Jordan type properties of birational and biregular automorphism groups of varieties

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Contents

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Introduction

Investigating transformation groups is a fruitful field of mathematical research. In this thesis we study the birational and the biregular automorphism groups of algebraic varieties. Our main focus will be on the boundedness and Jordan properties.

Definition 0.1 (Definition 2.9 in [Po11])**.** A group G is called bounded if there exists a constant $b = b(G) \in \mathbb{Z}^+$, only depending on G, such that the order of an arbitrary finite subgroup $H \leqq G$ is at most b.

Definition 0.2 (Definition 2.11 in [Po11])**.** A group G is called Jordan if there exists a constant $J = J(G) \in \mathbb{Z}^+$, only depending on G, such that every finite subgroup $H \leqq G$ has an Abelian subgroup $A \leqq H$ of index bounded above by J.

Both of these definitions were introduced by V. L. Popov ([Po11]). They are closely related, there are are various interactions between them (see Theorem 1.11).

The history of Jordan type properties of birational automorphism groups started with the paper of J.-P. Serre ([Se09]) in 2009. He proved that the Cremona group of rank two over an arbitrary field of characteristic zero enjoys the Jordan property. Also he raised the question, whether this holds for higher rank Cremona groups.
In 2011 V. L. Popov together with Yu. G. Zarhin proved that the birational automorphism group of a surface over a field of characteristic zero is Jordan, save the case when the surface is birational to the direct product of an elliptic curve and the projective line ([Po11], [Za14]).
Since then many authors have contributed to the subject ([BZ15a], [BZ15b], [BZ19], [Hu18], [MZ15], [Po11], [Po14], [PS14], [PS16], [PS18a], [Se09], [Za15]).

One of the central results of Jordan type theorems for birational automorphism groups is due to Yu. Prokhorov and C. Shramov ([PS16]). They affirmatively answered the question of J.-P. Serre and vastly generalized the result. Their proof is based on two pillars, on the Minimal Model Program and on the boundedness of Fano varieties (with mild singularities). At the time of their

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article, the theorem about boundedness of Fano varieties was known as the Borisov-Alekseev-Borisov conjecture. In 2016, C. Birkar verified the conjecture in his acclaimed article ([Bi16]), and by doing so, he completed the proof. Now we state the main theorem of [PS16].

Theorem 0.1. *Let d be a non-negative integer. There exists a constant $J = J(d) \in \mathbb{Z}^+$, only depending on d, such that if X is an arbitrary d dimensional rationally connected variety over some field k of characteristic zero, and $G \leqq \mathrm{Bir}(X)$ is an arbitrary finite subgroup of its birational automorphism group, then there exists an Abelian subgroup $A \leqq G$ with index at most J.*

The theorem is even stronger than stating that the birational automorphism group of a rationally connected variety is Jordan, as it finds Jordan constants for all at most d dimensional rationally connected varieties at once.

Another greatly influential article of the field also comes from the work of Yu. Prokhorov and C. Shramov ([PS14]). Amongst many interesting results they proved the following theorem.

Theorem 0.2. *The birational automorphism group of a non-uniruled variety over a field of characteristic zero is Jordan.*

Also in [PS14], Yu. Prokhorov and C. Shramov defined solvably Jordan groups (Definition 8.1 in [PS14]), and proved that the birational automorphism group of a variety is solvably Jordan (answering a question of D. Allcock).

Definition 0.3. A group G is called solvably Jordan or nilpotently Jordan of class at most c if there exists a constant $J = J(G) \in \mathbb{Z}^+$, only depending on G, such that every finite subgroup $H \leqq G$ contains a subgroup $K \leqq H$ such that the index of K in H is bounded by J, and K is solvable or nilpotent of class at most c, respectively.

Theorem 0.3. *The birational automorphism group of a variety over a field of characteristic zero is solvably Jordan.*

We have seen that the birational automorphism group of many varieties are Jordan, however even amongst surfaces one can find a counterexample. On the other hand, if we replace the Abelian property with the slightly weaker solvability property, then the birational automorphism group of every variety enjoys the solvably Jordan property. This naturally raises the question that how much we can weaken the condition of commutativity. One of the main results of this thesis is that the birational automorphism group of a d dimensional variety is nilpotently Jordan of class at most d (Theorem 3.1).

Theorem 0.4. *The birational automorphism group of a d dimensional variety over a field of characteristic zero is nilpotently Jordan of class at most d.*

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It is also worth to have a look at differential geometry for a moment. In the mid-nineties É. Ghys conjectured that the diffeomorphism group of a smooth compact real manifold is Jordan ([Gh97]). By the work of I. Mundet i Riera and others, in many cases the conjecture was verified ([MR10], [MR16], [MR18], [MRSC19], [Zi14]). However in 2014, B. Csikós, L. Pyber and E. Szabó found a counterexample ([CPS14]). Its construction was analogous to Yu. G. Zarhin's one. Hence, É. Ghys improved on his conjecture, and proposed the problem of showing that the diffeomorphism group of a compact real manifold is nilpotently Jordan ([Gh15]). As the first trace of evidence, I. Mundet i Riera and C. Saéz-Calvo showed that the diffeomorphism group of a 4-fold is nilpotently Jordan of class at most 2 ([MRSC19]).

Yu. Prokhorov and C. Shramov finish their article [PS14] with some questions. One of them asks whether the birational automorphism group of a conic bundle over an Abelian variety enjoys the Jordan property. It is a logical step to investigate these kind of varieties, as the main examples of varieties with non-Jordan birational automorphism group are provided by the direct product of a projective line and an Abelian variety. In [BZ15b] T. Bandman and Yu. G. Zarhin showed that the birational automorphism group of a non-trivial conic bundle over an Abelian variety is Jordan. One of the key steps in their proof was to show that the birational (hence the biregular) automorphism group of a non-trivial Brauer-Severi curve (the generic fibre of a non-trivial conic bundle) is bounded. In this thesis we generalize this result to automorphism groups of forms of admissible flag varieties (Theorem 4.1). (Later on we will precisely define what do we mean by admissibility (Definition 4.2). For the moment, we note that most flag varieties are admissible.) Our main result is the theorem below.

Theorem 0.5. *Let k be a field of characteristic zero, containing all roots of unity. Let the k-variety X be a form of an admissible flag variety. Then either the automorphism group $\mathrm{Aut}_k(X)$ is bounded, or X is birational to a direct product variety $Y \times \mathbb{P}^1$, in other words X is ruled.*

Another aspect of our motivation was that, we would have liked to study conditions which imply that the birational automorphism group of a rationally connected variety is bounded. We hope that by performing a smooth regularization (Lemma 3.2), we can delegate the question to investigating finite subgroups of the biregular automorphism group. The Minimal Model Program produces Mori fibrations for rationally connected varieties, whose generic fibres are Fano varieties over function fields. Hence Fano varieties over function fields could play an important role in an inductive argument. One of the more accessible examples of these kind of varieties are forms of flag varieties, therefore we chose to examine them.

The thesis is structured in the following way. In Chapter 1 we elaborate more on the history of Jordan type properties and sketch the proofs of some of the cornerstone theorems. In Chapter

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2 we collect results about some important objects of the field. First we have a look at the Minimal Model Program, then we summarize results about uniruled and rationally connected varieties and the maximal rationally connected fibration. Chapter 3 contains one of our main theorems, the theorem about the nilpotently Jordan property of the birational automorphism group, while Chapter 4 contains the other main theorem of this thesis, which is about the boundedness of the automorphism groups of forms of admissible flag varieties.

Unless stated otherwise, fields are of characteristic zero and we use the conventions of [Ha77] and [Ko96]. By a variety over a field k (which is not necessarily algebraically closed), we mean an integral separated scheme which is of finite type over the field k.

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Chapter 1

History

In this chapter we look through the history of Jordan type properties for transformation groups and elaborate on some of the proofs of the most important theorems. We mainly focus on the algebraic geometrical setting, however at the end of the chapter we will have an outlook to differential geometry as well.

The Cremona group $\mathrm{Cr}(n, k)$ of rank n over a field k is the group of the birational automorphisms of the projective space $\mathbb{P}^n_k$. Investigating it has a long and rich history. The general observation is that this group is very large, however it is more accessible on the level of its finite subgroups. Moreover the system of finite subgroups of $\mathrm{Cr}(n, k)$ share same similarities with the system of finite subgroups of the general linear group $\mathrm{GL}(n, k)$. One example of this phenomenon is the Jordan property which we will define and explore below.
C. Jordan proved the following beautiful theorem ([Jo878]).

Theorem 1.1. *Let n be a positive integer. There exists a constant $J = J(n) \in \mathbb{Z}^+$, only depending on n, such that if $G \leqq \mathrm{GL}(n, \mathbb{C})$ is a finite subgroup of the complex general linear group of degree n, then it has an Abelian subgroup $A \leqq G$ with index bounded (above) by J.*

Motivated by the theorem V. L. Popov introduced the concept of Jordan groups (Definition 2.11 in [Po11]).

Definition 1.1. A group G is called Jordan if there exists a constant $J = J(G) \in \mathbb{Z}^+$, only depending on G, such that every finite subgroup $H \leqq G$ has an Abelian subgroup $A \leqq H$ of index bounded by J.

Before moving forward it is worth to point out a couple of things about the theorem and the definition.

It does not make any difference in the above theorem and definition if we require A to be

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normal. Indeed, if $H_1 \leqq H$ is a subgroup of index I ($I \in \mathbb{Z}^+$), then H has a subgroup of index at most I^I which is normal. To see this, notice that H_1 has at most I many different conjugations, as their number is bounded by the index of the normalizer subgroup of H_1. The intersection of the subgroups conjugate to H_1 is clearly normal in H, and its index is bounded by I^I.

Informally, the Jordan property means that all finite subgroups of G are close to being Abelian. More precisely, if H is an arbitrary finite subgroup of the Jordan group G, the H can be built up as an extension of a "large" Abelian group (i.e. an Abelian normal subgroup of bounded index) and something "small" (i.e. a quotient group of bounded cardinality).

Theorem 1.1 holds for any field k of characteristic zero with the same constant J. Indeed, if $G \leqq \mathrm{GL}(n, k)$ is a finite subgroup, then, as G only uses finitely many elements from the ground field, it can be embedded to $\mathrm{GL}(n, l)$, where l is a finite extension of the prime field $\mathbb{Q}$. In turn, $\mathrm{GL}(n, l)$ can be embedded to $\mathrm{GL}(n, \mathbb{C})$, hence G is isomorphic to a finite subgroup of the complex linear group, which shows our claim.

On the other hand, in the characteristic p world (where p is a prime number), we have to be contented with a slightly weaker claim. Jordan's theorem still holds for the system of those finite subgroups whose order is relatively prime to the characteristic, however if we would like to treat all finite subgroups of a general linear group over a field of characteristic p, we have to pay special attention to the subgroups whose orders are divisible by p. For example, let k be an algebraically closed field of characteristic p, and denote by $\mathbb{F}_{p^m}$ the finite field with p^m elements ($m \in \mathbb{Z}^+$). The special linear groups $\mathrm{SL}(2, \mathbb{F}_{p^m})$ over the various finite fields $\mathbb{F}_{p^m}$ are all finite subgroups of the general linear group $\mathrm{GL}(2, k)$. They can have arbitrary large order, however their Abelian normal subgroups have at most two elements. Hence $\mathrm{GL}(2, k)$ is not Jordan in the sense of the above definition. See [BF66] for further discussion.

Several proofs are known to Theorem 1.1. C. Jordan's argument is based on analyzing the cardinalities and the "shapes" of the centralizer subgroups of various elements of $g \in G$, and then deriving an equation which involves the cardinality of G and the cardinalities of the centralizer subgroups. Besides the original source, a really nice and detailed treatment can be found in [Br11]. The modern way is due to Frobenius ([Fr911]). By Weyl's unitary trick (i.e. choosing a hermitian metric on $\mathbb{C}^n$ and averaging it by the help of the finite group G) we can assume that the finite group G is a subgroup of the unitary group. Then using the commutator shrinking property (which states that in the unitary group if two elements are sufficiently close to the identity element, then their commutator are even closer to the identity), one can show that the elements sufficiently close to the identity form a nilpotent normal subgroup. It can also be shown that the index of this group is bounded in terms of n. With a slight amount of extra work we can even find an Abelian subgroup of bounded index. The interested reader can look it up at [CR62], Theorem 36.13.

Investigating finite subgroups of the complex plane Cremona group $\mathrm{Cr}(2, \mathbb{C})$ was culminated in an essentially complete, however heavily involved classification by I. V. Dolgachev and V. A.

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Iskovskikh ([DI09]). In [Se09] J.-P. Serre examined the Cremona group of rank two over arbitrary fields, and he even found some far reaching structural results for the complex plane Cremona group (Theorem 5.3 in [Se09]).

Theorem 1.2. *The complex plane Cremona group* $\mathrm{Cr}(2, \mathbb{C})$ *is Jordan.*

Of course the result also holds over arbitrary fields of characteristic zero (with the same constant), and J.-P. Serre also proved the corresponding theorem for characteristic p fields and for the system of those finite subgroups whose cardinality is relatively prime to p. Moreover he raised the question whether higher rank Cremona groups enjoy the Jordan property (6.1 in [Se09]). This was answered affirmatively by Yu. Prokhorov and C. Shramov in [PS16]. The work of Yu. Prokhorov and C. Shramov relied on the Borisov-Alekseev-Borisov conjecture, which has later been verified by C. Birkar ([Bi16]).

Another possible way for generalizing Theorem 1.2 is considering the birational automorphism groups of complex algebraic surfaces. Using the fact that a surface has a smooth projective minimal model V. L. Popov was able to show that the birational automorphism group of a complex surface is Jordan, except the case when the surface is birational to the direct product of an elliptic curve and the projective line (Theorem 2.32 in [Po11]). This later case was handled by Yu. G. Zarhin in [Za14], he showed that this product variety provides a counterexample.

Up to nowadays still the product of the elliptic curve and the projective line is the main building block for finding counterexamples, so it is worth to spare a look at it. The n-Heisenberg group is a finite group which is isomorphic to the group of upper triangular 3×3 matrices with $\mathbb{Z}/n\mathbb{Z}$-valued entries which contains ones on their main diagonals. Note that the n-Heisenberg group is a central extension of the normal group $\mathbb{Z}/n\mathbb{Z}$ and the group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Also note that $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is a subgroup of the biregular automorphism group of an elliptic curve.
Fix an elliptic curve E. For every $n \in \mathbb{Z}^+$, we can find a line bundle such that the automorphism group of its total space contains the n-Heisenberg group, in such a way that the central subgroup $\mathbb{Z}/n\mathbb{Z}$ acts fibrewise via scalar multiplication on the fibres, while the quotient group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ descends to the given automorphism group of the base space E. Since the total space of a line bundle over E is birational to $E \times \mathbb{P}^1$, the above argument shows that $\mathrm{Bir}(E \times \mathbb{P}^1)$ contains every n-Heisenberg group. This prevents the birational automorphism group from enjoying the Jordan property.
An easy way to see this is that, first notice that the Heisenberg groups are not commutative. Secondly, if p is an arbitrary prime number, then the p-Heisenberg group is a finite p-group, hence its proper subgroups have index at least p. As the prime p can be arbitrary large, we cannot find a Jordan constant for a group which contains every p-Heisenberg group. So Theorem 2.23 in [Po11] combined with Theorem in 1.2 [Za14] gives us a the following theorem.

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Theorem 1.3. *Let S be a complex algebraic surface. The birational automorphism group* $\mathrm{Bir}(S)$ *is Jordan if and only if S is not birational to the product of an elliptic curve and the projective line.*

The case of curves are much more simple, for the sake of completeness we also state it (see Theorem 2.23 in [Po11]).

Theorem 1.4. *The birational automorphism group of a complex algebraic curve is Jordan.*

In [Po11] V. L. Popov also introduced another interesting class of groups (Definition 2.9 in[Po11]).

Definition 1.2. A group G is called bounded if there exists a constant $b = b(G) \in \mathbb{Z}^+$, only depending on G, such that the order of an arbitrary finite subgroup $H \leqq G$ is at most b.

Observe that boundedness implies the Jordan property. Moreover there are various interactions between the two properties. For example if G is an arbitrary group and $N \trianglelefteq G$ is a normal subgroup with the property that G/N is bounded, then G is Jordan if and only if N is Jordan (Lemma 2.11 in [Po11]).

As mentioned previously, using the arsenal of the Minimal Model Program (MMP) and boundedness of Fano varieties ([Bi16]) Yu. Prokhorov and C. Shramov were able to answer J.-P. Serre's question about the Jordan property of higher rank Cremona groups. Furthermore, they found a much more general result (Theorem 1.8 in [PS16]).

Theorem 1.5. *Let d be a non-negative integer. There exists a constant $J = J(d) \in \mathbb{Z}^+$, only depending on d, such that if X is an arbitrary d dimensional complex rationally connected variety, and $G \leqq$* $\mathrm{Bir}(X)$ *is an arbitrary finite subgroup of its birational automorphism group, then there exists an Abelian subgroup $A \leqq G$ with index bounded by J.*

As usual it does not make any difference if we state the theorem for complex rationally connected varieties or for rationally connected varieties over an arbitrary field of characteristic zero. Also we can require A to be normal in G.

Observe that the result of the theorem is even stronger then saying that the birational automorphisms group of a rationally connected variety is Jordan, as it finds the same Jordan constant for all (at most) d dimensional rationally connected varieties.

An important ingredient of the proof is the fact that Fano varieties (of fixed dimension) with mild singularities form a bounded family. This finiteness result is a starting point of an inductive argument. At the time when the article was written, this statement was known as the Borisov-Alexeev-Borisov (BAB) Conjecture. Since then it was proved by C. Birkar in his celebrated article

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[Bi16] (Theorem 1.1). (For a survey paper on the work of C. Birkar and its connection to the Jordan property, the interested reader can consult with [Ke19].)

The idea of the proof is the following. After regularizing X we can assume that the finite group G acts on it by regular automorphisms. After replacing G with a bounded index subgroup we can find a fixed point on X, which we denote by P. There is an induced faithful G action on the tangent space $\mathrm{T}_P X$, hence we can apply Jordan's theorem about general linear groups to G, which finishes the proof. Of course the hardest part of the argument is to show the existence of the fixed point. It is based on two pillars. By running the Minimal Model Program we can delegate the question to finding fixed points on lower dimensional rationally connected varieties, in other words the MMP allows us to perform induction on the dimension. Secondly, since Fano varieties form a bounded family we can find a fixed point on them after replacing the finite group G with a bounded index subgroup (where the bound only depends on the dimension of the Fano variety). This allows us to start the induction.
As the Jordan property of the birational automorphism groups of rationally connected varieties is one of the central results of the subject, we spend some time on having a closer look at how the proof works.

One of the important aspects of only dealing with finite subgroups of the birational automorphism group is that we can perform a smooth regularization (see Definition-Lemma 3.1 in [PS14]), i.e. we can replace X with a smooth projective variety birational to X on which G acts faithfully by regular automorphisms. So from now on, we can assume a biregular G-action.
The main auxiliary result which we are aiming for reads as follows (Theorem 4.2 of [PS14]).

Theorem 1.6. *Let d be a natural number. There exists a constant $R = R(d) \in \mathbb{Z}^+$, only depending on d, with the following property. If X is a rationally connected complex projective variety of dimension at most d, and $G \leqq \mathrm{Aut}(X)$ is an arbitrary finite subgroup of its automorphism group, then there exists a subgroup $H \leqq G \leqq \mathrm{Aut}(X)$ such that H has a fixed point in X, and the index of H in G is bounded by R.*

Besides that, this theorem is rather interesting on its own merits, as we have seen earlier, it almost immediately implies the Jordan property.

Note that, by the existence of G-equivariant resolution of singularities, it is enough to prove the theorem for smooth varieties. In the followings we show the main steps of the proof.

Let X be a smooth complex projective rationally connected variety endowed with a group action of a finite group $G \leqq \mathrm{Aut}(X)$. We will run a G-equivariant MMP on it, then investigate the possible terminal states of the MMP and the steps of the program. The terminal state is either a Fano variety or a Mori fibre space, in either cases we will find that, after replacing G

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with a bounded index subgroup, it contains a strictly smaller dimensional rationally connected G-equivariant closed subvariety. This subvariety can be pulled back to a strictly smaller dimensional rationally connected G-equivariant closed subvariety at each step of the Minimal Model Program, i.e. if $X_i \dashrightarrow X_{i+1}$ is the i-th morphism of the MMP run on X, and X_{i+1} contains a subvariety with the above properties, then so does X_i. (Note that, at these steps we do not need to replace G with a bounded index subgroups.) Finally we will find a strictly smaller dimensional rationally connected G-equivariant closed subvariety of X, and we can apply induction on the dimension. This will finish the proof.
The key lemma for pulling back rationally connected varieties is the following (Corollary 3.7 in [PS16]).

Lemma 1.1. *Let X and Y be complex normal quasi-projective varieties endowed with biregular actions of a finite group G. Assume that X has Kawamata log terminal singularities. Let $f : X \to Y$ be a surjective G-equivariant proper morphism with connected fibers, such that the anticanonical bundle of X is f-ample. Let $T \subsetneqq Y$ be a strictly smaller dimensional rationally connected G-equivariant closed subvariety of Y, then there exists a strictly smaller dimensional rationally connected G-equivariant closed subvariety $Z \subsetneqq X$, which dominates T.*

The lemma is designed to be applicable to the steps of a G-equivariant MMP. Indeed, the conditions on the singularities and on the morphisms are satisfied by the Mori fibration and the divisorial contractions of the MMP, meanwhile with a slight amount of extra work a similar statement can be derived for flips.

We can run a G-equivariant MMP on our smooth complex rationally connected variety by the famous result of C. Birkar, P. Cascini, C. D. Hacon and J. McKernan (Corollary 1.3.3 in [BCHM10]).
First have a look at Fano varieties as possible outputs of the MMP. Denote our Fano variety by F, and let $d = \dim X = \dim F$ be the dimension. By boundedness of Fano varieties a fixed power (denote it by m; m is bounded by some function of d) of the anticanonical divisor embeds any Fano variety of dimension at most d to a fixed dimensional (denote it by N; N is bounded by some function of d) projective space, where we also have a bound on the degree (again, bounded by some function of d) of the embedded Fano variety.
As the anticanonical divisor is functorial, the embedding of our Fano variety, also embeds G into the automorphism group of the projective space. Moreover, we can lift G to the general linear group $\mathrm{GL}(N+1, \mathbb{C})$ (as the G-action on the projective space derives from the G-action on $\mathrm{H}^0(F, -mK_F)$). Hence we can apply Jordan's theorem to G to find a bounded index Abelian subgroup of it, so we can assume that G is Abelian. Whence, we can find $(N+1)$ G-equivariant linearly independent hyperplanes of $\mathbb{P}^N$ (as G is a finite Abelian group). After intersecting some of them with F, we can find at most $\deg F$ many points on F which G permutes amongst themselves.

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Therefore a subgroup of G of index at most $(\deg F)!$ fixes a point on F. (Note that a point is a zero dimensional rationally connected variety.)
The other possible outcome for the MMP is a Mori fibre space $f : Y \to Z$, where Y is birational to X and $0 < \dim Z < \dim X$. In particular Z is rationally connected (as f is surjective and Y is rationally connected), hence we can apply induction on the dimension of the variety. So after replacing G with a bounded index subgroup, we can find a fixed point on Z. By Lemma 1.1 we can pull it back to a rationally connected closed subvariety of Y.
To sum it up, if Y is the output of the MMP, then, after replacing G with a bounded index subgroup, we can find a strictly smaller dimensional rationally connected G-invariant subvariety of Y. After a repeated application of Lemma 1.1 to the steps of the MMP, we can pull it back to X. So X contains a strictly smaller dimensional rationally connected G-invariant subvariety. We can apply induction on the dimension, to find the fixed point, and finish the proof.

To enclose our discussion about the proof of Theorem 1.5, we shortly sketch the proof of Lemma 1.1. It is based on two claims.
The first one is a theorem of T. Graber, J. Harris, and J. Starr (Corollary 1.3 in [GHS03]).

Theorem 1.7. *Let X and Y be proper complex varieties and let $f : X \to Y$ be a dominant morphism between them. If Y and the general fibers of f are rationally connected, then X is rationally connected.*

The second one is Lemma 3.4 in[PS16]. It is based on ideas from [HM07].

Lemma 1.2. *Let X and Y be complex normal quasi-projective varieties endowed with biregular actions of a finite group G. Assume that X has Kawamata log terminal singularities. Let $f : X \to Y$ be a surjective G-equivariant proper morphism with connected fibers, such that the anticanonical bundle of X is f-ample. Let $T \subsetneq Y$ be a strictly smaller dimensional G-equivariant closed subvariety of Y, then there exists a strictly smaller dimensional G-equivariant closed subvariety $Z \subsetneq X$, which dominates T and a general fiber of $f|_Z : Z \to T$ is rationally connected.*

Clearly the two claims immediately imply Lemma 1.1. Now we focus on the proof of Lemma 1.2.
The strategy is the following. We will construct T as a union of centres of non-Kawamata log terminal (non-klt) singularities. By general properties of centres of non-klt singularities and by the assumption that the anticanonical bundle of X is f-ample one can show that a general fibre of $f|_Z$ is rationally connected.
We can choose a G-invariant f-ample (non-complete) linear system such that the base locus of the linear system is T. Hence for some members of the linear system $H_1, H_2, ..., H_n$ we have $\cap H_i = T$. We can also require that, if H_i is one of the elements of $H_1, H_2, ..., H_n$, then so does gH_i for any

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$g \in G$. Let $D_Y = H_1 + ... + H_n$ (we can assume that $n \gg 1$), and let $D = f^*D_Y$. Consider the log pair (X, cD) where $c \in \mathbb{Q}$ is a parameter. If we raise the value of c, the singularities of (X, cD) can quickly get very bad over the points of T as $T \subseteqq H_i$ for every i. Let c be the log canonical treshold, then the union of non-klt centres lies above T and dominates it. As there is only finitely many non-klt centres, we can pick one, denoted by Z_0, which dominates T. Let $Z = \bigcup_{g \in G} gZ_0$.
Let S be the union of those non-klt centres which do not dominate T. Let $Y^\circ = Y \setminus f(S)$, $X^\circ = f^{-1}(Y^\circ)$ and $Z^\circ = Z \cap X^\circ$. Then Z° is a minimal G-centre of of non-klt singularities (i.e. a G-orbit of a minimal centre of non-klt singularities) of the log pair $(X^\circ, cD|_{X^\circ})$. By results on minimal centres of non-klt singularities one can show that the connected components of Z° are irreducible, moreover that the fibres of $f|_{Z^\circ}$ are connected. This in turn, implies that Z is irreducible. Hence Z is a G-equivariant subvariety which dominates T.
We are left with the task to show that the general fibers of $f|_Z$ are rationally connected. The usual form of Kawamata's subadjunction theorem tells us that if (X, E) is a strictly log canonical pair and Z is a minimal non-klt centre, then one can find a divisor E_Z on Z such that (Z, E_Z) is klt, moreover we have $(K_X + E)|_Z \sim_{\mathbb{Q}} K_Z + E_Z$. Yu. Prokhorov an C. Shramov proved a relative version of it, which can be applied to the general fibers of $f|_Z : Z \to T$. Using this relative version and the assumption that the anticanonical bundle of X is f-ample, they deduced that the general fibers of $f|_Z$ are varieties of Fano type. (We call a variety V Fano type if it is normal and there exists a $\mathbb{Q}$-divisor Δ on it, such that the pair (V, Δ) is klt, and $-(K_V + \Delta)$ is nef and big.) As varieties of Fano type are rationally connected, this finishes the proof.
This concludes our discussion on the proof of Theorem 1.5.

Another hugely important article of the subject [PS14] is also due to Yu. Prokhorov and C. Shramov. Amongst many interesting theorems they proved the ones below. The first one is Theorem 1.8 ii) of [PS14].

Theorem 1.8. *The birational automorphism group of a complex non-uniruled variety is Jordan.*

To put this theorem in context recall the concept of the maximal rationally connected (MRC) fibration. It builds up a complex variety as a fibration where the general fibers are rationally connected and the base is non-uniruled. Theorem 1.5 and Theorem 1.8 shows us the Jordan property hold for both of the building blocks.

Before discussing other results of [PS14] we introduce new Jordan type properties. The definition of solvably Jordan groups first appeared in [PS14] (Definition 8.1).

Definition 1.3. A group G is called solvably Jordan or nilpotently Jordan of class at most c if there exists a constant $J = J(G) \in \mathbb{Z}^+$, only depending on G, such that every finite subgroup $H \leqq G$ has a solvable subgroup or a nilpotent subgroup of class at most c $K \leqq H$ of index bounded by J, respectively.

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We can introduce yet another Jordan type property, which will prove itself to be very useful. It was first defined by T. Bandman and Yu. G. Zarhin (Definition 1.1 in [BZ15b]). Its importance was realized by A. Klyachko (see Theorem 1.11 for its properties).

Definition 1.4. A group G is called strongly Jordan if it is Jordan, and there exists a constant $r = r(G) \in \mathbb{Z}^+$, only depending on G, such that every finite Abelian subgroup $A \leqq G$ can be generated by r elements.

A second result of [PS14] which we would like to discuss is Proposition 8.6 of [PS14]. The questions, which the theorem answers, was raised by D. Allcock.

Theorem 1.9. *The birational automorphism group of a complex variety is solvably Jordan.*

We have seen that the birational automorphism group of a variety enjoys the Jordan property in many cases, however even amongst surfaces there is a counterexample. It is logical to examine what happens if the condition of commutativity is slightly weakened. The first candidate to look at is solvability. Theorem 1.9 tells us that the solvable Jordan property holds for the birational automorphism group of all varieties.
In this thesis we make another step in this direction, and show that if we replace solvability by the stronger nilpotency condition, then the theorem about the birational automorphism groups remains true (Theorem 3.1). We also give a bound on the nilpotency class.
As usual the field of complex numbers is not important, Theorem 1.8 and Theorem 1.9 hold over any field of characteristic zero.

A third theorem of [PS14] (Theorem 1.4 of [PS14]) which we would like to mention is slightly different in flavour.

Theorem 1.10. *Let k be a field which is finitely generated over $\mathbb{Q}$, and let X be a variety over k. Then the birational automorphism group of X is bounded.*

This answers a question of J.-P. Serre. He asked that if k is a finitely generated field over $\mathbb{Q}$, then is it possible to find a constant which bounds the order of the finite subgroups of the automorphism group of the field k.

The key for proving the above mentioned theorems is Proposition 6.2 in [PS14]. To properly state it, we need to introduce a couple of definitions.

If X is a normal projective variety (over a field of characteristic zero), then we denote its class group (i.e. the group of Weil divisors modulo linear equivalence) by $\mathrm{Cl}(X)$. We use $\mathrm{Cl}^0(X)$ to denote the subgroup of the class group which is formed by (the equivalence classes of) divisors algebraically equivalent to zero.

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Definition 1.5. Let X be a normal projective variety over a field of characteristic zero. The Neron-Severi group of X is the group of Weil divisors modulo algebraic equivalence. In formula $\mathrm{NS}(X) = \mathrm{Cl}(X)/\mathrm{Cl}^0(X)$. We use $\mathrm{NS}_{\mathbb{Q}}(X)$ to denote the Neron-Severi group with rational coefficients $\mathrm{NS}_{\mathbb{Q}}(X) = \mathrm{NS}(X) \otimes \mathbb{Q}$.

Note that if X is a smooth projective variety, then, by the Neron-Severi theorem, $\mathrm{NS}(X)$ is a finitely generated Abelian group. One can show that the same holds for the Neron-Severi group of normal projective varieties. Indeed, a resolution of singularities $\widetilde{X} \to X$ induces a surjective group homomorphism $\mathrm{NS}(\widetilde{X}) \to \mathrm{NS}(X)$ (where the equivalence classes of the exceptional divisors map to zero). Since $\mathrm{NS}(\widetilde{X})$ is finitely generated, so does $\mathrm{NS}(X)$.

Following an idea of C. Birkar, Yu. Prokhorov and C. Shramov introduced a very interesting concept, the concept of quasi-minimal models (Section 4 in [PS14]).

Definition 1.6. Let X be a variety over a field of characteristic zero, and let M be an effective $\mathbb{Q}$-divisor. M is $\mathbb{Q}$-movable if there exists $n \in \mathbb{Z}$ such that nM is an integral divisor, and the linear system generated by nM does not have a fixed component.

Definition 1.7. Let X be a projective variety with terminal singularities over a field of characteristic zero. X is a quasi-minimal model if the canonical divisor K_X is the limit of $\mathbb{Q}$-movable $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors M_j ($j \in \mathbb{Z}^+$) in the Neron-Severi group $\mathrm{NS}_{\mathbb{Q}}(X)$.

By the results of [BCHM10] it can be shown that if X is a non-uniruled variety, the there exists a quasi-minimal model birational to X.
Moreover any birational map between two quasi-minimal models is an isomorphism in codimension one. In particular, if X is a quasi-minimal model, then its birational automorphism group $\mathrm{Bir}(X)$ induces a group action on its Neron-Severi group $\mathrm{NS}(X)$.

Now we are ready to to state Proposition 6.2 of [PS14].

Proposition 1.1. *Let X be a smooth projective variety over a field of characteristic zero. Let $\phi : X \dashrightarrow Z$ be the maximal rationally connected (MRC) fibration, where Z is a quasi-minimal model. (Note that, we can require Z to be a quasi-minimal model, as the MRC fibration is defined up to birational equivalence. Moreover the base of the fibration is non-uniruled, hence it has a quasi-minimal model.) Let L be an ample divisor on Z. Finally, let ρ be the generic point of Z, and X_ρ be the generic fibre of ϕ. (Note that X_ρ is rationally connected.) If $G \leqq \mathrm{Bir}(X)$ is an arbitrary finite subgroup of the birational automorphism group, then we have the following short*

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exact sequence of groups.

$$1 \to G_\rho \to G \to G_Z \to 1$$
$$1 \to G_{alg} \to G_Z \to G_{NS} \to 1$$
$$1 \to G_L \to G_{alg} \to G_{Ab} \to 1$$

where we have the following properties

- G_ρ *is a finite subgroup of the birational automorphism group of the rationally connected variety* X_ρ*,*
- G_Z *is a finite subgroup of the birational automorphism group of the non-uniruled variety* Z*,*
- G_{NS} *is a finite subgroup of the automorphism group of the Neron-Severi group of* Z*, i.e.* $G_{NS} \leqq \mathrm{Aut}(\mathrm{NS}(Z))$*,*
- G_{alg} *is a finite group, which acts (not necessarily faithfully) on each of the algebraic equivalence classes of the Weil divisors of* Z*,*
- G_{Ab} *is a finite subgroup of the automorphism group of an Abelian variety of fixed dimension (here by automorphism we mean those transformations which respect the variety structure, but not necessarily preserve the group structure),*
- G_L *is a finite subgroup of the birational automorphism group* $\mathrm{Bir}(Z)$*, which preserve the equivalence class of* L *in the class group.*

Now we sketch the proof of the proposition.
Note that requiring X to be smooth and projective is just a technicality. We mainly interested in the properties of the birational automorphism group. Since we work in characteristic zero, we can replace an arbitrary variety with a smooth and projective one which is birational to it.
Since the MRC-fibration is functorial there is an induced birational G-action on the base Z which makes the rational map ϕ G-equivariant. Let G_Z be the image group $G_Z = \mathrm{Im}(G \to \mathrm{Bir}(Z))$. Clearly the kernel of $G \to G_Z$ acts via birational automorphisms on the generic fibre X_ρ. Choosing G_ρ to be the kernel $\mathrm{Ker}(G \to G_Z)$ establishes the first short exact sequence.
Since we choose Z to be a quasi-minimal model, G_Z acts on the Neron-Severi group. Let G_{NS} be the image group $G_{NS} = \mathrm{Im}(G_Z \to \mathrm{Aut}(\mathrm{NS}(Z)))$. Let G_{alg} be the kernel of this action, i.e. $G_{alg} = \mathrm{Ker}(G \to G_{NS})$. Clearly G_{alg} is a finite group which acts (not necessarily faithfully) on each of the algebraic equivalence classes of the Weil divisors of Z. This establishes the second short exact sequence.
Let $\mathrm{Cl}_L(Z) \subsetneqq \mathrm{Cl}(Z)$ be the subset of the class group formed by the (equivalence classes of) divisors algebraically equivalent to the fixed ample divisor L. $\mathrm{Cl}_L(Z)$ can be endowed with the structure

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of an Abelian variety (where $\dim \mathrm{Cl}_L(Z) = \dim H^1(X, \mathcal{O}_X)$). Clearly G_{alg} has an induced action on the variety $\mathrm{Cl}_L(Z)$, however it not necessarily preserves the group structure of it. Let G_{Ab} be the image group $G_{Ab} = \mathrm{Im}(G_{alg} \to \mathrm{Aut}(\mathrm{Cl}_L(Z)))$. Let G_L be the kernel of this action $G_L = \mathrm{Ker}(G_{alg} \to G_{Ab})$. Clearly, G_L is a finite subgroup of the birational automorphism group $\mathrm{Bir}(Z)$, which preserve the equivalence class of L. This establishes the third short exact sequence, and finishes the proof of the proposition.

We can analyze further the structures of the groups arising in Proposition 1.1 to draw the conclusions of Theorem 1.8 and Theorem 1.9. For Theorem 1.10, Proposition 1.1 is crucial, however there are some additional conclusions which are needed and which we do not expose here. (Still we decided to state the proposition for arbitrary fields, to illustrate that it is important for Theorem 1.10).

To make the description easier we extend the concept of Jordan, solvably Jordan and boundedness properties to families of groups (as it was done by Yu. Prokhorov and C. Shramov). A family of groups $\mathcal{G}$ is called uniformly Jordan, uniformly strongly Jordan, uniformly solvably Jordan or uniformly bounded if there exists a constant $C = C(\mathcal{G})$, only depending on the family, such that if $G \in \mathcal{G}$, then G is Jordan, strongly Jordan, solvably Jordan or bounded, respectively, with constant C (i.e. we can find a constant which works for every member of the family simultaneously).

Fix X, Z and L as in the proposition above. Let $\mathcal{G}_Z$ be the family of those groups which can arise in Proposition 1.1 as the group G_Z (i.e. it is the family of some finite subgroups of $\mathrm{Bir}(Z)$) for the various choices of the finite group G. Let $\mathcal{G}$, $\mathcal{G}_\rho$, $\mathcal{G}_{NS}$, $\mathcal{G}_{alg}$, $\mathcal{G}_{Ab}$ and $\mathcal{G}_L$ be the families of groups defined accordingly.

Since X_ρ is rationally connected, the family $\mathcal{G}_\rho$ is uniformly Jordan (Theorem 1.5).

One can show that the automorphism group of a finitely generated Abelian group is bounded. Since $\mathrm{NS}(Z)$ is a finitely generated Abelian group, this implies that the family $\mathcal{G}_{NS}$ is uniformly bounded.

The automorphism group of an Abelian variety is the extension of the normal Abelian group formed by the rational points of the Abelian variety and a subgroup of a general linear group with integer coefficients.

By a theorem of H. Minkowski if l is a finitely generated field extension of $\mathbb{Q}$ and m is an integer, then the general linear group $\mathrm{GL}(m, l)$ is bounded.

Moreover the n-torsion subgroup of a group formed by the rational points of a d-dimensional Abelian variety over an algebraically closed field of characteristic zero is $(\mathbb{Z}/n\mathbb{Z})^{2d}$. This implies that the finite subgroups of the group formed by the rational points of a d-dimensional Abelian variety can be generated by $2d$ elements.

Putting these together implies that the automorphism group of an Abelian variety is an extension of a normal Abelian group, whose finite subgroups can be generated by boundedly many elements, and a bounded group. Therefore the family $\mathcal{G}_{Ab}$ is uniformly strongly Jordan.

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Since L is ample, $X \cong \operatorname{Proj} \bigoplus \mathrm{H}^0(X, nL)$. Hence those birational automorphisms of X which preserve the equivalence class of L, defines biregular automorphisms. Moreover they also extends to automorphisms which makes the closed embedding $X \cong \operatorname{Proj} \bigoplus \mathrm{H}^0(X, nL) \hookrightarrow \mathbb{P}^0(\mathrm{H}^0(X, L)^*) \cong \mathbb{P}^N$ equivariant. Let $\mathrm{Bir}(Z, L)$ be the group of birational automorphisms which preserves the class of L. By the above considerations, $\mathrm{Bir}(Z, L)$ embeds into $\mathrm{PGL}(N+1, k)$. Assume that $\mathrm{Bir}(Z, L)$ is not finite, then the orbits of its one-paramter subgroups (which are isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$) are rational curves, which contradicts the assumption that Z is non-uniruled. Hence $\mathrm{Bir}(Z, L)$ is a finite group, and $\mathcal{G}_L$ is the family of its subgroups. In particular $\mathcal{G}_L$ is bounded.

As we have already seen there is a lot of interaction between the Jordan type and the boundedness properties. Here we state them more detailedly (see Section 2 and Lemma 8.4 of [PS14]).

Theorem 1.11. *Let $\mathcal{K}$, $\mathcal{G}$ and $\mathcal{Q}$ be families of groups. Assume that if $G \in \mathcal{G}$, then there exists $K \in \mathcal{K}$ and $Q \in \mathcal{Q}$ such that, G sits in the short exact sequence of groups*

$$1 \to K \to G \to Q \to 1.$$

1. *If the family $\mathcal{K}$ is uniformly Jordan and the family $\mathcal{Q}$ is uniformly bounded, then the family $\mathcal{G}$ is uniformly Jordan.*
2. *If the family $\mathcal{K}$ is uniformly strongly Jordan and the family $\mathcal{Q}$ is uniformly bounded, then the family $\mathcal{G}$ is uniformly strongly Jordan.*
3. *If the family $\mathcal{K}$ is uniformly bounded and the family $\mathcal{Q}$ is uniformly strongly Jordan, then the family $\mathcal{G}$ is uniformly strongly Jordan.*
4. *If the families $\mathcal{K}$ and $\mathcal{Q}$ are uniformly solvably Jordan, then the the family $\mathcal{G}$ is uniformly solvably Jordan.*

Using the above theorem we have the following. As $\mathcal{G}_L$ is uniformly bounded and $\mathcal{G}_{Ab}$ is uniformly strongly Jordan, $\mathcal{G}_{alg}$ is uniformly strongly Jordan.
As $\mathcal{G}_{alg}$ is uniformly strongly Jordan and $\mathcal{G}_{NS}$ is uniformly bounded, $\mathcal{G}_Z$ is uniformly strongly Jordan.
If X is non-uniruled, then Z and X are birational, hence $\mathcal{G}_Z$ is the family of the finite subgroups of the birational automorphism group $\mathrm{Bir}(X)$. Putting these together implies that birational automorphism group of a non-uniruled variety (over a field of characteristic zero) is strongly Jordan. This finishes the proof of Theorem 1.8.
Now, we know that the families $\mathcal{G}_\rho$ and $\mathcal{G}_Z$ are uniformly solvably Jordan (as they are even uniformly Jordan). Hence the family $\mathcal{G}$ is solvably Jordan. Hence $\mathrm{Bir}(X)$ is solvably Jordan. This

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finishes the proof of Theorem 1.9.

Yu. Prokhorov and C. Shramov enclose their article with a couple of questions. One of them asks about the Jordan property of the birational automorphism group of a conic bundle over an Abelian variety. It is a reasonable next step in the investigation. As a consequence of Theorem 1.3 we know that the birational automorphism group of a the product of an Abelian variety and the projective line does not enjoy the Jordan property. Therefore it is logical to deepen the examination in this direction and have a look on other $\mathbb{P}^1$-fibration over Abelian varieties.
The question was answered by T. Bandman and Yu. G. Zarhin in [BZ15b]. They found that the the birational automorphism group of a non-trivial conic bundle over an Abelian variety is Jordan. Their strategy was the following. Let $f : X \to A$ be a non-trivial conic bundle over the Abelian variety A, and let X_ρ be the generic fibre of f. Since f defines a non-trivial conic bundle, X_ρ is a non-trivial Brauer-Severi curve. An Abelian variety is non-uniruled, and the fibres of f are rationally connected, hence f is an MRC fibration. Let $G \leqq \mathrm{Bir}(X)$ be an arbitrary finite subgroup. Just as in Proposition 1.1, we have a short exact sequence of finite groups

$$1 \to G_\rho \to G \to G_A \to 1,$$

where G_ρ is a finite subgroup of the birational automorphism group of the Brauer-Severi curve X_ρ (note that $\mathrm{Bir}(X_\rho) = \mathrm{Aut}(X_\rho)$), while G_A is a finite subgroup of the birational automorphism group of the Abelian variety A.
Since an Abelian variety is non-uniruled, by Theorem 1.8, we know that the family of groups arise as G_A for the various choices of the finite group G is uniformly strongly Jordan. Hence by Theorem 1.11, to show that $\mathrm{Bir}(X)$ is Jordan it is enough to prove that the family of groups arise as G_ρ for the various choices of the finite group G is bounded. In other words, it is enough to show that the automorphism group of a non-trivial Brauer-Severi curve is bounded. Using linear algebra, T. Bandman and Yu. G. Zarhin were able to prove it, and found that the bound on the order of the finite subgroups is four.
In this thesis we generalize the result on the automorphism groups of Brauer-Severi curves. We show that, if X is a form of an admissible flag variety, then its automorphism group is bounded or X is ruled (Theorem 4.1). (We will precisely define later what we mean by admissibility. For now, we only note that, most flags are admissible.)
We were also motivated by the following problem, we wanted to find conditions which imply boundedness of the birational automorphism group of rationally connected varieties. Because of the Minimal Model Program it is worth to study automorphism groups of Fano varieties over function fields. One of the easiest examples of Fano varieties over function fields is forms of flag varieties.

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The question of examining Jordan type properties for automorphisms groups of different geometric structures flourishes. Besides the above mentioned results, there have been many different problems considered by various authors in the recent years. (In what follows, we assume that the ground field is of characteristic zero, if we do not say it otherwise.)
Using algebraic group theoretic methods S. Meng and D.-Q. Zhang showed that the automorphism group of a projective variety is Jordan ([MZ15]), while F. Hu proved the characteristic p counterpart of it ([Hu18]).
In [BZ15a] T. Bandman and Yu. G. Zarhin proved the Jordan property for the biregular automorphism groups of quasi-projective surfaces. They also proved it for the biregular automorphism group of certain higher dimensional quasi-projective varieties in [BZ19].
Yu. Prokhorov and C. Shramov classified three dimensional varieties with non-Jordan birational automorphism groups ([PS18a]). In that work the result on the boundedness of the automorphism groups of non-trivial Brauer-Severi curves was also proved to be useful. They also investigated bounds on the Jordan constants of birational automorphism groups of rationally connected 3-dimensional varieties ([PS17]), and the Abelian property and bounds on the number of generators of finite p-subgroups of the birational automorphism groups of rationally connected 3-dimensional varieties ([PS18b]). Furthermore, they also studied the automorphism groups of certain low dimensional compact complex manifolds ([PS18c], [PS19]).

The landscape is strikingly similar in differential geometry. The techniques are fairly different, still the results converge to similar directions. In the following we briefly review the history of the question of Jordan type properties of diffeomorphism groups of smooth compact real manifolds. (We note that there are many other interesting setups which were considered by differential geometers; for a very detailed account see the Introduction of [MR18].) As mentioned in [MR18], during the mid-nineties É. Ghys conjectured that the diffeomorphism group of a smooth compact real manifold is Jordan, and he proposed this problem in many of his talks ([Gh97]). The case of surfaces follows from the Riemann-Hurwitz formula (see [MR10]), the case of 3-folds are more involved. In [Zi14] B. P. Zimmermann proved the conjecture for them using the geometrization of compact 3-folds (which follows from the work of W. P. Thurston and G. Perelman). I. Mundet i Riera also verified the conjecture for several interesting cases, like tori, projective spaces, homology spheres and manifolds with non-zero Euler characteristic ([MR10],[MR16], [MR18]).
However in [CPS14] B. Csikós, L. Pyber and E. Szabó found a counterexample. Their construction was remarkably analogous to the one of Yu. G. Zarhin. They showed that, if the manifold M is diffeomorphic to the direct product of the two-sphere and the two-torus or to the total space of any other smooth orientable two-sphere bundle over the two-torus, then the diffeomorphism group contains n-Heisenberg groups for arbitrary large integers n. Hence $\mathrm{Diff}(M)$ cannot be Jordan. As a consequence, É. Ghys improved on his previous conjecture, and proposed the problem of showing that the diffeomorphism group of a compact real manifold is nilpotently Jordan ([Gh15]). As the

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first trace of evidence, I. Mundet i Riera and C. Saéz-Calvo showed that the diffeomorphism group of a 4-fold is nilpotently Jordan of class at most 2 ([MRSC19]). It is worth to mention that their proof uses the classification theorem of finite simple groups via the result of [MRT15].

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Chapter 2

Tools

In this chapter we collect results and tools which will be useful in the remainder of the thesis.

2.1 Minimal Model Program

As we could have seen in Chapter 1 the Minimal Model Program plays a central role in developing the theory of Jordan type properties for birational automorphism groups of varieties. Hence we dedicate a short section to summarize some of the results of this very powerful and beautiful theory. The section is mainly based on [Bi12], and we also used [Le17]. During the section we will work over the field of the complex numbers.

2.1.1 Overview

One of the ultimate goals of any branch of mathematics is classifying the objects of its investigation. In algebraic geometry we mainly interested in varieties. Therefore a central problem is to classify varieties up to isomorphisms. Since varieties show a great deal of diversity, it could be fruitful to consider other variants of the question. We could try to classify varieties up to birational equivalence.
By Hironaka's famous theorem on the resolution of singularities, every birational class contains a smooth projective variety. At this point, it could be worth to mention that, alongside with Kodaira's vanishing theorem, Hironaka's theorem are the main reasons behind that the MMP has much stronger results in characteristic zero, then in characteristic p. Without these two vital elements the field proved itself to be extremely difficult.

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Each birational class of curves contains exactly one smooth projective model. Furthermore they can be distinguished by the genus.
Recall that for a smooth projective variety X the genus is the dimension of the vector space of the global sections of the canonical line bundle ω_X. The Weil divisor (unique up to linear equivalence) corresponding to the canonical line bundle is the canonical divisor, it is denoted by K_X. During the MMP we heavily focus on the properties of the canonical divisor.
Smooth projective curves split into three categories according to their genera. If the genus is zero then the curve is isomorphic to the projective line; if it is one, then the curve is elliptic; while if the genus is larger or equal to two then the curve is of general type. These three categories behaves radically differently. In case of the projective line, the degree of the canonical divisor is negative and the curve admits a metric with constant positive curvature; in case of elliptic curves, the degree of the canonical divisor is zero, and the curves admit flat metrics; while in the general case, the degree of the canonical divisor is positive (hence it is ample), and the curves admits metric with constant negative curvature. We will see that, by the help of the MMP, there is hope to preserve some version of this trichotomy for higher dimensional varieties.

To understand the MMP for surfaces, let's recall Castelnuovo's theorem first. Let S be a smooth projective surface. If E is a curve on S such that E is isomorphic to the projective line, and its selfintersection is -1, then there exists a smooth surface S_1 and a birational morphism $f : S \to S_1$, which contracts exactly E. More precisely, $f : S \to S_1$ is the blowing-up of S_1 in the smooth point $f(E) = Q \in S_1$. So the theorem tells us that, if the curve E looks like an exceptional divisor of blowing-up a smooth point, then, indeed, we can construct the corresponding blowing-up morphism. We call E a -1-curve, and f the blowing down of S along E.
Let S be an arbitrary smooth projective surface. If S contains a -1-curve E, then we can blow it down, and replace S by the smooth surface S_1. We can repeat this process as long as we find a -1-curve. The process must terminate after a finite number of steps, since the Picard number drops by one after each step. Let S_0 be the output of the above described algorithm.
It turns out that S_0 has strong numerical properties. Moreover S_0 is either isomorphic to the projective plane, or it is a ruled surface over some curve, or its canonical divisor K_{S_0} is nef (i.e. it has non-negative intersection against any curve of S_0). S_0 can be classified even more precisely by the help of the Kodaira dimension.

Definition 2.1. Let X be a smooth projective variety. We call the non-negative integer $h^0(X, \omega_X^{\otimes m})$ the m-th plurigenus of X ($m \in \mathbb{Z}^+$).
The Kodaira dimension $\kappa(X)$ of the smooth projective variety X is defined as follows. If every plurigenus of X is zero, then $\kappa(X) = -\infty$, otherwise

$$\kappa(X) = \min \left\{ a \in \mathbb{Z}_0^+ \middle| 0 < \limsup_{m\to\infty} \frac{h^0(X, \omega_X^{\otimes m})}{m^a} \right\}.$$

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The Kodaira dimension is either $-\infty$, or it can take values between zero and the dimension of the variety.

If $\kappa(S_0) = -\infty$, then S_0 is either the projective plane or a $\mathbb{P}^1$-fibration over some curve (ruled surface). If $\kappa(S_0) = 0$, then S_0 is either a $K3$-surface, an Enrique surface, an Abelian variety or a hyperelliptic surface. In all of these cases, some power of the canonical line bundle is trivial, more precisely $12K_{S_0} = 0$. If $\kappa(S_0) = 1$, then S_0 is an elliptic surface, i.e. a surface fibred over a curve such that the general fibres are smooth elliptic curves. In this case, for a general fibre F, $K_F = 0$. If $\kappa(S_0) = 2$, then S_0 is of general type. Moreover if $\kappa(S_0) \geqq 0$, then the canonical divisor K_{S_0} is nef and a sufficiently large power of it is basepoint free.

As one might expect the case of higher dimensions are considerably more difficult. A fundamental problem is that we need to find an object which replaces the role of -1-curves. This question was solved by S. Mori, who introduced to concept of extremal rays. In case of a smooth projective variety one can think about it as follows. The Mori-Kleiman cone $\overline{\mathrm{NE}}(X)$ of a smooth projective variety X is the closure of the cone generated by the effective 1-cycles inside $\mathrm{H}^2(X, \mathbb{R})$. (It is important to take the closure.) An extremal ray is a one dimensional extremal face of this cone in the sense of convex geometry. The Cone Theorem tells us that, if K_X is negative against an extremal ray, then the ray contains an irreducible curve C, moreover there is a morphism which contracts exactly those curves which belong to the given ray in $\overline{\mathrm{NE}}(X)$.
Another great advantage of extremal rays is that, it helps us to find analogues of ruled surfaces which works well for our purposes in higher dimensions. A Mori fibration is a contraction $f : Y \to Z$ (between normal projective varieties) of a K_Y-negative extremal ray such that $\dim Z < \dim Y$.
To state the two main predictions of the MMP, we need to introduce the concept of minimal varieties. A minimal variety is a normal projective variety with mild singularities and (most importantly) with nef canonical divisor.

Conjecture 2.1 (Minimal Model)**.** Let X be a smooth projective variety.

- If $\kappa(X) = -\infty$, then X is birational to the total space of a Mori fibration.
- If $\kappa(X) \geqq 0$, then X is birational to a minimal variety.

Conjecture 2.2 (Abundance)**.** Let X be a minimal variety. There exists a normal projective variety Y, an ample divisor A on Y and a contraction morphism $f : X \to Y$ (i.e. a morphism with connected fibres) such that

$$mK_X = f^*A,$$

for some positive integer m. In particular,

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- a curve C on X is contracted to a point if and only if $K_X.C = 0$,
- the dimension of Y is equal to the Kodaira dimension of X, $\dim Y = \kappa(X) \geqq 0$.

Note that the variety Y in the Abundance Conjecture is nothing else but the canonical model of X, in formula $Y \cong \operatorname{Proj} \bigoplus \mathrm{H}^0(X, mK_X)$.
The two conjectures together imply that, up to birational equivalence every variety either admits a fibration, where for a general fibre F the canonical divisor K_F is either antiample (in case of a Mori fibre space), or it is a torsion, i.e. $K_F \sim_{\mathbb{Q}} 0$ (in case of $0 \leqq \kappa(X) < \dim X$), or up to birational equivalence (i.e. after replacing X with its canonical model) the canonical divisor K_X of X is ample (in case of $\kappa(X) = \dim X$). (Note that generally a variety admits many fibrations for which the general fibres have ample canonical divisors. So in the third case, where $\kappa(X) = \dim X$, we stated something much stronger, we stated that the canonical divisor of the variety is ample.)
The above discussion shows us that, if the conjectures hold, we preserved the trichotomy observed at curves. Up to birational equivalence every variety can be built up from varieties, for which the canonical divisor is either antiample, torsion or ample.
Another way to look at the Abundance Conjecture is that, it tells us that a numerical property implies a sectional one. This is quite rare, from intersection theory generally does not follow any kind of holomorphic information.
Also note that the above results hold for the classification of the minimal models of smooth surfaces.

There are many problems in higher dimensions which cannot be seen in the case of surfaces. Even if one starts with a smooth variety X, after contracting a K_X-negative extremal ray, we can end up with a singular variety. So the MMP needs to handle singularities.
Let $f : X \to Y$ be a contraction of a K_X-negative extremal ray. f can have three types. Either $\dim X > \dim Y$, in this case f is a Mori fibration. If the exceptional locus of f is one codimensional (i.e. f contracts at least one prime divisor), then f is a divisorial contraction. The most problematic case is when f is a small contraction, i.e. when the exceptional locus is at least two codimensional.
If $f : X \to Y$ is a small contraction, then the canonical divisor of Y cannot be $\mathbb{Q}$-Cartier. Since the MMP is governed by the numerical properties of the canonical divisor, it means that we cannot continue the algorithm of the MMP with Y. Therefore we need to do something else, we need to perform a flip.
So let $f : X \to Y$ be a small contraction of a K_X-negative extremal ray. The flip of this contraction f is the following diagram.

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d, "f"'] & X^+ \arrow[dl, "f^+"] \\ Y, & \end{tikzcd}$$

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where X^+ is a normal projective variety, $f^+ : X^+ \to Y$ is a birational morphism, which is also a contraction of small type, such that K_{X^+} is ample over Y. In particular it implies that $\phi : X \dashrightarrow X^+$ is a birational map which induces an isomorphism in codimension two.
Existence of flips (for log canonical pairs, see Definition 2.15) is a central question of the MMP. Now it is settled in many cases.
Note that, while divisorial contractions drop the Picard rank, hence they can be repeated only finitely many times, flips (i.e. replacing X by X^+) preserves the Picard rank.

Conjecture 2.3 (Termination of flips)**.** Under mild conditions there is no infinite sequence of flips.

Now we have collected many of the ingredients of the MMP. In the next section we describe what does it mean to run the MMP.

2.1.2 The algorithm

In this section we briefly sketch how the algorithm of the MMP works.

Let X be a projective normal variety with mild singularities. If the canonical divisor is nef, then the algorithm stops, and we found the minimal model.
If K_X is not nef then we can find a K_X-negative extremal ray in the Mori-Kleiman cone $\overline{\mathrm{NE}}(X)$. By the Cone Theorem, the extremal ray can be contracted. Let $f : X \to Y$ be the contraction of the extremal ray.
If $\dim X > \dim Y$, then we found a Mori fibration and the algorithm stops.
If f is a divisorial contraction, then we replace X by Y, and start the algorithm from the beginning.
If f is a contraction of small type, then we perform a flip $\phi : X \dashrightarrow X^+$. We replace X by X^+, and start the algorithm form the beginning.

To make the MMP work, one need to secure that the algorithm stops. Since the Picard number is lowered by each divisorial contraction, there could only be a bounded number of them. To show that there is no infinite sequence of flips, one could either prove Conjecture 2.3, or one could use some special kind of flips (like flips with scaling), and prove that they only produce finite length sequences.

Even if we are able to run the MMP, we still need to deal with the Abundance Conjecture to fully finish the program.

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2.1.3 Definitions and the Cone and Contraction Theorem

In this section we precisely define the objects which are needed to state some of the results of the MMP, and about which we talked in the previous sections. However we will work in greater generality then in the previous two sections. We will consider the relative setting (i.e. we allow a base scheme to be present), moreover instead of varieties we will consider pairs. These changes do not affect the general philosophy, however they allow greater flexibility for the MMP.
Recall that in the whole section we work over the field of complex numbers.

Definition 2.2. Let X and Y be varieties, and $f : X \to Y$ be a projective morphism between them. f is called a contraction if $f_*\mathcal{O}_X = \mathcal{O}_Y$. If f is a contraction, then it has connected fibres (Corollary 11.3 of Chapter 3 in [Ha77]).

Remark 2.1. By Stein factorization (Corollary 11.5 of Chapter 3 in [Ha77]), using the notion of the definition above, if Y is normal, then f is a contraction if and only if f has connected fibres.

Definition 2.3. Let X and Y be varieties and $f : X \to Y$ be a birational morphism between them. The exceptional locus of f is the set formed by those $x \in X$ for which the rational map f^{-1} is not regular in $f(x)$. It is denoted by $\mathrm{Exc}(f)$.

Definition 2.4. Let X and Z be normal varieties and $f : X \to Z$ be a projective morphism between them.

- An $\mathbb{R}$-divisor on X is an $\mathbb{R}$-linear combination of prime divisors on X.
- An $\mathbb{R}$-Cartier divisor on X is an $\mathbb{R}$-linear combination of Cartier divisors on X.
- Two $\mathbb{R}$-divisors D_1 and D_2 on X are linearly equivalent over Z, if their difference is an $\mathbb{R}$-linear combination of principal divisors on X and a pullback of an $\mathbb{R}$-linear divisor on Z. It is denoted by $D_1 \sim_{\mathbb{R},Z} D_2$.
- Two $\mathbb{R}$-divisors D_1 and D_2 on X are numerically equivalent over Z, if their difference is an $\mathbb{R}$-divisor which has zero intersection number against every curve contained in a fiber of f. It is denoted by $D_1 \equiv_Z D_2$.
- An $\mathbb{R}$-divisor on X is a ample over Z (or f-ample) if it is an $\mathbb{R}^+$-linear combination of ample divisors on X over Z (in the usual sense).
- An $\mathbb{R}$-divisor on X is nef over Z (or f-nef), if it has non-negative intersection against any curve contained in a fiber of f.

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- An $\mathbb{R}$-divisor D on X is big over Z (or f-big) if

$$0 < \limsup_{m\to\infty} \frac{h^0(X_f, \lfloor mD|_{X_f}\rfloor)}{m^{\dim X_f}},$$

  where X_f is the generic fibre of f. Equivalently, D is f-big if $D \sim_{\mathbb{R},Z} A + G$, where A is an ample $\mathbb{R}$-divisor on X over Z and G is an effective $\mathbb{R}$-divisor on X.
- An $\mathbb{R}$-divisor D on X is semiample over Z, if there exists a normal variety Y, an ample $\mathbb{R}$-divisor A over Z on Y and a projective morphism $g : X \to Y$ over Z such that $D \sim_{\mathbb{R},Z} g^*A$.
- An $\mathbb{R}$-divisor on X is pseudo-effective over Z (or f-pseudo-effective) if its numerical class over Z (i.e. its numerical class against the curves contained in the fibres of f) is the limit of the numerical classes of f-big $\mathbb{R}$-divisors on X.

All the definitions make sense if we use the field of the rational numbers $\mathbb{Q}$ instead of the field of the real numbers $\mathbb{R}$.

Definition 2.5. Let X be a normal variety. We call X $\mathbb{Q}$-factorial, if every divisor on X is $\mathbb{Q}$-Cartier.

Remark 2.2. If X is a smooth variety, then it is $\mathbb{Q}$-factorial.

Remark 2.3. One can show that if $f : X \to Y$ is a birational morphism between varieties, and Y is normal and $\mathbb{Q}$-factorial, then every irreducible component of $\mathrm{Exc}(f)$ is one codimensional (Remark 1.40 in [De01]).

Definition 2.6. Let X be a normal variety. Denote by U the largest open subvariety of X which is smooth. Define the canonical divisor K_X as the closure of the canonical divisor K_U of U.

Definition 2.7. Let X be a normal variety. Denote by U the largest open subvariety of X which is smooth, and let $i : U \hookrightarrow X$ be the corresponding open immersion. For a divisor D on X, define $\mathcal{O}_X(D)$ as $i_*\mathcal{O}_U(D|_U)$. This gives a reflexive sheaf.

Remark 2.4. Since we required X to be normal in the previous two definitions, the singular locus $X_{sing} = X \setminus U$ has codimension at least two. Therefore a divisor is uniquely defined by its restriction to U. Moreover it can be shown that, if X is projective, then $\mathcal{O}_X(K_X)$ agrees with the dualizing sheaf (Proposition 5.75 in [KM98]).

Of course the definition of $\mathbb{Q}$-Cartier divisors, do not use the fact that the ground field is $\mathbb{C}$. In the next definition, we define one of the central objects of birational geometry, Fano varieties.

Definition 2.8. Let F be a normal projective variety over an (arbitrary) ground field k. F is called Fano, if its canonical divisor is $\mathbb{Q}$-Cartier and antiample.

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Definition 2.9. Let X and Z be normal quasi-projective varieties, and let $f : X \to Z$ be a projective morphism between them.

- Let $\mathrm{Z}_1(X/Z)$ be the free Abelian group generated by the curves of X, which are contracted to a point by f.
- Two $\mathbb{R}$-1-cycles $V_1, V_2 \in \mathrm{Z}_1(X/Z) \otimes \mathbb{R}$ are numerically equivalent over Z, if for any $\mathbb{R}$-Cartier divisor D on X, we have $D.V_1 = D.V_2$. It is denoted by $V_1 \equiv_Z V_2$.
- The Neron-Severi space $\mathrm{N}_1(X/Z)$ is the $\mathbb{R}$-vector space of the $\mathbb{R}$-1-cycles modulo numerical equivalence over Z. In formula $\mathrm{N}_1(X/Z) = (\mathrm{Z}_1(X/Z) \otimes \mathbb{R})/ \equiv_Z$. By the Neron-Severi theorem, it is finite dimensional.
- The Mori-Kleiman cone $\overline{\mathrm{NE}}(X/Z)$ is the closure of the cone generated by the effective $\mathbb{R}$-1-cycles inside the Neron-Severi space $\mathrm{N}_1(X/Z)$.
- Let $\mathrm{N}^1(X/Z)$ be the $\mathbb{R}$-vector space of the $\mathbb{R}$-Cartier divisors modulo numerical equivalence over Z. In formula $\mathrm{N}^1(X/Z) = (\mathrm{Pic}(X) \otimes \mathbb{R})/ \equiv_Z$.
- The intersection gives a pairing between $\mathrm{N}_1(X/Z)$ and $\mathrm{N}^1(X/Z)$. Hence they are both finite dimensional $\mathbb{R}$-vector spaces of the same dimension. This dimension is called the relative Picard number of X over Z, and it is denoted by $\rho(X/Z)$.

Remark 2.5. If D is an $\mathbb{R}$-Cartier divisor on X, then we use the notation $\overline{\mathrm{NE}}(X/Z)_{D>0}$ for $\overline{\mathrm{NE}}(X/Z)_{D>0} = \{C \in \overline{\mathrm{NE}}(X/Z) | D.C > 0\}$. Governed by similar logic we can use the notation $\overline{\mathrm{NE}}(X/Z)_{D<0}$ as well.

Definition 2.10. Let C be a convex cone a in $\mathbb{R}^n$ ($n \in \mathbb{Z}_0^+$) with vertex in the origin. A subcone $F \subsetneqq C$ is called an extremal face if $\forall x, y \in C$ $x + y \in F$ implies that $x, y \in F$. An extremal face is called an extremal ray if it is one dimensional.

Remark 2.6. Note that there is a partial correspondence between extremal faces of $\overline{\mathrm{NE}}(X/Z)$ and contraction morphisms over the base scheme Z.
Let $g : X \to Y$ be a contraction over the base scheme Z to a normal quasi-projective variety Y. If a divisor D on X is the pullback of some divisor A on Y, which is ample over Z, then D is nef over Z (as it is semiample) and it is numerically trivial on exactly those curves which are contracted by g. In other words the hyperplane defined by D in the Neron-Severi space $\mathrm{N}_1(X/Z)$ contains exactly one extremal face F of the Mori-Kleiman cone $\overline{\mathrm{NE}}(X/Z)$, and g contracts those curves whose numerical classes belong to F.
Unfortunately, not all extremal faces can be contracted (some of them cannot even be represented by effective curves), however the Cone Theorem will show us that K_X-negative extremal rays can be contracted.

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Definition 2.11. Let X and Z be normal quasi-projective varieties, and let $f : X \to Z$ be a projective morphism between them. Let R be an extremal ray of the Mori-Kleiman cone $\overline{\mathrm{NE}}(X/Z)$. A morphism $\phi : X \to Y$ over the base variety Z is called a contraction of the extremal ray R, if

- Y is a normal quasi-projective variety over Z,
- f is a contraction,
- most importantly, for any curve $C \subsetneqq X$, C maps to a point of Y, if and only if the numerical class of C belongs to R.

Definition 2.12. We call $(X/Z, B)$ a pair if

- X is a normal quasi-projective variety over the base variety Z; the base variety Z is normal and quasi-projective, and there is a projective (structure) morphism $f : X \to Z$ between X and Z,
- B is an $\mathbb{R}$-divisor, and the coefficients of its prime divisors are drawn from the interval $[0, 1]$,
- $K_X + B$ is $\mathbb{R}$-Cartier.

B is called the boundary divisor. If the base variety $Z = \operatorname{Spec}\mathbb{C}$, or we are not interested in its presence, we simply use the notation (X, B). (Note that, if the base variety is $\operatorname{Spec}\mathbb{C}$, then X is projective.)

Definition 2.13. Let X be a normal variety and let D be a $\mathbb{R}$-divisor on it. A projective birational morphism $f : X \to Y$ is called a log resolution of X and D, if

- Y is a smooth variety,
- $\mathrm{Exc}(f)$ is a divisor,
- $\mathrm{Exc}(f) \cup (f^{-1})_* \operatorname{Supp} D$ is a simple normal crossing divisor.

X and D are log smooth, if the identity map gives a log resolution.

Remark 2.7. By Hironaka's theorem log resolutions exist.

Definition 2.14. Let (X, B) be a pair, and let $f : Y \to X$ be a log resolution of it. If we choose the canonical divisors K_X and K_Y (from their linear equivalence classes) in a such a way that $f_* K_Y = K_X$, then we can write K_Y as $K_Y = f^*(K_X + B) + A$ for some $\mathbb{R}$-divisor A on Y.
The discrepancy of a prime divisor E on Y is the coefficient of E in A. One can show that the discrepancy only depends on the prime divisor E and the boundary divisor B, and on the variety X. In particular it does not depend on the chosen resolution. We denote the discrepancy of the prime divisor E by $\mathrm{d}(E, X, B)$.

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Remark 2.8. Since discrepancies do not depend on the chosen log resolutions, and the possible prime divisors (which can occur on log resolutions) are determined by the valuation rings of $k(X)|\mathbb{C}$, it makes sense to talk about the discrepancies of prime divisors (without mentioning log resolutions).

Remark 2.9. The main reason for considering pairs is the adjunction formula. In its simplest form it states that if X is a smooth variety and S is a smooth subvariety of X, then $(K_X + S)|_S = K_S$. This opens the possibility to use induction on the dimension of the variety.
Also note that the prime divisors of A are either exceptional divisors or the birational transformations of the components of B.

Definition 2.15. Let (X, B) be a pair.

- (X, B) has terminal singularities if for any log resolution $f : Y \to X$ and for any exceptional prime divisor E on Y, we have $\mathrm{d}(E, X, B) > 0$.
- (X, B) has canonical singularities if for any log resolution $f : Y \to X$ and for any exceptional prime divisor E on Y, we have $\mathrm{d}(E, X, B) \geqq 0$.
- (X, B) has purely log terminal singularities if for any log resolution $f : Y \to X$ and for any exceptional prime divisor E on Y, we have $\mathrm{d}(E, X, B) > -1$ (in this case we call the pair (X, B) plt).
- (X, B) has Kawamata log terminal singularities if for any log resolution $f : Y \to X$ and for any prime divisor E on Y, we have $\mathrm{d}(E, X, B) > -1$ (in this case we call the pair (X, B) klt).
- (X, B) has ϵ-log canonical singularities (for some real number $0 < \epsilon \leqq 1$) if for any log resolution $f : Y \to X$ and for any prime divisor E on Y, we have $\mathrm{d}(E, X, B) \geqq -1 + \epsilon$ (in this case we call the pair (X, B) ϵ-lc).
- (X, B) has log canonical singularities if for any log resolution $f : Y \to X$ and for any prime divisor E on Y, we have $\mathrm{d}(E, X, B) \geqq -1$ (in this case we call the pair (X, B) lc).
- (X, B) has divisorially log terminal singularities if there exists a log resolution $g : V \to X$ such that for any prime divisor G on Y, we have $\mathrm{d}(G, X, B) > -1$ (in this case we call the pair (X, B) dlt).

Remark 2.10. Note that the only difference between plt and klt singularities is that, the definition of a klt pair requires that the boundary divisor has coefficients (for its prime divisors) in the interval $[0, 1)$, while in the case of plt pairs the coefficients can be drawn from $[0, 1]$. As the next lemma shows it has some consequences.

Lemma 2.1. *Let (X, B) be a pair, and let $f : Y \to X$ be an arbitrary log resolution of it.*

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- *(X,B) has terminal singularities if* $\mathrm{d}(E,X,B)>0$ *for any exceptional prime divisor E on Y.*
- *(X,B) has canonical singularities if* $\mathrm{d}(E,X,B)\geqq 0$ *for any exceptional prime divisor E on Y.*
- *(X,B) has Kawamata log terminal singularities if* $\mathrm{d}(E,X,B)>-1$ *for any prime divisor E on Y.*
- *(X,B) has ϵ-log canonical singularities if* $\mathrm{d}(E,X,B)\geqq -1+\epsilon$ *for any prime divisor E on Y ($\epsilon\in\mathbb{R}, 0<\epsilon\leqq 1$).*
- *(X,B) has log canonical singularities if* $\mathrm{d}(E,X,B)\geqq -1$ *for any prime divisor E on Y.*

Remark 2.11. The above lemma does not hold for plt and dlt pairs.

Remark 2.12. Let X and Y be normal quasi-projective varieties and let $f:Y\to X$ be a birational projective morphism between them. Let D be a Cartier divisor on X, let $E_1,E_2,...,E_m$ be exceptional prime divisors on Y, and let $a_1,a_2,...,a_m$ be positive integers ($m\in\mathbb{Z}^+$). Then $\mathrm{H}^0(X,D)\cong\mathrm{H}^0(Y,f^*D+\sum a_iE_i)$ canonically.
Hence if $(X,0)$ is a terminal or a canonical pair, and $f:Y\to X$ is any log resolution, then $\mathrm{H}^0(X,rK_X)\cong\mathrm{H}^0(Y,rK_Y)$ canonically ($r\in\mathbb{Z}^+$). In particular X and Y have isomorphic canonical rings and canonical models.
Also note that, if (X,B) is an lc pair and $f:X\to Y$ is a log resolution, then we can uniquely find effective $\mathbb{R}$-divisors $\Gamma\geqq 0$ and $E\geqq 0$ such that $f_*\Gamma=B$, $f_*E=0$ and $K_Y+\Gamma=f^*(K_X+B)+E$ (where $f_*K_Y=K_X$). Hence, by the same argument as above, the pairs (X,B) and (Y,Γ) have canonically isomorphic log canonical rings

$$R(X,K_X+B)=\bigoplus_{r=0}^{\infty}\mathrm{H}^0(X,\lfloor r(K_X+B)\rfloor)\cong\bigoplus_{r=0}^{\infty}\mathrm{H}^0(Y,\lfloor r(K_Y+\Gamma)\rfloor)=R(Y,K_Y+\Gamma)$$

and log canonical models.
On the other hand if (X,B) is a pair and $f:Y\to X$ is a log resolution such that there exists a prime divisor E on Y with $\mathrm{d}(E,X,B)<-1$, then for any positive integer l we can find a log resolution $g:V\to X$ and an exceptional prime divisor G on V with $\mathrm{d}(G,X,B)<-l$.

Remark 2.13. Smooth varieties have terminal singularities. During the MMP the discrepancies cannot decrease (this means that in some sense singularities improve), hence a minimal model or a Mori fibre space (the possible terminal states of the MMP) of a smooth variety has terminal singularities. This explains the name.
On the other hand, one can show that canonical models have canonical singularities, hence the name.

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Theorem 2.1 (Cone and Contraction Theorem)**.** *Let $(X/Z, B)$ be a klt pair with a rational boundary divisor. There exists a countable set of $K_X + B$-negative extremal rays $\{R_i | i \in I\}$ (I is some index set of countable cardinality) with the following properties.*

- $\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+B>0} + \sum_{i\in I} R_i$.
- *If A is an ample divisor on X over Z, and $\varepsilon \in \mathbb{R}^+$ is a (small) positive number, then there exists a finite subset $F \subseteqq I$ such that $\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+B+\varepsilon A>0} + \sum_{f\in F} R_f$, in other words $\{R_i\}$ is discrete in $\overline{\mathrm{NE}}(X/Z)_{K_X+B<0}$.*
- *R_i can be contracted ($\forall i \in I$).*
- *$\forall i \in I$ there exists a curve C_i on X, such that the numerical class of C_i belongs to R_i, and $0 \leqq -(K_X + B).C_i < 2 \dim X$.*
- *If $f : X \to Y$ is the contraction belonging to the $(K_X + B)$-negative extremal ray R, and L is a Cartier divisor on X such that $L.R = 0$, then there exists a Cartier divisor M on Y such that $L \sim_Z f^*M$.*

Remark 2.14. The theorem describes the Mori-Kleiman cone. Its main feature is that it looks nicely on the $(K_X + B)$-negative side: it is generated by only countable many rays which can only accumulate at the hyperplane defined by the $\mathbb{R}$-Cartier divisor $K_X + B$. Note that, a similar theorem can be stated for lc pairs (Theorem 1.4 in [Fu11]).

Definition 2.16. Let X and Y be normal quasi-projective varieties, and let $f : X \to Y$ be a contraction between them.

- f is a divisorial contraction, if it is a birational projective morphism such that the codimension of $\mathrm{Exc}(f)$ is one.
- f is a small contraction, if it is a birational projective morphism such that the codimension of $\mathrm{Exc}(f)$ is at least two.
- f is a fibration if $\dim X > \dim Y$.

Definition 2.17. Let $(X/Z, B)$ an lc pair, and let $f : X \to Y$ be a small contraction of a $(K_X + B)$-negative extremal ray. A log flip of this flipping contraction is the following diagram.

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d, "f"'] & X^+ \arrow[dl, "f^+"] \\ Y, & \end{tikzcd}$$

- X^+ is a normal quasi-projective variety, which is projective over the base variety Z,

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- f^+ is a birational contraction over the base variety Z, such that the codimension of $\mathrm{Exc}(f^+)$ is at least two,
- $K_{X^+} + B^+$ is ample over Y, where B^+ is the birational transformation of the boundary B, i.e. $B^+ = \phi_* B$.

Definition 2.18 (Minimal Model, Mori Fibre Space)**.** Let $(X/Z, B)$ and $(Y/Z, B_Y)$ be lc pairs and let $\phi : X \dashrightarrow Y$ be a birational map between X and Y, which satisfies the following properties.

- ϕ^{-1} does not contract any divisor,
- B_Y is the birational transformation of B, i.e. $B_Y = \phi_* B$,
- if E is a prime divisor (of a log resolution of (X, B)), then $\mathrm{d}(E, X, B) \leqq \mathrm{d}(E, Y, B_Y)$, with strict inequality if E is a prime divisor contracted by ϕ.

Then,

- $(Y/Z, B_Y)$ is a minimal model if $K_Y + B_Y$ is nef,
- $(Y/Z, B_Y)$ is a Mori fibre space, if there exists a $(K_Y + B_Y)$-negative extremal ray such that for the corresponding contraction $g : Y \to T$, $\dim T < \dim X$.

Remark 2.15. Any birational map between normal varieties which are projective over some base scheme is defined outside a codimension two set. Hence it makes sense to talk about the divisors which are contracted by ϕ or ϕ^{-1}.

Remark 2.16. As noted earlier if X is smooth, then $(X, 0)$ is terminal. If the MMP can be run on it, then its terminal state $(Y, 0)$ is terminal as well because of the condition on the discrepancies. In particular, if Y results a Mori fibre space, then the generic fibre of $g : Y \to T$ is terminal. Indeed, by adjunction, the general fibres of g are terminal, hence so does the generic fibre. (Terminal singularities for varieties over arbitrary fields of characteristic zero are defined the same way as for varieties over the field of the complex numbers.)

Definition 2.19 (Log Minimal Model Program)**.** Let $(X/Z, B)$ be an lc pair. The following algorithm is called the Log Minimal Model Program if all steps exist. If $K_X + B$ is nef over Z, then the algorithm stops, and we found a minimal model for $(X/Z, B)$. If $K_X + B$ is not nef, then we can find a $(K_X + B)$-negative extremal ray, and there exists a corresponding contraction $f : X \to Y$. If $\dim X > \dim Y$, then we found a Mori fibre space and the algorithm stops. If f is a divisorial contraction, then we replace $(X/Z, B)$ with $(Y/Z, B_Y)$, where $B_Y = f_* B$, and start the algorithm form the beginning. If f is a small contraction, then we perform a flip, and replace $(X/Z, B)$ with $(X^+/Z, B^+)$, and start the algorithm form the beginning.

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2.1.4 Conjectures and results of the Minimal Model Program

In this section we collect some conjectures and results of the MMP.

Conjecture 2.4 (Minimal Model)**.** Let $(X/Z, B)$ an lc pair. There exists a minimal model or a Mori fibre space for $(X/Z, B)$.

Conjecture 2.5 (Abundance)**.** Let $(X/Z, B)$ an lc pair such that $K_X + B$ is nef over Z, then $K_X + B$ is semiample over Z.

Conjecture 2.6. Let $(X/Z, B)$ an lc pair such that $K_X + B$ is a $\mathbb{Q}$-divisor. Then the log canonical algebra

$$R(X/Z, K_X + B) = \bigoplus_{m \geqq 0} f_* \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)$$

is a finitely generated $\mathcal{O}_Z$-algebra, where $f : X \to Z$ is the given structure morphism.

Conjecture 2.7 (Termination of flips)**.** Any sequence of flips for a log canonical pair terminates.

The following is the main result in the note of C. Birkar (Theorem 10.1 in [Bi12]). It is based on the theorems of [BCHM10].

Theorem 2.2. *Let $(X/Z, B)$ a klt pair such that the boundary B is a big divisor. Then,*

- *if $K_X + B$ is pseudo-effective over Z, then $(X/Z, B)$ has a minimal model $(Y/Z, B_Y)$. Moreover abundance holds, i.e. $K_Y + B_Y$ is semiample over Z.*
- *if $K_X + B$ is not pseudo-effective over Z, then $(X/Z, B)$ has a Mori fibre space.*

The next two corollaries of Theorem 2.2 can also be found in [Bi12] (Corollary 10.2 and Corollary 10.3).

Corollary 2.1. *Let $(X/Z, B)$ a klt pair such that $K_X + B$ is a $\mathbb{Q}$-divisor. Then the log canonical algebra*

$$R(X/Z, K_X + B) = \bigoplus_{m \geqq 0} f_* \mathcal{O}_X(\lfloor m(K_X + B) \rfloor)$$

is a finitely generated $\mathcal{O}_Z$-algebra, where $f : X \to Z$ is the given structure morphism.

Corollary 2.2. *Log flips exists for klt pairs.*

The main result of [BCHM10] gives the following corollary for smooth varieties (Corollary 1.1.1 in [BCHM10]).

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Theorem 2.3. *Let X be a smooth variety of general type, i.e. let X be a smooth variety such that K_X is big. Then,*

- *X has a minimal model,*
- *X has a canonical model,*
- *the canonical ring $R(X, K_X) = \bigoplus_{m \geqq 0} \mathrm{H}^0(X, mK_X)$ is finitely generated.*

For non-pseudo effective divisors Corollary 1.3.2 in [BCHM10] gives the following result.

Theorem 2.4. *Let $(X/Z, B)$ a klt pair such that X is $\mathbb{Q}$-factorial, and $K_X + B$ is not pseudo-effective over Z. Then we can run a Minimal Model Program on $(X/Z, B)$ which results a Mori fibre space.*

Remark 2.17. The above theorem does not claim that the MMP always results a Mori fiber space. It claims that if we choose the steps of the MMP carefully enough, then the algorithm terminates at a Mori fibre space.

We will use the following corollary of Theorem 2.4 in this thesis.

Corollary 2.3. *Let X and Z be smooth projective varieties, and $f : X \to Z$ be a projective morphism between them. If K_X is not f-pseudo-effective, then we can run a MMP on X over Z and end up with a Mori fiber space $g : X \to Y$ over Z. In particular, the generic fibre of g is a Fano variety with terminal singularities.*

Indeed, we can run the MMP on X over Z and end up with a Mori fibre space by Theorem 2.4. In Remark 2.16, we have already seen that the generic fibre of g is terminal. By adjunction the general fibres of g are Fano varieties, hence the generic fibre of g is a Fano variety as well. This proves Corollary 2.3.

2.2 Uniruled and rationally connected varieties

It is an important question to identify classes of varieties which looks like the projective space $\mathbb{P}^n$. The first natural candidates are rational varieties. However it turns out that many varieties which share a lot of common properties with the projective space fail to be rational.
Uniruledness try to capture the phenomena that a variety is covered by a family of rational curves, while rational connectedness try to capture the phenomena that any two points of the variety can be connected by a rational curve.
Both of these classes can be studied by the help of rational curves, and they behave very well. They are invariant under birational transformations, moreover if the variety is smooth and the

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ground field is algebraically closed, then they are invariant under smooth deformations, they can be detected locally by a single rational curve, furthermore they imply sectional properties of algebraic tensors. In fact, conjecturally, they are equivalent with sectional properties of algebraic tensors. In addition, because of the maximal rationally connected fibration, up to birational equivalence, any variety can be viewed as a fibred space where the base is non-uniruled (hence it lack of rational curves) and the fibres are rationally connected.
The material of this section is based on Chapter 4 of [De01], [De11] and Chapter 4 of [Ko96]. During this section we work over a ground field of characteristic zero.

2.2.1 Uniruled varieties

As we have discussed previously uniruled varieties are worth to study since they provides a possible way to generalize the notion of rationality. On the other hand they are of primal interest from the point of view of the MMP. Their canonical divisors are not pseudo-effective (hence their Kodaira dimensions are $-\infty$), and the MMP produces Mori fibrations for them. Usually, for a given uniruled variety, the possible Mori fibre space structures are not unique, and it is very hard to figure out which part of them fall into the same birational equivalence category. Hence it seem reasonable to investigate uniruled varieties directly.

Let k be a field of characteristic zero. Unless stated otherwise we assume that the ground field is k during this section.

Definition 2.20. A n-dimensional variety X is called uniruled, if there exists an $(n-1)$-dimensional variety Y and a dominant rational map $f : \mathbb{P}^1 \times Y \dashrightarrow X$.

Remark 2.18. Uniruledness is invariant under birational transformations.

Remark 2.19. If there exists a variety Z and a dominant rational map $f : \mathbb{P}^1 \times Z \dashrightarrow X$ such that there exists a closed (but not necessarily k-rational!) point $z \in Z$ such that the rational map $\mathbb{P}^1 \times \{z\} \dashrightarrow X$ is non-constant, then X is uniruled. Indeed, we can shrink Z to make it affine (hence quasi-projective). The condition on the non-constant rational map secures us that, if we take a general $(\dim X - 1)$-dimensional linear space section of Z, we end up with a variety Z_1 for which $\mathbb{P} \times Z_1 \dashrightarrow X$ is dominant, so X is uniruled.

Remark 2.20. In the definition of uniruledness, if X is proper, then we can require f to be a dominant morphism.
Indeed, without loss of generality, we can assume that Y is a smooth projective variety. A rational map from a normal variety to a proper one is defined outside a codimension two subset.
So let $U \subsetneq \mathbb{P}^1 \times Y$ be he largest open subvariety where the rational map f is defined. Its

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complement $Z = (\mathbb{P}^1 \times Y) \setminus U$ has codimension at least two. Hence Z projects to a proper closed subvariety of Y, denote it by V. Let Y_1 be the open subvariety $Y_1 = Y \setminus V$. Then $f : \mathbb{P}^1 \times Y_1 \to X$ defines a morphism.

The next proposition is the content of Chapter 2.1 of [De01].

Theorem 2.5. *Let X be a projective variety and A an ample divisor on it. For an arbitrary positive integer d, there exists a quasi-projective scheme $\mathrm{Mor}_d(\mathbb{P}^1, X)$ which parametrizes the morphisms $\mathbb{P}^1 \to X$ with degree d with respect to the given ample divisor A.*

Remark 2.21. It is important to point out that $\mathrm{Mor}_d(\mathbb{P}^1, X)$ parametrizes morphisms of degree d, not their image. An r-sheeted branched cover of a degree e one-dimensional closed subvariety has degree er as a morphism.

Definition 2.21. Using the notation of the previous theorem, we introduce a couple of objects.

- Let $\mathrm{ev}_d : \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to X$ be the evaluation map.
- Let $\mathrm{Mor}(\mathbb{P}^1, X)$ denote the scheme which parametrizes the morphisms $\mathbb{P}^1 \to X$, in formula $\mathrm{Mor}(\mathbb{P}^1, X) = \bigsqcup_d \mathrm{Mor}_d(\mathbb{P}^1, X)$. Let $\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X) \to X$ be the corresponding evaluation map.
- Let $\mathrm{M}_d(\mathbb{P}^1, X)$ denote the quasi-projective scheme which parametrizes the morphims $\mathbb{P}^1 \to X$ of degree at most d, in formula $\mathrm{M}_d(\mathbb{P}^1, X) = \bigsqcup_{e \leqq d} \mathrm{Mor}_e(\mathbb{P}^1, X)$. Let $\mathrm{ev}_{\leqq d} : \mathbb{P}^1 \times \mathrm{M}_d(\mathbb{P}^1, X) \to X$ be the corresponding evaluation map.

Lemma 2.2. *Let X be a projective variety and A an ample divisor on it. Let d be an arbitrary positive integer. The image of the evaluation map $\mathrm{ev}_{\leqq d} : \mathbb{P}^1 \times \mathrm{M}_d(\mathbb{P}^1, X) \to X$ is closed in X.*

This is Lemma 3.7 in[De11]. The proof is based on the following idea. Rational curves of degree at most d on X, can only degenerate into union of lower degree rational curves.
Hence the analytic picture is the following. Let V be the set of those points which are contained in a rational curve of degree at most d. Let x be a point in the boundary of V. Take a sequence in V whose limit is x. Take rational curves of degree at most d through every point in the given sequence. A limit (of a subsequence) of these rational curves contains x, and it is a union of rational curves. Hence the claim follows.

Remark 2.22. Let X be a projective variety, endowed with some ample divisor A. X is uniruled if and only if there exists a positive integer $d \in \mathbb{Z}^+$ such that the evaluation map $\mathrm{ev}_d : \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to X$ is dominant.
Assume X is uniruled. Let $f : \mathbb{P}^1 \times Y \to X$ be the dominant morphism constructed in Remark 2.20. It factors as $\mathbb{P}^1 \times Y \to \bigsqcup_d \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to X$ by the universal property. As Y is connected, f factors as $\mathbb{P}^1 \times Y \to \mathbb{P}^1 \times \mathrm{Mor}_{d_0}(\mathbb{P}^1, X) \to X$ for some $d_0 \in \mathbb{Z}^+$. Hence ev_{d_0} is dominant.

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Assume that ev_d is dominant. After taking the reduced scheme structure on an irreducible component of $\mathrm{Mor}_{d_0}(\mathbb{P}^1, X)$, we can find a variety Z, such that there exists a morphism $\mathbb{P} \times Z \to X$, which is dominant, and for some closed point $z \in Z$, $\mathbb{P}^1 \times \{z\} \to X$ is a non-constant morphism. Remark 2.19 finishes the proof.

Proposition 2.1. *Let the ground field k be algebraically closed. If X is a uniruled projective variety, then through every closed point of X there is a projective line of degree at most d (for some $d \in \mathbb{Z}^+$).*

Proof. By Remark 2.22 ev_d is dominant. Hence by Lemma 2.2, $\mathrm{ev}_{\leqq d}$ is surjective. This implies the claim. ☐

Remark 2.23. If k is not algebraically closed, then $\mathrm{Mor}(\mathbb{P}^1, X)$ might not have k-rational points, hence $\mathrm{ev}_{\leqq d}$ does not give morphisms from $\mathbb{P}^1_k$.

Proposition 2.2. *Let the ground field k be uncountable and algebraically closed. A variety is uniruled if and only if there exists a rational curve (the image of $\mathbb{P}^1$ by a non-constant rational map) through any general closed point of the variety.*

Proof. If X is uniruled, then it is birational to a projective uniruled variety, which implies one direction of the claim by Proposition 2.1.
Assume that there is a projective line through a general point of X. Shrink X to an affine variety, then take its projective closure. So we can assume that X is projective. By the assumption $\mathrm{ev} : \bigsqcup_d \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to X$ is surjective. So X is covered by the union of the images of $\mathrm{ev}_{\leqq d}$, which are closed sets by Lemma 2.2. Since k is of uncountable cardinality, one of the closed sets must be equal to the whole space X, hence there exists d_0 such that $\mathrm{ev}_{\leqq d_0}$ is surjective. This in turn implies that ev_{d_1} is dominant for some $(d_1 \leqq d_0)$. By Remark 2.22 X is uniruled, this finishes the proof. ☐

Remark 2.24. $\mathbb{A}^n$ provides an example of a uniruled variety which does not contain any projective line. It is uniruled, hence it contains many rational curves.

Remark 2.25. F. Bogomolov and Yu. Tschinkel constructed varieties over the algebraic closure of finite fields, such that every closed point lies on a rational curve, and the variety is not uniruled (Theorem 1.1 in [BT05]). It shows that to secure that a variety is uniruled it is not enough to require that every point lies on a rational curve. (Note that in this example the ground field is of characteristic p.)

Remark 2.26. Proposition 2.2 and Remark 2.25 illustrate the power of Definition 2.20. On the one hand it gives back our geometric intuition. On the other hand, it can exclude varieties from

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the uniruled class even if they are covered with rational curves. The problem in this case is that, there is no algebraic family of rational curves covering the variety. To put it another way, the definition excludes those projective varieties from the uniruled class for which, there cannot be found collection of bounded degree rational curves which covers the variety.
Another great advantage of Definition 2.20 is that, it can deal with varieties which does not even have rational points.

Remark 2.27. If one generalizes the notion of ruled varieties to schemes (see Definition 1.1 in Chapter 4 of [Ko96]), then one can show that a variety X is uniruled if and only if $X_{\overline{k}} = X \times \operatorname{Spec} \overline{k}$ is uniruled. (Note that $X_{\overline{k}}$ need not to be a variety in general, it need not to be integral.)
The basic idea is that, it is enough to consider projective varieties. They are uniruled if and only if the evaluation map is dominant for a given degree. The scheme $\operatorname{Mor}_d(\mathbb{P}^1, X)$ behaves well under changing the ground field, i.e. $\operatorname{Mor}_d(\mathbb{P}^1, X) \times \operatorname{Spec} \overline{k} \cong \operatorname{Mor}_d(\mathbb{P}^1_{\overline{k}}, X_{\overline{k}})$. Hence $\operatorname{ev}_d : \mathbb{P}^1 \times \operatorname{Mor}_d(\mathbb{P}^1, X) \to X$ is dominant if and only if $\operatorname{ev}_{d,\overline{k}} : \mathbb{P}^1_{\overline{k}} \times \operatorname{Mor}_d(\mathbb{P}^1_{\overline{k}}, X_{\overline{k}}) \to X_{\overline{k}}$ is dominant, which implies the claim.

We can also define ruled varieties which will be an important class for us in this thesis.

Definition 2.22. A d-dimensional variety X is called ruled, if there exists a $(d-1)$-dimensional variety Y and a birational map $f : \mathbb{P}^1 \times Y \dashrightarrow X$.

Remark 2.28. Unlike uniruledness, ruledness is not a geometric notion. As we will see later, there exists varieties such that X is not ruled, but $X_{\overline{k}} = X \times \operatorname{Spec} \overline{k}$ is ruled. We will show that non-trivial Brauer-Severi curves and surfaces provides example for this phenomenon. (See Theorem 4.2 and Remark 4.16.)

In the remainder of this section the ground field k will be algebraically closed (and of characteristic zero).

Definition 2.23. Let X be a smooth variety over the algebraically closed field k. Let $f : \mathbb{P}^1 \to X$ be a non-constant morphism of the projective line to X.

- f is called free, if the vector bundle $f^* \operatorname{T} X$ is generated by global sections.
- f is called very free, if the vector bundle $f^* \operatorname{T} X \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is generated by global sections.
- f is called r-free ($r \in \mathbb{Z}^+$), if the vector bundle $f^* \operatorname{T} X \otimes \mathcal{O}_{\mathbb{P}^1}(-r)$ is generated by global sections.

Remark 2.29. We know that any vector bundle over $\mathbb{P}^1$ is the direct sum of line bundles. Hence $f^* \operatorname{T} X \cong \bigoplus_i \mathcal{O}(a_i)$ for some integers a_i. f is free if $a_i \geqq 0$, very free if $a_i \geqq 1$ and r-free if $a_i \geqq r$ $(\forall i)$.

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Remark 2.30. Let f be a free rational curve, and denote its image by C. Then $K_X.C \leqq -2$. Indeed, using image factorization and resolution of singularities for curves, f can be factorized as $f = h \circ g_r$, where $g_r : \mathbb{P}^1 \to \mathbb{P}^1$ is a morphism of degree r, and $h : \mathbb{P}^1 \to X$ is a generically injective morphism from the projective line. f is free if and only if h is free. On the other hand $\mathcal{O}_{\mathbb{P}_1}(2) \cong \mathrm{T}\,\mathbb{P}^1 \leqq g^*\,\mathrm{T}\,X$ as h is generically injective. Hence $K_X.C = -\deg(g^*\,\mathrm{T}\,X) \leqq -2$.

Remark 2.31. Note that, if $g_r : \mathbb{P}^1 \to \mathbb{P}^1$ is a morphism of degree r and f is a very free morphism, then $f \circ g_r$ is r-free.

An explicit calculation on tangent spaces shows the following proposition (Proposition 4.8 and Proposition 4.9 in [De11] for the first claim, and Corollary 3.5.3 of Chapter 2 of [Ko96] for the second claim).

Proposition 2.3. *Let X be a smooth projective variety over the algebraically closed field k. Let $f : \mathbb{P}^1 \to X$ be a non-constant morphism of the projective line to X. Let* $\mathrm{red}(\mathrm{Mor}(\mathbb{P}^1, X))$ *the reduced closed subscheme of* $\mathrm{Mor}(\mathbb{P}^1, X)$ *(supported on the whole space). Denote by* $\mathrm{ev}_{\mathrm{red}}$ *the corresponding evaluation morphism.*

- *The rational curve f is free if and only if* $\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X) \to X$ *is smooth along* $\mathbb{P}^1 \times \{f\}$.
- *Moreover if* $\mathrm{ev}_{\mathrm{red}} : \mathbb{P}^1 \times \mathrm{red}(\mathrm{Mor}(\mathbb{P}^1, X)) \to X$ *is smooth at (f, p), for some $p \in \mathbb{P}^1$, then f is free.*

Remark 2.32. As smoothness is an open property, freeness is an open property as well. Hence we can deform free rational curve into free rational curves.

Remark 2.33. For an arbitrary positive integer d, we can use the notation $\mathrm{red}(\mathrm{Mor}_d(\mathbb{P}^1, X))$ and $\mathrm{ev}_{d,\mathrm{red}} : \mathbb{P}^1 \times \mathrm{red}(\mathrm{Mor}_d(\mathbb{P}^1, X)) \to X$ accordingly. As $\mathrm{Mor}(\mathbb{P}^1, X) = \bigsqcup_d \mathrm{Mor}_d(\mathbb{P}^1, X)$ is a disjoint union, Proposition 2.3 holds for them as well.

Theorem 2.6. *Let X be a smooth projective variety over the algebraically closed field k. X is uniruled if and only if there exists a free rational rational curve $f : \mathbb{P}^1 \to X$.*

Proof. Assume that the morphism f gives a free rational curve. Let d be the degree of f. Then by Proposition 2.3 (and by the remark following it) ev_d is smooth at (f, p) for any $p \in \mathbb{P}^1$. Smoothness is an open property, hence there exists an open dense subscheme $U \subsetneqq \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)$ such that $\mathrm{ev}_d|_U$ is smooth. A smooth morphism is dominant, hence ev_d is dominant. By Remark 2.22 it implies one direction of the claim.
Let X be uniruled. Then ev_d is dominant for some positive integer d (Remark 2.22). Hence $\mathrm{ev}_{d,\mathrm{red}}$ is dominant as well. Let V be an irreducible component of $\mathbb{P}^1 \times \mathrm{red}(\mathrm{Mor}_d(\mathbb{P}^1, X))$, for which $\mathrm{ev}_{d,\mathrm{red}}|_V$ is dominant. V is a variety, therefore by Generic Smoothness (Lemma 10.5 of Chapter 3 in [Ha77]), there exists an open dense subvariety $U \subsetneqq V$ such that $\mathrm{ev}_{d,\mathrm{red}}|_U$ is smooth. This

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implies that $\mathrm{ev}_{d,\mathrm{red}}$ is smooth along a non-empty open subvariety U_1 of $\mathbb{P}^1 \times \mathrm{red}(\mathrm{Mor}_d(\mathbb{P}^1, X))$ ($U_1 \subseteqq U$). Let (p, f) be an arbitrary closed point of U_1. By Proposition 2.3 and Remark 2.33 f is free. This finishes the proof. □

The moral of the proof is the following. The evaluation map is smooth at (f, p) if and only if f is free. If f is free, then the evaluation map is smooth in a non-empty neighbourhood, hence it is dominant, hence X is uniruled. If X is uniruled, then the evaluation map is dominant, hence by Generic Smoothness it is smooth in a non-empty neighbourhood, which implies the existence of a free rational curve.

Theorem 2.7. *Let X and T be varieties over the algebraically closed field k. Let $f : X \to T$ be a smooth projective morphism between them. Assume that for a closed point $t \in T$, the fibre X_t of f is uniruled. Then all fibres of f are uniruled.*

We can use free rational curves to prove this theorem. Let $B \subseteqq T$ be the set for which the fibers are uniruled. It is enough to show that B is open and closed.
Roughly speaking, deformations of a free rational curve lying in the fibre X_t give free rational curves lying in the neighbouring fibres. This shows openness. A limiting argument finds rational curves passing through the general points of fibers lying over the boundary of B, this shows closedness (at least when k is uncountable).

Proposition 2.4. *Let X be a uniruled smooth projective variety over the algebraically closed field k. The images of free rational curves cover a dense open subset of X.*

Proof. Let f be a free rational curve of degree d. Then ev_d is smooth at (p, f) for any $p \in \mathbb{P}^1$. Hence ev_d is smooth in a neighbourhood $U \subseteqq \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)$ of (p, f), hence $\mathrm{ev}_d|_U$ is dominant. Since the image of the dominant morphism $\mathrm{ev}_d|_U$ is constructible, it contains an open dense set $V \subseteqq X$. Hence $V \subseteqq X$ is covered by the image of free rational curves (freeness follows from Proposition 2.3). This prove the claim. □

Theorem 2.8. *Let X be a smooth projective variety over the algebraically closed field k. All plurigenera of X vanishes, i.e. $\mathrm{H}^0(X, mK_X) = 0$ ($\forall m \in \mathbb{Z}^+$). In other words, $\kappa(X) = -\infty$.*

Proof. Let $x \in X$ be a general closed point. Let $f : \mathbb{P}^1 \to X$ be a free rational curve passing through x (Proposition 2.4), and let C be the image of f. Decompose f as $h \circ g_r$, where $g_r : \mathbb{P}^1 \to \mathbb{P}^1$ is a degree r morphism, and $h : \mathbb{P}^1 \to X$ is a generically injective morphism. We have seen that $K_X.C \leqq -2$ (Remark 2.30). $\deg f^*\omega_X^{\otimes m} = r \deg h^*\omega_X^{\otimes m} = rmK_X.C \leqq -2$, hence $f^*\omega_X^{\otimes m}$ cannot have any global sections on $\mathbb{P}^1$ except the zero section. Therefore all global sections of $\omega_X^{\otimes m}$ vanish along C. In particular, all global sections of $\omega_X^{\otimes m}$ vanish at x. As it holds for all general points of X, a global section of $\omega_X^{\otimes m}$ vanishes along a dense open set, hence it is zero. So $\mathrm{H}^0(X, mK_X) = 0$. □

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S. Mori conjectured that, the converse also holds.

Conjecture 2.8. Let X be a positive dimensional smooth projective variety over the algebraically closed field k. If $\kappa(X) = -\infty$, then X is uniruled.

Remark 2.34. As noted in [BDPP13] the conjecture can be break into two parts.

- If K_X is pseudo-effective, then $\kappa(X) \geqq 0$.
- If K_X is not pseudo-effective (which implies that $\kappa(X) = -\infty$), then X is uniruled.

The second problem is answered positively in [BDPP13] by analytic methods.

2.2.2 Rationally connected varieties

The notion of rationally connected varieties is born out of the search for properties which makes a variety very similar to the projective space. It turned out that instead of global properties we need to focus on a special one. Namely that $\mathbb{P}^n$ contains a lot of rational curves. Even any two points of $\mathbb{P}^n$ can be connected by them.

Let k be a field of characteristic zero. Unless stated otherwise, we assume that the ground field is k during this section.

First we introduce a notation.

Definition 2.24. Let U, Y and X be k-schemes, and let $g : U \to Y$ and $u : U \to X$ be morphisms between them. We use the notation $u^{(2)}$ for the morphism induced by u form $U \times_Y U$ to $X \times X$, in formula $u^{(2)} : U \times_Y U \to X \times X$.
In particular, let Y be a scheme, $U = \mathbb{P}^1 \times Y$ and $g : \mathbb{P}^1 \times Y \to Y$ be the natural projection. Then $U \times_Y U \cong \mathbb{P}^1 \times \mathbb{P}^1 \times Y$ and $u^{(2)} : \mathbb{P}^1 \times \mathbb{P}^1 \times Y \to X \times X$

Remark 2.35. For example, if X is a projective variety, then we have $\mathrm{ev}^{(2)} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X) \to X \times X$.

Definition 2.25. Let X be a variety. We call X rationally connected, if it is proper, and there exists a variety Y and a morphism $e : \mathbb{P}^1 \times Y \to X$ such that $e^{(2)} : \mathbb{P}^1 \times \mathbb{P}^1 \times Y \to X \times X$ is dominant.

Remark 2.36. As we required X to be proper, we could have allowed e to only be a rational map, and we would have arrived to an equivalent definition. Indeed, assume that e is rational. As we have seen in Remark 2.20, Y can be replaced with Y_1 such that, they are birational, and e defines a morphism $e_1 : \mathbb{P}^1 \times Y^1 \to X$ such that ${e_1}^{(2)}$ remains dominant.

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Remark 2.37. Because of the previous remark, if X_1 and X_2 are proper birationally equivalent varieties, then X_1 is rationally connected if and only if X_2 is rationally connected.
Moreover, if a rationally connected variety dominates a proper variety, then the proper one is rationally connected as well.
Also observe that, rationally connectedness implies uniruledness.

Remark 2.38. The definition intuitively means that a *general pair of points* can be connected by a rational curve. We required e to be a morphism in the definition to emphasize that, we would like the image of projective lines to connect the points not only their rational images. In particular it excludes spaces like $\mathbb{A}^n$ (of course it is not proper either).

Proposition 2.5. *Let the ground field k be uncountable and algebraically closed. A proper variety is rationally connected if and only if for a general pair of points there exists a proper rational curve (the image of $\mathbb{P}^1$ by a morphism) containing them.*

Proof. The image of a dominant morphism between varieties contains a dense open set, as it is constructible, this implies one direction. If any general pair of points can be joined by a projective line, then $\mathrm{ev}^{(2)}$ is dominant, hence ${\mathrm{ev}_d}^{(2)}$ is dominant for some positive integer d, as k is uncountable, this implies the other direction of the claim. □

Remark 2.39. F. Bogomolov and Yu. Tschinkel constructed varieties over the algebraic closure of finite fields, such that every pair of closed point lies on a chain of proper rational curves, and the variety is not rationally chain connected (Section 5 in [BT05]). (See Remark 2.25 and Remark 2.26 for similar results.)

Theorem 2.9. *Let X be a projective variety, let $x \in X$ be a closed point and A an ample divisor on it. For an arbitrary positive integer d, there exists a quasi-projective scheme $\mathrm{Mor}_d(\mathbb{P}^1, X, 0 \mapsto x)$ which parametrizes morphisms $\mathbb{P}^1 \to X$, with degree d with respect to the given ample divisor A, which takes $0 = [0:1] \in \mathbb{P}^1$ to $x \in X$.*

Definition 2.26. Using the notation of the previous theorem, we introduce a couple of objects.

- Let $\mathrm{ev}_{d,x} : \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X, 0 \mapsto x) \to X$ be the evaluation map.
- Let $\mathrm{Mor}(\mathbb{P}^1, X, 0 \mapsto x)$ denote the scheme which parametrizes the morphisms $\mathbb{P}^1 \to X$, which take $0 = [0:1] \in \mathbb{P}^1$ to $x \in X$, in formula $\mathrm{Mor}(\mathbb{P}^1, X, 0 \mapsto x) = \bigsqcup_d \mathrm{Mor}_d(\mathbb{P}^1, X, 0 \mapsto x)$. Let $\mathrm{ev}_x : \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X, 0 \mapsto x) \to X$ be the corresponding evaluation map.

Just like in Proposition 2.3, an explicit calculation on tangent spaces shows the following Proposition (Proposition 4.8 and Proposition 4.9 in [De11] for the first claim, and Corollary 3.5.3 of Chapter 2 of [Ko96] for the second claim).

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Proposition 2.6. *Let X be a smooth projective variety over the algebraically closed field k, let $x \in X$ be a closed point. Let $f : \mathbb{P}^1 \to X$ be a non-constant morphism of the projective line to X, which takes $0 = [0:1] \in \mathbb{P}^1$ to $x \in X$. Let* $\mathrm{red}(\mathrm{Mor}(\mathbb{P}^1, X), 0 \mapsto x)$ *the reduced closed subscheme of* $\mathrm{Mor}(\mathbb{P}^1, X, 0 \mapsto x)$ *(supported on the whole space). Denote by* $\mathrm{ev}_{\mathrm{red},x}$ *the corresponding evaluation morphism.*

- *The rational curve f is very free if and only if* $\mathrm{ev}_x : \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X, 0 \mapsto x) \to X$ *is smooth along* $(\mathbb{P}^1 - \{0\}) \times \{f\}$.
- *If* $\mathrm{ev}_{\mathrm{red},x} : \mathbb{P}^1 \times \mathrm{red}(\mathrm{Mor}(\mathbb{P}^1, X, 0 \mapsto x)) \to X$ *is smooth at (f, p), for some $p \in \mathbb{P}^1 - \{0\}$, then f is very free.*

Remark 2.40. As smoothness is an open property, very freeness is an open property. So Proposition 2.6 tell us that, a very free rational curve can be deformed to a very free rational curve, even if we fix the image of one of its points. Actually, a similar statement can be formulated for r-free rational curves a well. Then we can fix the image of r many points. (Recall that covering a very free rational curve r-times gives an r-free rational curve.)

Remark 2.41. For an arbitrary positive integer d, we can use the notation $\mathrm{red}(\mathrm{Mor}_d(\mathbb{P}^1, X, 0 \mapsto x))$ and $\mathrm{ev}_{d,x,\mathrm{red}} : \mathrm{red}(\mathrm{Mor}_d(\mathbb{P}^1, X, 0 \mapsto x)) \to X$ accordingly. As $\mathrm{Mor}(\mathbb{P}^1, X, 0 \mapsto x) = \coprod_d \mathrm{Mor}_d(\mathbb{P}^1, X, 0 \mapsto x)$ is a disjoint union, Proposition 2.6 holds for them as well.

Lemma 2.3. *Let X be a smooth projective variety over the algebraically closed field k. X is rationally connected if and only if, there exists a positive integer d such that* $\mathrm{ev}_{d,x} : \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X, 0 \mapsto x) \to X$ *is dominant for a general closed point $x \in X$.*

Proof. Let $e : \mathbb{P}^1 \times Y \to X$ be the morphism in the definition of rationally connectedness. First notice that the morphism defined by the closed points of Y are of constant degree. Indeed, $e^{(2)}$ factorizes as $\mathbb{P}^1 \times \mathbb{P}^1 \times Y \to \coprod_d \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to X \times X$. Since Y is connected it factorizes as $\mathbb{P}^1 \times \mathbb{P}^1 \times Y \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to X \times X$ for some positive integer d. This means that the morphism defined by Y have degree d. Let U be an open subvariety in the image of the dominant morphism $e^{(2)}$. Let $i : U \hookrightarrow X \times X$ and let $p = \pi_2 \circ i : U \hookrightarrow X \times X \to X$. The morphism p is dominant. Hence there exists an open subset $V \subsetneqq X$, such that fibres over V has dimension equal to $\dim X$. In other words, if $x \in V$, then the image of some morphism of degree d from the projective line joins x with a dense open set of X (the points of the fibre of p over x). Since the automorphism group of the projective line is transitive, $\mathrm{ev}_{d,x} : \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X, 0 \mapsto x) \to X$ is dominant if x is general (i.e. for $x \in V$). This finishes one direction of the proof.
Assume that $\mathrm{ev}_{d,x}$ is dominant for a general $x \in X$. Let $Z \subsetneqq X \times X$ be the closure of the image of ${\mathrm{ev}_d}^{(2)}$ endowed with the reduced subscheme structure. Denote by $j : Z \hookrightarrow X \times X$ the closed immersion of Z. By assumption the fibres of $\pi_2 \circ j : Z \hookrightarrow X \times X \to X$ are equal to X for a general

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closed point $x \in X$. Hence $\dim Z = \dim X \times X$, therefore $Z = X \times X$, i. e. ${\mathrm{ev}_d}^{(2)}$ is dominant. As $\mathrm{Mor}_d(\mathbb{P}^1, X)$ is a quasi-projective scheme, this implies that X is rationally connected. □

The following theorem is the analogue of Theorem 2.6 for rationally connected varieties. The proof is almost the same.

Theorem 2.10. *Let X be a smooth projective variety over the algebraically closed field k. X is rationally connected if and only if there exists a very free rational rational curve $f : \mathbb{P}^1 \to X$.*

Proof. Assume that the morphism f gives a very free rational curve such that $f(0) = x$ for some $x \in X$. Let d be the degree of f. Then by Proposition 2.6 $\mathrm{ev}_{d,x}$ is smooth at (f, p) for any $p \in \mathbb{P}^1 - \{0\}$. Smoothness is an open property, hence there exists an open dense subscheme $U \subseteqq \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X, 0 \mapsto x)$ such that $\mathrm{ev}_{d,x}|_U$ is smooth. A smooth morphism is dominant, hence $\mathrm{ev}_{d,x}|_U$ is dominant, therefore $\mathrm{ev}_{d,x}$ is dominant. Let $V \subseteqq X$ be a dense open subset contained in the image of the dominant morphism $\mathrm{ev}_{d,x}|_U$. If $(p, f) \in U$ (for some $p \in \mathbb{P}^1 - \{0\}$), then f is very free by Proposition 2.6. Hence if $y \in V$ is a closed point, then there exists a very free morphism whose image passes through y. Since the automorphism group $\mathrm{Aut}\,\mathbb{P}^1$ is transitive, for any closed point $y \in V$ there exists very free morphism which maps $0 \in \mathbb{P}^1$ to y. So the above argument shows that $\mathrm{ev}_{d,y}$ is dominant for $y \in V$. Then by Lemma 2.3, we proved one direction of the claim.
Let X be rationally connected. Then $\mathrm{ev}_{d,x}$ is dominant for some general point $x \in X$ and for some positive integer d (Lemma 2.3). Hence $\mathrm{ev}_{d,x,\mathrm{red}}$ is dominant as well. Let V be an irreducible component of $\mathbb{P}^1 \times \mathrm{red}(\mathrm{Mor}_d(\mathbb{P}^1, X, 0 \mapsto x))$, for which $\mathrm{ev}_{d,x,\mathrm{red}}|_V$ is dominant. V is a variety, therefore by Generic Smoothness (Lemma 10.5 of Chapter 3 in [Ha77]), there exists an open dense subvariety $U \subseteqq V$ such that $\mathrm{ev}_{d,x,\mathrm{red}}|_U$ is smooth. This implies that $\mathrm{ev}_{d,x,\mathrm{red}}$ is smooth along a non-empty open subvariety U_1 of $\mathbb{P}^1 \times \mathrm{red}(\mathrm{Mor}(\mathbb{P}^1, X, 0 \mapsto x))$ ($U_1 \subseteqq U$). Let (p, f) be arbitrary closed point in U_1 such that $0 \neq p$. By Proposition 2.6 f is very free. This finishes the proof. □

Remark 2.42. The proof of the above theorem shows that through a general point of rationally connected smooth projective variety there is a very free rational curve. However there is a much stronger result (Theorem 3.9 in Chapter 4 of [Ko96]).

Theorem 2.11. *Let X be a rationally connected smooth projective variety over the algebraically closed field k. Let $x_1, x_2, ..., x_m \in X$ be an arbitrary finite set of points (where m is an arbitrary positive integer). There exists a morphism $f : \mathbb{P}^1 \to X$ such that*

- *f is very free,*
- *the image of f contains the given finite set of points $x_1, x_2, ..., x_m \in X$,*
- *f is an immersion if $\dim X = 2$ and an embedding if $\dim X \geqq 3$.*

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One of the tricks of the proof is that, an r-sheeted branched cover of a very free rational curve gives an r-free rational curve. Also notice that, from the MMP for surfaces, we know that for smooth projective surfaces over algebraically closed fields the notion of rational and rationally connected varieties are equivalent.

For smooth projective varieties the notions of rationally connectedness and rationally chain connectedness are equivalent. This is the content of the theorem below (Theorem 3.10 in Chapter 4 of [Ko96]). It has very important consequences in deformation theory. The proof is rather technical, we do not present it.

Theorem 2.12. *Let X be a smooth projective variety over the algebraically closed field k. X is rationally connected if and only if any two closed points can be connected by a chain of proper rational curves, i.e. there exists a morphism $f : C \to X$, where C is a connected curve such that the irreducible components of C (with the reduced scheme structure) are isomorphic to the projective line, moreover the image of C contains the given two points.*

The counterpart of Theorem 2.7 for rationally connected varieties is the theorem below (Theorem 3.11 in Chapter 4 of [Ko96]).

Theorem 2.13. *Let X and T be varieties over the algebraically closed field k. Let $f : X \to T$ be a smooth projective morphism between them. Assume that for a closed point $t \in T$, the fibre X_t of f is rationally connected. Then all fibres of f are rationally connected.*

Finally we have the counterpart of Theorem 2.8.

Theorem 2.14. *Let X be a smooth projective variety over the algebraically closed field k. The global sections of the tensor powers of the Kähler differentials vanish, i.e. $\mathrm{H}^0(X, (\Omega^1_X)^{\otimes m}) = 0$ ($\forall m \in \mathbb{Z}^+$).*

Proof. Let $x \in X$ be a general closed point. Let $f : \mathbb{P}^1 \to X$ be a very free rational curve passing through x (Theorem 2.11), and let C be the image of f. $f^*(\Omega^1_X)^{\otimes m} = (f^*TX)^{-\otimes m}$, hence by the definition of very freeness, it is isomorphic to a direct sum $\bigoplus_{i=1\ldots q} \mathcal{O}(-a_i)$, where a_i is a positive integer ($\forall 1 \leqq i \leqq q, q \in \mathbb{Z}^+$). Hence $f^*(\Omega^1_X)^{\otimes m}$ cannot have any global sections on $\mathbb{P}^1$, except the zero section. Therefore all global sections of $(\Omega^1_X)^{\otimes m}$ vanish along C. In particular, all global sections of $(\Omega^1_X)^{\otimes m}$ vanish at x. As it holds for all general points of X, a global section of $(\Omega^1_X)^{\otimes m}$ vanishes along a dense open set, hence it is zero. So $\mathrm{H}^0(X, (\Omega^1_X)^{\otimes m}) = 0$. □

D. Mumford conjectured that, the converse also holds.

Conjecture 2.9. Let X be a smooth projective variety over the algebraically closed field k. If $\mathrm{H}^0(X, (\Omega^1_X)^{\otimes m}) = 0$ $\forall m \in \mathbb{Z}^+$, then X is rationally connected.

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One of the most important classes of rationally connected varieties are Fano varieties (see Theorem 2.13 of Chapter 5 of [Ko96]). We recall a couple of theorem about them to enclose the section.

Theorem 2.15. *Let F be a smooth Fano variety over the algebraically closed field k. Then F is rationally connected.*

Fano type varieties are also rationally connected as proved recently by C. D. Hacon and J. McKernan (Corollary 1.13 in [HM07]). Recall the definition of pairs (Definition 2.12; also note that we defined them for varieties over the field of complex numbers).

Theorem 2.16. *Let (X, B) be a klt pair such that the boundary B is a $\mathbb{Q}$-divisor, and $-(K_X+B)$ is nef and big. Then X is rationally connected.*

Moreover Fano varieties with mild singularities form a bounded family. That was the long standing Borisov-Alekseev-Borisov conjecture, which has been proved recently in the famous article by C. Birkar (Theorem 1.1 in [Bi16]).

Theorem 2.17. *Let d be a non-negative integer, and ϵ be a positive real number. The collection formed by those at most d-dimensional complex projective varieties X for which there exists an $\mathbb{R}$-boundary divisor B such that*

- *(X, B) be is an ϵ-lc pair,*
- *$-(K_X + B)$ is nef and big*

is a bounded family.

In this thesis we will use the following corollary of the previous theorem. (Recall that terminal singularities can be defined for varieties over arbitrary fields of characteristic zero the same way as they are defined for complex varieties.)

Corollary 2.4. *Let d be a non-negative integer. There exist positive integers $n = n(d)$ and $m = m(d)$, only depending on d, such that if*

- *k is an arbitrary field of characteristic zero,*
- *F is an arbitrary Fano variety over the ground field k, with terminal singularities,*

then

- *$-mK_F$ is very ample and,*
- *$\dim_k \mathrm{H}^0(F, -mK_F) \leqq n$.*

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Proof. Fix k and F with the properties described by the theorem. There exists a finitely generated field extension $l_0|\mathbb{Q}$ and a Fano variety F_0 over l_0 such that $F \cong F_0 \times_{l_0} \operatorname{Spec} k$. Consider an embedding of fields $l_0 \hookrightarrow \mathbb{C}$, and let $F_1 \cong F_0 \times_{l_0} \operatorname{Spec} \mathbb{C}$. Since complex Fano varieties with terminal singularities of bounded dimension form a bounded family (Theorem2.17), there exist constants $m = m(d), n = n(d) \in \mathbb{N}$, only depending on d, such that m-th power of the anticanonical divisor embeds F_1 into the n_1-dimensional complex projective space, where $n_1 \leqq n + 1$. Since the m-th power of the anticanonical divisor is defined over any field, this embedding is defined over any field, in particularly over k. So we have a closed embedding of the form $F \hookrightarrow \mathbb{P}^{n_1}_k \cong \mathbb{P}(\mathrm{H}^0(X, -mK_F)^*)$. This, in particular, implies that $-mK_F$ is very ample and $\dim_k \mathrm{H}^0(X, -mK_F) \leqq n$. □

2.2.3 Maximal Rationally Connected Fibration

In this section we work over the field of complex numbers to avoid technical difficulties. Let X be a smooth variety. Call two closed points of X equivalent if they can be joined by a chain of rational curves. It turns out that, under some mild conditions we can find an open subvariety X_0 such that, there is a fibration $\phi_0 : X_0 \to Z_0$ for which the fibres are more or less the equivalence classes of the above defined relation. Hence the fibres are rationally chain connected. Since we work over the field of complex numbers, we can shrink Z_0 and assume ϕ_0 to be smooth (by Generic Smoothness on the base, Corollary 10.7 of Chapter 3 in [Ha77]). Hence the fibres of ϕ_0 are rationally connected, as the notion of rationally connectedness and rationally chain connectedness coincide for smooth varieties (Theorem 2.12). This gives us the maximal rationally connected fibration.

The maximal rationally connected fibration exists and is unique up to birational equivalence by Theorem 5.2 and Theorem 5.4 of Chapter 4 in [Ko96].

Theorem 2.18. *Let X be a smooth proper complex variety. The pair (Z, ϕ) is called the maximal rationally connected (MRC) fibration if*

- *Z is a complex variety,*
- *$\phi : X \dashrightarrow Z$ is a dominant rational map,*
- *there exist open subvarieties X_0 of X and Z_0 of Z such that ϕ descends to a proper morphism between them $\phi_0 : X_0 \to Z_0$ with rationally connected fibres,*
- *if (W, ψ) is another pair satisfying the three properties above, then ϕ can be factorized through ψ. More precisely, there exists a rational map $\tau : W \dashrightarrow Z$ such that $\phi = \tau \circ \psi$.*

The MRC fibration exists and is unique up to birational equivalence.

Moreover the MRC fibration is functorial in the following sense (Theorem 5.5 of Chapter 4 in [Ko96]).

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Theorem 2.19. *Let X_1 and X_2 be smooth proper complex varieties. Let (Z_1, ϕ_1) and (Z_2, ϕ_2) be the corresponding the maximal rationally connected fibrations. If $f : X_1 \dashrightarrow X_2$ is a dominant map, then there exists a rational map $g : Z_1 \dashrightarrow Z_2$ such that $g \circ \phi_1 = \phi_2 \circ f$.*

In particular we have the following corollary, which immediately follows from Theorem 2.19.

Corollary 2.5. *Let X be a smooth proper complex variety and (Z, ϕ) be its MRC fibration. Let G be a finite group which acts on X by birational transformations. There is an induced G action on Z by birational transformations which makes the rational map ϕ G-equivariant.*

By Complement 5.2.1 of Chapter 4 of [Ko96], we can characterize the MRC fibration the following way. (The original definition works over any fields, for this one we need that the ground field is uncountable.)

Theorem 2.20. *Let X be a smooth proper complex variety. The pair (Z, ϕ) is the maximal rationally connected (MRC) fibration if*

- *Z is a complex variety,*
- *$\phi : X \dashrightarrow Z$ is a dominant rational map,*
- *there exist open subvarieties X_0 of X and Z_0 of Z such that ϕ descends to a proper morphism between them $\phi_0 : X_0 \to Z_0$ with rationally connected fibres,*
- *if $z \in Z_0$ is a very general point and $C \subsetneqq X$ is a rational curve (i.e. the image of $\mathbb{P}^1$ by a non-constant rational map) passing through z, then C is contained in the fibre of the morphism $\phi : X_0 \to Z_0$ over z.*

The base of the MRC fibration is non-uniruled, this follows from the work of T. Graber, J. Harris and J. Starr. The next theorem is Corollary 1.3 in [GHS03].

Theorem 2.21. *Let X and Y be proper complex varieties and let $f : X \to Y$ be a dominant morphism between them. If Y and the general fibers of f are rationally connected, then X is rationally connected.*

Theorem 2.22. *Let X be an arbitrary smooth proper complex variety and (Z, ϕ) be its MRC fibration. Then Z is non-uniruled.*

Proof. We can assume that Z is projective. Pick a very general (in the sense of Theorem 2.20) closed point $z \in Z$. Let $f : \mathbb{P}^1 \to Z$ be a morphism whose image passes through z. Let $C \subsetneqq Z$ be the image of f (endowed with the reduced scheme structure). Let $V_0 \subsetneqq X_0$ be the inverse image of $C \cap Y_0$ by ϕ_0 (see Theorem 2.18 for the notion), and let $V \subsetneqq X$ be its closure endowed with the reduced scheme structure. We will show that V is rationally connected (clearly it is proper). Let

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Y be the irreducible component of the product $\mathbb{P}^1 \times_Z X_0$ which dominates $\mathbb{P}^1$ (where X_0 is the open subvariety of X in the definition of the MRC fibration) endowed with the reduced scheme structure. Let $p : Y \to \mathbb{P}^1$ be the projection map. Replace Y with a birational projective variety, then resolve the indeterminacies of p. As the next step resolve the singularities of Y. Hence we can assume that $p : Y \to \mathbb{P}^1$ is a dominant morphism of smooth projective varieties. Y is birational to the closed subvariety $V \subsetneqq X$. This implies that the fibres of $Y \to \mathbb{P}^1$ are rationally connected. Hence by Theorem 2.21, Y is rationally connected. As Y and V are birational proper varieties, V is rationally connected. Therefore $V \subsetneqq X$ contains many rational curves passing through the fibre over z, but not lying in the fibre over z. (Note that a proper rational curve of V not lying in any fibre of $\phi|_V$ projects dominantly to C.) This contradicts Theorem 2.20 and finishes the proof. □

Another interesting application of Theorem 2.21 is that S. Mori's conjecture implies D. Mumford's one.

Theorem 2.23. *S. Mori's conjectures (Conjecture 2.8)) implies D. Mumford's conjecture (Conjecture 2.9).*

Proof. Let X be a d-dimensional smooth complex variety ($d \in \mathbb{Z}^+$) and (Z, ϕ) be its MRC fibration. We can assume that Z is smooth and projective. We will show that if X is not rationally connected, then $\mathrm{H}^0(X, (\Omega_X^1)^{\otimes N}) \neq 0$ for some positive integer N.
If X is not rationally connected, then by Theorem 2.21, Z is a positive dimensional non-uniruled variety. Let $e = \dim Z$ ($e \in \mathbb{Z}^+$). Hence by S. Mori's conjecture, there exists a non-zero global section $\sigma \in \mathrm{H}^0(Z, \omega_Z^{\otimes m})$ for some $m \in \mathbb{Z}^+$. Since X is smooth and Z is projective, ϕ is defined in codimension one, hence we can pull back $\omega_Z^{\otimes m}$ to define $\phi^*\omega_Z^{\otimes m}$ on X, and $\phi^*\sigma$ gives a non-zero global section of it. Moreover $\mathrm{H}^0(X, \phi^*\omega_Z^{\otimes m})$ embeds into $\mathrm{H}^0(X, (\Omega_X^e)^{\otimes m})$ by Generic Smoothness on the base (Corollary 10.7 of Chapter 3 in [Ha77]). (Indeed, for a smooth morphism it clearly holds, and $\phi_0|_{\phi_0^{-1}(V)} : \phi_0^{-1}(V) \to V$ is smooth for some open dense subvariety $V \subsetneqq Z$.) On the other hand, we have an injection of sheaves $(\Omega_X^e)^{\otimes m} \hookrightarrow (\Omega_X^1)^{\otimes em}$. Hence $\phi^*\sigma$ defines a non-zero global section of $(\Omega_X^1)^{\otimes em}$, therefore $\mathrm{H}^0(X, (\Omega_X^1)^{\otimes em}) \neq 0$. □

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Chapter 3

Finite subgroups of the birational automorphism group are 'almost' nilpotent

As we have seen before, the birational autmorphism group of many varieties are Jordan ([BZ15a], [BZ15b], [BZ19], [MZ15], [Po11], [Po14], [PS14], [PS16], [PS18a], [Se09]). However, even amongst surfaces the direct product of an elliptic curve and the projective line provides a counterexample to this property (Theorem 1.2 in [Za14]). Yu. Prokhorov and C. Shramov proved that the birational automorphism group of a variety is solvably Jordan (Proposition of 8.6 in [PS14]). The picture is remarkable similar in differential geometry. In many cases it is proved that compact real manifolds have Jordan diffeomorhism groups ([MR10], [MR16], [MR18], [MRSC19], [Zi14]), however B. Csikós, L. Pyber and E. Szabó found a counterexample (Theorem 1 in [CPS14]). Their construction was analogous to Yu. G. Zarhin's one. Moreover E. Ghys conjectured that the diffeomorphism group of a compact real manifold is nilpotently Jordan ([Gh15]). As the first trace of evidence, I. Mundet i Riera and C. Saéz-Calvo showed that the diffeomorphism group of a 4-fold is nilpotently Jordan of class at most 2 (Theorem 1.1 in [MRSC19]). Motivated by these antecedents, we investigate the nilpotently Jordan property for birational automorphism groups of varieties.

The material of this chapter is based on [Gu19].

3.1 Introduction

First, let us recall the definitions of various Jordan type properties.

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Definition 3.1. A group G is called Jordan, solvably Jordan or nilpotently Jordan of class at most c ($c \in \mathbb{N}$) if there exists a constant $J = J(G) \in \mathbb{Z}^+$, only depending on G, such that every finite subgroup $H \leqq G$ has a subgroup $K \leqq H$ such that $|H : K| \leqq J$ and K is Abelian, solvable or nilpotent of class at most c, respectively.

(The notion of Jordan groups and solvably Jordan groups was introduced by V. L. Popov (Definition 2.1 in [Po11]) and Yu. Prokhorov and C. Shramov (Definition 8.1 in [PS14]), respectively.)

Theorem 3.1. *The birational automorphism group of a d dimensional variety over a field of characteristic zero is nilpotently Jordan of class at most d.*

We show in Lemma 3.1 that it is enough to consider varieties over the field of the complex numbers.

The idea of the proof stems from the following picture. Let X be a d dimensional complex variety. We can assume that X is smooth and projective. Let $G \leqq \mathrm{Bir}(X)$ be an arbitrary finite subgroup. Consider the MRC (maximal rationally connected) fibration $\phi : X \dashrightarrow Z$ (Theorem 2.18). Because of the functoriality of the MRC fibration, a birational G-action is induced on Z, making ϕ G-equivariant. After a smooth regularization (Lemma 3.2) we can assume that both X and Z are smooth and projective, G acts on them by regular automorphisms and ϕ is a G-equivariant morphism. Since the general fibres of ϕ are rationally connected, we can run a G-equivariant relative Minimal Model Program over Z on X (Theorem 3.3). It results a G-equivariant Mori fibre space $\varrho : W \to Y$ over Z.

$$\begin{tikzcd} X \arrow[r, dashed, "\cong"] \arrow[dr, "\phi"'] & W \arrow[r, "\varrho"] \arrow[d] & Y \arrow[dl, "\psi"] \\ & Z & \end{tikzcd}$$

We can understand the G-action on X by analyzing the G-actions on $\psi : Y \to Z$ and on $\varrho : W \to Y$. We will apply induction on the relative dimension $e = \dim X - \dim Z$ to achieve this (Theorem 3.9). Actually, we will prove a slightly stronger theorem then Theorem 3.1 and will show that $\mathrm{Bir}(X)$ is nilpotently Jordan of class at most $(e+1)$. The base of the induction is when $e = 0$. Then X is non-uniruled and a theorem of Yu. Prokhorov and C. Shramov (Theorem 1.8 in [PS14]) shows us that the birational automorphism group of X is Jordan.
Otherwise, the inductive hypothesis will show us that $H = \mathrm{Im}(G \to \mathrm{Aut}_{\mathbb{C}}(Y))$ has a bounded index nilpotent subgroup of class at most e. To perform the inductive step, we will take a closer look at the G-action on the generic fibre $W_\eta \to \mathrm{Spec}\, k(Y)$. We will use two key ingredients. The first one is based on the boundedness of Fano varieties, and will allow us to embed G into the semilinear group $\mathrm{GL}(n, k(Y)) \rtimes \mathrm{Aut}_{\mathbb{C}}(k(Y))$, where n is bounded in terms of e (Proposition

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3.4). The second one is a Jordan type theorem on certain finite subgroups of a semilinear group (Theorem 3.7). Putting these together will finish the proof.

The chapter is organized in the following way. In Section 3.2 we recall the definition and some basic facts about nilpotent groups. In Section 3.3 we collect results about finite birational group actions on varieties. Section 3.4 deals with the proof of the Jordan type theorem on semilinear groups (Theorem 3.7). Finally, in Section 3.5 we prove our main theorem.

3.2 Some group theory

We recall the definition of nilpotent groups and some of their basic properties.

Definition 3.2. Let G be a group. Let $\mathrm{Z}_0(G) = 1$ and define $\mathrm{Z}_{i+1}(G)$ as the preimage of $\mathrm{Z}(G/\mathrm{Z}_i(G))$ under the natural quotient group homomorphism $G \to G/\mathrm{Z}_i(G)$ $(i \in \mathbb{N})$. The series of groups $1 = \mathrm{Z}_0(G) \leqq \mathrm{Z}_1(G) \leqq \mathrm{Z}_2(G) \leqq \ldots$ is called the upper central series of G.
Let $\gamma_0(G) = G$ and let $\gamma_{i+1}(G) = [\gamma_i(G), G]$ ($i \in \mathbb{N}$, and $[,]$ denotes the commutator operation). The series of groups $G = \gamma_0(G) \geqq \gamma_1(G) \geqq \gamma_2(G) \geqq \ldots$ is called the lower central series of G.
G is called nilpotent if one (hence both) of the following equivalent conditions hold:

- There exists $n \in \mathbb{N}$ such that $\mathrm{Z}_n(G) = G$.
- There exists $n \in \mathbb{N}$ such that $\gamma_n(G) = 1$.

If G is a nontrivial nilpotent group, then there exists a natural number c for which $\mathrm{Z}_c(G) = G$, $\mathrm{Z}_{c-1}(G) \neq G$ and $\gamma_c(G) = 1$, $\gamma_{c-1}(G) \neq 1$ holds. c is called the nilpotency class of G. (If G is trivial, then its nilpotency class is zero.)

Remark 3.1. Note that $\mathrm{Z}_1(G)$ is the centre of the group G, while $\gamma_1(G)$ is the commutator subgroup. A non-trivial group G is nilpotent of class one if and only if it is Abelian.
Nilpotency is the property between the Abelian and the solvable properties. The Abelian property implies nilpotency, while nilpotency implies solvability.

The following proposition describes one of the key features of nilpotent groups. They can be built up by successive central extensions.

Proposition 3.1. *Let G be a group and $A \leqq \mathrm{Z}(G)$ be a central subgroup of G. If G/A is nilpotent of class at most c, then G is nilpotent of class at most $(c+1)$.*

We will use also the two properties below about nilpotent groups.

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Proposition 3.2. *Let G be a nilpotent group of class at most n. Fix $n-1$ arbitrary elements in G, denote them by $g_1, g_2, ...g_{n-1}$, and let $1 \leqq j \leqq n$ be an arbitrary integer. The map φ_j defined by the help of iterated commutators of length $(n-1)$*

$$\varphi_j : G \to \gamma_{n-1}(G)$$
$$g \mapsto [[...[[[[...[[g_1, g_2], g_3]...], g_{j-1}], g], g_j]...], g_{n-1}]$$

gives a group homomorphism.

Proposition 3.3. *Let G be a group. G is nilpotent of class at most n if and only if $\forall g_1, g_2, ..., g_{n+1} \in G$: $[[...[[g_1, g_2], g_3]...], g_{n+1}] = 1$.*

Remark 3.2. Typical examples of nilpotent groups are finite p-groups (where p is a prime number). If we restrict our attention to finite nilpotent groups, even more can be said. (Recall that a Sylow p-subgroup of a finite group is a maximal p-group contained in the group.) A finite group is nilpotent if and only if it is the direct product of its Sylow subgroups (Theorem 6.12 in [CR62]).

3.3 Finite group actions on varieties

In this section we introduce techniques which help us to solve partial cases of our problem and help us to build up the full solution from the special cases.

3.3.1 Reduction to the field of the complex numbers

We show that it is enough to prove Theorem 3.1 over the field of the complex numbers.

Lemma 3.1. *It is enough to prove Theorem 3.1 over the field of the complex numbers.*

Proof. Let k be a field of characteristic zero and X be a variety over k. First assume that X is geometrically integral. We can fix a finitely generated field extension $l_0|\mathbb{Q}$ and an l_0-variety X_0 such that $X \cong X_0 \times_{l_0} \operatorname{Spec} k$. Fix a field embedding $l_0 \hookrightarrow \mathbb{C}$ and let $X^* \cong X_0 \times_{l_0} \operatorname{Spec} \mathbb{C}$. For an arbitrary finite subgroup $G \leqq \operatorname{Bir}(X)$ we can find a finitely generated field extension $l_1|l_0$ such that the elements of G can be defined as birational transformations over the field l_1. Hence $G \leqq \operatorname{Bir}(X_1)$, where $X_1 \cong X_0 \times_{l_0} \operatorname{Spec} l_1$. We can extend the fixed field embedding $l_0 \hookrightarrow \mathbb{C}$ to a field embedding $l_1 \hookrightarrow \mathbb{C}$. Therefore $X^* \cong X_0 \times_{l_0} \operatorname{Spec} \mathbb{C} \cong X_1 \times_{l_1} \operatorname{Spec} \mathbb{C}$, and we can embed G into the birational automorphism group of the complex variety X^*. As the birational class of the complex variety X^* only depends on the birational class of the variety X, it is enough to examine complex varieties. This proves the lemma when X is geometrically integral.

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If X is not geometrically integral then, it is still geometrically reduced as we work in characteristic zero. We can still construct the l_0-variety X_0, the l_1-scheme X_1 and the complex scheme X^*. Therefore $G \leqq \mathrm{Bir}(X)$ embeds into $\mathrm{Bir}(X^*)$ just like in the geometrically integral case. There exists a constant only depending on the birational class of X such that X^* has C many irreducible components. Therefore a finite subgroup $H \leqq G$ of index at most $C!$ leaves all irreducible components of X^* invariant. Hence H has a nilpotent subgroup of class at most d of bounded index by the complex case. The bound on the index only depends on the birational classes of the components of X^*, hence it only depeneds on the birational class of X. Therefore G has a nilpotent subgroup of class at most d of bounded index, where the bound on the index only depends on the birational class of X. This finishes the proof. □

3.3.2 Jordan property

Let us recall a theorem of Yu. Prokhorov and C. Shramov (Theorem 1.8 in [PS14] and Theorem 1.8 in [PS16]). It will serve us as a starting point of an inductive argument in the proof of our main theorem and will be an important ingredient when we look for bounds on the number of generators of finite subgroups of the birational automorphism group (Theorem 3.4).

Theorem 3.2. *Let X be variety over a field of characteristic zero. Assume that X is either non-uniruled or rationally connected. Then the birational automorphism group of X is Jordan (in other words, it is nilpotently Jordan of class at most 1).*

3.3.3 Smooth regularization

The next lemma is a slight extension of the well-known (smooth) regularization of finite group actions on varieties (Lemma-Definition 3.1. in[PS14]).

Lemma 3.2. *Let X and Z be complex varieties and $\phi : X \dashrightarrow Z$ be a dominant rational map between them. Let G be a finite group which acts by birational automorphisms on X and Z in such a way that ϕ is G-equivariant. There exist smooth projective varieties X^* and Z^* with regular G-actions on them and a G-equivariant projective morphism $\phi^* : X^* \to Z^*$ such that X^* is G-equivariantly birational to X, Z^* is G-equivariantly birational to Z and ϕ^* is G-equivariantly birational to ϕ. In other words, we have a G-equivariant commutative diagram.*

$$\begin{tikzcd} X \arrow[r, dashed, "\cong"] \arrow[d, dashed, "\phi"] & X^* \arrow[d, "\phi^*"] \\ Z \arrow[r, dashed, "\cong"] & Z^* \end{tikzcd}$$

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Proof. Let $k(Z) \leqq k(X)$ be the field extension corresponding to the function fields of Z and X, induced by ϕ. Take the induced G-action on this field extension and let $k(Z)^G \leqq k(X)^G$ be the field extension of the G-invariant elements. Consider a projective model of it, i.e. let $\varrho_1 : X_1 \to Z_1$ be a (projective) morphism, where X_1 and Z_1 are projective varieties such that $k(X_1) \cong k(X)^G$ and $k(Z_1) \cong k(Z)^G$, and $\varrho_1 : X_1 \to Z_1$ induces the field extension $k(Z_1) \cong k(Z)^G \leqq k(X)^G \cong k(X_1)$. By normalizing X_1 in the function field $k(X)$ and Z_1 in the function field $k(Z)$ we get projective varieties X_2 and Z_2, moreover ϱ_1 induces a G-equivariant morphism $\varrho_2 : X_2 \to Z_2$ between them. As the next step, we can take a G-equivariant resolution of singularities $\widetilde{Z_2} \to Z_2$. After replacing Z_2 by $\widetilde{Z_2}$ and X_2 by the irreducible component of $X_2 \times_{Z_2} \widetilde{Z_2}$ which dominates $\widetilde{Z_2}$, we can assume that Z_2 is smooth. Hence G-equivarianlty resolving the singularities of X_2 finishes the proof. $\square$

3.3.4 Minimal Model Program and boundedness of Fano varieties

Applying the results of the famous article by C. Birkar, P. Cascini, C. D. Hacon and J. McKernan ([BCHM10]) enables us to use the arsenal of the Minimal Model Program. As a consequence, we can examine rationally connected varieties (fibres) with the help of Fano varieties (fibres). For the later we can use boundedness results because of yet another famous theorem by C. Birkar ([Bi16]). (This theorem was previously known as the BAB Conjecture).

Theorem 3.3. *Let X and Z be smooth projective complex varieties such that $\dim Z < \dim X$. Let $\phi : X \to Z$ be a dominant morphism between them with rationally connected general fibres. Let G be a finite group which acts by regular automorphisms on X and Z in such a way that ϕ is G-equivariant. We can run a G-equivariant Minimal Model Program (MMP) on X relative to Z which results a Mori fibre space. In particular, the Minimal Model Program gives a G-equivariant commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\cong"] \arrow[dr, "\phi"'] & W \arrow[r] \arrow[d] & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

where W is G-equivariantly birational to X, $\dim Y < \dim X$ and the generic fibre of the morphism between W and Y is a Fano variety with terminal singularities.

Proof. By Corollary 2.3, we can run a relative MMP on $\phi : X \to Z$ (which results a Mori fibre space) if the canonical divisor of X is not ϕ-pseudo-effective. It can be done equivariantly if we have finite group actions. (See Chapter 2.2 in [KM98] and Section 4 of [PS14] for further discussions on the topic.) So, it remains to show that the canonical divisor of X is not ϕ-pseudo-effective.

By generic smoothness, a general fibre of ϕ is a smooth rationally connected projective complex

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variety. Therefore if x is a general closed point of a general fibre F, then there exists a free rational curve C_x running through x, lying entirely in the fibre F (Proposition 2.4). Since C_x is a free rational curve, $C_x.K_X \leqq -2$ (Remark 2.30). Since the inequality holds for every general closed point of every general fibre, K_X cannot be ϕ-pseudo-effective. $\square$

The lemmas and the theorems above open the door for us to use induction on the relative dimension of the MRC fibration while proving Theorem 3.1. So we only need to deal with Fano varieties of bounded dimensions.

Proposition 3.4. *Let e be a natural number. There exists a constant $n = n(e) \in \mathbb{N}$, only depending on e, with the following property. If*

- *k is a field of characteristic zero,*
- *F is a Fano variety over the field k of dimension at most e with terminal singularities,*
- *G is a finite group which acts faithfully on F by regular automorphisms of the $\mathbb{Q}$-scheme F, and acts on $\operatorname{Spec} k$ by regular automorphisms of the $\mathbb{Q}$-scheme $\operatorname{Spec} k$, in such a way that the structure morphism $F \to \operatorname{Spec} k$ is G-equivariant,*

then G can be embedded into the semilinear group $\Gamma\mathrm{L}(n,k) \cong \mathrm{GL}(n,k) \rtimes \operatorname{Aut} k$ in such a way that composition $G \hookrightarrow \Gamma\mathrm{L}(n,k) \twoheadrightarrow \operatorname{Aut} k$ corresponds to the G-action on $\operatorname{Spec} k$.

Proof. Fix k, F and G with the properties described by the theorem. By Corollary 2.4 there exist constants $n = n(e), m = m(e) \in \mathbb{N}$, only depending on e, such that m-th power of the anticanonical divisor is very ample and $\dim_k \mathrm{H}^0(F, -mK_F) \leqq n$. So we have a closed embedding of the form $F \hookrightarrow \mathbb{P}^{n_1}_k \cong \mathbb{P}(\mathrm{H}^0(X, -mK_F)^*)$, for some non-negative integer $n_1 \leqq n-1$.
By the functorial property of a (fixed) power of the anticanonical divisor, an equivariant G-action is induced on the commutative diagram below.

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d] & \mathbb{P}(\mathrm{H}^0(X, -mK_F)^*) \arrow[dl] \\ \operatorname{Spec} k & \end{tikzcd}$$

Since $F \hookrightarrow \mathbb{P}(\mathrm{H}^0(X, -mK_F)^*)$ is a closed embedding, the semilinear action of G on the vector space $\mathrm{H}^0(X, -mK_F)$ is faithful. Hence G embeds into $\Gamma\mathrm{L}(\mathrm{H}^0(X, -mK_F))$. Clearly $G \to \operatorname{Aut} k$ corresponds to the G-action on $\operatorname{Spec} k$. Since $\dim_k \mathrm{H}^0(X, -mK_F) \leqq n$, G embeds into $\Gamma\mathrm{L}(V)$ for some n-dimensional k-vector space V in such a way that $G \to \operatorname{Aut} k$ corresponds to the G-action on $\operatorname{Spec} k$. This finishes the proof. $\square$

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3.3.5 Bound on the minimal size of a generating set of some groups

Now we turn our attention on finding bounds on the minimal size of a generating set of finite subgroups of the birational automorphism group of varieties. It will be important for us when we will investigate commutator relations (Lemma 3.5), and it will be crucial to have a bound on the size of a generating set of the group.
The next theorem and its proof are essentially due to Y. Prokhorov and C. Shramov. (We use the word essentially as they only considered the case of finite Abelian subgroups (Remark 6.9 of [PS14]).) It is also important to note that the proof of Remark 6.9 of [PS14] uses the result of C. Birkar about the boundedness of Fano varieties (Theorem 1.1 in [Bi16]).

Theorem 3.4. *Let X be a variety over a field of characteristic zero. There exists a constant $m = m(X) \in \mathbb{Z}^+$, only depending on the birational class of X, such that if $G \leqq \mathrm{Bir}(X)$ is an arbitrary finite subgroup of the birational automorphism group, then G can be generated by m elements.*

Proof. First we show the theorem in the special cases when X is either non-uniruled or rationally connected. By Remark 6.9 of [PS14] and Theorem 2.17, there exists a constant $m = m(X) \in \mathbb{Z}^+$, only depending on the birational class of X, such that if $A \leqq \mathrm{Bir}(X)$ is an arbitrary finite Abelian subgroup of the birational automorphism group, then A can be generated by m elements. Since $\mathrm{Bir}(X)$ is Jordan when X is non-uniruled or rationally connected (Theorem 3.2), the result on the finite Abelian groups implies the claim of the theorem in both of these special cases.
Now let X be arbitrary. Arguing as in the proof of Lemma 3.1 we can assume that X is a complex variety. Consider the MRC fibration $\phi : X \dashrightarrow Z$. By Lemma 3.2 we can assume that both X and Z are smooth projective varieties, and G acts on them by regular automorphisms. Let ρ be the generic point of Z, and let X_ρ be the generic fibre of ϕ. X_ρ is a rationally connected variety over the function field $k(Z)$.
Let $G_\rho \leqq G$ be the maximal subgroup of G acting fibrewise. G_ρ has a natural faithful action on X_ρ, while $G/G_\rho = G_Z$ has a natural faithful action on Z. This gives a short exact sequence of groups

$$1 \to G_\rho \to G \to G_Z \to 1.$$

By the rationally connected case there exists a constant $m_1(X_\rho)$, only depending on the birational class of X_ρ, such that G_ρ can be generated by $m_1(X_\rho)$ elements. By the non-uniruled case there exists a constant $m_2(Z)$, only depending on the birational class of Z, such that G_Z can be generated by $m_2(Z)$ elements. So G can be generated by $m(X_\rho, Z) = m_1(X_\rho) + m_2(Z)$ elements. Since $m(X_\rho, Z)$ only depends on the birational classes of X_ρ and Z, and both of the birational classes of X_ρ and Z only depend on the birational class of X, this finishes the proof. $\square$

In case of rationally connected varieties we will use a slightly stronger version of the theorem.

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To prove it, first recall the theorem of Yu. Prokhorov and C. Shramov about fixed points of rationally connected varieties (Theorem 4.2 of [PS14]).

Theorem 3.5. *Let e be a natural number. There exists a constant $R = R(e) \in \mathbb{Z}^+$, only depending on e, with the following property. If X is a rationally connected complex projective variety of dimension at most e, and $G \leqq \mathrm{Aut}(X)$ is an arbitrary finite subgroup of its automorphism group, then there exists a subgroup $H \leqq G \leqq \mathrm{Aut}(X)$ such that H has a fixed point in X, and the index of H in G is bounded by R.*

Theorem 3.6. *Let e be a natural number. There exists a constant $m = m(e) \in \mathbb{Z}^+$, only depending on e, with the following property. If k is an arbitrary field of characteristic zero, X is a rationally connected variety over k of dimension at most e, and $G \leqq \mathrm{Bir}(X)$ is an arbitrary finite subgroup of the birational automorphism group, then G can be generated by m elements.*

Proof. Fix k, X and G with the properties described by the theorem. Arguing as in the proof of Lemma 3.1, we can assume that k is the field of the complex numbers.
Using Lemma 3.2, we can assume that X is smooth and projective and G is a finite subgroup of the biregular automorphism group $\mathrm{Aut}(X)$.
By Theorem 3.5, we can assume that G has a fixed point in X. Denote it by P.
By Lemma 4 of [Po14] G acts faithfully on the tangent space of the fixed point P. So G can be embedded into $\mathrm{GL}(\mathrm{T}_P\, X)$, whence G can be embedded into $\mathrm{GL}(e, \mathbb{C})$. Therefore the claim of the theorem follows from Lemma 3.3. This finishes the proof. □

3.4 The general semilinear group

This section contains the group theoretic ingredient of the proof of the main theorem.

Theorem 3.7. *Let c, n and m be positive integers. Let F be the family of those finite groups G which have the following properties.*

- *There exists a field k of characteristic zero containing all roots of unity such that G is a subgroup of the semilinear group $\Gamma\mathrm{L}(n, k) \cong \mathrm{GL}(n) \rtimes \mathrm{Aut}\, k$.*
- *Every subgroup of G can be generated by m elements.*
- *The image of the composite group homomorphism $G \hookrightarrow \Gamma\mathrm{L}(n, k) \twoheadrightarrow \mathrm{Aut}\, k$, denoted by Γ, is nilpotent of class at most c ($c \in \mathbb{N}$) and fixes all roots of unity.*

There exists a constant $C = C(c, n, m) \in \mathbb{Z}^+$, only depending on c, n and m, such that every finite group G belonging to F contains a nilpotent subgroup $H \leqq G$ with nilpotency class at most $(c+1)$ and with index at most C.

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First, we recall a slightly strengthened version of Jordan's theorem.

Theorem 3.8. *Let n be a positive integer. There exists a constant $J = J(n) \in \mathbb{Z}^+$, only depending on n, such that if a finite group G is a subgroup of a general linear group $\mathrm{GL}(n, k)$, where k is a field of characteristic zero, then G contains a characteristic Abelian subgroup $A \leqq G$ of index at most J.*

Remark 3.3. The only claim of the above theorem which does not follow immediately from Theorem 2.3 in [Br11] is that we require the Abelian subgroup of bounded index $A \leqq G$ to be characteristic (i.e. invariant under all automorphisms of G) instead of being normal (i.e. invariant under the inner automorphisms of G). In the following we will prove some lemmas which help us to deduce the above variant of the theorem from the one which can be found in [Br11].

Lemma 3.3. *Let n be a positive integer. There exists a constant $r = r(n) \in \mathbb{Z}^+$, only depending on n, such that if a finite group G is a subgroup of a general linear group $\mathrm{GL}(n, k)$, where k is a field of characteristic zero, then G can be generated by r elements.*

Proof. It is enough to prove the lemma when k is algebraically closed, so we can assume it. By Theorem 2.3 in [Br11], G contains a diagonalizable subgroup of bounded index. Since finite diagonal groups of $\mathrm{GL}(n, k)$ can be generated by n elements, the lemma follows. □

Lemma 3.4. *Let J and r be positive integers. There exists a constant $L = L(J, r) \in \mathbb{N}$, only depending on r and J, such that if G is a finite group which can be generated by r elements, then G has at most L many subgroups of index J.*

Proof. Fix an arbitrary finite group G which can be generated by r elements. We can construct an injective map of sets from the set of index J subgroups of G to the set of group homomorphisms from G to the symmetric group of degree J. Since G can be generated by r elements the later set has boundedly many elements, hence the former set has boundedly many elements as well. So we only left with the task of constructing such an injective map.
Let S be a set with J elements. We can identify the symmetric group of degree J, denoted by Sym_J, with the symmetry group of the set S. Fix an arbitrary element $x \in S$. For every index J subgroup $K \leqq G$, fix a bijection μ_K between the set of the left cosets of K and the set S, subject to the following condition, K is mapped to the fixed element x, i.e. $\mu_K(K) = x$. Let $H \leqq G$ be an arbitrary subgroup of index J. G acts on the set of the left cosets of H by left multiplication. Using the bijection μ_H, this induces a group homomorphism $\phi_H : G \to \mathrm{Sym}_J$. The constructed assignment is injective as the stabilizator subgroup of x in the image group $\mathrm{Im}\, \phi_H$ uniquely determines H. □

Proof of Theorem 4.4. Let k be an arbitrary field of characteristic zero, and let G be an arbitrary finite subgroup of $\mathrm{GL}(n, k)$. By Theorem 2.3 in [Br11] G contains an Abelian subgroup $A \leqq G$ of

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index bounded by $J_0 = J_0(n)$. Consider the set S of the smallest index Abelian subgroups of G. By Lemma 3.3 and Lemma 3.4 there exists a constant $L = L(n)$, only depending on n, such that S has at most L many elements. Take the intersection of the subgroups contained in S, it gives a characteristic Abelian subgroup of index at most J_0^L. □

Next we prove a lemma about nilpotent groups.

Lemma 3.5. *Let c, J and m be positive integers. There exists a constant $C = C(c, J, m) \in \mathbb{N}$, only depending on c, J and m, such that if*

- *G is a nilpotent group of class at most $(c+1)$,*
- *G can be generated by m elements,*
- *the cardinality of $\gamma_c(G)$ is at most J,*

then G has a nilpotent subgroup $H \leqq G$ of class at most c whose index is bounded by C.

Proof. Fix a generating system $g_1, ..., g_m \in G$. Consider the group homomorphisms (Proposition 3.2)

$$\varphi_{i_1,i_2,\ldots,i_c} : G \to \gamma_c(G)$$
$$g \mapsto [[[...[[g_{i_1}, g_{i_2}], g_{i_3}]...], g_{i_c}], g],$$

where $1 \leqq i_1, i_2, ..., i_c \leqq m$, i.e. for every ordered length c sequence of the generators we assign a group homomorphism using the iterated commutators. Let H be the intersection of the kernels.

$$H = \bigcap_{1 \leqq i_1, i_2, \ldots, .i_c \leqq m} \operatorname{Ker} \varphi_{i_1,i_2,\ldots,i_c}$$

Using the fact that the length c iterated commutators give group homomorphisms in every variable if we fix the other variables (Proposition 3.2), one can show that all the length c iterated commutators of H vanish. Hence H is nilpotent of class at most c (Proposition 3.3).
On the other hand H is the intersection of m^c many subgroups of index at most $|\gamma_c(G)| \leqq J$. Hence the index of H is bounded in terms of c, J and m. This finishes the proof. □

Now we are ready to prove the main theorem of the chapter.

Proof of Theorem 3.7. Let k be an arbitrary field of characteristic zero containing all roots of unity, and let G be an arbitrary finite subgroup of $\Gamma\mathrm{L}(n, k)$ belonging to F. Consider the short exact sequence of groups given by

$$1 \to N \to G \to \Gamma \to 1,$$

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where $N = \mathrm{GL}(n,k) \cap G$ and $\Gamma = \mathrm{Im}(G \to \mathrm{Aut}\, k)$. By Theorem 4.4, N contains a characteristic Abelian subgroup of index bounded by $J = J(n) \in \mathbb{Z}^+$. Since A is characteristic in N and N is normal in G, A is a normal subgroup of G.
Consider the natural semilinear action of G on the vector space $V = k^n$. Since A is a finite Abelian subgroup of $\mathrm{GL}(V)$ and the ground field k contains all roots of unity, A decomposes V into common eigenspaces of its elements: $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$ $(r \leqq n)$. As A is normal in G, G respects this decomposition, i.e. G acts on the set of linear subspaces $\{V_1, V_2, \dots, V_r\}$ by permutations. The kernel of this group action, denoted by G_1, is a bounded index subgroup of G (indeed $|G : G_1| \leqq r! \leqq n!$). Furthermore, A is central in G_1, i.e. $A \leqq \mathrm{Z}(G_1)$. To see this, notice that on an arbitrary fixed eigenspace V_i $(1 \leqq i \leqq r)$ A acts by scalar matrices in such a way that all scalars are drawn from the set of the roots of unity. Since G_1 leaves V_i invariant by definition and $\mathrm{Im}(G_1 \to \mathrm{Aut}\, k)$ fixes all roots of unity, our claim follows. After replacing G with the bounded index subgroup G_1, we can assume that $A \leqq \mathrm{Z}(G)$.
As A is a central subgroup of G, we can consider the quotient group $\overline{G} = G/A$. By Proposition 3.1, we only need to prove that $\overline{G}$ has a bounded index nilpotent subgroup of class at most c. Our strategy will be that, first we prove that $\overline{G}$ has a bounded index nilpotent subgroup of class at most $(c+1)$, then we will apply Lemma 3.5.
Let $\overline{N} = N/A$, and consider the short exact sequence of groups

$$1 \to \overline{N} \to \overline{G} \to \Gamma \to 1.$$

The number of elements of $\overline{N}$ is bounded by $J(n)$, by the definition of A, and Γ is nilpotent of class at most c, by the definition of G.
$\overline{G}$ acts on $\overline{N}$ by conjugation, and the kernel of this action is the centralizer group $\mathrm{C}_{\overline{G}}(\overline{N}) = \{g \in \overline{G} |\ ng = gn\ \forall n \in \overline{N}\}$. Therefore $\overline{G}/\,\mathrm{C}_{\overline{G}}(\overline{N})$ embeds into the automorphism group of $\overline{N}$ which has cardinality at most $J!$. Hence $\mathrm{C}_{\overline{G}}(\overline{N})$ has bounded index in $\overline{G}$. Hence, after replacing $\overline{G}$ with $\mathrm{C}_{\overline{G}}(\overline{N})$, $\overline{N}$ with $\overline{N} \cap \mathrm{C}_{\overline{G}}(\overline{N})$ and Γ with the image group $\mathrm{Im}(\mathrm{C}_{\overline{G}}(\overline{N}) \to \Gamma)$, we can assume that $\overline{G}$ is the central extension of the Abelian group $\overline{N}$ and nilpotent group Γ whose nilpotency class is at most c. Therefore we can assume that $\overline{G}$ is nilpotent of class at most $(c+1)$ (Proposition 3.1). Notice that $\gamma_c(\overline{G})$ maps to $\gamma_c(\Gamma) = 1$, which implies that the former group is contained in $\overline{N}$. So $|\gamma_c(\overline{G})| \leqq |\overline{N}| \leqq J$. Hence we are in the position to apply Lemma 3.5, which finishes the proof. □

Remark 3.4. In the above proof we only used the assumption that G can be generated by m elements via Lemma 3.5. So if we omit this condition from Theorem 3.7, we can still prove that there exists a constant $D = D(n) \in \mathbb{Z}^+$, only depending on n (not even on c), such that if G belongs to the corresponding family of groups, then G contains a nilpotent subgroup $H \leqq G$ with nilpotency class at most $(c+2)$ and with index at most D.

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3.5 Proof of the Main Theorem

Using the techniques developed in the previous sections, we will prove our main theorem.

Theorem 3.9. *Fix a non-uniruled complex variety Z_0. Let F_{Z_0} be the collection of 5-tuples (X, Z, ϕ, G, e), where*

- *X is a complex variety,*
- *Z is a complex variety, which is birational to Z_0,*
- *$\phi : X \dashrightarrow Z$ is a dominant rational map such that there exist open subvarieties X_1 of X and Z_1 of Z such that ϕ descends to a proper morphism between them $\phi_1 : X_1 \to Z_1$ with rationally connected fibres,*
- *$G \leqq \mathrm{Bir}(X)$ is a finite group of the birational automorphism group of X, which also acts by birational automorphisms on Z in such a way that ϕ is G-equivariant,*
- *$e \in \mathbb{N}$ is the relative dimension $e = \dim X - \dim Z$.*

Then the following claims hold.

- *There exist constants $\{m_{Z_0}(e) \in \mathbb{Z}^+ | \, e \in \mathbb{N}\}$, only depending on e and the birational class of Z_0, such that if the 5-tuple (X, Z, ϕ, G, e) belongs to F_{Z_0}, then G can be generated by $m_{Z_0}(e)$ elements.*
- *There exist constants $\{J_{Z_0}(e) \in \mathbb{Z}^+ | \, e \in \mathbb{N}\}$, only depending on e and the birational class of Z_0, such that if the 5-tuple (X, Z, ϕ, G, e) belongs to F_{Z_0}, then G has a nilpotent subgroup $H \leqq G$ of nilpotency class at most $(e + 1)$ and index at most $J_{Z_0}(e)$.*

Proof. (Proof of the First Claim) Let (X, Z, ϕ, G, e) be an arbitrary 5-tuple belonging to F_{Z_0}. By Lemma 3.2 we can assume that both X and Z are smooth projective varieties, and G acts on them by regular automorphisms. Let ρ be the generic point of Z, and let X_ρ be the generic fibre of ϕ. X_ρ is a rationally connected variety of dimension e over the function field $k(Z)$.
Let $G_\rho \leqq G$ be the maximal subgroup of G acting fibrewise. G_ρ has a natural faithful action on X_ρ, while $G/G_\rho = G_Z$ has a natural faithful action on Z. This gives a short exact sequence of groups

$$1 \to G_\rho \to G \to G_Z \to 1.$$

By Theorem 3.6 there exists a constant $m_1(e)$, only depending on e, such that G_ρ can be generated by $m_1(e)$ elements. By Theorem 3.4 there exists a constant $m_2(Z)$, only depending on the birational class of Z, such that G_Z can be generated by $m_2(Z)$ elements. So G can be generated by $m_{Z_0}(e) = m_1(e) + m_2(Z)$ elements. Since $m_{Z_0}(e)$ only depends on e and the birational class of Z_0, this finishes the proof of the first claim.

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(Proof the Second Claim) We will apply induction on e. If $e = 0$, then X and Z_0 are birational, hence $G \leqq \mathrm{Bir}(Z_0)$ and the claim of the theorem follows from Theorem 3.2. So we can assume that $e > 0$ and the claim of the theorem holds if the relative dimension is strictly smaller than e. Let (X, Z, ϕ, G, e) be a 5-tuple belonging to F_{Z_0}. After regularizing ϕ in the sense of Lemma 3.2, we may assume that X and Z are smooth projective varieties, G acts on them by regular automorphisms and ϕ is a G-equivariant (projective) morphism.
Hence by Theorem 3.3, we can run a relative G-equivariant MMP on $\phi : X \to Z$. It results a G-equivariant commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\cong"] \arrow[dr, "\phi"'] & W \arrow[r, "\varrho"] \arrow[d] & Y \arrow[dl, "\psi"] \\ & Z & \end{tikzcd}$$

where $\varrho : W \to Y$ is a Mori fibre space and $\psi : Y \to Z$ is a dominant morphism with rationally connected general fibres (as so does ϕ). Let H be the image of $G \to \mathrm{Aut}_{\mathbb{C}}(Y)$, and let f be the relative dimension $f = \dim Y - \dim Z$. The 5-tuple (Y, Z, ψ, H, f) clearly belongs to F_{Z_0}. Moreover, since $f < e$, we can use the inductive hypothesis. Let $H_1 \leqq H$ be the nilpotent subgroup of nilpotency class at most $(f+1)$ and index at most $J_{Z_0}(f)$. After replacing H with its bounded index subgroup H_1 (and G with the preimage of H_1), we can assume that H is nilpotent of class at most e.
Let $\eta \cong \mathrm{Spec}\, k(Y)$ be the generic point of Y, and let W_η be the generic fibre of ϱ. Since $\varrho : W \to Y$ is a Mori fibre space, W_η is a Fano variety over $k(Y)$ with terminal singularities. Furthermore, G acts on the structure morphism $W_\eta \to \mathrm{Spec}\, k(Y)$ equivariantly by scheme automorphisms. Hence we can apply Proposition 3.4, and we can embed G into $\Gamma\mathrm{L}(n, k(Y)) \cong \mathrm{GL}(n, k(Y)) \rtimes \mathrm{Aut}\, k(Y)$ where $n = n(e)$ only depends on e (since $\dim W_\eta \leqq e$). Moreover, the image group $\Gamma = \mathrm{Im}(G \hookrightarrow \Gamma\mathrm{L}(n, k(Y)) \twoheadrightarrow \mathrm{Aut}\, k(Y))$ corresponds to the G-action on $\mathrm{Spec}\, k(Y)$, therefore it corresponds to the H-action on Y. Hence Γ fixes all roots of unity, as Y is a complex variety, and Γ is nilpotent of class at most e, as so does H. Furthermore, by the first claim of the theorem, every subgroup of G can be generated by $m = m_{Z_0}(e)$ elements (where m only depends on e and the birational class of Z_0). So we are in the position to apply Theorem 3.7 to the group G, which finishes the proof. □

To finish the chapter, we prove our main theorem.

Proof of Theorem 3.1. Let X be a d dimensional complex variety. We can assume that X is smooth and projective. We can also assume that X is uniruled by Theorem 3.2. Let $G \leqq \mathrm{Bir}(X)$ be an arbitrary finite subgroup of the birational automorphism group of X. Let $\phi : X \dashrightarrow Z$ be the MRC fibration, and let $e = \dim X - \dim Z$ be the relative dimension. By the functoriality

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of the MRC fibration (Corollary 2.5), G acts on the base Z by birational automorphisms making the rational map ϕ G-equivariant. Hence the 5-tuple (X, Z, ϕ, G, e) belongs to the collection F_Z defined in the previous theorem. Therefore G has a nilpotent subgroup of class at most $(e+1)$ and index at most $J_Z(e)$. Since $e < d$ (as X is uniruled), moreover the relative dimension e and the birational class of the base Z only depends on the birational class of X, the theorem follows. $\square$

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Chapter 4

Boundedness properties of automorphism groups of forms of flag varieties

The material of this chapter is based on [Gu18].

4.1 Introduction

Before stating our main theorem we need to introduce a couple of definitions and notations. Let us recall the definition of bounded groups.

Definition 4.1 (Definition 2.9 in [Po11])**.** A group G is called bounded if there exists a constant $C \in \mathbb{N}$ such that every finite subgroup of G has smaller cardinality than C.

Let V be a finite dimensional vector space (over an arbitrary field). A flag is a strictly increasing sequence of linear subspaces of V (with respect to the the containment order). By $\mathrm{Fl}(d_1 < d_2 < ... < d_r, V)$ or simply by $\mathrm{Fl}(\mathbf{d}, V)$ we denote the flag variety of the sequence of linear subspaces of V (flags) of dimensions determined by the strictly increasing sequence of non-negative integers $\mathbf{d} = (d_1, d_2, ..., d_r)$, where $d_r \leqq \dim V$. We also use the notation $\mathrm{Fl}(\mathbf{d} < \mathbf{e}, V)$ governed by similar logic, using the strictly increasing sequence of non-negative integers $\mathbf{d} < \mathbf{e} = (d_1, ..., d_p, e_1, ..., e_q)$ ($e_q \leqq \dim V$). If $d_1 \geqq n$ then the notation $\mathbf{d} - n$ stands for the strictly increasing sequence of non-negative integers $\mathbf{d} - n = (d_1 - n, ..., d_r - n)$.
If no confusion can arise we omit the specification of the vector space or the strictly increasing sequence of non-negative integers or both of them. When we say $\mathrm{Fl}(\mathbf{d}, V)$ is a flag variety, we implicitly assume that V is a vector space over some field and $\mathbf{d} = (d_1, ..., d_r)$ is a strictly increasing

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sequence of non-negative integers, where $d_r \leqq \dim V$.

Definition 4.2. We call a flag variety admissible, if its automorphism group is the projective general linear group, otherwise we call it non-admissible.

Later on we will see that a flag variety is admissible unless it is isomorphic to a flag variety $\mathrm{Fl}(d_1 < ... < d_r, V)$, where $0 < d_1$, $d_r < \dim V$, $\dim V \geqq 3$ and $\forall i = 1, ..., r \quad d_i + d_{r+1-i} = \dim V$. Notice that the conditions $0 < d_1$ and $d_r < \dim V$ are technical assumptions, they do not exclude any isomorphism class of flag varieties. The automorphism group of the non-admissible flag variety $\mathrm{Fl}(\mathbf{d}, V)$ is $\mathrm{PGL}(V) \rtimes \mathbb{Z}/2\mathbb{Z}$. (See Theorem 4.4 for further details.)

Definition 4.3. Let k be a field. The k-variety X is a form of a flag variety if $X \times \operatorname{Spec} \overline{k} \cong \mathrm{Fl}(\mathbf{d}, V_{\overline{k}})$ where $\overline{k}$ is the algebraic closure of k, $V_{\overline{k}}$ is a finite dimensional $\overline{k}$-vector space, and $\mathbf{d}$ is a strictly increasing sequence of non-negative integers.

Now we are ready to state the main theorem of the chapter.

Theorem 4.1. *Let k be a field of characteristic zero, containing all roots of unity. Let the k-variety X be a form of an admissible flag variety. Then either the automorphism group $\mathrm{Aut}_k(X)$ is bounded, or X is birational to a direct product variety $Y \times \mathbb{P}^1$, in other words X is ruled.*

Before moving further we recall the definition of Jordan groups.

Definition 4.4 (Definition 2.1 in [Po11])**.** A group G is called Jordan if there exists a constant $J \in \mathbb{N}$ such that for every finite subgroup $H \leqq G$ there exists an Abelian normal subgroup $A \leqq H$ such that $|H : A| < J$.

In [BZ15b] T. Bandman and Yu. G. Zarhin answered a question of Yu. Prokhorov and C. Shramov ([PS14]) by showing that the birational automorphism group of a conic bundle over a non-uniruled base is Jordan when it is not birational to the trivial $\mathbb{P}^1$-bundle over the non-uniruled base. One of the major steps in their proof was to show that the birational (and hence the biregular) automorphism group of a non-trivial Brauer-Severi curve is bounded. This follows from our theorem as a special case. (They also showed that the cardinalities of the finite subgroups of the automorphism group are bounded by four.)
The result on the boundedness of the automorphism groups of non-trivial Brauer-Severi curves was also used by Yu. Prokhorov and C. Shramov when they classified three dimensional varieties with non-Jordan birational automorphism groups ([PS18a]).

Another aspect of our motivation is that, we would like to investigate conditions which imply that the birational automorphism group of a rationally connected variety is bounded. We hope

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that by regularizing actions of finite subgroups of the birational automorphism groups (Lemma 3.2), and by the help of the Minimal Model Program, this question can be reduced to studying finite subgroups of the automorphism groups of Fano varieties over function fields. As a special case we investigated boundedness properties of automorphism groups of forms of flag varieties.

The definitions of the Jordan and the boundedness properties was introduced by V. L. Popov. They are closely related. Boundedness implies the Jordan property, while a typical strategy for proving that a group is Jordan is to show that the group sits in an exact sequence where the normal subgroup is Jordan and the quotient group is bounded ([Po11], Lemma 2.11). A survey of results concerning these properties of groups and the relations between them can be found in [Po14] and in Section 2 of [PS14].
Research about investigating Jordan properties for birational and biregular automorphism groups of varieties was initiated by J.-P. Serre in [Se09] and V. L. Popov in [Po11]. Recently many authors have contributed to the subject ([BZ15a], [BZ15b], [BZ19], [Hu18], [MZ15], [Po11], [Po14], [PS14], [PS16], [PS18a], [Se09], [Za15]).

The idea of our proof is the following. A form of a flag variety can be viewed as a flag variety equipped with a twisted Galois action. The automorphism group of the form embeds into the automorphism group of the flag variety, and its action commutes with the twisted Galois action. If the automorphism group of the form is not bounded, then the commutation imposes condition on the twisted Galois action. Using this, we may construct a Galois equivariant rational map from the flag variety to a smaller dimensional variety. It turns out that this rational map induces a vector bundle structure on the open subset of the flag variety where the map is defined and the twisted Galois action respects the vector bundle structure. By results of Galois descent, we descend the vector bundle structure to an open subvariety of the form. This proves our theorem.

We use the admissibility hypothesis to construct the *Galois equivarant* rational map from our flag variety to a smaller dimensional variety. Although the rational map can be constructed anyway, we use the admissibility condition when we endow the target space with a Galois action which makes the rational map equivaraint. For a more detailed discussion see Remark 4.8 and Remark 4.13.

In general, it is a very hard question to decide whether a variety is ruled or not. Amongst forms of (admissible) flag varieties we can find examples to both cases.
Indeed, flag varieties are rational, therefore they are ruled. On the other hand non-trivial Brauer-Severi curves and surfaces provide examples of non-ruled forms of admissible flag varieties. Non-trivial Brauer-Severi curves are non-ruled essentially as a consequence of their definition, while the case of non-trivial Brauer-Severi surfaces will be explored in Section 4.6. Here we only state

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the corresponding theorem.

Theorem 4.2. *Let k be a field of arbitrary characteristic. Let X be a Brauer-Severi surface over k. X is ruled if and only if it is trivial.*

The chapter is organized in the following way. In Section 4.2 we recall the necessary knowledge about automorphism groups of flag varieties and Galois descent. In Section 4.3 we construct the rational maps which will give us the vector bundle structure. It is followed by Section 4.4, where we analyze the effect of the commuting group actions when the automorphism group of the form of the flag variety is not bounded. Finally, Section 4.5 contains the proof of our theorem. We enclose our chapter with a discussion on Brauer-Severi surfaces in Section 4.6.

4.1.1 Conventions

To make the reading of the chapter easier, we collect our conventions here. Unless explicitly stated otherwise all fields are assumed to be of characteristic 0. For a field k we use $\overline{k}$ to denote its (fixed) algebraic closure.
By a vector space we mean a finite dimensional vector space. Sometimes in the notation of a vector space we make explicit the field over which the vector space is defined. When we say V_k is a vector space, we mean that V_k is a vector space defined over the field k.
Let V be a vector space over a field k. By $\mathrm{Lin}_k(V)$ we denote the k-linear automorphism group of V. (During the article we will encounter situations, where V is a vector space over a field l, where $k \leqq l$, however we need to consider its k-linear automorphism group.)
By a variety we mean a separated, integral scheme of finite type over a field.
Let X be an arbitrary scheme over a field k. By $\mathrm{Aut}_k(X)$ we denote the k-scheme automorphism group of X. (During the article we will encounter situations, where X is a variety over a field l, where $k \leqq l$, and we need to consider its k-scheme automorphism group.)
Let X be a variety, by $\mathrm{Bir}(X)$ we denote the birational automorphism group of X.

4.2 Preliminaries

4.2.1 Automorphism groups of flag varieties

In this section we collect results about automorphism groups of flag varieties. First, we recall the definition of the automorphism group scheme.

Definition 4.5. Let X be a scheme over a base scheme S. Consider the assignment $T \mapsto \mathrm{Aut}_T(X \times T)$ between S-schemes and abstract groups. It gives rise to a contravariant functor

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$A_X : (Sch/S)^{op} \to \underline{Gr}$ from the category of S-schemes to the category of groups. ($(Sch/S)^{op}$ denotes the opposite category of the category of S-schemes). If A_X can be represented by an S-scheme Y, then we call Y the automorphism group scheme of X, and denote it by $Aut_S(X)$ or simply by $Aut(X)$. (In case of $S = \operatorname{Spec} k$, for some field k, we also use the notation $Aut_k(X)$.)

Remark 4.1. Note that the definition implies that $Aut_T(X \times T) \cong Aut_S(X) \times T$ for any S-scheme T (by the adjoint property of restriction and extension of scalars).
It is also worth pointing out that an immediate consequence of the definition is the following. For a k-scheme X, if $Aut(X)$ exists, then the group of its k-rational points is isomorphic to the automorphism group of X, in formula $(Aut_k(X))(k) \cong \mathrm{Aut}_k(X)$.

The following theorem of H. Matsumura and F. Oort secures the existence of the automorphism group schemes for flag varieties (Theorem 3.7 in [MO67]).

Theorem 4.3. *Let k be a field of arbitrary characteristic, and let X be a proper k-scheme. The automorphism group scheme $Aut(X)$ exists and it is of locally finite type over k.*

Armed with the concept of automorphism group schemes, we can make our first step towards describing the automorphism groups of flag varieties.

Proposition 4.1. *Let k be a field, V be a k-vector space and $\mathrm{Fl}(\mathbf{d}, V)$ be a k-flag variety. The group scheme $PGL(V)$ is a closed subscheme of $Aut_k(\mathrm{Fl}(\mathbf{d}, V))$.*

Proof. Clearly the functor of points of the group scheme of the projective general linear group $\mathrm{Hom}(-, PGL(V))$ is a subfunctor of $A_{\mathrm{Fl}(\mathbf{d},V)}$ defined in Definition 4.5. Therefore we have a morphism of group schemes $\varphi : PGL(V) \to Aut_k(\mathrm{Fl}(\mathbf{d}, V))$.
The kernel of φ is trivial. Indeed, $\mathrm{PGL}(V \otimes l)$ embeds into $\mathrm{Aut}_l(\mathrm{Fl}(\mathbf{d}, V) \times \operatorname{Spec} l) \cong \mathrm{Aut}_l(\mathrm{Fl}(\mathbf{d}, V \otimes l))$ for any field extension $l|k$. Therefore the kernel has a unique rational point over any field. Since we work in characteristic 0, this implies that the kernel is trivial (by smoothness).
Since the kernel is trivial and φ is a smooth morphism (as the characteristic is 0), φ is a closed immersion (Lemma 38.7.8, [Stack]). □

Remark 4.2. Consider flag varieties of the form $\mathrm{Fl}(d_1 < ... < d_r, V)$, where $0 < d_1$, $d_r < \dim V$, $\dim V \geqq 3$ and $\forall i = 1, ..., r$ $d_i + d_{r+1-i} = \dim V$. (Notice that the conditions $0 < d_1$ and $d_r < \dim V$ are technical assumptions, they do not exclude any isomorphism class of flag varieties.)
In the next theorem we will show that non-admissible flag varieties are exactly flag varieties of the above form. In this remark we will construct an order two automorphism for them, called τ, which lies outside $\mathrm{PGL}(V)$ and normalizes it. This strengthens the previous proposition, since the existence of τ implies that in case of flag varieties of the above form $PGL(V) \rtimes \mathbb{Z}/2\mathbb{Z}$ is a closed subscheme of the automorphism group scheme.
The involution τ can be constructed in the following way. For an arbitrary flag variety $\mathrm{Fl}(\mathbf{e}, W)$

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(not necessarily of the form considered in the beginning of the remark) we can examine the dual map:

$$* : \mathrm{Fl}(e_1 < e_2 < ... < e_q, W) \to \mathrm{Fl}(m - e_q < m - e_{q-1} < ... < m - e_1, W^*)$$
$$U_1 < U_2 < ... < U_q \mapsto {U_q}^\perp < {U_{q-1}}^\perp < ... < {U_1}^\perp,$$

where $m = \dim W$, W^* is the dual space of W and for an arbitrary linear subspace $U \leqq W$ $U^\perp = \{\varphi \in W^* | \varphi|_U \equiv 0\}$ is the annihilator subspace.
Consider a flag variety $\mathrm{Fl}(\mathbf{d}, V)$ of the form introduced in the beginning of the remark, and fix a linear automorphism $j_0 : V^* \to V$ such that j_0^{-1} maps a (fixed) basis of V to its dual basis (V^* denotes the dual space of V). j_0 induces an isomorphism $j : \mathrm{Fl}(\mathbf{d}, V^*) \to \mathrm{Fl}(\mathbf{d}, V)$. With a little amount of work it can be checked that the automorphism $\tau = j \circ *$ is an involution outside the projective general linear group, and that τ normalizes the projective general linear group. (If $\dim V = 2$, then τ would be an element of the projective general linear group.)

Our next tool is the result of H. Tango (Theorem 2 in [Ta76]). By the use of Schubert calculus he gave a description of the automorphism groups of flag varieties over algebraically closed fields (of arbitrary characteristic).
Just as in the previous remark, when we state the next theorem we will use the technical assumption that a flag does not contain the trivial linear subspace and the whole vector space (i.e. $0 < d_1$ and $d_r < \dim V$).

Theorem 4.4. *Let k be a field, V be a k-vector space and let $\mathbf{d}$ denote a strictly increasing sequence of integers $d_1 < ... < d_r$, where $0 < d_1$ and $d_r < \dim V$. The automorphism group of the k-flag variety $\mathrm{Fl}(\mathbf{d}, V)$ is $\mathrm{PGL}(V)$ (with its natural action on the variety), except the case when $3 \leqq \dim V$ and $d_i + d_{r-i+1} = \dim V$ for all $i = 1, ..., r$. In this later case the automorphism group is $\mathrm{PGL}(V) \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Proof. When k is algebraically closed, this is Tango's theorem (Theorem 2 in[Ta76]).
As a first step towards describing the case when k is not algebraically closed, we prove a stronger version of the theorem. We prove that the automorphism group scheme has the form which naturally corresponds to the form of the automorphism group described by the theorem, i.e. it is $PGL(V)$ or a $PGL(V) \rtimes \mathbb{Z}/2\mathbb{Z}$ accordingly.
Observe that the automorphism group scheme of a complex flag variety has the desired form. Indeed, by Tango's theorem, the automorphism group of a complex flag variety is either $\mathrm{PGL}(V)$ or $\mathrm{PGL}(V) \rtimes \mathbb{Z}/2\mathbb{Z}$. Therefore the group of the closed points of the automorphism group scheme of a complex flag variety gives back the groups described by our theorem. Combining this fact with the result of Proposition 4.1, which states that $PGL(V)$ is a closed subscheme of the automorphism group scheme of a complex flag variety, we can conclude our claim for the automorphism group

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scheme of an arbitrary $\mathbb{C}$-flag variety.
As the next step, note that it is enough to show that the our claim for automorphism group schemes holds for flag varieties over $\mathbb{Q}$. Indeed, let $\mathrm{Fl}(\mathbf{d}, V_k)$ be an arbitrary flag variety over an arbitrary field k. By choosing a basis of V_k, we can find a $\mathbb{Q}$-vector space $W_\mathbb{Q}$ such that $W_\mathbb{Q} \otimes k \cong V_k$ and $\mathrm{Fl}(\mathbf{d}, V_k) \cong \mathrm{Fl}(\mathbf{d}, W_\mathbb{Q}) \times \mathrm{Spec}\, k$. Hence Remark 4.1 implies that $Aut_k(\mathrm{Fl}(\mathbf{d}, V_k)) \cong Aut_\mathbb{Q}(\mathrm{Fl}(\mathbf{d}, W_\mathbb{Q})) \times \mathrm{Spec}\, k$. The result follows, as $PGL(W_\mathbb{Q}) \times \mathrm{Spec}\, k \cong PGL(V_k)$.
Let $\mathrm{Fl}(\mathbf{d}, U_\mathbb{Q})$ be an arbitrary flag variety over $\mathbb{Q}$. Since base changing the ground field does not affect dimensions, the group schemes $PGL(U_\mathbb{Q})$ and $Aut_\mathbb{Q}(\mathrm{Fl}(\mathbf{d}, U_\mathbb{Q}))$ has the same dimension by the complex case. As $PGL(U_\mathbb{Q})$ is connected, and it is a closed group scheme of $Aut_\mathbb{Q}(\mathrm{Fl}(\mathbf{d}, U_\mathbb{Q}))$ (Proposition 4.1), we conclude that it is the identity component.
A similar logic applies to the number of connected components. Indeed, the number of connected components cannot decrease after base changing the ground field. Hence using the case of complex flag varieties and the result of Remark 4.2, our claim for the automorphism group schemes follows.
Now we can turn back our attention to the automorphism groups. We conclude our proof by taking rational points of the automorphism group schemes and using Remark 4.1. □

Remark 4.3. Notice that Theorem 4.4 gives a new characterization of admissible flag varieties. This new characterization only uses dimensions of the linear subspaces of a flag of the variety and the dimension of the under lying vector space.

Remark 4.4. The projective general linear group has a natural action on the set of d-dimensional linear subspaces of the underlying vector space (for every fixed d, where $0 \leqq d \leqq n$ and n is the dimension of the underlying vector space). These actions are compatible with the action of the projective general linear group on the flags of the vector space. Sometimes we use this observation without further notice. A similar statement holds for twisted Galois actions on admissible flag varieties (check Remark 4.8).
The automorphism group $\mathrm{PGL}(V) \rtimes \mathbb{Z}/2\mathbb{Z}$ of a non-admissible flag variety also has a natural action on the set of the union of d-dimensional and $(n-d)$-dimensional linear subspaces of the underlying vector space. However some group elements swap the dimensions. A similar kind of claim can be formulated for the twisted Galois actions on non-admissible flag varieties (check Remark 4.8).

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4.2.2 Galois descent

We collect results about Galois descent and fields in general. First we start with a couple of technical claims. The next lemma can be proved by standard techniques using the finiteness condition built in the definition of a variety.

Lemma 4.1. *Let k be a field. Let X and Y be $\overline{k}$-varieties and $\varphi : X \to Y$ be a morphism between them. There exists a finite Galois extension $l|k$ such that X, Y and φ are defined over l.*

More precisely, there exists X', Y' l-varieties and $\varphi' : X' \to Y'$ morphism between them such that $X' \times \operatorname{Spec} \overline{k} \cong X$, $Y' \times \operatorname{Spec} \overline{k} \cong Y$ and $\varphi' \times id \cong \varphi$.

Remark 4.5. Let the k-variety X be a form of a flag variety. By Lemma 4.1, we can find a finite Galois extension $l|k$ such that $X \times \operatorname{Spec} l \cong \mathrm{Fl}(\mathbf{d}, W_l)$. Indeed, applying the lemma to the isomorphism between $X \times \operatorname{Spec} \overline{k}$ and $\mathrm{Fl}(\mathbf{d}, V_{\overline{k}})$ proves the claim.

Remark 4.6. If the k-variety X is a form of a flag variety, then X is projective. Indeed if $X \times \operatorname{Spec} \overline{k}$ is projective, then the same holds for X as well (Proposition 14.55 in [GW10]).

Definition 4.6. Let k be a field, and let the k-variety X be a form of a flag variety. If $l|k$ is a field extension such that $X \times \operatorname{Spec} l \cong \mathrm{Fl}(\mathbf{d}, V_l)$, then we call l a splitting field for X. By Remark 4.5 $l|k$ can chosen to be a finite Galois extension.

Now we turn our attention to results about descents. This part of the section is mainly based on [Ja00].

Definition 4.7. Let $l|k$ be a Galois extension with Galois group Γ. We call a pair (X, T) a quasi-projective l-scheme equipped with a twisted Galois action, if X is a quasi-projective l-scheme and $T : \Gamma \to \mathrm{Aut}_k(X)$ is a group homomorphism satisfying the following commutative diagram (for every $\sigma \in \Gamma$):

$$\begin{tikzcd} X \arrow[r, "T(\sigma)"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} l \arrow[r, "S(\sigma)"] & \operatorname{Spec} l \end{tikzcd}$$

where $S(\sigma) : \operatorname{Spec} l \to \operatorname{Spec} l$ is the morphism of schemes induced by $\sigma^{-1} : l \to l$.
If no confusion can arise we denote the pair (X, T) simply by X. (Observe that $S(\sigma)$ is induced by σ^{-1} since there is an *antiequivalence* of categories between affine schemes and rings. Therefore using the inverse is necessary to define an action of the Galois group.)

The following theorem can be found in [Ja00] (Theorem 2.2.b).

Theorem 4.5. *Let $l|k$ be a finite Galois extension with Galois group Γ. There is an equivalence between the category of quasi-projective k-schemes and the category of quasi-projective l-schemes equipped with a twisted Γ-action. The equivalence functor is given by $X \mapsto X \times \operatorname{Spec} l$.*

Remark 4.7. Since the theorem is about equivalence of categories, it also says that Galois equivariant morphisms descend to morphisms of the underlying k-schemes.

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Definition 4.8. Let $l|k$ be a Galois extension with Galois group Γ and let V_l be an n-dimensional vector space over l. Let $b = (v_1, ..., v_n)$ be a basis of V_l. There is a twisted Galois action $A_b : \Gamma \to \mathrm{Lin}_k(V_l)$ defined by

$$A_b(\sigma) : V_l \to V_l$$
$$v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n \mapsto \sigma(\alpha_1) v_1 + \sigma(\alpha_2) v_2 + ... + \sigma(\alpha_n) v_n,$$

where the α_i's are coefficients from the field l $(i = 1, ..., n)$ and $\sigma \in \Gamma$ is an arbitrary element of the Galois group. For a flag variety $\mathrm{Fl}(\mathbf{d}, V_l)$ this induces a twisted Galois action, denoted by $B_b : \Gamma \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{d}, V_l))$.

$$\begin{tikzcd}
\mathrm{Fl}(\mathbf{d}, V_l) \arrow[r, "B_b(\sigma)"] \arrow[d] & \mathrm{Fl}(\mathbf{d}, V_l) \arrow[d] \\
\operatorname{Spec} l \arrow[r, "S(\sigma)"] & \operatorname{Spec} l
\end{tikzcd}$$

It might seem counterintuitive that the diagram contains $S(\sigma)$, which is induced by σ^{-1}. However after realizing that this means we pull back functions using σ^{-1} , we can also realize that it forces us to use σ when we want to 'push forward' scalars.
Notice that if T is an arbitrary twisted Galois action on $\mathrm{Fl}(\mathbf{d}, V_l)$, then for every $\sigma \in \Gamma$ the morphism $T(\sigma)B_b(\sigma)^{-1}$ is an element of the automorphism group of the flag variety, therefore $T(\sigma)$ can be written as $T(\sigma) = a_\sigma \circ B_b(\sigma)$ where $a_\sigma \in \mathrm{Aut}_l(\mathrm{Fl}(\mathbf{d}, V_l))$. (Of course a_σ also depends on the basis b, although we decided to omit it in the notation.)

Remark 4.8. Let $l|k$ be a Galois extension with Galois group Γ, V_l be an l-vector space and $\mathrm{Fl}(\mathbf{d}, V_l)$ be an l-flag variety with a twisted Galois action $T : \Gamma \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{d}, V_l))$. Choose a basis of V_l, denote it by b. We saw in the previous definition that $T(\sigma) = a_\sigma \circ B_b(\sigma)$.
Assume that the flag variety is admissible, then a_σ is an element of $\mathrm{PGL}(V_l)$, therefore it has a natural action on the set of linear subspaces of V_l. $B_b(\sigma)$ can also be endowed with a natural action on the set of linear subspaces of V_l (via $A_b(\sigma)$). This enables us to endow $T(\sigma)$ with a natural action on the set of linear subspaces of V_l . Moreover, this action is compatible with the action of $T(\sigma)$ on the flags. In formula $T(\sigma)(Z_1 < .. < Z_r) = (c_\sigma \circ A_b(\sigma))(Z_1) < ... < (c_\sigma \circ A_b(\sigma))(Z_r)$ for any flag $Z_1 < ... < Z_r \in \mathrm{Fl}(\mathbf{d}, V_l)$. Sometimes we use this observation without further notice.
If the flag variety is non-admissible then we can also formulate a similar claim. However if $a_\sigma \notin \mathrm{PGL}(V)$ then the d_i-dimensional linear subspace of the image flag $T(\sigma)(Z_1 < .. < Z_r)$ depends on the $(\dim V - d_i)$-dimensional linear subspace of the flag $Z_1 < ... < Z_r$. The existence of these 'dimension-swapping' morphisms can pose problems when we try to construct Galois equivariant morphisms from non-admissible flag varieties.

Remark 4.9. If $l|k$ is a finite Galois extension (such that $l \leqq \overline{k}$) with Galois group Γ and V_l is an l-vector space, then $\{\sigma \mapsto a_\sigma\}$ gives an element in the first group cohomology $\mathrm{H}^1(\Gamma, \mathrm{Aut}_l(\mathrm{Fl}(\mathbf{d}, V_l))$.

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The elements of the first group cohomology are in 1-to-1 correspondence with the forms of $\mathrm{Fl}(\mathbf{d}, V_l) \times \operatorname{Spec} \overline{k} \cong \mathrm{Fl}(\mathbf{d}, V_l \otimes \overline{k})$ split by l. For further informations on this, see Theorem 14.88 in[GW10]. Also Theorem 3.6 and Theorem 4.5 in [Ja00] give results of similar flavour in the case of Brauer-Severi varieties.

Theorem 4.6. *Let $l|k$ be a finite Galois extension with Galois group Γ. Let X, Y be quasi-projective l-schemes equipped with twisted Galois actions, and let $\phi : X \to Y$ be a Galois equivariant morphism of l-schemes such that the triple (X, Y, ϕ) forms a vector bundle. Moreover, let the Galois action respect the vector bundle structure (respect the addition and twist the multiplication by scalar operations). Then there exist X', Y' quasi-projective k-schemes and $\phi' : X' \to Y'$ morphism of k-schemes such that (X', Y', ϕ') forms a vector bundle and $X' \times \operatorname{Spec} l \cong X$, $Y' \times \operatorname{Spec} l \cong Y$, $\phi' \times id \cong \phi$.*

Proof. By Theorem 2.2.c in [Ja00], a locally free sheaf of finite rank $\mathcal{E}$ equipped with a Galois action compatible with the Galois action on the underlying quasi-projective l-scheme Y comes from a locally free sheaf (of the same rank) on the quasi-projective k-scheme Y', where $Y' \times \operatorname{Spec} l \cong Y$. Since there is a 1-to-1 canonical correspondence between finite rank vector bundles and locally free sheaves of finite rank, the result follows. □

4.3 Rational maps of flag varieties

Let k be a field and $\overline{k}$ be its algebraic closure. Let V be a vector space over $\overline{k}$. Assume $V = V_1 \oplus V_2$ is a direct sum decomposition, $\dim V = n$ and $\dim V_i = n_i$ $(i = 1, 2)$. Consider the strictly increasing sequences of non-negative integers $\mathbf{d} = (d_1, d_2, ..., d_p)$ and $\mathbf{e} = (e_1, e_2, ..., e_q)$, where $d_p \leqq n_1 < e_1$ and $e_q \leqq n$. We are going to investigate the rational maps

$$\begin{gathered} \phi_1 : \mathrm{Fl}(\mathbf{d}, V) \dashrightarrow \mathrm{Fl}(\mathbf{d}, V_1) \\ Z_1 < ... < Z_p \mapsto pr(Z_1) < ... < pr(Z_p) \end{gathered}$$

where $pr : V \to V_1$ is the projection along V_2, Z_i's $(i = 1, ..., p)$ are the vector spaces forming the flag $(\dim Z_i = d_i)$,

$$\begin{gathered} \phi_2 : \mathrm{Fl}(\mathbf{e}, V) \dashrightarrow \mathrm{Fl}(\mathbf{e} - n_1, V_2) \\ W_1 < ... < W_q \mapsto W_1 \cap V_2 < ... < W_q \cap V_2 \end{gathered}$$

where W_j's $(j = 1, ..., q)$ are the vector spaces forming the flag $(\dim W_j = e_j)$,

$$\begin{gathered} \psi : \mathrm{Fl}(\mathbf{d} < \mathbf{e}, V) \dashrightarrow \mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2) \\ Z_1 < ... < Z_p < W_1 < ... < W_q \mapsto (pr(Z_1) < ... < pr(Z_p), W_1 \cap V_2 < ... < W_q \cap V_2) \end{gathered}$$

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where Z_i's ($i = 1, ..., p$) and W_j's ($j = 1, ..., q$) are the vector spaces forming the flag ($\dim Z_i = d_i$, $\dim W_j = e_j$).
Clearly all of these are rational maps. ϕ_1 is defined on the open subvariety

$$U_1 = \{Z_1 < ... < Z_p \in \mathrm{Fl}(\mathbf{d}, V) | Z_p \cap V_2 = \{0\}\},$$

ϕ_2 is defined on the open subvariety

$$U_2 = \{W_1 < ... < W_q \in \mathrm{Fl}(\mathbf{e}, V) | W_1 \pitchfork V_2\} = \{W_1 < ... < W_q \in \mathrm{Fl}(\mathbf{e}, \mathbf{V}) | W_1 + V_2 = V\},$$

and ψ is defined on the open subvariety

$$U = \{Z_1 < ... < Z_p < W_1 < ... < W_q \in \mathrm{Fl}(\mathbf{d} < \mathbf{e}, V) | Z_p \cap V_2 = \{0\}, W_1 + V_2 = V\}.$$

We can check that U_1, U_2 and U are open subvarieties. Indeed, let α_p be the tautological vector bundle on the flag variety $\mathrm{Fl}(\mathbf{d}, V)$ corresponding to the d_p-dimensional linear subspaces of V, and let β_1 be the tautological vector bundle on the flag variety $\mathrm{Fl}(\mathbf{e}, V)$ corresponding to the e_1-dimensional linear subspaces of V. Let ρ be the global section of the hom-vector bundle $Hom_{\overline{k}}(\alpha_p, V/V_2)$ induced by the projection $V \to V/V_2$, and let τ be the global section of the hom-vector bundle $Hom_{\overline{k}}(\beta_1, V/V_2)$ induced by the projection $V \to V/V_2$. U_1 is the open locus where ρ has maximal rank, while U_2 is the open locus where τ has maximal rank. Combining the above arguments, we can also show that U is an open subvariety.

Proposition 4.2. *Using the notation introduced in this section, the following holds. The triples* $(U_1, \mathrm{Fl}(\mathbf{d}, V_1), \phi_1)$*,* $(U_2, \mathrm{Fl}(\mathbf{e} - n_1, V_2), \phi_2)$ *and* $(U, \mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2), \psi)$ *form vector bundles.*

Proof. To see this, first, consider the fiber of ϕ_1 over an arbitrary flag $S_1 < ... < S_p \in \mathrm{Fl}(\mathbf{d}, V_1)$. Notice that if $Z_1 < ... < Z_p$ is in the fiber, then it is uniquely determined by Z_p. Indeed *pr* induces and isomorphism between Z_p and S_p, so there is a unique linear subspace of Z_p which maps to S_i ($i = 1, ..., p$).
The d_p-dimensional linear subspaces of V which are mapped to S_p are parametrized by $\mathrm{Hom}(S_p, V_2)$. If $f \in \mathrm{Hom}(S_p, V_2)$, then the graph of f considered as a linear subspace of V determines Z_p. More precisely

$$Z_p = \{v + f(v) | v \in S_p\}. \tag{4.3.1}$$

On the other hand, Z_p gives an element in $\mathrm{Hom}(S_p, V_2)$ by the composition $pr' \circ t$, where $t : S_p \to Z_p$ is the inverse of the linear isomorphism between Z_p and S_p induced by pr and $pr' : V \to V_2$ is the projection along V_1. These two constructions are inverse to each other, which shows our claim on the fiber.
The argument can be globalized. It shows that U_1 is isomorphic to the total space of the vector

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bundle corresponding to the locally trivial sheaf of finite rank $Hom_{\mathcal{O}}(\gamma_p, V_2 \otimes \mathcal{O})$, where γ_p is the sheaf of sections of the tautological bundle of the flag variety $\mathrm{Fl}(\mathbf{d}, V_1)$ corresponding to the d_p-dimensional linear subspaces and $\mathcal{O}$ is the structure sheaf of $\mathrm{Fl}(\mathbf{d}, V_1)$.
A similar argument shows that U_2 is the total space of the vector bundle corresponding to the locally trivial sheaf of finite rank $Hom_{\mathcal{O}}(V_1 \otimes \mathcal{O}, (V_2 \otimes \mathcal{O})/\eta_1)$, where η_1 is the sheaf of sections of the tautological bundle of the flag variety $\mathrm{Fl}(\mathbf{e} - n_1, V_2)$ corresponding to the $e_1 - n_1$-dimensional linear subspaces and $\mathcal{O}$ is the structure sheaf of $\mathrm{Fl}(\mathbf{e}-n_1, V_2)$. (By the properties of η_1, $(V_2 \otimes \mathcal{O})/\eta_1$ is a locally trivial sheaf of finite rank).
Indeed, again, notice first that an element $W_1 < ... < W_q \in \mathrm{Fl}(\mathbf{e}, V)$, which is in the fiber over $T_1 < ... < T_q \in \mathrm{Fl}(\mathbf{e} - n_1, V_2)$, is uniquely determined by W_1. Since W_j should contain both W_1 and T_j, moreover $W_1 \cap T_j = W_1 \cap V_2 = T_1$, we have $W_j = W_1 + T_j$ by dimension counting $(j = 1, ..., q)$.
The e_1-dimensional linear subspaces $W_1 < V$, such that $W_1 \cap V_2 = T_1$, are parametrized by $\mathrm{Hom}(V_1, V_2/T_1)$. For $g \in \mathrm{Hom}(V_1, V_2/T_1)$ consider the linear subspace

$$W_1' = \{u(v) + g(v) \in V/T_1 | v \in V_1\} \tag{4.3.2}$$

of the quotient space V/T_1, where we use u to denote the quotient morphism $u : V \to V/T_1$. Finally, let

$$W_1 = u^{-1}(W_1'). \tag{4.3.3}$$

Conversely, assume W_1 is given. Identify V_1 , V_2/T_1 and W_1/T_1 with linear subspaces of V/T_1. Let $p_1 : V/T_1 \to V_1$ be the projection along V_2/T_1, and $p_2 : V/T_1 \to V_2/T_1$ be the projection along V_1. p_1 induces an isomorphism $q_1 : W_1/T_1 \to V_1$. Let $g \in \mathrm{Hom}(V_1, V_2/T_1)$ be $g = p_2 \circ q_1^{-1}$. These two constructions are inverse to each other. The argument globalizes. This proves our claim.
For ψ we can use similar constructions. The fiber over $(S_1 < ... < S_p, T_1 < ... < T_q)$ is parametrized by a linear subspace $E < \mathrm{Hom}(S_p, V_2) \times \mathrm{Hom}(V_1, V_2/T_1)$ for which the constructions, described in the previous paragraphs, yield linear subspaces Z_p and W_1 satisfying $Z_p < W_1$.
This condition is equivalent to $Z_p \leqq W_1$ by dimension counting, which in turn is equivalent to $Z_p + T_1 \leqq W_1$. Using the projection $u : V \to V/T_1$, our condition is $u(Z_p) \leqq u(W_1)$. By the construction of Z_p and W_1 from $(f, g) \in \mathrm{Hom}(S_p, V_2) \times \mathrm{Hom}(V_1, V_2/T_1)$, the condition is equivalent to

$$(u + u \circ f)(S_p) \leqq (u + g)(V_1).$$

Consider the identification $V/T_1 = V_1 \oplus V_2/T_1$.

$$\{(v, u \circ f(v)) \in V_1 \oplus V_2/T_1 | v \in S_p\} = (u + u \circ f)(S_p) \leqq \\ (u + g)(V_1) = \{(v, g(v)) \in V_1 \oplus V_2/T_1 | v \in V_1\}$$

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This is equivalent to $u \circ f = g \circ i$, where $i : S_p \to V_1$ is the inclusion map. Let F be the surjective map of linear spaces given by

$$\begin{gathered} F : \mathrm{Hom}(S_p, V_2) \times \mathrm{Hom}(V_1, V_2/T_1) \to \mathrm{Hom}(S_p, V_2/T_1) \\ (f, g) \mapsto u \circ f - g \circ i. \end{gathered}$$

Then $E = \operatorname{Ker} F$. Once again, this construction globalizes. $U \subset \mathrm{Fl}(\mathbf{d} < \mathbf{e}, V)$ is the total space of the vector bundle corresponding to a locally trivial sheaf of finite rank $\mathcal{E}$. ($\mathcal{E}$ is the kernel of a surjective morphism of locally trivial sheaves of finite rank, hence it is locally trivial of finite rank.) □

Remark 4.10. Let A_1, A_2 and A be the complements of the open subvarieties U_1, U_2 and U in the appropriate flag varieties and endow them with the reduced scheme structure. A short calculation shows that

$$A_1 = \{Z_1 < ... < Z_p \in \mathrm{Fl}(\mathbf{d}, V) | \dim(Z_p \cap V_2) > 0\},$$

$$A_2 = \{W_1 < ... < W_q \in \mathrm{Fl}(\mathbf{e}, V) | \dim(W_1 \cap V_2) > e_1 - n_1\},$$

$$A = \{Z_1 < ... < Z_p < W_1 < ... < W_q \in \mathrm{Fl}(\mathbf{d} < \mathbf{e}, V) | \\ \dim(Z_p \cap V_2) > 0 \text{ or } \dim(W_1 \cap V_2) > e_1 - n_1\}.$$

Hence A_1, A_2 and A are union of Schubert cells. Recall that an a-dimensional Schubert cell is isomorphic to the a-dimensional affine space $\mathbb{A}^a$. (For more details on Schubert cells the interested reader can consult with Chapter 10.2 in [Fu97].)

Remark 4.11. By Lemma 4.1 we can find a finite Galois extension $l|k$ (where $l \leqq \overline{k}$) such that ϕ_1, ϕ_2, ψ, U_1, U_2, U and A_1, A_2, A are defined over l. Moreover we can require that, the decompositions of A_1, A_2 and A into the union of Schubert cells exist over the field l. In particular, this implies that, the sets of l-rational points are dense in A_1, A_2 and A (as the same claim holds for the affine spaces).
Furthermore, since a vector bundle structure over a variety can be defined only using finitely many elements from the ground field, we can secure that Proposition 4.2 also holds over l.
During Section 4.5 we will work over a finite Galois extension $l|k$ and use the notation introduced in this section (more precisely its corresponding counterpart which is defined over the field l).

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4.4 Group actions on forms of flag varieties

We recall some theorems about birational automorphism groups. To start with, we recall the notion of strongly Jordan groups. It first appeared in [BZ15b].

Definition 4.9 (Definition 1.1 in[BZ15b])**.** A group G is called strongly Jordan if it is Jordan, and there exists a constant $r \in \mathbb{N}$ such that every finite Abelian subgroup $A \leqq G$ can be generated by r elements, in other words the rank of an arbitrary finite Abelian subgroup is smaller than r.

Theorem 4.7. *Let X be a variety. If X is either rationally connected or non-uniruled, then the birational automorphism group* $\mathrm{Bir}(X)$ *is strongly Jordan.*

Proof. If X is rationally connected then the birational automorphism group is Jordan by Theorem 1.8 of [PS16] and Theorem 1.1 of [Bi16]. If X is non-uniruled then the birational automorphism group is Jordan by Theorem 1.8 of [PS14].
Furthermore, Remark 6.9 of [PS14] and Theorem 1.1 of [Bi16] shows that the ranks of the finite Abelian subgroups of the birational automorphism group of an arbitrary variety is bounded by a constant depending only on the variety.
Putting together these results proves the theorem. ☐

Theorem 4.8. *Let X be a variety. Let $G \leqq \mathrm{Bir}(X)$ be an arbitrary subgroup of the birational automorphism group. Assume that G is not bounded. Then there exist elements of G of finite and arbitrary large order.*

Proof. Assume that there exists a constant $N \in \mathbb{N}$ such that if $g \in G$ is an element of finite order, then the order of g is smaller than N. We will show that this implies the boundedness of G.
By Proposition 6.2 of [PS14] for an arbitrary variety X (using the MRC-fibration) we can fix a rationally connected variety X_{rc} over some function field and a non-uniruled variety X_{nu} over the ground field such that an arbitrary finite subgroup $G_0 \leqq G(\leqq \mathrm{Bir}(X))$ is an extension of finite groups G_{rc} and G_{nu}, where $G_{rc} \leqq \mathrm{Bir}(X_{rc})$ and $G_{nu} \leqq \mathrm{Bir}(X_{nu})$. (Note that, if the MRC-fibration is trivial, then either the group G_{rc} or the group G_{nu} is trivial, which does not pose any problem in our argument.)
We know that $\mathrm{Bir}(X_{rc})$ and $\mathrm{Bir}(X_{nu})$ are strongly Jordan groups. Denote the corresponding Jordan constants by J_{rc} and J_{nu} respectively, and denote the constants bounding the ranks of finite Abelian subgroups by r_{rc} and r_{nu} respectively. Since X_{rc} and X_{nu} only depend on X, J_{rc}, J_{nu} and r_{rc}, r_{nu} only depend on X as well.
We will use the following easy observation. Let A be a finite Abelian group. Assume that A can be generated by r elements and the order of an arbitrary element $a \in A$ is smaller than N. Then the cardinality of A is smaller than r^N.
G_{rc} is isomorphic to a finite subgroup of G_0, therefore the order of an arbitrary element of G_{rc} is

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smaller than N. G_{rc} has an Abelian subgroup of rank at most r_{rc} and of index smaller than J_{rc}. Hence $|G_{rc}| < J_{rc} r_{rc}^N$.
G_{nu} is the homomorphic image of G_0, therefore the order of an arbitrary element of G_{nu} is smaller than N. G_{nu} has an Abelian subgroup of rank at most r_{nu} and of index smaller than J_{nu}. Hence $|G_{nu}| < J_{nu} r_{nu}^N$.
Therefore $|G_0| < J_{rc} J_{nu} (r_{rc} r_{nu})^N$. Since G_0 was an arbitrary finite subgroup of G, and all constants depend only on X, G is bounded. This contradiction finishes the proof. □

Definition 4.10. Let k be a field and let the k-variety X be a form of a flag variety. Let l be a splitting field for X such that $l|k$ is a Galois extension. Fix an isomorphism $\varphi : \mathrm{Fl}(\mathbf{d}, V) \to X \times \operatorname{Spec} l$. Let $T : \mathrm{Gal}(l|k) \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{d}, V))$ be the twisted Galois action defined by $T(\sigma) = \varphi^{-1} \circ (id \times S(\sigma)) \circ \varphi$, where σ is an arbitrary element of $\mathrm{Gal}(l|k)$ and $S(\sigma) : \operatorname{Spec} l \to \operatorname{Spec} l$ is induced by $\sigma^{-1} : l \to l$. We call T the Galois action corresponding to φ.

Let the variety X be a form of an admissible flag variety and assume that its automorphism group is not bounded. In the next lemma we will examine the effect of the commutation of the automorphism group of X (viewed as a subgroup of the automorphism group of the corresponding flag variety) and the corresponding twisted Galois action.

Lemma 4.2. *Let k be a field containing all roots of unity. Let the k-variety X be a form of an admissible flag variety. Let l be a splitting field for X such that $l|k$ is a Galois extension. Assume that the automorphism group $\mathrm{Aut}_k(X)$ is not bounded. Let $X \times \operatorname{Spec} l \cong \mathrm{Fl}(\mathbf{d}, V)$ (where V is an l-vector space), and let $T : \mathrm{Gal}(l|k) \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{d}, V))$ be the corresponding twisted Galois action. We can choose a basis b of V such that it splits as $b = b_1 \cup b_2$ ($b_1, b_2 \neq \emptyset$), giving rise to a direct sum decomposition $V = V_1 \oplus V_2$ such that $\forall \sigma \in \mathrm{Gal}(l|k)$: $T(\sigma) = a_\sigma \circ B_b(\sigma)$ (see Definition 4.8), where $a_\sigma \in \mathrm{Aut}_l(\mathrm{Fl}(\mathbf{d}, V)) = \mathrm{PGL}(V)$ respects this decomposition, i.e. an arbitrary lift $c_\sigma \in \mathrm{GL}(V)$ of a_σ is contained in $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) < \mathrm{GL}(V)$.*

Proof. The isomorphism $X \times \operatorname{Spec} l \cong \mathrm{Fl}(\mathbf{d}, V)$ induces an isomorphism $\mathrm{Aut}_l(X \times \operatorname{Spec} l) \cong \mathrm{Aut}_l(\mathrm{Fl}(\mathbf{d}, V)) = \mathrm{PGL}(V)$. Let $n = \dim V$, and fix a finite order element $g \in \mathrm{Aut}_k(X)$ with order larger than $n!$. It exists by the previous theorem. g can be viewed as an element in $\mathrm{PGL}(V)$ since $\mathrm{Aut}_k(X) \leqq \mathrm{Aut}_l(X \times \operatorname{Spec} l)$.
Let h be a fixed lift of g to $\mathrm{GL}(V)$ such that the order of h is equal to the order of g, it exists since k contains all roots of unity. Notice that h is of finite order, hence it is semisimple (since we are in characteristic 0). Let b be a basis of V consisting of eigenvectors of h. Again, this basis exists as k contains all roots of unity and h is of finite order (hence its eigenvalues are roots of unity).
Let $V \cong V_{\lambda_1} \oplus V_{\lambda_2} \oplus ... \oplus V_{\lambda_r}$ be the direct sum decomposition corresponding to the eigenspaces of $h^{n!}$. Since $g^{n!} \neq 1$, the linear transformation $h^{n!}$ cannot be a scalar multiply of the identity, therefore it has at least two distinct eigenspaces, i.e. $r \geqq 2$. Let $V_1 = V_{\lambda_1}$ and $V_2 = V_{\lambda_2} \oplus ... \oplus V_{\lambda_r}$.

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The basis b splits as $b_1 \cup b_2$, where b_i is a basis of V_i. Indeed, an eigenspace of $h^{n!}$ is a direct sum of the eigenspaces of h.
Moreover, since h is chosen to be of finite order: $h \circ A_b(\sigma) = A_b(\sigma) \circ h$, as k contains all roots of unity by assumption.
The action of g on $X \times \operatorname{Spec} l$ commutes with the natural Galois action. Indeed the action of g derives from a group action on X, while the natural Galois action derives form a group action on $\operatorname{Spec} l$. Using the isomorphism $\operatorname{Aut}_l(X \times \operatorname{Spec} l) \cong \operatorname{PGL}(V)$, this leads us to $g \circ (a_\sigma \circ B_b(\sigma)) = (a_\sigma \circ B_b(\sigma)) \circ g$. Since h and $A_b(\sigma)$ commutes, the same holds for g and $B_b(\sigma)$, hence $g \circ a_\sigma = a_\sigma \circ g \in \operatorname{PGL}(V)$.
Lift this equation to $\operatorname{GL}(V)$: $hc_\sigma = \nu_\sigma c_\sigma h$, where c_σ is an arbitrary lift of a_σ, and $\nu_\sigma \in l$ only depends on σ (as we keep h fixed throughout our argument).
Let $v_1, v_2, ..., v_n \in V$ be a basis consisting of eigenvectors of h, i.e. $hv_i = \mu_i v_i$ ($\mu_i \in l$; $i = 1, ..., n$). Consider the basis $c_\sigma v_1, c_\sigma v_2, ..., c_\sigma v_n$, it is also a basis consisting of eigenvectors of h. Indeed, $h(c_\sigma v_i) = \nu_\sigma c_\sigma h v_i = \nu_\sigma \mu_i (c_\sigma v_i)$. Since the eigenvalues of h are uniquely determined, multiplication with ν_σ must permute them. Therefore ν_σ is a root of unity, with order less than or equal to n ($\forall \sigma \in \operatorname{Gal}(l|k)$). Hence $h^{n!} c_\sigma = c_\sigma h^{n!}$. Therefore $c_\sigma \in \operatorname{GL}(V_1) \times \operatorname{GL}(V_2) < \operatorname{GL}(V)$ ($\forall \sigma \in \operatorname{Gal}(l|k)$). $\square$

Remark 4.12. A similar, but much more technical, statement can be formulated including the case of non-admissible flag varieties. Since we will not use it, we decided only to state the simpler version which applies to admissible flags.

4.5 Proof of the Main Theorem

In this section we prove Theorem 4.1. The strategy for the proof is the following. Instead of working with X, we will consider a flag variety equipped with a twisted Galois action. Using the splitting established in Lemma 4.2 and the constructions introduced in Section 4.3, we will build a Galois equivariant morphism from the flag variety to a lower dimensional variety, which is isomorphic to the Galois equivariant projection morphism of a vector bundle. Finally, by the use of Galois descent, we achieve the desired result.

If $\operatorname{Aut}_k(X)$ is bounded, then the claim of the main theorem (Theorem 4.1) holds, so in the followings we assume otherwise.

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4.5.1 Setup of the proof of the Main Theorem

Notation

Let k be a field of characteristic 0, containing all roots of unity. Let the k-variety X be a form of an admissible flag variety.
We will use the notations of ϕ_1, ϕ_2, ψ and U_1, U_2, U introduced in Section 4.3 (see also Remark 4.11).
Let l be a splitting field for X such that $l|k$ is a finite Galois extension (see Definition 4.6). Let $X \times \operatorname{Spec} l \cong \mathrm{Fl}(\mathbf{d_0}, V)$ (where V is an l-vector space), and let $T : \mathrm{Gal}(l|k) \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{d_0}, V))$ be the corresponding Galois action (see Definition 4.10). Let b be the basis of V established in Lemma 4.2, $b = b_1 \cup b_2$ and $V = V_1 \oplus V_2$ be the corresponding decompositions. By enlarging l if necessary, we can assume that ϕ_1, ϕ_2, ψ and U_1, U_2, U are defined over l (see Remark 4.11), Proposition 4.2 holds over l, moreover we can require that the sets of l-rational points in the complements of U_1, U_2 and U are dense (see Remark 4.11). Let

$$A = A_b : \mathrm{Gal}(l|k) \to \mathrm{Lin}_k(V),$$
$$B = B_b : \mathrm{Gal}(l|k) \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{d_0}, V))$$

be the corresponding twisted Galois actions (see Definition 4.8). Finally, let $n = \dim V$ and $n_i = \dim V_i$ $(i = 1, 2)$.
There are three different cases depending on the sequence $\mathbf{d_0} = (d_{0,1} < d_{0,2} < ... < d_{0,r})$ and on $\dim V_1 = n_1$. Case 1: $d_{0,r} \leqq n_1$, Case 2: $n_1 < d_{0,1}$ and Case 3: $d_{0,1} \leqq n_1 < d_{0,r}$. All of them should be handled similarly.

In the followings we will explicitly deal with Case 3. This contains all the necessary techniques and calculations involved in Case 1 and Case 2. At the end of each step we remark some of the necessary changes to deal with the other cases.

Construction of the Galois actions on the target spaces

Let's assume Case 3. We will investigate ψ, at the end of the section we will note the changes for the other two cases. Split $\mathbf{d_0}$ as $\mathbf{d} = (d_1 < ... < d_p)$ and $\mathbf{e} = (e_1 < ... < e_q)$, where $d_p \leqq n_1 < e_1$ and $\mathbf{d_0} = (d_1 < ... < d_p < e_1 ... < e_q)$. The basis b_1 and b_2 induce the following actions.

$$A_1 = A_{b_1} : \mathrm{Gal}(l|k) \to \mathrm{Lin}_k(V_1)$$
$$B_1 = B_{b_1} : \mathrm{Gal}(l|k) \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{d}, V_1))$$
$$A_2 = A_{b_2} : \mathrm{Gal}(l|k) \to \mathrm{Lin}_k(V_2)$$
$$B_2 = B_{b_2} : \mathrm{Gal}(l|k) \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{e} - n_1, V_2))$$

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By Lemma 4.2 $\forall\sigma \in \mathrm{Gal}(l|k)$: $T(\sigma) = a_\sigma \circ B_b(\sigma)$ ($a_\sigma \in \mathrm{PGL}(V)$), and an arbitrary lift of a_σ, denoted by c_σ, splits, i.e. $c_\sigma \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. For every $\sigma \in \mathrm{Gal}(l|k)$ fix a lift c_σ, and let $c_{\sigma,1} \in \mathrm{GL}(V_1)$ and $c_{\sigma,2} \in \mathrm{GL}(V_2)$ be its components. Let $a_{\sigma,1} \in \mathrm{PGL}(V_1)$ and $a_{\sigma,2} \in \mathrm{PGL}(V_2)$ be the images of $c_{\sigma,1}$ and $c_{\sigma,2}$ respectively. Since all steps in our construction was compatible with the decomposition $V = V_1 \oplus V_2$,

$$\begin{gathered}
Q_1 : \mathrm{Gal}(l|k) \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{d}, V_1)) \\
\sigma \mapsto a_{\sigma,1} \circ B_1(\sigma), \\
Q_2 : \mathrm{Gal}(l|k) \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{e} - n_1, V_2)) \\
\sigma \mapsto a_{\sigma,2} \circ B_2(\sigma)
\end{gathered}$$

define twisted Galois actions for $\mathrm{Fl}(\mathbf{d}, V_1)$ and $\mathrm{Fl}(\mathbf{e} - n_1, V_2)$ respectively. Putting them together

$$\begin{gathered}
Q : \mathrm{Gal}(l|k) \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2)) \\
\sigma \mapsto (a_{\sigma,1} \circ B_1(\sigma)) \times (a_{\sigma,2} \circ B_2(\sigma))
\end{gathered}$$

defines a twisted Galois action on $\mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2)$.
In Case 1 and Case 2 we do not need to introduce the notations $\mathbf{d}$ and $\mathbf{e}$. In Case 1 we need to consider a Q_1-like action on $\mathrm{Fl}(\mathbf{d_0}, V_1)$ (we denote it by R_1), while in Case 2 we need to consider a Q_2-like action on $\mathrm{Fl}(\mathbf{d_0} - n_1, V_2)$ (we denote it by R_2).

4.5.2 Steps of the proof of the Main Theorem

Galois equivariance of the rational maps ϕ_1, ϕ_2 and ψ

To show the Galois equivariance of ϕ_1, ϕ_2 and ψ we need to check two things, the invariance of the open subvariety where the rational maps are defined (U_1, U_2 and U, respectively), and the equivariance of the corresponding morphisms from the open subvarieties to the target spaces.

Lemma 4.3. *The open subvarieties U_1, U_2 and U are invariant under the Galois actions, i.e. $\forall\sigma \in \mathrm{Gal}(l|k)$ $T(\sigma)U_i = U_i$ $(i = 1, 2)$ and $T(\sigma)U = U$.*

Proof. We consider the case of U, the proof for the other two cases are almost verbatim.
Let $\sigma \in \mathrm{Gal}(l|k)$ be an arbitrary element of the Galois group. First notice that l-rational points of a flag variety can be identified with the flags of the underlying vector space.
We will show that an l-rational point (i.e. a flag) belongs to U if and only if it belongs to $T(\sigma)U$. Notice that the l-rational points of the open subvariety $U \subset \mathrm{Fl}(\mathbf{d} < \mathbf{e}, V)$ are given by

$$\{Z_1 < ... < Z_p < W_1 < ... < W_q | Z_p \cap V_2 = \{0\}, W_1 + V_2 = V\}.$$

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V_1, V_2 and V are invariant under the natural actions of c_σ and $A(\sigma)$ by construction. Hence

$$\begin{aligned} T(\sigma)(Z_1 < ... < Z_p < W_1 < ... < W_q) = \\ (c_\sigma \circ A(\sigma))(Z_1) < ... < (c_\sigma \circ A(\sigma))(Z_p) < \\ (c_\sigma \circ A(\sigma))(W_1) < ... < (c_\sigma \circ A(\sigma))(W_q) \end{aligned}$$

satisfies the defining equation of the l-rational points of U if and only if $Z_1 < ... < Z_p < W_1 < ... < W_q \in U$.
Consider the complements of U and $T(\sigma)U$ as topological subspaces in the underlying topological space of $\mathrm{Fl}(\mathbf{d} < \mathbf{e}, V)$. They are homeomorphic Zariski closed sets, moreover they contain exactly the same set of l-rational points. We have chosen the field l in such a way that the l-rational points in the complement of U form a dense set. Putting these together implies that complement of U and $T(\sigma)U$ are equal. Hence $T(\sigma)U = U$ as open subvarieties. □

Recall the definitions of the Galois actions R_1, R_2 and Q from Section 4.5.1.

Lemma 4.4.

1. *The morphism $\phi_1 : U_1 \to \mathrm{Fl}(\mathbf{d_0}, V_1)$ is equivariant for the twisted Galois actions T and R_1.*

2. *The morphism $\phi_2 : U_2 \to \mathrm{Fl}(\mathbf{d_0} - n_1, V_2)$ is equivariant for the twisted Galois actions T and R_2.*

3. *The morphism $\psi : U \to \mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2)$ is equivariant for the twisted Galois actions T and Q.*

Proof. First notice that l-flag varieties can be covered by affine spaces $\mathbb{A}_l^m$, where m is the appropriate dimension. Therefore the l-rational points form a dense set (as the same holds for $\mathbb{A}_l^m$). Hence, to show that two morphisms whose domains and target spaces are built up from open subvarieties of l-flag varieties are equal, it is enough to show that they are equal on l-rational points, which can be identified with flags. (Also note that checking Galois equivariance is equivalent to checking equality of morphisms.)
From now on we will deal with the case of ψ and note the necessary changes at the end of the proof for the other two cases. The twisted Galois action T on the l-rational points (i.e. on the flags) is given by the formula

$$\begin{aligned} T(\sigma)(Z_1 < ... < Z_p < W_1 < ... < W_q) = \\ (c_\sigma \circ A(\sigma))(Z_1) < ... < (c_\sigma \circ A(\sigma))(Z_p) < \\ (c_\sigma \circ A(\sigma))(W_1) < ... < (c_\sigma \circ A(\sigma))(W_q) \end{aligned}$$

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where $Z_1 < ... < Z_p < W_1 < ... < W_q$ is an arbitrary flag of the open subvariety U and $\sigma \in \mathrm{Gal}(l|k)$ is an arbitrary element of the Galois group. While the twisted Galois action Q is given by the formula

$$\begin{aligned} Q(\sigma)(S_1 < ... < S_p, T_1 < ... < T_q) = \\ ((c_{\sigma,1} \circ A_1(\sigma))(S_1) < ... < (c_{\sigma,1} \circ A_1(\sigma))(S_p), \\ (c_{\sigma,2} \circ A_2(\sigma))(T_1) < ... < (c_{\sigma,2} \circ A_2(\sigma))(T_q)) \end{aligned}$$

where $(S_1 < ... < S_p, T_1 < ... < T_q)$ is an arbitrary l-rational point of the product variety $\mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2)$ and $\sigma \in \mathrm{Gal}(l|k)$ is an arbitrary element of the Galois group. Comparing these equations with the definition of ψ shows that for verifying the Galois equivariance of ψ it is enough to check that the followings hold.

$$\begin{aligned} pr \circ (c_\sigma \circ A(\sigma))(Z) = (c_{\sigma,1} \circ A_1(\sigma)) \circ pr(Z) \\ (c_\sigma \circ A(\sigma))(W) \cap V_2 = (c_{\sigma,2} \circ A_2(\sigma))(W \cap V_2), \end{aligned}$$

where Z and W are arbitrary linear subspaces of V and $pr : V \to V_1$ is the projection along V_2. For the first equation, consider an arbitrary vector $v \in V$. It can be written as $v = v_1 + v_2$ where $v_i \in V_i$ $(i = 1, 2)$.

$$(pr \circ c_\sigma \circ A(\sigma))(v) = (c_{\sigma,1} \circ A_1(\sigma))(v_1) = (c_{\sigma,1} \circ A_1(\sigma) \circ pr)(v)$$

Hence the first equation is satisfied. For the second one, let $W \leqq V$ be an arbitrary linear subspace.

$$\begin{aligned} (c_\sigma \circ A(\sigma))(W) \cap V_2 = (c_\sigma \circ A(\sigma))(W) \cap (c_\sigma \circ A(\sigma))(V_2) = \\ (c_\sigma \circ A(\sigma))(W \cap V_2) = (c_{\sigma,2} \circ A_2(\sigma))(W \cap V_2), \end{aligned}$$

where we used that V_2 is invariant under $c_\sigma \circ A(\sigma)$ and that $c_\sigma \circ A(\sigma)$ is a bijection from V to V. Hence the second equation is satisfied too, which shows that ψ is Galois equivariant.
For the case of ϕ_1 we need to perform the steps corresponding to the $Z_1 < ... < Z_p$-part of the above argument, meanwhile for the case of ϕ_2 we need to perform the steps corresponding to the $W_1 < ... < W_q$-part. □

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Galois equivariance of the vector bundle structure

Lemma 4.5. *The vector bundles* $(U_1, \mathrm{Fl}(\mathbf{d_0}, V_1), \phi_1)$, $(U_2, \mathrm{Fl}(\mathbf{d_0} - n_1, V_2), \phi_2)$ *and* $(U, \mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2), \psi)$ *are Galois equivariant. In other words, the Galois actions respect the addition and twist (by the corresponding element of the Galois group) the multiplication by scalar operations.*

Proof. Again using the fact that l-rational points of l-flag varieties form a dense set, it is enough to check that the vector bundle structure is respected on the l-rational points, i.e. on the flags. As usual we assume the case of $(U, \mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2), \psi)$ and note the necessary changes for the other two cases at the end of the proof.
Let $Z_1 < ... < Z_p < W_1 < ... < W_q \in U$ be an arbitrary flag lying over $(S_1 < ... < S_p, T_1 < ... < T_q) \in \mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2)$. As we have seen before, its image under $T(\sigma)$ ($\sigma \in \mathrm{Gal}(l|k)$) is the flag

$$(c_\sigma \circ A(\sigma))(Z_1) < ... < (c_\sigma \circ A(\sigma))(Z_p) < (c_\sigma \circ A(\sigma))(W_1) < ... < (c_\sigma \circ A(\sigma))(W_q) \in U$$

which lies over

$$\begin{aligned}((c_{\sigma,1} \circ A_1(\sigma))(S_1) < ... < (c_{\sigma,1} \circ A_1(\sigma))(S_p),& \\ (c_{\sigma,2} \circ A_2(\sigma))(T_1) < ... < (c_{\sigma,2} \circ A_2(\sigma))(T_q)) \in \mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2).&\end{aligned}$$

Using these formulas and the equations (4.3.1), (4.3.2) and (4.3.3) which construct the flag $Z_1 < ... < Z_p < W_1 < ... < W_q \in U$ from $(f, g) \in E < \mathrm{Hom}(S_p, V_2) \times \mathrm{Hom}(V_1, V_2/T_1)$ (and the corresponding equations for the image of the flag), we can see that the image of the flag corresponds to

$$\begin{gathered}(c_{\sigma,2} \circ A_2(\sigma)) \circ f \circ ((c_{\sigma,1} \circ A_1(\sigma))^{-1} \in \mathrm{Hom}((c_{\sigma,1} \circ A_1(\sigma))(S_p), V_2) \\ \overline{(c_{\sigma,2} \circ A_2(\sigma))} \circ g \circ ((c_{\sigma,1} \circ A_1(\sigma))^{-1} \in \mathrm{Hom}(V_1, V_2/(c_{\sigma,2} \circ A_2(\sigma))(T_1)),\end{gathered}$$

where

$$\overline{(c_{\sigma,2} \circ A_2(\sigma))} : V_2/T_1 \to V_2/(c_{\sigma,2} \circ A_2(\sigma))(T_1)$$

is the σ-linear homomorphism induced by $c_{\sigma,2} \circ A_2(\sigma) : V_2 \to V_2$. Therefore we have a Galois action on the vector bundle structure which respects the addition and twists the multiplication by scalar operations. (Observe that the formula of the action does not depend on the choice of the lift c_σ as both $c_{\sigma,1}$ and $c_{\sigma,2}$ derives from the same lift.)
Again for the other two cases we only need to carry out half of the proof. For the case of $(U_1, \mathrm{Fl}(\mathbf{d_0}, V_1), \phi_1)$ we need the part which corresponds to $Z_1 < ... < Z_p$ and f, for the case of $(U_2, \mathrm{Fl}(\mathbf{d_0} - n_1, V_2), \phi_2)$ we need the other half which corresponds to $W_1 < ... < W_q$ and g. □

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Galois descent

We can finish our proof by the help of Galois descent.

Proof of Theorem 4.1. For Case 3 we can summarize the results of the previous lemmas in the following way. We constructed a Galois equivariant commutative diagram of l-varieties, where ψ is a Galois equivariant projection of a vector bundle structure.

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "\psi"] & \mathrm{Fl}(\mathbf{d} < \mathbf{e}, V) \arrow[dl, dashed] \\ \mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2) & \end{tikzcd}$$

Generally (including all three cases) we can say that, there is a Galois equivariant commutative diagram of l-varieties

$$\begin{tikzcd} W \arrow[r, hook, "i"] \arrow[d, "\pi"] & \mathrm{Fl}(\mathbf{d_0}, V) \arrow[dl, dashed] \\ Y & \end{tikzcd}$$

where W is an open subvariety of $\mathrm{Fl}(\mathbf{d_0}, V)$ and (W, Y, π) is a Galois equivariant vector bundle (Lemma 4.5).

Finally, we can use results of Galois descent (Theorem 4.5 and Theorem 4.6) to achieve a commutative diagram over k with the same properties.

$$\begin{tikzcd} W^* \arrow[r, hook] \arrow[d, "\pi^*"] & X \arrow[dl, dashed] \\ Y^* & \end{tikzcd}$$

W^* is an open subvariety of X, therefore they are birational. Since W^* is a vector bundle over Y^*, W^* is birational to $\mathbb{P}^m \times Y^*$ for some $m > 0$. Putting these together shows that X is birational to $\mathbb{P}^m \times Y^*$. □

Remark 4.13. Now we can reflect on the role of the admissibility condition. In the proof above we showed that we can endow $\mathrm{Fl}(\mathbf{d}, V_1)$, $\mathrm{Fl}(\mathbf{e} - n_1, V_2)$ and $\mathrm{Fl}(\mathbf{d}, V_1) \times \mathrm{Fl}(\mathbf{e} - n_1, V_2)$ with twisted Galois actions which makes the morphism ϕ_1, ϕ_2 and ψ Galois equivariant. If $\mathrm{Fl}(\mathbf{d}, V)$ is non-admissible then $\mathrm{Fl}(\mathbf{d}, V_1)$ and $\mathrm{Fl}(\mathbf{e} - n_1, V_2)$ must be admissible. In this case there exists a pair (T, σ) where $T : \mathrm{Gal}(l|k) \to \mathrm{Aut}_k(\mathrm{Fl}(\mathbf{d}, V))$ is a twisted Galois action and $\sigma \in \mathrm{Gal}(l|k)$ is such that $T(\sigma) = a_\sigma \circ B_b(\sigma)$ is 'dimension-swapping', i.e $a_\sigma \notin \mathrm{PGL}(V)$ (see Remark 4.8). Because of this dimension-swap we cannot construct Galois actions on the target spaces which makes the morphisms ϕ_1, ϕ_2 and ψ Galois equivariant.

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4.6 Brauer-Severi surfaces

We analyze the question of non-ruledness of non-trivial Brauer-Severi surfaces. We need to introduce a couple of new concepts. We will not explain them in full detail, the interested reader is referred to [GS06], [Ja00] and [Ko16] for further information on the subject. During this section we can relax the condition that the ground field is of characteristic zero.

Definition 4.11. Let k be a field, and let X and Y be Brauer-Severi varieties over k with dimensions n and m respectively. Let $\varphi : X \dashrightarrow Y$ be a rational map. φ is called twisted linear if it is linear over $\overline{k}$, i.e. the composite map $\mathbb{P}^n \cong X \times \operatorname{Spec} \overline{k} \dashrightarrow Y \times \operatorname{Spec} \overline{k} \cong \mathbb{P}^m$ is linear.
We call the pair (X, φ) a twisted linear subvariety of Y if the composite map is induced by a linear injection. (Notice that in the case of a twisted linear subvariety φ can be extended to a morphism of varieties.) If no confusion can arise we denote the pair (X, φ) simply by X.
We call X a minimal twisted linear subvariety of Y, if it is a twisted linear subvariety and has minimal dimension amongst the twisted linear subvarieties. By Theorem 28 in [Ko16] the isomorphism class of a minimal twisted linear subvariety is well defined.
We call the Brauer-Severi variety Y minimal if the only twisted linear subvariety of Y is itself (up to isomorphism).
For an arbitrary Brauer-Severi variety P we will denote a fixed minimal twisted linear subvariety by $P^{\min}$.

Lemma 4.6. *Let k be a field and let X be a Brauer-Severi curve or a Brauer-Severi surface over k. X is non-trivial if and only if X is minimal.*

Proof. If X is a non-minimal Brauer-Severi curve then X has a 0-dimensional twisted linear subvariety, i.e. X has a k-rational point. Then by Châtelet's theorem X is trivial (Theorem 5.1.3 in [GS06]).
If X is a non-minimal Brauer-Severi surface then X either has a k-rational point or a one codimensional twisted linear subvariety. In both cases X is trivial by versions of Châtelet's theorem. The other directions are trivial. □

Definition 4.12 (Definition-Lemma 31 in [Ko16])**.** We call two Brauer-Severi varieties X and Y similar or Brauer equivalent if $X^{\min} \cong Y^{\min}$.

Remark 4.14. There is a canonical correspondence between central simple algebras and Brauer-Severi varieties over a given field k (Theorem 5.1 in[Ja00]). We can also introduce the Brauer equivalence relation on the central simple algebras in a natural way. The canonical correspondence between central simple algebras and Brauer-Severi varieties respects these equivalence relations.
Furthermore we can endow the central simple algebras with operations (tensor product and taking the opposite algebra), which respect the Brauer equivalence relation and turn the equivalence

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classes into a commutative group, called the Brauer group (Chapter 2.4 in [GS06]).
A similar construction can be carried out purely geometrically.

Theorem 4.9. *We can introduce operations on Brauer-Severi varieties which turns the Brauer equivalence classes into a commutative group which is naturally isomorphic to the Brauer group. (For further details see Section 4 and 5 of [Ko16].)*

Remark 4.15. If X and Y are Brauer-Severi varieties we will use the notation $X \otimes Y$ for the binary operation introduced in Theorem 4.9. We will use the notation $X^{\otimes m}$ to denote the m-fold 'product' of X with itself ($m \in \mathbb{Z}^+$).

Theorem 4.10 (Amitsur's theorem, Proposition 45 in [Ko16])**.** *Let X and Q be Brauer-Severi varieties. The following two conditions are equivalent: Q is similar to $X^{\otimes m}$ for some positive integer m; there is a rational map $\varphi : X \dashrightarrow Q$.*

Definition 4.13. Let k be a field and X be a projective k-variety. The index of X is the greatest common divisor of the degrees of all 0-cycles on X. It is denoted by $\operatorname{ind}(X)$.

Lemma 4.7 (Lemma 51 in [Ko16])**.** *Let X be a Brauer-Severi variety and m be a positive integer, then the index of $X^{\otimes m}$ divides the index of X.*

Theorem 4.11 (Theorem 53 in [Ko16])**.** *Let X be a Brauer-Severi variety. Then* $\operatorname{ind}(X) = \operatorname{ind}(X^{\min}) = \dim X^{\min} + 1$.

Lemma 4.8. *Let k be a field and X be a Brauer-Severi surface over k. If X is ruled then either X is trivial or there exists a rational map $\varphi : X \dashrightarrow Q$, where Q is a non-trivial Brauer-Severi curve.*

Proof. If X is ruled then it is birational to $\mathbb{P}^1_k \times Q$, where Q is a smooth projective curve. Notice that if Q is birational to $\mathbb{P}^1_k$, then X has k-rational points, therefore X is trivial by Châtelet's theorem. So we can assume that Q is not isomorphic to the projective line.
Denote $Q \times \operatorname{Spec} \overline{k}$ by $Q_{\overline{k}}$. Since X is a Brauer-Severi surface, $\mathbb{P}^1_{\overline{k}} \times Q_{\overline{k}}$ is rational.
Therefore we can take a general rational curve $c : \mathbb{P}^1_{\overline{k}} \to \mathbb{P}^1_{\overline{k}} \times Q_{\overline{k}}$ (i.e. we can take a general morphism of the projective line to $\mathbb{P}^1_{\overline{k}} \times Q_{\overline{k}}$) and we can compose c with the canonical projection $\mathbb{P}^1_{\overline{k}} \to \mathbb{P}^1_{\overline{k}} \times Q_{\overline{k}} \to Q_{\overline{k}}$. Since c is general, the composite is dominant (i.e. the rational curve does not lie in a fiber over $Q_{\overline{k}}$). Hence we get a non-trivial morphism from a projective line to the smooth curve $Q_{\overline{k}}$. This implies that $Q_{\overline{k}}$ is isomorphic to the projective line, i.e. Q is a Brauer-Severi curve. As Q is not isomorphic to the projective line, Q is non-trivial.
The composite $X \dashrightarrow \mathbb{P}^1_k \times Q \to Q$, where the first map is the birational isomorphism giving the ruledness and the second is the canonical projection, gives a rational map $X \dashrightarrow Q$ from X to a non-trivial Brauer-Severi curve. □

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Proof of Theorem 4.2. If the Brauer-Severi surface X is trivial, then it is ruled.
Assume that X is non-trivial and ruled. By the previous lemma there is a rational map $\varphi : X \dashrightarrow Q$, where Q is a non-trivial Brauer-Severi curve.
By Amitsur's theorem (Theorem 4.10) Q is similar to $X^{\otimes m}$ for some positive integer m. Hence $Q^{\min} \cong (X^{\otimes m})^{\min}$ by the definition of similarity.
We can consider indices:

$$2 = \dim Q + 1 = \dim Q^{\min} + 1 = \operatorname{ind}(Q^{\min}) = \operatorname{ind}((X^{\otimes m})^{\min}) = \operatorname{ind}(X^{\otimes m}),$$

by Lemma 4.6 and by Theorem 4.11. On the other hand $\operatorname{ind}(X^{\otimes m})$ divides $\operatorname{ind}(X)$ by Lemma 4.7, and

$$\operatorname{ind}(X) = \dim X^{\min} + 1 = \dim X + 1 = 3,$$

by Theorem 4.11 and Lemma 4.6. Since 2 does not divide 3, we arrived to a contradiction. Hence a non-trivial Brauer-Severi surface cannot be ruled. This finishes the proof. □

Remark 4.16. Brauer-Severi surfaces corresponds canonically to degree three central simple algebras (Theorem 5.1 in[Ja00]). (The degree of a central simple algebra is the square root of its dimension, it is a positive integer.) By Wedderburn's theorem degree three central simple algebras are cyclic algebras (Chapter 15.6 in [Pi82]). Moreover, if k is a field of characteristic zero containing all roots of unity, then cyclic algebras over k are given by the following presentation:

$$k < x_1, x_2 | x_1^m = a, x_2^m = b, x_1 x_2 = \omega x_2 x_1 >,$$

where $m \in \mathbb{Z}^+$, $a, b \in k^*$ and ω is a primitive m-th root of unity (Corollary 2.5.5 in [GS06]).
We call a central simple algebra over k split if it is isomorphic to a matrix ring over k. It is equivalent with the corresponding Brauer-Severi variety being trivial. A cyclic algebra of the above presentation (where k is a field of characteristic zero containing all roots of unity) is split if and only if b is a norm from the field extension $k(\sqrt[m]{a})|k$ (Corollary 4.7.7 in [GS06]).
Putting these together, one can show that

$$\mathbb{C}(t_1, t_2) < x_1, x_2 | x_1^3 = t_1, x_2^3 = t_2, x_1 x_2 = e^{2\pi i/3} x_2 x_1 >$$

corresponds to a non-trivial Brauer-Severi surface over a field of characteristic zero containing all roots of unity.

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Bibliography

[BZ15a] T. Bandman, Y. G. Zarhin. *Jordan groups and algebraic surfaces.* Transform. Groups **20** (2015), no. 2, 327–334.

[BZ15b] T. Bandman, Y. G. Zarhin. *Jordan groups, conic bundles and abelian varieties.* ArXiv e-print (2015), 1512.01744.

[BZ19] T. Bandman, Y. G. Zarhin. *Jordan properties of automorphism groups of certain open algebraic varieties.* Transformation Groups **24** (2019), no. 3, 721–739.

[BT05] F. Bogomolov, Yu. Tschinkel *Rational curves and points on K3 surfaces.* Am. J. of Math., **127** (2005), no.4, 825–835.

[Bi12] C. Birkar. *Lectures on birational geometry.* ArXiv e-print (2012), 1210.2670.

[Bi16] C. Birkar. *Singularities of linear systems and boundedness of Fano varieties.* ArXiv e-print (2016), 1609.05543.

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon, J. McKernan. *Existence of minimal models for varieties of log general type.* J. Am. Math. Soc., **23** (2010), no. 2, 405–468.

[BDPP13] S. Boucksom, J.-P. Demailly, M. Paun , T. Peternell 2004. *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension.* J. Agebraic Geom. **22** (2013), no. 2, 201–248.

[BF66] R. Brauer, W. Feit. *An analogue of Jordan's theorem in characteritic p.* Annals of Math., **84** (1966), no.2, 119–131.

[Br11] E. Breuillard. *An exposition of Jordan's original proof of his theorem on finite subgroups of* $\mathrm{GL}_n(\mathbb{C})$. e-print (2011), http://www.math.u-psud.fr/ breuilla/Jordan.pdf

[CPS14] B. Csikós, L. Pyber, E. Szabó. *Diffeomorphism groups of compact 4-manifolds are not always Jordan.* ArXiv e-print (2014), 1411.7524.

[CR62] C. W. Curtis, I. Reiner. *Representation Theory of Finite Groups and Associative Algebras.* Pure and Applied Mathematics, Vol. 11, Interscience Publishers, Wiley & Sons, New York-London, 1962.

CEU eTD Collection

[De01] O. Debarre. *Higher Dimensional Algebraic Geometry.* Universitext. Springer-Verlag, New York, 2001.

[De11] O. Debarre. *Rational curves on algebraic varieties.* e-print (2011), https://www.math.ens.fr/ debarre/NotesGAEL.pdf

[DI09] I. V. Dolgachev, V. A. Iskovskikh (2009) *Finite Subgroups of the Plane Cremona Group.* In: *Algebra, Arithmetic, and Geometry.* Progress in Mathematics, Vol. 269, Birkhäuser Boston, 2009, 443–548.

[Fr911] G. Frobenius. *Uber den von L. Bieberbach gefundenen Beweis eines Satzes von C. Jordan* Sitzber. Preuss. Akad. Wiss. (1911), 241–248.

[Fu11] O. Fujino. *Non-vanishing theorem for log canonical pairs.* J. Algebraic Geom., **20** (2011), no. 4, 771–783.

[Fu97] W. Fulton. *Young Tableaux: with Applications to Representation Theory and Geometry.* Cambridge University Press, London, 1997.

[Gh97] É. Ghys. The following talks: *Groups of diffeomorphisms.* Colóquio brasileiro de matemáticas, Rio de Janeiro (Brasil), July 1997.
The structure of groups acting on manifolds. Annual meeting of the Royal Mathematical Society, Southampton (UK), March 1999.
Some open problems concerning group actions. Groups acting on low dimensional manifolds, Les Diablerets (Switzerland), March 2002.
Some open problems in foliation theory. Foliations 2006, Tokyo (Japan), September 2006.

[Gh15] É. Ghys. Talk: *My favourite groups.* IMPA, Rio de Janeiro (Brasil), April 2015. 9:29 at the video of https://www.youtube.com/watch?v=OdUMl3rIGAk

[Gu18] A. Guld. *Boundedness properties of automorphism groups of forms of flag varieties.* (to appear in Transform. Groups) ArXiv e-print (2018), 1806.05400.

[Gu19] A. Guld. *Finite subgroups of the birational autormophism group are 'almost' nilpotent.* ArXiv e-print (2019), 1903.07206.

[GHS03] T. Graber, J. Harris, J. Starr. *Families of rationally connected varieties.* J. Am. Math. Soc., **16** (2003), no. 1, 57–67.

[GS06] P. Gille, T. Szamuely. *Central Simple Algebras and Galois Cohomology.* Cambridge Studies in Advanced Mathematics, Vol. 101, Cambridge University Press, Cambridge, 2006.

[GW10] U. Görtz, T. Wedhorn. *Algebraic Geometry I.* Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010.

CEU eTD Collection

[HM07] C. D. Hacon, J. McKernan. *On Shokurov's rational connectedness conjecture.* Duke Math. J., **138** (2007), no. 1, 119–136.

[Ha77] R. Hartshorne.*Algebraic Geometry.*, Springer-Verlag, 1977.

[Hu18] F. Hu. *Jordan property for algebraic groups and automorphism groups of projective varieties in arbitrary characteristic.* ArXiv e-print (2018), 1804.10946.

[Ja00] J. Jahnel. *The Brauer-Severi variety associated with a central simple algebra.* Linear Algebraic Groups and Related Structures **52** (2000), 1-60.

[Jo878] C. Jordan. *Mémoire sur les equations différentielle linéaire à intégrale algébrique.* J. Reine Angew. Math. **84** (1878), 89–215.

[Ke19] S. Kebekus. *Boundedness results for singular Fano varieties and applications to Cremona groups.* ArXiv e-print (2019), 1812.04506.

[Ko16] J. Kollár. *Severi-Brauer varieties; a geometric treatment.* ArXiv e-print (2016), 1606.04368.

[Ko96] J. Kollár. *Rational Curves on Algebraic Varieties.* Egrebnisse der Math., Vol. 32, Springer Verlag, Berlin 1995.

[KM98] J. Kollár, S. Mori. *Birational Geometry of Algebraic Varieties.* Cambridge Tracts in Mathematics, Vol. 134, Cambridge University Press, Cambridge 1998.

[Le17] B. Lehmann. *A snapshot of the Minimal Model Program.* Surveys on Recent Developments in Algebraic Geometry **95** (2017), no. 1, 1–32.

[MO67] H. Matsumura, F. Oort. *Representability of group functors, and automorphisms of algebraic schemes.* Inventiones Math. **4** (1967), no. 1, 1–25.

[MR10] I. Mundet i Riera. *Jordan's theorem for the diffeomorphism group of some manifolds.* Proc. Amer. Math. Soc., **138** (2010), no. 6, 2253–2262.

[MR16] I. Mundet i Riera. *Finite group actions on 4-manifolds with nonzero Euler characteristic.* Mathematische Zeitschrift, **282** (2016), 25–42.

[MR18] I. Mundet i Riera. *Finite group actions on homology spheres and manifolds with nonzero Euler characteristic.* ArXiv e-print (2018), 1403.0383.

[MRSC19] I. Mundet i Riera, C. Saéz-Calvo. *Which finite groups acts smoothly on a given 4-manifold?* ArXiv e-print (2019), 1901.04223.

[MRT15] I. Mundet i Riera, A. Turull. *Boosting an analogue of Jordan's theorem for finite groups.* Advances in Mathematics **272** (2015), 820–836.

[MZ15] S. Meng, D.-Q. Zhang. *Jordan property for non-linear algebraic groups and projective varieties.* Amer. J. Math. **140** (2018), no. 4, 1133–1145.

CEU eTD Collection

[Pi82] R. S. Pierce. *Associative Algebras.* Graduate Texts in Mathematics, Vol. 88, Springer Verlag, Berlin-New York, 1982.

[Po11] V. L. Popov. *On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties.* In: *Affine Algebraic Geometry: The Russell Festschrift.* CRM Proceedings and Lecture Notes, Vol. 54, Amer. Math. Soc., 2011, 289–311.

[Po14] V. L. Popov. *Jordan groups and automorphism groups of algebraic varieties.* In: *Automorphisms in Birational and Affine Geometry.* Lecture Notes in Mathematics and Statistics, Vol. 79, Springer, Heidelberg, 2014, 185–213.

[PS14] Yu. Prokhorov, C. Shramov. *Jordan Property for groups of birational selfmaps.* Compositio Math. **150** (2014), no. 12, 2054–2072.

[PS16] Yu. Prokhorov, C. Shramov. *Jordan property for Cremona groups.* Amer. J. Math. **138** (2016), no. 2, 403–418.

[PS17] Yu. Prokhorov and C. Shramov. *Jordan constant for Cremona group of rank 3.* Moscow Math. J., **17** (2018), no. 3, 457–509.

[PS18a] Yu. Prokhorov, C. Shramov. *Finite groups of birational selfmaps of threefolds.* Math. Res. Lett. **25** (2018), no. 3, 957–972.

[PS18b] Yu. Prokhorov and C. Shramov *p-subgroups in the space Cremona group.* Math. Nachrichten, **291** (2018), no. 8–9, 1374–1389.

[PS18c] Yu. Prokhorov, C. Shramov. *Automorphism groups of Inoue and Kodaira surfaces.* ArXiv e-print (2018), 1812.02400.

[PS19] Yu. Prokhorov, C. Shramov. *Automorphism groups of Moishezon threefolds.* ArXiv e-print (2019), 1905.09548.

[Se09] J.-P. Serre. *A Minkowski-style bound for the orders of the finite subgroups of the Cremona group of rank 2 over an arbitrary field.* Moscow Math. J. **9** (2009), no. 1, 183–198.

[Stack] A. J. de Jong et al. *Stacks Project.* Open source project, https://stacks.math.columbia.edu/

[Ta76] H. Tango. *On the automorphisms of flag manifolds.* Bull. Kyoto Univ. Ed. Ser. B, **49** (1976), 1–9.

[Za14] Y.G. Zarhin. *Theta groups and products of abelian and rational varieties.* Proc. Edinb. Math. Soc. **57** (2014), no. 1, 299–304.

[Za15] Yu. G. Zarhin. *Jordan groups and elliptic ruled surfaces.* Transform. Groups **20** (2015), no. 2, 557–572.

CEU eTD Collection

[Zi14] B. P. Zimmermann. *On Jordan type bounds for finite groups acting on compact 3-manifolds.* Arch. Math., **103** (2014), no. 2, 195–200.

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