Equality in some Geometric Inequalities

Pavlos Kalantzopoulos

Supervisor:

Károly Böröczky

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Central European University

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Abstract

A simply way to prove Bollobas-Thomason inequality is via the Geometric Brascamp-Lieb inequality due to Liakopoulos. The same author found the dual Bollobas-Thomason as an application of the Reverse Brascamp-Lieb inequality. Here we show the equality case of the Bollobas-Thomason inequality, via the characterization of the equality case of Geometric Brascamp-Lieb inequality due to Valdimarsson. In addition, we give a partial characterization of the equality case of the Reverse Brascamp-Lieb inequality. This allows us to have the equality case of the dual Bollobas-Thomason inequality.

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1 Introduction

For a proper linear subspace E of $\mathbb{R}^n$ ($E \neq \mathbb{R}^n$ and $E \neq \{o\}$), let P_E denote the orthogonal projection into E. We write $e_1, \dots, e_n$ to denote an orthonomal basis of $\mathbb{R}^n$. For a compact set $K \subset \mathbb{R}^n$ with aff $K = m$, we write $|K|$ to denote the m-dimensional Lebesgue measure of K.

The starting point of my thesis is the classical Loomis-Whitney inequality [45].

Theorem 1 (Loomis, Whitney) *If $K \subset \mathbb{R}^n$ is compact and affinely spans $\mathbb{R}^n$, then*

$$|K|^{n-1} \leq \prod_{i=1}^{k} |P_{e_i^\perp} K|, \tag{1}$$

with equality if and only if $K = \oplus_{i=1}^n K_i$ where $\operatorname{aff} K_i$ is a line parallel to e_i.

Meyer [53] provided a dual form of the Loomis-Whitney inequality where equality holds for affine crosspolytopes.

Theorem 2 (Meyer) *If $K \subset \mathbb{R}^n$ is compact convex with $o \in \operatorname{int} K$, then*

$$|K|^{n-1} \geq \frac{n!}{n^n} \prod_{i=1}^{k} |K \cap e_i^\perp|, \tag{2}$$

with equality if and only if $K = \operatorname{conv}\{\pm\lambda_i e_i\}_{i=1}^n$ for $\lambda_i > 0$, $i = 1, \dots, n$.

We note that various Reverse and dual Loomis-Whitney type inequalities are proved by S. Campi, R. Gardner, P. Gronchi [47].

To consider a genarization of the Loomis-Whitney inequality and its dual form, we set $[n] := \{1, \dots, n\}$, and for a non-empty proper subset $\sigma \subset [n]$, we define $E_\sigma = \operatorname{lin}\{e_i\}_{i \in \sigma}$. For $s \geq 1$, we say that the not necessarily distinct proper non-empty subsets $\sigma_1, \dots, \sigma_k \subset [n]$ form an s-uniform cover of $[n]$ if each $j \in [n]$ is contained in exactly s of $\sigma_1, \dots, \sigma_k$.

The Bollobas-Thomason inequality [11] reads as follows.

Theorem 3 (Bollobas, Thomason) *If $K \subset \mathbb{R}^n$ is compact and affinely spans $\mathbb{R}^n$, and $\sigma_1, \dots, \sigma_k \subset [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$, then*

$$|K|^s \leq \prod_{i=1}^{k} |P_{E_{\sigma_i}} K|. \tag{3}$$

We note that additional the case when $k = n$, $s = n - 1$, and hence when we may assume that $\sigma_i = [n] \backslash e_i$, is the Loomis-Whitney inequality Therem 1.

Liakopoulos [44] managed to prove a dual form of the Bollobas-Thomason inequality. For a finite set σ, we write $|\sigma|$ to denote its cardinality.

Theorem 4 (Liakopoulos) *If $K \subset \mathbb{R}^n$ is compact convex with $o \in \operatorname{int} K$, and $\sigma_1, \dots, \sigma_k \subset [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$, then*

$$|K|^s \geq \frac{\prod_{i=1}^k |\sigma_i|!}{(n!)^s} \cdot \prod_{i=1}^{k} |K \cap E_{\sigma_i}|. \tag{4}$$

However, unlike for Loomis-Whitney inequality and its dual form, neither the equality cases of the Bollobas-Thomason inequality nor of its dual are known. The characterization of the equality cases of Theorem 3 and Theorem 4 is the main focus of this thesis.

Let $s \geq 1$, and let $\sigma_1, \ldots, \sigma_k \subset [n]$ be an s-uniform cover of $[n]$. We say that $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l \subset [n]$ form a 1-uniform cover of $[n]$ induced by the s-uniform cover $\sigma_1, \ldots, \sigma_k$ if $\{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l\}$ consists of all non-empty distinct subsets of $[n]$ of the form $\cap_{i=1}^{k} \sigma_i^{\varepsilon(i)}$ where $\varepsilon(i) \in \{0,1\}$ and $\sigma_i^0 = \sigma_i$ and $\sigma_i^1 = [n] \setminus \sigma_i$. We observe that $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l \subset [n]$ actually form a 1-uniform cover of $[n]$; namely, $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l$ is a partition of $[n]$.

Theorem 5 *Let $K \subset \mathbb{R}^n$ be compact and affinely span $\mathbb{R}^n$, and let $\sigma_1, \ldots, \sigma_k \subset [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$. Then equality holds in* (3) *if and only if $K = \oplus_{i=1}^{l} P_{E_{\tilde{\sigma}_i}} K$ where $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l$ is the 1-uniform cover of $[n]$ induced by $\sigma_1, \ldots, \sigma_k$.*

Concerning the dual Bollobas-Thomason inequality Theorem 4, we have a similar result.

Theorem 6 *Let $K \subset \mathbb{R}^n$ be compact convex with $o \in \text{int}K$, and let $\sigma_1, \ldots, \sigma_k \subset [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$. Then equality holds in* (4) *if and only if $K = \text{conv}\{K \cap F_{\tilde{\sigma}_i}\}_{i=1}^{l}$ where $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l$ is the 1-uniform cover of $[n]$ induced by $\sigma_1, \ldots, \sigma_k$.*

According to Liakopoulos [44] (see also Section 7), a simply way to prove Theorem 3 and Theorem 4 is via the Geometric Brascamp-Lieb inequality Theorem 7 and its Reverse form Theorem 8. In particular, we prove the equality case Theorem 5 of the Bollobas-Thomason inequality via the characterization of the equality case Theorem 9 due to by Valdimarsson [59] of the Brascamp-Lieb inequality. In addition, we prove Theorem 10 characterizing the equality case of the Reverse Brascamp-Lieb inequality in a special case that yields the understanding of equality in the dual Bollobas-Thomason inequality.

We say that the proper linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ form a Geometric Brascamp-Lieb data if they satisfy

$$\sum_{i=1}^{k} c_i P_{E_i} = I_n. \tag{5}$$

The name "Geometric Brascamp-Lieb data" comes from the following theorem, originating in the work of Brascamp, Lieb [14].

Theorem 7 (Brascamp, Lieb) *For the proper linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfying* (5)*, and for non-negative $f_i \in L_1(E_i)$, we have*

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(P_{E_i} x)^{c_i} \, dx \leq \prod_{i=1}^{k} \left(\int_{E_i} f_i \right)^{c_i}. \tag{6}$$

For the Brascamp-Lieb inequality Theorem 7, Brascamp, Lieb [14] proved the rank one case when $\dim E_i = 1$ for $i = 1, \ldots, k$, and Lieb [46] proved the general case. We note that equality holds in Theorem 7 if $f_i(x) = e^{-\pi \|x\|^2}$ for $i = 1, \ldots, k$; and hence, each f_i is a Gaussian density. Actually, Theorem 7, which is an important special case of the general Brascamp-Lieb inequality, is named Geometric Brascamp-Lieb inequality by Bennett, Carbery, Christ, Tao [10]. The form Geometric Brascamp-Lieb inequality of the otherwise more general Brascamp-Lieb inequality was discovered by Ball [2, 3].

Answering a conjecture by Ball, a Reverse form of the Geometric Brascamp-Lieb inequality was proved by Barthe [5]. We write $\int_{\mathbb{R}^n}^{*}$ to denote outer integral for a possibly non-integrable function.

Theorem 8 (Barthe) *For the proper linear subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfying* (5)*, and for non-negative $f_i \in L_1(E_i)$, we have*

$$\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i}\, dx \geq \prod_{i=1}^k \left(\int_{E_i} f_i \right)^{c_i}. \tag{7}$$

Let $E_1, \dots, E_k$ the proper linear subspaces of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfy (5). Valdimarsson [59] introduced the so called independent subspaces and the dependent space. We write J to denote the set of 2^k functions $\{1, \dots, k\} \to \{0, 1\}$. If $\varepsilon \in J$, then let $F_\varepsilon = \cap_{i=1}^k E^{\varepsilon(i)}$ where $E_i^0 = E_i$ and $E_i^1 = E_i^\perp$ for $i = 1, \dots, k$. We write J_0 to denote the subset of $\varepsilon \in J$ such that $\dim F_\varepsilon \geq 1$, and such an F_ε is called independent following Valdimarsson [59]. Readily F_ε and $F_{\tilde{\varepsilon}}$ are orthogonal if $\varepsilon \neq \tilde{\varepsilon}$ for $\varepsilon, \tilde{\varepsilon} \in J_0$. In addition, we write $F_{\rm dep}$ to denote the orthogonal component of $\oplus_{\varepsilon \in J_0} F_\varepsilon$. In particular, $\mathbb{R}^n$ can be written as a direct sum of pairwise orthogonal linear subspaces in the form

$$\mathbb{R}^n = \left(\oplus_{\varepsilon \in J_0} F_{(\varepsilon)} \right) \oplus F_{\rm dep}. \tag{8}$$

Here it is possible that $J_0 = \emptyset$, and hence $\mathbb{R}^n = F_{\rm dep}$, or $F_{\rm dep} = \{o\}$, and hence $\mathbb{R}^n = \oplus_{\varepsilon \in J_0} F_\varepsilon$ in that case.

Now we quote the special case of Valdimarsson's [59] characterization of the equality case of the Brascamp-Lieb inequality that we need to handle the Bollobás-Thomason inequality.

Theorem 9 (Valdimarsson) *For the proper linear subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfying* (5) *and $F_{\rm dep} = \{o\}$, let us assume that equality holds in* (6) *for non-negative $f_i \in L_1(E_i)$, $i = 1, \dots, k$, with positive integral. Writing $F_1, \dots, F_l$ to denote the independent subspaces, there exist $\theta_i > 0$ for $i = 1, \dots, k$ and $h_j : F_j \to [0, \infty)$ for $j = 1, \dots, l$ such that*

$$f_i(x) = \theta_i \prod_{F_j \subset E_i} h_j(P_{F_j}(x)) \quad \text{for Lebesgue a.a. } x \in E_i.$$

Theorem 10 clarifies the equality conditions in the Reverse Brascamp-Lieb inequality in some special cases that cover say the recent dual Bollobas-Thomason inequality Theorem 6. We say that a function $h : \mathbb{R}^n \to [0, \infty)$ is log-concave if $h((1-\lambda)x + \lambda y) \geq h(x)^{1-\lambda} h(y)^\lambda$ for any $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$; or in other words, $h = e^{-W}$ for a convex function $W : \mathbb{R}^n \to (-\infty, \infty]$.

Theorem 10 *For the proper linear subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfying* (5) *and $F_{\rm dep} = \{o\}$, let us assume that equality holds in* (7) *for non-negative $f_i \in L_1(E_i)$, $i = 1, \dots, k$, with positive integral. Writing $F_1, \dots, F_l$ to denote the independent subspaces, there exist $\theta_i > 0$ and $w_i \in E_i$ for $i = 1, \dots, k$ and log-concave $h_j : F_j \to [0, \infty)$ for $j = 1, \dots, l$ such that*

$$f_i(x) = \theta_i \prod_{F_j \subset E_i} h_j(P_{F_j}(x - w_i)) \quad \text{for Lebesgue a.a. } x \in E_i.$$

Theorem 10 explains the term "independent subspaces" because the functions h_j are chosen freely and independently of each other on F_j.

2 The determinantal inequality and structure theory for rank one Geometric Brascamp-Lieb data

We first discuss the basic properties of a set of vectors $u_1, \ldots, u_n \in S^{n-1}$ and constants $c_1, \ldots, c_n > 0$ occurring in the Geometric Brascamp-Lieb inequality; namely, satisfying

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = I_n. \tag{9}$$

This section just retells the story of Section 2 of Barthe [5] in the language of Bennett, Carbery, Christ, Tao [10].

Lemma 11 *For $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfying (9), we have*

(i) $\sum_{i=1}^{k} c_i = n$;

(ii) $\sum_{i=1}^{k} c_i \langle u_i, x\rangle^2 = \|x\|^2$ for all $x \in \mathbb{R}^n$;

(iii) $c_i \leq 1$ for $i = 1, \ldots, k$ with equality if and only if $u_j \in u_i^\perp$ for $j \neq i$;

(iv) $u_1, \ldots, u_k$ spans $\mathbb{R}^n$, and $k = n$ if and only if $u_1, \ldots, u_n$ is an orthonormal basis of $\mathbb{R}^n$ and $c_1 = \ldots = c_n = 1$;

(v) if L is a proper linear subspace of $\mathbb{R}^n$, then

$$\sum_{u_i \in L} c_i \leq \dim L,$$

with equality if and only if $u_1, \ldots, u_k \subset L \cup L^\perp$.

Remark If $\sum_{u_i \in L} c_i = \dim L$ in (v), then $\mathrm{lin}\{u_i : u_i \in L\} = L$ and $\mathrm{lin}\{u_i : u_i \in L^\perp\} = L^\perp$.
Proof: Here (i) follows from comparing the traces of the two sides of (9), and (ii) is just an equivalent form of (9). To prove $c_j \leq 1$ with the characterization of equality, we substitute $x = u_j$ into (ii).

Turning to (iv), let us assume that $u_1, \ldots, u_n \in S^{n-1}$ and $c_1, \ldots, c_n > 0$ satisfy (9). We consider $w_j \in S^{n-1}$ for $j = 1, \ldots, n$ such that $\langle w_j, u_i\rangle = 0$, $i = 1, \ldots, n$, and (ii) shows that $u_j = \pm w_j$ and $c_j = 1$.

For (v), let $v_i = \sqrt{c_i}\, u_i$ for $i = 1, \ldots, k$, and we observe that (ii) is equivalent with

$$\sum_{i=1}^{k} \langle v_i, x\rangle^2 = \|x\|^2 \;\text{ for all } x \in \mathbb{R}^n \tag{10}$$

where (i) yields that

$$\sum_{i=1}^{k} \|v_i\|^2 = n. \tag{11}$$

If $u_i \notin L$, then let $\tilde{v}_i = P_{L^\perp} v_i$. We deduce that if $x \in L^\perp$, then

$$\|x\|^2 = \sum_{i=1}^{k} \langle v_i, x\rangle^2 = \sum_{u_i \notin L} \langle \tilde{v}_i, x\rangle^2. \tag{12}$$

It follows from (i) and (ii) (compare (10) and (11)) applied in $L^\perp$ instead of $\mathbb{R}^n$ that

$$\dim L^\perp = \sum_{u_i \notin L} \|\tilde{v}_i\|^2 \leq \sum_{u_i \notin L} \|v_i\|^2 = \sum_{u_i \notin L} c_i. \tag{13}$$

In turn, we conclude the inequality in (v). If equality holds in (v), then $\|v_i\| = \|\tilde{v}_i\|$ whenever $u_i \notin L$; therefore, $u_1, \ldots, u_k \subset L \cup L^\perp$. $\Box$

Let $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfy (9). Following Bennett, Carbery, Christ, Tao [10], we say that a non-zero linear subspace V is a critical subspace with respect to $u_1, \ldots, u_k$ and $c_1, \ldots, c_k$ if

$$\sum_{u_i \in V} c_i = \dim V.$$

In particular, $\mathbb{R}^n$ is a critical subspace according to Lemma 11. We say that a critical subspace V is indecomposable if V has no proper critical linear subspace. In addition, we say that a non-empty subset $\mathcal{U} \subset \{u_1, \ldots, u_k\}$ is indecomposable if $\mathrm{lin}\,\mathcal{U}$ is an indecomposable critical subspace.

In order to understand the equality case of the rank one Brascamp-Lieb inequality, Barthe [5] indicated an equivalence relation on $\{u_1, \ldots, u_k\}$. First, we write that $u_i \bowtie u_j$ if there exists a subset $\mathcal{U} \subset \{u_1, \ldots, u_k\}$ of cardinality $n-1$ such that both $\{u_i\} \cup \mathcal{U}$ and $\{u_j\} \cup \mathcal{U}$ are independent. We define $\sim$ to be the transitive completion of $\bowtie$ on $\{u_1, \ldots, u_k\}$, and hence $\sim$ is an equivalence relation on $\{u_1, \ldots, u_k\}$.

Lemma 12 *For $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfying* (9)*, we have*

(i) a proper linear subspace $V \subset \mathbb{R}^n$ is critical if and only if $\{u_1, \ldots, u_k\} \subset V \cup V^\perp$;

(ii) if V, W are proper critical subspaces with $V \cap W \neq \{o\}$, then $V^\perp$, $V \cap W$ and $V + W$ are critical subspaces;

(iii) the equivalence classes with respect to $\sim$ are the indecomposable subsets of $\{u_1, \ldots, u_k\}$;

(iv) the proper indecomposable critical subspaces are pairwise orthogonal, and any critical subspace is the sum of some indecomposable critical subspaces.

Proof: (i) directly follows from Lemma 11 (v), and in turn (i) yields (ii).

We prove (iii) and and first half of (iv) simultatinuously. We say that a subset $\mathcal{D} \subset \{u_1, \ldots, u_k\}$ is minimally dependent if $\mathcal{D}$ is dependent and no proper subset of $\mathcal{D}$ is dependent. Since $u_1, \ldots, u_k$ spans $\mathbb{R}^n$, $u_i \bowtie u_j$ for $i \neq j$ is equivalent with the existence of a minimal dependent set $\mathcal{D} \subset \{u_1, \ldots, u_k\}$ satisfying $u_i, u_j \in \mathcal{D}$. This new formulation shows that if $V_1, \ldots, V_m$ are the linear hulls of the equivalence classes with respect to $\sim$, then $V_1, \ldots, V_m$ are complementary; or in words, $\dim V_1 + \ldots + \dim V_m = n$.

We deduce from Lemma 11 (v) that each V_i is a critical subspace, and if $i \neq j$, then V_i and V_j are orthogonal.

Next let $\mathcal{U} \subset \{u_1, \ldots, u_k\}$ be an indecomposable set, and let $V = \mathrm{lin}\,\mathcal{U}$. We write $I \subset \{1, \ldots, m\}$ to denote the set of indices i such that $V_i \cap \mathcal{U} \neq \emptyset$. Since V is a critical subspace, we deduce from Lemma 11 (v) that $V_i \cap V$ is a critical subspace for $i \in I$, as well; therefore, I consists of a unique index p as $\mathcal{U}$ is indecomposable. In particular, $V = V_p$.

It follows from Lemma 11 (v) that $\{u_1, \ldots, u_k\} \subset V \cup V^\perp$; therefore, there exists no minimally dependent subset of $\{u_1, \ldots, u_k\}$ intersecting both $\mathcal{U}$ and its complement. We conclude that $V = V_p$.

Finally, the second half of (iv) follows from (i) and (ii). $\square$

Proposition 13 *For $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$ satisfying (9), if $t_i > 0$ for $i = 1, \dots, k$, then*

$$\det\left(\sum_{i=1}^{k} c_i t_i u_i \otimes u_i\right) \geq \prod_{i=1}^{k} t_i^{c_i}. \tag{14}$$

Equality holds in (14) if and only if $t_i = t_j$ for any u_i and u_j lying in the same indecomposable subset of $\{u_1, \dots, u_k\}$.

Proof: To simplify expressions, let $v_i = \sqrt{c_i} u_i$ for $i = 1, \dots, k$.

In this argument, I always denotes some subset of $\{1, \dots, k\}$ of cardinality n. For $I = \{i_1, \dots, i_n\}$, we define

$$d_I := \det[v_{i_1}, \dots, v_{i_n}]^2 \quad \text{and} \quad t_I := t_{i_1} \cdots t_{i_n}.$$

For the $n \times k$ matrices $M = [v_1, \dots, v_k]$ and $\widetilde{M} = [\sqrt{t_1}\, v_1, \dots, \sqrt{t_k}\, v_k]$, we have

$$MM^T = I_n \text{ and } \widetilde{M}\widetilde{M}^T = \sum_{i=1}^{k} t_i v_i \otimes v_i. \tag{15}$$

It follows from the Cauchy-Binet formula that

$$\sum_I d_I = 1 \quad \text{and} \quad \det\left(\sum_{i=1}^{k} t_i v_i \otimes v_i\right) = \sum_I t_I d_I,$$

where the summations extend over all sets $I \subset \{1, \dots, k\}$ of cardinality n. It follows that the discrete measure μ on the n element subsets of $\{1, \dots, k\}$ defined by $\mu(\{I\}) = d_I$ is a probability measure. We deduce from inequality between the arithmetic and geometric mean that

$$\det\left(\sum_{i=1}^{k} t_i v_i \otimes v_i\right) = \sum_I t_I d_I \geq \prod_I t_I^{d_I}. \tag{16}$$

The factor t_i occurs in $\prod_I t_I^{d_I}$ exactly $\sum_{I,\, i \in I} d_I$ times. Moreover, the Cauchy-Binet formula applied to the vectors $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$ implies

$$\sum_{I,\, i \in I} d_I = \sum_I d_I - \sum_{I,\, i \notin I} d_I = 1 - \det\left(\sum_{j \neq i} v_j \otimes v_j\right)$$
$$= 1 - \det\left(\mathrm{Id}_n - v_i \otimes v_i\right) = \langle v_i, v_i \rangle = c_i.$$

Substituting this into (16) yields (14).

We now assume that equality holds in (14). Since equality holds in (16) when applying arithmetic and geometric mean, all the t_I are the same for any subset I of $\{1, \dots, k\}$ of cardinality n with $d_I \neq 0$. It follows that $t_i = t_j$ whenever $u_i \bowtie u_j$, and in turn we deduce that $t_i = t_j$ whenever u_i and u_j lie in the same indecomposable set by Lemma 12 (i).

On the other hand, Lemma 12 (ii) yields that if $t_i = t_j$ whenever u_i and u_j lie in the same indecomposable set, then equality holds in (14). $\square$

Combining Lemma 12 and Proposition 13 leads to the following:

Corollary 14 *For $u_i \in S^{n-1}$ and $c_i, t_i > 0$, $i = 1, \ldots, k$ satisfying* (9), *equality holds in (14) if and only if there exist pairwise orthogonal linear subspaces $V_1, \ldots, V_m$, $m \geq 1$, such that $\{u_1, \ldots, u_k\} \subset V_1 \cup \ldots \cup V_m$ and $t_i = t_j$ whenever u_i and u_j lie in the same V_p for some $p \in \{1, \ldots, m\}$.*

3 The determinantal inequality corresponding to the higher rank Brascamp-Lieb inequality

We consider proper linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfying the Geometric Brascamp-Lieb condition

$$\sum_{i=1}^{k} c_i P_{E_i} = I_n. \tag{17}$$

We now connect (17) to (9). For $i = 1, \ldots, k$, let $\dim E_i = n_i$ and let $u_1^{(i)}, \ldots, u_{n_i}^{(i)}$ be any orthonormal basis of E_i. In addition, for $i = 1, \ldots, k$, we consider the $n \times n_i$ matrix $M_i = \sqrt{c_i}[u_1^{(i)}, \ldots, u_{n_i}^{(i)}]$. We deduce that

$$c_i P_{E_i} = M_i M_i^T = \sum_{j=1}^{n_i} c_i u_j^{(i)} \otimes u_j^{(i)} \text{ for } i = 1, \ldots, k; \tag{18}$$

$$I_n = \sum_{i=1}^{k} c_i P_{E_i} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} c_i u_j^{(i)} \otimes u_j^{(i)} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} c_j^{(i)} u_j^{(i)} \otimes u_j^{(i)} \tag{19}$$

and hence $u_j^{(i)} \in S^{n-1}$ and $c_j^{(i)} = c_i > 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$ form a Geometric Brascamp-Lieb data like in (9).

Lemma 15 *For proper linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfying* (17)*,*

(i) if $x \in \mathbb{R}^n$, then $\sum_{i=1}^{k} c_i \|P_{E_i} x\|^2 = \|x\|^2$;

(ii) if $V \subset \mathbb{R}^n$ is a proper linear subset, then

$$\sum_{E_i \cap V \neq \{o\}} c_i \dim(E_i \cap V) \leq \dim V \tag{20}$$

where equality holds if and only if $E_i = (E_i \cap V) + (E_i \cap V^\perp)$ for $i = 1, \ldots, k$; or equivalently, when $V = (E_i \cap V) + (E_i^\perp \cap V)$ for $i = 1, \ldots, k$.

Proof: For $i = 1, \ldots, k$, let $\dim E_i = n_i$ and let $u_1^{(i)}, \ldots, u_{n_i}^{(i)}$ be any orthonormal basis of E_i such that if $V \cap E_i \neq \{o\}$, then $u_1^{(i)}, \ldots, u_{m_i}^{(i)}$ is any orthonormal basis of $V \cap E_i$ where $m_i \leq n_i$.

For any $x \in \mathbb{R}^n$ and $i = 1, \ldots, k$, we have $\|P_{E_i} x\|^2 = \sum_{j=1}^{n_i} \langle u_j^{(i)}, x \rangle^2$, thus Lemma 11 (ii) yields (i).

Concerning (ii), Lemma 11 (v) yields (20). On the other hand, if equality holds in (20), then V is a critical subspace for the rank one Geometric Brascamp-Lieb data $u_j^{(i)} \in S^{n-1}$ and $c_j^{(i)} = c_i > 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$ satisfying (19). Thus Lemma 15 (ii) follows from Lemma 11 (v). □

Following Bennett, Carbery, Christ, Tao [10], we say that a non-zero linear subspace V is a critical subspace with respect to the proper linear subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfying (17) if

$$\sum_{E_i \cap V \neq \{o\}} c_i \dim(E_i \cap V) = \dim V.$$

In particular, $\mathbb{R}^n$ is a critical subspace by calculating traces of both sides of (17). For a proper linear subspace $V \subset \mathbb{R}^n$, Lemma 15 yields that V is critical if and only if $V^\perp$ is critical, which is turn equivalent saying that

$$E_i = (E_i \cap V) + (E_i \cap V^\perp) \text{ for } i = 1, \dots, k; \tag{21}$$

or in other words,

$$V = (E_i \cap V) + (E_i^\perp \cap V) \text{ for } i = 1, \dots, k. \tag{22}$$

Again, a critical subspace V is indecomposable if V has no proper critical linear subspace, and we call the Geometric Brascamp-Lieb data of proper linear subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfying (17) indecomposable if there exists no proper critical subspace.

The following was pointed out in Valdimarsson [59].

Lemma 16 *If $E_1, \dots, E_k$ are proper linear subspaces of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfying* (17)*, and V, W are proper critical subspaces with $V \cap W \neq \{o\}$, then $V^\perp$, $V \cap W$ and $V + W$ are critical subspaces.*

Proof: The fact that $V^\perp$ is also critical follows directly from (21).

Concerning $V \cap W$, we need to prove that if $i = 1, \dots, k$, then

$$(V \cap W) \cap E_i + (V \cap W)^\perp \cap E_i = E_i. \tag{23}$$

For a linear subspace $L \subset E_i$, we write $L^{\perp_i} = L^\perp \cap E_i$ to denote the orthogonal complement within E_i. We observe that as V and W are critical subspaces, we have $(V \cap E_i)^{\perp_i} = V^\perp \cap E_i$ and $(W \cap E_i)^{\perp_i} = W^\perp \cap E_i$. It follows from the identity $(V \cap W)^\perp = V^\perp + W^\perp$ that

$$\begin{aligned}
E_i &\supset (V \cap W) \cap E_i + (V \cap W)^\perp \cap E_i = (V \cap E_i) \cap (W \cap E_i) + (V^\perp + W^\perp) \cap E_i \\
&\supset (V \cap E_i) \cap (W \cap E_i) + (V^\perp \cap E_i) + (W^\perp \cap E_i) \\
&= (V \cap E_i) \cap (W \cap E_i) + (V \cap E_i)^{\perp_i} + (W \cap E_i)^{\perp_i} \\
&= (V \cap E_i) \cap (W \cap E_i) + [(V \cap E_i) \cap (W \cap E_i)]^{\perp_i} = E_i,
\end{aligned}$$

yielding (23).

Finally, $V + W$ is also critical as $V + W = (V^\perp \cap W^\perp)^\perp$. □

We deduce from Lemma 16 that any crtical subspace can be decomposed inro indecomposable ones.

Corollary 17 *If $E_1, \dots, E_k$ are proper linear subspaces of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfy* (17)*, and W is a critical subspace or $W = \mathbb{R}^n$, then there exist pairwise orthogonal indecomposable critical subspaces $V_1, \dots, V_m$, $m \geq 1$, such that $W = V_1 + \ldots + V_m$ (possibly $m = 1$ and $W = V_1$).*

For a non-zero linear subspace $L \subset \mathbb{R}^n$, we say that a linear transformation $A : L \to L$ is positive definite if $\langle Ax, y \rangle = \langle x, Ay \rangle$ and $\langle x, Ax \rangle > 0$ for any $x, y \in L \backslash \{o\}$. The following is indicated in Barthe [5].

Proposition 18 (Barthe) *For proper linear subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfying* (17)*, if $A_i : E_i \to E_i$ is a positive definite linear transformation for $i = 1, \dots, k$, then*

$$\det\left(\sum_{i=1}^{k} c_i A_i P_{E_i}\right) \geq \prod_{i=1}^{k} (\det A_i)^{c_i}. \tag{24}$$

Equality holds in (24) *if and only if there exist linear subspaces $V_1, \dots, V_m$ where $V_1 = \mathbb{R}^n$ if $m = 1$ and $V_1, \dots, V_m$ are pairwise orthogonal indecomposable critical subspaces spanning $\mathbb{R}^n$ if $m \geq 2$, and $\lambda_1, \dots, \lambda_m > 0$ such that each E_i is spanned by the subspaces $E_i \cap V_j$ for $j = 1, \dots, m$, and if $E_i \cap V_j \neq \{o\}$, then $E_i \cap V_j$ is an eigenspace of A_i with eigenvalue λ_j.*

Proof: For $i = 1, \dots, k$, let $\dim E_i = n_i$, let $u_1^{(i)}, \dots, u_{n_i}^{(i)}$ be an orthonormal basis of E_i consisting of eigenvectors of A_i, and let $\lambda_j^{(i)} > 0$ be the eigenvalue of A_i corresponding to $u_j^{(i)}$. In particular $\det A_i = \prod_{j=1}^{n_i} \lambda_j^{(i)}$ for $i = 1, \dots, k$. In addition, for $i = 1, \dots, k$, we set $M_i = \sqrt{c_i}[u_1^{(i)}, \dots, u_{n_i}^{(i)}]$ and B_i to be the positive definite transformation with $A_i = B_i B_i$, and hence

$$c_i A_i P_{E_i} = (M_i B_i)(M_i B_i)^T = \sum_{j=1}^{n_i} c_i \lambda_j^{(i)} u_j^{(i)} \otimes u_j^{(i)}.$$

We deduce from Lemma 13 and (19) that

$$\begin{aligned} \det\left(\sum_{i=1}^{k} c_i A_i P_{E_i}\right) &= \det\left(\sum_{i=1}^{k} \sum_{j=1}^{n_i} c_i \lambda_j^{(i)} u_j^{(i)} \otimes u_j^{(i)}\right) \\ &\geq \prod_{i=1}^{k} \left(\prod_{j=1}^{n_i} \lambda_j^{(i)}\right)^{c_i} = \prod_{i=1}^{k} (\det A_i)^{c_i}. \end{aligned} \tag{25}$$

Finally, if we have equality in (24), and hence also in (25), then Corollary 14 implies that there exist pairwise orthogonal critical subspaces $V_1, \dots, V_m$, $m \geq 1$ spanning $\mathbb{R}^n$ and $\lambda_1, \dots, \lambda_m > 0$ (where $V_1 = \mathbb{R}^n$ if $m = 1$) such that if $E_i \cap V_j \neq \{o\}$, then $E_i \cap V_j$ is an eigenspace of A_i with eigenvalue λ_j. We conclude from (21) that each V_j is a critical subspace, and from Corollary 17 that each V_j can be assumed to be indecomposable. Finally, (21) yields that each E_i is spanned by the subspaces $E_i \cap V_j$ for $j = 1, \dots, m$. □

4 Further structural theory of a Geometric Brascamp-Lieb data

This section describes the structure of the Brascamp-Lieb data consisting of proper linear subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfying (17) based on Valdimarsson [59]. We deduce from Lemma 15 (i) and (22) that if V is a critical subspace, then writing $P_{E_i \cap V}^{(V)}$ to denote the restriction of $P_{E_i \cap V}$ onto V, we have

$$\sum_{E_i \cap V \neq \{o\}} c_i P_{E_i}^{(V)} = I_V \tag{26}$$

where I_V denotes the identity transformation on V.

If each E_i in the Geometric Brascamp-Lieb data is one dimensional, then Lemma 12 (iii) says that lower dimensional indecomposable critical subspaces are pairwise orthogonal, and hence there exists a unique decomposition of $\mathbb{R}^n$ as a direct some of indecomposable critical subspaces. This is a very useful property in light of Proposition 18. However, the uniqueness of a decomposition of $\mathbb{R}^n$ into indecomposable critical subspaces does not hold in general for a Geometric Brascamp-Lieb data if some E_i is of higher dimension (see examples in Valdimarsson [59]).

In general, the structure of a Geometric Brascamp-Lieb Data is described by Valdimarsson [59]. We write J to denote the set of 2^k functions $\{1,\ldots,k\} \to \{0,1\}$. If $\varepsilon \in J$, then let $F_{(\varepsilon)} = \cap_{i=1}^k E^{(\varepsilon(i))}$ where $E_i^{(0)} = E_i$ and $E_i^{(1)} = E_i^\perp$ for $i = 1,\ldots,k$. We write J_0 to denote the subset of $\varepsilon \in J$ such that $\dim F_{(\varepsilon)} \geq 1$, and such an $F_{(\varepsilon)}$ is called independent following Valdimarsson [59]. Readily $F_{(\varepsilon)}$ and $F_{(\tilde{\varepsilon})}$ are orthogonal if $\varepsilon \neq \tilde{\varepsilon}$ for $\varepsilon, \tilde{\varepsilon} \in J_0$. In addition, we write $F_{\rm dep}$ to denote the orthogonal component of $\oplus_{\varepsilon\in J_0} F_{(\varepsilon)}$. In particular, $\mathbb{R}^n$ can be written as a direct sum of pairwise orthogonal linear subspaces in the form

$$\mathbb{R}^n = \left(\oplus_{\varepsilon\in J_0} F_{(\varepsilon)}\right) \oplus F_{\rm dep}. \tag{27}$$

Here it is possible that $J_0 = \emptyset$, and hence $\mathbb{R}^n = F_{\rm dep}$, or $F_{\rm dep} = \{o\}$, and hence $\mathbb{R}^n = \oplus_{\varepsilon\in J_0} F_{(\varepsilon)}$ in that case. We deduce from (21) that

$$\text{each independendent subspace } F_{(\varepsilon)},\ \varepsilon \in J_0, \text{ and } F_{\rm dep} \text{ are critical subspaces.} \tag{28}$$

It follows from Lemma 15 (i) that

$$\cap_{i=1}^k E_i = \{o\} \ \text{ and } \ \cap_{i=1}^k E_i^\perp = \{o\}. \tag{29}$$

Therefore J_0 does not contain the two constant functions in J.

Lemma 10 in Valdimarsson [59] states the following crucial property of independent subspaces and general critical subspaces.

Lemma 19 (Valdimarsson) *If the proper linear subspaces $E_1,\ldots,E_k$ of $\mathbb{R}^n$ and $c_1,\ldots,c_k > 0$ satisfy* (17)*, $F_{(\varepsilon)}$, $\varepsilon \in J_0$, is an independent subspace and V is a critical subspace, then*

$$V = \left(V \cap F_{(\varepsilon)}\right) + \left(V \cap F_{(\varepsilon)}^\perp\right)$$

Lemma 20 *If the proper linear subspaces $E_1,\ldots,E_k$ of $\mathbb{R}^n$ and $c_1,\ldots,c_k > 0$ satisfy* (17)*, and V is an indecomposable critical subspace, then either $V \subset F_{\rm dep}$, or there exists independent subspace $F_{(\varepsilon)} \supset V$, $\varepsilon \in J_0$.*

Proof: We observe that the intersection of V with any critical subspasce is either $\{o\}$ or V by Lemma 16, therefore combining Lemma 19 with (27) and (28) yields the statement. □

5 Optimal transportation and the Reverse Brascamp-Lieb inequality

For a C^2 function φ on $\mathbb{R}^n$, we write $D\varphi$ the first derivative and $D^2\varphi$ the Hessian of φ. Combining Corollary 2.30, Corollary 2.32, Theorem 4.10 and Theorem 4.13 in Villani [60] on the Brenier type based on McCann [51, 52] for the first two, and on Caffarelli [16, 17, 18] for the last two theorems, we deduce the following:

Theorem 21 (Brenier,McCann,Caffarelli) *If f and g are C^1 positive probability density function on $\mathbb{R}^n$, then there exists a C^2 convex function φ on $\mathbb{R}^n$ (unique up to additive constant) such that $D\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is bijective and*

$$f(x) = g(D\varphi(x)) \cdot \det D^2\varphi(x) \text{ for } x \in \mathbb{R}^n. \tag{30}$$

In particular, the derivative $D\varphi$ of the convex potential is a transportation map between the measures determined by f_1 and f_2.

Proof of Theorem 8 based on Barthe [5]. First we assume that each f_i is a C^1 positive probability density function on $\mathbb{R}^n$, and let us consider the Gaussian densiy $g_i(x) = e^{-\pi\|x\|^2}$ for $x \in E_i$. According to Theorem 21, if $i = 1, \ldots, k$, then there exists a C^2 convex function φ_i on E_i such that for the C^1 transportation map $T_i = \nabla\varphi_i$, we have

$$g_i(x) = \det \nabla T_i(x) \cdot f_i(T_i(x)) \text{ for all } x \in E_i. \tag{31}$$

It follows from (34) that $\nabla T_i = D^2\varphi_i(x)$ is positive definite symmetric matrix for all $x \in E_i$. For the C^1 transformation $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Theta(y) = \sum_{i=1}^{k} c_i T_i\left(P_{E_i} y\right), \qquad y \in \mathbb{R}^n,$$

its differential

$$d\Theta(y) = \sum_{i=1}^{k} c_i \nabla T_i\left(P_{E_i} y\right)$$

is positive definite by Proposition 18. It follows that $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ is injective (see [5]). Therefore Proposition 18, (34) and Lemma 15 (i) imply

$$\begin{aligned}
&\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^{k} f_i(x_i)^{c_i}\, dx \\
&\geq \int_{\mathbb{R}^n}^{*} \left(\sup_{\Theta(y)=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^{k} f_i(x_i)^{c_i} \right) \det\left(d\Theta(y)\right)\, dy \\
&\geq \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} f_i\left(T_i\left(P_{E_i} y\right)\right)^{c_i} \right) \det\left(\sum_{i=1}^{k} c_i \nabla T_i\left(P_{E_i} y\right) \right) dy \\
&\geq \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} f_i\left(T_i\left(P_{E_i} y\right)\right)^{c_i} \right) \prod_{i=1}^{k} \left(\nabla T_i\left(P_{E_i} y\right)\right)^{c_i}\, dy \qquad (32) \\
&= \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} g_i\left(P_{E_i} y\right)^{c_i} \right) dy = \int_{\mathbb{R}^n} e^{-\pi\|y\|^2}\, dy = 1.
\end{aligned}$$

Finally, the reverse Brascamp-Lieb inequality (7) for arbitrary non-negative integrable functions f_i follows by scaling and approximation (see Barthe [5]). □

Given proper linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfying (17), we say that the non-negative integrable functions $f_1, \ldots, f_k$ with positive integrals are extremizers if equality holds in (7). In order to ensure that we only deal with positive smooth functions, we use convolutions. More precisely, Lemma 2 in Barthe [5] states the following.

Lemma 22 *Given proper linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfying* (17)*, if $f_1, \ldots, f_k$ and $g_1, \ldots, g_k$ are extremizers in the Reverse Brascamp-Lieb inequality* (7)*, then the same holds for $f_1 * g_1, \ldots, f_k * g_k$.*

Given proper linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfying (17), let us discuss the translation invariance of the Reverse Brascamp-Lieb inequality. For non-negative integrable function f_i on E_i, $i = 1, \ldots, k$, let us define

$$F(x) = \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i}.$$

We observe that for any $e_i \in E_i$, defining $\tilde{f}_i(x) = f_i(x + e_i)$ for $x \in E_i$, $i = 1, \ldots, k$, we have

$$\widetilde{F}(x) = \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^k \tilde{f}_i(x_i)^{c_i} = F\left(x - \sum_{i=1}^k c_i e_i\right). \tag{33}$$

Proposition 23 *For the proper linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfying* (5) *and $F_{\rm dep} = \{o\}$, let us assume that equality holds in* (7) *for non-negative $f_i \in L_1(E_i)$, $i = 1, \ldots, k$, with positive integral. Then writing $F_1, \ldots, F_l$ to denote the independent subspaces, there exist integrable $h_{ij} : F_j \to [0, \infty)$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$ with $F_j \subset E_i$ such that*

$$f_i(x) = \prod_{F_j \subset E_i} h_{ij}(P_{F_j} x) \quad \textit{for } x \in E_i.$$

Proof: For $i = 1, \ldots, k$ and $x \in E_i$, let $g_i(x) = e^{-\pi \|x\|^2}$, and hence g_i is a probability distribution on E_i, and $g_1, \ldots, g_k$ are extremizers in the Reverse Brascamp-Lieb inequality (7). Let $f_1, \ldots, f_k$ be extremizers in (7). We may assume that each f_i is a probability distribution on E_i, $i = 1, \ldots, k$.

Case 1 Each f_i is positive and C^1.

As in the proof of Theorem 8 above, let φ_i be Brenier's C^2 convex potential on E_i such that

$$g_i(x) = \det D^2\varphi_i(x) \cdot f_i(D\varphi_i(x)) \text{ for all } x \in E_i. \tag{34}$$

We write $T_i = D\varphi_i : E_i \to E_i$ and $\nabla T_i = D^2\varphi_i$ to denote the transportation map and its derivative, respectively, for $i = 1, \ldots, k$ where ∇T_i is positive definite. According to (33), we may assume that

$$T_i(o) = o \quad \text{for } i = 1, \ldots, k. \tag{35}$$

If equality holds in (7), then equality holds in the determinantal inequality in (32), therefore we apply the equality case of Proposition 18. In particular, for any $x \in \mathbb{R}^n$, there exist $m_x \geq 1$ and linear subspaces $V_{1,x}, \ldots, V_{m_x,x}$ where $V_1 = \mathbb{R}^n$ if $m_x = 1$ and $V_{1,x}, \ldots, V_{m_x,x}$ are pairwise orthogonal indecomposable critical subspaces spanning $\mathbb{R}^n$ if $m_x \geq 2$, and $\lambda_{1,x}, \ldots, \lambda_{m_x,x} > 0$ such that if $E_i \cap V_{j,x} \neq \{o\}$, then writing $\widetilde{P}_{i,j,x}$ to denote the orthogonal projection into $E_i \cap V_{j,x}$, we have

$$\nabla T_i(\widetilde{P}_{i,j,x} x)|_{E_i \cap V_{j,x}} = \lambda_{j,x} I_{E_i \cap V_{j,x}}; \tag{36}$$

and in addition, each E_i satisfies

$$E_i = \oplus_{E_i \cap V_{j,x} \neq \{o\}} E_i \cap V_{j,x}. \tag{37}$$

Let us consider a fixed E_i, $i \in \{1, \ldots, k\}$. It follows from (37) and Lemma 20 that if $E_i \cap F_p \neq \{o\}$ for $p \in \{0, \ldots, l\}$ and $x \in \mathbb{R}^n$, then $F_p \subset E_i$, and

$$F_p = \oplus_{\substack{E_i \cap V_{j,x} \neq \{o\} \\ V_{j,x} \subset F_p}} E_i \cap V_{j,x};$$

therefore, (36) yields that if $y \in E_i$, then

$$\nabla T_i(y)(F_p) = F_p. \tag{38}$$

Since applying again (37) and Lemma 20 yields that

$$E_i = \oplus_{E_i \cap F_p \neq \{o\}} F_p, \tag{39}$$

we deduce from combining (38) and (39) that for any $y \in E_i$, we have

$$\nabla T_i(y) = \oplus_{E_i \cap F_p \neq \{o\}} \nabla T_i(y)|_{F_p}. \tag{40}$$

In turn, (40) and $T_i(o) = o$ (*cf.* (35)) imply that if $y \in E_i$, then

$$T_i(y) = \oplus_{E_i \cap F_p \neq \{o\}} T_i(P_{F_p} y). \tag{41}$$

It follows from (41) that there exist $\theta_i > 0$ and positive integrable functions h_{ip} on F_p whenever $E_i \cap F_p \neq \{o\}$ for $i \in \{1, \ldots, k\}$ and $p \in \{1, \ldots, l\}$ such that if $y \in E_i$, then

$$f_i(y) = \theta_i \prod_{\substack{F_p \subset E_i \\ p \geq 1}} h_{ip}(P_{F_p} y) \tag{42}$$

Case 2 $f_1, \ldots, f_k$ are any extremizers in the Reverse Brascamp-Lieb inequality (7).

According to Lemma 22, $f_i * g_i$ are positive and C^1 extremizers, and hence they are of the of form as in (42). The use of Fourier transform shows that the original $f_1, \ldots, f_k$ are of the same form except for the fact that functions h_{ip} may not be positive on $\mathbb{R}^n$. $\square$

6 The equality case of the Reverse Brascamp Lieb inequaltiy (partial)

Given Proposition 23, the only extra ingredient we need is the Prekopa-Leindler inequality Theorem 24 (proved in various forms by Prekopa [55, 56], Leindler [43] and Borell [12]) whose equality case was clarified by Dubuc [20] (see the survey Gardner [25]). In turn, the Prekopa-Leindler inequality (43) is of the very similar structure like the Brascamp-Lieb inequality (7).

Theorem 24 (Prekopa, Leindler) *For $\lambda_1, \ldots, \lambda_m \in (0,1)$ with $\lambda_1 + \ldots + \lambda_m = 1$ and integrable $\varphi_1, \ldots, \varphi_m : \mathbb{R}^n \to \mathbb{R}_+$, we have*

$$\int_{\mathbb{R}^n}^{*} \sup_{x = \sum_{i=1}^m \lambda_i x_i,\, x_i \in \mathbb{R}^n} \prod_{i=1}^m \varphi_i(x_i)^{\lambda_i}\, dx \geq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \varphi_i \right)^{\lambda_i}, \tag{43}$$

and if equality holds and the left hand side is positive and finite, then there exist a log-concave function φ and $a_i > 0$ and $b_i \in \mathbb{R}^n$ for $i = 1, \ldots, m$ such that $\sum_{i=1}^m \lambda_i b_i = o$ and

$$\varphi_i(x) = a_i\, \varphi(x - b_i)$$

for Lebesgue almost all $x \in \mathbb{R}^n$, $i = 1, \ldots, m$.

Proof of Theorem 10 Our starting point is the statement and notation of Proposition 23, and hence let $F_1, \ldots, F_l$ be the independent critical subspaces. It follows from (33) that we may assume that $b_i = 0$ for $i = 1, \ldots, k$.

First we verify that for each F_j, $j = 1, \ldots, l$, we have

$$\sum_{F_j \subset E_i} c_i = 1. \tag{44}$$

For this, let $x \in F_j \backslash \{o\}$. We observe that for any E_i, either $F_j \subset E_i$, and hence $P_{E_i} x = x$, or $F_j \subset E_i^{\perp}$, and hence $P_{E_i} x = o$. We deduce from (5) that

$$x = \sum_{i=1}^{k} c_i P_{E_i} x = \left(\sum_{F_j \subset E_i} c_i \right) \cdot x,$$

which formula in turn implies (44).

Since $F_1 \oplus \ldots \oplus F_l = \mathbb{R}^n$ and $F_1, \ldots, F_l$ are critical subspaces, (21) yields for $i = 1, \ldots, k$ that

$$E_i = \bigoplus_{F_j \subset E_i} F_j; \tag{45}$$

therefore, the Fubini theorem implies

$$\int_{E_i} f_i = \prod_{F_j \subset E_i} \int_{F_j} h_{ij}(x)\, dx \tag{46}$$

On the other hand, using again $F_1 \oplus \ldots \oplus F_l = \mathbb{R}^n$, we deduce that if $x = \sum_{j=1}^{l} z_j$ where $z_j \in F_j$ for $j \geq 1$, then $z_j = P_{F_j} x$. It follows from (45) that for any $x \in \mathbb{R}^n$, we have

$$\sup_{\substack{x = \sum_{i=1}^{k} c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^{k} f_i(x_i)^{c_i} = \prod_{j=1}^{l} \left(\sup_{\substack{P_{F_j} x = \sum_{F_j \subset E_i} c_i x_{ji}, \\ x_{ji} \in F_j}} \prod_{F_j \subset E_i} h_{ij}(x_{ji})^{c_i} \right). \tag{47}$$

We deduce from (44) and the Prekopa-Leindler inequality Theorem 24 that for fixed $j \in \{1, \ldots, l\}$, we have

$$\int_{\mathbb{R}^n}^{*} \sup_{\substack{P_{F_j} x = \sum_{F_j \subset E_i} c_i x_{ji}, \\ x_{ji} \in F_j}} \prod_{F_j \subset E_i} h_{ij}(x_{ji})^{c_i}\, dx \geq \prod_{F_j \subset E_i} \left(\int_{F_j} h_{ij}(x)\, dx \right)^{c_i}. \tag{48}$$

Now in the case of the special functions f_i of Proposition 23, combining (48) with (46), (47) and the Fubini Theorem yields the Reverse Brascamp-Lieb inequality (7). On the other hand, if equality holds in (7), then equality holds in (48) for $j = 1, \dots, l$. According to the equality case of the Prekopa-Leindler inequality Theorem 24, for any fixed $j \in \{1, \dots, l\}$, there exists a log-concave function h_j on F_j, and there exists $a_{ij} > 0$ and $w_{ij} \in F_j$ for any $i \in \{1, \dots, m\}$ with $F_j \subset E_i$ such that

$$h_{ij}(x) = a_{ij} h_j(x - w_{ij}) \text{ for Lebesgue a.a. } x \in F_j.$$

In turn, we conclude Theorem 10 by choosing

$$w_i = \sum_{F_j \subset E_i} w_{ij}$$

for any $i \in \{1, \dots, k\}$. □

7 The equality cases of the Bollobas-Thomason inequality and in its dual

We will denote with $\sigma_i^0 = \sigma_i$ and $\sigma_i^1 = [n] \setminus \sigma_i$. When we write $\tilde{\sigma_1}, \dots, \tilde{\sigma_l}$ for the induced cover from $\sigma_1, \dots, \sigma_k$, we assume that the sets are distinct.

Lemma 25 *For $s \geq 1$, let $\sigma_1, \dots, \sigma_k \subset [n]$ form an s-uniform cover of $[n]$, and let $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ be the 1-uniform cover of $[n]$ induced by $\sigma_1, \dots, \sigma_k$. Then*

(i) for any fixed orthonormal basis $e_1, \dots, e_n$, the subspaces $E_{\sigma_i} := \{e_j : i \in \sigma_i\}$ satisfy

$$\sum_{i=1}^{k} \frac{1}{s} P_{E_{\sigma_i}} = I_n \tag{49}$$

i.e. form a geometric Brascamp Lieb data.

(ii) the elements $\tilde{\sigma_i}$ have the following form: there is $r \in [n]$ so that,

$$\tilde{\sigma_i} := \bigcap_{r \in \sigma_i} \sigma_i^0 \cap \bigcap_{r \notin \sigma_i} \sigma_i^1 \tag{50}$$

(iii) the subspaces $F_{\tilde{\sigma}_i} := \text{lin}\{e_j : j \in \tilde{\sigma_i}\}$ are the independent subspaces of the data (49) *and $F_{\text{dep}} = \{o\}$.*

Proof:

(i) Since $\sigma_1, \dots, \sigma_k$ form a s-uniform cover, every $e_i \in \mathbb{R}^n$ is contained in exactly s of $E_{\sigma_1}, \dots, E_{\sigma_k}$. So (i) follows.

(ii) Let $\sigma_1, \dots, \sigma_k$ be just subsets of $[n]$. We take a $I \subseteq [k]$ of cardinality s, and we consider the set

$$A_I := \bigcap_{i \in I} \sigma_i^0 \cap \bigcap_{i \notin I} \sigma_i^1.$$

If, after a replacement of 0 by 1 (1 by 0) in the left (right) big intersection we have that the new A_I is not empty, then there is $\tau \in [n]$ so that τ is contained in exactly $s - 1$ ($s + 1$) from

$\sigma_1, \dots, \sigma_k$. Now with the additional property that $\sigma_1, \dots, \sigma_k \subset [n]$ form an s-uniform cover of $[n]$, we have that any $\tilde{\sigma_i}$ has the form of A_I, and also for some $r \in [n]$

$$I \subseteq \{i \in [k] : r \in \sigma_i\}$$

Since both cardinalities of the above sets is s we conclude to (50).

(iii) If we prove the independence of the subspaces, then immediate we have that $F_{\text{dep}} = \{o\}$ since for each $r \in [n]$ we have that $r \in A_{I_r}$ where $I_r = \{i \in [k] : r \in \sigma_i\}$, namely one of the subspaces $F_{\tilde{\sigma}_1}, \dots, F_{\tilde{\sigma}_l}$ contains e_r and so they span $\mathbb{R}^n$. Now the independance follows from the easy observation,

$$\cap_{j=1}^{k} (\text{lin}\{e_i : i \in \sigma_j\})^{\varepsilon(j)} = \text{lin}\{e_i : i \in \cap_{j=1}^{k} \sigma_j^{\varepsilon(j)}\}$$

where, when ε takes the value 1, the left ε is the orthogonal complement in $\mathbb{R}^n$ and the right ε is the complement in $[n]$.

□

Let us introduce the notation that we use when handling both the Bollobas-Thomason inequality and its dual. Let $\sigma_1, \dots, \sigma_k$ be the s cover of $[n]$ occuring in Theorem 5 and Theorem 6, and hence $E_i = E_{\sigma_i}$, $i = 1, \dots, k$, satisfies

$$\sum_{i=1}^{k} \frac{1}{s} \cdot P_{E_{\sigma_i}} = I_n. \tag{51}$$

Let $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ be the 1-uniform cover of $[n]$ induced by $\sigma_1, \dots, \sigma_k$. It follows that

$$F_j = E_{\tilde{\sigma}_j} \quad \text{for } j = 1, \dots, l \text{ are the independent subspaces,} \tag{52}$$

$$F_{\text{dep}} = \{o\}. \tag{53}$$

For any $i \in \{1, \dots, k\}$, we set

$$I_i = \{j \in \{1, \dots, l\} : F_j \subset E_i\},$$

and for any $j \in \{1, \dots, l\}$, we set

$$J_j = \{i \in \{1, \dots, k\} : F_j \subset E_i\}.$$

For the reader's convenience, we restate Theorem 3 and Theorem 5 as Theorem 26, and Theorem 4 and Theorem 6 as Theorem 27.

Theorem 26 (Bollobas, Thomason) *If $K \subset \mathbb{R}^n$ is compact and affinely spans $\mathbb{R}^n$, and $\sigma_1, \dots, \sigma_k \subset [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$, then*

$$|K|^s \leq \prod_{i=1}^{k} |P_{E_{\sigma_i}} K|. \tag{54}$$

Equality holds if and only if $K = \oplus_{i=1}^{l} P_{F_{\tilde{\sigma}_i}} K$ where $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ is the 1-uniform cover of $[n]$ induced by $\sigma_1, \dots, \sigma_k$ and $F_{\tilde{\sigma}_i}$ is the linear hull of the e_i's with indeces from $\tilde{\sigma}_i$.

Proof: We denote with $E_i := E_{\sigma_i}$, where from Lemma 25 (i) these subspaces compose a geometric data. We start with a proof of Bollobas-Thomason inequality. It follows directly from the Brascamp-Lieb inequality as

$$|K| = \int_{\mathbb{R}^n} 1_K(x)\,dx \le \int_{\mathbb{R}^n} \prod_{i=1}^{k} 1_{P_{E_i}(K)}(P_{E_i}(x))^{\frac{1}{s}}\,dx$$

$$\le \prod_{i=1}^{k} \Big(\int_{E_i} 1_{P_{E_i}(K)} \Big)^{\frac{1}{s}} = \prod_{i=1}^{k} |P_{E_i}(K)|^{\frac{1}{s}} \tag{55}$$

where the first inequality is from the monotonicity of the integral while the second is Brasmap-Lieb inequality Theorem 7. Now, if equality holds in (55), then on the one hand,

$$1_K(x) = \prod_{i=1}^{k} 1_{P_{E_i}(K)}(P_{E_i}(x))$$

and on the other hand, if $F_1, \ldots, F_l$ are the independent subspaces of the data, which from Lemma 25 (iii) they span $\mathbb{R}^n$, namely $F_{\text{dep}} = \{0\}$, by Theorem 9 there are integrable functions $h_j : F_j \to \mathbb{R}$, such that, for Lebesgue a.a. $x_i \in E_i$

$$1_{P_{E_i}K}(x_i) = \theta_i \prod_{j \in I_i} h_j(P_{F_j}(x_i))$$

Therefore from the previous two, we have for $x \in \mathbb{R}^n$

$$1_K(x) = \prod_{i=1}^{k} \theta_i \prod_{j \in I_i} h_j(P_{F_j}(P_{E_i}(x)))$$

Now, since for $j \in I_i$ we have $F_j \subset E_i$ we can delete the P_{E_i} on the above product. Thus, for $\theta = \prod_{i=1}^{k} \theta_i$, we have for Lebesgue a.a. $x \in \mathbb{R}^n$

$$1_K(x) = \theta \prod_{i=1}^{k} \prod_{j \in I_i} h_j(P_{F_j}(x)) = \theta \prod_{j=1}^{l} h_j(P_{F_j}(x))^{|J_j|}. \tag{56}$$

Now, for $x \in K$ the last product on above is constant, so

$$\theta = \frac{1}{\prod_{i=1}^{l} h_j(P_{F_j}(x_0))^{|J_j|}} \tag{57}$$

for some $x_o \in K$. For $j = 1, \ldots, l$ we set $\varphi_j : F_j \to \mathbb{R}^n$, by

$$\varphi_j(x) = \frac{h_j(x + P_{F_j}(x_0))^{|J_j|}}{h_j(P_{F_j}(x_0))^{|J_j|}}.$$

We see that $\varphi_j(o) = 1$ and also (56) and (57) yields

$$1_{K-x_0}(x) = \prod_{j=1}^{l} \varphi_j(P_{F_j}(x)) \tag{58}$$

For $m \in \{1, \dots, l\}$, taking $x \in F_m$ in (58) (and hence $\varphi_j(P_{F_j}(x)) = 1$ for $j \neq m$) shows that

$$1_{K-x_0}(y) = \varphi_m(y),$$

for Lebesgue a.a. $y \in F_m$. Therefore (58) and the ortgonality of the F_j's,

$$K - x_0 = \bigcap_{j=1}^{l} P_{F_j}^{-1}(P_{F_j}(K - x_o)) = \bigoplus_{j=1}^{l} P_{F_j}(K - x_o),$$

completing the proof of Theorem 26. □

To prove Theorem 27, we use two small observations. First if M is any convex body with $o \in \operatorname{int} M$, then

$$\int_{\mathbb{R}^n} e^{-\|x\|_M}\, dx = \int_0^\infty e^{-r} n r^{n-1} |M|\, dr = n!|M|. \tag{59}$$

Secondly, if F_j are pairwise orthogonal subspaces and $M = \operatorname{conv}\{M_1, \dots, M_l\}$ where $M_j \subset F_j$ is a $\dim F_j$-dimensional compact convex set with $o \in \operatorname{relint} M_j$, then for any $x \in \mathbb{R}^n$

$$\|x\|_M = \sum_{i=1}^{l} \|P_{F_j} x\|_{M_j}. \tag{60}$$

In addition, we often use the fact, for a subspace F of $\mathbb{R}^n$ and $x \in F$, then $\|x\|_K = \|x\|_{K \cap F}$.

Theorem 27 (Liakopoulos) *If $K \subset \mathbb{R}^n$ is compact convex with $o \in \operatorname{int} K$, and $\sigma_1, \dots, \sigma_k \subset [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$, then*

$$|K|^s \geq \frac{\prod_{i=1}^{k} |\sigma_i|!}{(n!)^s} \cdot \prod_{i=1}^{k} |K \cap E_{\sigma_i}|. \tag{61}$$

Equality holds if and only if $K = \operatorname{conv}\{E_{\tilde{\sigma}_i} \cap K\}_{i=1}^{l}$ where $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ is the 1-uniform cover of $[n]$ induced by $\sigma_1, \dots, \sigma_k$.

Proof: We define

$$f(x) = e^{-\|x\|_K}, \tag{62}$$

which is a log-concave function with $f(o) = 1$, and satisfying (*cf* (59))

$$\int_{\mathbb{R}^n} f(y)^n\, dy = \int_{R^n} e^{-n\|y\|_K}\, dy = \int_{R^n} e^{-\|y\|_{\frac{1}{n}K}} = n! \left|\frac{1}{n} K\right| = \frac{n!}{n^n} \cdot |K|. \tag{63}$$

We claim that

$$n^n \int_{\mathbb{R}^n} f(y)^n\, dy \geq \prod_{i=1}^{k} \Big(\int_{E_i} f(x_i)\, dx_i \Big)^{1/s}. \tag{64}$$

Equating the traces of the two sides of (49), we deduce that, $d_i := |\sigma_i| = \dim E_i$

$$\sum_{i=1}^{k} \frac{d_i}{sn} = 1. \tag{65}$$

For $z = \sum_{i=1}^{k} \frac{1}{s} x_i$ with $x_i \in E_i$, the log-concavity of f and its definition (62), imply

$$f(z/n) \geq \prod_{i=1}^{k} f(x_i/d_i)^{\frac{d_i}{ns}} = \prod_{i=1}^{k} f(x_i)^{\frac{1}{ns}}. \tag{66}$$

Now, the monotonicity of the integral, and Reverse Brascamp Lieb inequality, give

$$\int_{\mathbb{R}^n} f(z/n)^n \, dz \geq \int_{\mathbb{R}^n}^{*} \sup_{z=\sum_{i=1}^{k} \frac{1}{s} x_i,\, x_i \in E_i} \prod_{i=1}^{k} f(x_i)^{1/s} \, dz \geq \prod_{i=1}^{k} \Big(\int_{E_i} f(x_i) \, dx_i \Big)^{1/s}. \tag{67}$$

Making the change of variable $y = z/n$ we conclude to (64). Computing the right hand side of (64), we have

$$\int_{E_i} f(x_i) \, dx_i = \int_{E_i} e^{-\|x_i\|_K} \, dx_i = \int_{E_i} e^{-\|x_i\|_{K \cap E_i}} \, dx_i = d_i! |K \cap E_i|. \tag{68}$$

Therefore, (63), (64) and (68) yield (61).

Let us assume that equality holds in (61), and hence we have two equalities in (67). We set

$$M = \operatorname{conv}\{K \cap F_j\}_{1 \leq j \leq l}.$$

Clearly, $K \supseteq M$. For the other inclusion, we start with $z \in \operatorname{int} K$, namely $\|z\|_K < 1$. Equality in the first inequality in (67) means,

$$\Big(e^{-\|z/n\|_K} \Big)^n = \sup_{z=\sum_{i=1}^{k} \frac{1}{s} x_i,\, x_i \in E_i} \prod_{i=1}^{k} e^{-\|x_i\|_K 1/s},$$

or in other words,

$$\|z\|_K = \frac{1}{s} \cdot \inf_{z=\sum_{i=1}^{k} \frac{1}{s} x_i,\, x_i \in E_i} \sum_{i=1}^{k} \|x_i\|_K = \inf_{z=\sum_{i=1}^{k} y_i,\, y_i \in E_i} \sum_{i=1}^{k} \|y_i\|_K. \tag{69}$$

We deduce that there exist $y_i \in E_i$, $i = 1, \ldots, k$ such that

$$z = \sum_{i=1}^{k} y_i \ \text{ and } \ \sum_{i=1}^{k} \|y_i\|_K < 1, \tag{70}$$

Therefore, from (70), then (60) and after the triangle inequality for $\| \cdot \|_{K \cap F_j}$, we have

$$\|z\|_M = \left\| \sum_{i=1}^{k} \sum_{j \in I_i} P_{F_j} y_i \right\|_M = \sum_{i=1}^{k} \left\| \sum_{i \in I_i} P_{F_j} y_i \right\|_{K \cap F_j} \leq \sum_{i=1}^{k} \sum_{i \in I_i} \left\| P_{F_j} y_i \right\|_{K \cap F_j}. \tag{71}$$

It suffices to show that

$$K \cap E_i = \operatorname{conv}\{K \cap F_j\}_{j \in I_i} \tag{72}$$

because then, from (71), applying (60) and (70), we have

$$\|z\|_M \leq \sum_{j=1}^{l} \sum_{i \in J_j} \left\| P_{F_j} y_i \right\|_{K \cap F_j} = \sum_{i=1}^{k} \|y_i\|_{K \cap E_i} < 1,$$

which means $z \in M$. Now, to show (72), we start with the equality case of the Reverse Brascamp-Lieb inequality which has been applied in (67). From Theorem 10, there exist $\theta_i > 0$ and $w_i \in E_i$ and log-concave $h_j : F_j \to [0, \infty)$, namely $h_j = e^{-\varphi_j}$ for a convex functon φ_j, such that

$$e^{-\|x_i\|_{K\cap E_i}} = \theta_i \prod_{j\in I_i} h_j(P_{F_j}(x_i - w_i)). \tag{73}$$

for Lebesgue a.a. $x_i \in E_i$. For $i \in [k]$ and $j \in I_i$ we set, $\psi_{ij} : F_j \to \mathbb{R}$ by

$$\psi_{ij}(x) = \varphi_j\left(x - P_{F_j}w_i\right) - \varphi_j\left(-P_{F_j}w_i\right) + \frac{\ln\theta_i}{|I_i|}.$$

We see

$$\psi_{ij}(o) = 0 \text{ and } \psi_{ij} \text{ is convex on } F_j. \tag{74}$$

and also (73) yields, for $x \in E_i$

$$e^{-\|x\|_{K\cap E_i}} = \exp\left(-\sum_{j\in I_i}\psi_{ij}(P_{F_j}x)\right). \tag{75}$$

For $x \in F_j$, we apply λx to (75) with $\lambda > 0$, and we have from $\psi_{im}(o) = 0$ for $m \in I_i \backslash \{j\}$ that

$$\psi_{ij}(\lambda x) = \lambda\psi_{ij}(x) \text{ and } \psi_{ij}(x) > 0. \tag{76}$$

We deduce from (74) and (76) that ψ_{ij} is a norm. Therefore, $\psi_{ij}(x) = \|x\|_{C_{ij}}$ for some $(\dim F_j)$-dimensional compact convex set $C_{ij} \subset F_j$ with $o \in \operatorname{relint} C_{ij}$. Now (75) becomes,

$$\|x\|_{K\cap E_i} = \sum_{j\in I_i} \|P_{F_j}x\|_{C_{ij}}$$

and hence by (60) we conclude to

$$K \cap E_i = \operatorname{conv}\{C_{ij}\}_{j\in I_i}.$$

In particular, if $i \in [k]$ and $j \in I_i$, then $C_{ij} = (K \cap E_i) \cap F_j = K \cap F_j$, and hence we have (72) and the proof is finished.

□

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