Bounded Intervals with Many Primes

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A thesis presented to Central European University (CEU) in partial fulfilment of the requirements for the degree of Master of Science in Mathematics and Its Applications

Budapest, Hungary
May, 2020

Contents

Abstract

This thesis analyzes the methodology used by James Maynard in his paper named "Small gaps between primes" where he proved there are infinitely many intervals of length 600 which contain at least 2 primes, and of length Cm^3e^{4m} for some constant $C > 0$ and any $m \geq 1$ which contain at least $m + 1$ primes. In the thesis, I present Maynard's method following the exposition of Terence Tao [9], adding further details, expanding calculations, and giving explanatory examples.

Acknowledgement

First of all, I would like to thank my supervisor Dr. Peter Maga. He was always ready to help whenever I had any trouble or question. I would also like to thank all my professors, especially Prof. Karoly Boroczky at Central European University for the valuable courses, continuous help and support. Moreover, I would like to thank all my mathematics professors, especially Mr. Elchin Hasanalizade at ADA University for their continuous help, support, and motivation. Last but not least, I must express my very profound gratitude to my family and friends who have always been by my side with their help, support, and motivation.

Chapter 1

Introduction

Number theory is one of the most important and at the same time interesting branches of mathematics. What makes it even more interesting is that there are certain problems in number theory which are very hard to solve although their statements are easy to understand. One of the most famous such problems is the twin prime conjecture which states there are infinitely many primes p such that $p + 2$ is also prime. Despite of trying to prove the conjecture over many years, it is still an open problem. In the last two decades, however, several groundbreaking results were proven in the area. The goal of this thesis is to present Maynard's one.

In the exposition, we follow Tao's blog post [9]. Section 1.1 gives a brief history about the problem without defining any mathematical concept or definition. Section 1.2 introduces the problem and defines it in a more mathematical way.

1.1 Historical Review

In 1846, French mathematician Alphonse de Polignac stated that every even number can be written as the difference of two primes in infinitely many different ways. Twin prime conjecture is a special case of this statement where the even number is 2. One of the biggest results concerning about this conjecture was the well-known prime number theorem. The prime number theorem [7, p. 179-180] states that

$$\pi(x) \sim \frac{x}{\log x}$$

where $\pi(x)$ is the number of primes up to x. It follows from the prime number theorem that the average gap between two consecutive primes p_n and p_{n+1} is $\sim \log p_n$, in particular,

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} \leq 1$$

where p_n denotes n^{th} prime. In 1915, Norwegian mathematician Viggo Brun [2] made an important progress on twin prime conjecture by proving even if there are infinitely many twin primes, they are rare i.e. up to x there are at most $Cx/(\log x)^2$ for some constant $C > 0$. Later in 1939, Hungarian mathematician Paul Erdős [4] published a paper named "The Difference of Consecutive Primes" where he improved the consequence of the prime number theorem and proved

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} < 1-c, \quad \text{for some } c > 0.$$

Erdős's result was the very first unconditional bound strictly less than 1 for the quantity although before him, Hardy and Littlewood and later Rankin considered the same problem and proved it is less or equal to 2/3 and 3/5 respectively assuming Riemann hypothesis.

The next big breakthrough came up in 2005 when Daniel Goldston, Janos Pintz, and Cem Yıldırım [5] started a new approach to this problem. Instead of pairs, one might consider k_0-tuples of integers close to each other for some $k_0 > 2$ and can try to prove that two members of the k_0-tuple are primes infinitely often. This idea turned out to be extremely successful. In this path, they published a paper called "Primes in Tuples I" where they improved Erdős's result and proved

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0.$$

In other words, there exists arbitrarily large x such that $(x, x+\epsilon \log x]$ contains at least two primes for any $\epsilon > 0$. Later in April, 2013, Yitang Zhang [11] proved in a groundbreaking paper that there are infinitely many prime tuples which have difference less or equal to 70000000 i.e.

$$\liminf_{n\to\infty} (p_{n+1}-p_n) \leq 70000000.$$

Of course, the importance of this result was not about the bound itself, it was rather about showing that the gap between two consecutive primes is infinitely often smaller than a finite number. The same year in November, James Maynard [6] published a paper called "Small Gaps between Primes" where he improved Zhang's result and proved

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \leq 600.$$

Nonetheless, what made Maynard's technique even more powerful was that using this technique, he not only proved it for prime tuples, but also he could prove the finite gaps between prime m-tuples i.e.

$$\liminf_{n\to\infty}(p_{n+m}-p_n) < \infty$$

for every fixed $m \in \mathbf{N}$. In this thesis, we will prove Maynard's result by following Tao's post [9]. In fact, it is not the best result so far found; in 2014, a group of number theorists united under the name Polymath8b [8] to further improve Maynard's result, and they achieved the best result so far obtained by proving

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \leq 246.$$

1.2 Definition of the Problem

For all $m \in \mathbf{N}$ define

$$H_m := \liminf_{n\to\infty}(p_{n+m}-p_n), \tag{1.1}$$

where p_n denotes the n^{th} prime. In other words, H_m is the least quantity with the property that there are infinitely many intervals length H_m which contains at least $m+1$ primes. With this notation in mind, Maynard's result shows that $H_1 \leq 600$, and the best result so far found by Polymath8b shows that $H_1 \leq 246$. In fact, the twin prime conjecture is equivalent to $H_1 = 2$.

Let an admissible k_0-tuple be defined as a increasing tuple $\mathcal{H} = (h_1, \ldots, h_{k_0})$ of integers such that for every prime p there exist an $n \in \mathbf{N}$ and $i \in 1, \ldots, k_0$ such that $h_i \not\equiv n \pmod p$. As an example of an admissible set we can pick the set $(0,2)$ as $n = 1$ does the job since both 0 and 2 are incongruent to 1 modulo to p for any prime p. On the other hand, the set $(0,1)$ is not an admissible set as for $p = 2$, there is no such an n since for every number n, either n or $n-1$ is divisible by 2. Now we define $DHL[k_0, j_0]$ for $1 \leq j_0 \leq k_0$ where DHL stands for "Dickson-Hardy-Littlewood" to be as following:

Conjecture. *If $\mathcal{H}$ is an admissible k_0-tuple, then there are infinitely many translates $n+\mathcal{H}$ which contains at least j_0 primes.*

Then the prime tuple conjecture is the assertion that $DHL[k_0, j_0]$ holds for $\forall k_0, j_0 \in \mathbf{N}$. On the other hand, if we can show that $DHL[k_0, m+1]$ holds, then obviously, we get that $H_m \leq h_{k_0} - h_1$ whenever $(h_1, \ldots, h_{k_0})$ is an admissible k_0-tuple.

In the subsequent chapters, we give a criterion for DHL and show the criterion holds for some sieve. Then we introduce Maynard's theorem, and using this criterion, we find an admissible set of length 600 to give the bound for H_1. Lastly, again using the criterion and doing some further calculations, we bound H_m for all $m \geq 1$ which proves the theorem.

Chapter 2

Main Ingredients for the Proof

This chapter gives the main ingredients for the proof of the theorem. Having these results in hand, the proof of the theorem for case $m = 1$ will follow immediately, however, for $m \geq 2$, we will need a little more work which will be done in Chapter 3. Section 2.1 gives the criterion for DHL to be held and proves it. Section 2.2 introduces the sieve asymptotics and in the subsections, we show that indeed, the defined sieve holds the hypotheses of the criterion.

2.1 Criterion for DHL

Let us define

$$w := \lfloor \log \log \log x \rfloor$$

and $W := \prod_{p<w} p$ where p stands for a prime (the convention is meant throughout the whole thesis). Let $\theta(n)$ be defined as a quantity $\log n$ if n is prime, and 0 otherwise. We take all $O()$, $o()$ or $\ll$ asymptotically with x tending to infinity unless otherwise stated. Then we have the following criterion for DHL.

Lemma 1. [9, Lemma 4] *Let $k_0 \geq 2$ and $m \geq 1$ be fixed integers. Suppose that for each fixed admissible k_0-tuple $\mathcal{H}$ and each congruence class $b \pmod W$ such that $b + h$ is coprime to W for all $h \in \mathcal{H}$, there exists a non-negative weight function $\nu \colon \mathbf{N} \to \mathbf{R}^+$, fixed quantities $\alpha, \beta > 0$ depending only on k_0 and quantities $B, R > 0$ depending on x with the upper bound*

$$\sum_{x \leq n \leq 2x : n \equiv b \ (W)} \nu(n) \leq (\alpha + o(1)) B \frac{x}{W}, \tag{2.1}$$

the lower bound

$$\sum_{x\le n\le 2x: n\equiv b\ (W)} \nu(n)\theta(n+h_i) \ge (\beta - o(1))B\frac{x}{W}\log R \tag{2.2}$$

for all $h_i \in \mathcal{H}$, and the key inequality

$$\frac{\log R}{\log x} > \frac{m}{k_0}\frac{\alpha}{\beta}. \tag{2.3}$$

Then $DHL[k_0, m+1]$ holds.

Proof. Consider the expression

$$\sum_{x\le n\le 2x: n\equiv b\ (W)} \nu(n)\left(\sum_{i=1}^{k_0}\theta(n+h_i) - m\log 3x\right),$$

First of all, using the hyphotesis (2.1), (2.2), and (2.3), we show that this expression is positive for sufficiently large x.

$$\begin{aligned}
&\sum_{x\le n\le 2x: n\equiv b\ (W)} \nu(n)\left(\sum_{i=1}^{k_0}\theta(n+h_i) - m\log 3x\right)\\
&= \sum_{x\le n\le 2x: n\equiv b\ (W)} \nu(n)\sum_{i=1}^{k_0}\theta(n+h_i) - \sum_{x\le n\le 2x: n\equiv b\ (W)} \nu(n) m\log 3x\\
&= \sum_{i=1}^{k_0}\sum_{x\le n\le 2x: n\equiv b\ (W)} \nu(n)\theta(n+h_i) - m\log 3x\sum_{x\le n\le 2x: n\equiv b\ (W)} \nu(n)\\
&\ge k_0(\beta - o(1))B\frac{x}{W}\log R - m(\log 3x)(\alpha + o(1))B\frac{x}{W}\\
&= B\frac{x}{W}k_0\beta\log x\left(\frac{\log R}{\log x} - o(1) - \frac{m\alpha}{k_0\beta} - \frac{m\alpha\log 3}{k_0\beta\log x} - o(1)\right).
\end{aligned}$$

Thus using hypothesis (2.3), and taking x sufficiently large, we can make the error term and the negative quantity smaller than the difference $\frac{\log R}{\log x} - \frac{m\alpha}{k_0\beta}$ which gives the positiveness of the expression. On the other hand, as we have $n + h_i < 3x$ for sufficiently large x, the expression is positive only if the set $n + h_1, \ldots, n + h_{k_0}$ contains at least $m + 1$ primes. □

Hence our aim in the rest of the chapter is to find a suitable weight function ν which satisfies the given hypotheses.

2.2 Sieve Asymptotics

Let's introduce the hypothesis $EH[\theta]$ for a given parameter $0 < \theta < 1$: if $Q \leq x^\theta$, and $A \geq 1$ is any fixed number, then

$$\sum_{q \leq Q} \sup_{a \in (\mathbf{Z}/q\mathbf{Z})^\times} \left| \Delta(\Lambda 1_{[x,2x]}; a\ (q)) \right| \ll x \log^{-A} x \tag{2.4}$$

where

$$\Delta(\alpha; a\ (q)) := \sum_{n \equiv a\ (q)} \alpha(n) - \frac{1}{\phi(q)} \sum_{(n,q)=1} \alpha(n).$$

The Elliott-Halberstam conjecture [3] states $EH[\theta]$ holds for all $0 < \theta < 1$. Bombieri-Vinogradov [1], [10] theorem is that $EH[\theta]$ holds for all $0 < \theta < 1/2$.

Lemma 2. [9, Proposition 5] *Suppose there exists $\theta \in (0,1)$ such that $EH[\theta]$ holds, and set $R = x^{c/2}$ for some fixed $c \in (0,\theta)$. Let $f : [0,+\infty)^{k_0} \to \mathbf{R}$ be a fixed symmetric[1] smooth function supported on the simplex*

$$\Delta_{k_0} := \{(t_1, \ldots, t_{k_0}) \in [0,+\infty)^{k_0} : t_1 + \ldots + t_{k_0} \leq 1\}.$$

Then we can find a non-negative weight function ν satisfying the bounds (2.1), (2.2) *with*

$$B := \left(\frac{W}{\phi(W)}\right)^{k_0} \frac{1}{\log^{k_0} R}, \tag{2.5}$$

$$\alpha := \int_{\Delta_{k_0}} f_{1,\ldots,k_0}(t_1, \ldots, t_{k_0})^2 \ dt_1 \ldots dt_{k_0}, \tag{2.6}$$

$$\beta := \int_{\Delta_{k_0-1}} f_{1,\ldots,k_0-1}(t_1, \ldots, t_{k_0-1}, 0)^2 \ dt_1 \ldots dt_{k_0-1} \tag{2.7}$$

where

$$f_{i_1,\ldots,i_j}(t_1, \ldots, t_n) := \frac{\partial^j}{\partial t_{i_1} \ldots \partial t_{i_j}} f(t_1, \ldots, t_n)$$

stands for the mixed partial derivatives of f.

We prove Lemma (2) by constructing the weight function ν. Namely, the sieve we are

[1] By symmetric, we mean a function which is invariant under any permutation of the coordinates of its argument.

going to use is defined as following:

$$\nu(n) := \left(\sum_{\substack{d_1,\ldots,d_{k_0}\in\mathcal{S}: \\ d_i|n+h_i \text{ for all } i=1,\ldots,k_0}} \left(\prod_{i=1}^{k_0}\mu(d_i)\right) f\left(\frac{\log d_1}{\log R},\ldots,\frac{\log d_{k_0}}{\log R}\right)\right)^2 \tag{2.8}$$

where $\mathcal{S}$ denotes the square-free integers, and μ is the Möbius function.

Our aim is to show that with this ν, α and β defined in (2.6) and (2.7) satisfy the bounds (2.1) and (2.2) defined in Lemma (1).

2.2.1 Proof of the Existence for α

In this section, we prove that with the predefined weight function ν, the α defined in Lemma (2) satisfies hypothesis (2.1) in Lemma (1).

Using (2.8) and interchanging sums, we can rewrite the left hand side of (2.1) as following:

$$\sum_{d_1,\ldots,d_{k_0},d'_1,\ldots,d'_{k_0}\in\mathcal{S}} \left(\prod_{j=1}^{k_0}\mu(d_j)\mu(d'_j)\right)$$

$$f\left(\frac{\log d_1}{\log R},\ldots,\frac{\log d_{k_0}}{\log R}\right) f\left(\frac{\log d'_1}{\log R},\ldots,\frac{\log d'_{k_0}}{\log R}\right)$$

$$\sum_{\substack{x\le n\le 2x: n\equiv b\ (W); \\ [d_j,d'_j]|n+h_j \text{ for all } j=1,\ldots,k_0}} 1.$$

Now we show that $n+h_0,\ldots,n+h_{k_0}$ have no common factor. Suppose $n+h_i$ and $n+h_j$ for $i\neq j$ have a common factor, then consider:

$$\gcd(n+h_i,n+h_j)\le |h_i-h_j| < w.$$

We get that if they have a common factor p, then it is less than w, hence it divides W. Because of the choice of $b \pmod W$, if $n\equiv b \pmod W$, for such primes $p|W$,

$$n+h\equiv b+h\not\equiv 0 \pmod p$$

which gives contradiction. Now as we have $n+h_0,\ldots,n+h_{k_0}$ have no common factor, clearly, the inner sum vanishes if $[d_1,d'_1],\ldots,[d_{k_0},d'_{k_0}]$ are not coprime. For the case they are coprime, the inner sum can be estimated by

$$\frac{x}{W[d_1, d_1'] \dots [d_{k_0}, d_{k_0}']} + O(1).$$

By the definition, the function f vanishes outside the simplex

$$\Delta_{k_0} := \{(t_1, \dots, t_{k_0}) \in [0, +\infty)^{k_0} : t_1 + \dots + t_{k_0} \leq 1\}.$$

Hence it will vanish unless we have $d_1 \dots d_{k_0}, d_1' \dots d_{k_0}' \leq R$. With this in mind, the contribution of the error term $O(1)$ can be bounded by

$$O\left(\sum_{d_1,\dots,d_{k_0}, d_1',\dots,d_{k_0}' : d_1 \dots d_{k_0}, d_1' \dots d_{k_0}' \leq R} 1\right) = O\left(\left(\sum_{d_1,\dots,d_{k_0} : d_1 \dots d_{k_0} \leq R} 1\right)^2\right),$$

In fact, the sum shows the number of ways that the numbers up to R can be written as a product of k_0 numbers. We compute the sum using the following lemma.

Lemma 3. *Let $\tau_k(n)$ the number of ways that a number n can be written as a product of $k \geq 1$ numbers. Then we have*

$$\sum_{n \leq x} \tau_k(n) \ll x \log^{O(1)} x$$

where $O(1)$ depends on k.

Proof. We prove the lemma by induction. For $k = 1$, it is trivial as we have

$$\sum_{n \leq x} \tau_1(n) = \sum_{n \leq x} 1 = x + O(1) = x \log^{O(1)} x.$$

Now we assume the statement holds for $k - 1$ and prove it for k

$$\begin{aligned}\sum_{n \leq x} \tau_k(n) &= \sum_{n \leq x} \sum_{d \mid n} \tau_{k-1}\left(\frac{n}{d}\right) = \sum_{d \leq x} \sum_{\substack{n \leq x, \\ d \mid n}} \tau_{k-1}\left(\frac{n}{d}\right) = \sum_{d \leq x} \sum_{m \leq \frac{x}{d}} \tau_{k-1}(m) \\ &\ll \sum_{d \leq x} \left(\frac{x}{d} \log^{O(1)} \frac{x}{d}\right) \leq \sum_{d \leq x} \frac{x}{d} \log^{O(1)} x = x \log^{O(1)x} \sum_{d \leq x} \frac{1}{d} \ll x \log^{O(1)} x.\end{aligned}$$

This completes the proof. □

Hence using the lemma we get the summand in the error term is bounded by $\ll R \log^{O(1)} R$, and the error term can be nicely bounded by $R^2 \log^{O(1)} R$. Since $R = x^{c/2}$ and $c < 1/2 < 1$, we get the contribution of the error term is negligible. Thus to conclude the proof of (2.1), recalling (2.1) and (2.5), it suffices to show

$$\sum_{\substack{d_1,\dots,d_{k_0},d'_1,\dots,d'_{k_0}\in\mathcal{S}: \\ [d_1,d'_1],\dots,[d_{k_0},d'_{k_0}] \text{ coprime}}} \left(\prod_{j=1}^{k_0}\mu(d_j)\mu(d'_j)\right)\frac{f\left(\frac{\log d_1}{\log R},\dots,\frac{\log d_{k_0}}{\log R}\right) f\left(\frac{\log d'_1}{\log R},\dots,\frac{\log d'_{k_0}}{\log R}\right)}{[d_1,d'_1]\dots[d_k,d'_k]} \tag{2.9}$$
$$= (\alpha+o(1))\left(\frac{W}{\phi(W)}\right)^{k_0}\frac{1}{\log^{k_0} R}.$$

Now we extend the function $f:[0,+\infty)^{k_0}\to\mathbf{R}$ to a smooth compactly supported function $f:\mathbf{R}^{k_0}\to\mathbf{R}$, and we continue referring it f as well. Using Fourier inversion, we express f in the following form:

$$f(t_1,\dots,t_{k_0}) = \int_{\mathbf{R}^{k_0}} \eta(\vec{s})e^{-\sum_{j=1}^{k_0}(1+is_j)t_j}\ d\vec{s} \tag{2.10}$$

where $\vec{s}:=(s_1,\dots,s_{k_0})$ and $\eta:\mathbf{R}^k\to\mathbf{C}$ is a smooth function with rapid decay bounds

$$|\eta(\vec{s})| \ll (1+|\vec{s}|)^{-A} \tag{2.11}$$

for any fixed $A>0$. Hence using (2.10), the left hand side of (2.9) can be rewritten as

$$\int_{\mathbf{R}^{k_0}}\int_{\mathbf{R}^{k_0}} \eta(\vec{s})\eta(\vec{s}')H(\vec{s},\vec{s}')\ d\vec{s}d\vec{s}' \tag{2.12}$$

where $\vec{s}':=(s'_1,\dots,s'_{k_0})$ and

$$H(\vec{s},\vec{s}') := \sum_{\substack{d_1,\dots,d_{k_0},d'_1,\dots,d'_{k_0}\in\mathcal{S}: \\ [d_1,d'_1],\dots,[d_{k_0},d'_{k_0}] \text{ coprime}}} \left(\prod_{j=1}^{k_0}\mu(d_j)\mu(d'_j)\right)\frac{\prod_{j=1}^{k_0} d_j^{-(1+is_j)/\log R}(d'_j)^{-(1+is'_j)/\log R}}{[d_1,d'_1]\dots[d_{k_0},d'_{k_0}]}. \tag{2.13}$$

We can factorize $H(\vec{s},\vec{s}')$ as an Euler product

$$H(\vec{s},\vec{s}') = \prod_{p>w}\left(1-\sum_{j=1}^{k_0}\left(p^{-1-(1+is_j)/\log R}+p^{-1-(1+is'_j)/\log R}-p^{-1-(1+is_j)/\log R-(1+is'_j)/\log R}\right)\right).$$

In particular, we have the crude bound

$$|H(\vec{s},\vec{s}')| \leq \prod_{p>w}(1+3k_0p^{-1/\log R}) \leq \prod_{p>w}\left(1+p^{-1/\log R}\right)^{3k_0} \leq \left(\prod_{p>w}\left(1+p^{-1/\log R}\right)\right)^{3k_0}$$
$$\leq \left(\zeta\left(1+\frac{1}{\log R}\right)\right)^{3k_0} \ll (\log R)^{3k_0} = \log^{O(1)} R.$$

Hence combining this with (2.11), we get that the contribution to (2.12) is negligible when we take $|\vec{s}| \geq \sqrt{\log R}$ or $|\vec{s}'| \geq \sqrt{\log R}$. Therefore, we restrict our integral in (2.12) to the region $|\vec{s}|, |\vec{s}'| \leq \sqrt{\log R}$. Recalling that $\zeta(s) = \frac{1}{s-1}(1+o(1))$ for $\Re s > 1$, s tending to 0 and using the bound $W = O(\log\log x)$, we have the following Euler product approximations in this region:

$$\begin{aligned}\prod_{p>w}\left(1-p^{-1-(1+is_j)/\log R}\right) &= \zeta\left(1+\frac{1+is_j}{\log R}\right)^{-1}\prod_{p\leq w}\left(1-p^{-1-(1+is_j)/\log R}\right)^{-1} \\ &= (1+o(1))\left(\frac{1+is_j}{\log R}\right)\prod_{p\leq w}\left(1-p^{-1}\right)^{-1} \\ &= (1+o(1))\frac{W}{\phi(W)}\left(\frac{1+is_j}{\log R}\right).\end{aligned} \tag{2.14}$$

Also, we can prove the following identity by simply multiplying the left hand side of the equation and using $p > w$ and w tends to infinity:

$$\begin{aligned}&\prod_{j=1}^{k_0}\frac{\left(1-p^{-1-(1+is_j)/\log R}\right)\left(1-p^{-1-(1+is'_j)/\log R}\right)}{1-p^{-1-(1+is_j)/\log R-(1+is'_j)/\log R}} \\ &= \prod_{j=1}^{k_0}\left(1-p^{-1-(1+is_j)/\log R}\right)\left(1-p^{-1-(1+is'_j)/\log R}\right)\left(1+p^{-1-(1+is_j)/\log R-(1+is'_j)/\log R}+o\left(\frac{1}{p^2}\right)\right) \\ &= 1-\sum_{j=1}^{k_0}\left(p^{-1-(1+is_j)/\log R}+p^{-1-(1+is'_j)/\log R}-p^{-1-(1+is_j)/\log R-(1+is'_j)/\log R}\right)+o\left(\frac{1}{p^2}\right).\end{aligned} \tag{2.15}$$

Thus using (2.14) and (2.15) and recalling $\sum_{p>w}\frac{1}{p^2}=o(1)$, we get

$$\begin{aligned}H(\vec{s},\vec{s}')&=(1+o(1))\prod_{j=1}^{k_0}\prod_{p>w}\frac{\left(1-p^{-1-(1+is_j)/\log R}\right)\left(1-p^{-1-(1+is'_j)/\log R}\right)}{1-p^{-1-(1+is_j)/\log R-(1+is'_j)/\log R}}\\&=(1+o(1))\left(\frac{W}{\phi(W)}\right)^{k_0}\frac{1}{\log^{k_0}R}\prod_{j=1}^{k_0}\frac{(1+is_j)(1+is'_j)}{1+is_j+1+is'_j}.\end{aligned}$$

Now we again use (2.11) to get rid of the error term and remove our restriction to $|\vec{s}|,|\vec{s}'|\leq\sqrt{\log R}$. Hence we get that it suffices to prove

$$\int_{\mathbf{R}^{k_0}}\int_{\mathbf{R}^{k_0}}\eta(\vec{s})\eta(\vec{s}')\prod_{j=1}^{k_0}\frac{(1+is_j)(1+is'_j)}{1+is_j+1+is'_j}\;d\vec{s}d\vec{s}'=\alpha.$$

To prove this, we repeatedly differentiate (2.10) under the integral sign and get

$$f_{1,\ldots,k_0}(t_1,\ldots,t_{k_0})=(-1)^{k_0}\int_{\mathbf{R}^{k_0}}\eta(\vec{s})e^{-\sum_{j=1}^{k_0}(1+is_j)t_j}\prod_{j=1}^{k_0}(1+is_j)\;d\vec{s}.$$

Hence taking the square of both sides

$$\begin{aligned}f_{1,\ldots,k_0}(t_1,\ldots,t_{k_0})^2=\int_{\mathbf{R}^{k_0}}\int_{\mathbf{R}^{k_0}}&\eta(\vec{s})\eta(\vec{s}')e^{-\sum_{j=1}^{k_0}(1+is_j+1+is'_j)t_j}\\&\prod_{j=1}^{k_0}(1+is_j)(1+is'_j)\;d\vec{s}d\vec{s}';\end{aligned}$$

and to conclude the proof of (2.1), we simply integrate this by using Fubini's theorem for $t_1,\ldots,t_{k_0}\in[0,+\infty)$ and use (2.6).

2.2.2 Proof of the Existence for β

In this section, we prove that with the predefined weight function ν, the β defined in Lemma (2) satisfies hypothesis (2.2) in Lemma (1). Using the symmetry hypothesis on f, it suffices to prove for the case $i=k_0$ as all other cases follow similarly. Again using (2.8), we can rewrite the left hand side of (2.2) as following:

$$\sum_{d_1,\ldots,d_{k_0},d'_1,\ldots,d'_{k_0}\in\mathcal{S}}\left(\prod_{j=1}^{k_0}\mu(d_j)\mu(d'_j)\right)$$

$$f\left(\frac{\log d_1}{\log R},\dots,\frac{\log d_{k_0}}{\log R}\right) f\left(\frac{\log d'_1}{\log R},\dots,\frac{\log d'_{k_0}}{\log R}\right)$$

$$\sum_{\substack{x\le n\le 2x: n\equiv b\ (W);\\ [d_j,d'_j]\mid n+h_j \text{ for all } j=1,\dots,k_0}} \theta(n+h_{k_0}).$$

By definition, $\theta(n)$ vanishes unless n is prime, and here observing that $n+k_0$ is comparable to x, thus it exceeds R, we get that the inner sum vanishes unless $d_{k_0} = d'_{k_0} = 1$. In this case, we have

$$\sum_{d_1,\dots,d_{k_0-1},d'_1,\dots,d'_{k_0-1}\in\mathcal{S}} \left(\prod_{j=1}^{k_0-1}\mu(d_j)\mu(d'_j)\right)$$

$$f\left(\frac{\log d_1}{\log R},\dots,\frac{\log d_{k_0-1}}{\log R},0\right) f\left(\frac{\log d'_1}{\log R},\dots,\frac{\log d'_{k_0-1}}{\log R},0\right)$$

$$\sum_{\substack{x\le n\le 2x: n\equiv b\ (W);\\ [d_j,d'_j]\mid n+h_j \text{ for all } j=1,\dots,k_0-1}} \theta(n+h_{k_0}).$$

As in the previous section, the inner sum vanishes unless $[d_1,d'_1],\dots,[d_{k_0-1},d'_{k_0-1}]$ are coprime, and one of the functions involving f vanishes unless

$$d_1\dots d_{k_0-1}, d'_1\dots d'_{k_0-1} \le R. \tag{2.16}$$

Now we define the discrepancy for any modulus q as

$$E(q) := \sup_{a\in(\mathbf{Z}/q\mathbf{Z})^\times} \left| \sum_{x\le n\le 2x: n+h_{k_0}=a\ (q)} \theta(n+h_{k_0}) - \frac{x}{\phi(q)} \right|. \tag{2.17}$$

Here we can use (2.4), and to replace the Λ function with θ function, we observe that

$$\sum_{n\le x}\Lambda(n) = \sum_{n\le x}\theta(n) + O\left(x^{1/2}\log^2 x\right)$$

and using prime number theorem, we have

$$\sum_{n\le x}\Lambda(n) = x + O\left(xe^{-c_1\sqrt{\log x}}\right) \quad \text{for some } c_1>0.$$

The error terms are negligible, and $R = x^{c/2}$ and $0 < c < \theta$, hence we can use Bombieri-Vinogradov theorem to get

$$\sum_{q \le WR^2} E(q) \ll x \log^{-A} x \tag{2.18}$$

for any fixed $A > 0$. On the other hand, using the Chinese remainder theorem, the sum

$$\sum_{\substack{x \le n \le 2x : n \equiv b\ (W); \\ [d_j, d'_j] | n + h_j \text{ for all } j = 1, \ldots, k_0 - 1}} \theta(n + h_{k_0})$$

can be written as

$$\sum_{x \le n \le 2x : n + h_{k_0} \equiv a\ (q)} \theta(n + h_{k_0})$$

where

$$q := W \prod_{j=0}^{k_0 - 1} [d_j, d'_j] \tag{2.19}$$

and $a(q)$ is a primitive residue class, and by (2.16), we have $q \le WR^2$. By (2.18), we have

$$\sum_{x \le n \le 2x : n + h_{k_0} \equiv a\ (q)} \theta(n + h_{k_0}) = \frac{x}{\phi(W) \prod_{j=0}^{k_0 - 1} \phi([d_j, d'_j])} + O\left(E\left(W \prod_{j=0}^{k_0 - 1} [d_j, d'_j] \right) \right).$$

Now we look at the error term which we can bound by

$$O\left(\sum_{d_1, \ldots, d_{k_0 - 1}, d'_1, \ldots, d'_{k_0 - 1} : d_1 \ldots d_{k_0 - 1}, d'_1 \ldots d'_{k_0 - 1} \le R} E\left(W \prod_{j=0}^{k_0 - 1} [d_j, d'_j] \right) \right).$$

Here using the fact that there are $O(\tau(q)^{O(1)})$ ways of choosing $d_1, \ldots, d_{k_0 - 1}, d'_1, \ldots, d'_{k_0 - 1}$ which satisfies (2.19), and again recalling $q \le WR^2$, we can bound the expression by

$$\ll \sum_{q \le WR^2} \tau(q)^{O(1)} E(q).$$

Here using Cauchy-Schwarz inequality and (2.18), we have

$$= \left(\left(\sum_{q\le WR^2} \tau(q)^{O(1)} E(q)\right)^2\right)^{1/2} \ll \left(\sum_{q\le WR^2} \tau(q)^{O(1)} \sum_{q\le WR^2} E(q)^2\right)^{1/2}$$
$$\ll \left(\sum_{q\le WR^2} \tau(q)^{O(1)} E(q) \sum_{q\le WR^2} E(q)\right)^{1/2} \ll (x\log^{-A} x)^{1/2} \left(\sum_{q\le WR^2} \tau(q)^{O(1)} E(q)\right)^{1/2} \tag{2.20}$$

for any fixed A. On the other hand, we have the crude bound $E(q) \ll \frac{x}{q}\log^{O(1)} x$. Substituting this in (2.20), we get

$$\ll \left(x\log^{-A} x\right)^{1/2} \left(\sum_{q\le WR^2} \tau(q)^{O(1)} \frac{x}{q}\log^{O(1)} x\right)^{1/2}$$
$$\ll \left(x\log^{-A} x\right)^{1/2} \left(x\log^{O(1)} x\right)^{1/2} \left(\sum_{q\le WR^2} \frac{\tau(q)^{O(1)}}{q}\right)^{1/2}.$$

Now we wish to bound $\sum_{q\le WR^2} \frac{\tau(q)^{O(1)}}{q}$. Consider the following expression [7, equation 2.31]

$$\sum_{q\le y} \tau(q)^{O(1)} \ll y\log^{O(1)} y.$$

Using this, we also have

$$\sum_{y/2\le q\le y} \tau(q)^{O(1)} \ll y\log^{O(1)} y.$$

Then

$$\frac{2}{y}\sum_{y/2\le q\le y} \tau(q)^{O(1)} \ll 2\log^{O(1)} y.$$

Then

$$\sum_{y/2\le q\le y} \frac{\tau(q)^{O(1)}}{q} \le \sum_{y/2\le q\le y} \frac{\tau(q)^{O(1)}}{y/2} \ll 2\log^{O(1)} y.$$

Applying this in the $\log_2 y$ many intervals $[y/2, y]$, $[y/4, y/2]$, etc. and picking up $\ll \log^{O(1)} y$ in each of them, their sum is at most $\log^{O(1)} y$. Substituting this back, we bound the error by

$$\ll \left(x\log^{-A} x\right)^{1/2} \left(x\log^{O(1)} x\right)^{1/2} \left(\log^{O(1)}\left(WR^2\right)\right)^{1/2}.$$

Hence we get the error term is negligible. Thus to conclude the proof of (2.2), again recalling (2.2) and (2.5), it suffices to prove

$$\sum_{d_1,\dots,d_{k_0-1},d'_1,\dots,d'_{k_0-1}\in\mathcal{S}} \left(\prod_{j=1}^{k_0-1}\mu(d_j)\mu(d'_j)\right)$$

$$\frac{f\left(\frac{\log d_1}{\log R},\dots,\frac{\log d_{k_0-1}}{\log R},0\right) f\left(\frac{\log d'_1}{\log R},\dots,\frac{\log d'_{k_0-1}}{\log R},0\right)}{\prod_{j=0}^{k_0-1}\phi([d_j,d'_j])}$$
$$= (\beta - o(1))\left(\frac{W}{\phi(W)}\right)^{k_0-1}\frac{1}{\log^{k_0-1}R}.$$

From this point on, we can proceed in the same way as we did in Section 2.2.1 to prove (2.9) by replacing k_0 and $f(t_1,\dots,t_{k_0})$ with k_0-1 and $f(t_1,\dots,t_{k_0-1},0)$ respectively. However, the only notable difference comes with the presence of Euler totient function which primarily changes the $H(\vec{s},\vec{s}')$ function which was defined in (2.13). Let's now define a new function

$$\tilde{H}(\vec{s},\vec{s}') := \sum_{\substack{d_1,\dots,d_{k_0},d'_1,\dots,d'_{k_0}\in\mathcal{S}:\\ [d_1,d'_1],\dots,[d_{k_0},d'_{k_0}]\text{ coprime}}} \left(\prod_{j=1}^{k_0}\mu(d_j)\mu(d'_j)\right)\frac{\prod_{j=1}^{k_0} d_j^{-(1+is_j)/\log R}(d'_j)^{-(1+is'_j)/\log R}}{\phi([d_1,d'_1])\dots\phi([d_{k_0},d'_{k_0}])}. \tag{2.21}$$

Let's fix a set of $d_1,\dots,d_{k_0}$ and $d'_1,\dots,d'_{k_0}$ and consider the prime factorization of $\prod_{i=1}^{k_0}[d_i,d'_i]$. Recalling $[d_1,d'_1],\dots,[d_{k_0},d'_{k_0}]$ are all coprime and d_i and d'_i for $1\le i\le k_0$ are square free integers, we get each prime occurs only once in the prime factorization of the product. Hence we may suppose

$$[d_1,d'_1]\dots[d_{k_0},d'_{k_0}] = p_1\dots p_l \quad \text{for some } l\in\mathbf{N}.$$

Then we get

$$\phi([d_1,d'_1])\dots\phi([d_{k_0},d'_{k_0}]) = (p_1-1)\dots(p_l-1).$$

Hence we can write the summand in (2.21) for each fixed set $d_1,\dots,d_{k_0}$ and $d'_1,\dots,d'_{k_0}$ as

$$\left(\prod_{j=1}^{k_0}\mu(d_j)\mu(d'_j)\right)\frac{\prod_{j=1}^{k_0} d_j^{-(1+is_j)/\log R}(d'_j)^{-(1+is'_j)/\log R}}{(p_1-1)\dots(p_l-1)}. \tag{2.22}$$

Using $\frac{1}{p-1}=\frac{1}{p}+O(\frac{1}{p^2})$, we get the following bound for (2.22)

$$\left(\prod_{j=1}^{k_0}\mu(d_j)\mu(d'_j)\right)\left(\prod_{j=1}^{k_0} d_j^{-(1+is_j)/\log R}(d'_j)^{-(1+is'_j)/\log R}\right)\left(\prod_{j=1}^{l}\left(\frac{1}{p_j}+O\left(\frac{1}{p_j^2}\right)\right)\right)$$

$$= \left(\prod_{j=1}^{k_0} \mu(d_j)\mu(d_j')\right) \left(\prod_{j=1}^{k_0} d_j^{-(1+is_j)/\log R}(d_j')^{-(1+is_j')/\log R}\right) \left(\frac{1}{p_1 \dots p_l} + \sum_{j=1}^{l} O\left(\frac{1}{p_j^2}\right)\right).$$

Finally, using $\sum_{p>w} \frac{1}{p^2} = o(1)$, we get

$$\tilde{H}(\vec{s}, \vec{s}') = H(\vec{s}, \vec{s}') + o(1).$$

Hence we conclude the error term coming with the presence of Euler totient function is negligible. This completes the proof.

Chapter 3

Proof of the Theorem

This chapter introduces the main theorem and gives the proofs for different cases. Section 3.1 introduces the theorem in two different versions and gives a Corollary of Lemma (2) with a proof. The following subsections show how the theorem follows from the corollary for the cases $m = 1$ and $m \geq 2$ respectively.

3.1 Theorem Statement

Let's recall (1.1). Then with this notation, we have

Theorem 1. *Unconditionally, we have the following bounds:*

- $H_1 \leq 600$.
- $H_m \leq Cm^3 e^{4m}$ *for an absolute constant* C *and any* $m \geq 1$.

Using the DHL notation introduced in the conjecture, we can rewrite the theorem in the following form.

Theorem 2 (DHL version)**.** *Unconditionally, we have the following bounds:*

- $DHL[105, 2]$.
- $DHL[k_0, m+1]$ *for sufficiently large* k_0 *and* $4m < \log k_0 - 2\log\log k_0 - 2$.

In this section, we are going to prove a corollary of Lemma (2), and using this corollary, we will prove the theorem for different cases in the subsections.

Let the quantity M_{k_0} be defined as

$$M_{k_0} := \sup_f k_0 \frac{\int_{\Delta_{k_0-1}} f_{1,\dots,k_0-1}(t_1, \dots, t_{k_0-1}, 0)^2 \; dt_1 \dots dt_{k_0-1}}{\int_{\Delta_{k_0}} f_{1,\dots,k_0}(t_1, \dots, t_{k_0})^2 \; dt_1 \dots dt_{k_0}} \tag{3.1}$$

where f ranges over all smooth symmetric functions $f : [0, +\infty)^{k_0} \to \mathbf{R}$ that are supported on the simplex

$$\Delta_{k_0} := \{(t_1, \ldots, t_{k_0}) \in [0, +\infty)^{k_0} : t_1 + \ldots + t_{k_0} \leq 1\}.$$

Then using this, we have the following corollary.

Corollary. *Let $EH[\theta]$ hold for some $\theta \in (0, 1)$, and let $k_0 \geq 2$, $m \geq 1$ be integers such that*

$$M_{k_0} > \frac{2m}{\theta}. \tag{3.2}$$

Then $DHL[k_0, m+1]$ holds.

Proof. We show (3.2) implies that for some f, (2.3) holds, hence the hypothesis of Lemma (1) are satisfied. Note that the denominator and numerator of (3.1) show α and β which were defined in (2.6) and (2.7) respectively. We choose f such that $\frac{k_0\beta}{\alpha}$ is close enough to M_{k_0} such that $\frac{k_0\beta}{\alpha} > \frac{2m}{\theta}$. Using this f we can write α and β by (2.6) and (2.7) as

$$\alpha = \int_{\Delta_{k_0}} f_{1,\ldots,k_0}(t_1, \ldots, t_{k_0})^2 \; dt_1 \ldots dt_{k_0},$$

$$\beta = \int_{\Delta_{k_0-1}} f_{1,\ldots,k_0-1}(t_1, \ldots, t_{k_0-1}, 0)^2 \; dt_1 \ldots dt_{k_0-1}.$$

Then

$$k_0 \frac{\beta}{\alpha} > \frac{2m}{\theta}$$

which is equivalent to

$$\frac{\theta}{2} > \frac{m}{k_0} \frac{\alpha}{\beta}. \tag{3.3}$$

On the other hand, recalling $R = x^{c/2}$ and $0 < c < \theta$, we can write the left hand side of (2.3) as

$$\frac{\log R}{\log x} = \frac{\log x^{c/2}}{\log x} = \frac{c}{2}.$$

We can take c as close to θ as we please to get

$$\frac{\log R}{\log x} = \frac{c}{2} > \frac{m}{k_0} \frac{\alpha}{\beta}$$

which is (2.3). This concludes the proof. □

Substituting $F := f_{1,\ldots,k_0}$ and using the fundamental theorem of calculus and approximation argument to remove the smoothness hypothesis on F, we can write (3.1) as

$$M_{k_0} := \sup_F k_0 \frac{\int_{\Delta_{k_0-1}} (\int_0^\infty F(t_1, \ldots, t_{k_0-1}, t_{k_0})\ dt_{k_0})^2\ dt_1 \ldots dt_{k_0-1}}{\int_{\Delta_{k_0}} F(t_1, \ldots, t_{k_0})^2\ dt_1 \ldots dt_{k_0}}$$

where F ranges over all bounded measurable functions supported on the simplex.

3.1.1 Proof for Case $m = 1$

We are going to use the corollary for the case $k_0 = 105$. Namely, we have

$$M_{105} > 4. \tag{3.4}$$

We do not prove this inequality in this thesis, but give a sketch of the proof. This can be proven by taking

$$F = 1_{\Delta_{k_0}} \sum_{i=1}^{d} a_i (1 - P_1)^{b_i} P_2^{c_i}$$

for different real coefficients a_i and non-negative integers b_i and c_i, where

$$P_1(t_1, \ldots, t_{k_0}) := t_1 + \ldots + t_{k_0}$$

and

$$P_2(t_1, \ldots, t_{k_0}) := t_1^2 + \ldots + t_{k_0}^2.$$

Doing some further calculations which were done in Maynard's original paper and using the computer, we can get the result.

Now in (3.2), we can set $m = 1$. Since we have strict inequality in (3.4), by Bombieri-Vinogradov theorem, we can take θ as close to $1/2$ as we please to satisfy (3.1). Hence by the corollary, we get $DHL[105, 2]$ holds. This proves the first part of the Theorem (2).

To prove the first part of the Theorem (1), we have the following admissible 105-tuple

$$\begin{aligned}(0, 10, 12, 24, 28, 30, 34, 42, 48, 52, 54, 64, 70, 72, 78, 82, 90, 94, 100, 112, 114, 118,\\ 120, 124, 132, 138, 148, 154, 168, 174, 178, 180, 184, 190, 192, 202, 204, 208, 220,\\ 222, 232, 234, 250, 252, 258, 262, 264, 268, 280, 288, 294, 300, 310, 322, 324, 328,\\ 330, 334, 342, 352, 358, 360, 364, 372, 378, 384, 390, 394, 400, 402, 408, 412, 418,\\ 420, 430, 432, 442, 444, 450, 454, 462, 468, 472, 478, 484, 490, 492, 498, 504, 510,\\ 528, 532, 534, 538, 544, 558, 562, 570, 574, 580, 582, 588, 594, 598, 600)\end{aligned}$$

found by Engelsma. Using this and recalling $H_m = h_{k_0} - h_1$, we get $H_1 \leq 600$. This concludes the proof.

Remark. In fact, we can show $M_5 > 2$. Hence if we assume Elliott-Halberstam conjecture, we can take θ as close to 1 as we please and get $DHL[5,2]$ holds. Taking admissible 5-tuple $(0,2,6,8,12)$, we get $H_1 \leq 12$.

3.1.2 Proof for Case $m \geq 2$

In this paper, as we follow Tao's blog post, we do not prove the second part of the theorem. Instead, we are going to prove a slightly weaker version which Tao proves in his post [9]. Namely, we prove unconditionally, we have

$$DHL[k_0, m+1] \text{ for sufficiently large } k_0 \text{ and } 4m < \log k_0 - 4\log\log k_0 - O(1). \tag{3.5}$$

As the bound $4m < \log k_0 - 4\log\log k_0 - O(1)$ is satisfied if $k_0 \geq Cm^4 e^{4m}$ for sufficiently large C and m large enough. Also, using the fact that one can establish an admissible k_0-tuple of length $O(k_0 \log k_0)$ by simple taking first k_0 primes greater than k_0 which is trivially an admissible k_0-tuple. Then in other words, unconditionally, we have

$$H_m \leq Cm^5 e^{4m} \quad \text{for an adjusted absolute constant } C \text{ and } m \geq 1.$$

Now we prove (3.5). In fact, after the corollary, we want to show

$$M_{k_0} > \log k_0 - 4\log\log k_0 - O(1). \tag{3.6}$$

From this point on using the same reasoning as in Section 3.1.1, we conclude (3.5). Here we use the following function

$$F = 1_{\Delta_{k_0}} F_0$$

where F_0 is the tensor product

$$F_0(t_1, \ldots, t_k) = \prod_{i=1}^{k_0} k_0^{1/2} g(k_0 t_i)$$

and $g : [0, +\infty) \to \mathbf{R}$ is supported on some interval $[0, T]$ with the normalization

$$\int_0^\infty g(t)^2 \, dt = 1. \tag{3.7}$$

Hence the function F is clearly symmetric and supported on the simplex

$$\Delta_{k_0} := \{(t_1, \ldots, t_{k_0}) \in [0, +\infty)^{k_0} : t_1 + \ldots + t_{k_0} \leq 1\}.$$

Now let's consider the fraction

$$k_0 \frac{\int_{\Delta_{k_0-1}} \left(\int_0^\infty F(t_1, \ldots, t_{k_0-1}, t_{k_0})\ dt_{k_0}\right)^2\ dt_1 \ldots dt_{k_0-1}}{\int_{\Delta_{k_0}} F(t_1, \ldots, t_{k_0})^2\ dt_1 \ldots dt_{k_0}}.$$

Bounding F by F_0 and using Fubini's theorem and (3.7), we can nicely bound the denominator by

$$\int_{\Delta_{k_0}} F(t_1, \ldots, t_{k_0})^2\ dt_1 \ldots dt_{k_0} \leq 1.$$

Thus we get the following lower bound for M_{k_0}

$$M_{k_0} \geq k_0 \int_{\Delta_{k_0-1}} \left(\int_0^\infty F(t_1, \ldots, t_{k_0-1}, t_{k_0})\ dt_{k_0}\right)^2\ dt_1 \ldots dt_{k_0-1}.$$

Now we consider the inner integral

$$\int_0^\infty F(t_1, \ldots, t_{k_0-1}, t_{k_0})\ dt_{k_0} = \int_0^\infty 1_{\Delta_{k_0}} \prod_{i=1}^{k_0} k_0^{1/2} g(k_0 t_i)\ dt_{k_0}. \tag{3.8}$$

We observe that if $t_{k_0} > \frac{T}{k_0}$, then $k_0 t_{k_0} > T$, and as g is supported on the interval $[0, T]$, we get $g(k_0 t_{k_0}) = 0$. Hence we are interested only in the region $t_{k_0} \leq \frac{T}{k_0}$. But in this region, the simplex Δ_{k_0} is equivalent to

$$t_1 + \ldots + t_{k_0-1} \leq 1 - \frac{T}{k_0}.$$

Hence we can write (3.8) as

$$\begin{aligned}\int_0^\infty F(t_1, \ldots, t_{k_0-1}, t_{k_0})\ dt_{k_0} &= \left(\prod_{i=1}^{k_0-1} k_0^{1/2} g(k_0 t_i)\right) k_0^{1/2} \int_0^\infty g(k_0 t_{k_0})\ dt_{k_0} \\ &= \left(\prod_{i=1}^{k_0-1} k_0^{1/2} g(k_0 t_i)\right) k_0^{-1/2} \int_0^\infty g(t)\ dt\end{aligned}$$

whenever we have $t_1 + \ldots + t_{k_0-1} \le 1 - \frac{T}{k_0}$. Hence we get

$$\begin{aligned} M_{k_0} &\ge k_0 \int_{t_1+\ldots+t_{k_0-1}\le 1-\frac{T}{k_0}} \left(\left(\prod_{i=1}^{k_0-1} k_0^{1/2} g(k_0 t_i) \right) k_0^{-1/2} \int_0^\infty g(t)\ dt \right)^2 dt_1 \ldots dt_{k_0-1} \\ &= \left(\int_0^\infty g(t)\ dt \right)^2 \int_{t_1+\ldots+t_{k_0-1}\le 1-\frac{T}{k_0}} \left(\prod_{i=0}^{k_0-1} k_0^{1/2} g(k_0 t_i) \right)^2 dt_1 \ldots dt_{k_0-1}. \end{aligned}$$

From now on, we use probabilistic methods to lower bound the last quantity. Let $X_1, \ldots, X_{k_0-1}$ be independent, identically distributed non-negative real random variables with probability density $g(t)^2$; this is well-defined due to the normalization on g. We observe that $\left(\prod_{i=1}^{k_0-1} k_0^{1/2} g(k_0 t_i)\right)^2$ is the joint probability density of $\frac{1}{k_0}(X_1, \ldots, X_{k_0-1})$. Thus we get

$$M_{k_0} \ge \left(\int_0^\infty g(t)\ dt \right)^2 \mathbf{P}(X_1 + \ldots + X_{k_0-1} \le k_0 - T).$$

Here we use Chebyshev's inequality to lower bound the probability. We assume the mean $(k_0-1)\mu$ of $X_1, \ldots, X_{k_0-1}$, where $\mu := \int_0^T t g(t)^2\ dt$, is less than $k_0 - T$ or the even stronger

$$(k_0 - 1)\mu < k_0 - T. \tag{3.9}$$

The variance of $X_1, \ldots, X_{k_0-1}$ is $k_0 - 1$ times the variance of a single X_i which we can bound by

$$\mathrm{Var}(X_i) \le \mathbf{E} X_i^2 \le T \mathbf{E} X_i = T\mu.$$

Hence by Chebyshev's inequality, we get

$$\begin{aligned} \mathbf{P}(X_1 + \ldots + X_{k_0-1} \le k_0 - T) &= 1 - \mathbf{P}(X_1 + \ldots + X_{k_0-1} > k_0 - T) \\ &= 1 - \mathbf{P}(X_1 + \ldots + X_{k_0-1} - (k_0-1)\mu > k_0 - T - (k_0-1)\mu) \\ &\ge 1 - \mathbf{P}(|X_1 + \ldots + X_{k_0-1} - (k_0-1)\mu| > k_0 - T - (k_0-1)\mu) \\ &\ge 1 - \frac{(k_0-1)T\mu}{(k_0 - T - (k_0-1)\mu)^2} \end{aligned}$$

where we use (3.9) to avoid 0 division in the last inequality. Using $k_0 - 1 < k_0$ and assuming $k_0 - T > k_0\mu$, we can make some simplifications:

$$\mathbf{P}(X_1 + \ldots + X_{k_0-1} \le k_0 - T) \ge 1 - \frac{k_0 T \mu}{(k_0 - T - k_0\mu)^2}.$$

Also, dividing both numerator and denominator by k_0 and assuming $\mu \le 1$, we can further simplify and get

$$\mathbf{P}(X_1 + \ldots + X_{k_0-1} \le k_0 - T) \ge 1 - \frac{T}{k_0(1 - T/k_0 - \mu)^2}.$$

Hence we get

$$M_{k_0} \ge \left(\int_0^T g(t)\ dt \right)^2 \left(1 - \frac{T}{k_0(1 - T/k_0 - \mu)^2} \right) \tag{3.10}$$

with $g : [0, T] \to \mathbf{R}$ satisfying (3.7) and

$$\mu = \int_0^T t g(t)^2\ dt < 1 - \frac{T}{k_0}$$

which also implies (3.9). In particular, we set the function g of the form

$$g(t) = \frac{c}{1 + At}$$

with $A := \log k_0$ and $T := k_0 \log^{-3} k_0$. In order to satisfy (3.7), we need to have

$$\begin{aligned} 1 = \int_0^\infty g(t)^2\ dt = \int_0^T g(t)^2\ dt = \int_0^T \left(\frac{c}{1 + At} \right)^2\ dt = c^2 \int_0^T \frac{1}{(1 + At)^2}\ dt \\ = c^2 \left[-\frac{1}{A} \frac{1}{(1 + At)} \right]_0^T = c^2 \frac{T}{1 + AT}. \end{aligned}.$$

Hence we get

$$c^2 = \frac{1 + AT}{T} = A + \frac{1}{T} = \log k_0 + \frac{\log^3 k_0}{k_0} = \log k_0 + O(1).$$

Thus we have

$$c = \log^{1/2} k_0 + O(\log^{-1/2} k_0).$$

We also compute μ as

$$
\begin{aligned}
\mu &= \int_0^T t\left(\frac{c}{1+At}\right)^2 \, dt = c^2\int_0^T \frac{t}{(1+At)^2}\, dt = \frac{c^2}{A^2}\left[\log(1+At)+\frac{1}{1+At}\right]_0^T \\
&= \frac{c^2}{A^2}\left(\log(1+AT)+\frac{1}{1+AT}-1\right) = \frac{c^2}{A^2}\log(1+AT)+\frac{1}{A^2T}-\frac{c^2}{A^2} \\
&= \frac{1+\frac{\log k_0}{k_0}}{\log k_0}\log(1+k_0\log^{-2}k_0)+\frac{\log k_0}{k_0}-\frac{1+\frac{\log k_0}{k_0}}{\log k_0} \\
&= \left(\frac{1}{\log k_0}+\frac{1}{k_0}\right)\left(\log k_0 - 2\log\log k_0 + O\left(\log(k_0\log^{-2}k_0)\right)\right)+\frac{\log k_0}{k_0}-\frac{1}{\log k_0}-\frac{1}{k_0} \\
&= 1-\frac{2\log\log k_0}{\log k_0}+\frac{\log k_0}{k_0}-\frac{2\log\log k_0}{k_0}+\frac{\log k_0}{k_0}-\frac{1}{\log k_0}-\frac{1}{k_0}+O\left(\frac{1}{\log k_0}\right) \\
&= 1-\frac{2\log\log k_0}{\log k_0}+O\left(\frac{1}{\log k_0}\right).
\end{aligned}
$$

Substituting this back, we get

$$
\frac{T}{k_0(1-T/k_0-\mu)^2} = O\left(\frac{1}{\log k_0}\right).
$$

Lastly, for the integral of g, we have

$$
\begin{aligned}
\int_0^T g(t)\, dt = \int_0^T \frac{c}{1+At}\, dt &= \frac{c}{A}\log\left[1+At\right]_0^T = \frac{c}{A}\log(1+AT) \\
&= \frac{c}{A}\log(1+k_0\log^{-2}k_0) \ll \frac{c}{A}(\log k_0 - 2\log\log k_0) \\
&= \log^{1/2}k_0 - 2\log^{-1/2}k_0\log\log k_0 + O(\log^{-1/2}k_0).
\end{aligned}
$$

Putting these back to (3.10), we get the result. This completes the proof.

Remark. Again, if we assume Elliott-Halberstam conjecture, we can take θ as close to 1 as we please and get $DHL[k_0, m+1]$ holds for sufficiently large k_0 and $2m < \log k_0 - 4\log\log k_0 - O(1)$.

Remark. We have $M_{k_0} \leq \frac{k_0}{k_0-1}\log k_0$ for every $k_0 \in \mathbf{N}$ by Corollary 6.4 of [8]. In particular, if we set $k_0 = 50$, we get $M_{50} < 4$, hence we fail to prove $DHL[50, 2]$ with this method. However, the latest result [8] uses some modification (based on Zhang's ideas [11]) to prove $DHL[50, 2]$ which is equivalent to $H_1 \leq 246$.

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