An Insight into Foulkes Conjecture

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A thesis submitted in fulfilment of the requirements for the degree of Practical Mathematics Specialization Program.

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Declaration of Authorship

I, "Madireddi Sai Praveen", declare that this thesis entitled, "An Insight into Foulkes Conjecture" and the work presented in it are my own.

I confirm that:

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Abstract

Foulkes module F_a^b is the permutation module of the set of partition of ab elements into size a each. $F_a^b \cong 1_{S_a wr S_b} \uparrow^{S_{ab}}$. The study goes back to 1942, when Thrall computed the structure of F_2^b and F_b^2. In 1950, Foulkes while analysing the structure of F_m^n for some specific m and n observed that F_n^m can be embedded in F_m^n when $m < n$ and thus conjectured that if $a < b$, F_b^a can be embedded in F_a^b. I will describe the progress on Foulkes conjecture and briefly explain the methods used by Tom Mckay and Eugenio Giannelli to prove their results. I will also explain the structure of $F_a^b \downarrow_{S_K \times S_{ab-k}}$ as a direct sum of permutation modules and study the structure of such permutation modules.

Contents

1 Representation Theory of S_n

1.1 Representation Theory of Finite Groups

Definition 1. A representation Φ of a finite group G is an homomorphism from $G \to GL(V)$, where $GL(V)$ is the automorphism group of V over field K.The degree of Φ is defined to be the dimesion of V.

Let $\Phi : G \to GL(V)$. Then G acts on V by

$$gv = \phi(g)v, \text{ where } g \in G \text{ and } v \in V. \tag{1}$$

Thus, V is a KG module.
If W is subspace of V then W is submodule of V, if

$$hw \in W, \ \forall \ h \in G \text{ and } w \in W. \tag{2}$$

Definition 2. For a KG module V, V is *reducible* if $\exists$ a nontrival submodule W. If V is not reducible then V is *irreducible* or *simple*.

Let G be a finite group and V be an infinite dimensional KG module. Choose an element v_1 in V. Consider the submodule V_1 generated by $\{gv_1 : \text{for } g \in G\}$. Clearly V_1 is finite dimensional non trivial subspace of V. Therefore V is reducible.

Let V and W be KG modules. G acts on $V \oplus W$ as

$$g(v + w) = gv + gw, \ \forall \ g \text{ in } G, \ v \text{ in } V\text{and } w \text{ in } W \tag{3}$$

$V \oplus_{KG} W$ is a KG mdoule isomorphic to $V \oplus W$ under the G action described in above.

V is a *semisimple* or *completely reducible* KG module if for each every submodule U of V, $\exists$ a complement W such that $V = U \oplus_{KG} W$. Let V be semisimple. Then

$$V = \oplus_{KG} V_i \tag{4}$$

where each V_i is irreducible.

Theorem 3. Maschhke's Theorem: Suppose $char(K) \nmid |G|$, then every finite dimensional KG module is semisimple.

If $char(K) \bigm| |G|$, then let $\epsilon : KG \to K$, given by $\epsilon(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g$. ϵ is a KG module homomorphism, thus $ker(\epsilon)$ is KG submodule. Since $char(K) \bigm| \mid G \mid$, $t = \sum_{g\in G} g \in Ker(\epsilon)$. If V is any submodule of KG, then for $v = \sum_{g\in G} b_g g$, $tv = (\sum_{g\in G} b_g)t$, Thus, $tv \in Ker(\epsilon)$. Therefore KG is not semisimple, since $Ker(\epsilon)$ does not have a complement in KG.

Definition 4. Let $H \leqslant G$ and V be a KG module. Then H acts naturally on V. The KH module which comes from this action is called V_H.

Definition 5. Let $H \leqslant G$ and V be a KH module with the set of basis elements $\{v_i \mid i \in [1, 2..n]\}$. If the index of H in G is t and $h_1, h_2, h_3..., h_t$ are the coset representatives, then let h_jV be the vector space generated by formal basis $\{h_jv_i \mid i \in [1, 2..n]\}$. $h_iV \cong_K V$. The *Induced module* V^G is defined as

$$V^G \cong_K \oplus h_jV \tag{5}$$

If for a given g in G, $gh_i = h_jh$ for some h in H, then V^G is a KG module under the following action.

$$g(h_iv) = h_j(hv), \; \forall \; v \text{ in } V. \tag{6}$$

A *class function* f is a map from $G \to \mathbb{C}$ which is constant on conjugacy classes of G. That is $f(hgh^{-1}) = f(g)$ for each h in G. Let CG denote the space of class functions of G over $\mathbb{C}$.

For any two class functions f and h we can define an inner product $\langle ., . \rangle_G$ by

$$\langle f, h \rangle_G = \frac{1}{|G|} \cdot \sum_{g \in G} f(g) \cdot \overline{h(g)} \tag{7}$$

If V is $\mathbb{C}G$ module and Φ is the corresponding representation then we define $\phi : G \to \mathbb{C}$ the *character* of Φ by

$$\phi(g) \;=\; trace(\Phi(g)) \tag{8}$$

Clearly, ϕ is a class function. ϕ is an *irreducible character* if the corresponding representation Φ is irreducible.

Let Φ_1 and Φ_2 be representations of G. Then Φ_1 and Φ_2 are equivalent if $\exists \; T : V_1 \to V_2$, such that T is invertible and $T \cdot \Phi_1(g) \cdot T^{-1} = \Phi_2(g)$ for all g in G.

If ϕ_1 and ϕ_2 are two irreducible $\mathbb{C}G$ characters. With the help of **Schur's Lemma** we get the **Orthogonality Relation**. If Φ_1 and Φ_2 are the representations corresponding to ϕ_1 and ϕ_2, then

$$(\phi_1, \phi_2)_G = 0 \;\text{ if } \Phi_1 \text{ is not equivalent to } \Phi_2 \tag{9}$$

and

$$(\phi_1, \phi_2)_G = 1 \;\text{ if } \Phi_1 \text{ and } \Phi_2 \text{ are equivalent} \tag{10}$$

One of the consequences is that complex characters of irreducible representations form orthonormal basis for CG, and thus,

Corollary 6. The number of irreducble represenations of G over $\mathbb{C}$ is equal to number of conjugacy classes.

Frobenius Reciprocity Theorem is very useful in determining the irreducible charcaters of G given the irreducible characters of $H \leqslant G$

Corollary 7. Frobenius Reciprocity
Let $H \leqslant G$. ϕ and ψ are complex characters of H and G respectively. Let ϕ^G be the character of the representation Φ^G and let ψ_H be the character of Ψ_H. Then

$$\langle \phi, \psi_H \rangle_H = \langle \phi^G, \psi \rangle_G \tag{11}$$

1.2 Symmetric Groups

Let $N = \{1, 2, ...n\}$. The symmetric group S_n is the set of permutations of N, with composotion as the group operation. Suppose $\sigma \in S_n$, then for any $g \in S_n$,

$$\sigma(i) = k \iff g \cdot \sigma \cdot g^{-1}(g(i)) = g(k). \tag{12}$$

A cyclic permutation $\sigma_t = (a_1 a_2 .. a_t)$ is defined as

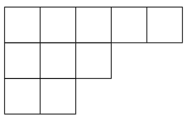
$$\sigma_t(a_i) = a_{i+1} \text{when } i \text{ is less then } t \tag{13}$$

and $\sigma_t(a_t) = a_1$. Any permutation σ is a disjoint product of cyclic permutation σ_t.

The type of a permutation is a tuple which represents the size of each disjoint cycle. It is easy to see that for each partition $\lambda = (\lambda_1, \lambda_2 ... \lambda_t)$ of n, we have a permutation of type λ.

Thus, by the above discussion, the number of conjugacy classes of S_n is equal to the number of partitions of n.

Let $\lambda = (\lambda_1, \lambda_2 ... \lambda_t)$ be a partition of n. A composition $(\lambda_1, \lambda_2, .., \lambda_t)$ of n is a partition $\lambda_1 \geq \lambda_2 \geq .. \lambda_t$. Let $\lambda = (\lambda_1, \lambda_2 ... \lambda_t)$ be a partition of. *Young diagram* of λ is a 2 dimensional diagram with n boxes put together such that, there are t rows, and j^{th} row has λ_j boxes.
For example, Young diagram of λ = (5,3,2) is,



On the set of Young diagrams we have a partial order called the *dominance order.*

Definition 8. *Domninance Order*
Let $\lambda = (\lambda_1, \lambda_2, ... \lambda_{t_1})$ and $\mu = (\mu_1, \mu_2, ... \mu_{t_2})$ be partitions of n, then we say that $\lambda \trianglerighteq \mu$ or λ dominates μ, if

$$\sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i \text{ for all } j \text{ in } \mathbb{N} \tag{14}$$

For a young digaram of λ we can fill the boxes with numbers from $\{1, 2..., n\}$. There are $n!$ various combinations in which we can fill the boxes. Each such combination is called a *Young Tableau.*

For example let $\mu = (3, 1, 1)$ be partition of 5, then

$$t_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \qquad t_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

t_1 and t_2 are Young tableaux of shape μ.

Let t be a tableau of shape μ. $\sigma \in S_n$ is a row stablizer if $\sigma(j)$ is in the row as j in t. The subgroup of row stabalizers is R_t. Similarly, we can define the set of column stabalizers C_t.

Definition 9. Let $\mu = (\mu_1, \mu_2, .., \mu_r)$ be a partition of n and t a tableau of shape μ. $R_t = S_{\mu_1} \times S_{\mu_2} \times .. \times S_{\mu_r}$. The *Young Tabloid* $\{t\}$ is the subpartition of $[1, 2..n]$ with parts the rows of t. Clearly, it is determined by $R_t \leqslant S_n$ and S_n acts on it naturally.

For example:

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$$

$$\{t\} = \begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$$

The action of S_n over μ tabloids gives rise to a permutation module M^μ.

$$M^\mu \cong 1_{R_t} \uparrow^{S_n} \tag{15}$$

Now, let t be a μ tableau and $k_t = \sum_{\sigma \in C_t} sgn(\sigma)\sigma$. Then the polytabloid e_t is an element of M^μ as,

$$e_t = k_t e_t \tag{16}$$

The permutation module of S_n over the action on the set of μ polytabloids is called the *Specht Module* S^μ. This is a cyclic module since

$$\sigma \cdot k_t \cdot \sigma^{-1} = k_{\sigma t}. \tag{17}$$

On M^μ, we can define a non-singular, symmetric and S_n invariant bilinear form $\langle ., . \rangle$, by defining it on the tabloids. If $\{t_1\}$ and $\{t_2\}$ are M^μ tabloids.

$$\langle \{t_1\}, \{t_2\} \rangle = 1, \text{ if } \{t_1\} = \{t_2\} \tag{18}$$

$$\langle \{t_1\}, \{t_2\} \rangle = 0, \text{ otherwise} \tag{19}$$

With the help of this bilinear form, we get the following result

Submodule Theorem
Let $S^{\mu^\perp}$ be the orthogonal space to S^μ with respect to the bilenear form of M^μ. If $U \subseteq M^\mu$, then either $S^\mu \subseteq U$ or $U \subseteq S^{\mu^\perp}$.

With the help of submodule thereom, we can in turn prove that if $S^\mu \cap S^{\mu^\perp} \neq S^\mu$, then $S^\mu \cap S^{\mu^\perp}$ is the unique maximal submodule of S^μ and thus,

$$\frac{S^\mu}{S^\mu \cap S^{\mu^\perp}} \ \textit{is irreducible}. \tag{20}$$

If $char(F) = 0$, then $S^\mu \cap S^{\mu^\perp} = 0$, thus S^μ is irreducible.

Another major consequence of Submodule Theorem is,

Corollary 10. Suppose $char(F) = 0$. If we have a non trivial homomorphism $S^\lambda \to M^\mu$, then $\lambda \trianglerighteq \mu$

Thus, S^μ is an irreducible $\mathbb{C}G$ mdoule and for two distinct partitions μ and λ, $S^\mu \ncong S^\lambda$ as a consequence of previous corollary.

We define a total ordering $\leq$ on the set of μ tabloids, given by
$\{t_1\} \leq \{t_2\}$, if and only if, $\exists\ i$ such that

1) when $j \geq i$, j is the same row of $\{t_1\}$ and $\{t_2\}$.
2) i is in the higher row of $\{t_1\}$ then $\{t_2\}$.

Definition 11. Standard Tableau: A tableau t is standard, if the entries in t are increasing in each row from left to right and each column from top to bottom.

An example of a standard tableau is

$$t = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

If t_s is standard tableau, then e_{t_s} is standard polytabloid.

Any element v in S^μ is a linear combination of standard polytabloids [1]. Let t_s be standard tabloid. Then, $\sigma t_s \leq t_s$. Thus, the set of standard polytabloids e_{t_s} is linearly independent. Therefore

The dimension of Specht module S^μ is the number of standard μ tableaux.

Studying properties of Specht modules is an interesting topic in itself. How does Specht module behave when we restrict it to subgroups? In the rest of the chapter we focus on this question.

Theorem 12. Branching Theorem:
Let S^μ be the specht module of S_n. Then

$$S^\mu \downarrow_{S_{n-1}} = \oplus S^{\mu'},$$

where a Young diagram of $\mu^{'}$ is obtained by removing one box from a corner in such a way that we get a Young diagram of partition μ' of $n-1$.

The proof of the branching theorem is complicated. James[9.3] [1] is a good reference.

Consider the subgroups $S_k \times S_{n-k}$ of S_n, what about the induced module $S^{\mu_k} \otimes S^{\mu_{n-k}} \uparrow^{S_n}$. Thanks to the **Littlewood Richardson Principle**, we know the exact way to compute the given induced module. For that let us introduce the concept of *sequences*.

A sequence a just a string of n integers. A sequence s is said to be of type ν, where ν is partition of n, if each number i occurs ν_i times in the sequence s.

In a sequence s each element can be determined as good or bad in the following way.

1) All $1's$ are good.
2) An element $i+1$ at position j is good if and only if the number of good i^s is strictly greater then the number of good $(i+1)^s$ at postions before j.

For example,

$$\begin{array}{ccccc} 2 & 1 & 2 & 3 & 2 \\ \times & \checkmark & \checkmark & \checkmark & \times \end{array}$$

Suppose ν' and ν are partitions of n. Then $s(\nu', \nu)$ is the set of sequences of type ν in which there are atleast ν_i^{prime} entries of good i.

Definition 13. Let λ and ν be partition of S_n and S_m respectively and $[\lambda]$ be the young diagram of λ. Let $(S^\lambda)^{[\nu',\nu]}$ be the KS_{m+n} module,

$$(S^\lambda)^{[\nu',\nu]} = \oplus a_\mu S^\mu \tag{21}$$

where $a_\mu = 0$, unless $\lambda_i < \mu_i$ for all i, in which case a_μ is the number of ways of filling the boxes in $[\mu]/[\lambda]$ such that,

1) The entries in each row are increasing from left to right.

2) The entries in each column are increasing from top to bottom.

3) When read from right to left in succesive rows, the sequence obetained belongs to $s(\nu', \nu)$.

Theorem 14. Littlewood Richardson Principle:

$$S^\lambda \otimes S^\nu \uparrow^{S_n} = (S^\lambda)^{[\nu,\nu']} \tag{22}$$

The proof of Littlewood Richardson Principle uses the concept of sequences and ordering among various types of sequences. For proof refer to James[16.2] [1].

The next chapter will introduce a special kind of S_n module called the Foulkes Module. As seen in the Littlewood Richardson Principle, we can compute the irreducible coefficients of $S^{\mu_k} \otimes S^{\mu_{n-k}} \uparrow^{S_n}$. What about the irreducible summands of $(1_{S_a\, wr\, S_b} \uparrow^{S_{ab}})$? Our goal is to understand.

2 Foulkes Module

Let $a, b \in \mathbb{N}$, The *Foulkes Module* is the permutation module

$$F_a^b = 1_{S_a wr S_b} :\uparrow^{S_{ab}} . \tag{23}$$

The Foulkes module can be visualized in many ways. For example,

1) Consider the set T of ab elements. Let

$$H_a^b = \{h_a^b \mid \text{where } h_a^b \text{ is partition of } T \text{ into b sets of size } a \text{ each}\}. \tag{24}$$

It is easy to see that the natural action of S_{ab} on H_a^b gives rise to the permutation module isomorphic to F_a^b.

2) A function f in n variables, is symmetric if $f(x_1, x_2, .., x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, .., x_{\sigma(n)})$, where $\sigma \in S_n$. In general, a function f', is symmetric if $f'(x_1, x_2, .., x_n, ..) = f'(x_{\sigma(1)}, x_{\sigma(2)}, .., x_{\sigma(n)}, ..)$, where σ is a permutation of $\mathbb{N}$
Let $\alpha = (\alpha_1, \alpha_2, .., \alpha_l)$ be a partiton of k and $x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \cdot x_l^{\alpha_l}$. The symmetric function f of degree k is obtained by symmetrization of the monomial x^α for some $\alpha \vdash k$. The ring of symmetric functions Λ is a generated by the elementery symmetric functions $f_k = \sum x^\alpha$, where $\alpha = (1^k)$.

Definition 15. Let $x = (x_1, x_2, .., x_l)$ and $\lambda = (\lambda_1, \lambda_2, .., \lambda_l)$ be a weakly decreasing sequence of non negative integers. Let $M_\lambda = [x_i^{\lambda_j}]$ and $a_\lambda = det(M_\lambda)$. If $\delta = (l-1, l-2, .., 0)$. Then the *Schur function* s_λ is defined as

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta} \tag{25}$$

An important generating set of Λ is the set of Schur functions.

$$\Lambda = \langle s_\mu \big| \mu \vdash n, n \in \mathbb{N}\rangle \tag{26}$$

Definition 16. The Plethysm [2] of symmetric functions f and $g = \sum c_\alpha \, x^\alpha$, $\alpha = (\alpha_1, \alpha_2, ...)$, is defined as

$$f \circ g = f(y_1, y_2, ..) \tag{27}$$

where y_i is given by the following

$$\Pi(1 + y_i t) = \Pi(1 + x^\alpha t)^{c_\alpha} \tag{28}$$

Let ϕ_m and ϕ_n be characters of S_m and S_n repectively, then we can define multiplication by

$$\phi_m * \phi_n \;=\; (\phi_m \times \phi_n) \uparrow^{S_{mn}} . \tag{29}$$

Let χ^λ be the character of S^λ where λ is a partition of n. The *characteristic map* [2] *ch* $\oplus_{n\in\mathbb{N}} C(S_n) \to \Lambda$ is defined on the generators:

$$ch(\chi^\lambda) = s_\lambda \tag{30}$$

The characteristic map is an isomorphism and $ch(f_a^b) = s_{(b)} \circ s_{(a)}$, where f_a^b is the character of F_a^b. Thus, Plethysms can be used to study Foulkes module.

An important object of study in Representation of Symmetric Groups and in turn Algebraic Combinatrics is the structure of Foulkes Module. Till date, we only know the structure of F^b_a, when a or b is 2 or 3 [3] [4].

An interesting question in the study of Foulkes module is the *Foulkes Conjecture*, which states that, if $a < b$, F^a_b can be embedded in F^b_a.

The orgin of this problem goes back to 1950 [3], when H.O. Foulkes, while computuing the coefficents in the plethysm of Schur functions, observed such a pattern. Thrall [5], which studying the plethysm of $s_{(2)}$ and $s_{(a)}$ as a sum of Schur functions, computed the coefficients and proved the conjecture when $a = 2$.

In 2000, S.C. Dent [4], computed the irreducible constituents of Foulkes Module and thus proved the conjecture for $a = 3$.

Let λ be a partiton of n and λ' be the conjugate partition. The standard map, $St : M^\lambda \to M^{\lambda'}$ is defined as,

$$St(\{t\}) = \sum_{g \in R_t} g\{t'\} \tag{31}$$

where, $\{t\}$ is a λ tableaux, and t' is the λ' tableaux, where (i, j)'th entry of t is the same as (j, i)'th entry of t'.

Let, $N(R_t)$ be the normalizer of R_t is S_n. Consider the permutation module $F^\lambda = 1_{N(R_t)} \uparrow^{S_n}$. Then, $F^\lambda \subseteq M^\lambda$ and moreover,

$$St(M^\lambda) \subseteq F^{\lambda'} \tag{32}$$

The S.W.S conjecture due to Siemens, Wagner and Stanley [6], states that the standard map is injective on F^λ whenever λ dominates λ'. In general the conjecture false. However Tom Mckay [7], proved it under a certain condition.
The conditions are as follows.

1) Let μ be the Young diagram obtained by removing the leftmost column of λ. S.W.S conjecture is true for F^μ.
2) In each box of the left most column, number of boxes below it is not larger then the number of boxes below it.

Note that $F^{(a^b)}$ is isomorphic to Foulkes module F^b_a. S.W.S conjecture is true for $\lambda = (4^4)$. Therefore, Foulkes Conjeture is true when $a = 4$. Now, suppose S.W.S conjecture is true for $\lambda = (a^a)$ for some a then Foulkes Conjecture is true for the same a. In 2015, Cheung, Ikenmeyer and Mkrtchyan [8] showed that the standard map is inejctive on $F^{(6^5)}$, thus proving the Foulkes Conjecture for $a = 5$. One important conclusion that came out of their paper is that the standard map is not injective for $\lambda = (5^5)$. One advantage of the standard map is that, by computing it for a fixed a and b, we might be able to prove Foulkes Conjecture for a, b_1, where b_1 is greater then b.

Even when S.W.S conjecture fails, so one particular map is not injective. It still could be true that F^λ is a submodule of $F^{\lambda'}$ when λ dominates λ'.

Giannelli [9], in 2013, proved that hook characters don't belong in the Foulkes character. In other words,

$$\langle f_a^b, \chi^{(n-r,1^r)}\rangle = 0 \tag{33}$$

Suppose, α is a partition of m with l parts, such that $k > l$, and $ab - k - m\ \alpha_1 + 1$. Let $[k : \alpha]$ be the partition $(ab - k - m, \alpha_1 + 1, \alpha_2 + 1, .., \alpha_t + 1, 1^{k-t})$, which is the partition α submerged inside the hook $(ab - k - m, 1^k)$. Then, in the same paper Giannelli proved that, for $n = \sum_{j=2}^{t} \alpha_j$, if $\alpha_1 < \frac{1}{2}(k-n)(k-n+1)$,

$$\langle f_a^b, \chi^{[k,\alpha]}\rangle \ = \ 0. \tag{34}$$

He analysed the decomposition of $F_a^b \downarrow_{S_k \times S_{ab-k}}$ to the direct sum of permutation modules.

Let Ω be the set of partitions of k such that $\mu \in \Omega$ if and only if μ is a subpartion of (a^b). Let $P_\lambda = \{A_b^\lambda | A_b^\lambda \in H_a^b$ such that type $A_b^\lambda \cap \{1, 2, ...k\}$ is $\lambda\}$ and V^λ be the permutation module of P_λ under the action of $S_k \times S_{n-k}$

$$F^{a^b}|_{S_k \times S_{n-k}} = \oplus_{\lambda \in \Omega} V^\lambda \tag{35}$$

Theorem 17. Let a = 2, then

$$V^{(1^b)} \cong \oplus_\lambda S^\lambda \otimes S^\lambda \tag{36}$$

for λ partiton of b.

Proof. First we shall prove that there is a submodule U of $V^{(1^b)}$ such that $U \cong S^\lambda \otimes S^\lambda$.
Let t_1 and t_2 be two standard λ tableaux such that $\{\lambda_k + 1, \lambda_k + 2, ..., \lambda_{k+1}\}$ and $\{\lambda_k + 1 + b, \lambda_k + 2 + b, ..., \lambda_{k+1} + b\}$ belong to $k-1$'st row of t_1 and t_2 respectively.

Let C_{t_1} and C_{t_2} be the sets of column stabilizers of t_1 and t_2 respectively. Let $e_{t_1} = k_t t_1$ and e_{t_2} be the standard polytabloid. Consider the homomorphism
$\phi_\lambda : S_1^\lambda \otimes S_2^\lambda \longrightarrow V^{(1^b)}$, given by

$$e_{t_1} \otimes e_{t_2} \longrightarrow k_{t_2} \cdot k_{t_1}\{(1, b+1), (2, b+2), ...(b, 2 \cdot b)\} \tag{37}$$

Since $e_{t_1} \otimes e_{t_2}$ generates the irreducible module $S_1^\lambda \otimes S_2^\lambda$, therefore if the image of the map above is nonzero, then it has to be isomporphic to $S^\lambda \otimes S^\lambda$.
Consider the map ϕ_C: $C_{t_2} \to C_{t_1}$, given by

$$\tau \to \sigma \tag{38}$$

$$\tau(b+i) = b+k \iff \sigma(i) = k \tag{39}$$

Now, the action of $\tau \in C_{t_2}$ to the element $F = \{(1, b+1), (2, b+2)...(b, 2 \cdot b)\}$ is the same as action of $\phi_C(\tau)$ on F and thus the action on $V^{(1^b)}$. Let τ in C_{t_2}, then

$$sgn(\tau)\tau(k_{t_1}F) = \big(\sum_{\sigma \in C_{t_1}} (sgn(\tau)\tau) \cdot sgn(\sigma)\sigma\big)F \tag{40}$$

$$= \Big(\sum_{\sigma \in C_{t_1}} (\phi_C(\tau) sgn(\phi_C(\tau))) \cdot (sgn(\sigma)\sigma) \Big) F \tag{41}$$

$$= \Big(\sum_{(\phi_C(\tau))\cdot(\sigma) \in C_{t_1}} sgn((\phi_C(\tau)) \cdot (\sigma))(\phi_C(\tau)) \cdot (\sigma) \Big) F \tag{42}$$

Therefore $k_{t_2} \cdot k_{t_1} F$ is not zero, since $k_{t_1} F$ is nonzero. thus the image of ϕ_λ is non zero and therefore $U = \mathrm{Im}(\phi_\lambda) \cong S^\lambda \otimes S^\lambda$

We obtained that for each partion λ of b, $S^\lambda \otimes S^\lambda$ is embedded in $V^{(1^b)}$. These are nonisomorphic, therefore

$$\oplus_\lambda S^\lambda \otimes S^\lambda \subseteq V^{(1^b)}. \tag{43}$$

On the other hand

$$dim(\oplus_\lambda S^\lambda \otimes S^\lambda) = b! = dim(V^{(1^b)}) \tag{44}$$

Therefore,

$$\oplus_\lambda S^\lambda \otimes S^\lambda \cong V^{(1^b)} \tag{45}$$

□

The nice structure of $V^{(1^b)}$ for $a = 2$, gives rise to the question of whether we can generalize for larger a. The rest of the chapter focuses on this question.
Consider the submodule $U_\epsilon \subseteq V^{(1^b)}$ as follows.

$$U_\epsilon = \langle \sum_{\sigma \in S_G} sgn(\sigma)\sigma\{(1, X_1), (2, X_2), (3, X_3)...(b, (X_b))\}\rangle \tag{46}$$

where S_G is the permutation group of $\{1, 2, ..b\}$ and X_i is the set $\{b + (i-1) \cdot a, b + (i-1) \cdot a + 1, .., b + i \cdot a - 1\}$.

Proposition 18. If $\epsilon \otimes S^\mu$ is embedded in $V^{(1^b)}$, then $\epsilon \otimes S^\mu$ is embedded in U_ϵ, where ϵ is the sign character.

Proof. Any non trivial element u of $\epsilon \otimes S^\mu$ as a submodule of V$^{(1^b)}$ can be expressed as a linear combination of of elements of type $\{(1, Y_1), (2, Y_2), (3, Y_3)...(b, (Y_b))\}$, where Y_i's are disjoint subsets of $\{b+1, b+2, ..., ab\}$, of size a each. Choose an element $\{(1, Z_1), (2, Z_2), (3, Z_3)...(b, (Z_b))\}$ such that its coefficient in u is not 0. Since

$$\{(1, Z_1), (2, Z_2), (3, Z_3)...(b, (Z_b))\} = \sigma\{(1, X_1), (2, X_2), (3, X_3)...(b, (X_b))\} \tag{47}$$

for some $\sigma \in$ permutation group of $\{b+1, b+2, ..., ab\}$ and $\tau u = \mathrm{sgn}(\tau)u$ for any τ in the permutation group of $\{1, 2, .., b\}$. Therefore the coefficient of $\mathrm{sgn}(\sigma)$ σ $\{(1, Z_1), (2, Z_2), (3, Z_3)...(b, (Z_b))\}$ is the same as the coefficient of $\{(1, Z_1), (2, Z_2), (3, Z_3)...(b, (Z_b))\}$ in u for any σ in S_n. Thus,

$$\epsilon \otimes S^\mu \subseteq U_\epsilon \tag{48}$$

□

Consider the $S_{a-1}wrS_b$ module U generated by $\sum_{\tau\in S_b} \mathrm{sgn}(\tau)\{X_{\tau(1)}, X_{\tau(2)}, .., X_{\tau(b)}\}$. Now, clearly $U \cong Inf_{S_b}^{S_{a-1}wrS_b}\ \epsilon$ and

$$V_\epsilon = Inf_{S_b}^{S_{a-1}wrS_b}\epsilon \uparrow^{S_{(a-1)\cdot b)}} = \langle \sum_{\tau\in S_b} sgn(\tau)\{X_{\tau(1)}, X_{\tau(2)}, .., X_{\tau(b)}\}\rangle \tag{49}$$

The previous equation is true because the induced module of a cyclic module is cyclic with the same generator.

Theorem 19. $\epsilon \otimes \mathrm{V}_\epsilon \cong \mathrm{U}_\epsilon$

Proof. ϵ is generated by $e_{(1^b)}$. Then, the map $\phi^a_\epsilon : \epsilon \otimes V_\epsilon \longrightarrow U_\epsilon$, given by

$$e_{(1^b)} \otimes \sum_{\tau\in S_b} sgn(\tau)\{X_{\tau(1)}, X_{\tau(2)}, .., X_{\tau(b)}\} \longrightarrow \sum_{\sigma\in S_G} sgn(\sigma)\sigma\{(1, X_1), (2, X_2), (3, X_3)...(b, (X_b))\} \tag{50}$$

Since both are cyclic modules and ϕ^a_ϵ maps one generator to the other thus ϕ^a_ϵ is $K\ S_k \otimes S_{n-k}$ module homomorphism. Since dimension of $\epsilon \otimes V_\epsilon$ is equal to dimension of U_ϵ. Therefore $\epsilon \otimes V_\epsilon \cong U_\epsilon$. □

Further, Paget and Wildon [10] have computed the minimal irreducicle constituents of Foulkes Module. More importantly, when a is even $S^{(a^b)}$ is a submodule of F^b_a.

Suppose λ and μ are partitions of k and $n-k$ respectively. Then,

$$\langle f^b_a \downarrow_{S_k\times S_{n-k}}, \chi^\lambda \times \chi^\mu\rangle = 0 \tag{51}$$

implies that, if every constituent of $S^\lambda \otimes S^\mu \uparrow^{S_n}$ is not a submodule of F^b_a. Thus, studying the restrictions of Foulkes Module can be helpful in proving Foulkes Conjecture.

I am trying to study the restrictions of Foulkes Module to elementary abelian subgroups, and further with the help of elementary abelian subgroups, studying Foulkes Module restricted to Sylow-p subgroups.

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