical and theoretical papers. In addition to the cited papers in the introduction, we provide further references on relevant overlapping generation models. At the same time, we review the empirical findings.

[5] Wong (2018) provides empirical estimates of age-specific consumption elasticities to interest rate shocks. The consumption of younger people is twice as responsive to interest rate shocks than that of older people, and explains about two-third of the aggregate response. The consumption responses are driven by homeowners who refinance or enter new loans after the interest rate declines. This implies that under an older demographic structure, aggregate consumption will respond less to monetary policy shocks.

[6] Additionally, according to the literature, there are other channels as well: the old consume fewer durable goods and housing, and spend more on health than the young; and the old also tend to shift their portfolio towards safer assets (e.g., government bonds). These channels are not directly modeled in our framework.

A number of empirical papers investigate the impact of population aging and demographics on the rate of inflation. The findings are contradictory. Many report that aging puts a downward pressure on inflation. For example, Kim, Lee, and Yoon (2014), based on a panel dataset covering 30 OECD economies between 1960 and 2013, find that population growth affects the inflation rate positively. Lindh and Malmberg (1998) and Lindh and Malmberg (2000) estimate the relation between inflation and aging also on OECD data between 1960-1994 for 20 countries, and claim that increases in the population of net savers dampen inflation, while net consumers have a positive effect on it (young retirees fan inflation as they start consuming out of accumulated pension claims). At the same time, Juselius and Takats (2015) and Juselius and Takats (2016) introduce the age-structure-inflation puzzle, that is, they report that aging increases inflation. Specifically, based on a panel of 22 advanced economies over the period 1955-2010, they show that both young and old dependents are inflationary, whereas the working-age population is disinflationary. In our paper, which is based on data from several advanced economies, we will present evidence that aging is associated with lower inflation. Moreover, in a panel regression of OECD countries, we provide evidence on the positive impact of aging on inflation volatility.

Several authors have explored the effect of aging on monetary policy transmission. Imam (2013) and Imam (2014) report that, in graying societies, more aggressive monetary policy is needed because such policy becomes less effective with aging. The author, based on Bayesian estimation techniques for the U.S., Canada, Japan, the U.K., and Germany, confirms the weakening of monetary policy effectiveness over time, and provides evidence - using dynamic panel-OLS techniques - that this is attributable to demographic changes. Relatedly, Deok Ryong and Dong-Eun (2017) perform a panel-VAR analysis using OECD data between 1995 and 2014, reporting that monetary policies lose their effectiveness considerably as aging hits the economy.

Additionally, structural models are considered, on the one hand, when examining the long-run inflationary, interest rate, and savings rate impacts of aging and, on the other, when exploring the transition path between the steady states of less and grayer economies, or the impulse responses of monetary policy shocks in an aged society. The findings are somewhat mixed, although most authors claim that aging has a significant impact on nominal variables and monetary policy transmission - that is, it increases inflation and reduces the effectiveness of monetary policy transmission. Nonetheless, we are not aware of any multi-period general equilibrium model with overlapping generations in this literature; the available models are two-period or partial equilibrium models, or they lack important overlapping generation or pension aspects.

Furthermore, to the best of our knowledge, the interplay between aging and optimal monetary policies has not yet been explored in the literature at all.

Fujiwara and Teranishi (2017), based on a closed-economy overlapping generations framework a la Gertler (1999), find that longevity lowers the natural interest rate, but aging does not significantly alter the impulse responses of monetary policy shocks. Nevertheless, as pointed out by Ripatti (2008), in the absence of aging-related fiscal policy and social security aspects, the study does not explore the links between demographics, pensions, and monetary policy. Hence, the results arise from the fact that the retirees' population share is not large enough, even in an aged society. Similarly, Kara and von Thadden (2010) report that demographic changes, while contributing slowly over time to a decline in the equilibrium interest rate, are not visible enough within the time horizon relevant for policy-making to require monetary policy reactions. They develop a small-scale DSGE model, calibrated for the euro area, which embeds a demographic structure within a monetary policy framework by extending Gertler (1999) such that the short-run dynamics are similar to the paradigm summarized in Woodford (2003). Nonetheless, they do not treat monetary policy transmission in their paper.

At the same time, Carvalho and Ferrero (2014), based on a calibrated model of a dynamic monetary economy with a life-cycle structure a la Gertler (1999) for Japan, point out that an increase in life expectancy puts a downward pressure on the effective real interest rate. Kantur (2013), in a two-period OLG New Keynesian model, and Carvalho, Ferrero, and Nechio (2016), in a life-cycle model calibrated for developed economies, also come to the conclusion that the natural rate of interest decreases as the old-age dependency ratio increases. Kantur (2013) also claims that the effectiveness of monetary policy decreases due to a decrease in interest rate sensitivity of the society as the population ages. Similarly, Auerbach, Kotlikoff, Hagemann, and Nicoletti (1989) and Auerbach, Cai, and Kotlikoff (1991), using three different types of models, among others a life-cycle model, and Rios-Rull (2001) - based on Spanish data - report that aging negatively influences the long-term savings rate. Anderson, Botman, and Hunt (2014), based on the IMF's Global Integrated Monetary and Fiscal (GIMF) model, find that aging causes deflation, mainly via slowing growth and falling land prices. Baesel and McMillan (1990) claim that as the Baby Boomers age, it is reasonable to expect that the unemployment rate and the real interest rates will become lower. Further, Miles (1999), in a general equilibrium model with overlapping generations for the UK and Europe, show that there will be a radical decline in saving rates as a result of an increase in the old-age dependency ratio. Other examples in this stream of the literature include Miles (2002), Katagiri, Konishi, and Ueda (2014), and Faruqee and Muhleisen (2003).

2.3 The Model

Demographics and overlapping generations are modeled following Baksa and Munkacsi (2016). We distinguish two cohorts, that is, the 20+ generation is divided into working and retired agents. In each period the number of working-age population changes by an exogenous net fertility rate. In addition, there is a given probability of retirement, with no age-specific retirement probabilities within the cohort. The number of the retired increases because some of the young retire, while we also assume a cohort-specific probability of death (**?**).

Workers decide how much to consume and save, and supply labor; they receive labor income and dividends from the firms. Only the young cohort pays taxes to the government, which is used to finance the pension system and other public expenditures. Retired agents receive pension benefits from the pay-as-you-go (PAYG) pension system: upon each agent's retirement, the government calculates a level of pension benefits that is assumed to be fixed in real terms (or, to put it differently, in nominal terms the government adjusts it with the inflation rate every year). Retired households decide how much of their previous income they wish to spend on consumption, or save. Since state-contingent bonds are assumed, young agents are able to fully insure themselves against any possible survival outcomes.

The production block follows standard neoclassical assumptions and Calvo-style price rigidity. The firms are responsible for physical capital accumulation and investment activity, and they demand labor. In this model, we do not assume productivity growth on the balanced-growth path.

The paper examines the role of monetary policy and short-run cyclical behavior. In the baseline scenario, we consider a standard Taylor-type reaction function, and at a later stage we test the model's robustness using various policy rules. The rest of this section describes the main technical assumptions and the parametrization of the framework.

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2.3.1 Demographics

First of all, we need to define the demographic structure of the economy. Total population (N_t) is equal to the sum of the old (the retired) (N_t^O) and the young (workers) (N_t^Y):

$$N_t = N_t^O + N_t^Y \tag{2.1}$$

Agents become retired with ω^Y probability and n is the net fertility rate, i.e. the share of the new-coming workers:

$$N_t^Y = (1 - \omega_{t-1}^Y)N_{t-1}^Y + n_t N_{t-1}^Y \tag{2.2}$$

Only $1 - \omega^O$ of the retired survive and live in the next period:

$$N_t^O = (1 - \omega_{t-1}^O)N_{t-1}^O + \omega_{t-1}^Y N_{t-1}^Y \tag{2.3}$$

Similarly to other standard general equilibrium models with population growth, we focus on the relative shares of different cohorts, and not on their levels. s_t denotes the ratio of the number of old and young people (i.e., the old-age dependency ratio), while s_t^Y is the share of young people in the whole population. Based on the above assumptions, we can express the ratios as a function of the survival probabilities and the fertility rate:

$$s_t = \frac{N_t^O}{N_t^Y} = \frac{(1 - \omega_{t-1}^O)}{(1 - \omega_{t-1}^Y + n_t)} s_{t-1} + \frac{\omega_{t-1}^Y}{(1 - \omega_{t-1}^Y + n_t)} \tag{2.4}$$

$$s_t^Y = \frac{N_t^Y}{N_t} = \frac{1}{1 + s_t} \tag{2.5}$$

The young, the old, and total population growth can then be expressed as functions of the relative ratios and survival probabilities:

$$1 + g_t^{N,Y} = \frac{N_t^Y}{N_{t-1}^Y} = 1 - \omega_{t-1}^Y + n_t \tag{2.6}$$

$$1 + g_t^{N,O} = \frac{N_t^O}{N_{t-1}^O} = (1 - \omega_{t-1}^O) + \frac{\omega_{t-1}^Y}{s_{t-1}} \tag{2.7}$$

$$1 + g_t^{N} = \frac{N_t}{N_{t-1}} = (1 + g_t^{N,Y})\frac{1 + s_t}{1 + s_{t-1}} \tag{2.8}$$

2.3.2 Households

Households optimize their lifetime utility; nevertheless, several individuals, that is, overlapping generations, live. In the next sections, we present the individuals' decisions and their consumption functions, and calculate the cohort-level aggregate variables as well. The solution for the households' problem is based on backward induction, which means that we start with the retired individuals' optimization, and, conditional on the expected behavior of retired agents, we can also solve the young households' optimization problem.

Retired Households

'Retired' agent i of retired cohort a is one individual who retired a years ago. Each agent maximizes the following Bellman equation:

$$V^O(B^O_{a-1,t-1}(i)) = \max\left\{\frac{1}{1-\gamma}\left\{C^O_{a,t}(i)\right\}^{1-\gamma} + \beta E_t(1-\omega^O_t)V^O(B^O_{a,t}(i))\right\}$$

subject to this budget constraint:

$$C^O_{a,t}(i) + (1-\omega^O_t)B^O_{a,t}(i) = (1+r_{t-1})B^O_{a-1,t-1}(i) + TR^{YO}_{a,t}(i)$$

where $C^O(i)$ denotes the individual consumption of the retired agent, $B^O(i)$ is individual savings, γ is the inverse of the intertemporal elasticity of substitution, β is the discount factor, r is the real interest rate, and $TR^{YO}(i)$ is the level of retirement benefits which was calculated at the time of retirement.

The first-order conditions imply the Euler equation that describes the substitution between the current and future individual retired consumption levels:

$$\beta E_t \frac{\left(C^O_{a+1,t+1}(i)\right)^{-\gamma}}{\left(C^O_{a,t}(i)\right)^{-\gamma}}(1+r_t) = 1 \tag{2.9}$$

After some simplifications and introducing some additional variables, the final version of the consumption of agent i of cohort a at time t is as follows:

$$C^O_{a,t}(i) = MPC^O_t TR^{YO}_{a,t}(i)\Omega^O_t + MPC^O_t(1+r_{t-1})B^O_{a-1,t-1}(i) \tag{2.10}$$

$$\Omega^O_t = 1 + E_t \frac{1-\omega^O_t}{1+r_t}\Omega^O_{t+1} \tag{2.11}$$

$$\frac{1}{MPC^O_t} = 1 + E_t(1-\omega^O_t)(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\frac{1}{MPC^O_{t+1}} \tag{2.12}$$

Here, $TR^{YO}_{n,t+n}(i) = TR^{YO}_{0,t}(i)$ for all $n > 0$. Following the conventions, we introduce the marginal propensity to consume of the retired cohort as MPC^O. The cohort-level consumption (C^O) and savings (B^O) are as follows (for the technical details, see the technical appendix):

$$C^O_t = MPC^O_t TR_t \Omega^O_t + (1+r_{t-1})MPC^O_t(\omega^Y_{t-1}B^Y_{t-1} + B^O_{t-1}) \tag{2.13}$$

where the $TR_t\Omega^O_t$ is the present value of the expected retirement benefits. The retired cohort's consumption depends on the present value of expected revenues and accumulated wealth from the previous periods, and the marginal propensity to consume shows the proportion of lifetime income spent on consumption, which is a function of the real interest rate and the survival probability.

Young Households

'Young' agent i of cohort b is one individual of its cohort who started to work (was born) b years ago. The dynamic optimization problem of the young can be described by the following Bellman equation:

$$\begin{aligned} V^Y_t(B^Y_{b-1,t-1}(i)) \quad = \quad & \max\left\{\frac{1}{1-\gamma}\left\{C^Y_{b,t}(i)^\sigma\left(1-L_{b,t}(i)\right)^{1-\sigma}\right\}^{1-\gamma}\right. \\ & \left. + E_t\left((1-\omega^Y_t)\beta V^Y_{t+1}(B^Y_{b,t}(i)) + \omega^Y_t\beta V^O_{t+1}(B^{YO}_{b,t}(i))\right)\right\} \end{aligned}$$

while the budget constraint is:

$$C^Y_{b,t}(i) + (1-\omega^Y_t)B^Y_{b,t}(i) + \omega^Y_t B^{YO}_{b,t}(i) = (1+r_{t-1})B^Y_{b-1,t-1}(i) \quad + \quad w_t L_{b,t}(i) + Profit_{b,t}(i) - Tax_{b,t}(i)$$

Here, $C^Y(i)$ denotes the young individual's consumption, $L(i)$ is her labor supply, σ is the weight of consumption in the one-period utility function, $Tax(i)$ is the lump-sum tax, $Profit(i)$ denotes the dividend from firms, and w is the real wage. A young agent saves for two possible future states; we assume state contingent bonds, for saving young, the workers save into $B^{Y}(i)$; and for the next period retired-self, the worker today saves into $B^{YO}(i)$. We assume full insurance; this means that any outcome will happen in a future period, and the workers' previous period savings are transferred into their own young-self or retired-self account.

Due to the presence of state-contingent bonds, the optimization problem of young households results in two Euler equations, that is, young households are able to insure themselves against the future retired status as well:

$$\beta E_t \frac{\left(C^Y_{b+1,t+1}(i)^{\sigma}\left(1-L_{b+1,t+1}(i)\right)^{1-\sigma}\right)^{-\gamma} C^Y_{b+1,t+1}(i)^{\sigma-1}\left(1-L_{b+1,t+1}(i)\right)^{1-\sigma}}{\left(C^Y_{b,t}(i)^{\sigma}\left(1-L_{b,t}(i)\right)^{1-\sigma}\right)^{-\gamma} C^Y_{b,t}(i)^{\sigma-1}\left(1-L_{b,t}(i)\right)^{1-\sigma}}(1+r_t) = 1 \tag{2.14}$$

$$\beta E_t \frac{\left(C^O_{b+1,t+1}(i)\right)^{-\gamma}}{\left(C^Y_{b,t}(i)^{\sigma}\left(1-L_{b,t}(i)\right)^{1-\sigma}\right)^{-\gamma} \sigma C^Y_{b,t}(i)^{\sigma-1}\left(1-L_{b,t}(i)\right)^{1-\sigma}}(1+r_t) = 1 \tag{2.15}$$

In addition, the young decide about their labor supply:

$$\frac{C^Y_{b,t}(i)}{1-L_{b,t}(i)} = \frac{\sigma}{1-\sigma} w_t \tag{2.16}$$

After some simplifications and by introducing additional variables, we can write up the young individual's consumption function as follows:

$$C^Y_{b,t}(i) = MPC^Y_t Inc^Y_{b,t}(i) + MPC^Y_t(1+r_{t-1})B^Y_{b-1,t-1}(i) \tag{2.17}$$

$$\begin{aligned} Inc^Y_{b,t}(i) &= w_t + Profit_{b,t}(i) - Tax_{b,t}(i) + \\ &+ E_t \frac{\omega^Y_t}{1+r_t} TR^{YO}_{0,t+1}(i)\Omega_{t+1} + E_t \frac{1-\omega^Y_t}{1+r_t} Inc^Y_{b,t}(i) \end{aligned} \tag{2.18}$$

$$\frac{1}{MPC^Y_t} = \frac{1}{\sigma} + E_t(1+r_t)^{\frac{1}{\gamma}-1}\left[(1-\omega^Y_t)\Lambda^Y_t \frac{1}{MPC^Y_{t+1}} + \omega^Y_t \Lambda^{YO}_t \frac{1}{MPC^O_{t+1}}\right] \tag{2.19}$$

$$\Lambda^Y_t = E_t \beta^{\frac{1}{\gamma}} \left(\frac{w_{t+1}}{w_t}\right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \tag{2.20}$$

$$\Lambda^{YO}_t = E_t \left\{\frac{\beta}{\sigma}\right\}^{\frac{1}{\gamma}} \left(\frac{1}{\frac{\sigma}{1-\sigma} w_t}\right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \tag{2.21}$$

Here, MPC^Y is the marginal propensity of the young to consume, and $Inc^Y(i)$ is the discounted sum of the current and present value of expected incomes and future pension benefits.

Based on the individual consumption function, we can express the aggregate consumption function and the aggregate income function as follows:

$$\begin{aligned} C_t^Y &= MPC_t^Y Inc_t^Y + (1+r_{t-1})MPC_t^Y(1-\omega_{t-1}^Y)B_{t-1}^Y && (2.22)\\ Inc_t^Y &= w_t N_t^Y + Profit_t - Tax_t + E_t\frac{1}{1+r_t}TR_{t+1}^{YO}\Omega_{t+1}^O + \\ &+ E_t\frac{1-\omega_t^Y}{(1+r_t)(1+g_{t+1}^{N,Y})}Inc_{t+1}^Y && (2.23)\end{aligned}$$

Since there are two distinct cohorts, and both can accumulate risk-free bonds, one of the budget constraints closes the model; the aggregate young budget constraint is as follows:

$$C_t^Y + B_t^Y = w_t L_t + Profit_t - Tax_t + (1+r_{t-1})(1-\omega_{t-1}^Y)B_{t-1}^Y \quad (2.24)$$

2.3.3 Firms

We assume monopolistic competition and price stickiness a la Calvo: in every period, a fraction of $1-\omega^P$ of firms has a chance to change the nominal price level; the rest of them can only adjust the previously agreed prices by the previous period's aggregate inflation rate (Calvo 1983; Milani 2005). The optimization problem is conditional on the expectation of the Calvo lottery. As a result, all firms maximize the present value of current and expected future profit flows subject to the production function, the demand function, and the capital accumulation equation. In other words, the firm j maximizes

$$E_t\sum_{n=0}^{\infty}\omega^{P^n}\prod_{k=1}^{n}\frac{1-\omega_{t+k-1}^Y}{1+i_{t+k-1}}\left(P_t^*(j)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}Y_{t+n}(j) - V_{t+n}L_{t+n}(j) - P_{t+n}Inv_{t+n}(j)\right)$$

subject to:

$$\begin{aligned} Y_t(j) &= e^{-\epsilon_t^P}A_t K_{t-1}(j)^{\alpha}L_t(j)^{1-\alpha}\\ Y_t(j) &= \left(\frac{P_t^*(j)}{P_t}\right)^{-\varphi_P}Y_t\\ K_t(j) &= Inv_t(j)\left(1-S\left(\frac{Inv_t(j)}{Inv_{t-1}(j)}\right)\right)+(1-\delta)K_{t-1}(j)\\ S\left(\frac{Inv_t(j)}{(1+g_t^N)Inv_{t-1}(j)}\right) &= \frac{\phi_{Inv}(1+\xi_t^{Inv})}{2}\left(\frac{Inv_t(j)}{(1+g_t^N)Inv_{t-1}(j)}-1\right)^2\end{aligned}$$

Here, i denotes the nominal interest rate, and the young households are the owners of the firms; thus, their survival probability is also taken into account. $P^*(j)$ denotes the individual optimal nominal prices, which are adjusted by the inflation rate. Further, $Inv(j)$ is the real investment variable, which is multiplied by the aggregate price index; α is the capital share in the Cobb-Douglas production function; δ is the depreciation rate of capital; φ_P is the price elasticity for $Y(j)$ demanded individual products; γ_P is the degree of price indexation; and $S(\cdot)$ is the convex investment adjustment cost function, with ϕ_{Inv} cost parameter and ξ^{Inv} investment shock. ϵ_t^P denotes the supply shock, which is only defined in a sticky price equilibrium. We also assume nominal wage stickiness a la Calvo, and V denotes the relevant nominal wage index.

Taking the first-order conditions, we can derive the usual input demand functions and marginal cost functions:

$$\alpha \frac{Y_t(j)}{K_{t-1}(j)} mc_t = r_t^K \tag{2.25}$$

$$(1-\alpha) \frac{Y_t(j)}{L_t(j)} mc_t = v_t \tag{2.26}$$

$$mc_t = e^{\epsilon_t^P} \frac{1}{A_t} \left(\frac{r_t^K}{\alpha}\right)^{\alpha} \left(\frac{v_t}{1-\alpha}\right)^{1-\alpha} \tag{2.27}$$

where r^K denotes the marginal product of capital, for simplification we use this variable in the rest of the paper. Because the firms are responsible for physical capital accumulation, the set of first-order conditions also contain the Tobin-Q equation and a no-arbitrage condition:

$$Q_t \left(1 - S\left(\frac{Inv_t(j)}{(1+g_t^N)Inv_{t-1}(j)}\right) - S'\left(\frac{Inv_t(j)}{(1+g_t^N)Inv_{t-1}(j)}\right)\frac{Inv_t(j)}{Inv_{t-1}(j)}\right) =$$

$$= 1 - E_t \frac{1-\omega_t^Y}{1+r_t} Q_{t+1} S'\left(\frac{Inv_{t+1}(j)}{(1+g_{t+1}^N)Inv_t(j)}\right)\left(\frac{Inv_{t+1}(j)}{Inv_t(j)}\right)^2 \tag{2.28}$$

$$E_t(1-\omega_t^Y)\left(r_{t+1}^K + Q_{t+1}(1-\delta)\right) = Q_t(1+r_t) \tag{2.29}$$

Next, the monopolistic firm sets the optimal price (p^*) as follows:

$$p_t^* = \frac{\varphi_P}{\varphi_P - 1}\frac{\mathcal{Z}_t^1}{\mathcal{Z}_t^2} \tag{2.30}$$

$$\mathcal{Z}_t^1 = p_t^{*-\varphi_P} Y_t mc_t + E_t \left(\frac{p_t^*}{p_{t+1}^*}\frac{(1+\pi_t)^{\gamma_P}}{1+\pi_{t+1}}\right)^{-\varphi_P} \omega_P (1-\omega_t^Y)\frac{1+\pi_{t+1}}{1+i_t}\mathcal{Z}_{t+1}^1 \tag{2.31}$$

$$\mathcal{Z}_t^2 = p_t^{*-\varphi_P} Y_t + E_t \left(\frac{p_t^*}{p_{t+1}^*}\frac{(1+\pi_t)^{\gamma_P}}{1+\pi_{t+1}}\right)^{-\varphi_P} \frac{(1+\pi_t)^{\gamma_P}}{1+i_t}\omega_P (1-\omega_t^Y)\mathcal{Z}_{t+1}^2 \tag{2.32}$$

Based on the definition of the price index, we can express the optimal individual relative price as a function of actual and previous inflation rates:

$$1 = (1-\omega_P) p_t^{*1-\varphi_P} + \omega_P \left(\frac{(1+\pi_{t-1})^{\gamma_P}}{1+\pi_t}\right)^{1-\varphi_P} \tag{2.33}$$

The effective labor cost for the firms differs from the workers' real wage. We assume that workers supply their labor force to a labor unions and that these unions, with their monopolistic power, set the profit-maximizing effective nominal wage. However, not all unions are able to set their optimal wage in each periods. Next, the optimization problem of the labor unions can be given as follows:

$$E_t \sum_{n=0}^{\infty} \omega_V^n \prod_{k=1}^{n} \frac{1-\omega_{t+k-1}^Y}{1+i_{t+k-1}} \left(V_t^*(j) \left(\frac{V_{t+n-1}}{V_{t-1}} \right)^{\gamma_V} L_{t+n}(j) - W_{t+n} L_{t+n}(j) \right)$$

subject to

$$L_t(j) \quad = \quad \left(\frac{V_t^*(j)}{V_t} \right)^{-\varphi_W} L_t$$

where ω_V is the fraction of unions that are not able to set their prices in a given period. $V^*(j)$ denotes the individual optimal nominal wage adjusted by the wage inflation rate. φ_W is the wage elasticity for labor, and γ_V is the degree of price indexation. The monopolistic labor union sets the optimal nominal wage as follows:

$$v_t^* \quad = \quad \frac{\varphi_V}{\varphi_V - 1} \frac{\mathcal{W}_t^1}{\mathcal{W}_t^2} \tag{2.34}$$

$$\mathcal{W}_t^1 \quad = \quad \left(\frac{v_t^*}{v_t} \right)^{-\varphi_W} L_t w_t + E_t \left(\frac{v_t^*}{v_{t+1}^*} \frac{(1+\pi_t^V)^{\gamma_V}}{1+\pi_{t+1}^V} \right)^{-\varphi_V} \frac{\omega_V (1-\omega_t^Y)(1+\pi_{t+1})}{1+i_t} \mathcal{W}_{t+1}^1 \tag{2.35}$$

$$\mathcal{W}_t^2 \quad = \quad \left(\frac{v_t^*}{v_t} \right)^{-\varphi_V} L_t + E_t \left(\frac{v_t^*}{v_{t+1}^*} \frac{(1+\pi_t^V)^{\gamma_V}}{1+\pi_{t+1}^V} \right)^{-\varphi_V} \frac{\omega_V (1-\omega_t^Y) \left(1+\pi_t^V\right)^{\gamma_V}}{1+i_t} \mathcal{W}_{t+1}^2 \tag{2.36}$$

Similarly, based on the definition of the price index, we can express the optimal individual relative wage as a function of actual and previous wage inflation and real aggregate wage indices:

$$1 = (1-\omega_V) \left(\frac{v_t^*}{v_t} \right)^{1-\varphi_V} + \omega_V \left(\frac{\left(1+\pi_{t-1}^V\right)^{\gamma_V}}{1+\pi_t^V} \right)^{1-\varphi_V} \tag{2.37}$$

where the π^V is the nominal wage inflation, which can be given as:

$$1+\pi_t^V = (1+\pi_t) \frac{v_t}{v_{t-1}} \tag{2.38}$$

Here, v denotes the union wage in real terms. Last, the profits of firms and the labor union go to the young:

$$Profit_t \quad = \quad Y_t - w_t L_t - Inv_t \tag{2.39}$$

2.3.4 Fiscal Policy

The role of fiscal policy is limited in this paper. The government is responsible for providing pay-as-you-go pension benefits, and it finances its expenditures by taxing the young cohort or issuing public debt:

$$Debt_t + Tax_t = TR_t + Gov_t + (1 + r_{t-1})Debt_{t-1} \tag{2.40}$$

where the $Debt$ is the level of public debt, and we chose a Tax lump-sum tax rule that sustains the initial $\frac{Debt}{Y}$ ratio:

$$Tax_t = TR_t + Gov_t + (1 + r_{t-1})Debt_{t-1} - \left\{\frac{Debt}{Y}\right\}^{Target} Y_t \tag{2.41}$$

In the PAYG-regime individual (i)'s pension benefits in the year of retirement t are based on the replacement rate ν and the pre-retirement labor income:

$$TR^{YO}_{0,t}(i) = \nu w_{t-1} L_{b-1,t-1}(i) \tag{2.42}$$

The aggregated version of the pension rules is:

$$TR^{YO}_t \quad = \quad \nu \omega^Y_{t-1} w_{t-1} L_{t-1} \tag{2.43}$$

Furthermore, the total pension expenditure of all retired agents can be described as a function of pension benefits and survival probabilities:

$$TR_t = \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^O_{t-k}) TR^{YO}_{t-n}. \tag{2.44}$$

which can be simply rewritten in recursive form as:

$$TR_t = TR^{YO}_t + (1 - \omega^O_{t-1}) TR_{t-1} \tag{2.45}$$

2.3.5 Monetary Policy

Initially, the central bank follows a simple rule with inflation reaction and interest rate smoothing:

$$1 + i_t \quad = \quad (1 + i_{t-1})^{\rho_i} \left((1 + r_t^n) \, (1 + \pi_t)^{\phi_\pi} \right)^{1-\rho_i} e^{\epsilon_t^i} \tag{2.46}$$

where ϵ_t^i is the monetary policy shock. By assumption, the central bank raises the interest rate if inflation exceeds its steady-state level. Since prices are sticky, the central bank is able to influence the real economy's performance via the interest rate channel. r^n is the natural interest rate that is consistent with the flexible-price version of the model. The Fisher-identity expresses the relationship between the nominal and real interest rates:

$$1 + i_t \quad = \quad E_t(1 + r_t)(1 + \pi_{t+1}) \tag{2.47}$$

2.3.6 Market Clearing Conditions

Finally, we need to clarify the market clearing conditions. Since there is no government debt, the sum of the two cohorts' savings should be equal with public debt.

$$Debt_t \quad = \quad B_t^Y + B_t^O \tag{2.48}$$

In the goods market, all supplied goods are equal to the demanded consumption goods, investment, and government expenditure:

$$Y_t \quad = \quad C_t^Y + C_t^O + Inv_t + Gov_t \tag{2.49}$$

2.4 Parametrization

In this paper, we are concerned with general patterns. Thus, we parametrize the model by choosing typical parameter values in the DSGE literature. Table 2.1 shows the chosen parameter values. Specifically, we use the Pessoa model, that is, the Portuguese version of the GIMF model (Almeida, Castro, Félix, J'ulio, and Maria (2013)), and the New Area-Wide Model (Christoffel, Coenen, and Warne (2008)).

Based on the Pessoa model, we calibrated the households' time preference (β), risk aversion

(γ) and weight of consumption (σ) in the utility function. The price and wage elasticities (φ and φ_W), capital share (α), depreciation rate (δ), Calvo parameters (ω_P and ω_W) and indexation parameters (γ_P and γ_W), investment adjustment cost (ϕ_{Inv}), inflationary reaction (ϕ_π) and interest smoothing (ρ_i) of the monetary policy rule are taken from the New Area-Wide Model. Regarding the calibration of the demographic parameters, we consider data on Germany, Portugal, Spain and Slovakia published by the Eurostat. Particularly, we choose the fertility rate, probability of retirement and mortality rate such that we match on the average probabilities and the old-age dependency ratio in the 80s. The replacement rate and the size of the pension system are based on Istenič, Hammer, Šeme, Lotrič Dolinar, and Sambt (2016), who measure the overall income redistribution from the worker to the retired cohort. The government consumption-to-GDP ratio is calibrated to the government individual consumption of European countries, while the steady-state public debt-to-GDP ratio is the Maastricht criterium, that is, 60 percent.

As we express all variables in per capita values and solve the steady state of the normalized model, we need a candidate for the steady-state value of the equilibrium real interest rate. Conditional on this assumed level of the real interest rate, we can calculate all the other endogenous variables. As a final step, we check whether the financial market equilibrium condition holds; if it does not, another starting value for the steady-state real interest rate is chosen, and the process starts again.[7]

In this paper, we deal with short-run cyclical fluctuations, where the sizes of the discount rates, the interest rate, and labor supply elasticities play a crucial role. They reflect the relative importance of different channels for different instruments used in the model. First, the interest rate elasticity from the non-separable utility function (i.e. the intertemporal substitution) can be derived as follows:

$$\varepsilon = -\frac{U_{C(i)}}{U_{C(i),C(i)}C(i)} = \frac{1}{1+\sigma(\gamma-1)} \tag{2.50}$$

Retired agents do not supply labor ($\sigma = 1$), which means that the ε-s are cohort-specific. The parametrized values are based on typical values in the DSGE literature, and the dynamic responses are relatively close to those estimated by Wong (2018) for the old and the young, respectively.

[7] The same approach was followed in Baksa and Munkacsi (2016)

Next, the Frisch labor supply elasticity can only be calculated for the young cohort:

$$\eta = \frac{U_{L(i)}}{L(i)\left(U_{L(i),L(i)} - \frac{U^2_{C(i),L(i)}}{U_{C(i),C(i)}}\right)} = \frac{s^Y - \tilde{L}}{\tilde{L}}\frac{1-\sigma(1-\gamma)}{\gamma} \tag{2.51}$$

where $\tilde{L}$ is the normalized level of labor. Our specification is consistent with the available microeconomic estimates (such as that of Whalen and Reichling (2017)). Namely, the Frisch elasticity is a function of the steady-state demographic and labor market variables. Thus, aging directly changes the relative size of cohorts and the supply of labor: the labor force shrinks and the labor market becomes tighter with aging. Relatedly, labor supply becomes less elastic (the Frisch-elasticity decreases as the old-age dependency ratio increases), and the volatility of real wages is intensified (the scarcer the labor supply, the more profound the real wage reaction to shocks) (Table 2.2).

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Table 2.1: Model parameters

Name	Notation	Value	Comments and source
Demography			
Probability of retirement	ω^Y	0.005	Quaterly frequency, Eurostat Population data
Probability of death	ω^O	0.022	Quaterly frequency, Eurostat Population data
Fertility rate	n	0.007	Quaterly frequency, Eurostat Population data
Households			
Discount rate	β	0.996	Quarterly frequency, Almeida, Castro, Félix, J'ulio, and Maria (2013)
Weight of consumption utility	σ	0.73	Only for youngs, Almeida, Castro, Félix, J'ulio, and Maria (2013)
Inverse of intertemporal elasticity	γ	2.00	Same for both cohort, Almeida, Castro, Félix, J'ulio, and Maria (2013)
Firms			
Price elasticity	φ_P	3.86	Calibrated mark-up: 1.35, Christoffel, Coenen, and Warne (2008)
Wage elasticity	φ_W	4.33	Calibrated mark-up: 1.30, Christoffel, Coenen, and Warne (2008)
Capital share	α	0.36	Christoffel, Coenen, and Warne (2008)
Capital depreciation	δ	0.03	Quarterly frequency, Christoffel, Coenen, and Warne (2008)
Investment adjustmen cost	ϕ_{Inv}	4.00	Christoffel, Coenen, and Warne (2008)
Share of price setters (Calvo)	ω_P	0.75	Quarterly frequency, Christoffel, Coenen, and Warne (2008)
Price indexation	γ_P	0.75	Christoffel, Coenen, and Warne (2008)
Share of union wage setters (Calvo)	ω_V	0.75	Quarterly frequency, Christoffel, Coenen, and Warne (2008)
Nominal wage indexation	γ_V	0.75	Christoffel, Coenen, and Warne (2008)
Monetary policy			
Interest rate smoothing	ρ^i	0.90	Quarterly frequency, Christoffel, Coenen, and Warne (2008)
Reaction to inflation	ϕ_π	1.70	Christoffel, Coenen, and Warne (2008)
Fiscal policy			
Replacement rate	ν	1.62	Calibrated from $\frac{TR}{Y} = 0.12$, Istenič, Hammer, Šeme, Lotrič Dolinar, and Sambt (2016)
Gov. cons.	Gov	0.17	Calibrated from $\frac{Gov}{Y} = 0.10$, Eurostat SNA
Public Debt to GDP	$\frac{Debt}{Y}$	2.40	Quarterly frequency, Maastricht criteria

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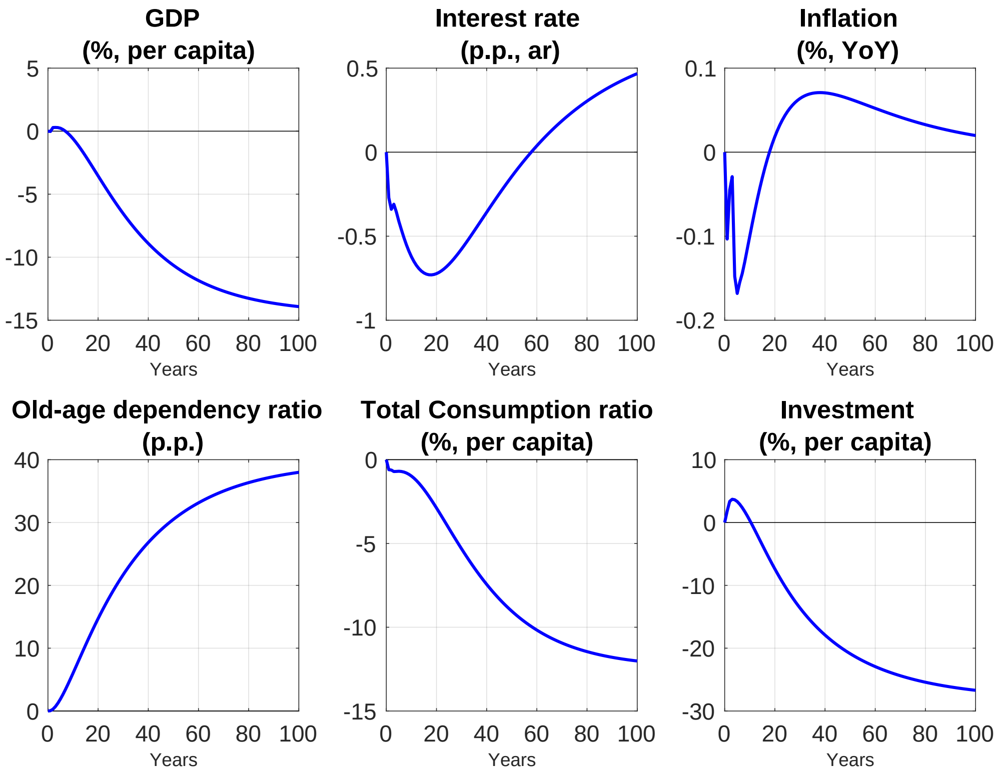
2.5 A Permanent Demographic Shift

In this paper, we mainly consider (i) how aging affects the short-run cyclical behavior of the macroeconomy, and (ii) how optimal monetary policies change with aging. Before doing so in Sections 2.6 and 2.7, though, in this section we demonstrate a transition from a young society to an old one. Specifically, Table 2.2 shows the steady-state comparison of a young and an old society when the fertility rate decreases from 3% to 1.7%, the probability of death decreases from 9% to 3.65% in 20 years, and the old-age dependency ratio gradually increases from 20 to 60 percent, while Figure 2.1[8] the transition path between the two steady states under the PAYG system, the results are consistent with Baksa (2019).

Figure 2.1: Transition from a young society to an old one



[8]"ar" is an abbreviation for annualized rate.

Due to the longer lifetime horizon (decreasing ω^O), retired households accumulate more savings. At the same time, the PAYG pension system is required to provide a higher amount of pension benefits, which can be financed by taxes deducted from the young workers. So, young agents face an increasing financing need from the pension system, and, due to consumption smoothing (and to minimize their own sacrifice), their indebtedness increases. The higher private indebtedness and the higher financing needs of the government with an extended pension system elevates the long-term real interest rate Baksa (2019).

Table 2.2: Steady-state comparison: young and gray societies

Name	Notation	Young society	Graying society
Demography			
Old-age dependency ratio	s	0.20	0.60
Share of the young	s^Y	0.83	0.63
Households			
Consumption to GDP: Total	$\frac{C}{Y}$	0.69	0.71
Consumption to GDP: Young	$\frac{C^Y}{Y}$	0.56	0.33
Consumption to GDP: Old	$\frac{C^O}{Y}$	0.13	0.37
Bonds to GDP: Old	$\frac{B^O}{Y}$	0.44	3.14
Bonds to GDP: Young	$\frac{B^Y}{Y}$	1.96	-0.74
Interest rate elasticity: Young	ε^Y	0.58	0.58
Interest rate elasticity: Old	ε^O	0.50	0.50
Frisch elasticity	η	0.49	0.29
Firms			
GDP	Y	1.67	1.43
Investment to GDP	$\frac{Inv}{Y}$	0.21	0.18
Capital to GDP	$\frac{K}{Y}$	7.71	7.32
Profit to GDP	$\frac{Profit}{Y}$	0.42	0.46
Monetary policy			
Real interest rate	r	0.00	0.01
Fiscal policy			
Transfer to GDP	$\frac{TR}{Y}$	0.12	0.35
Public consumption to GDP	$\frac{Gov}{Y}$	0.10	0.12

Regarding the supply side of the economy, via lower fertility and a shrinking labor force, the firms accommodate to the new situation: they realize the decrease in demand, so they gradually disinvest and reduce prices (generate disinflation or deflation) to offset the loss in profits. However, in the medium term, instead of investing in physical capital, they hire relatively more labor. The decreasing free labor capacities increase the real wage, and in the tighter labor market the real wage becomes more volatile and more responsive to short-run shocks as the Frisch elasticity decreases. Due to the labor shortages, real wages rise, and inflation goes back to its steady-state level. The central bank first decreases the interest rate to compensate for the downward pressure on inflation. However, in the long run, it gradually normalizes the policy rate, since inflation goes back to its steady-state level. The lower level of physical capital stock and the increased marginal product of capital are consistent with the higher equilibrium interest rate in the new steady state.

2.6 Cyclical Behavior and Monetary Policy Transmission

We now turn to presenting the dynamic properties of the model. In a young and an old society, we compare the short-run cyclical reactions of the macroeconomy to a monetary policy shock, a government expenditure shock, and a supply shock. We examine one-off, temporary shocks only, that is, all the impulse response functions converge back to the demographic-specific steady state.[9]

2.6.1 Monetary Policy Shock

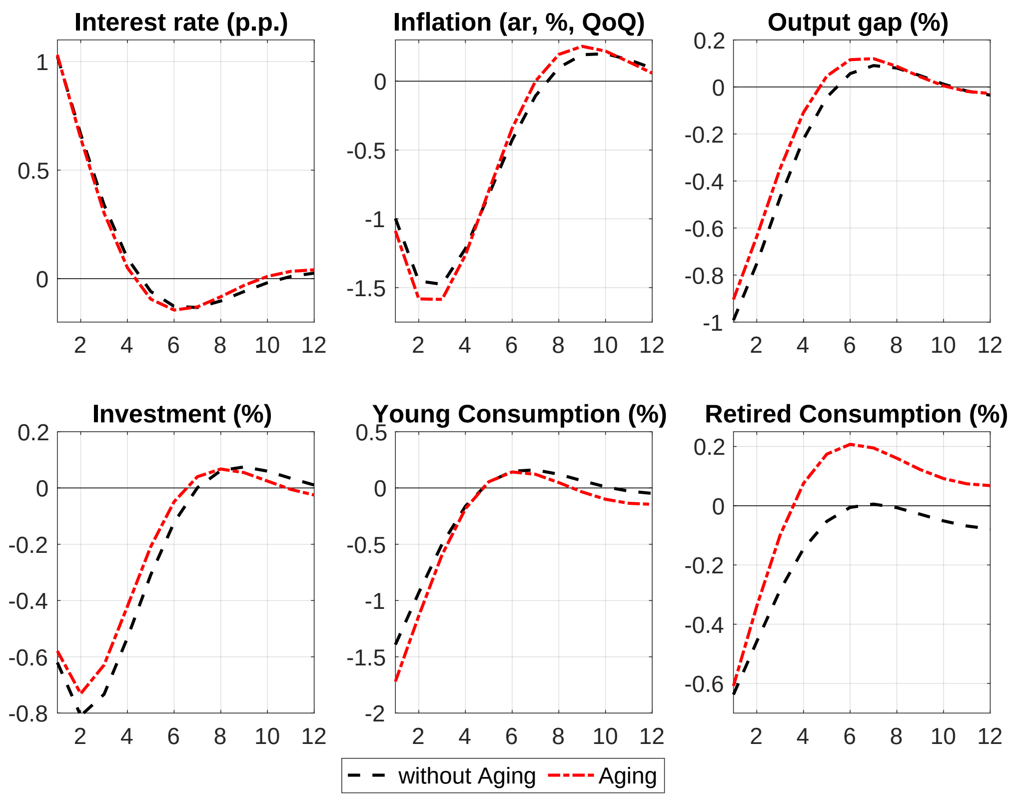
It is well known that if the central bank deviates from its systematic Taylor-rule, that is, if it increases the nominal interest rate, then, because of price stickiness, it also changes the real interest rate - and so can influence the performance of the real economy (Figure 2.2). Tighter monetary policy conditions force firms to postpone investment, and households to start accumulating more savings or decreasing their credit stock. Due to the shrinking aggregate demand, share $1-\omega^P$ of the firms decrease their prices to avoid (reduce) the loss in profits. Due to price stickiness and price indexation, the nominal adjustment is gradual, and disinflation takes more than 4 quarters. The central bank observes the shrinking aggregate demand and decreases the nominal interest rate back to its steady-state level, and the whole economy stabilizes at its initial steady state.

Aging generates differences in the output gap reaction (the deviation of GDP from its flexible-price equilibrium level). This is also reflected in the fact that the young and the old respond differently. Specifically, in an old society, the retired agents hold more savings, while the workers have more debt during their longer lifetime. Additionally, in a young society, the monetary restriction creates an incentive for postponing consumption and increasing savings. However, in an old society, the interest rate hike has an even more asymmetric effect: the old can interpret

[9]In all simulations, we assume that in young societies (on Figures 2-4 labeled *without Aging*) the old-age dependency ratio is 20%, and in old societies (on Figures 2-4 labeled *Aging*) 60%. On the same figures is an abbreviation for the annualized rate.

the shock as additional income; thus, later, they increase consumption by a larger amount, while the young face higher credit costs and decrease consumption more. Because aging also changes the relative size of cohorts, aggregate consumption and output gap fall less in gray societies. Hence, we find that aging reallocates the asset position among generations, which makes monetary policy less effective - defined as a larger drop in the inflation rate to the same monetary tightening - and aggregate demand less elastic to interest rate changes. Our finding is consistent with that of Wong (2018) who, based on micro-level cohort-specific US data, shows that the old react less to expansionary monetary policy shocks.

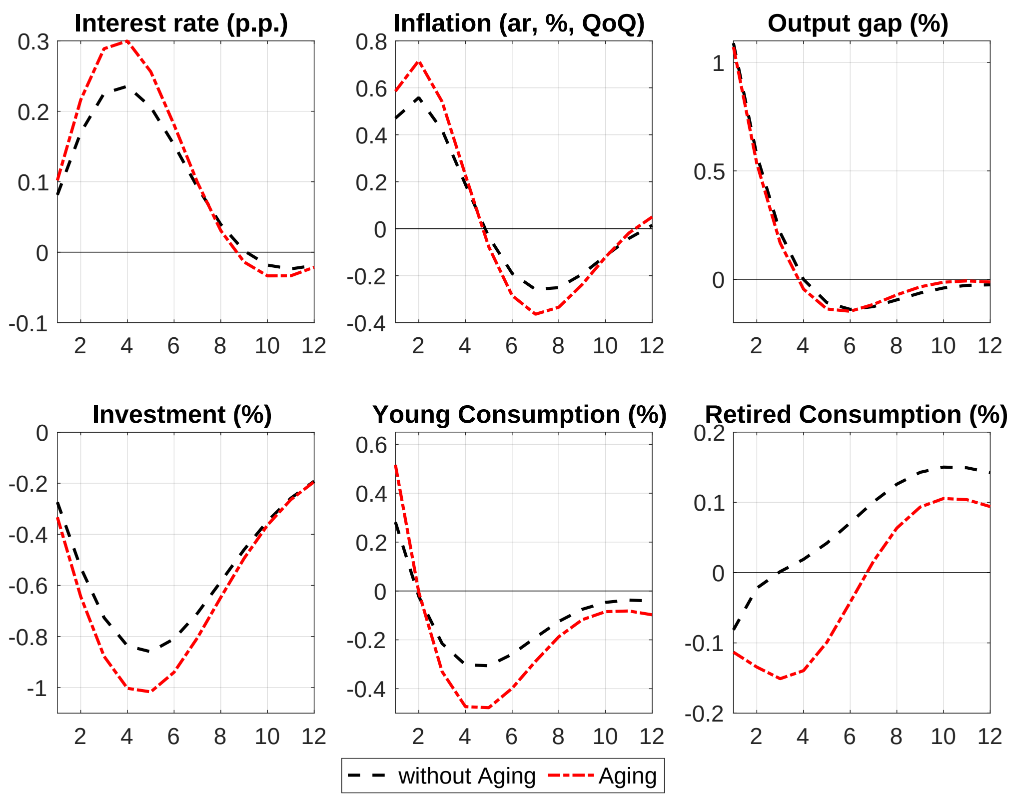
Figure 2.2: Impulse responses of a monetary policy shock



2.6.2 Government Expenditure Shock

The government increases public expenditure by one percent of the steady-state GDP (Figure 2.3). As a result, firms increase production to satisfy the extra demand. A higher level of production requires more labor, so firms increase wages to attract more workers. To offset the increase in production costs and the loss in profits, price-setting firms increase their prices. The central bank launches a tightening cycle, and holds the interest rate elevated until the demand-side inflationary pressure disappears and inflation goes back to its original steady-state level. Due to the increase in wages, young households consume more. Old households consume less because of the increasing interest rate and credit demand (from the young households). Later on, though, the higher interest rates push young consumption down and old consumption up. Aging also changes the relative size of cohorts, and decreases the available labor force in the economy. In an old society, the labor supply is more inelastic (the Frisch-elasticity is lower). Hence, firms are forced to increase wages more than in a young society. This additional increase in wages amplifies the increase in marginal costs and inflation, too. As a consequence, in an old society, the central bank needs to raise the policy rate by a larger amount than it does in a young society. A stronger monetary policy reaction in a grayer society forces the young to give up more consumption.

Figure 2.3: Impulse responses of a government expenditure shock



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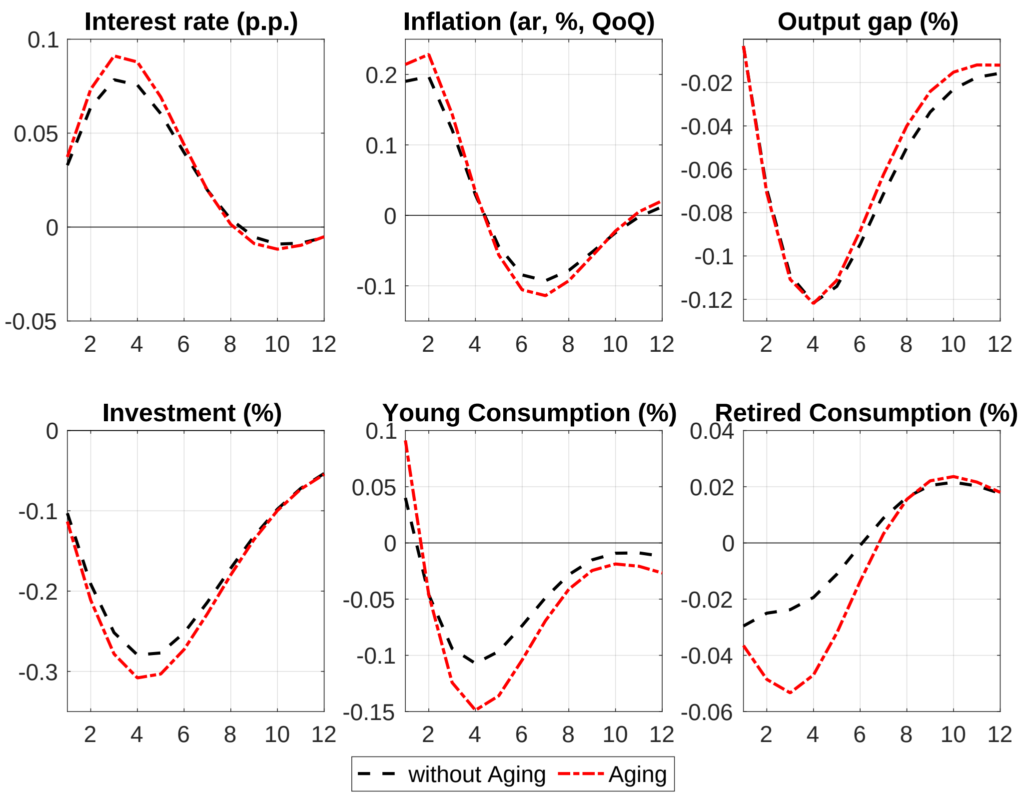
2.6.3 Supply Shock

The main channel of the supply shock is that price-setting firms decide to increase prices to improve profitability, or offset profit losses (Figure 2.4). Due to higher inflation, the central bank increases the nominal interest rate, which forces households and firms to postpone expenditure items (consumption and investment, respectively). Observing the fall in domestic demand, firms decide to give up on further price increases; consequently, inflation starts to normalize. As a result, the central bank reduces the policy rate, and the economy stabilizes around its original steady state.

Figure 2.4: Impulse responses of a supply shock



Although the size of the shocks in old and young societies are the same, the immediate impact on nominal and real variables differs. The central bank reacts to offset the positive inflationary expectations (and more negative real interest rate), and discourages households from reallocating future consumption to present consumption. In a young society, both households and firms are aware of a possible monetary tightening in the near future, and, due to the expected decline in aggregate demand, the overall increase in inflation is lower than what the shock would imply. At the same time, the young hold more debt, and, at the time of the shock, the temporary negative real interest rate depreciates young households' credit stock. Thus, young households initially increase their consumption. This effect in an old society is larger (in an old society, the young are more indebted than the old); hence, the jump in young consumption is more significant at the beginning. Due to the relatively higher demand in an old society, firms are able to increase their prices by a larger amount. In an old society, the overall inflationary pressure is also higher; the central bank thus needs to be more responsive. This later requires a higher sacrifice (in terms of consumption goods) from the young.

2.6.4 Demographic structure and inflation volatility

In the previous sections, we found that, with aging, the rate of inflation decreases while its volatility increases, and that aging makes monetary policy less effective. Regarding the impact of aging on the rate of inflation, several papers claim that aging leads to a downward pressure on inflation, while some report the opposite. Using data on developed economies (the U.S., the U.K., Germany, France, Japan, and Portugal)[10] between 1981 and 2018, we find that, from the 1990s, the periods of disinflation correspond with the periods of increases in the old-age dependency ratio (henceforth OADR) (Figure 2.5).[11] This is in line with what our model suggests when increasing the old-age dependency ratio in the steady state.

Table 2.3: Descriptive statistics of observed variables: OECD countries between 1993 and 2018

		Selected intervals			
	All obs.	1993-1999	2000-2007	2008-2013	2014-2018
$mean\left(CPI^{CORE}\right)$	3.162	6.058	2.998	1.939	1.527
$std\left(CPI^{CORE}\right)$	0.282	0.372	0.286	0.267	0.190
$mean(OADR)$	24.811	22.597	23.415	25.479	28.943
$std\,(Y)$	2.758	2.486	3.190	3.152	1.862
Number of countries	34	34	34	34	34

[10] Japan is already heavily hit by aging; Portugal will be among the countries with the highest old-age dependency ratio in Europe in a few decades; the US, the UK, Germany and France - although less affected by aging than Japan or Portugal - follow different monetary policy strategies (with France less affected by aging than Germany).

[11] Inflation data comes from the OECD while data on the old-age dependency ratio comes from the UN.

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Figure 2.5: Core inflation and old-age dependency ratio

Regarding the impact of aging on the volatility of inflation, we also test the model's hypothesis using an unbalanced panel dataset on developed OECD countries between 1993 and 2018. Table 2.3 contains the descriptive statistics of the observed variables. We run a fixed-effect (FE) estimation with country- and time-fixed effects.[12] For this, we need the time-variant volatility of inflation: first, we remove the trend component of non-food non-energy CPI by an HP-filter with $\lambda = 150000$; then, we calculate the yearly average of the standard deviation from the trend, which is interpreted as the time-variant volatility of the cyclical inflation. Next, we run the following FE regression:

$$\sigma(cpi\_core)_{it} = \alpha_i + \delta_t + \beta \cdot OADR_{it} + \gamma \cdot X_{it} + u_{it} \tag{2.52}$$

where $\sigma(cpi\_core)_{it}$ is the standard deviation of non-food non-energy CPI in country i and year t, α_i is the country-fixed effect, δ_t is the time-fixed effect, $OADR_{it}$ is the old-age dependency ratio, X contains other controls (lag, output gap volatility and the year-on-year changes of

[12] In R with plm package (**?**). We calculated and report robust standard errors.

inflation)[13], and u_{it} denotes the error term.

The estimation results are consistent with the model's intuitions. Namely, all of the estimations suggest a significant and positive relationship between the volatility of inflation and the level of the old-age dependency ratio. Moreover, the impact is sizeable: in our sample, the average inflation volatility is 0.28, and the old-age dependency ratio increased by 6 percentage point on average over the past 3 decades; the FE regressions suggest that a 6 percentage point increase in the old-age dependency ratio increases inflation volatility by around 0.04-0.11 cumulatively.

Table 2.4: Volatility of inflation and old-dependency ratio

Models	FE (1)	FE (2)	FE (3)	FE (4)	FE (5)	FE (6)
$\sigma(CPI\_CORE)_{i,t-1}$				0.218 (0.065)***	0.207 (0.061)***	0.104 (0.078)
$OADR_{it}$	0.014 (0.007)*	0.012 (0.007)*	0.007 (0.004)*	0.014 (0.007)**	0.013 (0.006)**	0.008 (0.004)**
$\sigma(Y)_{it}$		0.018 (0.008)**	0.014 (0.007)**		0.015 (0.007)**	0.011 (0.005)**
CPI_CORE_{it}			0.024 (0.008)***			0.025 (0.008)***
Country FE	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes
N. obs	830	830	830	796	796	796
N. of countries	34	34	34	34	34	34
R^2	0.287	0.298	0.393	0.327	0.335	0.415

*** is $p < 0.01$, ** is $p < 0.05$, * is $p < 0.1$

2.7 Welfare and Optimal Monetary Policy

Aging also influences the size of welfare losses. To demonstrate that, first we need to calculate the conditional and unconditional variances of the endogenous variables which are used as inputs in the welfare loss function. The optimal monetary policy rule is the rule which minimizes the loss in social welfare.[14]

2.7.1 The Model's State-Space Form

Based on the log-linear version of the model, one can express the following forward-looking system of equations which are conditional on the model's steady state, in particular on the

[13]Data on the output gap was downloaded from the OECD.

[14]Calculations are done by the Optimal Simple Rule toolbox of Dynare.

demographic structure:

$$A(\Theta)\xi_t = B(\Theta)\xi_{t-1} + C(\Theta)E_t\xi_{t+1} + D(\Theta)\varepsilon_t \tag{2.53}$$

where ξ is the vector of endogenous variables, ε is the vector of structural shocks with given variances, and Θ is the set of deep parameters including the steady-state levels of the endogenous variables. A, B, C, and D matrices consist of the linearized equations. Using the method of undetermined coefficients, one can express the state-space form of the forward-looking model as follows:

$$\xi_t = \Phi(\Theta)\xi_{t-1} + \Gamma(\Theta)\varepsilon_t \tag{2.54}$$

Here, Φ and Γ are matrices which are combinations of A, B, C, and D. Using the state-space form of the model, we can express the conditional covariances as follows:

$$\Xi_t = \Phi(\Theta)\Xi_{t-1}\Phi(\Theta)' + \Gamma(\Theta)\Omega\Gamma(\Theta)' \tag{2.55}$$

Here, $\xi_t\xi_t' = \Xi_t$ are conditional covariances of the endogenous variables, while $\varepsilon_t\varepsilon_t' = \Omega$ is a diagonal covariance matrix of the structural shocks.mIterating this equation forward ($t \to \infty$), we can express the unconditional covariances as follows:

$$vec(\Xi) = (I - \Phi(\Theta) \otimes \Phi(\Theta))^{-1} vec(\Gamma(\Theta)\Omega\Gamma(\Theta)') \tag{2.56}$$

where I is an identity matrix which has the same size as $\Phi(\Theta) \otimes \Phi(\Theta)$. For the optimal policy exercise with the simulated method of moments, we calibrated the structural shocks in such a way that the model's structural shocks capture the standard deviation of relevant variables of the New Area-Wide model (Christoffel, Coenen, and Warne (2008)); the results are reported in Table 2.5[15].

[15]We calibrated the initial young society version of the model. Each variable is expressed as the percent or percentage point deviation from its steady-state level (the deviation of endogenous variable x is noted by $\hat{x}$), the interest rate and inflations are annualized, and the consumption is the sum of the old and young cohort consumption.

Table 2.5: Calibrated standard deviation of variables and structural shocks

Variable	Historical value	Estimated (Christoffel, Coenen, and Warne (2008))	Calibrated value (this paper)
std($\hat{Y}$)	0.48	0.84	0.544
std($\hat{C}$)	0.48	0.74	0.409
std($\hat{Inv}$)	1.35	2.76	1.350
std($\hat{\pi}^4$)	0.36	0.49	0.721
std($\hat{i}^4$)	0.038	0.02	0.325

Structural shock	Shock value
std(ε_t^{Gov})	0.119
std(ε_t^{P})	0.503
std(ε_t^{Inv})	65.367
std(ε_t^{i})	0.019

2.7.2 Welfare Loss Functions and Monetary Policy Rules

In order to determine the optimal monetary policy rule, that is, the rule which minimizes the weighted sum of unconditional variances, we first need to specify the social welfare loss function. We have examined two different welfare loss functions.

The presence of overlapping generations and that of the non-representative features of the model make the model very complicated and hard to interpret welfare based on the agents' utility function. Therefore, in the first approach, we assume a simple welfare loss function which assigns a particular weight to the unconditional variances of inflation, output gap, and the difference between today's and yesterday's nominal interest rates. The latter is particularly relevant as it reflects the central bank's concern about financial stability: the central bank does not want to choose a policy which increases the volatility in the financial markets or discourages savings/credits. The weights are taken from Adolfson, Laseen, Linde, and Svensson (2011) and represent the importance of each objective in the central bank's decision-making:

$$\mathbf{L} = var(\pi_t) + 0.5 \cdot var(\hat{Y}_t) + 0.2 \cdot var(i_t - i_{t-1}) \tag{2.57}$$

Here, $\hat{Y}$ is the output gap defined as the deviation of GDP from its flexible price level.

In the second experiment, we derive the second-order approximation of the utility function and assume that the central bank in different stages of the population aging sets a policy rule which minimizes the weighted average of the two utilities[16]:

$$\mathcal{W}_t = s_t^Y \tilde{U}_t^Y(i) + (1 - s_t^Y)\tilde{U}_t^O(i) \tag{2.58}$$

where the utilities are expressed as follows:

$$\begin{aligned}
\tilde{U}_t^O(i) &= -(1-\gamma)U^O(i)\frac{\gamma-1}{2}\hat{C}_t^O(i)^2 &(2.59)\\
\tilde{U}_t^Y(i) &= -\sigma(1-\gamma)U^Y(i)\frac{\sigma(\gamma-1)}{2}\hat{C}_t^Y(i)^2 + \\
&- (1-\sigma)(1-\gamma)U^Y(i)\frac{L(i)}{1-L(i)}\left(\frac{\varphi_P}{2\lambda_P}\left(\pi_t - \gamma_P\pi_{t-1}\right)^2 + \frac{\varphi_W}{2\lambda_V}\left(\pi_t^V - \gamma_P\pi_{t-1}^V\right)^2\right) + \\
&- (1-\sigma)(1-\gamma)U^Y(i)\frac{L(i)}{1-L(i)}\frac{1}{2}\left(1 + (\sigma + \gamma(1-\sigma))\frac{L(i)}{1-L(i)}\right)\hat{L}_t(i)^2 \\
&- \sigma(1-\sigma)(1-\gamma)^2U^Y(i)\frac{L(i)}{1-L(i)}\hat{C}_t^Y(i)\hat{L}_t(i) &(2.60)
\end{aligned}$$

Here, the approximated levels of individual consumption and individual labor are given by:

$$\begin{aligned}
C_t^Y(i) &= \frac{C_t^Y}{N_t^Y} = \frac{\tilde{C}_t^Y}{s_t^Y} &(2.61)\\
L_t(i) &= \frac{L_t}{N_t^Y} = \frac{\tilde{L}_t}{s_t^Y} &(2.62)\\
C_t^O(i) &= \frac{C_t^O}{N_t^O} = \frac{\tilde{C}_t^O}{1-s_t^Y} &(2.63)
\end{aligned}$$

Several monetary policy rules are examined: (i) the pure inflation targeting rule (including a forward-looking IT), (ii) flexible inflation targeting with output or employment reaction (including a forward-looking version as well), (iii) price level targeting, and (iv) nominal GDP or nominal wage bill targeting. There are significant differences between the two welfare loss functions, namely, the utility-based function does not contain output gap. However, the variation in GDP mostly depends on the level of employment, so the optimized rules should be different in order to have reasonable reactions and the procedure can find the optimum of the likelihood functions. Therefore, in the flexible inflation targeting rule using the utility-based welfare loss function, we assume the employment gap instead of output gap, and, following the same idea,

[16] The detailed derivation of the utility-based welfare loss function can be found in the Appendix.

the nominal wage bill instead of the nominal GDP. Formally, the rules are defined as follows:

$$i_t \quad = \quad \rho i_{t-1} + (1-\rho)\tilde{r}_t^n + (1-\rho)\begin{cases} \phi_{\pi_t}\pi_t & \text{Pure IT} \\ \phi_{E_t\pi_{t+1}}E_t\pi_{t+1} & \text{Pure IT \& Fwd} \\ \phi_{\pi_t}\pi_t + \phi_{\hat{Y}_t}\hat{Y}_t \text{ or } \phi_{\hat{L}_t}\hat{L}_t & \text{Flex. IT} \\ \phi_{E_t\pi_{t+1}}\pi_{t+1} + \phi_{\hat{Y}_t}\hat{Y}_t \text{ or } \phi_{\hat{L}_t}\hat{L}_t & \text{Flex. IT \& Fwd} \\ \phi_{\hat{P}_t}\hat{P}_t & \text{Price level targeting} \\ \phi_{\widehat{P_tY_t}}\widehat{P_tY_t} \text{ or } \phi_{\widehat{V_tL_t}}\widehat{V_tL_t} & \text{Nom. GDP or wage bill targ.} \end{cases} \tag{2.64}$$

Conditional on the selected rules and the demographic structure, we look for the reaction parameters which minimize the total welfare losses described above.

2.7.3 Optimal Monetary Policy

First of all, we compare welfare losses with different demographic structures but with the same non-optimal monetary policy rule (same non-optimal parameters) and show how the increase in the old-age dependency ratio influences them (Table 2.6 - 2.7). Table 2.8 shows the changes of the weights in the utility-based welfare loss function. The old-age dependency ratio is a function of the retirement probability, the survival probability, and the net fertility rate. During the aging process, the latter two are assumed to change; in the table below, we present the cases where the two probabilities change at the same time (alternative combinations of the two rates and their welfare values can be found in the Appendix). As aging becomes more prevalent, that is, the old-age dependency ratio increases, the volatility of the inflation rate and that of the interest rate increase. Despite the increasing volatility of young consumption and employment, the decreasing weight of young households generates lower output gap volatility. Initial losses are normalized to $s = 0.2$, where s denotes the old-age dependency ratio and the optimal values are always normalized by their own initial loss values. Regardless of the population aging, the nominal variables have significantly larger weights in the utility-based loss function. This implies that the central bank should react strongly to nominal variables if it wishes to maximize utility-based welfare losses.

Table 2.6: Old-age dependency ratios and welfare levels with the baseline monetary policy rules when using the ad hoc welfare loss function

	Old-age dependency ratios			
	$s = 0.20$	$s = 0.36$	$s = 0.49$	$s = 0.60$
Initial losses	1.00	0.98	0.98	0.99
Inflation:	0.18	0.20	0.22	0.24
Output gap:	0.82	0.78	0.76	0.74
Interest rate:	0.00	0.00	0.00	0.00

Table 2.7: Old-age dependency ratios and welfare levels with the baseline monetary policy rules when using the utility-based welfare loss function

	Old-age dependency ratios			
	$s = 0.20$	$s = 0.36$	$s = 0.49$	$s = 0.60$
Initial losses	1.00	1.36	2.02	3.20
Inflation:	0.14	0.18	0.24	0.34
Wage Inflation:	0.25	0.40	0.68	1.22
Young Consumption:	0.06	0.08	0.11	0.18
Old Consumption:	0.00	0.00	0.01	0.02
Labor:	0.58	0.75	1.05	1.55
Young Cons. Labor cov:	−0.04	−0.05	−0.07	−0.09

Table 2.8: Old-age dependency ratios and age-dependent weights of the utility-based welfare loss function

	Old-age dependency ratios			
	$s = 0.20$	$s = 0.36$	$s = 0.49$	$s = 0.60$
Inflation:	11.00	12.68	15.54	19.87
Wage Inflation:	12.36	14.24	17.46	22.32
Young Consumption:	0.27	0.26	0.27	0.29
Old Consumption:	0.06	0.10	0.12	0.13
Labor:	0.77	1.01	1.41	2.08
Young Cons. Labor cov:	−0.35	−0.41	−0.50	−0.64

Next, optimal monetary policy reaction functions (Table 2.9 - 2.10) are explored. Thus, for any given OADR, we calculate the minimum social welfare loss with the different monetary-policy regimes and the related monetary-policy rule parameters. Our main finding is that in a younger society, the flexible inflation targeting regime, with a strong reaction to expected inflation and a somewhat weaker reaction to the output gap, is the most effective rule, that is, the loss-minimizing option, if the monetary policy minimizes the ad hoc welfare loss function. Under the utility-based welfare loss function, the flexible inflation targeting performs well. Nonetheless, nominal wage bill targeting is the best option, although the improvement in welfare compared to flexible inflation targeting is not significant. In grayer societies, however, although the flexible inflation targeting regime could be one of the favorable options, the central bank should react mildly to the output gap or employment gap and more strongly to the level of inflation.

Table 2.9: Old-age dependency ratios and optimal reaction parameters in six monetary policy regimes when using the ad hoc welfare loss function

		Old-age dependency ratios			
		$s = 0.20$	$s = 0.36$	$s = 0.49$	$s = 0.60$
Initial rule	Initial loss	1.00	0.98	0.98	0.99
	ϕ_{π_t}	1.70	1.70	1.70	1.70
Pure IT	Optimized loss	0.99	0.98	0.97	0.95
	ϕ_{π_t}	2.34	2.59	2.96	3.50
Pure IT & Fwd	Optimized loss	0.92	0.93	0.92	0.92
	$\phi_{E_t \pi_{t+1}}$	3.86	4.26	4.83	5.64
Flex. IT	Optimized loss	0.91	0.91	0.91	0.91
	ϕ_{π_t}	4.40	4.67	5.12	5.81
	$\phi_{\hat{Y}_t}$	0.86	0.86	0.88	0.93
Flex. IT & Fwd	Optimized loss	0.89	0.90	0.90	0.89
	$\phi_{E_t \pi_{t+1}}$	5.19	5.53	6.05	6.83
	$\phi_{\hat{Y}_t}$	0.65	0.62	0.59	0.56
Price level targ.	Optimized loss	1.10	1.10	1.09	1.07
	$\phi_{\hat{P}_t}$	0.08	0.14	0.23	0.39
Nominal GDP targ.	Optimized loss	0.94	0.94	0.93	0.92
	$\phi_{\widehat{P_t Y_t}}$	32.46	36.89	47.49	62.52

The nominal wage bill targeting rule is the most effective rule under utility maximization. To understand the reasons for this, we should check the components of nominal GDP and nominal wage bill targeting separately. Nominal GDP targeting is basically a combination of price and nominal wage level targeting and output gap or employment gap stabilization, meaning that second-round effects on aggregate demand are also taken into account. Price level targeting generates higher welfare losses than nominal GDP targeting does. While price level targeting is quite hawkish, the central bank tolerates a larger economic sacrifice to achieve price-level stability. This kind of dual-mandate, like nominal GDP targeting or wage bill targeting, guarantees that the welfare loss is minimized when both the price level and the output gap or employment gap vary as little as possible. However, the reaction to the nominal wage bill decreases in aging societies; this decline is consistent with the milder reaction to the employment gap in other rules under utility-based welfare maximization. Due to the decreasing Frisch elasticity in aging societies, for the same amount of labor increase, wages and inflation increase more. Thus, the monetary policy reacts less to the employment gap so as not to contribute to the higher volatility of the nominal variables.

Table 2.10: Old-age dependency ratios and optimal reaction parameters in six monetary policy regimes by the utility-based welfare loss function

		Old-age dependency ratios			
		$s = 0.20$	$s = 0.36$	$s = 0.49$	$s = 0.60$
Initial rule	Initial loss	1.00	1.36	2.02	3.20
	ϕ_{π_t}	1.70	1.70	1.70	1.70
Pure IT	Optimized loss	0.32	0.29	0.28	0.31
	ϕ_{π_t}	30.57	32.74	34.08	31.93
Pure IT & Fwd	Optimized loss	0.28	0.26	0.16	0.16
	$\phi_{E_t\pi_{t+1}}$	51.26	54.19	55.31	50.12
Flex. IT	Optimized loss	0.25	0.24	0.24	0.28
	ϕ_{π_t}	11.13	15.02	17.43	17.10
	$\phi_{\hat{L}_t}$	17.09	15.25	11.09	6.52
Flex. IT & Fwd	Optimized loss	0.25	0.23	0.24	0.27
	$\phi_{E_t\pi_{t+1}}$	14.71	19.66	24.16	24.84
	$\phi_{\hat{L}_t}$	15.92	14.38	11.18	6.90
Price level targ.	Optimized loss	0.45	0.42	0.40	0.41
	$\phi_{\hat{P}_t}$	13.83	15.89	17.73	17.47
Nominal Wage Bill targ.	Optimized loss	0.24	0.22	0.22	0.27
	$\phi_{\widehat{V_tL_t}}$	31.73	31.70	25.48	15.29

2.8 Conclusion

In this paper, we explored the monetary consequences of aging in a multi-period DGE model with OLG agents. Specifically, we examined how aging affects (i) inflation in the longer term, (ii) the short-run cyclical behavior of the economy, including monetary policy transmission, and (iii) optimal monetary policy rules. The stylized facts and empirical estimations were consistent with the message of the model, and we found that the rate of inflation decreases, while its volatility increases with aging, aging makes monetary policy less effective, and an IT regime with a high inflation reaction is optimal in the case of high, but still reasonable, OADRs.

In Section 2.6, we reported that the same size shock causes higher inflation volatility in an older society. Not only inflation, but the macroeconomy in general, reacts differently to shocks - including monetary policy shocks. The transmission of monetary policy changes, and monetary policy becomes less effective. The most important channel is that as agents live longer and their planning horizon becomes longer, their savings position changes: the young are willing to borrow more, while the retired accumulate more savings to guarantee their consumption over a longer time horizon. Hence, when the interest rate changes, it has different implications for the young and the old: higher interest rates imply an extra cost for the young, who are indebted, while the old generate more income. Additionally, the young and the old also make different consumption-savings decisions (as the old are more patient than the young), and there are labor market implications as well (the labor market becomes tighter and real wages react more as the labor force shrinks). In Section 2.7, we discussed that aging, via higher inflation, also increases social welfare loss. To avoid that, central banks with inflation-targeting regimes should react more strongly to nominal variables: they should increase the nominal interest rate by a larger amount, given the same size shock. Under some circumstances, nominal GDP targeting or nominal wage bill targeting is a more effective rule than the common practice of inflation targeting. Aging, a clear concern for fiscal economists, ought to be a concern for central bankers as well.

To the best of our knowledge, our paper is the first which (i) estimates the impact of aging on the volatility of inflation, (ii) explores the impact of aging on the short-run cyclical behavior of the macroeconomy, including monetary policy transmission, in an aged society, using a multi-period dynamic general equilibrium model with overlapping generations, and (iii) examines optimal monetary policy strategies in the presence of aging. Given the scarce literature in this field, we call for more research to better understand the implications of aging for central banks, central bankers, and the elderly.

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Chapter 3

Convergence Stories of Post-Socialist Central-Eastern European Countries

3.1 Introduction

In this paper, we aim to examine the growth and convergence process of five Central and Eastern European member states of the European Union(the Czech Republic, Hungary, Poland, Slovenia and Slovakia - henceforth, CEE) through the lens of the stochastic neoclassical growth model. We follow Aguiar and Gopinath (2007) and García-Cicco, Pancrazi, and Uribe (2010), who estimate similar models for Latin-American countries (Mexico and Argentina). We believe that the five CEE countries are a good laboratory for the neoclassical model. They are emerging economies that are highly open both to international trade and external finance. Their performance is broadly in line with the predictions of the neoclassical model, where convergence is driven by improvements in total factor productivity (TFP) and capital accumulation. Openness allows countries to finance some of their additional investment and consumption from abroad, which is exactly what happened in the CEE countries after transition in the 1990s. Also, after the introduction of market reforms in the early 1990s, the CEE economies have reasonably similar institutions to the advanced market economies of Western Europe, the natural reference group.

The literature has identified two main shocks that drive stochastic growth in small, open economies like the CEE countries. Aguiar and Gopinath (2007) compare Mexico and Canada, and conclude that in the former shocks to trend productivity growth are more important than in the latter. The main reason is that in emerging economies, such as Mexico, the trade balance is counter-cyclical. Transitory TFP shocks imply a pro-cyclical trade balance, since households want to save part of the temporary windfall gains. Permanent and lasting trend shocks, by contrast, imply improving growth performance for a while, leading to increases in current and future permanent income. In that case, households want to consume some of the future gains now, which implies a trade deficit.

García-Cicco, Pancrazi, and Uribe (2010) note that Aguiar and Gopinath (2007) ignore the role of financial frictions and shocks. In particular, they argue that external financing conditions - which can be taken as exogenous for small, open emerging countries - are important growth determinants. They estimate a financial frictions augmented RBC model on a century of Argentine data, and conclude that including interest-premium shocks in the estimation greatly reduces the importance of trend-productivity shocks. Increases in interest premia induce recessions and improve the trade balance at the same time; thus, they can also explain the counter-cyclicality of the latter. Moreover, in the absence of financial frictions the trade balance is a random walk, which is at odds with the data in emerging economies. That said, García-Cicco, Pancrazi, and Uribe (2010) find that growth volatility is mainly due to transitory technology shocks, at least in Argentina and Mexico.

Other papers have also followed up on the technology vs. interest premium debate. Naoussi and Tripier (2010) and Guerron-Quintana (2013) show that a common trend-productivity component better explains medium-term GDP growth volatility in African countries than financial shocks. In contrast, Taştan (2013) finds that, in Turkey, financial shocks are more important. Many papers investigate the role of financial intermediation. Zhao (2013) builds a model where agents face liquidity constraints, and it is changes in liquidity that lead to fluctuations in the risk premium. Minetti and Peng (2013) assumes asymmetric information between domestic and foreign creditors, which becomes effective when income prospects worsen. This leads to a large response in external financing, which increases country risk and the effective foreign interest rate.

We contribute to this literature in a number of ways. First, we reevaluate the findings of Aguiar and Gopinath (2007) and García-Cicco, Pancrazi, and Uribe (2010) in the context of the CEE countries. We find that while interest premium shocks are important for understanding GDP components, persistent shocks to productivity are the most critical contributors to the volatility of GDP growth. In other words, productivity has a strong random walk component, and even transitory technology shocks are estimated to be very persistent. The latter result also casts doubt on whether transitory technology shocks can be separately identified, especially once we include hours in the estimation and allow for labor market disturbances. To paraphrase Aguiar and Gopinath (2007), in our countries *the trend is the financial frictions augmented cycle.*

Second, and perhaps most interestingly, we estimate the exogenous driving forces of economic growth in a panel. While the time series are short, using a panel of five countries gives us degrees of freedom to identify the underlying shock processes. The panel allows us to separate "global" shocks that affect all countries from "local" shocks that are specific to a country. We show that the global components of both the trend-productivity shock and the interest-premium shock have a useful economic interpretation. In particular, the global trend component co-moves very strongly with the growth rate of the "old" European Union countries (EU 15). The implicit common interest rate component also tracks the EU 15 average real interest rate until 2008, but diverges from it sharply afterwards. This finding is consistent with a narrative of the financial crisis in which wedges opened up both between the financial markets of advanced and emerging countries, and between benchmark interest rates and corporate/household lending rates. Our results also relate to Aizenman, Jinjarak, Estrada, and Tian (2018), who find that the role of global factors in the level and volatility of economic growth increases after the financial crisis in emerging economies.

We make a number of additional methodological contributions, mostly related to the model setup. We include external consumption habits, which is an alternative to the estimated very persistent preference shock used by García-Cicco, Pancrazi, and Uribe (2010). We use adjustment costs to investment instead of capital, and we add an investment-specific shock. This specification was shown to capture investment dynamics better in a business cycle setting (Christiano, Eichenbaum, and Evans (2005)). When we plot our estimated investment-specific shock, we find an interesting co-movement between the shock and the magnitude of European Union funds flowing into the CEE countries. This suggests that in addition to the growth and financial environments, external funds were a major determinant of investment dynamics.

We use a labor market specification that is growth-consistent and does not require the inclusion of an ad-hoc trend in the value of leisure. As we later explain, this necessitates adding a working capital channel and using a gross output production function to get reasonable predictions for interest-premium shocks. In addition to the technical reasons, we also think that the working capital channel is an important propagation mechanism of changes in financial conditions. Finally, we use total labor hours as an observable, and we add a labor supply shock to the estimation. On the one hand, observing hours should make the identification of technology shocks more precise. On the other hand, changes in labor market regulation and taxes were important in the CEE countries.

Our model is deliberately simple. We want to focus on a few key mechanisms that influence medium term growth, so we omit other channels such as sticky prices and explicit monetary policy. We do, however, estimate a simple form of wage rigidity, which the data strongly reject.

The paper proceeds as follows. In Section 3.2, we present the basic stylized facts of growth in the CEE countries in the 1996-2017 period. In Section 3.3, we describe the stochastic growth model. In Section 3.4, we estimate the stochastic version of the model, present results from a variance decomposition exercise, and evaluate the potential role of wage rigidity. Using the estimation results, Section 3.5 presents interesting findings that we believe strongly validate our estimation results. Finally, Section 3.6 concludes and discusses future avenues for research.

3.2 Growth in the post-socialist Central-European countries

Before presenting the model, we describe the main tendencies of the macroeconomic data. Our narrative is based on IMF Article IV reports, which are useful for the interpretation of the estimation results.[1] Figure 3.1 plots the evolution of real GDP for the five countries. Lightly shaded periods are country-specific recessions or growth slowdowns, while the two darkly shaded periods are the global financial crisis and the subsequent European crisis, which were common to all countries.

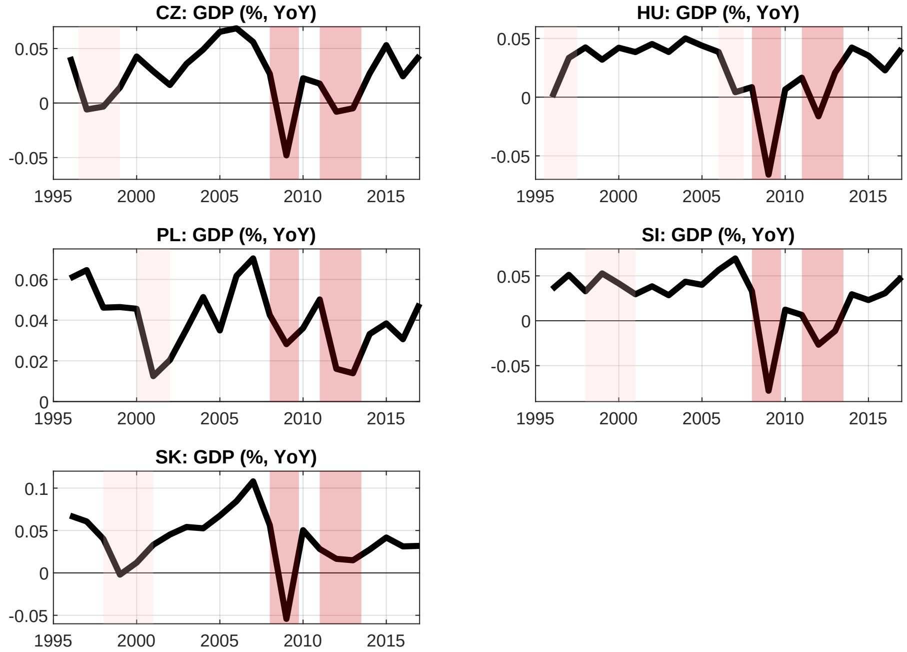
The collapse of socialist regimes fundamentally changed the economic performance of the CEE countries. From the middle of the 1990s, economic growth was boosted by structural reforms, a positive future economic outlook and supportive external economic developments. Massive inflows of foreign direct investment generated current account deficits, but the favorable investment climate and the good timing of the fiscal consolidation eliminated pressures on the country risk.

The convergence period started from second half of the 1990s in the CEE countries. Albeit with different timing, all the countries implemented fiscal consolidation, economic liberalization, labor market reform and a privatization plan i.e., the Polish Balcerowicz plan in 1990, Bokros package in Hungary from 1995 [2], Czech structural reforms after the currency crisis in 1997, and in

[1] Czech Republic: http://www.imf.org/en/Countries/CZE, Hungary: http://www.imf.org/en/Countries/HUN, Poland: http://www.imf.org/en/Countries/POL, Slovenia: https://www.imf.org/en/Countries/SVN, Slovakia: http://www.imf.org/en/Countries/SVK.

[2] https://www.mnb.hu/letoltes/wp1998-5.pdf

Figure 3.1: Growth in the CEE countries



The figure shows real GDP growth in the five CEE countries. Shaded regions indicate common recessions or slowdowns. Source: Eurostat.

Slovakia after the end of the Meciar era [3]. All consolidation and reform plans followed the same pattern, which led to a temporary decline in GDP growth, but later until the financial crisis all countries experienced robust and quick growth. Due to the successful structural reforms, compared to the other post-socialist countries, the CEE countries were more resilient to the Russian crisis that occurred at the end of 1990s: this is because the CEE countries were able to reorient their foreign trade to the Western European markets.

The 'Great Moderation' was a period before the financial crisis with low inflation and low interest rates in developed economies that also supported the growth of the CEE region until the financial crisis. Due to the ex-ante positive growth outlook and favorable financial conditions, the growth of domestic demand was supported by cheap credit expansion, and all the CEE countries experienced significant current account deficits. Elekdag and Wu (2013) document that both global and domestic factors play a role in emerging market credit booms. Accordingly, we further aim to separate local and global factors in explaining macroeconomic developments before and after the crisis in the CEE economies.

[3] http://www.finance.gov.sk/en/Components/CategoryDocuments/s_LoadDocument.aspx?categoryId=347&documentId=433

In 2009, as a consequence of the financial crisis, the CEE countries - with the exception of Poland - experienced a significant drop in economic growth and were forced to start a strong deleveraging process. After the crisis, all the countries recorded historically unprecedented trade surpluses. Deleveraging came with significant economic sacrifice and slowed down the growth of domestic demand components. Additionally, the Eurozone debt crisis in 2012, that is, the second wave of the financial crisis, worsened the outlook further. As Hungary was the most indebted country, Hungarian deleveraging was the most dramatic and long-lasting. In the past decade, the growth of private investment has remained subdued. Large inflows of EU funds, especially from 2010, compensated somewhat for external adjustment costs. The precautionary motive still dominates households' consumption expenditure. In Hungary, for example, in 2016, real private consumption was still below its pre-crisis level. Other CEE countries were less heavily indebted, so they were more resilient and recovered more quickly after the economic slack. The other partial exception is Slovenia, which experienced a severe banking crisis in 2013-2014.

The convergence stories of the past two decades are a mix of global (common or region-specific) and local (country-specific) events. To explore the stories and key shocks behind the data, we need to apply a structural model to decompose the data into different innovations. During the pre-crisis period, productivity growth and favorable financial conditions led to strong economic growth, but the crisis and the post-crisis deleveraging transformed the previous patterns. Without structural models, however, it is hard to say anything about the core mechanism and compare the different growth stories.

3.3 The model

We use a modified version of the stochastic, neoclassical growth model described in García-Cicco, Pancrazi, and Uribe (2010), or GPU henceforth. Ours is a one-sector, small open economy, where output is used for household consumption, capital investment, net exports and government consumption. Production requires labor and capital. Final good and factor markets are competitive, with flexible prices. The engine of growth is exogenous improvements in productivity; we specify the productivity process later. For simplicity, and given the demographics of the Visegrad countries, we assume that there is no population growth.

It is well known that aggregate variables are more persistent than the basic neoclassical model predicts, even at the annual frequency (Christiano, Eichenbaum, and Evans (2005)). In our case, this is an important issue, since the estimation starts at an arbitrary initial condition, determined by data availability (typically 1995). As we discussed in the previous section, the behavior of consumption and investment are heavily influenced by the exact timing of economic transition in each country. For this reason, we add a few real rigidities to the basic model which capture the slow adjustment of the main macro variables. We therefore assume external habits in consumption and adjustment costs to investment.

An important deviation from GPU is that while they assume GHH preferences (Greenwood, Hercowitz, and Huffman (1988)), we opt for a more standard specification (King, Plosser, and Rebelo (1988); henceforth KPR). The reason for this is that GHH preferences have a counter-factual prediction for labor hours in catching-up economies. As we show below, our preference specification does not suffer from this issue, but a drawback is that interest-premium shocks become expansionary (they are contractionary under GHH preferences). Therefore, we assume a working capital channel, which was shown to provide useful amplification for financial shocks (Mendoza (2010)). We work with a gross output production function, and impose financing requirements on intermediate inputs as well as the wage bill. Overall, we are able to construct a production structure that leads to plausible predictions both along the medium-run transition path and along the short-run business cycle.[4]

3.3.1 Households

The representative household solves the following problem:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log\left(C_t - \chi \bar{C}_{t-1}\right) - \frac{\theta_t h_t^{\omega}}{\omega} \right]$$

$$\text{s.t.} \quad C_t + D_t = W_t h_t + \frac{D_{t+1}}{R_t} + \Pi_t - \Xi_t,$$

where C_t is consumption, h_t is hours worked, D_{t+1} is foreign debt carried into the next period, R_t is the gross interest rate on debt, and Ξ_t is lump-sum taxes that finance government spending.[5] Households earn wages (W), and profits (Π) from the representative firm that they own. Note that consumption is subject to external habit formation ($\bar{C}_{t-1}$).

There are three structural shocks that affect household decisions. First, we take taxes (govern-

[4]Another paper that employs the working capital channel to explain stylized labor market facts in emerging economies is Altug and Kabaca (2017).

[5]We assume that government consumption is purely wasteful. Equivalently, we could include it in the utility function in an additively separable form.

ment spending) to be exogenous and random:

$$\log \Xi_t = (1 - \rho_\xi) \log \bar{\Xi} + \rho_\xi \log \Xi_{t-1} + \nu_t^\xi. \tag{3.1}$$

Second, the interest rate on foreign bonds is subject to exogenous disturbances. The interest rate also has an endogenous component, which depends on the external indebtedness of the economy (Schmitt-Grohé and Uribe (2003)):

$$R_t = \bar{R} + \psi \left(e^{D_t/Y_t - d_y} - 1 \right) + e^{\epsilon_{r,t}} - 1, \tag{3.2}$$

where

$$\epsilon_{r,t} = \rho_r \epsilon_{r,t-1} + \nu_{r,t}. \tag{3.3}$$

Finally, labor supply - or more broadly, the labor market - is influenced by an exogenous term θ_t, given as:

$$\log \theta_t = (1 - \rho_h) \log \bar{\theta} + \rho_h \log \theta_{t-1} + \nu_t^h \tag{3.4}$$

The first-order conditions of the problem are given as follows:

$$\frac{1}{C_t - \chi \bar{C}_{t-1}} = \Lambda_t \tag{3.5}$$

$$\theta_t h_t^\eta = \Lambda_t W_t \tag{3.6}$$

$$\Lambda_t = \beta R_t \mathbb{E}_t \Lambda_{t+1}, \tag{3.7}$$

where Λ_t is the Lagrange multiplier associated with the budget constraint. The final condition is the budget constraint, which was presented above.

3.3.2 Firms

Factor, intermediate and final good markets are perfectly competitive. We start with the specification of gross output for the representative firm:

$$Y_t^G = \left[\Upsilon e^{a_t} K_t^\alpha \left(X_t h_t \right)^{1-\alpha} \right]^{1-\mu} M_t^\mu,$$

where K_t is capital input, and M_t is the amount of intermediate inputs. The variable X_t represents the stochastic trend component of productivity, which evolves according to the following process:

$$\frac{X_t}{X_{t-1}} = g_t \tag{3.8}$$

$$\log g_t = (1-\rho_g)\log \bar{g} + \rho_g \log g_{t-1} + \nu_t^g. \tag{3.9}$$

In addition, we include a transitory productivity shock a_t, as in Aguiar and Gopinath (2007) and García-Cicco, Pancrazi, and Uribe (2010). Note that Υ is a constant which is included so that we can choose units conveniently (see below).

Firm profits are give as follows:

$$\Pi_t = \left[\Upsilon e^{a_t} K_t^{\alpha} \left(X_t h_t\right)^{1-\alpha}\right]^{1-\mu} M_t^{\mu} - R_t M_t - R_t W_t h_t - I_t,$$

where I_t stands for gross investment, and

$$K_{t+1} = (1-\delta) K_t + \left[1 - \frac{\phi}{2}\left(\frac{e^{\tilde{\epsilon}_{i,t}} I_t}{I_{t-1}} - \bar{g}\right)^2\right] I_t. \tag{3.10}$$

Note that we impose a working capital financing requirement on the wage bill and on intermediate inputs, so that the firm has to pre-finance these fully. We also add a shock to gross investment ($\tilde{\epsilon}_{i,t}$). The purpose of this shock is to capture the non-market driven changes of the private investment, i.e.: the government subsidy programs such as EU funding; and this shock also absorbs all other short run cyclical movements that can not be explained by the porftolio allocation mechanism.

We derive the first-order conditions in two steps. First, we optimize out the use of intermediate inputs, which leads to

$$R_t M_t = \mu Y_t^G \tag{3.11}$$

Plugging this back into the gross output production function, we can express total production in terms of value added:

$$Y_t^G = \left(\frac{\mu}{R_t}\right)^{\frac{\mu}{1-\mu}} \Upsilon Y_t, \tag{3.12}$$

where $Y_t = e^{a_t} K_t^{\alpha} \left(X_t h_t\right)^{1-\alpha}$. Combining this expression with (3.11) and substituting it into

(3.3.2), we can rewrite profits as

$$\begin{aligned}\Pi_t &= (1-\mu)\, Y_t^G - R_t W_t h_t - I_t \\ &= (1-\mu)\left(\frac{\mu}{R_t}\right)^{\frac{\mu}{1-\mu}} \Upsilon Y_t - R_t W_t h_t - I_t \\ &= R_t^{\frac{\mu}{\mu-1}} Y_t - R_t W_t h_t - I_t,\end{aligned}$$

where the second equality follows from a convenient normalization, $\Upsilon(1-\mu)\,\mu^{\frac{\mu}{1-\mu}} = 1$.

The second step in solving the representative firm's problem is to find the labor demand and capital investment. Using the derivation above, we can state the problem in value added form as follows:

$$\begin{aligned}\max \Pi_0 &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0}\left[R_t^{\frac{\mu}{\mu-1}} e^{a_t} K_t^{\alpha} (X_t h_t)^{1-\alpha} - R_t W_t h_t - I_t\right] \\ \text{s.t.} \quad K_{t+1} &= (1-\delta) K_t + \left[1 - \frac{\phi}{2}\left(\frac{I_t e^{\tilde{\epsilon}_{i,t}}}{I_{t-1}} - \bar{g}\right)^2\right] I_t,\end{aligned}$$

where the stochastic discount factor reflects that households are the ultimate owners of firms. Using q_t for the usual Tobin's q multiplier for the capital accumulation constraint, the first-order conditions are given by the following equations:

$$R_t^{\frac{1}{1-\mu}} W_t h_t = (1-\alpha)\, Y_t \tag{3.13}$$

$$q_t = \beta \mathbb{E}_t \left[R_{t+1}^{\frac{\mu}{\mu-1}} \frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta)\, q_{t+1}\right] \frac{\Lambda_{t+1}}{\Lambda_t} \tag{3.14}$$

$$\begin{aligned}1 &= q_t \left[1 - \frac{\phi}{2}\left(\frac{I_t e^{\tilde{\epsilon}_{i,t}}}{I_{t-1}} - \bar{g}\right)^2 e^{\tilde{\epsilon}_{i,t}} - \phi\left(\frac{I_t e^{\tilde{\epsilon}_{i,t}}}{I_{t-1}} - \bar{g}\right)\frac{I_t e^{\tilde{\epsilon}_{i,t}}}{I_{t-1}}\right] \\ &\quad + \beta \mathbb{E}_t q_{t+1} \phi \left(\frac{I_{t+1} e^{\tilde{\epsilon}_{i,t+1}}}{I_t} - \bar{g}\right)\left(\frac{I_{t+1} e^{\tilde{\epsilon}_{i,t+1}}}{I_t}\right)^2 \frac{\Lambda_{t+1}}{\Lambda_t}.\end{aligned} \tag{3.15}$$

3.3.3 Equilibrium

Combining the household and firm first-order conditions, along with the aggregate resource constraint, the evolution of the model economy is given by the following set of equations:

$$
\begin{aligned}
\theta_t h_t^{\omega} &= \frac{(1-\alpha)\, Y_t \Lambda_t}{R_t^{\frac{1}{1-\mu}}} \\
\Lambda_t &= \frac{1}{C_t - \chi \bar{C}_{t-1}} \\
1 &= \beta R_t \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \\
q_t &= \beta \mathbb{E}_t \left[R_{t+1}^{\frac{\mu}{\mu-1}} \frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta)\, q_{t+1} \right] \frac{\Lambda_{t+1}}{\Lambda_t} \\
1 &= q_t \left[1 - \frac{\phi}{2} \left(\frac{I_t e^{\tilde{\epsilon}_{i,t}}}{I_{t-1}} - \bar{g} \right)^2 e^{\tilde{\epsilon}_{i,t}} - \phi \left(\frac{I_t e^{\tilde{\epsilon}_{i,t}}}{I_{t-1}} - \bar{g} \right) \frac{I_t}{I_{t-1}} e^{\tilde{\epsilon}_{i,t}} \right] \\
&\quad + \beta \mathbb{E}_t Q_{t+1} \phi \left(\frac{I_{t+1} e^{\tilde{\epsilon}_{i,t+1}}}{I_t} - \bar{g} \right) \left(\frac{I_{t+1} e^{\tilde{\epsilon}_{i,t+1}}}{I_t} \right)^2 \frac{\Lambda_{t+1}}{\Lambda_t} \\
Y_t &= C_t + I_t + D_t - \frac{D_{t+1}}{R_t} + \Xi_t \\
K_{t+1} &= (1-\delta)\, K_t + \left[1 - \frac{\phi}{2} \left(\frac{I_t}{I_{t-1}} - \bar{g} \right)^2 \right] I_t \\
Y_t &= e^{a_t} K_t^{\alpha} \left(X_t h_t \right)^{1-\alpha} \\
R_t &= \bar{R} + \psi \left(e^{D_{t+1}/Y_t - d_y} - 1 \right) + e^{\epsilon_t^r} - 1
\end{aligned}
$$

The stochastic processes for the structural shocks were defined above.

The system is not stationary, since productivity has a stochastic trend. We introduce variables in effective form that are constant in the deterministic steady state: $c_t = C_t/X_t$, $i_t = I_t/X_t$, $y_t = Y_t/X_t$, $k_{t+1} = K_{t+1}/X_t$, $d_{t+1} = D_{t+1}/X_t$, $\xi_t = \Xi_t/X_t$ and $\lambda_t = X_t \Lambda_t$. Using these new variables, the equilibrium system is given as:

$$
\begin{aligned}
\theta_t h_t^{\omega} &= \frac{(1-\alpha)\, y_t \lambda_t}{R_t^{\frac{1}{1-\mu}}} \\
\lambda_t &= \frac{1}{c_t - (\chi/g_t)\,\bar{c}_{t-1}} \\
1 &= \beta R_t \mathbb{E}_t \frac{1}{g_{t+1}} \frac{\lambda_{t+1}}{\lambda_t} \\
q_t &= \beta \mathbb{E}_t \frac{1}{g_{t+1}} \left[R_{t+1}^{\frac{\mu}{\mu-1}} \frac{\alpha g_{t+1} y_{t+1}}{k_{t+1}} + (1-\delta)\, q_{t+1} \right] \frac{\lambda_{t+1}}{\lambda_t} \\
1 &= q_t \left[1 - \frac{\phi}{2} \left(g_t \frac{i_t e^{\tilde{\epsilon}_{i,t}}}{i_{t-1}} - \bar{g} \right)^2 - \phi \left(g_t \frac{i_t e^{\tilde{\epsilon}_{i,t}}}{i_{t-1}} - \bar{g} \right) \frac{g_t i_t e^{\tilde{\epsilon}_{i,t}}}{i_{t-1}} \right] \\
&\quad + \beta \mathbb{E}_t q_{t+1} \phi \left(g_{t+1} \frac{i_{t+1} e^{\tilde{\epsilon}_{i,t+1}}}{i_t} - \bar{g} \right) \left(\frac{g_{t+1} i_{t+1} e^{\tilde{\epsilon}_{i,t+1}}}{i_t} \right)^2 \frac{\lambda_{t+1}}{\lambda_t} \\
y_t &= c_t + i_t + \xi_t + tb_t \\
tb_t &= \frac{d_t}{g_t} - \frac{d_{t+1}}{R_t} \\
k_{t+1} &= (1-\delta)\, \frac{k_t}{g_t} + \left[1 - \frac{\phi}{2} \left(g_t \frac{i_t e^{\tilde{\epsilon}_{i,t}}}{i_{t-1}} - \bar{g} \right)^2 \right] i_t \\
y_t &= e^{a_t} \left(\frac{k_t}{g_t} \right)^{\alpha} h_t^{1-\alpha} \\
R_t &= \bar{R} + \psi \left(e^{d_{t+1}/y_t - d_y} - 1 \right) + e^{\epsilon_t^r} - 1,
\end{aligned}
$$

where tb_t is the normalized trade balance.

3.3.4 Interest-premium shocks and the labor market

Before we move on to the estimation, we present some basic results that motivated our modeling choices. It is well known that the labor market is central to the behavior of the RBC model. In our model, the labor market equilibrium condition is written as follows:

$$\theta_t h_t^{\omega} = \frac{(1-\alpha)\, Y_t \Lambda_t}{R_t^{\frac{1}{1-\mu}}}.$$

Assume for simplicity that there are no habits in consumption, in which case $\Lambda_t = 1/C_t$, and the condition can be written as:

$$\theta_t h_t^{\omega} = \frac{1-\alpha}{R_t^{\frac{1}{1-\mu}} \cdot (C_t/Y_t)}. \tag{3.16}$$

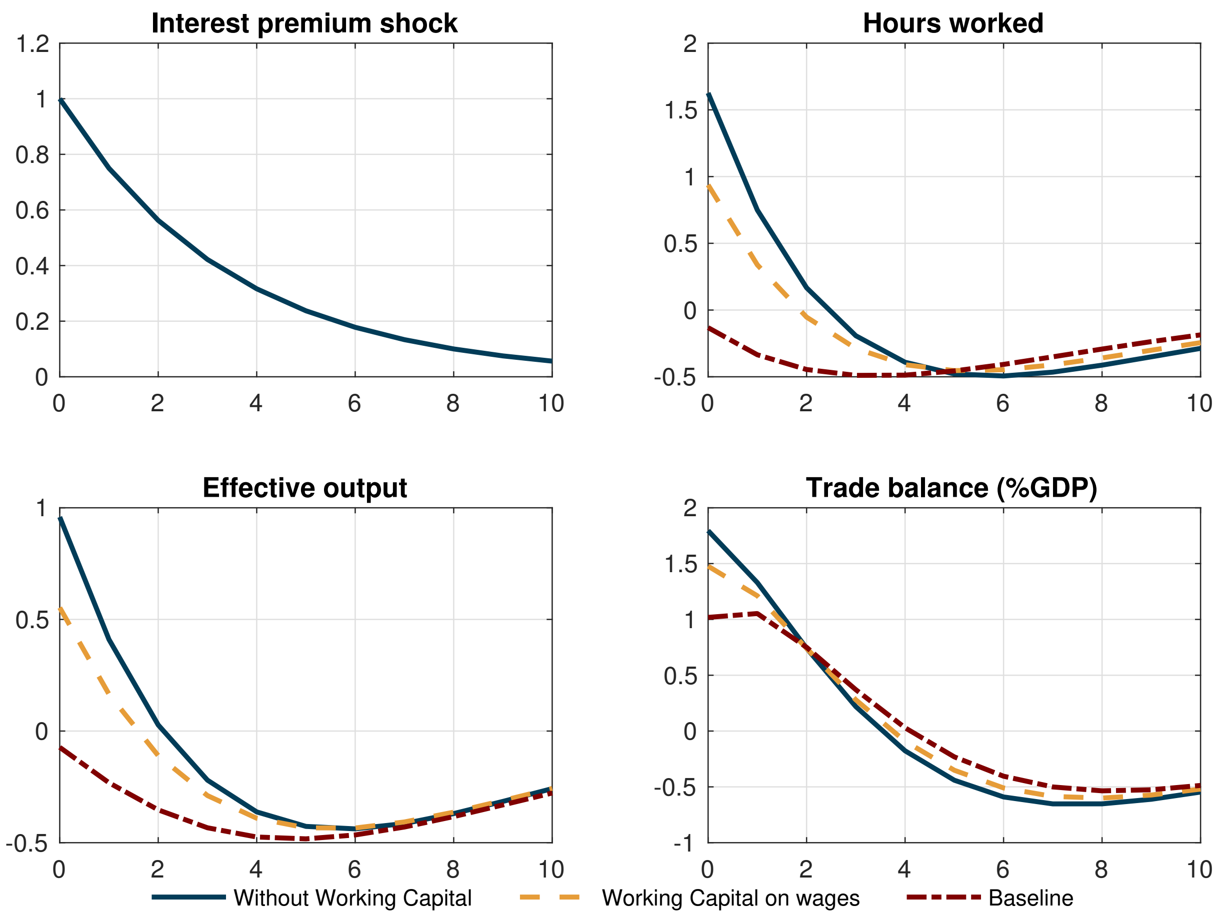
The advantage of this specification is that it is consistent with the existence of a balanced growth path (BGP), since the consumption-output ratio C_t/Y_t is constant along the BGP. In our context, however, there is a disadvantage as well. In the absence of a working capital channel, hours are negatively correlated with the consumption-output ratio. This means that a positive interest-premium shock, which leads to a decline in C_t/Y_t and in I_t/Y_t, results in an increase in hours worked and output. The fact that labor supply increases to a negative income shock is not necessarily unreasonable. Nonetheless, the general equilibrium outcome that a tightening of financing conditions is expansionary in an open economy is implausible. To counter the labor supply effect, we introduce a working capital channel to create a negative labor demand effect. The net effect of a risk premium shock on hours worked depends on the relative strength of the labor demand and labor supply channels. In our specification, the first one dominates, and an increase in the risk premium is contractionary. It is important to emphasize that having a working capital channel is not enough for this result. The cost increase has to be significant, which cannot be achieved with a standard value added production function alone. Using a gross output concept and imposing a pre-financing condition on intermediate inputs is necessary to create a strong enough cost channel. This result was also found in Mendoza (2010), who constructs a more elaborate model of financial frictions where default is possible.

Figure 3.2 illustrates these results. We plot the effect of a 1 percentage point temporary increase in the risk premium shock. The three lines correspond to (i) the baseline with gross output and working capital requirement for intermediate inputs and wages, (ii) a scenario where only wages need to be pre-financed, and (iii) a scenario without a working capital channel. Both hours worked and output rise on impact, unless intermediate inputs are included in the working capital requirement. With a value added production function, the working capital channel is simply not strong enough to counter the increase in labor supply. Notice, however, that the trade balance is not particularly sensitive to the working capital specification.

An alternative to the working capital channel is to use GHH preferences, as found in García-Cicco, Pancrazi, and Uribe (2010). Under the GHH specification (again, ignoring habits and assuming a unitary intertemporal elasticity of substitution), period utility is written as

$$u_t = \log\left(C_t - \frac{\theta_t X_t h_t^{\omega}}{\omega}\right),$$

Figure 3.2: The effects of a risk premium shock



The figure shows model simulations without a working capital channel, with working capital financing imposed only on wages, or with working capital financing both on wages and intermediate inputs (the baseline).

and the labor market equilibrium condition is

$$\theta_t h_t^{\omega} = (1 - \alpha)\, y_t. \tag{3.17}$$

In this case, an increase in the risk premium is contractionary, even without a working capital requirement. The cost of capital goes up, which decreases output and wages, and this leads to a decline in labor supply, since there is no opposing income effect.

There are a few reasons why we do not use the GHH specification. First, the mechanism is not particularly convincing, and it is at odds with recent models of financial frictions, which emphasize labor demand. Second, in a GHH setup, hours are very strongly linked to effective output. With our calibrated value of $\omega = 1.6$, a 10% increase in output is associated with a 6% rise in total hours. Taken seriously, this implies that in converging economies (with effective output well below the steady state) hours should increase significantly along the transition path. This is again a prediction that is counterintuitive and at odds with the data. Finally, when we estimated the model with a GHH specification, the overall fit and the various diagnostics proved to be much worse than under the KPR setup with working capital.[6] The shock decompositions, by contrast, were fairly similar to the ones we present below.

3.4 Shock estimation

In order to estimate the stochastic shocks, we log-linearize the equilibrium conditions in (**??**) around the deterministic steady state. Observable are the growth rates of GDP, consumption, investment and hours, and the trade balance-GDP ratio. To the greatest extent possible, we use raw data. Thus, we only demean the growth rates (except for hours, which are stationary both in the model and in the data) with the country-specific average growth rates of GDP per capita. This is the simplest way to remove additional growth that comes from economic transition.

Notably, we do not use observed interest rates in the estimation. This is standard in RBC-type models, and the main reason is that the real interest rate that is relevant for household and firm decisions might differ substantially from real interest rates calculated from policy or money market rates. In fact, one of our goals is to compare our implicit, model-based interest rate to an observed time series. As we show in Section 3.5, this turns out be a quite illuminating exercise.

The main challenge for the estimation is that we have a short time series, namely, annual observations between 1996 and 2017. While using quarterly data is possible, the advantage of higher frequency comes with the cost of additional noise. Since our purpose is to learn about the growth process and slow-moving shocks, we think the annual frequency is more suitable for our purposes. To capture quarterly dynamics reasonably well, additional nominal and real rigidities are needed, which would make the model much more complicated. An important assumption behind our exercise is that these frictions are less important for annual data, and can be captured without explicitly modeling monetary policy and exchange rates.

We gain degrees of freedom through two main strategies. First, we estimate the model on a panel of five Central- and Eastern European (CEE) countries. These economies share a largely common economic history, and they are all transitioning from central planning to market economies. They all joined the European Union in 2004, and have been converging to the "old" member states for most of the sample period. We assume that the structural parameters and the shock

[6] The results of this exercise are available from the authors upon request.

autoregressive parameters are common across the five countries, but we allow for country-specific shock innovations (see the details below). These assumptions are routinely made in reduced form panel studies that use country-level data. In fact, our specification is more flexible, since we allow for time-varying "fixed effects" in the form of the country-specific shock innovations.[7]
Second, in our baseline specification, we calibrate most of the model parameters, and focus on the shock processes. The static parameters are easy to set using steady state conditions. There are three dynamic parameters where this is not possible: the debt sensitivity of the interest rate (ψ), the investment adjustment cost (ϕ), and the strength of consumption habit (χ). It is well known that DSGE models suffer from serious identification problems (Canova and Sala (2009)). When we tried to estimate all three parameters, the MH chains were not converging, so we have little trust in the results. Experimentation reveals that, at most, one dynamic parameter can be estimated reliably. Given our interest in risk premium shocks, we chose to estimate the debt elasticity parameter, and set the other two to standard values from the literature. Results for other specifications are available upon request: the main conclusions remain robust.
Turning to the shock processes, we assume that trend productivity shock and interest-premium shock innovations contain both common and country-specific components:

$$\log g_t^j = \rho_g \log g_{t-1}^j + \nu_{g,t} + \nu_{g,t}^j \tag{3.18}$$

$$\epsilon_{r,t}^j = \rho_r \log \epsilon_{r,t-1}^j + \nu_{r,t} + \nu_{r,t}^j, \tag{3.19}$$

where j indexes countries. The innovations ν_g and ν_r represent the external growth and financial environments, which are likely to be important determinants of growth in the CEE countries. All other shocks are assumed only to have local innovations.[8] As discussed above, we impose the same autoregressive parameter for the five countries for each shock. This is partly because these economies have a similar structure, and also because when estimating country-specific AR(1) terms, we cannot reject the hypothesis that they are the same across economies. This might be the consequence of the short time series, which is an external constraint we cannot do much about. Even with the common AR term, we believe that our specification gives us enough

[7] We also tried adding the Baltic countries to our sample, but the results were much more noisy. We suspect this is because the three Baltic economies have much less in common with our CEE countries. They are all much smaller, they were part of the Soviet Union, and they have much stronger economic links with Scandinavia than the CEE countries, whose main economic partner is Germany.

[8] Note that in the log–linear representation of the model, we rescale the investment shock to $\epsilon_{i,t} = -\phi \bar{g}^2 (1 - \beta\rho) \tilde{\epsilon}_{i,t}$. This also implies that a positive innovation to $\epsilon_{i,t}$ is associated with a decrease in the cost of investment, that is, the shock is expansionary.

flexibility to uncover common and country-specific drivers of the main macro series in question. Returning to the structural parameters, we follow standard practice and use equations in the deterministic steady state to calibrate as many parameters as possible. The steady state conditions are given as follows:

$$\begin{aligned}
\bar{R} &= \frac{\bar{g}}{\beta} \\
\frac{\bar{k}}{\bar{g}\bar{y}} &= \frac{\bar{i}/\bar{y}}{\bar{g} - 1 + \delta} \\
\alpha &= \frac{\bar{g}/\beta - 1 + \delta}{\bar{R}^{\frac{\mu}{\mu-1}}} \frac{\bar{k}}{\bar{g}\bar{y}} \\
\frac{\bar{k}}{\bar{g}\bar{h}} &= \left(\frac{\bar{k}}{\bar{g}\bar{y}}\right)^{\frac{1}{1-\alpha}} \\
\frac{\bar{y}}{\bar{h}} &= \left(\frac{\bar{k}}{\bar{g}\bar{h}}\right)^{\frac{\alpha}{1-\alpha}} \\
\frac{\bar{tb}}{\bar{y}} &= \left(\frac{1}{\bar{g}} - \frac{1}{\bar{R}}\right)\frac{\bar{d}}{\bar{y}} \\
\frac{\bar{c}}{\bar{y}} &= 1 - \frac{\bar{i}}{\bar{y}} - \frac{\bar{\xi}}{\bar{y}} - \frac{\bar{tb}}{\bar{y}} \\
\bar{h} &= \left[\frac{1-\alpha}{\bar{\theta}\bar{R}^{\frac{1}{1-\mu}}\left(1 - \chi/\bar{g}\right)\bar{c}/\bar{y}}\right]^{\frac{1}{\omega}}.
\end{aligned}$$

We set the discount factor to $\beta = 0.98$, and the long-run growth rate to $\bar{g} = 1.0159$, where the latter is the average per capita value for the EU 15 countries in the sample period. The long-run average interest rate is given as the ratio of the two values. We set the steady state investment-GDP ratio to the sample average for each country using the chain-linked volumes for investment and GDP.[9] Assuming a depreciation rate of $\delta = 0.05$, we can then compute the adjusted capital-output ratio $\bar{k}/\left(\bar{g}\bar{y}\right)$. This yields the capital share parameter α, which we allow to be country specific, and long-run GDP per hours.

The other expenditure items are set as follows. We impose a uniform government spending share of $\bar{\Xi}/\bar{Y} = 0.1$, which is in line with the data for the CEE countries. Since the long-run debt level is exogenous in the model, and average data from a short sample can be very misleading for these values, we simply set $\bar{d}/\bar{y} = 0$ for all three countries. This means that the long-run trade balance is also zero. Plugging the investment share, the trade balance and the share of government spending into the GDP identity then yields the steady state consumption-output

[9] An alternative is to use the nominal ratio, which is better from a statistical point of view. In the CEE countries, however, the relative price of investment declined significantly over the sample period. This means that investment expenditure in nominal terms does not adequately reflect the time series of the physical units of capital being created.

ratio.

Recall that we use a gross output production function. Thus, we also need to calibrate the share of intermediate inputs (μ). We use data from Eurostat on gross output and value added. Our measure of intermediate share is simply given by

$$\mu = 1 - \frac{1}{T}\sum_{t=1}^{T}\frac{Y_t}{Y_t^G},$$

where Y and Y^G are observations of value added and gross output for the total economy at current prices.

We normalize the average level of hours to $\bar{h} = 0.3$, which is a standard value in the literature. This is without loss of generality, and the only role of this normalization is to pin down the parameter $\bar{\theta}$, as can be seen from the last steady state condition. We need a value for the inverse of the Frisch elasticity of labor supply, where we follow García-Cicco, Pancrazi, and Uribe (2010) and use $\omega = 1.6$. This leads to an elasticity of 1/0.6, which is in line with the parameterization of RBC models that rely on an elastic labor supply to deliver volatilities for GDP and its components in line with the data.

Table 3.1: Calibrated parameters

Parameter name		Value				
		Common				
Discount factor	β	0.98				
Depreciation rate	δ	0.05				
Consumption habit	χ	0.5				
Investment cost	ϕ	2				
Frisch elasticity	ω	1.6				
Steady state debt/GDP	d_y	0				
		Country specific				
		CZ	HU	PL	SI	SK
Capital share	α	0.394	0.313	0.279	0.327	0.365
Share of intermediates	μ	0.603	0.575	0.553	0.544	0.606
Value of leisure	θ	12.14	12.67	12.90	12.71	12.29

For the two dynamic parameters discussed above, we set $\chi = 0.5$, which is equivalent to a quarterly value of 0.84. This is at the high end of DSGE estimates, but, given our annual frequency, significantly lower values would mean habits are unimportant. For the investment adjustment cost parameter, we use $\phi = 2$. This is in line with Smets and Wouters (2002), whose mean estimate for the Euro Area is around 7 in a quarterly setting.

Table 3.1 summarizes the calibrated parameters. Most of them are common across countries, except for the capital share, the share of intermediates, and the value of leisure. We now turn to the estimation of the shock processes.

3.4.1 Estimation results

The model is estimated using Bayesian techniques (An and Schorfheide (2007)). We impose flat (uniform) priors on all shock persistences on the $[0, 0.99]$ interval, and assume that these parameters are the same across the five countries. We allow, however, for country-specific innovations as described in the previous section. We use flat priors for all the standard deviations of the - global or local - innovations, with a range of $[0, 0.2]$.

Data includes chain-linked annual growth rates for GDP, gross fixed capital formation and actual individual consumption for the CEE countries, downloaded from Eurostat. The trade balance is measured by the ratio of net exports to GDP at current prices (source: Eurostat). We use the growth rate of total hours to measure labor input, also downloaded from Eurostat. The sample period is 1996-2017 for all countries.

Table 3.4 contains the prior distributions and the estimation results. The shock processes are fairly precisely estimated. Except for the trend shock, the shocks are quite persistent, but clearly identified within the bounds. It is noteworthy to emphasize that although our sample period is short and we use flat priors, the data is informative about the parameter values.

Notice that the data has a hard time disentangling the two technology shock components, namely, the transitory shock a_t and the trend shock g_t. The former is extremely persistent, while the latter is not. This suggests that productivity growth in the CEE countries might be a random walk. Indeed, when we omit the transitory technology shock from the estimation entirely, the main results are almost identical. To preserve comparability with Aguiar and Gopinath (2007) and García-Cicco, Pancrazi, and Uribe (2010), we present results with the technology shock included.

Table 3.2: Bayesian estimation priors and results

	Prior mean	Post. mean	90% conf. int.	Prior

			AR(1) parameters			
ψ	0.05	0.0121	0.0041	0.0192	Uniform	$0-0.1$
ρ_a	0.495	0.856	0.7998	0.911	Uniform	$0-0.99$
ρ_g	0.495	0.0945	0.0045	0.1584	Uniform	$0-0.99$
ρ_r	0.495	0.5555	0.4711	0.6254	Uniform	$0-0.99$
ρ_ξ	0.495	0.8131	0.7173	0.9092	Uniform	$0-0.99$
ρ_i	0.495	0.6203	0.4326	0.8179	Uniform	$0-0.99$
ρ_h	0.495	0.9624	0.9292	0.99	Uniform	$0-0.99$
			Standard deviations			
			Global			
ν_g	0.1	0.0281	0.0189	0.0372	Uniform	$0-0.2$
ν_r	0.1	0.0082	0.0027	0.0131	Uniform	$0-0.2$
			Czech Republic			
ν_a^{cz}	0.1	0.0152	0.0086	0.0235	Uniform	$0-0.2$
ν_g^{cz}	0.1	0.0253	0.0148	0.0364	Uniform	$0-0.2$
ν_r^{cz}	0.1	0.0084	0.0039	0.0131	Uniform	$0-0.2$
ν_ξ^{cz}	0.1	0.1421	0.1048	0.1781	Uniform	$0-0.2$
ν_i^{cz}	0.1	0.0845	0.0585	0.112	Uniform	$0-0.2$
ν_h^{cz}	0.1	0.0166	0.0077	0.0265	Uniform	$0-0.2$
			Hungary			
ν_a^{hu}	0.1	0.0063	0	0.0116	Uniform	$0-0.2$
ν_g^{hu}	0.1	0.0195	0.0097	0.029	Uniform	$0-0.2$
ν_r^{hu}	0.1	0.021	0.013	0.0281	Uniform	$0-0.2$
ν_ξ^{hu}	0.1	0.1653	0.1406	0.1974	Uniform	$0-0.2$
ν_i^{hu}	0.1	0.1224	0.0878	0.1588	Uniform	$0-0.2$
ν_h^{hu}	0.1	0.036	0.0247	0.0467	Uniform	$0-0.2$
			Poland			
ν_a^{pl}	0.1	0.0049	0	0.0099	Uniform	$0-0.2$
ν_g^{pl}	0.1	0.0222	0.013	0.0315	Uniform	$0-0.2$
ν_r^{pl}	0.1	0.0096	0.0042	0.0151	Uniform	$0-0.2$
ν_ξ^{pl}	0.1	0.1196	0.0869	0.1526	Uniform	$0-0.2$
ν_i^{pl}	0.1	0.0996	0.0699	0.1296	Uniform	$0-0.2$
ν_h^{pl}	0.1	0.0386	0.0232	0.055	Uniform	$0-0.2$
			Slovenia			
ν_a^{si}	0.1	0.0132	0.0046	0.0224	Uniform	$0-0.2$
ν_g^{si}	0.1	0.0144	0.0012	0.025	Uniform	$0-0.2$
ν_r^{si}	0.1	0.0168	0.0109	0.0225	Uniform	$0-0.2$
ν_ξ^{si}	0.1	0.165	0.138	0.1979	Uniform	$0-0.2$
ν_i^{si}	0.1	0.1079	0.0763	0.1402	Uniform	$0-0.2$
ν_h^{si}	0.1	0.0257	0.0109	0.0423	Uniform	$0-0.2$
			Slovakia			
ν_a^{sk}	0.1	0.0151	0.0092	0.0219	Uniform	$0-0.2$

ν_g^{sk}	0.1	0.0198	0.0073	0.0318	Uniform	$0-0.2$
ν_r^{sk}	0.1	0.0181	0.0117	0.0238	Uniform	$0-0.2$
ν_ξ^{sk}	0.1	0.163	0.1378	0.1973	Uniform	$0-0.2$
ν_i^{sk}	0.1	0.1526	0.1165	0.1905	Uniform	$0-0.2$
ν_h^{sk}	0.1	0.0263	0.0156	0.0372	Uniform	$0-0.2$

3.4.2 Variance decomposition

Our main exercise is to decompose the growth rates of GDP, the demand side components and the trade-balance to GDP ratio into contributions of various shock innovations. We have 7 items: global trend (ν_g), local trend (ν_g^j), global premium (ν_r), local premium (ν_r^j), technology level (ν_a^j), government (ν_{gov}^j), investment (ν_i^j) and labor (ν_h^j) shocks. The variance decomposition shows the relative importance of the estimated structural shocks. Table 3.3 presents the results of the exercise where we simulate the model using the estimated shock persistences and standard deviations.

Global and local growth shocks explain 38-68% of the volatility of GDP growth, while transitory technology shocks are also important(with the exceptions of Hungary and Poland). Moreover, we estimate the transitory shock to be very persistent, which makes it difficult to disentangle from "true" growth shocks. Therefore, the distinct role of productivity shocks is consistent with the idea that, during the pre-crisis period, growth expectations played a major role in the CEE economies. Changes in the external growth environment and in income expectations seem to have been the main drivers of aggregate GDP growth. Labor supply shocks were also important, especially in Hungary and Poland. Poland implemented its labor market reform in the middle of the 2000s. After the financial crisis, Hungarian interventions made the labor market more flexible, increased the labor supply and thus had a positive effect on Hungarian GDP growth.

Consumption and investment are also functions of productivity growth, but the picture is heterogeneous and other factors like premium shocks and investment shocks have a large impact. Premium shocks influence the composition of aggregate demand, and they are also partly behind the volatility of the trade-balance. The Hungarian economy was fueled by cheap credit before the crisis, and it had to go through a significant balance sheet adjustment post-crisis. The local premium shock is also important for Slovenian and Slovakian domestic demand, especially for consumption growth.

In the past decade, investment might also have been strongly affected by the inflow of EU

Table 3.3: Variance decomposition

	ν_g	ν_r	ν_a^j	ν_g^j	ν_r^j	ν_ξ^j	ν_i^j	ν_h^j
				GDP growth				
CZ	26.6	0.87	41.86	21.45	0.9	1.06	2.63	4.63
HU	33.95	0.67	8.33	16.31	4.42	1.87	4.68	29.77
PL	34.01	0.42	4.75	21.19	0.58	1	2.56	35.5
SI	33.17	0.36	35.85	8.72	1.53	1.81	4.02	14.55
SK	25.16	0.9	37.57	12.47	4.37	1.33	6.93	11.28
				Consumption growth				
CZ	31.1	15.72	6.16	25.08	16.38	1.76	1.04	2.76
HU	19.64	7.98	0.57	9.44	52.4	1.06	0.69	8.23
PL	32.83	12.19	0.53	20.46	16.7	0.8	0.55	15.94
SI	25.9	10.73	3.4	6.81	45.28	1.48	0.84	5.54
SK	20.82	9.75	3.81	10.32	47.55	1.36	1.75	4.63
				Investment growth				
CZ	15.39	5.02	3.82	12.41	5.23	0.4	57	0.74
HU	9.2	2.4	0.37	4.42	15.73	0.21	65.2	2.47
PL	14.26	3.24	0.31	8.89	4.44	0.13	64.09	4.63
SI	11.92	2.79	1.93	3.14	11.78	0.24	66.67	1.53
SK	6.46	2.06	1.6	3.2	10.05	0.2	75.58	0.85
				Trade Balance to GDP				
CZ	6.06	9.74	11.27	4.88	10.15	6	51.04	0.87
HU	3.21	5.15	1.45	1.54	33.85	4.97	46.67	3.15
PL	5.89	9.94	1.88	3.67	13.62	5.2	51.73	8.07
SI	4.11	6.45	7.14	1.08	27.2	5.79	46.43	1.8
SK	2.45	3.94	5.13	1.22	19.19	3.41	63.65	1.02

The table shows the shock variance decomposition of the four key variables based on model simulations. The simulations use the baseline calibration, together with the posterior means of shock persistences and standard errors.

transfers. It is intriguing interesting that investment specific shocks not only drive investment growth itself, but also the volatility of the trade balance. While the model does not have an explicit role for external funds, the investment-specific shocks can capture at least some of these extra developments. We provide some suggestive evidence in the next section that this is indeed the case.

Overall, our results indicate that both trend-productivity shocks and interest-premium shocks are important to understand the growth experience of the CEE countries. As in Aguiar and Gopinath (2007), but in contrast to García-Cicco, Pancrazi, and Uribe (2010), we find that the volatility of GDP growth is mainly driven by shocks to the trend component of productivity. Nonetheless, interest-premium shocks are important for understanding the evolution of the main GDP components, and consumption growth in particular. In addition, labor market shocks and especially investment specific shocks have played a significant role. Restricting attention to productivity and interest-premium shocks might thus be too restrictive, at least in the context of the CEE economies.

3.4.3 Wage rigidity

We experimented with many variations on the baseline estimation strategy, and our main conclusions remain robust. We discussed above that estimating two additional parameters - the investment adjustment cost (ϕ) and the habit parameter (χ) - turned out to be fruitless, since the model diagnostics showed a failure for the Metropolis-Hastings algorithm. We also experimented with quarterly data, but the results were very noisy, probably reflecting the fact that our simple model lacks many rigidities that are necessary to fit high-frequency data. Since our goal in this paper is to interpret longer-term trends, we opted to keep the simple model structure and use annual data.

It is unclear whether nominal rigidities are relevant at the annual frequency. If the answer is yes, wage rigidity is the most likely candidate (Blanchard and Galí (2010); Olivei and Tenreyro (2010)). Fortunately, it is easy to modify our simple framework to accommodate real wage rigidity. We sketch the modification below, and also report on the empirical results. To summarize, the relevance of wage rigidity is strongly rejected by the data.

As in the much of the literature (Erceg, Henderson, and Levin (2000)), we assume that households supply differentiated labor to the representative firm, which uses a CES aggregate as labor input:

$$N_t = \left[\int_0^1 N_{i,t}^{1-\frac{1}{\sigma_w}} di \right]^{\frac{\sigma_w}{\sigma_w - 1}}.$$

The demand function for individual labor types is easily derived as:

$$N_{i,t} = \left(\frac{W_{i,t}}{W_t} \right)^{-\sigma_w} N_t.$$

Households set wages to maximize their sub-utility from leisure, given demand for their labor variety:

$$\max \mathbb{E}_t \sum_{t=0}^{\infty} \beta \left[\Lambda_{i,t} \left(\frac{W_{i,t}}{W_t} \right)^{-\sigma_w} W_{i,t} N_t - \theta_t \frac{(W_{i,t}/W_t)^{-\sigma_w(1+\omega)}}{1+\omega} N_t^{1+\omega} - \frac{\varphi}{2} \left(\frac{W_{i,t}}{W_{i,t-1}} - \bar{g} \right)^2, \right]$$

where we assume that wage setting is subject to quadratic adjustment costs. The first-order condition is given by:

$$\begin{aligned} (\sigma_w - 1) N_{i,t} \Lambda_{i,t} W_{i,t} = \sigma_w \theta_t N_{i,t}^{1+\omega} - \varphi \left(\frac{W_{i,t}}{W_{i,t-1}} - \bar{g} \right) \frac{W_t}{W_{i,t-1}} \\ + \beta \varphi \mathbb{E}_t \Lambda_{i,t+1} \left(\frac{W_{i,t+1}}{W_{i,t}} - \bar{g} \right) \frac{W_{i,t+1}}{W_{i,t}^2} W_t, \end{aligned}$$

which after normalization and log-linearization simplifies to:

$$\hat{w}_t - \hat{w}_{t-1} = \beta \mathbb{E}_t \left(\hat{w}_{t+1} - \hat{w}_t \right) + \frac{\sigma_w \bar{\theta} \bar{N}^{1+\omega}}{\varphi \bar{g}^2} \left(\omega \hat{N}_t - \hat{\lambda}_t - \hat{w}_t + \hat{\theta}_t \right) + \beta \mathbb{E}_t \hat{g}_{t+1} - \hat{g}_t.$$

Wage rigidity can amplify the impact of shocks on employment and output, which may have been important during the adjustment period following the financial crisis. Ultimately, it is an empirical question whether the data support this mechanism. We therefore re-estimated the model where the earlier labor supply condition is replaced by this real wage Phillips curve; the other equations remain unaffected.

Empirical evidence for many countries (Babecky, Caju, Kosma, Lawless, Messina, and Room (2010)) suggest that firms adjust their wages annually. This is equivalent to a quarterly Calvo parameter of about 0.75. At the annual frequency, this translates to a Calvo parameter of about 0.3. We are using quadratic adjustment costs, so we link the parameter φ to the equivalent Calvo coefficient in the estimation using the formal correspondence between our wage Phillips curve and the identical equation in Erceg, Henderson, and Levin (2000). We impose a uniform prior on the Calvo equivalent, with support between $[0 - 0.99]$.

The posterior mean is 0.0015, with a tight posterior confidence interval of $0.0000 - 0.0034$. The hypothesis that real wage rigidity is an important channel at the growth frequency is very strongly rejected by the data. The other parameter estimates are very similar to the baseline case. We therefore conclude that while potentially important for some countries and episodes, wage rigidity does not seem to be relevant for the general growth experience of the CEE economies in our sample period.

3.5 External model validation

After presenting the main results, we discuss additional findings that - while interesting in their own right - provide strong external validation for the estimation exercise. We compare the estimated global components of the trend shock and the interest-premium shock to an observable (EU) time series. We also study the investment specific shock innovations and relate them to EU funding after the 2004 period. As a final robustness check, we observe the long-term interest rate differentials and re-estimate the model.

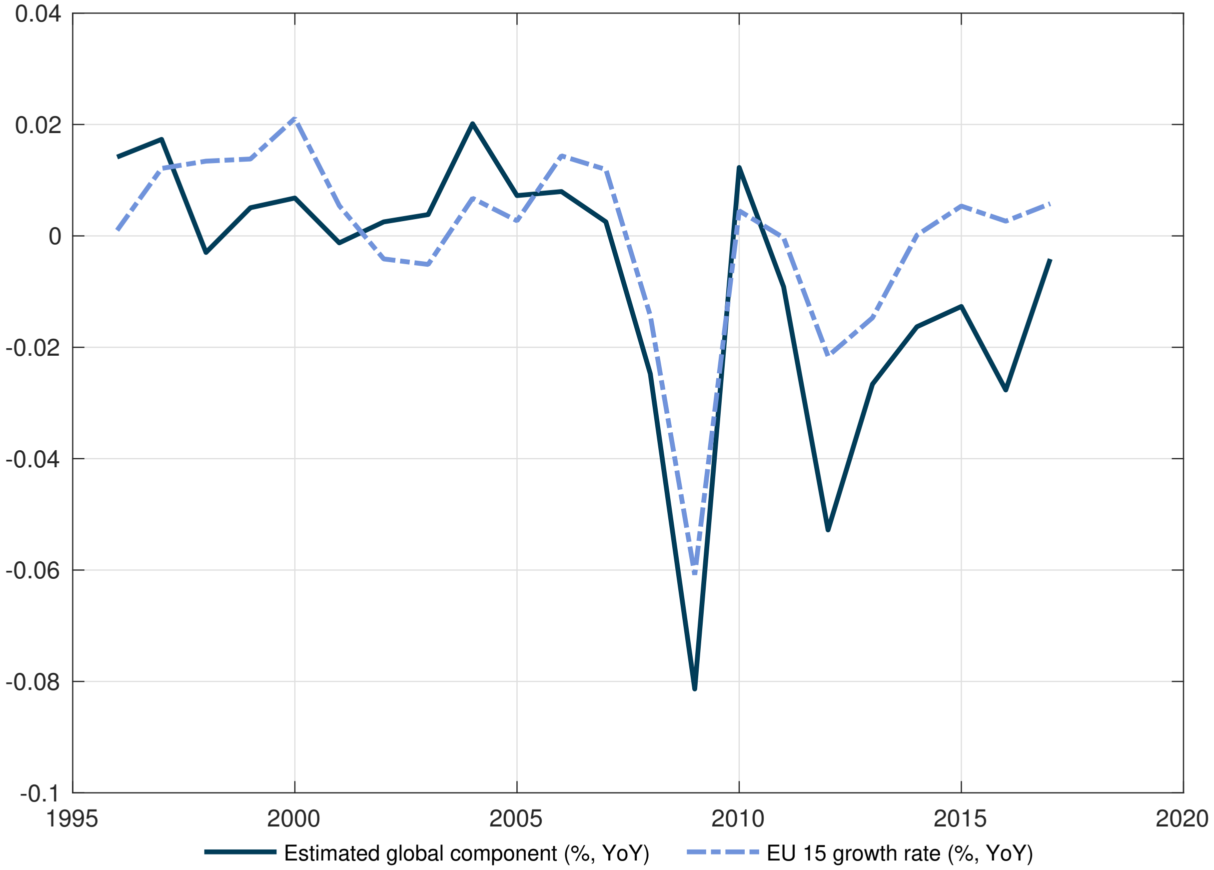
3.5.1 Trend growth and investment

First, we take a closer look at the estimated trend productivity shock. In particular, we want to examine the global component that the estimation uncovered, $\nu_{g,t}$. Figure 3.3 plots the global trend innovation against the growth rate of the EU 15 countries.[10] The rationale for this is that the CEE countries overwhelmingly trade with other EU countries, and the EU 15 represents the "core" economy of the group. Thus, we expect that external common growth shocks are highly correlated with the growth rate of the EU 15 countries. This is indeed what we find, as Figure 3.3 shows. The global innovation $\nu_{g,t}$ tracks EU 15 growth very closely, especially since 2004, when the CEE countries joined the European Union. While only suggestive, this result gives us confidence that the estimation procedure "makes sense".

To further investigate the role of the European Union in the growth process of the CEE countries, we now turn to investment. As we saw above, investment-specific shocks are significant determinants of the volatilities of investment and the trade balance. For the latter, this is especially true after 2004, when the CEE countries joined the European Union. A possible explanation for this is the presence of EU structural funds, which have become a significant source of invest-

[10] These are the "old" EU member states before the expansion of 2004: Austria, Belgium, Denmark, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, Netherlands, Portugal, Spain, Sweden, United Kingdom.

Figure 3.3: Global growth shock component and EU growth



The figure shows the estimated common ("global") innovations to trend productivity growth $\nu_{g,t}$ together with the real GDP growth rate of the EU 15 countries.

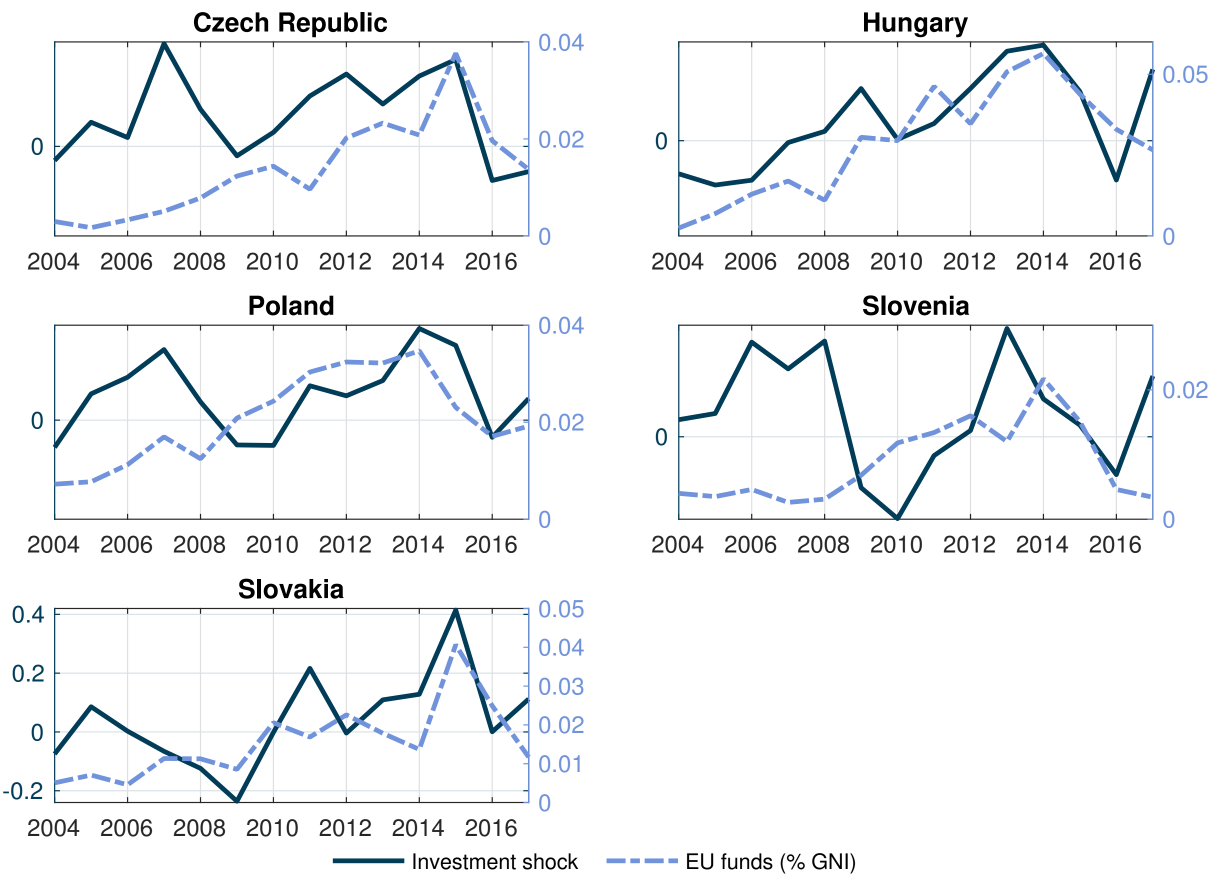
ment spending in the five countries. Overall, net EU payments are on the scale of 2-5% of gross national income, especially since 2010.

A detailed study of the role of EU funds is beyond the scope of this paper. For a proper understanding, we would need to build a model with external transfers and a government sector with its separate investment activity. Instead, we provide some preliminary evidence that EU funds are likely to be an important explanation for the idiosyncratic behavior of investment captured by its specific shock. Figure 3.4 plots the estimated investment shock ($\epsilon_{i,t}$, left scale) against European Union funds received annually (as a share of GNI, right scale). Data is available from the European Union since 2004, when the CEE countries became members.[11]

Once we adjust for the differences in measurement units, the two time series are remarkably similar in our countries. This is particularly the case in Hungary, Poland and Slovakia, but the trends and the main turning points line up quite well in the other two economies, especially after

[11] http://ec.europa.eu/budget/figures/interactive/index_en.cfm

Figure 3.4: EU funding and investment specific shocks



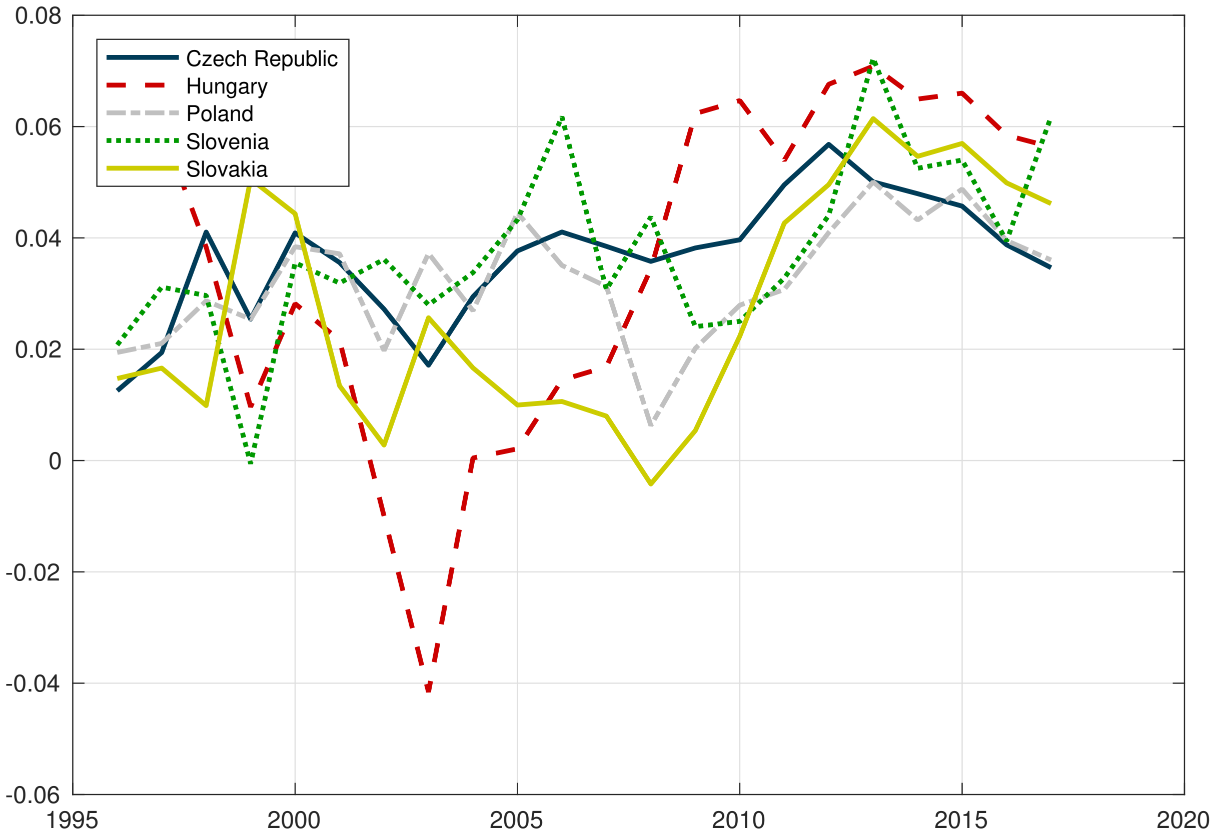
The figure shows the estimated investment specific shock ($\epsilon_{i,t}^{j}$) for each country (left scale), together with net EU funds received as a share of GNI (right scale).

2009, when the bulk of EU funding started to arrive. While these charts are only suggestive, it is reassuring that the estimation recovers a stochastic shock that can be given a convincing empirical interpretation.

3.5.2 Implicit nterest rate and interest premium

In our estimation, we do not use observed interest rates; rather, we back them out from the evolution of GDP components. It is interesting to see whether these implicit interest rates "make sense", that is, whether their paths are in line with our prior expectations. We would like to find the following patterns: high values in the 1990s, a gradual decline before the financial crisis (especially in the 2004-2008) period, and increased heterogeneity after the crisis. For the latter period, we expect interest rate increases for more heavily indebted countries (Hungary), and decreases for less-indebted countries (the Czech Republic, and to a lesser extent Poland and Slovakia). It is important to note that our implicit interest rates condense price and non-price information that are relevant for intertemporal consumption and investment decisions, and thus can be quite different from the policy rate. This is especially important after the financial crisis, when quantitative restrictions on credit became much more prevalent, and low headline interest rates may mask high effective borrowing rates by households and small enterprises.

Figure 3.5: Estimated implicit interest rates



The figure shows the estimated implicit real interest rates for each country.

Figure 3.5 presents the results. The implicit interest rate have been most stable in the Czech Republic, the richest and most stable economy in the group. Slovenia, the other economic leader until recently, had a deep banking crisis in 2013, which is reflected in the sharp rise in the implicit interest rate. The two countries that joined the Eurozone by the time of the financial crisis (Slovenia and Slovakia) experienced the lowest implicit rates in 2009-2010. Poland, which was the only country to escape recession after 2009, shows the lowest increase in the interest rate after 2009. The experience of Hungary is the most dramatic. In the pre-crisis period, Hungary enjoyed a positive investment climate and became the most heavily indebted economy; subsequently, it was most exposed to financial market tightening and balance sheet adjustment. This is reflected reflected in the very low estimated rates until 2007, and the steep rise that started just before the crisis, and continued afterwards. By 2010, Hungarian implicit rates have become the highest in the group, and remained high until 2017.

Recall that similarly to the stochastic productivity trend, we estimated the interest rate innovations with a global and local component. We expect the global component to pick up changes in external financial conditions that effected all CEE countries similarly. As before, we use the EU 15 countries as a benchmark to see if the global interest rate component is related to the evolution of a real interest rate observed in the relevant external financial market. For this purpose, we use the short-run real interest rate for the EU 15 countries, downloaded from the AMECO database[12]. This is a GDP weighted average of the 15 countries, and uses the GDP deflator as its measure of inflation.

To construct a "global" implicit interest rate relevant for the Visegrad countries, we use the following procedure. Let r_t^g indicate the implicit interest rate that only includes the global innovation. We define this interest rate as follows:

$$r_t^g = \rho_r \rho_{t-1}^g + \nu_{r,t}, \tag{3.20}$$

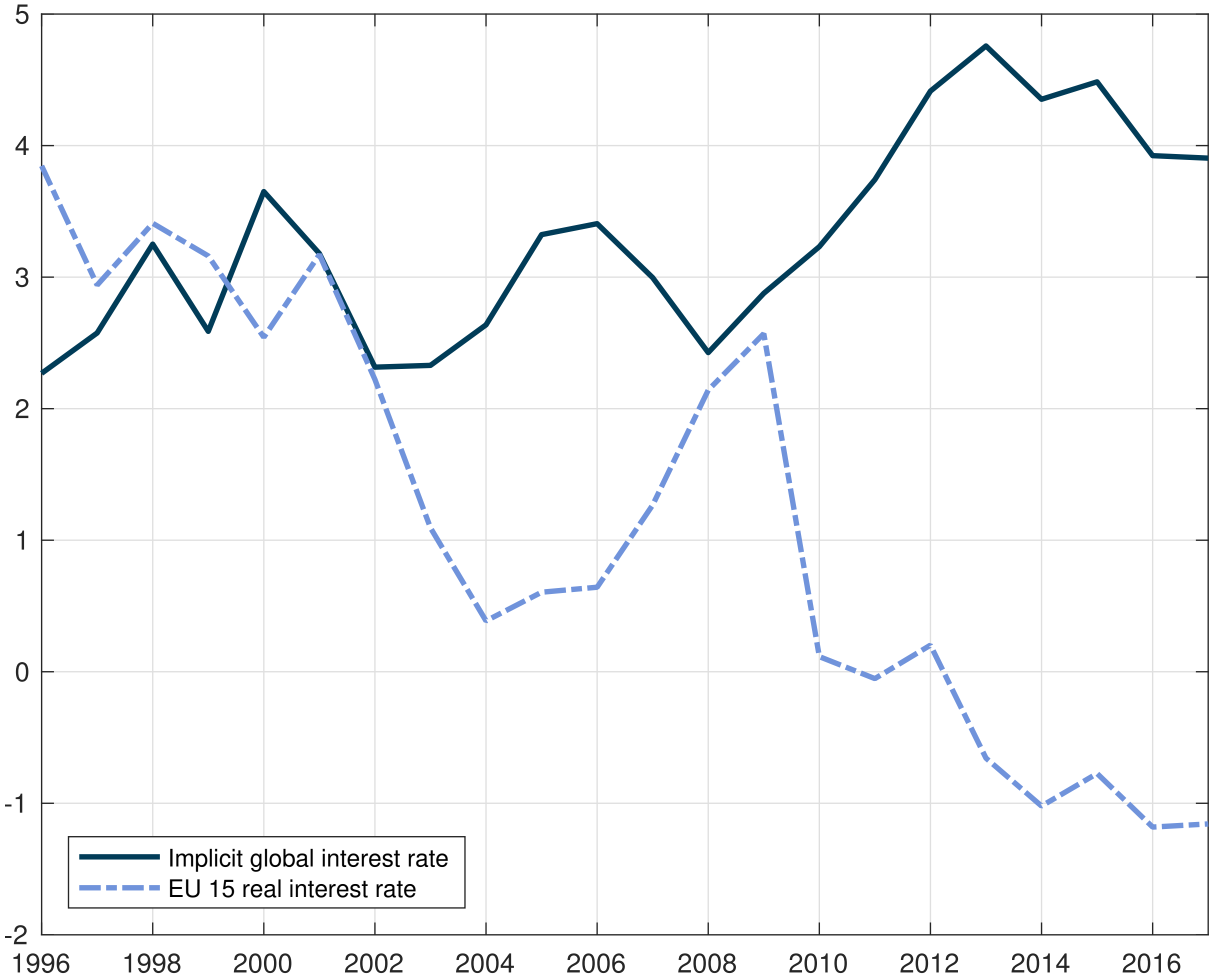
where ρ_r is the (common) estimated persistence of the interest premium shock, and $\nu_{r,t}$ is the estimated global component of the shock innovation. Our sample starts in 1996, and we simply assume that $r_{1995}^g = \bar{r}$.[13]

Figure 3.6 presents the estimated implicit global interest rate and the real interest rate in the EU 15 countries. Two key patterns stand out. First, before the financial crisis (2008), the global component tracks the actual EU 15 real interest rate until about 2001. The common component remains stable until 2008, despite the fact that the EU 15 real rate declines, and then rises. Second, the two series diverge dramatically from 2008. It is beyond the scope of this paper to examine the reasons, but we offer two (probably complementary) possible explanations. On the one hand, the implicit interest rate influencing investment and savings might have diverged from the money market rate during and after the crisis. This can happen for various reasons, such as an increase in the risk premium associated with household and corporate lending, or an increase in credit rationing and other non-price restrictions. On the other hand, the "global" rate relevant for the CEE countries might have gone up relative to the EU 15 countries, due to

[12]http://ec.europa.eu/economy_finance/ameco/user/serie/SelectSerie.cfm

[13]We could have followed a similar procedure when comparing the global trend component to growth in the EU 15. Since the estimated trend shock is close to a random walk, this does not matter for the trend shock. The interest-premium shock is persistent, however, so it is important to take into account autoregressive behavior.

Figure 3.6: Global interest rate and the observed EU 15 real interest rate



The figure shows the observed average real interest rate for the EU 15 countries (source:AMECO), together with the estimated common ("global") interest rate for the Visegrad countries.

the general increase in risk aversion and the flight to safety by investors away from emerging markets.

3.5.3 Observing the interest rate differentials

In our model, as a robustness check, we also tried to observe the long-term real interest rate. However, this rate is not necessarily consistent with the logic of the neoclassical model. In these models, the real interest rate expresses those effective equilibrium yields that are driven by the external financing premium and marginal product of capital. Thus, this unobserved rate contains price and non-price related components as well, that is, the non-price related components are the unobserved credit constraint and all conditions that directly influence the financial intermediaries. We collected the long-term real interest rate[14] from the AMECO database from 2002, calculated the difference between the country-level rate and Eurozone long-term real interest rate, and compared the long-term real interest rate differentials (see Figure 3.7). In the previous estimation, we calculated the level of the real interest rate by adding together the steady-state level and dynamic components; in this estimation, the observed variable is the deviation from the Eurozone interest rate.

The relative positions of the interest rate differentials are similar to the estimated implicit interest rate: before the crisis the monetary conditions were relatively easy, and, after the crisis, the strong deleveraging increased the financing costs. However, there are differences in the dynamics, and the volatility of the observed interest rate differential is smaller than the implied rates from the previous estimation.

Figure 3.7: Observed interest rate differentials

The figure shows the long-term interest rate differentials between the Eurozone and selected member states.

[14] 10Y interest rate minus inflation expectation.

Table 3.4: Bayesian estimation priors and results

	Prior mean	Post. mean	90% conf. int.		Prior	
			AR(1) parameters			
ψ	0.05	0.0098	0.0039	0.0148	Uniform	$0-0.1$
ρ_a	0.495	0.8148	0.7588	0.8855	Uniform	$0-0.99$
ρ_g	0.495	0.2437	0.1659	0.3633	Uniform	$0-0.99$
ρ_r	0.495	0.2431	0.1562	0.3410	Uniform	$0-0.99$
ρ_ξ	0.495	0.9262	0.8618	0.99	Uniform	$0-0.99$
ρ_i	0.495	0.2666	0.1295	0.4018	Uniform	$0-0.99$
ρ_h	0.495	0.9881	0.9855	0.99	Uniform	$0-0.99$
			Standard deviations			
			Global			
ν_g	0.1	0.0334	0.0229	0.0483	Uniform	$0-0.2$
ν_r	0.1	0.0075	0.0032	0.0120	Uniform	$0-0.2$
			Czech Republic			
ν_a^{cz}	0.1	0.0174	0.0121	0.0229	Uniform	$0-0.2$
ν_g^{cz}	0.1	0.0125	0.0003	0.0215	Uniform	$0-0.2$
ν_r^{cz}	0.1	0.0097	0.0056	0.0143	Uniform	$0-0.2$
ν_ξ^{cz}	0.1	0.1198	0.0927	0.1434	Uniform	$0-0.2$
ν_i^{cz}	0.1	0.1018	0.0798	0.1214	Uniform	$0-0.2$
ν_h^{cz}	0.1	0.0429	0.0334	0.0537	Uniform	$0-0.2$
			Hungary			
ν_a^{hu}	0.1	0.0199	0.0129	0.0274	Uniform	$0-0.2$
ν_g^{hu}	0.1	0.0270	0.0178	0.0370	Uniform	$0-0.2$
ν_r^{hu}	0.1	0.0139	0.0099	0.0179	Uniform	$0-0.2$
ν_ξ^{hu}	0.1	0.1715	0.1460	0.1999	Uniform	$0-0.2$
ν_i^{hu}	0.1	0.1707	0.1433	0.2000	Uniform	$0-0.2$
ν_h^{hu}	0.1	0.0901	0.0603	0.1187	Uniform	$0-0.2$
			Poland			
ν_a^{pl}	0.1	0.0116	0.0074	0.0157	Uniform	$0-0.2$
ν_g^{pl}	0.1	0.0233	0.0117	0.0337	Uniform	$0-0.2$
ν_r^{pl}	0.1	0.0161	0.0107	0.0213	Uniform	$0-0.2$
ν_ξ^{pl}	0.1	0.1538	0.1246	0.1956	Uniform	$0-0.2$
ν_i^{pl}	0.1	0.1680	0.1424	0.2000	Uniform	$0-0.2$
ν_h^{pl}	0.1	0.0445	0.0328	0.0553	Uniform	$0-0.2$
			Slovenia			
ν_a^{si}	0.1	0.0221	0.0157	0.0297	Uniform	$0-0.2$
ν_g^{si}	0.1	0.0268	0.0188	0.0359	Uniform	$0-0.2$
ν_r^{si}	0.1	0.0104	0.0068	0.0138	Uniform	$0-0.2$
ν_ξ^{si}	0.1	0.1778	0.1605	0.2000	Uniform	$0-0.2$

ν_i^{si}	0.1	0.1565	0.1175	0.1917	Uniform	$0-0.2$
ν_h^{si}	0.1	0.0463	0.0328	0.0584	Uniform	$0-0.2$
			Slovakia			
ν_a^{sk}	0.1	0.0590	0.0201	0.0951	Uniform	$0-0.2$
ν_g^{sk}	0.1	0.1249	0.0853	0.1535	Uniform	$0-0.2$
ν_r^{sk}	0.1	0.0156	0.0112	0.0204	Uniform	$0-0.2$
ν_ξ^{sk}	0.1	0.1618	0.1240	0.1927	Uniform	$0-0.2$
ν_i^{sk}	0.1	0.1581	0.1248	0.1856	Uniform	$0-0.2$
ν_h^{sk}	0.1	0.0650	0.0498	0.0806	Uniform	$0-0.2$

Table 3.4 shows the results of the estimation. Most of the estimated parameters and standard deviations are very close to the benchmark estimation. From the variance decomposition (see Table 3.5), the role of global and local trend shocks remain important factors for the GDP and demand growth. The country-level investment-specific shocks still explain most of the country-level investment growth. Once the real interest rate is observed, the role of the risk premium (local and global as well) is more limited, and, instead of the risk premium shock, the transitory technology shock is another contributor to the domestic demand decomposition. Table 3.6 compares the variance decomposition of the unobserved from the benchmark and observed real interest rate from the alternative estimations. In both estimations, global and local premium shocks are the key determinants of the interest rate volatility. However, if we observe the real interest rate, the overall volatility of the rates and the contribution of the estimated risk premium shocks is lower. Hence, in the alternative estimation, the less volatile risk premium shocks are insufficient to explain the distribution of the domestic demands. In this sense, adding the real interest rate to the estimation puts more weight on the *the trend is the financial frictions augmented cycle* and the technology-driven interpretation of the economic fluctuations of the emerging economies.

Table 3.5: Results with observed interest rate: Variance decomposition

	ν_g	ν_r	ν_a^j	ν_g^j	ν_r^j	ν_ξ^j	ν_i^j	ν_h^j
				GDP growth				
CZ	25.47	3.37	43.3	3.55	5.61	1.17	0.66	16.85
HU	13.26	1.67	27.11	8.67	5.68	1.27	0.63	41.72
PL	27.07	3	18.48	13.17	13.73	2.1	1.01	21.45
SI	18.54	1.61	46.99	11.97	3.09	1.89	0.82	15.1
SK	2.75	0.41	51.92	38.49	1.76	0.23	0.15	4.3
				Consumption growth				
CZ	50.82	4.94	4.8	7.09	8.22	2.43	0.15	21.55
HU	25.07	1.98	2.61	16.4	6.73	1.63	0.1	45.49
PL	41.49	3	1.39	20.19	13.74	1.84	0.12	18.22
SI	38.86	3.12	5.15	25.08	5.98	2.93	0.16	18.71
SK	5.7	0.51	5.77	79.86	2.22	0.43	0.03	5.48
				Investment growth				
CZ	32.68	0.78	5	4.56	1.29	0.15	50.68	4.87
HU	15.61	0.3	2.73	10.21	1.01	0.06	58.15	11.92
PL	19.27	0.32	1.06	9.37	1.49	0.04	64.69	3.77
SI	19.1	0.34	3.8	12.33	0.64	0.07	60.01	3.71
SK	4.8	0.11	8.39	67.3	0.48	0..03	17.14	1.75
				Trade Balance to GDP				
CZ	25.44	0.69	27.57	3.55	1.14	4.34	29.27	8
HU	12.09	0.34	23.42	7.91	1.16	4.78	30.05	20.25
PL	18.64	0.52	14.68	9.07	2.39	6.44	39.89	8.37
SI	14.8	0.35	31.57	9.56	0.66	5.94	31.05	6.07
SK	3.04	0.09	42.63	42.67	0.39	0.99	7.81	2.38

The table shows the shock variance decomposition of the four key variables based on model simulations. The simulations use the baseline calibration, together with the posterior means of shock persistences and standard errors.

Table 3.6: Variance decomposition: Real interest rate in two specifications

	ν_g	ν_r	ν_a^j	ν_g^j	ν_r^j	ν_ξ^j	ν_i^j	ν_h^j
			Unobserved real interest rate					
CZ	1.14	46.47	0.73	0.92	48.43	0.3	1.81	0.2
HU	0.23	13	0.05	0.11	85.39	0.12	0.74	0.35
PL	0.62	40.51	0.1	0.38	55.49	0.2	1.26	1.44
SI	0.38	18.75	0.29	0.1	79.11	0.17	0.94	0.25
SK	0.36	16.51	0.28	0.18	80.51	0.14	1.83	0.19
			Observed real interest rate					
CZ	2.68	35.25	0.87	0.37	58.69	0.38	0.42	1.34
HU	1.22	20.95	0.88	0.8	71.34	0.45	0.47	3.9
PL	0.87	17.4	0.28	0.42	79.59	0.3	0.31	0.83
SI	2.04	31.57	1.62	1.32	60.48	0.77	0.66	1.55
SK	1.01	14.76	4.53	14.18	63.54	0.28	0.36	1.33

The table shows the shock variance decomposition of the four key variables based on model simulations. The simulations use the baseline calibration, together with the posterior means of shock persistences and standard errors.

3.6 Conclusion

In this paper, we used a version of the neoclassical growth model to understand the stochastic growth process of the post-socialist Central-Eastern-European economies. We estimated a version of the model with simple financial frictions and a working capital channel. We found that trend and persistent productivity shocks are the most important components behind fluctuations in GDP growth. Interest-premium shocks are crucial to understand consumption growth, and to a lesser extent other GDP components. Labor- and investment-specific shocks are important as well.

We allowed for a common component for the trend and interest-premium shocks. We showed that these can be related to observed EU 15 time series. We also found some preliminary evidence that EU funds played a major role in investment dynamics after 2004. Studying the role of EU funding in more detail is an important future research direction.

Many other questions remain, including the role of government spending and investment, the possibility of a structural break associated with the global financial crisis, and the role of expectations about future growth prospects. Our results nevertheless show that the stochastic neoclassical growth model, augmented with a few key frictions, is a useful tool to examine the growth process of emerging economies, and the CEE countries in particular.

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Summary and conclusion

This thesis consists of a series of papers, two of which investigate the role of demographic aging in macroeconomic behavior in Gertler-type overlapping generation models, the third estimating the shocks of the convergence process of five post-socialist Central-Eastern European economies.

Chapter 1 demonstrates that population aging does not necessarily co-exist with a decreasing interest rate, especially if the economic agent has bounded rational expectations. In accordance with the theoretical findings, the empirical results suggest that the interest rate decreases only in those countries where the agent expectation is close to the rational expectations, that is, agents have a long planning horizon, a relatively high level of patience, or are relatively financially literate.

Chapter 2 compares the cyclical properties of aging and young societies. Demographic aging redistributes and changes the wealth position of households. Aging impacts the monetary transmission mechanism in two ways: (1) the central bank becomes less efficient in influencing the output gap, and (2) the shrinking labor force increases the volatility of production costs and nominal variables. These changes affect the optimal monetary policies. Thus, if the central bank follows a simple optimal rule, it should react more strongly to nominal variables in an aging society than it does in a young one.

In the final chapter, we find that the real economic convergence of Central-Eastern European countries was driven by global and local productivity shocks. Nonetheless, financial shocks could be important for the composition of domestic demand. Post-financial crisis, investment-specific shocks (due to the inflow of European funds) and labor market shocks (after some labor market reforms) have become important factors in economic growth.

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Appendix A

Technical Appendix of the OLG model

In this technical appendix, first, we focus on solving the optimizing problems of the young and old generations; thus, we describe the pay-as-you-go pension system and the price and wage setting equations of the firms. At the end, we provide the normalized - detrended by population growth - equations and the steady state calculation of the model. Regarding any other technical detail, further information is available from the authors upon request.

A.1 Demography and overlapping generations

A.1.1 Demography

Total population (N_t) is equal to the sum of the number of old (retired) (N_t^O) and young (worker) people (N_t^Y):

$$\begin{aligned} N_t &= N_t^O + N_t^Y \\ N_t^Y &= (1-\omega_{t-1}^Y)N_{t-1}^Y + n_t N_{t-1}^Y \\ N_t^O &= (1-\omega_{t-1}^O)N_{t-1}^O + \omega_{t-1}^Y N_{t-1}^Y \end{aligned}$$

s_t denotes the ratio of the number of old and young people, while s_t^Y denotes the share of young people in the whole population:

$$\begin{aligned} s_t &= \frac{N_t^O}{N_t^Y} = \frac{(1-\omega_{t-1}^O)N_{t-1}^O + \omega_{t-1}^Y N_{t-1}^Y}{N_t^Y} = (1-\omega_{t-1}^O)\frac{N_{t-1}^O}{N_{t-1}^Y}\frac{N_{t-1}^Y}{N_t^Y} + \omega_{t-1}^Y \frac{N_{t-1}^Y}{N_t^Y} \\ &= \frac{(1-\omega_{t-1}^O)}{(1-\omega_{t-1}^Y + n_t)} s_{t-1} + \frac{\omega_{t-1}^Y}{(1-\omega_{t-1}^Y + n_t)} \\ s_t^Y &= \frac{N_t^Y}{N_t} = \frac{N_t^Y}{N_t^Y + N_t^O} = \frac{1}{1+\frac{N_t^O}{N_t^Y}} = \frac{1}{1+s_t} \end{aligned}$$

Then, we can express the growth rate of each cohort:

$$\begin{aligned} 1+g_t^{N,Y} &= \frac{N_t^Y}{N_{t-1}^Y} = \frac{(1-\omega_{t-1}^Y)N_{t-1}^Y + n_t N_{t-1}^Y}{N_{t-1}^Y} = 1-\omega_{t-1}^Y + n_t \\ 1+g_t^{N,O} &= \frac{N_t^O}{N_{t-1}^O} = \frac{(1-\omega_{t-1}^O)N_{t-1}^O + \omega_{t-1}^Y N_{t-1}^Y}{N_{t-1}^O} = (1-\omega_{t-1}^O) + \frac{\omega_{t-1}^Y}{s_{t-1}} \end{aligned}$$

Finally, population (and the BGP) growth follows as:

$$\begin{aligned} 1+g_t &= 1+g_t^N = \frac{N_t^Y + N_t^O}{N_{t-1}^Y + N_{t-1}^O} = \frac{\frac{N_t^Y}{N_{t-1}^Y} + \frac{N_t^O}{N_{t-1}^Y}}{\frac{N_{t-1}^Y}{N_{t-1}^Y} + \frac{N_{t-1}^O}{N_{t-1}^Y}} = \frac{1+g_t^{N,Y} + \frac{N_t^O}{N_{t-1}^Y}}{1+s_{t-1}} = \\ &= \frac{1+g_t^{N,Y} + \frac{N_t^O}{N_{t-1}^Y}\frac{N_t^Y}{N_t^Y}}{1+s_{t-1}} = \frac{1+g_t^{N,Y} + s_t(1+g_t^{N,Y})}{1+s_{t-1}} = (1+g_t^{N,Y})\frac{1+s_t}{1+s_{t-1}} \end{aligned}$$

A.1.2 Retired generation

First order conditions of a retired agent

'Retired' agent i of retired cohort a is one individual who retired a years ago. Each agent maximises the following Bellman equation:

$$V^O(B_{a-1,t-1}^O(i)) = \max\left\{\frac{1}{1-\gamma}C_{a,t}^O(i)^{1-\gamma} + \beta E_t(1-\omega_t^O)V^O(B_{a,t}^O(i))\right\}$$

subject to this budget constraint:

$$C_{a,t}^O(i) + (1-\omega_t^O)B_{a,t}^O(i) = (1+r_{t-1})B_{a-1,t-1}^O(i) + TR_{a,t}^{YO}(i)$$

where O denotes the retired cohort, the $TR_{a,t}^{YO}(i)$ is the pension benefit that was set by the government a years ago. Here, we assume that $TR_{n,t+n}^{YO}(i) = TR_{0,t}^{YO}(i)\ \forall n > 0$, i.e. the government in the pay-as-you-go pension system sets the real value of the benefit at the time of retirement, and provides the same real amount until the pensioner passes away. $C^O_{a,t}(i)$ is the level of individual consumption, and $B^O_{a,t}(i)$ is the individual risk-free bond.

First-order conditions:

$$\begin{aligned} C^O_{a,t}(i) &: \quad C^O_{a,t}(i)^{-\gamma} + \lambda^O_{a,t} = 0 \\ B^O_{a,t}(i) &: \quad \beta E_t(1-\omega^O_t)V^O_{B^O_{a,t}(i)} + E_t(1-\omega^O_t)\lambda^O_{a,t} = 0 \end{aligned}$$

One-period-ahead Envelope theorem:

$$E_t V_{B^O_{a,t}(i)} = -E_t \lambda^O_{a+1,t+1}(1+r_t)$$

The first-order conditions imply the Euler equation:

$$\beta E_t \frac{C^O_{a,t}(i)^\gamma}{C^O_{a+1,t+1}(i)^\gamma}(1+r_t) = 1$$

which can be rearranged:

$$E_t C^O_{a+1,t+1}(i) = C^O_{a,t}(i)\beta^{\frac{1}{\gamma}}(1+r_t)^{\frac{1}{\gamma}}$$

Based on the Euler-equation, all future individual retired consumptions follow:

$$E_t C^O_{a+n,t+n}(i) = C^O_{a,t}(i)\beta^{\frac{n}{\gamma}} E_t \prod_{k=1}^{n}(1+r_{t+k-1})^{\frac{1}{\gamma}}$$

Individual consumption of a retired agent

First, we derive the intertemporal budget constraint from the one-period budget constraint:

$$E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^O_{t+k-1})C^O_{a+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} = E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^O_{t+k-1})TR^{YO}_{a+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\ +(1+r_{t-1})B^O_{a-1,t-1}(i)$$

if $k > n$ and $r_{t+k} = 0$.

We can use the Euler equation for future consumptions:

$$\begin{aligned}
&E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^O_{t+k-1}) C^O_{a,t}(i) \beta^{\frac{n}{\gamma}} \prod_{k=1}^{n}(1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=1}^{n}(1+r_{t+k-1})} = \\
&= E_t \sum_{n=0}^{\infty} \frac{(1-\omega^O_{t+k-1})^n TR^{YO}_{a+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} + +(1+r_{t-1}) B^O_{a-1,t-1}(i)
\end{aligned}$$

If we rearrange, we get consumption of agent i of cohort a at time t as a function of the present value of pension benefits, other income and initial wealth:

$$\begin{aligned}
C^O_{a,t}(i) = & \frac{E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^O_{t+k-1}) TR^{YO}_{a+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})}}{E_t \sum_{n=0}^{\infty} \beta^{\frac{n}{\gamma}} \prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1}} + \\
& + \frac{(1+r_{t-1}) B^O_{a-1,t-1}(i)}{E_t \sum_{n=0}^{\infty} \beta^{\frac{n}{\gamma}} \prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1}}
\end{aligned}$$

Finally, using the assumption that $TR^{YO}_{n,t+n}(i) = TR^{YO}_{0,t}(i) \ \forall n > 0$:

$$\begin{aligned}
C^O_{a,t}(i) = & TR^{YO}_{a,t}(i) \frac{E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})}}{E_t \sum_{n=0}^{\infty} \beta^{\frac{n}{\gamma}} \prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1}} + \\
& + \frac{(1+r_{t-1}) B^O_{a-1,t-1}(i)}{E_t \sum_{n=0}^{\infty} \beta^{\frac{n}{\gamma}} \prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1}}
\end{aligned}$$

The denominators are the function of non-individual variables; thus, we can then we can denote it as an aggregate variable:

$$\begin{aligned}
\frac{1}{MPC^O_t} &= E_t \sum_{n=0}^{\infty} \beta^{\frac{n}{\gamma}} \prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1} \\
&= 1 + E_t \sum_{n=1}^{\infty} \beta^{\frac{n}{\gamma}} \prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1} \\
&= 1 + \beta^{\frac{1}{\gamma}}(1-\omega^O_t)(1+r_t)^{\frac{1}{\gamma}-1} E_t \sum_{n=1}^{\infty} \beta^{\frac{n}{\gamma}} \prod_{k=2}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1} \\
&= 1 + \beta^{\frac{1}{\gamma}}(1-\omega^O_t)(1+r_t)^{\frac{1}{\gamma}-1} E_t \frac{1}{MPC^O_{t+1}}
\end{aligned}$$

Using the same recursive substitution for the future expected pension benefit, the consumption function of agent i of cohort a at time t is:

$$C^O_{a,t}(i) = MPC^O_t \Omega^O_t TR^{YO}_{a,t}(i) + MPC^O_t (1+r_{t-1}) B^O_{a-1,t-1}(i)$$

where Ω^O is the discount factor of the retired households:

$$\Omega_t^O = 1 + E_t \frac{1-\omega_t^O}{1+r_t} \Omega_{t+1}^O$$

Aggregate consumption of the retired cohort

Aggregate consumption is equal to the sum of pension benefits, other income and initial wealth:

$$\begin{aligned} \sum_{a=0}^{\infty} N_{a,t}^O(i) C_{a,t}^O(i) &= MPC_t^O \Omega_t^O \sum_{a=0}^{\infty} N_{a,t}^O(i) TR_{a,t}^{YO}(i) + \\ &+ MPC_t^O (1+r_{t-1}) \sum_{a=0}^{\infty} N_{a,t}^O(i) B_{a-1,t-1}^O(i) \end{aligned}$$

First, the number of old people declines over time:

$$\begin{aligned} N_{a+1,t}^O &= (1-\omega_{t-1}^O) N_{a,t-1}^O \\ N_{a+2,t}^O &= (1-\omega_{t-1}^O)(1-\omega_{t-2}^O) N_{a,t-2}^O \\ &\vdots \end{aligned}$$

and

$$N_t^O = \sum_{a=0}^{\infty} N_{a,t}^O$$

Now, we can express aggregate pension income in period t of those who retire at period t, one period before, etc.:

$$\begin{aligned} N_{0,t}^O TR_{0,t}^{YO}(i) &= TR_t^{YO} \\ N_{1,t}^O TR_{1,t}^{YO}(i) &= (1-\omega_{t-1}^O) N_{0,t-1}^O TR_{0,t-1}^{YO}(i) = (1-\omega_{t-1}^O) TR_{t-1}^{YO} \\ N_{2,t}^O TR_{2,t}^{YO}(i) &= (1-\omega_{t-1}^O)(1-\omega_{t-2}^O) N_{0,t-2}^O TR_{0,t-2}^{YO}(i) = (1-\omega_{t-1}^O)(1-\omega_{t-2}^O) TR_{t-2}^{YO} \\ &\vdots \end{aligned}$$

using $TR_{n,t+n}^{YO}(i) = TR_{0,t}^{YO}(i)\ \forall n > 0$ again.

Then, adding up all pensions implies:

$$\begin{aligned} TR_t &\equiv \sum_{a=0}^{\infty} N_{a,t}^O(i) TR_{a,t}^{YO}(i) = TR_t^{YO} + (1-\omega_{t-1}^O) TR_{t-1}^{YO} + ... \\ &= TR_t^{YO} + (1-\omega_{t-1}^O) TR_{t-1} \end{aligned}$$

Now, aggregate consumption of the retired cohort cohort is defined as:

$$C_t^O = \sum_{a=0}^{\infty} N_{a,t}^O C_{a,t}^O(i)$$

while total savings of the retired is:

$$\sum_{a=0}^{\infty} N_{a,t}^O B_{a,t-1}^O(i) = N_{0,t}^O B_{0,t-1}^O(i) + \sum_{a=1}^{\infty} N_{a,t}^O B_{a,t-1}^O(i)$$

Here, we need to be careful with the just-retired agents: they were young one period before without knowing about their next period retirement. We can use the law of large numbers to get the following expression: $N_{0,t}^O = \omega_{t-1}^Y N_{t-1}^Y$:

$$N_{0,t}^O B_{0,t-1}^O(i) = N_{0,t}^O \sum_{b=1}^{\infty} B_{b,t-1}^{Y,last}(i) \simeq \omega_{t-1}^Y N_{t-1}^Y \frac{B_{t-1}^Y}{N_{t-1}^Y}$$

where the *last* refers to the fact that those who retire today spent their last year in the young cohort in the previous year.

Then, from $t-1$ to t it is easy to see that: $\sum_{a=1}^{\infty} N_{a,t}^O = \sum_{a=1}^{\infty}(1-\omega_{t-1}^O) N_{a-1,t-1}^O$ which implies that

$$\sum_{a=0}^{\infty} N_{a,t}^O B_{a,t-1}^O(i) = \omega_{t-1}^Y B_{t-1}^Y + \sum_{a=1}^{\infty}(1-\omega_{t-1}^O) N_{a-1,t-1}^O B_{a,t-1}^O(i)$$

Here, the second term means that only those retired agents accummulate savings who expect to survive the next period. Hence, the amount of aggregate old-age savings from the previous period is $B_{t-1}^O = \sum_{a=1}^{\infty}(1-\omega_{t-1}^O) N_{a-1,t-1}^O B_{a,t-1}^O(i)$. Then, overall savings of the retired cohort in period t can be expressed easily by adding just-retired savings from the previous period's young cohorts:

$$\sum_{a=0}^{\infty} N_{a,t}^{O} B_{a,t-1}^{O}(i) = \omega_{t-1}^{Y} B_{t-1}^{Y} + B_{t-1}^{O}$$

As a last step, we put together all parts of the equation, so, aggregate consumption of formal goods of the retired cohort is:

$$C_t^O = MPC_t^O \Omega_t^O TR_t + (1 + r_{t-1}) MPC_t^O (\omega_{t-1}^{Y} B_{t-1}^{Y} + B_{t-1}^{O}).$$

A.1.3 Young generation

First-order conditions of a young agent

'Young' agent i of young cohort b is one individual of its cohort who started to work (was born) b years ago. The Bellman equation of a young individual is:

$$\begin{aligned} V_t^Y(B_{b-1,t-1}^{Y}(i)) = \max \Bigg\{ \frac{1}{1-\gamma} \left\{ C_{b,t}^{Y}(i)^{\sigma} \left(1 - L_{b,t}(i)\right)^{1-\sigma} \right\}^{1-\gamma} \\ + E_t \beta \left((1 - \omega_t^Y) V_{t+1}^{Y}(B_{b,t}^{Y}(i)) + \omega_t^Y V_{t+1}^{O}(B_{b,t}^{YO}(i)) \right) \Bigg\} \end{aligned}$$

while the budget constraint is:

$$\begin{aligned} & C_{b,t}^{Y}(i) + (1 - \omega_t^Y) B_{b,t}^{Y}(i) + \omega_t^Y B_{b,t}^{YO}(i) = \\ & = (1 + r_{t-1}) B_{b-1,t-1}^{Y}(i) + w_t L_{b,t}(i) + Profit_{b,t}(i) - Tax_{b,t}(i) \end{aligned}$$

where $C^Y(i)$ denotes the young individual's consumption, $L(i)$ is her labor supply, σ is the weight of consumption in the one-period utility function, β is the cohort-specific discount factor, $Tax(i)$ is the lump-sum tax, $Profit(i)$ denotes the dividend from firms, and w is the real wage. A young agent saves for two possible future states, we assume state contingent bonds, for saving young the workers save into $B^Y(i)$; and for the next period retired-self the worker today saves into $B^{YO}(i)$. We assume full insurance, which means that any outcome will happen in the future period, and the workers' previous period savings are transferred into their own young self or retired self account.

First order conditions:

$$
\begin{aligned}
C^Y_{b,t}(i) &: \left\{C^Y_{b,t}(i)^\sigma\left(1-L_{b,t}(i)\right)^{1-\sigma}\right\}^{-\gamma}\sigma C^Y_{b,t}(i)^{\sigma-1}\left(1-L_{b,t}(i)\right)^{1-\sigma}+\lambda^Y_{b,t}=0\\
L_{b,t}(i) &: -\left\{C^Y_{b,t}(i)^\sigma\left(1-L_{b,t}(i)\right)^{1-\sigma}\right\}^{-\gamma}(1-\sigma)C^Y_{b,t}(i)^\sigma\left(1-L_{b,t}(i)\right)^{-\sigma}-w_t\lambda^Y_{b,t}=0\\
B^Y_{b,t}(i) &: \beta E_t(1-\omega^Y_t)V^Y_{B^Y_{b,t}}+E_t(1-\omega^Y_t)\lambda^Y_{b,t}=0\\
B^{YO}_{b,t}(i) &: \beta E_t\omega^Y_t V^Y_{B^{YO}_{b,t}}+E_t\omega^Y_t\lambda^Y_{b,t}=0
\end{aligned}
$$

One-period-ahead Envelope theorem:

$$E_tV_{B^Y_{b,t}}=-E_t\lambda^Y_{b+1,t+1}(1+r_t)$$

Also, from the retired agent's optimization we know that:

$$E_tV_{B^{YO}_{b,t}}=-E_t\lambda^O_{0,t+1}(1+r_t)=-E_t\lambda^O_{b+1,t+1}(1+r_t)$$

where $E_t\lambda^O_{0,t+1}=E_t\lambda^O_{b+1,t+1}$ because someone who was young in t gets retired in $t+1$. Thus, the Euler equations of the young individual are:

$$
\begin{aligned}
\beta E_t\frac{\left(C^Y_{b+1,t+1}(i)^\sigma\left(1-L_{b+1,t+1}(i)\right)^{1-\sigma}\right)^{-\gamma}C^Y_{b+1,t+1}(i)^{\sigma-1}\left(1-L_{b+1,t+1}(i)\right)^{1-\sigma}}{\left(C^Y_{b,t}(i)^\sigma\left(1-L_{b,t}(i)\right)^{1-\sigma}\right)^{-\gamma}C^Y_{b,t}(i)^{\sigma-1}\left(1-L_{b,t}(i)\right)^{1-\sigma}}(1+r_t) &= 1\\
\beta E_t\frac{\left(C^O_{b+1,t+1}(i)\right)^{-\gamma}}{\left(C^Y_{b,t}(i)^\sigma\left(1-L_{b,t}(i)\right)^{1-\sigma}\right)^{-\gamma}\sigma C^Y_{b,t}(i)^{\sigma-1}\left(1-L_{b,t}(i)\right)^{1-\sigma}}(1+r_t) &= 1
\end{aligned}
$$

Expressing the leisure from labor supply:

$$
\begin{aligned}
\frac{C^Y_{b,t}(i)}{1-L_{b,t}(i)} &= \frac{\sigma}{1-\sigma}w_t\\
1-L_{b,t}(i) &= \left(\frac{\sigma}{1-\sigma}w_t\right)^{-1}C^Y_{b,t}(i)
\end{aligned}
$$

Plugging back in Euler equations:

$$
\begin{aligned}
\beta E_t\frac{C^Y_{b+1,t+1}(i)^{-\gamma}\left(\frac{\sigma}{1-\sigma}w_{t+1}\right)^{\gamma(1-\sigma)-(1-\sigma)}}{C^Y_{b,t}(i)^{-\gamma}\left(\frac{\sigma}{1-\sigma}w_t\right)^{\gamma(1-\sigma)-(1-\sigma)}}(1+r_t) &= 1\\
\frac{\beta}{\sigma}E_t\frac{C^O_{b+1,t+1}(i)^{-\gamma}}{C^Y_{b,t}(i)^{-\gamma}\left(\frac{\sigma}{1-\sigma}w_t\right)^{\gamma(1-\sigma)-(1-\sigma)}}(1+r_t) &= 1
\end{aligned}
$$

Rearranging:

$$
\begin{aligned}
E_t C^{Y}_{b+1,t+1}(i) &= C^{Y}_{b,t}(i)\beta^{\frac{1}{\gamma}}(1+r_t)^{\frac{1}{\gamma}}\Lambda^{Y}_{t} \\
E_t C^{O}_{0,t+1}(i) &= C^{Y}_{b,t}(i)\beta^{\frac{1}{\gamma}}(1+r_t)^{\frac{1}{\gamma}}\Lambda^{YO}_{t}
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda^{Y}_{t} &= E_t\left(\frac{w_{t+1}}{w_t}\right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \\
\Lambda^{YO}_{t} &= \left(\frac{1}{\sigma}\right)^{\frac{1}{\gamma}}\left(\frac{1}{\frac{\sigma}{1-\sigma}w_t}\right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)}
\end{aligned}
$$

Additionally, we can express each period's consumption as a function of period-t consumption and the discount rate:

$$
E_t C^{Y}_{b+n,t+n}(i) = C^{Y}_{b,t}(i)\beta^{\frac{n}{\gamma}} E_t \prod_{k=1}^{n}(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda^{Y}_{t+k-1}
$$

Individual consumption of a young agent

First of all, we would like to stress that one needs to be careful when deriving the young agent's individual consumption because old-age incomes and expenditures must be taken into account, too. Moreover, the young agents also consider the probability of retirement, for instance, in period t the probability that a young agent becomes retired in period $t+1$ is ω^{Y}_{t}, while the probability that the same agent becomes retired in period $t+2$ is $(1-\omega^{Y}_{t})\omega^{Y}_{t+1}$. So, the first term of the left-hand side of this equation shows the stream of lifetime consumption if the agent stays young; thus, from the second term onwards she retires with some probability in each period:

$$
\begin{aligned}
&E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1}) C^Y_{b+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\
&+E_t \omega^Y_t \left(\sum_{n=1}^{\infty} \frac{\prod_{k=2}^{n}(1-\omega^O_{t+k-1}) C^O_{n-1,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + \\
&+E_t (1-\omega^Y_t)\omega^Y_{t+1} \left(\sum_{n=2}^{\infty} \frac{\prod_{k=3}^{n}(1-\omega^O_{t+k-1}) C^O_{n-2,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + ... \\
&= E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1}) \left[w_{t+n} L_{b+n,t+n}(i) + Profit_{b+n,t+n}(i) - Tax_{b+n,t+n}(i) \right]}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\
&+(1+r_{t-1}) B^Y_{b-1,t-1}(i) + \\
&+E_t \omega^Y_t \sum_{n=1}^{\infty} TR^{YO}_{n-1,t+n}(i) \frac{\prod_{k=2}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\
&+E_t (1-\omega^Y_t)\omega^Y_{t+1} \sum_{n=2}^{\infty} TR^{YO}_{n-2,t+n}(i) \frac{\prod_{k=3}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\
&+E_t (1-\omega^Y_t)(1-\omega^Y_{t+1})\omega^Y_{t+2} \sum_{n=3}^{\infty} TR^{YO}_{n-3,t+n}(i) \frac{\prod_{k=4}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + ...
\end{aligned}
$$

Based on labor supply curve, we can express labor income as a function of real wage and consumption:

$$
w_t L_{b,t}(i) = w_t - \frac{1-\sigma}{\sigma} C_{b,t}(i) = w_t + C_{b,t}(i) - \frac{1}{\sigma} C_{b,t}(i)
$$

We can substitute out the labor income and rearrange the equations:

$$
\begin{aligned}
&\frac{1}{\sigma} E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1}) C^Y_{b+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\
&+E_t \omega^Y_t \left(\sum_{n=1}^{\infty} \frac{\prod_{k=2}^{n}(1-\omega^O_{t+k-1}) C^O_{n-1,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + \\
&+E_t (1-\omega^Y_t)\omega^Y_{t+1} \left(\sum_{n=2}^{\infty} \frac{\prod_{k=3}^{n}(1-\omega^O_{t+k-1}) C^O_{n-2,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + ... \\
&= E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1}) \left[w_{t+n} + Profit_{b+n,t+n}(i) - Tax_{b+n,t+n}(i) \right]}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\
&+(1+r_{t-1}) B^Y_{b-1,t-1}(i) + \\
&+E_t \omega^Y_t \sum_{n=1}^{\infty} TR^{YO}_{n-1,t+n}(i) \frac{\prod_{k=2}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\
&+E_t (1-\omega^Y_t)\omega^Y_{t+1} \sum_{n=2}^{\infty} TR^{YO}_{n-2,t+n}(i) \frac{\prod_{k=3}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + ...
\end{aligned}
$$

Based on the Euler equations, we can express expected future consumptions. Let's consider an agent who is young in period t; thus, her consumption functions in the next periods after retiring are:

$$E_t C^O_{n,t+n+1}(i) = E_t C^O_{0,t+1}(i)\beta^{\frac{n}{\gamma}} \prod_{k=2}^{n} (1+r_{t+k-1})^{\frac{1}{\gamma}}$$

On the other hand, if the agent stays young in period $t+1$ and retires after that, her future old-age consumptions are:

$$E_t C^O_{n,t+n+2}(i) = E_t C^O_{0,t+2}(i)\beta^{\frac{n}{\gamma}} \prod_{k=3}^{n} (1+r_{t+k-1})^{\frac{1}{\gamma}}$$

Now, we plug them in the intertemporal budget constraint:

$$\begin{aligned}
&\frac{1}{\sigma} E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1})C^Y_{b+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\
&+E_t\omega^Y_t \left(\sum_{n=1}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=2}^{n}(1-\omega^O_{t+k-1})C^O_{0,t+1}(i)(1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + \\
&+E_t(1-\omega^Y_t)\omega^Y_{t+1} \left(\sum_{n=2}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=3}^{n}(1-\omega^O_{t+k-1})C^O_{0,t+2}(i)(1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + ... \\
&= E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1})\left[w_{b+n,t+n} + Profit_{b+n,t+n}(i) - Tax_{b+n,t+n}(i)\right]}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\
&+(1+r_{t-1})B^Y_{b-1,t-1}(i) + \\
&+E_t\omega^Y_t \sum_{n=1}^{\infty} TR^{YO}_{n-1,t+n}(i) \frac{\prod_{k=2}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\
&+E_t(1-\omega^Y_t)\omega^Y_{t+1} \sum_{n=2}^{\infty} TR^{YO}_{n-2,t+n}(i) \frac{\prod_{k=3}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + ...
\end{aligned}$$

After that, we use the other Euler equation (the one that shows the substitution between period-t

young and period-$t+1$ old-age consumption):

$$\frac{1}{\sigma} E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega_{t+k-1}^{Y}) C_{b+n,t+n}^{Y}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t \omega_t^{Y} \left(\sum_{n=1}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=2}^{n}(1-\omega_{t+k-1}^{O}) C_{b,t}^{Y}(i)(1+r_t)^{\frac{1}{\gamma}} \Lambda_t^{YO} (1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) +$$
$$+E_t (1-\omega_t^{Y}) \omega_{t+1}^{Y} \left(\sum_{n=2}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=3}^{n}(1-\omega_{t+k-1}^{O}) C_{b+1,t+1}^{Y}(i)(1+r_{t+1})^{\frac{1}{\gamma}} \Lambda_{t+1}^{YO} (1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + ...$$
$$= E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega_{t+k-1}^{Y}) \left[w_{b+n,t+n} + Profit_{b+n,t+n}(i) - Tax_{b+n,t+n}(i) \right]}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+(1+r_{t-1}) B_{b-1,t-1}^{Y}(i) +$$
$$+E_t \omega_t^{Y} \sum_{n=1}^{\infty} TR_{n-1,t+n}^{YO}(i) \frac{\prod_{k=2}^{n}(1-\omega_{t+k-1}^{O})}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t (1-\omega_t^{Y}) \omega_{t+1}^{Y} \sum_{n=2}^{\infty} TR_{n-2,t+n}^{YO}(i) \frac{\prod_{k=3}^{n}(1-\omega_{t+k-1}^{O})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + ...$$

Concentrating on consumptions:

$$\frac{1}{\sigma} E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega_{t+k-1}^{Y}) C_{b+n,t+n}^{Y}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t \omega_t^{Y} \left(\sum_{n=1}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=2}^{n}(1-\omega_{t+k-1}^{O}) C_{b,t}^{Y}(i)(1+r_t)^{\frac{1}{\gamma}} \Lambda_t^{YO} (1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) +$$
$$+E_t (1-\omega_t^{Y}) \omega_{t+1}^{Y} \left(\sum_{n=2}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=3}^{n}(1-\omega_{t+k-1}^{O}) C_{b+1,t+1}^{Y}(i)(1+r_{t+1})^{\frac{1}{\gamma}} \Lambda_{t+1}^{YO} (1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + ...$$

We rearrange:

$$\frac{1}{\sigma} C_{b,t}^{Y}(i) + C_{b,t}^{Y}(i) E_t \omega_t^{Y} \left(\sum_{n=1}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=2}^{n}(1-\omega_{t+k-1}^{O})(1+r_t)^{\frac{1}{\gamma}} \Lambda_t^{YO} (1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) +$$
$$+E_t \frac{1}{\sigma} C_{b+1,t+1}^{Y}(i) \frac{(1-\omega_t^{Y})}{(1+r_t)} +$$
$$+E_t C_{b+1,t+1}^{Y}(i)(1-\omega_t^{Y}) \omega_{t+1}^{Y} \left(\sum_{n=2}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=3}^{n}(1-\omega_{t+k-1}^{O})(1+r_{t+1})^{\frac{1}{\gamma}} \Lambda_{t+1}^{YO} (1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + ...$$

Simplifying before recursive substitution:

$$
\begin{aligned}
&\frac{1}{\sigma}C^Y_{b,t}(i) + C^Y_{b,t}(i)E_t\frac{\omega^Y_t(1+r_t)^{\frac{1}{\gamma}}\beta^{\frac{1}{\gamma}}\Lambda^{YO}_t}{1+r_t}\left(\sum_{n=1}^{\infty}\frac{\beta^{\frac{n}{\gamma}}\prod_{k=2}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=2}^{n}(1+r_{t+k-1})}\right)+\\
&+E_t\frac{1}{\sigma}C^Y_{b+1,t+1}(i)\frac{(1-\omega^Y_t)}{(1+r_t)}+\\
&+E_tC^Y_{b+1,t+1}(i)(1-\omega^Y_t)\frac{\omega^Y_{t+1}(1+r_{t+1})^{\frac{1}{\gamma}}\beta^{\frac{1}{\gamma}}\Lambda^{YO}_{t+1}}{(1+r_t)(1+r_{t+1})}\left(\sum_{n=2}^{\infty}\frac{\beta^{\frac{n}{\gamma}}\prod_{k=3}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}}}{\prod_{k=3}^{n}(1+r_{t+k-1})}\right)+...
\end{aligned}
$$

Now, we can use $\frac{1}{MPC^O_{t+1}}$ from the retired agents' optimization:

$$
\begin{aligned}
&C^Y_{b,t}(i)\left[\frac{1}{\sigma}+E_t\beta^{\frac{1}{\gamma}}\omega^Y_t(1+r_t)^{\frac{1}{\gamma}-1}\Lambda^{YO}_t\frac{1}{MPC^O_{t+1}}\right]+\\
&+E_tC^Y_{b+1,t+1}(i)\frac{(1-\omega^Y_t)}{(1+r_t)}\left[\frac{1}{\sigma}+\beta^{\frac{1}{\gamma}}\omega^Y_{t+1}(1+r_{t+1})^{\frac{1}{\gamma}-1}\Lambda^{YO}_{t+1}\frac{1}{MPC^O_{t+2}}\right]+...
\end{aligned}
$$

And, using the Euler equation again (to have period-t consumption only):

$$
E_tC^Y_{b+n,t+n}(i) = C^Y_{b,t}(i)\beta^{\frac{n}{\gamma}}E_t\prod_{k=1}^{n}(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda^Y_{t+k-1}
$$

Finally:

$$
\begin{aligned}
&C^Y_{b,t}(i)\left[\frac{1}{\sigma}+E_t\omega^Y_t(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\Lambda^{YO}_t\frac{1}{MPC^O_{t+1}}\right]+\\
&C^Y_{b,t}(i)E_t(1+r_t)^{\frac{1}{\gamma}}\beta^{\frac{1}{\gamma}}\Lambda^Y_t\frac{(1-\omega^Y_t)}{(1+r_t)}\left[\frac{1}{\sigma}+\omega^Y_{t+1}(1+r_{t+1})^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\Lambda^{YO}_{t+1}\frac{1}{MPC^O_{t+2}}\right]+\\
&C^Y_{b,t}(i)E_t(1+r_t)^{\frac{1}{\gamma}}\beta^{\frac{2}{\gamma}}\Lambda^Y_t(1+r_{t+1})^{\frac{1}{\gamma}}\Lambda^Y_{t+1}\frac{(1-\omega^Y_t)(1-\omega^Y_{t+1})}{(1+r_t)(1+r_{t+1})}\left[\frac{1}{\sigma}+\omega^Y_{t+2}(1+r_{t+2})^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\Lambda^{YO}_{t+2}\frac{1}{MPC^O_{t+3}}\right]+...
\end{aligned}
$$

which is equal to:

$$
C^Y_{b,t}(i)\frac{1}{MPC^Y_t}
$$

where:

$$
\frac{1}{MPC^Y_t} = \frac{1}{\sigma}+E_t\beta^{\frac{1}{\gamma}}(1+r_t)^{\frac{1}{\gamma}-1}\left[(1-\omega^Y_t)\Lambda^Y_t\frac{1}{MPC^Y_{t+1}}+\omega^Y_t\Lambda^{YO}_t\frac{1}{MPC^O_{t+1}}\right]
$$

Similarly, the young agent's budget constraint contains old-age income items, i.e., expected revenues from the pension fund.

$$\mathcal{I}^{YO}_{b,t}(i) = E_t\omega^Y_t\Omega^O_{t+1}TR^{YO}_{0,t+1}(i) + E_t\frac{(1-\omega^Y_t)\omega^Y_{t+1}}{(1+r_{t+1})}\Omega^O_{t+2}TR^{YO}_{0,t+2}(i) +$$
$$+E_t\frac{(1-\omega^Y_t)(1-\omega^Y_{t+1})\omega^Y_{t+2}}{(1+r_{t+1})(1+r_{t+2})}\Omega^O_{t+3}TR^{PG,YO}_{0,t+3}(i) + ...$$

Again, we use that $TR^{PG,YO}_{n,t+n}(i) = TR^{PG,YO}_{0,t}(i)\ \forall n > 0$. In a recursive way, it is:

$$\mathcal{I}^{YO}_{b,t}(i) = E_t\omega^Y_t TR^{YO}_{0,t+1}(i)\Omega_{t+1} + E_t\frac{(1-\omega^Y_t)}{(1+r_{t+1})}\mathcal{I}^{YO}_{b+1,t+1}(i)$$

Furthermore, young-age income is:

$$\begin{aligned}\mathcal{I}^{Y}_{b,t}(i) &= E_t\sum_{n=0}^{\infty}\frac{\prod_{k=1}^{\infty}(1-\omega^Y_{t+k-1})^n\left[w_{t+n} + Profit_{b+n,t+n}(i) - Tax_{b+n,t+n}(i)\right]}{\prod_{k=1}^{n}(1+r_{t+k-1})} = \\ &= w_t + Profit_{b,t}(i) - Tax_{b,t}(i) + E_t\frac{1-\omega^Y_t}{1+r_t}\mathcal{I}^Y_{b+1,t+1}(i)\end{aligned}$$

If we add the present value of young income and expected pension benefits, we can introduce a new variable:

$$\begin{aligned}Inc^Y_{b,t}(i) &= \mathcal{I}^Y_{b,t}(i) + \frac{\mathcal{I}^{YO}_{b,t}(i)}{1+r_t} \\ &= w_t + Profit_{b,t}(i) - Tax_{b,t}(i) + E_t\frac{\omega^Y_t}{1+r_t}TR^{YO}_{0,t+1}(i)\Omega_{t+1} + E_t\frac{1-\omega^Y_t}{1+r_t}Inc^Y_{b+1,t+1}(i)\end{aligned}$$

Thus, the individual consumption function of agent i of cohort b in period t is:

$$C^Y_{b,t}(i) = MPC^Y_t Inc^Y_{b,t}(i) + (1+r_{t-1})MPC^Y_t B^Y_{b-1,t-1}(i)$$

Introducing a new variable for life-time income, and using marginal propensity to consume:

$$\begin{aligned}C^Y_{b,t}(i) &= MPC^Y_t Inc^Y_{b,t}(i) + MPC^Y_t(1+r_{t-1})B^Y_{b-1,t-1}(i) \\ Inc^Y_{b,t}(i) &= w_t + Profit_{b,t}(i) - Tax_{b,t}(i) + E_t\frac{\omega^Y_t}{1+r_t}TR^{YO}_{0,t+1}(i)\Omega_{t+1} + E_t\frac{1-\omega^Y_t}{1+r_t}Inc^Y_{b+1,t+1}(i) \\ \frac{1}{MPC^Y_t} &= \frac{1}{\sigma} + E_t(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\left[(1-\omega^Y_t)\Lambda^Y_t\frac{1}{MPC^Y_{t+1}} + \omega^Y_t\Lambda^{YO}_t\frac{1}{MPC^O_{t+1}}\right]\end{aligned}$$

Aggregate consumption of the young cohort

As a first step, we need to express the total number of young people. If $N_{b,t}^Y$ is the number of b-year old workers, the total number of workers is:

$$N_t^Y = \sum_{b=0}^{\infty} N_{b,t}^Y$$

Following the previous idea, we sum up all consumptions, incomes and savings:

$$\sum_{b=0}^{\infty} N_{b,t}^Y C_{b,t}^Y(i) = MPC_t^Y \sum_{b=0}^{\infty} N_{b,t}^Y Inc_{b,t}^Y(i) + (1+r_{t-1}) MPC_t^Y \sum_{b=0}^{\infty} N_{b,t}^Y B_{b-1,t-1}^Y(i)$$

where we note that the new young workers in time t have zero savings from the previous period.

$$\sum_{b=0}^{\infty} N_{b,t}^Y C_{b,t}^Y(i) = MPC_t^Y \sum_{b=0}^{\infty} N_{b,t}^Y Inc_{b,t}^Y(i) + (1+r_{t-1}) MPC_t^Y \sum_{b=1}^{\infty} N_{b,t}^Y \frac{N_{b-1,t-1}^Y}{N_{b-1,t-1}^Y} B_{b-1,t-1}^Y(i)$$

Rearranging gives us:

$$\sum_{b=0}^{\infty} N_{b,t}^Y C_{b,t}^Y(i) = MPC_t^Y \sum_{b=0}^{\infty} N_{b,t}^Y Inc_{b,t}^Y(i) + (1+r_{t-1}) MPC_t^Y (1-\omega_{t-1}^Y) \sum_{b=1}^{\infty} N_{b-1,t-1}^Y B_{b-1,t-1}^Y(i)$$

Aggregate values are defined as:

$$\begin{aligned} C_t^Y &\equiv \sum_{b=0}^{\infty} N_{b,t}^Y C_{b,t}^Y(i) \\ B_{t-1}^Y &\equiv \sum_{b=1}^{\infty} N_{b-1,t-1}^Y B_{b-1,t-1}^Y(i) \\ Inc_t^Y &\equiv \sum_{b=0}^{\infty} N_{b,t}^Y Inc_{b,t}^Y(i) \end{aligned}$$

It is important to note that in each period, independently from the survival probabilities, each young agent saves for the next period; hence, the overall savings $B_{t-1}^Y = \sum_{b=1}^{\infty} N_{b-1,t-1}^Y B_{b-1,t-1}^Y(i)$ are divided among those who remain young and get retired.

As a result, the aggregate consumption functions are:

$$C_t^Y = MPC_t^Y Inc_t^Y + (1+r_{t-1}) MPC_t^Y (1-\omega_{t-1}^Y) B_{t-1}^Y$$

Now we need to aggregate the supporting variables as well. First of all, we rename individual contemporary income as follows:

$$Inc^{Y}_{b,t}(i) = \mathcal{I}^{Y}_{b,t}(i) + \frac{1}{1+r_t}\mathcal{I}^{YO}_{b,t}(i)$$

Aggregating gives us:

$$Inc^{Y}_{t} = \mathcal{I}^{Y}_{t} + \frac{1}{1+r_t}\mathcal{I}^{YO}_{t}$$

After aggregating an rearranging we get:

$$\sum_{b=0}^{\infty} N^{Y}_{b,t}\mathcal{I}^{Y}_{b,t}(i) = \sum_{b=0}^{\infty} N^{Y}_{b,t}\left(w_t + Profit_{b,t}(i) - Tax_{b+n,t+n}(i)\right) + E_t\frac{(1-\omega^{Y}_{t})}{1+r_t}\sum_{b=0}^{\infty} N^{Y}_{b,t}\mathcal{I}^{Y}_{b+1,t+1}(i)$$
$$= \sum_{b=0}^{\infty} N^{Y}_{b,t}\left(w_t + Profit_{b,t}(i) - Tax_{b,t}(i)\right) + E_t\frac{1}{1+r_t}\sum_{b=0}^{\infty} N^{Y}_{b+1,t+1}\mathcal{I}^{Y}_{b+1,t+1}(i)$$

Because $\mathcal{I}^{Y}_{t+1}$ contains the income of the new-born people as well, the last term can be rearranged, using the law of large numbers as follows:

$$E_t\sum_{b=0}^{\infty} N^{Y}_{b+1,t+1}\mathcal{I}^{Y}_{b+1,t+1}(i) = E_t\mathcal{I}^{Y}_{t+1} - E_t N^{Y}_{b,t+1}\mathcal{I}^{Y}_{b,t+1}(i) =$$
$$= E_t\mathcal{I}^{Y}_{t+1}\left(1 - \frac{N^{Y}_{b,t+1}}{N^{Y}_{t+1}}\right) = E_t\mathcal{I}^{Y}_{t+1}\left(1 - \frac{n_t N^{Y}_{t}}{N^{Y}_{t+1}}\right)$$

Then, total young income is:

$$\mathcal{I}^{Y}_{t} = w_t N^{Y}_{t} + Profit_t - Tax_t + E_t\frac{1-\omega^{Y}_{t}}{(1+r_t)(1+g^{N,Y}_{t+1})}\mathcal{I}^{Y}_{t+1}$$

A similar exercise can be done for pension benefits. First, we define $\mathcal{I}^{YO}_{t}$ which can be rearranged as:

$$\mathcal{I}^{YO}_{t} = \sum_{b=0}^{\infty} N^{Y}_{b,t}\mathcal{I}^{YO}_{b,t}(i) = E_t\omega^{Y}_{t}\sum_{b=0}^{\infty} N^{Y}_{b,t}E_t TR^{YO}_{0,t+1}(i)\Omega^{O}_{t+1} +$$
$$+\frac{(1-\omega^{Y}_{t})}{(1+r_{t+1})}\sum_{b=0}^{\infty} N^{Y}_{b,t}\mathcal{I}^{YO}_{b+1,t+1}(i) = E_t N^{O}_{0,t+1} TR^{YO}_{0,t+1}(i)\Omega^{O}_{t+1} +$$
$$+E_t\frac{1}{(1+r_{t+1})}\sum_{b=0}^{\infty} N^{Y}_{b+1,t+1}\mathcal{I}^{YO}_{b+1,t+1}(i)$$

Now, similarly to total young income, the last term can be expressed as:

$$E_t \sum_{b=0}^{\infty} N^Y_{b+1,t+1} \mathcal{I}^{YO}_{b+1,t+1}(i) = E_t \frac{1-\omega^Y_t}{1+g^{N,Y}_{t+1}} \mathcal{I}^{YO}_{t+1}$$

Also, we know that the following expression holds:

$$E_t N^O_{0,t+1} TR^{YO}_{0,t+1}(i) \Omega^O_{t+1} = E_t TR^{YO}_{t+1} \Omega^O_{t+1}$$

Finally, the expected income of the young after getting retired is

$$\mathcal{I}^{YO}_t = E_t TR^{YO}_{t+1} \Omega^O_{t+1} + E_t \frac{1-\omega^Y_t}{(1+r_{t+1})(1+g^{N,Y}_{t+1})} \mathcal{I}^{YO}_{t+1}$$

Based on the derivation above, we can express the aggregate version of the young household's income as:

$$Inc^Y_t = w_t N^Y_t + Profit_t - Tax_t + E_t \frac{TR^{YO}_{t+1}}{1+r_t} \Omega^O_{t+1} + E_t \frac{1-\omega^Y_t}{(1+r_t)(1+g^{N,Y}_{t+1})} Inc^Y_{t+1}$$

Aggregating the young households' budget constraints

The individual budget constraint of a young agent is as follows:

$$\begin{aligned} &C^Y_{b,t}(i) + (1-\omega^Y_t) B^Y_{b,t}(i) + \omega^Y_t B^{Y*}_{b,t}(i) = \\ &= w_t L_{b,t}(i) + Profit_{b,t}(i) - Tax_{b,t}(i) + (1+r_{t-1}) B^Y_{b-1,t-1}(i) \end{aligned}$$

Aggregating implies:

$$\begin{aligned} &\sum_{b=0}^{\infty} N^Y_{b,t} C^{Y,F}_{b,t}(i) + \sum_{b=0}^{\infty} N^Y_{b,t} (1-\omega^Y_t) B^Y_{b,t}(i) + \sum_{b=0}^{\infty} N^Y_{b,t} \omega^Y_t B^{Y*}_{b,t}(i) = \\ &= \sum_{b=0}^{\infty} N^Y_{b,t} \left(w_t L_{b,t}(i) + Profit_{b,t}(i) - Tax_{b,t}(i) \right) + (1+r_{t-1}) \sum_{b=1}^{\infty} N^Y_{b,t} B^Y_{b-1,t-1}(i) \end{aligned}$$

where the definition of aggregate savings is:

$$\sum_{b=1}^{\infty} N^Y_{b,t} B^Y_{b-1,t-1}(i) = \sum_{b=1}^{\infty} (1-\omega^Y_{t-1}) N^Y_{b-1,t-1} B^Y_{b-1,t-1}(i)$$

After aggregation, there is no difference between the B^Y_t and B^{Y*}_t. So, we can easily express aggregate budget constraint:

$$C_t^Y + B_t^Y = w_t L_t + Profit_t - Tax_t + (1 + r_{t-1})(1 - \omega_{t-1}^Y) B_{t-1}^Y$$

A.2 Fiscal policy and pay-as-you-go pension plan

Pension system

To account for the overall expenditure of the pension system, we need to count the number of just-retired and retired agents. The number of just-retired agents (those who were young one period before) is:

$$N_{0,t}^O = \sum_{b=1}^{\infty} \omega_{t-1}^Y N_{b-1,t-1}^Y$$

and the total number of retired agents in period t is the just-retired agents plus those who survived the previous periods:

$$N_t^O = N_{0,t}^O + (1 - \omega_t^O) N_{1,t-1}^O + (1 - \omega_t^O)(1 - \omega_{t-1}^O) N_{2,t-2}^O + ...$$

Individual's (i) pension in the year of retirement t is based on replacement rate ν_t and the previous period income:

$$TR_{0,t}^{YO}(i) = \nu_t w_{t-1} L_{b-1,t-1}(i)$$

We need to use the following expressions to aggregate:

$$\begin{aligned} N_{0,t}^O TR_{0,t}^{YO}(i) &= \nu_t N_{0,t}^O w_{t-1} L_{b-1,t-1}(i) = \nu_t \omega_{t-1}^Y \sum_{b=1}^{\infty} N_{b-1,t-1}^Y w_{t-1} L_{b-1,t-1}(i) \\ TR_t^{YO} &= \nu_t \omega_{t-1}^Y w_{t-1} L_{t-1} \end{aligned}$$

Furthermore, the total pension expenditure of all retired people is as follows:

$$TR_t = TR_t^{YO} + (1 - \omega_{t-1}^O) TR_{t-1}^{YO} + (1 - \omega_{t-1}^O)(1 - \omega_{t-2}^O) TR_{t-2}^{YO} + ...$$

which can be rewritten as:

$$TR_t = TR_t^{YO} + (1 - \omega_{t-1}^{O})TR_{t-1}$$

Remainder of the fiscal sector

The government budget constraint is as follows:

$$Debt_t + Tax_t \quad = \quad Gov_t + TR_t + (1 + r_{t-1})Debt_{t-1}$$

where *Gov* denotes the other - exogenous - current expenditures, *Debt* is the level of public debt. The government wants to stabilizes the public debt-to-GDP ratio by adjusting the level of lump-sum taxes:

$$Tax_t \quad = \quad Gov_t + TR_t + (1 + r_{t-1})Debt_{t-1} - \left\{ \frac{Debt}{Y} \right\}^{Target} Y_t$$

The households finance government debt and the bond market equilibrium is the following:

$$Debt_t \quad = \quad B_t^Y + B_t^O$$

A.3 Firms' optimization

The young households own the firms and the labor union; hence, in their optimization, we take into account their survival probability.

Production and nominal price setting

The firms produce differentiated products, and, due to their monopolistic power, they are able to set optimal prices. However, nominal price setting is only available for $1 - \omega_P$ fraction of the firms in a given period. Hence, their optimal price settings conditional on that from the next period the firms are not able to set the optimal price (Calvo, 1983). Those who cannot set prices adjust them by the previous period's inflation:

$$\begin{aligned}
\mathcal{L} &= \sum_{n=0}^{\infty} \omega_P^n \Delta_{t,t+n} \left(P_{t+n}(i)Y_{t+n}(i) - V_{t+n}L_{t+n}(i) - P_{t+n}Inv_{t+n}(i)\right) + \\
&+ \sum_{n=0}^{\infty} \Delta_{t,t+n} MC_{t+n} \left(A_{t+n}K_{t-1+n}(i)^{\alpha} L_{t+n}(i)^{1-\alpha} - Y_{t+n}(i)\right) + \\
&+ \sum_{n=0}^{\infty} \Delta_{t,t+n} Q_{t+n} \left(Inv_{t+n}(i)\left(1 - S\left(\frac{Inv_{t+n}(i)}{(1+g_{t+n})Inv_{t+n-1}(i)}\right)\right) - K_{t+n}(i) + (1-\delta)K_{t+n-1}(i)\right)
\end{aligned}$$

where $P_{t+n}(i) = P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}$ and optimizing firms set $P_t(i)$ with respect to the following demand function:

$$Y_{t+n}(i) = \left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi} Y_t$$

The discount factor takes into account current and future probability of survival and the nominal interest rates:

$$\Delta_{t,t+n} = \prod_{k=1}^{n} \frac{1 - \omega_{t+k-1}^{Y}}{1 + i_{t+k-1}}$$

The firms are responsible for capital accumulation. The $S(\cdot)$ function denotes the investment adjustment cost, and any changes in investment which differ from the balanced growth path is costly. Q is the nominal Tobin-Q. V denotes the aggregate nominal wage index, that comes from the labor union (below we give more detailed description of the labor union). We can write up the optimization problem in a more compact form:

$$\begin{aligned}
\mathcal{L} &= \sum_{n=0}^{\infty} \omega_P^n \Delta_{t,t+n} \left(P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P} \left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi} Y_{t+n} - V_{t+n}L_{t+n}(i) - P_{t+n}Inv_{t+n}(i)\right) + \\
&+ \sum_{n=0}^{\infty} \Delta_{t,t+n} MC_{t+n} \left(A_{t+n}K_{t-1+n}(i)^{\alpha} L_{t+n}(i)^{1-\alpha} - \left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi} Y_{t+n}\right) + \\
&+ \sum_{n=0}^{\infty} \Delta_{t,t+n} Q_{t+n} \left(Inv_{t+n}(i)\left(1 - S(\cdot)\right) - K_{t+n}(i) + (1-\delta)K_{t+n-1}(i)\right)
\end{aligned}$$

The firms decide about the profit-maximizing nominal price level, the level of capital, investment and labor input. The first-order conditions are the following:

$$
\begin{aligned}
P_t^*(i) \quad &: \quad \sum_{n=0}^{\infty}\omega_P^n\Delta_{t,t+n}\Bigg(\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}\left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi}Y_{t+n} - \\
&- \quad \varphi\left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi-1}\frac{P_t^*(i)}{P_{t+n}}\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}Y_{t+n} + \\
&+ \quad \varphi MC_{t+n}\left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi-1}\frac{1}{P_{t+n}}\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}Y_{t+n}\Bigg) = 0 \\
Inv_t(i) \quad &: \quad (-1)P_t + Q_t\left(1 - S(\cdot) - S'(\cdot)\frac{Inv_t(i)}{(1+g_t)Inv_{t-1}(i)}\right) + \Delta_{t,t+1}Q_{t+1}Inv_{t+1}(i)S'(\cdot)\frac{Inv_{t+1}(i)}{(1+g_{t+1})Inv_t^2(i)} = 0 \\
K_t(i) \quad &: \quad \Delta_{t,t+1}MC_{t+1}\alpha A_{t+1}K_t(i)^{\alpha-1}L_{t+1}(i)^{1-\alpha} - Q_t + Q_{t+1}\Delta_{t,t+1}(1-\delta) = 0 \\
L_t(i) \quad &: \quad MC_t(1-\alpha)A_tK_{t-1}(i)^{\alpha}L_t(i)^{-\alpha} - V_t = 0
\end{aligned}
$$

If we rearrange the conditions, we can get the usual formulas:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\omega_P^n\Delta_{t,t+n}\left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi}Y_{t+n}\left(P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P} - \frac{\varphi}{\varphi-1}MC_{t+n}\right) = 0 \\
&q_t\left(1 - S(\cdot) - S'(\cdot)\frac{Inv_t(i)}{(1+g_t)Inv_{t-1}(i)}\right) + \frac{1-\omega_t^Y}{1+r_t}q_{t+1}\frac{S'(\cdot)}{1+g_{t+1}}\left(\frac{Inv_{t+1}(i)}{Inv_t(i)}\right)^2 = 1 \\
&(1-\omega_t^Y)\left(mc_{t+1}\alpha A_{t+1}K_t(i)^{\alpha-1}L_{t+1}(i)^{1-\alpha} + q_{t+1}(1-\delta)\right) = q_t(1+r_t) \\
&mc_t(1-\alpha)A_tK_{t-1}(i)^{\alpha}L_t(i)^{-\alpha} = v_t
\end{aligned}
$$

We can write up the pricing equation:

$$
\frac{P_t^*(i)}{P_t} = \frac{\varphi}{\varphi-1}\frac{\sum_{n=0}^{\infty}\omega_P^n\Delta_{t,t+n}\left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi}Y_{t+n}\frac{MC_{t+n}}{P_t}}{\sum_{n=0}^{\infty}\omega_P^n\Delta_{t,t+n}\left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi}Y_{t+n}\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}
$$

We can express the optimal relative price equations by recursive equations:

$$
\begin{aligned}
\mathcal{Z}_t^1 &= \sum_{n=0}^{\infty}\omega_P^n\Delta_{t,t+n}\left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi}Y_{t+n}\frac{MC_{t+n}}{P_t} \\
\mathcal{Z}_t^2 &= \sum_{n=0}^{\infty}\omega_P^n\Delta_{t,t+n}\left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi}Y_{t+n}\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}
\end{aligned}
$$

Which can be written as:

$$\begin{aligned}
\mathcal{Z}_t^1 &= \left(\frac{P_t^*(i)}{P_t}\right)^{-\varphi} Y_t mc_t + \sum_{n=1}^{\infty} \omega_P^n \Delta_{t,t+n} \left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi} Y_{t+n} \frac{MC_{t+n}}{P_t} \\
\mathcal{Z}_t^2 &= \left(\frac{P_t^*(i)}{P_t}\right)^{-\varphi} Y_t + \sum_{n=1}^{\infty} \omega_P^n \Delta_{t,t+n} \left(\frac{P_t^*(i)\left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi} Y_{t+n} \left(\frac{P_{t+n-1}}{P_{t-1}}\right)^{\gamma_P}
\end{aligned}$$

Stepping one-period ahead:

$$\begin{aligned}
\mathcal{Z}_{t+1}^1 &= \sum_{n=0}^{\infty} \omega_P^n \Delta_{t+1,t+n+1} \left(\frac{P_{t+1}^*(i)\left(\frac{P_{t+n-1+1}}{P_t}\right)^{\gamma_P}}{P_{t+n+1}}\right)^{-\varphi} Y_{t+n+1} \frac{MC_{t+n+1}}{P_{t+1}} \\
\mathcal{Z}_{t+1}^2 &= \sum_{n=0}^{\infty} \omega_P^n \Delta_{t+1,t+n+1} \left(\frac{P_{t+1}^*(i)\left(\frac{P_{t+n-1+1}}{P_t}\right)^{\gamma_P}}{P_{t+n+1}}\right)^{-\varphi} Y_{t+n+1} \left(\frac{P_{t+n-1+1}}{P_t}\right)^{\gamma_P}
\end{aligned}$$

The sum starts from $t+1$:

$$\begin{aligned}
\mathcal{Z}_{t+1}^1 &= \sum_{n=1}^{\infty} \omega_P^{n-1} \Delta_{t+1,t+n} \left(\frac{P_{t+1}^*(i)\left(\frac{P_{t+n-1}}{P_t}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi} Y_{t+n} \frac{MC_{t+n}}{P_{t+1}} \\
\mathcal{Z}_{t+1}^2 &= \sum_{n=1}^{\infty} \omega_P^{n-1} \Delta_{t+1,t+n} \left(\frac{P_{t+1}^*(i)\left(\frac{P_{t+n-1}}{P_t}\right)^{\gamma_P}}{P_{t+n}}\right)^{-\varphi} Y_{t+n} \left(\frac{P_{t+n-1}}{P_t}\right)^{\gamma_P}
\end{aligned}$$

Since all price setting firms follow the same optimization, then the optimal price can be written as $p_t^* = \frac{P_t^*}{P_t}$, where the recursive equations are:

$$\begin{aligned}
p_t^* &= \frac{\varphi}{\varphi-1}\frac{\mathcal{Z}_t^1}{\mathcal{Z}_t^2} \\
\mathcal{Z}_t^1 &= p_t^{*-\varphi} Y_t mc_t + \left(\frac{p_t^*}{p_{t+1}^*}\frac{(1+\pi_t)^{\gamma_P}}{1+\pi_{t+1}}\right)^{-\varphi} \omega_P (1-\omega_t^Y)\frac{1+\pi_{t+1}}{1+i_t}\mathcal{Z}_{t+1}^1 \\
\mathcal{Z}_t^2 &= p_t^{*-\varphi} Y_t + \left(\frac{p_t^*}{p_{t+1}^*}\frac{(1+\pi_t)^{\gamma_P}}{1+\pi_{t+1}}\right)^{-\varphi} (1+\pi_t)^{\gamma_P}\, \omega_P \frac{1-\omega_t^Y}{1+i_t}\mathcal{Z}_{t+1}^2
\end{aligned}$$

The price index is the following follows:

$$
\begin{aligned}
P_t &= \left((1-\omega_P) P_t^{*1-\varphi} + \omega_P \left(P_{t-1} \left(\frac{P_{t-1}}{P_{t-2}} \right)^{\gamma_P} \right)^{1-\varphi} \right)^{\frac{1}{1-\varphi}} \\
1 &= \left((1-\omega_P) p_t^{*1-\varphi} + \omega_P \left(\frac{(1+\pi_{t-1})^{\gamma_P}}{1+\pi_t} \right)^{1-\varphi} \right)^{\frac{1}{1-\varphi}}
\end{aligned}
$$

As a simplification, we can introduce a new variable for the marginal product of capital which modifies the no-arbitrage condition:

$$
\begin{aligned}
r_t^K &= mc_t \alpha A_t K_{t-1}(i)^{\alpha-1} L_t(i)^{1-\alpha} = \alpha \frac{Y_t}{K_{t-1}} mc_t \\
q_t &= \frac{1-\omega_t^Y}{1+r_t} \left(r_{t+1}^K + q_{t+1}(1-\delta) \right)
\end{aligned}
$$

If we substitute out the capital and labor from the production function, we can get the marginal cost function:

$$
mc_t = \frac{1}{A_t} \left(\frac{r_t^K}{\alpha} \right)^{\alpha} \left(\frac{v_t}{1-\alpha} \right)^{1-\alpha}
$$

Nominal wage rigidity

We assume that labor unions are able to set nominal wages a la Calvo with indexation:

$$
L_t(j) = \left(\frac{V_t(j)}{V_t} \right)^{-\varphi_w} L_t
$$

where $V_t(j)$ is individual wage cost, V_t nominal wage index. The unions hire young households (for W_t nominal wage) and try to maximize the following value function:

$$
\sum_{n=0}^{\infty} \omega_w^n \Delta_{t,t+n} \left(V_t^*(j) \left(\frac{V_{t+n-1}}{V_{t-1}} \right)^{\gamma_w} L_{t+n}(j) - W_{t+n} L_{t+n}(j) \right)
$$

Substituting out the labor demand function:

$$\sum_{n=0}^{\infty}\omega_w^n\Delta_{t,t+n}\left(V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}\left(\frac{V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}\right)^{-\varphi_w}L_{t+n}-W_{t+n}\left(\frac{V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}\right)^{-\varphi_w}L_{t+n}\right)$$

Taking the first-order condition:

$$\begin{aligned}&\sum_{n=0}^{\infty}\omega_w^n\Delta_{t,t+n}\Bigg(\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}\left(\frac{V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}\right)^{-\varphi_w}L_{t+n}+\\&-\varphi_w V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}\left(\frac{V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}\right)^{-\varphi_w-1}\frac{\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}L_{t+n}-\\&+\varphi_w W_{t+n}\left(\frac{V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}\right)^{-\varphi_w-1}\frac{\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}L_{t+n}\Bigg)=0\end{aligned}$$

Multiplying with $V_t^*(j)$ and rearranging:

$$\sum_{n=0}^{\infty}\omega_w^n\Delta_{t,t+n}\left(\frac{V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}\right)^{-\varphi_w}L_{t+n}\left(V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}-\frac{\varphi_w}{\varphi_w-1}W_{t+n}\right)=0$$

We can express the individual wage:

$$\frac{V_t^*(j)}{P_t}=\frac{\varphi_w}{\varphi_w-1}\frac{\frac{1}{P_t}\sum_{n=0}^{\infty}\omega_w^n\Delta_{t,t+n}\left(\frac{V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}\right)^{-\varphi_w}L_{t+n}W_{t+n}}{\sum_{n=0}^{\infty}\omega_w^n\Delta_{t,t+n}\left(\frac{V_t^*(j)\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}\right)^{-\varphi_w}L_{t+n}\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}$$

All wage-setters follow the same optimization problem, so we can use the aggregate optimal wage notation:

$$v_t^*=\frac{\varphi_w}{\varphi_w-1}\frac{\mathcal{W}_t^1}{\mathcal{W}_t^2}$$

where $\mathcal{W}_t^1$ and $\mathcal{W}_t^2$ are given by the following recursive substition:

$$\begin{aligned}
\mathcal{W}_t^1 &= \frac{1}{P_t}\sum_{n=0}^{\infty}\omega_w^n \Delta_{t,t+n}\left(\frac{V_t^*\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}\right)^{-\varphi_w} L_{t+n}W_{t+n} \\
&= \left(\frac{v_t^*}{v_t}\right)^{-\varphi_w} L_t w_t + \left(\frac{v_t^*}{v_{t+1}^*}\frac{\left(1+\pi_t^V\right)^{\gamma_w}}{1+\pi_{t+1}^V}\right)^{-\varphi_w} \omega_w \frac{1-\omega_t^Y}{1+i_t}(1+\pi_{t+1})\mathcal{W}_{t+1}^1
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_t^2 &= \sum_{n=0}^{\infty}\omega_w^n \Delta_{t,t+n}\left(\frac{V_t^*\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w}}{V_{t+n}}\right)^{-\varphi_w} L_{t+n}\left(\frac{V_{t+n-1}}{V_{t-1}}\right)^{\gamma_w} \\
&= \left(\frac{v_t^*}{v_t}\right)^{-\varphi_w} L_t + \left(\frac{v_t^*}{v_{t+1}^*}\frac{\left(1+\pi_t^V\right)^{\gamma_w}}{1+\pi_{t+1}^V}\right)^{-\varphi_w} \omega_w \frac{1-\omega_t^Y}{1+i_t}\left(1+\pi_t^V\right)^{\gamma_w}\mathcal{W}_{t+1}^2
\end{aligned}$$

The aggregate wage index is as follows:

$$\begin{aligned}
V_t^{1-\varphi_w} &= (1-\omega_w)V_t^{*1-\varphi_w} + \omega_w\left(V_{t-1}\left(\frac{V_{t-1}}{V_{t-2}}\right)^{\gamma_w}\right)^{1-\varphi_w} \\
1 &= (1-\omega_w)\left(\frac{v_t^*}{v_t}\right)^{1-\varphi_w} + \omega_w\left(\frac{(1+\pi_{t-1}^V)^{\gamma_w}}{1+\pi_t^V}\right)^{1-\varphi_w}
\end{aligned}$$

where wage inflation is defined as:

$$1+\pi_t^V = \frac{v_t}{v_{t-1}}(1+\pi_t)$$

A.4 Monetary policy

The behavior of the central bank can be described by a Taylor-type monetary policy rule:

$$1+i_t = (1+i_{t-1})^{\rho_i}\left(\left(1+r_t^n\right)\left(1+\pi_t\right)^{\phi_\pi}\right)^{1-\rho_i} e^{\varepsilon_t^i}$$

where ρ_i is interest smoothing, ϕ_π denotes the inflationary reaction, and ε_t^i assigns the monetary policy shock. Once the inflation reaches the target again, the central bank should set the interest rate at its flexible price equilibrium level.

The real interest rate is defined by the Fisher identity:

$$1+i_t = E_t(1+r_t)(1+\pi_{t+1})$$

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A.5 Checking the equilibrium conditions

The budget constraint of the young households:

$$C_t^Y + B_t^Y = w_t L_t + Profit_t - Tax_t + (1 + r_{t-1})(1 - \omega_{t-1}^Y) B_{t-1}^Y$$

The budget constraint of the old households:

$$C_t^O + B_t^O = TR_t + (1 + r_{t-1}) \left[\omega_{t-1}^Y B_{t-1}^Y + B_{t-1}^O \right]$$

The profits of the firm and labor union:

$$\begin{aligned} Profit_t &= Profit_t^L + Profit_t^Y \\ Profit_t^L &= v_t L_t - w_t L_t \\ Profit_t^Y &= Y_t - v_t L_t - Inv_t \end{aligned}$$

The government budget constraint:

$$Debt_t + Tax_t = Gov_t + TR_t + (1 + r_{t-1}) Debt_{t-1}$$

If we add the households budget constraint:

$$C_t^Y + C_t^O + B_t^Y + B_t^O = w_t L_t + Profit_t - Tax_t + TR_t + (1 + r_{t-1}) \left[B_{t-1}^Y + B_{t-1}^O \right]$$

Substituting the profits:

$$C_t^Y + C_t^O + B_t^Y + B_t^O = Y_t - Inv_t - Tax_t + TR_t + (1 + r_{t-1}) \left[B_{t-1}^Y + B_{t-1}^O \right]$$

From the bonds market equilibrium, we know that total savings are equal to the public debt:

$$C_t^Y + C_t^O + Debt_t = Y_t - Inv_t - Tax_t + TR_t + (1 + r_{t-1}) Debt_{t-1}$$

Finally we can substitute out the lump-sum taxes from the government budget constraint:

$$Y_t = C_t^Y + C_t^O + Inv_t + Gov_t$$

A.6 Normalized model equations

Each variable must be detrended: individual variables are normalized by population (N_t) because there is only population growth in the model. This section lists all the final equations of the model: detrended variables are denoted by $\tilde{x}_t$.

Demography:

$$\begin{aligned}
s_t &= \frac{(1-\omega^O_{t-1})}{(1-\omega^Y_{t-1}+n_t)} s_{t-1} + \frac{\omega^Y_{t-1}}{(1-\omega^Y_{t-1}+n_t)} \\
s^Y_t &= \frac{1}{1+s_t} \\
1+g^Y_t &= 1-\omega^Y_{t-1}+n_t \\
1+g^O_t &= (1-\omega^O_{t-1}) + \frac{\omega^Y_{t-1}}{s_{t-1}} \\
1+g_t &= 1+g^N_t = (1+g^{N,Y}_t)\frac{1+s_t}{1+s_{t-1}}
\end{aligned}$$

Households:

$$\begin{aligned}
\tilde{C}^O_t &= MPC^O_t \tilde{TR}_t \Omega^O_t + MPC^O_t \frac{1+r_{t-1}}{1+g_t}\left[\omega^Y_{t-1}\tilde{B}^Y_{t-1} + \tilde{B}^O_{t-1}\right] \\
\frac{1}{MPC^O_t} &= 1+\beta^{\frac{1}{\gamma}}(1-\omega^O_t)(1+r_t)^{\frac{1}{\gamma}-1}E_t\frac{1}{MPC^O_{t+1}} \\
\Omega^O_t &= 1+E_t\frac{1-\omega^O_t}{1+r_t}\Omega^O_{t+1} \\
\tilde{C}^Y_t &= MPC^Y_t \tilde{Inc}^Y_t + MPC^Y_t\frac{1+r_{t-1}}{1+g_t}(1-\omega^Y_{t-1})\tilde{B}^Y_{t-1} \\
\frac{\tilde{C}^Y_t}{s^Y_t-\tilde{L}_t} &= \frac{\sigma}{1-\sigma}w_t \\
\tilde{Inc}^Y_t &= w_t s^Y_t + \tilde{Profit}_t - \tilde{Tax}_t + E_t(1+g^N_{t+1})\frac{\tilde{TR}^{YO}_{t+1}}{1+r_t}\Omega^O_{t+1} + E_t\frac{1-\omega^Y_t}{1+r_t}\frac{1+s_{t+1}}{1+s_t}\tilde{Inc}^Y_{t+1} \\
\frac{1}{MPC^Y_t} &= \frac{1}{\sigma}+\beta^{\frac{1}{\gamma}}(1+r_t)^{\frac{1}{\gamma}-1}E_t\left((1-\omega^Y_t)\Lambda^Y_t\frac{1}{MPC^Y_{t+1}} + \omega^Y_t\Lambda^{YO}_t\frac{1}{MPC^O_{t+1}}\right) \\
\Lambda^Y_t &= E_t\left(\frac{w_{t+1}}{w_t}\right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \\
\Lambda^{YO}_t &= E_t\left(\frac{1}{\sigma}\right)^{\frac{1}{\gamma}}\left(\frac{1}{\frac{\sigma}{1-\sigma}w_t}\right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \\
\tilde{C}^Y_t+\tilde{B}^Y_t &= w_t\tilde{L}_t + \tilde{Profit}_t - \tilde{Tax}_t + \frac{1+r_{t-1}}{1+g_t}(1-\omega^Y_{t-1})\tilde{B}^Y_{t-1}
\end{aligned}$$

Wage setting:

$$
\begin{aligned}
v_t^* &= \frac{\varphi_w}{\varphi_w - 1}\frac{\tilde{\mathcal{W}}_t^1}{\tilde{\mathcal{W}}_t^2} \\
\tilde{\mathcal{W}}_t^1 &= \left(\frac{v_t^*}{v_t}\right)^{-\varphi_w}\tilde{L}_t w_t + \left(\frac{v_t^*}{v_{t+1}^*}\frac{\left(1+\pi_t^V\right)^{\gamma_w}}{1+\pi_{t+1}^V}\right)^{-\varphi_w}\omega_w\frac{1-\omega_t^Y}{1+i_t}(1+\pi_{t+1})(1+g_{t+1})\tilde{\mathcal{W}}_{t+1}^1 \\
\tilde{\mathcal{W}}_t^2 &= \left(\frac{v_t^*}{v_t}\right)^{-\varphi_w}\tilde{L}_t + \left(\frac{v_t^*}{v_{t+1}^*}\frac{\left(1+\pi_t^V\right)^{\gamma_w}}{1+\pi_{t+1}^V}\right)^{-\varphi_w}\omega_w\frac{1-\omega_t^Y}{1+i_t}\left(1+\pi_t^V\right)^{\gamma_w}(1+g_{t+1})\tilde{\mathcal{W}}_t^2 \\
1 &= (1-\omega_w)\left(\frac{v_t^*}{v_t}\right)^{1-\varphi_w} + \omega_w\left(\frac{(1+\pi_{t-1}^V)^{\gamma_w}}{1+\pi_t^V}\right)^{1-\varphi_w} \\
1+\pi_t^V &= \frac{v_t}{v_{t-1}}(1+\pi_t)
\end{aligned}
$$

Firms:

$$
\begin{aligned}
p_t^* &= \frac{\varphi}{\varphi-1}\frac{\tilde{\mathcal{Z}}_t^1}{\tilde{\mathcal{Z}}_t^2} \\
\tilde{\mathcal{Z}}_t^1 &= p_t^{*-\varphi}\tilde{Y}_t mc_t + \left(\frac{p_t^*}{p_{t+1}^*}\frac{(1+\pi_t)^{\gamma_P}}{1+\pi_{t+1}}\right)^{-\varphi}\omega_P(1-\omega_t^Y)\frac{1+\pi_{t+1}}{1+i_t}(1+g_{t+1})\tilde{\mathcal{Z}}_{t+1}^1 \\
\tilde{\mathcal{Z}}_t^2 &= p_t^{*-\varphi}\tilde{Y}_t + \left(\frac{p_t^*}{p_{t+1}^*}\frac{(1+\pi_t)^{\gamma_P}}{1+\pi_{t+1}}\right)^{-\varphi}(1+\pi_t)^{\gamma_P}\omega_P\frac{1-\omega_t^Y}{1+i_t}(1+g_{t+1})\tilde{\mathcal{Z}}_{t+1}^2 \\
1 &= (1-\omega_P)p_t^{*1-\varphi} + \omega_P\left(\frac{(1+\pi_{t-1})^{\gamma_P}}{1+\pi_t}\right)^{1-\varphi} \\
1 &= q_t\left(1 - S\left(\frac{\tilde{Inv}_t}{\tilde{Inv}_{t-1}}\right) - S'\left(\frac{\tilde{Inv}_t}{\tilde{Inv}_{t-1}}\right)\frac{\tilde{Inv}_t}{\tilde{Inv}_{t-1}}\right) + \\
&+ \frac{1-\omega_t^Y}{1+r_t}q_{t+1}(1+g_{t+1})S'\left(\frac{\tilde{Inv}_{t+1}}{\tilde{Inv}_t}\right)\left(\frac{\tilde{Inv}_{t+1}}{\tilde{Inv}_t}\right)^2 \\
q_t &= \frac{1-\omega_t^Y}{1+r_t}\left(r_{t+1}^K + q_{t+1}(1-\delta)\right) \\
r_t^K &= \alpha\frac{\tilde{Y}_t}{\tilde{K}_{t-1}}mc_t(1+g_t) \\
v_t &= (1-\alpha)\frac{\tilde{Y}_t}{\tilde{L}_t}mc_t \\
mc_t &= \frac{1}{A_t}\left(\frac{r_t^K}{\alpha}\right)^{\alpha}\left(\frac{v_t}{1-\alpha}\right)^{1-\alpha} \\
\tilde{K}_t &= \tilde{Inv}_t\left(1 - S\left(\frac{\tilde{Inv}_t}{\tilde{Inv}_{t-1}}\right)\right) + \frac{1-\delta}{1+g_t}\tilde{K}_{t-1} \\
\tilde{Profit}_t &= \tilde{Y}_t - w_t\tilde{L}_t - \tilde{Inv}_t
\end{aligned}
$$

Fiscal policy and pension system:

$$\begin{aligned}
\tilde{TR}_t^{YO} &= \nu_t \frac{\omega_{t-1}^Y}{1+g_t} w_{t-1} \tilde{L}_{t-1} \\
\tilde{TR}_t &= \tilde{TR}_t^{YO} + \frac{1-\omega_{t-1}^O}{1+g_t} \tilde{TR}_{t-1} \\
\tilde{Debt}_t + \tilde{Tax}_t &= \tilde{Gov}_t + \tilde{TR}_t + \frac{1+r_{t-1}}{1+g_t} \tilde{Debt}_{t-1} \\
\tilde{Tax}_t &= \tilde{Gov}_t + \tilde{TR}_t + \frac{1+r_{t-1}}{1+g_t} \tilde{Debt}_{t-1} - \left\{ \frac{Debt}{Y} \right\}^{Target} \tilde{Y}_t \\
\tilde{Debt}_t &= \tilde{B}_t^Y + \tilde{B}_t^O
\end{aligned}$$

Monetary policy:

$$\begin{aligned}
1 + i_t &= (1 + i_{t-1})^{\rho_i} \left((1 + r_t^n) (1 + \pi_t)^{\phi_\pi} \right)^{1-\rho_i} e^{\varepsilon_t^i} \\
1 + i_t &= E_t (1 + r_t)(1 + \pi_{t+1})
\end{aligned}$$

Market clearing:

$$\tilde{Y}_t = \tilde{C}_t^Y + \tilde{C}_t^O + \tilde{Inv}_t + \tilde{Gov}_t$$

A.7 Steady state of the model

To be able to calculate the steady-state solution we need to specify initial guesses for r and $\tilde{Y}$ then the rest of the variables and equations can be solved numerically. As a function of the initial guesses, we can determine the variables of production, labor market and those of the government and pension system. Finally, we turn to the consumption and savings functions. At the end, using the market clearing equations and labor supply curve, we can check whether our initial guesses are correct.

First, the demographic equations are:

$$\begin{aligned}
s &= \frac{\omega^Y}{(1-\omega^Y+n)} \left(1 - \frac{(1-\omega^O)}{(1-\omega^Y+n)} \right)^{-1} \\
s^Y &= \frac{1}{1+s} \\
1 + g^{N,O} &= 1 - \omega^O + \frac{\omega^Y}{s} \\
1 + g &= 1 + g^N = 1 + g^{N,Y} = 1 - \omega^Y + n
\end{aligned}$$

Then, we need to guess an initial value for r which is verified by the Newton-Raphson algorithm. Assuming $\pi = 0$ in the steady state implies:

$$i = r$$

The Tobin-Q (q) is one in the steady-state equilibrium, so the initial assumption for r and the no-arbitrage condition imply the steady-state value of the marginal product of capital:

$$r^K = \frac{1+r}{1-\omega^Y} - 1 + \delta$$

The firms' supply curve in the steady state gives us the marginal cost as the inverse of the markup:

$$mc = \frac{\varphi - 1}{\varphi}$$

Based on the marginal cost function we can calculate the real wage:

$$v = (1-\alpha)\left[A \cdot mc \left(\frac{\alpha}{r^K}\right)^{\alpha}\right]^{\frac{1}{1-\alpha}}$$

We can calculate the capital and labor per production ratios from the input demand functions of the firms:

$$\begin{aligned}
\frac{\tilde{K}}{\tilde{Y}} &= \frac{\alpha}{r^K} mc(1+g) \\
\frac{\tilde{L}}{\tilde{Y}} &= \frac{1-\alpha}{v} mc
\end{aligned}$$

$\frac{\tilde{K}}{\tilde{Y}}$ also implies $\frac{\tilde{Inv}}{\tilde{Y}}$:

$$\frac{\tilde{Inv}}{\tilde{Y}} = \frac{\tilde{K}}{\tilde{Y}}\left(1 - \frac{1-\delta}{1+g}\right)$$

The wage setting equations imply the real wage of the households:

$$w = \frac{\varphi_w - 1}{\varphi_w} v$$

We can express Λ^Y and Λ^{YO} as follows:

$$\begin{aligned}
\Lambda^Y &= 1 \\
\Lambda^{YO} &= \left(\frac{1}{\sigma}\right)^{\frac{1}{\gamma}} \left(\frac{1}{\frac{\sigma}{1-\sigma}w}\right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)}
\end{aligned}$$

Using the assumption of replacement ratio and labor market variables, we can calculate the steady state pension expenditures. Using the pension expenditures, the assumptions for public debt-to-GDP ratio, and government expenditure-to-GDP ratios we can calculate the equilibrium level of the tax burden:

$$\begin{aligned}
\frac{\tilde{TR}^{YO}}{\tilde{Y}} &= \nu \frac{\omega^Y}{1+g} w \frac{\tilde{L}}{\tilde{Y}} \\
\frac{\tilde{TR}}{\tilde{Y}} &= \frac{\tilde{TR}^{YO}}{\tilde{Y}} \left(1 - \frac{1-\omega^O}{1+g}\right)^{-1} \\
\frac{\tilde{Tax}}{\tilde{Y}} &= \frac{\tilde{Gov}}{\tilde{Y}} + \frac{\tilde{TR}}{\tilde{Y}} + \left(\frac{1+r}{1+g} - 1\right) \frac{\tilde{Debt}}{\tilde{Y}}
\end{aligned}$$

We can also express the marginal propensities as a function of the real interest rate, discount factor and survival probabilities:

$$\begin{aligned}
MPC^O &= 1 - (1-\omega^O)(1+r)^{\frac{1}{\gamma}-1} \beta^{\frac{1}{\gamma}} \\
MPC^Y &= \left(1 - \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1}{\gamma}-1} (1-\omega^Y)\Lambda^Y\right) \left(\frac{1}{\sigma} + \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1}{\gamma}-1} \omega^Y \Lambda^{YO} \frac{1}{MPC^O}\right)^{-1}
\end{aligned}$$

The pensioners' discount factor in the steady state is the following:

$$\Omega^O = \left(1 - \frac{1-\omega^O}{1+r}\right)^{-1}$$

We can express the young households' expected lifetime income-to-GDP ratio by using an initial guess of $\tilde{Y}$:

$$\frac{\tilde{Inc}^Y}{\tilde{Y}} = \left(1 - \frac{1-\omega^Y}{1+r}\right)^{-1} \left(\frac{ws^Y}{\tilde{Y}} + \frac{\tilde{Profit}}{\tilde{Y}} - \frac{\tilde{Tax}}{\tilde{Y}} + (1+g^N) \frac{\tilde{TR}^{YO}}{\tilde{Y}} \frac{\Omega^O}{1+r}\right)$$

Based on the young consumption function, one can substitute out the young consumption to

GDP in the budget constraint and express the young bond-to-GDP ratio:

$$\frac{\tilde{B}^Y}{\tilde{Y}} = \left(w\frac{\tilde{L}}{\tilde{Y}} + \frac{Pr\tilde{o}fit}{\tilde{Y}} - \frac{T\tilde{a}x}{\tilde{Y}} - MPC^Y \frac{I\tilde{n}c^Y}{\tilde{Y}} \right) \left(1 + (MPC^Y - 1)\frac{1+r}{1+g}(1-\omega^Y) \right)^{-1}$$

Now, we can express the old households' bond-to-GDP ratio from the bond market equilibrium:

$$\frac{\tilde{B}^O}{\tilde{Y}} = \frac{D\tilde{e}bt}{\tilde{Y}} - \frac{\tilde{B}^Y}{\tilde{Y}}$$

And, based on the consumption functions, we can calculate the consumption-to-GDP ratios:

$$\begin{aligned}
\frac{\tilde{C}^O}{\tilde{Y}} &= MPC^O \frac{\tilde{TR}}{\tilde{Y}}\Omega^O + MPC^O \frac{1+r}{1+g}\left[\omega^Y \frac{\tilde{B}^Y}{\tilde{Y}} + \frac{\tilde{B}^O}{\tilde{Y}} \right] \\
\frac{\tilde{C}^Y}{\tilde{Y}} &= MPC^Y \frac{I\tilde{n}c^Y}{\tilde{Y}} + MPC^Y \frac{1+r}{1+g}(1-\omega^Y)\frac{\tilde{B}^Y}{\tilde{Y}}
\end{aligned}$$

Finally, we need to check if the initial assumptions for r and $\tilde{Y}$ are correct. This means that we need to check if the two unused constraints (the market equilibrium condition and labor supply curve) are satisfied. Otherwise, the algorithm should choose another initial value until the two conditions are satisfied.

$$\begin{aligned}
1 &\stackrel{?}{=} \frac{\tilde{C^Y}}{\tilde{Y}} + \frac{\tilde{C^O}}{\tilde{Y}} + \frac{I\tilde{n}v}{\tilde{Y}} + \frac{G\tilde{o}v}{\tilde{Y}} \\
\frac{\tilde{C}^Y}{\tilde{Y}} &\stackrel{?}{=} \frac{\sigma}{1-\sigma} w \left(\frac{s^Y}{\tilde{Y}} - \frac{\tilde{L}}{\tilde{Y}} \right)
\end{aligned}$$

If we have the right initial values, we can calcuate the levels of all the normalized variables, and run the simulations.

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Appendix B

Appendix of Chapter 1

The appendix contains the summary of the model equations, the derivations of the simplified model, the parameters and the initial steady-state ratios, the steady-state calculations of bounded rationality equilibrium.

B.1 List of the model equations

Demography:

$$\begin{aligned}
s_t &= \frac{(1-\omega^O_{t-1})}{(1-\omega^Y_{t-1}+n_t)} s_{t-1} + \frac{\omega^Y_{t-1}}{(1-\omega^Y_{t-1}+n_t)} \\
s^Y_t &= \frac{1}{1+s_t} \\
1+g^{N,Y}_t &= 1-\omega^Y_{t-1}+n_t \\
1+g^{N,O}_t &= (1-\omega^O_{t-1}) + \frac{\omega^Y_{t-1}}{s_{t-1}} \\
1+g^N_t &= (1+g^{N,Y}_t)\frac{1+s_t}{1+s_{t-1}}
\end{aligned}$$

Retired households:

$$\begin{aligned}
\tilde{C^O_t} &= MPC^O_t \tilde{TR}_t \Omega^O_t + MPC^O_t \frac{(1+r_{t-1})}{1+g^N_t}\left(\omega^Y_{t-1}\tilde{B^Y_{t-1}} + \tilde{B^O_{t-1}}\right) \\
\Omega^O_t &= 1 + E_t \frac{1-\omega^O_t}{1+r_t}\Omega^O_{t+1} \\
\frac{1}{MPC^O_t} &= 1 + E_t(1-\omega^O_t)(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\frac{1}{MPC^O_{t+1}}
\end{aligned}$$

Worker households:

$$
\begin{aligned}
\tilde{C_t^Y} &= MPC_t^Y \tilde{Inc_t} + MPC_t^Y \frac{(1+r_{t-1})(1-\omega_{t-1}^Y)}{1+g_t^N} \tilde{B_{t-1}^Y} \\
\tilde{Inc_t} &= w_t s_t^Y + \tilde{Profit_t} - \tilde{Tax_t} + E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \tilde{Inc_{t+1}} \\
\frac{\tilde{C_t^Y}}{s_t^Y - \tilde{L}_t} &= \frac{\sigma}{1-\sigma} w_t \\
\frac{1}{MPC_t^Y} &= \frac{1}{\sigma} + E_t (1+r_t)^{\frac{1}{\gamma}-1} \left[(1-\omega_t^Y) \Lambda_t^Y \frac{1}{MPC_{t+1}^Y} + \omega_t^Y \Lambda_t^{YO} \frac{1}{MPC_{t+1}^O} \right] \\
\Lambda_t^Y &= \beta^{\frac{1}{\gamma}} \left(E_t \frac{w_{t+1}}{w_t} \right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \\
\Lambda_t^{YO} &= \left\{ \frac{\beta}{\sigma} \right\}^{\frac{1}{\gamma}} \left(\frac{1}{\frac{\sigma}{1-\sigma} w_t} \right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \\
\tilde{B_t^Y} &= w_t \tilde{L}_t + \tilde{Profit_t} - \tilde{Tax_t} - \tilde{C_t^Y} + \frac{(1+r_{t-1})}{1+g_t^N} (1-\omega_{t-1}^Y) \tilde{B_{t-1}^Y}
\end{aligned}
$$

Firms:

$$
\begin{aligned}
\tilde{Y}_t &= A_t \left(\frac{\tilde{K}_{t-1}}{1+g_t^N} \right)^{\alpha} \tilde{L}_t^{1-\alpha} \\
\tilde{K}_t &= \tilde{Inv_t} + (1-\delta) \frac{\tilde{K}_{t-1}}{1+g_t^N} \\
w_t &= (1-\alpha) \frac{\tilde{Y}_t}{\tilde{L}_t} \\
1+r_t &= E_t \alpha (1+g_{t+1}^N) \frac{\tilde{Y}_{t+1}}{\tilde{K}_t} + (1-\delta) \\
\tilde{Profit_t} &= \tilde{Y}_t - w_t \tilde{L}_t - \tilde{Inv_t}
\end{aligned}
$$

Fiscal policy:

$$
\begin{aligned}
\tilde{Debt_t} &= \tilde{TR_t} + \tilde{Gov_t} - \tilde{Tax_t} + \frac{(1+r_{t-1})}{1+g_t} \tilde{Debt_{t-1}} \\
\tilde{TR_t} &= \nu \frac{\omega_{t-1}^Y}{1+g_t^N} w_{t-1} \tilde{L}_{t-1} + \frac{(1-\omega_{t-1}^O)}{1+g_t^N} \tilde{TR_{t-1}}
\end{aligned}
$$

Equilibrium conditions:

$$
\begin{aligned}
\tilde{Debt_t} &= \tilde{B}_t^Y + \tilde{B}_t^O \\
\tilde{Y}_t &= \tilde{C_t^Y} + \tilde{C_t^O} + \tilde{Inv_t} + \tilde{Gov_t}
\end{aligned}
$$

B.2 Simplified model

The life-time income can be expressed as:

$$\tilde{Inc}_t = w_t s_t^Y + \tilde{Profit}_t - \tilde{Tax}_t + E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \tilde{Inc}_{t+1}$$

Plug in $\tilde{Profit}_t$:

$$\tilde{Inc}_t = w_t s_t^Y + \tilde{Y}_t - \tilde{Inv}_t - w_t \tilde{L}_t - \tilde{Tax}_t + E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \tilde{Inc}_{t+1}$$

Based on labor supply curve we can substitute out $w_t s_t^Y - w_t \tilde{L}_t$ with $\frac{1-\sigma}{\sigma} \tilde{C}_t^Y$, and in the next step we can also substitute out $\tilde{Y}_t$ with demand components in the goods market equilibrium:

$$\tilde{Inc}_t = \frac{1}{\sigma} \tilde{C}_t^Y + \tilde{C}_t^O + \tilde{Gov}_t - \tilde{Tax}_t + E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \tilde{Inc}_{t+1}$$

From the workers' consumption function we can express $\tilde{Inc}_t$:

$$\tilde{Inc}_t = \frac{\tilde{C}_t^Y}{MPC_t^Y} - \frac{(1+r_{t-1})(1-\omega_{t-1}^Y)}{1+g_t^N} \tilde{B}_{t-1}^Y$$

Plugging it back into the previous equation, and substituting out $\tilde{Gov}_t - \tilde{Tax}_t$ from the government budget constraint we get:

$$\frac{\tilde{C}_t^Y}{MPC_t^Y} - \frac{(1+r_{t-1})(1-\omega_{t-1}^Y)}{1+g_t^N} \tilde{B}_{t-1}^Y = \frac{1}{\sigma} \tilde{C}_t^Y + \tilde{C}_t^O + \tilde{Debt}_t - \tilde{TR}_t - \frac{(1+r_{t-1})}{1+g_t} \tilde{Debt}_{t-1} +$$
$$+ E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \left(\frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y} - \frac{(1+r_t)(1-\omega_t^Y)}{1+g_{t+1}^N} \tilde{B}_t^Y \right)$$

For the next steps we need to use the bonds market equilibrium and the retireds' budget constraint:

$$\tilde{Debt}_t = \tilde{B}_t^Y + \tilde{B}_t^O$$
$$\tilde{C}_t^O + \tilde{B}_t^O = \tilde{TR}_t + \frac{1+r_{t-1}}{1+g_t^N} \left(\omega_{t-1}^Y \tilde{B}_{t-1}^Y + \tilde{B}_{t-1}^O \right)$$

Substituting out $\tilde{Debt}_t$ from the bonds market equilibrium and $\tilde{C}_t^O - \tilde{TR}_t$ from the retired budget constraint, we can get the "dynamic IS-curve":

$$\frac{\tilde{C}_t^Y}{MPC_t^Y} = \frac{\tilde{C}_t^Y}{\sigma} + \tilde{B}_t^Y\left(1 - \frac{(1-\omega_t^Y)^2}{1+g_{t+1}^{N,Y}}\right) + E_t\frac{\omega_t^Y}{1+r_t}\nu\alpha\tilde{Y}_t\Omega_{t+1}^O + E_t\frac{1-\omega_t^Y}{1+r_t}\frac{1+s_{t+1}}{1+s_t}\frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y}$$

To determine the demand function for the workers' bond, we need to rearrange the workers' budget constraint, substitute out the $\tilde{Profit}_t$ and $\tilde{Y}_t$ from the goods market equilibrium

$$\begin{aligned}
\tilde{C_t^Y} + \tilde{B_t^Y} + \tilde{Tax}_t &= w_t\tilde{L}_t + \tilde{Profit}_t + \frac{(1+r_{t-1})}{1+g_t^N}(1-\omega_{t-1}^Y)B_{t-1}^{\tilde{Y}} \\
\tilde{B}_t^Y &= \tilde{C}_t^O + \tilde{Gov}_t - \tilde{Tax}_t + \frac{(1+r_{t-1})}{1+g_t^N}(1-\omega_{t-1}^Y)\tilde{B}_{t-1}^Y
\end{aligned}$$

As a next step we can substitute out the retirees' consumption from their consumption function:

$$\begin{aligned}
\tilde{B}_t^Y &= MPC_t^O\tilde{TR}_t\Omega_t^O + MPC_t^O\frac{(1+r_{t-1})}{1+g_t^N}\left(\omega_{t-1}^Y\tilde{B}_{t-1}^Y + \tilde{B}_{t-1}^O\right) + \\
&\quad + \tilde{Gov}_t - \tilde{Tax}_t + \frac{(1+r_{t-1})}{1+g_t^N}(1-\omega_{t-1}^Y)\tilde{B}_{t-1}^Y
\end{aligned}$$

We can substitute out $\tilde{B}_t^O$ from the bonds market equilibrium, and from the government budget constraint we can express $\tilde{Gov}_t - \tilde{Tax}_t$. Rearranging the equation we got the demand function for the workers' bond:

$$\tilde{B}_t^Y = \tilde{Debt}_t - (1 - MPC_t^O\Omega_t^O)\tilde{TR}_t - \frac{(1+r_{t-1})}{1+g_t^N}(1-MPC_t^O)\left[\tilde{Debt}_{t-1} - (1-\omega_{t-1}^Y)\tilde{B}_{t-1}^Y\right]$$

B.3 Bounded rationality and steady-state calculations

I assume level-k thinking for the workers, level-l thinking for the retired. Based on the formula we can rewrite the equation of MPC^O:

$$\frac{1}{MPC_t^{O,l}} = \sum_{i=1}^{l} \beta^{\frac{i-1}{\gamma}} \prod_{h=1}^{i-1} (1-\omega_{t+h-1}^O)(1+r_{t+h-1})^{\frac{1}{\gamma}-1} + \beta^{\frac{l}{\gamma}} \prod_{h=1}^{l} (1-\omega_{t+h-1}^O)(1+r_{t+h-1})^{\frac{1}{\gamma}-1} \frac{1}{MPC^{O,*}}$$

where $MPC^{O,*}$ is the initial (original steady-state) value of the MPC^O. And for the Ω^O

$$\Omega_t^{O,l} = \sum_{i=1}^{l} \prod_{h=1}^{i-1} \frac{1-\omega_{t+h-1}^O}{1+r_{t+h-1}} + \prod_{h=1}^{l} \frac{1-\omega_{t+h-1}^O}{1+r_{t+h-1}} \Omega_t^{O,*}$$

where $\Omega^{O,*}$ is the initial (original steady-state) value of the Ω^O.

The MPC^Y can be written as

$$\begin{aligned} \frac{1}{MPC_t^{Y,k}} &= \sum_{i=1}^{k} \prod_{h=1}^{i-1} (1-\omega_{t+h-1}^Y)(1+r_{t+h-1})^{\frac{1}{\gamma}-1} \Lambda_{t+h-1}^Y \left(\frac{1}{\sigma} + \omega_{t+i-1}^Y (1+r_{t+i-1})^{\frac{1}{\gamma}-1} \Lambda_{t+i-1}^{YO} \frac{1}{MPC_{t+i}^{O,k}} \right) + \\ &+ \prod_{h=1}^{k} (1-\omega_{t+h-1}^Y)(1+r_{t+h-1})^{\frac{1}{\gamma}-1} \Lambda_{t+h-1}^Y \frac{1}{MPC^{Y,*}} \end{aligned}$$

where $MPC^{Y,*}$ is the initial (steady-state) value of the MPC^Y. The dynamic IS-curve can be given by the following:

$$\left(\frac{1}{MPC_t^Y} - \frac{1}{\sigma} \right) \tilde{C}_t^Y = \tilde{B}_t^Y \left(1 - \frac{(1-\omega_t^Y)^2}{1+g_{t+1}^{N,Y}} \right) + E_t \frac{\omega_t^Y \nu \alpha \tilde{Y}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y}$$

If we rearrange and express $\tilde{C}_t^Y$ as the function of forward-looking terms:

$$\begin{aligned} \tilde{C}_t^{Y,k} &= \sum_{i=1}^{k} \left[\prod_{h=1}^{i-1} \frac{\sigma MPC_{t+h-1}^Y}{\sigma - MPC_{t+h-1}^Y} \frac{(1-\omega_{t+h-1}^Y)(1+s_{t+h})}{(1+r_{t+h-1})(1+s_t)} \frac{1}{MPC_{t+h}^Y} \right] \frac{\sigma MPC_{t+i-1}^Y}{\sigma - MPC_{t+i-1}^Y} \left(\tilde{B}_{t+i-1}^Y \left(1 - \frac{(1-\omega_{t+i-1}^Y)^2}{1+g_{t+i}^{N,Y}} \right) + \right. \\ &+ \left. \frac{\omega_{t+i}^Y \nu \alpha \tilde{Y}_{t+i-1} \Omega_{t+i}^O}{1+r_{t+i-1}} \right) + \left[\prod_{h=1}^{k} \frac{\sigma MPC_{t+h-1}^Y}{\sigma - MPC_{t+h-1}^Y} \frac{(1-\omega_{t+h-1}^Y)(1+s_{t+h})}{(1+r_{t+h-1})(1+s_t)} \frac{\tilde{C^Y}^*}{MPC_{t+h}^Y} \right] \end{aligned}$$

By the steady state calculations, the products are simplified into geometric sums, but all steady-state variables that have expectation terms depend on the initial steady-state values also.

The retireds MPC^O in the steady state can be given as

$$\frac{1}{MPC^{O,l}} = \sum_{i=1}^{l}\left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{i-1} + \left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{l}\frac{1}{MPC^{O,*}}$$

$$\frac{1}{MPC^{O,l}} = \frac{1-\left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{l}}{1-(1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}} + \left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{l}\frac{1}{MPC^{O,*}}$$

$$MPC^{O,l} = \left(\frac{1-\left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{l}}{1-(1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}} + \left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{l}\frac{1}{MPC^{O,*}}\right)^{-1}$$

According to the rational expectation theory, the $l\to\infty$ and $k\to\infty$, then MPC^O is independent from $MPC^{O,*}$. Despite bounded rationality, we get the same results if we assume $MPC^{O,*} = MPC^O$. For the initial steady-state calibration, where $MPC^{O,*} = MPC^O$ condition is satisfied, the rational expectation equilibrium is a valid solution of the bounded rational equilibrium. This can be applied for the other forward-looking equations also.
The discount factor can be written as

$$\Omega^{O,l} = \sum_{i=1}^{l}\left(\frac{1-\omega^O}{1+r}\right)^{i-1} + \left(\frac{1-\omega^O}{1+r}\right)^{l}\Omega^{O,*}$$

$$\Omega^{O,l} = \frac{1-\left(\frac{1-\omega^O}{1+r}\right)^{l}}{1-\frac{1-\omega^O}{1+r}} + \left(\frac{1-\omega^O}{1+r}\right)^{l}\Omega^{O,*}$$

The bonds market equilibrium in the steady state can be expressed as the following:

$$\tilde{B^Y} = \frac{\tilde{Debt}\left(1-\frac{(1+r)}{1+g}(1-MPC^{O,l})\right) + (MPC^{O,l}\Omega^{O,O,l}(r)-1)\tilde{TR}}{1-\frac{(1+r)}{1+g}(1-MPC^{O,l})(1-\omega^Y)}$$

For the young households' equation we need to differentiate the marginal propensity to consume and discount factor since the young and old have different level of thinking. The workers' MPC^Y is the following

$$\frac{1}{MPC^{Y,k}} = \left(\frac{1}{\sigma}+\frac{\omega^Y(1+r)^{\frac{1}{\gamma}-1}\Lambda^{YO}}{MPC^{O,k}}\right)\sum_{i=1}^{k}\left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{i-1} + \left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{k}\frac{1}{MPC^{Y,*}}$$

$$\frac{1}{MPC^{Y,k}} = \left(\frac{1}{\sigma}+\frac{\omega^Y(1+r)^{\frac{1}{\gamma}-1}\Lambda^{YO}}{MPC^{O,k}}\right)\frac{1-\left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{k}}{1-\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)} + \left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{k}\frac{1}{MPC^{Y,*}}$$

$$\frac{1}{MPC^{Y,k}} = \left(\left(\frac{1}{\sigma}+\frac{\omega^Y(1+r)^{\frac{1}{\gamma}-1}\Lambda^{YO}}{MPC^{O,k}}\right)\frac{1-\left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{k}}{1-\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)} + \left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{k}\frac{1}{MPC^{Y,*}}\right)^{-1}$$

Based on the workers' consumption function, we can express the steady-state version of workers' consumption function:

$$\tilde{C}^{Y,k} = \frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\left(\tilde{B}^{Y}\left(1-\frac{(1-\omega^{Y})^{2}}{1+g^{N,Y}}\right)+\frac{\omega^{Y}\nu\alpha\tilde{Y}\Omega^{O,k}}{1+r}\right)\sum_{i=1}^{k}\left[\frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\frac{1-\omega^{Y}}{1+r}\frac{1}{MPC^{Y,k}}\right]^{i-1} + \\ + \left[\frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\frac{1-\omega^{Y}}{1+r}\frac{\tilde{C}^{Y,*}}{MPC^{Y,k}}\right]^{k}$$

After some rearranging:

$$\tilde{C}^{Y,k} = \frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\left(\tilde{B}^{Y}\left(1-\frac{(1-\omega^{Y})^{2}}{1+g^{N,Y}}\right)+\frac{\omega^{Y}\nu\alpha\tilde{Y}\Omega^{O,k}}{1+r}\right)\frac{1-\left[\frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\frac{1-\omega^{Y}}{1+r}\frac{1}{MPC^{Y,k}}\right]^{k}}{1-\frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\frac{1-\omega^{Y}}{1+r}\frac{1}{MPC^{Y,k}}} + \\ + \left[\frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\frac{1-\omega^{Y}}{1+r}\frac{1}{MPC^{Y,k}}\right]^{k}\tilde{C}^{Y,*}$$

B.4 Rational expectations and decomposition of aging shock

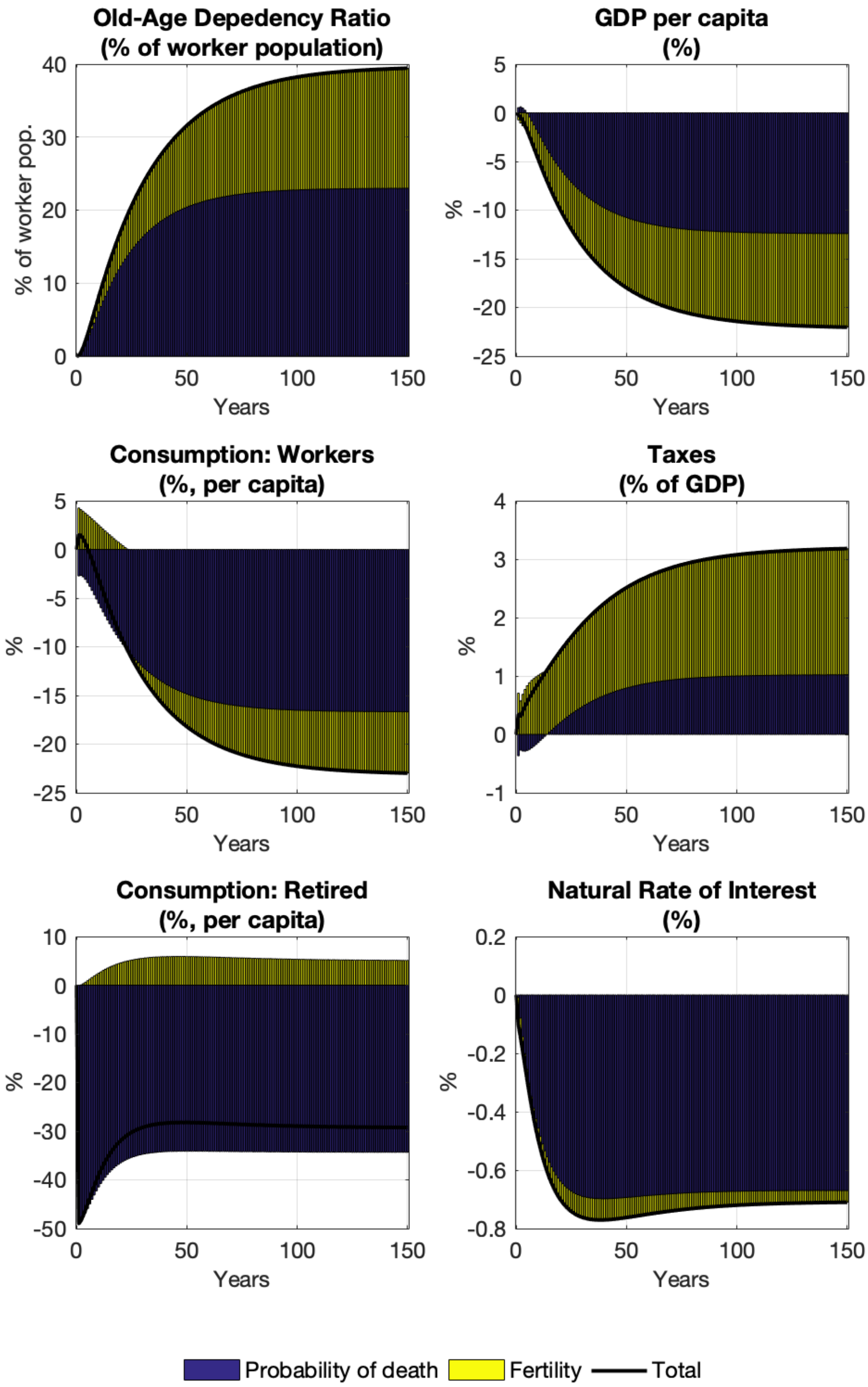

Figure B.1: Population aging and transitional dynamics in with $\frac{TR}{Y} = 0$

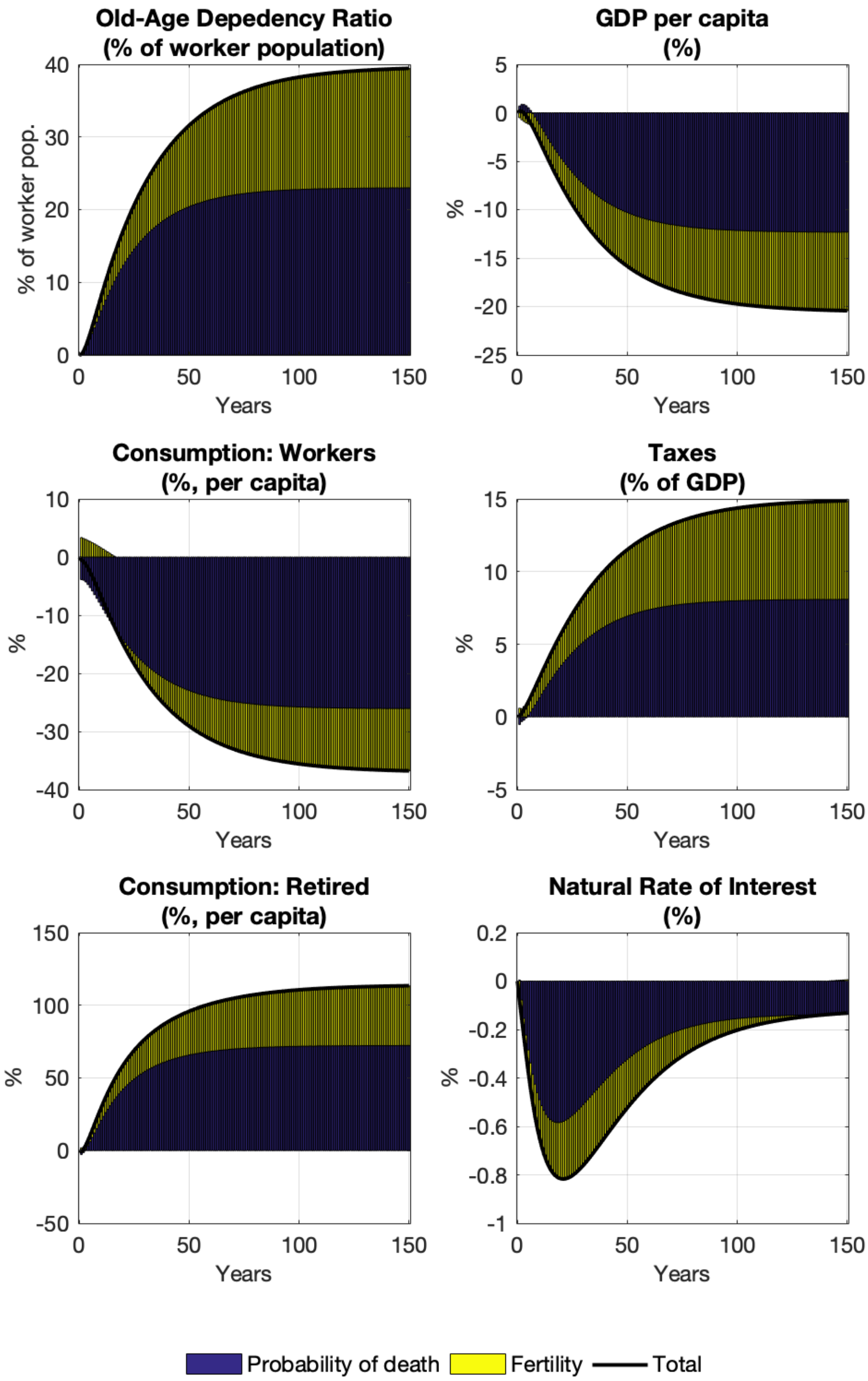

Figure B.2: Population aging and transitional dynamics in with $\frac{TR}{Y} = 0.06$

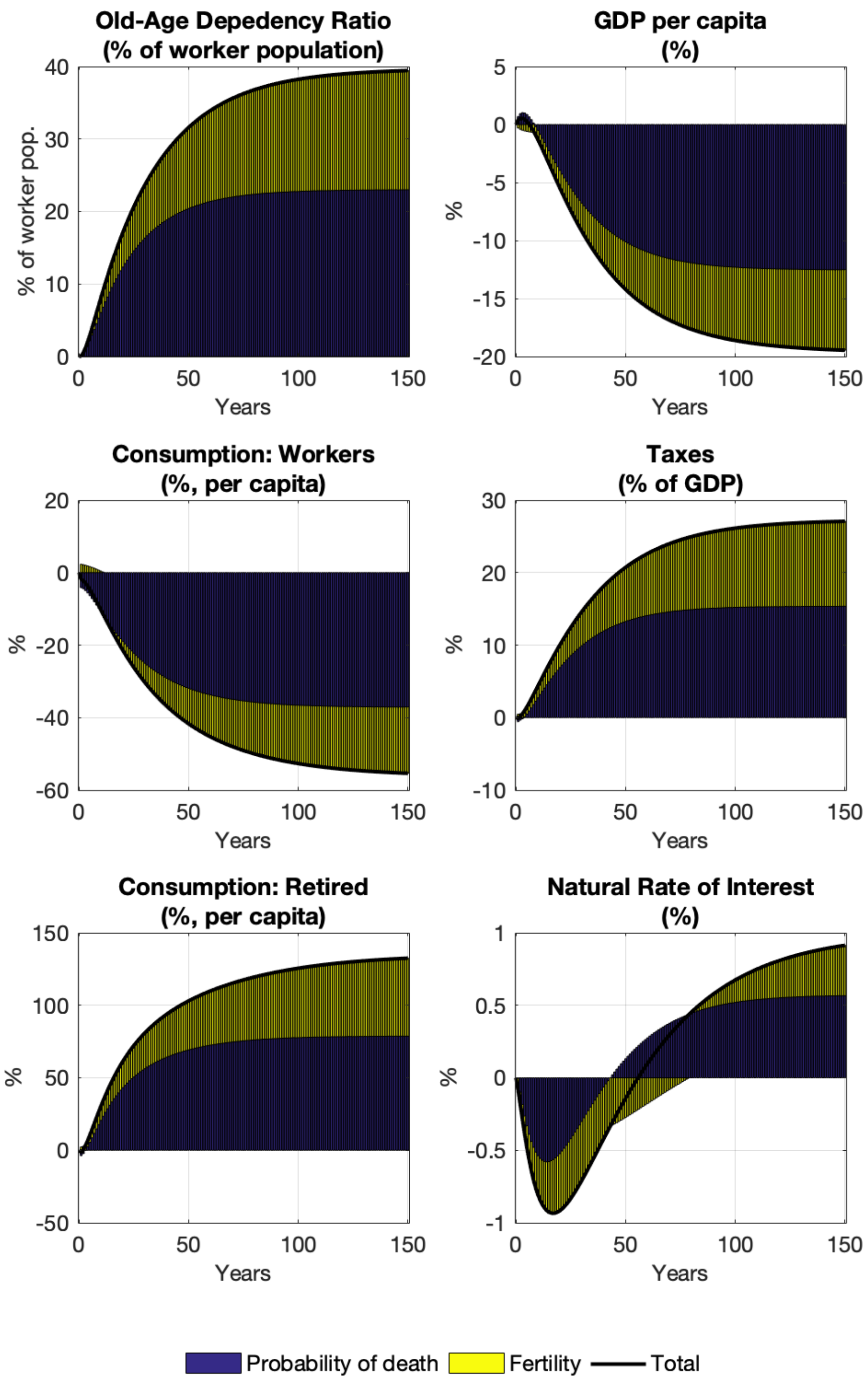

Figure B.3: Population aging and transitional dynamics in with $\frac{TR}{Y} = 0.118$

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Appendix C

Appendix of Chapter 2

C.1 Second-order approximation of utility based welfare loss function

In this appendix, we derive the second order approximation of the households' utility function. The social welfare function is defined as the weighted average of contemporaneous cohort level welfare functions. The central bank minimizes the population weighted average of utilities:

$$\mathcal{W}_t = s_t^Y \tilde{U}_t^Y(i) + (1 - s_t^Y)\tilde{U}_t^O(i)$$

where the $\mathcal{W}_t$ denotes the social welfare function, $\tilde{U}_t^Y(i)$ and $\tilde{U}_t^O(i)$ are the approximated welfare of the young and retired households, respectively.

Retired generation

'Retired' agent i of retired cohort is one individual:

$$U_t^O(i) = \frac{1}{1-\gamma}\left\{C_t^O(i)\right\}^{1-\gamma}$$

Second-order approximation:

$$U_t^O(i) = U^O(i) + U_{C^O}^O\left(C_t^O(i) - C^O(i)\right) + \frac{1}{2}U_{C^O(i),C^O(i)}^O\left(C_t^O(i) - C^O(i)\right)^2 + o(2)$$
$$U_{C^O(i)}^O = C^O(i)^{-\gamma}$$
$$U_{C^O(i),C^O(i)}^O = -\gamma C^O(i)^{-\gamma-1}$$

or

$$U_t^O(i) = U^O(i) + U^O_{C^O(i)} C^O(i) \frac{C_t^O(i) - C^O(i)}{C^O(i)} + \frac{1}{2} U^O_{C^O(i),C^O(i)} C^O(i)^2 \left(\frac{C_t^O(i) - C^O(i)}{C^O(i)} \right)^2$$

We can also express the second-order approximation of consumption changes:

$$\frac{C_t^O(i) - C^O}{C^O} = \hat{C}_t^O(i) + \frac{1}{2} \hat{C}_t^O(i)^2 + o(3)$$

Putting them together:

$$U_t^O(i) = U^O(i) + U^O_{C^O(i)} C^O(i) \left(\hat{C}_t^O(i) + \frac{1}{2} \hat{C}_t^O(i)^2 + \frac{1}{2} \frac{U^O_{C^O(i),C^O(i)} C^O(i)^2}{U^O_{C^O(i)} C^O(i)} \left(\frac{C_t^O(i) - C^O(i)}{C^O(i)} \right)^2 \right)$$

$$U_t^O(i) = U^O(i) + U^O_{C^O(i)} C^O(i) \left(\hat{C}_t^O(i) + \frac{1-\gamma}{2} \hat{C}_t^O(i)^2 \right)$$

where $\hat{C}_t^O(i)^2$ are the unconditional variance of the retired households consumption, we assume that the monetary policy minimizes the variance of the variable:

$$\tilde{U}_t^O(i) = -(1-\gamma) U^O(i) \frac{\gamma - 1}{2} \hat{C}_t^O(i)^2$$

As a simplification, we assume that the individual consumption is equal with the average consumption of the pensioners:

$$C_t^O(i) \approx \frac{C_t^O}{N_t^O} = \frac{\tilde{C}_t^O}{1 - s_t^Y}$$

Young generation

The second-order approximation of young generation's utility function can be given as

$$\begin{aligned} U_t^Y(i) &= U^Y(i) + U^Y_{C^Y(i)} \left(C_t^Y(i) - C^Y(i) \right) + \frac{1}{2} U^Y_{C^Y(i),C^Y(i)} \left(C_t^Y(i) - C^Y(i) \right)^2 + \\ &+ U^Y_{L(i)} \left(L_t(i) - L(i) \right) + \frac{1}{2} U^Y_{L(i),L(i)} \left(L_t(i) - L(i) \right)^2 \\ &+ U^Y_{C^Y,L(i)} \left(C_t^Y(i) - C^Y(i) \right) \left(L_t(i) - L(i) \right) \end{aligned}$$

As a first step we need to calculate the derivatives of the following utility function:

$$U^Y(i) = \frac{1}{1-\gamma} \left\{ C^Y(i)^\sigma \left(1 - L(i)\right)^{1-\sigma} \right\}^{1-\gamma}$$

The first partial derivative of consumption:

$$\begin{aligned}
U^Y_{C^Y(i)} &= \left\{C^Y(i)^{\sigma}\left(1-L(i)\right)^{(1-\sigma)}\right\}^{-\gamma}\sigma C^Y(i)^{\sigma-1}\left(1-L(i)\right)^{(1-\sigma)} \\
&= \sigma C^Y(i)^{\sigma-1-\gamma\sigma}\left(1-L(i)\right)^{(1-\sigma)(1-\gamma)} \\
&= \sigma(1-\gamma)\frac{U^Y(i)}{C^Y(i)}
\end{aligned}$$

The second partial derivative of consumption:

$$\begin{aligned}
U^Y_{C^Y(i),C^Y(i)} &= \sigma(\sigma-1-\gamma\sigma)C^Y(i)^{\sigma-2-\gamma\sigma}\left(1-L(i)\right)^{(1-\sigma)(1-\gamma)} \\
&= \sigma(\sigma-1-\gamma\sigma)(1-\gamma)\frac{U^Y(i)}{C^Y(i)^2}
\end{aligned}$$

The first partial derivative of labor:

$$\begin{aligned}
U^Y_{L(i)} &= -\left\{C^Y(i)^{\sigma}\left(1-L(i)\right)^{(1-\sigma)}\right\}^{-\gamma}(1-\sigma)C^Y(i)^{\sigma}\left(1-L(i)\right)^{-\sigma} \\
&= -(1-\sigma)C^Y(i)^{\sigma-\gamma\sigma}\left(1-L(i)\right)^{-\sigma-\gamma(1-\sigma)} \\
&= -(1-\sigma)(1-\gamma)\frac{U^Y(i)}{1-L(i)}
\end{aligned}$$

The second partial derivative of labor:

$$\begin{aligned}
U^Y_{L(i),L(i)} &= -(1-\sigma)(\sigma+\gamma(1-\sigma))C^Y(i)^{\sigma-\gamma\sigma}\left(1-L(i)\right)^{-\sigma-\gamma(1-\sigma)-1} \\
&= -(1-\sigma)(\sigma+\gamma(1-\sigma))(1-\gamma)\frac{U^Y(i)}{(1-L(i))^2}
\end{aligned}$$

Due to the non-separable utility functions, the cross partial derivatives are non-zero:

$$\begin{aligned}
U^Y_{L(i),C^Y(i)} &= -\sigma(1-\sigma)(1-\gamma)C^Y(i)^{\sigma-1-\gamma\sigma}\left(1-L(i)\right)^{(1-\sigma)(1-\gamma)-1} \\
&= -\sigma(1-\sigma)(1-\gamma)^2\frac{U^Y(i)}{C^Y(i)(1-L(i))}
\end{aligned}$$

Based on the partial derivatives above we can express the following ratios that can be used later

$$\begin{aligned}
\frac{U^Y_{C^Y(i),C^Y(i)}C^Y(i)}{U^Y_{C^Y(i)}} &= \sigma-1-\gamma\sigma = -(\sigma(\gamma-1)+1) \\
\frac{U^Y_{L(i),L(i)}L(i)}{U^Y_{L(i)}} &= (\sigma+\gamma(1-\sigma))\frac{L(i)}{1-L(i)}
\end{aligned}$$

We can rewrite the approximated individual utility function:

$$\begin{aligned}
U_t^Y(i) &= U^Y(i) + U^Y_{C^Y(i)} C^Y(i) \left\{ \frac{C_t^Y(i) - C^Y(i)}{C^Y(i)} + \frac{1}{2} \frac{U^Y_{C^Y(i),C^Y(i)} C^Y(i)}{U^Y_{C^Y(i)}} \left(\frac{C_t^Y(i) - C^Y(i)}{C^Y(i)} \right)^2 \right\} + \\
&+ U^Y_{L(i)} L(i) \left\{ \frac{L_t(i) - L(i)}{L(i)} + \frac{1}{2} \frac{U^Y_{L(i),L(i)} L(i)}{U^Y_{L(i)}} \left(\frac{L_t(i) - L(i)}{L(i)} \right)^2 \right\} \\
&+ U^Y_{C^Y(i),L(i)} \left(C_t^Y(i) - C^Y(i) \right) \left(L_t(i) - L(i) \right)
\end{aligned}$$

Rearranging the equation above:

$$\begin{aligned}
U_t^Y(i) &= U^Y(i) + U^Y_{C^Y(i)} C^Y(i) \left(\hat{C}_t^Y(i) - \frac{\sigma(\gamma-1)}{2} \hat{C}_t^Y(i)^2 \right) + \\
&+ U_L^Y(i) L(i) \left(\hat{L}_t(i) + \frac{1}{2} \left(1 + (\sigma + \gamma(1-\sigma)) \frac{L(i)}{1-L(i)} \right) \hat{L}_t(i)^2 \right) \\
&+ U^Y_{C^Y(i),L(i)} C^Y(i) L(i) \hat{C}_t^Y(i) \hat{L}_t(i)
\end{aligned}$$

We can use the results for $U^Y_{C^Y(i)}$, $U^Y_{L(i)}$ and $U^Y_{C^Y(i),L(i)}$ to simplify it further:

$$\begin{aligned}
U_t^Y(i) &= U^Y(i) + \sigma(1-\gamma) U^Y(i) \left(\hat{C}_t^Y(i) - \frac{\sigma(\gamma-1)}{2} \hat{C}_t^Y(i)^2 \right) + \\
&- (1-\sigma)(1-\gamma) U^Y(i) \frac{L(i)}{1-L(i)} \left(\hat{L}_t(i) + \frac{1}{2} \left(1 + (\sigma + \gamma(1-\sigma)) \frac{L(i)}{1-L(i)} \right) \hat{L}_t(i)^2 \right) \\
&- \sigma(1-\sigma)(1-\gamma)^2 U^Y(i) \frac{L(i)}{1-L(i)} \hat{C}_t^Y(i) \hat{L}_t(i)
\end{aligned}$$

The labor demand and wage and price dispersion:

$$\begin{aligned}
\hat{L}_t &= \log \int_0^1 \left(\frac{V_t(j)}{V_t} \right)^{-\varphi_W} dj + \log \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\varphi} dj + \hat{Y}_t + \hat{mc}_t - \hat{v}_t \\
&= \frac{\varphi_W}{2} var(V_t(j)) + \frac{\varphi_P}{2} var(P_t(j)) + \hat{Y}_t + \hat{mc}_t - \hat{v}_t
\end{aligned}$$

The variances can given from Woodford (2003):

$$\frac{\varphi_P}{2} var(P_t(j)) = \frac{\varphi_P}{2\lambda_P} \left(\pi_t - \gamma_P \pi_{t-1} \right)^2$$
$$\frac{\varphi_W}{2} var(V_t(j)) = \frac{\varphi_W}{2\lambda_V} \left(\pi_t^V - \gamma_P \pi_{t-1}^V \right)^2$$

The central bank minimizes the volatility of the variables:

$$
\begin{aligned}
\tilde{U}_t^Y(i) &= -\sigma(1-\gamma)U^Y(i)\frac{\sigma(\gamma-1)}{2}\hat{C}_t^Y(i)^2 + \\
&- (1-\sigma)(1-\gamma)U^Y(i)\frac{L(i)}{1-L(i)}\left(\frac{\varphi_P}{2\lambda_P}\left(\pi_t-\gamma_P\pi_{t-1}\right)^2+\frac{\varphi_W}{2\lambda_V}\left(\pi_t^V-\gamma_P\pi_{t-1}^V\right)^2\right) + \\
&- (1-\sigma)(1-\gamma)U^Y(i)\frac{L(i)}{1-L(i)}\frac{1}{2}\left(1+(\sigma+\gamma(1-\sigma))\frac{L(i)}{1-L(i)}\right)\hat{L}_t(i)^2 \\
&- \sigma(1-\sigma)(1-\gamma)^2U^Y(i)\frac{L(i)}{1-L(i)}\hat{C}_t^Y(i)\hat{L}_t(i)
\end{aligned}
$$

where the approximated level of individual consumption and individual labor can be given by:

$$
\begin{aligned}
C_t^Y(i) &\approx \frac{C_t^Y}{N_t^Y} = \frac{\tilde{C}_t^Y}{s_t^Y} \\
L_t(i) &\approx \frac{L_t}{N_t^Y} = \frac{\tilde{L}_t}{s_t^Y}
\end{aligned}
$$

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C.2 Welfare Functions and Optimal Reactions

Figure C.1: Welfare functions with the baseline monetary policy rules by ad hoc welfare leos

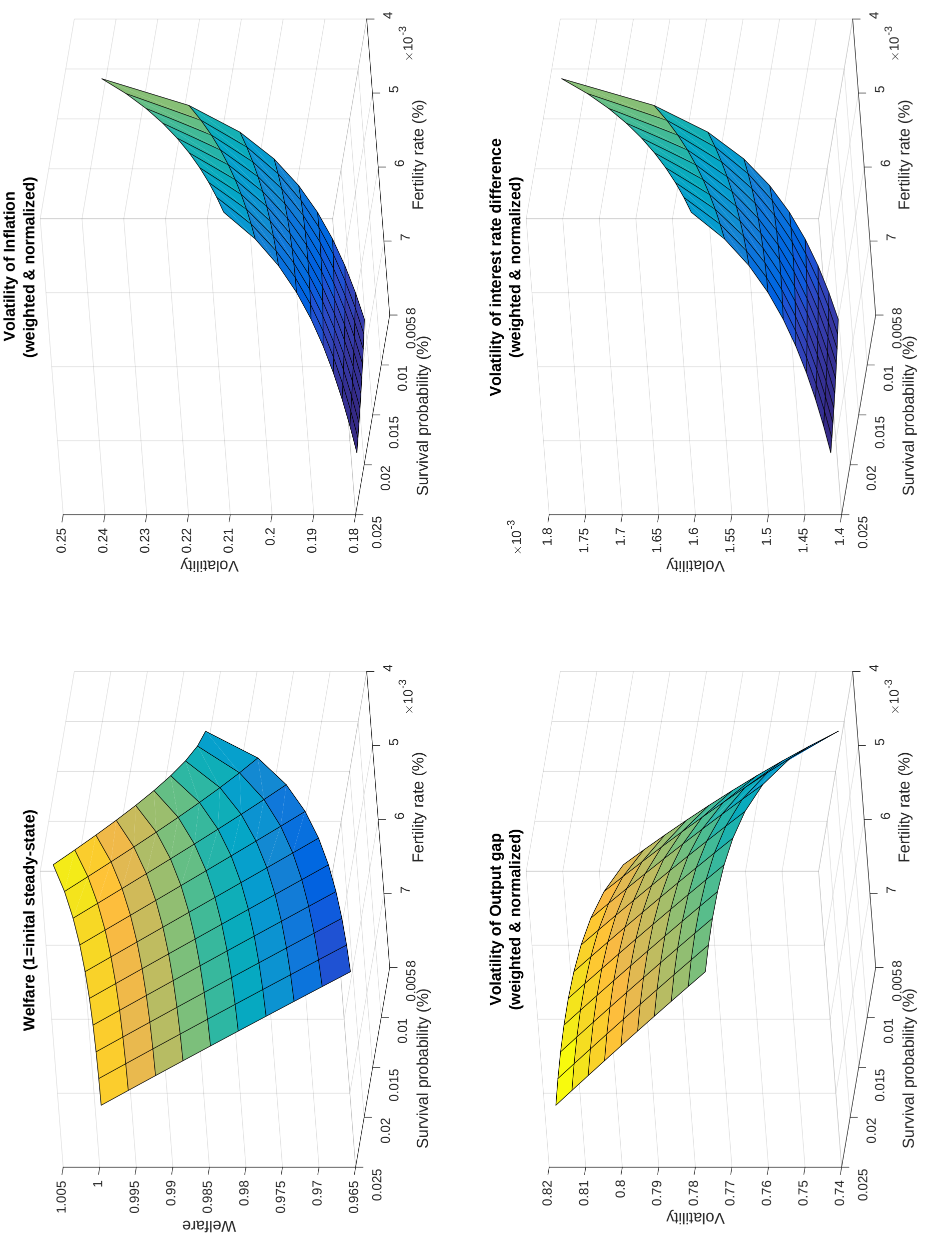

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Figure C.2: Optimal reactions by ad hoc welfare loss (1)

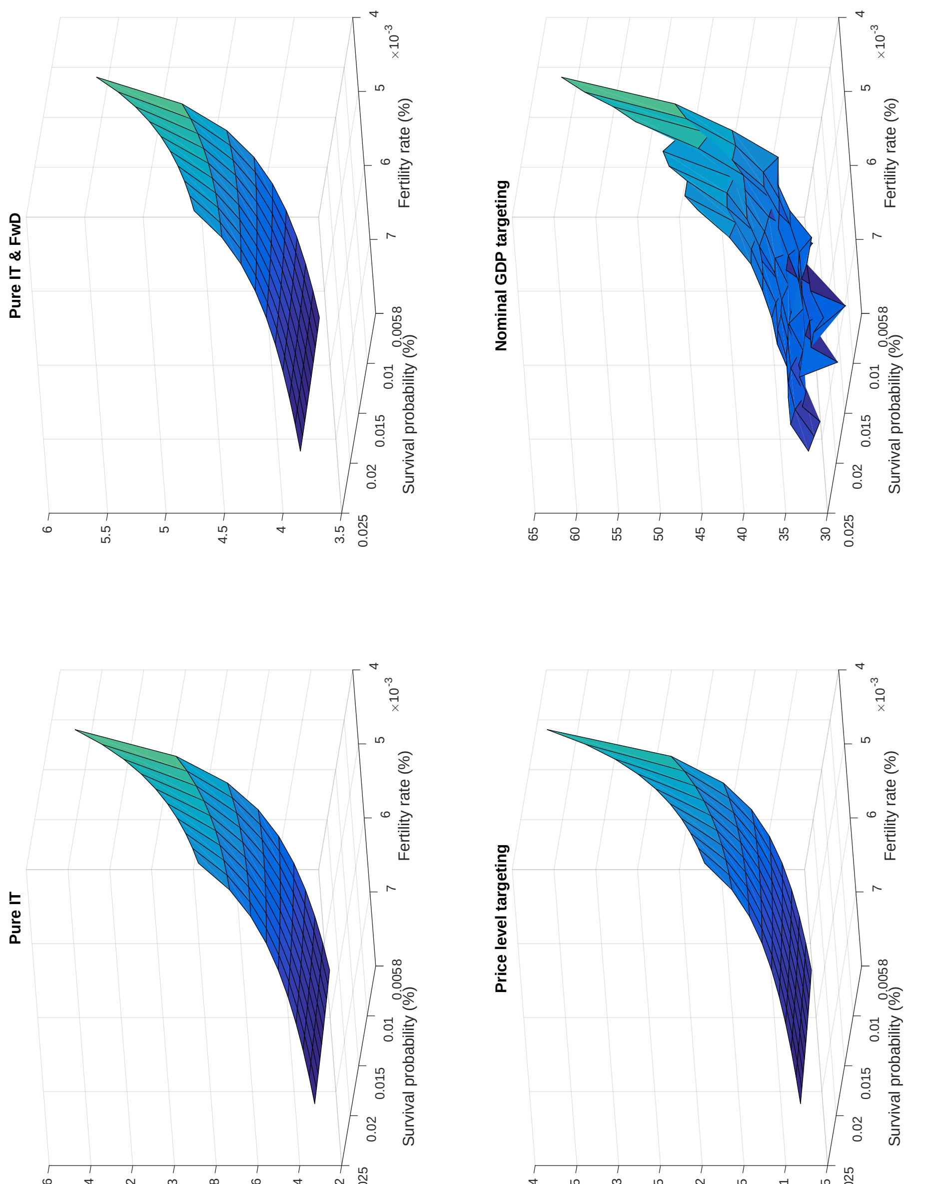

Figure C.3: Optimal reactions by ad hoc welfare loss (2)

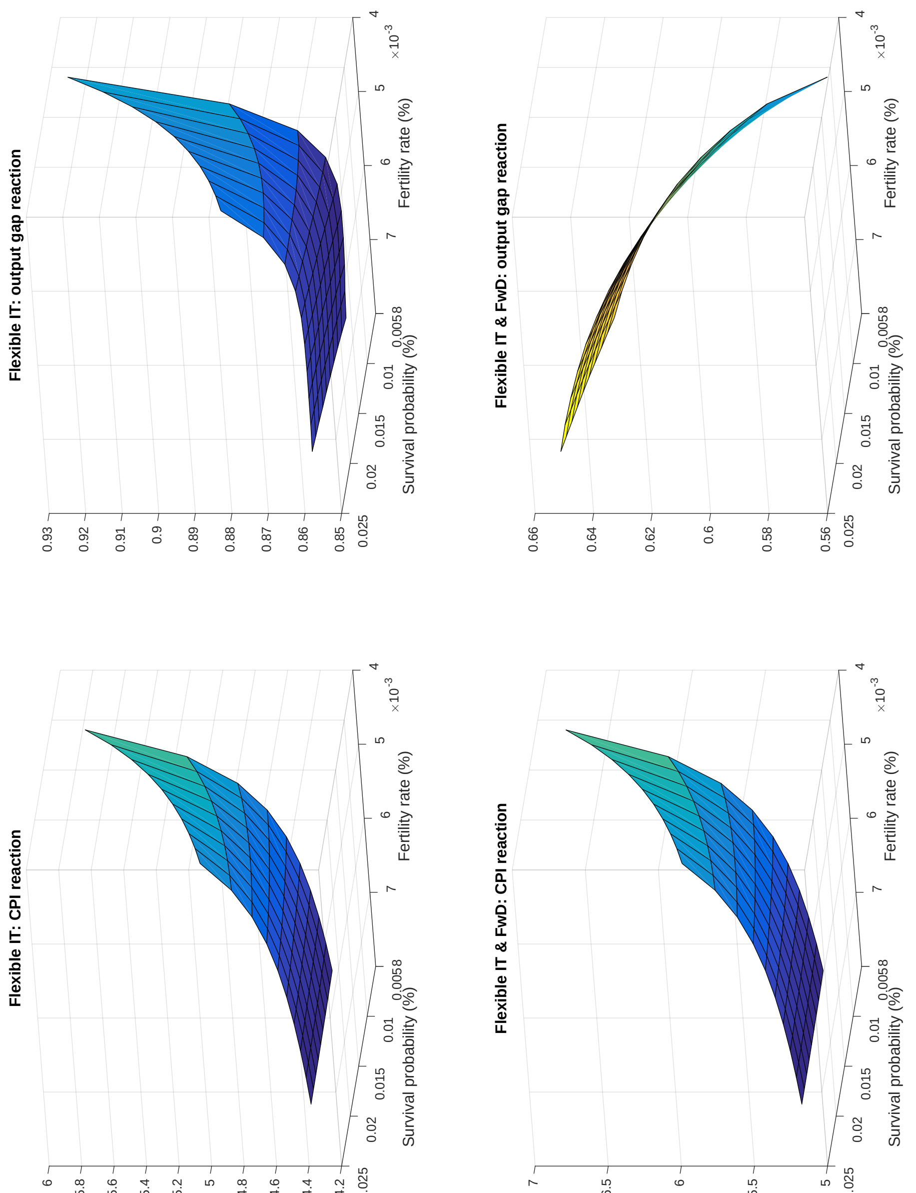

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Figure C.4: Welfare functions with the baseline monetary policy rules by utility function

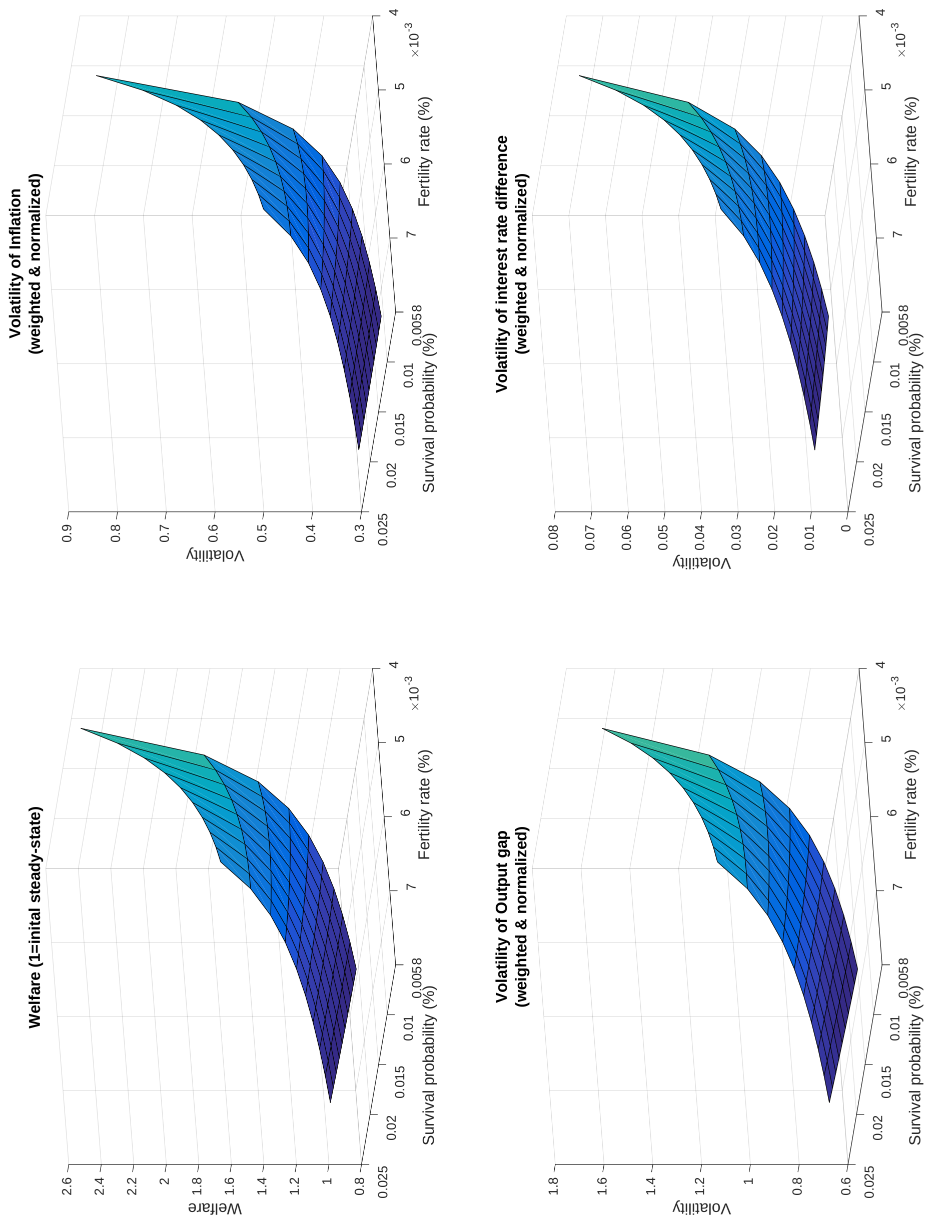

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Figure C.5: Optimal reactions by utility function (1)

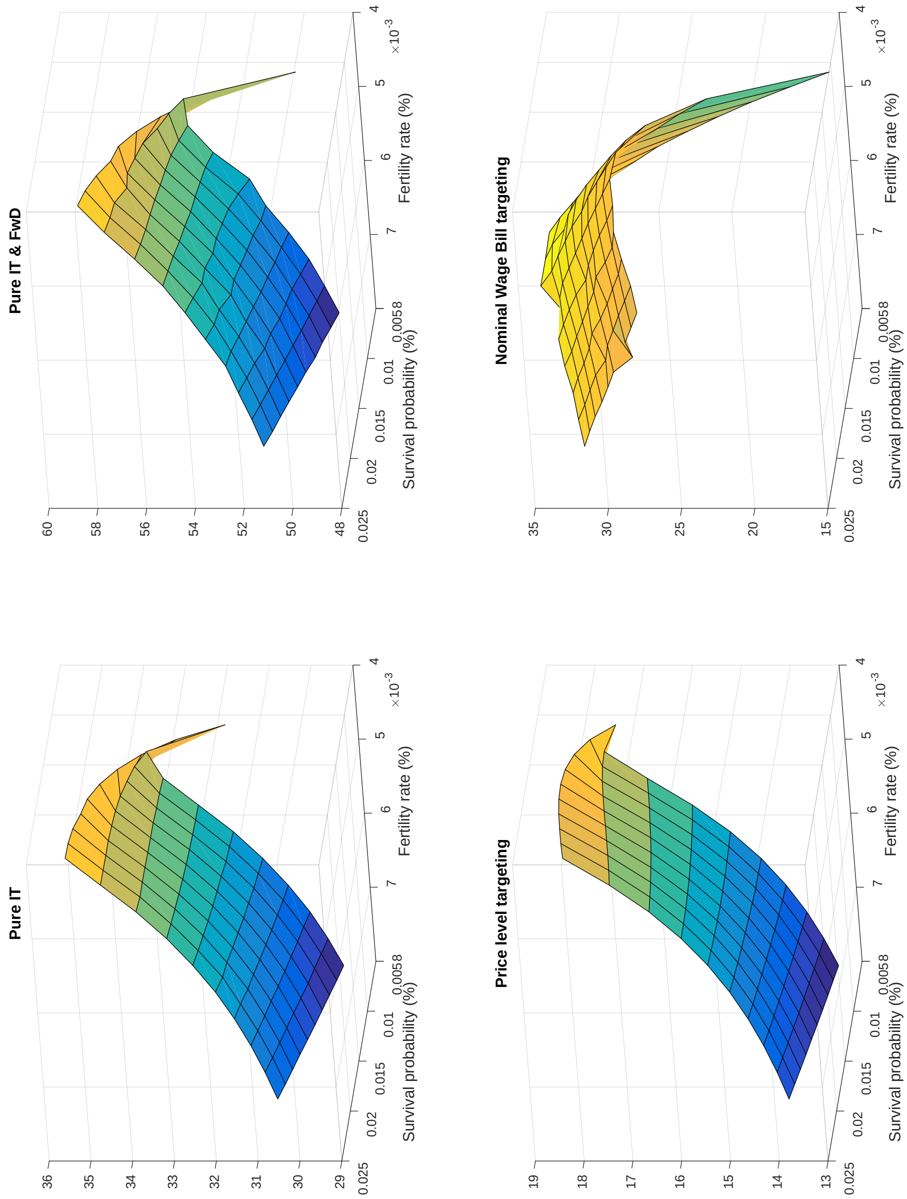

Figure C.6: Optimal reactions by utility function (2)