Graphs, groups and measures

Doctoral Thesis

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Declaration

I, the undersigned Ferenc Bencs, candidate for the degree of Doctor of Philosophy at the Central European University Department of Mathematics and its Applications, declare herewith that the present thesis is based on my research and only such external information as properly credited in notes and bibliography.

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1

Introduction

In this thesis, we will investigate some combinatorial and algebraic objects through polynomial or probability theory. In the first part, we will investigate a kind of generating function for certain combinatorial objects. In the second part, we will examine a probability generalization of normal subgroups (the invariant random subgroups) in some self-similar groups.

As a usual strategy for investigating combinatorial objects, for example, the matchings in a graph, independent sets in a graph, or the number of colorings, etc., one introduces a polynomial that encodes the interesting quantities. In the case of finite objects, this encoding can be given as a valuation or as the coefficients of a polynomial. As a motivation, we choose the already well-studied problem of the number of matchings. We call an edge set of a graph to be a matching if no two chosen edges share the same endpoint. Let $m_k(G)$ be the number of matchings of size k in the graph, then

$$M(G,x) = \sum_{k=0}^{\nu(G)} m_k(G)x^k$$

is the modified matching polynomial of the graph G. We can also think about this object as the normalization constant for the random matching $\mathbb{M}_x$ in G, where we pick a random matching $M \subseteq E(G)$ proportional to $x^{|M|}$. In statistical physics, this probability measure appears as the stationary distribution of a Markov process on the matchings also known as Glauber dynamics. Also, observe that $\mathbb{M}_x$ is converging in distribution to the uniform maximal matching as $x \to \infty$.

As the main point of interest, on the one hand, one would like to see if there are any characteristics of this probability measure that change abrupt (i.e. if there is a phase transition) as we tune the parameter over all $x > 0$. On the other hand, the second question is whether we can approximate the 'energy/entropy' of the matchings, that is

$$H(\mathbb{M}_x) = \frac{\log(M(G,x))}{|V(G)|}?$$

Since counting matchings in a graph is a self-reducible problem, therefore by [43], the approximation $M(G,x)$ (or $H(\mathbb{M}_x)$) is equivalent to the existence of a full polynomial time almost uniform sampler of M_x (called FPAUS). This shows a connection between the previous two questions. Thus, a natural construction of an FPAUS is to show that the above mentioned Markov process has nice mixing properties [44, 53].

Another possible approach for capturing the lack of phase transitions is due to Barvinok's Taylor series approach [6] that was used in the work of Patel and Regts [56]. Their designed approximation algorithm (FPTAS) is based on the observation that the modified matching polynomial of a class of graphs has no complex roots that accumulate to $[0,\infty)$. According to the celebrated theorem of Heilmann and Lieb [41], we know that $M(G,x)$ has only negative real roots. Therefore, for any reasonable graph family, we will not observe a phase transition. Because of the success of the investigation of the location of zeros of the matching polynomial, it might be worth to investigate other graph polyomials as well. In this thesis, these polynomials will be the independence polynomial and the partition function of the Potts model.

In the first chapter of this thesis, we will investigate the roots of the independence polynomial, that is also known as the partition function of the hard-core lattice gas model. This polynomial is defined similarly to the matching polynomial, but now the coefficient of x^k will count the number of independent sets of size k in G. As a part of this chapter, we will give a relaxation of a theorem of Chudnovsky and Seymour [23], and give a unified approach to prove whether graphs have only real-rooted independence polynomial.

Then, in the next chapter, we consider a different partition function, the so-called Potts model on q states. We will prove that if the number of colors q is at least $e\Delta(G)+1$, then the Potts model has no phase transition as $b \in [0,1]$. In particular, we will prove that the number of q colorings is approximable for graphs of degree at most Δ

The second part of the thesis will resemble the first part. To investigate some substructures of permutations, we will encode them into a polynomial, and prove some of its properties. Fix a pattern $\sigma \in S_k$, and fix some positions $I \subseteq [n]$, then we are interested in the number and the growth rate of permutations $\pi \in S_n$, such that we start to see the subpermutation σ only at the position $i \in I$. In this thesis, we will focus on the descending positions. For a fixed permutation $\sigma \in S_n$, we define the descending positions of σ as the set of indices $\{i \in [n-1] \mid \sigma_i > \sigma_{i+1}\}$. We denote the number of elements of S_n that have descending positions exactly at I by $d(I,n)$. This can be regarded as a function of n, and it was proven to be a polynomial in n (see [52]) of degree $\max(I)$. Similarly to the chromatic polynomial, the coefficients of $d(I,n)$ have combinatorial meanings in different polynomial bases. We will also investigate the location of the complex zeros of these descending polynomials.

The last part of the thesis is related to group theory, where we will prove some interesting phenomena. We will consider self-similar groups acting on the infinite d-ary tree, and we will show that it has continuum many distinct ergodic invariant random subgroups, that form a conjugation invariant probability measure on the space of subgroups. The key idea is to understand the action of the subgroups on the boundary of the tree, and gain information on the IRS. For instance, we will prove that if the action on the boundary gives finite orbit-closure classes, then the IRS is just a random conjugation of a finite indexed subgroup.

2

Zeros of independence polynomials

In this chapter, we will examine the independence polynomial, that is defined for every graph G as

$$I(G,x) = \sum_{k\geq 0} i_k(G)x^k,$$

where $i_k(G)$ denotes the number of independent sets of G of size k. We set $i_0(G) = 1$, since the empty set is always an independent set by definition. We can think about $I(G,x)$ as the generator function or weighted sum of independent sets of G, that is

$$I(G,x) = \sum_{A\in\mathcal{F}(G)} x^{|A|},$$

where $\mathcal{F}(G)$ is the set of subsets of G that are independent sets.

It is clear that this polynomial is not zero for any $x \geq 0$, thus, one might wonder about the root that is the closest to 0. It turns out that the shortest complex root $-\beta(G)$ for a connected graph is well defined (by that we mean it is a unique shortest complex root), moreover, it is always negative (so $\beta(G) > 0$). Also, this parameter has a nice monotonicity property:

Theorem 2.1. *[32, 38] Let G be a connected graph. Then $I(G,x)$ has a zero in the interval $[-1,0)$, and let $-\beta(G)$ be the largest among them. Then $-\beta(G)$ is a simple zero of $I(G,x)$, and if $\xi \neq -\beta(G)$ is a zero of $I(G,x)$, then $\beta(G) < |\xi|$.*

Theorem 2.2. *[24] Let G be a connected graph and H be a proper subgraph of G. Then $\beta(H) > \beta(G)$.*

The proof of this monotonicity result depends on the observation that the Taylor series of the fraction of independence polynomials of an induced subgraph and the whole graph have alternating integer coefficients (in particular, it is a growth function of some elements in a trace monoid [18, 32]).

Theorem 2.3. *[24] Let G be a connected graph and H be an induced subgraph, then in the following series*

$$\frac{I(H,x)}{I(G,x)} = \sum_{k\geq 0}(-1)^k r_k(G,H)x^k,$$

for each $k \geq 0$ the coefficients $r_k(G,H)$ are positive integers.

The previously mentioned theorems play the key role to prove that for a graph G of maximum degree at most d the ball of radius $\beta_d = \frac{(d-1)^{d-1}}{d^d}$ doesn't contain any roots of $I(G,x)$ (see [62]). In other words, the ball of radius β_d around 0 (called Shearer's disk) is a zero-free region for graphs of degree at most d.

For another theorem about the location of the roots of $I(G,x)$ depending on some structures of G, let us recall the matching polynomial as one of our main motivations.

For a graph G the matching polynomial is defined in a similar way as the independence polynomial:

$$\mu(G,x) = \sum_{k\geq 0}(-1)^k m_k(G)x^{n-2k},$$

where $m_0(G) = 1$, and if $k \geq 1$, then $m_k(G)$ is the number of matchings with k edges. One could also define as a transformation of an independence polynomial, that is

$$\mu(G,x) = x^n I(L(G), -x^{-2}),$$

where $L(G)$ denotes the line-graph of G.

When Heilmann and Lieb [41] introduced the theory of matching polynomials, they already noticed that matching polynomials show strong analogies with orthogonal polynomials, and in some instances they are indeed orthogonal polynomials. For instance, one can show that,

$$\mu(C_n,x) = 2\cdot T_n^{(1)}\left(\frac{x}{2}\right),$$
$$\mu(P_n,x) = T_n^{(2)}\left(\frac{x}{2}\right),$$
$$\mu(K_n,x) = 2^{-n/2}H_n\left(\frac{x}{\sqrt{2}}\right)$$

where $T_n^{(1)}$, $T_n^{(2)}$ are the Chebyshev polynomials of the first and second kind, H_n is the Hermitian polynomial; and P_n, C_n and K_n are respectively the path, the cycle and the complete graph on n vertices. In [41] the following Christoffel–Darboux identities were proved:

Theorem 2.4. *Let G be a graph and $u, v \in V(G)$. Let $\mathcal{P}_{u,v}$ be the set of paths from u to v, and for a path $P \in \mathcal{P}_{u,v}$ let us denote by $G - P$ the subgraph induced by the complement of $V(P) \subseteq V(G)$. Then*

$$\mu(G-u,x)\mu(G-v,x) - \mu(G,x)\mu(G-u-v,x) = \sum_{P \in \mathcal{P}_{u,v}} \mu(G-P,x)^2.$$

Theorem 2.5. *Let G be a graph and $u \in V(G)$. Let $\mathcal{P}_u$ be the set of paths starting from u. Then*

$$\mu(G,x)\mu(G-u,y) - \mu(G-u,x)\mu(G,y) =$$
$$= (x-y) \sum_{P \in \mathcal{P}_u} \mu(G-P,x)\mu(G-P,y).$$

We remark that Theorem 2.5 provides a fast proof of the fact that all matching polynomials have only real zeros.

In the first part of this chapter, we prove analogous statements for independence polynomials and give a few corollaries along with Theorem 2.11, that is a slight extension of a theorem of Chudnovsky and Seymour [23], which states that the independence polynomial of a claw-free graph has only real roots.

Theorem 2.11. *Let G be a graph and let $\mathcal{B}(G)$ be the set of connected induced subgraphs of G, and denote $bd(G) = \max_{(A,B;F) \in \mathcal{B}(G)}(||A| - |B||)$. Then $I(G,x)$ doesn't have any root in*

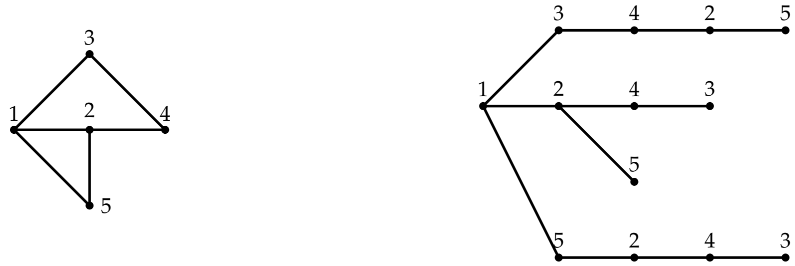
$$\left\{ z \in \mathbb{C} \;\middle|\; |arg(z)| < \frac{\pi}{bd(G)} \right\}.$$

A graph is claw-free if it does not contain any induced $K_{1,3}$ (see Figure 2.2), the complete bipartite graph on $1+3$ vertices. We also have to remark that this extends the result of Heilmann and Lieb since the line-graph $L(G)$ of any graph is known to be claw-free.

It arises as a natural question, which trees have real rooted independence polynomials. If we just check the condition of the Chudnovsky and Seymour theorem [23], we get that claw-free trees can only be paths.

Again our motivation arises from the theory of matching polynomials. For the matching polynomial, it is well known that for any finite graph G and $u \in V(G)$ there exists a rooted tree (T, r), such that

$$\frac{\mu(G-u,x)}{\mu(G,x)} = \frac{\mu(T-r,x)}{\mu(T,x)} \tag{2.1}$$



(a) A graph G with labeled vertices.

(b) Path tree of G from vertex 1. The labels of the vertices denote endpoints of paths.

Figure 2.1: A graph with its path tree.

A well-known construction for T is the path-tree [36] (a.k.a. Godsil tree), which is the tree on paths of G starting from u, and the edges are the strict inclusions. (For an example, see Figure 2.1.)

In the middle part of this chapter, we will prove an "independence version" of this theorem through a quite similar construction. More precisely, we will show that there exists a rooted tree (T', r), such that

$$\frac{I(G-u,x)}{I(G,x)} = \frac{I(T'-r,x)}{I(T',x)}.$$

We will call the constructed tree a stable-path tree. This construction already appeared in the work of Scott and Sokal (see [62]) and a variant of this construction in the work of Weitz (see [76]). We will see that the key property of a stable-path tree is that its independence polynomial is a product of independence polynomials of some induced subgraphs of G.

This construction gives us an approach that unifies most literature attempts to show whether a tree has real-rooted independence polynomial. The construction also establishes that there is no difference if we restrict the roots of independence polynomials of graphs of degree at most d for trees of a degree at most d.

In the last part of this chapter, we will define a new graph polynomial and show its connection to the independence polynomial. The adjoint polynomial of a graph G, that is

$$h(G,x) = \sum_{k \leq 0} (-1)^k a_k(G) x^k,$$

where n is the number of vertices of G and $a_k(G)$ is the number of ways to partition the vertices of G into $n-k$ many complete subgraphs. The adjoint polynomial was introduced by R. Liu [49] and it is studied in a series of papers ([14, 15, 78–80]).

The adjoint polynomial shows a strong connection with the chromatic polynomial [60]. More precisely the chromatic polynomial of the complement graph $\overline{G}$ of G is

$$ch(\overline{G}, x) = \sum_{k=1}^{n} a_k(G)x(x-1)\dots(x-k+1).$$

The adjoint polynomial shows certain nice analytic properties. For instance, it has a real zero whose modulus is the largest among all zeros. Zhao showed ([78]) that the adjoint polynomial always has a real zero, furthermore, Csikvári proved ([25]) that the largest real zero has the largest modulus among all zeros. He also showed that the absolute value of the largest real zero is at most $4(\Delta - 1)$, where Δ is the largest degree of the graph G.

In this part, we will establish a connection between the independence and adjoint polynomial (similar as between the independence and matching polynomial), in the sense that for any graph G we will construct an auxiliary graph $\widehat{G}$, such that

$$h(G, x) = x^n I(\widehat{G}, -1/x).$$

This correspondence will enable us to use the rich theory of independence polynomials to study the adjoint polynomials. In particular, we give new proofs of the aforementioned results Liu and Csikvári.

Throughout of this chapter we will use the following notations.

- For a graph G and $u, v \in V(G)$, let $d_G(u, v)$ denote the length of the shortest path from u to v in G, if it exists, or else let it be ∞.
- For $H \subseteq G$, let $N[H] = \{v \in V(G) \mid \exists u \in V(H), d_G(u, v) \leq 1\}$ be the closure of H.
- If $S \subseteq V(G)$, then $G[S]$ denotes the induced subgraph of G on the vertex set S, and $G - S$ denotes $G[V(G) - S]$.
- Let us denote the set of induced connected, bipartite subgraphs of G by $\mathcal{B}(G)$.
- Let $\mathcal{B}_u(G) = \{H \in \mathcal{B} \mid u \in V(H)\}$, and for a graph $H \in \mathcal{B}_u(G)$, let $A_u(H)$ be the color class containing u, and let $B_u(H) = V(H) \setminus A(H)$, $|A_u(H)| = a_u(H)$ and $|B_u(H)| = b_u(H)$.
- For $u \neq v \in V(G)$ let $\mathcal{B}_{u,v}(G) = \mathcal{B}_u(G) \cap \mathcal{B}_v(G)$.

If G is clear from the context, we simply write $\mathcal{B}$, $\mathcal{B}_u$ or $\mathcal{B}_{u,v}$ instead of $\mathcal{B}(G)$, $\mathcal{B}_u(G)$ or $\mathcal{B}_{u,v}(G)$.

1 Christoffel–Darboux type identities for the independence polynomial

The aim of this section is to prove analogous statements of the Christoffel-Darboux identities for independence polynomials and to give a new proof and a possible relaxation of a theorem of Chudnovsky and Seymour [23], which states that the independence polynomial of a claw-free graph has only real roots.

We will prove the following theorems.

Theorem 2.6. *Let G be a graph, and $u, v \in V(G)$. Then*

$$I(G-u,x)I(G-v,x) - I(G,x)I(G-u-v,x) =$$
$$= \sum_{H \in \mathcal{B}_{u,v}} (-1)^{d_H(u,v)+1} x^{|V(H)|} I(G-N[H],x)^2.$$

Theorem 2.7. *Let G be a graph, and $u \in V(G)$. Then*

$$I(G,x)I(G-u,y) - I(G-u,x)I(G,y) =$$
$$= \sum_{H \in \mathcal{B}_u} I(G-N[H],x)I(G-N[H],y)(x^{a_u(H)}y^{b_u(H)} - x^{b_u(H)}y^{a_u(H)}).$$

As it appears, one might try to prove all the previous theorems by examining the union of two special sets in G, that are either matchings (Theorems 2.4 and 2.5) or independent sets (Theorems 2.6 and 2.7). In particular, the union of two matchings is a collection of cycles and paths, whereas the union of two independent sets induces a bipartite graph.

A special case of Theorem 2.6 is related to the so-called Merrifield–Simmons conjecture. This conjecture asserts that for every graph G and $u, v \in V(G)$, the sign of

$$I(G-u,1)I(G-v,1) - I(G,1)I(G-u-v,1)$$

depends only on the parity of the distance of u and v in G. This was claimed to be true without proof in their book [54], and became known as the Merrifield–Simmons conjecture. This conjecture turned out to be false for general graphs, as it was pointed out in [40]. On the other hand, the conjecture is true for bipartite graphs [71]. Now we see that Theorem 2.6 implies a slight generalization of this result. Suppose that the length of every path in G from u to v has the same parity, then $(-1)^{d_G(u,v)} = (-1)^{d_H(u,v)}$ for every $H \in \mathcal{B}_{u,v}$. In particular, if G is bipartite, then for every $u, v \in V(G)$ the parity of all paths from u to v are the same.

Corollary 2.8. *Let G be a bipartite graph, and $u, v \in V(G)$ and $x \in \mathbb{R}^+$. Then*

$$\operatorname{sgn}\big[I(G-u,x)I(G-v,x) - I(G,x)I(G-\{u,v\},x)\big] = \begin{cases} 1 & \text{if } d_G(u,v) \text{ is odd;} \\ 0 & \text{if } d_G(u,v) = \infty; \\ -1 & \text{if } d_G(u,v) \text{ is even.} \end{cases}$$

For an $H \in \mathcal{B}(G)$, let us denote one of its color classes by $A(H)$ and the other by $B(H)$. Let $a(H) = |A(H)|$ and $b(H) = |B(H)|$. The following statements are consequences of Theorem 2.6 and 2.7 using the facts (see, e.g. [47]) that

$$I'(G,x) = \sum_{u \in V(G)} I(G - N[u], x),$$

and

$$I(G,x) = I(G-u,x) + xI(G-N[u],x).$$

Corollary 2.9. *Let G be a graph and $u \in V(G)$. Then*

$$xI'(G-u,x)I(G,x) - xI(G-u,x)I'(G,x) =$$

$$= \sum_{H \in \mathcal{B}_u} (b_u(H) - a_u(H)) x^{|V(H)|} I(G-N[H],x)^2,$$

and

$$x^2 I'(G,x)^2 - x^2 I''(G,x)I(G,x) - xI'(G,x)I(G,x) =$$
$$= -\sum_{H \in \mathcal{B}} (a(H) - b(H))^2 x^{|V(H)|} I(G-N[H],x)^2.$$

Theorem 2.10. *Let G be a graph. Then*

$$xI'(G,x)I(G,y) - yI(G,x)I'(G,y) =$$

$$= \sum_{H \in \mathcal{B}} (a(H) - b(H)) I(G-N[H],x) I(G-N[H],y) (x^{a(H)} y^{b(H)} - x^{b(H)} y^{a(H)}).$$

At this point we would like to emphasize the difference between the use of $a_u(H)$ and $a(H)$. In case of $a(H)$ it is allowed to choose any of the color classes of the bipartite graph H. It also means that the choice of $A(H)$ and $B(H)$ does not affect the two previous formulae. However, in the case of $a_u(H)$, we have a fixed vertex u, and $a_u(H)$ denotes the color class that contains u.

Let us recall that, for a fixed graph G the quantity $\operatorname{bd}(G)$ is the maximum of $|a(H) - b(H)|$, where the maximum is taken over all induced connected bipartite subgraph of G.

The proof of Corollary 2.9 can be found in [37] for matching polynomials and goes quite similarly for independence polynomials. Therefore we will not give the detailed proof of it. In the proof of Theorem 2.10 we will follow an argument similar to the one given in [37] for matching polynomials.

Then as an application of Theorem 2.10, we will prove the following theorem that generalize the theorem of Chudnovsky and Seymour [23], see Subsection 1.3.

Theorem 2.11. *Let G be a graph. Then $I(G,x)$ doesn't have any root in*

$$\left\{ z \in \mathbb{C} \;\middle|\; |arg(z)| < \frac{\pi}{bd(G)} \right\}.$$

Another similar proof of this theorem can be found in [46], which concerns the generalization of the "Mehler-formula" of orthogonal polynomials for independence and matching polynomials.

This section is organized as follows. In the next section we prove Theorem 2.6, Theorem 2.7 and Theorem 2.10. In the third section we prove Theorem 2.11.

1.1 Proof of the Christoffel–Darboux identities

The main idea of the proofs is that we think about $I(G,x)$ as a generating function of the independent subsets of G. We mean by that

$$I(G,x) = \sum_{A \in \mathcal{F}(G)} x^{|A|},$$

where $\mathcal{F}(G)$ is the set of independent subsets in G. In other words, we give weight $x^{|A|}$ to each $A \in \mathcal{F}(G)$, and write $I(G,x)$ for the total weight of $\mathcal{F}(G)$.

As a corollary we see that for any two graphs G, H, the polynomial $I(G,x)I(H,x)$ can be thought of as a generating function of pairs of independent subsets from G and H, that is

$$I(G,x)I(H,x) = \sum_{(A,B) \in \mathcal{F}(G) \times \mathcal{F}(H)} x^{|A|+|B|}.$$

By considering the difference of two products of the similar form, we will find pairs appearing in each summation with the same weight, therefore those terms can be simultaneously eliminated. As we will see, it might happen that we use this argument repeatedly.

We would also like to remark that if we take two independent subsets of G, then their union induces a bipartite subgraph of G.

Proof of Theorem 2.6. Let $\mathcal{F}(G)$ be the set of the independent sets of G.

Let

$$\mathcal{F}_1 = \mathcal{F}(G-u) \times \mathcal{F}(G-v),$$
$$\mathcal{F}_2 = \mathcal{F}(G) \times \mathcal{F}(G-u-v).$$

By definition $I(G,x)$ is equal to $\sum\limits_{A \in \mathcal{F}(G)} x^{|A|}$. Then the left hand side of the identity yields:

$$\left(\sum_{(A,B)\in\mathcal{F}_1} x^{|A|+|B|} \right) - \left(\sum_{(A,B)\in\mathcal{F}_2} x^{|A|+|B|} \right). \tag{2.2}$$

If $A, B \in \mathcal{F}(G)$ and $z \in A \cap B$, then $d_{G[A\cup B]}(z) = 0$ and $G[A \triangle B]$ is a bipartite graph with color classes $A \setminus B$ and $B \setminus A$.

Let us observe the following equivalences:

- $(A,B) \in \mathcal{F}_1$ and $u, v \notin A \cup B \Leftrightarrow (A,B) \in \mathcal{F}_2$ and $u, v \notin A \cup B$
- $(A,B) \in \mathcal{F}_1$ and $u \notin A \cup B \ni v \Leftrightarrow (A,B) \in \mathcal{F}_2$ and $u \notin A \cup B \ni v$
- $(A,B) \in \mathcal{F}_1$ and $v \notin A \cup B \ni u \Leftrightarrow (B,A) \in \mathcal{F}_2$ and $v \notin A \cup B \ni u$

Now we can reformulate (2.2) with the following notations.

Let

$$\mathcal{F}_1' = \{(A,B) \in \mathcal{F}_1 \mid u, v \in A \cup B\},$$
$$\mathcal{F}_2' = \{(A,B) \in \mathcal{F}_2 \mid u, v \in A \cup B\}.$$

Then by the previous equivalences we have a natural bijection between $\mathcal{F}_1 \setminus \mathcal{F}_1'$ and $\mathcal{F}_2 \setminus \mathcal{F}_2'$, therefore these terms eliminate each other. Thus, (2.2) is equal to:

$$\left(\sum_{(A,B)\in\mathcal{F}_1'} x^{|A|+|B|} \right) - \left(\sum_{(A,B)\in\mathcal{F}_2'} x^{|A|+|B|} \right). \tag{2.3}$$

Suppose that $(A,B) \in \mathcal{F}_1'$ and u and v are not in the same connected component of $G[A \cup B]$. Let us switch the colors only in the component of v, by which we mean that the new independent sets A' and B' are

$$A' = A \triangle \{w \in A \cup B \mid d_{G[A\cup B]}(v,w) < \infty\},$$
$$B' = B \triangle \{w \in A \cup B \mid d_{G[A\cup B]}(v,w) < \infty\}.$$

Then $u, v \in A' \in \mathcal{F}(G)$ and $u, v \notin B' \in \mathcal{F}(G)$, thus, $(A', B') \in \mathcal{F}_2'$ and $|A| + |B| = |A'| + |B'|$. It is easy to see that every pair is canceled, where u and v are not in the same connected component.

Let

$$\mathcal{F}_1'' = \{(A, B) \in \mathcal{F}_1' \mid d_{G[A \cup B]}(u, v) < \infty\},$$
$$\mathcal{F}_2'' = \{(A, B) \in \mathcal{F}_2' \mid d_{G[A \cup B]}(u, v) < \infty\}.$$

We can therefore rewrite (2.3) as

$$\left(\sum_{(A,B) \in \mathcal{F}_1''} x^{|A|+|B|} \right) - \left(\sum_{(A,B) \in \mathcal{F}_2''} x^{|A|+|B|} \right). \tag{2.4}$$

Let us observe, that if $(A, B) \in \mathcal{F}_1''$, then $d_{G[A \cup B]}(u, v)$ is odd, and if $(A, B) \in \mathcal{F}_2''$, then $d_{G[A \cup B]}(u, v)$ is even. Therefore if A, B are independent sets of G, their union contains u and v, and they are in the same component of the induced graph, then we can decide whether $(A, B) \in \mathcal{F}_1''$ or $(A, B) \in \mathcal{F}_2''$. For $(A, B) \in \mathcal{F}_1'' \cup \mathcal{F}_2''$, let $P(A, B)$ be the connected component of u and v in the induced graph $G[A \cup B]$. Thus, (2.4) is equal to

$$\left(\sum_{(A,B) \in \mathcal{F}_1''} (-1)^{d_{G[A \cup B]}(u,v)+1} x^{|A|+|B|} \right) + \left(\sum_{(A,B) \in \mathcal{F}_2''} (-1)^{d_{G[A \cup B]}(u,v)+1} x^{|A|+|B|} \right) =$$
$$= \left(\sum_{(A,B) \in \mathcal{F}_1'' \cup \mathcal{F}_2''} (-1)^{d_{P(A,B)}(u,v)+1} x^{|P(A,B)|} x^{|A|+|B|-|P(A,B)|} \right). \tag{2.5}$$

We can rearrange the summation by first collecting the possible subsets of vertices of G of the form $P(A, B)$ for $(A, B) \in \mathcal{F}_1'' \cup \mathcal{F}_2''$. Observe that $P(A, B)$ is an element of $\mathcal{B}_{u,v}$, moreover $A \setminus P(A, B)$ and $B \setminus P(A, B)$ are independent subsets in $G - N[P(A, B)]$. Then (2.5) is equal to

$$\sum_{H \in \mathcal{B}_{u,v}} (-1)^{d_H(u,v)+1} x^{|V(H)|} \left(\sum_{A,B \in \mathcal{F}(G-N[H])} x^{|A|+|B|} \right) =$$
$$= \sum_{H \in \mathcal{B}_{u,v}} (-1)^{d_H(u,v)+1} x^{|V(H)|} I(G - N[H], x)^2.$$

□

Proof of Theorem 2.7. We will use the same argument as in the previous proof. Let $\mathcal{F}(G)$ be the set of independent sets and let $\mathcal{F}' = \mathcal{F}(G) \times \mathcal{F}(G-u)$. Then the left hand side is equal to

$$\sum_{(A,B)\in\mathcal{F}'} x^{|A|}y^{|B|} - \sum_{(A,B)\in\mathcal{F}'} x^{|B|}y^{|A|}. \tag{2.6}$$

If $(A,B) \in \mathcal{F}'$ and $u \notin A$, then $(B,A) \in \mathcal{F}'$. Let $\mathcal{F}'' = \mathcal{F}' \setminus (\mathcal{F}(G-u) \times \mathcal{F}(G-u))$.

Then (2.6) is equal to

$$\sum_{(A,B)\in\mathcal{F}''} x^{|A|}y^{|B|} - \sum_{(A,B)\in\mathcal{F}''} x^{|B|}y^{|A|}. \tag{2.7}$$

Note that for all $(A,B) \in \mathcal{F}''$, u is always in A and $u \notin A \cap B$.

Let $P(A,B)$ be the connected component of the graph induced by the set $A \cup B$ that contains u. Then for the first sum we can write the following:

$$\begin{aligned}\sum_{(A,B)\in\mathcal{F}''} x^{|A|}y^{|B|} &= \\ &= \sum_{(A,B)\in\mathcal{F}''} x^{a_u(P(A,B))}y^{b_u(P(A,B))}x^{|A|-a_u(P(A,B))}y^{|B|-b_u(P(A,B))}.\end{aligned} \tag{2.8}$$

Similarly as before, we can rearrange the summation by first collecting all subsets of vertices of G of the form $P(A,B)$ for $(A,B) \in \mathcal{F}''$. Thus, (2.8) is equal to

$$\begin{aligned}&\sum_{H\in\mathcal{B}_u} x^{a_u(H)}y^{b_u(H)} \sum_{K,L\in\mathcal{F}(G-N[H])} x^{|K|}y^{|L|} = \\ &= \sum_{H\in\mathcal{B}_u} x^{a_u(H)}y^{b_u(H)} I(G-N[H],x)I(G-N[H],y).\end{aligned}$$

We get the same formula for the second sum.

$$\begin{aligned}\sum_{(A,B)\in\mathcal{F}''} x^{|B|}y^{|A|} &= \\ &= \sum_{H\in\mathcal{B}_u} y^{a_u(H)}x^{b_u(H)} I(G-N[H],y)I(G-N[H],x)\end{aligned}$$

Then (2.7) is equal to

$$\sum_{H\in\mathcal{B}_u} (x^{a_u(H)}y^{b_u(H)} - y^{a_u(H)}x^{b_u(H)})I(G-N[H],x)I(G-N[H],y).$$

$\square$

Proof of Theorem 2.10. We use the facts that

$$I'(G,x) = \sum_{u \in V(G)} I(G - N[u], x),$$

and

$$I(G,x) = I(G-u,x) + xI(G - N[u], x).$$

Let $n = |V(G)|$. By combining these two formulae we get

$$\sum_{u \in V(G)} I(G-u,x) = nI(G,x) - xI'(G,x).$$

Let us sum both sides of the identity of Theorem 2.7 for all $u \in V(G)$ and apply the above identities. For the left hand side, we get

$$\begin{aligned}
&\sum_{u \in V(G)} (I(G,x)I(G-u,y) - I(G-u,x)I(G,y)) = \\
&\quad = I(G,x) \sum_{u \in V(G)} I(G-u,y) - I(G,y) \sum_{u \in V(G)} I(G-u,x) = \\
&\quad = I(G,x)(nI(G,y) - yI'(G,y)) - I(G,y)(nI(G,x) - xI'(G,x)) = \\
&\quad = xI'(G,x)I(G,y) - yI(G,x)I'(G,y).
\end{aligned}$$

Now we sum up the right hand side, which is

$$\begin{aligned}
&\sum_{u \in V(G)} \sum_{H \in \mathcal{B}_u} (x^{a_u(H)} y^{b_u(H)} - y^{a_u(H)} x^{b_u(H)}) I(G - N[H], x) I(G - N[H], y) = \\
&\quad = \sum_{H \in \mathcal{B}} (a(H) - b(H))(x^{a(H)} y^{b(H)} - y^{a(H)} x^{b(H)}) I(G - N[H], x) I(G - N[H], y).
\end{aligned}$$

□

1.2 Proof of Theorem 2.11

We will prove Theorem 2.11 by induction on the number of vertices of G, using Theorem 2.10 to prove the induction step. Also the statement is true for the graph with one vertex, since its independence polynomial is $1 + x$. So let G be a graph on $n \geq 2$ vertices, and assume that the statement is true for graphs with at most $n - 1$ vertices.

Suppose by contradiction there is a graph G on n vertices, that doesn't satisfies the statement. So there exists complex number ξ, such that $0 < \arg(\xi) < \frac{\pi}{\mathrm{bd}(G)}$, since

$I(G,x)$ has positive coefficients. Then using the identity of 2.10 for $x = \xi$ and $y = \bar{\xi}$, we obtain that

$$\xi I'(G,\xi)I(G,\bar{\xi}) - \bar{\xi}I(G,\xi)I'(G,\bar{\xi}) = \sum_{H \in \mathcal{B}} 2(a(H)-b(H))|\xi|^{2b(H)}|I(G-N[H],\xi)|^2 \Im(\xi^{a(H)-b(H)})i \tag{2.9}$$

First of all the left hand side is 0, since $I(G,x)$ has real coefficients. Moreover, $G - N[H]$ is an induced subgraph of G on fewer vertices and $\mathrm{bd}(G - N[H]) \leq \mathrm{bd}(G)$, so by the induction hypothesis $I(G - N[H], \xi) \neq 0$. Since $d\Im(\xi^d) > 0$ for any $0 < |d| \leq \mathrm{bd}(G)$ and $\mathbb{B}$ contains one vertex subgraphs, therefore the right hand side all the members of the product is positive multiple of i, this contradiction proves the induction step.

1.3 Roots of some graph families

In this section, we will give a few applications of Theorem 2.11, that will cover the Chudnovsky-Seymour theorem about claw-free graphs and fork-free graphs.

In [16], the authors investigated graphs G (called stable graphs), such that all complex roots of $I(G,x)$ are in the left half plane. In other words, there is no complex root whose argument's absolute value is smaller than $\frac{\pi}{2}$. We immediately see by Theorem 2.11, that if $\mathrm{bd}(G) \leq 2$, then G is stable. Since $\mathrm{bd}(G) \leq 2$ is a hereditary property of graphs, one might characterize this class of graphs with forbidden induced subgraphs. This list will consist of infinitely many trees that are 2 claws connected by a path, where the distance between the centers of the claws is even (possibly 0).

Our next corollary can be viewed as a generalization of Proposition 2.1 of [16] for arbitrary $\alpha(G)$.

Corollary 2.12. *Let G be a graph. Then $I(G,x)$ doesn't have any root in*

$$\left\{ z \in \mathbb{C} \;\middle|\; |arg(z)| < \frac{\pi}{\alpha(G)-1} \right\},$$

where $\alpha(G)$ is the independence number (i.e. the degree of $I(G,x)$).

Proof. If we take any connected induced bipartite subgraph of G with color classes A, B, then A and B are independent sets of G. So we have that $1 \leq |A| \leq \alpha(G)$ and $1 \leq |B| \leq \alpha(G)$, so $||A| - |B|| \leq \alpha(G) - 1$. So we proved that $\mathrm{bd}(G) \leq \alpha(G) - 1$. □

Next, we prove the Chudnovsky-Seymour theorem about claw-free graphs [23].

Corollary 2.13. *The independence polynomial of a claw-free graph G has only real roots.*

Proof. It is enough to show that if H is an induced connected bipartite subgraph of G, then $||A| - |B|| \leq 1$. Since H is induced subgraph of a claw-free graph, therefore H is as well. But it implies that a degree of a vertex in a claw-free bipartite graph is at most 2, i.e. H is either a path of an even cycle. In both cases the difference of the color classes is at most 1. So we proved, that $\mathrm{bd}(G) \leq 1$, i.e. $I(G, x)$ has no root in $\mathbb{C} \setminus \mathbb{R}^-$. $\square$

A possible extension of the previous is to forbid the fork graph (a tree on 5 vertices is in Figure 2.2). It was proposed as a question in [29], whether it is possible to design an FPTAS based on the the work of [56] to approximate $I(G, x)$ for fork-free graphs. In order follow their work, first, one would need to find a open neighborhood of $[0, \infty)$ that doesn't contain any roots of the independence polynomial of a fork-free graph. Before giving a partially positive answer, we have a warning example for the existence of such a region. If we take the sequence of complete graphs (they are also claw-free), and their independence polynomials, that are $I(K_n, x) = 1 + nx$, then their roots converge to 0.

However, if we restrict our attention to bounded degree graphs, then by a result of Scott and Sokal [62], we get an open zero-free region around 0. In the next statement, we establish an open zero-free region for bounded degree fork-free graphs containing $(0, \infty)$.

(a) The claw graph $K_{1,3}$. **(b)** The fork graph.

Figure 2.2: The claw and fork graphs.

Corollary 2.14. *Let G be a fork-free graph of degree at most d. Then $I(G, x)$ doesn't have any root in*

$$\left\{ z \in \mathbb{C} \;\middle|\; |arg(z)| < \frac{\pi}{d-1} \right\}.$$

Proof. It is enough to show that if H is a connected bipartite graph of degree at most d, then $||A| - |B|| \leq d - 1$. One could use the classification of connected bipartite graphs (see [29]) to obtain the statement. For the sake of completeness we will give a different proof.

Let us assume that there exists a fork-free connected bipartite graph H of degree at most d with color classes A, B, such that $|A| \geq |B| + d$. Since H is connected, therefore there exists vertex $u \in B$ with degree at least 3. If there would be a vertex $v \in B$, such that the distance between u, v is at least 4, then we would have a fork. Therefore we may assume that for any $v \in B$ the distance is 2 from u. If there would be a vertex $v \in B$, such that $|N(u) \setminus N(v)| \geq 2$, then we would also able to find a fork. So we may assume that for any $v \in B$, $|N(u) \setminus N(v)| \leq 1$. It immediatly implies that for any $v \in B$ the neighborhood has size at least 2 and also $|N(v) \setminus N(u)| \leq 1$.

But we claim that it is impossible, since

$$d + |B| \leq |A| = |\cup_{v \in B} N(v)| \leq |N(u)| + \sum_{u \neq v \in B} |N(v) \setminus N(u)| \leq d + |B| - 1.$$

So we obtained that for any fork-free connnected bipartite graph of degree at most d has difference at most $d - 1$ between its sides, i.e. $\mathrm{bd}(G) \leq d - 1$. $\square$

2 Independence polynomial of some trees

In this section, we study the independence polynomials of trees. For trees, it is a well known conjecture that the sequence $(i_k(T))_{k \geq 0}$ is unimodal [3].

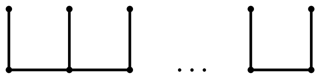
Recall that a sequence $(b_k)_{k=0}^n$ is unimodal ([68]), if there exists an index k, such that

$$b_0 \leq b_1 \leq \cdots \leq b_{k-1} \leq b_k \geq b_{k+1} \geq \cdots \geq b_n.$$

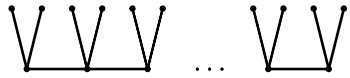
A stronger property for positive sequences is the so called log-concavity: for any i such that $0 < i < n$, we have $b_i^2 \geq b_{i-1} b_{i+1}$. An even stronger property is the real-rootedness of the polynomial $p(x) = \sum_{i=0}^n b_i x^i$ (any complex zero of the polynomial is real). This prompted many mathematicians to study trees with real-rooted independence polynomials. In this section, we show a general method to construct such trees or prove real-rootedness.

In particular, we will give a new proof for real-rootedness of the independence polynomials of certain families of trees, which includes centipedes (Zhu's theorem, see [82]), caterpillars (Wang and Zhu's theorem, see [75]), and we will prove a conjecture of Galvin and Hilyard about the real-rootedness of the independence polynomial of the Fibonacci trees (Conj. 6.1. of [35]).

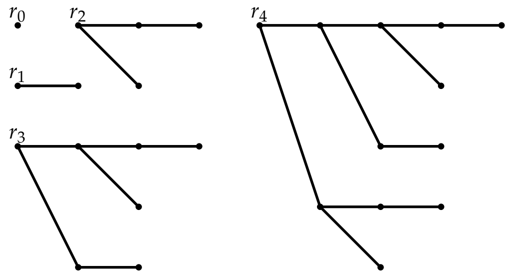
Recall that the n-centipede W_n is a graph (Fig. 2.3a), such that we take a path on n vertices and we hang 1 pendant edge from each vertex of it. Similarly, the n-caterpillar H_n is the graph (Fig. 2.3b) obtained by taking a path on n vertices and by hanging 2 pendant edges from each vertex of it. The Fibonacci trees were defined by Wagner [74] as follows (Fig. 2.3c): let $F_0 = K_1$ and $F_1 = K_2$ with roots $r_0 \in V(F_0)$ and $r_1 \in V(F_1)$. Then for $n \geq 2$ the nth Fibonacci tree F_n is obtained from the disjoint union of F_{n-1},



(a) The centipede



(b) The caterpillar



(c) The first 5 Fibonacci trees

Figure 2.3: Some families of trees

F_{n-2} and a new vertex, labeled by r_n and connecting r_n to the roots of F_{n-1} and F_{n-2}. Define r_n as the root of F_n.

Motivated by theorems for matching polynomials as described in the beginning of this chapter we will show that there exists a rooted tree (T', r), such that

$$\frac{I(G-u,x)}{I(G,x)} = \frac{I(T'-r,x)}{I(T',x)}.$$

We will call the constructed tree a stable-path tree. We will see that the key property of a stable-path tree is that its independence polynomial is a product of independence polynomials of some induced subgraphs of G. We will realize all the previously mentioned trees as stable-path trees of some graph G. Therefore, to understand the roots of those trees, it is enough to understand the location of the roots of the induced subgraphs of G.

In particular, in Corollary 2.13 we proved that claw-free graphs have only real-rooted independence polynomial. Since any induced subgraph of a claw-free graph is also claw-free, this enables us to conclude that any stable-path tree of a claw-free graph has real-rooted independence polynomial. In Section 2.2, we will construct claw-free graphs such that their stable-path trees will be n-centipedes, n-caterpillars and Fibonacci trees. In the same section we will give further applications of this method.

This chapter is organized as follows: in the next section we will define stable-path trees of graphs, and we will prove some properties of it. In the last section we will prove real-rootedness of independence polynomials of certain graphs.

2.1 Tree of stable paths

In this section we will give two variants of the definition of the stable-path tree, where the first one is a special case of the latter one. For the applications it is enough to get familiar with the first definition. But first let us recall the following properties of the independence polynomial, which we will use intensively in the proofs. For proof see [47].

Lemma 2.15. *Let G be a graph with connected components $G_1, \dots, G_k$, and let $u \in V(G)$ be a fixed vertex. Then*

$$I(G,x) = I(G-u,x) + xI(G-N_G[u],x)$$
$$I(G,x) = \prod_{i=1}^{k} I(G_i,x)$$

Definition 2.16 (Tree of stable paths)**.** *Let G be a graph, where we have a total ordering $\prec$ on $V(G)$ and let $u \in V(G)$ fixed. Then we define a tree $(T^{<}_{G,u}, \bar{u})$ as follows. Let us denote by $N(u) = \{u_1 \prec \dots \prec u_d\}$, and let*

$$G^i = G[V(G) \setminus \{u, u_1, v_2, \dots, u_{i-1}\}]$$
$$(T^i, r^i) = (T^{<}_{G^i,u_i}, \bar{u}_i),$$

where we take the induced ordering of the vertices on $V(G^i)$ for $1 \le i \le d$. Consider the disjoint unions of T^i with roots r^i and a new vertex with label $\bar{u}$, and add edges $(\bar{u}, r^i)$ for $1 \le i \le d$. In this way we gain a tree $T^{<}_{G,u}$ and let $\bar{u}$ be the root of this tree. See an example in Fig 2.4.

(a) A graph G with labeled vertices.

(b) The graph $T^{<}_{G,1}$. The labels of the vertices denote end-points of stable-paths.

Figure 2.4: A graph with its stable-path tree. The ordering on the vertices of G is induced by its labeling

Theorem 2.17. *Let G be a graph, $u \in V(G)$. Then for $T = T^{<}_{G,u}$ we have that*

$$\frac{I(G-u,x)}{I(G,x)} = \frac{I(T-\overline{u},x)}{I(T,x)},$$

Proof. We will prove the statement by induction on the number of vertices of G. If G has exactly one vertex, then $T^{<}_{G,u}$ is constructed to be a graph with one vertex.

Let $N(u) = \{u_1 \prec \cdots \prec u_d\}$, and then let $G^i = G[V(G) \setminus \{u, u_1, v_2, \ldots, u_{i-1}\}]$ and $(T^i, r^i) = (T_{G^i,u_i}, \overline{u}_i)$ for $1 \leq i \leq d$ as in the definition. Then

$$\frac{I(G,x)}{I(G-u,x)} = \frac{I(G-u,x) + xI(G-N[u],x)}{I(G-u,x)} = 1 + \frac{xI(G-N[u],x)}{I(G-u,x)} =$$
$$1 + x\frac{I(G-u-u_1,x)I(G-u-\{u_1,u_2\},x)\ldots I(G-u-\{u_1,\ldots,u_k\},x)}{I(G-u,x)I(G-u-u_1)\ldots I(G-u-\{u_1,\ldots,u_{k-1}\})} =$$
$$1 + x\frac{I(G^1-u_1,x)}{I(G^1,x)}\frac{I(G^2-u_2,x)}{I(G^2,x)}\cdots\frac{I(G^d-u_d,x)}{I(G^d,x)} =$$
$$1 + x\frac{I(T^1-r^1,x)}{I(T^1,x)}\frac{I(T^2-r^2,x)}{I(T^2,x)}\cdots\frac{I(T^d-r^d,x)}{I(T^d,x)} =$$
$$\frac{I(T-r,x) + xI(T-N[r],x)}{I(T-r,x)} = \frac{I(T,x)}{I(T-r,x)}.$$

□

We would like to remark that in all applications it will be enough to use this definition, however, for the completeness we will give a a more general form.

The following construction already appeared in the work of Scott and Sokal (see [62]), where they called the this tree as pruned SAW-tree.

Definition 2.18 (Tree of σ-stable paths)**.** *Let $\mathcal{P}_u$ be the set of paths from u in G, and let*

$$A_{G,u} = \{(P,e) \in \mathcal{P}_u \times E(G) \mid P = (v_0, \ldots, v_k), v_k \in e\}.$$

A function $\sigma : A_{G,u} \to \mathbb{R}$ is called deep decision if it satisfies the following condition: whenever $(P,e), (P,f) \in A_{G,u}$ and $\sigma(P,e) = \sigma(P,f)$, then $e = f$. Then a path $P = (v_0, v_1, \ldots, v_k)$ from u is σ-stable, if whenever $(v_i, v_j) \in E(G)$ and $i+1 < j$, then $\sigma(P', (v_i, v_{i+1})) < \sigma(P', (v_i, v_j))$, where $P' = (v_0, \ldots, v_i)$ is a subpath. If the path $P = (u, v_1, \ldots, v_k)$ is stable with respect to σ, then $P' = (u, v_1, \ldots, v_{k-1})$ is also stable with respect to σ.

Let $T^{\sigma}_{G,u}$ be a tree, whose vertices are the σ-stable paths from u, and the edges correspond to the strict inclusion. In that tree the path (u) (with length 0) appears, which we will denote by $\overline{u}$.

To see the relation between the two definitions, let us assume, that G has a total ordering on its vertices, so we may assume, that $(V(G), \prec) = (\{1, \ldots, n\}, <)$. Then for a $(P,e) \in A_{G,u}$, such that $P = (u, v_1, \ldots, v_k)$ and $e = (v_k, v_{k+1})$ let $\sigma(P,e) = v_{k+1}$. Then it is easy to check that $T^{\sigma}_{G,u} = T^{<}_{G,u}$. Indeed the second definition is a generalization of the first one.

For the completeness we will prove Theorem 2.17 also for the generalized σ-stable-path tree.

Theorem 2.19. *Let G be a graph, $u \in V(G)$ and let $\sigma : A_{G,u} \to \mathbb{R}$ be a deep decision. Then for $T = T^{\sigma}_{G,u}$ we have that*

$$\frac{I(G-u,x)}{I(G,x)} = \frac{I(T-\overline{u},x)}{I(T,x)},$$

Proof. We will prove the statement by induction on the number of vertices. If G has exactly one vertex, then $T^{\sigma}(G,u)$ is constructed to be a graph with one vertex.

Furthermore we may assume that G is connected, since if $G_1,\dots,G_k$ are the connected components of G, where $u \in V(G_1)$, then by using the multiplicity of the independence polynomial, we have:

$$\frac{I(G-u,x)}{I(G,x)} = \frac{I(G_1-u,x)I(G_2,x)\dots I(G_k,x)}{I(G_1,x)I(G_2,x)\dots I(G_k,x)} = \frac{I(G_1-u,x)}{I(G_1,x)}.$$

and by $A_{G,u} = A_{G_1,u}$ we have that $T^{\sigma}(G_1,u) = T^{\sigma}(G,u)$, which is the appropriate tree.

For the rest assume that G is connected. Then let $N(u) = \{u_1,\dots,u_d\}$ in such a way, such that $\sigma(\overline{u},(u,u_i)) < \sigma(\overline{u},(u,u_j))$, whenever $1 \le i < j \le d$. Then for any $1 \le i \le d$ and for any $(P,e) \in A_{G-\{u,u_1,\dots,u_{i-1}\},u_i}$ let σ_i be defined as follows (where $P = (u_i,v_1,\dots,v_k)$):

$$\sigma_i(P,e) = \sigma((u,u_i,v_1,\dots v_k),e).$$

Then

$$\frac{I(G,x)}{I(G-u,x)} = \frac{I(G-u,x)+xI(G-N[u],x)}{I(G-u,x)} = 1+\frac{xI(G-N[u],x)}{I(G-u,x)} =$$
$$1+x\frac{I(G-u-u_1,x)I(G-u-\{u_1,u_2\},x)\dots I(G-u-\{u_1,\dots,u_d\},x)}{I(G-u,x)I(G-u-u_1)\dots I(G-u-\{u_1,\dots,u_{d-1}\})} =$$
$$1+x\frac{I(G-u-u_1,x)}{I(G-u,x)}\frac{I(G-u-\{u_1,u_2\},x)}{I(G-u-u_1)}\cdots\frac{I(G-u-\{u_1,\dots,u_d\},x)}{I(G-u-\{u_1,\dots,u_{d-1}\})} =$$
$$1+x\frac{I(T^{\sigma_1}_{G-u,u_1}-\overline{u_1},x)}{I(T^{\sigma_1}_{G-u,u_1},x)}\frac{I(T^{\sigma_2}_{G-u-u_1,u_2}-\overline{u_2},x)}{I(T^{\sigma_2}_{G-u-u_1,u_2},x)}\cdots\frac{I(T^{\sigma_d}_{G-u-\{u_1\dots u_{d-1}\},u_d}-\overline{u_d},x)}{I(T^{\sigma_d}_{G-u-\{u_1\dots u_{d-1}\},d_k},x)} =$$
$$\frac{I(T,x)}{I(T-r,x)},$$

where T is a tree that is obtained from a star with k leaves, whose root is r, and the ith leaf is glued to the root of $T^{\sigma_i}_{G-u-\{u_1\dots u_{i-1}\},u_i}$.

On the other hand, this T is isomorphic to $T^{\sigma}_{G,u}$, since any σ-stable path $P = (u, u_i, v_1, \dots, v_k)$ (specially, if $1 \leq j < i$, then $u_j \notin \{v_1, \dots, v_k\}$) the path $P' = (u_i, v_1, \dots, v_k)$ is σ_i-stable. And for any σ_i-stable path $P' = (u_i, v_1, \dots, v_k)$ is a $P = (u, u_i, v_1, \dots, v_k)$ σ-stable path. So

$$\frac{I(T-r,x)}{I(T,x)} = \frac{I(T^{\sigma}_{G,u} - \overline{u}, x)}{I(T^{\sigma}_{G,u}, x)}$$

□

We would like to remark that Weitz's construction of the self-avoiding path tree is a special case of the previously defined stable-path tree of a deep decision. Let $\phi : E(G) \to \{1, \dots, m\}$ bijection, where $m = |E(G)|$. Then for a $(P, e) \in A_{G,u}$ let $\sigma(P, e) = \phi(e)$. Then $T^{\sigma}_{G,u}$ is the Weitz-tree.

Remark 2.20. *Observe that if we have a deep decision for a connected graph, then we can perform the DFS-algorithm with respect to σ, in the following way. Whenever we arrive into the vertex v along the path P and there is an unvisited neighbor of v, then we will move to that unvisited vertex w for which $\sigma(P, (v, w))$ is the smallest.*

Formally, let us assume, that there is a given connected graph G, $u \in V(G)$ and a deep decision σ from u. Then one can construct a spanning tree $F_{G,u,\sigma}$ (call as σ-DFS tree of G) as follows. Let $G_1, \dots, G_k$ be a the connected components of $G - u$, $u_i = \operatorname{argmin}_{v \in V(G_i) \cap N_G(u)}(\sigma(u, (u, v)))$ for $1 \leq i \leq k$ and the functions $\sigma_i : A_{G_i,u_i} \to \mathbb{R}$ are

$$\sigma_i((u_i, v_1, \dots, v_k), e) = \sigma((u, u_i, v_1, \dots, v_k), e).$$

Then we gain $F_{G,u,\sigma}$ as we take the disjoint union of F_{G_i,u_i,σ_i} for $1 \leq i \leq k$ and we connect a new vertex called u with u_i for $1 \leq i \leq k$.

By induction we can prove the following properties of a stable-path tree.

Proposition 2.21. *Let G be a connected graph, $u \in V(G)$, σ a deep decision, and let F be a σ-DFS tree. Denote by $\overline{F}$ the set of paths from u in F (they are σ-stable paths). Then*

1. *there exists a sequence $G_1, \dots, G_k$ of induced subgraphs of G, such that*

$$I(T^{\sigma}_{G,u}, x) = I(G, x) I(G_1, x) \dots I(G_k, x),$$

2. *and*

$$I(G, x) = \frac{I(T^{\sigma}_{G,u})}{I(T^{\sigma}_{G,u} - \overline{F}, x)}.$$

Proof. We will prove the first part by induction on the number of vertices of G. The proof of the second part goes similarly. From the proof of the previous theorem (and with its notations) we know that

$$I(T^{\sigma}_{G,u},x) = \frac{I(G,x)}{I(G-u,x)} I(T^{\sigma}_{G,u} - \overline{u},x) =$$
$$\frac{I(G,x)}{I(G-u,x)} I(T^{\sigma_1}_{G-u,u_1},x) I(T^{\sigma_2}_{G-\{u,u_1\},u_2},x) \dots I(T^{\sigma_d}_{G-\{u,u_1,\dots u_{d-1}\},u_d},x) =$$
$$\frac{I(G,x)}{I(G-u,x)} \prod_{i=1}^{d} \prod_{j=0}^{l_i} I(G^i_j,x),$$

where G^i_0 is the connected component of $G - \{u, u_1, \dots, u_{i-1}\}$, which contains u_i; and each G^i_j is an induced subgraph of G^i_0. So each G^i_j is an induced subgraph of G. Let $\{H_1, \dots, H_t\}$ the set of connected components of $G - u$, and

$$I = \{ \min_{u_i \in V(H_j)} (i) \mid 1 \le j \le t\}.$$

By definition of I we have that the set $\{G^i_0 \mid i \in I\}$ is the set of connected components of $G - u$. This implies that the product $\prod_{i \in I} I(G^i_0, x) = I(G - u, x)$, therefore

$$I(T^{\sigma}_{G,u},x) = I(G,x) \prod_{i \in I'} I(G^i_0,x) \prod_{i=1}^{d} \prod_{j=1}^{l_i} I(G^i_j,x), \tag{2.10}$$

where $I' = \{1, \dots, d\} \setminus I$. □

Remark 2.22. *Sometimes, it is useful to follow the induction to determine explicitly the multiplicites of the subgraphs occuring in the formula (2.10).*

2.2 Applications of stable-path tree

In this section we will present various applications of the following corollary of Proposition 2.21:

Corollary 2.23. *Let G be a graph, $v \in V(G)$, and let σ be a deep decision. If G is a claw-free graph, then $I(T^{\sigma}_{G,u},x)$ is real-rooted. Moreover $I(G,x)$ divides $I(T^{\sigma}_{G,u},x)$.*

Proof. Assume that G is a claw-free graph. Then by Proposition 2.21 we have a sequence of induced subgraphs $G_1, \dots, G_k$ of G, such that

$$I(T^{\sigma}_{G,u}) = I(G,x) \prod_{i=1}^{k} I(G_i,x).$$

Since each G_i is an induced subgraph of a claw-free graph, therefore it is also claw-free. Then by Corollary 2.13, we have that each polynomial $I(G_i, x)$ and the polynomial $I(G, x)$ are real-rooted, so their product is also real rooted. $\square$

In this section will show some applications of this corollary. In all applications, of Corollary 2.23 the vertices of G will be labelled by integers. This labeling will induce a total order on the vertices in the most natural way, the order of two vertices will be the order of their labels.

Trees with real-rooted independence polynomial

In this subsection we will show that some families of trees have real-rooted independence polynomials.

Definition 2.24. *Let us recall that, the n-centipede W_n is a graph such that we take a path on n vertices and we hang 1 pendant edge from each vertex of it.*

The n-caterpillar H_n is a graph such that we take a path on n vertices and we hang 2 pendant edges from each vertex of it.

The Fibonacci tree $F_0 = K_1$ and $F_1 = K_2$ with roots $r_0 \in V(F_0)$ and $r_1 \in V(F_1)$. Then for $n \geq 2$ the nth Fibonacci tree F_n is obtained from the disjoint union of F_{n-1}, F_{n-2} and a new vertex, labeled as r_n, and connecting r_n to the roots of F_{n-1} and F_{n-2}. Define r_n as the root of F_n.

The proof of the real-rootedness of the independence polynomial of W_n was in [82], then a unified proof for W_n and H_n appeared in [75]. The statement for F_n was verified in [35] for $n \leq 22$, and conjectured for arbitrary n. Our proofs will follow the following strategy: for each mentioned T tree we will define a claw-free graph $\widetilde{G}$ with integer labels, such that the stable-path tree of $\widetilde{G}$ from one of its vertex will be isomorphic to T.

Proposition 2.25. *For any n, the independence polynomial of W_n is real-rooted, hence log-concave and unimodal.*

Proof. Let $\widetilde{W}_n$ be a graph (Fig. 2.5), such that we take a path on $\{1, \ldots, n\}$ and we attach a triangle to every $(2k+1)$th edge of the path. If n is odd, then we attach a pendant edge to n. Also label all the new vertices by numbers bigger than n.

These graphs are claw-free, and

$$T^{<}_{\widetilde{W}_n,1} \cong W_n.$$

(a) The graph $\widetilde{W}_n$, when n is even **(b)** The graph $\widetilde{W}_n$, when n is odd

Figure 2.5: The graph family $\widetilde{W}_n$

Therefore by Corollary 2.23 we have the desired statement. □

Proposition 2.26. *For any n, the independence polynomial of H_n are real-rooted, hence log-concave and unimodal.*

Proof. Let $\widetilde{H}_n$ be a graph (Fig. 2.6), such that we take a path on $\{0, \ldots, n+1\}$ and we attach a triangle to each edge, which is not the first or the last. Also label all the new vertices with numbers bigger than n.

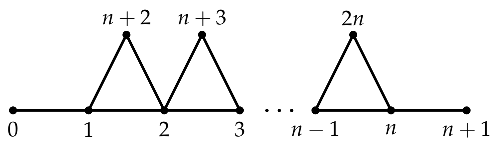


Figure 2.6: The graph $\widetilde{H}_n$

These graphs are claw-free, and

$$T^{<}_{\widetilde{H}_n,0} \cong H_n.$$

Therefore by Corollary 2.23 we have the desired statement. □

Proposition 2.27. *For any n, the independence polynomial of F_n are real-rooted, hence log-concave and unimodal.*

Proof. Let $\widetilde{F}_n$ be a graph (Fig. 2.7), such that we take the set $\{0, \ldots, n-1\}$ and we connect i and j if $0 < |i - j| \leq 2$.

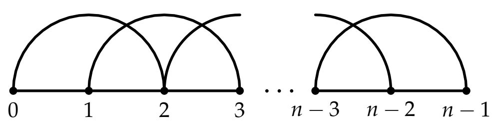


Figure 2.7: The graph $\widetilde{F}_n$

These graphs are claw-free, and

$$T^{<}_{\widetilde{F}_n,0} \cong F_n.$$

Therefore by Corollary 2.23 we have the desired statement. □

Remark 2.28. *If someone carefully examine the formula (2.10), then one might get the following identities:*

$$I(W_n, x) = I(\widetilde{W}_n)(1+x)^{\lfloor n/2 \rfloor},$$
$$I(H_n, x) = I(\widetilde{H}_n)(1+x)^{n-2},$$
$$I(F_n, x) = \prod_{k=0}^{n} I(\widetilde{F}_k, x)^{f_{n-k}},$$

where $f_0 = 1$*,* $f_1 = 0$ *and* $f_n = f_{n-1} + f_{n-2}$ *for* $n \geq 1$*.*

Some real-rooted graph families

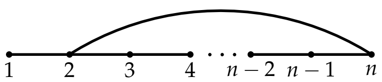
In this subsection we show another approach to verify real-rootedness of independence polynomials of some graphs. The idea is that for a graph G we construct a stable-path tree T, which is real-rooted. Then by Corollary 2.23 we know that $I(G, x)$ divides $I(T, x)$, so it means that $I(G, x)$ is also real-rooted.

Definition 2.29. *Let us define the following graph families.*

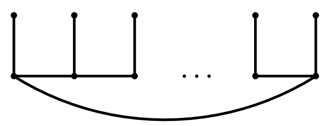
The nth apple graph A_n *is a graph (Fig. 2.8a), such that we take a path on* $\{1, \ldots, n\}$*, and we add the edge* $(2, n)$*.*

The n-sunlet graph N_n *is a graph (Fig. 2.8b), such that we take a cycle on* $\{1, \ldots, n\}$*, and we attach a new vertex to each vertex of the cycle. Also label all the new vertices with numbers bigger than n.*

Let M_n *be a graph (Fig. 2.9), such that we take a path on* $\{1, \ldots, n\}$*, and we attach 2 triangles to any* $2k+1$*th edge of the path. If n is odd, then we attach 2 pendant edges to n. For the new vertices choose different numbers greater than n as labels.*



(a) The apple graph (A_n)



(b) The sunlet graph (N_n)

Figure 2.8: Some graph families

A proof for real-rootedness of the independence polynomial of M_n and N_n was given in [75].

(a) The graph M_n, when n is even

(b) The graph M_n, when n is odd

Figure 2.9: The graph family M_n

Proposition 2.30. *For any n, the independence polynomial of M_n is real-rooted, hence log-concave and unimodal.*

Proof. By Proposition 2.26 we have that H_n has real-rooted independence polynomial. However we can see that

$$T^{<}_{M_n,1} \cong H_n.$$

By Corollary 2.23 we know that $I(M_n, x)$ divides $I(H_n, x)$, which implies, that $I(M_n, x)$ is real-rooted polynomial. □

Proposition 2.31. *For any n, the independence polynomial of N_n is real-rooted, hence log-concave and unimodal.*

Proof. By Proposition 2.25 we have that W_n has real-rooted independence polynomial. However we can see that

$$T^{<}_{N_n,1} \cong W_{2n-1}.$$

By Corollary 2.23 we know that $I(N_n, x)$ divides $I(W_{2n-1}, x)$, which implies, that $I(N_n, x)$ is real-rooted polynomial. □

Proposition 2.32. *For any $n \geq 4$, the independence polynomial of A_n is real-rooted, hence log-concave and unimodal.*

Proof. Let $\widetilde{A}_n$ be a graph (Fig. 2.10), such that we take a path on $\{1, \ldots, n\}$, and add the edge $(2, 4)$.

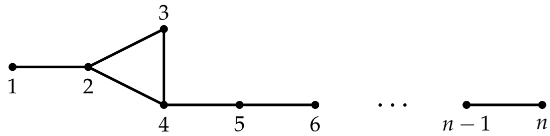


Figure 2.10: The graph $\widetilde{A}_n$

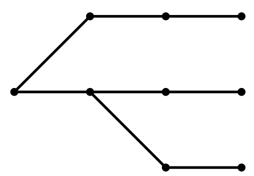


Figure 2.11: A tree T with real-rooted independence polynomial, which is not a stable-path tree of any non-tree graph

Since $\widetilde{A}_n$ is a claw-free graph, so for any $n \geq 4$ we have that $T_{\widetilde{A}_n,1}$ has a real-rooted independence polynomial. However we can see that

$$T^{<}_{\widetilde{A}_n,1} \cong T^{<}_{A_n,1},$$

which means that $I(T^{<}_{A_n,1}, x)$ is real-rooted. By Corollary 2.23 we know that $I(A_n, x)$ divides $I(T^{<}_{A_n,1}, x)$, which implies, that $I(A_n, x)$ is real-rooted polynomial. $\square$

2.3 Further examples

We would like to remark that this method is capable of proving the real-rootedness of the independence polynomial of the ladder graph (Thm. 5.1. of [81]), the polyphenyl ortho-chain ($\bar{O}_n$ of [4]), k-ary analogue of the Fibonacci tree (Remark of [74]).

One might ask that it is true that any tree with real-rooted independence polynomial is a stable path tree of a non-tree graph G. The answer is no, as the following example shows:

Let T be a tree on 9 vertices as on the Figure 2.11 and assume that there exists a graph G, a deep decision σ and a vertex $u \in V(G)$, such that $T = T^{\sigma}_{G,u}$. Then the independence polynomial of T is

$$I(T, x) = (1 + 3x + x^2)(1 + 5x + 6x^2 + x^3) + x(1 + 2x)^3 =$$
$$(1 + x)(1 + 8x + 20x^2 + 16x^3 + x^4),$$

where the factors are real-rooted and irreducible polynomials in $\mathbb{Q}[x]$. By Proposition 2.21 we have that $I(G, x)$ divides $I(T, x)$, and clearly G cannot be K_1 or the empty graph, therefore $I(G, x)$ should be $1 + 8x + 20x^2 + 16x^3 + x^4$. However it can be proved, that there is no such a graph G.

3 The adjoint polynomial as an independence polynomial

Let us recall from the beginning of this chapter the adjoint polynomial of a graph G, that is

$$h(G,x) = \sum_{k=1}^{n} (-1)^{n-k} a_k(G) x^k,$$

where $a_k(G)$ denotes the number of ways one can cover all vertices of the graph G by exactly k disjoint cliques of G. From the definition it is clear that $a_n(G) = 1$ and $a_{n-1}(G) = e(G)$, the number of edges, where the number of vertices of G is n.

Based on the aforementioned results (see Chapter 2) of [14, 15, 25, 49, 60, 78–80], there might be a connection between the independence polynomials and the adjoint polynomials.

In this section, we will show that there is indeed such a connection between the two graph polynomials. We will prove the following theorem:

Theorem 2.33. *For any graph G there exists a graph $\widehat{G}$, such that*

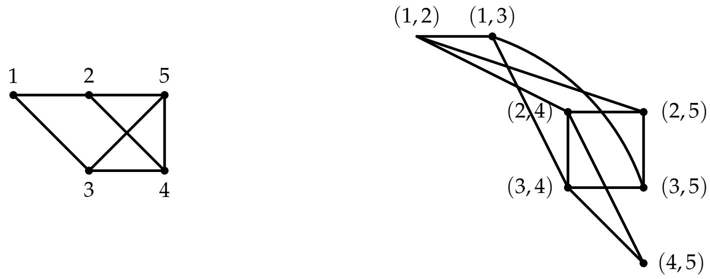
$$h(G,x) = x^n I(\widehat{G}, -1/x).$$

This correspondence will enable us to use the rich theory of independence polynomials to study the adjoint polynomials. In particular, we give new proofs of the aforementioned results Liu and Csikvári, and have another look at the proof of Brown and Erey [17]. For details see Section 3.2.

In Section 3.1, we will give the construction of $\widehat{G}$ and prove Theorem 2.33, moreover, we will show that $\widehat{G}$ can be taken as a spanning subgraph of the line graph of G. Recall that the line graph $L(G)$ for a graph G is a graph on the edge set of G, and there is an edge between two vertices of the line graph if they share a common vertex. This enables us to establish a connection with the matching polynomial of the graph G. For definition of the matching polynomial and applications of this connection see Section 3.2.

3.1 The construction

In this section, we will construct $\widehat{G}$ and prove Theorem 2.33.



(a) A graph G

(b) The graph $\widehat{G}$

Figure 2.12: An example for the construction

Let $\mathcal{K}(G)$ denote the set of clique covers of G, that is

$$\{\{S_1,\dots,S_k\} \subseteq \mathcal{P}(V(G)) \mid \cup_{i=1}^{k} S_i = V(G), \text{ if } 1 \le i \ne j \le n, \text{ then } S_i \cap S_j = \emptyset,\ G[S_i] \text{ is a complete graph}\},$$

then one can write the adjoint polynomial of G in the following way:

$$h(G,x) = \sum_{Q \in \mathcal{K}(G)} (-1)^{n-|Q|} x^{|Q|}.$$

We will also use the notation

$$h^*(G,x) = x^n h(G,1/x) = \sum_{k=0}^{n-1} (-1)^k a_{n-k}(G) x^k.$$

Let us choose an arbitrary ordering on the vertices of G, that is $V(G) = \{u_1,\dots,u_n\}$. Then we construct a $\widehat{G}$ graph as follows. Let $V(\widehat{G}) = \{(u_i,u_j) \in E(G) \mid 1 \le i < j \le n\}$, and let $(u_i,u_j) \ne (u_k,u_l) \in V(\widehat{G})$ be two vertices. We may assume that $j \le l$, then the two vertices are connected, if and only if ($i = k$) or ($j = k$) or ($j = l$ and $(u_i,u_k) \notin E(G)$). Clearly $\widehat{G}$ is a subgraph of the line graph of G. In the next theorem we show that the independence polynomial of $\widehat{G}$ actually equals to $h^*(G,x)$. For example see Figure 2.12.

Proposition 2.34. *Let G be a graph and let us choose an ordering of the vertices. Then the constructed $\widehat{G}$ graph satisfies that*

$$h^*(G,x) = I(\widehat{G},-x).$$

Moreover if $e = (u_{n-1},u_n) \in E(G)$, then $\widehat{G-e} \subseteq \widehat{G}$.

Proof. The second statement is clear from the construction above. In order to prove the first statement we will show that there is a bijection between independent sets of $\widehat{G}$ and $\mathcal{K} = \mathcal{K}(G)$. More precisely, if we let $\mathcal{I} = \mathcal{I}(\widehat{G})$ denote the set of independent sets of $\widehat{G}$, then there exists a bijection $\phi : \mathcal{K} \to \mathcal{I}$ such that for any $Q \in \mathcal{K}$ we have $|\phi(Q)| = n - |Q|$.

Let $Q = \{S_1, \dots, S_k\} \in \mathcal{K}$. For $1 \leq i \leq k$ let $f(i)$ denote the maximal index in S_i, that is $f(i) = \max\{1 \leq j \leq n \mid u_j \in S_i\}$. Then $\phi(Q)$ will be the union of clique edges having one endpoint as a maximally indexed vertex, that is

$$\phi(Q) = \bigcup_{1 \leq i \leq k} \{(u_j, u_{f(i)}) \mid u_j \in S_i,\ j \neq f(i)\} \subseteq V(\widehat{G}).$$

First we show that $\phi(Q)$ is an independent set in $\widehat{G}$. Let us define the sets $F_i = \{(u_j, u_{f(i)}) \mid u_j \in S_i,\ j \neq f(i)\}$ for $1 \leq i \leq k$. If $1 \leq i < i' \leq k$, then there is no edge between F_i and $F_{i'}$ in $\widehat{G}$, since Q is a partition of $V(G)$. Also the set F_i is independent for $1 \leq i \leq k$, since if $(u_j, u_{f(i)}) \neq (u_{j'}, u_{f(i)}) \in F_i \subseteq E(G[S_i])$, then $(u_j, u_{j'}) \in E(G[S_i])$, because S_i is a clique.

Furthermore we see that $|F_i| = |S_i| - 1$, so

$$|\phi(Q)| = \sum_{1 \leq i \leq k} |F_i| = \sum_{1 \leq i \leq k} (|S_i| - 1) = n - |Q|$$

For the surjectivity of ϕ let $I \in \mathcal{I}$ be fixed, and let $K_i = \{(u_j, u_i) \in I \mid j < i\}$ for $1 \leq i \leq n$. Then

$$Q' = \big\{\{u_i\} \cup \{u_j \mid (u_j, u_i) \in K_i\} \mid K_i \neq \emptyset\big\}$$

is a set of pairwise disjoint subsets of $V(G)$, where each subset induces a clique in G. Therefore

$$Q = Q' \cup \big\{\{u_i\} \mid u_i \notin \cup Q'\big\}$$

is a partition of $V(G)$ where each part induces a clique in G. Moreover we have that

$$\phi(Q) = \bigcup_{K_i \neq \emptyset} \{(u_j, u_i) \in I \mid j < i\} = I.$$

So ϕ is a bijection.

Now the statement follows as:

$$h^*(G, x) = \sum_{Q \in \mathcal{K}} (-x)^{n-|Q|} = \sum_{Q \in \mathcal{K}} (-x)^{|\phi(Q)|} = \sum_{I \in \mathcal{I}} (-x)^{|I|} = I(\widehat{G}, -x).$$

□

3.2 About "the largest" root of the adjoint polynomial

In this section we will give a various applications of Theorem 2.33 and the construction. First we collect some results on independence polynomials of graphs.

We will also need some results on a modified version of the matching polynomial. Let

$$\overline{M}(G,x) = \sum_{k\geq 0}(-1)^k m_k(G)x^{n-k},$$

where $m_k(G)$ denotes the number of matchings with k edges in G. Note that

$$\overline{M}(G,x) = x^n I(L(G), -1/x).$$

The following theorem is due to Heilmann and Lieb.

Theorem 2.35. *[41] All zeros of $\overline{M}(G,x)$ are real, and the largest zero $t(G)$ is at most $4(\Delta - 1)$, where Δ is the largest real root.*

Now we will present our new proofs for various results of Liu and Csikvári.

Corollary 2.36. *[25, 78] Let G be a connected graph. Then $h(G,x)$ has a real zero, and let $\gamma(G)$ be the largest among them. Then $\gamma(G)$ is a simple zero of $h(G,x)$, and if $\xi \neq \gamma(G)$ is a zero of $h(G,x)$, then $\gamma(G) > |\xi|$.*

Proof. Choose an ordering of the vertices of G, and construct $\widehat{G}$. From the construction of $\widehat{G}$ it is clear that $\widehat{G}$ is a connected graph, so $\beta(\widehat{G}) > 0$ is a simple simple of $I(\widehat{G}, -x) = h^*(G,x)$, thus, $\beta(\widehat{G}) = \gamma(G)^{-1}$. The rest is the consequence of Theorem 2.1. □

Corollary 2.37. *[25] Let G be a connected graph. Then $\gamma(G) \leq t(G)$. In particular $\gamma(G) \leq 4(\Delta - 1)$.*

Proof. The first inequality follows from the fact that $\widehat{G} \subseteq L(G)$. Indeed this implies that

$$\gamma(G)^{-1} = \beta(\widehat{G}) \geq \beta(L(G)) = t(G)^{-1},$$

where the inequality follows from Theorem 2.2, and the equalities follow from the identities $h^*(G,x) = x^n I(\widehat{G}, -1/x)$ and $\overline{M}(G,x) = x^n I(L(G), -1/x)$. The second inequality follows from the first inequality and Theorem 2.35. □

Corollary 2.38. *[25] Let H be a proper subgraph of G. Then $\gamma(H) < \gamma(G)$.*

Proof. Suppose that H can be obtained from G by deleting the edges $\{e_1, \dots, e_k\}$ and then deleting the isolated vertices $\{v_1, \dots, v_l\}$. Let $G_i = G - \{e_1, \dots, e_i\}$ for $1 \leq i \leq$

k, and $G_0 = G$. Since G_i and G_{i+1} differ only in one edge, then we can choose an ordering of the vertices of G_i, such that that e_i is the edge between the last two vertices. Then $\widehat{G_{i+1}} = \widehat{G_i - e}$. This implies that $\gamma(G_{i+1}) < \gamma(G_i)$ and so $\gamma(G_k) < \gamma(G_{k-1}) < \dots \gamma(G_0) = \gamma(G)$.

Since $h(H \cup K_1, x) = (-x)h(H, x)$ and $\gamma(H) > 0$, we have that $\gamma(H \cup K_1) = \gamma(H)$. So $\gamma(G_k) = \gamma(G_k - v_1) = \dots = \gamma(G_k - \{v_1, \dots, v_l\}) = \gamma(H)$. $\square$

Corollary 2.39. *[25] Let G be a connected graph and H be a subgraph, then in the following series*

$$\frac{h^*(H,x)}{h^*(G,x)} = \sum_{k \geq 0} s_k(G,H)x^k$$

for each $k \geq 0$ *the coefficients* $s_k(G,H)$ *are positive integers.*

Proof. Direct consequence of Theorem 2.3. $\square$

Another consequence is that it gives an insight into the proof of the theorem of [17]. Brown and Erey proved (using our notations) that if G is a graph on n vertices and $\chi(\overline{G}) \geq n-3$, then $h(G,x)$ has only real roots. Using some reduction and classification of graphs with chromatic number $\chi(G) \geq n-3$ in their paper, they reduced the problem to prove real rootedness of $h(G,x)$ only for graphs G, where G has n vertices and a subset $S \subseteq V(G)$, such that $|S| \leq 3$ and for any $u \in V(G) \setminus S$ the neighborhood of u is strictly contained in S. If one would follow the construction of $\widehat{G}$ for such a graph G, then it would turn out that $\widehat{G}$ is a claw-free graph, that has real-rooted independence polynomial according to Corollary 2.13.

3

On zero-free regions for the anti-ferromagnetic Potts model on bounded-degree graphs

The Potts model is an important object in statistical physics generalizing the Ising model for magnetism. The partition function of the Potts model captures much information about the model and its study connects several different areas including statistical physics, probability theory, combinatorics and theoretical computer science.

Every graph G (throughout the chapter we will always assume graphs are simple) has an associated Potts model partition function defined as follows. Fix $k \in \mathbb{N}$, which will be the number of states or colors. We will consider all functions $\phi : V \to [k] := \{1, \dots, k\}$ and often refer to $\phi(v)$ as the color of v. For our given graph $G = (V, E)$, we associate a variable $w_e \in \mathbb{C}$ to each edge $e \in E$. The k-state *partition function of the Potts model* for G is a polynomial in the variables $(w_e)_{e \in E}$ given by

$$\mathbf{Z}(G; k, (w_e)_{e \in E}) := \sum_{\phi: V \to [k]} \prod_{\substack{uv \in E \\ \phi(u) = \phi(v)}} w_{uv}.$$

If k and the w_e are clear from the context we simply write $\mathbf{Z}(G)$. One often considers the 'univariate' special case when all w_e are equal to some $w \in \mathbb{C}$, in which case we write $\mathbf{Z}(G; k, w)$ for the partition function. We note that in statistical physics one parametrizes $w_e = e^{\beta J_e}$ with β the inverse temperature and J_e the coupling constant. The model is called *anti-ferromagnetic* if $w_e \in (0, 1)$ (i.e. $J_e < 0$) for each $e \in E$ and *ferromagnetic* if $w_e > 1$ (i.e. $J_e > 0$) for each $e \in E$.

The study of the location of the complex zeros of the partition function is originally motivated by a seminal result of Lee and Yang [77], roughly saying that absence of complex zeros near a point on the real axis implies that the model does not undergo a phase transition at this point. Another motivation is the algorithmic computation of partition functions which has recently been linked to the location of the complex zeros. We discuss this theme in more detail after stating our main result: a new zero-free region for the multivariate anti-ferromagnetic Potts model, which will be proved in Section 3.

Theorem 3.1. *For each $\Delta \in \mathbb{N}$ there exists a constant $c_\Delta \leq e$ and an open set $U \subset \mathbb{C}$ containing the real interval $[0,1)$ such that the following holds. For all graphs G of maximum degree at most Δ, all integers $k \geq k_\Delta := \lceil c_\Delta \cdot \Delta + 1 \rceil$, and for all $(w_e)_{e\in E}$ such that $w_e \in U$ for each $e \in E$, we have*

$$\mathbf{Z}(G; k, (w_e)_{e\in E}) \neq 0.$$

See Table 3.1 below for better bounds on c_Δ and k_Δ for small values of Δ.

Remark 3.2. *We can in fact guarantee an open set U containing the* closed *interval $[0,1]$ under the same conditions as in the theorem above. It is however more convenient to work with $[0,1)$. In Remark 3.12 we indicate how to extend our results to the closed interval.*

We moreover note that while we work with simple graphs in the chapter, our result also holds for graphs with multiple edges (loops are not allowed). Our proof of Theorem 3.1 only requires a tiny change to accommodate for this. We leave this for the reader.

Δ	3	4	5	6	7	8	9	10	11	12
c_Δ	1.485	1.749	1.939	2.081	2.193	2.283	2.357	2.419	2.472	2.517
k_Δ	6	8	11	14	17	20	23	26	29	32

Table 3.1: Upper bounds on c_Δ and the resulting bounds on k_Δ for small Δ.

Related work

There are several results concerning zero-free regions of the partition function of the Potts model, some of which we discuss below. See e.g. [19–21, 30, 63] for results on the location of the (Fisher) zeros of the partition function of the anti-ferromagnetic Potts model on several lattices, and [6, 31, 42, 64] for results on general (bounded degree) graphs. Let us say a few words on the latter results and connect these to our present work.

The partition function of the Potts model is a special case of the random cluster model of Fortuin and Kasteleyn [33] which, for a graph $G = (V, E)$ and variables q and

$(v_e)_{e\in E}$, is given by

$$Z(G;q,(v_e)_{e\in E}) := \sum_{F\subseteq E} q^{k(F)} \prod_{e\in F} v_e,$$

where $k(F)$ denotes the number of components of the graph (V,F). Indeed, taking $q = k$ and $v_e = w_e - 1$ for each edge e, it turns out that $Z(G;q,(v_e)_{e\in E}) = \mathbf{Z}(G;k,(w_e)_{e\in E})$; see [65] for more details and for the connection with the Tutte polynomial.

Almost twenty years ago Sokal [64] proved that for any graph G of maximum degree $\Delta \in \mathbb{N}$ there exists a constant $C \leq 7.964$ such that if $|1+v_e| \leq 1$ for each edge e, then for any $q \in \mathbb{C}$ such that $|q| \geq C\Delta$ one has $Z(G;q,(v_e)_{e\in E}) \neq 0$. The bound on the constant C was improved to $C \leq 6.907$ by Procacci and Fernández [31]. See also [42] for results when the condition $|1+v| \leq 1$ is removed. In our setting, Sokal's result implies that $\mathbf{Z}(G;k,(w_e)_{e\in E}) \neq 0$ for any integer $k > C\Delta$ when every w_e lies in the unit disk.

Our main result may be seen as an improvement upon the constant C, though in a more restricted setting where, instead of demanding that $\mathbf{Z}(G;k,(w_e)_{e\in E})$ is nonzero in the unit disk, we demand that $\mathbf{Z}(G;k,(w_e)_{e\in E})$ is nonzero in an open region containing $[0,1)$. Interestingly, our method of proof is completely different from the approach in [31, 42, 64], which is based on cluster expansion techniques from statistical physics. We prove our results by induction using some basic facts from geometry and convexity, building on an approach developed by Barvinok [6]. Previously, Barvinok used this approach in [6, Theorem 7.1.4] (improving on [8]) to show that for each positive integer Δ there exists a constant $\delta_\Delta > 0$ (one may choose e.g. $\delta_3 = 0.18$, $\delta_4 = 0.13$, and in general $\delta_\Delta = \Omega(1/\Delta)$) such that for any positive integer k and any graph G of maximum degree at most Δ one has

$$\mathbf{Z}(G;k,(w_e)_{e\in E}) \neq 0 \text{ provided } |1-w_e| \leq \delta_\Delta \text{ for each edge } e. \tag{3.1}$$

In fact this result is proved in much greater generality, but we have stated it here just for the Potts model.

While the approach in [6] seems crucially to require that w_e is close to 1, here we present ideas that allow us to extend the approach in a way that bypasses this requirement. As such the approach may be applicable to other types of models.

Algorithmic applications

Barvinok [6] recently developed an approach to design efficient approximation algorithms based on absence of complex zeros in certain domains. This gives an additional motivation for studying the location of of complex zeros of partition functions. While it is typically #P-hard to compute the partition function of the Potts model exactly one may hope to find efficient approximation algorithms (although for certain choices of parameters it is known to be NP-hard to approximate the partition function of the Potts model [34]).

Combining Theorem 3.1 with Barvinok's approach and results from [56], we obtain the following corollary. We discuss how the corollary is obtained at the end of this section.

Corollary 3.3. *Let $\Delta \in \mathbb{N}$, $w \in [0,1]$ and let $k \geq c_\Delta \cdot \Delta + 1$. Then there exists a deterministic algorithm which given an n-vertex graph of maximum degree at most Δ computes a number ξ satisfying*

$$e^{-\varepsilon} \leq \frac{\mathbf{Z}(G;k,w)}{\xi} \leq e^{\varepsilon}$$

in time polynomial in n/ε.

Corollary 3.3 gives us a fully polynomial time approximation scheme (FPTAS) for computing the partition function of the anti-ferromagnetic Potts model (for the right choice of parameters). In the case when $w = 0$, $\mathbf{Z}(G;k,w)$ is the number of proper k-colorings of G and so the corollary gives an FPTAS for computing the number of proper k colorings when $k \geq k_\Delta^{\min} > c_\Delta \cdot \Delta + 1$. Lu and Yin [51] gave an FPTAS for this problem when $k \geq 2.58\Delta + 1$; we improve their bound for $\Delta = 3, \ldots, 11$. We remark that for $\Delta = 3$ there is in fact an FPTAS for counting the number of 4-colorings [50]. Moreover, there exists an efficient randomized algorithm due to Vigoda [73], which is based on Markov chain Monte Carlo methods, that only requires $k > (11/6)\Delta$ colors. See [22] for a very recent small improvement on the constant $11/6$.

Proof sketch of Corollary 3.3. We first sketch Barvinok's algorithmic approach applied to the partition function of the Potts model from which Corollary 3.3 is derived. Suppose we wish to evaluate $\mathbf{Z}(G;k,w)$ at some point $w \in [0,1)$ for some graph G of maximum degree at most Δ and positive integer $k \geq c_\Delta \cdot \Delta$. The first step is to define a univariate polynomial $q(z) := \mathbf{Z}(G;k,1+z(w-1))$. We then wish to compute $q(1)$.

By Theorem 3.1 combined with (3.1) (cf. Remark 3.2) there exists an open region U' that contains $[0,1]$ on which q does not vanish. Then we take a disk D of radius slightly larger than 1 and a fixed polynomial p such that $p(0) = 0$ and $p(1) = 1$ and such that D is mapped into U' by p; see [6, Section 2.2] for details. We next define another polynomial f on D by $f(z) = q(p(z))$. Then f does not vanish on D and hence $\log(f(z))$ is analytic on D and has a convergent Taylor series. To approximate $f(1) = \mathbf{Z}(G;k,w)$ we truncate the Taylor series of $\log(f(z))$ (see [6, Lemma 2.2.1] for details on where exactly to truncate the Taylor series to get a good approximation), and then we compute these Taylor coefficients.

To compute the Taylor coefficients of $\log(f(z))$ it turns out that it is suffices to compute the low order coefficients of the polynomial q, since these can be combined with the coefficients of the polynomial p to obtain the low order coefficients of f, from which one can deduce the Taylor coefficients of $\log(f(z))$ via the Newton identities; see [56, Section 2]. By Theorem 3.2 from [56] the low order coefficients of q can be computed in polynomial time, since, up to an easy to compute multiplicative constant, q is a *bounded induced graph counting polynomial* ([56, Definition 3.1]), as is proved in greater generality in [56, Section 6]. $\square$

Organization of the chapter

In the next section we set up some notation and discuss some preliminaries that we need in the proof of our main theorem. This proof is inspired by Barvinok's proof of (3.1), and has a similar flavor. It is based on induction with a somewhat lengthy and technical induction hypothesis. For this reason we give a brief sketch of our approach in the next section. Section 2 then contains an induction for Theorem 3.1. This induction contain a condition that is checked in Section 3. The proof of Theorem 3.1 follows upon combining the results of Sections 2 and 3; see the remark after the statement of Proposition 3.10. In Section 4 we slightly modify our induction hypotheses and add another condition to it that allows us to improve our bounds for small values of Δ. We close with some concluding remarks in Section 5.

1 Preliminaries, notation and main idea of the proofs

In order to prove our results, we will need to work more generally with the partition function of the Potts model with boundary conditions. For a list $W = w_1 \dots w_m$ of distinct vertices of V and a list $L = \ell_1 \dots \ell_m$ of pre-assigned colors in $[k]$ for the vertices in W the *restricted partition function* $\mathbf{Z}_L^W(G)$ is defined by

$$\mathbf{Z}_L^W(G) := \sum_{\substack{\phi: V \to [k] \\ \phi \text{ respects } (W,L)}} \prod_{\substack{uv \in E \\ \phi(u) = \phi(v)}} w_{uv},$$

where we say that ϕ respects (W, L) if for all $i = 1 \dots, m$ we have $\phi(w_i) = \ell_i$. We call the vertices $w_1, \dots, w_m$ *fixed* and refer to the remaining vertices in V as *free* vertices. The length of W (resp. L), written $|W|$ (resp. $|L|$) is the length of the list. Given a list of distinct vertices $W' = w_1 \dots w_m$, and a vertex u (distinct from $w_1, \dots, w_m$) we write $W = W'u$ for the concatenated list $W = w_1 \dots w_m u$ and we use similar notation $L'\ell$ for concatenation of lists of colors. We write $\deg(v)$ for the degree of a vertex v and we write $G \setminus uv$ $(G - u)$ for the graph obtained from G by removing the edge uv (by removing the vertex u).

In our proofs we often view the restricted partition functions $\mathbf{Z}_L^W(G)$ as vectors in $\mathbb{C} \simeq \mathbb{R}^2$. The following lemma of Barvinok turns out to be very convenient.

Lemma 3.4 (Barvinok [6, Lemma 3.6.3]). *Let $u_1, \dots, u_k \in \mathbb{R}^2$ be non-zero vectors such that the angle between any vectors u_i and u_j is at most α for some $\alpha \in [0, 2\pi/3)$. Then the u_i all lie in a cone of angle at most α and*

$$\Big| \sum_{j=1}^{k} u_j \Big| \geq \cos(\alpha/2) \sum_{j=1}^{k} |u_j| \,.$$

Let us now try to explain our approach. It starts with Barvinok's approach from [6, Section 7.2.3] tailored to the partition function of the Potts model. Fix a vertex v of the graph G. Then $\mathbf{Z}(G) = \sum_{i=1}^{k} \mathbf{Z}_i^v(G)$. If we can prove that the pairwise angles between $\mathbf{Z}_i^v(G)$ and $\mathbf{Z}_j^v(G)$ for all $i, j \in [k]$ are bounded above by $2\pi/3$ then one can conclude by the Lemma 3.4 that $\mathbf{Z}(G) \neq 0$. So the idea is to show (using induction on list size) that for any list W of distinct vertices of G and L of pre-assigned colors from $[k]$ where $|W| = |L|$ we have for any vertex $v \notin W$ that the pairwise angles between $\mathbf{Z}_{L,W,v;}^{W,L,i;}(G)$ and $\mathbf{Z}_{L,W,v;}^{W,L,j;}(G)$ are bounded by some $\alpha < 2\pi/3$.

To obtain information about $\mathbf{Z}_{L,W,v;}^{W,L,i;}(G)$, the next step is to fix the neighbors of v and apply a suitably chosen induction hypothesis to all of these neighbors combined with some kind of telescoping argument. Suppose for the moment that the degree of v is 1, and let u be the unique neighbor of v. Then

$$\mathbf{Z}_{L,W,v;}^{W,L,j;}(G) = \sum_{i=1}^{k} \mathbf{Z}_{L,;i;W,v,u;}^{W,;u;L,j,i;}(G) = \sum_{i \neq j} \mathbf{Z}_{L,;i;W,v,u;}^{W,;u;L,j,i;}(G \setminus uv) + w_{uv} \mathbf{Z}_{L,;j;W,v,u;}^{W,;u;L,j,j;}(G \setminus uv). \quad (3.2)$$

To compare $\mathbf{Z}_{L,W,v;}^{W,L,j;}(G)$ with $\mathbf{Z}_{L,;W,v;}^{W,L,j';}(G)$, Barvinok shows that if w_{uv} is sufficiently close to 1, then their angle is not too big (if $w_{uv} = 1$ then they are equal) and then the induction can continue.

We however allow w_{uv} to be arbitrarily close to zero, so we need an additional idea: in the induction hypothesis, besides the condition that the angle between two vectors $\mathbf{Z}_{L,;i;W,v,u;}^{W,;u;L,j,i;}(G)$ and $\mathbf{Z}_{L,;i';W,v,u;}^{W,;u;L,j,i';}(G)$ is small, we add the condition that their lengths should not be too far apart. This leads to complications, but fortunately they can be overcome with some additional ideas. We refer to the next section for the induction statement and the details of the proofs. We next collect some tools that we will use.

We will need the following simple geometric facts, which follow from the sine law and cosine law for triangles.

Proposition 3.5. *Let u and u' be non-zero vectors in $\mathbb{R}^2$.*

(i) *If the angle between u and u' is at most $\pi/3$, then $|u - u'| \leq \max\{|u|, |u'|\}$.*

(ii) *The angle γ between u and u' satisfies $\sin\gamma \leq |u - u'|/|u'|$.*

For $r > 0$ and $a \in \mathbb{C}$ we denote by $B(a, r) \subseteq \mathbb{C}$ the open disk of radius r centered at a. For $d \in \mathbb{N}$ we denote

$$B(a,r)^d := \{b_1 \dots b_d \mid b_i \in B(a,r) \text{ for } i = 1, \dots, d\} \subseteq \mathbb{C}.$$

We will need the Grace–Szegő–Walsh coincidence theorem, which we state here just for disks. Recall that a polynomial p in variables $x_1, \dots, x_d$ is called *multi-affine* if for each variable its degree in p at most one.

Lemma 3.6 (Grace–Szegő–Walsh). *Let p be a multi-affine polynomial in the variables $x_1, \dots, x_d$. Suppose that p is symmetric under permuting the variables. Then for any disk $B \subset \mathbb{C}$, if $\zeta_1, \dots, \zeta_d \in B$, then there exists $\zeta \in B$ such that*

$$p(\zeta, \dots, \zeta) = p(\zeta_1, \dots, \zeta_d).$$

We refer the reader to [59, Theorem 3.4.1b] for a proof of this result, background and related results. Using the previous result we can show convexity of the set $B(1,r)^d \subseteq \mathbb{C}$ for certain choices of r and d.

Lemma 3.7. *Let $d \in \mathbb{N}$. Then for any $0 < r < 1/d$ the set $B(1,r)^d$ is convex.*

Proof. Define $f : \mathbb{C} \to \mathbb{C}$ by $z \mapsto (1 + rz)^d$. Then, by Lemma 3.6, $B(1,r)^d$ is the image of $B(0,1)$ under f. We compute the ratio

$$\frac{f''(z)}{f'(z)} = r(d-1)\,(1+zr)^{-1}.$$

The norm of this ratio is, for any $z \in B(0,1)$, strictly upper bounded by 1, since $r < 1/d$. This implies that for all $z \in B(0,1)$,

$$\Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0.$$

A classical result cf. [28, Section 2.5] now implies that the image of $B(0,1)$ under f is a convex set. $\square$

In Section 4, we will also need the following geometric lemma, which we prove below.

Lemma 3.8. *Let u and u' be non-zero vectors in $\mathbb{R}^2$ and $r \geq 1$ real number such that the angle between u and u' is at most $\phi < \pi/3$ and*

$$r^{-1} \leq \frac{|u|}{|u'|} \leq r.$$

Then

$$|u - u'| \leq \max\left\{2\sin(\phi/2), \sqrt{1 + r^{-2} - 2r^{-1}\cos\phi}\right\} \cdot \max\left\{|u|, |u'|\right\}.$$

Proof. Without loss of generality assume that $|u'| \geq |u|$ and $\arg(u) - \arg(u') = \phi \geq 0$. Then we can assume that u' is the point A in Figure 3.1, the length OA is $|u'|$, the length OD is $r^{-1}|u'|$, and that u lies in the shaded area which we denote by U.

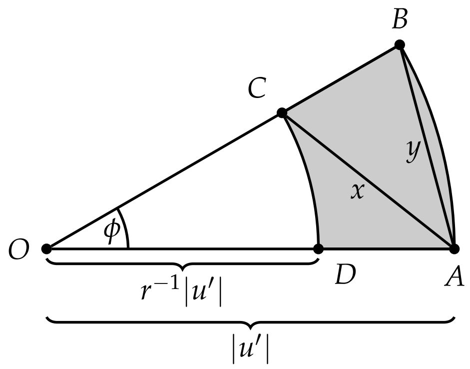


Figure 3.1: A diagram for the proof of Lemma 3.8. The shaded area is U.

The diameter of U is an upper bound on $|u - u'|$, and it is not hard to see that the diameter of U is the maximum of the distances between any pair of the points A, B, C, and D. By symmetry and by the triangle inequality one can see that this maximum is achieved by x or y. In order to calculate these lengths we apply the cosine law in the triangles OAC and OAB (and a half-angle formula). □

2 An induction for Theorem 3.1

Let $G = (V, E)$ be a graph together with complex weights $w = (w_e)_{e \in E}$ assigned to the edges, a list of distinct vertices W, and a list of pre-assigned colors L with $|W| = |L|$ (i.e. each vertex in the list W is colored with the corresponding color from the list L). Recall that the vertices in W are called fixed and those in $V \setminus W$ are called free.

Let $\varepsilon > 0$ be given. We say a neighbor v of a vertex $u \in V$ is a *bad neighbor* of u if $|w_{uv}| \leq \varepsilon$. We say a color $\ell \in [k]$ is *good* for a vertex $u \in V$ if every fixed neighbor of u is not colored ℓ; we call ℓ *bad* if u has at least one fixed, bad neighbor colored ℓ. We call a color *neutral* if it is neither good nor bad. Note that the definition of good, neutral and bad colors also applies if u is fixed. We denote the set of good colors by $\mathcal{G}(G,W,L,u)$, the set of neutral colors by $\mathcal{N}(G,W,L,u,\varepsilon)$ and the set of bad colors by $\mathcal{B}(G,W,L,u,\varepsilon)$. We will also write $m(G,W,L,u,\varepsilon,\ell)$ for the number of fixed bad neighbors of u with color ℓ. When G, W, L, u, and ε are clear from the context we will write e.g. $\mathcal{G} = \mathcal{G}(G,W,L,u)$, $\mathcal{B} = \mathcal{B}(G,W,L,u,\varepsilon)$, $\mathcal{N} = [k] \setminus (\mathcal{G} \cup \mathcal{B})$, and $m(\ell) = m(G,W,L,u,\varepsilon,\ell)$.

For a graph $G = (V,E)$ we call $W \subseteq V$ a *leaf-independent set* if W is an independent set and every vertex in W has degree exactly 1. In particular this means every vertex in W has exactly one neighbor in $V \setminus W$.

Theorem 3.9. *Let $\Delta \in \mathbb{N}_{\geq 3}$. Suppose that $k > \Delta$ and $0 < \varepsilon < 1$ are such that there exists a positive constant $K < 1/(\Delta-1)$ with $\theta := \arcsin(K) \in (0, \frac{\pi}{3(\Delta-1+\varepsilon)})$ such that for each $d = 0,\dots,\Delta-1$, with $b = \Delta - d$,*

$$0 < \frac{(1+\varepsilon)^2}{(k-b)(1-K)^d - \varepsilon b} \leq K. \tag{3.3}$$

Then for each graph $G = (V,E)$ of maximum degree at most Δ and every $w = (w_e)_{e\in E}$ satisfying for each $e \in E$ that

(i) $|w_e| \leq \varepsilon$, or

(ii) $|\arg(w_e)| \leq \varepsilon\theta$ and $\varepsilon < |w_e| \leq 1$,

the following statements hold for $\mathbf{Z}(G) = \mathbf{Z}(G;k,w)$.

A *For all lists W of distinct vertices of G such that W forms a leaf-independent set in G and for all lists of pre-assigned colors L of length $|W|$, $\mathbf{Z}_L^W(G) \neq 0$.*

B *For all lists $W = W'u$ of distinct vertices of G such that W is a leaf-independent set and for any two lists $L'\ell$ and $L'\ell'$ of length $|W|$:*

(i) *the angle between the vectors $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ and $\mathbf{Z}^{W';L',\ell';}_{L'';W',u;}(G)$ is at most θ,*

(ii)

$$\frac{\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)}{\mathbf{Z}^{W';L',\ell';}_{L'';W',u;}(G)} \in B(1,K). \tag{3.4}$$

C *For all lists $W = W'u$ of distinct vertices such that the initial segment W' forms a leaf-independent set in G and for all lists of pre-assigned colors L' of length $|W'|$, the* *following holds. Write $\mathcal{G} = \mathcal{G}(G, W', L', u)$ and $\mathcal{N} = \mathcal{N}(G, W', L', u, \varepsilon)$, let d be the number of free neighbors of u, and let $b = \Delta - d$. Then*

(i) *for any $\ell \in \mathcal{G} \cup \mathcal{N}$, $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G) \neq 0$,*

(ii) *for any $\ell, \ell' \in \mathcal{G} \cup \mathcal{N}$, the angle between the vectors $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ and $\mathbf{Z}^{W';L',\ell';}_{L'';W',u;}(G)$ is at most $(d + b\varepsilon)\theta$,*

(iii) *for any $\ell, j \in \mathcal{G}$,*

$$\frac{\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)}{\mathbf{Z}^{W';L',j;}_{L';W',u;}(G)} \in B(1, K)^d. \tag{3.5}$$

2.1 Proof

We prove that **A**, **B**, and **C** hold by induction on the number of free vertices of a graph. The base case consists of graphs with no free vertices. Clearly **A** and **B** hold in this case as they are both vacuous: if there are no free vertices then $W = V$ but then W cannot be a leaf-independent set.

For statement **C** we note that since there are no free vertices, $V \setminus W' = \{u\}$, and hence G must be a star with center u. Part **C(i)** follows since when $\ell \in \mathcal{G} \cup \mathcal{N}$ we have that $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ is a product over nonzero edge-values. Part **(ii)** follows since changing the color of u from ℓ to $j \in \mathcal{G} \cup \mathcal{N}$, we can obtain $\mathbf{Z}^{W';L',j;}_{L';W',u;}(G)$ from $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ by multiplying and dividing by at most $\deg(u)$ factors w_{uv} with $\arg(w_{uv}) \leq \varepsilon\theta$; hence the restricted partition function changes in angle by at most $\Delta\varepsilon\theta$. Part **(iii)** follows similarly, as when there are no free vertices we must have $d = 0$, and changing the color of u from j to ℓ does not change the value of the restricted partition function since both colors are good. Hence $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)/\mathbf{Z}^{W';L',j;}_{L';W',u;}(G) = 1 \in B(1, K)^d$.

Now let us assume that statements **A**, **B**, and **C** hold for all graphs with $f \geq 0$ free vertices. We wish to prove the statements for graphs with $f + 1$ free vertices. We start by proving **A**.

Proof of A

Let u be any free vertex. We proceed using the fact that $|\mathbf{Z}^W_L(G)| = |\sum_{j=1}^k \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)|$. Let $\mathcal{G} = \mathcal{G}(G, W, L, u)$, $\mathcal{B} = \mathcal{B}(G, W, L, u, \varepsilon)$, $\mathcal{N} = [k] \setminus (\mathcal{G} \cup \mathcal{B})$ and $\hat{b} = |\mathcal{B}|$. Let d be the number of free neighbors of u and let $b = \Delta - d$. Note that $\hat{b} \leq b$ and $|\mathcal{G}| \geq k - b$. After fixing u to any $j \in [k]$ we have one less free vertex, and hence can apply **C** using induction as necessary.

There are two cases to consider. If $\hat{b} = 0$ then by induction using **C(i)** we have that the $\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)$ are non-zero and by **C(ii)** the angle between any two of the $\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)$ is at most $\Delta\theta < \frac{\Delta}{\Delta-1}\pi/3 \leq 3/2 \cdot \pi/3 = \pi/2$. So the $\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)$ all lie in some cone of angle at most $\pi/2$. In particular their sum must be in that cone and nonzero.

If $\hat{b} > 0$ then u must have at least one fixed neighbor, and hence $d \leq \Delta - 1$. Let H be the graph obtained from G by deleting all fixed neighbors of u, i.e. $H = G - (N_G(u) \cap W)$, and let $W' = W \setminus N_G(u)$ and L' be the sublist of L corresponding to the vertices in W'. Observe that by definition for any $j \in [k]$ we have

$$\mathbf{Z}^{W,L,j;}_{L,W,u;}(G) = \mathbf{Z}^{W';L',j;}_{L';W',u;}(H) \cdot \prod_{\substack{v' \in W \cap N_G(u) \\ \text{s.t. } L(v')=j}} w_{uv'}, \tag{3.6}$$

where by $L(v')$ we mean the color that the list L pre-assigns to the vertex v'. In particular, if $j \in \mathcal{G}$, then $\mathbf{Z}^{W,L,j;}_{L,W,u;}(G) = \mathbf{Z}^{W';L',j;}_{L';W',u;}(H)$. Note also that by construction u has no fixed neighbors in the graph H and hence any color is good for u in H. Let

$$M := \max\left\{ \left|\mathbf{Z}^{W';L',j;}_{L';W',u;}(H)\right| : j \in [k] \right\},$$

and assume that $j_M \in [k]$ achieves the maximum above. Note that $M > 0$ by induction using **C(i)**. We then have by the triangle inequality

$$\begin{aligned} |\mathbf{Z}^{W}_{L}(G)/M| = \left| \sum_{j=1}^{k} \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)/\mathbf{Z}^{W';L',j_M;}_{L'_M;W',u;}(H) \right| &\geq \left| \sum_{j \in \mathcal{G} \cup \mathcal{N}} \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)/\mathbf{Z}^{W';L',j_M;}_{L'_M;W',u;}(H) \right| \\ &\quad - \sum_{j \in \mathcal{B}} |\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)/\mathbf{Z}^{W';L',j_M;}_{L'_M;W',u;}(H)|. \end{aligned}$$

Since by induction using **C(ii)** the pairwise angles between the $\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)$ for $j \in \mathcal{G} \cup \mathcal{N}$ are bounded by $(d + b\varepsilon)\theta \leq (\Delta - 1 + \varepsilon)\theta \leq \pi/3$ these vectors lie in a cone of angle at most $\pi/3$ and therefore,

$$\left| \sum_{j \in \mathcal{G} \cup \mathcal{N}} \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)/\mathbf{Z}^{W';L',j_M;}_{L'_M;W',u;}(H) \right| \geq \left| \sum_{j \in \mathcal{G}} \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)/\mathbf{Z}^{W';L',j_M;}_{L'_M;W',u;}(H) \right|.$$

By induction using **C(iii)**, the numbers

$$\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)/\mathbf{Z}^{W';L',j_M;}_{L'_M;W',u;}(H) = \mathbf{Z}^{W';L',j;}_{L';W',u;}(H)/\mathbf{Z}^{W';L',j_M;}_{L'_M;W',u;}(H)$$

are contained in $B(1,K)^d$, for $j \in \mathcal{G}$. By Lemma 3.7 this is a convex set, as $K < 1/d$. Therefore, $\sum_{j \in \mathcal{G}} \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)/\mathbf{Z}^{W';L',j_M;}_{L'_M;W',u;}(H) \in |\mathcal{G}| \cdot B(1,K)^d$, which implies by convexity of $B(1,K)^d$ that

$$\left| \sum_{j \in \mathcal{G}} \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)/\mathbf{Z}^{W';L',j_M;}_{L'_M;W',u;}(H) \right| \geq |\mathcal{G}| \cdot (1-K)^d.$$

By (3.6) and the definition of $\mathcal{B}$, we have that for each $j \in \mathcal{B}$

$$|\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)|/|\mathbf{Z}^{W',L',j_M;}_{L'_M;W',u;}(H)| \leq \varepsilon.$$

Combining these inequalities we arrive at

$$|\mathbf{Z}^{W}_{L}(G)/M| \geq (k-b)(1-K)^d - \varepsilon\hat{b} \geq (k-b)(1-K)^d - \varepsilon b.$$

Now the conditions (3.3) give that $\mathbf{Z}^{W}_{L}(G) \neq 0$.

Next we will prove **B**.

Proof of B

Since $W = W'u$ is a leaf-independent set, $\deg(u) = 1$ and the unique neighbor of u, which we call v, is free. We start by introducing some notation.

We define complex numbers z_j for $j \in [k]$ by

$$z_j := \mathbf{Z}^{W',v;L',\ell,j;}_{L',j;W',u,v;}(G \setminus uv) = \mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G \setminus uv)\,, \tag{3.7}$$

where the second equality holds because u is isolated in $G \setminus uv$. Let $w := w_{uv}$ and define complex numbers x_j and y_j for $j \in [k]$ by

$$x_j = \begin{cases} z_j & \text{if } j \neq \ell; \\ wz_\ell & \text{if } j = \ell\,, \end{cases} \qquad\qquad y_j = \begin{cases} z_j & \text{if } j \neq \ell'; \\ wz_{\ell'} & \text{if } j = \ell'\,. \end{cases}$$

Let $x = \sum_{j=1}^k x_j$ and $y = \sum_{j=1}^k y_j$. Observe that $x = \mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ and $y = \mathbf{Z}^{W';L',\ell';}_{L'';W',u;}(G)$, and that we may apply induction to the restricted partition function evaluations represented by the z_j because there are f free vertices in $G \setminus uv$ when the vertices in $W'uv$ are fixed.

For **B(i)** and **(ii)** we wish to bound the angle between x and y and to constrain the ratio x/y respectively. To do this we first bound $|y|$ and $|x-y|$.

We note for later that by the definition of x and y, we have $x - y = (w-1)z_\ell + (1-w)z_{\ell'} = (1-w)(z_{\ell'} - z_\ell)$. Also $|1-w| \leq 1+\varepsilon$ by conditions (i) and (ii) in the statement of the theorem so $|x-y| \leq (1+\varepsilon)|z_{\ell'} - z_\ell|$.

Let $\mathcal{G} = \mathcal{G}(G, W'u, L'\ell', v)$, $\mathcal{B} = \mathcal{B}(G, W'u, L'\ell', v, \varepsilon)$, $\mathcal{N} = [k] \setminus (\mathcal{G} \cup \mathcal{B})$, $\hat{b} = |\mathcal{B}|$, and suppose v has d free neighbors (in G when $W'u$ is fixed). Let H be the graph obtained from G by deleting all fixed neighbors of v, i.e. $H = G - (N_G(v) \cap W)$, and let $W'' = W \setminus N_G(v)$ and L'' be the sublist of L corresponding to the vertices in W''. Observe that by definition for any $j \in [k]$ we have

$$z_j = \mathbf{Z}^{W';L',j;}_{L';W',v;}(G-u) = \mathbf{Z}^{W'v;L'',j;}_{L'j;W'',v;}(H) \cdot \prod_{\substack{v' \in W' \cap N_G(v) \\ \text{s.t. } L'(v')=j}} w_{vv'}, \tag{3.8}$$

where by $L'(v')$ we mean the color that the list L' pre-assigns to the vertex v'. In particular, if $j \in \mathcal{G}$, then $\mathbf{Z}^{W';L',j;}_{L';W',v;}(G-u) = \mathbf{Z}^{W'v;L'',j;}_{L'j;W'',v;}(H)$. Note also that by construction v has no fixed neighbors in the graph H. Now write $b = \Delta - d$. Note that $d \leq \Delta - 1$, and define M, j^* by

$$M := \max\left\{\left|\mathbf{Z}^{W'v;L'',j;}_{L'j;W'',v;}(H)\right| : j \in [k]\right\} = |\mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H)|.$$

We perform a similar calculation to the case $\hat{b} > 0$ of **A** to show that

$$|y/M| \geq (k-b)(1-K)^d - \hat{b}\varepsilon. \tag{3.9}$$

To see this we have by the triangle inequality,

$$\left|y/\mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H)\right| = \left|\sum_{j=1}^{k} \mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)/\mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H)\right| \geq$$
$$\left|\sum_{j \in \mathcal{G} \cup \mathcal{N}} \mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)/\mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H)\right| \quad - \sum_{j \in \mathcal{B}} \left|\mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)/\mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H)\right|.$$

As before, by (3.8) and by induction using **C(ii)** the pairwise angles of the summands in the sum over $\mathcal{G} \cup \mathcal{N}$ is at most $(d + b\varepsilon)\theta \leq \pi/3$. This implies that these numbers lie in a cone of angle at most $\pi/3$, which implies that

$$\left|\sum_{j \in \mathcal{G} \cup \mathcal{N}} \mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)/\mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H)\right| \geq \left|\sum_{j \in \mathcal{G}} \mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)/\mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H)\right|.$$

Now for any $j \in \mathcal{G}$, we have that

$$\mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)/\mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H) = \mathbf{Z}^{W'v;L'',j;}_{L'j;W'',v;}(H)/\mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H) \in B(1,K)^d$$

by induction using **C(iii)**. As this set is convex, we have

$$\left|\sum_{j \in \mathcal{G}} \mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)/\mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H)\right| \geq |\mathcal{G}|(1-K)^d.$$

Since for any $j \in \mathcal{B}$ we have $m(j) \geq 1$, it follows by (3.8) and the definition of the y_j that

$$\sum_{j \in \mathcal{B}} \left| \mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G) / \mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H) \right| \leq \hat{b}\varepsilon^{m(j)} \leq \hat{b}\varepsilon.$$

Combining these two bounds we obtain (3.9).

We next claim that

$$\frac{|x-y|}{|y|} < K. \tag{3.10}$$

To prove this we need to distinguish two cases, depending on whether or not ℓ or ℓ' is a bad color in $G-u$ for the vertex v. We first introduce further notation. Let $\widehat{\mathcal{G}} = \mathcal{G}(G-u, W', L', v)$, $\widehat{\mathcal{B}} = \mathcal{B}(G-u, W', L', v, \varepsilon)$, $\widehat{\mathcal{N}} = \mathcal{N}(G-u, W', L', v, \varepsilon)$, and let $\widehat{m}(j)$ be the number of bad neighbors of v in $G-u$ with pre-assigned color j. Note that v has $d \leq \Delta - 1$ free neighbors in $G-u$. We now come to the two cases: either both $\ell, \ell' \in \widehat{\mathcal{G}} \cup \widehat{\mathcal{N}}$, or at least one is in $\widehat{\mathcal{B}}$. In the first case, by induction using **C(ii)** for $\mathbf{Z}^{W';L',j;}_{L';W',v;}(G-u) = \mathbf{Z}^{W';L',j;}_{L';W',v;}(G \setminus uv) = z_j$, the angle between z_ℓ and $z_{\ell'}$ is at most $(d + b\varepsilon)\theta \leq (\Delta - 1 + \varepsilon)\theta \leq \pi/3$, and hence we have $|z_{\ell'} - z_\ell| \leq \max\{|z_\ell|, |z_{\ell'}|\}$ by Proposition 3.5. Putting the established bounds together, we have

$$\frac{|x-y|}{|y|} \leq \frac{(1+\varepsilon)|z_{\ell'} - z_\ell|}{|y|} \leq (1+\varepsilon) \max_{j \in \{\ell, \ell'\}} \frac{|z_j|/M}{|y|/M} \leq \frac{1+\varepsilon}{(k-b)(1-K)^d - \varepsilon b} < K, \tag{3.11}$$

where the second inequality follows using (3.9) and the definition of M, and the final inequality follows from the condition (3.3). Hence (3.10) holds when $\ell, \ell' \in \widehat{\mathcal{G}} \cup \widehat{\mathcal{N}}$.

For the other case, when at least one of ℓ, ℓ' is in $\widehat{\mathcal{B}}$, we use the triangle inequality and (3.8) to obtain

$$\frac{|z_\ell - z_{\ell'}|}{M} \leq \frac{|z_\ell|}{M} + \frac{|z_{\ell'}|}{M} \leq \left(\varepsilon^{\widehat{m}(\ell)} + \varepsilon^{\widehat{m}(\ell')}\right) \leq (1+\varepsilon),$$

since at least one of $\widehat{m}(\ell)$ and $\widehat{m}(\ell')$ is at least 1 in this case. Therefore, using (3.9),

$$\frac{|x-y|}{|y|} \leq \frac{(1+\varepsilon)|z_{\ell'} - z_\ell|/M}{|y|/M} \leq \frac{(1+\varepsilon)^2}{(k-b)(1-K)^d - \varepsilon b} < K, \tag{3.12}$$

where the final inequality comes from the condition (3.3), establishing (3.10).

Now, by Proposition 3.5, the angle γ between x and y satisfies $\sin\gamma \leq |x-y|/|y| < K$, and we conclude that $\gamma \leq \arcsin(K) = \theta$ as required for **B(i)**. Additionally, we have

$$\frac{x}{y} = \frac{y + x - y}{y} = 1 + \frac{x-y}{y} \in B(1, K),$$

since $|x-y|/|y| < K$. This gives **B(ii)**. We now turn to **C**.

Proof of C

We start with **(i)**, that is we will show that for any $\ell \in \mathcal{G}$, $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G) \neq 0$. Since we have already proved **A** and **B** for the case of $f+1$ free vertices and since we have $f+1$ free vertices for $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$, we might hope to immediately apply **A**; the only problem is that $W'u$ is not a leaf- independent set, so we will modify G first.

Let $v_1, \ldots, v_d$ be the free neighbors of u. We construct a new graph H from G by adding vertices $u_1, \ldots, u_d$ to G and replacing each edge uv_i with u_iv_i for $i = 1, \ldots, d$, while keeping all other edges of G unchanged (so note that u is only adjacent to its fixed neighbors in H). Each edge e of H is assigned value w'_e where $w'_e = w_e$ if e is an edge of G and $w'_{u_iv_i} = w_{uv_i}$ for the new edges uv_i. See Figure 3.2 for an illustrative example.

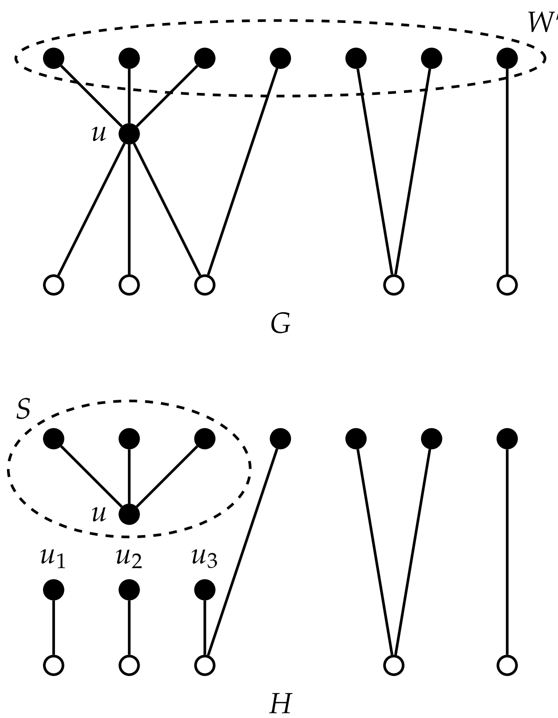


Figure 3.2: An illustration of the construction of H (below) from G (above) in the proof of **C**. Note that W' forms a leaf-independent set, but that we do not require that $W'u$ has this property.

Then by construction we have

$$\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G) = \mathbf{Z}^{W',u_1,\ldots,u_d;L',\ell,\ell,\ldots,\ell;}_{L',\ell,\ldots,\ell;W',u,u_1,\ldots,u_d;}(H). \tag{3.13}$$

Notice that in H, the vertex u together with its neighbors form a star S that is disconnected from the rest of H (and all vertices of S are in $W = W'u$ so they are fixed).

Thus H is the disjoint union of S and some graph $\widehat{H}$. Thus the partition function $z := \mathbf{Z}_{L',\ell,\dots,\ell;W',u,u_1,\dots,u_d;}^{W',u_1,\dots,u_d;L',\ell,\ell,\dots,\ell;}(H)$ factors as

$$\mathbf{Z}_{L',\ell,\dots,\ell;W',u,u_1,\dots,u_d;}^{W',u_1,\dots,u_d;L',\ell,\ell,\dots,\ell;}(H) = \mathbf{Z}_{L',\dots,\ell;W',u_1,\dots,u_d;}^{W'1,\dots,u_d;L',\ell,\dots,\ell;}(\widehat{H}) \cdot \mathbf{Z}_{L';W',u;}^{W';L',\ell;}(S); \tag{3.14}$$

here we abuse notation by having a list $W'u_1 \dots u_d$ (resp. $W'u$) that may contain vertices not in $\widehat{H}$ (resp. S); such vertices and their corresponding color should simply be ignored.

The fixed vertices in $\widehat{H}$ form a leaf-independent set, so we can apply **A** to conclude that the first factor above is nonzero. It is also clear that second factor above is nonzero because all vertices in S are fixed and $\ell \in \mathcal{G} \cup \mathcal{N}$. Hence $z \neq 0$ as required.

To prove part **(ii)**, we will apply **B** to $\widehat{H}$ with $W'u_1 \dots u_d$ fixed, which (as above) is possible since we already proved **B** for $f+1$ free vertices and $W'u_1 \dots u_d$ restricted to $\widehat{H}$ is a leaf-independent set. By **B(i)** the angle between

$$\mathbf{Z}_{L',\dots,\ell,\ell';W',u_1,\dots,u_{d-1},u_d;}^{W'1,\dots,u_{d-1},u_d;L',\ell,\dots,\ell,\ell';}(\widehat{H}) \qquad \text{and} \qquad \mathbf{Z}_{L',\dots,\ell,\ell;W',u_1,\dots,u_{d-1},u_d;}^{W'1,\dots,u_{d-1},u_d;L',\ell,\dots,\ell,\ell;}(\widehat{H})$$

is at most θ. Continuing to change the label of each u_i one step at the time, we conclude that the angle between

$$\mathbf{Z}_{L'',\dots,\ell';W',u_1,\dots,u_d;}^{W'1,\dots,u_d;L',\ell',\dots,\ell';}(\widehat{H}) \qquad \text{and} \qquad \mathbf{Z}_{L',\dots,\ell;W',u_1,\dots,u_d;}^{W'1,\dots,u_d;L',\ell,\dots,\ell;}(\widehat{H})$$

is at most $d\theta$. We next notice that since for **(ii)** we assume $\ell, \ell' \in \mathcal{G} \cup \mathcal{N}$, changing the color of u from ℓ to ℓ' can only change $\mathbf{Z}_{L';W',u;}^{W';L',\ell;}(S)$ by $\deg_S(u) \leq \Delta - d = b$ factors, each of argument at most $\varepsilon\theta$ thus giving a total change of angle by at most $b\varepsilon\theta$. Hence by (3.14), we therefore conclude that the angle between $\mathbf{Z}_{L';W',u;}^{W';L',\ell;}(G)$ and $\mathbf{Z}_{L'';W',u;}^{W';L',\ell';}(G)$ is at most $d\theta + b\varepsilon\theta$.

To prove **(iii)** we observe that we can write for any $j, \ell \in [k]$ the telescoping product:

$$\frac{\mathbf{Z}_{L',\dots,\ell;W',u_1,\dots,u_d;}^{W'1,\dots,u_d;L',\ell,\dots,\ell;}(\widehat{H})}{\mathbf{Z}_{L',\dots,j;W',u_1,\dots,u_d;}^{W'1,\dots,u_d;L',j,\dots,j;}(\widehat{H})} = \frac{\mathbf{Z}_{L',\dots,\ell;W',u_1,\dots,u_d;}^{W'1,\dots,u_d;L',\ell,\dots,\ell;}(\widehat{H})}{\mathbf{Z}_{L',\dots,\ell,j;W',u_1,\dots,u_{d-1},u_d;}^{W'1,\dots,u_{d-1},u_d;L',\ell,\dots,\ell,j;}(\widehat{H})} \cdots \frac{\mathbf{Z}_{L',j,\dots,j;W',u_1,u_2,\dots,u_d;}^{W'1,u_2,\dots,u_d;L',\ell,j,\dots,j;}(\widehat{H})}{\mathbf{Z}_{L',\dots,j;W',u_1,\dots,u_d;}^{W'1,\dots,u_d;L',j,\dots,j;}(\widehat{H})}. \tag{3.15}$$

By **B(ii)**, each of these factors is contained in $B(1,K)$ and hence

$$\frac{\mathbf{Z}_{L',\dots,\ell;W',u_1,\dots,u_d;}^{W'1,\dots,u_d;L',\ell,\dots,\ell;}(\widehat{H})}{\mathbf{Z}_{L',\dots,j;W',u_1,\dots,u_d;}^{W'1,\dots,u_d;L',j,\dots,j;}(\widehat{H})} \in B(1,K)^d.$$

Finally note that since $\ell, j \in \mathcal{G}$ then $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G) = \mathbf{Z}^{W'{}_{1,\ldots,u_d};L',\ell,\ldots,\ell;}_{L',\ldots,\ell;W',u_1,\ldots,u_d;}(\widehat{H})$ by (3.13) and (3.14) (since $\mathbf{Z}(S) = 1$ in this case), and similarly for j. So we deduce that the ratio $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)/\mathbf{Z}^{W';L',j;}_{L';W',u;}(G)$ is contained in $B(1,K)^d$, as desired. This completes the proof.

3 Finding constants for Theorems 3.1

Recall that we denote the base of the natural logarithm by e.

Proposition 3.10. *Let $\Delta \in \mathbb{N}_{\geq 3}$ and $k \geq e\Delta + 1$. Then there exists $\varepsilon \in (0,1)$ such that with $K = 1/\Delta$, for each $d = 0, \ldots, \Delta - 1$, with $b = \Delta - d$,*

$$0 < \frac{(1+\varepsilon)^2}{(k-b)(1-K)^d - \varepsilon b} \leq K, \tag{3.16}$$

and additionally $\arcsin(K) \leq \frac{\pi}{3(\Delta-1+\varepsilon)}$.

Note that (3.16) is precisely the condition (3.3) required in the hypothesis of Theorem 3.9. Therefore combining this proposition with Theorem 3.9 proves Theorem 3.1 for $c_\Delta = e$ (we only need part **A**).

Proof. We first observe that once ε is set to zero in (3.16) the condition states: for all $d = 0, \ldots, \Delta - 1$,

$$\frac{1}{(k+d-\Delta)(1-K)^d} \leq K. \tag{3.17}$$

We will show that (3.17) is satisfied with strict inequality provided $k \geq e\Delta + 1$ when $K = 1/\Delta$. Since the expression involving ε in (3.16) is a continuous, increasing function of ε, there exists an $\varepsilon > 0$ for which (3.16) is satisfied. Moreover, for $K = 1/\Delta$ we also have $\arcsin(K) < \frac{\pi}{3(\Delta-1)}$ for any $\Delta \geq 3$. So for $\varepsilon > 0$ small enough all conditions will be satisfied.

Noting that we assume $\Delta \geq 3$, let us define for $d \geq 0$ the function

$$f_\Delta(d) := \Delta \left(\frac{\Delta}{\Delta - 1} \right)^d - d.$$

We observe that condition (3.17) with $K = 1/\Delta$ is satisfied with strict inequality provided $f_\Delta(d) < k - \Delta$ for each $d = 0, \ldots, \Delta - 1$.

We first claim that $f_\Delta(d)$ as a function of d is convex on $\mathbb{R}_{\geq 0}$. Indeed, its second derivative in d is given by

$$f''_\Delta(d) = \log^2 \left(\frac{\Delta}{\Delta - 1} \right) (f_d(\Delta) + d) > 0.$$

This implies that f_Δ attains its maximum on $[0, \Delta - 1]$ at a boundary point: either when $d = 0$ or when $d = \Delta - 1$. In fact the maximum is attained at $d = \Delta - 1$, since we observe

$$f_\Delta(\Delta - 1) = \Delta \sum_{i=0}^{\Delta-1} \binom{\Delta - 1}{i} \left(\frac{1}{\Delta - 1}\right)^i - (\Delta - 1) \geq 2\Delta - (\Delta - 1) > \Delta = f_\Delta(0),$$

where the penultimate inequality holds by taking the first two terms in the sum. It now remains to check that $f_\Delta(\Delta - 1) < k - \Delta$ whenever $k \geq e\Delta + 1$. This holds since

$$f_\Delta(\Delta - 1) + \Delta - 1 = \Delta \left(1 + \frac{1}{\Delta - 1}\right)^{\Delta - 1} < \Delta e.$$

Thus we obtained that if $k \geq e\Delta + 1$, we can choose $K = 1/\Delta$, such that all the conditions in (3.17) are satisfied with strict inequality. This finishes the proof. □

Remark 3.11. *We could have given a slightly tighter analysis by parametrizing $K = x/\Delta$ in the proof given above. However, it is not difficult to show that as $\Delta \to \infty$ the optimal choice for x converges to* 1*. In the next section we give better bounds for small values of Δ by adding additional constraints and using a computer to find the optimal value of K.*

Remark 3.12. *We note that if one replaces (3.3) by*

$$0 < \frac{(1+\varepsilon)^{2+b}}{(1-K)^d(k-b) - \varepsilon b(1+\varepsilon)^b} \leq K \tag{3.18}$$

for all $d = 0, \ldots, \Delta - 1$ and $b = \Delta - d$, one can give essentially the same proof (where only **C(iii)** *needs to be modified) to conclude that in Theorem 3.1 we can in fact guarantee an open set containing the closed interval* $[0, 1]$.

4 Improvements for small values of Δ

In the previous section we showed that we can take $c_\Delta \leq e$ for each $\Delta \geq 3$ in the statement of Theorem 3.1. In this section we prove the second part of Theorem 3.1, by showing that for small values of Δ, we can improve the bound on c_Δ. We do this by proving a slightly different version of Theorem 3.9 in which we constrain the ratios of the restricted partition functions to lie in slightly different sets.

We first define a function f by

$$f(d, K, \phi) = \max\left(2\sin(\phi/2), \sqrt{1 + (1+K)^{-2d} - 2\cos(\phi)(1+K)^{-d}}\right),$$

where d is a positive integer, $K \in (0, 1)$, and ϕ is an angle.

Theorem 3.13. *Let $\Delta \in \mathbb{N}_{\geq 3}$. Suppose that $k > \Delta$ and $0 < \varepsilon < 1$ are such that there exist constants $K \in (0,1)$ with $\theta := \arcsin(K) \in (0, \frac{\pi}{3(\Delta-1+\varepsilon)})$ satisfying, for each $d = 0, \ldots, \Delta-1$ with $b := \Delta - d$, that*

$$\frac{(1+\varepsilon)^2(1+K)^d}{\cos((d+b\varepsilon)\theta/2)(k-b) - \varepsilon b(1+K)^d} \leq K \qquad \text{for } d = 0, \ldots, \Delta-2; \tag{3.19}$$

$$\frac{(1+\varepsilon)(1+K)^d}{\cos((d+b\varepsilon)\theta/2)(k-b) - \varepsilon b(1+K)^d} \leq \frac{K}{f(d,K,(d+b\varepsilon)\theta)} \qquad \text{for } d = 0, \ldots, \Delta-1. \tag{3.20}$$

Then for each graph $G = (V,E)$ of maximum degree at most Δ and every $w = (w_e)_{e \in E}$ satisfying for each $e \in E$ that

(i) $|w_e| \leq \varepsilon$, *or*

(ii) $|\arg(w_e)| \leq \varepsilon\theta$ *and* $\varepsilon < |w_e| \leq 1$,

the following statements hold for $\mathbf{Z}(G) = \mathbf{Z}(G;k,w)$.

A′ *For all lists W of distinct vertices of G such that W forms a leaf-independent set in G and for all lists of pre-assigned colors L of length $|W|$, $\mathbf{Z}^W_L(G) \neq 0$.*

B′ *For all lists $W = W'u$ of distinct vertices of G such that W is a leaf-independent set and for any two lists $L'\ell$ and $L'\ell'$ of length $|W|$:*

(i) *the angle between the vectors $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ and $\mathbf{Z}^{W';L',\ell';}_{L'';W',u;}(G)$ is at most θ,*

(ii)

$$\frac{|\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)|}{|\mathbf{Z}^{W';L',\ell';}_{L'';W',u;}(G)|} \leq 1 + K. \tag{3.21}$$

C′ *For all lists $W = W'u$ of distinct vertices such that the initial segment W' forms a leaf-independent set in G and for all lists of pre-assigned colors L' of length $|W'|$, the following holds. Write $\mathcal{G} = \mathcal{G}(G,W',L',u)$ and $\mathcal{N} = \mathcal{N}(G,W',L',u,\varepsilon)$, let d be the number of free neighbors of u, let $b = \Delta - d$, and let $m(\ell)$ be the number of fixed, bad neighbors of u with pre-assigned color ℓ. Then*

(i) *for any $\ell \in \mathcal{G} \cup \mathcal{N}$, $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G) \neq 0$,*

(ii) *for any $\ell, \ell' \in \mathcal{G} \cup \mathcal{N}$, the angle between the vectors $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ and $\mathbf{Z}^{W';L',\ell';}_{L'';W',u;}(G)$ is at most $(d+b\varepsilon)\theta$,*

(iii) *for any $\ell \in [k]$ and $j \in \mathcal{G}$*

$$\frac{|\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)|}{|\mathbf{Z}^{W';L',j;}_{L';W',u;}(G)|} \leq \varepsilon^{m(\ell)}(1+K)^d. \tag{3.22}$$

The proof is almost the same as the proof of Theorem 3.9: we essentially replace a convexity argument by an application of Lemma 3.4. For convenience of the reader, we give a full proof.

4.1 Proof

We prove that **A′**, **B′**, and **C′** hold by induction on the number of free vertices of a graph. The base case consists of graphs with no free vertices. Clearly **A′** and **B′** hold in this case as they are both vacuous: if there are no free vertices then $W = V$, but then W cannot be a leaf-independent set.

For statement **C′** we note that since there are no free vertices, $V \setminus W' = \{u\}$, and hence G must be a star with center u. Part **C′(i)** follows since when $\ell \in \mathcal{G} \cup \mathcal{N}$ we have that $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ is a product over nonzero edge-values. Part **(ii)** follows since, by changing the color of u from ℓ to $j \in \mathcal{G} \cup \mathcal{N}$, we obtain $\mathbf{Z}^{W';L',j;}_{L';W',u;}(G)$ from $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ by multiplying and dividing by at most $\deg(u)$ factors w_{uv} with $\arg(w_{uv}) \leq \varepsilon\theta$; hence the restricted partition function changes in angle by at most $\Delta\varepsilon\theta \leq (d + b\varepsilon)\theta$ (since $d = 0$ and so $b = \Delta$). Part **(iii)** follows similarly, as when there are no free vertices we must have $d = 0$, and changing the color of u from j to ℓ corresponds to multiplying $\mathbf{Z}^{W';L',j;}_{L';W',u;}(G)$ by at most $\deg(u)$ factors w_{uv} all satisfying $|w_{uv}| \leq 1$ and $m(\ell)$ of the factors satisfying $|w_{uv}| \leq \varepsilon$.

Now let us assume that statements **A′**, **B′**, and **C′** hold for all graphs with $f \geq 0$ free vertices. We wish to prove the statements for graphs with $f + 1$ free vertices. We start by proving **A′**.

Proof of A′

Let u be any free vertex. We proceed using the fact that $|\mathbf{Z}^{W}_{L}(G)| = |\sum_{j=1}^{k} \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)|$. Let $\mathcal{G} = \mathcal{G}(G, W, L, u)$, $\mathcal{B} = \mathcal{B}(G, W, L, u, \varepsilon)$, $\mathcal{N} = [k] \setminus (\mathcal{G} \cup \mathcal{B})$ and $\hat{b} = |\mathcal{B}|$. Let d be the number of free neighbors of u and let $b = \Delta - d$. Note that $\hat{b} \leq b$ and $|\mathcal{G}| \geq k - b$. After fixing u to any $j \in [k]$ we have one less free vertex, and hence can apply **C′** using induction as necessary.

There are two cases to consider. If $\hat{b} = 0$ then by induction using **C′(i)** we have that the $\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)$ are non-zero and by **C′(ii)** the angle between any two of the $\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)$ is at most $\Delta\theta < \pi/2$. Hence by Lemma 3.4

$$|\mathbf{Z}^{W}_{L}(G)| = \Big|\sum_{j=1}^{k} \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)\Big| \geq \cos(\pi/4) \sum_{j=1}^{k} \big|\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)\big| > 0\,.$$

If $\hat{b} > 0$ then u must have at least one fixed neighbor, and hence $d \leq \Delta - 1$. Let

$$M := \min\left\{\left|\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)\right| : j \in \mathcal{G}\right\},$$
$$m := \max\left\{\left|\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)\right| : j \in \mathcal{B}\right\},$$

and assume that $j_M \in \mathcal{G}$ achieves the minimum above and $j_m \in \mathcal{B}$ achieves the maximum. Note that $M > 0$ by induction using **C′(i)**. Note further by induction using **C′(iii)** that

$$m \leq \varepsilon^{m(j_m)}(1+K)^d |\mathbf{Z}^{W',L',j_M;}_{L'_M;W',u;}(G)| \leq \varepsilon(1+K)^d M, \tag{3.23}$$

where we used that $m(j_m) \geq 1$ since $j_m \in \mathcal{B}$. We then have

$$\begin{aligned}
|\mathbf{Z}^W_L(G)| = \Big|\sum_{j=1}^{k} \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)\Big| &\geq \Big|\sum_{j\in\mathcal{G}\cup\mathcal{N}} \mathbf{Z}^{W,L,j;}_{L,W,u;}(G)\Big| - \sum_{j\in\mathcal{B}} |\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)| \\
&\geq \cos((d+b\varepsilon)\theta/2) \sum_{j\in\mathcal{G}\cup\mathcal{N}} |\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)| - \sum_{j\in\mathcal{B}} |\mathbf{Z}^{W,L,j;}_{L,W,u;}(G)| \\
&\geq M|\mathcal{G}|\cos((d+b\varepsilon)\theta/2) - m|\mathcal{B}| \\
&\geq M\big[(k-b)\cos((d+b\varepsilon)\theta/2) - b\varepsilon(1+K)^d\big],
\end{aligned}$$

where the first inequality is the triangle inequality, the second uses **C′(ii)** and Lemma 3.4, the third uses the definition of M and m, and the fourth follows from (3.23). Now the conditions (3.19) give that $\mathbf{Z}^W_L(G) \neq 0$ (recalling we have $d \leq \Delta - 1$ and noting the denominator in (3.19) must be positive).

Next we will prove **B′**.

Proof of B′

The proof starts in exactly the same way as the proof of **B**. Recall that $\deg(u) = 1$ and that its unique neighbor, which we call v, is free. We start by introducing some notation.

We define complex numbers z_j for $j \in [k]$ by

$$z_j := \mathbf{Z}^{W',v;L',\ell,j;}_{L',j;W',u,v;}(G \setminus uv) = \mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G \setminus uv) = \mathbf{Z}^{W';L',j;}_{L';W',v;}(G-u), \tag{3.24}$$

where the equalities follow because u is isolated in $G \setminus uv$ and so makes no contribution to the partition function. Let $w := w_{uv}$ and define complex numbers x_j and y_j for $j \in [k]$

by

$$x_j = \begin{cases} z_j & \text{if } j \neq \ell; \\ wz_\ell & \text{if } j = \ell, \end{cases} \qquad\qquad y_j = \begin{cases} z_j & \text{if } j \neq \ell'; \\ wz_{\ell'} & \text{if } j = \ell'. \end{cases}$$

Let $x = \sum_{j=1}^{k} x_j$ and $y = \sum_{j=1}^{k} y_j$. Observe that $x = \mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ and $y = \mathbf{Z}^{W';L',\ell';}_{L'';W',u;}(G)$, and that we may apply induction to the restricted partition function evaluations represented by the z_j because there are f free vertices in $G \setminus uv$ when the vertices in $W'uv$ are fixed.

For **B′(i)** and **(ii)** we wish to bound the angle between x and y and the ratio $|x|/|y|$ respectively. To do this we first bound $|y|$ and $|x-y|$.

We note for later that by the definition of x and y, we have $x - y = (w-1)z_\ell + (1-w)z_{\ell'} = (1-w)(z_{\ell'} - z_\ell)$. Also $|1-w| \leq 1+\varepsilon$ by conditions (i) and (ii) in the statement of the theorem so $|x-y| \leq (1+\varepsilon)|z_{\ell'} - z_\ell|$.

Let $\mathcal{G} = \mathcal{G}(G, W'u, L'\ell', v)$, $\mathcal{B} = \mathcal{B}(G, W'u, L'\ell', v, \varepsilon)$, $\mathcal{N} = [k] \setminus (\mathcal{G} \cup \mathcal{B})$, $\hat{b} = |\mathcal{B}|$, and suppose v has d free neighbors (in G when $W'u$ is fixed), so $d = \Delta - 1$. Let H be the graph obtained from G by deleting all fixed neighbors of v, i.e. $H = G - (N_G(v) \cap W)$, and let $W'' = W \setminus N_G(v)$ and L'' be the sublist of L corresponding to the vertices in W''. Observe that by definition for any $j \in [k]$, we have the similar identities

$$z_j := \mathbf{Z}^{W';L',j;}_{L';W',v;}(G-u) = \mathbf{Z}^{W'v;L'',j;}_{L'j;W'',v;}(H) \cdot \prod_{\substack{v' \in W' \cap N_G(v) \\ \text{s.t. } L'(v')=j}} w_{vv'}, \tag{3.25}$$

$$\mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G) = \mathbf{Z}^{W'v;L'',j;}_{L'j;W'',v;}(H) \cdot \prod_{\substack{v' \in W \cap N_G(v) \\ \text{s.t. } L'(v')=j}} w_{vv'}, \tag{3.26}$$

where by $L'(v')$ we mean the color that the list L' assigns to the vertex v'. In particular, if $j \in \mathcal{G}$, we have $\mathbf{Z}^{W';L',j;}_{L';W',v;}(G-u) = \mathbf{Z}^{W'v;L'',j;}_{L'j;W'',v;}(H)$.

Now write $b = \Delta - d$. Note that $d \leq \Delta - 1$, and define M, j^*, and C by

$$M := \min\left\{ \left|\mathbf{Z}^{W'v;L'',j;}_{L'j;W'',v;}(H)\right| : j \in [k] \right\} = \mathbf{Z}^{W'v;L'',j^*;}_{L'j^*;W'',v;}(H);$$
$$C := (k-b)\cos((d+b\varepsilon)\theta/2) - b\varepsilon(1+K)^d.$$

Note that for all $j \in [k]$, we have

$$M \leq |\mathbf{Z}^{W'v;L'',j;}_{L'j;W'',v;}(H)| \leq M(1+K)^d, \tag{3.27}$$

where the upper bound follows by induction using **C′(iii)** (here H with fixed vertices $W''v$ has fewer free vertices than G with fixed vertices W) and noting that all colors in $[k]$ are good for v in H. We perform a similar calculation to the case $\hat{b} > 0$ of **A′** to bound $|y|$. We have

$$\begin{aligned}
|y| = \left|\mathbf{Z}^{W';L',\ell';}_{L'';W',u;}(G)\right| &\geq \Big|\sum_{j\in\mathcal{G}\cup\mathcal{N}} \mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)\Big| - \sum_{j\in\mathcal{B}} |\mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)| \\
&\geq \cos((d+b\varepsilon)\theta/2)\sum_{j\in\mathcal{G}} |\mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)| - \sum_{j\in\mathcal{B}} |\mathbf{Z}^{W',v;L',\ell',j;}_{L'',j;W',u,v;}(G)| \\
&\geq \cos((d+b\varepsilon)\theta/2)\sum_{j\in\mathcal{G}} |\mathbf{Z}^{W';L',j;}_{L';W',v;}(H)| - \varepsilon\sum_{j\in\mathcal{B}} |\mathbf{Z}^{W';L',j;}_{L';W',v;}(H)| \\
&\geq M|\mathcal{G}|\cos((d+b\varepsilon)\theta/2) - M|\mathcal{B}|\varepsilon(1+K)^d \geq M\cdot C\,;
\end{aligned}$$

here the first inequality is the triangle inequality, the second follows from Lemma 3.4 and induction using **C′(ii)**, the third follows from (3.26) and the fourth follows from (3.27).

We next claim that

$$\frac{|x-y|}{|y|} \leq K. \tag{3.28}$$

To prove this, we will need to distinguish three cases, for which we now introduce the notation. Let $\widehat{\mathcal{G}} = \mathcal{G}(G-u, W', L', v)$, $\widehat{\mathcal{B}} = \mathcal{B}(G-u, W', L', v, \varepsilon)$, $\widehat{\mathcal{N}} = \mathcal{N}(G-u, W', L', v, \varepsilon)$, and let $\widehat{m}(j)$ be the number of bad neighbors of v in $G-u$ with pre-assigned color j. Then, by (3.25) and (3.27) we have for any j,

$$|z_j| \leq \varepsilon^{\widehat{m}(j)}|\mathbf{Z}^{W'v;L'',j;}_{L'j;W'',v;}(H)| \leq \varepsilon^{\widehat{m}(j)}M(1+K)^d. \tag{3.29}$$

We now come to the three cases: either (a) $\ell, \ell' \in \widehat{\mathcal{G}}$, or (b) $\ell \in \widehat{\mathcal{B}}$ or $\ell' \in \widehat{\mathcal{B}}$, or (c) both ℓ and ℓ' are contained in $\widehat{\mathcal{G}} \cup \widehat{\mathcal{N}}$ and one of them is not contained in $\widehat{\mathcal{G}}$.

In case (a), by induction using **C′** for $\mathbf{Z}^{W';L',j;}_{L';W',v;}(G-u) = \mathbf{Z}^{W';L',j;}_{L';W',v;}(G\setminus uv) = z_j$, the angle between z_ℓ and $z_{\ell'}$ is at most $(d+b\varepsilon)\theta \leq (\Delta - 1 + \varepsilon)\theta \leq \pi/3$ (by **C′(ii)**) and $(1+K)^{-d} \leq |z_\ell/z_{\ell'}| \leq (1+K)^d$ by **C′(iii)**, so we can apply Lemma 3.8 to conclude that

$$|z_{\ell'} - z_\ell| \leq f(d, K, (d+b\varepsilon)\theta)\cdot\max\{|z_\ell|, |z_{\ell'}|\} \leq (1+K)^d M,$$

where the final inequality follows by (3.29). Then

$$\frac{|x-y|}{|y|} \leq \frac{(1+\varepsilon)|z_{\ell'} - z_\ell|}{|y|} \leq (1+\varepsilon)\frac{f(d,K,(d+b\varepsilon)\theta)M(1+K)^d}{M\cdot C} < K,$$

where the final inequality comes from the condition (3.20).

For case (b), at least one of ℓ, ℓ' is in $\widehat{\mathcal{B}}$, so we know that $d \leq \Delta - 2$ (as the degree of v in H is at most $\Delta - 1$ and v has at least one fixed neighbor in H). We use the triangle inequality to obtain

$$|z_\ell - z_{\ell'}| \leq |z_\ell| + |z_{\ell'}| \leq \left(\varepsilon^{\widehat{m}(\ell)} + \varepsilon^{\widehat{m}(\ell')}\right) M(1+K)^d \leq (1+\varepsilon) M(1+K)^d,$$

using (3.29) for $j \in \{\ell, \ell'\}$ and that at least one of $\widehat{m}(\ell)$ and $\widehat{m}(\ell')$ is at least 1 in this case. Therefore

$$\frac{|x-y|}{|y|} = \frac{(1+\varepsilon)|z_{\ell'} - z_\ell|}{|y|} \leq \frac{(1+\varepsilon)^2 M(1+K)^d}{MC} < K,$$

where the final inequality comes from the condition (3.19) (noting that we only need the condition for $d = 1, \ldots, \Delta - 2$).

In case (c), both ℓ and ℓ' are contained in $\widehat{\mathcal{G}} \cup \widehat{\mathcal{N}}$ and at least one of them is contained in $\widehat{N}$, and so $d \leq \Delta - 2$ (as in case (b)). By induction using **C′** for $\mathbf{Z}^{W';L',j;}_{L';W',v;}(G-u) = \mathbf{Z}^{W';L',j;}_{L';W',v;}(G \setminus uv) = z_j$, the angle between z_ℓ and $z_{\ell'}$ is at most $(d+b\varepsilon)\theta \leq (\Delta - 1 + \varepsilon)\theta \leq \pi/3$ by **C′(ii)**. Thus using Proposition 3.5 and (3.29), we obtain

$$|z_\ell - z_{\ell'}| \leq \max\{|z_\ell|, |z_{\ell'}|\} \leq M(1+K)^d,$$

and so as before

$$\frac{|x-y|}{|y|} = \frac{(1+\varepsilon)|z_{\ell'} - z_\ell|}{|y|} \leq \frac{(1+\varepsilon)(1+K)^d M}{MC} < K,$$

where the final inequality comes from the condition (3.19) (noting that we only need the condition for $d = 1, \ldots, \Delta - 2$). This establishes (3.28) in all cases.

Now, by Proposition 3.5, the angle γ between x and y satisfies $\sin\gamma \leq |x-y|/|y| \leq K$, and we conclude that $\gamma \leq \arcsin(K) = \theta$ as required for **B′(i)**. Additionally, by the triangle inequality we have

$$\frac{|x|}{|y|} \leq \frac{|y| + |x-y|}{|y|} \leq 1 + K,$$

which gives **B′(ii)**. We now turn to **C′**.

Proof of C′

We start with **(i)**, that is we will show that for any $\ell \in \mathcal{G} \cup \mathcal{N}$, $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G) \neq 0$. Since we have already proved **A′** and **B′** for the case of $f+1$ free vertices and since we have

$f+1$ free vertices for $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$, we might hope to immediately apply **A′**; the only problem is that $W'u$ may not a leaf- independent set, so we will modify G first.

Let $v_1,\ldots,v_d$ be the free neighbors of u. We construct a new graph H from G by adding vertices $u_1,\ldots,u_d$ to G and replacing each edge uv_i with u_iv_i for $i=1,\ldots,d$, while keeping all other edges of G unchanged (so note that u is only adjacent to its fixed neighbors in H). Each edge e of H is assigned value w'_e where $w'_e = w_e$ if e is an edge of G and $w'_{u_iv_i} = w_{uv_i}$ for the new edges uv_i. See Figure 3.2 for an illustrative example.

Then by construction we have

$$\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G) = \mathbf{Z}^{W',u_1,\ldots,u_d;L',\ell,\ell,\ldots,\ell;}_{L',\ell,\ldots,\ell;W',u,u_1,\ldots,u_d;}(H). \tag{3.30}$$

Notice that in H, the vertex u together with its neighbors form a star S that is disconnected from the rest of H (and all vertices of S are in $W = W'u$ so they are fixed). Thus H is the disjoint union of S and some graph $\widehat{H}$. Thus the partition function $z := \mathbf{Z}^{W',u_1,\ldots,u_d;L',\ell,\ell,\ldots,\ell;}_{L',\ell,\ldots,\ell;W',u,u_1,\ldots,u_d;}(H)$ factors as

$$\mathbf{Z}^{W',u_1,\ldots,u_d;L',\ell,\ell,\ldots,\ell;}_{L',\ell,\ldots,\ell;W',u,u_1,\ldots,u_d;}(H) = \mathbf{Z}^{W'{}_1,\ldots,u_d;L',\ell,\ldots,\ell;}_{L',\ldots,\ell;W',u_1,\ldots,u_d;}(\widehat{H}) \cdot \mathbf{Z}^{W';L',\ell;}_{L';W',u;}(S); \tag{3.31}$$

here we abuse notation by having a list $W'u_1\ldots u_d$ (resp. $W'u$) that may contain vertices not in $\widehat{H}$ (resp. S); such vertices and their corresponding color should simply be ignored.

The fixed vertices in $\widehat{H}$ form a leaf-independent set, so we can apply $\mathbf{A}'$ to conclude that the first factor above is nonzero. It is also clear that second factor above is nonzero because all vertices in S are fixed and $\ell \in \mathcal{G} \cup \mathcal{N}$. Hence $z \neq 0$ as required.

To prove part **(ii)**, we will apply **B′** to $\widehat{H}$ with $W'uu_1\ldots u_d$ fixed, which (as above) is possible since we already proved **B′** for $f+1$ free vertices and $W'uu_1\ldots u_d$ restricted to $\widehat{H}$ is a leaf-independent set. By **B′(i)** the angle between

$$\mathbf{Z}^{W'{}_1,\ldots,u_{d-1},u_d;L',\ell,\ldots,\ell,\ell';}_{L',\ldots,\ell,\ell';W',u_1,\ldots,u_{d-1},u_d;}(\widehat{H}) \qquad \text{and} \qquad \mathbf{Z}^{W'{}_1,\ldots,u_{d-1},u_d;L',\ell,\ldots,\ell,\ell;}_{L',\ldots,\ell,\ell;W',u_1,\ldots,u_{d-1},u_d;}(\widehat{H})$$

is at most θ. Continuing to change the label of each u_i one step at the time, we conclude that the angle between

$$\mathbf{Z}^{W'{}_1,\ldots,u_d;L',\ell',\ldots,\ell';}_{L'',\ldots,\ell';W',u_1,\ldots,u_d;}(\widehat{H}) \qquad \text{and} \qquad \mathbf{Z}^{W'{}_1,\ldots,u_d;L',\ell,\ldots,\ell;}_{L',\ldots,\ell;W',u_1,\ldots,u_d;}(\widehat{H})$$

is at most $d\theta$. We next notice that since for **(ii)** we assume $\ell,\ell' \in \mathcal{G} \cup \mathcal{N}$, changing the color of u from ℓ to ℓ' can only change $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(S)$ by $\deg(u) \leq \Delta - d = b$ factors, each of argument at most $\varepsilon\theta$ thus giving a total change of angle by at most $b\varepsilon\theta$. Hence by

(3.31), we therefore conclude that the angle between $\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)$ and $\mathbf{Z}^{W';L',\ell';}_{L'';W',u;}(G)$ is at most $d\theta + b\varepsilon\theta$.

To prove **(iii)** we observe that we can write for any $j,\ell \in [k]$ the telescoping product,

$$\frac{\mathbf{Z}^{W'{}_1,\dots,u_d;L',\ell,\dots,\ell;}_{L',\dots,\ell;W',u_1,\dots,u_d;}(\widehat{H})}{\mathbf{Z}^{W'{}_1,\dots,u_d;L',j,\dots,j;}_{L',\dots,j;W',u_1,\dots,u_d;}(\widehat{H})} = \frac{\mathbf{Z}^{W'{}_1,\dots,u_d;L',\ell,\dots,\ell;}_{L',\dots,\ell;W',u_1,\dots,u_d;}(\widehat{H})}{\mathbf{Z}^{W'{}_1,\dots,u_{d-1},u_d;L',\ell,\dots,\ell,j;}_{L',\dots,\ell,j;W',u_1,\dots,u_{d-1},u_d;}(\widehat{H})} \cdots \frac{\mathbf{Z}^{W'{}_1,u_2,\dots,u_d;L',\ell,j,\dots,j;}_{L',j,\dots,j;W',u_1,u_2,\dots,u_d;}(\widehat{H})}{\mathbf{Z}^{W'{}_1,\dots,u_d;L',j,\dots,j;}_{L',\dots,j;W',u_1,\dots,u_d;}(\widehat{H})}, \tag{3.32}$$

and consequently by **B'(ii)**, it follows that

$$(1+K)^{-d} \le \frac{|\mathbf{Z}^{W'{}_1,\dots,u_d;L',\ell,\dots,\ell;}_{L',\dots,\ell;W',u_1,\dots,u_d;}(\widehat{H})|}{|\mathbf{Z}^{W'{}_1,\dots,u_d;L',j,\dots,j;}_{L',\dots,j;W',u_1,\dots,u_d;}(\widehat{H})|} \le (1+K)^d. \tag{3.33}$$

Next we observe that in S when changing the color of u from $\ell \in [k]$ to a good color $j \in \mathcal{G}$, we have

$$\left|\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(S)\right| \le \varepsilon^{m(\ell)}\left|\mathbf{Z}^{W';L',j;}_{L';W',u;}(S)\right|,$$

and so by (3.31) we have

$$\left|\mathbf{Z}^{W',u_1,\dots,u_d;L',\ell,j,\dots,j;}_{L',j,\dots,j;W',u,u_1,\dots,u_d;}(H)\right| \le \varepsilon^{m(\ell)}\left|\mathbf{Z}^{W',u_1,\dots,u_d;L',j,j,\dots,j;}_{L',j,\dots,j;W',u,u_1,\dots,u_d;}(H)\right|,$$

Combining the above inequality with (3.33), we obtain **(iii)** as required:

$$\frac{|\mathbf{Z}^{W';L',\ell;}_{L';W',u;}(G)|}{|\mathbf{Z}^{W';L',j;}_{L';W',u;}(G)|} = \frac{|\mathbf{Z}^{W',u_1,\dots,u_d;L',\ell,\ell,\dots,\ell;}_{L',\ell,\dots,\ell;W',u,u_1,\dots,u_d;}(H)|}{|\mathbf{Z}^{W',u_1,\dots,u_d;L',j,j,\dots,j;}_{L',j,\dots,j;W',u,u_1,\dots,u_d;}(H)|} \le \varepsilon^{m(\ell)}(1+K)^d.$$

This completes the proof.

In Table 3.2 we list the improvements that Theorem 3.13 gives; we give the improved values for k_Δ together with values of K and θ that allow the reader to check that the conditions of Theorem 3.13 are met with strict inequality for $\varepsilon = 0$.

5 Concluding remarks and questions

In the present chapter we have established that if k is an integer satisfying $k \ge e\Delta + 1$, then there exists an open set U containing $[0,1)$ such that the for any graph G of maximum degree at most Δ and $w \in U$, $\mathbf{Z}(G;k,w) \ne 0$. For small values of Δ we have shown that we can significantly improve on e. We raise the following question.

Δ	K	θ	k_Δ
3	0.4124	0.4251	6
4	0.2900	0.2943	8
5	0.2224	0.2244	11
6	0.1814	0.1826	14
7	0.1536	0.1543	17
8	0.1334	0.1339	20
9	0.1179	0.1183	23
10	0.1057	0.1060	26
11	0.0959	0.0961	29
12	0.0877	0.0879	32
13	0.0800	0.0802	35

Table 3.2: Bounds for the number of colors and values for K and θ for small values of Δ for Theorem 3.13.

Question 3.14. *Is it true that for each $\Delta \in \mathbb{N}_{\geq 3}$ there exists an open set $U = U_\Delta$ containing $[0,1]$ such that for any integer k satisfying $k \geq \Delta + 1$ and any graph G of maximum degree at most Δ and $w \in U$, $\mathbf{Z}(G;k,w) \neq 0$?*

As mentioned in the introduction, Barvinok's approach for proving zero-free regions for partition functions has been used for several types of partition functions, see [6–8, 61]. This of course raises the question as to which of these partition functions our ideas could be applied. In particular it would be interesting to apply our ideas to partition functions of edge-coloring models (a.k.a. tensor networks, or Holant problems). This framework may be useful to study the zeros of the Potts model on line graphs.

Implicit in our proof of Theorem 3.9 is an iteration of a complex-valued dynamical system, which for $k = 2$ coincides with the dynamical system analyzed in [48, 57]. Given the recent success of the use of methods from the field of complex dynamical systems to identify zero-fee regions and the location of zeros of the partition function of the hardcore model [58] and the partition function of the Ising model [48, 57], it seems natural to study this dynamical system. We intend to expand on this in future work.

4

Descent polynomial

Denote the group of permutations on $[n] = \{1, \ldots, n\}$ by $\mathcal{S}_n$ and for a permutation $\pi \in \mathcal{S}_n$, the set of descending position is

$$Des(\pi) = \{i \in [n-1] \mid \pi_i > \pi_{i+1}\}.$$

We would like to investigate the number of permutations with a fixed descent set. More precisely, for a finite $I \subseteq \mathbb{Z}^+$ let $m = \max(I \cup \{0\})$. Then for $n > m$ we can count the number of permutations with descent set I, that we will denote by

$$d(I, n) = |D(I, n)| = |\{\pi \in \mathcal{S}_n \mid Des(\pi) = I\}|.$$

This function was shown to be a degree m polynomial in n by MacMahon in [52]. In order to investigate this polynomial we extend the domain to $\mathbb{C}$, and for this chapter we call $d(I, n)$ the descent polynomial of I.

This polynomial was recently studied in the article of Diaz-Lopez, Harris, Insko, Omar and Sagan [26], where the authors found a new recursion which was motivated by the peak polynomial. The paper investigated the roots of descent polynomials and their coefficients in different bases. In this chapter we will answer a few conjectures of [26].

The coefficient sequence $a_k(I)$ is defined uniquely through the following equation

$$d(I, n) = \sum_{k=0}^{m} a_k(I) \binom{n-m}{k}.$$

In [26] it was shown that the sequence $a_k(I)$ is non-negative, since it counts some combinatorial objects. By taking a transformation of this sequence we were able to apply Stanley's theorem about the statistics of heights of a fixed element in a poset. As a result we prove

Theorem 4.11. *If $I \neq \emptyset$, then the sequence $\{a_k(I)\}_{k=0}^m$ is log-concave, that means that for any $0 < k < m$ we have*

$$a_{k-1}(I)a_{k+1}(I) \leq a_k^2(I).$$

As a corollary of the proof of Theorem 4.11 we get a bound on the roots of $d(I,n)$:

Theorem 4.14. *If $I \neq \emptyset$ and $d(I,z_0) = 0$ for some $z_0 \in \mathbb{C}$, then $|z_0| \leq m$.*

As in [26] we will also consider the $c_k(I)$ coefficient sequence, that is defined by the following equation

$$d(I,n) = \sum_{k=0}^{m} (-1)^{m-k} c_k(I) \binom{n+1}{k}.$$

By using a new recursion from [26] we prove that

Proposition 4.6. *If $I \neq \emptyset$, then for any $0 \leq k \leq m$ the coefficient $c_k(I) \geq 0$.*

In the last section we will establish zero-free regions for descent polynomials. In particular we will prove the following.

Theorem 4.20. *If $I \neq \emptyset$ and $d(I,z_0) = 0$ for some $z_0 \in \mathbb{C}$, then $|z_0 - m| \leq m+1$. In particular, $\Re z_0 \geq -1$.*

This chapter is organized as follows. In the next section we will define two sequences, $a_k(I)$ and $c_k(I)$, we recall the two main recursions for the descent polynomial and we introduce one of our main key ingredients. Then in Section 2 we will prove a conjecture concerning the sequence $c_k(I)$ and some consequences. In Section 3 we will prove a conjecture concerning the sequence $a_k(I)$, then in Section 4 we prove some bounds on the roots.

1 Preliminaries

In this section we will recall some recursions of the descent polynomial and we will establish some related coefficient sequences by choosing different bases for the polynomials.

First of all, for the rest of the chapter we will always denote a finite subset of $\mathbb{Z}^+$ by I, and $m(I)$ is the maximal element of $I \cup \{0\}$. If it is clear from the context, $m(I)$ will be denoted by m.

Let us define the coefficients $a_k(I), c_k(I)$ for any I with maximal element m and $k \in \mathbb{N}$ through the following expressions:

$$d(I,n) = \sum_{k=0}^{m} a_k(I)\binom{n-m}{k} = \sum_{k=0}^{m} (-1)^{m-k} c_k(I)\binom{n+1}{k},$$

if $k \leq m$, and $c_k(I) = a_k(I) = 0$, if $k > m$. Observe that they are well-defined, since $\{\binom{n-m}{k}\}_{k\in\mathbb{N}}$ and also $\{\binom{n+1}{k}\}_{k\in\mathbb{N}}$ form a base of the space of one-variable polynomials. For later on, we will refer to the first and second bases as "a-base" and "c-base", respectively. We will also consider an other base that is also a Newton-base.

As it turns out, these coefficients are integers, moreover, they are non-negative. To be more precise, in [26] it has been proved that $a_k(I)$ counts some combinatorial objects (i.e. they are non-negative integers), and $c_0(I)$ is non-negative. The authors of [26] also conjectured that each $c_k(I) \geq 0$, and for a proof of the affirmative answer see Proposition 4.6.

Next, we would like to establish two recurrences for the descent polynomial, which will be intensively used in several proofs. Before that, we need the following notations. For an $\emptyset \neq I = \{i_1, \dots, i_l\}$ and $1 \leq t \leq l$, let

$$\begin{aligned}
I^- =& I - \{i_l\},\\
I_t =& \{i_1, \dots, i_{t-1}, i_t - 1, \dots, i_l - 1\} - \{0\},\\
\widehat{I}_t =& \{i_1, \dots, i_{t-1}, i_{t+1} - 1, \dots, i_l - 1\},\\
I' =& \{i_j \mid i_j - 1 \notin I\},\\
I'' =& I' - \{1\}.
\end{aligned}$$

For the rest, $m(I)$ denotes the maximal element of a non-empty set $I \cup \{0\}$. If it is clear from the context, we will denote this element by m .

Proposition 4.1. *If $I \neq \emptyset$, then*

$$d(I,n) = \binom{n}{m} d(I^-, m) - d(I^-, n)$$

In contrast to the simplicity of this recursion, the disadvantage is that the descent polynomial of I is a difference of two polynomials. In [26], the authors found an other way to write $d(I,n)$ as a sum of polynomials (Thm 2.4. of [26]). Now we will state an equivalent form, which will fit our purposes better, and we also give its proof.

Corollary 4.2. *If $I \neq \emptyset$, then*

$$d(I,n+1) = \tag{4.1}$$
$$d(I,n) + \sum_{i_t \in I'' \setminus \{m\}} d(I_t,n) + \sum_{i_t \in I' \setminus \{m\}} d(\hat{I}_t,n) + d(I^-,m-1)\binom{n}{m-1}.$$

Proof. Let us recall the formula of Theorem 2.4. of [26]:

$$d(I,n+1) = d(I,n) + \sum_{i_t \in I''} d(I_t,n) + \sum_{i_t \in I'} d(\hat{I}_t,n). \tag{4.2}$$

If $I = \{1\}$, then trivially (4.1) is true. For $I \neq \{1\}$ we will distinguish two cases.

If $m \notin I'$ (and also $m \notin I''$), then by definition it means that $m-1 \in I$. But it means that $m-1 \in I^-$ and
$$d(I^-,m-1)\binom{n}{m-1} = 0.$$
Therefore the right hand side of (4.1) is the same as the right hand side of (4.2).

If $m \in I'$ (and also $m \in I''$), then $i_l = m$, $\hat{I}_l = I^- \cup \{m-1\}$ and $I_l = I^-$. Now take the difference of the right hand sides of (4.1) and (4.2), that is

$$d(I^-,m-1)\binom{n}{m-1} - d(I_l,n) - d(\hat{I}_l,n) =$$
$$d(I^-,m-1)\binom{n}{m-1} - \left(d(\hat{I}_l^-,n)\binom{n}{m-1} - d(I^-,n)\right) - d(I^-,n) =$$
$$d(I^-,m-1)\binom{n}{m-1} - d(I^-,n)\binom{n}{m-1} = 0.$$

Therefore the two equations have to be equal. □

As a conjecture in [26] it arose that the coefficient sequence $\{a_k(I)\}_{k=0}^m$ is log-concave. We mean by that that for any $0 < k < m$ we have

$$a_{k-1}(I)a_{k+1}(I) \leq a_k(I)^2.$$

In particular, the sequence $\{a_k(I)\}_{k=0}^m$ is unimodal.

Our main tool to attack this problem will be a result of Stanley about the height of a certain element of a finite poset in all linear extensions. So let P be a finite poset and

$v \in P$ a fixed element, and denote the set of order-preserving bijection from P to the chain $[1, 2, \ldots, |P|]$ by $\mathrm{Ext}(P)$. Then, the height polynomial of v in P defined as

$$h_{P,v}(x) = \sum_{\phi \in \mathrm{Ext}(P)} x^{\phi(v)-1} = \sum_{k=0}^{|P|-1} h_k(P, v)x^k.$$

In other words $h_k(P, v)$ counts how many linear extensions P has, such that below v there are exactly k many elements.

In special cases, when all comparable elements from v (except for v) are bigger in P, we can reformulate $h_k(P, v)$ as it counts how many linear extensions P has, such that below v there are exactly k many incomparable elements. For such a case, we could combine two results of Stanley to obtain the following theorem.

Theorem 4.3. *Let P be a finite poset, and $v \in P$ be fixed. Then the coefficient sequence $\{h_k(P, v)\}_{k=0}^{|P|-1}$ is log-concave. Moreover if all comparable elements with v are bigger than v in P, then $\{h_k(P, v)\}_{k=0}^{|P|-1}$ is a decreasing, log-concave sequence.*

Proof. The first part of the theorem is Corollary 3.3. of [66]. For the second part we use fact that $h_k(P, v)$ can be interpreted as the number of linear extensions such that there are k many smaller than v incomparable elements in the extension. Then by Theorem 6.5. of [67] we obtain the desired statement. □

We will use this theorem in a special case. For any I we define a poset P_I on $[u_1, \ldots, u_{m+1}]$, as $u_i > u_{i+1}$ if $i \in I$ and $u_i < u_{i+1}$ if $i \notin I$. Observe that any comparable element with x_{m+1} is bigger in P_I, therefore the sequence $\{h_k(P_I, u_{m+1})\}_{k=0}^{m}$ is decreasing and log-concave. We would like to remark that any linear extension of P_I can be viewed as an element of $D(I, m+1)$. In that way we can write that

$$h(I, x) = h_{P_I, u_{m+1}}(x) = \sum_{\pi \in D(I, m+1)} x^{\pi_{m+1}-1}.$$

2 Descent polynomial in "c-base"

The aim of the section is to give an affirmative answer for Conjecture 3.7. of [26], and give some immediate consequences on the coefficients and evaluation. For corollaries considering the roots of $d(I, n)$ see Section 4. We would like to remark at that point that the proof will be just an algebraic manipulation, not a "combinatorial" proof. However, giving such a proof could imply some kind of "combinatorial reciprocity" for descent polynomials.

First, we will translate the recursion of Corollary 4.2 to the terms of $c_k(I)$.

Lemma 4.4. *If $I \neq \emptyset$ and $0 \leq k \leq m-1$, then*

$$c_{k+1}(I) = \sum_{i_t \in I'' \setminus \{m\}} c_k(I_t) + \sum_{i_t \in I' \setminus \{m\}} c_k(\widehat{I_t}) + d(I^-, m-1).$$

Proof. The idea is that we rewrite the equation of 4.2 as

$$d(I, n+1) - d(I, n) = \sum_{i_t \in I'' \setminus \{m\}} d(I_t, n) + \sum_{i_t \in I' \setminus \{m\}} d(\hat{I}_t, n) + d(I^-, m-1)\binom{n}{m-1},$$

and express both sides in c-base, then compare the coefficients of $\binom{n+1}{k}$.

The left side can be written as

$$\begin{gathered} d(I, n+1) - d(I, n) = \\ \sum_{k=0}^{m} c_k(I)(-1)^{m-k}\binom{n+2}{k} - \sum_{k=0}^{m} c_k(I)(-1)^{m-k}\binom{n+1}{k} = \\ \sum_{k=1}^{m} c_k(I)(-1)^{m-k}\binom{n+1}{k-1} = \\ \sum_{k=0}^{m-1} c_{k+1}(I)(-1)^{m-k-1}\binom{n+1}{k}. \end{gathered}$$

Next we use the famous Chu-Vandermonde's identity:

$$\binom{n}{m-1} = \sum_{k=0}^{m-1} \binom{n+1}{k}\binom{-1}{m-1-k} = \sum_{k=0}^{m-1} (-1)^{m-1-k}\binom{n+1}{k}.$$

Therefore the right hand side can be written as:

$$\begin{gathered} \sum_{i_t \in I'' \setminus \{m\}} d(I_t, n) + \sum_{i_t \in I' \setminus \{m\}} d(\widehat{I_t}, n) + d(I^-, m-1)\binom{n}{m-1} = \\ \sum_{k=0}^{m-1} (-1)^{m-1-k} \left(\sum_{i_t \in I'' \setminus \{m\}} c_k(I_t) + \sum_{i_t \in I' \setminus \{m\}} c_k(\widehat{I_t}) + d(I^-, m-1) \right) \binom{n+1}{k}. \end{gathered}$$

We gain that for any $0 \leq k \leq m-1$,

$$\begin{aligned} &(-1)^{m-k-1} c_{k+1}(I) = \\ &\qquad (-1)^{m-1-k} \left(\sum_{i_t \in I'' \setminus \{m\}} c_k(I_t) + \sum_{i_t \in I' \setminus \{m\}} c_k(\widehat{I_t}) + (-1)^{m-1} d(I^-, m-1) \right). \end{aligned}$$

By multiplying both sides by $(-1)^{m-k-1}$ we get the desired statement. $\square$

Similarly, we can rephrase Proposition 4.1, but we leave the proof for the readers.

Lemma 4.5. *If $I \neq \emptyset$ and $0 \leq k \leq m$, then*

$$c_k(I) = d(I^-, m) - (-1)^{m-m^-} c_k(I^-), \tag{4.3}$$

where $m^- = m(I^-)$.

The next theorem settles Conjecture 3.7 of [26]. We would like to point out that the non-negativity of $c_0(I)$ has already been proven in [26], and one can use it to find a shortcut in the proof. However, we will give a self-contained proof.

Theorem 4.6. *For any I and $0 \leq k \leq m$, the coefficient $c_k(I)$ is a non-negative integer.*

Proof. We will proceed by induction on m. If $m = 0$, then $I = \emptyset$, thus,

$$d(I, n) = 1,$$

therefore $c_0(I) = 1 \geq 0$.

If $m = 1$ or $|I| = 1$, then $I = \{m\}$ and

$$d(I,n) = \binom{n}{m} - 1 = \sum_{k=0}^{m} \binom{n+1}{k}\binom{-1}{m-k} - \binom{n+1}{0} =$$
$$\sum_{k=1}^{m} (-1)^{m-k}\binom{n+1}{k} + (-1)^{m-0}\binom{n+1}{0}(1-(-1)^m).$$

We obtained that

$$c_k(I) = \begin{cases} 1 & \text{if } 0 < k \leq m \\ 2 & \text{if } k = 0 \text{ and } m \text{ is odd} \\ 0 & \text{if } k = 0 \text{ and } m \text{ is even} \end{cases}$$

For the rest of the proof, we assume that the size of I is at least 2. Therefore $m > 1$, and $m^- = \max(I^-) > 0$. Since for any $i_t \in I''$ (and $i_t \in I'$) the maximum of I_t (and $\widehat{I}_t$) is exactly $m-1$, we can use induction on them, i.e. $c_k(I_t) \geq 0$ integer ($c_k(\widehat{I}_t) \geq 0$ integer). On the other hand, $d(I^-, m-1)$ counts permutations with descent set I^-, so $d(I^-, m-1) \geq 0$ integer. Now by Lemma 4.4 and by the previous paragraph we have for any $k \geq 1$ that

$$c_k(I) = \sum_{i_t \in I' \setminus \{m\}} c_{k-1}(I_t) + \sum_{i_t \in I'' \setminus \{m\}} c_{k-1}(\widehat{I}_t) + d(I^-, m-1) \geq 0. \tag{4.4}$$

What remains is to prove that $c_0(I) \geq 0$. This is exactly the statement of Proposition 3.10. of [26], but for the completeness we also give its proof.

We consider two cases. If $m-1 \in I$, then by (4.3)

$$c_0(I) = d(I^-, m) - (-1)^{m-(m-1)} c_0(I^-) = d(I^-, m) + c_0(I^-) \geq 1 + 0 > 0,$$

since $m > \max(I^-)$.

If $m-1 \notin I$, then by (4.4),

$$c_1(I) \geq d(I^-, m-1) \geq 1.$$

On the other hand, we can express $d(I,0)$ in two ways. The first equality is by Lemma 3.8. of [26], the second is by the definition of $c_k(I)$.

$$(-1)^{\#I} = d(I,0) = \sum_{k=0}^{m} (-1)^{m-k} c_k(I) \binom{1}{k} = (-1)^m (c_0(I) - c_1(I)),$$

therefore

$$c_0(I) = c_1(I) + (-1)^{\#I+m} \geq 1 + (-1) = 0.$$

□

As a corollary we will see that the values of the polynomial $d(I,n)$ at negative integers are of the same sign. This phenomenon is kind of similar to a "combinatorial reciprocity", by which we mean that there exists a sequence of "nice sets" A_n parametrized by n, such that $(-1)^m d(I,-n) = |A_n|$. We think that either proving the previous theorem using combinatorial arguments or finding a combinatorial reciprocity for $d(I,n)$ could provide an answer for the other.

Corollary 4.7. *Let n be a positive integer, then*

$$(-1)^m d(I,-n) \geq 0.$$

Moreover if $n > 1$ positive integer, then $(-1)^m d(I,-n) > 0$.

Proof. Assume that $n = 1$. Then

$$(-1)^m d(I,-1) = \sum_{k=0}^{m} (-1)^{-k} c_k(I) \binom{-1+1}{k} = (-1)^0 c_0(I) \binom{0}{0} = c_0(I),$$

and by the previous proposition we know that $c_0(I) \geq 0$.

$$(-1)^m d(I,-n) = \sum_{k=0}^{m} c_k(I)(-1)^{-k}\binom{-n+1}{k} =$$
$$\sum_{k=0}^{m} c_k(I)(-1)^{-k}(-1)^k\binom{n+k-2}{k} = \sum_{k=0}^{m} c_k(I)\binom{n+k-2}{k} > 0$$

□

We would like to remark that in Section 4 we will prove that in particular there is no root of $d(I,n)$ on the half-line $(-\infty,-1)$, that is, for any real number $z_0 \in (-\infty,-1)$, the expression $(-1)^m d(I,z_0)$ is always positive.

Moreover if we carefully follow the previous proofs, then one might observe that $d(I,-1) = 0$ iff $c_0(I) = 0$ iff $I = \{m\}$ where m is even or $I = [m-2] \cup \{m\}$.

3 Descent polynomial in "a-base"

In this section we would like to investigate the coefficients $a_k(I)$. In order to do that, we will need to understand the coefficients of $d(I,n)$ in the base of $\{\binom{n-m+k}{k+1}\}_{k=-1}^{m-1}$, which is defined by the following equation

$$d(I,n) = \bar{a}_{-1}(I)\binom{n-m-1}{0} + \bar{a}_0(I)\binom{n-m}{1} + \cdots + \bar{a}_{m-1}(I)\binom{n-1}{m}.$$

Observe that $\bar{a}_{-1}(I) = 0$, since

$$0 = d(I,m) = \bar{a}_{-1}(I)\binom{-1}{0} + \sum_{k=0}^{m-1} \bar{a}_k(I)\binom{k}{k+1} = \bar{a}_{-1}(I),$$

therefore later on, we will concentrate on the coefficients $\bar{a}_k(I)$ for $0 \leq k \leq m-1$. As it will turn out, all these coefficients are non-negative integers, moreover, each of them counts some combinatorial objects.

On the other hand, this new coefficient sequence is closely related to the coefficients $a_k(I)$. To show the connection, we introduce two polynomials

$$a(I,x) = \sum_{k=0}^{m} a_k(I)x^k,$$
$$\bar{a}(I,x) = \sum_{k=0}^{m-1} \bar{a}_k(I)x^k.$$

First we will show that $\bar{a}_k(I) = h_{m-k}(P_I, u_{m+1})$, i.e. $\bar{a}_k(I)$ counts the number of permutations from $D(I, m+1)$, such that there are $(k+1)$ elements above u_{m+1}.

Proposition 4.8. *If $I \neq \emptyset$ and $0 \leq k \leq m-1$, then*

$$\bar{a}_k(I) = h_{m-k}(P_I, u_{m+1}).$$

Proof. We will show that if $n > m$, then

$$d(I,n) = \sum_{k=0}^{m-1} h_{m-k}(P_I, u_{m+1}) \binom{n-m+k}{k+1}.$$

It is enough, since $\{\binom{n-m+k}{k+1}\}_{k=-1}^{m-1}$ is a base in the space of polynomials of degree at most m.

Let us define the sets $B_k(I,n) = \{\pi \in D(I,n) \mid \pi_{m+1} = k\}$ for $1 \leq k \leq n$. For any $\pi \in D(I,n)$ the last descent is between m and $m+1$, therefore $\pi_m > \pi_{m+1} < \pi_{m+2} < \cdots < \pi_n \leq n$, i.e. $\pi_{m+1} \leq m$. Therefore $B_k(I,n) = \emptyset$ for any $m < k \leq n$, and $D(I,n)$ is a disjoint union of the sets $B_k(I,n)$ for $1 \leq k \leq m$. Also observe that $|B_k(I, m+1)| = h_k(P_I, u_{m+1})$.

We claim that

$$|B_k(I,n)| = |B_k(I,m+1) \times \binom{[k+1,n]}{m+1-k}| = |B_k(I,m+1)| \binom{n-k}{m+1-k}.$$

To prove the first equality we establish a bijection. If $\pi \in B_k(I,n)$, then let $E_\pi = \{1 \leq i \leq m \mid \pi_i > k\}$, $V_\pi = \{\pi_i \mid i \in E_\pi\}$ and $\pi|_{m+1} \in B_k(I, m+1)$ the unique induced linear ordering on the first $m+1$ element. As before, for any $l > m+1$ the value π_l is bigger than π_{m+1}, therefore $|E_\pi| = m+1-k$ and $V_\pi \subseteq [k+1, n]$ has size $m+1-k$. So let $f : B_k(I,n) \to B_k(I, m+1) \times \binom{[k+1,n]}{m+1-k}$ defined as

$$f(\pi) = (\pi|_{m+1}, V_\pi).$$

Checking whether the function f is a bijection is left to the readers.

Putting the pieces together, we have

$$\begin{aligned} d(I,n) = |D(I,n)| = |\cup_{k=1}^m B_k(I,n)| = \sum_{k=1}^m |B_k(I,n)| = \\ \sum_{k=1}^m |B_k(I,m+1) \times \binom{[k+1,n]}{m+1-k}| = \\ \sum_{k=1}^m |B_k(I,m+1)| \binom{n-k}{m+1-k} = \end{aligned}$$

$$\sum_{k=1}^{m} h_k(P_I, u_{m+1}) \binom{n-k}{m+1-k} =$$
$$\sum_{l=0}^{m-1} h_{m-l}(P_I, u_{m+1}) \binom{n-m+l}{l+1}.$$

□

Corollary 4.9. *If $I \neq \emptyset$, then the sequence $\overline{a}_0(I), \overline{a}_1(I), \ldots, \overline{a}_{m-1}(I)$ is a monotone increasing, log-concave sequence of non-negative integers.*

Proof. By the previous proposition we know that this sequence is the same as

$$\{h_{m-k}(P_I, u_{m+1})\}_{k=1}^{m},$$

which is clearly a sequence of non-negative integers. Moreover, by Theorem 4.3, it is log-concave and monotone decreasing. □

We just want to remark that since the polynomial $\overline{a}(I,x)$ has a monotone coefficient sequence, all of its roots are contained in the unit disk (see Figure 4.1).

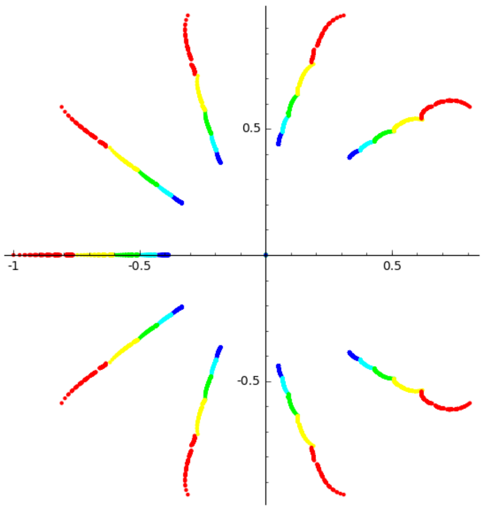


Figure 4.1: The roots of $\bar{a}(I,n)$ where I has the form $I = J \cup [10, 11, \ldots, 10+k]$ for some $k = 0, \ldots, 4$ and $J \subseteq [8]$. Different colors mark different values of k.

Our next goal is to establish a connection between the coefficients $a_k(I)$ and $\overline{a}_k(I)$.

Proposition 4.10. *If $I \neq \emptyset$, then*

$$a(I,x) = x\overline{a}(I, x+1)$$

Proof. By definition we see that

$$d(I,n) = \sum_{k=0}^{m-1} \overline{a}_k(I)\binom{n-m+k}{k+1} = \sum_{k=0}^{m-1} \overline{a}_k(I) \sum_{l=0}^{k+1} \binom{n-m}{l}\binom{k}{k+1-l} =$$
$$\sum_{k=0}^{m-1} \overline{a}_k(I) \sum_{l=1}^{k+1} \binom{n-m}{l}\binom{k}{l-1} = \sum_{l=1}^{m} \binom{n-m}{l} \left(\sum_{k=l-1}^{m-1} \overline{a}_k(I)\binom{k}{l-1} \right),$$

which means that $a_l(I) = \sum_{k=l-1}^{m-1} \overline{a}_l(I)\binom{k}{l-1}$ for $1 \le l \le m$, i.e.

$$a(I,x) = \sum_{l=1}^{m} x^l \left(\sum_{k=l-1}^{m-1} \overline{a}_k(I)\binom{k}{l-1} \right)$$

On the other hand, let us calculate the coefficients of $x\overline{a}(I,x+1)$.

$$x\overline{a}(I,x+1) = x\left(\sum_{k=0}^{m-1} \overline{a}_k(I)(x+1)^k \right) =$$
$$x\left(\sum_{k=0}^{m-1} \overline{a}_k(I) \sum_{l=0}^{k} \binom{k}{l} x^l \right) = x\left(\sum_{l=0}^{m-1} x^l \sum_{k=l}^{m-1} \overline{a}_k(I)\binom{k}{l} \right) =$$
$$\sum_{l=0}^{m-1} x^{l+1} \sum_{k=l}^{m-1} \overline{a}_k(I)\binom{k}{l} =$$
$$\sum_{l=1}^{m} x^l \sum_{k=l-1}^{m-1} \overline{a}_k(I)\binom{k}{l-1} = a(I,x).$$

□

As a corollary of two previous propositions, we will give a proof of Conjecture 3.4 of [26].

Corollary 4.11. *If $I \neq \emptyset$, then the sequence $a_0(I), a_1(I), \ldots, a_m(I)$ is a log-concave sequence of non-negative integers.*

Proof. By Corollary 4.9 we know that the coefficient sequence of the polynomial $\overline{a}(I,x)$ is log-concave, and by monotonicity, it is clearly without internal zeros. Therefore by the fundamental theorem of [13], the coefficient sequence of the polynomial $\overline{a}(I,x+1)$ is log-concave. Since multiplication with an x only shifts the coefficient sequence, $x\overline{a}(I,x+1) = a(I,x)$ also has a log-concave coefficient sequence. □

4 On the roots of $d(I,n)$

In this section we will prove four propositions about the locations of the roots of $d(I,n)$, two are for general I, and two are for some special ones. The first result is obtained by the technique of Theorem 4.16. of [26] based on the non-negativity of the coefficients $c_k(I)$. In the second, we will prove a linear bound in m for the length of the roots of $d(I,n)$, which will be based on the monotonicity of the coefficients $\overline{a}_k(I)$. For the third we use similar arguments as in the proof of the second statement. In the fourth we will prove a real-rootedness for some special I using Neumaier's Gershgorin type result.

First we will recall some basic notations from [26]. Let R_m be the region described by Theorem 4.16. of [26], that is $R_m = S_m \cup \overline{S_m}$ and

$$S_m = \left\{ z \in \mathbb{C} \;\middle|\; \arg(z) \geq 0 \text{ and } \sum_{i=1}^{m} \arg(z - i + 1) < \pi \right\}.$$

Then we have the following corollary of Proposition 4.6.

Corollary 4.12. *Let I be a finite set of positive integers. Than any element of $(m-2) - R_m$ is not a root of $d(I,z)$. In particular, if z_0 is a real root of $d(I,z)$, then $z_0 \geq -1$.*

Proof. Let $z \in \mathbb{C}$ be a complex number such that

$$S = \left\{ (-1)^0 (z+1)_{\downarrow 0}, \ldots, (-1)^m (z+1)_{\downarrow m} \right\}$$

is non-negatively independent, i.e.

$$S = \left\{ (-1-z)_{\uparrow 0}, \ldots, (-1-z)_{\uparrow m} \right\}$$

is in an open half plane H, such that $1 \in H$. But this is equivalent to the fact that the points

$$S' = \left\{ (m-2-z)_{\downarrow m} (-1-z)^{-1}_{\uparrow 0}, \ldots, (m-2-z)_{\downarrow m} (-1-z)^{-1}_{\uparrow m} \right\}$$

are in H, which is the same set as

$$S' = \left\{ (m-2-z)_{\downarrow m}, (m-2-z)_{\downarrow m-1} \ldots, (m-2-z)_{\downarrow 0} \right\}.$$

But by Theorem 4.16. of [26], we know that this set lies on an open half-plane iff $m-2-z \in R_m$.

Therefore S is an open half plane iff $m-2-z \in R_m$ iff $z \in (m-2) - R_m$.

The last statement can be obtained from the fact that $(m-1, \infty) \subseteq R_m$. □

The following lemma will be useful in the upcoming proofs.

Lemma 4.13. *Let $m > 0$ integer given and assume that $|z| > m$. Then the lengths*

$$\left|\binom{z-m+k}{k}\right|$$

are increasing for $k = 0, \dots, m$.

In particular, if $\alpha_0, \dots, \alpha_m \in \mathbb{R}$, $\alpha_m \neq 0$, $\sum_{i=0}^{m-1} |\alpha_i| \leq |\alpha_m|$ and $|z| > m$, then

$$\left|\alpha_m \binom{z}{m}\right| > \left|\sum_{k=0}^{m-1} \alpha_k \binom{z-m+k}{k}\right|.$$

$$\sum_{i=0}^{m} \alpha_k \binom{z-m+k}{k} \neq 0$$

Proof. Let $0 \leq k \leq m-1$ be fixed. Then to see that the lengths are increasing we have to consider the ratio of two consecutive elements:

$$\left|\binom{z-m+k+1}{k+1}\right| \left|\binom{z-m+k}{k}\right|^{-1} =$$
$$= \frac{|z-m+k+1|}{k+1} \geq \frac{|z|-m+k+1}{k+1} > 1$$

Therefore the sequence is increasing.

To see the second statement let us define $C = \sum_{i=0}^{m-1} |\alpha_i|$. If $C = 0$, then the statement is trivially true. If $C \neq 0$, then the vector

$$v = \left|\sum_{k=0}^{m-1} \frac{\alpha_k}{C} \binom{z-m+k}{k}\right| = \left|\sum_{k=0}^{m-1} \frac{|\alpha_k|}{C} \mathrm{sign}(\alpha_i) \binom{z-m+k}{k}\right|$$

is a convex combination of the vectors $\left\{\mathrm{sign}(\alpha_k)\binom{z-m+k}{k}\right\}_{k=0}^{m-1}$. Hence

$$|v| \leq \left|\binom{z-1}{m-1}\right|,$$

and

$$\left|\sum_{k=0}^{m-1} \alpha_k \binom{z-m+k}{k}\right| = C|v| \leq C \left|\binom{z-1}{m-1}\right| < \alpha_m \left|\binom{z}{m}\right|$$

□

Corollary 4.14. *If z_0 is a root of $d(I,z)$, then $|z_0| \leq m$.*

Proof. Let us consider the polynomial $p(z) = (z-1)\bar{a}(I,z)$, and let p_i (resp. $\bar{a}_i$) be the coefficient of z^i in p (resp. $\bar{a}$), i.e.

$$p(z) = \sum_{i=0}^{m} p_i z^i \qquad \bar{a}(I,z) = \sum_{i=0}^{m-1} \bar{a}_i z^i.$$

The relation of p and $\bar{a}$ translates as follows:

$$p_i = \begin{cases} \bar{a}_{m-1} & \text{if } i = m \\ \bar{a}_{i-1} - \bar{a}_i & \text{if } 0 < i < m \\ -\bar{a}_0 & \text{if } i = 0 \end{cases}$$

and

$$d(I,n) = \sum_{k=0}^{m} p_k \binom{n-m+k}{k}.$$

Since the coefficient sequence of $\bar{a}(I,z)$ is non-decreasing by Corollary 4.9, therefore all coefficients of p except p_m are non-positive and their sum is 0. In other words for any $k \in \{0,1,\ldots,m-1\}$:

$$|p_k| = -p_k$$

and

$$\sum_{k=0}^{m-1} |p_k| = -\sum_{k=0}^{m-1} p_k = a_{m-1} = p_m > 0.$$

Therefore by Lemma 4.13, if $|z| > 0$, then

$$d(I,z) = \sum_{k=0}^{m} p_k \binom{z-m+k}{k} \neq 0.$$

□

In the previous proof we did not use the fact that $\bar{a}_k(I)$ is a log-concave sequence, which would be interesting if one could make use of it. Our next goal is to prove Theorem 4.20. In order to prove it, we have to distinguish a few cases depending on the number of consecutive elements ending at $\max(I)$. For simplicity, first we will consider the case, when the distance of the last two elements is at least 2.

Proposition 4.15. *If $I = \{i_1, \dots, i_l\}$ for some $l \geq 1$, such that $|I| = 1$ or $i_l - i_{l-1} \geq 2$. If $d(I, z_0) = 0$, then*

$$|m - 1 - z_0| \leq m.$$

In particular $\Re z_0 \geq -1$.

Proof. Let us consider $p(n) = (-1)^m d(I, m-1-n)$ using coefficients $c_k(I)$.

$$d(I, -(n-m+1)) = \sum_{k=0}^{m} (-1)^{m-k} c_k(I) \binom{-(n-m+1)+1}{k} =$$
$$\sum_{k=0}^{m} (-1)^{m-k} c_k(I)(-1)^k \binom{n-m+k-1}{k} =$$
$$(-1)^m \sum_{k=0}^{m} c_k(I) \binom{n-m+k-1}{k}.$$

It might be familiar from the proof of Corollary 4.14. As before we expend $p(n)$ in base $\{\binom{n-m+k}{k}\}_{k \in \mathbb{N}}$.

$$p(n) = \sum_{k=0}^{m} c_k(I) \binom{n-m+k-1}{k} =$$
$$\sum_{k=1}^{m} c_k(I) \left(\binom{n-m+k}{k} - \binom{n-m+k-1}{k-1} \right) + c_0(I) \binom{n-m-1}{0} =$$
$$c_m \binom{n}{m} + \sum_{k=0}^{m-1} (c_k(I) - c_{k+1}(I)) \binom{n-m+k}{k} =$$
$$\sum_{k=0}^{m} \tilde{c}_k(I) \binom{n-m+k}{k}.$$

Now we claim that $\sum_{k=0}^{m-1} |\tilde{c}_k(I)| \leq c_m(I)$. To prove that, we use induction on $|I|$ and m, and we use the recursion of Lemma 4.4. If $I = \{m\}$, then it can be easily checked.

So for the rest assume, that the statement is true for sets of size at most $l-1$ and with maximal element at most $m-1$. Let $|I| = l \geq 2$ with $i_l = m$ and assume that $i_l - i_{l-1} \geq 2$. Then

$$\sum_{k=0}^{m-1} |c_k(I) - c_{k+1}(I)| =$$
$$|c_0(I) - c_1(I)| + \sum_{k=1}^{m-1} \left| \sum_{t \in I'' \setminus \{m\}} c_{k-1}(I_t) - c_k(I_t) + \sum_{t \in I' \setminus \{m\}} c_{k-1}(\hat{I}_t) - c_k(\hat{I}_t) \right| \leq$$

$$1+\sum_{k=0}^{m-2}\sum_{t\in I''\setminus\{m\}}|c_k(I_t)-c_{k+1}(I_t)|+\sum_{k=0}^{m-2}\sum_{t\in I'\setminus\{m\}}|c_k(\hat{I}_t)-c_{k+1}(\hat{I}_t)|$$

For any $t \in I'' \setminus \{m\}$ the two largest elements of I_t will be $i_{t-1}-1$ and $i_t - 1 = m-1$, so their difference is at least 2, therefore we can use inductive hypothesis. If $t \in I' \setminus \{m\}$, then either $\hat{I}_t$ has exactly one element, or $|\hat{I}_t| > 1$. In this second case the largest element of $\hat{I}_t$ is $i_t - 1 = m-1$ and the second largest is i_{t-2} or $i_{t-1}-1$. Clearly in each cases the inductive hypothesis is true, therefore

$$\sum_{k=0}^{m-1}|c_k(I)-c_{k+1}(I)| \leq 1+\sum_{t\in I''\setminus\{m\}} c_{m-1}(I_t)+\sum_{t\in I'\setminus\{m\}} c_{m-1}(\hat{I}_t) =$$
$$1+c_m(I)-d(I^-,m-1)\leq c_m(I)=\tilde{c}_m(I).$$

The last inequality is true, since $m-1 > \max(I^-)$.

So we obtained that $\sum_{k=0}^{m-1}|\tilde{c}_k(I)| \leq c_m(I)$, therefore by Lemma 4.13, if $|z| > m$, then

$$0\neq \sum_{k=0}^{m}\tilde{c}_k(I)\binom{z-m+k}{k}=p(z)=(-1)^m d(I,m-1-z),$$

equivalently if $|m-1-z_0| > m$, then $d(I,z_0)\neq 0$.

□

We would like to remark two facts about the previous proof. First of all the introduced "new" coefficients, $\tilde{c}_k(I)$, are exactly

$$\tilde{c}_k(I)=d(I^c,k)=\begin{cases}(-1)^{m+|[k+1,\infty)\cap I|+k}d(I\cap[k-1],k) & \text{if } k\in I\\ 0 & \text{otherwise}\end{cases},$$

where $I^c = [m]\setminus I$, therefore

$$d(I,n)=\sum_{k=0}^{m}(-1)^{m-k}\tilde{c}_k(I)\binom{n}{k}=\sum_{k=0}^{m}(-1)^{m-k}d(I^c,k)\binom{n}{k}.$$

Secondly we can not extend the proof for any I, because the crucial statement, that was $\sum_{k=0}^{m-1}|c_k(I)-c_{k+1}(I)|\leq c_m(I)$, is not true for any $I\subseteq\mathbb{Z}^+$. (E.g. $I=\{1,2,3,4,5\}$)

From now on we would like to understand the roots of I's with "non-trivial endings". To analyze these cases we introduce for the rest of the chapter the following notation: for any finite set $I\subseteq\mathbb{Z}^+$ and $t\in\mathbb{N}$ let $I^t = I\cup\{m+1,m+2,\ldots,m+t\}$.

Proposition 4.16. *For any $\emptyset \neq I$ such that $m-1 \notin I$. Then if $t = 1,2,3,4$, then there exists an $m_0 = m_0(t)$, such that if $m \geq m_0$ and $d(I^t, z_0) = 0$, then*

$$|m+t-1-z_0| \leq m+t.$$

Proof. Let us consider $d(I^t, n)$ in base $\{\binom{n}{k}\}_{k\in\mathbb{N}}$. Then

$$d(I^t, n) = \sum_{k=0}^{m+t} (-1)^{m+t-k} d(I^c, k) \binom{n}{k},$$

where $I^c = (I^t)^c = [m+t] \setminus I^t = [m] \setminus I$.

We claim that if $t \in \{1,2,3,4\}$ and m sufficiently large, then for any $m \leq k < m+t$ we have

$$2d(I^c, k) \leq d(I^c, k+1). \tag{4.5}$$

To see that let us observe that all the roots $\xi_1, \ldots, \xi_{m-1}$ of $d(I^c, n)$ are in a ball of radious $m-1$ around 0 by Corollary 4.14. Without loss of generality let us assume that $\xi_1 = \max(I^c) = m-1$. Then

$$\begin{aligned}
\frac{d(I^c,k)}{d(I^c,k+1)} = \left|\frac{d(I^c,k)}{d(I^c,k+1)}\right| &= \frac{(k-\xi_1)\prod_{i=2}^{m-1}|k-\xi_i|}{(k+1-\xi_1)\prod_{i=2}^{m-1}|k+1-\xi_i|} \\
\frac{k-m+1}{k-m+2}\prod_{i=2}^{m-1}\frac{|k-\xi_i|}{|k+1-\xi_i|} &\leq \frac{k-m+1}{k-m+2}\prod_{i=2}^{m-1}\frac{k+m-1}{k+m} \\
\leq \frac{t}{t+1}\left(\frac{2m+t-2}{2m+t-1}\right)^{m-2} &= \frac{t}{t+1}\left(1-\frac{1}{2m+t-1}\right)^{m-2} \to \frac{t}{t+1}e^{-0.5}
\end{aligned}$$

Since $\frac{t}{t+1}e^{-0.5} < 1/2$, therefore we get that for any $t \in \{1,2,3,4\}$ there exists an $m_0 = m_0(t)$, such that $\forall m \geq m_0$ and for any $m \leq k < m+t$ we have $2d(I^c, k) \leq d(I^c, k+1)$. In particular $2^{m+t-k} d(I^c, k) \leq d(I^c, m+t)$.

To finish the proof let us assume that $m \geq m_0$ for some fixed $t \in \{1,2,3,4\}$. Then consider the following polynomial $p(n) = (-1)^{m+t} d(I, m+t-1-n)$ as in the previos proof

$$\begin{aligned}
p(n) &= (-1)^{m+t} \sum_{k=0}^{m+t} (-1)^{m+t-k} d(I^c, k) \binom{m+t-1-n}{k} \\
&= \sum_{k=0}^{m+t} d(I^c, k) \binom{n-(m+t)+k}{k}
\end{aligned}$$

Assume that z_0 is a zero of $p(n)$ with length at least $m+t$ i.e.

$$d(I^c, m+t)\binom{z_0}{m+t} = \sum_{k=0}^{m+t-1}(-d(I^c,k))\binom{z_0-(m+t)+k}{k}$$

By the previous proof we get that $\sum_{k=0}^{m-1}|d(I^c,k)| \le d(I^c,m)$, therefore

$$\begin{aligned} C = \sum_{k=0}^{m+t-1}|-d(I^c,k)| &\le d(I^c,m) + \sum_{k=m}^{m+t-1} d(I^c,k) \\ &\le 2^{-t}d(I^c,m+t) + \sum_{k=m}^{m+t-1} 2^{-(m+t-k)}d(I^c,m+t) \\ &= d(I^c,m+t). \end{aligned}$$

But it means that $\frac{d(I^c,m+t)}{C}\binom{z_0}{m+t}$ is a convex combination of $\mathcal{F} = \{\epsilon_k\binom{z_0-(m+t)+k}{k}\}_{k=0}^{m+t-1}$, where $\epsilon_k = \operatorname{sgn}(-d(I^c,k))$. However this is a contradiction, since $\frac{d(I^c,m+t)}{C} \ge 1$ and $\binom{z_0}{m+t}$ is strictly longer than any member of the set $\mathcal{F}$.

□

Trivial upper bounds on m_0 is the smallest m_0', such that for any $m \in [m_0', \infty)$ we have

$$\frac{t}{t+1}\left(1-\frac{1}{2m+t-1}\right)^{m-2} < 1/2. \tag{4.6}$$

These values can be found in the following Table 4.1.

Lemma 4.17. *For any $\emptyset \ne I$, such that $m-1 \notin I$ and*

$$(m-1)(2m+1) \le \binom{t+m-1}{t}, \tag{4.7}$$

then

$$d(I^c,m)(2m+1) \le d(I^c,m+t)$$

Proof. First of all

$$d(I^c,m) = d(I,m) \le d(I^-,m-1)(m-1) = d((I^c)^-,m-1)(m-1),$$

because any $\pi \in D(I,m)$ can be written uniquely as an element in $D(I^-,m-1) \times [1,m-1]$.

On the other hand

$$d(I^c, m+t) \geq \binom{t+m-1}{t} d((I^c)^-, m-1),$$

because the left hand side counts the number of elements in $D(I^c, m+t)$, while the right hand side is the number of elements π in $D(I^c, m+t)$, such that $\pi_m = 1$.

Combining these inequalities and using the hypothesis we get the desired statement. $\square$

Proposition 4.18. *For any $\emptyset \neq I$ such that $m-1 \notin I$. If*

$$(m-1)(2m+1) \leq \binom{t+m-1}{t}$$

and $d(I^t, z_0) = 0$, then

$$|m+t-z_0| \leq m+t+1.$$

Proof. Let us consider the polynomial $p(n) = (-1)^{m+t} d(I^t, m+t-n)$

$$\begin{aligned} p(n) =& (-1)^{m+t} d(I^t, m+t-n) = \sum_{k=0}^{m+t} d(I^c, k) \binom{-t-m+n+k-1}{k} = \\ &= \sum_{k=0}^{m-1} d(I^c, k) \binom{n-m-t+k-1}{k} + \sum_{k=m}^{m+t} d(I^c, k) \binom{n-m-t+k-1}{k} \\ &= u(n) + \sum_{k=m}^{m+t} d(I^c, k) \binom{n-m-t+k-1}{k}. \end{aligned}$$

As a result of the proof of Proposition 4.15 we get that if $|z| > m$, then

$$|u(z+t+1)| = \left| \sum_{k=0}^{m-1} d(I^c, k) \binom{z-m+k}{k} \right| < \left| d(I^c, m) \binom{z}{m} \right|.$$

So if $|z| > m+t+1$, then $|z-(t+1)| > m$ and therefore

$$\begin{aligned} |u(z)| &\leq d(I^c, m) \left| \binom{z-t-1}{m} \right| \\ &\leq d(I^c, m) \left(\left| \binom{z-t}{m} \right| + \left| \binom{z-t-1}{m-1} \right| \right) \\ &= d(I^c, m) \left(\left| \frac{(m+t)\dots(m+1)}{z(z-1)\dots(z-t+1)} \right| + \left| \frac{(m+t)\dots m}{z(z-1)\dots(z-t)} \right| \right) \left| \binom{z}{m+t} \right| \\ &< d(I^c, m) \left(\frac{2m+1}{t+m+1} \right) \left| \binom{z}{m+t} \right| \end{aligned}$$

Let us assume that $p(z) = 0$ and $|z| > m + t + 1$, therefore

$$d(I^c, m+t)\binom{z}{m+t} = \sum_{k=m-1}^{m+t-1} (d(I^c, k+1) - d(I^c, k))\binom{z-m-t+k}{k} + u(z),$$

equivalently

$$\binom{z}{m+t} = \sum_{k=m-1}^{m+t-1} \frac{d(I^c, k+1) - d(I^c, k)}{d(I^c, m+t)}\binom{z-m-t+k}{k} + \frac{1}{d(I^c, m+t)}u(z).$$

Observe that the summation on the right hand side is a convex combination of some complex numbers, therefore its length can be bounded from above by the length of the longest vector, that is

$$\begin{aligned}\left|\sum_{k=m-1}^{m+t-1} \frac{d(I^c, k+1) - d(I^c, k)}{d(I^c, m+t)}\binom{z-m-t+k}{k} + \frac{1}{d(I^c, m+t)}u(z)\right| \leq \\ \left|\binom{z-m-t+m+t-1}{m+t-1}\right| + |u(z)| \\ < \frac{t+m}{t+m+1}\left|\binom{z}{m+t}\right| + \frac{d(I^c, m)}{d(I^c, m+t)}\left(\frac{2m+1}{t+m+1}\right)\left|\binom{z}{m+t}\right| \\ = \left(\frac{t+m}{t+m+1} + \frac{d(I^c, m)}{d(I^c, m+t)}\left(\frac{2m+1}{t+m+1}\right)\right)\left|\binom{z}{m+t}\right|\end{aligned}$$

We claim that

$$\frac{t+m}{t+m+1} + \frac{d(I^c, m)}{d(I^c, m+t)}\left(\frac{2m+1}{t+m+1}\right) \leq 1$$

equivalently

$$d(I^c, m)(2m+1) \leq d(I^c, m+t), \tag{4.8}$$

but this is exactly the statement of Lemma 4.17. Therefore we get that

$$\left|\binom{z}{m+t}\right| < \left(\frac{t+m}{t+m+1} + \frac{d(I^c, m)}{d(I^c, m+t)}\left(\frac{2m+1}{t+m+1}\right)\right)\left|\binom{z}{m+t}\right| \leq \left|\binom{z}{m+t}\right|,$$

and that is a contradiction. So we obtained that any root of $p(n)$ has length at most $m + t + 1$. Therefore if

$$0 = d(I^t, z_0) = d(m+t-(m+t-z_0)) = (-1)^{m+t}p(m+t-z_0),$$

then $|m + t - z_0| \leq m + t + 1$ □

Remark 4.19. *With some easy calculation one could get the smallest value $m_0(t)$, for each t, such that the conditions of the corresponding proposition is satisfied for any $m \geq m_0(t)$.*

Specifically it means that if $\max(I) > 10$*, then one of the conditions are satisfied. For* $\max(I) \leq 10$ *we refer to Figure 4.2, where we included all the possible roots of* $d(I,n)$*, depending on* $m = \max(I)$ *and regions ball (blue) of radius* m *around* 0*, ball (blue) of radius* $m+1$ *around* m *and ball (red) of radius* $(m+1)/2$ *around* $(m-1)/2$*.*

Observe that in Proposition 4.18 the crucial inequality was (4.8), and checking this condition for the these 84 cases we end up with 16 cases when (4.8) is not satisfied.

t	Corollary 4.15	Condition (4.6)	Condition (4.7)	Condition (4.8)
0	**1**	-	-	-
1	-	**3**	-	-
2	-	**6**	-	-
3	-	14	**8**	(3)
4	-	53	**3**	(2)
≥ 5	-	-	**1**	(1)

Table 4.1: Smallest values for $m_0(t)$, such that the corresponding conditions are satisfied for any $m \geq m_0(t)$. There are 84 I's, that do not satisfy any of the first 3 conditions, and there are 16 of them, that do not satisfy any of the 4 conditions.

By combining the previous four propositions and checking the uncovered cases of the table (see Figure 4.2) we obtaine the following theorem.

Theorem 4.20. *For any* $\emptyset \neq I$ *if* $d(I,z_0) = 0$*, then*

1. $|z_0| \leq m$
2. $|m - z_0| \leq m+1$

In particular, $-1 \leq \Re z_0 \leq m$

As the previous theorem shows, all the complex roots of $d(I^t,n)$ have their real parts in between -1 and $m+t$. In the following proposition we will show that if t is large enough, then all the roots of $d(I^t,n)$ are real.

Proposition 4.21. *Let* $I \neq \emptyset$*, such that* $m-1 \notin I$*. Then there exists a* $t_0 = t_0(I) \in \mathbb{N}$*, such that for any* $t > t_0$ *and* $v \in \{-1,0,\ldots,m+t\} \setminus \{m-1\}$ *there exists a unique root of* $d(I^t,n)$ *of distance 1/4 from* v*. In particular the roots of* $d(I^t,n)$ *are contained in the interval* $[-1,m+t]$*.*

Proof. The proof is based on Neumaier's Gershgorin type results on the location of roots of polynomials. For further reference see [55]. Let

$$p_t(n) = \frac{d(I^t, n)}{\prod_{i=1}^{t}(n-(m+i))}$$

and

$$T(n) = n(n-1)\dots(n-m+2)(n-m),$$

and let us fix the value of t.

Then the leading coefficient of p_t is

$$\frac{d(I^{t-1}, m+t)}{(m+t)!},$$

it has degree m, and for $v = 0, \dots, m-2, m$

$$|\alpha_v| = \frac{|d(I,v)|(m-1-v)}{v!(m-v)!\prod_{i=1}^{t}(m-v+i)} = \frac{|d(I,v)|(m-1-v)}{v!(m+t-v)!}.$$

Therefore

$$|r_v| = \frac{m}{2}\frac{|d(I,v)|(m-1-v)}{v!(m+t-v)!}\frac{(m+t)!}{d(I^{t-1},m+t)} = \frac{m}{2}\frac{|d(I,v)|(m-1-v)}{d(I^{t-1},m+t)}\binom{m+t}{v}.$$

If we are able to prove that $|r_v| \to 0$ as $t \to \infty$ for any $v = 0, \dots, m-2, m$, then we would be done.

In order to prove that we observe that

$$d(I^{t-1}, m+t) \geq d(I^-, m-1)\binom{m+t-1}{t},$$

since the set of permutations of $D(I^{t-1}, m+t)$ with the largest element at position m has size $d(I^-, m-1)\binom{m+t-1}{t}$. To see that, choose the largest element $m+t$ into the mth position, and take an arbitrary subset of $\{1, \dots, m+t-1\}$ after the mth position in a decreasing order, and take the rest as $D(I^-, m-1)$ on the first $m-1$ position through an order-preserving bijection of the base-set.

Therefore

$$|r_v| \leq \frac{m(m-1-v)}{2} \frac{|d(I,v)|}{d(I^-,m-1)} \frac{\binom{m+t}{v}}{\binom{m+t-1}{t}} =$$
$$\frac{m(m-1-v)}{2} \frac{|d(I,v)|}{d(I^-,m-1)} \frac{(m+t)(m-1)!}{v!} \frac{t!}{(m+t-v)!} =$$
$$C_{v,m} \frac{(m+t)t!}{(t+m-v)!}.$$

If $v = m$, then $|r_v| = 0$, since $d(I,m) = 0$.

If $v \in \{0, \ldots, m-2\}$, then

$$|r_v| \leq C_{v,m} \frac{a_{v,m}(t)}{b_{v,m}(t)},$$

where $a_{v,m}(t) = t + m$ is a polynomial of degree 1, and $b_{v,m}(t) = \prod_{i=1}^{(m-v)}(t+i)$ is a polynomial of degree at least 2. Therefore $C_{v,m}\frac{a_v(t)}{b_v(t)} \to 0$ as $t \to \infty$, i.e. $|r_v| \to 0$.

□

5 Some remarks and further directions

We described an interesting phenomenon in Section 2, namely that $c_k(I)$ and $(-1)^m d(I,-n)$ are non-negative integers. This result suggests that there might be some combinatorial proofs for them.

Question 4.22. *What do the coefficients $c_k(I)$ and evaluations $(-1)^m d(I,-n)$ count?*

There are two conjectures about the roots of the descent polynomial:

Proposition 4.23 (Conjecture 4.3. of [26])**.** *If z_0 is a root of $d(I,n)$, then*

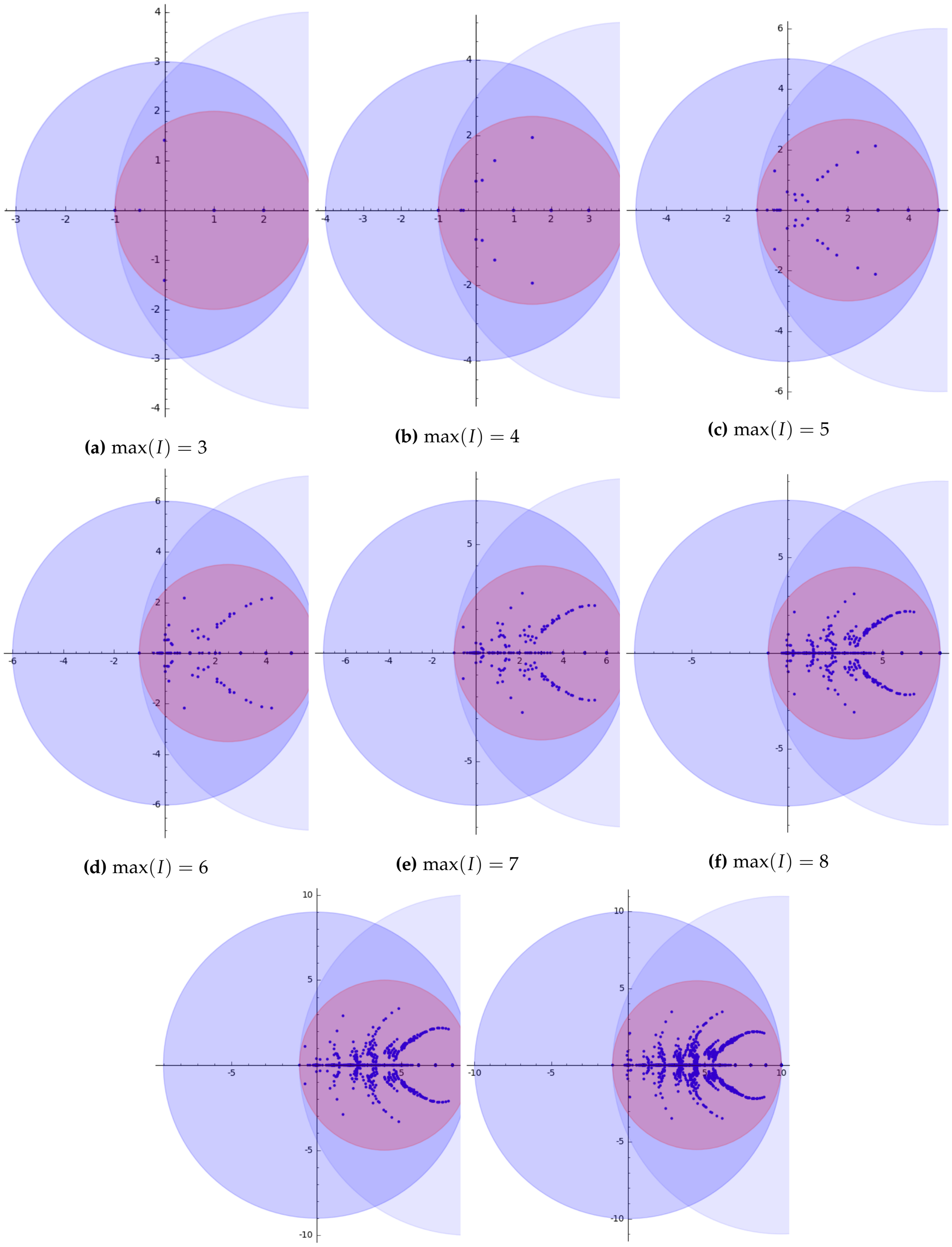
- $|z_0| \leq m$,
- $\Re z_0 \geq -1$.

This conjecture can be viewed as a special case of Theorem 4.20. As a common generalization of the two parts we conjecture that (motivated by numerical computations for $m \leq 13$ (e.g. see red regions on Figure 4.2), by a proof for the case $|I| = 1$ and by Proposition 4.21) the roots of $d(I,m)$ will be in a disk with the endpoints of one of its diameters being -1 and m. More precisely:

Conjecture 4.24. *If $d(I, z_0) = 0$, then $|z_0 - \frac{m-1}{2}| \leq \frac{m+1}{2}$.*

Similarly to the descent polynomial, instead of counting permutations with described descent set, one could ask for the number of permutations with described positions of peaks (i.e. $\pi_{i-1} < \pi_i > \pi_{i+1}$). As it turns out, this peak-counting function is not a polynomial. However, it can be written as a product of a polynomial and an exponential function in a "natural way". (See the precise definition in [10]). This polynomial is the so-called peak polynomial. This polynomial behaves quite similarly to the descent one, thus it is natural to ask whether there is a deeper connection between them, or whether we can prove similar propositions to the already obtained ones. In line with this we propose a conjecture about the coefficients in a base similar to $\bar{a}_k(I)$.

Conjecture 4.25. *For the peak-polynomial the coefficients in base $\{\binom{n-m+k}{k+1}\}_{k \in \mathbb{N}}$ form a symmetric, log-concave sequence of non-negative integers.*



(a) max$(I) = 3$ (b) max$(I) = 4$ (c) max$(I) = 5$

(d) max$(I) = 6$ (e) max$(I) = 7$ (f) max$(I) = 8$

(g) max$(I) = 9$ (h) max$(I) = 10$

Figure 4.2: Roots of $d(I,n)$ for $m = \max(I) \in \{3,\ldots,10\}$ and regions: ball (blue) of radius m around 0, ball (blue) of radius $m+1$ around m and ball (red) of radius $(m+1)/2$ around $(m-1)/2$

5

Invariant random subgroups of groups acting on rooted trees

For a countable discrete group Γ let Sub_Γ denote the compact space of subgroups $H \leq \Gamma$, with the topology induced by the product topology on $\{0,1\}^\Gamma$. The group Γ acts on Sub_Γ by conjugation. An *invariant random subgroup* (IRS) of Γ is a Borel probability measure on Sub_Γ that is invariant with respect to the action of Γ.

Examples include Dirac measures on normal subgroups and uniform random conjugates of finite index subgroups. More generally, for any p.m.p. action $\Gamma \curvearrowright (X, \mu)$ on a Borel probability space (X, μ), the stabilizer $\mathrm{Stab}_\Gamma(x)$ of a μ-random point x defines an IRS of Γ. Abért, Glasner and Virág [1] proved that all IRS's of Γ can be realized this way.

A number of recent papers have been studying the IRS's of certain countable discrete groups. Vershik [72] characterized the ergodic IRS's of the group $\mathrm{FSym}(\mathbb{N})$ of finitary permutations of a countable set. In [2] the authors investigate IRS's in lattices of Lie groups. Bowen [11] and Bowen-Grigorchuk-Shavchenko [12] showed that there exists a large "zoo" of IRS's of non-abelian free groups and the lamplighter groups $(\mathbb{Z}/p\mathbb{Z})^n \wr \mathbb{Z}$ respectively. Thomas and Tucker-Drob [69, 70] classified the ergodic IRS's of strictly diagonal limits of finite symmetric groups and inductive limits of finite alternating groups. Dudko and Medynets [27] extend this in certain cases to blockdiagonal limits of finite symmetric groups.

In this chapter we study the IRS's of groups of automorphisms of rooted trees. Let T be the infinite d-ary rooted tree, and let $\mathrm{Aut}(T)$ denote the group of automorphisms of T.

An *elementary* automorphism applies a permutation to the children of a given vertex, and moves the underlying subtrees accordingly. The group of *finitary* automorphisms $\mathrm{Aut}_f(T)$ is generated by the elementary automorphisms. The *finitary alternating automorphism group* $\mathrm{Alt}_f(T)$ is the one generated by even elementary automorphisms.

The group $\mathrm{Aut}(T)$ comes together with a natural measure preserving action. The *boundary* of T – denoted ∂T – is the space of infinite rays of T. It is a compact metric space with a continuous $\mathrm{Aut}(T)$ action and an ergodic invariant measure $\mu_{\partial T}$.

For some natural classes of groups IRS's tend to behave like normal subgroups. In [2] the Margulis Normal Subgroup Theorem is extended to IRS's, it is shown that every nontrivial ergodic IRS of a lattice in a higher rank simple Lie group is a random conjugate of a finite index subgroup. On the other hand, the finitary alternating permutation group FAlt($\mathbb{N}$) is simple, in particular it has no finite index subgroups, but as Vershik shows in [72] it admits continuum many ergodic IRS's.

The group $\mathrm{Alt}_f(T)$ is an interesting mixture of these two worlds. It is both locally finite and residually finite, and all its nontrivial normal subgroups are level stabilizers. The Margulis Normal Subgroup Theorem does not extend to IRS's, as the stabilizer of a random boundary point gives an infinite index ergodic IRS. However, once we restrict our attention to IRS's without fixed points, the picture changes.

Theorem 5.1. *Let H be a fixed point free ergodic IRS of $\mathrm{Alt}_f(T)$, with $d \geq 5$. Then H is the uniform random conjugate of a finite index subgroup. In other words H contains a level stabilizer.*

Note, that an IRS H is *fixed point free* if it has no fixed points on ∂T almost surely. In general let $\mathrm{Fix}(H)$ denote the closed subset of fixed points of H on ∂T.

When we do not assume fixed point freeness IRS's of $\mathrm{Alt}_f(T)$ start to behave like the ones in FAlt($\mathbb{N}$). In the case of FAlt($\mathbb{N}$), any nontrivial ergodic IRS contains a specific (random) subgroup that arises by partitioning the base space in an invariant random way and then taking the direct sum of deterministic subgroups on the parts. We proceed to define a (random) subgroup of $\mathrm{Alt}_f(T)$ which highly resembles these subgroups.

Every closed subset $C \subseteq \partial T$ corresponds to a rooted subtree T_C with no leaves. The complement of T_C in T is a union of subtrees $T_0, T_1, \dots$ as in Figure 5.1. Choose an integer m_i for each T_i, and let $\mathcal{L}_{m_i}(T_i)$ stand for the m_i^{th} level of the tree T_i. We define $L\big(C, (m_i)\big)$ to be the direct sum of level stabilizers in the T_i:

$$L\big(C, (m_i)\big) = \bigoplus_{i \in \mathbb{N}} \mathrm{Stab}_{\mathrm{Alt}_f(T)}\big(\mathcal{L}_{m_i}(T_i)\big).$$

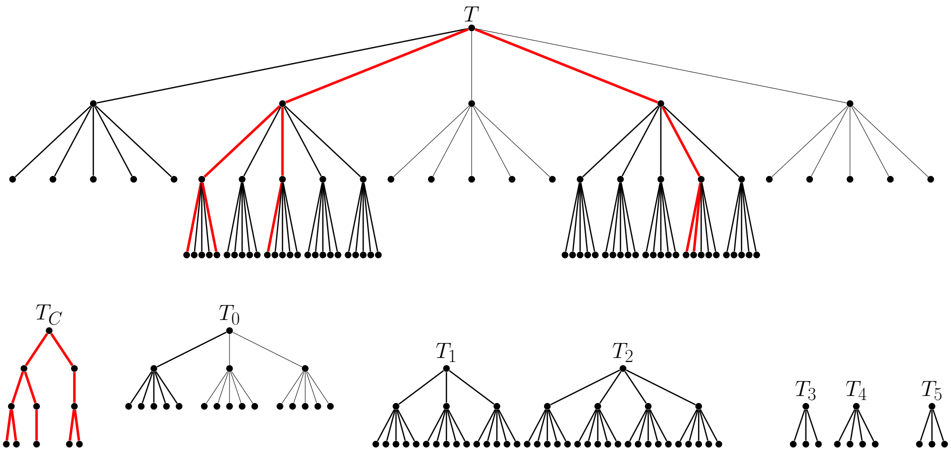


Figure 5.1: Decomposition of T with respect to C

It is easy to see that $\mathrm{Fix}\big(L\big(C,(m_i)\big)\big) = C$. We call such an $L\big(C,(m_i)\big)$ a *generalized congruence subgroup* with respect to the fixed point set C.

Let $\widetilde{C}$ be the translate of C with a Haar-random element from the compact group $\mathrm{Alt}(T) = \overline{\mathrm{Alt}_f(T)}$. Then $L\big(\widetilde{C},(m_i)\big)$ becomes an ergodic IRS of $\mathrm{Alt}_f(T)$ with fixed point set $\widetilde{C}$.

Theorem 5.2. *Let H be an ergodic IRS of $\mathrm{Alt}_f(T)$, with $d \geq 5$. Then $\mathrm{Fix}(H)$ is the Haar-random translate of a fixed closed subset C. Moreover, there exists (m_i) such that the generalized congruence subgroup $L\big(\mathrm{Fix}(H),(m_i)\big)$ is contained in H almost surely.*

We can exploit our methods to prove new results on branch groups as well. We postpone the formal definition of branch groups to Section 1. The examples to keep in mind are the groups $\mathrm{Aut}_f(T)$, $\mathrm{Alt}_f(T)$ and groups defined by finite automata, such as the first Grigorchuk group $\mathfrak{G}$.

In [9] Benli, Grigorchuk and Nagnibeda exhibit a group of intermediate growth with continuum many distinct atomless (continuous) ergodic IRS's. In the ergodic case being atomless means that the measure is not supported on a finite set. We extend this result to weakly branch groups in general.

Theorem 5.3. *Every weakly branch group admits continuum many distinct atomless ergodic IRS's.*

A key ingredient in Theorems 5.1 and 5.2 is to analyze the orbit-closures of IRS's on ∂T. For any subgroup $L \leq \mathrm{Aut}(T)$ taking the closures of orbits of L gives an equivalence relation on ∂T, that is L acts minimally on each class. It turns out that nontrivial orbit-closures of IRS's are necessarily clopen.

Theorem 5.4. *Let H be an ergodic IRS of a countable regular branch group Γ. Then almost surely all orbit-closures of H on ∂T that are not fixed points are clopen. In particular if H is fixed point free, then H has finitely many orbit-closures on ∂T almost surely.*

In a group Γ the *rigid stabilizer* of a vertex $v \in V(T)$ is the set $\mathrm{Rst}_\Gamma(v) \subseteq \Gamma$ of automorphisms that fix all vertices except the descendants of v. The rigid stabilizer of the level $\mathcal{L}_n$ is

$$\mathrm{Rst}_\Gamma(\mathcal{L}_n) = \prod_{v \in \mathcal{L}_n} \mathrm{Rst}_\Gamma(v).$$

In [39, Theorem 4] Grigorchuk showed that nontrivial normal subgroups in branch groups contain the derived subgroup $\mathrm{Rst}'_\Gamma\big(\mathcal{L}_m(T)\big)$ for some $m \in \mathbb{N}$. Our next theorem can be thought of as a generalization of this statement for finitary regular branch groups.

Using the decomposition of T with respect to C above we can define a *generalized rigid level stabilizer* $L(C, m_i)$ by taking the direct sum of the rigid level stabilizers $\mathrm{Rst}_\Gamma\big(\mathcal{L}_{m_i}(T_i)\big)$ instead of the $\mathrm{Stab}_\Gamma\big(\mathcal{L}_{m_i}(T_i)\big)$ we used before. The next theorem generalizes Theorem 5.1 and Theorem 5.2 for *finitary* regular branch groups.

Theorem 5.5. *Let Γ be a finitary regular branch group, and let H be a nontrivial ergodic IRS of Γ. Then $\mathrm{Fix}(H)$ is the Haar-random translate of a closed subset C with an element from $\overline{\Gamma}$. Also there exists (m_i) such that H almost surely contains the derived subgroup $L'\big(\mathrm{Fix}(H), (m_i)\big)$ of a generalized rigid level stabilizer. In particular if H is fixed point free, then H almost surely contains $\mathrm{Rst}'_\Gamma\big(\mathcal{L}_m(T)\big)$ for some $m \in \mathbb{N}$.*

Already in the case of $\mathrm{Aut}_f(T)$ with $d = 2$ the abelianization of $\mathrm{Aut}_f(T)$ equals $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. This itself gives rise to a lot of IRS's, which makes the following consequence of Theorem 5.5 somewhat surprising.

Theorem 5.6. *All ergodic fixed point free IRS's in finitary regular branch groups are supported on finitely many subgroups, and therefore are the uniform random conjugates of a subgroup with finite index normalizer.*

One can think of Theorem 5.6 as a dual of Theorem 5.3. Also note that merely containing $\mathrm{Rst}'_\Gamma\big(\mathcal{L}_m(T)\big)$ does not imply finite index normalizer.

In the Grigorchuk group $\mathfrak{G}$ the elements are not finitary. In this case our methods yield a weaker result on the closures of IRS's.

Theorem 5.7. *Let Γ be a countable regular branch group, and let H be a nontrivial ergodic IRS of Γ. Then there exists (m_i) such that $\overline{H}$ contains the derived subgroup $L'\big(\mathrm{Fix}(H), (m_i)\big)$*

of a generalized rigid level stabilizer almost surely, where the elements of the rigid stabilizers in $L\big(\mathrm{Fix}(H), (m_i)\big)$ *can be chosen from* $\overline{\Gamma}$ *instead of* Γ.

However, classifying IRS's of the discrete Grigorchuk group $\mathfrak{G}$ is still open.

Problem 5.8. *What are the (fixed point free) ergodic IRS's of the first Grigorchuk group* $\mathfrak{G}$*? Is it true, that a fixed point free ergodic IRS of* $\mathfrak{G}$ *contains a congruence subgroup almost surely?*

The structure of the chapter is as follows. We introduce the basic notions of the chapter in Section 1 and state some lemmas leading towards Theorem 5.4. In Section 2 we investigate the actions of IRS's on the boundary and prove Theorems 5.3 and 5.4. Section 3 is dedicated to understanding the structure of IRS's in finitary regular branch groups and proving Theorem 5.5. We show how Theorems 5.6 and 5.7 follow from our earlier results in Section 4. In the Appendix we prove a few technical details that we postpone during the exposition.

1 Preliminaries

In this section we introduce the basic notions discussed in the chapter. Notation mostly follows [5], which we recommend as an introduction to automorphisms of rooted trees and branch groups.

1.1 Automorphisms of rooted trees

Let T be a locally finite tree rooted at o, and let d_T denote the graph distance on T. For any vertex v the *parent* of v is the unique neighbor u of v with $d_T(u,o) = d_T(v,o) - 1$. Accordingly, the *children* of u are all the neighbors v of u with $d_T(v,o) = d_T(u,o)$. Similarly we use the phrases *ancestors* and *descendants* of a vertex v to refer to vertices that can be reached from v by taking some number of steps towards or away from the root respectively. The n^{th} level of T is the set of vertices $\mathcal{L}_n = \{v \in V(T) \mid d_T(v,o) = n\}$.

To effectively talk about automorphisms of a rooted tree T one has to distinguish the vertices. For any vertex v we fix an ordering of the children of v. In the case of the d-ary tree this corresponds to thinking of T as the set of finite length words Y^* over the alphabet Y with d letters. The empty word represents the root, and the parent of any word $w_1 w_2 \dots w_n$ is $w_1 w_2 \dots w_{n-1}$. Being an ancestor of v corresponds to being a prefix of the word corresponding to v.

An automorphism of T (which preserves the root) corresponds to a permutation of the words which preserves the prefix relation. For an element $\gamma \in \mathrm{Aut}(T)$ and a word $w \in Y^*$ we denote by w^γ the image of w under γ. For a letter $y \in Y$ we have $(wy)^\gamma = w^\gamma y'$ where y' is a uniquely determined letter in Y. The map $y \mapsto y'$ is a permutation of Y, we refer to it as the *vertex permutation* of γ at w and denote it $(w)\gamma$.

Considering all the vertex permutations $\big((w)\gamma\big)_{w\in Y^*}$ gives us the *portrait* of γ, which is a decoration of the vertices of T with elements from the symmetric group S_d. In turn any assignment of these vertex permutations – that is, every possible portrait – gives an automorphism of T. Note that one has to perform these vertex permutations "from bottom to top".

An automorphism γ is finitary, if it has finitely many nontrivial vertex permutations. It is alternating, if all are from the alternating group A_d.

Let $S_d^{\mathrm{wr}(n)}$ denote the n-times iterated permutational wreath product of the symmetric group S_d. That is, let $[d] = \{1, \ldots, d\}$ and set

$$S_d^{\mathrm{wr}(n)} = \underbrace{((S_d \wr_{[d]} \ldots) \wr_{[d]} S_d) \wr_{[d]} S_d}_{n}.$$

Then $S_d^{\mathrm{wr}(n)}$ is isomorphic to the automorphism group of the d-ary rooted tree of depth n. These groups can be embedded in $\mathrm{Aut}(T)$ as acting on the first n levels. The group Aut_f is the union of these embedded finite groups. The full automorphism group $\mathrm{Aut}(T)$ however is isomorphic to the projective limit $\varprojlim S_d^{\mathrm{wr}(n)}$ with the projections being the natural restrictions of the permutations.

The groups $\mathrm{Alt}_f(T)$ and $\mathrm{Alt}(T)$ are in a similar relationship with the finite groups $A_d^{\mathrm{wr}(n)}$.

1.2 The boundary of T

The *boundary* of T is the set of infinite paths starting from o, and is denoted ∂T. For two distinct paths $p_1 = (u_0, u_1, \ldots)$ and $p_2 = (v_0, v_1, \ldots)$ with $u_n, v_n \in \mathcal{L}_n$ their distance is defined to be

$$d_{\partial T}(p_1, p_2) = \frac{1}{2^k}, \text{ where } k = \max\{n \mid u_n = v_n\}.$$

Two infinite paths are close it they have a long common initial segment. This distance turns ∂T into a compact, totally disconnected metric space.

The *shadow* of v on ∂T, denoted by $\mathrm{Sh}(v)$ is the set of paths passing through v. Similarly the shadow of v on $\mathcal{L}_n$ is the set $\mathrm{Sh}_{\mathcal{L}_n}(v)$ of descendants of v in $\mathcal{L}_n$. The sets $\mathrm{Sh}(v)$ form a basis for the topology of ∂T. Define the probability measure $\mu_{\partial T}$ by setting its value on this basis:

$$\mu_{\partial T}\big(\mathrm{Sh}(v)\big) = \frac{1}{d^n} \text{ for every } v \in \mathcal{L}_n.$$

A $\mu_{\partial T}$-random point of ∂T is a random infinite word $(w_1 w_2 \ldots)$ with each letter chosen uniformly from the set Y.

As $\gamma \in \mathrm{Aut}(T)$ permutes the vertices, it induces a bijection on ∂T, so we have an action of $\mathrm{Aut}(T)$ on ∂T. This action is by isometries and preserves the measure $\mu_{\partial T}$.

The objects in relation of the tree considered in this chapter include vertices $v \in V(T)$, points $x \in \partial T$, closed subsets $C \subseteq \partial T$ and later 3-colorings of the vertices $\varphi : V(T) \to \{r, g, b\}$. For any such object z let z^γ denote its translate by γ.

1.3 Topology on $\mathrm{Aut}(T)$

We equip $\mathrm{Aut}(T)$ with the topology of pointwise convergence. This can be metrized by the following distance:

$$d_{\mathrm{Aut}(T)}(\gamma_1, \gamma_2) = \frac{1}{2^k}, \text{ where } k = \max\{n \mid \gamma_1|_{\mathcal{L}_n} = \gamma_2|_{\mathcal{L}_n}\}.$$

Two automorphisms are close if they act the same way on a deep level of T. This metric turns $\mathrm{Aut}(T)$ into a compact, totally disconnected group.

The action $\mathrm{Aut}(T) \curvearrowright \partial T$ is continuous in the first coordinate as well. For any subgroup $H \leq \mathrm{Aut}(T)$ the set $\mathrm{Fix}(H)$ is closed in ∂T, and similarly for any set $C \subseteq \partial T$ its pointwise stabilizer $\mathrm{Stab}_\Gamma(C)$ is closed in $\mathrm{Aut}(T)$.

For a subgroup $\Gamma \leq \mathrm{Aut}(T)$ its closure $\overline{\Gamma}$ is a closed subgroup of $\mathrm{Aut}(T)$, and therefore it is compact. We note that $\overline{\mathrm{Aut}_f(T)} = \mathrm{Aut}(T)$ and $\overline{\mathrm{Alt}_f(T)} = \mathrm{Alt}(T)$. Even though the groups Γ we are considering are discrete, their closures in $\mathrm{Aut}(T)$ always carry a unique Haar probability measure.

For any object z in relation to the tree we write $\widetilde{z}$ for its Haar random translate, that is z^γ where $\gamma \in \overline{\Gamma}$ is chosen randomly according to the Haar measure.

1.4 Fixed points and orbit-closures in ∂T

We aim to understand the IRS's of Γ through their actions on ∂T. The first step is to look at the set of fixed points. The boundary $(\partial T, d_{\partial T})$ is a compact metric space, so let $(\mathcal{C}, d_H)$ denote the compact space of closed subsets of ∂T with the Hausdorff metric.

Lemma 5.9. *The map $H \mapsto \mathrm{Fix}(H)$ is a measurable and Γ-equivariant map from Sub_Γ to $(\mathcal{C}, d_H)$.*

Equivariance is trivial, while the proof of the measurability is a standard argument. We postpone it to the Appendix.

Lemma 5.9 implies that the fixed points of the IRS constitute a Γ-invariant random closed subset of ∂T. We will also consider the orbit-closures of the subgroup on ∂T. For a subgroup $H \le \mathrm{Aut}(T)$ let $\mathcal{O}_H$ denote the set of orbit-closures of the action $H \curvearrowright \partial T$. It is easy to see that $\mathcal{O}_H$ is a partition of ∂T into closed subsets. Note that all fixed points are orbit-closures. Denote by $\mathcal{O}$ the space of all possible orbit-closure partitions on ∂T, i.e. $\mathcal{O} = \{\mathcal{O}_H \mid H \le \mathrm{Aut}(T)\}$. This $\mathcal{O}$ is a subset of all the possible partitions of ∂T.

As earlier, we would like to argue that the map $H \mapsto \mathcal{O}_H$ is a measurable map, with respect to the appropriate measurable structure on $\mathcal{O}$. This allows us to associate to our IRS a Γ-invariant random partition (into closed subsets) of ∂T. We will then analyze these invariant random objects on the boundary.

To this end we introduce a metric on the space $\mathcal{O}$. Denote by $\mathcal{O}_{H,\mathcal{L}_n}$ the partition of $\mathcal{L}_n$ into H-orbits. As $\mathcal{L}_n$ is finite, there is no need to take closure here.

Definition. *Let $P = \mathcal{O}_H \in \mathcal{O}$ be the orbit-closure partition of H and $n \in \mathbb{N}$. Then let P_n be the orbit-structure of H on $\mathcal{L}_n$, i.e.*

$$P_n = \mathcal{O}_{H,\mathcal{L}_n}.$$

For $P \neq Q \in \mathcal{O}$ let

$$d_{\mathcal{O}}(P,Q) = \min_{n\in\mathbb{N}} \left\{ \frac{1}{2^n} \mid P_n = Q_n \right\}.$$

Observe that if $P_n = Q_n$, then $P_{n-1} = Q_{n-1}$, so the above distance measures how deep one has to go in the tree to see that two partitions are distinct. This definition turns $(\mathcal{O}, d_{\mathcal{O}})$ into a metric space. To check that distinct points cannot have zero distance we argue that if $x = (v_0, v_1, \ldots)$ and $y = (u_0, u_1, \ldots)$ are two rays such that v_n and u_n are in the same orbit in $\mathcal{L}_n$ for all n, then y is indeed in the closure of the orbit of x.

The group $\mathrm{Aut}(T)$ acts on $\mathcal{O}$ in a natural way by shifting the sets of the partition. The resulting partition is again in $\mathcal{O}$ because $(\mathcal{O}_H)^\gamma = \mathcal{O}_{H^\gamma}$ for $\gamma \in \mathrm{Aut}(T)$.

Lemma 5.10. *The map $H \mapsto \mathcal{O}_H$ is measurable and Γ-equivariant.*

Again, equivariance is obvious, and measurability is proved in the Appendix.

1.5 Invariant random objects on ∂T

Now we study invariant random closed subsets and partitions on the boundary. We show that the invariance can be extended to $\overline{\Gamma}$, which carries a Haar measure. Ergodic objects turn out to be random translates according to this Haar measure.

Lemma 5.11. *Every Γ-invariant random closed subset of ∂T is in fact $\overline{\Gamma}$-invariant. Similarly a Γ-invariant random $P \subseteq \mathcal{O}$ is $\overline{\Gamma}$-invariant.*

Proof. Let $P(\mathcal{C})$ denote the set of probability measures on $\mathcal{C}$. The action of $\overline{\Gamma}$ on ∂T gives rise to a translation action on $(\mathcal{C}, d_H)$, which in turn gives rise to an action on $P(\mathcal{C})$.

We claim that this action $\overline{\Gamma} \times P(\mathcal{C}) \to P(\mathcal{C})$ is continuous in both coordinates with respect to the pointwise convergence topology on $\overline{\Gamma}$ and the weak star topology on $P(\mathcal{C})$.

The weak topology on $P(\mathcal{C})$ is metrizable by the Lévy - Prokhorov metric, which is defined as follows:

$$\pi(\mu,\nu) = \inf\{\varepsilon > 0 \mid \mu(A) \leq \nu(A^\varepsilon) + \varepsilon, \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for all } A \subseteq \mathcal{C} \text{ Borel}\}.$$

Here A^ε denotes the set elements of $\mathcal{C}$ with d_H distance at most ε from A.

If γ_1 and γ_2 agree on the first n levels of T, then for every $x \in \partial T$ we have $d(\gamma_1 x, \gamma_2 x) \leq 1/2^n$. This implies, that for a compact set $C \in \mathcal{C}$ we have $d_H(\gamma_1 C, \gamma_2 C) \leq 1/2^n$. This in turn implies that for all $A \subseteq \mathcal{C}$ Borel we have $\gamma_1 A \subseteq \gamma_2 A^{1/2^n}$ and vice versa.

This means, that $\big((\gamma_1)_*\mu\big)(A) = \mu(\gamma_1^{-1}A) \leq \mu(\gamma_2^{-1}A^{1/2^n}) = \big((\gamma_2)_*\mu\big)(A^{1/2^n})$, so as a consequence $\pi\big((\gamma_1)_*\mu, (\gamma_2)_*\mu\big) \leq 1/2^n$. That is, the action is continuous in the first coordinate.

Continuity in the second coordinate is an easy exercise, as it turns out that the elements of $\overline{\Gamma}$ act by isometries on $(\partial T, d_{\partial T})$, $(\mathcal{C}, d_H)$ and $\big(P(\mathcal{C}), \pi\big)$ respectively.

As Γ is a dense subset of $\overline{\Gamma}$, by continuity we can say that if some $\mu \in P(\mathcal{C})$ is Γ-invariant then it is also $\overline{\Gamma}$-invariant, thus proving the statement for invariant random closed subsets.

The proof for invariant random partitions follows the exact same steps after substituting $(\mathcal{C}, d_H)$ with $(\mathcal{O}, d_{\mathcal{O}})$ everywhere. $\square$

Remark. *The fact that the same lemma holds with the same proof for closed subsets and partitions is not a coincidence. A closed subset C can be thought of as a partition into the two sets C and C^c (the complement might not be closed). While it is not generally true that this partition is in $\mathcal{O}$ – it might not arise as an orbit-closure partition of some $H \subseteq \mathrm{Aut}(T)$ – but it still can be approximated on the finite levels. Indeed define C_n to be the set of vertices v on $\mathcal{L}_n$ with $\mathrm{Sh}(v) \cap C \neq \emptyset$. The C_n correspond to the $(1/2^n)$-neighborhoods of C in ∂T, and so the Hausdorff distance of closed subsets coincides with the distance $d_{\mathcal{O}}$ we could define using these C_n.*

Lemma 5.12. *Any ergodic $\overline{\Gamma}$-invariant random closed subset of ∂T is the γ translate of a fixed closed subset C, where $\gamma \in \overline{\Gamma}$ is a uniform random element chosen according to the Haar measure. Similarly an ergodic $\overline{\Gamma}$-invariant partition from $\mathcal{O}$ is the Haar-random translate of some fixed $P \in \mathcal{O}$.*

Proof. We introduce an equivalence relation on closed subsets of ∂T: we say that $C_1 \sim C_2$ if and only if there is an automorphism $\gamma \in \overline{\Gamma}$ such that $C_1^\gamma = C_2$. Let $[C]$ denote the equivalence class of C.

Define the following metric on equivalence classes that measures how well one can overlap two arbitrary sets from the classes:

$$d([C_1], [C_2]) = \min_{\gamma \in \overline{\Gamma}} \{ d_H(C_1^\gamma, C_2) \}.$$

The minimum exists by compactness of $\overline{\Gamma}$, and standard arguments using the fact that $\overline{\Gamma}$ acts by isometries on $(\mathcal{C}, d_H)$ show that this is well defined and indeed a metric.

The function $C \to [C]$ is measurable (in fact continuous) and $\overline{\Gamma}$-invariant, hence it is almost surely a constant by the ergodicity of the measure.

In other words the measure is concentrated on one equivalence class, say $[C]$. However $[C]$ is a homogeneous space of $\overline{\Gamma}$, i.e. the action of $\overline{\Gamma}$ on $[C]$ is the same as $\overline{\Gamma} \curvearrowright \overline{\Gamma}/\mathrm{Stab}_{\overline{\Gamma}}(C)$. $\mathrm{Stab}_{\overline{\Gamma}}(C)$ is a closed and therefore compact subgroup of $\overline{\Gamma}$, and as such $\overline{\Gamma}/\mathrm{Stab}_{\overline{\Gamma}}(C)$ carries a unique invariant measure. Of course picking a random translate of C is an invariant measure, so the two must coincide.

The result for partitions again follows word for word, by writing P for C and $(\mathcal{O}, d_{\mathcal{O}})$ for $(\mathcal{C}, d_H)$ everywhere. $\square$

Remark. *A way to put the previous lemmas into a general framework is the following: let G be a metrizable compact group acting continuously on a compact metric space (X, d) and let*

$\Gamma \leq G$ be a dense subgroup. Then any Γ-invariant measure on X is also G-invariant. Moreover if the metric d is G-compatible, then any ergodic Γ-invariant measure on X has the distribution of a random G-translate of a fix element in X.

If the IRS $H \leq \Gamma$ is ergodic, then so is the associated invariant random closed subset. This means that $\mathrm{Fix}(H)$ is the random translate of a fixed closed subset C. Similarly $\mathcal{O}_H$ is the random translate of some partition P.

1.6 Branch Groups

For a vertex v of T let T_v denote the induced subtree of T on v and its descendants. We denote by $\mathrm{Stab}_\Gamma(v)$ the stabilizer of v in Γ. Every $\gamma \in \mathrm{Stab}_\Gamma(v)$ acts on T_v by an automorphism, which we denote γ_v. Then $U_v = \{\gamma_v \mid \gamma \in \mathrm{Stab}_\Gamma(v)\}$ is a subgroup of $\mathrm{Aut}(T_v)$. U_v is the group of automorphisms of T_v that are realized by some element of Γ.

The trees we are considering are regular, so T_v is canonically isomorphic to T. (The isomorphism preserves the ordering of the vertices on each level. If we think of the vertices as finite words over a fixed alphabet, then this isomorphism just deletes the initial segment of each word in T_v.) This identification of the trees allows us to compare the action of G on T to the action of U_v on T_v. In particular we say that Γ is a *fractal* group, if U_v is equal to G for all $v \in V(T)$ (under the above identification of the trees they act on).

For a vertex $v \in V(T)$ let $\mathrm{Rst}_\Gamma(v)$ denote the *rigid stabilizer* of v, that is the subgroup of elements of Γ that fix every vertex except the descendants of v. Clearly $\mathrm{Rst}_\Gamma(v) \leq \mathrm{Stab}_\Gamma(v)$. For a subset of vertices $V \subseteq \mathcal{L}_n$ the rigid stabilizer of the set is $\mathrm{Rst}_\Gamma(V) = \prod_{v \in V} \mathrm{Rst}_\Gamma(v)$.

Throughout the chapter we will be able to prove statements in varying levels of generality, so we introduce several notions of branching. In all cases we assume Γ to be transitive on all levels. We say that Γ is *weakly branch*, if all rigid vertex stabilizers $\mathrm{Rst}_\Gamma(v)$ are nontrivial. We say that the group Γ is *branch*, if for all n the rigid level stabilizer $\mathrm{Rst}_\Gamma(\mathcal{L}_n)$ is a finite index subgroup of Γ. Finally we define *regular branch groups*.

Definition. *Suppose the fractal group Γ has a finite index subgroup K. The group K^d is a subgroup of* $\mathrm{Aut}(T)$*, each component acting independently on T_{v_i} where $\{v_1, \ldots, v_d\}$ are the vertices on $\mathcal{L}_1$. We say that Γ is a* regular branch group *over K, if K contains K^d as a finite index subgroup.*

In fractal groups the action on any subtree T_v is the same as on T, however for some $v_1, v_2 \in \mathcal{L}_n$ we might not be able to move T_{v_1} and T_{v_2} independently. This independence (up to finite index) is what we required in the definition above. The following lemma is straightforward and we leave the proof to the reader.

Lemma 5.13. *Having finite index and being a direct product remains to be true after taking closures:*

1. *Let $K \subseteq \Gamma$ be a subgroup of finite index. Then $\overline{K}$ is a finite index subgroup of $\overline{\Gamma}$;*
2. $\underbrace{\overline{K \times \cdots \times K}}_{d} = \underbrace{\overline{K} \times \cdots \times \overline{K}}_{d}$.

2 Fixed points and orbit-closures of IRS's

In this section we prove Theorems 5.3 and 5.4. To a closed subset C on the boundary one can associate two natural subgroup of Γ, the pointwise stabilizer of C and the setwise stabilizer of C. The pointwise stabilizer gives us a big "zoo" of IRS's when we choose C as a $\overline{\Gamma}$-invariant closed subset, proving Theorem 5.3. The setwise stabilizer will play a key role in the proof of Theorem 5.4.

In order to investigate these stabilizers we introduce a coloring to encode C on the tree T. The coloring will help analyzing the Haar random translate $\widetilde{C}$.

2.1 Closed subsets of the boundary

To every closed subset of the boundary $C \subseteq \partial T$ we associate a vertex coloring $\varphi : V(T) \to \{r, g, b\}$ with 3 colors: red, green and blue. If a vertex has its shadow completely in C, then color it red. If it has its entire shadow in the complement of C, then color it blue. Otherwise color it green.

$$\varphi(v) = \begin{cases} r, & \text{if } \mathrm{Sh}(v) \subseteq C; \\ b, & \text{if } \mathrm{Sh}(v) \cap C = \emptyset; \\ g, & \text{otherwise.} \end{cases}$$

All descendants of a red vertex are red, and similarly all descendants of blue vertices are blue. On the other hand all ancestors of a green vertex are green.

C being clopen is equivalent to saying that after some level all vertices are either red or blue. So if C is not clopen, then there are green vertices on all the levels. Using König's lemma we see that there is an infinite ray with vertices colored green. This ray corresponds to a boundary point of C. As the complement of C is open, we get that every vertex on this infinite ray has a blue descendant.

Lemma 5.14. *Let $\Gamma \subseteq \mathrm{Aut}(T)$ be a group of automorphisms that is transitive on every level. Let $\varphi : V(T) \to \{r, g, b\}$ be a vertex coloring with the colors red, green and blue, and suppose it satisfies the above properties, namely:*

1. *descendants of red and blue vertices are red and blue respectively;*
2. *ancestors of green vertices are green; (This formally follows from 1.)*
3. *there is an infinite ray $(u_0, u_1, \dots)$ of green vertices such that for each u_i there exists some descendant of u_i which is blue.*

Then φ has infinitely many Γ-translates.

Proof. The root u_0 is colored green. It has a blue descendant, say on the level n_1. We denote this blue descendant w_{n_1}. By the transitivity assumption there is some $\gamma_1 \in \Gamma$, such that $\gamma_1(w_{n_1}) = u_{n_1}$. Furthermore, u_{n_1} has a blue descendant, on some level n_2, we denote it w_{n_2}. We choose $\gamma_2 \in \Gamma$ such that $\gamma_1(w_{n_1}) = u_{n_1}$, and so on. One can easily check, that moving φ with the different γ_i yields different colorings. Indeed $\varphi^{\gamma_i}(u_{n_j}) = g$ for all $j < i$, and $\varphi^{\gamma_i}(u_{n_i}) = b$, and this shows that the φ^{γ_i} are pairwise distinct. See Figure 5.2. $\square$

Corollary 5.15. *Let Γ and φ be as in Lemma* 5.14*. Then the uniform (Haar) random $\overline{\Gamma}$-translate of φ is an atomless measure on the space of all 3-vertex-colorings.*

Proof. If there was some translate φ^g, $g \in \overline{\Gamma}$ which occurred with positive probability, then all its Γ-translates would occur with the same positive probability. Furthermore φ^g would also satisfy the assumptions of Lemma 5.14, which then implies that it has infinitely many Γ-translates, and they would have infinite total measure, which is a contradiction. $\square$

Corollary 5.16. *If C is not clopen, then its random $\overline{\Gamma}$-translate $\widetilde{C}$ is an atomless measure on $(\mathcal{C}, d_H)$.*

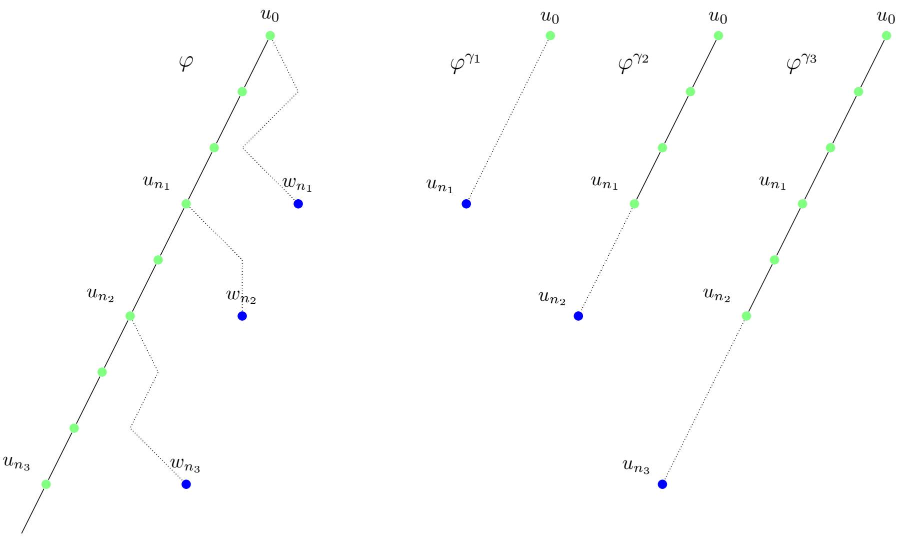


Figure 5.2: Distinct colorings

2.2 Continuum many distinct atomless ergodic IRS's in weakly branch groups

Proof of Theorem 5.3. We argue that for any closed subset $C \subseteq \partial T$ the random subgroup $\mathrm{Stab}_\Gamma(\widetilde{C})$ is an ergodic IRS. This follows from $\widetilde{C}$ being an ergodic invariant random closed subset.

We also claim that if $[C_1] \neq [C_2]$, then the corresponding IRS's are distinct. To prove this we first observe that in weakly branch groups taking the stabilizer $\mathrm{Stab}_\Gamma(C)$ of a closed subset C, and then looking at the fixed points of that subset we get back C.

Lemma 5.17. *For any $C \subseteq \partial T$ closed we have* $\mathrm{Fix}\big(\mathrm{Stab}_\Gamma(C)\big) = C$.

Proof. The key idea – present in [9, Proposition 8] and earlier works credited there – is to show, that for any $x \notin C$, with $x = (u_0, u_1, \ldots)$ we can find some n large enough such that $\mathrm{Sh}(u_n) \cap C = \emptyset$, and some $\gamma \in \mathrm{Rst}_\Gamma(u_n)$ with $x^\gamma \neq x$.

Indeed such an n exists as the complement of C is open. By weak branching there exists some $\gamma_0 \in \mathrm{Rst}_\Gamma(u_n)$ moving some descendant of u denoted v to $v' \neq v$ on $\mathcal{L}_m$, $m \geq n$. By transitivity we can find some $\eta \in \mathrm{Stab}_\Gamma(u_n)$ with $v = u_m^\eta$. Now $u_m^{\eta\gamma_0\eta^{-1}} = (v')^{\eta^{-1}} \neq u_m$, so $\gamma = \eta\gamma_0\eta^{-1} \in \mathrm{Rst}_\Gamma(u_n)$, and $x^\gamma \neq x$ as witnessed on $\mathcal{L}_m$.

As $\mathrm{Sh}(u_n) \cap C = \emptyset$ we have $\mathrm{Rst}_\Gamma(u_n) \subseteq \mathrm{Stab}_\Gamma(C)$. The existence of γ shows that $x \notin \mathrm{Fix}\big(\mathrm{Stab}_\Gamma(C)\big)$, which implies $\mathrm{Fix}\big(\mathrm{Stab}_\Gamma(C)\big) \subseteq C$, which is the nontrivial inclusion. □

To show that $[C_1] \neq [C_2]$ implies that $\mathrm{Stab}_\Gamma(\widetilde{C_1})$ and $\mathrm{Stab}_\Gamma(\widetilde{C_2})$ distinct simply consider the function $H \mapsto [\mathrm{Fix}(H)]$ on ergodic IRS's. Using Lemma 5.17 we have

$$\big[\mathrm{Fix}\big(\mathrm{Stab}_\Gamma(\widetilde{C_1})\big)\big] = [\widetilde{C_1}] = [C_1].$$

This implies that the constructed IRS are distinct. By Corollary 5.16 we know that if C is not clopen then $\widetilde{C}$ is atomless. Then Lemma 5.17 implies that $\mathrm{Stab}_\Gamma\widetilde{C}$ is an atomless IRS.

There are continuum many non-$\overline{\Gamma}$-equivalent closed (but not clopen) subsets of ∂T, as one can construct a closed subset C_r with $\mu_{\partial T}(C_r) = r$ for any $r \in [0,1]$, and if r is irrational then C is not clopen. $\square$

2.3 Random colorings in regular branch groups

We proceed to prove a stronger versions of Corollary 5.15 for the case when the group is regular branch.

Let Γ be a regular branch group over K. Consider the finite index subgroup $\overline{K^{d^n}} \leq \mathrm{Stab}_{\overline{\Gamma}}(\mathcal{L}_n) = \cap_{v\in\mathcal{L}_n}\mathrm{Stab}_{\overline{\Gamma}}(v)$, and let $\{t_1,\dots,t_l\}$ be a transversal to $\overline{K^{d^n}}$ in $\overline{\Gamma}$. We can think of a random element γ of $\overline{\Gamma}$ as $\gamma = \gamma_0 \cdot k$, where γ_0 is chosen uniformly from the transversal and k is chosen according to the Haar measure on $\overline{K^{d^n}}$.

Take φ to be a 3-vertex-coloring as in Lemma 5.14. Let $\widetilde{\varphi} = \varphi^\gamma$ denote the translate of φ by the Haar random group element γ. Conditioning on $\gamma_0 = t_i$ we get a conditional distribution $(\widetilde{\varphi}|\gamma_0 = t_i)$. Note that this random coloring is always the same up to the n^{th} level, and t_i already determines where the random translate of the infinite green ray $(u_0, u_1, \dots)$ intersects $\mathcal{L}_n$, namely at $v = u_n^{t_i}$.

Lemma 5.18. *The restriction of the random coloring $(\widetilde{\varphi}|\gamma_0 = t_i)$ to T_v is atomless.*

Proof. The coloring φ^{t_i} restricted to T_v satisfies the assumptions of Lemma 5.14. Hence its Γ-orbit is infinite. As K is finite index in Γ, its K orbit is also infinite. Hence when it is randomly translated with an element from $\overline{K}$ (corresponding to v in $\overline{K}^{d^n}$) the resulting random coloring is atomless. $\square$

Lemma 5.19. *Fix any isomorphism $f : V(T_v) \to V(T_{v'})$ between T_v and $T_{v'}$ for some $v' \in \mathcal{L}_n$. Then the probability that f respects the colorings we get by restricting $(\widetilde{\varphi}|\gamma_0 = t_i)$ to T_v and $T_{v'}$ respectively is 0.*

$$\mathbb{P}\Big[\big((\widetilde{\varphi}|\gamma_0 = t_i)|_{T_v}\big)^f = (\widetilde{\varphi}|\gamma_0 = t_i)|_{T_{v'}}\Big] = 0.$$

Proof. The restricted colorings $\varphi^{t_i}|_{T_v}$ and $\varphi^{t_i}|_{T_{v'}}$ are translated by the random elements $k_1, k_2 \in \overline{K}$ respectively. These k_1 and k_2 are independent since they are two coordinates of a Haar random element from $\overline{K}^{d^n}$. Furthermore we know from Lemma 5.18 that $(\varphi^{t_i}|_{T_v})^{k_1}$ is atomless, and hence $\left((\varphi^{t_i}|_{T_v})^{k_1}\right)^f$ is also atomless. This together with the independence of k_1 and k_2 implies that

$$\mathbb{P}\left[\left((\varphi^{t_i}|_{T_v})^{k_1}\right)^f = (\varphi^{t_i}|_{T_{v'}})^{k_2}\right] = 0.$$

□

2.4 Proof of Theorem 5.4

The idea of the proof is to show that taking the *setwise* stabilizer of a Haar random translate $\widetilde{C}$ of a closed but not clopen subset C has a fixed point in $\widetilde{C}$. With some considerations one can apply this to the orbit-closures of H, which are setwise stabilized by H.

Proposition 5.20. *Let Γ be a countable regular branch group over K. Suppose C is a closed subset of ∂T. Consider the IRS $L \leq \Gamma$ obtained by taking the setwise stabilizer of $\widetilde{C}$, which is the uniform $\overline{\Gamma}$-translate of C. If C is not clopen, then L has a fixed point in $\widetilde{C}$ almost surely.*

Proof. Associate the coloring $\varphi : V(T) \to \{r, g, b\}$ to C as before: vertices with shadows contained in C are colored red, vertices with shadows in the complement are colored blue, everything else is colored green. As automorphisms move the set C the coloring moves with it.

Choose a point $x_0 \in \partial T$ which is on the boundary of C, that is $x_0 \in C \setminus \operatorname{int}(C)$. Being a boundary point means that every vertex on the path $(u_0, u_1, \ldots)$ corresponding to x_0 is green, and we can find a blue vertex among the descendants of u_i for all i.

Let $\widetilde{C}$, $\widetilde{\varphi}$ and $\widetilde{x}_0$ denote the uniform random translates of C, φ and x_0 respectively.

Fix an element $\eta \in \Gamma$. We will study the probability that η stabilizes $\widetilde{C}$ and does not fix $\widetilde{x}_0$, and conclude that it is 0. If η stabilizes $\widetilde{C}$ then it preserves $\widetilde{\varphi}$.

First assume η is finitary, that is we can find a level n with vertices $\mathcal{L}_n = \{v_1, \ldots, v_{d^n}\}$ such that η moves the subtrees T_{v_i} hanging off the n^{th} level rigidly. The condition that $\widetilde{x}_0$ is moved has to be witnessed on $\mathcal{L}_n$. Assume $v_1, \ldots, v_l$ are moved by η and $v_{l+1}, \ldots, v_{d^n}$ are fixed.

Let us assume that $\widetilde{\varphi}$ is preserved by η, and the ray corresponding to $\widetilde{x}_0$ is moved by η. Then $\widetilde{x}_0 \cap \mathcal{L}_n = v_i$ for some $i \leq l$ with $v_j = \eta v_i \neq v_i$. Conditioning on this v_i we are looking for the probability that $(\widetilde{\varphi}|_{T_{v_i}})^\eta = \widetilde{\varphi}|_{T_{v_j}}$. However, as the colorings are uniform random translates on T_{v_i} and T_{v_j} respectively, the probability of the coloring appearing in the exact same way under two points is 0. In the case of $\Gamma = \mathrm{Alt}_f(T)$ this is an easy consequence of Corollary 5.15. There are finitely many choices of v_i, so the probability of moving $\widetilde{x}_0$ while stabilizing $\widetilde{C}$ is 0.

As Γ is countable this means that with probability 1 the whole setwise stabilizer of $\widetilde{C}$ fixes $\widetilde{x}_0$.

In the general case when Γ is a regular branch group we condition on $\gamma_0 = t_i$ as in Lemma 5.19, and with f the canonical isomorphism between T_{v_i} and T_{v_j} we conclude that the conditional probability of η preserving $(\widetilde{\varphi}|\gamma_0 = t_i)$ is 0. There are finitely many choices for t_i, so again we conclude that the probability of moving $\widetilde{x}_0$ while stabilizing $\widetilde{C}$ is 0.

When η is not finitary there are two points where the above argument fails:

1) the trees T_{v_i} are not moved rigidly;

2) the n^{th} level might not witness that $\widetilde{x}_0$ is moved by η.

Notice however that 1) is not a real problem as we have fixed η and this fixes an isomorphism between T_{v_i} and T_{v_j}. The full generality of Lemma 5.19 (with $f = \eta|_{T_{v_i} \to T_{v_j}}$) ensures that the probability of randomizing η-compatible colorings for T_{v_i} and T_{v_j} is 0 even if γ is not finitary.

To work our way around 2) we notice that the probability of $\widetilde{x}_0$ being moved by η but this not being witnessed on $\mathcal{L}_n$ tends to 0 as $n \to \infty$. The set of fixed points of η is the decreasing intersection of the shadows of its fixed points on the finite levels. So the probability of $\widetilde{x}_0$ being in the shadow of the fixed points of $\mathcal{L}_n$ but outside $\mathrm{Fix}(\eta)$ converges to 0. This means that repeating the argument for all $n \in \mathbb{N}$ we get $\mathbb{P}[\eta \text{ moves } \widetilde{x}_0, \text{ but preserves } \widetilde{\varphi}] = 0$. $\square$

Proof of Theorem 5.4. By ergodicity and Lemma 5.12 we know that there exists a $P \in \mathcal{O}$, such that $\widetilde{P}$ has the same distribution as $\mathcal{O}_H$. Let us choose a closed set C which is not a single point from the partition P. We aim to use Proposition 5.20 to conclude that C is clopen. For that we will couple H and $\widetilde{C}$ such that $H \leq L$ holds almost surely, where L is the setwise stabilizer IRS of $\widetilde{C}$. Then H moving all points of C implies the same for L, which then through Proposition 5.20 implies that C is clopen.

Let $X = (P, C) \in \mathcal{O} \times \mathcal{C}$. Consider the diagonal action of $\overline{\Gamma}$ on $\mathcal{O} \times \mathcal{C}$. Let $\widetilde{X}$ be the Haar random translate of X. This way we obtained that the first coordinate of $\widetilde{X}$ has the same distribution as $\mathcal{O}_H$, the second coordinate has the same distribution as $\widetilde{C}$, and the second coordinate is always a closed subset in the partition given by the first coordinate.

Now we use the transfer theorem (see Theorem 6.10. of [45]) to obtain a random element C_H of $\mathcal{C}$, such that $(\mathcal{O}_H, C_H) \stackrel{d}{=} \widetilde{X}$. The first coordinate of $\widetilde{X}$ always contains the second, therefore $C_H \in \mathcal{O}_H$ and clearly $C_H \stackrel{d}{=} \widetilde{C}$. Choosing L to be the setwise stabilizer of C_H concludes the proof. $\square$

3 IRS's in regular branch groups

Our goal is to understand all IRS's H of Γ. Let $\widetilde{C} = \mathrm{Fix}(H)$. Lemmas 5.11 and 5.12 tell us that $\widetilde{C}$ is the γ translate of a fixed closed subset $C \subseteq \partial T$, where $\gamma \in \overline{\Gamma}$ is Haar random. First we exhibit some concrete examples which are worth to keep in mind and to motivate the decomposition of the tree in Subsection 3.2. We study the action of H on the parts in Subsection 3.3. The last two subsections contain the proof of the main theorem of the chapter.

3.1 Examples

We show a few examples to keep in mind. For simplicity let $d = 5$, and $\Gamma = \mathrm{Alt}_f(T)$. Recall that in this group the normal subgroups are the level stabilizers $\mathrm{Stab}_\Gamma(\mathcal{L}_n)$, and the quotients are the finite groups $A_d^{\mathrm{wr}(n)}$.

Example 5.21. *Pick $n \in \mathbb{N}$, and a finite subgroup $L \leq A_d^{\mathrm{wr}(n)}$. Let $\widetilde{L}$ be the uniform random conjugate of L in $A_d^{\mathrm{wr}(n)}$, and H be the preimage of $\widetilde{L}$ under the quotient map, that is $H = \widetilde{L} \cdot \mathrm{Stab}_\Gamma(\mathcal{L}_n)$. Then H is an ergodic fixed point free IRS of Γ. Note that this construction also works if G is only eventually d-ary, i.e. vertices on the first few levels might have different number of children.*

Theorem 5.1 states that all ergodic fixed point free IRS of $\mathrm{Alt}_f(T)$ are listed in Example 5.21. We give a very broad outline of the proof for this case in the hope that it makes the subsequent proof of the stronger Theorem 5.5 more transparent and motivates Proposition 5.26 that we state beforehand.

Outline of proof of Theorem 5.1. By Theorem 5.4 we know that an ergodic fixed point free IRS H has finitely many clopen orbit-closures on the boundary. A deep enough level

$\mathcal{L}_{k_0}$ witnesses this partition into clopen sets, and H acts transitively on the different parts on each $\mathcal{L}_n$ with $n \geq k_0$.

This means that we can find fixed elements supported above some level $\mathcal{L}_k$ ($k \geq k_0$) generating the orbits on $\mathcal{L}_{k_0}$ that are in H with positive probability. In the finite groups $A_d^{\mathrm{wr}(n)}$ (with $n \geq k$) this property translates to having a fixed subgroup L containing many conjugates of fixed elements. One can show that if n is sufficiently large this implies L containing a whole level stabilizer $\mathrm{Stab}_{A_d^{\mathrm{wr}(n)}}(\mathcal{L}_m)$ for some $m \geq k$ which does not depend on the choice of n.

Using that $\mathrm{Alt}_f(T)$ is the union of the $A_d^{\mathrm{wr}(n)}$ with some additional analysis of ergodic components one can show that actually H contains $\mathrm{Stab}_{\mathrm{Alt}_f(T)}(\mathcal{L}_m)$ almost surely. $\square$

Example 5.22. *Pick a random point $x \in \partial T$, this will be the single fixed point of the IRS H. Deleting the edges of the ray $(u_0, u_1, \ldots)$ corresponding to x from T we get infinitely many disjoint trees, where the roots u_n have degree* 4*, while the rest of the vertices have* 5 *children. Pick any fixed point free IRS for each of these trees as in example 5.21, randomize them independently and take their direct sum to be H. This construction works with other random fixed point sets instead of a single point as well.*

Example 5.23. *A modification of the previous example is the following. Let $x \in \partial T$ be random as before, and do the exact same thing for all the trees hanging of the ray $(u_0, u_1, \ldots)$ except for the first two, T_1 and T_2 rooted at u_0 and u_1 respectively. The finitary alternating automorphism groups of these trees are $\mathrm{Alt}_f(T) \wr A_4$. Now pick an (ergodic) fixed point free IRS of the finitary alternating and bi-root-preserving automorphism group of $T_1 \cup T_2$, which is $(\mathrm{Alt}_f(T) \wr A_4) \times (\mathrm{Alt}_f(T) \wr A_4)$, and use this to randomize H on $T_1 \cup T_2$. We will show that this is different from the previous examples. When we pick an IRS of $(\mathrm{Alt}_f(T) \wr A_4) \times (\mathrm{Alt}_f(T) \wr A_4)$ we pick some $n \in \mathbb{N}$, assume that the stabilizers of the n^{th} levels in T_1 and T_2 are in the IRS, and pick a random conjugate of some $L \leq (\mathrm{Alt}_f(T) \wr A_4) \times (\mathrm{Alt}_f(T) \wr A_4)$ to extend the stabilizer. If we pick for example $L = \{(\gamma, \gamma) \mid \gamma \in (\mathrm{Alt}_f(T) \wr A_4)\}$, then the IRS we construct will not be the direct product of IRS's on the two components, because the "top" parts of the subgroups are coupled together. Taking a random conjugate of L makes the coupling random as well, but nonetheless in every realization of H there is some nontrivial dependence between the actions of H on T_1 and T_2.*

3.2 Decomposition of T

The set of fixed points $\widetilde{C}$ corresponds to a subtree $T_{\widetilde{C}}$, which is the union of all the rays corresponding to the points of $\widetilde{C}$. All elements of H fix all vertices of the tree $T_{\widetilde{C}}$, so understanding H requires us to focus on the rest of T.

We will decompose T according to the subtree $T_{\widetilde{C}}$. Note that the following decomposition is slightly different to the one in the introduction as it is easier to work with.

On $\mathcal{L}_n$ denote the set of fixed vertices $F_n = V(T_{\widetilde{C}}) \cap \mathcal{L}_n$. Remove all edges $E(T_{\widetilde{C}})$ from T, the remaining graph T' is a union of trees.

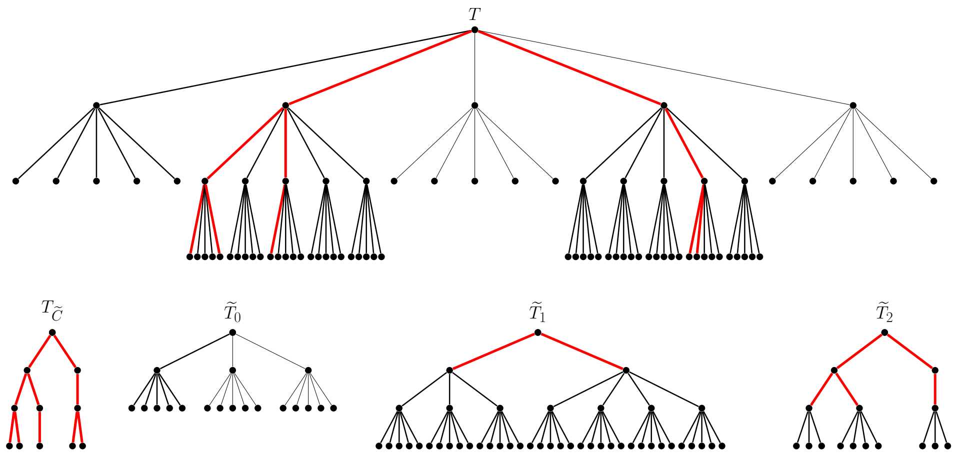


Figure 5.3: Decomposition of T with respect to $\widetilde{C}$

Let $\widetilde{T}_0$ be the connected component of T' containing the root of T. In other words it is the tree starting at the root in T'. In general let $\widetilde{T}_n$ be constructed as follows. The first n levels on $\widetilde{T}_n$ will be the same as the first n levels of $T_{\widetilde{C}}$, and beyond that select the connected components of T' containing the vertices of F_n. The vertices of $\widetilde{T}_n$ are exactly the vertices of T that can be reached from the root by taking n steps in $T_{\widetilde{C}}$ and then some number of steps in T'. See Figure 5.3.

The boundary ∂T decomposes as well. Clearly $\partial T_{\widetilde{C}} = \widetilde{C}$, and

$$\partial T = \widetilde{C} \cup \partial \widetilde{T}_0 \cup \partial \widetilde{T}_1 \cup \dots$$

Each $\partial \widetilde{T}_i$ is H-invariant, and a clopen and therefore compact subset of ∂T. It is the union of clopen orbit-closures from $\mathcal{O}_H$ because of Theorem 5.4, so it is the union of fintely many.

In the remaining part of this section we will prove that for any $C \in \mathcal{O}_H$ there exists some number $m^* \in \mathbb{N}$ and a subset $C_{m^*} \subseteq \mathcal{L}_{m^*}$ with $\mathrm{Sh}(C_{m^*}) = C$ such that $\mathrm{Rst}'_\Gamma(C_{m^*}) \leq H$. This m^* does not depend on the realization of $\mathcal{O}_H$, only on the equivalence class $[C]$.

Using this for the finitely many orbit-closures that constitute $\partial \widetilde{T}_i$ and taking a maximum yields that for some $m_i \geq i$ we have $\mathrm{Rst}'_\Gamma(\mathcal{L}_{m_i}(\widetilde{T}_i)) \subseteq H$. Knowing this for all i

yields

$$\bigoplus_{i\in\mathbb{N}} \mathrm{Rst}'_\Gamma\big(\mathcal{L}_{m_i}(\widetilde{T}_i)\big) \subseteq H,$$

which is equivalent to the statement of Theorem 5.5.

3.3 The action of H on the $\widetilde{T}_i$

Before we turn to proving Theorem 5.5 we argue that all IRS's resemble the previous examples in the sense that their projections on the $\widetilde{T}_n$ are fixed point free IRS's in $\mathrm{Stab}_\Gamma(\widetilde{T}_n)$.

While the $\widetilde{T}_n$ are random, the isomorphism type of each $\widetilde{T}_n$ is always the same because of ergodicity, and $\widetilde{T}_n$ can appear in finitely many Γ-equivalent ways in T. Let $T_n^1, T_n^2, \ldots T_n^{l(n)}$ denote the possible realizations of $\widetilde{T}_n$, and note that $\mathbb{P}[\widetilde{T}_n = T_n^i]$ is the same for all $i \in \{1, \ldots, l(n)\}$.

Let $\varphi_n : H \to \mathrm{Stab}_\Gamma(\widetilde{T}_n)$ denote the restriction function:

$$\varphi_n(h) = h|_{\widetilde{T}_n}.$$

The function φ_n is also random, but it only depends on $\widetilde{T}_n$, so once we condition H on $\widetilde{T}_n$ the function φ_n is well defined.

Proposition 5.24. *The random subgroup $\varphi_n\big((H \mid \widetilde{T}_n = T_n^i)\big)$ is a fixed point free IRS in $\mathrm{Stab}_\Gamma(T_n^i)$.*

Proof. For a fixed subgroup $L \leq \Gamma$ let $T_n(L)$ denote the deterministic subtree defined the same way as $\widetilde{T}_n$ was for H. The set $\{L \leq \Gamma \mid T_n(L) = T_n^i\}$ is invariant under the conjugation action of $\mathrm{Stab}_\Gamma(T_n^i) \leq \Gamma$, so the invariance of the random subgroup H implies the invariance of the conditioned subgroup $(H \mid \widetilde{T}_n = T_n^i)$. This IRS is fixed point free because all fixed points of H are in $T_{\widetilde{C}}$. □

Remark. *One might be tempted to prove the more general Theorem 5.5 by first proving the more transparent fixed point free case and then using Proposition 5.24 on the individual subtrees, where H acts fixed point freely. However, we do not see this approach to work. Instead with some mild additional technical difficulties we present the proof for the more general case.*

3.4 IRS's in finite subgroups of Γ

Let Γ_n stand for the elements of Γ that only have nontrivial vertex permutations above $\mathcal{L}_n$.

Lemma 5.25. *For n large enough we have $[\Gamma_n : (K \cap \Gamma_n)] \leq [\Gamma : K]$.*

Proof. Fix a transversal for K. All elements in the transversal are finitary, so choose n such that all are supported above $\mathcal{L}_n$. Then the translates of $(K \cap \Gamma_n)$ with this transversal cover Γ_n. $\square$

Let $\gamma \in \Gamma$, and $v \in \mathcal{L}_k$. The *section* of γ at v is the automorphism $[\gamma]_v$ we get by restricting the portrait of γ to the rooted subtree T_v consisting of v and its descendants. That is, the vertex permutations of $[\gamma]_v$ are $(u)\gamma$ for every $u \in T_v$ and the identity permutation otherwise. We think of $[\gamma]_v$ as the automorphism on T_v carried out by γ before all the vertex permutations above the level $\mathcal{L}_k$ take place.

Suppose $s \in \Gamma_k$, and let $L \subseteq \Gamma_n$ where $k < n$. Let $\widetilde{L}$ denote the uniform random Γ_n-conjugate of L, which is an IRS of Γ_n. Furthermore, assume that $\mathbb{P}[s \in \widetilde{L}] \geq c > 0$, which is equivalent to

$$\frac{\left|\{\gamma \in \Gamma_n \mid s^\gamma \in L\}\right|}{|\Gamma_n|} \geq c.$$

Let $R \subseteq \Gamma_n$ be a transversal for the subgroup $\mathrm{Rst}_{\Gamma_n}(\mathcal{L}_k)$. By choosing the optimal one, we can find $\bar{\gamma} \in R$ such that

$$\frac{\left|\left\{(\sigma_{v_1}, \ldots, \sigma_{v_{d^k}}) \in \mathrm{Rst}_{\Gamma_n}(\mathcal{L}_k) \;\middle|\; s^{\bar{\gamma}(\sigma_{v_1}, \ldots, \sigma_{v_{d^k}})} \in L\right\}\right|}{|\mathrm{Rst}_{\Gamma_n}(\mathcal{L}_k)|} \geq c. \tag{5.1}$$

Here $(\sigma_{v_1}, \ldots, \sigma_{v_{d^k}})$ stands for the element of $\mathrm{Rst}_{\Gamma_n}(\mathcal{L}_k)$ that pointwise fixes $\mathcal{L}_k$, and has sections $\sigma_{v_i} \in \mathrm{Rst}_{\Gamma_n}(v_i)$ at the vertices $v_i \in \mathcal{L}_k$.

Let $\bar{s} = s^{\bar{\gamma}}$, and let the cycles of $\bar{s}$ on $\mathcal{L}_k$ be $C_1, \ldots C_r$, and let $C_i = (u_1^i u_2^i \ldots u_{l(i)}^i)$, $l(i)$ denotes the length of the cycle C_i, and $\bar{s}(u_j^i) = u_{j+1}^i$. We use the convention that $u_{l(i)+1}^i = u_1^i$. Assume that $l(1) \geq l(2) \geq \ldots \geq l(r)$ and let t be the largest index for which $l(t) \geq 3$. Then $C = C_1 \cup \ldots \cup C_t \subseteq \mathcal{L}_k$ is the union of $\bar{s}$-orbits of length at least 3 on $\mathcal{L}_k$.

The next proposition shows that if n is large enough, then L has to contain the double commutator of some rigid level stabilizer under C, where the depth of this level does not depend on n.

Proposition 5.26. *Let k, s and c be fixed. Then there exists some $m > k$ and $n_0 > m$ such that for any $n \geq n_0$, L and corresponding $\bar{\gamma}$ satisfying (5.1) above we have $\mathrm{Rst}''_{\Gamma_n}\left(\mathrm{Sh}_{\mathcal{L}_m}(C)\right) \subseteq L$.*

Proof. Let $\sigma = (\sigma_{v_1}, \ldots, \sigma_{v_{d^k}})$. Fix σ_{v_i} for all $v_i \notin C$, and let the rest of the coordinates $\sigma_{u^i_j}$ vary over $\mathrm{Rst}_{\Gamma_n}(u^i_j)$. Choosing a maximum over all choices of the fixed σ_{v_i} we can assume that

$$\frac{\left|\left\{(\sigma_{u^i_j})_{i,j=1}^{r,l(i)} \in \mathrm{Rst}_{\Gamma_n}(C) \mid \bar{s}^\sigma \in L\right\}\right|}{|\mathrm{Rst}_{\Gamma_n}(C)|} \geq c.$$

Consider the conjugates $\bar{s}^\sigma$, more precisely what their sections are at the vertices u^i_j:

$$\left[\bar{s}^{(\sigma_{v_1},\ldots,\sigma_{v_{d^k}})}\right]_{u^i_j} = \sigma_{u^i_j} \cdot (\sigma_{u^i_{j+1}})^{-1}. \tag{5.2}$$

Fix one $\eta = (\eta_{v_1}, \ldots, \eta_{v_{d^{n-1}}}) \in \mathrm{Rst}_{\Gamma_n}(\mathcal{L}_k)$ with $\eta_{v_i} = \sigma_{v_i}$ for all $v_i \notin C$ and $\bar{s}^\eta \in L$. Let $\sigma_{u^i_j}$ run through $\mathrm{Rst}_{\Gamma_n}(u^i_j)$, and consider $\bar{s}^\sigma \cdot (\bar{s}^\eta)^{-1}$. All these elements fix $\mathcal{L}_k$ pointwise, and their sections are

$$\left[\bar{s}^\sigma \cdot (\bar{s}^\eta)^{-1}\right]_{u^i_j} = \sigma_{u^i_j} \cdot (\sigma_{u^i_{j+1}})^{-1} \cdot \left(\eta_{u^i_j} \cdot (\eta_{u^i_{j+1}})^{-1}\right)^{-1}.$$

Observe that the sections are trivial over $v_i \notin C$.

We will discard one vertex from each C_i, and focus on the sections we see on the rest. Let $D_i = C_i \setminus \{u^i_1\}$.

Consider the sections of $\bar{s}^\sigma$ at the vertices in D_i as the sections $(\sigma_{u^i_1}, \ldots, \sigma_{u^i_{l(i)}})$ run through $\mathrm{Rst}_{\Gamma_n}(C_i)$. We claim that the sections $\left([\bar{s}^\sigma]_{u^i_2}, \ldots, [\bar{s}^\sigma]_{u^i_{l(i)}}\right)$ run through $\mathrm{Rst}_{\Gamma_n}(D_i)$.

Indeed, given any sections $\left([\bar{s}^\sigma]_{u^i_j}\right)_{j=2}^{l(i)}$, and any choice of $\sigma_{u^i_2}$ we can sequentially choose the $\sigma_{u^i_{j+1}}$ according to (5.2) to get the given sections at $j = 2, 3, \ldots, l(i)$. The last choice is $\sigma_{u^i_1}$, which ensures $[\bar{s}^\sigma]_{u^i_{l(i)}}$ is correct. The last remaining section $[\bar{s}^\sigma]_{u^i_1}$ is already determined at this point, so we cannot hope to surject onto the whole $\mathrm{Rst}_{\Gamma_n}(C_i)$.

We can do this independently for each D_i. Let

$$D = \bigcup_i D_i.$$

The sections of $\bar{s}^{\sigma}$ over the index set D give $\mathrm{Rst}_{\Gamma_n}(D)$ as the sections $(\sigma_{u^i_j})$ run through $\mathrm{Rst}_{\Gamma_n}(C)$.

The fact that a fixed positive proportion of these conjugates are in L ensures that when we consider $\bar{s}^{\sigma} \cdot (\bar{s}^{\eta})^{-1}$ we get that a fixed proportion of the elements of $\mathrm{Rst}_{\Gamma_n}(D)$ are seen in L_0, where $L_0 \subseteq \mathrm{Stab}_L(\mathcal{L}_k)$ is the set of elements with trivial sections outside C. Let $\pi_D : \mathrm{Stab}_L(\mathcal{L}_k) \to \Gamma^D_{n-k}$ denote the projection to the coordinates in D. Formally we get

$$\left|\pi_D(L_0)\right| \geq c \cdot |\mathrm{Rst}_{\Gamma_n}(D)|.$$

We have $\pi_D(L_0) \leq (\Gamma_{n-k})^{|D|}$. Since $(K \cap \Gamma_{n-k})^{|D|} \leq \mathrm{Rst}_{\Gamma_n}(D)$, using Lemma 5.25 we get that the index of $\pi_D(L_0)$ in $\Gamma^{|D|}_{n-k}$ is bounded:

$$\left[(\Gamma_{n-k})^{|D|} : \pi_D(L_0)\right] \leq \left\lceil \frac{1}{c} \right\rceil \cdot [\Gamma : K]^{|D|}.$$

This means we can find some $N \lhd (\Gamma_{n-k})^{|D|}$ such that $N \leq \pi_D(L_0)$ and

$$\left[(\Gamma_{n-k})^{|D|} : N\right] \leq \left(\left\lceil \frac{1}{c} \right\rceil \cdot [\Gamma : K]^{|D|}\right)!$$

The bound on the index of N does not depend on n, only on k, s and c. The bounded index ensures, that we can find some m_0 such that for each index $u \in D$ we can find an element $\varphi \in N$ such that $\pi_u(\varphi) \notin \mathrm{Stab}_{\Gamma_{n_k}}\left(\mathcal{L}_{m_0}(T_u)\right)$. Let $m = k + m_0$. Choose $n_0 > m$ such that Γ_{n_0-k} acts transitively on $\mathcal{L}_{m_0}(T_u)$.

Using Grigorchuk's standard argument from [5, Lemma 5.3] and [39, Theorem 4] we pick some $w \in \mathcal{L}_{m_0}(T_u)$ not fixed by φ, elements f and g from $\mathrm{Rst}_{\Gamma_n}(uw)$ and argue that the commutator $[[\varphi, f], g] = [f, g]$ is in N. This shows $\mathrm{Rst}'_{\Gamma_n}(uw) \subseteq N$. If $n \geq n_0$ then G_{n-k} is transitive on $\mathcal{L}_{m_0}(T_u)$, so we get $\mathrm{Rst}'_{\Gamma_n}\left(\mathcal{L}_{m_0}(T_u)\right) \subseteq N$.

Repeating the argument of the previous paragraph for all $u \in D$ we get

$$\mathrm{Rst}'_{\Gamma_n}\left(\mathrm{Sh}_{\mathcal{L}_m}(D)\right) \subseteq N \subseteq \pi_D(L_0).$$

We now repeat this discussion, but we discard different points from the orbits: let $E_i = (C_i) \setminus \{u^i_2\}$ and $E = \bigcup_i E_i$. We have

$$\mathrm{Rst}'_{\Gamma_n}\left(\mathrm{Sh}_{\mathcal{L}_m}(E)\right) \subseteq \pi_E(L_0).$$

We claim that $\mathrm{Rst}''_{\Gamma_n}\big(\mathrm{Sh}_{\mathcal{L}_m}(D\cap E)\big) \subseteq L$. Indeed, let $u^i_j \in C_i$, $j \neq 1,2$. By the above we see that for any $\varphi \in \mathrm{Rst}'_{\Gamma_n}\big(\mathrm{Sh}_{\mathcal{L}_m}(u^i_j)\big)$ we have $h_1 \in L_0$ such that $\pi_D(h_1)_{u^i_j} = \varphi$ and all other coordinates of $\pi_D(h_1)$ are the identity. Similarly we have $h_2 \in L_0$ such that $\pi_E(h_2)_{u^i_j} = \psi$ and all other coordinates of $\pi_E(h_1)$ are the identity. Since $\mathcal{L}_k \setminus D$ and $\mathcal{L}_k \setminus E$ are disjoint the commutator $[h_1, h_2] \in L_0$ has all identity coordinates except for the one corresponding to u^i_j which is $[\varphi, \psi]$.

We have managed to take care of the points u^i_j where $j \neq 1,2$. To cover the remaining points as well we need one more way to discard points from the orbits. Namely F, where we discard the third vertex u^i_3 from every C_i. Using the fact that $(D\cap E)\cup(E\cap F)\cup(D\cap F) = C$ we get that $\mathrm{Rst}''_{\Gamma_n}\big(\mathrm{Sh}_{\mathcal{L}_m}(C)\big) \subseteq L$, which finishes the proof. □

3.5 Proof of the main result

Proof of Theorem 5.5. During the proof we will have to choose deeper and deeper levels in T. For the convenience of the reader we summarized these choices in Figure 5.4.

Let $\Gamma_n \subseteq \Gamma$ denote the elements of Γ that are supported on the first n levels. Suppose that H is an ergodic IRS of Γ.

By Theorem 5.4 we know that the all nontrivial orbit-closures of H on ∂T are clopen. For every clopen set C there exists a smallest integer k_C such that C is the union of shadows of points on $\mathcal{L}_{k_C}$. Clearly k_C does not change when C is translated by some automorphism. For the random subgroup H and a fixed $k_0 \in \mathbb{N}$ we can collect the clopen sets C from $\mathcal{O}_H$ with $k_C < k_0$, let C_{H,k_0} be the union of these. This set moves together with H when conjugating by some $\gamma \in \Gamma$:

$$C_{H^\gamma,k_0} = (C_{H,k_0})^\gamma.$$

For $n \geq k_0$ let $V_n \subset \mathcal{L}_n$ be the set of points whose shadow make up C_{H,k_0}. As C_{H,k_0} moves with H, so does V_n. V_{k_0} is a union of orbits of H, let those orbits be denoted $V^i_{k_0}$, where $i \in \{1,\ldots,j\}$ and

$$V_{k_0} = \bigcup_{i=1}^{j} V^i_{k_0}.$$

Let $V^i_n = \mathrm{Sh}_{\mathcal{L}_n}(V^i_{k_0})$. The fact that H acts minimally on the components of C_{H,k_0} translates to saying that H acts transitively on each V^i_n. Notice that since we collected clopen sets C with k_C strictly less then k_0 we ensured that $V^i_{k_0}$ contains at least d points for all i.

For every realization of H we can choose finitely many elements of H that already show that H acts transitively on the $V^i_{k_0}$. These finitely many elements are all finitary, so there is some n_H, which might depend on the realization of H, such that all those finitely many elements are in Γ_{n_H}.

This function n_H is not necessarily conjugation-invariant, so it need not be constant merely by ergodicity. However one can find some $k \geq k_0$ such that the $V^i_{k_0}$ are distinct orbits of $H_k = H \cap \Gamma_k$ on $\mathcal{L}_{k_0}$ with probability $1-\varepsilon$. This k is a deterministic number, it does not depend on the realization of H.

Enlist all the possible subsets $S_1, \dots, S_N$ of Γ_k that generate a realization of the $V^i_{k_0}$ as orbits on $\mathcal{L}_{k_0}$. Clearly there are finitely many. The probability that $S_i \subseteq H$ cannot always be 0, otherwise we would contradict the previous paragraph. So we can find some finite set S of elements of Γ_k and some sets $U^i_{k_0} \subseteq \mathcal{L}_{k_0}$ such that the $U^i_{k_0}$ are a realization of the $V^i_{k_0}$, S is in H with probability $p > 0$ and the $U^i_{k_0}$ are orbits of S.

By replacing S with $\langle S \rangle$ we may assume that S is a subgroup of Γ_k, as $S \subseteq H$ and $\langle S \rangle \subseteq H$ are the same events.

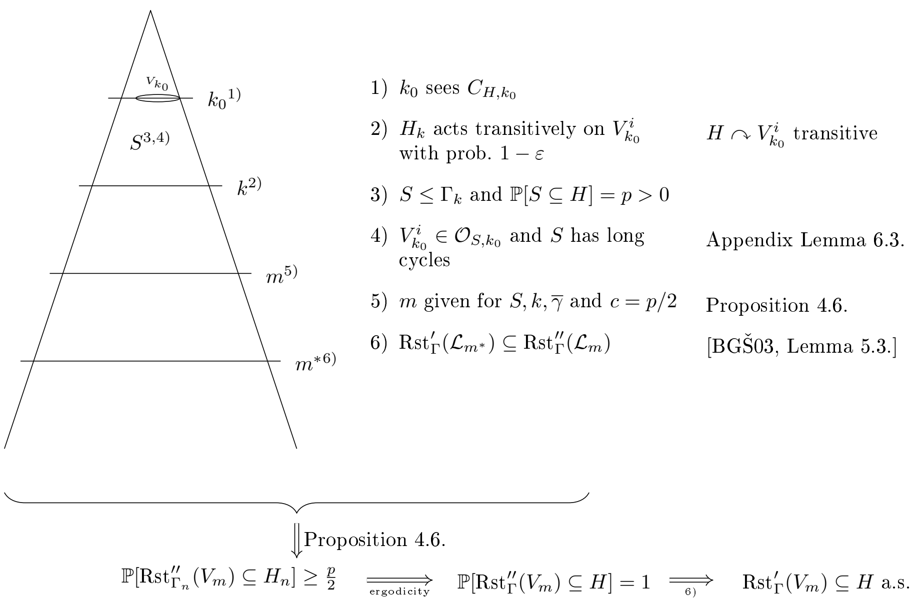


Figure 5.4: Choice of levels

As $|U^i_{k_0}| \geq d$, we know that all vertices of U_{k_0} are moved by some $s \in S$. As a consequence the same holds for U_k: for every vertex $v \in U_k$ there is some $s \in S$ such that $v \neq v^s$. However, we will need a stronger technical assumption on S to make our argument work. We will assume that for every $v \in U_k$ we can find some $s \in S$ such that v, v^s and v^{s^2} are distinct, that is v is part of a cycle of length at least 3 in the cycle decomposition of s. In Lemma 5.31 in the Appendix we show that one can indeed find such a k and S.

Let $H_n = H \cap \Gamma_n$, for $n \geq k$. The random subgroup H_n is clearly an IRS of Γ_n, however it need not be ergodic, i.e. the uniform random conjugate of a fixed subgroup in Γ_n. As $S \leq \Gamma_k \leq \Gamma_n$ we have $\mathbb{P}[S \leq H_n] = p$.

Lemma 5.27. *In the ergodic decomposition of H_n the measure of components that contain S with probability at least $p/2$ is at least $p/2$.*

Proof. Denote the ergodic components of H_n by $H_n^1, \dots, H_n^r$. Assume H_n^i has weight q_i in the decomposition, and contains S with probability p_i. By ordering appropriately we can also assume $p_1, \dots, p_l \leq p/2$ and $p_{l+1}, \dots p_r < p/2$.

$$p = \sum_{i=1}^{r} q_i p_i = \left(\sum_{i=1}^{l} q_i\right) \cdot 1 + \left(\sum_{i=l+1}^{r} q_i\right) \cdot \frac{p}{2} \leq \left(\sum_{i=1}^{l} q_i\right) + \frac{p}{2},$$

$$\frac{p}{2} \leq \sum_{i=1}^{l} q_i.$$

So the weight of components containing S with probability at least $p/2$ is at least $p/2$. ☐

Choose an ergodic component of H_n which contains S with probability at least $p/2$. This ergodic component is the uniform random conjugate of a fixed subgroup $L \leq \Gamma_n$.

We have $\mathbb{P}[S \in \widetilde{L}] \geq \frac{p}{2} > 0$. In other words L contains at least a $p/2$ proportion of the Γ_n-conjugates of S. By a "maximum is at least as large as the average" argument we can find some $\bar{\gamma}$ from the transversal of $\mathrm{Rst}_{\Gamma_n}(\mathcal{L}_k)$ such that

$$\frac{\left|\left\{(\sigma_{v_1}, \dots, \sigma_{v_{d^k}}) \in \mathrm{Rst}_{\Gamma_n}(\mathcal{L}_k) \mid S^{\bar{\gamma}(\sigma_{v_1}, \dots, \sigma_{v_{d^k}})} \in L\right\}\right|}{|\mathrm{Rst}_{\Gamma_n}(\mathcal{L}_k)|} \geq \frac{p}{2}.$$

We now use Proposition 5.26 for all $s \in S$ with k, $\bar{\gamma}$ defined above and $c = \frac{p}{2}$. As the cycles of length at least 3 of elements of $S^{\bar{\gamma}}$ cover $(U_k)^{\bar{\gamma}}$ we get that for some fixed m and large enough n we have

$$\mathrm{Rst}''_{\Gamma_n}\left((U_m)^{\bar{\gamma}}\right) \subseteq L.$$

It is clear that $(U_{k_0})^\gamma$ is the realization of V_{k_0} corresponding to the realization L of H_n, so we (almost surely) have $\mathrm{Rst}''_{\Gamma_n}(V_m) \subseteq \widetilde{L}$. By Lemma 5.27 this means that

$$\mathbb{P}\left[\mathrm{Rst}''_{\Gamma_n}(V_m) \subseteq H_n\right] \geq \frac{p}{2}.$$

As $\left(\mathrm{Rst}''_{\Gamma_n}(V_m) \subseteq H_n\right) \Leftrightarrow \left(\mathrm{Rst}''_{\Gamma_n}(V_m) \subseteq H\right)$ we have

$$\mathbb{P}\left[\mathrm{Rst}''_{\Gamma_n}(V_m) \subseteq H\right] \geq \frac{p}{2}.$$

We get this for all n large enough. Since $\mathrm{Rst}''_{\Gamma_n}(V_m) \subseteq \mathrm{Rst}''_{\Gamma_{n+1}}(V_m)$ the events in question form a decreasing chain, and for the intersection we get

$$\mathbb{P}\left[\mathrm{Rst}''_{\Gamma}(V_m) \subseteq H\right] \geq \frac{p}{2}.$$

As H is ergodic the above implies

$$\mathbb{P}\left[\mathrm{Rst}''_{\Gamma}(V_m) \subseteq H\right] = 1.$$

Clearly $\mathrm{Rst}''_{\Gamma}(\mathcal{L}_m) \lhd \Gamma$, so using [5, Lemma 5.3] we can find some $m^* \geq m$ such that $\mathrm{Rst}'_{\Gamma}(\mathcal{L}_{m^*}) \subseteq \mathrm{Rst}''_{\Gamma}(\mathcal{L}_m)$. This also means that $\mathrm{Rst}'_{\Gamma}(V_{m^*}) \subseteq \mathrm{Rst}''_{\Gamma}(V_m)$, so

$$\mathbb{P}\left[\mathrm{Rst}'_{\Gamma}(V_{m^*}) \subseteq H\right] = 1.$$

The number m^* only depended on the IRS H and the choice of k_0. Repeating this argument for all $k_0 \in \mathbb{N}$ covers all clopen sets from $\mathcal{O}_H$, which as discussed in part 3.2 proves Theorem 5.5. □

4 Corollaries of Theorem 5.5

In this section we prove Theorem 5.6 and sketch the proof of Theorem 5.7.

4.1 Fixed point free IRS's

To motivate the following result let us recall Theorem 5.1, which states that any ergodic IRS of $\mathrm{Alt}_f(T)$ with $d \geq 5$ contains a whole level stabilizer, in particular H is a random conjugate of a finite indexed subgroup. In other words the measure defining the IRS is atomic. As it turns out the fixed point free case of Theorem 5.5 implies this for fixed point free ergodic IRS's of countable, finitary regular branch groups as well.

Proof of Theorem 5.6. By Theorem 5.5 we know that an ergodic almost surely fixed point free IRS H contains $\mathrm{Rst}'_\Gamma(\mathcal{L}_m)$ for some $m \in \mathbb{N}$.

IRS's of Γ containing the normal subgroup $\mathrm{Rst}'_\Gamma(\mathcal{L}_m)$ are in one-to-one correspondence with IRS's of the quotient $G = \Gamma/\mathrm{Rst}'_\Gamma(\mathcal{L}_m)$, which in this case is of the form $A \rtimes F$ where A is the abelian group $\mathrm{Rst}_\Gamma(\mathcal{L}_m)/\mathrm{Rst}'_\Gamma(\mathcal{L}_m)$, and F is the finite group $\Gamma/\mathrm{Rst}_\Gamma(\mathcal{L}_m)$. As Γ is assumed to finitary both Γ and G are countable.

Let $\widehat{H} = H/\mathrm{Rst}'_\Gamma(\mathcal{L}_m) \leq G$ be the image of H in G. It is an ergodic IRS of G. Let $\widehat{H}_0 = \widehat{H} \cap A$, which is also an ergodic IRS of G. We see that $\widehat{H}_0 \subseteq A$ is an ergodic random subgroup with distribution invariant under conjugation by elements of G. As A is abelian and F finite, it is clearly the uniform random F-conjugate of some subgroup $L_0 \leq A$. This shows that $\widehat{H}_0$ can only obtain finitely many possible values.

We claim that once $\widehat{H}_0$ is fixed, there are only countably many possible choices for $\widehat{H}$. Indeed we have to choose a coset of $\widehat{H}_0$ in G for all $f \in F$, which can do in only countably many different ways.

This shows that the support of $\widehat{H}$ is countable, but there is no ergodic invariant measure on a countably infinite set, so the support is finite. This proves the theorem. □

4.2 IRS's in non-finitary branch groups

In this subsection we will sketch the proof of Theorem 5.7. This theorem is not a direct consequence (as far as we see) of Theorem 5.5, but one can alter the proof to obtain the desired theorem. First of all let us fix $\pi_n : \Gamma \to S_d^{\mathrm{wr}(n)}$ to be the projection from Γ to the automorphism group of the d-ary tree of depth n, which is the restriction of elements to the first n levels. The main conceptional difference is that we are trying to understand the group Γ through the groups $\pi_n(\Gamma)$ instead of Γ_n. The statement that we conclude in this case is weaker.

Our aim is to present only the spine of the proof, as the reasoning is very similar to the proof of Theorem 5.5 and we leave the details to the reader. In fact some technical details such as the ergodicity of H_n and the fact that H_n already acts transitively on the V_n^i makes this proof easier.

Proof of Theorem 5.7. Let $G_n = \pi_n(\Gamma)$, fix $k \in \mathbb{N}$ and let $C_{H,k}$ be the union of clopen orbit-closures C from $\mathcal{O}_H$ in ∂T with $k_C < k$. For any $n \geq k$ let $V_n \subseteq \mathcal{L}_n$ be the set of points whose shadow make up $C_{H,k}$. We can decompose V_k into H-orbits, denoted by

$$V_k = \cup_{i=1}^{j} V_k^i.$$

Observe that for any realization of H one can find at most $|V_k|$ many elements in H that already show that H acts transitively on each V_k^i. This means that we can find an $S \subset \Gamma$ of size at most $|V_k|$, such that S generates a realization of V_k on $\mathcal{L}_k$ and

$$\mathbb{P}[S \subseteq H] = p > 0.$$

Denote by U_k^i the realization of V_k^i generated by S. As before we can ensure that for any $v \in U_k$ there is an $s \in S$, such that v, v^s and v^{s^2} are distinct by replacing k and S if necessary. (See the Remark after the proof of Lemma 5.31 in the Appendix.)

For every n let $H_n = \pi_n(H) \leq G_n$. The random subgroup H_n is an ergodic IRS of G_n, therefore there exists an $L_n \leq G_n$ such that H_n is an uniform random conjugate of L_n. Since

$$\mathbb{P}[\pi_n(S) \subseteq \widetilde{L_n}] = \mathbb{P}[\pi_n(S) \subseteq H_n] \geq p,$$

we have an element $\overline{\gamma}$ from the transversal of $\mathrm{Rst}_{G_n}(\mathcal{L}_k)$ in G_n such that

$$\frac{\left|\left\{(\sigma_{v_1}, \ldots, \sigma_{v_{d^k}}) \in \mathrm{Rst}_{G_n}(\mathcal{L}_k) \mid \pi_n(S)^{\bar{\gamma}(\sigma_{v_1}, \ldots, \sigma_{v_{d^k}})} \in L_n\right\}\right|}{|\mathrm{Rst}_{G_n}(\mathcal{L}_k)|} \geq p.$$

By following the argument in Proposition 5.26 but replacing Γ_n by G_n one can prove that there exists some m such that for any n large enough

$$\mathrm{Rst}''_{G_n}((U_m)^{\bar{\gamma}}) \subseteq L_n.$$

Therefore

$$\mathbb{P}[\mathrm{Rst}''_{G_n}(V_m) \subseteq H_n] \geq p > 0,$$

which by ergodicity implies

$$\mathbb{P}[\mathrm{Rst}''_{G_n}(V_m) \subseteq H_n] = 1.$$

Again we can find an $m^* \geq m$, such that $\mathrm{Rst}'_{\Gamma}(V_{m^*}) \subseteq \mathrm{Rst}''_{\Gamma}(V_m)$, therefore

$$\mathbb{P}[\pi_n(\mathrm{Rst}'_{\Gamma}(V_{m^*})) \subseteq \pi_n(H)] = 1.$$

This means that for any $g \in \mathrm{Rst}'_\Gamma(V_{m^*})$ there exists a sequence $h_n \in H$, such that $\pi_n(h_n) = \pi_n(g)$, which implies that $\overline{\mathrm{Rst}'_\Gamma(V_{m^*})} \subseteq \overline{H}$ with probability 1.

On the other hand $\overline{\mathrm{Rst}'_\Gamma(V_{m^*})} \supseteq \overline{\mathrm{Rst}_\Gamma(V_{m^*})}'$. We claim that $\mathrm{Rst}_{\overline{\Gamma}}(V_{m^*}) = \overline{\mathrm{Rst}_\Gamma(V_{m^*})}$. Indeed, $\mathrm{Rst}_{\overline{\Gamma}}(\mathcal{L}_{m^*})$ is finite index in $\overline{\Gamma}$ which implies that it is open. Using this one can show that $\mathrm{Rst}_{\overline{\Gamma}}(\mathcal{L}_{m^*}) = \overline{\mathrm{Rst}_\Gamma(\mathcal{L}_{m^*})}$, which implies the same for $V_{m^*} \subseteq \mathcal{L}_{m^*}$.

Putting this together we get

$$\mathrm{Rst}'_{\overline{\Gamma}}(V_{m^*}) \subseteq \overline{H}$$

with probability 1. □

Note that this result on closures is possibly weaker than our earlier results. It is not clear even in the fixed point free case in $\mathrm{Alt}_f(T)$ if for some $L \leq \mathrm{Alt}_f(T)$ the closure $\overline{L}$ containing a level stabilizer implies the same for L.

Problem 5.28. *Let $L \leq \mathrm{Alt}_f(T)$ be a subgroup such that $\pi_n(L) = A_d^{\mathrm{wr}(n)}$ for all n. Does it follow that $L = \mathrm{Alt}_f(T)$?*

In other words: is there a subgroup $L \neq \mathrm{Alt}_f(T)$ which is dense in $\mathrm{Alt}(T)$? We saw that this cannot happen with positive probability when L is invariant random.

The answer to Problem 5.28 is negative in the case of $\mathrm{Aut}(T)$. In the case of the binary tree let L be the subgroup of elements with an even number of nontrivial vertex permutations. Generally for arbitrary d let L be the subgroup of elements whose vertex permutations multiply up to an alternating element. This L is not the whole group, yet dense in $\mathrm{Aut}(T)$. Of course the really relevant question in this case would involve the containment of derived subgroups of level stabilizers.

5 Appendix

In this section we prove the technical statements that we postponed during the rest of the chapter.

5.1 Measurability of maps

Proof of Lemma 5.9. A closed subset C can be approximated on the finite levels. Define C_n to be the set of vertices v on $\mathcal{L}_n$ with $\mathrm{Sh}(v) \cap C \neq \emptyset$. The C_n correspond to the $(1/2^n)$-neighborhoods of C in ∂T.

We show that any preimage of a ball in $(\mathcal{C}, d_H)$ is measurable in Sub_Γ. Let $C \in \mathcal{C}$ and $n \in \mathbb{N}$ be fixed. Then the ball

$$B_{1/2^n}(C) = \{C' \in \mathcal{C} \mid C_n = C'_n\},$$

therefore its preimage is

$$X = \{H \in \mathrm{Sub}_\Gamma \mid \mathrm{Fix}(H)_n = C_n\}.$$

We say that a finite subset $S \subseteq \Gamma$ witnesses C_n, if the subgroup they generate has no fixed points in $\mathcal{L}_n \setminus C_n$. Clearly every C_n has a witness of cardinality at most $|\mathcal{L}_n \setminus C_n|$. Let W_{C_n} be the set of possible witnesses of C_n of size at most $|\mathcal{L}_n \setminus C_n|$:

$$W_{C_n} = \left\{S \subseteq \Gamma \;\middle|\; |S| \leq |\mathcal{L}_n \setminus C_n| \text{ and } S \text{ has no fixed points in } \mathcal{L}_n \setminus C_n\right\}.$$

Let us define $F_{C_n} \subseteq \Gamma$ to be the set of forbidden group elements, which do not fix C_n. These are the elements that cannot be in any $H \in X$.

Observe that both W_{C_n} and F_{C_n} are countable, since Γ is countable, and X can be obtained as

$$X = \bigcup_{S \in W_{C_n}} \bigcap_{g \in F_{C_n}} \{H \in \mathrm{Sub}_\Gamma \mid S \subseteq H, g \notin H\}.$$

The sets $\{H \in \mathrm{Sub}_\Gamma \mid S \subseteq H, g \notin H\}$ are cylinder sets in the topology of Sub_Γ, so the above expression shows that X is measurable. □

Proof of Lemma 5.10. To prove that the map is measurable, it is enough to show that any preimage of a ball is measurable in Sub_Γ. So let $P \in \mathcal{O}$ and $n \in \mathbb{N}$ be fixed. Then the ball

$$B_{1/2^n}(P) = \{Q \in \mathcal{O} \mid Q_n = P_n\},$$

therefore its preimage is

$$X = \{H \in \mathrm{Sub}_\Gamma \mid (\mathcal{O}_H)_n = P_n\}.$$

We say that a finite subset $S \subseteq \Gamma$ witnesses P_n, if the subgroup they generate induces the same orbits on $\mathcal{L}_n$, that is $\mathcal{O}_{\langle S \rangle, \mathcal{L}_n} = P_n$. Clearly every P_n has a witness of cardinality at most $|\mathcal{L}_n|$. Let W_{P_n} be the set of possible witnesses of P_n of size at most $|\mathcal{L}_n|$:

$$W_{P_n} = \left\{ S \subseteq \Gamma \;\middle|\; |S| \leq |\mathcal{L}_n| \text{ and } \mathcal{O}_{\langle S \rangle, \mathcal{L}_n} = P_n \right\}.$$

Let us define $F_{P_n} \subseteq \Gamma$ to be the set of forbidden group elements, which do not preserve P_n. In other words these are the elements that cannot be in any $H \in X$.

Observe that both W_{P_n} and F_{P_n} are countable, since Γ is countable, and X can be obtained as

$$X = \bigcup_{S \in W_{P_n}} \bigcap_{g \in F_{P_n}} \{H \in \mathrm{Sub}_\Gamma \mid S \subseteq H, g \notin H\}.$$

The sets $\{H \in \mathrm{Sub}_\Gamma \mid S \subseteq H, g \notin H\}$ are cylinder sets in the topology of Sub_Γ, so the above expression shows that X is measurable. $\square$

5.2 Technical assumption in Theorem 5.5

First we prove a lemma on intersection probabilities.

Lemma 5.29. *Let $B_1, \dots B_r$ be measurable subsets of the standard probability space (X, μ) with $\mu(B_j) = p$ for all j, and $r = \left\lceil \frac{2}{p} \right\rceil$. Then there is some pair (j, l) such that $\mu(B_j \cap B_l) \geq \frac{p^3}{6}$.*

Proof. Let χ_B denote the characteristic function of the measurable set B. Let D_l denote the set of points in X that are covered by at least l sets from $B_1, \dots B_r$. Then

$$\sum_{j=1}^{r} \chi_{B_j} = \sum_{l=1}^{r} \chi_{D_l},$$

$$\int_X \sum_{j=1}^{r} \chi_{B_j} \, d\mu = \sum_{j=1}^{r} \mu(B_j) = rp,$$

$$rp = \int_X \sum_{l=1}^{r} \chi_{D_l} \, d\mu = \sum_{l=1}^{r} \mu(D_l).$$

We have $D_1 \supseteq D_2 \dots \supseteq D_r$, so $1 \geq \mu(D_1) \geq \mu(D_2) \dots \geq \mu(D_r)$.

$$rp = \sum_{l=1}^{r} \mu(D_l) \leq 1 + (r-1)\mu(D_2).$$

$$\mu(D_2) \geq \frac{rp-1}{r-1}.$$

The set D_2 is covered by the $B_j \cap B_l$, so

$$\max_{j,l} \mu(B_j \cap B_l) \geq \frac{\mu(D_2)}{\binom{r}{2}} \geq \frac{rp-1}{\binom{r}{2}(r-1)} \geq \frac{1}{\left(\frac{\left(\frac{2}{p}+1\right)\left(\frac{2}{p}\right)}{2}\right)\left(\frac{2}{p}\right)} \geq \frac{p^3}{2(p+2)} \geq \frac{p^3}{6}.$$

□

We also prove that one can find a lot of elements of order at least 3 in weakly branch groups.

Lemma 5.30. *Let G be a weakly branch group. Then for any $v \in T$ there is a $g \in \mathrm{Rst}_G(v)$ of order at least 3.*

Proof. As G is weakly branch, $\mathrm{Rst}_G(v)$ is not the trivial group. By contradiction let us assume that any nontrivial element of $\mathrm{Rst}_G(v)$ has order 2. Let a nontrivial element $g \in \mathrm{Rst}_G(v)$ be fixed. We can find descendants $u_1 \neq u_2$ of v, such that $u_1^g = u_2$. Let us choose a nontrivial element $h \in \mathrm{Rst}_G(u_1)$. As $h \in \mathrm{Rst}_G(v)$, by our assumption it has order 2.

We claim that $hg \in \mathrm{Rst}_G(v)$ has order at least three. To prove this we will find a vertex which has an orbit of size at least 3. Let $w_1 \neq w_2$ descendants of u_1, such that $w_1^h = w_2$. Since g maps descendants of u_1 to descendants of u_2, we have $w_1^{hg} = w_2^g = t_2 \neq w_1, w_2$. Then $w_1^{(hg)^2} = t_2^{hg} = t_2^g = w_2 \neq w_1$. We see that w_1, w_1^{hg} and $w_1^{(hg)^2}$ are pairwise distinct, therefore the order of hg is at least 3. □

Now we will prove that the technical assumption we assumed in the proof of Theorem 5.5 can be satisfied. We remind the reader that in the setting of Theorem 5.5 the following were established:

(1) The random sets $(V_1^{k_0}, \dots V_j^{k_0})$ are orbits of H on $\mathcal{L}_{k_0}$;

(2) $\mathrm{Sh}(V_1^{k_0}), \dots, \mathrm{Sh}(V_j^{k_0})$ are orbit-closures of H on ∂T and their union is C_{H,k_0} almost surely;

(3) $S \leq \Gamma_k$ is a finite subgroups with $p = \mathbb{P}[S \subseteq H]$ positive;

(4) $(U_1^{k_0}, \dots U_j^{k_0})$ are a realization of $(V_1^{k_0}, \dots V_j^{k_0})$, and S acts transitively on the $U_i^{k_0}$.

(5) $V_i^n = \mathrm{Sh}_{\mathcal{L}_n}(V_i^{k_0})$ and $U_i^n = \mathrm{Sh}_{\mathcal{L}_n}(U_i^{k_0})$.

Lemma 5.31. *By possibly replacing k, S and p we can assume that for every $u \in U_i^k$ we can find some $s \in S$ such that u, u^s and u^{s^2} are distinct.*

Remark. *In the case when d is not a power of 2 it can be shown that Lemma 5.31 is implied by the earlier properties, simply because a transitive permutation group with all nontrivial elements being fixed point free and of order 2 can only exist on 2^k points. For the case when d is a power of 2 however we can only show Lemma 5.31 by a probabilistic argument and by increasing k and S if necessary.*

Proof. Assume that there is an $s \in S$ which admits a long cycle – that is a cycle of length at least 3 – on U_i^k for some i. In this first case we define k' such that $H_{k'}$ acts transitively on all the V_i^k with probability $1 - \frac{p}{2}$. Then

$$\mathbb{P}[S \subseteq H_{k'} \text{ and } H_{k'} \text{ is transitive on the } V_i^k] \geq \frac{p}{2} > 0.$$

If $S \subseteq H_{k'}$ then the V_i^k are realized as the U_i^k. Now we enlist all subsets S_l' in $\Gamma_{k'}$ that contain S and act transitively on the U_i^k. There are finitely many, so we can find some S' with

$$\mathbb{P}[S' \subseteq H_{k'}] \geq p' > 0.$$

We can assume S' to be a subgroup, and by having $s \in S'$ we will show that long cycles of S' cover U_i^k. Indeed, by conjugating s one can move the cycle around in U_i^k, and by the transitivity of S' we get that the whole of U_i^k is covered. This in turn implies that long cycles of S' cover $U_i^{k'}$ as well.

If on the other hand S acts on U_i^k by involutions, we will increase k and S while keeping p positive such that the first case holds.

Let $r = \left\lceil \frac{2}{p} \right\rceil$. Furthermore let $k' > k$ such that the shadow of a vertex $v \in U_i^k$ on $\mathcal{L}_{k'}$ contains at least r vertices, namely $\{v_1, v_2, \ldots, v_r, \ldots\} \subseteq U_i^{k'}$. Let $\gamma_1, \gamma_2, \ldots, \gamma_r \in \Gamma_{k'+t}$ such that $\gamma_i \in \mathrm{Rst}_{\Gamma_{k'+t}}(v_i)$ and γ_i has order at least 3 by Lemma 5.30, i.e. γ_i has a long cycle on $\mathcal{L}_{k'+t}$.

As H is an IRS we have

$$\mathbb{P}[S^{\gamma_j} \subseteq H] = \mathbb{P}[S \subseteq H] = p.$$

By Lemma 5.29 we can find some j, l such that

$$\mathbb{P}\left[(S^{\gamma_j} \cup S^{\gamma_l}) \subseteq H\right] \geq \frac{p^3}{6}.$$

Set $S' = \langle S^{\gamma_j} \cup S^{\gamma_l} \rangle$. Pick some $s \in S$ which moves $v \in \mathcal{L}_k$. It is easy to check that $s^{\gamma_j} \cdot (s^{\gamma_l})^{-1} \in S' \cap \mathrm{Rst}_{\Gamma_{k'+t}}(\mathcal{L}_k)$ has nontrivial sections only at v_j, v_l, v_j^s and v_l^s, and these sections are some conjugates of γ_j and γ_l, and therefore $s^{\gamma_j} \cdot (s^{\gamma_l})^{-1}$ has a long cycle on $\mathcal{L}_{k'+t}$. So replacing S by S', k by $k' + t$ and p by $\frac{p^3}{6}$ we get to the first case.

Repeating the argument for the first case at most j times we make sure that all U_k^i are covered by long cycles, which finishes the proof. □

Remark. *For the proof of Theorem 5.7 one can modify this proof such that instead of Γ_n, $H_k = \Gamma_n \cap H$ and $S \subseteq \Gamma_n$ we use $G_n = \pi_n(\Gamma)$, $H_n = \pi_n(H)$ and $\pi_n(S) \subseteq G_n$. Another difference is that H_n automatically acts transitively on all the V_i^n, so there is no need to distinguish between k_0 and k.*

6

Summary

In the first chapter, we investigated the partition function of the hard-core model, the independence polyonomial. We have shown some algebraic identities, that were motivated by the theory of matching polynomials, and we used them to obtain information about the location of the roots of the independence polynomial of graphs. By introducing a new parameter $\mathrm{bd}(G)$, we gave a relaxation of a theorem of Chudnovsky and Seymour and a zero-free region for bounded degree fork-free graphs. Moreover, we gave a new method to prove real-rootedness of the independence polynomials of certain families of trees, and a 'new connection' between the theory of independence polynomials and the theory of chromatic polynomials.

In the second chapter, we gave zero-free regions for the partition function of the anti-ferromagnetic Potts model on bounded degree graphs. In particular, we showed that for any $\Delta \in \mathbb{N}$ and any $k \geq e\Delta + 1$, there exists an open set U in the complex plane that contains the interval $[0, 1)$ such that $\mathbf{Z}(G; k, w) \neq 0$ for any $w \in U$ and any graph G of maximum degree at most Δ. For small values of Δ, we were able to give better results.

In the third chapter, we defined the descent polynomial, and we proved some conjectures concerning the coefficient sequences of $d(I, n)$. As a corollary, we described some zero-free regions for the descent polynomial.

In the last chapter, we worked on invariant random subgroups in groups acting on rooted trees. Our main concern was $\mathrm{Alt}_f(T)$ being the group of finitary even automorphisms of the d-ary rooted tree T. We proved that a nontrivial ergodic IRS of $\mathrm{Alt}_f(T)$ that acts without fixed points on the boundary of T contains a level stabilizer, in particular, it is the random conjugate of a finite index subgroup.

Applying the technique to branch groups, we proved that an ergodic IRS in a finitary regular branch group contains the derived subgroup of a generalized rigid level stabilizer. We also proved that every weakly branch group has continuum many distinct atomless ergodic IRSs. This extends a result of Benli, Grigorchuk, and Nagnibeda who show that there exists a group of intermediate growth with this property.

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