Sheaf Cohomology and its Applications in Analytic Geometry

by

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Declaration

As a work carried out at Central European University in partial fulfilment of the requirements for a Master of Science in Mathematics, I hereby declare that the work contained in this thesis is my original work. The work done by others has been acknowledged and referenced accordingly.

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Abstract

The purpose of this project is to explore the notion of sheaves of abelian groups and sheaf cohomology over a topological space, and apply them to investigate analytic varieties and several problems in analytic geometry. Any sheaf of abelian groups over a topological space X can be considered as an algebraic object as it is a collection of abelian groups, parametrized by the space X. It also can be considered as a topological object by lifting the topology of X under the natural projection given by the parameters of the collection. From this, we apply homological algebra to construct sheaf cohomology, which helps us to investigate the global sections of sheaves. For computation of sheaf cohomology, we describe two methods. The first one is based on fine resolutions, which is useful in case of paracompact Hausdorff spaces. The second one uses Čech cohomology, which is very useful when we have natural open coverings of the space X. As an application, we investigate the notion of Stein varieties in terms of sheaf cohomology. Furthermore, we analyze two problems in the theory of holomorphic functions in several variables, namely, the *additive and multiplicative Cousin's problems.*

Contents

1 Introduction

The notion of sheaves and sheaf cohomology was introduced by *Jean Leray* in 1945 and since then, it became a crucial tool in many areas of mathematics such as analytic geometry, algebraic geometry and differential geometry. The idea is to investigate local algebraic data available on a topological space, and to decide whether those local data can be glued together to get global ones. Consider for instance two open subsets U and V of the complex line $\mathbb{C}^1$ such that $U \cap V \neq \emptyset$, and consider holomorphic functions $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$. If $f = g$ on $U \cap V$, then they can be glued in a trivial way to give a holomorphic function on $U \cup V$. It is more convenient to work with class of functions to guarantee the existence of such holomorphic extension. At every point $z \in \mathbb{C}$, we associate the set of germs of holomorphic functions at z, denoted by $\mathcal{O}_z$. This set consists of class of holomorphic functions in a neighborhood of z associated with an equivalence relation $\sim_z$. By definition, the relation $\sim_z$ is defined by $f \sim_z g$ if and only if there exists an open neighborhood of z in which f and g agree. It is clear that $\mathcal{O}_z$ has a ring structure with the usual addition and multiplication of functions. The gluing property allows us to glue the local data $\{\mathcal{O}_z, z \in \mathbb{C}\}$ and gives us a big space above the complex line, called *the sheaf of germs of holomorphic functions over* $\mathbb{C}$. It is denoted by $\mathcal{O}$. The gluing can be considered as a procedure of giving on $\mathcal{O}$, in a continuous manner, a topology lifted from $\mathbb{C}$, under the natural projection $\pi : \mathcal{O} \to \mathbb{C}$ given by $\pi(\mathcal{O}_z) = z$.

At any point $z \in \mathbb{C}$, the units $\mathcal{O}_z^*$ of the ring $\mathcal{O}_z$ consist of the class of functions, which do not vanish at z. These units can also be glued and give us a space called *the sheaf of germs of nowhere vanishing holomorphic functions on* $\mathbb{C}$, denoted by $\mathcal{O}^*$. Under the exponential map, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2\pi i} \mathcal{O} \xrightarrow{exp} \mathcal{O}^* \longrightarrow 1.$$

One can ask whether we still have exactness in the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2\pi i} \mathcal{O}(U) \xrightarrow{exp} \mathcal{O}^*(U) \longrightarrow 1,$$

where $U \subset \mathbb{C}$ is an open subset. The exactness of this sequence means that any nowhere vanishing holomorphic function on U is the image under the exponential map of a holomorphic function on U. The use of sheaf cohomology allows us to measure the exactness of this sequence. Our goal in this project is to explore the notion of sheaf cohomology, and to analyze various problems in analytic geometry.

In the first chapter, we give an overview of the notion of analytic varieties based on [1]. The notion of germs of analytic varieties and germs of holomorphic functions allow us to study analytic varieties locally. Based on their local nice properties, most problems are solved locally, and we would like to investigate such problems globally. This gives us an insight into the discussion of sheaf cohomology. In particular, the gluing properties on a holomorphic line bundle motivates the definition of sheaves. Then we explore the notion and basic properties of sheaves over a topological space, following [2], [3], [4] and [5].

In the next chapter, we describe the notion of sheaf cohomology, based on [2], [6] and [3]. This is an application of homological algebra. We explore the notion of spectral sequences to be able to compute the cohomology of a cochain complex. This study also allows us to compare the cohomology of two different cochain complexes. We define cohomology groups with coefficients in a sheaf in terms of *Godement resolutions*. Using this, we can consider those sheaves, which have vanishing higher cohomology groups. They are known as acyclic sheaves. We use such sheaves to compute sheaf cohomology by using the notion of acyclic resolutions. We can perform this by showing that sheaf cohomology is independent of the choice of an acyclic resolution. The Godement resolution is a special case of the so called *flasque resolutions*, which are particular cases of acyclic resolutions. Any flasque resolution is formed by a cochain complex of the so called *flasque sheaves*, which are defined by the property that the sections on arbitrary open sets always extend to the whole space. Another family of sheaves, called *fine sheaves* are very useful in case of paracompact Hausdorff spaces. They have a technical property on each locally finite open covering of the space. By studying a more larger family of sheaves, called *soft sheaves*, one can prove that fine sheaves are acyclic. This result allows us to compute the sheaf cohomology groups from any fine resolution.

We also describe another method of computation, called Čech cohomology. It is based on locally finite open coverings of the space and it is useful in the case, where the space is locally contractible. The notion of Leray covering on an open covering allows us to directly compute the cohomology of the space with coefficients in a sheaf, using Čech cohomology associated to that covering.

As an application, we investigate two problems known as *additive and multiplicative Cousin's problems*. The first one targets the existence of a meromorphic function with prescribed poles, and the second one allows us to study the variety, by investigating the Picard groups on them. Based on [2] and [7], we explore the notion of analytic coherent sheaves and describe a class of analytic varieties called *Stein varieties*. In such varieties, all higher cohomology groups with coefficients in any analytic coherent sheaf, vanish. In particular, on Stein manifolds, we explain why the additive Cousin's problem is solvable. Also in the multiplicative case, we have a nice cohomological sufficient condition for the solvability of the problem.

2 Local Properties and Sheaf Theory

2.1 Notion of Analytic Varieties

In this section, let n be a fixed non-negative integer and $\mathbb{C}^n$ the n-dimensional complex affine space with the usual topology. Also, for each open set $U \subset \mathbb{C}^n$, let $\mathcal{O}_n(U)$ be the algebra of holomorphic functions on U.

Definition 2.1.1. A subset $X \subset \mathbb{C}^n$ is called *an analytic subvariety of* $\mathbb{C}^n$ if it is locally defined by common zeros of finitely many holomorphic functions. More precisely, for each point $x \in X$, we can find an open neighborhood U_x of x and holomorphic functions $f_{1,x}, \dots, f_{k,x} \in \mathcal{O}_n(U_x)$ such that

$$X \cap U_x = \{x \in U_x \mid f_{1,x}(x) = \cdots = f_{k,x}(x) = 0\}.$$

In particular, an analytic subvariety, which is locally defined by a single non-zero holomorphic function is called an *analytic hypersurface of* $\mathbb{C}^n$. Of course, open sets of $\mathbb{C}^n$ are analytic subvarieties since they can be defined by the zero function.

On analytic subvarieties containing a point $x \in \mathbb{C}^n$, we define the equivalence relation $\sim_x$. We say that $X_1 \sim_x X_2$ if and only if there is an open neighborhood U_x of x such that $X_1 \cap U_x = X_2 \cap U_x$. An equivalence class (X, x) is called *a germ of analytic varieties at x*.

Since our objects of study depend on local properties, our study will focus on local algebraic properties.

Definition 2.1.2. For each point $x \in \mathbb{C}^n$, let $\mathcal{O}_{n,x}$ be the algebra of class of holomorphic functions defined in a neighborhood of x, where two functions f and g are identified if and only if we can find an open neighborhood U_x of x such that $f|_{U_x} = g|_{U_x}$. An equivalence class $\mathbf{f}$ with representative f is called *the germ of f at x*.

By translation, it is enough to study the germs at the origin. The Laurent series expansion of holomorphic functions allow us to identify $\mathcal{O}_{n,0}$ with the algebra of convergence power series in n variables $\mathbb{C}\{x_1, \dots, x_n\}$. This is of course a local ring, with maximal ideal consisting of those germs, which vanish at 0. Furthermore, we have the following strong result.

Theorem 2.1.3. *The algebra $\mathbb{C}\{x_1, \dots, x_n\}$ is a noetherian and unique factorization domain.*

Proof. See on page 7 of *Robert Gunning*'s book [1] . □

To each ideal I of $\mathcal{O}_{n,0}$, we associate the germ of analytic variety $(\mathscr{V}_0(I), 0)$, where

$$\mathscr{V}_0(I) = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } \mathbf{f} \in I\}.$$

Conversely, to a germ of variety $(X, 0)$, we associate the ideal

$$\mathscr{I}_0(X) = \{\mathbf{f} \in \mathcal{O}_{n,0} \mid f(x) = 0 \text{ for all } x \in X\}.$$

We can relate germs of subvarieties at 0 and the ideals of $\mathcal{O}_{n,0}$ as follow.

Proposition 2.1.4. The assignments $\mathscr{V}_0$ and $\mathscr{I}_0$ satisfy the following properties:

(i) for any inclusion $X_1 \subset X_2$ of analytic varieties, $\mathscr{I}(X_2) \subseteq \mathscr{I}(X_1)$;

(ii) for any inclusion $I \subset J$ of ideals of $\mathcal{O}_{n,0}$, $\mathscr{V}(J) \subseteq \mathscr{V}(I)$;

(iii) for any germ $(X, 0)$ of analytic variety, we have $(X, 0) = (\mathscr{V}_0(\mathscr{I}_0(X)), 0)$;

(iv) two analytic subvarieties X_1 and X_2 are equivalent at 0 if and only if $\mathscr{I}_0(X_1) = \mathscr{I}_0(X_2)$;

(v) for any germs $(X_1, 0)$ and $(X_2, 0)$, we have $\mathscr{I}_0(X_1 \cup X_2) = \mathscr{I}_0(X_1) \cap \mathscr{I}_0(X_2)$;

(vi) for any ideals I, J of $\mathcal{O}_{n,0}$, we have $\mathscr{V}_0(IJ) = \mathscr{V}_0(I \cap J) = \mathscr{V}_0(I) \cup \mathscr{V}_0(J)$ and $\mathscr{V}_0(I + J) = \mathscr{V}_0(I) \cap \mathscr{V}_0(J)$.

Proof. See on page 14 of *Robert Gunning's* book [1]. □

With this proposition, the factorization in $\mathcal{O}_{n,0}$ can be translated into factorization of germs of analytic subvarieties at 0 through the map $\mathscr{V}_0$ and this leads us into the discussion of irreducible germs.

Definition 2.1.5. A germ $(X, 0)$ of an analytic subvariety of $\mathbb{C}^n$ is called *irreducible* if it cannot be written as $X = X_1 \cup X_2$, where X_1 and X_2 are two non-empty germs of analytic subvarieties at 0, and each of them is not contained in the other one.

Through the map $\mathscr{I}_0$, we have the following correspondance:

Proposition 2.1.6. A germ $(X, 0)$ is irreducible if and only if $\mathscr{I}_0(X)$ is a prime ideal.

Proof. See on page 15 of *Robert Gunning's* book [1]. □

The noetherian property of the ring $\mathcal{O}_{n,0}$, together with the correspondence given by $\mathscr{V}_0$ and $\mathscr{I}_0$ in Proposition 2.1.4 allow us to write uniquely a germ of an analytic variety as finite union of maximal irreducible germs, due to the following theorem.

Theorem 2.1.7. *For any germ $(X, 0)$ of an analytic variety, there exist finitely many irreducible germs, $(X_1, 0), \dots, (X_k, 0)$, which are uniquely determined up to permutation, such that $X = \bigcup_{i=1}^{k} X_i$ around the origin and $X_i \not\subseteq X_j$ for $i \neq j$.*

Proof. See on page 15 of *Robert Gunning's* book [1]. □

Example 2.1.8. Let $X \subset \mathbb{C}^n$ be an analytic hypersurface, which contains the origin, and let f a holomorphic function, which defines X around the origin. Since $\mathcal{O}_{n,0}$ is a unique factorization domain, we can write $\mathbf{f}$ as product of irreducible elements $\mathbf{f}_1, \dots, \mathbf{f}_k$. Around the origin, we have $X = \mathscr{V}_0(\mathbf{f}) = \bigcup_{i=1}^{k} \mathscr{V}_0(\mathbf{f}_i)$, and by Proposition 2.1.6, all $\mathscr{V}_0(\mathbf{f}_i)$ are irreducible. So they define the irreducible maximal germs of X around the origin.

Let X be an analytic subvariety of $\mathbb{C}^n$. The notion of holomorphic and meromorphic functions on X constitutes the main tool in order to describe X.

Definition 2.1.9. A complex-valued function f on X is called *holomorphic on* X if it is locally defined by the restriction on X of holomorphic functions. That is, for each point $x \in X$, there exist an open neighborhood $U_x \subset \mathbb{C}^n$ of x and a holomorphic function $f_x \in \mathcal{O}_n(U_x)$ such that $f|_{X\cap U_x} = f_x|_{X\cap U_x}$.

As before, for each open set $X \cap U \subset X$, we denote by $\mathcal{O}_X(X \cap U)$ the algebra of holomorphic functions on $U \cap X$ and for each point $x \in X$, we denote by $\mathcal{O}_{X,x}$ the algebra of germs of holomorphic functions on X at x. It is clear that $\mathscr{I}_x(X)$ is just the kernel of the restriction homomorphism $\mathbf{f} \longmapsto \mathbf{f}|_X$ from $\mathcal{O}_{n,x}$ to $\mathcal{O}_{X,x}$, which is surjective by definition. So we have the identification $\mathcal{O}_{X,x} \cong \mathcal{O}_{n,x}/\mathscr{I}_x(X)$.

The function f is called *meromorphic on* X if at each point $x \in X$, there exist an open neighborhood $U_x \subset \mathbb{C}^n$ of x and holomorphic functions $f_x, g_x \in \mathcal{O}_X(X \cap U_x)$ such that the germ $\mathbf{g}_x$ is a not a zero divisor in $\mathcal{O}_{X,x}$ and $f|_{X\cap U_x} = f_x/g_x$.

To avoid the embedding problems into different affine spaces, we consider maps between analytic subvarieties.

Definition 2.1.10. Let $X \subset \mathbb{C}^m$ and $Y \subset \mathbb{C}^n$ be analytic subvarieties. A mapping

$$f = (f_1, \ldots, f_n) : X \to Y$$

is called *a holomorphic mapping* if each f_i is holomorphic on X. It is furthermore called *a biholomorphic mapping* if it has a holomorphic inverse.

The following theorem shows that an analytic subvariety is uniquely determined by its algebra of holomorphic functions.

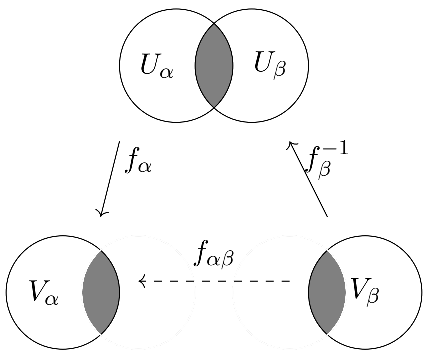
Theorem 2.1.11. *A mapping $f : X \to Y$ is biholomorphic if and only if for each point $x \in X$, $\mathcal{O}_{X,x}$ is isomorphic to $\mathcal{O}_{Y,f(x)}$ as $\mathbb{C}$-algebras.*

Proof. See on page 23 of *Robert Gunning's* book [1]. □

Also, through the notion of biholomorphic mappings, we can generalize the notion of analytic subvarieties as follow.

Definition 2.1.12. A second-countable Hausdorff topological space X is called *an analytic variety* if there exist an open covering $\{U_\alpha\}$ of X and homeomorphisms $f_\alpha : U_\alpha \to V_\alpha$, where each V_α is an analytic subvariety of $\mathbb{C}^n$, such that the following condition holds:

If $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, then the mapping $f_{\alpha\beta} := f_\alpha \circ f_\beta^{-1} : f_\beta(U_{\alpha\beta}) \to f_\alpha(U_{\alpha\beta})$ is a biholomorphic.



The family $\{U_\alpha, f_\alpha\}$ is called a *coordinate covering of the variety* X. The mappings f_α are called *coordinate mappings* and those $f_{\alpha\beta}$ are called *coordinate transition mappings*. In particular, we say that X is a *complex manifold of dimension* n if the analytic subvarieties V_α are all open subsets of $\mathbb{C}^n$.

The notion of holomorphic functions on analytic varieties is defined as follow.

Definition 2.1.13. Let X be an analytic variety. A complex-valued function f on X is called *holomorphic (resp. meromorphic) on* X if for each local coordinate (U_α, f_α), the composition $f \circ f_\alpha^{-1} : V_\alpha \to \mathbb{C}$ is holomorphic (resp. meromorphic) on the analytic subvariety V_α.

For each open set $U \subset X$, we denote by $\mathcal{O}_X(U)$ (resp. $\mathcal{M}_X(U)$) the algebra of holomorphic (resp. meromorphic) functions on U. If X is an analytic variety, we define the notion of analytic subvarieties of X as in that of the affine space $\mathbb{C}^n$. That is, those subsets locally defined by common zeroes of finitely many holomorphic functions on X. One useful approach in studying holomorphic and meromorphic functions on X is by investigating the notion of line bundles over it.

2.2 Holomorphic Line Bundles over Analytic Varieties

Let X be an analytic variety. It is clear that the Cartesian product $X \times \mathbb{C}$ also has a structure of an analytic variety. The variety $X \times \mathbb{C}$, together with the natural projection $\pi : X \times \mathbb{C} \to X$ is called *the trivial holomorphic line bundle over* X. Holomorphic line bundles are just analytic varieties, which locally look like this trivial one as we explain in the definition below.

Definition 2.2.1. An analytic variety E is called a *holomorphic line bundle over* X if there exists a holomorphic function $\pi : E \to X$, called *projection*, such that the following conditions hold:

(i) for each $x \in X$, the fiber $\pi^{-1}(x)$ has a structure of one dimensional complex vector space;

(ii) there exist an open covering $\{U_\alpha\}$ of X and biholomorphic mappings of the form $\phi_\alpha = (\pi, \varphi_\alpha) : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ such that φ_α is linear in each fiber over U_α.

The family $\{U_\alpha, \phi_\alpha\}$ is called a *local trivialization of the line bundle* (E, X, π).

Definition 2.2.2. Let (E_1, X, π_1) and (E_2, X, π_2) be two holomorphic line bundles over X. A holomorphic mapping $\phi : E_1 \to E_2$ is called *a homomorphism of holomorphic line bundles over* X if $\pi_1 = \pi_2 \circ \phi$. It is called *an isomorphism if it has a homomorphism inverse.*

Let $\{U_\alpha, \phi_\alpha\}$ be a local trivialization of a holomorphic line bundle (E, X, π). On each intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have a holomorphic mapping $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \to GL(1, \mathbb{C}) = \mathbb{C}^*$. The mappings $\varphi_{\alpha\beta}$'s are called *the transition functions associated to the local trivialization* $\{U_\alpha, \varphi_\alpha\}$. It is clear that on each intersection $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$\varphi_{\alpha\beta} \cdot \varphi_{\beta\gamma} \cdot \varphi_{\gamma\alpha} = 1. \tag{2.2.1}$$

Conversely, assume that we have an open covering $\{U_\alpha\}$ of X and holomorphic functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$, which satisfy the condition (2.2.1). Then we can construct a holomorphic line bundle on X by setting $E = (\coprod_\alpha U_\alpha \times \mathbb{C})/\sim$, where we identify (x, z) with $(x, \varphi_{\alpha\beta}(x)z)$. With this construction, it is easy to see that the holomorphic line bundle constructed from any local trivialization is isomorphic to the original holomorphic line bundle. Therefore, we can redefine a line bundle over X as a family $\{U_\alpha, \varphi_{\alpha\beta}\}$ such that $\{U_\alpha\}$ is an open covering of X and the $\varphi_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$'s are holomorphic mappings satisfying (2.2.1).

Now, let $E_1 = \{U_\alpha, \varphi_{\alpha\beta}\}$ and $E_2 = \{V_\alpha, \psi_{\alpha\beta}\}$ be two holomorphic line bundles over X. We define the tensor product of E_1 and E_2 by $E_1 \otimes E_2 = \{U_\alpha \cap V_\alpha, \varphi_{\alpha\beta} \cdot \psi_{\alpha\beta}\}$. With this operation, it is clear that the isomorphism classes of holomorphic line bundles over X form an abelian group, called *the Picard group of* X, denoted by $Pic(X)$. The notion of sheaf cohomology in the next chapter will allow us to investigate holomorphic functions on X though line bundles.

2.3 Motivation Examples for Sheaves

We have seen that analytic varieties depend on local properties. The way of gluing local properties to get global ones, and by restricting global properties to local ones are the main techniques in sheaf theory. We motivate this principle by the following examples.

Example 2.3.1 (Holomorphic Continuation)**.** Let U and V be two open sets of an analytic variety X such that $U \cap V \neq \emptyset$, and consider holomorphic functions $f \in \mathcal{O}_X(U)$ and $g \in \mathcal{O}_X(V)$. The condition which guarantees that f is a holomorphic extension of g on U is that $f = g$ on $U \cap V$. We would like to know how far we can extend g holomorphically. It is more convenient to work with class of functions rather than a single one to study such holomorphic continuation. At every point $x \in X$, we associate the algebra of germs of holomorphic functions $\mathcal{O}_{X,x}$. In order to understand the continuation problems, we glue these local algebras and this gives us a big space above X, called *the sheaf of germs of holomorphic functions on* X, denoted by $\mathcal{O}_X$ or just $\mathcal{O}$ if there is no confusion. The study of this space is helpful in getting information about the holomorphic continuation on our original variety X.

Example 2.3.2 (Poincaré Problem)**.** Let X be an analytic variety. We have seen that a meromorphic function on X is locally defined by quotient of holomorphic functions. This means that global meromorphic functions on X are obtained by gluing quotients of holomorphic functions in a trivial way. The problem arises whether a global meromorphic function can be represented as a quotient of two global holomorphic functions. Or, for which variety a global meromorphic function can be represented as a quotient of two global holomorphic functions. The sheaf theoretical study of the space $\mathcal{M}_X = \coprod_{x \in X} \mathcal{M}_{X,x}$, called *the sheaf of germs of meromorphic functions on* X, is helpful to solve this problem.

Example 2.3.3 (Mittag Leffler Problem)**.** Let X be an open subset of the complex line $\mathbb{C}^1$. The set of poles of a meromorphic function on X is at most a discrete set of points. The problem arises for the inverse. That is, the existence of a meromorphic function in the case when we have a prescribed set of poles supported on an infinite discrete set of points. In terms of covering, if we have a discrete set $S \subset X$, then we can always find an open covering $\{U_s\}_{s \in S}$ of X, together with meromorphic functions $f_s \in \mathcal{M}(U_s)$ such that f_s has only s as a pole on U_s and $f_r - f_s \in \mathcal{O}(U_r \cap U_s)$ for $r \neq s$. So the problem is equivalent to the existence of a global meromoprhic function f such that $f - f_s \in \mathcal{O}(U_s)$ for all $s \in S$.

Let us now formalize the notion of sheaves in order to be able to globalize local properties as motivated by the above examples.

2.4 Elementary Properties of Sheaves

In this section, let X be a fixed topological space.

Definition 2.4.1. *A sheaf of abelian groups over* X is a topological space $\mathscr{S}$, together with the continuous surjective map $\pi : \mathscr{S} \to X$, which satisfy the following conditions:

(S1) π is a local homeomorphism;

(S2) $\mathscr{S}_x := \pi^{-1}(x)$ has a structure of abelian group, for each $x \in X$;

(S3) the group operation $(r, s) \mapsto r - s$ is continuous in $\mathscr{S} + \mathscr{S} := \{(r, s) \mid \pi(r) = \pi(s)\} \subset \mathscr{S} \times \mathscr{S}$ with the induced topology.

We define correspondingly *the sheaf of rings*, with continuous ring operations. For each $x \in X$, the group $\mathscr{S}_x$ is called *the stalk of the sheaf* $\mathscr{S}$ *at* x. The map π is called *projection*.

Example 2.4.2. Let A be an abelian group. For each $x \in X$, let $\mathscr{S}_x = A$. From this, we can identify the sheaf $\mathscr{S}$ with the space $X \times A$, where A is considered with the discrete topology and $X \times A$ is considered with the product topology. This sheaf is called *the constant sheaf associated to* A, denoted by $\underline{A}_X$ or just $\underline{A}$ if the space is clear from the context.

On a given subset $Y \subset X$, with the induced topology of the topology on X; a continuous map $f : Y \to \mathscr{S}$ satisfying $\pi \circ f = Id_Y$ is called *a section of the sheaf* $\mathscr{S}$ *over* Y. The group $\mathscr{S}(Y)$ is called *the group of sections of the sheaf* $\mathscr{S}$ *over* Y, where the group operation is defined by the pointwise addition on stalks. If $Y = X$, then $\mathscr{S}(Y)$ is called *the group of global sections of the sheaf* $\mathscr{S}$, and we denote it by $\Gamma(X, \mathscr{S})$.

Remark 2.4.3. It is clear that the property of the projection π as a local homeomorphism implies that every point of $\mathscr{S}$ lies in the image of some section over an open set of X.

Let us now analyze the behaviour of sheaves in terms of their sections. The topology on the sheaf $\mathscr{S}$ can be understood better by lifting the open sets of the base space X through sections, as discussed in [6]. We make this fact more precise in the following propositions.

Proposition 2.4.4. With the notations of the definitions above, the set of all $f(U)$, where U runs through the open sets of X and f runs through $\mathscr{S}(U)$, form a basis of the topology of $\mathscr{S}$.

Proof. Remark 2.4.3 implies that every open set of $\mathscr{S}$ can be written as union of image of sections on open sets of X. Let f be a section defined on some open set U of X and let $s \in f(U) \subset \mathscr{S}$ such that $x = \pi(s)$. Again, by Remark 2.4.3, we can find an open neighborhood $\mathcal{V}$ of s and an open neighborhood V of x such that π maps homeomorphically from $\mathcal{V}$ to V. The continuity of f implies that $f^{-1}(\mathcal{V})$ is open. Let $W = U \cap f^{-1}(\mathcal{V})$. We have $\pi|_{\mathcal{V}} \circ f|_W = Id_W$. So $f(U) \supset f(W) = (\pi|_{\mathcal{V}})^{-1}(W) \ni s$ and then $f(U)$ is open. $\square$

It is also clear from the above remark that if $f \in \mathscr{S}(U)$ and $g \in \mathscr{S}(V)$ such that $f(x) = g(x)$ for some $x \in U \cap V$, then there exists an open neighborhood of x on which f and g agree. From this, we can describe the stalk $\mathscr{S}_x$ at x as a class of sections defined at x associated with an equivalence relation $\sim_x$. The relation $\sim_x$ is defined by $f \sim_x g$ if and only if there exists an open neighborhood of x in which f and g agree. In other words, $\mathscr{S}_x$ is the direct limit of the system $\mathscr{S}(U)$ through the ordered filtration of open neighborhoods of x.

We have seen that a sheaf $\mathscr{S}$ can be reconstructed from the system of sections $\{\mathscr{S}(U)\}$, and with the usual restriction maps. Let us now generalize this concept to construct a sheaf.

Definition 2.4.5. A *presheaf of abelian groups over* X is a family of abelian groups $\{\mathscr{S}_U\}_{U\subset X,\text{ open}}$, together with a group homomorphism, called *restriction morphism* $\rho_V^U : \mathscr{S}_U \to \mathscr{S}_V$ whenever $V \subset U$, and satisfying the two conditions:

(P1) $\rho_U^U = id_{\mathscr{S}_U}$ for each open set U;

(P2) $\rho_W^V \circ \rho_V^U = \rho_W^U$, whenever $W \subset V \subset U$.

We write $f|_V$ instead of $\rho_V^U(f)$ when $V \subset U$ and $f \in \mathscr{S}_U$.

Let $\{\mathscr{S}_U, \rho_V^U\}$ be a presheaf of abelian groups over X. For each point $x \in X$, let us denote by $\mathcal{U}_x$ the ordered filtration of open neighborhoods of x. On the disjoint union of $\mathscr{S}_U$, $U \in \mathcal{U}_x$, we have an equivalence relation defined by $\mathscr{S}_U \ni f \sim_x g \in \mathscr{S}_V$ if and only if there exists $W \in \mathcal{U}_x$ such that $W \subset U \cap V$ and $f|_W = g|_W$. We define the stalk $\mathscr{S}_x$ at any point x to be the set of equivalence classes with respect to the relation $\sim_x$. Any two class representative elements always meet, through the restriction morphisms, at some $\mathscr{S}_W$ for a smaller open set $W \in \mathcal{U}_x$. This gives us the structure of abelian group on $\mathscr{S}_x$. For each $U \in \mathcal{U}_x$, we denote by ρ_x^U the group homomorphism, which sends an element of $\mathscr{S}_U$ to its equivalence class in $\mathscr{S}_x$. It is clear that if $x \in V \subset U$, then $\rho_x^V \circ \rho_V^U = \rho_x^U$.

Let us now give a topology on $\mathscr{S} = \coprod_{x\in X} \mathscr{S}_x$, so that it can be a sheaf corresponding to presheaf $\{\mathscr{S}_U, \rho_V^U\}$. For each $U \subset X$ and $f \in \mathscr{S}(U)$, consider the set

$$f^+(U) = \bigcup_{x\in U} \{\rho_x^U(f)\} \subset \mathscr{S} \tag{2.4.1}$$

Proposition 2.4.6. The family of subsets of the form as in (2.4.1) form a topological basis in $\mathscr{S}$.

Proof. Let $s \in f^+(U) \cap g^+(V)$ and $x = \pi(s)$. Clearly, $x \in U \cap V$ and $s = \rho_x^U(f) = \rho_x^V(g)$. So there exists an open neighborhood $W \subset U \cap V$ of x in which $\rho_W^U(f) = \rho_W^V(g) \in \mathscr{S}(W)$, and we denote this common new element by h. It is immediate that

$$s = \rho_x^W(h) \in h^+(W) \subset f^+(U) \cap g^+(V).$$

□

With this topology, it is clear that the map $\pi : \mathscr{S} \to X$, defined by $\pi(\mathscr{S}_x) = x$, is a local homeomorphism. So it is left to clarify the continuity of the group operations.

Proposition 2.4.7. The group operation

$$\begin{aligned} \ominus : \mathscr{S} + \mathscr{S} &\longrightarrow \mathscr{S} \\ (r, s) &\longmapsto r - s \end{aligned}$$

is continuous with respect to the topology constructed in Prosposition 2.4.6.

Proof. By Proposision 2.4.6, it suffice to prove that $\ominus^{-1}(f^+(U))$ is open, for each $f^+(U) \subset \mathscr{S}$ as in (2.4.1). Let $(r, s) \in \mathscr{S} + \mathscr{S}$ such that $r - s = \rho_x^U(f)$, for some $x \in X$. There exist an open set $V \in \mathcal{U}_x$, and elements $g, h \in \mathscr{S}_V$, which represent the classes r and s respectively. By restricting to a smaller enough open subset, we may assume that $V \subset U$. We have then the equality $\rho_x^V(g) - \rho_x^V(h) = \rho_x^V(f)$. This means that there exists $W \in \mathcal{U}_x$ such that $W \subset V$ and $\rho_W^V(g) - \rho_W^V(h) = \rho_W^V(f)$. This implies that $\ominus$ maps the open neighborhood $[g^+(W) \times h^+(W)] \cap \mathscr{S} + \mathscr{S}$ of (r, s) into $f^+(W) \subset f^+(U)$. □

Remark 2.4.8. Let $\mathscr{S}$ be a sheaf, which is constructed from a presheaf $\{\mathscr{S}_U, \rho_V^U\}$. A section of $\mathscr{S}$ can be written locally as $f^+(U)$ for some $f \in \mathscr{S}_U$ and from this we have the usual restriction of sections of $\mathscr{S}$, by $f^+(U)|_V = \bigcup_{x \in V} \{\rho_x^U(f)\}$, whenever $V \subset U$. This implies that the sheaf associated to the presheaf $\{\mathscr{S}(U)\}$ of sections is just $\mathscr{S}$ itself.

We have seen that any sheaf can be constructed from a presheaf, and that sheaf is uniquely determined by such presheaf by definition of its sections. If $\mathscr{S}$ is the sheaf associated to a presheaf $\{\mathscr{S}_U, \rho_V^U\}$, then we have a family of group homomorphisms $\theta_U : \mathscr{S}_U \to \mathscr{S}(U)$, which maps to each f the section $f^+(U)$. The condition for which $\{\mathscr{S}_U, \rho_V^U\}$ can be identified with the presheaf of sections of $\mathscr{S}$ is that each θ_U is an isomorphism. This isomorphism condition can be clarified entirely by the properties of the given presheaf as we clarify in the following discussion.

Definition 2.4.9. A presheaf $\{\mathscr{S}_U, \rho_V^U\}$ is called *complete* if it is isomorphic to the presheaf of sections of its associated sheaf, i.e each homomorphism $\theta_U : \mathscr{S}_U \to \mathscr{S}(U)$ is bijective.

The following propositions formulate the necessary and sufficient conditions for presheaves to be complete, due to [4].

Proposition 2.4.10. Let $U \subset X$ be an open set. Then θ_U is injective if and only if, for each $f \in \mathscr{S}_U$ and for any open cover $\{U_i\}$ of U, $f|_{U_i} = 0$ implies that $f = 0$.

Proof. If $f^+(U) = 0$, then for each $x \in U$, $\rho_x^U(f) = 0$. This means that there exists $V_x \in \mathcal{U}_x$ such that $f|_{V_x} = 0$. The collection of those V_x covers U, so $f = 0$. This proves the injectivity. Conversely, since $f|_{U_i} = 0$, $\rho_x^U(f) = \rho_x^{U_i}(f|_{U_i}) = 0$ for all $x \in U_i$. Since the U_i's cover U, $f^+(U) = 0$ and by the injectivity of θ_U, $f = 0$. $\square$

Proposition 2.4.11. Let $U \subset X$ be an open set and assume that θ_U is injective. Then it is surjective if and only if, for any open covering $\{U_i\}$ of U and any $\{f_i \in \mathscr{S}_{U_i}\}$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, there exists $f \in \mathscr{S}_U$ such that $f|_{U_i} = f_i$.

Proof. Let us denote by $U_{ij} = U_i \cap U_j$. For the direct implication, we have the family of local sections $\{f_i^+(U_i)\}$ in which $f_i^+(U_{ij}) = f_j^+(U_{ij})$. From this, the map $f^0(U)$ defined by $f^0(U_i) = f_i^+(U_i)$ is a well defined section on U. By surjectivity of θ_U, we can find $f \in \mathscr{S}_U$ such that $f^+(U) = f^0(U)$. So $(f|_{U_i} - f_i)^+(U_i) = 0$ for each i. By the injectivity of θ_U, we have that $f|_{U_i} = f_i$ for each i. Conversely, let $f \in \mathscr{S}(U)$. By Remark 2.4.8, we can find an open covering $\{U_i\}$ of U and a family $\{f_i \in \mathscr{S}_{U_i}\}$ such that $f|_{U_i} = f_i^+(U_i)$. This implies that $(f_i - f_j)^+(U_{ij}) = 0$ for all i, j. The injectivity of θ_U implies that $f_i = f_j$ on $U_{i,j}$. So we have a well define map $g \in \mathscr{S}_U$ such that $g|_{U_i} = f_i$. This means that $g^+(U) = f$ and we have the surjectivity. $\square$

The usual restriction of functions allows us to verify the completeness of a certain presheaf, as we see in the examples below.

Example 2.4.12. We associate to each open set $U \subset X$, the ring of all real valued continuous functions on U, denoted by $\mathcal{C}_X(U)$. With the usual restriction of functions, it is clear that the system $\{\mathcal{C}_X(U)\}$ defines a complete presheaf over X. The corresponding associated sheaf is called *the sheaf of germs of continuous functions on* X, denoted by $\mathcal{C}_X$. Similarly, if X has a smooth manifold structure, we have the so called *sheaf of germs of* C^∞ *functions on* X, denoted by $\mathcal{C}_X^\infty$.

Further interesting examples also come out with a notion of complex manifold structure on the space X.

Examples 2.4.13. (a) The *sheaf of germs of holomorphic and meromorphic functions on* X, denoted by $\mathcal{O}_X$ and $\mathcal{M}_X$ respectively.

(b) The *sheaf of germs of nowhere vanishing holomorphic functions* $\mathcal{O}_X^*$, and the *sheaf of germs of not identically zero meromorphic functions* $\mathcal{M}_X^*$ on X. Both are sheaf of abelian groups with multiplicative group operations.

(c) The *sheaf of germs of differential forms of degree* r *over* X, denoted by $\mathscr{E}_X^r$.

However, even with the usual restriction of functions, presheaves are not always complete, as we show in the following example.

Example 2.4.14. Let A be an abelian group. For each nonempty open set U of X, let

$$\underline{A}_U = \{\text{constant functions from U to A}\} \cong A.$$

Clearly, the family $\{\underline{A}_U\}$ is a presheaf on X with the usual restriction of functions, which are all equal to the identity map of A. The corresponding associated sheaf is just *the constant sheaf associated to* A. The presheaf of sections of the constant sheaf $\underline{A}$ is just the system defined by the locally constant functions from open sets to A, which is different from the presheaf $\{\underline{A}_U\}$.

We assume from now on that sheaves are at least sheaves of abelian groups over X. The notion of algebraic structure on stalks allows us to construct algebraic operations on sheaves, such as subsheaves, quotient sheaves, sheaf homomorphisms, sheaf kernels and sheaf images.

Definition 2.4.15. A sheaf $\mathscr{R}$ is called *a subsheaf* of a sheaf $\mathscr{S}$ if it is an open subset of $\mathscr{S}$ and for each $x \in X$, the stalk $\mathscr{R}_x$ is a subgroup of $\mathscr{S}_x$.

The condition in the subsheaf $\mathscr{R}$ to be open means that any section of $\mathscr{S}$, which cross any point r in $\mathscr{R}$ defines a section of $\mathscr{R}$ in a small open neighborhood of r.

Examples 2.4.16. Let $\mathscr{S}$ be a sheaf of abelian groups over X, the following are some examples of subsheaves of $\mathscr{S}$.

(a) The *trivial sheaf* 0, which is defined by the zeros $0_x \in \mathscr{S}_x$ for all $x \in X$.

(b) Consider an open set $U \subset X$, we define a sheaf $\mathscr{R}$ by

$$\mathscr{R}_x = \begin{cases} \mathscr{S}_x & \text{if } x \in U; \\ 0 & \text{if } x \notin U \end{cases}$$

It is clear that with the induced topology of $\mathscr{S}$, $\mathscr{R}$ is a subsheaf. We denote this subsheaf by $\mathscr{S}|_U$.

(c) Let $x \in X$ be fixed. We define a sheaf $\mathscr{R}$ by

$$\mathscr{R}_y = \begin{cases} \mathscr{S}_x & \text{if } y = x; \\ 0 & \text{if } y \neq x \end{cases}$$

Consider $\mathscr{R}$ with the induced topology of $\mathscr{S}$. It is clear that each stalk $\mathscr{R}_y$ is a subgroup of $\mathscr{S}_y$; but if X is Hausdorff, then this sheaf is not an open subset of $\mathscr{S}$. So it is not a subsheaf. This sheaf is called *the skyscraper sheaf at* x *with respect to the sheaf* $\mathscr{S}$.

In the notion of skyscraper sheaf at a point x, it seems that we kill the stalks of the sheaf $\mathscr{S}$ outside x to construct such a sheaf, we formalize this in terms of quotient sheaves.

Definition 2.4.17. Let $\mathscr{R}$ be a subsheaf of a sheaf $\mathscr{S}$, we define the quotient sheaf $\mathscr{S}/\mathscr{R}$ by

$$\mathscr{S}/\mathscr{R} = \bigcup_{x \in X} (\mathscr{S}/\mathscr{R})_x,$$

where $(\mathscr{S}/\mathscr{R})_x = \mathscr{S}_x/\mathscr{R}_x$ for all $x \in X$, together with the quotient topology defined by the natural projection from $\mathscr{S}$ to $\mathscr{S}/\mathscr{R}$.

Any global section of the quotient sheaf $\mathscr{S}/\mathscr{R}$ can be described locally of the form $f + \mathscr{R}(U)$ for some $f \in \mathscr{S}(U)$ and some open set $U \subset X$. We can glue a section $f + \mathscr{R}(U)$ with a section $g + \mathscr{R}(V)$ if and only if $f|_{U\cap V} - g|_{U\cap V} \in \mathscr{R}(U \cap V)$, to get an element of $(\mathscr{S}/\mathscr{R})(U \cup V)$.

Example 2.4.18. (a) If X is a T_1-space, then the skyscraper sheaf at any point x with respect to a sheaf $\mathscr{S}$ is just the sheaf $\mathscr{S}/(\mathscr{S}|_{X\setminus\{x\}})$.

(b) If X is a complex manifold, then it is clear that the sheaf $\mathcal{O}_X$ is a subsheaf of $\mathcal{M}_X$. The quotient sheaf $\mathcal{P}_X := \mathcal{M}_X/\mathcal{O}_X$ is called *the sheaf of germs of principal parts on* X.

(c) If X is a complex manifold, then it is clear that the multiplicative sheaf $\mathcal{O}_X^*$ is a subsheaf of $\mathcal{M}_X^*$. The quotient sheaf $\mathcal{D}_X := \mathcal{M}_X^*/\mathcal{O}_X^*$ is called *the sheaf of germs of divisors on* X.

Let $\mathscr{R}$ be a subsheaf of a sheaf $\mathscr{S}$. For each $x \in X$, we have a short exact sequence of abelian groups

$$0 \longrightarrow \mathscr{R}_x \longrightarrow \mathscr{S}_x \longrightarrow \mathscr{S}_x/\mathscr{R}_x \longrightarrow 0.$$

The collection of those sequences allow us to interpret the sections of the sheaf $\mathscr{S}$ in terms of sections of the subsheaf $\mathscr{R}$ and those of the quotient sheaf $\mathscr{S}/\mathscr{R}$. The notion of sheaf homomorphisms, sheaf kernels and images will clarify this vision.

Definition 2.4.19. A continuous map $\varphi : \mathscr{R} \to \mathscr{S}$ is called *a sheaf homomorphism*, if it maps homomorphically from each stalk $\mathscr{R}_x$ into the stalk $\mathscr{S}_x$. We denote by φ_x the restriction of φ in $\mathscr{R}_x$ for each $x \in X$.

Since we have the following commutative diagram

$$\begin{tikzcd}
\mathscr{R} \arrow[rr, "\varphi"] \arrow[dr, "\pi_r"'] & & \mathscr{S} \arrow[dl, "\pi_s"] \\
& X &
\end{tikzcd}$$

and since π_r and π_s are local homeomorphisms, then also φ is. In particular φ is open. Consequently, any sheaf homomorphism sends sections to sections. We denote by φ_X the induced group homomorphism on global sections, i.e $\varphi_X : \Gamma(X, \mathscr{R}) \to \Gamma(X, \mathscr{S})$.

Definition 2.4.20. Let $\varphi : \mathscr{R} \to \mathscr{S}$ be a sheaf homomorphism. We define the *sheaf kernel* and *image* of φ as follow:

$$\ker\varphi = \bigcup_{x\in X} (\ker\varphi)_x \text{ and } \operatorname{im}\varphi = \bigcup_{x\in X} (\operatorname{im}\varphi)_x,$$

where $(\ker\varphi)_x = \ker\varphi_x$ and $(\operatorname{im}\varphi)_x = \operatorname{im}\varphi_x$ for all $x \in X$. It is clear that $\ker\varphi = \varphi^{-1}(0)$, where 0 is the trivial subsheaf of $\mathscr{S}$, so it is open. Also, $\varphi(\mathscr{R})$ is open since φ is an open map. Therefore $\ker\varphi$ is a subsheaf of $\mathscr{R}$ and $\operatorname{im}\varphi$ is a subsheaf of $\mathscr{S}$, and we have the trivial identification $\mathscr{R}/\ker\varphi \cong \operatorname{im}\varphi$. We clarify this in terms of sheaf isomorphisms.

Definition 2.4.21. We say that a sheaf homomorphism $\varphi : \mathscr{R} \to \mathscr{S}$ is an isomorphism if there exists a sheaf homomorphism $\psi : \mathscr{S} \to \mathscr{R}$ such that $\varphi \circ \psi = \mathrm{Id}_{\mathscr{S}}$ and $\psi \circ \varphi = \mathrm{Id}_{\mathscr{R}}$. In case where $\ker\varphi = 0$, then we say that φ is *injective*, and if $\operatorname{im}\varphi = \mathscr{S}$, then we say that it is *surjective.*

It is clear by definition that φ is injective (resp. surjective) if and only if the induced homomorphism $\varphi_x : \mathscr{R}_x \to \mathscr{S}_x$, is injective (resp. surjective) for all $x \in X$. Since every sheaf homomorphism is a local homeomorphism, we have the following result:

Corollary 2.4.22. A sheaf homomorphism φ is an isomorphism if and only if it is injective and surjective.

One way to analyze the sections of a sheaf is to connect the sheaf by other sheaves through exact sequences of sheaves, as we motivate in the next discussion.

Definition 2.4.23. Given a sequence of sheaves and sheaf homomorphisms of the form

$$\cdots \longrightarrow \mathscr{S}_{i-1} \xrightarrow{\varphi_{i-1}} \mathscr{S}_i \xrightarrow{\varphi_i} \mathscr{S}_{i+1} \longrightarrow \cdots .$$

We say that this sequence is *exact at* $\mathscr{S}_i$ if $\ker\varphi_i = \operatorname{im}\varphi_{i-1}$. It is called *exact* if the exactness holds at every $\mathscr{S}_i$. Exactness can also be tracked from local exactness as follow.

Lemma 2.4.24. Any sequence

$$\mathscr{R} \xrightarrow{\varphi} \mathscr{S} \xrightarrow{\psi} \mathscr{T}$$

of sheaves over X is exact if and only if

$$\mathscr{R}_x \xrightarrow{\varphi_x} \mathscr{S}_x \xrightarrow{\psi_x} \mathscr{T}_x$$

is exact for all $x \in X$.

Proof. It is clear since $\ker\psi = \operatorname{im}\varphi$ if and only if

$$\begin{aligned}\ker\psi_x &= (\ker\psi)_x\\ &= (\operatorname{im}\varphi)_x\\ &= \operatorname{im}\varphi_x\end{aligned}$$

for all $x \in X$. □

The sections of sheaves can be interpreted using short exact sequences. The following result is due to [2], but we give a more detailed version for the proof.

Proposition 2.4.25. Given a short exact sequence

$$0 \longrightarrow \mathscr{R} \xrightarrow{\varphi} \mathscr{S} \xrightarrow{\psi} \mathscr{T} \longrightarrow 0 \tag{2.4.2}$$

of sheaves. Then

$$0 \longrightarrow \Gamma(X, \mathscr{R}) \xrightarrow{\varphi_X} \Gamma(X, \mathscr{S}) \xrightarrow{\psi_X} \Gamma(X, \mathscr{T}) \tag{2.4.3}$$

is exact.

Proof. The injectivity of φ implies that of φ_X. Indeed, if $\varphi_X(r) = 0$ for some $r \in \Gamma(X, \mathscr{R})$. This implies that $\varphi(r_x) = 0$ for all $x \in X$, where r_x is the value of the section r above x for each x. The condition on φ implies that $r_x = 0$ for all $x \in X$. So $r = 0$ and then φ_X is injective. The inclusion $\operatorname{im} \varphi_X \subset \ker \psi_X$ is clear since $\psi_X \circ \varphi_X = (\psi \circ \varphi)_X = 0$. For the reverse inclusion, let $s \in \Gamma(X, \mathscr{S})$ such that $\psi_X(s) = 0$. For each $x \in X$, we have $\psi(s_x) = 0$. By the exactness condition, we can find $r_x \in \mathscr{R}$ such that $\varphi(r_x) = s_x$ for all $x \in X$. So, with the local homeomorphism property of φ, the function r defined by $r(x) = r_x$ is a global section of the sheaf $\mathscr{R}$, which maps to s under φ_X. □

The homomorphism ψ_X is not always surjective, which means that even if we can catch all local stalks of $\mathscr{T}$ by ψ, we cannot catch all global sections of $\mathscr{T}$ under this map. We clarify this by the following example.

Example 2.4.26. Consider an open subset U of the complex plane $\mathbb{C}$. Since every nowhere vanishing holomorphic function on D can be expressed locally as the exponential of some holomorphic function, we have a short exact sequence of sheaves of the form

$$0 \longrightarrow \underline{\mathbb{Z}}|_U \xrightarrow{\times 2i\pi} \mathcal{O}|_U \xrightarrow{exp(\cdot)} \mathcal{O}^*|_U \longrightarrow 1.$$

Consider the open set $U = \mathbb{C}^*$ and $\mathrm{Id}_U(x) = x \in \mathcal{O}^*(U)$. The existence of $h \in \mathcal{O}(U)$ such that $exp(h) = \mathrm{Id}_U$ is equivalent to the existence of holomorphic logarithm defined on U. But such holomorphic logarithm does not exist, so the map $exp_U : \mathcal{O}(U) \to \mathcal{O}^*(U)$ is not surjective.

3 Sheaf Cohomology

From now on, all sheaves will be at least sheaves of abelian groups over a fixed topological space X. Our goal is to analyze the global sections of sheaves using homological algebra. Let us recall first some interesting results from this machinery.

3.1 Complexes

Let R be a fixed commutative ring with unit.

Definition 3.1.1. A *cochain complex* A^* is a sequence of R-modules, together with R-module homomorphisms of the form

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots$$

such that $d^2 = 0$. We write this cochain complex as (A^*, d) or just A^* if the so called *differential* d is well understood.

Let (A^*, d) be a cochain complex. The condition $d^2 = 0$ implies that $\mathcal{B}^n(A^*) := \operatorname{im}\{d : A^{n-1} \to A^n\}$ is always a submodule of $\mathcal{Z}^n(A^*) := \ker\{d : A^n \to A^{n+1}\}$, for all $n > 0$. Elements of A^n are called *n-cochains*, those of $\mathcal{Z}^n(A^*)$ are called *n-cocycles* and those of $\mathcal{B}^n(A^*)$ are called *n-coboundaries*. The *cohomology modules of* (A^*, d) are the R-modules defined by

$$H^n(A^*) = \begin{cases} \mathcal{Z}^n(A^*)/\mathcal{B}^n(A^*) & \text{if } n > 0; \\ \ker\{d : A^0 \to A^1\} & \text{if } n = 0. \end{cases}$$

We say that the cochain complex A^* is *exact at* A^n if $H^n(A^*) = 0$. It is called *exact* if $H^n(A^*) = 0$ for all $n \geq 0$.

The theory of homological algebra is designed to compute and study the cohomology of a given complex. The key point is to connect the cochain complex A^* with another cochain complex B^* through module homomorphisms, which realize some connections between their cohomology modules.

Definition 3.1.2. A *homomorphism of cochain complexes* $f : A^* \to B^*$ is a collection of homomorphisms of R-modules $f_n : A^n \to B^n$ such that each square diagram

$$\begin{tikzcd}
A^n \arrow[r, "d_A"] \arrow[d, "f_n"] & A^{n+1} \arrow[d, "f_{n+1}"] \\
B^n \arrow[r, "d_B"] & B^{n+1}
\end{tikzcd}$$

commutes.

Let $f : A^* \to B^*$ be a homomorphism of cochain complexes. If $x \in A^n$ is an n-cocycle, we have $d_B f_n(x) = f_n(d_A(x)) = 0$ and if x is an n-coboundary, then by writing $x = d_A(x')$ for some $x' \in A^{n-1}$, we have $f_n(x) = f_n d_A(x') = d_B(f_{n-1}(x'))$. So we have the inclusions $f_n(\mathcal{Z}^n(A^*)) \subset \mathcal{Z}^n(B^*)$ and $f_n(\mathcal{B}^n(A^*)) \subset \mathcal{B}^n(B^*)$, which implies that f induces naturally an R-module homomorphism $f_n^* : H^n(A^*) \to H^n(B^*)$ for each $n \geq 0$. Furthermore, if we have another homomorphism of cochain complexes $g : B^* \to C^*$, then $(g_n f_n)^* = g_n^* f_n^*$ for all $n \geq 0$.

If the sequences

$$0 \longrightarrow A^n \xrightarrow{f_n} B^n \xrightarrow{g_n} C^n \longrightarrow 0$$

are exact for all $n \geq 0$, then we say that the sequence

$$0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0 \tag{3.1.1}$$

is exact. Any sequence of this type is called *short exact sequence of cochain complexes.* The following result is a fundamental theorem, which connects the cohomologies of complexes.

Theorem 3.1.3. *Any short exact sequence of cochain complexes, as in (3.1.1), induces a long exact sequence of the form*

$$\begin{tikzcd}
0 \arrow[r] & H^0(A^*) \arrow[r, "f_0^*"] & H^0(B^*) \arrow[r, "g_0^*"] & H^0(C^*) \arrow[dll, rounded corners, "\partial_0"] \\
& H^1(A^*) \arrow[r, "f_1^*"] & H^1(B^*) \arrow[r, "g_1^*"] & H^1(C^*) \arrow[dll, rounded corners, "\partial_1"] \\
& H^2(A^*) \arrow[r, "f_2^*"] & \cdots .
\end{tikzcd}$$

in cohomology.

Proof. See on page 25 of *Robert Gunning's* book [2]. □

As a consequence of this, we have the so called *Nine Lemma* from a 3×3 complex diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A^3 \arrow[r] \arrow[u] & B^3 \arrow[r] \arrow[u] & C^3 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A^2 \arrow[r] \arrow[u] & B^2 \arrow[r] \arrow[u] & C^2 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A^1 \arrow[r] \arrow[u] & B^1 \arrow[r] \arrow[u] & C^1 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{3.1.2}$$

Corollary 3.1.4. Assume that all rows and two of the columns of (3.1.2) are exact. Then so is the one column left.

Let us now give another approach of computing the cohomology modules, which is very useful in the discussion of sheaf cohomology.

3.2 Spectral Sequences of Filtered Complexes

Throughout this section, let (A^*, d) be a fixed cochain complex of $R-$modules.

The notion of spectral sequences can be thought as a book with infinitely many pages, which starts at a page $k \in \mathbb{Z}$, and each page $r \geq k$ contains a double sequence of R-modules $\{E_r^{p,q}\}$, together with morphisms $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^2 = 0$. Also, the next page should be derived from the previous page by taking cohomology, i.e for each $r \geq k$,

$$E_{r+1}^{p,q} \cong \{x \in E_r^{p,q} \mid d_r x = 0\}/d_r E_r^{p-r,q+r-1}.$$

Such a data is called *an E_k-spectral sequence*. We say that an E_k-spectral sequence $\{E_r^{p,q}\}$ is *convergent* if for each pair p, q, there exists $\lambda = \lambda(p,q)$ such that $E_\lambda^{p,q} \cong E_{\lambda'}^{p,q}$ for all $\lambda' \geq \lambda$. This means that we do not get any new information at the position (p,q) after the page λ. We denote this limit term by $E_\infty^{p,q}$. The method of spectral sequences is used to compute the cohomology modules $H^n(A^*)$ by approximating them using the increasing pages of a convergent spectral sequence. One approach to construct an appropriate spectral sequence to that approximation is by using the notion of filtration of the cochain modules A^n.

Definition 3.2.1. Let A be an R-module. A *descending filtration* of A is a sequence $(F^pA)_{p\in\mathbb{Z}}$ of submodules of A such that $F^pA \supseteq F^{p+1}A$ for all p, $\bigcup_{p\in\mathbb{Z}} F^pA = A$ and $\bigcap_{p\in\mathbb{Z}} F^pA = 0$.

We say that a filtration of the R-module A is finite if it is formed by only finitely many distinct submodules. The notion of filtration on each cochain module A^n should satisfy the compatibility with the differential d, so that it induces a natural filtration of $H^n(A^*)$.

Definition 3.2.2. A *filtration* F of the cochain complex (A^*, d) consists of a descending filtration $(F^pA^n)_p$ of each cochain A^n, such that $d(F^pA^n) \subseteq F^pA^{n+1}$. We say that this filtration is *finite* if $(F^pA^n)_p$ is finite for all n.

Let F be a filtration of A^* and p a fixed integer. Since the differential d maps F^pA^n into F^pA^{n+1} for all n, we have a cochain complexes (F^pA^*, d) such that the inclusion $\iota_p : F^pA^* \to A^*$ is a homomorphism of cochain complexes. The image of the induced homomorphism in cohomology $\iota_p^* : H^n(F^pA^*) \to H^n(A^*)$ will be denoted by $F^pH^n(A^*)$. We also have an inclusion homomorphism $\iota_{p,p+1} : F^{p+1}A^* \to F^pA^*$, which satisfies $\iota_{p+1} = \iota_p \cdot \iota_{p+1}$. So in cohomology, we have $\iota_{p+1}^* = \iota_p^* \cdot \iota_{p+1}^*$, which implies that $F^{p+1}H^n(A^*) \subseteq F^pH^n(A^*)$. We have then a natural filtration of $H^n(A^*)$, called *the induced filtration of the filtration F on the cohomology modules of A^*.*

Through the quotient modules $E^p(A^n) = F^pA^n/F^{p+1}A^n$, the following theorem states the existence of an appropriate spectral sequence which approximates $H^n(A^*)$.

Theorem 3.2.3. *Let F be a finite filtration of (A^*, d). Then there exists a convergent E_0-spectral sequence $\{E_r^{p,q}\}$ such that*

$$E_0^{p,q}(A^*) = F^pA^{p+q}/F^{p+1}A^{p+q} \text{ and } E_\infty^{p,q}(A^*) = F^pH^{p+q}(A^*)/F^{p+1}H^{p+q}(A^*).$$

Proof. See on page 28 of *Robert Gunning's* book [2]. □

In practice, the cohomology modules can be computed in many different ways. One way is to use double complexes and some filtrations on them. In this way, the theory of spectral sequences leads us to the needed results.

Definition 3.2.4. A *double complex of R-modules* is a double sequence $A^{*,*} = \{A^{p,q}\}_{p,q\in\mathbb{Z}_{\geq 0}}$ of R-modules, together with differentials $d_v : A^{p,q} \to A^{p+1,q}$ and $d_h : A^{p,q} \to A^{p,q+1}$, satisfying one of the following conditions:

$$d_v^2 = d_h^2 = d_vd_h - d_hd_v = 0 \text{ (commutative), or} \tag{3.2.1}$$

$$d_v^2 = d_h^2 = d_vd_h + d_hd_v = 0 \text{ (anticommutative).} \tag{3.2.2}$$

The differentials d_v and d_h are referred as vertical and horizontal differentials as shown in (3.2.3).

Remark 3.2.5. If we have a double complex $(A^{*,*}, d_v, d_h)$ with anticommutative differentials, then we can make it as a double complex with commutative differentials by keeping d_v and setting $d_h' = (-1)^pd_h : A^{p,q} \to A^{p,q+1}$; or by keeping d_h and setting $d_v' = (-1)^qd_v : A^{p,q} \to A^{p+1,q}$. Conversely, we also can make in a similar way a double complex with anticommutative differentials from one with commutative differentials. The relations $d_v^2 = d_h^2 = 0$ implies that we have a cochain complex in each row and each column of the double complex, so we can take cohomologies. It is clear that those cohomologies do not depend on the new double complexes with sign modification of the differentials.

$$
\begin{tikzcd}
& \vdots & & \vdots & \vdots & \vdots & \\
0 \arrow[r] & H_h^0(A^{2,*}) \arrow[u, "d_v^*"] \arrow[rr, dashed] & & A^{2,0} \arrow[u, "d_v"] \arrow[r, "d_h"] & A^{2,1} \arrow[u, "d_v"] \arrow[r, "d_h"] & A^{2,2} \arrow[u, "d_v"] \arrow[r, "d_h"] & \cdots \\
0 \arrow[r] & H_h^0(A^{1,*}) \arrow[u, "d_v^*"] \arrow[rr, dashed] & & A^{1,0} \arrow[u, "d_v"] \arrow[r, "d_h"] & A^{1,1} \arrow[u, "d_v"] \arrow[r, "d_h"] & A^{1,2} \arrow[u, "d_v"] \arrow[r, "d_h"] & \cdots \\
0 \arrow[r] & H_h^0(A^{0,*}) \arrow[u, "d_v^*"] \arrow[rr, dashed] & & A^{0,0} \arrow[u, "d_v"] \arrow[r, "d_h"] & A^{0,1} \arrow[u, "d_v"] \arrow[r, "d_h"] & A^{0,2} \arrow[u, "d_v"] \arrow[r, "d_h"] & \cdots \\
& & & H_v^0(A^{*,0}) \arrow[u, dashed] \arrow[r, "d_h^*"] & H_v^0(A^{*,1}) \arrow[u, dashed] \arrow[r, "d_h^*"] & H_v^0(A^{*,2}) \arrow[u, dashed] \arrow[r, "d_h^*"] & \cdots \\
& & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{3.2.3}
$$

On a fixed p^{th} line of the double complex, we denote by $H_h^q(A^{p,*})$ the q^{th} cohomology module of the cochain complex $(A^{p,*}, d_h)$. By Remark 3.2.5, d_v can be viewed as a homomorphism of cochain complexes $d_v : (A^{p,*}, d_h) \to (A^{p+1,*}, d_h)$, so that it induces a homomorphism in cohomology $d_v^* : H_h^q(A^{p,*}) \to H_h^q(A^{p+1,*})$ for each $q \geq 0$.

$$
\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
& H_h^0(A^{p+1,*}) \arrow[u, "d_v^*"] & H_h^1(A^{p+1,*}) \arrow[u, "d_v^*"] & H_h^2(A^{p+1,*}) \arrow[u, "d_v^*"] & \cdots \\
p & H_h^0(A^{p,*}) \arrow[u, "d_v^*"] & H_h^1(A^{p,*}) \arrow[u, "d_v^*"] & H_h^2(A^{p,*}) \arrow[u, "d_v^*"] & \cdots \\
& \vdots \arrow[u, "d_v^*"] & \vdots \arrow[u, "d_v^*"] & \vdots \arrow[u, "d_v^*"] & \cdots \\
& & q & &
\end{tikzcd}
\tag{3.2.4}
$$

We also have $(d_v^*)^2 = 0$ since $d_v^2 = 0$. So for each fixed q, we obtain a new cochain complex $(H_h^q(A^{*,*}), d_v^*)$ as in (3.2.4), where we denote by $H_v^p(H_h^q(A^{*,*}))$ its p^{th} cohomology module. It is also clear that this cohomology module does not depend on the double complexes constructed in Remark 3.2.5.

Since d_v and d_h play a symmetrical role, we can repeat the procedure starting from the original double complex as above, but by taking the cohomology of the columns first. So, similarly, at each position (p, q), we have the cohomology module $H_h^q(H_v^p(A^{*,*}))$.

The anticommutativity condition on the differentials also implies that we have a notion of cochain complex on the antidiagonals of the double complex, by taking the differential $d = d_h + d_v$. We construct a filtration of this complex so that the cohomology modules $H_v^p(H_h^q(A^{*,*}))$ and $H_h^q(H_v^p(A^{*,*}))$ can be considered as an approximation of its cohomology modules. In particular, with some nice condition on the double complex, we can relate those cohomology modules defined above through the cohomology of this new cochain complex. The precise statement is the following.

Definition 3.2.6. The *total complex of a double complexes* $A^{*,*}$, denoted by $T^*(A)$, is the cochain complex defined by $T^n(A) = \sum\limits_{i+j=n} A^{i,j}$, with the total differential $d = d_h + d_v$.

For each n, there are two natural finite filtration of the module $T^n(A)$ defined as follow:

$$_1F^pT^n(A) = \begin{cases} 0 & \text{if } p < 0 \\ \sum_{\substack{i+j=n \\ i \geq p}} A^{i,j} & \text{if } 0 \leq p < n \\ T^n(A) & \text{if } p \geq n \end{cases} \text{ and } {}_2F^pT^n(A) = \begin{cases} 0 & \text{if } p < 0 \\ \sum_{\substack{i+j=n \\ j \geq p}} A^{i,j} & \text{if } 0 \leq p < n \\ T^n(A) & \text{if } p \geq n \end{cases}.$$

It is clear with those filtration that

$$_1E^p(T^{p+q}(A)) = {}_1F^pT^{p+q}(A)/{}_1F^{p+1}T^{p+q}(A) = A^{p,q} \text{ and}$$
$$_2E^p(T^{p+q}(A)) = {}_2F^pT^{p+q}(A)/{}_2F^{p+1}T^{p+q}(A) = A^{q,p}.$$

According to Theorem 3.2.3, there are two corresponding convergent E_0-spectral sequences $\{_1E_r^{p,q}\}$ and $\{_2E_r^{p,q}\}$ such that

$_kE_0^{p,q}(T^*(A)) = {}_kE^p(T^{p+q}(A))$ and $_kE_\infty^{p,q}(T^*(A)) = {}_kF^p(H^{p+q}(T^*(A)))/{}_kF^{p+1}(H^{p+q}(T^*(A)))$, for $k = 1, 2$.

Proposition 3.2.7. In those two convergent E_0-spectral sequences above, the modules on their second pages are given by

$$_1E_2^{p,q}(T^*(A)) \cong \begin{cases} 0 & \text{if } p < 0 \text{ or } q < 0, \\ H_v^p(H_h^q(A^{*,*})) & \text{otherwise;} \end{cases}$$
$$_2E_2^{p,q}(T^*(A)) \cong \begin{cases} 0 & \text{if } p < 0 \text{ or } q < 0, \\ H_h^p(H_v^q(A^{*,*})) & \text{otherwise.} \end{cases}$$

Proof. See on page 33 of *Robert Gunning's* book [2]. □

As a consequence of this proposition, if we assume that all the rows of the diagram (3.2.3) are exact, then we have $H_h^q(A^{p,*}) = 0$ for all p, q except at $q = 0$. Since there is just one non zero column left in the first page, we have $_1E_\infty^{p,q}(T^*(A)) = {}_1E_2^{p,q}(T^*(A))$ for all p, q. In particular, we have $H^n(T^*(A)) \cong {}_1E_\infty^{n,0}(T^*(A)) \cong H_v^n(H_h^0(A^{*,*}))$ for each $n \geq 0$. Similarly, if we have exactness in the columns of the diagram (3.2.3), then we have $H_v^p(A^{*,q}) = 0$ for all p, q except at $p = 0$. Since there is just one non zero row left in the first page, we have $_2E_\infty^{p,q}(T^*(A)) = {}_2E_2^{p,q}(T^*(A))$ for all p, q. In particular, we have $H^n(T^*(A)) \cong {}_2E_\infty^{n,0}(T^*(A)) \cong H_h^n(H_v^0(A^{*,*}))$ for each $n \geq 0$. We have then the following result.

Theorem 3.2.8. *Assume that all the rows and the columns of the diagram* (3.2.3) *are exact. Then the cohomologies of the cochain complexes on the first line and the first column are isomorphic.*

In the following section, we explore a mathematical tool, based on the notion of complexes and the results above. Through that tool, we can analyze locally solved problems in a topological space.

3.3 Cohomology with Coefficients in a Sheaf

Usually, geometrical problems, which can be solved locally on the space X, can be represented by surjective sheaf homomorphisms. The exponential image problem in Example 2.4.26 is a suggestive example for such problems. Consider a surjective sheaf homomorphism $\varphi : \mathscr{R} \to \mathscr{S}$. In this case, the corresponding problem has a global solution if and only if the induced group homomorphism φ_X on global sections is surjective. With the short exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow \mathscr{R} \xrightarrow{\varphi} \mathscr{S} \longrightarrow 0,$$

we always have the induced exact sequence

$$0 \longrightarrow \Gamma(X, \ker\varphi) \longrightarrow \Gamma(X, \mathscr{R}) \xrightarrow{\varphi_X} \Gamma(X, \mathscr{S}).$$

We would like to continue this sequence in terms of sheaf cohomology and that will give us a sufficient condition for the surjectivity of φ_X.

As in R-modules, we define correspondingly a *cochain complex $\mathscr{S}^*$ of sheaves*, where the differential d is a sheaf homomorphism. Any cochain complex of sheaves $(\mathscr{S}^*, d)$ induces naturally a cochain complex of abelian groups (hence $\mathbb{Z}$-modules), $(\Gamma(X, \mathscr{S}^*), d_X)$ on global sections, so that we can take cohomology. But we need a special type of cochain complex of sheaves, for which the theory runs properly.

Definition 3.3.1. Let $\mathscr{S}$ be a sheaf over X. A cochain complex of sheaves $(\mathscr{S}^*, d)$ is called *a resolution of the sheaf $\mathscr{S}$* if we have an exact sequence of sheaves of the form

$$0 \longrightarrow \mathscr{S} \longrightarrow \mathscr{S}^0 \xrightarrow{d} \mathscr{S}^1 \xrightarrow{d} \cdots .$$

Our goal is to recover information about $\mathscr{S}$ from the complex $(\mathscr{S}^*, d)$. Additionally, we would require the terms in the resolution to have specific nice sections.

Definition 3.3.2. A sheaf $\mathscr{S}$ over X is called *flasque* if the restriction map $\rho^X_U : \mathscr{S}(X) \to \mathscr{S}(U)$ is surjective for all open set $U \subset X$. A *flasque resolution of $\mathscr{S}$* is a resolution $(\mathscr{S}^*, d)$ of $\mathscr{S}$ such that $\mathscr{S}^q$ is flasque for all $q \geq 0$.

As above, we have an induced cochain complex on global sections $0 \longrightarrow \Gamma(X, \mathscr{S}) \longrightarrow \Gamma(X, \mathscr{S}^*)$, where

$$\Gamma(X, \mathscr{S}^*) = \Gamma(X, \mathscr{S}^0) \xrightarrow{d_X} \Gamma(X, \mathscr{S}^1) \xrightarrow{d_X} \Gamma(X, \mathscr{S}^2) \xrightarrow{d_X} \cdots .$$

Let us now construct a canonical resolution, called *Godement resolution* to get a flasque resolution of a given sheaf.

For each open set $U \subset X$, let $\mathrm{God}(\mathscr{S})(U)$ be the set of not necessarily continuous sections from U to $\mathscr{S}$. In other word, $\mathrm{God}(\mathscr{S})(U) = \prod_{p \in U} \mathscr{S}_x$. Clearly $\mathrm{God}(\mathscr{S})$ is a flasque sheaf on X with the usual restriction map and it contains $\mathscr{S}$ as a subsheaf. So we have an exact sequence of the form

$$0 \longrightarrow \mathscr{S} \longrightarrow \mathrm{God}(\mathscr{S}),$$

and we can extend it recursively as follow.

Let

$$\begin{cases} \mathrm{God}^n(\mathscr{S}) = \begin{cases} \mathrm{God}(\mathscr{S}) & \text{if } n = 0, \\ \mathrm{God}(Q^{n-1}(\mathscr{S})) & \text{if } n > 0; \end{cases} \\ Q^n(\mathscr{S}) = \begin{cases} \mathrm{God}(\mathscr{S})/\mathscr{S} & \text{if } n = 0, \\ \mathrm{God}^n(\mathscr{S})/Q^{n-1}(\mathscr{S}) & \text{if } n > 0. \end{cases} \end{cases}$$

Then, for any n, we have an exact sequence of the form

$$0 \longrightarrow \mathscr{S} \longrightarrow \mathrm{God}^0(\mathscr{S}) \longrightarrow \mathrm{God}^1(\mathscr{S}) \longrightarrow \cdots \longrightarrow \mathrm{God}^n(\mathscr{S}) \longrightarrow Q^n(\mathscr{S}) \longrightarrow 0,$$

and this repeated gives us the Godement resolution of the sheaf $\mathscr{S}$:

$$\mathrm{God}^*(\mathscr{S}) = \mathrm{God}^0(\mathscr{S}) \longrightarrow \mathrm{God}^1(\mathscr{S}) \longrightarrow \mathrm{God}^2(\mathscr{S}) \longrightarrow \cdots.$$

Definition 3.3.3. The q^{th} cohomology group of the cochain complex of abelian groups $\Gamma(X, \mathrm{God}^*(\mathscr{S}))$ is called *the q^{th} cohomology group of X with coefficients in the sheaf $\mathscr{S}$*, denoted by $H^q(X, \mathscr{S})$.

Remark 3.3.4. It is clear from Proposition 2.4.25, that we always have an isomorphism $H^0(X, \mathscr{S}) \cong \Gamma(X, \mathscr{S})$.

Now, we build a long exact sequence in terms of sheaf cohomology out of a short exact sequence of sheaves. The following results are due to [2], but we give a more detailed explanation for the proofs.

Lemma 3.3.5. Any sheaf homomorphism $\varphi : \mathscr{R} \to \mathscr{S}$ induces naturally a homomorphism of cochain $\varphi_* : \mathrm{God}^*(\mathscr{R}) \to \mathrm{God}^*(\mathscr{S})$.

Proof. We construct the morphisms φ_n for all $n \geq 0$ as follow. For each open set $U \subset X$, the group homomorphism $\varphi_U : \mathscr{R}(U) \to \mathscr{S}(U)$ induces naturally a group homomorphism $\varphi_{0,U} : \mathrm{God}^0(\mathscr{R})(U) \to \mathrm{God}^0(\mathscr{S})(U)$. So we have a morphism $\varphi_0 : \mathrm{God}^0(\mathscr{R}) \to \mathrm{God}^0(\mathscr{S})$. This φ_0 induces a well defined morphism through quotients since $\varphi_0(\mathscr{R}) = \varphi_0(Q^0(\mathscr{R})) \subset \mathscr{S} = Q^0(\mathscr{S})$. So we have a sheaf homomorphism ${\varphi'}_1 : Q^0(\mathscr{R}) \to Q^0(\mathscr{S})$, and each square of the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{R} \arrow[r] \arrow[d, "\varphi"] & \mathrm{God}^0(\mathscr{R}) \arrow[r] \arrow[d, "\varphi_0"] & Q^0(\mathscr{R}) \arrow[r] \arrow[d, "{\varphi'}_1"] & 0 \\ 0 \arrow[r] & \mathscr{S} \arrow[r] & \mathrm{God}^0(\mathscr{S}) \arrow[r] & Q_0(\mathscr{S}) \arrow[r] & 0 \end{tikzcd}$$

commutes. We apply this process to the sheaf homomorphism φ'_1 to get a sheaf homomorphism $\varphi_1 : \mathrm{God}^1(\mathscr{R}) \to \mathrm{God}^1(\mathscr{S})$ and $\varphi'_2 : Q^1(\mathscr{R}) \to Q^1(\mathscr{S})$. By using that process several times, we have a cochain homomorphism between the two Godement resolutions. □

Lemma 3.3.6. If

$$0 \longrightarrow \mathscr{R} \xrightarrow{\varphi} \mathscr{S} \xrightarrow{\psi} \mathscr{T} \longrightarrow 0$$

is a short exact sequence of sheaves, then for each open set $U \subset X$, the sequence

$$0 \longrightarrow \mathrm{God}^n(\mathscr{R})(U) \xrightarrow{\varphi_{n,U}} \mathrm{God}^n(\mathscr{S})(U) \xrightarrow{\psi_{n,U}} \mathrm{God}^n(\mathscr{T})(U) \longrightarrow 0$$

is exact as well for each n. In particular, the sequence

$$0 \longrightarrow \mathrm{God}^n(\mathscr{R}) \xrightarrow{\varphi_n} \mathrm{God}^n(\mathscr{S}) \xrightarrow{\psi_n} \mathrm{God}^n(\mathscr{T}) \longrightarrow 0,$$

is exact for each n.

Proof. We have that $\mathscr{S}/\mathscr{R} \cong \mathscr{T}$. So for each $x \in X$, $\mathscr{T}_x \cong (\mathscr{S}/\mathscr{R})_x \cong \mathscr{S}_x/\mathscr{R}_x$. Now, let U be an open set of X. By the discontinuity of the sections of God^0, we have

$$\begin{aligned}
\mathrm{God}^0(\mathscr{S})(U)/\,\mathrm{God}^0(\mathscr{R})(U) &= \prod_{x\in U}\mathscr{S}_x \Big/ \prod_{x\in U}\mathscr{R}_x \\
&\cong \prod_{x\in U}(\mathscr{S}_x/\mathscr{R}_x) \\
&\cong \prod_{x\in U}\mathscr{T}_x = \mathrm{God}^0(\mathscr{T})(U).
\end{aligned}$$

This means that the sequence

$$0 \longrightarrow \mathrm{God}^0(\mathscr{R})(U) \xrightarrow{\varphi_{0,U}} \mathrm{God}^0(\mathscr{S})(U) \xrightarrow{\psi_{0,U}} \mathrm{God}^0(\mathscr{T})(U) \longrightarrow 0.$$

is exact, and so is

$$0 \longrightarrow \mathrm{God}^0(\mathscr{R}) \xrightarrow{\varphi_0} \mathrm{God}^0(\mathscr{S}) \xrightarrow{\psi_0} \mathrm{God}^0(\mathscr{T}) \longrightarrow 0.$$

We have then a 3×3 diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{R} \arrow[r, "\varphi"] \arrow[d] & \mathscr{S} \arrow[r, "\psi"] \arrow[d] & \mathscr{T} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{God}^0(\mathscr{R}) \arrow[r, "\varphi_0"] \arrow[d] & \mathrm{God}^0(\mathscr{S}) \arrow[r, "\psi_0"] \arrow[d] & \mathrm{God}^0(\mathscr{T}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q^0(\mathscr{R}) \arrow[r, "{\varphi'}_1"] \arrow[d] & Q^0(\mathscr{S}) \arrow[r, "{\psi'}_1"] \arrow[d] & Q^0(\mathscr{T}) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd} \tag{3.3.1}$$

where all the columns and the rows are exact except the last row. But with the Nine-Lemma (3.1.2) applied to each stalk, the last row is also exact. We repeat this process again in this last row, and do it several times to have short exact sequences

$$0 \longrightarrow \mathrm{God}^n(\mathscr{R}) \xrightarrow{\varphi_n} \mathrm{God}^n(\mathscr{S}) \xrightarrow{\psi_n} \mathrm{God}^n(\mathscr{T}) \longrightarrow 0,$$

for each n. □

Theorem 3.3.7. *Any short exact sequence of sheaves*

$$0 \longrightarrow \mathscr{R} \xrightarrow{\varphi} \mathscr{S} \xrightarrow{\psi} \mathscr{T} \longrightarrow 0$$

induces an exact sequence of the cohomology groups of the form

$$\begin{aligned}
0 \longrightarrow H^0(X,\mathscr{R}) &\xrightarrow{\varphi_0^*} H^0(X,\mathscr{S}) \xrightarrow{\psi_0^*} H^0(X,\mathscr{T}) \xrightarrow{\partial_0} \\
H^1(X,\mathscr{R}) &\xrightarrow{\varphi_1^*} H^1(X,\mathscr{S}) \xrightarrow{\psi_1^*} H^1(X,\mathscr{T}) \xrightarrow{\partial_1} \\
H^2(X,\mathscr{R}) &\xrightarrow{\varphi_2^*} \cdots .
\end{aligned}$$

Proof. By Lemma 3.3.6, we have the short exact sequence

$$0 \longrightarrow \Gamma(X, \mathrm{God}^*(\mathscr{R})) \xrightarrow{\varphi_{*,X}} \Gamma(X, \mathrm{God}^*(\mathscr{S})) \xrightarrow{\psi_{*,X}} \Gamma(X, \mathrm{God}^*(\mathscr{T})) \longrightarrow 0.$$

So we just apply Theorem 3.1.3 to conclude the proof. □

With this theorem, if we have for instance, $H^1(X,\mathscr{R}) = 0$, then automatically we have the surjectivity of $\psi_o^* : H^0(X,\mathscr{S}) \to H^0(X,\mathscr{T})$ in global sections. Hence, any global section of $\mathscr{T}$ can be lifted to a global section of $\mathscr{S}$, under the sheaf homomorphism ψ.

Let us now describe a special type of sheaf resolutions, in which the sheaf cohomology is much easier to compute.

Definition 3.3.8. A sheaf $\mathscr{S}$ over X is called *acyclic* if $H^p(X,\mathscr{S}) = 0$ for all $p > 0$. *An acyclic resolution of a sheaf* $\mathscr{S}$ is a sheaf resolution of $\mathscr{S}$, which is formed by acyclic sheaves.

Any sheaf cohomology can be computed from any acyclic resolution according to the following theorem.

Theorem 3.3.9. *For any acyclic resolution* $(\mathscr{S}^*, d)$ *of a sheaf* $\mathscr{S}$*, we have the isomorphism* $H^q(X,\mathscr{S}) \cong H^q(\Gamma(X,\mathscr{S}^*))$ *for all* $q \geq 0$.

Proof. By taking the Godement resolution of each $\mathscr{S}^q$ and by lifting the differential d according to Lemma

3.3.5, we have the double complex $\{\mathrm{God}^p(\mathscr{S}^q)\}$ of sheaves as in (3.3.2).

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{God}^2(\mathscr{S}) \arrow[u] \arrow[r, dashed] & \mathrm{God}^2(\mathscr{S}^0) \arrow[u] \arrow[r, "d_2"] & \mathrm{God}^2(\mathscr{S}^1) \arrow[u] \arrow[r, "d_2"] & \mathrm{God}^2(\mathscr{S}^2) \arrow[u] \arrow[r, "d_2"] & \cdots \\
0 \arrow[r] & \mathrm{God}^1(\mathscr{S}) \arrow[u] \arrow[r, dashed] & \mathrm{God}^1(\mathscr{S}^0) \arrow[u] \arrow[r, "d_1"] & \mathrm{God}^1(\mathscr{S}^1) \arrow[u] \arrow[r, "d_1"] & \mathrm{God}^1(\mathscr{S}^2) \arrow[u] \arrow[r, "d_1"] & \cdots \\
0 \arrow[r] & \mathrm{God}^0(\mathscr{S}) \arrow[u] \arrow[r, dashed] & \mathrm{God}^0(\mathscr{S}^0) \arrow[u] \arrow[r, "d_0"] & \mathrm{God}^0(\mathscr{S}^1) \arrow[u] \arrow[r, "d_0"] & \mathrm{God}^0(\mathscr{S}^2) \arrow[u] \arrow[r, "d_0"] & \cdots \\
0 \arrow[r] & \mathscr{S} \arrow[u] \arrow[r] & \mathscr{S}^0 \arrow[u, dashed] \arrow[r, "d"] & \mathscr{S}^1 \arrow[u, dashed] \arrow[r, "d"] & \mathscr{S}^2 \arrow[u, dashed] \arrow[r, "d"] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{3.3.2}$$

By taking their global sections with the induced homomorphisms, we have exactness in all columns (except the first) of the diagram (3.3.2), since each $\mathscr{S}^q$ is acyclic. On the other side, according to Lemma 3.3.6, we also have exactness in all rows (except the first) of the diagram (3.3.2), when we take global sections. So we apply Theorem 3.2.8 to complete the proof. $\square$

The flasque sheaves range in the class of acyclic sheaves according to the following results. Let

$$0 \longrightarrow \mathscr{R} \xrightarrow{\varphi} \mathscr{S} \xrightarrow{\psi} \mathscr{T} \longrightarrow 0 \tag{3.3.3}$$

be a fixed short exact sequence of sheaves.

Lemma 3.3.10. If $\mathscr{R}$ is flasque. Then the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{R}) \xrightarrow{\varphi_X} \Gamma(X, \mathscr{S}) \xrightarrow{\psi_X} \Gamma(X, \mathscr{T}) \longrightarrow 0$$

is also exact.

Proof. See on page 42 of the *Robert Gunning's* book [2]. $\square$

Lemma 3.3.11. With the condition in (3.3.3), if $\mathscr{R}$ and $\mathscr{S}$ are flasque then $\mathscr{T}$ is also flasque.

Proof. Let U be an open set of X and $t \in \mathscr{T}(U)$. By Lemma 3.3.10 above, we can find $s \in \mathscr{S}(U)$ such that $\psi_U(s) = t$. Since $\mathscr{S}$ is flasque, s can be extended as a section $\tilde{s}$ defined on the whole space X, and also t is, since $\psi_X(\tilde{s})|_U = \psi_U(\tilde{s}|_U) = \psi_U(s) = t$. $\square$

Now, let us take a flasque sheaf $\mathscr{S}$ over X. In the Godement resolution of $\mathscr{S}$, we have a short exact sequence

$$0 \longrightarrow \mathscr{S} \longrightarrow \mathrm{God}^0(\mathscr{S}) \longrightarrow Q^0(\mathscr{S}) \longrightarrow 0.$$

The Lemma 3.3.11 implies that the sheaf $Q^0(\mathscr{S})$ is also flasque. By induction on n, with the short exact sequences

$$0 \longrightarrow Q^n(\mathscr{S}) \longrightarrow \mathrm{God}^n(\mathscr{S}) \longrightarrow Q^{n+1}(\mathscr{S}) \longrightarrow 0,$$

all $Q^n(\mathscr{S})$ are also flasque. By applying successively Lemma 3.3.10 on those short exact sequences, we also have exactness in global sections. This means that the cochain complex $\Gamma(X, \mathrm{God}^*(\mathscr{S}))$ is exact, and so $\mathscr{S}$ is acyclic. In particular, for any sheaf $\mathscr{S}$, all $\mathrm{God}^n(\mathscr{S})$ are acyclic.

In the case where the space X is paracompact and Hausdorff as well as topological manifold, sheaf cohomology can be computed using another type of acyclic resolutions, called *fine resolutions*.

3.4 Computation Using Fine Resolutions

The notion of *fine sheaves* is controlled by the locally finite open coverings of the topological space. Recall that an open covering $\mathcal{U} = \{U_\alpha\}$ of the space X is *locally finite* if each point has an open neighborhood, which intersects only finitely many of elements of $\mathcal{U}$. An open covering $\mathcal{V} = \{V_\beta\}$ is called *a refinement of* $\mathcal{U} = \{U_\alpha\}$ if there exists an index map r such that $V_\beta \subset U_{r(\beta)}$ for all β. In case, where X is paracompact, the study on any open covering can be brought into a locally finite one through refinement, since such refinement always exists by definition. It is also known as Shrinking Lemma that if $\{U_\alpha\}$ is a locally finite open covering in a paracompact Hausdorff space, then we can find open sets V_α satisfying $V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$ and $\{V_\alpha\}$ is still an open covering of X.

Definition 3.4.1. Let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open covering of X and $\mathscr{S}$ a sheaf on X. A *partition of unity of the sheaf $\mathscr{S}$ subordinated to the cover $\mathcal{U}$* is a family of sheaf homomorphisms $\{\eta_\alpha : \mathscr{S} \to \mathscr{S}\}_\alpha$ such that:

(i) $\eta_\alpha(\mathscr{S}_x) = 0$ for all $x \in X \setminus U_\alpha$;

(ii) $\sum_\alpha \eta_\alpha(s) = s$ for all $s \in \mathscr{S}$.

The sum in (ii) is well defined since it is just a finite sum by the locally finite condition. A sheaf $\mathscr{S}$ is called *fine* if it has a partition of unity subordinated to any locally finite open covering of X.

Examples 3.4.2. (a) If $\mathscr{S}$ be a sheaf over X, then $\mathrm{God}(\mathscr{S})$ is fine. Indeed, for a locally finite open covering $\{U_\alpha\}$ of the space X, consider the set

$$\Lambda = \{\{X_\alpha\} \mid X_\alpha \subset U_\alpha,\ X_\alpha \cap X_\beta = \emptyset \text{ for } \alpha \neq \beta\}.$$

It is clear that $\Lambda \neq \emptyset$. The relation defined by $\{Y_\alpha\} \leq \{X_\beta\}$ if and only if $Y_\alpha \subset X_\alpha$ for all α, makes Λ as a partial ordered set. For each chain $\{X_\alpha^0\} \leq \{X_\alpha^1\} \leq \{X_\alpha^2\} \leq \cdots$, $\{\bigcup_{i=0}^\infty X_\alpha^i\}$ is an upper bound. So by Zorn's lemma, Λ has a maximal element say $\{X_\alpha\}$. Assume that we can find an element $x \in X \setminus \bigcup_\alpha X_\alpha$. Then if we fix an index α_0, $\{X_\alpha\} < \{X_{\alpha_0} \cup \{x\}, X_\alpha\}_{\alpha \neq \alpha_0} \in \Lambda$, which contradicts the maximality of $\{X_\alpha\}$. So $\{X_\alpha\}$ covers X. It is clear that the $\{n_\alpha\}$ defined by

$$\eta_\alpha(s) := \begin{cases} s & \text{for all } s \in \coprod_{x \in X_\alpha} \mathscr{S}_x \\ 0 & \text{otherwise} \end{cases}$$

is a partition of unity subordinated to $\{U_\alpha\}$.

(b) It is also well known that if the space X is normal, then it has a continuous partition of unity subordinated to any locally finite open covering. That is, given such an open covering $\{U_\alpha\}$, we can find continuous functions $\varepsilon_\alpha : X \to [0,1]$ such that $\varepsilon_\alpha = 0$ in $X \setminus U_\alpha$ and $\sum_\alpha \varepsilon_\alpha(x) = 1$ for all $x \in X$. We can multiply the sections of the sheaf $\mathcal{C}_X$ with those ε_α to get a partition of unity of $\mathcal{C}_X$ subordinated to the open covering $\{U_\alpha\}$. Also if the space X is a C^∞ manifold, then a well known result from differential geometry says that one can find a C^∞ partition of unity subordinated to any locally finite open covering. Therefore $\mathcal{C}_X, \mathcal{C}_X^\infty$ and $\mathscr{E}_X^r$ are all fine sheaves.

Let us now clarify that fine sheaves over a paracompact Hausdorff space are acyclic. This allow us to compute sheaf cohomology from any fine resolution. In the case of flasque sheaves, sections on open sets can be extended to global sections. Fine sheaves also have such extension property but on closed subsets as we describe bellow.

Definition 3.4.3. A sheaf $\mathscr{S}$ over X is called *soft* if every section on any closed subset of X can be extended to a global section.

Let $\mathscr{S}$ be a fine sheaf over a paracompact space X and f a section of $\mathscr{S}$ over a closed subset $K \subset X$. By definition of sections, we can find an open covering $\{U_\alpha\}$ of K and sections $f_\alpha \in \mathscr{S}(U_\alpha)$ such that $f|_{U_\alpha \cap K} = f_\alpha|_{U_\alpha \cap K}$. We may add the extra open set $U_0 = X \setminus K$ to cover the whole space X and take the zero section $f_0 = 0$ with it. Let $\{V_\beta\}$ be a refinement of this covering of X. We may set $f_\beta = f_{r(\beta)}\big|_{V_\beta}$, where r is the index map in the refinement, so that $f|_{U_\beta \cap K} = f_\beta|_{U_\beta \cap K}$. So when we take a partition of unity $\{\eta_\beta\}$ subordinated to $\{V_\beta\}$, we have a well defined global section

$$g_\beta = \begin{cases} \eta_\beta(f_\beta) & \text{on } U_\beta \\ 0 & \text{on } X \setminus U_\beta. \end{cases}$$

The locally finiteness condition on the covering implies that $g := \sum_\beta g_\beta$ is a well defined global section and for any $x \in K$, we have

$$\begin{aligned} g(x) &= \sum_\beta \eta_\alpha(f_\beta(x)) \\ &= \sum_\beta \eta_\beta(f(x)) \\ &= f(x). \end{aligned}$$

So we summarise this in the following result.

Proposition 3.4.4. Every fine sheaf over a paracompact space is soft.

Now, assume that we have a short exact sequence

$$0 \longrightarrow \mathscr{R} \xrightarrow{\varphi} \mathscr{S} \xrightarrow{\psi} \mathscr{T} \longrightarrow 0 \tag{3.4.1}$$

of sheaves over a paracompact Hausdorff space X. We have similar results as in the case of flasque sheaves.

Lemma 3.4.5. If the sheaf $\mathscr{R}$ is soft, then the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{R}) \xrightarrow{\varphi_X} \Gamma(X, \mathscr{S}) \xrightarrow{\psi_X} \Gamma(X, \mathscr{T}) \longrightarrow 0$$

is exact. Furthermore, if the sheaf $\mathscr{S}$ is soft, then the sheaf $\mathscr{T}$ is also soft.

Proof. See on page 45 of *Robert Gunning's* book [2]. □

Consider now a fine sheaf $\mathscr{S}$ over a paracompact Hausdorff space X. Since all $\mathrm{God}^n(\mathscr{S})$ are fine, they are soft according to Proposition 3.4.4. With the short exact sequences

$$0 \longrightarrow \mathscr{S} \longrightarrow \mathrm{God}^0(\mathscr{S}) \longrightarrow Q^0(\mathscr{S}) \longrightarrow 0$$

and

$$0 \longrightarrow Q^n(\mathscr{S}) \longrightarrow \mathrm{God}^n(\mathscr{S}) \longrightarrow Q^{n+1}(\mathscr{S}) \longrightarrow 0,$$

we can apply Lemma 3.4.5 recursively on n to show that all $Q^n(\mathscr{S})$ are soft. Also, we have exactness in taking global sections on those sequences, which means that $\mathscr{S}$ is acyclic. Consequently, we can compute sheaf cohomology with a soft or fine resolution.

Example 3.4.6. Let X be a smooth manifold of dimension n. With the exterior derivative operator d, we have a cochain complex of sheaves of germs of differential forms $(\mathscr{E}_X^*, d)$ such that $\mathscr{E}_X^r = 0$ for all $r > n$. For any $f \in \mathscr{E}_X^0 = \mathcal{C}_X^\infty$, $df = 0$ if and only if f is locally constant. So this complex gives us a fine resolution of the constant sheaf $\underline{\mathbb{R}}$. The exact sequence

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathscr{E}_X^0 \xrightarrow{d} \mathscr{E}_X^1 \xrightarrow{d} \mathscr{E}_X^2 \longrightarrow \cdots \longrightarrow \mathscr{E}_X^{n-1} \xrightarrow{d} \mathscr{E}_X^n \longrightarrow 0$$

is called the *de Rham exact sequence of sheaves.* The cohomology of the cochain complex $(\Gamma(X, \mathscr{E}_X^*), d_X)$ is called the *de Rham cohomology of* X, denoted by $H_{dR}^*(X)$, which is then isomorphic to the cohomology $H^*(X, \underline{\mathbb{R}})$ of the space with coefficients in the constant sheaf $\underline{\mathbb{R}}$.

Another method in computing sheaf cohomology is through Čech cohomology, which is based on the sections of the sheaf on open coverings of the space.

3.5 Čech Cohomology

Let $(I, \leq)$ be a totally ordered set and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open covering of X.

Definition 3.5.1. Let $\sigma = (U_{\alpha_0}, U_{\alpha_1}, \cdots, U_{\alpha_q})$ be a $(q+1)$-tuple of elements of $\mathcal{U}$. We say that σ is *a q-simplex* if $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_q} \neq \emptyset$ and $\alpha_0 < \alpha_1 < \cdots < \alpha_q$.

The set of q-simplices of $\mathcal{U}$ is denoted by $\mathcal{U}^q$ and the collection of those set of simplices $\mathcal{U}^q$ is called *the nerve of the covering* $\mathcal{U}$, denoted by $N(\mathcal{U})$. Also, for each $\sigma = (U_{\alpha_0}, U_{\alpha_1}, \cdots, U_{\alpha_q}) \in \mathcal{U}^q$, the intersection $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}$ is called *the support of the q-simplex* σ and is denoted by $|\sigma|$. Out of this, we define the notion of Čech complexes associated to the open covering $\mathcal{U}$ as follow.

Definition 3.5.2. Let $\mathscr{S}$ be a sheaf over X. The group defined by $\check{C}^q(\mathcal{U}, \mathscr{S}) := \prod_{\sigma \in \mathcal{U}^q} \mathscr{S}(|\sigma|)$ is called the *q^{th} Čech cochain of $\mathcal{U}$ with coefficients in the sheaf $\mathscr{S}$*. In other words, an element of $\check{C}^q(\mathcal{U}, \mathscr{S})$ is a map, which associates to each element σ of $\mathcal{U}^q$ a section in $\mathscr{S}(|\sigma|)$.

We define a differential $\partial : \check{C}^q(\mathcal{U},\mathscr{S}) \to \check{C}^{q+1}(\mathcal{U},\mathscr{S})$ as follow. For each $f \in \check{C}^q(\mathcal{U},\mathscr{S})$ and $\sigma = (U_{\alpha_0}, U_{\alpha_1}, \cdots, U_{\alpha_{q+1}}) \in \mathcal{U}^q$,

$$\partial(f)(\sigma) = \sum_{i=0}^{q+1} (-1)^i f(\sigma_i)|_{|\sigma|}$$

where σ_i is obtained by dropping U_{α_i} from σ.

A straightforward calculation (see [2, p.45] for instance) shows that $\partial^2 = 0$. So we have a complex $\check{C}^*(\mathcal{U},\mathscr{S})$ called *the Čech cochain complex of the covering* $\mathcal{U}$ *with coefficient in the sheaf* $\mathscr{S}$. The cohomology group of this cochain complex is called *the Čech cohomology of the covering* $\mathcal{U}$ *with coefficients in the sheaf* $\mathscr{S}$. We denote correspondingly by $\check{H}^q(\mathcal{U},\mathscr{S})$ the q^{th} cohomology group of this cochain complex.

We always have a nice description of the zero dimensional Čech cohomology by the following proposition.

Proposition 3.5.3. We have the identification $\check{H}^0(\mathcal{U},\mathscr{S}) \cong \Gamma(X,\mathscr{S}) \cong H^0(X,\mathscr{S})$.

Proof. It is clear that a global section defines a 0-cocycle. Conversely, if f is a 0-cocycle, then f induces a section in U for all $U \in \mathcal{U}$ and on each pair U, V of elements of $\mathcal{U}$, with nonempty intersection, we have the relation $f(U)|_{U\cap V} = f(V)|_{U\cap V}$. So f defines a global section. □

Also, some further condition on the covering $\mathcal{U}$ allow us to identify the higher cohomology group of the space with Čech cohomology of the covering. This result works under the paracompactness condition on the space X.

Definition 3.5.4. The open covering $\mathcal{U}$ is called a *Leray covering* if $H^q(|\sigma|,\mathscr{S}) = 0$ for all $q > 0$ and $\sigma \in N(\mathcal{U})$.

Lemma 3.5.5. If $\mathscr{S}$ is a fine sheaf over a paracompact space X, then $\check{H}^q(\mathcal{U},\mathscr{S}) = 0$ for all $q > 0$.

Proof. See on page 54 of *Robert Gunning's* book [2]. □

Theorem 3.5.6. *Let* $\mathcal{U}$ *be a Leray covering of a paracompact space* X*. Then* $\check{H}^q(\mathcal{U},\mathscr{S}) \cong H^q(X,\mathscr{S})$ *for all* $q \geq 0$.

Proof. Take a fine resolution $(\mathscr{S}^*, d)$ of the sheaf $\mathscr{S}$. For each $q \geq 0$ and $\sigma \in \mathcal{U}^q$, the Leray condition means that we have an exact sequence

$$0 \longrightarrow \mathscr{S}(|\sigma|) \longrightarrow \mathscr{S}^0(|\sigma|) \xrightarrow{d_X^0} \mathscr{S}^1(|\sigma|) \xrightarrow{d_X^1} \mathscr{S}^2(|\sigma|) \xrightarrow{d_X^2} \cdots.$$

This implies the exactness on q^{th} Čech cochains

$$0 \longrightarrow \prod_{\sigma\in\mathcal{U}^q} \mathscr{S}(|\sigma|) \longrightarrow \prod_{\sigma\in\mathcal{U}^q} \mathscr{S}^0(|\sigma|) \xrightarrow{d_X^0} \prod_{\sigma\in\mathcal{U}^q} \mathscr{S}^1(|\sigma|) \xrightarrow{d_X^1} \prod_{\sigma\in\mathcal{U}^q} \mathscr{S}^2(|\sigma|) \xrightarrow{d_X^2} \cdots \tag{3.5.1}$$

We summarise this by the following diagram.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \check{C}^3(\mathcal{U},\mathscr{S}) \arrow[u, "\partial"] \arrow[r, dashed] & \check{C}^3(\mathcal{U},\mathscr{S}^0) \arrow[u, "\partial"] \arrow[r, "d_X^0"] & \check{C}^3(\mathcal{U},\mathscr{S}^1) \arrow[u, "\partial"] \arrow[r, "d_X^1"] & \check{C}^3(\mathcal{U},\mathscr{S}^2) \arrow[u, "\partial"] \arrow[r, "d_X^2"] & \cdots \\
0 \arrow[r] & \check{C}^2(\mathcal{U},\mathscr{S}) \arrow[u, "\partial"] \arrow[r, dashed] & \check{C}^2(\mathcal{U},\mathscr{S}^0) \arrow[u, "\partial"] \arrow[r, "d_X^0"] & \check{C}^2(\mathcal{U},\mathscr{S}^1) \arrow[u, "\partial"] \arrow[r, "d_X^1"] & \check{C}^2(\mathcal{U},\mathscr{S}^2) \arrow[u, "\partial"] \arrow[r, "d_X^2"] & \cdots \\
0 \arrow[r] & \check{C}^1(\mathcal{U},\mathscr{S}) \arrow[u, "\partial"] \arrow[r, dashed] & \check{C}^1(\mathcal{U},\mathscr{S}^0) \arrow[u, "\partial"] \arrow[r, "d_X^0"] & \check{C}^1(\mathcal{U},\mathscr{S}^1) \arrow[u, "\partial"] \arrow[r, "d_X^1"] & \check{C}^1(\mathcal{U},\mathscr{S}^2) \arrow[u, "\partial"] \arrow[r, "d_X^2"] & \cdots \\
0 \arrow[r] & \check{C}^0(\mathcal{U},\mathscr{S}) \arrow[u, "\partial"] \arrow[r, dashed] & \check{C}^0(\mathcal{U},\mathscr{S}^0) \arrow[u, "\partial"] \arrow[r, "d_X^0"] & \check{C}^0(\mathcal{U},\mathscr{S}^1) \arrow[u, "\partial"] \arrow[r, "d_X^1"] & \check{C}^0(\mathcal{U},\mathscr{S}^2) \arrow[u, "\partial"] \arrow[r, "d_X^2"] & \cdots \\
0 \arrow[r] & \Gamma(X,\mathscr{S}) \arrow[u] \arrow[r] & \Gamma(X,\mathscr{S}^0) \arrow[u, dashed] \arrow[r, "d_X^0"] & \Gamma(X,\mathscr{S}^1) \arrow[u, dashed] \arrow[r, "d_X^1"] & \Gamma(X,\mathscr{S}^2) \arrow[u, dashed] \arrow[r, "d_X^2"] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{3.5.2}
$$

It is clear that the differentials d_X and ∂ commutes. So we have a double complex

$$\check{C}^{*,*} = \{\check{C}^p(\mathcal{U},\mathscr{S}^q)\}_{p,q\geq 0}$$

with exact rows. Since all $\mathscr{S}^q$, $q \geq 0$ are fine, all the columns of $\check{C}^{*,*}$ are also exact by Lemma 3.5.5. We apply Theorem 3.2.8 to show that the cohomology of the first row of the diagram (3.5.2) is equal to that of the first column. □

In the following example, we discuss the relationship between Čech, sheaf and singular cohomology on locally contractible spaces such as topological manifolds.

Example 3.5.7. Let A be an abelian group. The group of singular q-cochains is defined by

$$C^q(X,A) = \mathrm{Hom}_{\mathbb{Z}}(C_q(X),A),$$

where $C_q(X)$ is the free abelian group generated by the continuous functions from the standard q-simplex Δ^q into X. The differential $\delta : C^q(X,A) \to C^{q+1}(X,A)$ is induced from the differential $\partial : C_{q+1}(X) \to C_q(X)$ defined by the alternative sum of the restriction of the functions in the faces of Δ^{q+1}. The *singular cohomology of the space X with coefficient in A*, denoted by $H^*_{\text{sing}}(X,A)$ is the cohomology of the singular cochain complex $((C^*(X,A)),\delta)$. As a result from the theory of singular cohomology, if the space X is contractible, then

$$H^q_{\text{sing}}(X,A) = \begin{cases} 0 & \text{if } q > 0; \\ A & \text{if } q = 0. \end{cases}$$

An inclusion of open sets $i : V \to U$ induces naturally a group homomorphism $i_* : C_q(V) \to C_q(U)$, which is clarified by the diagram

$$\begin{tikzcd} \Delta^q \arrow[r, dashed, "i\circ f"] \arrow[dr, "f"'] & U \\ & V \arrow[u, "i"'] \end{tikzcd}$$

In particular, we obtain a group homomorphism i^* from $C^q(U,A)$ to $C^q(V,A)$ on the duals. The homomorphism i^* satisfies the restriction conditions, so we can form a presheaf of singular q-cochains $\{C^q(U,A)\}$. We denote by $\mathscr{C}^q$ the corresponding associated sheaf, which is flasque if the space X is locally contractible according to [8]. The differential δ also induced a sheaf homomorphism $\delta : \mathscr{C}^q \to \mathscr{C}^{q+1}$, so that we have a cochain complex of flasque sheaves $(\mathscr{C}^*, \delta)$. For each open subset $U \subset X$, since $C_0(U)$ is just the free abelian group generated by the points of U, any $f \in C^0(U,A)$ can be considered as a function from U into A, and δf maps any path in U with end points a and b into $f(a) - f(b)$. So $\ker \delta : C^0(U,A) \to C^1(U,A)$ consists of the locally constant functions from U to A and then $\ker \delta : \mathscr{C}^0 \to \mathscr{C}^1$ is just the constant sheaf $\underline{A}$. Also, with the locally contractible condition on X, each point of X has an open neighborhood U such that $H^q_{\text{sing}}(U,A) = 0$ for all $q > 0$. This implies that we have an exact sequence of sheaves

$$0 \longrightarrow \underline{A} \longrightarrow \mathscr{C}^0 \xrightarrow{\delta} \mathscr{C}^1 \xrightarrow{\delta} \cdots,$$

and then by taking the global sections, we have an isomorphism

$$H^q(X, \underline{A}) \cong H^q(\Gamma(X, \mathscr{C}^*)) = H^q_{\text{sing}}(X, A),$$

for each $q \geq 0$. In addition, if we take a covering $\mathcal{U}$ of X, which is formed by contractible open sets, then by Theorem 3.5.6, we have an isomorphism $\check{H}^q(\mathcal{U}, X) \cong H^q_{\text{sing}}(X, A)$ for all $q \geq 0$.

With the examples 3.4.6 and 3.5.7, we conclude that if X is a smooth manifold, then

$$H^q_{dR}(X) \cong H^q(X, \underline{\mathbb{R}}) \cong H^q_{\text{sing}}(X, \mathbb{R})$$

for all $q \geq 0$. We may not obtain sheaf cohomology from Čech cohomology if the corresponding open covering is not Leray, according to the following example.

Example 3.5.8. Consider the unit sphere S^1 with the covering $\mathcal{U} = \{U_0, U_1\}$ defined by $U_0 = S^1 \setminus \{-1\}$ and $U_1 = S^1 \setminus \{1\}$. The Čech complex associated to this covering, with coefficients in the constant sheaf $\underline{\mathbb{R}}$, is given by

$$\mathbb{R} \oplus \mathbb{R} \xrightarrow{\partial} \mathbb{R} \longrightarrow 0$$

such that $\partial(a,b) = a - b$. So $\check{H}^1(\mathcal{U}, \underline{\mathbb{R}}) = \mathbb{R}/\operatorname{im}\partial = 0 \neq H^1_{\text{sing}}(X, \mathbb{R}) = \mathbb{R}$. This is because the intersection $U_0 \cap U_1$ is not contractible.

However, we have the following result for paracompact Hausdorff spaces.

Theorem 3.5.9. *Let $\mathscr{S}$ be a sheaf of abelian groups over a paracompact Hausdorff space X. Then, there exists an open covering $\mathcal{U}$ of X such that $H^q(X, \mathscr{S}) \cong \check{H}^q(\mathcal{U}, \mathscr{S})$ for all $q > 0$.*

Proof. See on page 38 of *Robert Gunning's* book [6]. $\square$

4 Applications in Analytic Geometry

In this chapter, let X be an analytic variety and $\mathcal{O}_X$ the sheaf of germs of holomorphic functions on X. We are going to show how sheaf cohomology works efficiently with problems in holomorphic and meromorphic functions in several variables. The notion of coherent sheaves will play a very important role here as we describe first.

4.1 Coherent Analytic Sheaves

Definition 4.1.1. A sheaf $\mathscr{S}$ of abelian groups over X is called *an $\mathcal{O}_X$-module* if for each open set $U \subset X$, the group $\mathscr{S}(U)$ has a structure of module over the algebra $\mathcal{O}_X(U)$, which is compatible with the restriction morphisms, i.e for each pair of open sets $V \subset U$, $u \in \mathcal{O}(U)$ and $f \in \mathscr{S}(U)$, $(u \cdot f)|_V = u|_V \cdot f|_V$.

The structure of modules on each open set, together with this compatibility condition induce naturally a structure of $\mathcal{O}_{X,x}$-module on the stalk $\mathscr{S}_x$ for each point $x \in X$. Also, if $\mathscr{R}$ is a subsheaf of an $\mathcal{O}_X$-module, then the structure of submodule on $\mathscr{R}$ as an $\mathcal{O}_X$-module requires the condition that each $\mathscr{R}(U)$ is a sub-$\mathcal{O}_X(U)$-module of $\mathscr{S}(U)$ for each open set U; and which inherits the compatibility condition on $\mathscr{S}$.

The operations on modules can be translated into that of $\mathcal{O}_X$-modules. In particular, we have the notion of direct sum:

Definition 4.1.2. Let $\mathscr{R}$ and $\mathscr{S}$ be two $\mathcal{O}_X$-modules. The collection of direct sums $\{\mathscr{R}(U) \oplus \mathscr{S}(U)\}$, together with the natural restriction of functions form a complete presheaf on X, the corresponding associated sheaf is called *the direct sum of $\mathscr{R}$ and $\mathscr{S}$*, and denoted by $\mathscr{R} \oplus \mathscr{S}$.

Note that $\mathscr{R} \oplus \mathscr{S}$ is indeed an $\mathcal{O}_X$-module. The completeness of the presheaf $\{\mathscr{R}(U) \oplus \mathscr{S}(U)\}$ allows us to identify $(\mathscr{R} \oplus \mathscr{S})(U)$ with $\mathscr{R}(U) \oplus \mathscr{S}(U)$. This implies that for each $x \in X$, the stalk of the direct sum at x is given by

$$(\mathscr{R} \oplus \mathscr{S})_x \cong \mathscr{R}_x \oplus \mathscr{S}_x\,.$$

The direct sums $\mathscr{S} \oplus \cdots \oplus \mathscr{S}$ of n copies of an $\mathcal{O}_X$-module $\mathscr{S}$ is denoted by $\mathscr{S}^n$, with convention $\mathscr{S}^0 = 0$. In particular, we have the sheaf $\mathcal{O}_X^n$, called *the free analytic sheaf of rank n*. To keep in mind the analytic structure on X, we say *analytic sheaf over X* instead of $\mathcal{O}_X$-module.

Now, let us take an arbitrary point $x \in X$, and an analytic sheaf $\mathscr{S}$ over X. If the stalk $\mathscr{S}_x$ is a finitely generated as $\mathcal{O}_x$-module, then of course, we can find finitely many sections $f_1, \ldots, f_p$ defined in a neighborhood of x such that $\mathscr{S}_x$ is generated by $f_1(x), \ldots, f_p(x)$ as an $\mathcal{O}_x$-module. Though those sections, we would like to share this information to those points close enough to x, not only at x. So we need to assume that for every $y \in X$, $\mathscr{S}_y$ is finitely generated as $\mathcal{O}_y$-module and the information move continuously through y. The simplest example of sheaf with that properties is the free analytic sheaf of of rank n, where each stalk $\mathcal{O}^n_{X,x}$ is clearly generated by the n global sections

$$e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, 0, \cdots, 0), \dots, e_n = (0, \cdots, 0, 1).$$

In this case, we say that the sheaf $\mathcal{O}^n_X$ is generated by the sections $e_1, \dots, e_n$. The following definition will formalize such discussion.

Definition 4.1.3. An analytic sheaf $\mathscr{S}$ over X is called *locally finitely generated* if for each point $x \in X$, there exist an open neighborhood U_x of x, in which we have an exact sequence of the form

$$\mathcal{O}^n_X|_{U_x} \xrightarrow{\varphi} \mathscr{S}|_{U_x} \longrightarrow 0.$$

In the open set U_x, the sheaf $\mathscr{S}|_{U_x}$ can be generated by n sections $f_1, \dots, f_n \in \mathscr{S}(U_x)$, which are the image under φ of $e_1, \dots, e_n$ respectively. For each point $y \in U_x$, we can find sections $g_1, \dots, g_m$, which generate $\mathscr{S}_y$. For y close enough to x, each g_i is defined at x. We can find sections u_{ij} of $\mathcal{O}_X$, defined in a neighborhood of x, such that $g_i(x) = \sum_{j=1}^n u_{ij}(x) \cdot f_j(x)$. So $\mathscr{S}_y$ is also generated by $f_1(y), \dots, f_n(y)$. The kernel sheaf $\ker\varphi \subset \mathcal{O}^n_X|_{U_x}$ is called *the sheaf of relations of the sections* $f_1, \dots, f_n$, and we denote it by $\ker(f_1, \dots, f_n)$. Logically, since both $\mathcal{O}^n_X|_{U_x}$ and $\mathscr{S}|_{U_x}$ are finitely generated, we may also require this property on the corresponding sheaf of relations.

On the opposite direction, if we have finitely many sections $f_1, \dots, f_n$ of the sheaf $\mathscr{S}$ on some open set U, then we can build the sheaf homomorphism $\varphi : \mathcal{O}^n_X|_U \to \mathscr{S}|_U$, defined by $\varphi(\lambda_i e_i) = \lambda_i f_i$.

Definition 4.1.4. An analytic sheaf $\mathscr{S}$ is called *coherent* if it is locally finitely generated and for every finitely many sections $f_1, \dots, f_n$ of $\mathscr{S}$ over an open set U, the sheaf of relations $\ker(f_1, \dots, f_n)$ is also locally finitely generated as an $\mathcal{O}_X|_U$-module. In other words, for every point x, we can find an appropriate open neighborhood U_x of x and an exact sequence of the form

$$\mathcal{O}^{p_1}_X|_{U_x} \longrightarrow \mathcal{O}^p_X|_{U_x} \longrightarrow \mathscr{S}|_{U_x} \longrightarrow 0.$$

If we have a submodule $\mathscr{R}$ of an analytic coherent sheaf $\mathscr{S}$, then it is coherent if and only if it is locally finitely generated. This equivalence holds since any sheaf homomorphism $\varphi : \mathcal{O}^p|_U \to \mathscr{R}|_U$ can be considered as a sheaf homomorphism $\tilde{\varphi} : \mathcal{O}^p|_U \to \mathscr{S}|_U$, and in this case we have $\ker\varphi = \ker\tilde{\varphi}$. In particular, if we have a homomorphism $\varphi : \mathscr{S} \to \mathscr{T}$ of coherent analytic sheaves, then clearly the sheaf image $\operatorname{im}\varphi$ is locally finitely generated, and so it is coherent. Furthermore, under the relation given by the short exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow \mathscr{S} \xrightarrow{\varphi} \operatorname{im}\varphi \longrightarrow 0,$$

$\ker\varphi$ is also coherent due to the following theorem.

Theorem 4.1.5. *Let*

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{S} \longrightarrow \mathscr{T} \longrightarrow 0,$$

be a short exact sequence of analytic sheaves over X. Then if two of them are coherent, then the one left is also coherent.

Proof. See on page 209 *Serre's* paper [4]. □

For complex manifolds, we have the following result due to *Kiyoshi Oka*:

Theorem 4.1.6 (Oka's Coherence Theorem)**.** *If X is a complex manifold, then $\mathcal{O}_X$ is coherent.*

Proof. See on page 145 of *L. Kaup and B. Kaup's* book [7]. □

If Y is an analytic subvariety of a complex manifold X, then the sheaf $\mathcal{O}_Y$ of germs of holomorphic functions on Y can be taken as an analytic sheaf over X by defining $\mathscr{S}(U) = \mathcal{O}_Y(Y \cap U)$ for each open set $U \subset X$ and we have a surjective sheaf homomorphism $\varphi : \mathcal{O}_X \to \mathcal{O}_Y$ given by the restriction of functions on Y. The kernel of this sheaf homomorphism is called *the ideal sheaf of Y over X*, denoted by $\mathscr{I}_{Y/X}$ and we have then a short exact sequence

$$0 \longrightarrow \mathscr{I}_{Y/X} \longrightarrow \mathcal{O}_X \xrightarrow{\varphi} \mathcal{O}_Y \longrightarrow 0. \tag{4.1.1}$$

The stalks of the ideal sheaf $\mathscr{I}_{Y/X}$ is given by

$$(\mathscr{I}_{Y/X})_x = \begin{cases} 0 & \text{if } x \in X \setminus Y \\ \mathscr{I}_x(Y) & \text{if } x \in Y \end{cases}$$

where $\mathscr{I}_x(Y)$ consists of the germs of holomorphic functions in a neighborhood of x, which vanish at x. The coherence also hold with this sheaf of ideal due to the following result.

Theorem 4.1.7 (Cartan's Theorem)**.** *If Y is an analytic subvariety of an analytic variety X, then the ideal sheaf $\mathscr{I}_{Y/X}$ is coherent as an $\mathcal{O}_X$-module.*

Proof. See on page 17 of *Robert Gunning's* book [2]. □

4.2 Stein Varieties

The notion of coherence discussed above allows us to control analytic varieties through sheaf cohomology. In this section, we will concentrate on those analytic varieties whose higher cohomology groups with coefficients in coherent analytic sheaves are trivial and we describe some examples of such varieties to motivate the discussion.

Definition 4.2.1. An analytic variety X is called *a Stein variety* if $H^q(X, \mathscr{S}) = 0$ for all $q > 0$ and for all coherent analytic sheaf $\mathscr{S}$ over X. Furthermore, if X has a complex manifold structure, then we say that it is *a Stein manifold*.

Let Y be an analytic subvariety of a stein variety X. Since the ideal sheaf of Y over X is coherent, we have $H^1(X, \mathscr{I}_{Y/X}) = 0$. So by long exact sequence induced by the short exact sequence (4.1.1) in cohomology, we have the exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{I}_{Y/X}) \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(Y, \mathcal{O}_Y) \longrightarrow 0$$

in global sections. So we have the following result:

Theorem 4.2.2. *Any holomorphic functions on the analytic subvariety Y of a Stein variety X is given by the restriction to Y of a holomorphic function defined on X.*

Assume now that we have two distinct points x_1 and x_2 in a Stein variety X. Then of course, the set of the two points $Y = \{x_1, x_2\}$ is an analytic subvariety of X and $\Gamma(Y, \mathcal{O}_Y) = \mathbb{C} \oplus \mathbb{C}$. So by the short exact sequence (4.2) in global sections, we can find a holomorphic function $f \in \Gamma(X, \mathcal{O}_X)$ such that $f(x_1) \neq f(x_2)$. In this case, we say that the global holomorphic functions on X *separate points*. We summarize this in the following theorem:

Theorem 4.2.3. *The global holomorphic functions on a Stein variety separate points.*

The property of non-separation by global holomorphic functions is then a criterion on an analytic variety to be a non-Stein variety.

Example 4.2.4. If X is a compact complex manifold, then by the maximum principles, $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$, so it will never separate points and by Theorem 4.2.3, X not a Stein variety.

If we have a discrete sequence of points $\{x_n\}_{n\in\mathbb{N}}$ of points in a Stein variety X, then of course $Y = \{x_n \mid n \in \mathbb{N}\}$ is an analytic subvariety of X. Since $\Gamma(X, \mathcal{O}_Y) = \mathbb{C}^{\mathbb{N}}$, by Theorem 4.2.2 we can find a holomorphic function on X such that $f(x_n) = n$. In particular we have $|f(x_n)| \to \infty$ as $n \to \infty$. We have then the following result on Stein varieties.

Theorem 4.2.5. *On every discrete sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of a Stein variety X, there exists a global holomorphic function on X such that $|f(x_n)| \to \infty$ as $n \to \infty$.*

To describe fully a criterion on analytic varieties to be Stein, we introduce the following notion.

Definition 4.2.6. An analytic variety X is called *holomorphically convex* if for each compact subset $K \subset X$, the so called *holomorphically convex hull of K* defined by

$$\hat{K} = \{x \in X \mid |f(x)| \leq \|f\|_K \text{ for all } f \in \Gamma(X, \mathcal{O}_X)\}$$

is also compact, where $\|f\|_K = \underset{x\in K}{sup}|f(x)|$.

It is clear from definition that $K \subset \hat{K}$ and $\hat{K}$ is a closed subset of X. Furthermore, it is impossible to have a non compact $\hat{K}$ in a Stein variety. Since otherwise, one can find an infinite discrete sequence $\{x_n\}$ of points of $\hat{K}$ and by Theorem 4.2.5, we have a holomorphic function f on X such that $|f(x_n)| \to \infty$ as $n \to \infty$. This contradicts the fact that $|f(x_n)| \leq \|f\|_K < \infty$ for all n. So we have the following result.

Theorem 4.2.7. *Every Stein variety is holomorphically convex.*

We have seen that Stein varieties are holomorphically separable and holomorphically convex. The following theorem shows that these are necessary and sufficient conditions on analytic varieties to be Stein.

Theorem 4.2.8. *An analytic variety, which is holomorphically separable and holomorphically convex is a Stein variety.*

Proof. See on page 143 of *Robert Gunning's* book [2]. □

Assume that X is an open subset of $\mathbb{C}^n$. Then we always have the holomorphically separability condition. If K is a compact subset of X, then $\hat{K}$ is bounded since for each point $(x_1, \ldots, x_n) \in \hat{K}$, each x_i is bounded by a maximal value of the coordinate function x_i on K. So the holomorphically convexity condition is equivalent to say that $\hat{K}$ is a closed subset of $\mathbb{C}^n$ for all compact subset $K \subset X$. This also means that $\hat{K}$

does not intersect with any open neighborhood of the boundary ∂X of X. In one dimensional case, we always have the notion of Stein variety on X according to the following theorem.

Theorem 4.2.9. *Any open subset of the complex line $\mathbb{C}$ is holomorphically convex, and then it is a Stein manifold.*

Proof. If K is a compact subset of an open set $U \subset \mathbb{C}$, then on each point $x_0 \in \partial X$, the function f_0 defined by $f_0(x) = (x - x_0)^{-1}$ is holomorphic on U, which tends to infinity when x closes to x_0, so x_0 have an open neighborhood which does not intersect with $\hat{K}$. □

This property can be extended into Cartesian product so that any open subset $X \subset \mathbb{C}^n$ formed by a Cartesian product of open subsets of $\mathbb{C}$ is a Stein variety.

Let us now describe some existence problems in the theory of holomorphic functions in several variables.

4.3 Additive Cousin's Problem

This problem is a generalization of the Mittag Leffler problem in higher dimensions. As the sheaf of germs of meromorphic $\mathcal{M}_X$ on X contains $\mathcal{O}_X$ as a subsheaf, we can consider their quotient.

Definition 4.3.1. The quotient sheaf $\mathcal{P}_X = \mathcal{M}_X/\mathcal{O}_X$ is called *the sheaf of germs of principal parts on X* and a global section of $\mathcal{P}_X$ is called a *Cousin* I *distribution on X*.

By definition, it is clear that a Cousin I distribution on X consists of an open covering $\{U_\alpha\}$ of X, together with meromorphic functions $f_\alpha \in \mathcal{M}_X(U_\alpha)$ such that $f_\alpha - f_\beta \in \mathcal{O}_X(U_\alpha \cap U_\beta)$. The additive Cousin's problem is about the existence of a global meromorphic function f on X such that $f - f_\alpha \in \mathcal{O}_X(U_\alpha)$ for a given Cousin I distribution $\{U_\alpha, f_\alpha\}$.

With the long exact sequence in cohomology, induced by the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{M}_X \longrightarrow \mathcal{P}_X \longrightarrow 0,$$

the condition that $H^1(X, \mathcal{O}_X) = 0$ is sufficient for the additive Cousin's problem to be solvable. In particular, we have the following result.

Theorem 4.3.2. *On Stein manifolds the additive Cousin's problem is solvable. In particular, the Mittag Leffler problem is always solvable.*

Proof. By Oka's Coherence Theorem, $\mathcal{O}_X$ is coherent. Since we are working on Stein manifolds, we have in particular that $H^1(X, \mathcal{O}_X) = 0$ and the problem is solvable due to the discussion above. Also, by Theorem 4.2.9, we have in particular that open subsets of the complex plane are Stein manifold. □

We also have a multiplicative analogue of this problem as we describe next.

4.4 Multiplicative Cousin's Problem

The sheaf of germs of not identically zero meromorphic functions $\mathcal{M}_X^*$ contains the sheaf of germs of nowhere vanishing holomorphic functions $\mathcal{O}_X^*$ as a subsheaf, so we can consider their quotient.

Definition 4.4.1. The quotient sheaf $\mathcal{D}_X^* = \mathcal{M}_X^*/\mathcal{O}_X^*$ is called *the sheaf of germs of divisors on* X and a global section of $\mathcal{D}_X^*$ is called *a divisor or a Cousin* II *distribution on* X.

It is clear that a divisor on X consists of an open covering $\{U_\alpha\}$ of X, together with some not identically zero meromorphic functions $f_\alpha \in \mathcal{M}_X^*(U_\alpha)$ such that $f_\alpha/f_\beta \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$. The multiplicative Cousin's problem is about the existence of a meromorphic function f on X such that $f/f_\alpha \in \mathcal{O}_X^*(U_\alpha)$ for a given divisor $\{U_\alpha, f_\alpha\}$.

With the long exact sequence in cohomology, induced by the short exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^* \xrightarrow{\pi} \mathcal{D}_X^* \longrightarrow 0, \tag{4.4.1}$$

the condition that $H^1(X, \mathcal{O}_X^*) = 0$ is sufficient for the multiplicative Cousin's problem to be solvable. This motivates the investigation of the first cohomology group $H^1(X, \mathcal{O}_X^*)$.

Proposition 4.4.2. There is a natural isomorphism $Pic(X) \cong H^1(X, \mathcal{O}_X^*)$.

Proof. Since we are working on paracompact Hausdorff spaces, we can always find an open covering $\mathcal{U} = \{U_\alpha\}$ of X such that $H^1(X, \mathcal{O}_X^*) \cong \check{H}^1(\mathcal{U}, \mathcal{O}_X^*)$ according to Theorem 3.5.9. If we have a cohomology class $[\xi]$ represented by a one cocycle $\xi = \{\xi_{\alpha\beta}\} \in \check{\mathcal{Z}}^1(\mathcal{U}, \mathcal{O}_X^*)$, then, by definition of one Čech cocycle, we have $\xi_{\alpha\beta} \cdot \xi_{\alpha\gamma}^{-1} \cdot \xi_{\beta\gamma} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$. This is equivalent to the conditions on transition functions for holomorphic line bundles. So we have a line bundle $E = \{U_\alpha, \xi_{\alpha\beta}\}$. It is easy to check that another representative corresponds to a line bundle isomorphic to E. The converse is trivial since any family of transition functions represents a Čech cocycle and we just take its cohomology class to give the inverse homomorphism. □

Consider now the short exact sequence given by the exponential map

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^* \longrightarrow 1. \tag{4.4.2}$$

In the corresponding induced long exact in cohomology, we have the exact sequence

$$H^1(X, \mathcal{O}_X) \xrightarrow{exp_1} H^1(X, \mathcal{O}_X^*) \xrightarrow{c} H^2(X, \underline{\mathbb{Z}}) \longrightarrow H^2(X, \mathcal{O}_X). \tag{4.4.3}$$

The homomorphism c is called *the characteristic homomorphism* and for each line bundle $\xi \in H^1(X, \mathcal{O}_X^*)$, $c(\xi) \in H^2(X, \underline{\mathbb{Z}})$ is called *the characteristic class or Chern class of the holomorphic line bundle* ξ.

For Stein manifolds, $H^2(X, \underline{\mathbb{Z}}) = 0$ is a sufficient condition for the multiplicative Cousin's problem to be solvable, according to the following result.

Theorem 4.4.3. *If* X *is a Stein manifold, then the characteristic homomorphism is an isomorphism.*

Proof. In the exact sequence (4.4.3), we have $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ since X is a Stein manifold. So c is an isomorphism. □

By combining the long exact sequence induced by (4.4.1) with that of (4.4.2), we have a sequence

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow \Gamma(X, \mathcal{M}_X^*) \xrightarrow{\pi_X} \Gamma(X, \mathcal{D}_X^*) \xrightarrow{\delta} H^1(X, \mathcal{O}_X^*) \xrightarrow{c} H^2(X, \underline{\mathbb{Z}}), \tag{4.4.4}$$

which is not necessarily exact. For any divisor $D \in \Gamma(X, \mathcal{D}_X^*)$, $c(\delta(D))$ is called *the Chern class of the divisor* D.

Remark 4.4.4. With the definition of the connecting homomorphism δ (can be seen in [2]), for any divisor $D = \{U_\alpha, f_\alpha\} \in \Gamma(X, \mathcal{D}_X^*)$, $\delta(D)$ is just the isomorphism class of line bundle with representative $\{U_\alpha, f_{\alpha\beta}\}$, where $f_{\alpha\beta} = f_\alpha / f_\beta$. It is called *the holomorphic line bundle associated to the divisor* D and usually denoted by $[D]$.

Now, let $\xi = \{U_\alpha, \xi_{\alpha\beta}\}$ be a holomorphic line bundle over X. For each open set $U \subset X$, we consider the algebra

$$\mathcal{O}_X(\xi)(U) = \left\{ \{f_\alpha\}_\alpha \mid f_\alpha \in \mathcal{O}_X(U \cap U_\alpha) \text{ and } f_\alpha|_{U \cap U_\alpha \cap U_\beta} = \xi_{\alpha\beta} \cdot f_\beta|_{U \cap U_\alpha \cap U_\beta} \right\} .$$

With the usual restriction of functions, it is clear that the system $\{\mathcal{O}_X(\xi)(U)\}_U$ is a complete presheaf over X. The corresponding associated sheaf is called *the sheaf of germs of holomorphic sections of the line bundle* ξ, and it is denoted by $\mathcal{O}_X(\xi)$. By the obvious multiplication $f \cdot \{f_\alpha\}_\alpha = \{f \cdot f_\alpha\}_\alpha$, it is clear that $\mathcal{O}_X(\xi)$ has a structure of $\mathcal{O}_X$-module. So it is an analytic sheaf over X.

Consider a fixed open set U_γ from the line bundle ξ. For each open set $U \subset U_\gamma$, one can easily check that the map $\varphi_U : \mathcal{O}_X(U) \to \mathcal{O}_X(\xi)(U)$, defined by $\varphi_U(f) = \{\xi_{\alpha\gamma} \cdot f\}_\alpha$ is an isomorphism of $\mathcal{O}_X(U)$-modules. So we have a sheaf isomorphism $\mathcal{O}_X(\xi)|_{U_\alpha} \cong \mathcal{O}_X|_{U_\alpha}$. The sheaf $\mathcal{O}_X(\xi)$ is then a locally free sheaf of rank one, and we have the following result:

Theorem 4.4.5. *If ξ is a holomorphic line bundle over a Stein manifold X, then $\mathcal{O}_X(\xi)$ is coherent.*

Assume now that X is a Stein manifold with connected components $\{X_i\}_{i\in I}$ and consider a holomorphic line bundle $\xi = \{U_\alpha, \xi_{\alpha\beta}\}$ over X. Also, for each i, let x_i be a point belong to X_i and we denote by Y the analytic subvariety of X consisting of the points x_i's. With the restriction of functions to Y, we have a surjective sheaf homomorphism $\varphi : \mathcal{O}_X(\xi) \to \mathcal{O}_Y(\xi|_Y)$, where $\xi|_Y = \{U_\alpha \cap Y, \xi_{\alpha\beta}|_Y\}$. Of course, we can identify $\Gamma(Y, \mathcal{O}_Y(\xi|_Y))$ with $\mathbb{C}^I$. Since the sheaf of ideal $\mathscr{I} = \ker\varphi$ is locally isomorphic to a sheaf of ideal of $\mathcal{O}_X$, it is also coherent. In paricular, we have $H^1(X, \mathscr{I}) = 0$. So with the short exact sequence of analytic sheaves

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X(\xi) \xrightarrow{\varphi} \mathcal{O}_Y(\xi|_Y) \longrightarrow 0,$$

we have the short exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{I}) \longrightarrow \Gamma(X, \mathcal{O}_X(\xi)) \xrightarrow{\varphi_X} \mathbb{C}^I \longrightarrow 0$$

in global sections. Therefore, we can find a global section $f \in \Gamma(X, \mathcal{O}_X(\xi))$ such that $f(x_i) \neq 0$ for all $i \in I$. By writing f as $\{f_\alpha\}_\alpha$, where $f_\alpha \in \mathcal{O}_X(U_\alpha)$, we have that $f_\alpha = \xi_{\alpha\beta} \cdot f_\beta$ on each $U_\alpha \cap U_\beta$ and then each f_α does not vanish identically at any connected components of X. So we have a well defined divisor $D = \{U_\alpha, f_\alpha\} \in \Gamma(X, \mathcal{D}_X^*)$, with associated line bundle $[D] = \xi$. The following result is then immediate.

Theorem 4.4.6. *On a Stein manifold X, we have an exact sequence*

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow \Gamma(X, \mathcal{M}_X^*) \xrightarrow{\pi_X} \Gamma(X, \mathcal{D}_X^*) \xrightarrow{\delta} H^1(X, \mathcal{O}_X^*) \cong H^2(X, \underline{\mathbb{Z}}) \longrightarrow 0.$$

5 Conclusion

With the language of category, we have presented the category of sheaves $Sh(X)$ over a topological space X, where the objects of the category are sheaves of abelian groups over X and the arrows of the category are sheaf homomorphisms. The notion of sheaves was built from collection of abelian groups on open subsets of the space and under the condition of presheaves. See [5], for instance, for the categorical point of view of sheaves. Since $Sh(X)$ is an abelian category, homological algebra works and we can talk about exact sequences of sheaves. The sheaf cohomology groups can be considered as covariant functors from $Sh(X)$ to the category of abelian groups $\mathcal{A}b$. We have seen several ways to compute the sheaf cohomology groups and this depends on the properties of the topological space. For any topological space, we can compute them using any acyclic resolution. In case where the space is paracompact Hausdorff, it is much easier to find an acyclic resolution based on the notion of fine resolutions. We also introduced another method of computation based on open coverings of the space, which is the Čech cohomology groups. Another way is to use Leray coverings and Leray sheaves in order to compute sheaf cohomology.

We also have presented some applications of sheaf cohomology in analytic geometry. We investigated those analytic varieties, with trivial higher cohomology groups with coefficients in any coherent analytic sheaf, known as Stein varieties. On these varieties, several nice properties hold and the additive Cousin's problem has a positive answer. Furthermore, the condition $H^2_{\text{sing}}(X, \mathbb{Z}) = 0$ was sufficient to obtain a positive answer for the multiplicative Cousin's problem as well. There are many further applications of sheaf cohomology in analytic geometry, algebraic geometry and differential geometry, which are important subjects for the future work to investigate. For further information about the characterization of Stein varieties and more examples, the interested readers can consult [2].

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