Coloring results in some extremal problems

Lucas Colucci

Supervisor: Prof. Ervin Győri

In partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics and its Applications

Department of Mathematics and its Applications
Central European University
Budapest, Hungary
November, 2019

Acknowledgments

First of all, I would like to thank my advisor, Ervin Győri. We first met in the Spring of 2013, when I was his student in his Extremal Combinatorics class at Budapest Semesters in Mathematics. He introduced me to the world of Extremal Graph Theory, and, since then, he was always a source of inspiration and guidance through my years in Budapest.

I am grateful to both Central European University and Alfréd Rényi Institute of Mathematics for the joint PhD program and for the support these institutions have offered me, and also for the year I spent as a junior research fellow in the Institute.

I would like to thank Péter L. Erdős and Tamás Róbert Mezei for the fruitful collaboration (together with Ervin) that led to the results of the first chapter, on terminal-pairability of graphs, and the two joint papers containing them.

I learned the subject of the second chapter, $L(2,1)$-labelings of graphs, at II Workshop Paulista em Otimização, Combinatória e Algoritmos (WoPOCA 2018). I would like to thank the organizers, especially Guilherme Mota, for having invited me to the event, which was financed by FAPESP (São Paulo Research Foundation) and CNPq (Brazilian National Council for Scientific and Technological Development).

I am very grateful to Professor Yoshiharu Kohayakawa for many reasons, since we first met when I was a freshman at Universidade of São Paulo (USP). I am especially thankful for the months I spent at USP under his supervision during the academic year 2017/2018. I learned the subject of the third chapter, edge colorings of graphs without monochromatic subgraphs, from him, during my time there.

Last but not least, I am thankful for all the support my family has given me through all these years. This thesis would never have been completed without their encouragement and love.

Abstract

This thesis deals with three types of problems in Graph Theory related, in different forms, to colorings of graphs.

In the first chapter, we study the well-known edge-disjoint paths problem in a setting introduced by Csaba, Faudree, Gyárfás, Lehel, and Schelp and give new bounds on the maximum degree of a multigraph D that guarantees that it is realizable in a complete bipartite base graph $K_{n,n}$ on the same vertex set, both assuming that D is bipartite with respect to the bipartition classes of $K_{n,n}$ and without this assumption. We also give sharp results on the number of edges that D may have to guarantee that $K_{n,n}$ is terminal-pairable with respect to D. This generalizes the work of Győri, Mezei and Mészáros on complete base graphs. In our proofs, edge colorings of multigraphs are used and we apply the celebrated result of Kahn on the list chromatic index of multigraphs along with other more classical results on edge colorings.

In the second chapter, the well-studied problem of $L(2,1)$-labelings of graphs, a vertex color where the colors satisfy some distance restrictions, is studied. We focus on oriented graphs, and prove an analogous result of Griggs and Yeh in this setting about bounds on the $L(2,1)$ number in terms of the maximum degree of the graph and related parameters. We introduce alternative versions of the $L(2,1)$-labeling problem and prove similar results for these new problems raised. Finally, we improve some results of Jiang, Shao and Vesel on the $L(2,1)$ number of product of oriented cycles.

In the third chapter, we consider the problem of Erdős and Rothschild of determining the maximum number of edge colorings without a monochromatic copy of a fixed subgraph that a graph on n vertices may admit. More especifically, we improve the results of Hoppen, Kohayakawa and Lefmann when the monochromatic forbidden subgraph is a star.

Contents

List of Figures

Introduction

A *vertex coloring* of a graph $G = (V(G), E(G))$ is an assignment of colors to the vertices of G in a way that adjacent vertices receive distinct colors. More specifically, a k*-coloring* of G is a function $c : V(G) \to S$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$ (sometimes the phrasing *proper coloring* or *valid coloring* is used to emphasize this property) and $|S| \leq k$ (we usually take S to be the set of integers from 1 to k, denoted succinctly by $[k]$). The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest value of k such that G admits a k-coloring. In other words, $\chi(G)$ is the least positive integer k such that $V(G)$ can be partitioned into k independent sets, i.e., sets of vertices pairwise not joined by an edge.

Coloring problems are one of the most studied kind of problems in Graph Theory. One reason for this is that it is both rich in applications (we will mention a few in the next paragraphs) and computationally hard: besides the simple cases $k = 1$ and $k = 2$ (a non-empty graph has $\chi(G) = 1$ iff it is edgeless and $\chi(G) = 2$ iff it has at least one edge and no odd cycle), for every $k \geq 3$, it is NP-complete to decide if a given graph admits a k-coloring (in fact, it is one of Karp's 21 NP-complete problems [76]). Also, for all $\varepsilon > 0$, approximating the chromatic number within $n^{1-\varepsilon}$ is NP-hard [128].

The starting point of the study of graph colorings is probably the well-known *Four Color Problem*, raised by Francis Guthrie in 1852: is it possible to color the regions of a map using four colors in a way that no two adjacent regions have the same color? This is equivalent to the statement that $\chi(G) \leq 4$ for every planar graph G. Despite its harmless look, this conjecture took more than a hundred years to be solved (by Appel and Haken in 1976 [6, 7, 8]) and it fostered the development of a good part of modern Graph Theory. We refer to the excellent book of Chartrand and Zhang [20] for a meticulous account of the Four Color Problem, and also as a comprehensive reference about graph colorings.

There are many applications of coloring the vertices of graphs besides coloring a map. Another class of applications are scheduling problems: suppose that an organization has some members that form committees. Each committee holds a meeting for a day, and some members may belong to more than one committee. What is the minimum number of days needed to hold all the meetings in a way that every member attends all the meetings of the committees they belong to? If we construct a graph G whose vertex set is the set of committees and in which two vertices are joined exactly if there is a member that belongs to both committees, then $\chi(G)$ is precisely the number we are looking for.

It is clear that if k vertices of G are joined to each other, then any valid vertex coloring assigns a different color for each one of those vertices. In other words, $\chi(G) \geq \omega(G)$, where $\omega(G)$ stands for the biggest k with the aforementioned property, known as the *clique number* of G. On the other hand, a greedy algorithm colors G with at most $\Delta(G)+1$ colors, so $\chi(G) \leq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of a vertex in G. These two bounds, however, are typically very far from each other and from $\chi(G)$ itself: indeed, if we select uniformly a random graph among all the graphs on the vertex set $[n]$, where n is a large integer, it is very likely (i.e., the probability of this event goes to 1 as n goes to infinity) that the chosen graph G has $\chi(G)$ close to $\frac{n}{2\log_2(n)}$, $\Delta(G)$ close to $\frac{n}{2}$ and $\omega(G)$ is close to $2\log_2(n)$ (see, for instance, [68]).

The classical theorem of Brooks, from 1941, characterizes precisely the graphs that match the upper bound:

Theorem 0.1 (Brooks [14])**.** *For a connected graph G, $\chi(G) \leq \Delta(G)$ unless G is an odd cycle or a complete graph.*

As for the lower bound, the graphs G that satisfy $\chi(H) = \omega(H)$ for all induced subgraphs $H \subseteq G$ are called *perfect graphs*. They form a fascinating class of graphs which is rich in applications and problems. It was defined in the sixties by Berge [13], who conjectured what is known nowadays as the Strong Perfect Graph Theorem, proved in 2006 by Chudnovsky, Robertson, Seymour and Thomas:

Theorem 0.2 (Chudnovsky, Robertson, Seymour and Thomas [21])**.** *A graph is perfect if and only if it does not contain an odd cycle of length at least five or its complement as an induced subgraph.*

One could ask if, in general, the chromatic number could be bounded as a function of ω, i.e., whether there exists a function f such that $\omega(G) \leq k$ implies $\chi(G) \leq f(k)$ for every graph G. This was disproved first by Zykov in 1949 [129] and by Mycielski in 1955 [93]. Both of them constructed families of triangle-free graphs ($\omega = 2$) and chromatic number arbitrarily large, followed by many other constructions:

The Zykov graphs Z_i are defined recursively as follows: $Z_1 = K_1$, the graph on one vertex; for each $i \geq 1$, the graph Z_{i+1} is obtained by taking i copies of Z_i and a set of $|V(Z_i)|^i$ new vertices. We label these vertices with the i-tuples $(v_1, \ldots, v_i)$, where $x_i \in V(Z_i)$ (notice that there exactly $|V(Z_i)|^i$ of them), and join the vertex $(x_1, \ldots, x_i)$ to the vertex x_1 in the first copy of Z_i, x_2 in the second copy of Z_i, and so on. It is not hard to check that all the Z_i are triangle-free and $\chi(Z_i) = i$.

Mycielski's construction is more economic in terms of number of vertices: let M_1 be the graph on one vertex; given $i \geq 1$, M_{i+1} is the graph obtained by taking a copy of M_i, a set S of $|V(M_i)|$ new vertices, say, $S = \{v* : v \in V(M_i)\}$, and a new vertex x. We join the vertex $v*$ to the neighbors of v in M_i, and join x to every vertex in S. Again, it is possible to prove that, for every $i \geq 1$, M_i is a triangle-free graph and $\chi(M_i) = i$.

Erdős and Hajnal gave another construction without the need of recursion, the so-called *shift graphs* SH_n: $V(SH_n) = \{(i,j) : 1 \leq i < j \leq n\}$ and the vertices (p,q) and (r,s) are adjacent only if $q = r$ or $p = s$. The shift graph SH_n has no triangles and $\chi(SH_n) = \lceil \log_2(n) \rceil$.

The well-known Kneser graphs $KG_{n,k}$ are defined as $V(KG_{n,k}) = \binom{[n]}{k}$, i.e, the k-subsets of an n-element ground set, and $E(G) = \{\{U, V\} : U \cap V = \emptyset\}$. It is a theorem of Lovász [86] that, whenever $n \geq 2k$, $\chi(KG_{n,k}) = n - 2k + 2$. Moreover, it is easy to see that if $n < 3k$, then $KG_{n,k}$ is triangle-free. Thus, for some choice of the parameters n and k, the Kneser graph is triangle-free and has chromatic number arbitrarily large.

Tutte's construction, which is better in the sense that the graphs not only do not contain triangles, but the length of their shortest cycle is at least six. However, the number of vertices of those graphs are enormous. The Tutte graph T_i is the graph on one vertex, and for $i \geq 1$, T_{i+1} is constructed from T_i in the following way: if T_i has n_i vertices, take $m_i = \binom{(n_i-1)i+1}{n_i}$ copies of T_i and a set S of $(n_i - 1)i + 1$ new vertices. This set has exactly m_i subsets of size n_i, so there is a bijection between these subsets and the copies of T_i. For each one of the n_i-subsets of S, we take its corresponding copy of T and place an arbitrary matching in between them. This is the graph T_{i+1}. It is possible to prove that, for every $i \geq 1$, T_i has no cycle of length three, four or five, and $\chi(T_i) = i$.

Later, this was widely generalized by Erdős, who proved the following celebrated theorem in 1959 using the probabilistic method:

Theorem 0.3 (Erdős [33])**.** *For every positive integers g and k, there is a graph G with $\chi(G) > k$ and no cycle of length less than g.*

The result above shows that, in a way, the chromatic number is a global property of a graph. Indeed, graphs with large girth look locally like trees, and hence there is no local reason for their chromatic number to be large. Its proof, as many probabilistic proofs, does not construct explicitly graphs with the required property. It took almost 10 years after Erdős's proof for the first such construction, due to Lovász [85], which uses hypergraphs. Another ten years later, Nešetřil and and Rödl [95] gave a simplified construction, which still uses hypergraphs. Only ten years after this constuction, almost thirty years after the original proof of Erdős, a hypergraph-free construction of the theorem was given by Kříž [81]. Another construction was recently found by Alon, Kostochka, Reiniger, West and Zhu [4]. Finally, we remark that certain expander graphs provide more economic constructions of such graphs. However, their constructions rely heavily on algebraic techniques. We refer to [62] for a comprehensive account of the subject.

Still, for some classes of graphs, χ is bounded as a function of ω. Gyárfás [54] defined the notion of χ-boundedness as follows: a family $\mathcal{F}$ of graph is χ-bounded if there is a function f such that for every $G \in \mathcal{F}$ and every induced subgraph H of G, it holds that $\chi(H) \leq f(\omega(H))$. By definition, perfect graphs are χ-bounded (with $f(x) = x$). There is a big list of papers and result in this area in the last few years. We mention an elegant conjecture of Gyárfás [53] and Sumner [111]:

Conjecture 0.4 (Gyárfás and Sumner)**.** *For every fixed tree T, the class of T-free graphs is χ-bounded.*

Another example of application of colorings is the following task distribution problem: in a school, there is a set of teachers and a set of classes. Each teacher is able to give some of the classes, each class lasts one hour and our goal is, assuming that we may have as many simultaneous classes as we wish, to minimize the total length of a timetable. This problem is easily stated as an *edge coloring* problem. If we consider the bipartite graph G in which one color class the vertices are the teachers, in the other, the vertices are the classes, and a teacher and a class are joined by an edge if the teacher is able to teach that class, the number we are looking for is the least number of colors needed to color the edges of G without edges of the same color sharing a vertex. Equivalently, this is the smallest number of classes in a partition of the edges of G into matchings. This parameter is called the *edge-chromatic number*, or the *chromatic index* of G, and it is denoted by $\chi'(G)$.

If $L(G)$ is the *line graph* of G, the graph such that $V(L(G)) = E(G)$ and two distinct edges $e, f \in E(G)$ are joined by an edge in $L(G)$ if and only if they share a common endpoint, we have $\chi'(G) = \chi(L(G))$. Despite this reduction of edge colorings to vertex colorings, χ and χ' behave quite differently. In particular, there is a striking difference in the range of values of these two parameters: while χ is very hard to compute and even to approximate, as mentioned above, $\chi'(G)$ may assume only two values according to $\Delta(G)$, as the celebrated theorem of Vizing states:

Theorem 0.5 (Vizing 1964 [119])**.** *For every graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Back to the scheduling problem, suppose now that there is the extra condition that each person has a different list of available days to meet. How can we arrange the meetings? In the terminology of Graph Theory, this is a *list coloring* of G: in addition to the original condition of a vertex coloring, each vertex v has a list $L(v)$ of admissible colors (the days that every member in this committee is able to meet). A list coloring is simply a vertex coloring c with the requirement that $c(v) \in L(v)$ for every $v \in V(G)$.

The *list chromatic number* (or *choosability*) of G, denoted by $ch(G)$ (or, less often, by $\chi_l(G)$), is the least integer k with the following property: for every assignment L of lists of size at least k to the vertices of G, there is a list coloring with respect to L. This concept was introduced in the seventies independently by Vizing [121] and Erdős, Rubin, and Taylor [34].

The constant assignment $L(v) = \{1, \ldots, k\}$ for every $v \in V(G)$ shows that $ch(G) \geq \chi(G)$. These difference between these numbers can be arbitrarily large, as there are bipartite graphs G with $ch(G) > k$ for every k.

On the other hand, the bound $ch(G) \leq \Delta(G)+1$ also holds, and even Brooks's theorem (Theorem 0.1) is still true for the list chromatic number. As for planar graphs, it is true that $ch(G) \leq 5$ for every planar graph G, as proved by Thomassen [116], but there are planar graphs G with $ch(G) = 5$ (as opposed to $\chi(G) \leq 4$, by the Four Color Theorem).

Another distinctive characteristic of the list chromatic number is that it grows with the average degree, which clearly is not the case for the usual chromatic number. More precisely:

Theorem 0.6 (Alon [2])**.** *There is a function $g(d)$ tending to infinity such that if G has average degree d, then $ch(G) > g(d)$.*

One can define the corresponding edge version of list colorings and define the *list edge chromatic number* (or *edge choosability*) of G accordingly. This number is denoted by $ch'(G)$. Again, we have $ch'(G) \geq \chi'(G)$. In this case, however, the equality was conjectured by many authors since the seventies (see [69] for historical references):

Conjecture 0.7 (List edge coloring conjecture)**.** *For every graph G, $ch'(G) = \chi'(G)$.*

This result was proved to be true for bipartite graphs by Galvin [50]. An asymptotic version was proved by Kahn [73], in the context of multigraphs:

Theorem 0.8 (Kahn [73])**.** *For any multigraph H*

$$\mathrm{ch}'(H) \leq (1 + o(1))\chi'(H).$$

1 Terminal-pairability of graphs (edge-disjoint paths problem)

1.1 Definitions and related problems

Let G be a graph (sometimes called the *base graph*) and D be a loopless multigraph with $V(D) = V(G)$ (sometimes called the *demand graph*). The *terminal-pairability problem*, also known as the *edge-disjoint paths problem*, asks whether we can replace each edge e of D by a path P_e in G joining the endpoints of e in a way that the P_e are pairwise edge-disjoint. If this is the case, the graph with vertex set $V(G)$ and edge set $\bigcup_{e \in E(D)} E(P_e)$ is called a *resolution* of D in G. We say, then, that G is *terminal-pairable* with respect to D, or, conversely, that D is *realizable* (or *resolvable*) in G. More generally, we say that G is terminal-pairable with respect to a family $\mathcal{F}$ of loopless multigraphs (and that $\mathcal{F}$ is realizable in G) if D is realizable in G for every $D \in \mathcal{F}$. Figure 1.1 depicts an example when the base graph is a K_5, and each color represents a path in the resolution.

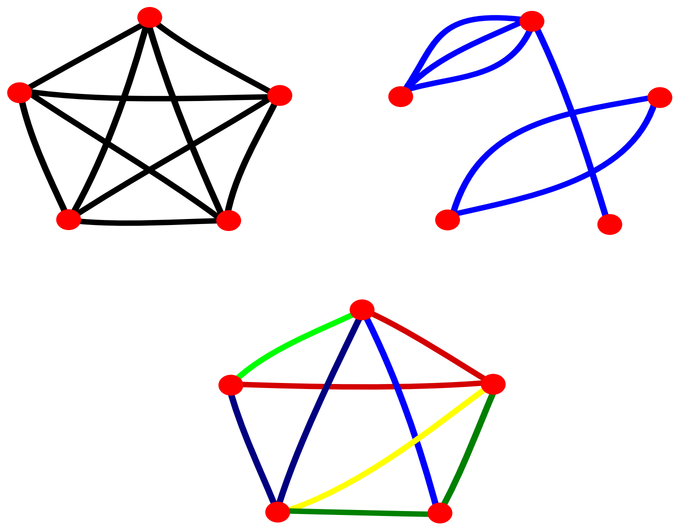

Figure 1.1: A base graph G, a demand graph D, and a resolution of D in G

There are a few well-studied problems related to the edge-disjoint paths problem. We mention them in the next paragraphs.

I. Menger's theorem, Mader's theorem, flows in networks

The study of connection of vertices by paths in graphs traces back to the first half of last century. One of the version of the classical theorem of Menger from 1927 states:

Theorem 1.1 (Menger [89]). *Let G be a graph and u and v two distinct vertices. Then the maximum number of pairwise edge-disjoint paths from u to v is equal to the minimum number of edges whose removal disconnects u and v.*

This theorem corresponds to the edge-disjoint paths problem when the edge set $E(D)$ of the demand graph consists of parallel edges joining a fixed pair of vertices. In particular, it says that a demand graph consisting of k parallel edges uv is realizable in G if and only if G satisfies the following condition: for every partition of $V(G)$ into two sets S and T such that $u \in S$ and $v \in T$, there are at least k edges with one endpoint in each set.

Menger's theorem was later generalized by Mader in 1978, replacing the two vertices u and v above by any subset of $V(G)$. To state his result, we need the following definitions, following the notation of [107]: for a subset $T = \{t_1, \ldots, t_k\} \subseteq V(G)$, a *$T$-path* is a path with endpoints in two distinct vertices of T and all other vertices in $V(G) - T$. Moreover, a collection $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of V is called a *T-partition* if $t_i \in A_i$ for every $i \in [k]$. Finally, for $X \subseteq V(G)$ let $d_G(X)$ denote the number of edges in G with exactly one endpoint in X.

Theorem 1.2 (Mader [87]). *Let G be a graph and $T = \{t_1, \ldots, t_k\} \subseteq V(G)$. The maximum number of edge-disjoint T-paths in G is equal to the minimum of*

$$|A_0| + \sum_{C \in \mathcal{C}} \left\lfloor \frac{1}{2} d_G(C) \right\rfloor$$

taken over all T-partitions $\mathcal{A} = \{A_1, \ldots, A_k\}$, where A_0 is the set of edges whose two endpoints are in A_i and A_j with $i \neq j$; $A = \cup_{i \in [k]} A_i$, and the elements of $\mathcal{C}$ are the vertex sets of the connected components of $G - A$.

In another direction, the first generalization of Menger's theorem to other demand graphs appeared in a paper of Seymour in 1980 [108]. First, he remarks that a simple construction reduces the case of demand graphs consisting of a multistar (a multigraph in which there is a vertex incident to every edge) to Menger's Theorem. Further, he establishes some technical characterization for a graph G to be terminal-pairable with respect to a demand graph D in the next simplest cases: when D contains two edges and when the edges of D cover at most three vertices, the latter using the result of Mader stated above.

Another theorem related to Menger's theorem is the max-flow min-cut theorem of Ford and Fulkerson. Let $D = (V, E)$ be a digraph and s and t be two distinct distinguished vertices of D that we call, respectively, the *source* and the *sink*. The *capacity* $c(e)$ of a (directed) edge e is a positive real number that represents the maximum amount of flow that may pass through it. An *$s - t$ flow* in D is a mapping f from $E(D)$ to the non-negative real numbers such that:

1. For every edge $e \in E(D)$, $f(e) \leq c(e)$,
2. For every vertex v in $V(D) - \{s, t\}$, the amount of flow that enters v is equal to the flow that leaves it:

$$\sum_{u:(u,v) \in E(D)} f((u,v)) = \sum_{w:(v,w) \in E(D)} f((v,w)).$$

The *value* of a flow f is defined to be $\sum_{v:(s,v)\in E(D)} f((s,v))$, and it is simple to check that it is equal to $\sum_{v:(v,t)\in E(D)} f((v,t))$.

An $s-t$ *cut* C is a partition of $V(D)$ into two sets S and T such that $s \in S$ and $t \in T$. The *capacity* of a cut C is defined to be the sum of the capacities of the edges with one endpoint in S and the other in T.

The theorem of Ford and Fulkerson states that:

Theorem 1.3 (Ford and Fulkerson [46])**.** *The maximum value of an $s-t$ flow is equal to the minimum capacity over all $s-t$ cuts.*

We refer to their book [47] for a comprehensive treatment of flows in networks.

Finally, we mention that there is a variation of the max flow problem which is more closely related to the edge-disjoint paths problem. It is called the *multiple-source unsplittable flow*, and it is defined as follows: a digraph G is given in which each edge e has capacity c_e. There are k commodities correspondig to triples $(s_i, t_i, c_i) \in V(G)\times V(G)\times \mathbb{R}^+$, where s_i is the *source*, t_i is the *sink* and c_i is the *demand* of the i-th commodity. The goal of the problem is to route the commodities along G without splitting the flow of any commodity (i.e., the i-th commodity must flow through a single $s_i - t_i$ path) and in a way that the flow in each edge e (the sum of the commodities flowing through it) does not exceed c_e (see [9], [11] and [12]). In case all the demands and capacities are equal and $(u, v) \in E(G)$ implies $(v, u) \in E(G)$ for all vertices u and v, the problem reduces to decide if the underlying undirected graph of G is terminal-pairable with respect to the multigraph obtained by creating an edge e_i joining the pair (s_i, t_i), $i \in [k]$.

II. Graph immersions

Let H and G be (multi)graphs. An *immersion* of H in G is a map ϕ with domain $V(H) \cup E(H)$ such that

- ϕ maps vertices of H into distinct vertices of G,
- a loop on a vertex $u \in V(H)$ is mapped to a cycle of G that contains $\phi(u)$,
- an edge uv $(u \neq v)$ of H is mapped to a path in G connecting $\phi(u)$ and $\phi(v)$,
- for every two distinct edges e_1 and e_2 of H, the images $\phi(e_1)$ and $\phi(e_2)$ are edge-disjoint in G.

Furthermore, if for every $v \in V(H)$ and $e \in E(H)$ we have $v \notin e \implies \phi(v) \notin V(\phi(e))$, ϕ is called a *strong immersion*.

In other words, a multigraph H admits an immersion in the graph G such that $|V(H)| = |V(G)|$ iff there is a bijection $f : V(H) \to V(G)$ such that G is terminal-pairable with respect to the multigraph H' such that $V(H') = V(G)$ and $E(H') = \{f(u)f(v) : uv \text{ is an edge of } H\}$. In the immersion we are free to identify the vertices of H and G (or, in general, a subgraph of G) before replacing the edges by paths. In the terminal-pairability problem, this identification is fixed. Clearly, the two problems are the same if G is the complete graph and H is a loopless multigraph on the same number of vertices. The following result is the best minimum degree condition known that forces a complete graph to be immersed in a graph G:

Theorem 1.4 (Dvořák and Yepremyan [32]). *For every positive integer t, if G is a graph with $\delta(G) \geq 11t + 7$, then there is a strong immersion of K_t in G.*

Robertson and Seymour proved the fundamental result that graph immersions form a well-quasi-ordering of graphs. In other words:

Theorem 1.5 (Robertson and Seymour [103]). *In any infinite sequence of graphs $(G_i)_{i=1}^{\infty}$, there are $i < j$ such that there is an immersion of G_i in G_j.*

Remark 1.6. *The authors mention in the same paper that they might have a proof for strong immersions as well, and state that "even if it was correct it was very much more complicated, and it is unlikely that we will write it down".*

The relation between the chromatic number of a graph and an immersion of a complete graph in it is also studied. Namely, there are the following conjectures:

Conjecture 1.7 (Lescure and Meyniel [84]). *If $\chi(G) \geq t$, then G contains a strong immersion of K_t.*

Conjecture 1.8 (Abu-Khzam and Langston [1]). *If $\chi(G) \geq t$, then G contains an immersion of K_t.*

Remark 1.9. *These two conjectures are analogous of Hadwiger's conjecture, which states that $\chi(G) \geq t$ implies that G contains K_t as a minor (i.e., K_t can be obtained by a sequence of edge contractions, vertex and edge deletions of G), and it is known to be true up to $t = 6$ (see [104]). The corresponding statement obtained by replacing "minor" by "topological minor" (i.e., G contains a subdivision of K_t as a subgraph) in Hadwiger's conjecture, formerly known as Hajós conjecture, is known to be false for all $t \geq 7$ (although it holds for graphs with girth at least 186, as proved in [83]).*

III. Weakly-k-linked graphs

The concept of weakly-linkedness of (multi)graphs was introduced in [115] by Thomassen as follows: a graph G with at least $2k$ vertices is called *weakly-k-linked* if for every choice of (not necessarily distinct) $2k$ vertices of G, $s_1, \ldots, s_k$, $t_1, \ldots, t_k$, there are k pairwise edge-disjoint paths $P_1, \ldots, P_k$ in G such that P_i connects s_i and t_i, $i \in [k]$. In our terminology, a graph is weakly-k-linked if every loopless multigraph with at most k edges is realizable in G.

It is clear that every weakly-k-linked graph is k-edge-connected, and in [115] Thomassen conjectured that in fact, for odd k, a graph is weakly-k-linked if and only if it is k-edge-connected. The search for the smallest function $f(k)$ such that every $f(k)$-edge-connected graph is weakly-k-linked was the object of study of a number of papers [97, 98, 61, 65], culminating in the following result, which is currently the best known:

Theorem 1.10 (Huck [65]). *Let G be a graph and k be a positive integer.*

- *If k is odd and G is $(k+1)$-edge-connected, then G is weakly-k-linked.*
- *If k is even and G is $(k+2)$-edge-connected, then G is weakly-k-linked.*

1.2 Algorithms versus sufficient conditions

The reader may wonder why there are two distinct names for our problem. This fact reflects the different directions of research in the area. We describe both perspectives in the next lines.

The phrasing "edge-disjoint paths problem" (EDP problem for short) was used originally in computer science, where the complexity of constructing the set of edge-disjoint paths is studied. In this direction, the decision version of EDP was first shown to be NP-complete by Even, Itai, and Shamir [41]. Robertson and Seymour [105] proved that for a fixed number of paths the problem is solvable in polynomial time, and the running time was later improved by Kawarabayashi, Kobayashi and Reed [77] (these results are about vertex-disjoint paths, but by moving to the line graph of G, edge-disjoint paths become vertex-disjoint). However, if the number of required paths is part of the input then the problem is NP-complete even for complete [80] and series-parallel graphs [96]. The problem is NP-hard even if $G+D$ (the graph obtained by taking the disjoint union of the edge sets) is Eulerian and D consists of at most three sets of parallel edges, as shown by Vygen [122]. If no restrictions are made on G, then the problem is NP-hard for one set of parallel edges which should be mapped to edge-disjoint paths of length exactly 3, see [5].

On the other hand, Csaba, Faudree, Gyárfás, Lehel and Schelp, in a paper in 1992 [29], coined the term "terminal-pairability" as they applied the edge-disjoint paths problem in a practical task of building a network of processors in a way that allows simultaneous communication with any pairing of the so-called terminal nodes, which are degree one processors. In the same paper, they introduced the concept of *path-pairability*: Let $G=(V,E)$ be a graph on an even number of vertices, and $\mathcal{M}$ be the set of matchings of the complete graph with vertex set V. G is said to be *path-pairable* if it is terminal-pairable with respect to $\mathcal{M}$. This is a fast growing line of research, and among the many results in the area, we mention the following question: what is the minimum maximum degree that a path-pairable graph G on n vertices may have? Faudree, Gyárfás and Lehel [43] proved that $\Delta(G) \geq c \log n / \log\log n$ for some constant c. As for upper bounds, path-pairable graphs on n vertices and $\Delta \sim 2\sqrt{n}$ and $\Delta \sim \sqrt{n}$ were constructed by Kubicka, Kubicki and Lehel [82] and Mészáros [90], respectively. More recently, Győri, Mezei and Mészáros [57] greatly improved the upper bound by constructing a path-pairable graph on n vertices and $\Delta \leq c \log n$, where $c < 8$. Finally, we mention a conjecture that traces back to the original paper of Csaba et al., and which would give another example of path-pairable graph with $\Delta < C \log n$, with an even smaller constant C:

Conjecture 1.11 ([29])**.** *For every $k \geq 1$, the $(2k+1)$-dimensional hypercube Q_{2k+1} is path-pairable.*

Back to terminal-pairability in general, considering the NP-hardness of the problem, we do not hope to give a condition which is both necessary and sufficient for D to be realizable in G. Instead, sufficient conditions for a graph D to be realized in G are sought after.

More specifically, Gyárfás, Lehel and Schelp in [29] and later Győri, Mezei and Mészáros in [56] dealt with the following question: given a fixed base graph G (in their case, the complete graph), what are some sufficient conditions for a demand graph D, in terms of its maximum degree or its number of edges, to be realizable in G? The second set of authors proved the following two results (the first one is an improvement on an early result from the first group of authors, and the second generalizes a result of [82]):

Theorem 1.12 (Győri, Mezei and Mészáros [56])**.** *If D is a loopless multigraph on n vertices such that $\Delta(D) \leq 2\lfloor n/6 \rfloor - 4$, then D is realizable in K_n.*

Theorem 1.13 (Győri, Mezei and Mészáros [56])**.** *If D is a loopless multigraph on n vertices, $e(D) \leq 2n-5$ and $\Delta(D) \leq n-1$, then D is realizable in K_n. Moreover, a demand graph consisting of the disjoint union of two sets of $n-2$ parallel edges shows that the bound is sharp.*

Remark 1.14. *The condition $\Delta(D) \leq n-1$ is clearly necessary, as for a demand graph D to be realizable in G we must have $d_D(v) \leq d_G(v)$ for every $v \in V(G) = V(D)$.*

While Theorem 1.13 is sharp, this is not the case for Theorem 1.12: on the one hand, it is easy to see that the result cannot hold if $\Delta(D) > n/2$, as shown by a demand graph obtained by replacing each edge in a one-factor of G by a set of $\Delta(D)$ parallel edges. It was conjectured in [29] and [42] that if $n \equiv 2 \pmod 4$, the upper bound of $n/2$ can be attained. This was proved to be false by Girão and Mészáros in a short and elegant proof that we present here for the sake of completeness:

Theorem 1.15 (Girão, Mészáros [51])**.** *If $q > 13/27n + O(1)$, there is a demand graph on n vertices such that $\Delta(D) = q$ and D is not realizable in K_n.*

Proof. We may assume that n is divisible by 3. Let q be an integer and D be the demand graph obtained by partitioning the n vertices into $n/3$ triples and placing $q/2$ parallel edges joining every pair of vertices in each triple (so D is a q-regular multigraph), i.e., the graph consists of $n/3$ edge-disjoint triangles in which every edge has multiplicity $q/2$.

Assume $\mathcal{P}$ is a collection of paths that corresponds to a resolution of D in K_n. Note that $e(D) = nq/2$ and that at most n demand edges can be realized using exactly one edge of K_n or using 2 edges within is triple, thus at least $nq/2 - n$ demand edges correspond to paths of length 2 or more in $\mathcal{P}$. In particular, if t denotes the number of paths of length 2 in $\mathcal{P}$ not lying within a triple, then the following condition holds due to simple edge counting:

$$n + 2t + 3(nq/2 - n - t) \leq n(n-1)/2,$$

that is, $t \geq n/2(3q - n - 3)$. Hence, if q is sufficiently large (in terms of n), then lots of demand edges must be realized through path of length 2 (we call it a "cherry").

For a triangle T_i, let α_i denote the number of demand edges not in T_i that are resolved in a cheery through any vertex of T_i. Also, let β_i be the number of demand edges of T_i that are resolved via a cherry with its middle vertex lying outside of T_i. Observe that by simple double-counting:

$$\sum_{i=1}^{n/3} \alpha_i + \beta_i \geq n(3q - n - 3)$$

and therefore there must exist a triangle T_i with $\alpha_i + \beta_i \geq 3(3q - n - 3)$.

Note that between two distinct triangles at most 4 edges can be solved via paths of length 2 (every cherry requires two edges between the triangles in K_n and we only have 9 of them). This implies that between T_i and any other triangle at most $4(n/3 - 1)$ demand edges can be solved via cherries. Hence, $4(n/3 - 1) \geq 3(3q - n - 3)$, which implies $q \leq 13n/27 + 5/9$.

□

1.3 Complete bipartite base graphs

The next natural step after taking $G = K_n$ is to consider $G = K_{n,n}$ or, more generally, a complete bipartite graph. In this section, we state and prove analogous results of Theorems 1.12 and 1.13 in this setting.

Given an edge $e \in E(D)$ with endvertices u and v, we define the *lifting of e to a vertex $w \in V(D)$*, as an operation which transforms D by deleting e and adding two new edges joining uw and vw; in case $u = w$ or $v = w$, the operation does not do anything. We stress that we do not use any information about G to perform a lifting and that the graph obtained using a lifting operation is still a demand graph.

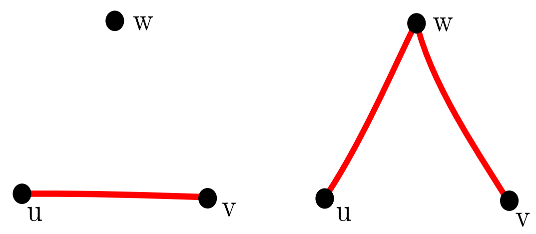

Figure 1.2: A lifting of an edge uv to a vertex w

Notice that the terminal-pairability problem defined by G and D is solvable if and only if there exists a series of liftings, which, applied successively to D, results in a subgraph of G, which is the resolution of D in G. The edge-disjoint paths can be recovered by assigning pairwise different labels to the edges of D, and performing the series of liftings so that new edges inherit the label of the edge they replace. Clearly, edges sharing the same label form a walk between the endpoints of the demand edge of the same label in D, and so there is also a such path.

There is a bipartite demand graph D with $\Delta(D) = \lceil n/3 \rceil + 1$ which is not realizable in $K_{n,n}$, namely the union of n pairs of vertices joined by $\lceil n/3 \rceil + 1$ parallel edges. Indeed, in any resolution of D in $K_{n,n}$, from each set of edges joining the same pair of vertices at most one edge is resolved into a path of length 1 (itself), while the rest of them must be replaced by paths of length at least 3. Therefore, every realization uses at least $n + 3 \cdot n \cdot \lceil n/3 \rceil \geq n^2 + n$ edges in a $K_{n,n}$, which is a contradiction.

Two different cases are considered: first, when the demand graph D is also bipartite with respect to the color classes of G; later, we studied the case when D may have edges inside the color classes of G as well.

1.3.1 Bipartite demand graphs

If D is also bipartite with respect to the color classes of $K_{n,n}$, a result of Gyárfás and Schelp [55] gave the first lower bound on the maximum degree that guarantees that D is realizable in $K_{n,n}$:

Theorem 1.16 (Gyárfás, Schelp [55])**.** *If a demand graph D is bipartite with respect to the color classes of $K_{n,n}$ and $\Delta(D) \leq n/12$, then D is realizable in $K_{n,n}$.*

We improve this result by a factor of 3, replacing $n/12$ by $n/4$ (asymptotically) in a slightly more general context:

Theorem 1.17 (Colucci, Erdős, Győri and Mezei [23])**.** *Let D be a bipartite demand graph whose two color classes A and B have sizes a and b, respectively. If $d(x) \leq (1 - o(1))b/4$ for all $x \in A$ and $d(y) \leq (1 - o(1))a/4$ for all $y \in B$, then D is resolvable in the complete bipartite graph with color classes A and B.*

On the other hand, we show that, for some reasonably broad class of bipartite graphs, every demand graph with maximum degree at most $n/3$ in this class is resolvable in $K_{n,n}$. More specifically, we proved the following:

Theorem 1.18 (Colucci, Erdős, Győri and Mezei [23])**.** *Assume that n is divisible by 3, and let D be a bipartite demand graph with base graph $K_{n,n}$, such that*

$$U = \biguplus_{i=1}^{3} U_i \text{ and } V = \biguplus_{i=1}^{3} V_i$$

are the two color classes of D with $|U_i| = |V_i| = \frac{n}{3}$ for $i = 1, 2, 3$, where $\biguplus$ denotes disjoin union of sets. If $\Delta(D) \leq \frac{n}{3}$ and for any $i \neq j$ there is no edge of D joining some vertex of U_i to some vertex of V_j, then D is resolvable in $K_{n,n}$.

As for the upper bound, no analogous of Theorem 1.15 is known. Girão and Mészáros conjecture that the trivial upper bound can be improved:

Conjecture 1.19 ([51])**.** *There is $\varepsilon > 0$ such that there is a demand graph D for $K_{n,n}$ with $\Delta(D) \leq (1/3 - \varepsilon)n$ and D is not realizable in $K_{n,n}$.*

The extremal number of edges, as in the case of complete base graphs, can be determined precisely. We proved that:

Theorem 1.20 (Colucci, Erdős, Győri and Mezei [23])**.** *Let $n \geq 4$ and D be a bipartite demand graph with the base graph $K_{n,n}$. If D has at most $2n-2$ edges and $\Delta(D) \leq n$, then D is resolvable in $K_{n,n}$.*

Remark 1.21. *Again, the maximum degree condition is trivially required, since at most n edge-disjoint paths can start from a vertex v.*

Remark 1.22. *The following demand graph D shows that the result above is sharp: a pair of vertices joined by n edges, another pair of vertices joined by $n-1$ edges, and $2n-4$ isolated vertices. Indeed, in any resolution of D, one of the paths corresponding to one of the n edges joining the first pair of vertices passes through a vertex of the pair of vertices joined by $n-1$ edges, implying that this vertex has degree $\geq n+1$ in the resolution, a contradiction.*

Proofs of the Theorems 1.17, 1.18 and 1.20

Before proving Theorems 1.17 and 1.18, we will recall some definitions and lemmas about multigraphs.

Let H be a loopless multigraph. The *chromatic index* or the *edge chromatic number*, denoted by $\chi'(H)$, is the minimum number of colors required to properly color the edges of a graph H. Similarly, the *list chromatic index* or the *list edge chromatic number*, denoted by $\mathrm{ch}'(H)$, is the smallest integer k such that if for each edge of G there is a list of k different colors given, then there exists a proper coloring of the edges of H where each edge gets its color from its list. The maximum multiplicity, denoted by $\mu(H)$, is the maximum number of edges joining the same pair of vertices in H. The number of edges joining a vertex $x \in V(H)$ to a subset $A \subseteq V(H)$ of vertices is denoted by $e_H(x, A)$. The (non-multi) set of neighbors of x in H is denoted by $N_H(x)$. For other notations the reader is referred to [30]. We are going to apply the following well-known results:

Theorem 1.23 (Kőnig [79])**.** *For any bipartite multigraph H we have $\chi'(H) = \Delta(H)$, or, in other words, the edge set of H can be decomposed into $\Delta(H)$ matchings.*

Theorem 1.24 (Vizing [120])**.** *For any multigraph H*

$$\chi'(H) \leq \Delta(H) + \mu(H).$$

Theorem 1.25 (Kahn [74])**.** *For any multigraph H*

$$\mathrm{ch}'(H) \leq (1 + o(1))\chi'(H).$$

Finally, we remark that, even though in our theorems the demand graphs are bipartite, in the proofs we may transform them into non-bipartite ones.

Proof of Theorem 1.18. Let D_i be the (bipartite) subgraph of D induced by $U_i \cup V_i$ for $i = 1,2,3$. As parallel edges are allowed in D, without loss of generality, we may assume that D_i is $\lfloor \frac{n}{3} \rfloor$-regular. By Kőnig's theorem, $E(D_i)$ can be partitioned into matchings $M_{i,1}, M_{i,2}, \ldots, M_{i,\lfloor \frac{n}{3} \rfloor}$, each of size $|U_i|$. We derive D'_i from D by lifting the edges of $M_{i,j}$ to the j^{th} vertex of U_i for each $j = 1,2,\ldots,\lfloor \frac{n}{3} \rfloor$. Firstly, all the edges of D'_i between U_i and V_i have multiplicity 1. Secondly, observe, that $D'_i[U_i]$ is $2(\lfloor \frac{n}{3} \rfloor - 1)$-regular and $\mu(D'_i[U_i]) = 2$.

Applying Vizing's theorem we get $\chi'(D'_i[U_i]) \le \Delta(D'_i[U_i]) + \mu(D'_i[U_i]) = 2\lfloor \frac{n}{3} \rfloor$, so let $c_i : E(D'_i[U_i]) \to \{1,2,\ldots,2\lfloor \frac{n}{3} \rfloor\}$ be a proper-coloring of $D'_i[U_i]$. Let D' be the (disjoint) union of D'_1, D'_2, D'_3. We derive D'' from D' by lifting each edge of $c_i^{-1}(j)$ to the j^{th} vertex of $V_{i+1} \cup V_{i+2}$ (take the indices cyclically). Observe that D'' is a simple bipartite graph, whose color classes are still U and V, and it is obtained from D via a series of liftings, therefore it is a resolution of D. □

Proof of Theorem 1.17. Let us assume that $a \ge b$ and $A = \{v_1, \ldots, v_a\}$. By adding edges, if necessary, we may assume that D is semiregular with degrees Δ_A and Δ_B, where $|E(D)| = a \cdot \Delta_A = b \cdot \Delta_B$. As D is bipartite, by Kőnig's theorem we have $\chi'(D) = \Delta(D) = \Delta_B$, which means that we can split the edges of D into Δ_B matchings of size b, say $M_1, M_2, \ldots, M_{\Delta_B}$.

We claim that by splitting these matchings appropriately, we can get a partition of the edges of D into matchings $M'_1, M'_2, \ldots, M'_a$, each of size Δ_A. Pick Δ_A edges of M_1 arbitrarily to get M'_1 and continue picking sets of Δ_A edges of M_1 that are disjoint from the previously chosen sets, until less than $b/4$ edges of M_1 are available. Put the remaining edges into a new M'_i; it is easy to see that these edges intersect at most $b/2$ edges of M_2, so we can pick some of these edges of M_2 to fill up M'_i to the appropriate size. Continue this procedure until less than $b/4$ edges remain in M_{Δ_B}. However, as $a = |E(D)|/\Delta_A$, this means that actually all the edges in M_{Δ_B} are used up as well, thus our claim is proven.

For each $1 \le i \le a$, we lift the edges of M'_i to v_i. Let us call the resulting demand graph D'. In D' there are no multiple edges between A and B, $\mu(D'[A]) \le 2$, $e_{D'}(v_i, A) \le 2\Delta_A$ and $e_{D'}(v_i, B) = \Delta_A$ for all $v_i \in A$.

To each edge e with end vertices v_i and v_j, we associate a list $L(e)$ of vertices of B, to which we can lift e to without creating multiple edges:

$$L(e) = V(B) \setminus ((N_{D'}(v_i) \cap B) \cup (N_{D'}(v_j) \cap B).$$

We have $|L(e)| \ge b - e_{D'}(v_i, B) - e_{D'}(v_j, B) \ge b - 2\Delta_A$. By Kahn's theorem (Theorem 1.25), $\mathrm{ch}'(D'[A]) \le (1 + o(1))\chi'(D'[A])$. Furthermore, by Vizing's theorem (Theorem 1.24), $\chi'(D'[A]) \le \Delta(D'[A]) + \mu(D'[A]) \le 2\Delta_A + 2$. By the assumptions made in the statement of the theorem on Δ_A, we have $\mathrm{ch}'(A) \le |L(e)|$ for each edge e in $E(D'[A])$.

Thus, there is a proper list edge coloring c which maps each $e \in E(D'[A])$ to an element of $L(e)$. Finally, we lift every edge $e \in E(D'[A])$ to $c(e)$. As we do not create multiple edges between A and B, the resulting graph is a resolution of D. $\square$

Proof of Theorem 1.20. We proceed by mathematical induction on n. It is easy to check that the result holds for $n = 4, 5$ by a straightforward case analysis.

Let A and B be the color classes of D, each of cardinality n. In the induction step we lift some edges in D in such a way that the resulting graph D' is still bipartite with the same color classes and there exists a subset $Z \subset V(D')$ such that

1. $|Z \cap A| = |Z \cap B|$ holds,
2. $\geq |Z|$ edges of D' are incident to vertices of Z,
3. $\Delta(D'[(A \cup B) \setminus Z]) \leq n - |Z|/2$, and
4. there are no multiple edges incident to vertices of Z in D'.

The first three conditions guarantee that we can invoke the inductive hypothesis on $D'[(A \cup B) \setminus Z]$, to conclude that $D'[(A \cup B) \setminus Z]$ is resolvable. The fourth condition now implies that D' is resolvable as well, which in turn implies the same for D.

Since we want to keep D' bipartite with the same color classes as D, we define the *edge-lifting* of an edge $e \in E(D)$, with end vertices $u \in A$ and $v \in B$, to xy, whenever $\{u, v, x, y\}$ are four different vertices and $x \in A$ and $y \in B$: the operation adds a copy of xy, uy, and xv to D and then deletes e. Note that an edge-lifting operation can also be obtained as a composition of two liftings (one to x and then to y).

Assume now that $n \geq 6$ and let D be a demand graph on $2n-2$ edges (we may assume that by adding edges between two vertices of degree less than n in distinct classes). Let

$$X = \{v \in A \cup B \ : \ d(v) = n\}.$$

As we have $2n - 2$ edges, it is clear that X meets both A and B in at most one vertex, so $|X| \leq 2$. Furthermore, each color class has either at least one isolated vertex or at least two vertices of degree 1.

We distinguish four major cases.

Case 1 $u_1, u_2 \in A$ and $v_1, v_2 \in B$ are two-two isolated vertices in A and B.

Let $Y = \{v \in A \cup B \ : \ d(v) \geq n - 1\}$ and set $Z = \{u_1, u_2, v_1, v_2\}$. Suppose there exists a set $F \subset E(D)$ of four edges, which cover every vertex of D at most twice, cover every element of Y at least once, and cover every element of X exactly twice. It is easy to see that there is an ordering $F = \{e_1, e_2, e_3, e_4\}$ of these edges, so that edge-lifting e_1 to u_1v_1, e_2 to u_1v_2, e_3 to u_2v_2, and e_4 to u_2v_1 does not create multiple edges. Therefore, given the existence of F, we can invoke the inductive hypothesis and conclude that D is resolvable in $K_{n,n}$.

Notice that

$$\sum_{v\in Y} d(v) - |E(D[Y])| \leq |E(D)| = 2n-2,$$

and $\Delta(D[Y]) \leq n$. Depending on the cardinality of $|Y|$, we distinguish 5 subcases.

Case 1.1 $|Y| = 4$.

We have $|E(D[Y])| = 2n-2$. If there is a C_4 in $D[Y]$, then the edges of the cycle are a good choice for F. Otherwise we can pair the vertices of Y in such a way that the pairs are joined by at least $n-2$ edges each; choose two edges from each pair, and let their set be F.

Case 1.2 $|Y| = 3$.

We have $n \geq |E(D[Y])| \geq n-1$. Without loss of generality, we may suppose that $A \cap Y = \{a_1\}$ and $B \cap Y = \{b_1, b_2\}$, and that $e(a_1, b_1) \geq (n-1)/2 \geq 2$. Therefore $e(b_2, V(D) \setminus Y) \geq (n-1)/2 \geq 2$ as well. Choose two edges joining a_1 to b_1 and two edges joining b_2 to $V(D) \setminus Y$, and let their set be F.

Case 1.3 $|Y| = 2$.

If both $e(A \cap Y, V(D) \setminus Y) \geq 2$ and $e(B \cap Y, V(D) \setminus Y) \geq 2$, then choose from the respective sets two-two edges; this is a good choice for F. Otherwise $|E(D[Y])| \geq n-2$, therefore there are at most $n+2$ edges incident on Y, or in other words, $V(D) \setminus Y$ induces at least $n-4 \geq 2$ edges. Choose two edges from both $D[Y]$ and $D[V(D) \setminus Y]$, and let their set be F.

Case 1.4 $|Y| = 1$.

There is a vertex v to which Y is joined by at least two edges (there are two isolated vertices in both color classes). The vertex v and Y cover at most $2n-4$ edges, so select two edges not intersecting either v or Y into F, plus two edges joining v and Y; let their set be F.

Case 1.5 $|Y| = 0$.

There are two vertices joined by at least two edges, as otherwise D is the resolution of itself. We can proceed exactly as in the $|Y| = 1$ case.

From now on, without loss of generality, we may assume that there is at most one isolated vertex in one of the classes.

Case 2 X is empty.

Case 2.1 We have a vertex x of degree 1 in one of the classes, say, A.

We can find a vertex y in B that is either isolated or has degree 1 and is not joined to x. In the first case, we edge-lift an edge $e \in E(D)$, which is not incident to x or to the neighbor of x, to xy and let $Z = \{x, y\}$. In the latter case, simply let $D' = D$ and $Z = \{x, y\}$.

Case 2.2 There is no vertex of degree one in D.

We must have at least one isolated vertex in each class. Furthermore, the average degree of the remaining vertices in each class is $(2n-2)/(n-1) = 2$, so we either have another isolated vertex or every remaining vertex has degree exactly two.

Recall that we may assume that there is at most one isolated vertex in one of the classes.

Case 2.2.1 There is a vertex of degree two without multiple edges.

Put this vertex and an isolated vertex from the other class into Z, and invoke the inductive argument.

Case 2.2.2 There are two isolated vertices, a and b, in one of the classes.

Without loss of generality, we may assume that $a, b \in A$. All but one vertex of B has degree two and we may assume that all of them have parallel edges. In this case, let u and v be the vertices in A with highest degrees, and let z be a neighbor of u and w be a neighbor of v. We edge-lift uz to aw, wv to bz and let $Z = \{a, b, z, w\}$.

Case 2.2.3 There is only one isolated vertex in both A and B.

We may assume that every remaining vertex has degree two and is the endpoint of two parallel edges. In this case, let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be the vertices of D, with a_i and b_i connected by two edges for each $1 \leq i \leq n-1$ and a_n and b_n isolated. In this setting, we construct a resolution of D by edge-lifting a copy of the edge a_ib_i to $a_{i+1}b_{i+2}$ for each $1 \leq i \leq n-2$, and edge-lifting a copy of $a_{n-1}b_{n-1}$ to a_nb_1.

Case 3 $|X| = 1$.

Let $z \in A$ be the only vertex of degree n in D. Notice that, in this case, there is no vertex of degree $n-1$ in A and there exists at least one isolated vertex in A. Let us call it v.

Case 3.1 There is a vertex u of degree 1 in B.

We have two cases: if it is joined to z, we edge-lift a copy of an edge xy independent from uz to uv and let $Z = \{u, v\}$. If u is not joined to z, we simply edge-lift any edge incident on z to uv and let $Z = \{u, v\}$.

Case 3.2 There is no vertex of degree 1 in B.

There must be an isolated vertex u in this class, and the average degree of the remaining vertices is $(2n-2)/(n-1) = 2$. Therefore, either every remaining vertex has degree exactly two or there is another isolated vertex in B.

Case 3.2.1 Every vertex in B except u has degree two.

Either one of them has no adjacent multiple edges or the neighborhood of each of them consists of two parallel edges. In the first case, let x be a vertex without multiplicity. We simply edge-lift an edge of z to uv and let $Z = \{x, v\}$. In the latter case, the degree of each vertex is even, and, as the average degree of the vertices in $A/\{v, z\}$ is 1, we must have another isolated vertex v' in A. Let a be a neighbor of z and b be a vertex of B not joined to z, let z' be its neighbor. We edge-lift a copy of az to bv, bz' to av' and let $Z = \{v, v', a, b\}$.

Case 3.2.2 There is another isolated vertex u' in B.

The remaining vertices of A have average degree $(n-2)/(n-2) = 1$, so all of them have degree one (recall that there is at most one isolated vertex in one of the classes). In the first case, just take a neighbor x of z that has a non-neighbor y of degree one in A (it does exist because z has at least two neighbors). Edge-lift the edge zx to uy and let $Z = \{u, y\}$.

Case 4 $|X| = 2$.

Let $z_1 \in A$ and $z_2 \in B$ be the vertices of degree n. Notice that z_1 and z_2 must be joined by at least two edges and that there is no other vertex of degree n or $n-1$ in D. Furthermore, in each class, we must have an isolated vertex, $v_1 \in A$ and $v_2 \in B$, and the average degree of the remaining vertices is $(n-2)/(n-2) = 1$, so in each class either we have another isolated vertex or all the remaining vertices have degree one.

Recall that there is at most one isolated vertex in one of the classes, say B. All vertices except z_2 and v_2 have degree one, then either we have a vertex x of degree one which is not joined to z_1, or z_1 is joined to every vertex of positive degree in B. In the first case, edge-lift a copy of z_1z_2 to v_1x and let $Z = \{x, v_1\}$. In the latter case, the neighborhood of z_1 consists of $n-2$ simple edges connecting it to the vertices of degree one in B and one double edge joining z_1 and z_2. Simply edge-lift one copy of this double edge to v_1v_2 and let $Z = \{z_1, v_2\}$ (as z_1 has no multiplicities now).

Our case analysis is now complete, as is the proof of Theorem 1.20.

□

1.3.2 Non-bipartite demand graphs

In this section, we extend the results above, namely Theorems 1.17 and 1.20, for the case when the condition that the demand graph is bipartite is dropped.

We were able to prove a result similar to Theorem 1.17 in case G is a complete symmetric bipartite graph. Namely, we proved that:

Theorem 1.26 (Colucci, Erdős, Győri and Mezei [24])**.** *Let D be a demand graph with $V(D) = V(K_{n,n})$, where D is not necessarily bipartite. If*

$$\Delta(D) \leq (1 - o(1)) \cdot \frac{n}{4}$$

as $n \to \infty$, then D is realizable in $K_{n,n}$.

Furthermore, if D has many edges inside the color classes of $K_{n,n}$, we were able to give a better (bigger) bound on $\Delta(D)$:

Theorem 1.27. *Let D be a (not necessarily bipartite) demand graph for $K_{n,n}$, the complete symmetric bipartite graph with color classes A and B. If*

$$\Delta(D) \leq (1 - o(1)) \cdot \left(\frac{2n}{7} - \frac{3}{7} \cdot \frac{e(D[A,B])}{n} \right)$$

as $n \to \infty$, then D is realizable in $K_{n,n}$.

Finally, as an analogous of Theorem 1.20, we prove the following:

Theorem 1.28 (Colucci, Erdős, Győri and Mezei [24])**.** *Let $n \geq 2$ and D be a demand graph for $K_{n,n}$, the complete symmetric bipartite graph with color classes A and B. If D has at most $2n-3$ edges and $\Delta(D) \leq n$, then D is realizable in $K_{n,n}$.*

Remark 1.29. *Again, the assumption $\Delta(D) \leq n$ is trivially necessary, as at any given vertex, there can be at most n edge-disjoint paths that terminate there. Moreover, this result is sharp as well, as shown by the demand graph on $2n-2$ edges consisting of two bundles of $n-1$ edges, where one of the bundles joins an arbitrary pair of vertices in A, while the other bundle joins a pair in B.*

Proofs of the Theorems 1.26, 1.27 and 1.28

For the proof of these theorems, we will need the following well-knows results, in addition to the ones in the previous section:

Proposition 1.30 (Equitable edge coloring)**.** *If H is a multigraph and $\chi'(H) \leq k$ for some integer k, then there is an equitable edge coloring of H with exactly k colors, i.e., a proper edge coloring in which the sizes of any two color classes differ by at most one.*

Proposition 1.31 (Greedy edge coloring)**.** *For any multigraph H we have*

$$\chi'(H) \leq \mathrm{ch}'(H) \leq 2\Delta(H) - 1.$$

Theorem 1.32 (Shannon [109])**.** *For any multigraph H, its chromatic index satisfies*

$$\chi'(H) \leq \frac{3}{2}\Delta(H).$$

Also, we need to prove a technical proposition before the proofs of Theorems 1.26 and 1.27:

Proposition 1.33. *If D is a demand graph on the vertex set $V(K_{n,n})$ and $\Delta(D) \leq n/4$, then there exists a proper edge $2\lfloor n/2 \rfloor$-coloring of $D[A,B] \cup D[B]$, which induces an equitable $2\lfloor n/2 \rfloor$-coloring on $D[B]$ and an almost equitable (the difference between the sizes of two color classes is ≤ 2) coloring on $D[A,B]$.*

Proof. Observe that (by Proposition 1.31)

$$\chi'(D[B]) \le 2\Delta(D) \le \frac{1}{2}n,$$
$$\chi'(D[A,B]) \le 2\Delta(D) \le \frac{1}{2}n.$$

By Proposition 1.30, there is a partition of $E(D[B])$ into $\lfloor n/2 \rfloor$ matchings of size $\lfloor e(D[B])/n \rfloor$ and $\lceil e(D[B])/n \rceil$, say $M_1, \ldots, M_{\lfloor n/2 \rfloor}$, so that $|M_i| \ge |M_j|$ for $i < j$. Similarly, there is a partition of $E(D[A,B])$ into $\lfloor n/2 \rfloor$ matchings of size $\lfloor e(D[A,B])/n \rfloor$ and $\lceil e(D[A,B])/n \rceil$, say $N_1, \ldots, N_{\lfloor n/2 \rfloor}$, so that $|N_i| \le |N_j|$ for $i < j$. It is sufficient to prove now that for all $i = 1, \ldots, \lfloor n/2 \rfloor$, there exists a 2-coloring of $M_i \cup N_i$ which is induces an equitable 2-coloring on $D[B]$ and induces an almost equitable 2-coloring on $D[A,B]$.

Fix i. Observe, that $M_i \cup N_i$ is the vertex disjoint union of some edges and paths composed of two or three edges that alternate between elements of M_i and N_i. The paths of two and three edges contain one edge of M_i exactly. Let the number of components of $M_i \cup N_i$ containing k edges be c_k.

Color the M_i edge of $\lfloor c_3/2 \rfloor$ of the path components of length three with color 1, and color the M_i edges of the remaining $\lceil c_3/2 \rceil$ paths of length three with color 2. Similarly, color the M_i edge of $\lceil c_2/2 \rceil$ paths of length two with color 1, and color the remaining $\lfloor c_2/2 \rfloor$ uncolored M_i edges in paths of length two with color 2.

The colors of edges of N_i intersecting colored M_i edges are now determined in a proper coloring using the colors $\{1, 2\}$. Let this proper partial 2-coloring be c. It trivially induces an equitable coloring on $D[B]$. On $D[A,B]$, we have $|c^{-1}(1) \cap D[A,B]| = 2\lceil c_3/2 \rceil + \lfloor c_2/2 \rfloor$ and $|c^{-1}(2) \cap D[A,B]| = 2\lfloor c_3/2 \rfloor + \lceil c_2/2 \rceil$, the difference of which is clearly at most 2. As the yet uncolored edges of $M_i \cup N_i$ are vertex disjoint, this partial coloring can be extended to a proper 2-coloring, which is equitable in M_i and almost equitable in N_i. □

Proof of Theorem 1.26. As D has an even number of vertices, we may assume that D is regular by adding edges, if necessary. Clearly, $e(D[A]) = e(D[B])$, $e(D) = e(D[A]) + e(D[A,B]) + e(D[B])$, and $e(D) = n \cdot \Delta(D)$.

Our proof consists of three steps. In the first step, we resolve the high multiplicity edges of $D[A]$, while leaving $D[A,B] \cup D[B]$ untouched. In the second step, we lift the edges of $D[B]$ to A, and resolve the multiplicities of $D[A,B]$. In the third step, we lift the edges induced by A to B, while preserving a simpleness of the bipartite subgraph induced by A and B, thus we end up with a graph which is a realization of D.

By Proposition 1.31, $\chi'(D[A]) \le n$, so Proposition 1.30 implies the existence of an equitable edge n-coloring c_1 of $D[A]$. We construct D' from D by lifting the elements of $c_1^{-1}(i)$ to a_i for all $i = 1, \ldots, n$. As c_1 is a proper coloring, $\mu(D'[A]) \le 2$. For any $a \in A$ and $b \in B$, we have the following estimates:

$$e_{D'}(a,A) \le e_D(a,A) + 2 \cdot \lceil e(D[A])/n \rceil,$$
$$e_{D'}(a,B) = e_D(a,B), \ e_{D'}(b,A) = e_D(b,A), \ e_{D'}(b,B) = e_D(b,B).$$

For the second step, we use Proposition 1.33 to take a proper edge n-coloring c_2 of $D[A,B] \cup D[B]$, which is an (almost) equitable n- or $(n-1)$-coloring if restricted to both $D[A,B]$ and $D[B]$. We get D'' from D' by lifting the elements of $c_2^{-1}(i)$ to a_i for all $i = 1, \dots, n$. As c_2 is a proper edge coloring, $D''[A,B]$ is simple, and $\mu(D''[A]) \leq \mu(D'[A]) + 2 \leq 4$. For any $a \in A$ and $b \in B$, we have the following estimates:

$$\begin{aligned} e_{D''}(a,A) &\leq e_{D'}(a,A) + e_{D'}(a,B) + \lceil e(D[A,B])/(n-1) \rceil + 1 \\ e_{D''}(a,B) &\leq \lceil e(D[A,B])/(n-1) \rceil + 1 + 2 \cdot \lceil e(D[B])/(n-1) \rceil \\ e_{D''}(b,A) &= \Delta(D), \ e_{D''}(b,B) = 0. \end{aligned}$$

To each edge $e \in E(D''[A])$ with end vertices a_i and a_j, we associate a list $L(e)$ of vertices of B, to which we can lift e to without creating multiple edges:

$$L(e) = B \setminus \left(N_{D''}(a_i) \cup N_{D''}(a_j) \right),$$

whose size is bounded from below

$$\begin{aligned} |L(e)| &\geq n - e_{D''}(a_i,B) - e_{D''}(a_j,B) \geq \\ &\geq n - 2 \cdot \lceil e(D[A,B])/(n-1) \rceil - 4 \cdot \lceil e(D[B])/(n-1) \rceil - 2. \end{aligned}$$

By Vizing's theorem (Theorem 1.24),

$$\begin{aligned} \chi'(D''[A]) &\leq \Delta(D''[A]) + \mu(D''[A]) \leq \\ &\leq \max_{a \in A} \left(e_{D'}(a,A) + e_{D'}(a,B) + \lceil e(D[A,B])/(n-1) \rceil + 1 \right) + 4 \leq \\ &\leq \Delta(D) + 2 \cdot \lceil e(D[A])/n \rceil + \lceil e(D[A,B])/(n-1) \rceil + 5. \end{aligned}$$

By Kahn's theorem (Theorem 1.25), $\mathrm{ch}'(D''[A]) \leq (1+o(1))\chi'(D''[A])$. We have $\mathrm{ch}'(D''[A]) \leq |L(e)|$ for each edge e in $E(D''[A])$, if

$$\begin{aligned} (1+o(1)) &\left(\Delta(D) + 2 \cdot \lceil e(D[A])/n \rceil + \lceil e(D[A,B])/(n-1) \rceil \right) \leq \\ &\leq n - 2 \lceil e(D[A,B])/(n-1) \rceil - 4 \cdot \lceil e(D[B])/(n-1) \rceil. \end{aligned}$$

This inequality holds, if

$$(1+o(1)) \left(\Delta(D) + 2 \cdot e(D[A])/n + 3 \cdot e(D[A,B])/n + 4 \cdot e(D[B])/n \right) \leq n.$$

Using our observations at the beginning of this proof, the previous inequality is a consequence of the regularity of D and

$$(1+o(1)) \cdot 4 \cdot \Delta(D) \leq n.$$

Thus, if the conditions of the statement of this theorem hold, there is a proper list edge coloring c_3 which maps each $e \in E(D''[A])$ to an element of $L(e)$. Finally, we lift every edge $e \in E(D''[A])$ to $c_3(e)$. As we do not create multiple edges between A and B, the resulting graph is a realization of D. □

Proof of Theorem 1.27. This proof is a slight variation on the previous proof. We do not lift edges of $D[A]$ to elements of A. Futhermore, instead of Vizing's theorem, Shannon's theorem (Theorem 1.32) will be used to bound the chromatic index of a graph induced by A.

We may assume that D is regular. For the first step, we use Proposition 1.33 to take a proper edge n-coloring c_1 of $D[A,B] \cup D[B]$, which is an (almost) equitable n- or $(n-1)$-coloring if restricted to $D[A,B]$ and $D[B]$. Lift $c_1^{-1}(i)$ to a_i for all $i = 1,\dots,n$ to get D' from D. Now $D'[A,B]$ is simple and $D'[B]$ is an empty graph on n-vertices. For any $a \in A$ and $b \in B$, we have the following estimates:

$$\begin{aligned}
&e_{D'}(a,A) \le e_D(a,A) + e_D(a,B) + \lceil e(D[A,B])/(n-1)\rceil + 1,\\
&e_{D'}(a,B) \le \lceil e(D[A,B])/(n-1)\rceil + 1 + 2\cdot \lceil e(D[B])/(n-1)\rceil,\\
&e_{D'}(b,A) = \Delta(D),\ e_{D'}(b,B) = 0.
\end{aligned}$$

To each edge $e \in E(D'[A])$ with end vertices a_i and a_j, we associate a list $L(e)$ of vertices of B, to which we can lift e to without creating multiple edges:

$$L(e) = B \setminus \left(N_{D'}(a_i) \cup N_{D'}(a_j)\right),$$

whose size is bounded from below

$$\begin{aligned}
|L(e)| &\ge n - e_{D'}(a_i,B) - e_{D'}(a_j,B) \ge\\
&\ge n - 2\lceil e(D[A,B])/(n-1)\rceil - 4\cdot\lceil e(D[B])/(n-1)\rceil - 2 \ge\\
&\ge n - (1+o(1))2\Delta(D).
\end{aligned}$$

By Shannon's theorem (Theorem 1.32),

$$\begin{aligned}
\chi'(D'[A]) &\le \frac{3}{2}\Delta(D'[A]) \le\\
&\le \frac{3}{2}\cdot \max_{a\in A}\left(e_D(a,A) + e_D(a,B) + \lceil e(D[A,B])/(n-1)\rceil + 1\right) \le\\
&\le (1+o(1))\cdot\frac{3}{2}\cdot\left(\Delta(D) + e(D[A,B])/n\right).
\end{aligned}$$

Furthermore, by Kahn's theorem (Theorem 1.25), $\mathrm{ch}'(D'[A]) \le (1+o(1))\chi'(D'[A])$. We have $\mathrm{ch}'(D'[A]) \le |L(e)|$ for each edge e in $E(D'[A])$, if

$$(1+o(1))\cdot\frac{3}{2}\cdot\left(\Delta(D) + e(D[A,B])/n\right) \le n - 2\Delta(D).$$

This holds, if

$$(1+o(1))\cdot\left(\frac{7}{2}\cdot\Delta(D) + \frac{3}{2}\cdot\frac{e(D[A,B])}{n}\right) \le n.$$

Thus, if the conditions of the statement of this theorem hold, there is a proper list edge coloring c_2 which maps each $e \in E(D'[A])$ to an element of $L(e)$. Finally, we lift every edge $e \in E(D'[A])$ to $c_2(e)$. As we do not create multiple edges between A and B, the resulting graph is a realization of D. □

Proof of Theorem 1.28. We apply induction on n. It is easy to check the result for $n \leq 3$, so let us assume from now on that $n \geq 4$. Since a subgraph of a realizable graph is realizable as well, it is enough to prove the result for demand graphs D on exactly $2n-3$ edges. Recall that A and B be are the color classes of $K_{n,n}$, and let

$$S = \{v \in A \cup B : d_D(v) \geq n-1\}.$$

Since D has $2n-3$ edges, it is clear that $|S| \leq 3$ and that for every pair of vertices in S there is at least one edge joining them.

For a vertex $v \in V(D)$, we denote by $d(v)$ its degree and by $\gamma_A(v)$, $\gamma_B(v)$ the number of neighbors of v in class A and B, respectively. Let $d'(v), \gamma'_A(v), \gamma'_B(v)$ denote the value of these quantities after resolution of a vertex in D; similarly, $d''(v)$, $\gamma''_A(v)$, $\gamma''_B(v)$ denotes the values after the resolution of a second vertex, and so on. We denote the multiplicity of an edge uv by $\mu(uv)$, and we call it *monochromatic* if u and v are in the same color class of D, and *crossing*, otherwise.

Notice that, for a vertex $v \in A$, we need precisely $d(v) - \gamma_B(v)$ vertices in $B \backslash N_B(v)$ (which can be freely chosen in this set) to lift all the multiple edges and monochromatic edges incident to v. After these liftings, which increased the number of edges of the graph by $d(v) - \gamma_B(v)$, all the edges incident to v have their other endpoint in B and are simple. Clearly, we have the same for a vertex in B, exchanging all the occurrences of A and B. We say in this case that v is *resolved*.

For the induction step, we will resolve $t = 1$ or 3 vertices in each color class of D (possibly making some liftings before), remove them from the graph, getting a smaller graph D', and apply the induction hypothesis on D'. It is clear that D is realizable if D' is. By the inductive hypothesis, D' is realizable if the following conditions hold:

1. $\Delta(D') \leq n - t$,
2. D' has at most $2(n-t)-3$ edges, i.e., there were at least $2t$ edges incident to the $2t$ removed vertices after their resolution.

Assume first that there are 3 vertices of degree n in D lying on the same color class (this can only happen if $n \geq 6$, since we must have $3n \leq \sum_{v \in D} d(v) = 4n-6$). In this case, all other vertices in D have degree at most $4n-6-3n = n-6$.

Let $x, y, z \in A$ be the vertices of degree n. As $e(D) = 2n-3$, it is clear that we have $\mu(xy) + \mu(xz) + \mu(yz) \geq n+3$ and that there are at least 6 isolated vertices in B. We choose three from them, say, a, b, c. Without loss of generality, we may assume that $\mu(xy) + \mu(xz) \geq 2/3 \cdot (n+3) \geq 6$.

We resolve x, y and z in this order. After resolving x, we have $\gamma'_B(y) \geq \mu(xy)$, and after resolving y, we have $\gamma''_B(z) \geq \mu(xz)$. In total, we add $d(x) - \gamma_B(x) + d'(y) - \gamma'_B(y) + d''(z) - \gamma''_B(z)$ edges to D, and we delete at least $d(x) + d'(y) + d''(z)$ edges when we remove x, y, z, a, b, c from D, so $e(D') \leq e(D) - (\gamma_B(x) + \gamma'_B(y) + \gamma''_B(z)) \leq e(d) - (\mu(xy) + \mu(xz)) \leq e(D) - 6$ edges, and we can apply induction since $\Delta(D') \leq n-3$.

From now on, we may assume that there are at most two vertices of degree n in a class.

Let u be a maximum degree vertex in D. We may assume that $u \in A$. We distinguish some cases based on the value of $\gamma_B(u)$:

Case 1 $\gamma_B(u) \geq 2$, or $\gamma_B(u) = 1$ and $N_A(u) \neq \emptyset$.

We resolve the demands of u first. Then, if $N_A(u) \neq \emptyset$, let $u' \in N_A(u)$ be a vertex of maximum degree in this set, and $v \in B$ be a vertex that was used for a lifting of an edge uu'. Otherwise, if $N_A(u) = \emptyset$, let v be an arbitrary neighbor of u in B. We resolve the vertex v using the available vertices in A for the lifts in increasing order of degree, and then delete u and v.

We claim that the remaining graph D' satisfies $\Delta(D') \leq n-1$. Indeed, the procedure above increases the degree of a vertex by one if it was used for a lift of either u or v, does not increase the degree of any other vertex in the graph, and decreases the degree of the neighbors of u in B by at least one. Since the vertices used for a lift of u are not joined to it, and hence have degree at most $n-2$ (recall that every vertex of degree at least $n-1$ is joined to a maximum degree vertex), no vertex in B has degree more than $n-1$ after the procedure. On the other hand, we could have a non-neighbor of v, $x \in A$, distinct from u and u', which has degree at least $n-1$ originally. This vertex would end up with degree at least n after the procedure in case it is used for a lift of v. The way we chose the vertices in A for lifts of v would imply, however, that $d(v) \leq n-2$, and so $4n - 6 = \sum_{v \in D} d(v) \geq d(u) + d(u') + d(x) + d(v) = 4n - 5$, a contradiction.

The liftings added $d(u) - \gamma_B(u) + d'(v) - \gamma'_A(v)$ edges to D, and we deleted $d(u) + d'(v) - 1$ edges when we remove u and v, so $e(D') \leq e(D) - (\gamma_B(u) + \gamma'_A(v) - 1) \leq e(D) - 2$, so we can apply the induction hypothesis on D'.

Case 2 $\gamma_B(u) = 1$ and $N_A(u) = \emptyset$.

Let u' be the neighbor of u in B. If u' has another neighbor distinct from u, we would have $d(u') > d(u)$, a contradiction. So uu' forms a bundle. Also, if there is any crossing edge vv' not belonging to this bundle, we resolve u first and then $v' \in B$ without using u in a lift (which is possible since we need $d'(v') - \gamma'_A(v') \leq n - 2 - 1 = n - 3$ vertices of A for the lifts). We are done by induction again after we deleting u and v, since $e(D') \leq e(D) - 2$ and $\Delta(D') \leq n - 1$.

Assume now that $E(D)$ consists of the bundle uu' and monochromatic edges not incident to u or u'. In this case, we take $a \neq u$ in A, $b \neq u'$ in B with smallest degree (by the number of edges, it is at most 3). Let e be an edge, say, in A, which is not incident to a. We lift e to b, and replace one copy of the edge uu' by the path $ubau'$. Then we resolve the multiple edges of a and b, and delete both of them. The remaining graph D' has $\Delta(D') \leq n - 1$ and two less edges than D, so we may apply the induction hypothesis on D'.

Case 3 $\gamma_B(u) = 0$.

Among the neighbors of u, let u' be one with the largest degree. Let us consider two cases:

Case 3.1 There is an edge e independent of uu'.

If e is a crossing edge, let $e = ab$. If not, let a and b be vertices in A and B, respectively, distinct from u, u' and the endpoints of e. In the first case, we lift uu' to b, and in the second, we also lift e to the vertex a or b which is in the opposite class of e. Then, we resolve the vertices a and b and delete them. In both cases, it is clear that the remaining graph D' satisfies $e(D') \leq e(D) - 2$ and $\Delta(D') \leq n - 1$, so the result follows from induction applied in D'.

Case 3.2 There is no edge independent from uu'.

As $e(D) = 2n - 3$ and $d(u), d(u') \leq n$, it follows that uu' is an edge of multiplicity at most 3. So, it is clear that there are two independent edges e and f such that u and u' are incident to e and f, respectively. Again, we let a, b be vertices in A and B not incident to e or f, and we lift both edges to b. After resolving and deleting a and b, we are left with D' with $\Delta(D') \leq n - 1$ and $e(D') \leq e(D) - 2$, so we are done by the induction hypothesis on D'.

□

1.4 Open problems

Our proofs of the degree versions (Theorems 1.17 and 1.26) reduced the gap between the lower and upper bound of the maximum degree of a demand graph to guarantee that it is realizable in $K_{n,n}$. However, it is still an open question to determine its asymptotic correct value. Besides this natural question, we propose some other open problems:

1.4.1 Monochromatic demand graphs in the bipartite case

We studied the terminal-pairability of $K_{n,n}$ when D is a bipartite demand graph and in the general case where it may have edges inside and in between the color classes A and B of $K_{n,n}$. What happens if we assume that all the edges of D lie inside the color classes?

First, if we assume that all the edges are inside one of the color classes, say, A, the problem is very simple: realizing D in $K_{n,n}$ is the same as coloring the edges D properly where the colors are the vertices of B, and then lifting each edge to its corresponding color. Theorem 1.32 guarantees that we can do it whenever $\Delta(G) \leq 2n/3$, and a demand graph consisting of three vertices with $n/3 + 1$ edges joining each pair of them shows that we cannot improve this result.

In case D may have edges inside both A and B, we are not able to give a better estimate than the one in Theorem 1.27. In this case, the result says that D is realizable as long as $\Delta(G) \leq (1+o(1))2n/7$. We conjecture that this bound can be improved. Namely, we conjecture that:

Conjecture 1.34. *If D is a demand graph for $K_{n,n}$ with all edges lying inside the color classes and $\Delta(D) \leq (1+o(1))n/2$, then D is realizable in $K_{n,n}$.*

1.4.2 Other base graphs

In general, if we are looking for conditions for a pair of (multi)graphs (G, D) that guarantees D to be realizable in G, a natural set of candidates, which in particular generalizes the maximum degree condition we had in some of our results, are the more general cut conditions. We could ask whether it is enough for D to be realizable in G that the size of every cut in D is at most some fraction of the size of the corresponding cut in G. Namely:

Problem 1.35. *Let G and D be (connected) multigraphs such that $V(G) = V(D) = [n]$. What is the minimum value of $f(n)$ such that the following implication holds?*

$$f(n) \leq \min_{\emptyset \neq X \subseteq V(G)} \frac{e_G(X, V(G)-X)}{e_D(X, V(D)-X)} \Rightarrow D \text{ is realizable in } G.$$

It is possible to show, by taking G to be an expander multigraph, that $f(n) \geq \Omega(\log n)$. If we insist that G should be a simple graph, a similar construction shows that $f(n) \geq \Omega(\log n/\log\log n)$.

2 $L(2,1)$-labelings of oriented graphs

2.1 $L(2,1)$-labelings

An important application of colorings of graphs is the well-known Channel Assignment Problem: radio channels are to be assigned to transmitters at several locations in a way that different channels have to be assinged to pairs of transmitters that are close, to avoid interference. This is the same as (vertex) coloring the graph whose vertices are the transmitters and two transmitters are joined if they are close to each other, where each color represents a channel. Metzger [91], Zoeliner and Beall [127] and Hale [58] were the first to deal with channel assignment using Graph Theory.

The following generalization of this problem was raised by Hale: suppose that, in case two transmitters interfere, not only their channels must be different, but also they cannot receive channels that differ in some specific values, to permit clear reception of the transmitted signals. More precisely, let T be a finite set of nonnegative integers containing 0 that represents the set of forbidden differences. We are looking for an assignment $c : V(G) \to \mathbb{N}$ such that $|c(u) - c(v)| \notin T$ whenever $uv \in E(G)$. In this case, c is called a *T-coloring* of G.

A $\{0\}$-coloring of a graph is simply an usual (proper) coloring. If we denote by $\chi_T(G)$ the minimum number of colors in a T-coloring of G, it is clear (by using sufficient large numbers as colors) that $\chi_T(G) = \chi(G)$. It is more meaningful, then, to consider the so called *T-span* of G, i.e., the minimum, taken over all the T-colorings of G, of the difference between the largest and the smallest colors used in the coloring. This number is denoted by $sp_T(G)$.

It is not hard to see that, if T contains 0 and $\max(T) = r$, then $sp_T(G) \le (\chi(G) - 1)(r + 1)$. Cozzens and Roberts improved it to the following:

Theorem 2.1 (Cozzens and Roberts [28])**.** *If G is a graph and T is a finite set of nonnegative integers containing 0, then $sp_T(G) \le (\chi(G) - 1)|T|$.*

The list coloring version of T-colorings was introduced by Tesman [113, 114]. For more results about T-colorings and related problems we refer to the survey paper of Roberts [102] and to the excellent books of Chartrand and Zhang [20] and Jensen and Toft [69].

Motivated by Roberts [102], Roger Yeh in his PhD thesis in 1990 [124] and later together with Jerrold Griggs in a paper of 1992 [52], introduced *$L(2,1)$-labelings of graphs*, the subject of this chapter:

An $L(2,1)$-labeling (also known as *$L(2,1)$-coloring*) of a graph G is an assignment $f : V(G) \to \{0, \dots, k\}$ of labels (or colors) to the vertices of G such that $|f(u)-f(v)| \ge 2$ if $uv \in E(G)$ and $|f(u) - f(v)| \ge 1$ if the distance between u and v is 2 in G. This property of f is called the $L(2,1)$ *condition*, and k is sometimes called the *span* of f. The least value of k such that G admits an $L(2,1)$-labeling $f : V(G) \to \{0, \dots, k\}$, which we sometimes call a *k-$L(2,1)$-labeling*, is called the *$L(2,1)$-number* of G and it is denoted by $\lambda(G)$. In their seminal paper, they collected a few elementary properties of $\lambda(G)$. The next theorem compiles some of them:

Theorem 2.2 (Griggs and Yeh [52]). *The following properties hold for the $L(2,1)$-number:*

1. *For every graph G, $\lambda(G) \geq \Delta(G) + 1$.*
2. *Let P_n denote the path on n vertices. Then $\lambda(P_2) = 2$; $\lambda(P_3) = \lambda(P_4) = 3$; $\lambda(P_n) = 4$ for $n \geq 5$.*
3. *Let C_n denote the cycle on n vertices. Then $\lambda(C_n) = 4$ for all $n \geq 3$.*
4. *If T is a tree, then $\lambda(T) = \Delta(T) + 1$ or $\lambda(T) = \Delta(T) + 2$.*
5. *For every graph G, $\lambda(G) \leq n + \chi(G) - 2$.*

Proof. We prove items 1, 4 and 5. Items 2 and 3 are simple case-by-case analysis (for cycles, the pattern in a optimal labeling depends on the remainder of the division of n by 3).

For the first, note that, if u is a vertex of maximum degree in G, then all of its $\Delta(G)$ neighbours must receive distinct colors. Furthermore, all these colors must be at least 2 apart from the color of u.

If T is a tree, the lower bound is implied by the paragraph above. As for the upper bound, we apply a greedy labeling in T, using an ordering $v_1, \dots, v_n$ of the vertices of T in a way that v_i has exactly one neighbor v_j with $j < i$: when we are to assign a color to a given vertex v, we have only 1 colored neighbor of v and at most $\Delta(T) - 1$ colored vertices with distance 2 from v. This means we can always find color among $\{0, \dots, \Delta(T) + 2\}$ to assign to v in a way that the $L(2,1)$ condition is satisfied for v and the already colored vertices.

Finally, let $S_1, \dots, S_{\chi(G)}$ be a partition of $V(G)$ into independent sets, where $S_i = \{v_{i,1}, \dots, v_{i,n_i}\}$ and $|S_i| = n_i$ for $1 \leq i \leq \chi(G)$. The following labeling to is an $L(2,1)$-labeling of G with span $n + \chi(G) - 2$: starting with a 0, f assigns consecutive integers to vertices inside a color class of G, and skips an integer when jumps from one color class to the next. Namely:

$$f(v_{i,j}) = \begin{cases} j - 1, \text{ if } i = 1; \\ i + j - 2 + \sum_{t=1}^{i-1} n_t, \text{ otherwise.} \end{cases}$$

□

The first item in the theorem above gives a sharp lower bound on $\lambda(G)$ in terms of $\Delta(G)$: for instance, $\lambda(G) = \Delta(G) + 1$ if G is a star. As for an upper bound, the authors proved an asymptotically sharp result, namely that $\lambda(G) \leq \Delta(G)^2 + O(\Delta(G))$. More precisely, they have the following result:

Theorem 2.3 (Griggs and Yeh [52]). *Let G be a graph with maximum degree Δ. Then:*

1. *$\lambda(G) \leq \Delta^2 + 2\Delta$;*

2. if G is 3-connected, then $\lambda(G) \leq \Delta^2 + 2\Delta - 3$;

3. if the diameter of G is 2, then $\lambda(G) \leq \Delta^2$.

Proof.

1. The first statement comes from an arbitrary greedy coloring of G: let $k = \Delta^2 + 2\Delta$ (so we have $\Delta^2 + 2\Delta + 1$ available colors), assume the graph is partially colored and let v be a vertex of G. It has at most Δ colored neighbors, and at most $\Delta(\Delta - 1) = \Delta^2 - \Delta$ colored vertices at distance 2 from it. As each neighbor blocks at most 3 colors (the color it is assigned and at most two neighboring colors), and each vertex at distance 2 blocks at most one color for v, we have at most $3\Delta + \Delta^2 - \Delta = \Delta^2 + 2\Delta$ forbidden colors to avoid, so we can find a suitable color for v.

2. Suppose now that G is 3-connected. If G is complete, it is clear that $\lambda(G) = 2\Delta$. Otherwise, there are three vertices u, v and w such that uv and vw are edges, but uw is not. Let $v_1 = u$, $v_2 = w$ and $v_3, \ldots, v_n$ be an ordering of the remaining vertices of G with the following property: for every $3 \leq i \leq n - 1$, there is $j > i$ such that $v_i v_j$ is an edge of G. This can be achieved, for instance, by ordering the v_i from the furthest to the closest to v (so $v_n = v$). Again, we color G greedly with respect to this order, starting by coloring $v_1 = u$ with color 0 and $v_2 = w$ with color 1. However, now a vertex v_i, $3 \leq i \leq n - 1$, has at most $\Delta - 1$ colored neighbors. The same argument in the paragraph above now shows that $k = 3(\Delta - 1) + \Delta(\Delta - 1) = \Delta^2 + 2\Delta - 3$ is enough to find a free color for v_i. Finally, as v has two neighbors colored 0 and 1, it is also possible to find a free color for it and finish the coloring of G.

3. If G has diameter 2 and $\Delta = 2$, then G is either a C_4, a C_5 or a P_3, and the result is easily checked. Assume now that $\Delta \geq 3$. If $\Delta \geq \frac{n-1}{2}$, item 5 of Theorem 2.2 together with Brook's Theorem imply that $\lambda(G) \leq n + \Delta - 2 \leq 2\Delta + 1 + \Delta - 2 = 3\Delta - 1 \leq \Delta^2$. On the other hand, if $\Delta < \frac{n-1}{2}$, then $\delta(\bar{G}) \geq n/2$, and by Dirac's Theorem [31], $\bar{G}$ contains a Hamilton path, say, $v_1 v_2 \ldots v_n$. The labeling $f(v_i) = i - 1$ for $1 \leq i \leq n$ is an $L(2,1)$-labeling of G with $k = n - 1$. As G has diameter two, we have $\Delta^2 \geq n - 1$, from where the result follows.

□

Remark 2.4. *The smallest bound of the Theorem above, namely Δ^2, is achieved by the so called* Moore graphs *of diameter two, which are C_5, the Petersen graph, and the Hoffman–Singleton graph, a 7-regular graph with 50 vertices and 175 edges, plus a hypothetical 57-regular graph with 3250 vertices and 92625 edges (whose existence is currently not known). No infinite family of graphs with $\lambda \geq \Delta^2 - C$, or even $\lambda \geq \Delta^2 - o(\Delta)$, is known.*

On the other hand, Yeh proved that there are graphs that match the upper bounds in Theorem 2.3 asymptotically. A *projective plane* is a set of points and a set of lines with the following incidence conditions:

1. Every pair of points is contained in exactly one line;
2. Every pair of lines intersect in exactly one point;
3. There are four points such that no line is incident with more than two of them.

It is known that (see, for instance, the book of Hughes and Piper [66]), if a finite projective plane exists, then there is an integer n, called its *order*, such that the plane has exactly n^2+n+1 points, n^2+n+1 lines, each point is incident to exactly $(n+1)$ lines, and each line is incident to exactly $(n+1)$ points. Finite projective planes are known to exist for every n which is a power of a prime.

From a projective plane Π, one can construct its *incidence graph*. It is a bipartite graph where one color class represent the points, the other represents the lines, and there is an edge join a point and a line precisely when they incide in Π. This graph is a $(n+1)$-regular bipartite graph on two color classes of n^2+n+1 vertices each, diameter three and the remarkable property that for each pair of vertices in a color class, there is exactly one path of lenght two joining them.

Theorem 2.5 (Yeh [124]). *Let G be a incidence graph of a projective plane. Then $\lambda(G) = \Delta(G)^2 - \Delta(G)$.*

Indeed, it is conjectured that the third upper bound in Theorem 2.3 is not only asymptotically correct, but exact:

Conjecture 2.6 (Griggs and Yeh [52]). *For every graph G with maximum degree $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$.*

This was proved to be the case for many classes of graphs, but it is open in general. The best result in this direction is due to Havet, Reed and Sereni, who proved, using deep probabilistic methods, that this result holds for graph with sufficiently large maximum degree:

Theorem 2.7 (Havet, Reed and Sereni [60]). *There is Δ_0 such that, if G is a graph with maximum degree $\Delta \geq \Delta_0$, then $\lambda(G) \leq \Delta^2$. In particular, there is a constant C such that, for every graph G, $\lambda(G) \leq \Delta^2 + C$.*

Following the seminal paper of Griggs and Yeh, a multitude of papers and results on $L(2,1)$ number of graphs appeared, mostly focusing on computing or estimating λ in specific classes, but the $L(2,1)$ number of different kinds of products of graphs, the number of edges of a graph with a given order and $L(2,1)$ number, and the list version of $L(2,1)$-labelings were also studied, among other variants. Furthermore, the more general $L(h,k)$*-labeling*, where adjacent vertices must receive colors with difference at least h, and vertices joined by a path of length two must receive colors with difference at least k was considered. It is not our intention to survey all these results here, as we are interested in the directed version of $L(2,1)$-labelings. We refer to the surveys of Yeh [125] and Calamoneri [16] for a comprehensive account of the area.

Rather, we will consider here the oriented version of $L(2,1)$-labeling, namely: given an oriented graph G (a directed graph where no loops, multiple or opposite directed edges are allowed), an $L(2,1)$-labeling of G is an assignment $f : V(G) \to \{0, \dots, k\}$ of labels (or colors) to the vertices of G such that $|f(u) - f(v)| \geq 2$ if $uv \in E(G)$ and $|f(u) - f(v)| \geq 1$ if there is a (directed) path of length two joining u and v is 2 in G. Again, the least value of k such that G admits an $L(2,1)$-labeling $f : V(G) \to \{0, \dots, k\}$ is called the $L(2,1)$-*number* of G, and it is denoted by $\overrightarrow{\lambda}(G)$.

Notice that, if G is an oriented graph and H is it underlying graph (the undirected graph obtained by ignoring the orientation of the edges), then the following inequality holds:

$$\overrightarrow{\lambda}(G) \leq \lambda(H) \tag{2.1}$$

We will see later that this bound is usually not sharp, and that indeed those two parameters may behave quite differently in some classes of graphs.

2.2 $L(2,1)$ number of paths, cycles and Cartesian product

The second and third items of Theorem 2.2 characterize completely the $L(2,1)$ number of cycles and paths. It is easy to see that the upper bound in inequality (2.1) is attained for these graphs: indeed, every $L(2,1)$-labeling of a cycle (resp. path) is also an $L(2,1)$-labeling of a directed cycle (resp. directed path) of the same length (with the obvious correspondence between the vertices), since the directed paths of length two in the directed graph are in correspondence with the undirected paths of length two in the undirected counterpart. This means that the results of this part of the theorem translate to the directed version:

Proposition 2.8.

1. $\overrightarrow{\lambda}(\overrightarrow{C_n}) = 4$ *for every* $n \geq 3$*;*
2. $\overrightarrow{\lambda}(\overrightarrow{P_2}) = 2$*,* $\overrightarrow{\lambda}(\overrightarrow{P_3}) = \overrightarrow{\lambda}(\overrightarrow{P_4}) = 3$ *and* $\overrightarrow{\lambda}(\overrightarrow{P_n}) = 4$ *for all* $n \geq 5$*.*

A natural next step is to study how the $L(2,1)$ number behaves when we combine two cycles or paths in some way. There are many different sorts of products of graphs (see [59] and [67] for good surveys on the topic). We will consider one of the most studied graph products in this section: the Cartesian product.

2.2.1 Cartesian product

The *Cartesian product* of two graphs (resp. digraphs) G and H is the graph (resp. digraph) $G\Box H$ such that $V(G\Box H) = V(G) \times V(H)$, and where there is an edge joining (a, x) and (b, y) if $ab \in E(G)$ and $x = y$, or if $a = b$ and $xy \in E(H)$ (resp. there is an edge pointing from (a, x) to (b, y) if $ab \in E(G)$ and $x = y$, or if $a = b$ and $xy \in E(H)$).

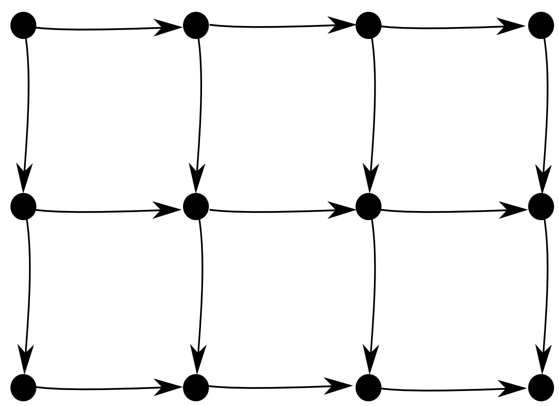

Figure 2.1: The Cartesian product of $\overrightarrow{P_3}$ and $\overrightarrow{P_4}$

The following theorem of Whittlesey, Georges and Mauro settles the question for the Cartesian product of two undirected paths:

Theorem 2.9 (Whittlesey, Georges and Mauro [123])**.** *Let* $m, n \geq 3$*. Then:*

1. $\lambda(P_2\Box P_2) = \lambda(C_4) = 4$*;*
2. $\lambda(P_2\Box P_n) = 5$*;*
3. $\lambda(P_m\Box P_n) = 6$.

As for the Cartesian product of cycles and paths, the following result is known:

Theorem 2.10 (Jha, Narayanan, Sood, Sundaram and Sunder [70]; Klavžar and Vesel [78])**.** *Let* $m \geq 3$ *and* $n \geq 4$*. Then:*

1. $\lambda(C_m\Box P_2) = \begin{cases} 5, & \textit{if } m \equiv 0 \pmod 3; \\ 6, & \textit{otherwise.} \end{cases}$
2. $\lambda(C_m\Box P_3) = \begin{cases} 7, & \textit{if } m = 4 \textit{ or } m = 5; \\ 6, & \textit{otherwise.} \end{cases}$
3. $\lambda(C_m\Box P_n) = \begin{cases} 6, & \textit{if } m \equiv 0 \pmod 7; \\ 7, & \textit{otherwise.} \end{cases}$

Finally, Schwarz and Troxell settled the Cartesian product of two cycles completely. The dependence of λ on the lengths of the cycles is a bit more subtle in this case, as shown in their theorem:

Theorem 2.11 (Schwarz and Troxell [106])**.** *Let $m, n \geq 3$. Then:*

$$\lambda(C_m \Box C_n) = \begin{cases} 6, & \text{if } m, n \equiv 0 \pmod 7; \\ 7, & \text{if } \{m, n\} \in A \cup B \cup C; \\ 8, & \text{otherwise}; \end{cases}$$

where $A = \{\{3, i\} : i \geq 3, i$ odd or $i \in \{4, 10\}\}$, $B = \{\{5, i\} : i \in \{5, 6, 9, 10, 13, 17\}\}$, and $C = \{\{6, 7\}, \{6, 11\}, \{7, 9\}, \{9, 10\}\}$.

As for the oriented version, much less is known. The following result, from 2018, was the best for product of two oriented cycles:

Theorem 2.12 (Jiang, Shao and Vesel [110])**.** *If $m, n \geq 40$, then $\overrightarrow{\lambda}(\overrightarrow{C_m} \Box \overrightarrow{C_n}) \leq 5$.*

Our result refines this theorem, determining precisely the value of $\overrightarrow{\lambda}$ in this range. Perhaps surprinsingly, it depends only on the value of the greatest common divisor of the lengths of the cycles:

Theorem 2.13 (Colucci and Győri [26])**.** *Let $m, n \geq 40$. Then:*

$$\overrightarrow{\lambda}(\overrightarrow{C_m} \Box \overrightarrow{C_n}) = \begin{cases} 4, & \text{if } \gcd(m, n) \geq 3; \\ 5, & \text{otherwise.} \end{cases}$$

Proof of Theorem 2.13

Let $S(m, n) = \{am + bn : a, b$ nonnegative integers, not both zero$\}$. A classical result of Sylvester [112] states that $t \in S(m, n)$ for all integers $t \geq (m-1)(n-1)$ that are divisible by $\gcd(m, n)$.

We start with a lemma which is a slightly stronger version of Lemma 5 from [110] that can be obtained with the same proof:

Lemma 2.14. *(Lemma 5 in [110]) For every m, n with m, $n \geq 3$ and every 4-$L(2,1)$-labeling f of $\overrightarrow{C_m} \Box \overrightarrow{C_n}$, the following periodicity condition holds:*

$$f(i, j) = f(i + 1 \bmod m, j - 1 \bmod n) \text{ for all } i \in [m], j \in [n]. \tag{2.2}$$

The following lemma combined the result of Sylvester mentioned above will also be useful for us:

Lemma 2.15. *(Lemmas 2 and 3 in [110]) Let m, n, $p \geq 3$ and t, $k \geq 1$ be integers. If $\overrightarrow{\lambda}(\overrightarrow{C_m} \Box \overrightarrow{C_n}) \leq k$ and $\overrightarrow{\lambda}(\overrightarrow{C_p} \Box \overrightarrow{C_n}) \leq k$, then $\overrightarrow{\lambda}(\overrightarrow{C_{m+tp}} \Box \overrightarrow{C_n}) \leq k$.*

In particular, if m and n are such that $\overrightarrow{\lambda}(\overrightarrow{C_m} \Box \overrightarrow{C_n}) \leq k$, $\overrightarrow{\lambda}(\overrightarrow{C_m} \Box \overrightarrow{C_m}) \leq k$ and $\overrightarrow{\lambda}(\overrightarrow{C_n} \Box \overrightarrow{C_n}) \leq k$, then $\overrightarrow{\lambda}(\overrightarrow{C_a} \Box \overrightarrow{C_b}) \leq k$ for all a, $b \in S(m, n)$, and hence for all a, $b \geq (m-1)(n-1)$ divisible by $\gcd(m, n)$.

After stating these two lemmas, we are able to give the proof:

For $m, n \geq 3$, let G denote the graph $\overrightarrow{C_m} \square \overrightarrow{C_n}$, i.e., $V(G) = [m] \times [n]$ and the directed edges of G point from (i, j) to $(i + 1 \bmod m, j)$ and to $(i, j + 1 \bmod n)$, for every $i \in [m], j \in [n]$. For a labeling f, we write $f(i, j)$ instead of $f((i, j))$ for short.

A directed P_3 subgraph of G shows, together with Proposition 2.8, that $\overrightarrow{\lambda}(\overrightarrow{C_m} \square \overrightarrow{C_n}) \geq 4$.

Let $d = \gcd(m, n)$ and assume first that $d \geq 3$. According to Lemma 2.15, it is enough to prove that $\overrightarrow{\lambda}(\overrightarrow{C_d} \square \overrightarrow{C_d}) = 4$. Any 4-$L(2,1)$-labeling f of $\overrightarrow{C_d}$ can be extended to a 4-$L(2,1)$-labeling f' of $\lambda(\overrightarrow{C}_d \square \overrightarrow{C_d})$ by setting $f'(i, j) = f(i - j \bmod d)$, and, by Theorem 2.2, we know that $\lambda(\overrightarrow{C_d}) = 4$.

On the other hand, assume for the sake of contradiction that $d \in \{1, 2\}$ and there is a 4-$L(2,1)$-labeling f of $\overrightarrow{C_m} \square \overrightarrow{C_n}$. In particular, $m \neq n$, so let us assume that $m > n$.

It is easy to check that, if $m \geq n+3$, f induces a valid 4-$L(2,1)$-labeling of $\overrightarrow{C_{m-n}} \square \overrightarrow{C_n}$. In fact, let $g(i, j) = f(i, j)$ for all $1 \leq i \leq m - n$ and $1 \leq j \leq n$. We claim that g is a 4-$L(2,1)$-labeling of $\overrightarrow{C_{m-n}} \square \overrightarrow{C_n}$, which, in particular, satisfies (2.2) as well.

Indeed, all we have to check is that the following conditions hold for g, since the other restrictions are inherited by f: $|g(m-n-1, j) - g(1, j)| \geq 1$, $|g(m-n, j) - g(1, j)| \geq 2$, $|g(m - n, j) - g(2, j)| \geq 1$, $|g(m - n, j) - g(1, j + 1 \bmod n)| \geq 2$, for every $j \in [n]$. All these conditions follow from $g(m-n-1, j) = f(m-n-1, j) = f(m-1, j+n \bmod n) = f(m - n - 1, j)$ and $g(m - n, j) = f(m - n, j) = f(m, j + n \bmod n) = f(m - 1, j)$, which follow from the application of (2.2) n times, together with the fact that f is a $L(2,1)$-labeling of $\overrightarrow{C_m} \square \overrightarrow{C_n}$.

Applying this argument consecutively, using the fact that $d = \gcd(m, n)$ and by the symmetry of the factors of the product, we conclude that f induces a 4-$L(2,1)$-labeling c of either $\overrightarrow{C_{k+1}} \square \overrightarrow{C_k}$ or $\overrightarrow{C_{k+2}} \square \overrightarrow{C_k}$, for some $k \geq 3$. This is a contradiction, since in this case we would have $c(1, 1) = c(2, k) = \cdots = c(k+1, 1)$ and $(k+1, 1)$ and $(1, 1)$ are joined by and edge or by a directed path of length two, respectively. $\square$

2.3 $L(2,1)$-labelings of other oriented graphs

Chang and Liaw studied the $L(2,1)$-labelings of oriented trees. Namely, in a paper of 2003, they proved the following result:

Theorem 2.16 (Chang and Liaw [19]). *Let T be an oriented tree. Then $\overrightarrow{\lambda}(T) \leq 4$. More precisely, let l be the length of a longest (directed) path in T. Then:*

1. *If $l = 1$, then $\overrightarrow{\lambda}(T) = 2$;*
2. *If $l = 2$, then $\overrightarrow{\lambda}(T) = 3$;*

3. If $l = 3$, then $\overrightarrow{\lambda}(T) \in \{3, 4\}$;

4. If $l \geq 4$, then $\overrightarrow{\lambda}(T) = 4$.

This result, unlike some of the result of the previous section, shows a pronounced difference between $\overrightarrow{\lambda}(T)$ and the $L(2,1)$ number of its underlying graph: by Theorem 2.2, $\lambda(T) \in \{\Delta(T)+1, \Delta(T)+2\}$ for every tree T.

Our following result, proved together with Győri, sheds some light on this phenomenon, providing an upper bound for $\overrightarrow{\lambda}(G)$ in terms of the maximum degree of the blocks, i.e., the biconnected components, of the underlying graph of G. Namely:

Theorem 2.17 (Colucci and Győri [25]). *Let G be an oriented graph with the following property: for every block B of its underlying graph, all the in- and out-degrees of the vertices of $G[B]$, the subgraph of G induced by $V(B)$, are bounded by k. Then $\overrightarrow{\lambda}(G) \leq 2k^2 + 6k$.*

Proof. We proceed by induction on the number of blocks of H, the underlying graph of G. If H has only one block (that is, it is 2-connected), it is clear that we can color G greedily using at most $2k^2 + 6k + 1$ colors, since the first (resp. second) directed neighborhood of any vertex v in G contains at most $2k$ (resp. $2k^2$) vertices, and each of those vertices forbids at most three (resp. one) colors for v.

On the other hand, if H contains at least two blocks, let v be a cut vertex with the property that at most one of the blocks containing v contains a cut vertex distinct from v. It is clear that such a vertex exists from the tree structure of the blocks of H. Let $B_1, \ldots, B_t$ be the blocks containing v such that v is the only cut vertex of B_i.

We apply induction on the graph $G' = G - \bigcup_{i=1}^{t}(V(B_i) \backslash \{v\})$ to get a coloring of it using at most $2k^2 + 6k + 1$ colors. We are left with the vertices of the blocks B_i (except v) to color.

Let A and B be, respectively, the set of uncolored vertices that point to and from v in G. It is clear that the size of any connected component in A and B is at most k and that the only paths joining these components pass through v. In this way, as v has at most $2k$ colored neighbors in G at this point, we have at least $2k^2 + 6k + 1 - 2k - 3 \geq 2k$ distinct free colors for the vertices in A and B. Let some of the free colors be $c_1 < c_2 < \cdots < c_{2k}$. We use colors $c_1, c_3, \ldots, c_{2k-1}$ for A and $c_2, c_4, \ldots, c_{2k}$ for B, coloring each vertex in a connected component with a distinct color.

Now that $A \cup B$ is colored, we have to color the vertices of $\bigcup_{i=1}^{t} B_i$ at distance at least two from v. We can color these vertices greedily as before, since its neighbors and second neighbors lie inside a block of H, in which the maximum degree is k. □

In particular, this result implies that oriented trees have bounded $L(2,1)$ number (with a slightly worse constant than the optimal value 4 given in Theorem 2.16), as the blocks of their underlying graph are edges. Namely, it proves that $\overrightarrow{\lambda}(T) \leq 8$ for every oriented tree T.

We also gave a construction that yields a lower bound asymptotically equal to half of the upper bound of Theorem 2.17:

Theorem 2.18 (Colucci and Győri [25])**.** *There is an oriented graph G such that its underlying graph is 2-connected, every in-degree and out-degree in G is bounded by $k + O(1)$ and $\overrightarrow{\lambda}(G) \geq k^2 + O(k)$.*

Proof. Let $V(G) = \mathbb{Z}_k^2$, the set of pairs of integers modulo k, where $k \geq 4$ is a positive integer. To simplify the notation, we write ab for the pair $(a, b) \in \mathbb{Z}_k^2$. The arcs of G are defined as follows:

i. $ab \to bc$, if $c > a$.

ii. $ab \to (b+1)c$, if $b < k-1$, $c \leq a$ and $c \neq a-1$.

iii. $ab \to a(b+1)$, if $b < k-1$ and $a \neq b+2$.

iv. $ab \to (a+1)b$, if $a < k-1$ and $a \neq b+1$.

It is easy to check that G does not contain opposite arcs and both the in-degree and out-degree of its vertices are bounded by $k+1$. Furthermore, it will be clear from the proof that its underlying graph is 2-connected.

Note that to prove that the theorem it suffices to show that, for every pair of vertices ab, cd with $a, b, c, d \notin \{0, k-1\}$, there is a directed path of length at most 2 from ab to cd or vice-versa. Therefore, we assume this condition holds in what follows.

We can find paths of length at most 2 joining ab and cd as follows:

1. If $a < c$ and $b < d$: $ab \to bc \to cd$.
2. If $a > c$ and $b > d$: $cd \to da \to ab$.
3. If $a < c$ and $b > d$: $cd \to (d+1)a \to ab$, except if:
   i. $c = a+1$: $ab \to (b+1)a \to (a+1)d$.
   ii. $b = d+1$: $cd \to (d+1)(a-1) \to a(d+1)$.
4. If $a > c$ and $b < d$: $ab \to (b+1)c \to cd$, except if:
   i. $a = c+1$: $(c+1)b \to (b+1)(c-1) \to cd$.
   ii. $d = b+1$: $ab \to (b+1)(c-1) \to c(b+1)$.
5. If $a = c$ and, say, $b < d$ (without loss of generality): $ab \to (b+1)a \to ad$, except if:
   i. $d = b+1$ and $a \neq b+2$: $ab \to a(b+1)$.
   ii. $d = b+1$ and $a = b+2$: $a(b+1) \to ab$.
6. If, say, $a < c$ (without loss of generality) and $b = d$: $ab \to b(c-1) \to cb$, except if:

i. $c = a+1$ and $a \neq b+1$: $ab \to (a+1)b$.

ii. $c = a+1$ and $a = b+1$: $(a+1)b \to ab$.

□

We conjecture that there is a construction that matches the upper bound asymptotically:

Conjecture 2.19. *There is an oriented graph G for which each indegree and outdegree is bounded by $(1+o(1))k$ and $\overrightarrow{\lambda}(G) \geq 2k^2 + O(k)$.*

A related notion to $L(2,1)$-labeling in oriented graphs is the *oriented coloring.* An oriented coloring of an oriented graph G is a function $c : V(G) \to [k]$ such that

1. $c(u) \neq c(v)$, if u and v are joined by an edge;
2. if uv is an edge, there is no edge xy with $c(x) = c(v)$ and $c(u) = c(x)$.

In other words, an oriented coloring is a proper vertex coloring that has no pair of color classes joined by two edges pointing to opposite directions. The *oriented chromatic number* of G, denoted by $\overrightarrow{\chi}(G)$ (or sometimes by $\chi_o(G)$), is the least k such that G admits an oriented coloring $c :\to [k]$. Equivalently, it is the least integer k such that $V(G)$ can be partitioned into k independent sets $S_1, \ldots, S_k$ in a way that for every $1 \leq i < j \leq k$, all edges joining S_i and S_j point in one fixed direction (either from S_i to S_j or the other way around).

It is easy to see that $\overrightarrow{\lambda}(G) \leq 2(\overrightarrow{\chi}(G) - 1)$, since subracting one and then doubling every color of an oriented coloring of G, we get an $L(2,1)$-labeling. Calamoneri and Sinaimeri [18], using results on the oriented chromatic number of oriented planar graphs [92, 94, 101, 118], noticed that this inequality implies that the $L(2,1)$ number of every oriented planar graph is bounded by a constant. They also gave some exact values and bounds for some classes of oriented graphs, as cactus, Halin graphs, wheels and prisms. Also, Calamoneri [17] studied the $L(2,1)$ number of some oriented grid graphs. Again, possibly with a bigger constant, Theorem 2.17 also proves that the $L(2,1)$ number of oriented cacti, prisms and grid graphs are bounded.

2.4 A new generalization of $L(2,1)$-labeling of oriented graphs

In this section, we introduce one more generalization of $L(2,1)$-labeling and prove analogous results of Theorems 2.17 and 2.18 in this new setting.

A path of length two admits three pairwise non-isomorphic orientations: $a \to b \to c$, $a \to b \leftarrow c$, and $a \leftarrow b \to c$; we call these paths P_1, P_2 and P_3, respectively. In this terminology, we can rephrase the definition of a $L(2,1)$-labeling of an oriented graph

G as follows: an assignment $f : V(G) \to \{0, \ldots, k\}$ such that $|f(u) - f(v)| \geq 2$, if $uv \in E(G)$; and $|f(u) - f(v)| \geq 1$, if there is a P_1 in G joining u and v.

We study the corresponding problems that arise when we replace P_1 in this definition by P_2 or P_3, or, even more generally, by a subset S of $\{P_1, P_2, P_3\}$. We denote the corresponding minimum value of k by $\lambda_S(G)$. Some of the choices of S lead us back to previous questions, namely, $\lambda_\emptyset(G) = 2\chi(G) - 1$; $\lambda_{\{P_1,P_2,P_3\}}(G) = \lambda(H)$, where H is the underlying graph of G; and $\lambda_{\{P_1\}}(G) = \overrightarrow{\lambda}(G)$. Also, by the symmetry of P_2 and P_3, we have just the following three cases left to consider: $S = \{P_2\}$, $S = \{P_2, P_3\}$ and $S = \{P_1, P_2\}$.

In each one of those cases, we are going to determine the order of magnitude, and, with one exception, the correct asymptotic value, of the maximum possible value of $\lambda_S(G)$ in terms of the maximum degree of G.

First, we consider $S = \{P_2\}$, i.e., when the only two path considered is $a \to b \leftarrow c$. We have the following asymptotically sharp result:

Theorem 2.20 (Colucci and Győri [25]). *Let G be an oriented graph such that $d_+(v) \leq k$ and $d_-(v) \leq k$ for all $v \in V(G)$. Then $\lambda_{\{P_2\}}(G) \leq k^2 + O(k)$, and there is a family of graphs that matches this upper bound asymptotically.*

Proof. We color G greedily with the colors $\{0, \ldots, k^2 + 5k\}$: given a vertex v, each of its at most $2k$ neighbors forbid at most 3 colors for v. Among the second neighbors, only the at most $k(k-1) = k^2 - k$ vertices that are joined by a P_2 to v forbid colors for v, at most one new color per vertex. In total, at most $3 \cdot 2k + k^2 - k = k^2 + 5k$ colors are forbidden for v.

As for the sharpness of the bound, the same construction as in the undirected case works. Let $G = (A, B, E)$ be the oriented bipartite incidence graph of a projective plane with point set A, line set B, $|A| = |B| = k$, and all the edges pointing from A to B. Both the in- and outdegrees of G are bounded by $(1 + o(1))\sqrt{k}$ and there is a P_2 joining every pair of vertices in A. Therefore, at least k different colors are needed in any valid labeling of G. $\square$

In the case $S = \{P_2, P_3\}$, we have the following result, which does not yield an asymptotic sharp bound, but a factor 2 for the ratio between the upper and lower estimates:

Theorem 2.21 (Colucci and Győri [25]). *Let G be an oriented graph such that $d_+(v) \leq k$ and $d_-(v) \leq k$ for all $v \in V(G)$. Then $\lambda_{\{P_2,P_3\}}(G) \leq 2k^2 + O(k)$. On the other hand, there is a family of graphs G with $d_+(v) = (1 + o(1))k$, $d_-(v) = (1 + o(1))k$ for every $v \in V(G)$ and $\lambda_{\{P_2,P_3\}}(G) \geq k^2 + O(k)$.*

The proof of the upper bound in Theorem 2.21 is obtained in a similar way as in Theorem 2.20, i.e., coloring the graph greedily, bounding the number of forbidden colors for a given vertex using the sizes of its first and second neighboorhoods. The lower bound comes from the very same construction in Theorem 2.20. We omit the details.

Finally, in the case $S = \{P_1, P_2\}$, we have a different upper bound and an asymptotically sharp construction, as stated in the following theorem:

Theorem 2.22 (Colucci and Győri [25]). *Let G be an oriented graph such that $d_+(v) \leq k$ and $d_-(v) \leq k$ for all $v \in V(G)$. Then $\lambda_{\{P_1,P_2\}}(G) \leq 3k^2 + O(k)$. Furthermore, there is a family of graphs that matches this bound asymptotically.*

Proof. Again, we apply the greedy algorithm as in Theorem 2.20 to get the upper bound.

On the other hand, consider the following construction: if $H = (A, B, E)$ is the bipartite incidence graph of a projective plane with point set A, line set B and $|A| = |B| = k$ with all edges oriented from A to B, let $H' = (A', B', E')$ and $H'' = (A'', B'', E'')$ be two copies of H with edges oriented from A' to B' and A'' to B'', respectively. For a vertex $p \in V(H)$, we denote by p' (resp. p'') its copy in H' (resp. H''), and we call p, p', p'' twin vertices. We construct an oriented graph G as follows: The vertex set of G is $V(G) = V(H) \cup V(H') \cup V(H'')$. The edge set of G is $E(G) = E(H) \cup E(H') \cup E(H'') \cup \{(l, p'), (l', p''), (l'', p) : l \in B, p \in A \text{ and } (p, l) \in E(H)\} \cup \{(p, p'), (p', p''), (p'', p) : p \in A\}$. In the graph G, all degrees are bounded by $(1 + o(1))\sqrt{k}$. Moreover, given two vertices p, q from $A \cup A' \cup A''$, either they are joined by a P_2 (in case both vertices come from the same set), by a P_1 (if they are in different sets and are not twin vertices) or by an edge (if they are twin vertices). This shows that a valid labeling of G must use at least $3k$ colors.

□

3 Edge colorings without a fixed monochromatic subgraph

3.1 Classical questions in Extremal Graph Theory: maximizing the number of edges

One of the most natural and well-studied parameters in Extremal Graph Theory is the **Turán number** (also known as the **extremal number**) of a graph H. It is denoted by $ex(n, H)$ and defined as

$$ex(n, H) = \max\{e(G) : G \text{ is a graph on } n \text{ vertices and } H \nsubseteq G\}.$$

A graph G is called and **extremal graph** for H if $e(G) = ex(|V(G)|, H)$.

This parameter is named after the Hungarian mathematician Pál Turán, whose theorem, published in 1941, generalizes the result from 1907 of Mantel [88] that $ex(n, K_3) = \lfloor n^2/4 \rfloor$ to all complete graphs and characterizes its extremal graphs:

Theorem 3.1 (Turán [117])**.** *For every $t \geq 2$,*

$$ex(n, K_t) = e(T(n, t-1)) \sim \left(1 - \frac{1}{t-1}\right) \frac{n^2}{2},$$

*where $T(n, r)$ is the so-called r-partite **Turán graph**: a complete r-partite graph with n vertices in which each color class contains either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ vertices. Furthermore, $T(n, t-1)$ is the* unique *graph on n vertices and $ex(n, K_t)$ edges.*

Turán's Theorem is the cornerstone of a huge body of work in Extremal Graph Theory and beyond, with literally thousands of papers and myriads of results tracing back to it, so it is out of the scope of the present text to survey all of them. However, it is worth mentioning a few theorems in the area to illustrate a recurring phenomenon in Extremal Graph Theory that will also manifest in the problems we are dealing in this thesis: the fact that the nature of the questions is surprisingly more intricate when we deal with forbidden bipartite graphs.

One of the first generalizations of Turán's theorem is the following result of Erdős and Simonovits. As it is based on an earlier result of Erdős and Stone [36], it is sometimes referred to as Erdős-Stone-Simonovits theorem in the literature:

Theorem 3.2 (Erdős and Simonovits [40])**.** *If H is a graph with $\chi(H) = t \geq 2$, then*

$$ex(n, H) = ex(n, K_t) + o(n^2).$$

The result above is an asymptotically sharp estimate for the extremal number of every non-bipartite graph, since by Turán's theorem we have $ex(n, K_t) = \Theta(n^2)$ whenever $t \geq 3$. For bipartite graphs H, on the other hand, it simply gives the crude upper bound $ex(n, H) = o(n^2)$, as the first term on the right-hand side vanishes in this case.

The problem of estimating the extremal number of bipartite graphs is much more challenging. Indeed, with few exceptional classes of bipartite graphs H, not even the order of magnitude of $ex(n,H)$ is known. Erdős and Simonovits [38] conjectured that for every bipartite graph H, there are constants c and $1 \leq \alpha < 2$ such that $ex(n,H) \sim cn^{\alpha}$, and, reciprocally, for every rational number $1 \leq \alpha < 2$ there is a bipartite graph H such that $ex(n,H) \sim cn^{\alpha}$ for some c. Bukh and Conlon [15] answered the latter question affirmatively if instead of forbidding a single bipartite graph we may forbid a finite collection of graphs. Some other results giving families of rational numbers α such that there is a graph H with $ex(n,H) \sim cn^{\alpha}$ appear in [71], [75] and [72]. We refer the reader to the survey of Füredi and Simonovits [48] for the history and results in this area.

3.2 Maximizing the number of colorings

In a paper of 1974, Erdős [37] raised the following generalization of Turán-type problems: for a fixed graph H, let $c_{r,H}(G)$ denote the number of (not necessarily proper) r-edge-colorings of a graph G, i.e., functions $c : E(G) \to [r] = \{1, \dots, r\}$, without a monochromatic copy of H as a subgraph. For instance, $c_{1,H}(G) = 0$ if G contains H as a subgraph, and $c_{r,P_3}(G)$, where P_3 denotes the path on three vertices, is the number of proper r-edge-colorings of G. We define $c_{r,H}(n)$ as

$$c_{r,H}(n) = \max\{c_{r,H}(G) : G \text{ is a graph on } n \text{ vertices}\}.$$

Also, similarly as in the previous section, we call a graph on n vertices G *extremal* with respect to r and H (or (r,H)-*extremal*) if $c_{r,H}(G) = c_{r,H}(n)$.

For every r, n and H, we have the following general bounds:

$$r^{ex(n,H)} \leq c_{r,H}(n) \leq r^{r \cdot ex(n,H)}. \tag{3.1}$$

The lower bound comes from the fact that any r-coloring of an H-free graph does not contain a monochromatic H, so in particular an extremal H-free graph has $r^{ex(n,H)}$ colorings without a monochromatic H. The upper bound follows from the fact that in any r-coloring of a graph on at least $r \cdot ex(n,H) + 1$ edges there is a monochromatic subgraph on at least $ex(n,H) + 1$ edges, by Pigeonhole Principle, and hence a monochromatic H. This means that every graph G with $c_{r,H}(G) > 0$ has at most $r \cdot ex(n,H)$ edges, so it admits at most $r^{r \cdot ex(n,H)}$ r-edge-colorings.

Considering the disjoint union of two (r,H)-extremal graphs on n and m vertices, it is easy to see, assuming H is a connected graph, that $c_{r,H}(n+m) \geq c_{r,H(n)} \cdot c_{r,H}(m)$ holds for all positive integers m and n (i.e., the function $c_{r,H}(n)$ is supermultiplicative). A lemma of Fekete [44] implies, then, that the following limit exists (it is either a positive real number or infinity):

$$b_{r,H} = lim_{n \to \infty} \; c_{r,H}(n)^{1/n}. \tag{3.2}$$

In [37], Erdős mentions that, together with Rothschild, he conjectures that, for every t, there is $n_0(t)$ such that $c_{2,K_t}(n) = 2^{ex(n,K_t)}$ for $n > n_0(t)$, and asks whether the upper bound in (3.1) could be replaced by $r^{(1+\epsilon)ex(n,H)}$ for all or almost all graphs H. The conjecture essentially says that an extremal graph for H in Turán's sense should also be "almost" extremal in this new coloring sense. We will see later that this is not true in general.

Eighteen years later, in 1992, the question reappears in a new collection of problems by Erdős [39], where he restates the conjecture for triangles and two colors, asking whether $c_{2,K_3}(n) \le 2^{\lfloor n^2/4 \rfloor}$.

Four years later, in 1996, Yuster published the first result in this area, answering Erdős question positively. He proved the following:

Theorem 3.3 (Yuster [126])**.** *For every $n \ge 6$,*

$$c_{2,K_3}(n) = 2^{\lfloor n^2/4 \rfloor}.$$

Furthermore, using the Regularity Lemma of Szemerédi, as usual, with an embedding lemma, he provided the following asymptotic result for complete graphs:

Theorem 3.4 (Yuster [126])**.** *For all $t \ge 4$,*

$$c_{2,K_t}(n) = 2^{ex(n,K_t)+o(n^2)}.$$

The exact conjecture of Erdős and Rothschild for complete graphs, without the $o(n^2)$ term in the result above, was only proved eight years later, in 2004, in a paper of Alon, Balogh, Keevash and Sudakov, where they discovered an interesting phenomenon: although the lower bound given by the extremal graph is exact for two or three colors, this is very far from the truth in case $r \ge 4$, in the sense that $c_{r,K_t}(n)$ is exponentially larger than $r^{ex(n,K_t)}$. More precisely, they proved the following:

Theorem 3.5 (Alon, Balogh, Keevash, Sudakov [3])**.** *For $r \in \{2,3\}$ and t and $n > n_0(r,t)$,*

$$c_{r,K_t}(n) = r^{ex(n,K_t)}.$$

On the other hand, for every $r \ge 4$ and $t \ge 3$, there is $c > 1$ such that, for $n > n_0(r,t)$,

$$c_{r,K_t}(n)/ex(n,K_t) > c^{n^2}.$$

In their proof, they combine a multicolored version of Szemerédi's Regularity Lemma with a stability result for the Turán graphs. It was one of the first examples where the Regularity Lemma was used to prove an exact result. Their results extend to edge-color-critical graphs as well.

Furthermore, they proved the following for more colors:

Theorem 3.6 (Alon, Balogh, Keevash, Sudakov [3])**.** *For every $r \geq 4$ and $t \geq 3$, the limit $f(r,t) = \lim_{n\to\infty} c_{r,K_t}(n)^{1/\binom{n}{2}}$ exists and satisfies*

$$r^{(t-2)/(t-1)} < f(r,t) \leq r.$$

Furthermore, $f(r,t) = r(t-2)/(t-1)(1+o(1))$, where the $o(1)$ term goes to zero as $t+r$ tends to infinity.

Finally, they proved that $c_{4,K_3}(n) = (3^{1/2}2^{1/4})^{\binom{n}{2}+o(n^2)}$ and $c_{4,K_4}(n) = (3^{8/9})^{\binom{n}{2}+o(n^2)}$. Similar results can be obtained replacing K_t by a graph H with $\chi(H) \geq 3$.

In a paper of 2012, Pikhurko and Yilma [100] removed the $o(n^2)$ in the bounds above, proving that, for large n, $c_{4,K_3}(n) \sim (3^{1/2}2^{1/4})^{\binom{n}{2}}$ and $c_{4,K_4}(n) \sim (3^{8/9})^{\binom{n}{2}}$, and that the only extremal graphs are indeed the ones conjectured by Alon et al., namely $T_4(n)$ and $T_9(n)$, respectively. A few years later, in 2016, they proved with Staden [99] that even if we forbid complete graphs of different sizes in each color, there is always an extremal graph which is complete multipartite, and wrote the problem as an optimization program.

Balogh, in 2006, was the first to address a non-monochromatic version of the problem. In fact, he proved the following:

Theorem 3.7 (Balogh [10])**.** *For every fixed 2-edge-coloring c of K_t that uses two colors, i.e., a surjective function $c : E(K_t) \to \{1,2\}$, and n sufficient large, the Turán graph $T(n,t-1)$ is the graph on n vertices with the largest number of edge-colorings without a K_t colored as c.*

In this case, however, three colors are already enough to change the behavior of the problem: for instance, the complete graph K_n has more colorings that avoid a rainbow triangle (there are $3(2^{\binom{n}{2}}-1) \approx 3\cdot 1.41^{n^2}$ such colorings that uses only two colors) than the complete bipartite graph $T_2(n)$, which admits only $3^{n^2/4} \approx 1.32^{n^2}$ such colorings.

3.3 Forbidding some monochromatic forests

3.3.1 Colorings without monochromatic matchings

Hoppen, Kohayakawa and Lefmann [63] were the first to study this problem forbidding a bipartite monochromatic graph. More specifically, they studied the number $c_{r,I_l}(n)$, where I_l stand for a matching on l edges. To state their result, we need two definitions:

Given integers $c \geq 1$ and $n \geq c+2$, let $G_{n,c}$ be the graph on the vertex set $[n]$ and such that the vertex set is divided into two classes, the first of size c, inducing a clique, the second of size $n-c$, inducing an independent set, and all the possible edges joining vertices in different classes. Moreover, given integers k, $l \geq 2$, let $c(k,l)$ be the quantity defined by

$$c(k,l) = \begin{cases} l-1, \text{ if } k \in \{2,3\}; \\ \lceil (l-1)k/3 \rceil, \text{ if } k \geq 4. \end{cases}$$

Theorem 3.8 (Hoppen, Kohayakawa and Lefmann [63]). *Let k, $l \geq 2$ be fixed integers. There exists $n_0 = n_0(k,l)$ such that, for $n \geq n_0$, we have $c_{k,I_l}(n) = c_{k,I_l}(G_{n,c(k,l)})$. Moreover, for $n \geq n_0$, the graph $G_{n,c(k,l)}$ is the unique (k, I_l)-extremal graph up to isomorphism.*

Similarly as Theorem 3.5, Theorem 3.8 shows that the (k,H)-extremal graphs are the H-free extremal graphs for $k = 2, 3$ but not for $k \geq 4$.

3.3.2 Colorings without monochromatic paths

The same set of authors, in a paper from 2014, started to investigate the question for other forbidden bipartite graphs. More especifically, they considered some families of trees, such as paths and stars. Perhaps surprisingly, the computation of $c_{r,H}(n)$ is a much harder problem for those graphs, even for small number of colors and relatively small paths and stars.

Those are graphs that have linear Turán number, i.e., $ex(n,H) = O(n)$. In particular, the bounds of (3.1) show that the limit in (3.2) is finite in this case.

They first consider the case of paths, giving some precise results for and P_3 or P_4. In the case of the path on three vertices, they prove the following results for two and three colors, respectively:

Theorem 3.9 (Hoppen, Kohayakawa and Lefmann [64]). *Let $n \geq 2$ be an integer. Then:*

$$c_{2,P_3}(n) = 2^{\lfloor n/2 \rfloor}.$$

Equality $c_{2,P_3}(G) = c_{2,P_3}(n)$ holds if and only if n is even and G consists of n pairwise independent edges, or n is odd and G either consists of $(n-1)/2$ pairwise independent edges and an isolated vertex or of $(n-3)/2$ pairwise independent edges and a path on the three remaining vertices.

Theorem 3.10 (Hoppen, Kohayakawa and Lefmann [64]). *Let $n \geq 2$ be a positive integer. For $n = 2$ we have $c_{3,P_3}(2) = 3$, and equality $c_{3,P_3}(G) = c_{3,P_3}(2)$ holds only for $G = K_2$. For $n = 3$ we have $c_{3,P_3}(3) = 3^3$, and equality $c_{3,P_3}(G) = c_{3,P_3}(3)$ is only achieved by $G = K_3$ or $G = P_3$. For $n \leq 4$, the function $c_{3,P_3}(n)$ is given by*

$$c_{3,P_3}(n) = \begin{cases} 18^{n/4}, & \text{if } n \equiv 0 \pmod 4; \\ 30 \cdot 18^{(n-5)/4}, & \text{if } n \equiv 1 \pmod 4; \\ 66 \cdot 18^{(n-6)/4}, & \text{if } n \equiv 2 \pmod 4; \\ 126 \cdot 18^{(n-7)/4}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

There is a single graph G on $n \geq 4$ vertices with $c_{3,P_3}(G) = c_{3,P_3}(n)$. For $4 \leq n \leq 7$ it is a cycle on n vertices. For $n \geq 8$ and $n \equiv 0 \pmod 4$, the graph G consists of $n/4$ pairwise vertex-disjoint 4-cycles. For $n = 4n' + i$, $n' \geq 2$ and $i \in \{1,2,3\}$, the graph G consists of n' vertex-disjoint cycles, $(n'-1)$ of which on four vertices and one of which on $(4+i)$ vertices.

Theorem 3.10 shows that, already for P_3, a different behavior shows up: three colors are enough to make a P_3-free extremal graph not (r, P_3)-extremal. Indeed, a P_3-free extremal graph is, for n even, a collection of $n/2$ matchings. This graph has $3^{n/2} \leq 1.74^n$ P_3-free colorings. On the other hand, when n is divisible by four, a collection of $n/4$ disjoint copies of C_4 has $18^{n/4} \geq 2.05^n$ P_3-free colorings.

For more colors, they give the following general upper bound:

Theorem 3.11 (Hoppen, Kohayakawa and Lefmann [64]). *For every $r \geq 4$ and n, $c_{r,P_3}(n) \leq (r!)^{n/2}$.*

Although they are not able to characterize or even conjecture what the (r, P_3)-extremal graphs are, they give constructions that yields much more colorings than the extremal graph: for instance, if the number r of colors is an odd multiple of three, let $m = r/3+1$, and consider a 1-factorization of the complete graph K_m with corresponding matchings $M_1, \ldots, M_{m-1}$, each of size $m/2$. If we split the set of colorings among the matchings M_i, $i = 1, \ldots, m-1$ in such a way that each matching receives three colors, and then color each edge in M_i with any of these three colors, we get $(3^{m/2})^{m-1}$ distinct r-colorings of K_m. Taking n/m disjoint copies of K_m, we get that

$$c_{r,P_3}(n) \geq ((3^{m/2})^{m-1})^{n/m} = 3^{rn/6},$$

which is much larger than then $r^{n/2}$ that a P_3-free extremal graph admits.

Another construction that works for $r = 4$ or $r = 5$ is to take $n/4$ disjoint copies of C_4 and, in each cycle, use two fixed colors to color one of the matchings and the two or three remaining colors to color the other matching. This gives more than $2^{n/4} \cdot 2^n = 2^{5n/4}$ (resp. $2^{n/4} \cdot 2^{n/2} \cdot 3^{n/2}$) colorings, again a bigger number than $4^{n/2}$ (resp. $5^{n/2}$) colorings of the P_3-free extremal graph.

For P_4, again for $r = 3$ there are graphs with more colorings that the P_4-free extremal graphs: on the one hand, an P_4 extremal graph on n vertices has either r^n or r^{n-1} colorings, depending if $n \equiv 0 \pmod 3$ or not; on the other hand, the following general proposition shows that, in particular, $c_{3,P_4}(n)$ is much bigger than $3^{ex(n,P_4)}$:

Proposition 3.12 (Hoppen, Kohayakawa and Lefmann [64]). *For integers $r \geq 1$ and $n \geq 4$, we have*

$$c_{r,P_4}(n) \geq (r!)^{n-r+1}.$$

In case of two colors, they prove that the P_4-free extremal is also extremal when n is divisible by three:

Theorem 3.13 (Hoppen, Kohayakawa and Lefmann [64]). *If n is a positive integer, then*

$$c_{2,P_4}(n) \leq 2^n.$$

Equality holds for graphs on n vertices if and only if $n \equiv 0 \pmod 3$ and G consists of $n/3$ pairwise vertex-disjoint triangles. Moreover, if an n-vertex graph G is not P_4-extremal, then $c_{2,P_4}(n) \leq 31 \cdot 2^{n-5}$.

If n is not divisible by three, they show a better construction than the extremal graph:

Proposition 3.14 (Hoppen, Kohayakawa and Lefmann [64]). *For any integer $n \geq 4$ with $n \not\equiv 0 \pmod 3$, we have*

$$c_{2,P_4}(n) \geq 9/8 \cdot 2^{n-1} > 2^{ex(n,P_4)}.$$

The behavior of c_{r,P_l} is not known for $r \geq 2$ and $l \geq 5$. The authors conjectured the following:

Conjecture 3.15. *For every $l \geq 5$, $b_{2,P_l} = 2$. On the other hand, for every fixed $r \geq 3$, there is $C > 1$ such that $c_{r,P_l}(n) > C^n \cdot r^{ex(n,P_l)}$.*

This conjecture is proven to be true for $r \geq 4$ by the following more general proposition:

Proposition 3.16 (Hoppen, Kohayakawa and Lefmann [64]). *Let $r \geq 4$ and let H be a connected graph with linear Turán number. Then there exists $C > 1$ such that $c_{r,H}(n) > C^n \cdot r^{ex(n,H)}$.*

3.4 Colorings without monochromatic stars

We consider now the number $c_{r,S_t}(n)$, where S_t stands for the star with t edges. The fact that $ex(n, S_t) = \lfloor n(t-1)/2 \rfloor$, together with (3.1) and (3.2), implies that

$$r^{\lfloor n(t-1)/2 \rfloor} \leq c_{r,S_t}(n) \leq r^{r \lfloor n(t-1)/2 \rfloor}$$

and

$$r^{\lfloor (t-1)/2 \rfloor} \leq b_{r,S_t} \leq r^{r \lfloor (t-1)/2 \rfloor}.$$

The following theorem improves both the upper and the lower bounds above. For that, first we consider the number $\chi_{r,t} = \prod_{i=0}^{r-1} \binom{(r-i)(t-1)}{t-1} = (r(t-1))!/(t-1)!^r$. This is the number of r-colorings of the star $S_{r(t-1)}$ without a monochromatic S_t.

Theorem 3.17 (Hoppen, Kohayakawa and Lefmann [64]). *For positive integers $r \geq 3$, n and r, we have*

$$b_{r,S_t} \leq (\chi_{r,t})^{1/2}.$$

As for the lower bound, the following asymptotic result is known:

Theorem 3.18 (Hoppen, Kohayakawa and Lefmann [64]). *Given a positive integer $r \geq 3$, we have*

$$b_{r,S_t} \geq r^{-(\sqrt{3}/4+o(1))r\sqrt{t\log t}} \cdot (\chi_{r,t})^{1/2},$$

where the $o(1)$ term above is with respect to $t \to \infty$.

This result implies that $c_{r,S_t}(n) \geq C^n \cdot r^{ex(n,S_t)}$ for some $C > 1$ and $t > t_0$. The graph $K_{t,t}$ shows that the same holds for all $r \geq 2$ and $t \geq 3$. In other words, for any fixed $r \geq 2$ and $t \geq 3$, if n is sufficient large, an S_t-free extremal graph is not (r, S_t)-extremal. This is not the case for $r = t = 2$ by Theorem 3.9.

The discussion above shows that the computation of $c_{r,S_t}(n)$ may be difficult even for small values of r and t. In particular, the value of $c_{2,S_3}(n)$, or even b_{2,S_3}, is known.

We were able to improve some of the current upper bounds when the forbidden graph is a star. We now state the best known uppper and lower bounds followed by our corresponding improvements on the upper bounds in each case.

First, we consider small forbidden stars (S_3 and S_4) and 2-colorings. For S_3, Hoppen, Kohayakawa and Lefmann had the following bounds:

Theorem 3.19 (Hoppen, Kohayakawa and Lefmann [64]). $b_{2,S_3} \leq \sqrt{6} \approx 2.45$. *On the other hand, the graph consisting of $n/6$ disjoint copies of the complete bipartite graph $K_{3,3}$ gives $b_{2,S_3} \geq \sqrt[6]{102} \approx 2.16$.*

We improve the upper bound above to:

Theorem 3.20 (Colucci, Győri and Methuku [27]). *There is a constant c such that $c_{2,S_3}(n) \leq c \cdot 18^{3n/10}$. In particular, $b_{2,S_3} \leq 18^{3/10} \approx 2.38$.*

Their result for S_4 is:

Theorem 3.21 (Hoppen, Kohayakawa and Lefmann [64]). $b_{2,S_4} \leq \sqrt{20} \approx 4.47$. *On the other hand, the graph consisting of the union of $n/10$ disjoint bipartite graphs $K_{5,5}$ gives $b_{2,S_4} \geq 3.61$.*

Our improved upper bound in this case is:

Theorem 3.22 (Colucci, Győri and Methuku [27]). $b_{2,S_4} \leq 200^{5/18} \approx 4.36$.

Next, we consider 2-colorings that forbid monochromatic big stars. Hoppen, Kohayakawa and Lefmann, in the same paper, proved the following:

Theorem 3.23 (Hoppen, Kohayakawa and Lefmann [64]). *For every t, $b_{2,S_t} \leq \binom{2t-2}{t-1}^{1/2}$. Furthermore, a certain complete bipartite graph gives $b_{2,S_t} \geq 2^{-(\sqrt{2}/2+o(1))\sqrt{t\log(t)}} \cdot \left(\binom{2t-2}{t-1}\right)^{1/2}$.*

We improve the upper bound for large t as follows:

Theorem 3.24 (Colucci, Győri and Methuku [27]). *For large values of t, we have:*

$$b_{2,S_t} \leq \left(\sqrt{2}\cdot\binom{2t-3}{t-2}\right)^{\frac{2t-3}{4t-7}}.$$

Finally, we fix the forbidden star to be S_3 and consider r-colorings. The bounds in Hoppen, Kohayakawa and Lefmann's paper are:

Theorem 3.25 (Hoppen, Kohayakawa and Lefmann [64]). *For every r, $b_{r,S_3} \leq \left(\frac{(2r)!}{2^r}\right)^{1/2}$.*

The new upper bound for this quantity that we prove here is:

Theorem 3.26 (Colucci, Győri and Methuku [27]). *If r is a sufficiently large integer, then*

$$b_{r,S_3} \leq \left(\frac{r(2r-1)!^2}{2^{2r-2}}\right)^{\frac{2r-1}{8r-6}} \sim \frac{\sqrt[8]{2}}{\sqrt[4]{e}}\cdot\left(\frac{(2r)!}{2^r}\right)^{1/2} \approx 0.85\cdot\left(\frac{(2r)!}{2^r}\right)^{1/2}.$$

3.4.1 A useful lemma

Given a graph G, we call an edge $e = uv \in E(G)$ an ab-edge ($a \leq b$) if $\{d(u), d(v)\} = \{a, b\}$. Furthermore, we denote by m_{ab} the number of ab-edges (sometimes we will write m_a instead of m_{aa} for short) and by v_a the number of vertices of degree a in G.

We now state and prove a simple lemma that will be used throughout the proofs of this paper.

Lemma 3.27. *For every $r \geq 2$, $t \geq 3$ and n, there is an (r, S_t)-extremal graph G on n vertices and a constant $c(r,t)$ with the following properties: $\Delta(G) \leq r(t-1)-1$, and $d(v) \geq \lceil \frac{r}{2} \rceil \cdot (t-1)$ holds for all but at most $c(r,t)$ vertices $v \in V(G)$.*

Proof. Let G be a graph on n vertices. If G has a vertex of degree at least $r(t-1)+1$, all of its r-edge colorings contain a monochromatic S_t, by Pigeonhole Principle, so $c_{r,S_t}(G) = 0$. Furthermore, if there is a vertex v of degree exactly $r(t-1)$, then for an edge e incident to v, the graph $G' = G - e$ has at least as many colorings as G. Indeed, every coloring of G induces a coloring of G' in an injective way, since the color of the other $(r-1)(t-1)-1$ edges incident to v define the color of the edge e uniquely.

On the other hand, if G has two vertices u, v of degree less than $\lceil \frac{r}{2} \rceil \cdot (t-1)$ not joined by an edge, the graph $G' = G + uv$ has at least as many good colorings as G, since in every partial coloring of G' that comes from a coloring of G, there is at least one free color for the edge uv. Therefore, we may assume that all such vertices induce a clique, which implies that there is at most a constant number of them.

□

3.4.2 Applying an entropy lemma

In this section, we will outline the general framework on which our proofs will rely. We start by stating a crucial lemma from [22]:

Lemma 3.28. *Let $\mathcal{F}$ be a family of vectors in $F_1 \times \cdots \times F_m$. Let $\mathcal{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$ be a collection of subsets of $M = \{1, \ldots, m\}$, and suppose that each element $i \in M$ belongs to at least k members of $\mathcal{G}$. For $j = 1, \ldots, n$ let $\mathcal{F}_j$ be the set of all projections of the members of $\mathcal{F}$ on $\mathcal{G}_j$. Then*

$$|\mathcal{F}|^k \leq \prod_{j=1}^{n} |\mathcal{F}_j|. \tag{3.3}$$

In our proofs, we will take $\mathcal{F}$ to be the set of r-edge-colorings of a graph G without monochromatic copies of S_t. It is a family of vectors in $[r]^{|E(G)|}$, where an edge-coloring $c : E(G) \to [r]$ is identified with the vector indexed by the edges of G whose value in entry $e \in E(G)$ is $c(e)$.

For each ab-edge e_i of G, we will take a set $\mathcal{G}_i$ to be the set of indices of e_i and the edges incident to it, and we take $2r(t-1)-2-(a+b)$ identical unit sets $\mathcal{G}_i^1, \ldots, \mathcal{G}_i^{2r(t-1)-2-(a+b)}$ containing the index of e_i. This choice guarantees that each edge is counted $2r(t-1)-3$ times among the sets in $\mathcal{G}$, so we may apply inequality (3.3) with $k = 2r(t-1)-3$.

Let us estimate now the size of the $\mathcal{F}_j$. It is the number of restrictions of r-edge-colorings of G without monochromatic S_t to the subgraph spanned by the edges in the set $\mathcal{G}_j$. The number of r-edge-colorings without monochromatic S_t of this subgraph is an upper bound for $|\mathcal{F}_j|$.

For the unit sets $\mathcal{G}_j^i$, it is clear that $|\mathcal{F}_j^i| \leq r$. Otherwise, let us denote by $f(x)$ the number of r-edge-colorings without monochromatic S_t of a star on x edges in which the color of exactly one edge is fixed. If we color an ab-edge e_i and then the stars hanging on its endpoints, we get $|\mathcal{F}_i| \leq rf(a)f(b)$.

Taking into account both types of sets, an ab-edge contributes to the right-hand side of (3.3) with a factor of $g(a,b) = r^{2r(t-1)-1-(a+b)} f(a)f(b)$.

Plugging this bound in (3.3), we get an optimization problem in terms of the number of ab-edges of G. This problem would be significantly simplified if we could assume that almost all edges of G are aa-edges.

This is indeed the case, since whenever we have a pair of independent ab-edges ($a \neq b$) $e = uv$ and $f = xy$, say, $d(u) = d(x) = a$ and $d(v) = d(y) = b$, such that ux and vy are not edges, we may consider the graph G' formed by G by deleting uv and xy and adding ux and vy. Note that G' has two less ab-edges, one more aa-edge and one more bb-edge than G. On the other hand, the upper bounds on the number of colorings of G and G' given by (3.3) are the same, since $g(a,b)^2 = g(a,a) \cdot g(b,b)$, and the degree of the endpoints of all other edges remain unchanged. Therefore, repeating this procedure as long as we can, we may assume that G has at most a constant number of ab-edges with $a \neq b$. In particular, we may rewrite (3.3) as

$$|\mathcal{F}|^{2r(t-1)-3} \le c \cdot \prod_{a=\lceil \frac{r}{2} \rceil \cdot (t-1)}^{r(t-1)-1} (r^{2r(t-1)-1-2a} f(a)^2)^{m_a}$$
$$= c' \cdot \prod_{a=\lceil \frac{r}{2} \rceil \cdot (t-1)}^{r(t-1)-1} (r^{2r(t-1)-1-2a} f(a)^2)^{a v_a/2}, \tag{3.4}$$

where the range of a in the product comes from Lemma 3.27.

By taking logarithms, it is clear that we are maximizing a linear function of the v_i. This means that the maximum is attained when all but one of the v_i are zero, and the exceptional v_i corresponds to the value that maximizes the function $g(a) = (r^{2r(t-1)-1-2a} f(a)^2)^a$.

3.4.3 Forbidding small stars in 2-edge-colorings

In this section, we prove Theorems 3.20 and 3.22. Following the setup in the previous section, the proofs are quite straightforward:

Proof of Theorem 3.20. By (3.4), we have the following bound:

$$|\mathcal{F}|^5 \le c \cdot \prod_{a=2}^{3} (2^{7-2a} f(a)^2)^{a v_a/2} \tag{3.5}$$
$$= c' \cdot 32^{v_2} \cdot 18^{3v_3/2}, \tag{3.6}$$

since $f(2) = 2$ and $f(3) = 3$ in this case. The fact that $32 < 18^{3/2} \approx 76$ concludes the proof.

□

Proof of Theorem 3.22. In this case, simple computations show that $f(3) = 4$, $f(4) = 7$ and $f(5) = 10$. Therefore, the bound (3.4) reads as

$$|\mathcal{F}|^9 \le c \cdot 512^{m_3} \cdot 392^{m_4} \cdot 200^{m_5} = c' \cdot 512^{3v_3/2} \cdot 392^{4v_4/2} \cdot 200^{5v_5/2}.$$

As $512^{3/2} \approx 11585$, $392^{4/2} = 153664$ and $200^{5/2} \approx 565685$, the maximum is achieved when $v_3 = v_4 = 0$ and $v_5 = n$, and the proof is complete.

□

3.4.4 Forbidding large monochromatic stars in 2-edge-colorings

In this section, we prove Theorem 3.24.

Proof of Theorem 3.24. In this case, $f(x) = \sum_{k=x-t}^{t-2} \binom{x-1}{k}$, since given a star on x edges with one edge colored with color c, we may choose at least $x-t$ and at most $t-2$ of the remaining $x-1$ edges to assign c without having a monochromatic S_t in any of the colors.

We are done, then, if we find the maximum of $g(a) = \left(2^{4t-5-2a}\left(\sum_{k=a-t}^{t-2}\binom{a-1}{k}\right)^2\right)^a$, for $t-1 \leq a \leq 2t-3$. We claim that, for t large enough, the maximum value of g is attained for $a = 2t-3$.

To prove this claim, we will use the following well-known bounds for large a and t:

$$\binom{2t-3}{t-2} \geq 0.9 \cdot \frac{2^{2t-3}}{\sqrt{\pi t}} \tag{3.7}$$

and

$$\binom{a-1}{\lceil \frac{a-1}{2} \rceil} \leq 1.01 \cdot \frac{2^{a-1}}{\sqrt{\pi a}}, \tag{3.8}$$

that are consequences of the well-known Stirling's formula: $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$.

The first one implies that

$$\begin{aligned} g(2t-3) &= \left(2\binom{2t-3}{t-2}^2\right)^{2t-3} \\ &> \left(\frac{0.9^2 \cdot 2^{4t-5}}{\pi t}\right)^{2t-3} \\ &> 2^{8t^2 - 2t\log_2 t - 25.92t + O(\log(t))}. \end{aligned}$$

Also, we have $f(a) \leq 2^{a-1}$, since $f(a)$ is a sum of binomial coefficients in the $(a-1)$-st row of Pascal's triangle. Hence,

$$g(a) \leq (2^{4t-5-2a}(2^{a-1})^2)^a = 2^{(4t-7)a}.$$

Suppose first that $a \leq 2t - \log_2 t$. Then the last inequality implies that

$$g(a) \leq 2^{(4t-7)(2t-\log_2 t)} = 2^{8t^2-4t\log_2 t + O(t)} \leq g(2t-3)$$

for large t.

On the other hand, if $2t - \log_2 t \leq a \leq 2t-4$, notice that, as the central binomial coefficient is the maximum in its row, we have

$$f(a) = \sum_{k=a-t}^{t-2} \binom{a-1}{k} \leq (2t-a-1)\binom{a-1}{\lceil \frac{a-1}{2} \rceil} \leq 1.01(2t-a-1)\frac{2^{a-1}}{\sqrt{\pi a}},$$

by (3.8).

The latter estimate implies that

$$\begin{aligned} g(a) &\leq (2^{4t-5-2a}(1.01(2t-a-1) \cdot 2^{a-1}/\sqrt{\pi a})^2)^a \\ &= 2^{a(4t-7+2\log_2(2t-a-1)+\log_2(1.01^2/\pi)-\log_2 a)}. \end{aligned}$$

By taking the derivative (for fixed t, with respect to a) of the function in the exponent, it is easy to see that this bound on g is increasing for $2t - \log_2 t \leq a \leq 2t-4$ and large t. Therefore, the maximum of the bound in this range is attained for $a = 2t-4$, which gives, for large t,

$$\begin{aligned} g(a) &\leq 2^{(2t-4)(4t-7+2\log_2(3)+\log_2(1.01^2/\pi)-\log_2(2t-4))} \\ &< 2^{8t^2-2t\log_2 t-26t+O(\log(t))} \\ &< g(2t-3). \end{aligned}$$

Now the fact that $g(2t-3) = \left(2\binom{2t-3}{t-2}^2\right)^{2t-3}$, together with (3.4), gives the result. $\square$

3.4.5 More colors

Finally, we prove Theorem 3.26.

Proof of Theorem 3.26. The bound in (3.4) can be written as

$$|\mathcal{F}|^{4r-3} \leq c \prod_{a=r}^{2r-1} (r^{4r-2a-1} f(a)^2)^{m_a} = c' \prod_{a=r}^{2r-1} (r^{4r-2a-1} f(a)^2)^{av_a/2}. \tag{3.9}$$

Again, all it is left to do is to prove that the maximum of $g(a) = (r^{4r-2a-1}f(a)^2)^a$ is obtained for $a = 2r-1$. With this result, our theorem follows by plugging $v_i = 0$ for $i < 2r-1$ and $v_{2r-1} = n$ in (3.9) and by the fact that $f(2r-1) = \frac{(2r-1)!}{2^{r-1}}$.

We have, from Stirling's formula,

$$g(2r-1) = \left(\frac{r(2r-1)!^2}{2^{2r-2}}\right)^{2r-1} = r^{8r^2-4(2-\log(2))\frac{r^2}{\log(r)}+o(\frac{r^2}{\log(r)})}.$$

We are going to bound $f(a)$ in two different ways and use each of the bounds for a different range of the value of a.

First, notice that $f(a) \leq r^{a-1}$, since this is the total number of r-colorings of a star with $a-1$ edges. This bound is enough if $a \leq 2r - 2r/\log(r)$. Indeed, in this case,

$$\begin{aligned} g(a) &\leq (r^{4r-2a-1} \cdot r^{2a-2})^a \\ &< r^{(4r-3)(2r-2\frac{r}{\log(r)})} \\ &= r^{8r^2 - 8\frac{r^2}{\log(r)} + O(r)} \\ &< g(2r-1), \end{aligned}$$

for large r.

Suppose now that that $a \geq 2r - 2r/\log(r)$. Let us divide the colorings counted by $f(a)$ according to the number of times each color appears on it. There are exactly $\frac{(a-1)!}{\prod_{i=1}^r c_i!}$ colorings where the color i appears exactly c_i times (without loss of generality: $c_1 \leq 1$; $c_i \leq 2$, for $i \geq 2$; $\sum_{i=1}^r c_i = a-1$). Notice that, in any valid coloring with $a \leq 2r-1$, at least $a-1-r$ colors appear twice, so $\frac{(a-1)!}{\prod_{i=1}^r c_i!} \leq \frac{(a-1)!}{2^{a-1-r}}$ holds for any choice of the c_i. The number of choices for the c_i satyisfing the condition above is bounded from above (being very rough) by $\binom{3r-a-2}{r-1}$ (see, for instance, Corollary 2.4 in [**?**]). Hence we have the following estimate for g:

$$g(a) \leq \left(r^{4r-2a-1} \binom{3r-a-2}{r-1}^2 \frac{(a-1)!^2}{2^{2a-2-2r}} \right)^a. \tag{3.10}$$

We will prove that the upper bound for $g(a)$ in (3.10) is increasing with a in this range, and that for $a = 2r-2$ it gives a value smaller than $g(2r-1)$.

Plugging $a = 2r-2$ in (3.10), we get

$$g(2r-2) \leq \left(\frac{r^7 (2r-3)!^2}{2^{2r-4}} \right)^{2r-2}.$$

So, Stirling's formula implies that

$$\begin{aligned} \frac{g(2r-1)}{g(2r-2)} &\geq \left(\frac{r(2r-1)!^2}{2^{2r-2}} \right)^{2r-1} \cdot \left(\frac{r^7(2r-3)!^2}{2^{2r-4}} \right)^{-(2r-2)} \\ &\sim c \cdot r^{c'} \cdot \frac{2^{6r}}{e^{4r}} \\ &> 1, \end{aligned}$$

for large r, since $2^6 > e^4$, where c and c' are positive constants.

To prove that the bound in (3.10) is increasing in this range, we first rewrite it as

$$g(a) \leq \left(r^{-1}2^{2r+2}\right)^a \cdot \left(\frac{r^{2r}(a-1)!}{(2r)^a}\binom{3r-a-2}{r-1}\right)^{2a}$$

The first term in the right-hand side of the inequality above is clearly increasing with a. Let us show that the second term, call it $h(a)^2$, grows with a as well. It is enough to prove $h(a+1)/h(a) \geq 1$. The following calculation shows that this is indeed the case:

$$\begin{aligned}
\frac{h(a+1)}{h(a)} &= \frac{r^{2r} \cdot a! \cdot a^a}{(2r)^{2a+1}} \cdot \binom{3r-a-2}{r-1} \cdot \left(\frac{2r-a-1}{3r-a-2}\right)^a \\
&\geq \frac{(2r-r/\log(r))! \cdot (2r-r/\log(r))^{(2r-r/\log(r))}}{2^{4r-3} \cdot r^{2r-3}} \cdot \frac{(r/\log(r)+1)^{2r-r/\log(r)}}{r^{2r-2}} \\
&\to \infty,
\end{aligned}$$

as $r \to \infty$, where we used the fact that $2r - r/\log(r) \leq a \leq 2r-2$ and that $\binom{3r-a-2}{r-1} \geq 1$, replacing a for either $2r-2$ or $2r - r/\log(r)$ to get the smallest bound possible.

□

3.4.6 Remarks and open problems

Our argument could be generalized by taking the sets $\mathcal{G}_j$ to include bigger neighborhoods of the edge e_j. However, in this case, new technical problems arise when we try to estimate the $|\mathcal{F}_i|$. Somewhat better results could be achieved, but we do not believe that they get substantially closer to the lower bounds.

We conjecture that $b_{2,S_3} = \sqrt[6]{102}$, i.e., the union of disjoint $K_{3,3}$'s is the graph with the largest number of 2-edge-colorings without monochromatic S_3. In general, for 2-colorings forbidding monochromatic stars of a fixed size, we think, in agreement with [64], that the extremal configuration is given by a collection of copies of a fixed (possibly complete bipartite) graph of constant size.

3.5 Forbidding other graphs

In the first sections of this chapter, we mentioned results for $c_{r,H}(n)$ where H was a nonbipartite graph (either complete or edge-critical). Subsequently, forests, i.e., (bipartite) graphs without cycles were considered. In this section, as a natural next step, we mention a few initial results for $c_{r,H}(n)$ for other graphs H.

3.5.1 Bipartite graphs containing a cycle, large number of colors

The following result of Ferber, McKinley and Samotij estimates the number of H-free graphs, where H is any graph containing a cycle.

Theorem 3.29 (Ferber, McKinley and Samotij [45]). *Let H be an arbitrary graph containing a cycle. Suppose that there are positive constants α and A such that $ex(n,H) \leq An^{\alpha}$ for all n. Then there exists a constant C depending only on α, A, and H such that for all n, the number of H-free graph on the vertex set $[n]$ is at most $2^{Cn^{\alpha}}$.*

In particular, if $ex(n,H) = An^{\alpha}$ for some α and A, this result implies that the number of H-free graphs on $[n]$ is at most $2^{C \cdot ex(n,H)}$. This is known for many classes of graphs and it is conjecture, as mentioned before, that it holds for every graph.

Since we can view an r-edge coloring of a graph on n vertices as the union of r H-free graphs on the same vertex set, we have $c_{r,H}(n) \leq 2^{Cr \cdot ex(n,H)}$ if H is a bipartite graph as above. This is better than the trivial bound $r^{r \cdot ex(n,H)}$ for large values of r, namely bigger than 2^C (personal communication with Y. Kohayakawa, 2018).

3.5.2 Four-cycle, two colors

This subsection consists of an unpublished ongoing joint project with Ervin Győri.

The smallest forbidden graph not studied in the literature, which happens to be the smallest bipartite graph with a cycle, is C_4. To the best of our knowledge, no estimates on $c_{2,C_4}(n)$, apart from the general ones from (3.1), are known.

The problem of estimating $c_{2,C_4}(n)$ is probably substantially harder than the corresponding question for complete graphs since two of the fundamental tools used in the forbidden complete graph case lack here: the Regularity Lemma and a stability theorem. Although there are versions of the former for sparse graphs, they cannot be applied directly as in the case of forbidden cliques. As for the latter, no sufficiently strong stability results are known for C_4, i.e., results that guarantee that a graph with slightly more than $ex(n, C_4)$ edges must contain many C_4s. Therefore, we believe a new technique is needed to settle the problem.

Nevertheless, we give some minor improvement on the upper bound given by (3.1), which is $c_{2,C_4}(n) \leq 2^{2ex(n,C_4)} = 2^{n^{3/2}+O(n)}$, since $ex(n,C_4) = \frac{1}{2}n^{3/2} + O(n)$ (see [35]). The lower bound, also given by (3.1), is $2^{n^{3/2}/2+O(n)} \geq 1.41^{n^{3/2}+O(n)}$.

The idea is, given a graph, to color some copies of small complete bipartite graphs ($K_{2,4}$, $K_{2,3}$ and $K_{2,2} = C_4$) in it, remove them, and then apply a trivial upper bound in the remaining edges. The following theorem of Füredi will guarantee that such copies exist:

Theorem 3.30 (Füredi [49]). *For any fixed $t \geq 2$,*

$$ex(n, K_{2,t}) = \frac{1}{2}\sqrt{t-1}n^{3/2} + O(n^{4/3}).$$

In particular, we are going to use that $ex(n,K_{2,4}) \sim \frac{\sqrt{3}}{2}n^{3/2}$ and $ex(n,K_{2,3}) \sim \frac{\sqrt{2}}{2}n^{3/2}$.

Proposition 3.31. $c_{2,C_4}(n) \leq 1.95^{n^{3/2}+o(n^{3/2})}$.

Proof. Let G be some $(2, C_4)$-extremal graph on n vertices. A graph with more than $2ex(n, C_4)$ does not have any good coloring by the pigeonhole principle, and a graph with less than $ex(n, C_4)$ has less colorings than an extremal C_4-free graph. Thus, $ex(n, C_4) \leq e(G) \leq 2ex(n, C_4)$. We may assume that $e(G) \sim 2ex(n, C_4) = n^{3/2} + O(n)$, since applying our argument with smaller $e(G)$ would give a smaller bound.

The extremal number of $K_{2,4}$ and $K_{2,3}$ imply that, ignoring lower order terms, G has $\frac{(e(G)-\sqrt{3}/2 \cdot n^{3/2})}{8} \leq \frac{2-\sqrt{3}}{16} n^{3/2}$ disjoint copies of $K_{2,4}$. Removing these copies, we are left with a graph on $\frac{\sqrt{3}}{2} n^{3/2}$ edges, so it has $\frac{\sqrt{3}/2-\sqrt{2}/2}{6} n^{3/2} = \frac{\sqrt{3}-\sqrt{2}}{12} n^{3/2}$ copies of $K_{2,3}$. The remaining graph has $\frac{\sqrt{2}}{2} n^{3/2}$ edges, so it has $\frac{\sqrt{2}-1}{8} n^{3/2}$ copies of C_4. Removing these copies, we are left with a graph on $\frac{1}{2} n^{3/2}$ edges.

It is a simple computation that the number of 2-edge-coloring of a $K_{2,4}$ (resp. $K_{2,3}$ and C_4) without a monochromatic C_4 is 128 (resp. 44 and 14). This give the following upper bound on the number of colorings of G, and, consequently, on $c_{2,C_4}(n)$:

$$c_{2,C_4}(n) \leq (128^{\frac{2-\sqrt{3}}{16}} \cdot 44^{\frac{\sqrt{3}-\sqrt{2}}{12}} \cdot 14^{\frac{\sqrt{2}-1}{8}} \cdot 2^{\frac{1}{2}})^{n^{3/2}+o(n^{3/2})} \approx 1.944^{n^{3/2}+o(n^{3/2})}.$$

$\square$

Bibliography

[1] Abu-Khzam, F. N., and Langston, M. A. Graph coloring and the immersion order. In *International Computing and Combinatorics Conference* (2003), Springer, pp. 394–403.

[2] Alon, N. Restricted colorings of graphs. *Surveys in combinatorics 187* (1993), 1–33.

[3] Alon, N., Balogh, J., Keevash, P., and Sudakov, B. The number of edge colorings with no monochromatic cliques. *Journal of the London Mathematical Society 70*, 2 (2004), 273–288.

[4] Alon, N., Kostochka, A., Reiniger, B., West, D. B., and Zhu, X. Coloring, sparseness and girth. *Israel Journal of Mathematics 214*, 1 (2016), 315–331.

[5] Alpert, H., and Iglesias, J. Length 3 edge-disjoint paths is NP-hard. *computational complexity 21*, 3 (Sep 2012), 511–513.

[6] Appel, K., and Haken, W. Every planar map is four colorable. *Bull. Amer. Math. Soc. 82*, 5 (1976), 711–712.

[7] Appel, K., Haken, W., and Koch, J. Every planar map is four colorable. *Illinois Journal of Mathematics 21* (1977), 439–657.

[8] Appel, K., Haken, W., and Koch, J. Every planar map is four colorable. Part II: Reducibility. *Illinois Journal of Mathematics 21*, 3 (1977), 491–567.

[9] Assad, A. A. Multicommodity network flows—a survey. *Networks 8*, 1 (1978), 37–91.

[10] Balogh, J. A remark on the number of edge colorings of graphs. *European Journal of Combinatorics 27*, 4 (2006), 565–573.

[11] Barnhart, C., Hane, C. A., and Vance, P. H. Integer multicommodity flow problems. In *International Conference on Integer Programming and Combinatorial Optimization* (1996), Springer, pp. 58–71.

[12] Belaidouni, M., and Ben-Ameur, W. On the minimum cost multiple-source unsplittable flow problem. *RAIRO-Operations Research 41*, 3 (2007), 253–273.

[13] Berge, C. Six papers on Graph Theory. Chapter perfect graphs, 1–21. *Calcutta: Indian Statistical Institute* (1963).

[14] Brooks, R. L. On colouring the nodes of a network. In *Mathematical Proceedings of the Cambridge Philosophical Society* (1941), vol. 37, Cambridge University Press, pp. 194–197.

[15] Bukh, B., and Conlon, D. Rational exponents in extremal graph theory. *Journal of the European Mathematical Society 20*, 7 (2018), 1747–1757.

[16] Calamoneri, T. The L(h,k)-labelling problem: an updated survey and annotated bibliography (2014). Available on `http://www.dsi.uniroma1.it/$\sim$calamo/PDF-FILES/survey.pdf`.

[17] Calamoneri, T. The $L(2,1)$-labeling problem on oriented regular grids. *The Computer Journal 54*, 11 (2011), 1869–1875.

[18] Calamoneri, T., and Sinaimeri, B. $L(2,1)$-labeling of oriented planar graphs. *Discrete Applied Mathematics 161*, 12 (2013), 1719–1725.

[19] Chang, G. J., Liaw, S.-C., et al. The $L(2,1)$-labeling problem on ditrees. *Ars Combinatoria 66* (2003), 23–31.

[20] Chartrand, G., and Zhang, P. *Chromatic graph theory.* Chapman and Hall/CRC, 2008.

[21] Chudnovsky, M., Robertson, N., Seymour, P., and Thomas, R. The strong perfect graph theorem. *Annals of mathematics* (2006), 51–229.

[22] Chung, F. R., Graham, R. L., Frankl, P., and Shearer, J. B. Some intersection theorems for ordered sets and graphs. *Journal of Combinatorial Theory, Series A 43*, 1 (1986), 23–37.

[23] Colucci, L., Erdős, P. L., Győri, E., and Mezei, T. R. Terminal-pairability in complete bipartite graphs. *Discrete Applied Mathematics 236* (2018), 459–463.

[24] Colucci, L., Erdős, P. L., Győri, E., and Mezei, T. R. Terminal-pairability in complete bipartite graphs with non-bipartite demands: Edge-disjoint paths in complete bipartite graphs. *Theoretical Computer Science 775* (2019), 16–25.

[25] Colucci, L., and Győri, E. On $L(2,1)$-labelings of oriented graphs. *Discussiones Mathematicae Graph Theory*, to appear.

[26] Colucci, L., and Győri, E. On $L(2,1)$-labelings of some products of oriented cycles. In preparation.

[27] Colucci, L., Győri, E., and Methuku, A. Edge colorings of graphs without monochromatic stars. *arXiv preprint arXiv:1903.04541* (2019).

[28] Cozzens, M., and Roberts, F. T-colorings of graphs and the channel assignment problem. *Congr. Numer. 35* (1982), 191–208.

[29] Csaba, L., Faudree, R. J., Gyárfás, A., Lehel, J., and Schelp, R. H. Networks communicating for each pairing of terminals. *Networks 22*, 7 (1992), 615–626.

[30] Diestel, R. *Graph theory*, fourth ed., vol. 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, 2010.

[31] Dirac, G. A. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society 3*, 1 (1952), 69–81.

[32] Dvořák, Z., and Yepremyan, L. Complete graph immersions and minimum degree. *Journal of Graph Theory 88*, 1 (2018), 211–221.

[33] Erdős, P. Graph theory and probability. *Canadian Journal of Mathematics 11* (1959), 34–38.

[34] Erdős, P., Rubin, A. L., and Taylor, H. Choosability in graphs. In *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium* (1979), vol. 26, pp. 125–157.

[35] Erdős, P., Rényi, A., and T. Sós, V. On a problem of graph theory. *Stud Sci. Math. Hung. 1* (1966), 215–235.

[36] Erdős, P., Stone, A. H., et al. On the structure of linear graphs. *Bull. Amer. Math. Soc 52*, 1087-1091 (1946), 1.

[37] Erdős, P. *Some new applications of probability methods to combinatorial analysis and graph theory.* University of Calgary, Department of Mathematics, Statistics and Computing . . . , 1974.

[38] Erdős, P. On the combinatorial problems which I would most like to see solved. *Combinatorica 1*, 1 (1981), 25–42.

[39] Erdős, P. Some of my favourite problems in various branches of combinatorics. *Le Matematiche 47*, 2 (1992), 231–240.

[40] Erdős, P., and Simonovits, M. A limit theorem in graph theory. In *Studia Sci. Math. Hung* (1965), Citeseer.

[41] Even, S., Itai, A., and Shamir, A. On the complexity of time table and multi-commodity flow problems. In *16th Annual Symposium on Foundations of Computer Science (sfcs 1975)* (1975), IEEE, pp. 184–193.

[42] Faudree, R. J. Properties in path-pairable graphs. *New Zealand Journal of Mathematics 21* (1992), 91–106.

[43] Faudree, R. J., Gyárfás, A., and Lehel, J. Path-pairable graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing 29* (1999), 145–158.

[44] Fekete, M. Über die verteilung der wurzeln bei gewissen algebraischen gleichungen mit ganzzahligen koeffizienten. *Mathematische Zeitschrift 17*, 1 (1923), 228–249.

[45] Ferber, A., McKinley, G. A., and Samotij, W. Supersaturated sparse graphs and hypergraphs. *arXiv preprint arXiv:1710.04517* (2017).

[46] Ford, L. R., and Fulkerson, D. R. Maximal flow through a network. In *Classic papers in combinatorics*. Springer, 2009, pp. 243–248.

[47] Ford Jr, L. R., and Fulkerson, D. R. *Flows in networks*. Princeton university press, 2015.

[48] Füredi, Z., and Simonovits, M. The history of degenerate (bipartite) extremal graph problems. In *Erdős Centennial*. Springer, 2013, pp. 169–264.

[49] Füredi, Z. New asymptotics for bipartite Turán numbers. *J. Combinatorial Theory, Series A 75* (1996), 141–144.

[50] Galvin, F. The list chromatic index of a bipartite multigraph. *Journal of Combinatorial Theory, Series B 63*, 1 (1995), 153–158.

[51] Girão, A., and Mészáros, G. An improved upper bound on the maximum degree of terminal-pairable complete graphs. *Discrete Mathematics 341*, 9 (2018), 2606–2607.

[52] Grig, J., and Yeh, R. Labeling graphs with a condition at distance two. *SIAM J. Discrete Math 5* (1992), 586–595.

[53] Gyárfás, A. On Ramsey covering-numbers. *Infinite and Finite Sets 2* (1975), 801–816.

[54] Gyárfás, A. *Problems from the world surrounding perfect graphs*. No. 177. MTA Számítástechnikai és Automatizálási Kutató Intézet, 1985.

[55] Gyárfás, A., and Schelp, R. H. A communication problem and directed triple systems. *Discrete applied mathematics 85*, 2 (1998), 139–147.

[56] Győri, E., Mezei, T. R., and Mészáros, G. Terminal-pairability in complete graphs. *arXiv preprint arXiv:1605.05857* (2016).

[57] Győri, E., Mezei, T. R., and Mészáros, G. Note on terminal-pairability in complete grid graphs. *Discrete Mathematics 340*, 5 (2017), 988–990.

[58] Hale, W. K. Frequency assignment: Theory and applications. *Proceedings of the IEEE 68*, 12 (1980), 1497–1514.

[59] Hammack, R., Imrich, W., and Klavžar, S. *Handbook of product graphs*. CRC press, 2011.

[60] HAVET, F., REED, B., AND SERENI, J.-S. L(2,1)-labelling of graphs. In *Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete Algorithms* (2008), Society for Industrial and Applied Mathematics, pp. 621–630.

[61] HIRATA, T., KUBOTA, K., AND SAITO, O. A sufficient condition for a graph to be weakly-k-linked. *Journal of Combinatorial Theory, Series B 36*, 1 (1984), 85–94.

[62] HOORY, S., LINIAL, N., AND WIGDERSON, A. Expander graphs and their applications. *Bulletin of the American Mathematical Society 43*, 4 (2006), 439–561.

[63] HOPPEN, C., KOHAYAKAWA, Y., AND LEFMANN, H. Edge colourings of graphs avoiding monochromatic matchings of a given size. *Combinatorics, Probability and Computing 21*, 1-2 (2012), 203–218.

[64] HOPPEN, C., KOHAYAKAWA, Y., AND LEFMANN, H. Edge-colorings of graphs avoiding fixed monochromatic subgraphs with linear turán number. *European Journal of Combinatorics 35* (2014), 354–373.

[65] HUCK, A. A sufficient condition for graphs to be weakly-k-linked. *Graphs and Combinatorics 7*, 4 (1991), 323–351.

[66] HUGHES, D. R., AND PIPER, F. *Design theory*, vol. 1187. CUP Archive, 1988.

[67] IMRICH, W., AND KLAVZAR, S. *Product graphs: structure and recognition.* Wiley, 2000.

[68] JANSON, S., LUCZAK, T., AND RUCINSKI, A. *Random graphs*, vol. 45. John Wiley & Sons, 2011.

[69] JENSEN, T. R., AND TOFT, B. *Graph coloring problems*, vol. 39. John Wiley & Sons, 2011.

[70] JHA, P. K., NARAYANAN, A., SOOD, P., SUNDARAM, K., AND SUNDER, V. On $L(2,1)$-labeling of the Cartesian product of a cycle and a path. *Ars Combinatoria 55* (2000), 81–90.

[71] JIANG, T., MA, J., AND YEPREMYAN, L. On Turán exponents of bipartite graphs. *arXiv preprint arXiv:1806.02838* (2018).

[72] JIANG, T., AND QIU, Y. Turán numbers of bipartite subdivisions. *arXiv preprint arXiv:1905.08994* (2019).

[73] KAHN, J. Asymptotics of the list-chromatic index for multigraphs. *Random Structures & Algorithms 17*, 2 (2000), 117–156.

[74] KAHN, J. Asymptotics of the list-chromatic index for multigraphs. *Random Structures Algorithms 17*, 2 (2000), 117–156.

[75] Kang, D. Y., Kim, J., and Liu, H. On the rational Turán exponents conjecture. *arXiv preprint arXiv:1811.06916* (2018).

[76] Karp, R. M. Reducibility among combinatorial problems. In *Complexity of computer computations*. Springer, 1972, pp. 85–103.

[77] Kawarabayashi, K.-i., Kobayashi, Y., and Reed, B. The disjoint paths problem in quadratic time. *Journal of Combinatorial Theory, Series B 102*, 2 (2012), 424–435.

[78] Klavžar, S., and Vesel, A. Computing graph invariants on rotagraphs using dynamic algorithm approach: the case of (2, 1)-colorings and independence numbers. *Discrete Applied Mathematics 129*, 2-3 (2003), 449–460.

[79] König, D. Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Math. Ann. 77*, 4 (1916), 453–465.

[80] Kosowski, A. The maximum edge-disjoint paths problem in complete graphs. *Theoretical Computer Science 399*, 1-2 (2008), 128–140.

[81] Kříž, I. A hypergraph-free construction of highly chromatic graphs without short cycles. *Combinatorica 9*, 2 (1989), 227–229.

[82] Kubicka, E., Kubicki, G., and Lehel, J. Path-pairable property for complete grids. *Combinatorics, graph theory, and algorithms. Vol. 2.* (1999), 577–586.

[83] Kühn, D., and Osthus, D. Topological minors in graphs of large girth. *Journal of Combinatorial Theory, Series B 86*, 2 (2002), 364–380.

[84] Lescure, F., and Meyniel, H. On a problem upon configurations contained in graphs with given chromatic number, graph theory in memory of ga dirac (sandbjerg, 1985), 325–331. *Ann. Discrete Math 41*.

[85] Lovász, L. On chromatic number of finite set-systems. *Acta Mathematica Hungarica 19*, 1-2 (1968), 59–67.

[86] Lovász, L. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A 25*, 3 (1978), 319–324.

[87] Mader, W. Über die maximalzahl kantendisjunktera-wege. *Archiv der Mathematik 30*, 1 (1978), 325–336.

[88] Mantel, W. Problem 28. *Wiskundige Opgaven 10* (1907), 60–61.

[89] Menger, K. Zur allgemeinen kurventheorie. *Fundamenta Mathematicae 10*, 1 (1927), 96–115.

[90] Mészáros, G. On path-pairability in the cartesian product of graphs. *Discussiones Mathematicae Graph Theory 36*, 3 (2016), 743–758.

[91] Metzger, B. Spectrum management technique. In *38th national ORSA meeting* (1970), vol. 460.

[92] Molloy, M., and Salavatipour, M. R. A bound on the chromatic number of the square of a planar graph. *Journal of Combinatorial Theory, Series B 94*, 2 (2005), 189–213.

[93] Mycielski, J. Sur le coloriage des graphes. In *Colloq. Math* (1955), vol. 3, p. 9.

[94] Nešetřil, J., Raspaud, A., and Sopena, E. Colorings and girth of oriented planar graphs. *Discrete Mathematics 165* (1997), 519–530.

[95] Nešetřil, J., and Rödl, V. A short proof of the existence of highly chromatic hypergraphs without short cycles. *Journal of Combinatorial Theory, Series B 27*, 2 (1979), 225–227.

[96] Nishizeki, T., Vygen, J., and Zhou, X. The edge-disjoint paths problem is NP-complete for series–parallel graphs. *Discrete Applied Mathematics 115*, 1-3 (2001), 177–186.

[97] Okamura, H. Every $4k$-edge-connected graph is weakly $3k$-linked. *Graphs and Combinatorics 6*, 2 (1990), 179–185.

[98] Okamura, H. Every $6k$-edge-connected graph is weakly $5k$-linked. *Memoirs of the Faculty of Engineering, Osaka City University 31* (1990), 157–169.

[99] Pikhurko, O., Staden, K., and Yilma, Z. B. The Erdős–Rothschild problem on edge-colourings with forbidden monochromatic cliques. In *Mathematical proceedings of the cambridge philosophical society* (2017), vol. 163, Cambridge University Press, pp. 341–356.

[100] Pikhurko, O., and Yilma, Z. B. The maximum number of K_3-free and K_4-free edge 4-colorings. *Journal of the London Mathematical Society 85*, 3 (2012), 593–615.

[101] Raspaud, A., and Sopena, E. Good and semi-strong colorings of oriented planar graphs. *Information Processing Letters 51*, 4 (1994), 171–174.

[102] Roberts, F. S. T-colorings of graphs: recent results and open problems. *Discrete mathematics 93*, 2-3 (1991), 229–245.

[103] Robertson, N., and Seymour, P. Graph minors XXIII. Nash-Williams' immersion conjecture. *Journal of Combinatorial Theory, Series B 100*, 2 (2010), 181–205.

[104] Robertson, N., Seymour, P., and Thomas, R. Hadwiger's conjecture for K_6-free graphs. *Combinatorica 13*, 3 (1993), 279–361.

[105] Robertson, N., and Seymour, P. D. Graph minors. XIII. The disjoint paths problem. *Journal of combinatorial theory, Series B 63*, 1 (1995), 65–110.

[106] Schwarz, C., and Troxell, D. S. L(2,1)-labelings of cartesian products of two cycles. *Discrete Applied Mathematics 154*, 10 (2006), 1522–1540.

[107] Sebő, A., and Szegő, L. The path-packing structure of graphs. In *International Conference on Integer Programming and Combinatorial Optimization* (2004), Springer, pp. 256–270.

[108] Seymour, P. D. Disjoint paths in graphs. *Discrete mathematics 29*, 3 (1980), 293–309.

[109] Shannon, C. E. A theorem on coloring the lines of a network. *Studies in Applied Mathematics 28*, 1-4 (1949), 148–152.

[110] Shao, Z., Jiang, H., and Vesel, A. $L(2,1)$-labeling of the cartesian and strong product of two directed cycles. *Mathematical Foundations of Computing 1*, 1 (2018), 49–61.

[111] Sumner, D. Subtrees of a graph and the chromatic number, in" the theory and applications of graphs," Kalamazoo, Michigan, 1980, 1981.

[112] Sylvester, J. J. Mathematical questions with their solutions. *Educational times 41*, 21 (1884), 6.

[113] Tesman, B. A. *T-colorings, list T-colorings, and set T-colorings of graphs.* PhD thesis, Department of Mathematics, Rutgers University New Brunswick, NJ, 1989.

[114] Tesman, B. A. List T-colorings of graphs. *Discrete Applied Mathematics 45*, 3 (1993), 277–289.

[115] Thomassen, C. 2-linked graphs. *European Journal of Combinatorics 1*, 4 (1980), 371–378.

[116] Thomassen, C. Every planar graph is 5-choosable. *Journal of Combinatorial Theory Series B 62*, 1 (1994), 180–181.

[117] Turán, P. On an extremal problem in graph theory. *Mat. Fiz. Lapok 48* (1941), 436–452.

[118] van den Heuvel, J., and McGuinness, S. Coloring the square of a planar graph. *Journal of Graph Theory 42*, 2 (2003), 110–124.

[119] Vizing, V. G. On an estimate of the chromatic class of a p-graph. *Discret Analiz 3* (1964), 25–30.

[120] Vizing, V. G. The chromatic class of a multigraph. *Kibernetika (Kiev) 1965*, 3 (1965), 29–39.

[121] VIZING, V. G. Vertex colorings with given colors. *Diskret. Analiz 29* (1976), 3–10.

[122] VYGEN, J. NP-completeness of some edge-disjoint paths problems. *Discrete Applied Mathematics 61*, 1 (1995), 83–90.

[123] WHITTLESEY, M. A., GEORGES, J. P., AND MAURO, D. W. On the λ-number of Q_n and related graphs. *SIAM Journal on Discrete Mathematics 8*, 4 (1995), 499–506.

[124] YEH, K.-C. *Labeling graphs with a condition at distance two.* PhD thesis, University of South Carolina, 1990.

[125] YEH, R. K. A survey on labeling graphs with a condition at distance two. *Discrete Mathematics 306*, 12 (2006), 1217–1231.

[126] YUSTER, R. The number of edge colorings with no monochromatic triangle. *Journal of Graph Theory 21*, 4 (1996), 441–452.

[127] ZOELINER, J. A., AND BEALL, C. L. A breakthrough in spectrum conserving frequency assignment technology. *IEEE Transactions on Electromagnetic Compatibility*, 3 (1977), 313–319.

[128] ZUCKERMAN, D. Linear degree extractors and the inapproximability of max clique and chromatic number. In *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing* (2006), ACM, pp. 681–690.

[129] ZYKOV, A. A. On some properties of linear complexes. *Matematicheskii sbornik 66*, 2 (1949), 163–188.