Abel maps on normal surface singularities

by

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Contents

Chapter 1

Introduction

1.1 Preliminary words

1.1.0.1. Let (X, o) a complex analytic normal surface singularity with link M. We consider a good resolution $\phi : \widetilde{X} \to X$ of the singular point o. Let $E := \phi^{-1}(o)$ be the exceptional divisor with irreducible components $\{E_v\}_{v\in\mathcal{V}}$, I the corresponding intersection form and Γ the dual resolution graph associated with ϕ. It is well known that Γ is connected and I is negative definite. Moreover, the link M is a rational homology sphere ($H_1(M, \mathbb{Q}) = 0$) if and only if the graph is a tree and the genus of E_v is 0 for all $v \in \mathcal{V}$.

The map ϕ identifies $\partial\widetilde{X}$ and M, hence Γ can be viewed as a plumbing graph, and M as the associated S^1-plumbed oriented 3-manifold, which is the boundary of the oriented plumbed 4-manifold obtained by plumbing disc-bundles. This second space can smoothly identified with $\widetilde{X}$.

In the theory of normal surface singularities in the last decade one of the major issues was to compare the analytic invariants with the topological ones. The topological invariants are computable just from the resolution graph Γ of the given singularity, or equivalently, from the link of the singularity (as invariants of oriented graph 3-manifolds).

The research aimed to provide topological formulae for several discrete analytic invariants, or at least topological candidates. Obviously, when we fix the topological type and vary the analytic structure most of these analytic invariants also can change. In particular, we have a hope to find purely topological formulae only in the case of special analytical families.

In [NN02], Némethi and Nicolaescu formulated the 'Seiberg-Witten invariant conjecture', which relates the analytic invariants (e.g. geometric genus) of (X, o) to the Seiberg-Witten invariants of the link, whenever the link is a rational homology sphere and the analytic type is $\mathbb{Q}$–Gorenstein. Though counterexamples were found among superisolated hypersiurface singularities, the validity of the conjecture was verified for a large class of singularities: e.g. for normal surface singularities which admit a good $\mathbb{C}^*$-action ([NN04]) and suspension singularities of type $g(x, y, x) = f(x, y) + z^n$ where f is an irreducible plane curve singularity ([NN03]). Furthermore, it remains true for splice quotient singularities [BN10, N12] and (some version of it) for Newton nondegenerate hypersurface singularities [NS16].

This conjecture connects singularity theory with low dimensional topology, since the Seiberg-Witten invariant is the normalized Euler-characteristic of the Seiberg-Witten monopole Floer homology of Kronheimer-Mrowka, or equivalently, of the Heegaard-Floer homology of Ozsváth-Szabó (or, of the lattice cohomology). On the other hand one has to fix always some special restrictions on the analytic type for the Seiberg witten invariant conjecture, because it clearly fails for many elliptic singularities with generic analytic type. Also the results above aims to calculate the cohomologies of line bundles with a fixed Chern class, and fixed class in the Picard group (for example the geometric genus is the h^1 of the trivial line bundle of $\widetilde{X}$).

There is an another topological candidate, which is the normalized Euler characteristic of the path lattice cohomology, see e.g. [NS16, NO17], denoted also by

MIN_γ. This is an upper bound for the geometric genus for every analytic structure. Moreover, for some special families of singularities even the equality holds, e.g. for superisolated or Newton nondegenerate hypersurface singularities [NS16].

However, again, there are resolution graphs, for which MIN_γ is not the geometric genus of any singularity corresponding to it, in particular, the upper bound is not sharp in general for any topological type [NO17].

1.1.0.2. One of our main motivations in this thesis is to investigate opposite type of problems: the determination of the geometric genus of a generic analytic type (generic with respect to a fixed topological type), or the determination of the cohomology of a line bundle on $\widetilde{X}$, which is generic in the Picard group $\mathrm{Pic}(\widetilde{X})$ with a fixed Chern class.

In the first case the answer clearly must depends just on the resolution graph, and, indeed, we succeed to prove a combinatorial formula for it.

Also, we will compute the cohomology of other 'natural' or 'special' line bundles of $\widetilde{X}$ whenever the analytical type is generic. Though in this case it is not a priori clear that the answer should be totally topological, we succeed again to provide topological formulae.

In it worth to mention that while the minimal possible geometric genus for a fixed topological type is determined in this work (it is realized by the geometric genus of generic singularities), the determination of the maximal possible geometric genus is still an open problem.

The main massage of these results is that while the geometric genus can change when we vary the analytic structure, there are combinatorial candidates for this value, and equality happens for special families of analytic types. However, if we take all possible analytic structures into consideration, then the possible values of the geometric genus $p_g(X, o)$ form an interval of integers.

1.1.0.3. The main machinery behind these results is the newly created theory of Abel maps, constructed for resolutions of normal surface singularities. They constitute certain analogy with the Brill-Noether theory of smooth projective curves.

Compared the theory of Abel maps of surface singularities with the classical Brill-Noether theory, though at many points the techniques and even the questions are rather different, there are several points, where we use the same ideas.

In the classical case one has a genus g complex algebraic curve C, and want to investigate the h^1 stratification of the line bundles $\mathrm{Pic}^d(C)$, where d is an arbitrary nonnegative integer. One of the main tools is to look at the Abel map $f : Sym^d(C) \to \mathrm{Pic}^d(C)$, which for unordered points $D = (p_1, \dots, p_d)$ (which might even coincide) associates the line bundle of the divisor $\sum_i p_i$.

There is a lot of analytic information coded in this map, for example for any effective divisor $D \in Sym^d(C)$ one has $h^1(C, f(D)) = g - \dim(\mathrm{im}(T_D f))$.

In the cases of normal surface singularities instead of the number d we should fix a Chern class for line bundles on $\widetilde{X}$, which should be an element $l' \in L' := H^2(\widetilde{X}, \mathbb{Z})$.

The next step is to find the analogue of the source space $Sym^d(C)$ and the analogue of the target space of the Abel map $\mathrm{Pic}^d(C)$.

Since $Sym^d(C)$ paramterises the degree d effective Cartier divisors on C, it would be reasonable to look at the space of effective Cartier divisors on $\widetilde{X}$. However, it turns out that this space is infinite dimensional, so we have to 'cut it off' somehow to a finite dimensional space. In order to this, we consider a (large) cycle Z supported on the exceptional divisor and we look at the space $\mathrm{ECa}^{l'}(Z)$ of effective Cartier divisors on Z with Chern class l'.

In fact, for any effective non-zero Z and Chern class l' the space $\mathrm{ECa}^{l'}(Z)$ is already constructed in the literature. In fact, we can regard Z as a projective algebraic scheme, in which situation $\mathrm{ECa}^{l'}(Z)$ was constructed by Grothendieck [Gro62], see also the article of Kleiman [Kl13] and the book of Mumford for curves on algebraic surfaces [Mu66]. In particular, $\mathrm{ECa}^{l'}(Z)$ is a quasiprojective variety. Though the existence of the space $\mathrm{ECa}^{l'}(Z)$ in this way is already established, we will provide several key properties valid in our particular situation. For example, a bit counterintuitively, even though the cycle Z has a nonreduced structure and singular points at the intersection of exceptional divisors, the space $\mathrm{ECa}^{l'}(Z)$ will always be smooth.

The aim of the thesis is to investigate the Abel map $\mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$ with special attention concerning key questions on normal surface singularities.

1.2 Summary of the main results

In this brief summary we wish to provide the major ideas and some of the major results of the thesis without technical details. The presentation will automatically provide the structure of the thesis as well.

1.2.1 Abel maps

The study of the Abel map of projective irreducible smooth curves was a crucial tool in the classical algebraic geometry and it remained so in the modern theory as well. Though in this work we will not use/apply very much this classical theory, in this introduction (and some places later) we will discuss some comparisons between the curve case and the theory of the present thesis established for normal surface singularities, mostly to emphasize the major conceptual differences and additional difficulties in the later case. (For the Abel map of curves one can consult [ACGH85] and the references therein.)

We wish to emphasize from the start that we are not generalizing the Abel construction from the curve case to the — smooth or singular — (quasi)projective surfaces: our goal is to develop its analogue valid in the context of a resolution of a complex normal surface singularity germ. This means that if (X, o) is such a singularity with a fixed good resolution $\widetilde{X} \to X$, then for any effective cycle Z supported on the reduced exceptional curve E and for any (possible) Chern class $l' \in H^2(\widetilde{X}, \mathbb{Z})$ we construct the space $\mathrm{ECa}^{l'}(Z)$ of effective Cartier divisors D supported on Z, whose associated line bundles $\mathcal{O}_Z(D)$ have first Chern class l'. Furthermore, we consider the space $\mathrm{Pic}^{l'}(Z) \subset H^1(\mathcal{O}_Z^*)$ of isomorphism classes of holomorphic line bundles with Chern class l' and the Abel map $c^{l'}(Z) : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$, $D \mapsto \mathcal{O}_Z(D)$. In this way, our Abel map is associated with non–reduced projective curves supported by the exceptional set of a good resolution of a normal surface singularity.

In particular, the combinatorial background is the combinatorics of the dual resolution graph Γ (or the intersection from $(\ ,\ )$ of the irreducible exceptional curves), that is, equivalently, the 3–dimensional link of the singularity. In fact, in order to run properly the theory, we will even assume that the link of the singularity is a rational homology sphere. This happens exactly when the resolution graph Γ represents a tree of rational curves. In this way, in all the discussions regarding the analytic types and properties we move the difficulties from the moduli space of each irreducible exceptional curve E_v (which is trivial in this case) to the analytic properties of their infinitesimal tubular neighbourhoods and their gluings (analytic plumbing).

The Abel map $c^{l'}$ behaves rather differently than the (projective) Abel map of reduced smooth curves, it shares more the properties of non–proper affine maps rather than the projective ones. This will also be clear from the next preliminary presentation of its source and target.

In fact, the space $\mathrm{ECa}^{l'}(Z)$ is already constructed in the literature. Note that by a theorem of Artin [A69, 3.8], there exists an affine algebraic variety Y and a point $y \in Y$ such that (Y, y) and (X, o) have isomorphic formal completions. Then, according to Hironaka [Hi65], (Y, y) and (X, o) are analytically isomorphic. In particular, we can regard Z as a projective algebraic scheme, in which situation $\mathrm{ECa}^{l'}(Z)$ was constructed by Grothendieck [Gro62], see also the article of Kleiman [Kl13] and the book of Mumford for curves on algebraic surfaces [Mu66]. In particular, $\mathrm{ECa}^{l'}(Z)$ is a quasiprojective variety. Though the existence of the space $\mathrm{ECa}^{l'}(Z)$ in this way is established, we will provide several key properties valid in our particular situation, including the local charts.

In Theorem 3.1.1.11 we prove the following.

Theorem I. *If $-l'$ belongs to the Lipman cone then the following facts hold.*

(1) $\mathrm{ECa}^{l'}(Z)$ is a smooth complex irreducible variety of dimension (l', Z).

(2) The natural restriction $r : \mathrm{ECa}^{l'}(Z) \to \mathrm{ECa}^{l'}(E)$ is a locally trivial fiber bundle with fiber isomorphic to an affine space. Moreover, the homotopy type of $\mathrm{ECa}^{l'}(Z)$ is independent of the choice of Z and it depends only on the topology of (X, o).

The affine fibers of $r : \mathrm{ECa}^{l'}(Z) \to \mathrm{ECa}^{l'}(E)$ can be considered as certain jet spaces in the local infinitesimal neighbourhoods of the local equations of the effective Cartier divisors. In fact, even $\mathrm{ECa}^{l'}(E)$ usually turns out to be non–projective too.

Note also that the base space $\mathrm{Pic}^{l'}(Z)$ is also noncompact, it is an affine space of dimension $h^1(\mathcal{O}_Z)$. (Here the assumption that the link is a rational homology sphere plays a role; otherwise $\mathrm{Pic}^{l'}(Z) \simeq H^1(\mathcal{O}_Z)/H^1(\widetilde{X}, \mathbb{Z})$ would have a complex torus component as well). This affine structure will be exploited deeply in the body of the paper.

We also mention that the Abel map itself is algebraic, and in fact its expression in local charts can be done explicitly via Laufer duality (integrating forms along divisors in $\widetilde{X}$).

Since the Abel map is not proper, its image usually is not closed, and it can be a rather complicated constructible set (it can be singular as well).

In order to show the presence of possible anomalies we list several examples based on the theory of elliptic and splice quotient singularities (certain familiarity with them might help essentially the reading).

We also show that all the fibers of $c^{l'}$ are smooth (irreducible, quasiprojective), however, their dimensions might jump. The dimension of $c^{-1}(\mathcal{L})$ ($\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$) is

$h^0(Z,\mathcal{L}) - h^0(\mathcal{O}_Z) = (l', Z) + h^1(Z,\mathcal{L}) - h^1(\mathcal{O}_Z)$. Any fiber appears as quotient by the algebraic free proper action of $H^0(\mathcal{O}_Z^*)$, which, as algebraic variety, has dimension $h^0(\mathcal{O}_Z)$. (This also shows a major difference with the curve cases, where the space of effective divisors associated with a bundle has the form $H^0(\mathcal{L})\setminus\{0\}$, and the action is the projectivization action of $\mathbb{C}^*$. In particular, the fibers are projective spaces.) The above relation makes the connection with another major problem/task of the theory, namely determination of possible values of $h^1(Z,\mathcal{L})$.

This 'h^1'–problem can be formulated even independently of the Abel map, let us fix a topological type (say, the resolution graph Γ), and we consider an arbitrary analytic type of singularity and its resolution supported by Γ. Then for fixed Chern class l' and cycle Z we can also consider all the possible line bundles $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$.

The challenge is to determine all the possible values of $h^1(Z,\mathcal{L})$, and understand/organize them in a conceptual way. This can be split in two major steps: in the first case one varies all the analytic structures (both of (X, o) and of the line bundles), in the second case one fixes an analytic structure (X, o) (and one of its resolutions $\widetilde{X}$) and one moves $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$. E.g., in this second case, one can ask for the stratification $\cup_k W_{l',k}$ of $\mathrm{Pic}^{l'}(Z) \simeq H^1(\mathcal{O}_Z)$ by $W_{l',k} = \{\mathcal{L} : h^1(\mathcal{L}) = k\}$. (These are the analogues of the Brill–Noether strata. For the Brill–Noether theory see [ACGH85, Fl10].) Or, one can search for the possible values k when $W_{l',k} \neq \emptyset$.

In the body of the thesis we will provided several bounds and partial results (with sharp lower bounds provided by generic structures).

Though the older previous results in normal surface singularities focus mostly on particular analytic structures (rational, elliptic, weighted homogeneous, splice quotient, etc), and to special line bundles (e.g. of type $\mathcal{O}_Z(l)$), we wish to treat the general case as well, e.g. the case of generic analytic structure or the generic line bundles.

Part of the results are reduced to the case of Abel maps which are dominant. This case is completely characterized in Theorem 3.2.1.1:

Theorem II. *Fix* $-l'$ *from the Lipman cone,* $Z \geq E$*, and consider* $c^{l'} : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$.

(1) $c^{l'}$ *is dominant if and only if* $\chi(-l') < \chi(-l'+l)$ *for all* $0 < l \leq Z$, $l \in L$. *In particular, the fact that* $c^{l'}$ *is dominant is independent of the analytic structure supported by* Γ *and it can be characterized topologically (and explicitly).*

(2) If $c^{l'}$ *is dominant then* $h^1(Z, \mathcal{L}) = 0$ *for generic* $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$.

For fixed and large Z (in which case $\mathrm{Pic}^{l'}(Z) = \mathrm{Pic}^{l'}(\widetilde{X})$) we introduce $\mathcal{S}'_{dom}$ as the set of those Chern classes l' for which $c^{-l'}$ is dominant, and we list several properties of it. It is a semigroup of the topological Lipman semigroup/cone $\mathcal{S}'$, and it has several properties of the analytic semigroups. The study of dominant maps emphasizes again the importance of the study of generic line bundles.

We will list several cohomological properties for the generic line bundle $\mathcal{L}_{gen}$ of $\mathrm{Pic}^{l'}$ (e.g. we determine its h^1 topologically, and we show that this value is a sharp lower bound for any $h^1(\mathcal{L})$). Similarly, the generic line bundle *of the image of the Abel map* $c^{l'}$ is also studied (its h^1 is the codimension of $\mathrm{im}(c^{l'})$ and it is also the sharp lower bound for any $h^1(\mathcal{L})$ with $\mathcal{L} \in \mathrm{im}(c^{l'})$). Upper bounds for $h^1(Z, \mathcal{L})$ are also established. E.g. Theorem 3.3.2.2 and Proposition 3.3.5.1 imply:

Theorem III. *Fix* $Z > 0$.

(I) Fix an arbitrary $l' \in L'$*. Then for any* $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ *one has*

$$h^1(Z, \mathcal{L}) \geq \chi(-l') - \min_{0 \leq l \leq Z,\, l \in L} \chi(-l'+l).$$

Furthermore, if $\mathcal{L}$ *is generic in* $\mathrm{Pic}^{l'}(Z)$ *then the inequality transforms into an equality.*

In particular, $h^*(Z, \mathcal{L})$ *is topological and explicitly computable from* L*, whenever* $\mathcal{L}$ *is generic.*

(II) For any $\mathcal{L} \in \mathrm{im}(c^{l'}) \subset \mathrm{Pic}^{l'}(Z)$ *one has*

$$h^1(Z, \mathcal{L}) \geq h^1(\mathcal{O}_Z) - \dim(\mathrm{im}(c^{l'})) = \mathrm{codim}(\mathrm{im}(c^{l'})).$$

Furthermore, equality holds whenever $\mathcal{L}$ *is generic in the image of* c*.*

The Abel map is compatible with additive structure of the divisors and multiplicative structure of the line bundles. The point is that if we consider a sufficiently high multiple of a Chern class (that is, we replace l' with nl' where $n \gg 0$), then the image of $c^{nl'}$ becomes an affine subspace for each n, and the associated vector subspaces (indexed by n) stabilize, and this stabilized vector subspace depends only on the 'dual-base-support' of l' (see Theorem 3.4.1.9).

This collection of stabilized linear subspaces (as a linear subspace arrangement) and their dimensions become the source of important new analytic invariants. E.g., the dimensions serve as correction terms in our new analytic surgery formulae (see e.g. Theorem 3.4.1.9). If the analytic structure of $(X, 0)$ is 'nice' (e.g. splice quotient), then these correction invariants can be connected with known analytic invariants computable from the Poincaré series of the divisorial filtrations, and in such cases the 'classical' surgery formulae can be recovered or improved.

Similarly as in the case of classical theory of curves we develop the 'duality picture' between divisors and differential forms. This not only describes the Abel map and its tangent map, but it gives a computational tool in concrete examples as well.

When a concrete basis of $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ (dual to $H^1(\mathcal{O}_{\widetilde{X}})$) can be explicitly determined, the Abel map also becomes more transparent, and several of the above listed problems have precise (sometimes even combinatorial) solutions. This is exemplified in the case of superisolated hypersurface singularities.

1.2.2 Generic analytic structures

The goal of this section is to provide topological formulae for several discrete analytic invariants whenever the analytic structure is generic (with respect to a fixed topological type). Regarding this problem very little is known in the present literature. The type of formulae of the topological characterizations and the proofs in the present work are based on the theory of Abel maps.

In order to formulate the invariants and the topological characterizations we need some notation. Let $\widetilde{X} \to X$ be a good resolution with irreducible exceptional curves $\{E_v\}_{v\in\mathcal{V}}$, with resolution graph Γ, negative definite intersection lattice $L = H_2(\widetilde{X}, \mathbb{Z})$, dual lattice $L' = H^2(\widetilde{X}, \mathbb{Z}) \simeq H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$, and discriminant group $H = L'/L$. We assume that the link M of (X, o) is a rational homology sphere, that is, Γ is a tree of rational E_v's. In such a case $H = H_1(M, \mathbb{Z})$ is finite. Usually Z will denote an effective cycle supported on the exceptional curve E. For any Chern class one defines the 'natural line bundle' $\mathcal{O}_{\widetilde{X}}(l') \in \mathrm{Pic}^{l'}(\widetilde{X})$, and its restrictions $\mathcal{O}_Z(l')$, cf. 2.1.4.

In the sequel we fix a topological type, that is, a resolution graph. The topological invariants are read from Γ, or equivalently, from L. The most elementary one is the 'Riemann–Roch' expression $\chi : L' \to \mathbb{Q}$ given by $\chi(l') := -(l', l' - Z_K)/2$, where $Z_K \in L'$ is the anticanonical cycle defined combinatorially by the adjunction formulae.

The list of analytic invariants, associated with a generic analytic type (with respect to the fixed graph), which are described in the present work topologically are the following: $h^1(\mathcal{O}_Z)$, $h^1(\mathcal{O}_Z(l'))$ (with certain restriction on the Chern class l'), — this last one applied for $Z \gg 0$ provides $h^1(\mathcal{O}_{\widetilde{X}})$ and $h^1(\mathcal{O}_{\widetilde{X}}(l'))$ too —, the cohomological cycle of natural line bundles, the multivariable Hilbert and Poincaré series associated with the divisorial filtration, the analytic semigroup, the maximal ideal cycle. See [CDGZ04, CDGZ08, Li69, N99b, N08, N12, O08, Re97] for the definitions and relationships between them, some definitions will be recalled.

Surprisingly, in all the topological characterization we need to use merely χ, however, it is really remarkable the level of complexity and subtlety of the combinatorial expressions/invariants carried by this 'simple' quadratic function. Definitely, this can happen due to the fact that we work over the lattices L and L', and the position of the lattice points with respect to the level sets of χ play the key role. It is a real challenge now to interpret these expressions in terms of lattice cohomology [N08b, N11] or other topological 3–manifold invariants.

Theorem IV. *Fix a resolution graph and assume that the analytic type of $\widetilde{X}$ is generic. Then the following identities hold:*

(a) For any effective cycle $Z \in L_{>0}$

$$h^1(\mathcal{O}_Z) = 1 - \min_{0<l\leq Z, l\in L}\{\chi(l)\}.$$

(b) If $l' = \sum_{v\in\mathcal{V}} l'_v E_v \in L'$ satisfies $l'_v < 0$ for any E_v in the support of Z then

$$h^1(Z, \mathcal{O}_Z(l')) = \chi(-l') - \min_{0\leq l\leq Z, l\in L}\{\chi(-l'+l)\}.$$

(For a characterization valid for more general Chern classes l' see section 4.4.)

(c) If $p_g(X,o) = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is the geometric genus of (X,o) then

$$p_g(X,o) = 1 - \min_{l\in L_{>0}}\{\chi(l)\} = -\min_{l\in L}\{\chi(l)\} + \begin{cases} 1 & \text{if } (X,o) \text{ is not rational,} \\ 0 & \text{else.} \end{cases}$$

(d) More generally, for any $l' \in L'$

$$h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(l')) = \chi(-l') - \min_{l\in L_{\geq 0}}\{\chi(-l'+l)\} + \begin{cases} 1 & \text{if } l' \in L_{\geq 0} \text{ and } (X,o) \text{ is not rational,} \\ 0 & \text{else.} \end{cases}$$

(e) Let $H(\mathbf{t}) = \sum_{l'\in L'} \mathfrak{h}(l')\mathbf{t}^{l'}$ be the multivariable equivariant Hilbert series associated

with the divisorial filtration. Write l' as r_h+l_0 for some $l_0 \in L$ and $r_h \in L'$ the unique representative of $h=[l']$ in the semi-open cube of L'. Then $\mathfrak{h}(r_h)=0$ for $l_0=0$. Furthermore, for $l_0>0$ and $h\neq 0$

$$\mathfrak{h}(l') = \min_{l\in L_{\geq 0}}\{\chi(l'+l)\} - \min_{l\in L_{\geq 0}}\{\chi(r_h+l)\}.$$

For $h=0$ and $l'=l_0>0$

$$\mathfrak{h}(l_0) = \min_{l\in L_{\geq 0}}\{\chi(l_0+l)\} - \min_{l\in L_{\geq 0}}\{\chi(l)\} + \begin{cases} 1 & \textit{if } (X,o) \textit{ is not rational,} \\ 0 & \textit{else.} \end{cases}$$

(f) Write the multivariable equivariant Poincaré series $P(\mathbf{t}) = -H(\mathbf{t})\cdot\prod_{v\in\mathcal{V}}(1-t_v^{-1})$ as $\sum_{l'\in\mathcal{S}'}\mathfrak{p}(l')\mathbf{t}^{l'}$. It is supported in the Lipman (antinef) cone, in particular in $L'_{\geq 0}$. Then $\mathfrak{p}(0)=1$ and for $l'>0$ one has

$$\mathfrak{p}(l') = \sum_{I\subset\mathcal{V}}(-1)^{|I|+1}\min_{l\in L_{\geq 0}}\chi(l'+l+E_I).$$

(g) Consider the analytic semigroup $\mathcal{S}'_{an} := \{l'\in L' : \mathcal{O}_{\widetilde{X}}(l')$ has no fixed components $\}$. Then

$$\mathcal{S}'_{an} = \{l' : \chi(l') < \chi(l'+l) \textit{ for any } l\in L_{>0}\}\cup\{0\}.$$

(h) Assume that Γ is a non–rational graph and set $\mathcal{M} = \{Z\in L_{>0} : \chi(Z) = \min_{l\in L}\chi(l)\}$.

Then the unique minimal element of $\mathcal{M}$ is the cohomological cycle, while the unique maximal element of $\mathcal{M}$ is the maximal ideal cycle of $\widetilde{X}$.

The results of the previous section show, that for any analytic singularity and resolution with fixed resolution graph, and for any $\mathcal{L}\in \mathrm{Pic}^{l'}(Z)$, one has $h^1(Z,\mathcal{L}) \geq \chi(-l') - \min_{0\leq l\leq Z,\, l\in L}\chi(-l'+l)$, and equality holds for a generic line bundle $\mathcal{L}_{gen} \in$

$\mathrm{Pic}^{l'}(Z)$.

In particular, for any analytic type, the cohomology numbers of $\mathcal{L}_{gen} \in \mathrm{Pic}^{l'}(Z)$ can be expressed combinatorially. Now, the expectation and our guiding principle is the following: for a generic analytic structure the natural line bundle $\mathcal{O}_Z(l')$ should have the same h^1 as the generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{l'}(Z)$ (associated with any analytic structure). This is the next key technical statement.

Theorem V. *Assume that $\widetilde{X}$ is generic. Under some (necessary) negativity restriction on the Chern class l' (see Theorem 4.3.1.1 and Remark 4.4.1.1(b)) the following facts hold.*

(I) The following facts are equivalent:

(a) $\mathcal{O}_Z(l') \in \mathrm{im}(c^{\tilde{l}})$, where $\mathcal{O}_Z(l')$ is the natural line bundle with Chern class l';

(b) $\mathcal{L}_{gen} \in \mathrm{im}(c^{\tilde{l}})$, where $\mathcal{L}_{gen}$ is a generic line bundle in $\mathrm{Pic}^{\tilde{l}}(Z)$ (that is, $c^{\tilde{l}}$ is dominant);

(c) $\mathcal{O}_Z(l') \in \mathrm{im}(c^{\tilde{l}})$, and for any $D \in (c^{\tilde{l}})^{-1}(\mathcal{O}_Z(l'))$ the tangent map $T_Dc^{\tilde{l}}: T_D\mathrm{ECa}^{\tilde{l}}(Z) \to T_{\mathcal{O}_Z(l')}\mathrm{Pic}^{\tilde{l}}(Z)$ is surjective.

(II) $h^i(Z, \mathcal{O}_Z(l')) = h^i(Z, \mathcal{L}_{gen})$ for $i = 0, 1$ and for a generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{\tilde{l}}(Z)$.

The proof is long and technical (the ‘hard’ part is (a)⇒(c)) and it uses the explicit description of tangent map of $c^{l'}$ in terms of Laufer duality (integration of forms along divisors).

By this result, if $\widetilde{X}$ has generic analytic structure, then the cohomology of natural line bundles can be expressed by the very same topological formula as $\mathcal{L}_{gen}$ with the same Chern class. Then all the formulae of Theorem IV above follow directly.

In the next paragraph we say a few words about ‘generic analytic type’.

1.2.3 Discussion regarding the 'generic analytic type'

Let us comment first what kind of difficulties appear in the definition and study of 'generic' analytic type. The point is that for a fixed topological type the moduli space of all analytic structures supported by that fixed topological type, is not yet described in the literature; hence, we cannot define our generic structure as a generic point of such a space. Laufer in [La73b] characterized those topological types which support only one analytic type, but about the general cases very little is known. Usually, generic structures — when they appeared — were introduced by certain ad-hoc definitions, or only in particular situations. In a slightly different direction a remarkable progress was made by Laufer (see e.g. [La73]) when he defined *local complete deformations* of (resolution of) singularities. This parameter space will be the major tool in our working definition as well.

However, even if one defines a certain 'genericity' notion by eliminating a discriminant from a parameter space (consisting of the pathological objects from the point of view of the discussion), the next hard major task is to exploit from the genericity some key geometric/numerical/cohomological properties. E.g., in the present work this is done via Theorem V.

Laufer in [La77] proved that a generic elliptic singularity has geometric genus $p_g = 1$, but except this almost no other example is known.

Wagreich already in 1970 in [Wa70] defined topologically the 'arithmetical genus' p_a of a normal surface singularity and for any non–rational germ (that is, when $p_g \neq 0$) he proved that $p_a \leq p_g$ (see [Wa70, p. 425]). Though in some (easy) cases was known that they agree, analyzing the existing proofs of the inequality (see e.g. the very short proof in [NO17]), one might think that this inequality for germs with complicated topological types probably is extremely week. However, the point is that in the present note we prove that (contrary to the first naive judgement) the generic analytic structure realizes exactly this p_a. For the other invariants (listed in Theorem

IV) even the corresponding candidates were not on the table.

In fact, even in this thesis we make the selection of a package of analytic invariants (organized around the cohomology of natural line bundles), for which we present the corresponding 'package of topological expressions', and we will treat, say, the Hilbert–Samuel function/multiplicity/embedded-dimension package in a forthcoming manuscript (with rather different type of combinatorial answers).

Usually when we have a parameter space for a family of geometric objects, the 'generic object' might depend essentially on the fact that what kind of geometrical problem we wish to solve, or, what kind of anomalies we wish to avoid. Accordingly, we determine a discriminant space of the non–wished objects, and generic means its complement. In the present work all the discrete analytic invariants we treat are basically guided by the cohomology groups of the natural line bundles (for their definition see [N07], [O04] or 2.1.4 here, they associate in a canonical way a line bundle to any given Chern class). Hence, the discriminant spaces (sitting in the base space of complete deformation spaces of Laufer [La73]) are defined as the 'jump loci' of the cohomology groups of the natural line bundles. We recall the needed results of Laufer regarding complete deformations of some $\widetilde{X}$, and we build on this our working definition of general analytic type.

Note that the natural line bundles are well–defined only if the link is a rational homology sphere. Furthermore, this assumption appeared in the case of Abel maps as well. Hence, we impose this topological restriction all along.

1.2.4 Dimensions of images of Abel maps

Fix a complex normal surface singularity (X, o) and let $\widetilde{X}$ be one of its good resolutions. We assume that the link of (X, o) is a rational homology sphere. Let's fix an effective cycle $Z \geq E$ and Chern class $l' \in -\mathcal{S}'$ and let's look at the Abel map $c^{l'}(Z) : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$.

The image of the Abel map consists of line bundles without fixed components.

The main goal of this section is the computation of $\dim \mathrm{im}(c^{l'}(Z))$ and the deduction of several new consequences. We consider these as necessary steps towards a long–term final goal: the development of the Brill–Noether theory of normal surface singularities.

Though the dimension (l', Z) (and the homotopy type) of the connected complex manifold $\mathrm{ECa}^{l'}(Z)$ is topological (i.e. it depends only on the link, or on the lattice L), the dimension $h^1(\mathcal{O}_Z)$ of the target affine space $\mathrm{Pic}^{l'}(Z)$ depends essentially on the analytic structure: if we fix the topological type (and Z), the cohomology group $H^1(\mathcal{O}_Z)$ usually depends on the chosen analytic structure supported by the fixed topological type. The same is true for both $\dim \mathrm{im}(c^{l'}(Z))$ and $\mathrm{codim}\, \mathrm{im}(c^{l'}(Z))$: though (surprisingly) there is a topological characterisation of those cases when $c^{l'}(Z))$ is dominant, oppositely, the cases e.g. when $c^{l'}(Z))$ is a point or it is a hypersurface have no such topological characterisations. In particular, both integers $\dim \mathrm{im}(c^{l'}(Z))$ and $\mathrm{codim}\, \mathrm{im}(c^{l'}(Z))$ are subtle analytical invariants. In fact, it turns out that $\mathrm{codim}\, \mathrm{im}(c^{l'}(Z))$ equals $h^1(Z, \mathcal{L}^{im}_{gen})$, where $\mathcal{L}^{im}_{gen}$ is a generic line bundle from $\mathrm{im}(c^{l'}(Z))$.

Maybe it is worth to emphasize that in the case of the Abel map associated with a smooth projective curve the dimension of the image is immediate (for this classical case consult e.g. [ACGH85, Fl10]). This (and almost any other comparison) shows the huge technical differences between the classical smooth curve cases and our situation (which, basically, is the Brill–Noether theory of a non–reduced exceptional curve supported by the exceptional set of a surface singularity resolution).

In the body of the thesis we present two inductive algorithm for the computation of $d_Z(l') := \dim \mathrm{im}(c^{l'}(Z))$. The induction follows a sequential blow up procedure starting from the resolution $\widetilde{X}$. Write $-l' = \sum_{v\in\mathcal{V}} a_v E^*_v \in \mathcal{S}' \setminus \{0\}$ (hence each $a_v \in \mathbb{Z}_{\geq 0}$). Then, for every $v \in \mathcal{V}$ with $a_v > 0$ we fix a_v generic points on E_v, say

p_{v,k_v}, $1 \le k_v \le a_v$. Starting from each p_{v,k_v} we consider a sequence of blowing ups: first we blow up p_{v,k_v} and we create the exceptional curve $F_{v,k_v,1}$, then we blow up a generic point of $F_{v,k_v,1}$ and we create $F_{v,k_v,2}$, and we do this, say, $\mathbf{s}_{v,k_v}$ times (an exact bound is given in 5.2.1). We proceed in this way with all points p_{v,k_v}, hence we get $\sum_v a_v$ chains of modifications. Hence, a set of integers $\mathbf{s} = \{\mathbf{s}_{v,k_v}\}_{v\in\mathcal{V},\ 1\le k_v\le a_v}$ provides a modification $\pi_{\mathbf{s}} : \widetilde{X}_{\mathbf{s}} \to \widetilde{X}$. In $\widetilde{X}_{\mathbf{s}}$ we find the exceptional curves $\cup_{v\in\mathcal{V}} E_v \cup \cup_{v,k_v} \cup_{1\le t\le \mathbf{s}_{v,k_v}} F_{v,k_v,t}$. At each level $\mathbf{s}$ we set $Z_{\mathbf{s}} := \pi^*_{\mathbf{s}}(Z)$ and $-l'_{\mathbf{s}} := \sum_{v,k_v} F^*_{v,k_v,\mathbf{s}_{v,k_v}}$ (in $L'(\widetilde{X}_{\mathbf{s}})$, where $F_{v,k_v,0} = E_v$). We also write $d_{\mathbf{s}} := \dim \operatorname{im}(c^{l'_{\mathbf{s}}}(Z_{\mathbf{s}}))$. Note that $d_{\mathbf{0}} = d_Z(l')$, and it turns out that $d_{\mathbf{s}} = 0$ whenever the entries of $\mathbf{s}$ are large enough. (Sometimes we abridge the pair (v, k_v) by (v, k).)

In order to run an induction, for any $\mathbf{s}$ and (v, k) let $\mathbf{s}^{v,k}$ denote that tuple which is obtained from $\mathbf{s}$ by increasing $\mathbf{s}_{v,k}$ by one. The inductive algorithm compares $d_{\mathbf{s}}$ with all possible $d_{\mathbf{s}^{v,k}}$.

Using the fact (cf. the proof of Theorem 5.6.1.1) that $\mathrm{ECa}^{l'_{\mathbf{s}^{v,k}}}(Z_{\mathbf{s}^{v,k}})$ is birational with a codimension one subspace of $\mathrm{ECa}^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})$, with some work we obtain

$$d_{\mathbf{s}} - d_{\mathbf{s}^{v,k}} \in \{0, 1\}. \tag{1.2.4.1}$$

A very subtle part of the theory is to identify all those pairs $(\mathbf{s}, \mathbf{s}^{v,k})$, where the gaps/jumps occur (that is, when the difference in (1.2.4.1) is 0 or 1). The identification of such places carries a deep analytic content (and even if in some cases it can be characterised topologically — e.g., in the case of a generic analytic structure —, it might be guided by rather complicated combinatorial patterns).

Example 1.2.4.2. To create a good intuition for such a phenomenon, let us recall the classical case of Weierstrass points. Let C be a smooth projective complex curve of genus g and let us fix a point $p \in C$. For any $s \in \mathbb{Z}_{\ge 0}$ consider $\ell(s) := h^0(C, \mathcal{O}_C(sp))$. Then $\ell(0) = 1$ and $\ell(2g-1+k) = g+k$ for $k \ge 0$. Moreover, $\ell(s) - \ell(s-1) \in \{0, 1\}$

for any $s \geq 0$. Those s values when this difference is 0 are called the gaps, there are g of them. For a generic point the gaps are $\{1, 2, \dots, g\}$, otherwise p is called a Weierstrass point. For Weierstrass points the set of gaps might depend on the choice of p and on the analytic structure of C. The characterization of all possible gap–sets is still unsettled.

In order to characterize completely our gaps/jump places, we will use *test functions*. For such a test function, say $\tau_{\mathbf{s}}$, we will require the following properties. Firstly, it is a function $\mathbf{s} \mapsto \tau_{\mathbf{s}} \in \mathbb{Z}_{\geq 0}$, such that $d_{\mathbf{s}} \leq \tau_{\mathbf{s}}$ for any $\mathbf{s}$.

Usually, $\tau_{\mathbf{s}}$ is defined by a weaker geometric construction, which approximates/bounds $\mathrm{im}(c^{l'}(Z))$, and which hopefully is easier to compute. Secondly, $t_{\mathbf{s}}$ satisfies the following remarkable *testing property* formulated by the next pattern theorem.

Pattern Theorem. *The sequence of integers $d_{\mathbf{s}}$ are determined inductively as follows:*

(1) $d_{\mathbf{s}} - d_{\mathbf{s}^{v,k}} \in \{0, 1\}$ (cf. (1.2.4.1)),

(2) if for some fixed $\mathbf{s}$ the numbers $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ are not the same, then $d_{\mathbf{s}} = \max_{v,k}\{\, d_{\mathbf{s}^{v,k}}\}$. In the case when all the numbers $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ are the same, then if this common value $d_{\mathbf{s}^{v,k}}$ equals $\tau_{\mathbf{s}}$, then $d_{\mathbf{s}} = \tau_{\mathbf{s}} = d_{\mathbf{s}^{v,k}}$; otherwise $d_{\mathbf{s}} = d_{\mathbf{s}^{v,k}} + 1$.

More precisely, we wish to determine from the collection $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ the term $d_{\mathbf{s}}$ (as a decreasing induction). Using *(1)* this is ambiguous only if all this numbers are the same, say d. In this case $d_{\mathbf{s}}$ can be d or $d+1$. Well, if the inequality (†) $d_{\mathbf{s}} \leq \tau_{\mathbf{s}}$ is not obstructed by the choice of $d_{\mathbf{s}} = d + 1$, then this value is taken. Otherwise it is d. That is, $d_{\mathbf{s}}$ is as large as it can be, modulo *(1)* and (†).

If the Pattern Theorem from above holds, then it turns out (see e.g. Corollary 5.2.1.8) that $d_{\mathbf{s}} = \min_{\mathbf{s} \leq \widetilde{\mathbf{s}}}\{|\widetilde{\mathbf{s}} - \mathbf{s}| + \tau_{\widetilde{\mathbf{s}}}\}$ for any $\mathbf{s}$. (Here $|\mathbf{s}| = \sum_{v,k} s_{v,k_v}$.) In particular,

$$d_Z(l') = d_{\mathbf{0}} = \min_{\mathbf{0} \leq \mathbf{s}}\{|\mathbf{s}| + \tau_{\mathbf{s}}\}. \tag{1.2.4.3}$$

Such type of formulas already appeared in the computation of $d_Z(l')$ for weighted homogeneous singularities (and specific l') in [NN18], case which lead us to the present general case. (The type of formula, and also the conceptual approach behind, can also be compared e.g. with Pflueger's formula regarding the dimension of the Brill–Noether varieties of a generic smooth projective curve C with fixed gonality, cf. [P16, JR17].) Nevertheless, the approach of the testing function (and the corresponding min–type close formulae) is the novelty of the results in the section.

1.2.5 The testing functions for $d_{\mathbf{s}}$

Obviously, the above theorem is valuable only if $\tau_{\mathbf{s}}$ is essentially different than $d_{\mathbf{s}}$ and also if it is computable from other different geometrical behaviours. It is also clear that not any upper bound $d_{\mathbf{s}} \leq \tau_{\mathbf{s}}$ satisfies the testing property *(2)*: this is satisfied only for bounds $\tau(\mathbf{s})$ with very structural relationship, symbiosis with the original $d_{\mathbf{s}}$. Hence it is not easy to find testing functions, they must 'testify' about some deep geometric property: even the existence of computable testing function(s) is really remarkable.

Our first test function is defined as follows. Consider again $Z \geq E$, $l' \in -\mathcal{S}'$ associated with a resolution $\widetilde{X}$, as above. Then, besides the Abel map $c^{l'}(Z)$ one can consider its 'multiples' $\{c^{nl'}(Z)\}_{n\geq 1}$. It turns out that $n \mapsto \dim \operatorname{im}(c^{nl'}(Z))$ is a non-decreasing sequence, $\operatorname{im}(c^{nl'}(Z))$ is an affine subspace for $n \gg 1$, whose dimension $e_Z(l')$ is independent of $n \gg 0$, and essentially it depends only on the E^*–support of l' (i.e., on $I \subset \mathcal{V}$, where $-l' = \sum_{v\in I} a_v E^*_v$ with all $\{a_v\}_{v\in I}$ nonzero). From construction $d_Z(l') \leq e_Z(l')$, however they usually are not the same.

Now, at any step of the tower $\widetilde{X}_{\mathbf{s}}$ one can consider this invariant $e_{Z_{\mathbf{s}}}(l'_{\mathbf{s}})$, an integer denoted by $e_{\mathbf{s}}$.

Theorem 5.2.1.6 (the 'first algorithm') guarantees that $e_{\mathbf{s}}$ is a testing function for $d_{\mathbf{s}}$.

The invariants $\{e_{\mathbf{s}}\}_{\mathbf{s}}$ are still hard to compute (cf. 5.3.1). However, the first algorithm is a necessary intermediate step for the second algorithm, valid for another testing function.

The advantage of the second testing function is that it is defined at the level of $\widetilde{X}$ only. It is based on Laufer's perfect pairing $H^1(\mathcal{O}_Z)\otimes\mathcal{G}_Z\to\mathbb{C}$, where $\mathcal{G}_Z$ denoted the space of classes of forms $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$.

$\mathcal{G}_Z$ has a natural divisorial filtration $\{\mathcal{G}_l\}_{0\leq l\leq Z}$, where $\mathcal{G}_l$ is generated by forms with pole $\leq l$. Its dimension (via Laufer duality) is $h^1(\mathcal{O}_l)$. (For more see [NN18] and 3.5.1 here.) Next, for any $\mathbf{s}$ define the cycle $l_{\mathbf{s}}\in L$ of $\widetilde{X}$ by

$$l_{\mathbf{s}}:=\min\Big\{\sum_{v\in\mathcal{V}}\min_{1\leq k_v\leq a_v}\{\mathbf{s}_{v,k_v}\}E_v, Z\Big\}\in L.$$

Set also $g_{\mathbf{s}}:=\dim\mathcal{G}_{l_{\mathbf{s}}}$ as well. It turns out (see 5.3.1) that $d_{\mathbf{s}}\leq e_{\mathbf{s}}\leq h^1(\mathcal{O}_Z)-g_{\mathbf{s}}$. Usually, the equality $e_{\mathbf{s}}=h^1(\mathcal{O}_Z)-g_{\mathbf{s}}$ rarely happens, however, it happens whenever the testing property requires it! Theorem 5.3.1.2 (the 'second algorithm') says that $h^1(\mathcal{O}_Z)-g_{\mathbf{s}}$ is a testing function for $d_{\mathbf{s}}$ indeed.

The cases of superisolated singularities is exemplified.

The second algorithm has several consequences. E.g., a 'numerical' one, cf. (5.3.1.6):

$$d_Z(l')=\min_{0\leq Z_1\leq Z}\{\,(l',Z_1)+h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})\,\},\ \text{or,}\ \operatorname{codim}\operatorname{im}(c^{l'}(Z))=\max_{0\leq Z_1\leq Z}\{\,h^1(\mathcal{O}_{Z_1})-(l',Z_1)\,\}.$$

The cycles Z_1 for which the above minimum is realized have several additional geometric properties (cf. Lemma 5.3.1.13 and 5.3.2). In particular, such a Z_1 imposes the following conceptual consequence:

Structure Theorem for the image of the Abel map. *Fix a resolution $\widetilde{X}$, a cycle $Z\geq E$ and a Chern class $l'\in-\mathcal{S}'$ as above. Then there exists an effective cycle $Z_1\leq Z$, such that: (i) the map $\mathrm{ECa}^{l'}(Z)\to H^1(Z_1)$ is birational onto its image,* *and (ii) the generic fibres of the restriction of* r, $r^{im}:\mathrm{im}(c^{l'}(Z))\to\mathrm{im}(c^{l'}(Z_1))$, *have dimension* $h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})$. *In particular, for any such* Z_1, *the space* $\mathrm{im}(c^{l'}(Z))$ *is birationally equivalent with an affine fibration over* $\mathrm{ECa}^{l'}(Z_1)$ *with affine fibers of dimension* $h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})$.

1.2.6 The case of generic analytic structure

In section 5.3.3 we prove that if $\widetilde{X}$ has a generic analytic structure (in the sense of [La73, NN18]), and $Z\geq E$ and $l'\in -\mathcal{S}'$ then both $\dim\mathrm{im}(c^{l'}(Z))$ and $\mathrm{codim}\,\mathrm{im}(c^{l'}(Z))$ are topological and we have:

$$\mathrm{codim}\,\mathrm{im}(c^{l'}(Z))=\max_{0\leq Z_1\leq Z}\big\{-(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\big\}. \tag{1.2.6.1}$$

The maximum at the right hand side is realized e.g. for the cohomology cycle of $\mathcal{L}^{im}_{gen}\in\mathrm{im}(c^{l'}(Z))\subset\mathrm{Pic}^{l'}(Z)$. Furthermore,

$$h^1(Z,\mathcal{L})\geq\max_{0\leq Z_1\leq Z}\big\{-(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\big\}$$

for any $\mathcal{L}\in\mathrm{im}(c^{l'}(Z))$ and equality holds for generic $\mathcal{L}^{im}_{gen}\in\mathrm{im}(c^{l'}(Z))$.

The identity (1.2.6.1), valid for a generic analytic structure of $\widetilde{X}$, extends to an optimal inequality valid for *any analytic structure*.

Theorem VI. *Consider an* arbitrary *normal surface singularity* (X,o), *its resolution* $\widetilde{X}$, $Z\geq E$ *and* $l'\in-\mathcal{S}'$. *Then* $\mathrm{codim}\,\mathrm{im}(c^{l'}(Z))=h^1(Z,\mathcal{L}^{im}_{gen})$ *satisfies*

$$\mathrm{codim}\,\mathrm{im}(c^{l'}(Z))\geq\max_{0\leq Z_1\leq Z}\big\{-(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\big\}. \tag{1.2.6.2}$$

In particular, for any $\mathcal{L}\in\mathrm{im}(c^{l'}(Z))$ *one also has*

$$h^1(Z,\mathcal{L})\geq h^1(Z,\mathcal{L}^{im}_{gen})=\mathrm{codim}\,\mathrm{im}(c^{l'}(Z))\geq\max_{0\leq Z_1\leq Z}\big\{-(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\big\}.$$

The right hand side of (1.2.6.2) is a sharp topological lower bound for $\operatorname{codim}\operatorname{im}(c^{l'}(Z))$. The inequality (1.2.6.2) can also be interpreted as the semi-continuity statement

$$\begin{aligned}\operatorname{codim}\operatorname{im}(c^{l'}(Z))(\text{arbitrary analytic structure}) \geq \\ \operatorname{codim}\operatorname{im}(c^{l'}(Z))(\text{generic analytic structure}).\end{aligned}$$

1.2.7 Generalization.

Sections 5.5 and 5.6 target generalizations of the previous parts, valid for $\{h^1(Z,\mathcal{L})\}_{\mathcal{L}\in \operatorname{im} c^{l'}(Z)}$, to the shifted case, valid for $\{h^1(Z,\mathcal{L}_0\otimes\mathcal{L})\}_{\mathcal{L}\in \operatorname{im} c^{l'}(Z)}$, where $\mathcal{L}_0\in \operatorname{Pic}^{l'_0}(Z)$ is a fixed bundle without fixed components. In order to run a parallel theory based on Abel maps, we have to create the *new Abel map* $c^{l'}_{\mathcal{L}_0}(Z):\operatorname{ECa}^{l'}(Z)\to \operatorname{Pic}^{l'}_{\mathcal{L}_0}(Z)$, where $\operatorname{Pic}^{l'}_{\mathcal{L}_0}(Z)$ is an affine space associated with the vector space $\operatorname{Pic}^{0}_{\mathcal{L}_0}(Z)\simeq H^1(Z,\mathcal{L}_0)$. ($\operatorname{Pic}^{l'}_{\mathcal{L}_0}(Z)$ appears also as an affine quotient of the classical $\operatorname{Pic}^{l'}(Z)$ as well.) Section 5.5 contains the definitions and the needed exact sequences. Section 5.6 contains the extension of the two algorithms to this situation.

1.2.8 Gorenstein singularities

Let us fix a numerically Gorenstein resolution graph Γ. Recall, that this means that $Z_K\in L$.

From [PPP11] we know, that if Γ is numerically Gorenstein, then there is a Gorenstein surface singularity with resolution $\widetilde{X}$ and resolution graph Γ. The Gorenstein property means that there exists a differential form $\omega\in H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z_K))$, such that ω has a pole on the exceptional divisor of order Z_K but it does not vanish anywhere in $\widetilde{X}\setminus E$. The construction in [PPP11] is given by a very special analytic plumbing.

In this thesis we describe a gluing construction, which for every numerically Gorenstein resolution graph Γ gives a Gorenstein singularity with resolution graph Γ, and furthermore, every Gorenstein singularity with resolution graph Γ can be given by this construction.

Chapter 2

Preliminaries

In the sequel $\#A$ denotes the cardinality of the finite set A.

2.1 Basic notations

In this section we review some basic facts about topological and analytical invariants of surface singularities, and we introduce the needed notations as well.

2.1.1 The resolution

Let (X, o) be the germ of a complex analytic normal surface singularity, and let us fix a good resolution $\phi : \widetilde{X} \to X$ of (X, o). We denote the exceptional curve $\phi^{-1}(o)$ by E, and let $\cup_{v\in\mathcal{V}} E_v$ be its irreducible components. Set also $E_I := \sum_{v\in I} E_v$ for any subset $I \subset \mathcal{V}$. The support of a cycle $l = \sum n_v E_v$ is defined as $|l| = \cup_{n_v \neq 0} E_v$. For more details see [La71, N07, N12, N99b, L13].

2.1.2 Topological invariants

Let Γ be the dual resolution graph associated with ϕ; it is a connected graph. Then $M := \partial\widetilde{X}$ can be identified with the link of (X, o), it is also the oriented plumbed 3–

manifold associated with Γ. It is known that (X,o) locally is homeomorphic with the real cone over M, and M contains the same information as Γ. We will assume that M *is a rational homology sphere*, or, equivalently, Γ is a tree and all genus decorations of Γ are zero. We use the same notation $\mathcal{V}$ for the set of vertices, and δ_v for the valency of a vertex v.

The lattice $L := H_2(\widetilde{X},\mathbb{Z})$ is endowed with the natural intersection form $(\,,\,)$, which is negative definite. L is freely generated by the classes of 2–spheres $\{E_v\}_{v\in\mathcal{V}}$. The dual lattice $L' := H^2(\widetilde{X},\mathbb{Z})$ is generated by the (anti)dual classes $\{E_v^*\}_{v\in\mathcal{V}}$ defined by $(E_v^*, E_w) = -\delta_{vw}$ (where δ_{vw} stays for the Kronecker symbol). The intersection form embeds L into L'. Then $H_1(M,\mathbb{Z}) \simeq L'/L$, and it is abridged by H. Usually one identifies L' with those rational cycles $l' \in L\otimes\mathbb{Q}$ for which $(l', L) \in \mathbb{Z}$, or, $L' = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$.

There is a natural (partial) ordering of L' and L: we write $l'_1 \geq l'_2$ if $l'_1 - l'_2 = \sum_v r_v E_v$ with all $r_v \geq 0$. We set $L_{\geq 0} = \{l \in L \,:\, l \geq 0\}$ and $L_{>0} = L_{\geq 0} \setminus \{0\}$.

Each class $h \in H = L'/L$ has a unique representative $r_h = \sum_v r_v E_v \in L'$ in the semi-open cube (i.e. each $r_v \in \mathbb{Q}\cap[0,1)$), such that its class $[r_h]$ is h.

All the E_v–coordinates of any E_u^* are strictly positive. We define the Lipman cone as $\mathcal{S}' := \{l' \in L' \,:\, (l', E_v) \leq 0 \text{ for all } v\}$. As a monoid it is generated over $\mathbb{Z}_{\geq 0}$ by $\{E_v^*\}_v$.

The *multivariable topological Poincaré series* is the Taylor expansion $Z(\mathbf{t}) = \sum_{l'} z(l')\mathbf{t}^{l'}$ at the origin of the rational function

$$\mathcal{Z}(\mathbf{t}) = \prod_{v\in\mathcal{V}} (1-\mathbf{t}^{E_v^*})^{\delta_v - 2}, \tag{2.1.2.1}$$

where $\mathbf{t}^{l'} := \prod_{v\in\mathcal{V}} t_v^{l'_v}$ for any $l' = \sum_{v\in\mathcal{V}} l'_v E_v \in L'$. By definition, $Z(\mathbf{t})$ is supported on $\mathcal{S}'$, hence $Z(\mathbf{t}) \in \mathbb{Z}[[\mathcal{S}']]$. It has a natural decomposition $Z(\mathbf{t}) = \sum_{h\in H} Z_h(\mathbf{t})$, where $Z_h(\mathbf{t}) = \sum_{[l']=h} z(l')\mathbf{t}^{l'}$. (Though the exponents of $\mathbf{t}^{l'}$ might be rational, that is, $Z(\mathbf{t}) \in \mathbb{Z}[[t_1^{1/d}, \ldots, t_{|\mathcal{V}|}^{1/d}]]$, where $d = \det(\Gamma)$, the right hand side of (2.1.2.1) still will be called 'rational function', and $\sum_{l'} z(l')\mathbf{t}^{l'}$ a 'series'.)

2.1.3 Analytic invariants

In this manuscript we focus mainly on the structure of the Picard group and the holomorphic line bundles on $\widetilde{X}$. The group $\mathrm{Pic}(\widetilde{X}) := H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}})$ of isomorphism classes of *holomorphic* line bundles on $\widetilde{X}$ appears in the exact sequence

$$0 \to \mathrm{Pic}^0(\widetilde{X}) \to \mathrm{Pic}(\widetilde{X}) \xrightarrow{c_1} L' \to 0, \tag{2.1.3.1}$$

where c_1 denotes the first Chern class. Here $\mathrm{Pic}^0(\widetilde{X}) = H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \simeq \mathbb{C}^{p_g}$, where p_g is the *geometric genus* of (X, o). (X, o) is called *rational* if $p_g(X, o) = 0$. Artin in [A62, A66] characterized rationality topologically via the graphs; such graphs are called 'rational'. By this criterion, Γ is rational if and only if $\chi(l) \geq 1$ for any effective non–zero cycle $l \in L_{>0}$. Here $\chi(l) = -(l, l - Z_K)/2$, where $Z_K \in L'$ is the (anti)canonical cycle identified by adjunction formulae $(-Z_K + E_v, E_v) + 2 = 0$ for all v.

2.1.4 Natural line bundles

Let us start again with a good resolution $\phi : (\widetilde{X}, E) \to (X, o)$ of a normal surface singularity with rational homology sphere link, and consider the cohomology exact sequence associated with the exponential exact sequence of sheaves

$$0 \to \mathrm{Pic}^0(\widetilde{X}) \xrightarrow{\epsilon} \mathrm{Pic}(\widetilde{X}) \xrightarrow{c_1} H^2(\widetilde{X}, \mathbb{Z}) \to 0. \tag{2.1.4.1}$$

Here $c_1(\mathcal{L}) \in H^2(\widetilde{X}, \mathbb{Z}) = L'$ is the first Chern class of $\mathcal{L}$. Then, see e.g. [O04, N07], there exists a unique homomorphism (split) $s : L' \to \mathrm{Pic}(\widetilde{X})$ of c_1 such that $c_1 \circ s = id$

and s restricted to L is $l \mapsto \mathcal{O}_{\widetilde{X}}(l)$. The line bundles $s(l')$ are called *natural line bundles* of $\widetilde{X}$, and are denoted by $\mathcal{O}_{\widetilde{X}}(l')$. For several definitions of them see [N07]. E.g., $\mathcal{L}$ is natural if and only if one of its power has the form $\mathcal{O}_{\widetilde{X}}(l)$ for some *integral* cycle $l \in L$ supported on E. Here we recall another construction from [O04, N07], which will be extended later to the deformations space of singularities.

Fix some $l' \in L'$ and let n be the order of its class in L'/L. Then nl' is an integral cycle; its reinterpretation as a divisor supported on E will be denoted by $\mathrm{div}(nl')$. We claim that there exists a divisor $D = D(l')$ in $\widetilde{X}$ such that one has a linear equivalence $nD \sim \mathrm{div}(nl')$ and $c_1(\mathcal{O}_{\widetilde{X}}(D)) = l'$. Furthermore, $D(l')$ is unique up to linear equivalence, hence $l' \mapsto \mathcal{O}_{\widetilde{X}}(D(l'))$ is the wished split of (2.1.4.1). Indeed, since c_1 is onto, there exists a divisor D_1 such that $c_1(\mathcal{O}_{\widetilde{X}}(D_1)) = l'$. Hence $\mathcal{O}_{\widetilde{X}}(nD_1 - \mathrm{div}(nl'))$ has the form $\epsilon(\mathcal{L})$ for some $\mathcal{L} \in \mathrm{Pic}^0(\widetilde{X}) = H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \mathbb{C}^{p_g}$. Define D_2 such that $\mathcal{O}_{\widetilde{X}}(D_2) = \frac{1}{n}\mathcal{L}$ in $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Then $D_1 - D_2$ works. The uniqueness follows from the fact that $\mathrm{Pic}^0(\widetilde{X})$ is torsion free.

The following warning is appropriate. Note that if $\widetilde{X}_1$ is a connected small convenient neighbourhood of the union of some of the exceptional divisors (hence $\widetilde{X}_1$ also stays as the resolution of the singularity obtained by contraction of that union of exceptional curves) then one can repeat the definition of natural line bundles at the level of $\widetilde{X}_1$ as well. However, the restriction to $\widetilde{X}_1$ of a natural line bundle of $\widetilde{X}$ (even of type $\mathcal{O}_{\widetilde{X}}(l)$ with l integral cycle supported on E) usually is not natural on $\widetilde{X}_1$: $\mathcal{O}_{\widetilde{X}}(l')|_{\widetilde{X}_1} \neq \mathcal{O}_{\widetilde{X}_1}(R(l'))$ (where $R : H^2(\widetilde{X}, \mathbb{Z}) \to H^2(\widetilde{X}_1, \mathbb{Z})$ is the natural restriction), though their Chern classes coincide.

In the sequel we will deal with the family of 'restricted natural line bundles' obtained by *restrictions of* $\mathcal{O}_{\widetilde{X}}(l')$. Even if we need to descend to a 'lower level' $\widetilde{X}_1$ with smaller exceptional curve, or to any cycle Z with support included in E (but not necessarily E) our 'restricted natural line bundles' will be associated with Chern classes $l' \in L' = L'(\widetilde{X})$ via the restrictions $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{X}_1)$ or $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(Z)$

of bundles of type $\mathcal{O}_{\widetilde{X}}(l') \in \mathrm{Pic}(\widetilde{X})$. This basically means that we fix a tower of singularities $\{\widetilde{X}_1\}_{\widetilde{X}_1\subset\widetilde{X}}$, or $\{\mathcal{O}_Z\}_{|Z|\subset E}$, determined by the 'top level' $\widetilde{X}$, and all the restricted natural line bundles, even at intermediate levels, are restrictions from the top level.

We use the notations $\mathcal{O}_{\widetilde{X}_1}(l') := \mathcal{O}_{\widetilde{X}}(l')|_{\widetilde{X}_1}$ and $\mathcal{O}_Z(l') := \mathcal{O}_{\widetilde{X}}(l')|_Z$ respectively.

2.1.4.2. One of our main interest is to understand the stratification $\{\mathcal{L} \in \mathrm{Pic}(\widetilde{X}) : h^1(\mathcal{L}) = k\}_{k\in\mathbb{Z}_{\geq 0}}$ of $\mathrm{Pic}(\widetilde{X})$. In the literature about $h^1(\mathcal{L})$ — for arbitrary $\mathcal{L}$ — very little is known. However, about the natural line bundles (of some special analytic structures (X, o)) recently several results were proved, see e.g. [CDGZ04, CDGZ08, N08, N11, N12]. Since some of these facts are used in several examples and play key role in the general presentation we review them in the next subsection.

2.1.4.3. The analytic multivariable Poincaré series is defined as follows [N12], see also [CDGZ04, CDGZ08]. For every $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ (respectively, for $Z \geq E$ and $\mathcal{L} \in \mathrm{Pic}(Z)$) one defines

$$p_{\mathcal{L}} := \sum_{I\subset\mathcal{V}} (-1)^{|I|+1} \dim \frac{H^0(\widetilde{X}, \mathcal{L})}{H^0(\widetilde{X}, \mathcal{L}(-E_I))} \quad \text{and}$$
$$p_{Z,\mathcal{L}} := \sum_{I\subset\mathcal{V}} (-1)^{|I|+1} \dim \frac{H^0(Z, \mathcal{L})}{H^0(Z - E_I, \mathcal{L}(-E_I))}.$$

For $Z \gg 0$ and $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ one has $p_{\mathcal{L}} = p_{Z,\mathcal{L}|_Z}$. If $(c_1(\mathcal{L}), E_v) < 0$ for some $v \in \mathcal{V}$, then $H^0(\widetilde{X}, \mathcal{L}(-E_{I\cup v})) \to H^0(\widetilde{X}, \mathcal{L}(-E_I))$ is an isomorphism for any $I \not\ni v$ (and similar isomorphism holds for any $Z \geq E$), hence

$$p_{\mathcal{L}} = p_{Z,\mathcal{L}} = 0 \quad \text{whenever} \quad c_1(\mathcal{L}) \notin -\mathcal{S}'. \tag{2.1.4.4}$$

At the level of $\widetilde{X}$ one defines a multivariable series as $P_{\mathcal{L}}(\mathbf{t}) := \sum_{l'\in L'} p_{\mathcal{L}(-l')}\mathbf{t}^{l'}$. It also has an H–decomposition $\sum_h P_{\mathcal{L},h}$, $P_{\mathcal{L},h} = \sum_{[l']=h} p_{\mathcal{L}(-l')}\mathbf{t}^{l'}$, according to the classes $[l'] \in H$ of the exponents of $\mathbf{t}^{l'}$. By (2.1.4.4) it is supported on $c_1(\mathcal{L}) + \mathcal{S}'$. We write

$P(\mathbf{t}) := P_{\mathcal{O}_{\widetilde{X}}}(\mathbf{t}) = \sum_{l'} p_{\mathcal{O}_{\widetilde{X}}(-l')}\mathbf{t}^{l'}$.

The first cohomology of the natural line bundles and the series $P(\mathbf{t})$ are linked by the following identity proved in [N12]: for any $l \in L$ one has

$$h^1(\widetilde{X}, \mathcal{O}(-r_h - l)) = -\sum_{a\in L,\, a\not\geq 0} p_{\mathcal{O}(-r_h-l-a)} + p_g(X_{ab}, o)_h + \chi(l) - (l, r_h). \quad (2.1.4.5)$$

2.1.4.6. Recently there is an intense activity in the comparison of the analytic invariant $P(\mathbf{t})$ and the topological $Z(\mathbf{t})$ (their coincidence imply e.g. the so-called Seiberg–Witten Invariant Conjecture [N11, N12]). For the equality of $P(\mathbf{t})$ and $Z(\mathbf{t})$ for certain families singularities (rational, weighted homogeneous, splice quotient) see e.g. [CDGZ04, CDGZ08, N08, N12] and the references therein.

We emphasize that in the previous results in the literature the main goal mostly was to characterize for special ('nice') analytic structures the sheaf–theoretical invariants $h^1(\mathcal{L})$ topologically, and those methods were applicable only for natural line bundles $\mathcal{L}$. In the present note our goal is to treat $h^1(\mathcal{L})$ for any line bundle and for any analytic structure.

2.1.5 Minimal cycle, maximal cycle

In the body of the article we will present several examples. In them we will use the following standard notations. We will write $Z_{min} \in L$ for the *minimal* (or fundamental) cycle of Artin, which is the minimal non–zero cycle of $\mathcal{S}' \cap L$ [A62, A66]. Yau's *maximal ideal cycle* $Z_{max} \in L$ is the divisorial part of the pullback of the maximal ideal $\mathfrak{m}_{X,o} \subset \mathcal{O}_{X,o}$, i.e. $\phi^*\mathfrak{m}_{X,o} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-Z_{max}) \cdot \mathcal{I}$, where $\mathcal{I}$ is an ideal sheaf with 0–dimensional support [Y80]. In general $Z_{min} \leq Z_{max}$. Z_{min} can be found by *Laufer's algorithm* [La72]. This algorithm also shows that $h^0(\mathcal{O}_{Z_{min}}) = 1$, hence $h^1(\mathcal{O}_{Z_{min}}) = 1 - \chi(Z_{min})$ is topological.

2.2 Laufer's results

2.2.1 Local deformation spaces

In this subsection we review some results of Laufer regarding deformations of the analytic structure on a resolution space of a normal surface singularity with fixed resolution graph (and deformations of non–reduced analytic spaces supported on exceptional curves) [La73].

First, let us fix a normal surface singularity (X, o) and a good resolution $\phi : (\widetilde{X}, E) \to (X, o)$ with reduced exceptional curve $E = \phi^{-1}(o)$, whose irreducible decomposition is $\cup_{v\in\mathcal{V}} E_v$ and dual graph Γ. Let $\mathcal{I}_v$ be the ideal sheaf of $E_v \subset \widetilde{X}$. Then for arbitrary positive integers $\{r_v\}_{v\in\mathcal{V}}$ one defines two objects, an analytic one and a topological (combinatorial) one. At analytic level, one sets the ideal sheaf $\mathcal{I}(r) := \prod_v \mathcal{I}_v^{r_v}$ and the non–reduces space $Z(r)$ with structure sheaf $\mathcal{O}_{Z(r)} := \mathcal{O}_{\widetilde{X}}/\mathcal{I}(r)$ supported on E.

The topological object is a graph decorated with multiplicities, denoted by $\Gamma(r)$. As a non–decorated graph $\Gamma(r)$ coincides with the graph Γ without decorations. Additionally each vertex v has a 'multiplicity decoration' r_v, and we put also the self–intersection decoration E_v^2 whenever $r_v > 1$. (Hence, the vertex v does not inherit the self–intersection decoration of v if $r_v = 1$). Note that the abstract 1–dimensional analytic space $Z(r)$ determines by its reduced structure the shape of the dual graph Γ, and by its non–reduced structure all the multiplicities $\{r_v\}_{v\in\mathcal{V}}$, and additionally, all the self–intersection numbers E_v^2 for those v's when $r_v > 1$ (see [La73, Lemma 3.1]).

We say that the space $Z(r)$ has topological type $\Gamma(r)$.

Clearly, the analytic structure of (X, o), hence of $\widetilde{X}$ too, determines each 1–dimensional non–reduced space $Z(r)$. The converse is also true in the following sense.

Theorem 2.2.1.1. *[La71, Th. 6.20],[La73, Prop. 3.8] (a) Consider an abstract*

1–dimensional space $Z(r)$, whose topological type $\Gamma(r)$ can be completed to a negative definite graph Γ (or, lattice L). Then there exists a 2–dimensional manifold $\widetilde{X}$ in which $Z(r)$ can be embedded with support E such that the intersection matrix inherited from the embedding $E \subset \widetilde{X}$ is the negative definite lattice L. In particular (since by Grauert theorem [GR62] the exceptional locus E in $\widetilde{X}$ can be contracted to a normal singularity), any such $Z(r)$ is always associated with a normal surface singularity (as above).

(b) Suppose that we have two singularities (X, o) and (X', o) with good resolutions as above with the same resolution graph Γ. Depending solely on Γ, the integers $\{r_v\}_v$ may be chosen so large that if $\mathcal{O}_{Z(r)} \simeq \mathcal{O}_{Z'(r)}$, then $E \subset \widetilde{X}$ and $E' \subset \widetilde{X}'$ have biholomorphically equivalent neighbourhoods via a map taking E to E'. (For a concrete estimate how large r should be see Theorem 6.20 in [La71].)

In particular, in the deformation theory of $\widetilde{X}$ it is enough to consider the deformations of non–reduced spaces of type $Z(r)$.

Fix a non–reduced 1–dimensional space $Z = Z(r)$ with topological type $\Gamma(r)$. Following Laufer and for technical reasons (partly motivated by further applications in the forthcoming continuations of the series of manuscripts) we also choose a closed subspace Y of Z (whose support can be smaller, it can be even empty). More precisely, (Z, Y) locally is isomorphic with $(\mathbb{C}\{x, y\}/(x^a y^b), \mathbb{C}\{x, y\}/(x^c y^d))$, where $a \geq c \geq 0$, $b \geq d \geq 0$, $a > 0$. The ideal of Y in $\mathcal{O}_Z$ is denoted by $\mathcal{I}_Y$.

Definition 2.2.1.2. [La73, Def. 2.1] A deformation of Z, fixing Y, consists of the following data:

(i) There exists an analytic space $\mathcal{Z}$ and a proper map $\lambda : \mathcal{Z} \to Q$, where Q is a manifold containing a distinguished point 0.

(ii) Over a point $q \in Q$ the fiber Z_q is the subspace of $\mathcal{Z}$ determined by the ideal sheaf $\lambda^*(\mathfrak{m}_q)$ (where $\mathfrak{m}_q$ is the maximal ideal of q). Z is isomorphic with Z_0, usually they are identified.

(iii) λ is a trivial deformation of Y (that is, there is a closed subspace $\mathcal{Y} \subset \mathcal{Z}$ and the restriction of λ to $\mathcal{Y}$ is a trivial deformation of Y).

(iv) λ is *locally trivial* in a way which extends the trivial deformation $\lambda|_{\mathcal{Y}}$. This means that for ant $q \in Q$ and $z \in \mathcal{Z}$ there exist a neighborhood W of z in $\mathcal{Z}$, a neighborhood V of z in Z_q, a neighborhood U of q in Q, and an isomorphism $\phi : W \to V \times U$ such that $\lambda|_W = pr_2 \circ \phi$ (compatibly with the trivialization of $\mathcal{Y}$ from (iii)), where pr_2 is the second projection; for more see [loc.cit.].

One verifies that under deformations (with connected base space) the topological type of the fibers Z_q, namely $\Gamma(r)$, stays constant (see [La73, Lemma 3.1]).

Definition 2.2.1.3. [La73, Def. 2.4] A deformation $\lambda : \mathcal{Z} \to Q$ of Z, fixing Y, is complete at 0 if, given any deformation $\tau : \mathcal{P} \to R$ of Z fixing Y, there is a neighbourhood R' of 0 in R and a holomorphic map $f : R' \to Q$ such that τ restricted to $\tau^{-1}(R')$ is the deformation $f^*\lambda$. Furthermore, λ is complete if it is complete at each point $q \in Q$.

Laufer proved the following results.

Theorem 2.2.1.4. *[La73, Theorems 2.1, 2.3, 3.4, 3.6] Let $\theta_{Z,Y} = \mathcal{H}om_Z(\Omega^1_Z, \mathcal{I}_Y)$ be the sheaf of germs of vector fields on Z, which vanish on Y, and let $\lambda : \mathcal{Z} \to Q$ be a deformation of Z, fixing Y.*

(a) If the Kodaira–Spencer map $\rho_0 : T_0Q \to H^1(Z, \theta_{Z,Y})$ is surjective then λ is complete at 0.

(b) If ρ_0 is surjective then ρ_q is surjective for all q sufficiently near to 0.

(c) There exists a deformation λ with ρ_0 bijective. In such a case in a neighbourhood U of 0 *the deformation is essentially unique, and the fiber above q is isomorphic to Z for only at most countably many q in U.*

2.2.1.5. Functoriality. Let Z' be a closed subspace of Z such that $\mathcal{I}_{Z'} \subset \mathcal{I}_Y \subset \mathcal{O}_Z$. Then there is a natural reduction of pairs $(\mathcal{O}_Z, \mathcal{O}_Y) \to (\mathcal{O}_{Z'}, \mathcal{O}_Y)$. Hence, any deformation $\lambda : \mathcal{Z} \to Q$ of Z fixing Y reduces to a deformation $\lambda' : \mathcal{Z}' \to Q$ of Z' fixing Y. Furthermore, if λ is complete then λ' is automatically complete as well (since $H^1(Z, \theta_{Z,Y}) \to H^1(Z', \theta_{Z',Y})$ is onto).

Chapter 3

Effective Cartier divisors and Abel maps

In this chapter we define and investigate the main properties of the space of effective Cartier divisors and Abel maps on normal surface singularities with some examples, like the case of superisolated singulaities.

3.1 Effective Cartier divisors

3.1.1 Basic definitions

For any $Z \in L_{>0}$ let $\mathrm{ECa}(Z)$ be the (moduli) space of analytic effective Cartier divisors on Z. Their supports are zero–dimensional in E. Taking the class of a Cartier divisor provides the *Abel map* $c : \mathrm{ECa}(Z) \to \mathrm{Pic}(Z)$. Let $\mathrm{ECa}^{l'}(Z)$ be the set of effective Cartier divisors with Chern class $l' \in L'(|Z|)$, that is, $\mathrm{ECa}^{l'}(Z) := c^{-1}(\mathrm{Pic}^{l'}(Z))$. Sometimes we denote the restriction of c by $c^{l'} : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$, $l' \in L'(|Z|)$. It is also convenient to use the simplified notation $\mathrm{ECa}^{l'}(Z) := \mathrm{ECa}^{R(l')}(Z)$ and $\mathrm{Pic}^{l'}(Z) := \mathrm{Pic}^{R(l')}(Z)$ for any $l' \in L'$ (where $R : L' \to L'(|Z|)$ is the restriction as above).

For any $Z_2 \geq Z_1 > 0$ (and $l' \in L'$) one has the commutative diagram

$$\begin{array}{ccc} \mathrm{ECa}^{l'}(Z_2) & \longrightarrow & \mathrm{Pic}^{l'}(Z_2) \\ \downarrow & & \downarrow \\ \mathrm{ECa}^{l'}(Z_1) & \longrightarrow & \mathrm{Pic}^{l'}(Z_1) \end{array} \tag{3.1.1.1}$$

Regarding the existence of $\mathrm{ECa}(Z)$ and the Abel map we note the following. First, by a theorem of Artin [A69, 3.8], there exists an affine algebraic variety Y and a point $y \in Y$ such that (Y, y) and (X, o) have isomorphic formal completions. Then, according to Hironaka [Hi65], (Y, y) and (X, o) are analytically isomorphic. In particular, we can regard Z as a projective algebraic scheme. In this algebraic context, $\mathrm{ECa}^{l'}(Z)$ — as an algebraic variety — together with the algebraic Abel map was constructed by Grothendieck [Gro62], see e.g. the article of Kleiman [Kl13] with several comments and citations and the book of Mumford for curves on algebraic surfaces [Mu66]. (Recall that $\mathrm{Pic}(Z) \simeq \mathbb{C}^{h^1(\mathcal{O}_Z)}$ is an affine space.) In particular,

$$c : \mathrm{ECa}(Z) \to \mathrm{Pic}(Z) \ \text{ is algebraic.}$$

(For concrete charts of $\mathrm{ECa}^{l'}(Z)$ see e.g. the proof of theorem 3.1.1.11 and for the Abel map in concrete charts see section 3.5.) Though these spaces are identified by the general theory, in the body of this note we verify directly several properties of them in order to illuminate the peculiarities of the present situation, e.g. we discuss the smoothness and the dimension of $\mathrm{ECa}^{l'}(Z)$ and the structure of the fibers of the Abel map: the related numerical invariants will be crucial in the further discussions. Doing this we develop several special properties of the Abel map in the language of invariants of normal surface singularities; these connections will be exploited deeply.

We write $\mathrm{ECa}(\widetilde{X})$ for the *set* of effective Cartier divisors on $\widetilde{X}$.

3.1.1.2. Let us fix $Z \in L$, $Z > 0$. As usual, we say that $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ has no fixed components if

$$H^0(Z,\mathcal{L})_{\rm reg} := H^0(Z,\mathcal{L}) \setminus \bigcup_{E_v\subset |Z|} H^0(Z-E_v,\mathcal{L}(-E_v)) \tag{3.1.1.3}$$

is non–empty. Here the inclusion of $H^0(Z-E_v,\mathcal{L}(-E_v))$ into $H^0(Z,\mathcal{L})$ is given by the long cohomological exact sequence associated with $0\to\mathcal{L}(-E_v)\to\mathcal{L}\to\mathcal{L}|_{E_v}\to 0$, and it represents the subspace of sections, whose fixed components contain E_v.

Note that $H^0(Z,\mathcal{L})$ is a module over the algebra $H^0(\mathcal{O}_Z)$, hence one has a natural action of $H^0(\mathcal{O}_Z^*)$ on $H^0(Z,\mathcal{L})_{\rm reg}$. For the next lemma see e.g. [Kl05, §3].

Lemma 3.1.1.4. *$\mathcal{L}\in {\rm Pic}^{l'}(Z)$ is in the image of $c^{l'}:{\rm ECa}^{l'}(Z)\to{\rm Pic}^{l'}(Z)$ if and only if $H^0(Z,\mathcal{L})_{\rm reg}\neq\emptyset$. In this case, $c^{-1}(\mathcal{L})=H^0(Z,\mathcal{L})_{\rm reg}/H^0(\mathcal{O}_Z^*)$.*

In the next discussion we assume $Z\geq E$ basically imposed by the easement of the presentation; everything can be adopted for any $Z>0$, see e.g. 3.2.1.4 or 3.3.1.

Note that $H^0(Z,\mathcal{L})_{\rm reg}\neq\emptyset \ \Rightarrow\ H^0(\mathcal{L}|_{E_v})\neq 0\ \forall\, v \Rightarrow\ (l',E_v)\geq 0\ \forall\, v\ \Rightarrow\ l'\in-\mathcal{S}'$. Conversely, if $l'=-\sum_v m_vE^*_v\in-\mathcal{S}'$ (for certain $m_v\in\mathbb{Z}_{\geq 0}$), and $l'\neq 0$, then one can construct for each E_v cuts (local complex discs considered as reduced divisors) in $\widetilde{X}$ intersecting E_v in a generic point and having with it intersection multiplicity m_v. Since $l'\neq 0$ their collection is nonempty, and it provides elements in ${\rm ECa}^{l'}(\widetilde{X})$ and ${\rm ECa}^{l'}(Z)$ respectively (the second one by restriction). However, this collection is empty whenever $l'=0$, hence this special case needs slightly more attention. By definition we declare that ${\rm ECa}^0(Z)$ is a space consisting of a point (what we can call the 'empty divisor'), ${\rm ECa}^0(Z)=\{\emptyset\}$, and $c^0:{\rm ECa}^0(Z)\to{\rm Pic}^0(Z)$ is defined as $c^0(\emptyset)=\mathcal{O}_Z$. Since for $l'=0$ any section from $H^0(Z,\mathcal{L})_{\rm reg}$ trivializes $\mathcal{L}$, one has:

$$H^0(Z,\mathcal{L})_{\rm reg}\neq\emptyset\ \Leftrightarrow\ \mathcal{L}=\mathcal{O}_Z\ \Leftrightarrow\ \mathcal{L}\in{\rm im}(c^0)\qquad (l'=0). \tag{3.1.1.5}$$

Therefore, the above discussions combined provide

$$\mathrm{ECa}^{l'}(Z) \neq \emptyset \quad \Leftrightarrow \quad l' \in -\mathcal{S}'. \tag{3.1.1.6}$$

The action of $H^0(\mathcal{O}_Z^*)$ can be analysed quite explicitly. Note that from the exact sequence

$$0 \to H^0(\mathcal{O}_{Z-E}(-E)) \to H^0(\mathcal{O}_Z) \xrightarrow{r_E} H^0(\mathcal{O}_E) = \mathbb{C} \to 0 \tag{3.1.1.7}$$

one gets that $H^0(\mathcal{O}_Z^*) = r_E^{-1}(\mathbb{C}^*) = H^0(\mathcal{O}_Z) \setminus H^0(\mathcal{O}_{Z-E}(-E))$. In particular, $H^0(\mathcal{O}_Z^*)$, as algebraic variety, has the dimension of the vector space $H^0(\mathcal{O}_Z)$, $\mathbb{P}H^0(\mathcal{O}_Z^*) := H^0(\mathcal{O}_Z^*)/\mathbb{C}^*$ as algebraic variety is isomorphic with $H^0(\mathcal{O}_{Z-E}(-E))$, and $H^0(Z,\mathcal{L})_{\mathrm{reg}}/H^0(\mathcal{O}_Z^*) = \mathbb{P}H^0(Z,\mathcal{L})_{\mathrm{reg}}/\mathbb{P}H^0(\mathcal{O}_Z^*)$. (Here, again, $\mathbb{P}H^0(Z,\mathcal{L})_{\mathrm{reg}}$ by definition denotes $H^0(Z,\mathcal{L})_{\mathrm{reg}}/\mathbb{C}^*$.)

Lemma 3.1.1.8. *Assume that $H^0(Z,\mathcal{L})_{\mathrm{reg}} \neq \emptyset$. Then*

(a) the action of $H^0(\mathcal{O}_Z^*)$ on $H^0(Z,\mathcal{L})_{\mathrm{reg}}$ is algebraic, free and proper;

(b) $\mathbb{P}H^0(Z,\mathcal{L})_{\mathrm{reg}}$ over $\mathbb{P}H^0(Z,\mathcal{L})_{\mathrm{reg}}/\mathbb{P}H^0(\mathcal{O}_Z^*)$ is a principal affine bundle.

Hence, the fiber $c^{-1}(\mathcal{L})$, $\mathcal{L} \in \mathrm{im}(c^{l'})$, is an irreducible quasiprojective smooth variety of dimension

$$h^0(Z,\mathcal{L}) - h^0(\mathcal{O}_Z) = (l', Z) + h^1(Z,\mathcal{L}) - h^1(\mathcal{O}_Z). \tag{3.1.1.9}$$

Proof. For $s \in H^0(Z,\mathcal{L})_{\mathrm{reg}}$ the multiplication by s, $\mathcal{O}_Z \xrightarrow{\cdot s} \mathcal{L}$, is injective, hence induces injections $H^0(\mathcal{O}_Z) \xrightarrow{\cdot s} H^0(\mathcal{L})$ and $H^0(\mathcal{O}_Z^*) \xrightarrow{\cdot s} H^0(\mathcal{L})_{\mathrm{reg}}$. Hence the action is free. Next we prove that the action of $\mathbb{P}H^0(\mathcal{O}_Z^*)$ on $\mathbb{P}H^0(Z,\mathcal{L})_{\mathrm{reg}}$ is proper.

Introduce hermitian metrics in both $H^0(\mathcal{O}_Z)$ and $H^0(Z,\mathcal{L})$. Write $H^0 := H^0(\mathcal{O}_{Z-E}(-E))$ in $H^0(\mathcal{O}_Z)$ and choose $h^\perp$ with $H^0(\mathcal{O}_Z) = H^0 \oplus \mathbb{C}\langle h^\perp\rangle$. Set also $B := \cap_v H^0(Z - E_v, \mathcal{L}(-E_v)) \subset H^0(Z,\mathcal{L})$ and let $B^\perp$ be its unitary complement in $H^0(Z,\mathcal{L})$. Note that $H^0(Z,\mathcal{L}) \setminus B$ is also stable with respect to the action of $H^0(\mathcal{O}_Z^*) = B \oplus \mathbb{C}\langle h^\perp\rangle \setminus B \oplus 0$. Since $H^0(Z,\mathcal{L})_{\mathrm{reg}}$ is open in $H^0(Z,\mathcal{L}) \setminus B$, it is enough to show that $H^0(\mathcal{O}_Z^*)$

acts properly on $H^0(Z,\mathcal{L})\setminus B$. Fix K compact in $H^0(Z,\mathcal{L})\setminus B$ and let K' be its lift to the unit sphere of $H^0(Z,\mathcal{L})$. We need to show that if $h=h^0+h^\perp\in H^0\oplus\mathbb{C}\langle h^\perp\rangle$ and $|h^0|\to\infty$, and $k\in K'$, then the components $(hk)_1+(hk)_2\in B\perp B^\perp$ of hk satisfy $|(hk)_1|/|(hk)_2|\to\infty$. For this note the following facts.

First, $H^0\cdot H^0(Z,\mathcal{L})\subset B$, hence $(h^0k)_2=0$. Next, since K' is compact, $|(h^\perp k)_1|$ and $|(h^\perp k)_2|$ are bounded from above. Finally, since $h^0k\neq 0$, for any h^0 in the unit sphere, the set $\{|h^0k|\}_k$ is bounded from below by a positive number. Hence, whenever $|h^0|\to\infty$ one also has

$$|(hk)_1|/|(hk)_2| = |(h^\perp k)_1+|h^0|\cdot(\frac{h^0}{|h^0|}\cdot k)|/|(h^\perp k)_2|\to\infty\ .$$

(a) implies (b) (since $\mathbb{P}H^0(\mathcal{O}_Z^*)\simeq H^0$ is an affine space) and the equality in (3.1.1.9) follows from Riemann–Roch formula. □

Example 3.1.1.10. Assume that (X,o) is rational, and $l'\in-\mathcal{S}'$. Then $\mathrm{Pic}^{l'}(Z)=0$, hence if $c_1(\mathcal{L})=l'$ then $\mathcal{L}=\mathcal{O}(l')$. Furthermore, $\mathcal{L}$ is basepoint free [Li69, Th. 12.1]. Thus $\mathrm{ECa}^{l'}(Z)=H^0(Z,\mathcal{L})_{\mathrm{reg}}/H^0(\mathcal{O}_Z^*)$ and since the action of $H^0(\mathcal{O}_Z^*)$ is free (cf. 3.1.1.8), $\mathrm{ECa}^{l'}(Z)$ is smooth. Since $h^1(Z,\mathcal{L})=h^1(\mathcal{O}_Z)=0$ (cf. [Li69, N99b]), the dimension of $\mathrm{ECa}^{l'}(Z)$ is (l',Z) (use (3.1.1.9)). Furthermore, its topological Euler characteristic is $\chi_{top}(\mathrm{ECa}^{l'}(Z))=\chi_{top}(\mathbb{P}H^0(Z,\mathcal{L})_{\mathrm{reg}})$, which is the coefficient $z(-l')$ of the multivariable series $Z(\mathbf{t})$ by [CDGZ08, N08, N12].

These facts generalize as follows.

Theorem 3.1.1.11. *If $l'\in-\mathcal{S}'$ then the following facts hold.*

(1) $\mathrm{ECa}^{l'}(Z)$ is a smooth complex (irreducible) variety of dimension (l',Z).

(2) The topological Euler characteristic of $\mathrm{ECa}^{l'}(Z)$ is $z(-l')$. In fact, the natural restriction $r:\mathrm{ECa}^{l'}(Z)\to\mathrm{ECa}^{l'}(E)$ is a locally trivial fiber bundle with fiber isomorphic to an affine space. Hence, the homotopy type of $\mathrm{ECa}^{l'}(Z)$ is independent of the choice of Z and it depends only on the topology of (X,o).

(3) $r : \mathrm{ECa}^{l'}(Z_2) \to \mathrm{ECa}^{l'}(Z_1)$ *is surjective for any* $Z_2 \geq Z_1$.

Proof. As we already said in the first paragraphs of 3.1.1, $\mathrm{ECa}^{l'}(Z)$ is an algebraic variety, cf. [Gro62, Kl13]. We need to construct in the neighbourhood of each Cartier divisor a smooth chart.

First assume that $Z = E$. Then $\mathrm{ECa}^{l'}(E)$ is independent of the self-intersections E_v^2, hence (keeping the analytic type of E, but) modifying the self–intersections into very negative integers, we can assume that the singularity is rational. In this modified case, $\mathrm{ECa}^{l'}(E) = \mathbb{P}(H^0(E, \mathcal{O}(l'))_{\mathrm{reg}})$, see Example 3.1.1.10. Note that $H^0(E, \mathcal{O}(l'))_{\mathrm{reg}}$ is also independent of the self–intersection numbers, hence, in any case, $\mathrm{ECa}^{l'}(E) = \mathbb{P}(H^0(E, \mathcal{O}(l'))_{\mathrm{reg}})$. In particular, $\mathrm{ECa}^{l'}(E)$ is smooth, irreducible and with the required dimension and Euler characteristic, cf. Example 3.1.1.10.

Let us provide some local charts of $\mathrm{ECa}^{l'}(E)$. Fix $D \in \mathrm{ECa}^{l'}(E)$ with support $\{p_i\}_i \subset E$.

If $p_i \in E_v$ is a smooth point of E, then there exists a local neighbourhood U_i of p_i in $\widetilde{X}$ with local coordinates (x, y) such that $\{x = 0\} = E \cap U_i$ and D in U_i is represented by the local Cartier equation $\{y^m\}$ for some $m \in \mathbb{Z}_{>0}$. Then a local neighbourhood $\mathcal{U}_i(E)$ of the divisor $\{y^m\}$ in $\mathrm{ECa}^{-mE_v^*}(E)$ is given by local Cartier divisors $\{y^m + f(y)\}$, where $f \in \mathcal{O}(E \cap U_i)$ is a small perturbation of the zero function, modulo the multiplicative action of $\mathcal{O}^*(E \cap U_i)$. Multiplying y^m by $1 + a_k y^k$ we get that perturbation of type $y^m + \sum_{k \geq 0} a_k y^{k+m}$ constitute the orbit of y^m (or, differently said, $\sum_{k \geq 0} a_k y^{k+m}$ is the tangent space of the orbit). Therefore, the smooth transversal slice to this orbit $(a_i)_{0 \leq i < m} \mapsto \{y^m + \sum_{i<m} a_i y^i\}$ ($|a_i| \ll 1$) provides a smooth chart $\mathcal{U}_i(E)$ of dimension $m = (-mE_v^*, E)$. Here, $-mE_v^*$ is the local contribution in the Chern class l'.

Similarly, if $p_i = E_u \cap E_v$, then there exists a neighbourhood U_i of p_i in $\widetilde{X}$ with local coordinates (x, y) such that $\{x = 0\} = U_i \cap E_v$ and $\{y = 0\} = U_i \cap E_u$, and D in U_i is represented by $\{x^n + y^m\}$ for certain $n, m \in \mathbb{Z}_{>0}$. [Indeed, any Cartier divisor in $\mathbb{C}[[x,y]]/(xy) \simeq \mathcal{O}_{E,p_i}$ can be represented by a local equation in U_i of this type.] Then, a local neighbourhood $\mathcal{U}_i(E)$ of x^n+y^m in $\mathrm{ECa}^{-mE_v^*-nE_u^*}(E)$ is given by $\{x^n+y^m+a_0+\sum_{i\geq 1} a_i x^i+\sum_{i\geq 1} b_i y^i\}$ modulo the action of $\mathcal{O}^*(E\cap U_i)$. The orbit of this action at x^n+y^m is $\{x^n+y^m+\sum_{i>n} a_i x^i+\sum_{i>m} b_i y^i+\lambda(x^n+y^m)\}$, it is smooth. A possible smooth slice of it is $\{x^n+y^m+a_0+\sum_{i=1}^n a_i x^i+\sum_{i=1}^m b_i y^i\}/\{a_n+b_m=0\}$, which is of dimension $(-mE_v^*-nE_u^*,E)$ (the local contribution into (l',E)).

Products of type $\mathcal{U}(D)=\prod_i \mathcal{U}_i(E)$ constitute a local neighbourhood of D in $\mathrm{ECa}^{l'}(E)$.

Consider now an arbitrary $Z\geq E$ and the restriction $r:\mathrm{ECa}^{l'}(Z)\to \mathrm{ECa}^{l'}(E)$. We show that $\mathrm{ECa}^{l'}(Z)$ can be covered by open sets of type $r^{-1}(\prod_i \mathcal{U}_i(E))=\prod_i r_i^{-1}(\mathcal{U}_i(E))$, where r_i is either the restriction $\mathrm{ECa}^{-mE_v^*}(Z)\to \mathrm{ECa}^{-mE_v^*}(E)$ or $\mathrm{ECa}^{-mE_v^*-nE_u^*}(Z)\to \mathrm{ECa}^{-mE_v^*-nE_u^*}(E)$, and each $r_i^{-1}(\mathcal{U}_i(E))$ is a product of $\mathcal{U}_i(E)$ and an affine space.

Indeed, assume first that p_i is a smooth point of E as above, $p_i\in E_v$, and let $N\geq 1$ be the multiplicity of Z along E_v. Then in U_i the local equation of Z is x^N and let us fix a Cartier divisor in $r^{-1}(\mathcal{U}_i(E))$ whose restriction is y^m, represented by $f:=y^m+xg(x,y)$ for some $g\in \mathcal{O}(U_i)/(x^{N-1})$, modulo $\mathcal{O}^*(U_i)/(x^N)$. Multiplication $f(1+a_iy^ix^{N-1})\equiv f+a_iy^{m+i}x^{N-1}$ shows that $f+y^mx^{N-1}\mathcal{O}(U_i) \pmod{(x^N)}$ is in the orbit. Using this fact, and multiplication by $1+a_iy^ix^{N-2}$ one shows that $f+y^mx^{N-2}\mathcal{O}(U_i) \pmod{(x^N)}$ is also in the orbit. By induction, we get that the orbit is $f+y^m\mathcal{O}(U_i) \pmod{(x^N)}$, and it is smooth. A transversal smooth cut (slice) can be parametrized by the chart $\{y^m+\sum_{i<N,\,j<m} a_{ij}x^iy^j\}$, which has dimension $(-mE_v^*,Z)=mN$. For $i>0$ the variables a_{ij} can be chosen as affine coordinates.

More conceptually, in this case, multiplication of f by $1+h$ gives $f+fh$ (mod (x^N)), hence the orbit is identified with $f+\mathrm{ideal}(f,x^N)$, which has a smooth section whose dimension is the codimension of $\mathrm{ideal}(f,x^N)$, that is, the intersection multiplicity $(f,x^N)_{p_i}=mN$.

Similar chart can be found in the case of $p_i=E_u\cap E_v$ as well. Let us use the previous notations, let us fix a divisor $f = x^n + y^m + xyg(x, y)$ whose restriction to E is $x^n + y^m$, and assume that in Z the multiplicities of $\{x = 0\}$ and $\{y = 0\}$ are N and M. Then the orbit is identified with $f + \text{ideal}(f, x^N y^M)$, which has a smooth transversal cut whose dimension is the intersection multiplicity $(f, x^N y^M)_{p_i} = mN + nM$. The $mN + nM$ coordinates of the cut cannot be chosen canonically. We invite the reader to check that these coordinates can be chosen in such a way that first we choose the $m + n$ (local) coordinates of the reduces part (as above in the case $Z = E$) then we can complete them with $m(N - 1) + n(M - 1)$ affine coordinates.

Taking product we obtain charts of type $\prod_i \mathcal{U}_i(Z) := r^{-1}(\prod_i \mathcal{U}_i(E)) = (\prod_i \mathcal{U}_i(E)) \times \mathbb{C}^{(l', Z-E)}$.

(3) follows from the description of the above charts. □

3.1.2 The tangent map of c. The smoothness of $c^{-1}(\mathcal{L})$.

Assume that $\mathcal{L} \in \text{Pic}^{l'}(Z)$ has no fixed components. Fix any $D \in c^{-1}(\mathcal{L}) \subset \text{ECa}^{l'}(Z)$, and let $s \in H^0(Z, \mathcal{L})$ be the section whose divisor is D. Then multiplication by s gives an exact sequence of sheaves

$$0 \to \mathcal{O}_Z \xrightarrow{\cdot s} \mathcal{L} \to \mathcal{O}_D \to 0. \tag{3.1.2.1}$$

Division by s identifies $\mathcal{L}$ by $\mathcal{O}_Z(D)$, hence the above exact sequence can be identified with the exact sequence $0 \to \mathcal{O}_Z \to \mathcal{O}_Z(D) \to \mathcal{O}_D(D) \to 0$ (this is a generalization of the so-called *Mittag–Lefler sequence*, defined for effective divisors on curves).

We emphasize that $\mathcal{O}_D$ is finitely supported. The dimension of $H^0(\mathcal{O}_D)$ is (l', Z).

Proposition 3.1.2.2. *The coboundary homomorphism $\delta^1_D : H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_Z)$ of the cohomological long exacts sequence of (3.1.2.1) can be identified with the tangent map*

$$T_D(c^{l'}) : T_D(ECa^{l'}(Z)) \to T_{\mathcal{L}}(\text{Pic}^{l'}(Z))$$

of $c^{l'}$ at D. Moreover, the Zariski tangent space $T_D(c^{-1}(\mathcal{L}))$ of $c^{-1}(\mathcal{L})$ at D is identified with its kernel, hence (by the cohomological long exact sequence) by $H^0(Z,\mathcal{L})/H^0(\mathcal{O}_Z)$. This shows that $\dim T_D(c^{-1}(\mathcal{L})) = \dim c^{-1}(\mathcal{L})$ at any $D \in c^{-1}(\mathcal{L})$ (cf. (3.1.1.9)), hence $c^{-1}(\mathcal{L})$ is smoothly embedded into $\mathrm{ECa}^{l'}(Z)$, and $c^{-1}(\mathcal{L})$, as a subscheme of $\mathrm{ECa}^{l'}(Z)$, can be identified with $H^0(Z,\mathcal{L})_{\mathrm{reg}}/H^0(\mathcal{O}_Z^*)$.

This fact reformulated shows that δ^1_D induced on $N_D(c^{-1}(\mathcal{L})) := T_D(ECa^{l'}(Z))/T_D(c^{-1}(\mathcal{L}))$, the normal space of $c^{-1}(\mathcal{L}) \subset ECa^{l'}$ at D, is injective.

Proof. See [Mu66, p. 164], or [Kl05, Remark 5.18], or [Kl13, §5]. □

Corollary 3.1.2.3. *If $\dim(\mathrm{ECa}^{l'}(Z)) = 1$ and $c^{l'}$ is not constant then $\mathrm{im}(c^{l'})$ is smooth.*

3.1.3 The special fibers of $c^{l'}$

Though all the fibers of $c^{l'}$ are smooth, still we wish to distinguish certain fibers of $c^{l'}$ with pathological behaviour. There are several types we can consider.

Definition 3.1.3.1. (a) $D \in \mathrm{ECa}^{l'}(Z)$ is called a critical divisor (point) if $\mathrm{rank}(T_Dc) < \mathrm{rank}(T_{D_{gen}}c)$, where $D_{gen} \in \mathrm{ECa}^{l'}(Z)$ is a generic divisor. If $(c^{l'})^{-1}(\mathcal{L})$ contains a critical divisor (point) then $\mathcal{L}$ is a called a critical bundle (value).

(b) We say that $\mathcal{L} \in \mathrm{im}(c^{l'})$ is T–typical ('tangent–map–typical') if the linear subspace $\mathrm{im}(T_D(c^{l'})) \subset T_{\mathcal{L}}\mathrm{Pic}^{l'}(Z)$ is independent of the choice of $D \in c^{-1}(\mathcal{L})$. Otherwise $\mathcal{L}$ is T–atypical.

The prototype of a map with a T–atypical value is the blowing up $c : B \to \mathbb{C}^2$ at the origin $0 \in \mathbb{C}^2$: then 0 is a T–atypical value. For such an example realized by a concrete $c^{l'}$ see 3.1.4.3.

Lemma 3.1.3.2. *For fixed l' and $\mathcal{L} \in \mathrm{im}(c^{l'})$ consider the following properties:*

(i) $\mathcal{L}$ is a T–atypical value of $c^{l'}$,

(ii) $\mathcal{L}$ *is a singular point of the closure* $\overline{\mathrm{im}(c^{l'})}$ *of the image of* $c^{l'}$ *(where* $\overline{\mathrm{im}(c^{l'})}$ *is taken with the reduced structure),*

(iii) $\dim((c^{l'})^{-1}(\mathcal{L}))$ *is strictly larger the the dimension of the generic fiber of* $c^{l'}$*,*

(iv) $\mathcal{L}$ *is critical bundle,*

(v) *any* $D \in (c^{l'})^{-1}(\mathcal{L})$ *is a critical divisor.*

Then (iii) $\Leftrightarrow$ *(iv)* $\Leftrightarrow$ *(v), (i)* $\Rightarrow$ *(iii) and (ii)* $\Rightarrow$ *(iii).*

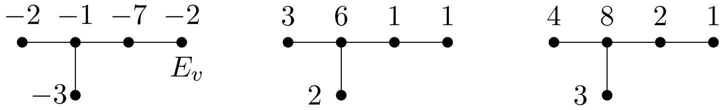
Proof. The equivalences *(iii)* $\Leftrightarrow$ *(iv)* $\Leftrightarrow$ *(v)* follow from Proposition 3.1.2.2. For *(i)* $\Rightarrow$ *(iii)* first notice that $c^{-1}(\mathcal{L})$ is smooth and irreducible, hence it is enough to verify the statement locally at a generic point of $c^{-1}(\mathcal{L})$. On the other hand, if *(iii)* is not true, that is, if (locally) $\mathrm{rank}(T_D c) = \mathrm{rank}(T_{D_{gen}} c)$, then c in that neighbourhood is a fibration, hence (locally) the normal bundle of $c^{-1}(\mathcal{L})$ is a pullback of a vector space V, hence (using also Proposition 3.1.2.2) $\mathrm{im}(T_D(c))$ is constant V.

(ii) $\Rightarrow$ *(iii).* Assume that *(iii)* is not true, hence, as in the previous case, $\mathrm{rank}(T_D c) = \mathrm{rank}(T_{D_{gen}} c)$ for any $D \in c^{-1}(\mathcal{L})$, and c in that neighbourhood is a fibration. $\mathrm{im}(c)$ is the image of the quotient space obtained from the total space by collapsing each fibers into a point. But for any $D \in c^{-1}(\mathcal{L})$ the space $N_D(c^{-1}(\mathcal{L}))$ is mapped by $T_D c$ injectively onto $\mathrm{im}(T_D c)$, and this image is independent of the choice of D (by the proof *(i)* $\Rightarrow$ *(iii)*). This shows that, in fact, $\mathrm{im}(c)$ is immersed at $\mathcal{L}$. Since the fiber $c^{-1}(\mathcal{L})$ is connected, $\mathrm{im}(c)$ is in fact embedded. Hence, $\mathrm{im}(c)$ is smooth at $\mathcal{L}$. □

3.1.4 Examples

Next we exemplify some typical anomalies of the map c.

Example 3.1.4.1. Fix a topological type of singularities (e.g. a resolution graph) and consider different analytic structures realizing it. Then not only the dimension of the target of $c : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$ (that is, $h^1(\mathcal{O}_Z)$) but also the **dimension of the image of** $c^{l'}$ **might depend on the analytic structure of** (X, o). Indeed, let us fix the following graph (picture from the left):



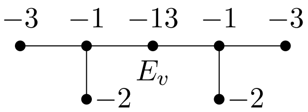
Then (X,o) is a numerically Gorenstein elliptic singularity with $1 \leq p_g \leq 2$; for details regarding elliptic singularities see [N99, N99b]. Set $-l' := Z_{min}$ (the minimal cycle, which equals E_v^*, the cycle shown in the middle diagram), and $Z = Z_K$ (the last diagram), hence $(Z,l') = 1$. Then $\mathrm{ECa}^{l'}(Z) = \mathbb{C}$, and $\mathrm{Pic}^{l'}(Z) = \mathbb{C}^{p_g}$. Write $\mathcal{L} = \mathcal{O}_{Z_K}(-Z_{min})$.

If $p_g = 2$ (hence (X,o) is Gorenstein) then $\mathcal{L}$ has no fixed components [N99, 5.4], and $h^1(Z,\mathcal{L}) = 1$ [N99, 2.20(d)]. Hence $\mathcal{L} \in \mathrm{im}(c)$ and $\dim c^{-1}(\mathcal{L}) = 0$ (use (3.1.1.9)). Therefore, $\dim \mathrm{im}(c) = 1$.

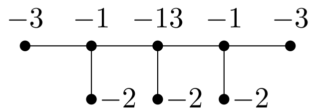
On the other hand, if $p_g = 1$, then $Z_{max} > Z_{min}$, see e.g. [N99, 2.20(f)]. Hence $\mathcal{L}$ has fixed components and $\mathcal{L} \notin \mathrm{im}(c)$. Since the fibers of c are connected (cf. 3.1.1.8), $c : \mathbb{C} \to \mathbb{C}$ (with $\mathcal{L} \notin \mathrm{im}(c)$) cannot be quasi–finite, hence c is constant and $\dim \mathrm{im}(c) = 0$. (This last statement can be deduced from Theorem 3.2.1.1 too, or from 3.4.3 *(i)* $\Leftrightarrow$ *(v)*, where we characterize completely the cases $\dim(\mathrm{im}(c^{l'})) = 0$.)

Example 3.1.4.2. The image of c usually is not closed. We construct such an example in two steps. First, assume that (X,o) is a singularity with topological type given by the graph Γ_1 from the left

Γ_1 :



Γ_2 :



Furthermore, assume that the minimal cycle Z_{min} equals the maximal ideal cycle Z_{max}. In particular, $\mathcal{O}(-Z_{min})$ has no fixed components. For a detailed study of this singularity (and any analytic type with the above graph) see [NO17]. Set $-l' = Z = Z_{min} = E_v^*$, and $\mathcal{L} := \mathcal{O}_Z(-Z)$. Since $(E_v^*, E_v^*) = -1$ (hence $\dim \mathrm{ECa}^{l'}(Z) = 1$), and the $Z(\mathbf{t})$–coefficient $z(E_v^*) = 0$ (hence $\chi(\mathrm{ECa}^{l'}(Z)) = 0$), one has $\mathrm{ECa}^{l'}(Z) = \mathbb{C}^*$. In fact, $\mathrm{ECa}^{l'}(Z)$ is the space of divisors corresponding to the points of $E_v^{reg} :=$

$E_v \setminus \mathrm{Sing}(E) \simeq \mathbb{C}^*$. Using (3.1.1.9) and [NO17, §4] (which shows $h^1(\mathcal{L}) = 1$) one obtains that $\dim c^{-1}(\mathcal{L}) = 0$. Furthermore, $\mathrm{Pic}^{l'}(Z) = \mathbb{C}^2$ (cf. 2.1.5), hence we get an injection $c : \mathbb{C}^* \hookrightarrow \mathbb{C}^2$. For any $q \in E_v^{reg} = \mathbb{C}^*$ we write $\mathcal{L}_q := c^{l'}(q) \in \mathrm{Pic}^{l'}(Z)$.

In fact, $\mathrm{im}(c^{l'})$ can be determined explicitly. Let Γ_l and Γ_r be the subgraphs consisting of the left/right cusp together with v. They determine minimally elliptic singularities with $p_g = 1$, and the corresponding restrictions provide the two coordinates in $\mathrm{Pic}^{l'}(Z)$. Applying [Ha77, 6.11.4] for these two coordinates we get that $\mathrm{im}(c^{l'})$ in some affine coordinates (z_1, z_2) has the form $z_1 z_2 = 1$.

Furthermore, this situation can be used to analyze another singularity (X', o), whose $\mathrm{im}(c')$ equals $\mathrm{im}(c) \setminus \{1 \text{ point}\}$. Fix an arbitrary point $p \in E_v^{reg}$, and glue to the resolution of (X, o) (associated with Γ_1) another irreducible (-2)–exceptional curve E'_p transversally to E_v at p. In this way we create the resolution of a new singularity (X', o) with exceptional curve $E' = \{E'_v\}_v \cup \{E'_p\}$ (with natural notations). The new graph is on the right hand side above.

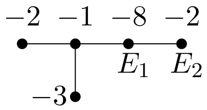
In the new situation we take $-l' = E'^*_v$ and $Z' := Z'_{min} = E'^*_p$. Then $\mathrm{ECa}^{l'}(Z')$ can be identified with $(E'_v)^{reg} = E_v^{reg} \setminus \{p\} = \mathbb{C}^* \setminus \{p\}$, and $c' : \mathbb{C}^* \setminus \{p\} \to \mathbb{C}^2$ with the restriction of c to $\mathbb{C}^* \setminus \{p\}$. (More precisely, for $q \in \mathbb{C}^* \setminus \{p\}$ one has $c'(q)|_Z = \mathcal{L}_q \otimes \mathcal{O}_Z(p)$.) Since c is injective, the image of c' cannot be closed. Via similar construction we can eliminate from the image of c any point.

Example 3.1.4.3. The map c usually is not a locally trivial fibration over its image, in fact, **the fibers of c usually are not even equidimensional**.

Consider the graph Γ_1 from Example 3.1.4.2. It can be realized also by a complete intersection (splice quotient) singularity with $p_g = 3$, cf. [NW90, NO17]. Set $-l' = 2Z_{min} = 2E_v^*$ and $Z = Z_{min}$. Then $\mathrm{ECa}^{l'}(Z)$ is the double symmetric product of E_v^{reg}, namely $\mathbb{C}^* \times \mathbb{C}^*/\mathbb{Z}_2 \simeq \mathbb{C}^* \times \mathbb{C}$. On the other hand, $\mathrm{Pic}^{l'}(Z) = \mathbb{C}^2$. (For numerical cohomological invariants see again [NO17].) It turns out that c is dominant (use e.g. Theorem 3.2.1.1(3)), hence c is birational, with all fibers connected. Since $Z_{max} =$

$2Z_{min}$, $\mathcal{L} = \mathcal{O}_Z(-2Z_{min})$ has no fixed components, hence $\mathcal{L} \in \mathrm{im}(c)$. Furthermore, $h^1(\mathcal{L}) = 1$ (see e.g. [NO17, (5.4)]), hence $\dim c^{-1}(\mathcal{L}) = 1$ by (3.1.1.9) (since $h^1(\mathcal{O}_Z) = 2$ and $(l', Z) = 2$). This can be seen in the following way as well. By Riemann–Roch $h^0(\mathcal{L}) = 2$ and $H^0(\mathcal{O}_Z)^* = \mathbb{C}^*$, hence by 3.1.1.8 $c^{-1}(\mathcal{L})$ is 1–dimensional. In particular, the fibers of c are not equidimensional. (Furthermore, one can show that $\mathrm{im}(c)$ is homeomorphic to $(\mathbb{C}^*)^2 \cup \{(0,0)\}$, where $(0,0)$ corresponds to $\mathcal{L}$. The map c has the following description. Take the blow up $b : B\mathbb{C}^2 \to \mathbb{C}^2$ of $\mathbb{C}^2$ at the origin. Let L_x and L_y be the strict transforms of $\{x = 0\}$ and $\{y = 0\}$. Then $\mathrm{ECa}^{l'}(Z)$ can be identified with $B\mathbb{C}^2 \setminus (L_x \cup L_y)$ and c with the restriction of b to this space.)

Example 3.1.4.4. Even if all the fibers have the same dimension (and by Theorem 3.1.2.2 they are smooth) **the topology of some fibers might jump**. Take for example the graph



It supports a non–numerically Gorenstein elliptic singularity. Recall that if C denotes the elliptic cycle (here it is supported on the union of all irreducible exceptional curves except E_2), and $(C, Z_{min}) < 0$, then the length of the elliptic sequence is one, cf. [Y79, Y80]. Hence, for any analytic realization, $p_g = 1$. Take $-l' = Z = Z_{min} = E_1^* + E_2^*$. A computation shows that $\mathrm{ECa}^{l'}(Z) = \mathbb{C}^2 \setminus \{0\}$. Then $c : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$ can be identified with the restriction to $\mathbb{C}^2 \setminus \{0\}$ of the linear projection $\mathbb{C}^2 \to \mathbb{C}$. Hence the generic fiber is $\mathbb{C}$ while there is a special fiber $\simeq \mathbb{C}^*$. By this correspondence $\mathrm{Pic}^{l'}(Z) = \mathbb{C}$ is identified by $E_1 \setminus E_{node}$. The generic fibers correspond to the divisors $\{p, q\}$, where $p \in E_1^{reg} \simeq \mathbb{C}^*$, and $q \in E_2^{reg} \simeq \mathbb{C}$; they are sent by c to $p \in E_1^{reg} \subset E_1 \setminus E_{node} \simeq \mathrm{Pic}^{l'}(Z)$. Since q can be any point on E_2^{reg}, the fibers are $\mathbb{C}$. On the other hand, any divisor given by a smooth cut at $E_1 \cap E_2$, transversal to both E_1 and E_2, (parametrized by the slope $\mathbb{C}^*$) is sent by c to $E_1 \cap E_2$, whose fiber is exactly this parameter space $\mathbb{C}^*$.

3.1.5 The topology of the fibers of c and the Poincaré series

Let us analyse again the fibers of $c : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$, $Z \geq E$. Assume that $\mathcal{L} \in \mathrm{im}(c)$. Then $\{H_v := H^0(Z - E_v, \mathcal{L}(-E_v))\}_{v \in \mathcal{V}}$ is a proper linear subspace arrangement in $H^0 := H^0(Z, \mathcal{L})$. For any subset $\emptyset \neq I \subset \mathcal{V}$ write $H_I := \cap_{v \in I} H_v$, and introduce also $H_\emptyset := H^0$. Note that the topological Euler characteristic satisfies $\chi_{top}(\mathbb{P}H_I) = \dim H_I$, hence by the inclusion–exclusion principle one obtains

$$\chi_{\mathrm{top}}(\mathbb{P}(H^0 \setminus \cup_v H_v)) = \sum_{I \subset \mathcal{V}} (-1)^{|I|} \dim H_I = \sum_I (-1)^{|I|+1} \mathrm{codim}(H_I \subset H^0). \quad (3.1.5.1)$$

In particular the analytic invariant $p_{Z,\mathcal{L}}$ (cf. 2.1.4.3) equals the topological Euler characteristic of the corresponding linear subspace arrangement complement, $p_{Z,\mathcal{L}} = \chi_{top}(\mathbb{P}(H^0(Z, \mathcal{L})_{\mathrm{reg}}))$. Using Lemmas 3.1.1.4 and 3.1.1.8 this reads as

$$p_{Z,\mathcal{L}} = \chi_{top}(c^{-1}(\mathcal{L})).$$

This fact links the coefficients of the topological series $Z(\mathbf{t})$ and the numerical analytical invariants $p_{Z,\mathcal{L}}$: the Euler characteristic of the total space $\mathrm{ECa}^{l'}$ is $z(-l')$, while the Euler characteristic of each fiber $c^{-1}(\mathcal{L})$ ($\mathcal{L} \in \mathrm{im}(c)$) is $p_{Z,\mathcal{L}}$.

Example 3.1.5.2. Assume that (X, o) is rational. Then $\mathrm{Pic}^{l'}(Z)$ is a point: if $c_1(\mathcal{L}) = l'$ then $\mathcal{L} = \mathcal{O}(l')$. Hence $\mathrm{ECa}^{l'}$ is the unique fiber $c^{-1}(\mathcal{O}(l'))$. Therefore, $z(-l') = p_{Z,\mathcal{O}(l')}$ ($l' \in -\mathcal{S}'$), or $Z(\mathbf{t}) = P_{Z,\mathcal{O}}(\mathbf{t})$. This generalizes the similar identity proved in [CDGZ04, CDGZ08, N08, N12] valid for $Z \gg 0$ (or, for $\widetilde{X}$).

This identity $Z(\mathbf{t}) = P_{\mathcal{O}_{\widetilde{X}}}(\mathbf{t})$ is valid for a more general family of singularities, namely for splice quotient singularities [N12, N08]. (This family was introduced by Neumann and Wahl in [NW05b, NW05]). This identity reinterpreted in our present language says that for any $-l' \in \mathcal{S}'$ and $Z \gg 0$ the Euler characteristic of the total space $\mathrm{ECa}^{l'}(Z)$ and the Euler characteristic of the very special fiber $c^{-1}(\mathcal{O}(l'))$ (over the unique natural line bundle) coincide.

Conjecture 3.1.5.3. *For a splice quotient singularity and $-l' \in \mathcal{S}'$ the fiber $c^{-1}(\mathcal{O}(l'))$ is a topological deformation retract of $\mathrm{ECa}^{l'}(Z)$.*

A detailed study of the Abel map in the case of splice quotient singularities will appear in one of the parts of the present series of articles.

In the present work we wish to focus (instead/besides of the '$P_{\mathcal{O}} = Z$ identity') on the more complex package of invariants provided by (all the fibers of) c. In particular, we analyse other, less specific fibers as well, e.g. the generic fibers over $\mathrm{im}(c)$.

3.2 When is $c^{l'}$ dominant?

3.2.1 Characterization result, the semigroup $\mathcal{S}'_{dom}$

In order to determine properties of line bundles $\mathcal{L} \in \mathrm{Pic}(Z)$ with given Chern class we need first to understand the situations when $c^{l'}$ is dominant.

Theorem 3.2.1.1. *Fix $l' \in -\mathcal{S}'$, $Z \geq E$ as above, and consider $c^{l'} : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$.*

(1) $c^{l'}$ is dominant if and only if $H^0(Z, \mathcal{L})_{\mathrm{reg}} \neq \emptyset$ for generic $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$.

(2) If $c^{l'}$ is dominant then $h^1(Z, \mathcal{L}) = 0$ for generic $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$.

(3) $c^{l'}$ is dominant if and only if $\chi(-l') < \chi(-l' + l)$ for all $0 < l \leq Z$, $l \in L$. In particular, the fact that $c^{l'}$ is dominant is independent of the analytic structure supported by Γ and it can be characterized topologically (and explicitly).

Proof. For *(1)* use Lemma 3.1.1.4. For *(2)* note that for c dominant the dimension of $\mathrm{ECa}^{l'}(Z)$ is the sum of the dimensions of the generic fiber and of the base (which equals $h^1(\mathcal{O}_Z)$). Hence, by (3.1.1.9) and 3.1.1.11*(1)*, $h^0(Z, \mathcal{L}) = \dim c^{-1}(\mathcal{L}) + h^0(Z) = (l', Z) - h^1(Z) + h^0(Z) = (l', Z) + \chi(Z) = \chi(Z, \mathcal{L})$.

(3) First note that for any cycle $l \in L$, $0 < l \leq Z$, and any $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$, one has

$$\chi(-l') \geq \chi(-l'+l) \ \Leftrightarrow \ \chi(Z, \mathcal{L}) \leq \chi(Z-l, \mathcal{L}(-l)), \tag{3.2.1.2}$$

where, by convention, $\chi(Z-l, \mathcal{L}(-l))$ is zero whenever $l = Z$.

Assume that c is dominant and the equivalent conditions from (3.2.1.2) are satisfied for some l, where $0 < l \leq Z$. Take a generic $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$. Hence $H^0(Z, \mathcal{L})_{\mathrm{reg}} \neq \emptyset$ (cf. part *(1)*) and $\chi(Z, \mathcal{L}) = h^0(Z, \mathcal{L})$ by part *(2)*. Hence $h^0(Z, \mathcal{L}) = \chi(Z, \mathcal{L}) \leq \chi(Z-l, \mathcal{L}(-l)) \leq h^0(Z-l, \mathcal{L}(-l))$. Therefore, by the cohomological exact sequence of $0 \to \mathcal{L}(-l)|_{Z-l} \to \mathcal{L}$, we necessarily have equality $H^0(Z-l, \mathcal{L}(-l)) = H^0(Z, \mathcal{L})$. Then for any E_v in the support of l we also have equality $H^0(Z-E_v, \mathcal{L}(-E_v)) = H^0(Z, \mathcal{L})$, hence $H^0(Z, \mathcal{L})_{\mathrm{reg}} = \emptyset$, which leads to a contradiction.

Assume that $\chi(-l') < \chi(-l'+l)$ for any $0 < l \leq Z$. This, for $l = Z$, implies $\chi(Z, \mathcal{L}) > 0$, hence necessarily $h^0(Z, \mathcal{L}) > 0$ for any $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$. If for a generic $\mathcal{L}$ one has $H^0(Z, \mathcal{L})_{\mathrm{reg}} = \emptyset$, then there exists E_v such that $H^0(Z, \mathcal{L}) = H^0(Z - E_v, \mathcal{L}(-E_v))$. If $H^0(Z-E_v, \mathcal{L}(-E_v))_{\mathrm{reg}} = \emptyset$ again, then we continue the procedure. In this way we obtain a cycle $0 < l \leq Z$ such that $H^0(Z-l, \mathcal{L}(-l)) = H^0(Z, \mathcal{L})$ and $H^0(Z-l, \mathcal{L}(-l))_{\mathrm{reg}} \neq \emptyset$. Note that for $\mathcal{L}$ generic $\mathcal{L}(-l)|_{Z-l} \in \mathrm{Pic}^{l'-l}(Z-l)$ is generic as well. Hence $c^{l'-l}$ is dominant and by *(1)–(2)* $h^1(Z-l, \mathcal{L}(-l)) = 0$. Therefore, $\chi(Z, \mathcal{L}) \leq h^0(Z, \mathcal{L}) = h^0(Z-l, \mathcal{L}(-l)) = \chi(Z-l, \mathcal{L}(-l))$, which by (3.2.1.2) reads as $\chi(-l') \geq \chi(-l'+l)$, a contradiction. $\square$

Example 3.2.1.3. The statement of Theorem 3.2.1.1*(3)* is non–trivial even for $l' = 0$. In this case, since ECa^0 is a point, c^0 is dominant if and only if $\mathrm{Pic}^0(Z)$ is a point, that is, $h^1(\mathcal{O}_Z) = 0$. Hence part *(3)* reads as the following topological characterization of the vanishing of $h^1(\mathcal{O}_Z)$: For any normal surface singularity and any cycle $Z > 0$, $h^1(\mathcal{O}_Z) = 0$ if and only if $\chi(l) > 0$ for any $0 < l \leq Z$. (This is a generalization of the rationality criterion of Artin [A62, A66], which corresponds to $Z \gg 0$.)

Remark 3.2.1.4. Above, we assumed that $Z \geq E$. This is not really necessary: if the support $|Z|$ of Z is smaller then one can restrict all the objects to $|Z|$, and the above statements (and also the next Theorem 3.3.2.2) remain valid. (Along the restriction, $\widetilde{X}$ will be replaced by a small convenient neighbourhood of $\cup_{E_v \subset |Z|} E_v$, and L by $\mathbb{Z}\langle E_v \rangle_{E_v \subset |Z|}$.)

3.2.1.5. The semigroup of dominant Chern classes ($Z \gg 0$). Theorem 3.2.1.1*(3)* motivates the introduction of the following combinatorial set

$$\mathcal{S}'_{dom} := \{-l' \mid \chi(-l') < \chi(-l'+l) \text{ for all } l \in L_{>0}\}.$$

By definition, $-l' \in \mathcal{S}'_{dom}$ if and only if $c^{l'}$ is dominant for $Z \gg 0$.

Sometimes it is more convenient to use the next equivalent form (note the sign modification):

$$\mathcal{S}'_{dom} = \{l' \mid \chi(l) > (l', l) \text{ for all } l \in L_{>0}\}. \tag{3.2.1.6}$$

Lemma 3.2.1.7. *$\mathcal{S}'_{dom}$ has the following properties:*

(i) $\mathcal{S}'_{dom} \subset \mathcal{S}'$.

(ii) $0 \in \mathcal{S}'_{dom}$ iff L is rational. More generally, for $I \subset \mathcal{V}$ and $n_v > 0$ for all $v \in I$, if $\sum_{v \in I} n_v E^*_v \in \mathcal{S}'_{dom}$ then the components of $\cup_{v \notin I} E_v$ are rational. Hence, in general, $\mathcal{S}' \setminus \mathcal{S}'_{dom}$ is infinite.

(iii) $\mathcal{S}' \cap (Z_K/2 + \mathcal{S}'_{\mathbb{Q}}) \subset \mathcal{S}'_{dom}$, where $\mathcal{S}'_{\mathbb{Q}} := \{l' \in L \otimes \mathbb{Q} \,:\, (l', E_v) \leq 0 \text{ for all } v\}$.

(iv) $\mathcal{S}'_{dom}$ is a semigroup (not necessarily with identity element).

(v) $\mathcal{S}'_{dom}$ is an $\mathcal{S}'$–module, that is, if $l'_1 \in \mathcal{S}'_{dom}$, $l'_2 \in \mathcal{S}'$ then $l'_1 + l'_2 \in \mathcal{S}'_{dom}$.

(vi) $\mathcal{S}'_{dom}$ is min*–stable, like $\mathcal{S}'$, that is, if $l'_1, l'_2 \in \mathcal{S}'_{dom}$ then $m := \min\{l'_1, l'_2\} \in \mathcal{S}'_{dom}$.*

Proof. For (i) use (3.1.1.6) or (3.2.1.6). (ii) follow from Artin's criterion. (iii) is clear. For $(iv)-(v)$ use (3.2.1.6): if $\chi(l) > (l'_1, l)$ and $0 > (l'_2, l)$ (cf. (i)), then $\chi(l) > (l'_1 + l'_2, l)$. Next we prove (vi).

We wish to show that $\chi(l) > (m, l)$ for any $l > 0$. Set $x_i = l'_i - m$ $(i = 1, 2)$. Assume first that $l \geq x_1$, and write $l = x_1 + z$. Then from the assumptions $\chi(x_1) \geq (m + x_2, x_1)$ (equality only if $x_1 = 0$) and $\chi(z) \geq (m + x_1, z)$ (equality only if $z = 0$). These added provide $\chi(l) > (m, l) + (x_1, x_2) \geq (m, l)$.

Next assume that $l \not\geq x_1$, and choose $u_1 > 0$ minimal, supported by the support of x_1, such that $l + u_1 \geq x_1$. Then the hypothesis applied for $l'_1 = m + x_1$ gives $\chi(l + u_1 - x_1) \geq (m + x_1, l + u_1 - x_1)$ (equality only if $l + u_1 - x_1 = 0$) and applied for $l'_2 = m + x_2$ gives $\chi(x_1 - u_1) \geq (m + x_2, x_1 - u_1)$ (equality only if $x_1 - u_1 = 0$). These added gives $\chi(l) - (m, l) > (u_1, l + u_1 - x_1) + (x_2, x_1 - u_1) \geq 0$. $\square$

Corollary 3.2.1.8. *(i) For any $-l' \in L'$ there exists a unique minimal $l_{dom} \in L_{\geq 0}$ with $-l' + l_{dom} \in \mathcal{S}'_{dom}$.*

(ii) l_{dom} can be found by the following algorithm (see the analogy with [La72]). We construct a computation sequence $\{z_i\}_{i=0}^{t}$, (where $z_{i+1} = z_i + E_{v(i)}$ for some $v(i) \in \mathcal{V}$) as follows. Fix a generic line bundle $\mathcal{L} \in \mathrm{Pic}^{l'}(\widetilde{X})$. Start with $z_0 = 0$. Assume that z_i is already constructed and consider $\mathcal{L}(-z_i)$. If $H^0(\mathcal{L}(-z_i))$ has no fixed components then stop and z_i is the last term z_t. If $H^0(\mathcal{L}(-z_i))$ has a fixed component, choose one of them, say $E_{v(i)}$, and write $z_{i+} := z_i + E_{v(i)}$ and repeat the algorithm. Then this procedure stops after finitely many steps and $z_t = l_{dom}$.

Proof. (i) Set $\mathcal{D} := (-l' + L_{\geq 0}) \cap \mathcal{S}'_{dom}$. Then $\mathcal{D} \neq \emptyset$ by 3.2.1.7(iii) and it has a unique minimal element by 3.2.1.7(vi).

(ii) We show inductively that $z_i \leq l_{dom}$ and the construction stops exactly when $z_i = l_{dom}$. Note that $z_0 = 0 \leq l_{dom}$. If $z_i = l_{dom}$ then $-l' + z_i \in \mathcal{S}'_{dom}$, hence by Theorem 3.2.1.1(1) $H^0(\mathcal{L}(-z_i))$ has no fixed components, hence we have to stop.

If, by induction $z_i < l_{dom}$, we have to show that the algorithm does not stop and $z_{i+1} \leq l_{dom}$ as well. Indeed, if $-l' + z_i < -l' + l_{dom}$ then $-l' + z_i \notin \mathcal{S}'_{dom}$ by the minimality of l_{dom}, hence by Theorem 3.2.1.1 $H^0(\mathcal{L}(-z_i))$ has fixed components. Hence the procedure continues. Note also that the generic section of $H^0(\mathcal{L}(-l_{dom}))$ has no fixed components, hence the fixed components of $H^0(\mathcal{L}(-z_i))$ should be supported on $l_{dom} - z_i$. Hence $z_i + E_{v(i)} \leq l_{dom}$. $\square$

Remark 3.2.1.9. Though $\mathcal{S}'_{dom}$ is defined above combinatorially/topologically, it shares (see e.g. *(iv)* and *(vi)*) several properties of an *analytic semigroup* associated with an analytic structure supported on Γ. This 'coincidence' will be clarified completely in the forthcoming part [NN18], where we prove that the analytic semigroup associated with the *generic analytic structure* is exactly $\mathcal{S}'_{dom} \cup \{0\}$.

3.3 Cohomology of line bundles and $\dim \mathrm{im}(c^{l'})$

3.3.1 Line bundles with $c_1(\mathcal{L}) \notin -\mathcal{S}'$.

Recall that by (3.1.1.6) $\mathrm{ECa}^{l'}(Z) \neq \emptyset$ iff $l' \in -\mathcal{S}'$. Hence any result based on the Abel map uses $l' \in -\mathcal{S}'$. E.g., in this section we establish a sharp lower bound for $h^1(Z, \mathcal{L})$ whenever $c_1(\mathcal{L}) = l' \in -\mathcal{S}'$. Before we provide that statement we wish to emphasise that this extends automatically to the case of all bundles $\mathcal{L}$, even if $c_1(\mathcal{L}) \notin -\mathcal{S}'$.

Indeed, it is known that for any $x \in L'$ there exist $s(x) = x + l \in L'$ with the following properties: (a) $s(x) \in \mathcal{S}'$, (b) $l \in L_{\geq 0}$, (c) $s(x)$ is minimal with properties (a)-(b). Furthermore, the cycle l can be determined explicitly using a generalized Laufer sequence [N07, Prop. 4.3.3]. One constructs a computation sequence $\{z_i\}_{i=0}^t$, $z_0 = 0$, $z_{i+1} = z_i + E_{v(i)}$ for some $v(i) \in \mathcal{V}$ inductively as follows. If $x + z_i \in \mathcal{S}'$ then one stops, and automatically $i = t$ and $z_i = l$. If there exists E_v with $(x + z_i, E_v) > 0$ then choose $E_{v(i)}$ as such an E_v, and one defines $z_{i+1} = z_i + E_{v(i)}$. Along the computation sequence $i \mapsto \chi(x + z_i)$ is decreasing. Furthermore, if $Z > l$, then the sequence applied for $x = -l' = -c_1(\mathcal{L})$, we get that $h^0(Z - z_i, \mathcal{L}(-z_i))$ is constant, and

$$h^1(Z, \mathcal{L}) = h^1(Z - l, \mathcal{L}(-l)) - \chi(\mathcal{L}|_l) \quad \text{and} \quad c_1(\mathcal{L}(-l)) \in -\mathcal{S}'. \qquad (3.3.1.1)$$

Here, clearly, $\chi(\mathcal{L}|_l) = (l', l) + \chi(l) = \chi(-l' + l) - \chi(-l')$. If $l \not\leq Z$, then one constructs a computation sequence inductively as follows: if $-l' + z_i \in \mathcal{S}'(|Z - z_i|)$ (the Lipman cone associated with the support $|Z - z_i|$) then one stops, otherwise there exists $E_{v(i)}$ (identified as above) supported on $Z - z_i$, which provides $z_{i+1} = z_i + E_{v(i)}$. In particular, for any $\mathcal{L} \in \mathrm{Pic}(Z)$, there exists $l \in L_{\geq 0}$ such that $-c_1(\mathcal{L}(-l)) \in \mathcal{S}'(|Z - l|)$, and (3.3.1.1) holds.

Summarized, the computation of any $h^1(Z, \mathcal{L})$, up to the topology of the graph, can be reduced to the case $-c_1(\mathcal{L}) \in \mathcal{S}'$ (maybe supported on a smaller set).

3.3.2 Semicontinuity.

We emphasise another specific fact as well: since $c^{l'}$ is not proper, the semicontinuity of the dimension of the fiber (with respect to the points of the target) does not follow automatically from the general theory. Nevertheless, we have the following result.

Lemma 3.3.2.1. *$h^0(Z, \mathcal{L})$ and $h^1(Z, \mathcal{L})$ are semicontinuous with respect to $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$. In particular, via (3.1.1.9), $\dim c^{-1}(\mathcal{L})$ is also semicontinuous with respect to $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$.*

Proof. Consider a covering by small balls $\{U_\alpha\}_\alpha$ of $\widetilde{X}$. Since $\mathcal{L}|_{U_\alpha}$ is trivial for any α and $\mathcal{L}$, $H^0(Z, \mathcal{L}) = \ker(\delta_{\mathcal{L}} : \oplus_\alpha H^0(\mathcal{O}_Z|_{U_\alpha}) \to \oplus_{\alpha \neq \beta} H^0(\mathcal{O}_Z|_{U_\alpha \cap U_\beta}))$, where the $\mathcal{L}$–dependence is codified in $\delta_{\mathcal{L}}$. But the corank of the linear map (hence, consequently $h^0(Z, \mathcal{L})$ too) is semicontinuous. The semicontinuity of $h^1(Z, \mathcal{L})$ follows by Riemann–Roch. □

We prove the following sharp semicontinuity inequality.

Theorem 3.3.2.2. *(1) Fix an arbitrary $l' \in L'$. Then for any $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ one has*

$$h^1(Z, \mathcal{L}) \geq \chi(-l') - \min_{0 \leq l \leq Z,\, l \in L} \chi(-l' + l), \quad \text{or, equivalently}$$
$$h^0(Z, \mathcal{L}) \geq \max_{0 \leq l \leq Z,\, l \in L} \chi(Z - l, \mathcal{L}(-l)) = \max_{0 \leq l \leq Z,\, l \in L} \{\, \chi(Z - l) + (Z - l, l' - l) \,\}. \tag{3.3.2.3}$$

Furthermore, if $\mathcal{L}$ is generic in $\mathrm{Pic}^{l'}(Z)$ then in both inequalities we have equality.

In particular, $h^*(Z,\mathcal{L})$ is topological and explicitly computable from L, whenever $\mathcal{L}$ is generic.

(2) Assume that $l' \in -\mathcal{S}'$ and $c^{l'}$ is not dominant. Then the inequalities in (3.3.2.3) are strict for any $\mathcal{L} \in \mathrm{im}(c^{l'})$.

Proof. (1) The two inequalities (and the corresponding equalities) are equivalent by Riemann–Roch. We will prove the statement for h^0. For any l and $\mathcal{L}$ (by a cohomological exact sequence) one has

$$h^0(Z,\mathcal{L}) \geq h^0(Z-l,\mathcal{L}(-l)) \geq \chi(Z-l,\mathcal{L}(-l)), \tag{3.3.2.4}$$

hence the inequality follows. We need to show the opposite inequality for $\mathcal{L}$ generic. Clearly, if $h^0(Z,\mathcal{L}) = 0$, then the opposite inequality follows (take e.g. $l = Z$). Hence, assume $h^0(Z,\mathcal{L}) \neq 0$. Then, as in the proof of Theorem 3.2.1.1, there exists $0 \leq l < Z$ such that $h^0(Z,\mathcal{L}) = h^0(Z-l,\mathcal{L}(-l))$ and $H^0(Z-l,\mathcal{L}(-l))_{\mathrm{reg}} \neq \emptyset$. In this case $l'-l \in -\mathcal{S}'$ by (3.1.1.6) and (by Theorem 3.2.1.1) $h^1(Z-l,\mathcal{L}(-l)) = 0$ as well. Hence $h^0(Z,\mathcal{L}) = \chi(Z-l,\mathcal{L}(-l)) \leq \max_{0\leq l\leq Z}\chi(Z-l,\mathcal{L}(-l))$.

(2) Assume that $h^0(Z,\mathcal{L}) = \max_{0\leq l\leq Z}\chi(Z-l,\mathcal{L}(-l))$. If the max at the right hand side can be realized by a certain $l_0 > 0$ then using (3.3.2.4) for l_0 we get that $h^0(Z,\mathcal{L}) = h^0(Z-l_0,\mathcal{L}(-l_0))$, hence $\mathcal{L}$ has fixed components, that is, $\mathcal{L} \notin \mathrm{im}(c^{l'})$. On the other hand, if the max is realized only by $l = 0$, then $c^{l'}$ is dominant by Theorem 3.2.1.1*(3)*. □

Since $H^1(\widetilde{X},\mathcal{L}) = \lim_{\leftarrow,Z} H^1(Z,\mathcal{L})$, cf. [Ha77, Th. 11.1], we obtain the following.

Corollary 3.3.2.5. *For $l' \in L'$ and any $\mathcal{L} \in \mathrm{Pic}^{l'}(\widetilde{X})$ one has $h^1(\widetilde{X},\mathcal{L}) \geq \chi(-l') - \min_{l\in L_{\geq 0}}\chi(-l'+l)$. Equality holds whenever $\mathcal{L}$ is generic in $\mathrm{Pic}^{l'}(\widetilde{X})$. Furthermore, if $l' \in -\mathcal{S}'$ and $c^{l'}$ is not dominant, then $h^1(\widetilde{X},\mathcal{L}) > \chi(-l') - \min_{l\in L_{\geq 0}}\chi(-l'+l)$ whenever $\mathcal{L} \in \mathrm{im}(c^{l'})$.*

Example 3.3.2.6. Assume that $l' = 0$ and $h^1(\mathcal{O}_Z) \neq 0$. Then c^0 is not dominant, hence $h^1(Z,\mathcal{L}) \geq -\min_{0\leq l\leq Z}\chi(l)$ for any $\mathcal{L}$, and $h^1(\mathcal{O}_Z) \geq 1-\min_{0\leq l\leq Z}\chi(l)$.

Moreover, for generic $\mathcal{L} \in \mathrm{Pic}^0(Z)$ one has $h^1(Z,\mathcal{L}) = -\min_{0\leq l\leq Z}\chi(l)$. This for $Z \gg 0$ and Γ elliptic reads as $h^1(\widetilde{X},\mathcal{L}) = 0$; this fact for minimally elliptic Γ was proved by Laufer in [La77], and for arbitrary elliptic case in [N99].

Example 3.3.2.7. Consider the situation of Corollary 3.3.2.5. For certain topological types one can find for any l' explicitly a cycle $l_{min} \in L_{\geq 0}$ which realizes $\min_{l\in L_{\geq 0}}\chi(-l'+l) = \chi(-l'+l_{min})$. Indeed, consider the construction $x \mapsto x+l = s(x)$ described in 3.3.1. Since χ is decreasing along the sequence, $(*)$ $\chi(s(x)) \leq \chi(x)$. Next, assume e.g. that the lattice has the property that $\chi(l) \geq 0$ for all $l \in L_{\geq 0}$ (hence the graph is either rational or elliptic). Then for any $s \in \mathcal{S}'$ one has $(**)$ $\chi(s) \leq \chi(s+l)$ for all $l \in L_{\geq 0}$.

We claim that for rational and elliptic singularities $\min_{l\in L_{\geq 0}}\chi(-l'+l) = \chi(s(-l'))$.

Indeed, by $(*)$ one has $\chi(-l'+l_{min}) \geq \chi(s(-l'+l_{min}))$, and by the universal property of the operator s one also has $s(-l'+l_{min}) \geq s(-l')$, hence by $(**)$ $\chi(s(-l'+l_{min})) \geq \chi(s(-l'))$.

In particular, for rational and elliptic germs $h^1(\widetilde{X},\mathcal{L}) = \chi(-l') - \chi(s(-l'))$ whenever $\mathcal{L}$ is generic.

See also Corollary 3.3.4.2, where we prove for any (X,o) the existence of a unique minimal cycle with the property of l_{min}.

3.3.3 The subset Van'

In parallel to $\mathcal{S}'_{dom}$ (see 3.2.1.5), Corollary 3.3.2.5 indicates another subset of L':

$$Van' := \{-l' \mid \chi(-l') \leq \chi(-l'+l) \text{ for all } l \in L_{\geq 0}\}. \tag{3.3.3.1}$$

This indexes those cycles $-l'$ for which $h^1(\widetilde{X},\mathcal{L}) = 0$ for generic $\mathcal{L} \in \mathrm{Pic}^{l'}(\widetilde{X})$.

For arbitrary line bundles $\mathcal{L} \in \mathrm{Pic}^{l'}(\widetilde{X})$ the existent vanishing theorems formulate sufficient (but usually not necessary) criterions. E.g., $h^1(\widetilde{X}, \mathcal{L}) = 0$ for any (X, o) whenever $-l' \in Z_K + \mathcal{S}'$ (this is the so-called Grauert-Riemenschneider vanishing) [GrRie70, La72, Ra72], or, for rational (X, o) whenever $-l' \in \mathcal{S}'$ (Lipman's Criterion) [Li69]. Even so, Corollary 3.3.2.5 provides a *necessary and sufficient* vanishing condition for *generic line bundles*, which, surprisingly, is independent of the analytic structure of (X, o). Van' lists precisely the corresponding Chern classes.

For rational singularities (since $h^1(\widetilde{X}, \mathcal{L})$ depends only on $c_1(\mathcal{L})$, cf. [N07, 4.3.3]), $h^1(\widetilde{X}, \mathcal{L}) = 0$ for *any* line bundle with fixed $c_1(\mathcal{L})$ exactly when $-c_1(\mathcal{L}) \in Van'$. This is not valid for more general singularities: $-l' \in Van'$ does not guarantee the vanishing $h^1(\widetilde{X}, \mathcal{L}) = 0$ for non–generic (hence for arbitrary) bundles. E.g., in the elliptic case, $0 \in Van'$, however $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = p_g > 0$.

Though most of the statements of the next lemma will not be needed in this first part of the series of articles, for completeness and further references we list some properties of Van' (which can be compared e.g. with those from Lemma 3.2.1.7). Note that a semigroup module structure of type *(iv)* usually is not studied/observed in vanishing theorems.

Lemma 3.3.3.2. *Van' satisfies the following properties:*

(i) $Van' \subset \{l' \,|\, (l', E_v) \leq 1$ for all $v\}$; in general $Van' \not\subset \mathcal{S}'$ (e.g. for rational singularities each $E_v \in Van'$), furthermore $\mathcal{S}'_{dom} \subset Van'$,

(ii) $0 \in Van'$ iff L is rational or elliptic,

(iii) Van' is not necessarily a semigroup ($2E_v \notin Van'$ if $|\mathcal{V}| > 1$, cf. (i)),

(iv) Van' is closed to the $\mathcal{S}'$–action,

(v) Van' is min*–stable,*

(vi) $Van' \setminus \mathcal{S}'$ might have infinitely many elements (e.g. if $E_v \in Van'$ then $E_v + \mathcal{S}' \subset Van'$ too),

(vii) Van' is not necessarily in the first quadrant, however $Van' \cap L$ is in the

first quadrant for a minimal resolution (hence for $\mathcal{L}$ generic and with $c_1(\mathcal{L}) \in L$, the vanishing $h^1(\widetilde{X}, \mathcal{L})$ implies $c_1(\mathcal{L}) \leq 0$).

Proof. For (i) take $l = E_v$ in (3.3.3.1), and check $h^1(\mathcal{O}(-E_v)) = 0$ for rational germs. For $(iv) - (v)$ repeat the arguments from the proof of 3.2.1.7. For (vii) note that if the graph consists of a (-1) (resp. (-2)) vertex then $-E$ (resp. $-E/2$) is in Van'. On the other hand, if $-l' = x_1 - x_2$, where $x_1, x_2 \in L_{\geq 0}$ have no common E_v in their supports, then $\chi(-l') \leq \chi(-l' + x_2)$ implies $\chi(-x_2) \leq 0$. But, in a minimal graph if $\chi(-x) \leq 0$ and $x \geq 0$ then $x = 0$. Indeed, take $E_v \subset |x|$ such that $(E_v, x) < 0$. Then $\chi(-x + E_v) \leq \chi(-x) \leq 0$. If we continue the procedure, in the last step we get $\chi(-E_w) \leq 0$ for some w, a fact which can happen only if E_w is a (-1)–curve. □

Remark 3.3.3.3. In Theorem 3.3.2.2 (see also Corollary 3.3.2.5 too) the set of 'generic' line bundles $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ which satisfy (3.3.2.3) with equality is not explicit. There exists an open Zariski set for which (3.3.2.3) holds with equality, but this usually is not the complement of $\mathrm{im}(c^{l'})$. In other words, the complement of $\mathrm{im}(c^{l'})$ might have a non–trivial stratification according to the values of $h^1(Z, \mathcal{L})$, and the Zariski open strata corresponds to the 'generic' bundles of Theorem 3.3.2.2.

Indeed, take the graph Γ_1 from Example 3.1.4.2, and consider the splice quotient analytic structure on it (for details see e.g. [NO17]). In particular, $p_g = 3$. Set $Z \gg 0$ (e.g. $Z = Z_K$), and $\mathcal{L} := \mathcal{O}_Z(-Z_{min})$. Since $h^1(\mathcal{O}_{Z_{min}}) = 2$ and $h^1(\widetilde{X}, \mathcal{O}(-Z_{min})) = 1$, one also has $h^1(Z, \mathcal{L}) = 1$. Note also that the maximal ideal cycle Z_{max} is $2Z_{min}$, hence $\mathcal{L} \notin \mathrm{im}(c^{-Z_{min}})$. On the other hand, $\min \chi = \chi(Z_{min}) = -1$, hence $h^1(Z, \mathcal{L}_{gen}) = 0$ for generic bundles $\mathcal{L}_{gen} \in \mathrm{Pic}^{-Z_{min}}(Z)$. Hence, the complement of $\mathrm{im}(c^{-Z_{min}})$ has a non–trivial h^1–stratification.

3.3.4 The cohomology cycle of line bundles

If (X, o) is a singularity with $p_g > 0$, then its cohomology cycle (associated with a fixed resolution ϕ) is the unique minimal cycle $Z_{coh} \in L_{>0}$ such that $p_g = h^1(Z_{coh}, \mathcal{O}_{\widetilde{X}})$.

We extend this definition as follows.

Proposition 3.3.4.1. *(a) Fix a line bundle $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ with $h^1(\widetilde{X}, \mathcal{L}) > 0$. The set $L_{\mathcal{L}} := \{l \in L_{>0} : h^1(l, \mathcal{L}) = h^1(\widetilde{X}, \mathcal{L})\}$ has a unique minimal element, denoted by $Z_{coh}(\mathcal{L})$, called the cohomological cycle of $\mathcal{L}$ (and of ϕ). It has the property that $h^1(l, \mathcal{L}) < h^1(\widetilde{X}, \mathcal{L})$ for any $l \not\geq Z_{coh}(\mathcal{L})$ ($l > 0$).*

(b) Fix $Z > 0$ and $\mathcal{L} \in \mathrm{Pic}(Z)$ with $h^1(Z, \mathcal{L}) > 0$. The set $L_{Z,\mathcal{L}} := \{l \in L,\ 0 < l \leq Z : h^1(l, \mathcal{L}) = h^1(Z, \mathcal{L})\}$ has a unique minimal element, denoted by $Z_{coh}(Z, \mathcal{L})$, called the cohomological cycle of $(Z, \mathcal{L})$. It has the property that $h^1(l, \mathcal{L}) < h^1(Z, \mathcal{L})$ for any $l \not\geq Z_{coh}(Z, \mathcal{L})$ ($0 < l \leq Z$).

Proof. The proof of [Re97, 4.8], valid for $\mathcal{O}_{\widetilde{X}}$, can be adopted to this situation as well. □

If $h^1(\widetilde{X}, \mathcal{L}) = 0$, then by convention $Z_{coh}(\mathcal{L}) = 0$.

Corollary 3.3.4.2. *(a) For any $l' \in L'$ consider the set*

$$L_{l'} := \{l_{min} \in L_{\geq 0} \mid \chi(-l' + l_{min}) = \min_{l \in L_{\geq 0}} \chi(-l' + l)\}.$$

Then $L_{l'}$ has a unique minimal element $Z_{coh}(l')$, which coincides with the cohomological cycle of any generic $\mathcal{L} \in \mathrm{Pic}^{l'}(\widetilde{X})$.

(b) For any $Z > 0$ and $l' \in L'$ consider the set

$$L_{Z,l'} := \{l_{min} \in L,\ 0 \leq l_{min} \leq Z, \mid \chi(-l' + l_{min}) = \min_{0 \leq l \leq Z,\, l \in L} \chi(-l' + l)\}.$$

Then $L_{Z,l'}$ has a unique minimal element $Z_{coh}(Z, l')$, which coincides with the cohomological cycle of any generic $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$.

Proof. Combine Theorem 3.3.2.2 and Proposition 3.3.4.1. □

Corollary 3.3.4.3. *1. Elements of type $-l' + Z_{coh}(l')$ ($l' \in L'$) belong to Van'.*

2. *If* $-l' \leq -l''$ *then* $-l' + Z_{coh}(l') \leq -l'' + Z_{coh}(l'')$ *as well. Furthermore, if* $-l' \leq -l'' \leq -l' + Z_{coh}(l')$ *then* $-l' + Z_{coh}(l') = -l'' + Z_{coh}(l'')$.

Example 3.3.4.4. Assume that L is numerically Gorenstein (that is, $Z_K \in L$). Then by [KN17, Lemma 6] (and $\chi(l) = \chi(Z_K - l)$) one gets $Z_{coh}(l' = 0) \leq Z_K/2$.

3.3.5 The dimension of $\mathrm{im}(c)$

For an arbitrary element $\mathcal{L}$ of the image $\mathrm{im}(c : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z))$ one has $\dim \mathrm{im}(c) + \dim c^{-1}(\mathcal{L}) \geq \dim \mathrm{ECa}^{l'}(Z) = (l', Z)$, with equality whenever $\mathcal{L}$ is a generic element *of the image* $\mathrm{im}(c)$. This combined with Lemma 3.1.1.8(b) gives the following.

Proposition 3.3.5.1. *For any* $\mathcal{L} \in \mathrm{im}(c^{l'}) \subset \mathrm{Pic}^{l'}(Z)$ *one has*

$$h^1(Z, \mathcal{L}) \geq h^1(\mathcal{O}_Z) - \dim(\mathrm{im}(c^{l'})) = \mathrm{codim}(\mathrm{im}(c^{l'})). \tag{3.3.5.2}$$

In (3.3.5.2) equality holds whenever $\mathcal{L}$ *is generic in the image of* c *(that is, generic with the property* $H^0(Z, \mathcal{L})_{\mathrm{reg}} \neq \emptyset$*). This fact and Theorem 3.3.2.2 applied for the generic element of* $\mathrm{im}(c)$ *imply*

$$\mathrm{codim}(\mathrm{im}(c^{l'})) \geq \chi(-l') - \min_{0 \leq l \leq Z} \chi(-l' + l). \tag{3.3.5.3}$$

Furthermore, if $c^{l'}$ *is not dominant then the inequality in (3.3.5.3) is strict.*

In general, the codimension of $\mathrm{im}(c)$ cannot be characterized topologically. Indeed, take e.g. $l' = 0$, then $\mathrm{im}(c)$ is a point with codimension $h^1(\mathcal{O}_Z)$. Moreover, by Example 3.1.4.1, the dimension of $\mathrm{im}(c)$ is not topological either.

3.3.6 Upper bounds for $h^1(Z,\mathcal{L})$.

Theorem 3.3.2.2 and Corollary 3.3.2.5 provide sharp lower bounds for $h^1(Z,\mathcal{L})$ and $h^1(\widetilde{X},\mathcal{L})$. A possible upper bound is given by the next proposition.

Proposition 3.3.6.1. *Fix $Z>0$ and an arbitrary $\mathcal{L}\in \mathrm{Pic}(Z)$ with $l'=c_1(\mathcal{L})\in -\mathcal{S}'$.*

(a) If $h^0(Z,\mathcal{L})=0$ then $h^1(Z,\mathcal{L})\leq -\chi(Z)<h^1(\mathcal{O}_Z)$.

(b) If $H^0(Z,\mathcal{L})_{\mathrm{reg}}\neq\emptyset$ then $h^1(Z,\mathcal{L})\leq h^1(\mathcal{O}_Z)$.

(c) In general, if $h^0(Z,\mathcal{L})\neq 0$ then

$$h^1(Z,\mathcal{L})\leq \max_{0\leq l\leq Z}\{\,h^1(\mathcal{O}_{Z-l})+\chi(-l')-\chi(-l'+l)\,\}\leq h^1(\mathcal{O}_Z)+\chi(-l')-\min_{0\leq l\leq Z}\chi(-l'+l). \tag{3.3.6.2}$$

In particular, by (3.3.2.3) and (3.3.6.2), $h^1(Z,\mathcal{L})$ takes values in an interval of length (at most) $h^1(\mathcal{O}_Z)$.

Note that $h^1(\mathcal{O}_Z)\leq \max_{0\leq l\leq Z}\{\,h^1(\mathcal{O}_{Z-l})+\chi(-l')-\chi(-l'+l)\,\}$ (take $l=0$). Hence *(b)* gives a better bound than *(c)* whenever $H^0(Z,\mathcal{L})_{\mathrm{reg}}\neq\emptyset$. (Examples with $h^1(Z,\mathcal{L})\not\leq h^1(\mathcal{O}_Z)$ exist even for $l'=0$, see e.g. Example 8.2.4 in part II [NN18], when we will treat the generic analytic structures).

Furthermore, *(c)* for $l'=0$ reads as $h^1(Z,\mathcal{L})\leq \max_{0\leq l\leq Z}\{\,h^1(\mathcal{O}_{Z-l})-\chi(l)\,\}$, which for $Z=Z_K\in L$ transforms into $h^1(Z_K,\mathcal{L})\leq \max_{0\leq l\leq Z_K}\{\,h^1(\mathcal{O}_l)-\chi(l)\,\}$ (use $\chi(Z_K-l)=\chi(l)$).

Proof. *(a)* $h^1(Z,\mathcal{L})=-\chi(Z,\mathcal{L})=-\chi(Z)-(Z,l')\leq -\chi(Z)=-h^0(\mathcal{O}_Z)+h^1(\mathcal{O}_Z)$.

(b) Multiplication by a generic $s\in H^0(Z,\mathcal{L})$ gives an exact sequence of sheaves $0\to\mathcal{O}_Z\to\mathcal{L}\to\mathcal{F}\to 0$, where $\mathcal{F}$ is Stein. Hence $H^1(\mathcal{O}_Z)\to H^1(Z,\mathcal{L})$ is onto and $h^1(Z,\mathcal{L})\leq h^1(\mathcal{O}_Z)$.

(c) If l is the fixed divisor of $\mathcal{L}$ supported on E, then from the exact sequence $0\to\mathcal{L}(-l)|_{Z-l}\to\mathcal{L}\to\mathcal{L}|_l\to 0$ we get $h^1(\mathcal{L})=h^1(Z-l,\mathcal{L}(-l))-\chi(\mathcal{L}|_l)$, and $\mathcal{L}(-l)|_{Z-l}$ has no fixed components. Hence $h^1(Z-l,\mathcal{L}(-l))\leq h^1(\mathcal{O}_{Z-l})$ by *(b)*. □

Remark 3.3.6.3. The inequality $h^1(Z, \mathcal{L}) \leq h^1(\mathcal{O}_Z)$, valid for the case when $\mathcal{L}$ has no fixed components, has the following geometric interpretation, cf. (3.1.1.9): $h^1(\mathcal{O}_Z) - h^1(Z, \mathcal{L}) = \mathrm{codim}(c^{-1}(\mathcal{L}) \subset \mathrm{ECa}^{l'}) \geq 0$. The inequality for $\mathcal{L} = \mathcal{O}(-l)$, $l \in L_{>0}$, was already proved in [OWY14, Th. 3.1].

3.3.7 The h^1–stratification of $\mathrm{Pic}^{l'}(Z)$.

Fix $Z > 0$, $l' \in -\mathcal{S}'$ and $k \in \mathbb{Z}$ with

$$\chi(-l') - \min_{0 \leq l \leq Z} \chi(-l' + l) \leq \ k \ \leq h^1(\mathcal{O}_Z) + \chi(-l') - \min_{0 \leq l \leq Z} \chi(-l' + l).$$

Definition 3.3.7.1. For any l' and k as above we set

$$W_{l',k} := \{\mathcal{L} \in \mathrm{Pic}^{l'}(Z) \, : \, h^1(Z, \mathcal{L}) = k\}. \tag{3.3.7.2}$$

From the semicontinuity lemma 3.3.2.1 we automatically have for the closure $\overline{W_{l',k}}$

$$\overline{W_{l',k}} \subset \{\mathcal{L} \in \mathrm{Pic}^{l'}(Z) \, : \, h^1(Z, \mathcal{L}) \geq k\}. \tag{3.3.7.3}$$

These sets constitute the analogs of the Brill–Noether strata defined for projective curves by the Brill–Noether theory, see [ACGH85, Fl10] and the references therein.

Lemmas 3.3.5.1 and 3.1.3.2 have the following consequences.

Corollary 3.3.7.4. *Fix $l' \in -\mathcal{S}'$. Then $\mathrm{im}(c^{l'}) \subset \overline{W_{l', \mathrm{codim}\, \mathrm{im}(c^{l'})}}$. Furthermore, the set of critical bundles of $c^{l'}$ are included in $\overline{W_{l', \mathrm{codim}\, \mathrm{im}(c^{l'})+1}}$.*

Example 3.3.7.5. If the fibers of $c^{l'}$ over $\mathrm{im}(c^{l'})$ are not equidimensional, then $\mathrm{im}(c^{l'})$ consists of more strata of type $W_{l',k}$ (see e.g. Example 3.1.4.3). But, even if the fibers over $\mathrm{im}(c^{l'})$ are equidimensional, hence $\mathrm{im}(c^{l'})$ consists of only one stratum, it can happen that $c^{l'}$ is not a (topological) locally trivial fibration over $\mathrm{im}(c^{l'})$, see e.g.

Example 3.1.4.4. In particular, $c^{l'}$ over a strata $W_{l',k}$ usually is not a (topological) locally trivial fibration.

3.4 'Multiplicative' structures. The 'stable' $\mathrm{im}(c^{l'})$.

3.4.1 Monoid structure of divisors

In this section we will exploit the additional natural additive structure $s^{l'_1,l'_2}(Z)$: $\mathrm{ECa}^{l'_1}(Z)\times\mathrm{ECa}^{l'_2}(Z)\to\mathrm{ECa}^{l'_1+l'_2}(Z)$ $(l'_1,l'_2\in-\mathcal{S}')$ provided by the sum of the divisors. (Sometimes we will abridge $s^{l'_1,l'_2}(Z)$ as s.)

Lemma 3.4.1.1. *$s^{l'_1,l'_2}(Z)$ is dominant and quasi–finite.*

Proof. An effective divisor decomposes in finitely many ways, hence the quasi–finiteness follows. Since the dimensions of the source and the target are equal, cf. Theorem 3.1.1.11, s is dominant. □

In general, s is not surjective. E.g., in Example 3.1.4.4, the elements of $c^{-1}(E_1\cap E_2)=\mathbb{C}^*$ are not in the image of $s^{E_1^*,E_2^*}(Z)$.

There is a parallel multiplication $\mathrm{Pic}^{l'_1}(Z)\times\mathrm{Pic}^{l'_2}(Z)\to\mathrm{Pic}^{l'_1+l'_2}(Z)$, $(\mathcal{L}_1,\mathcal{L}_2)\mapsto\mathcal{L}_1\otimes\mathcal{L}_2$. Clearly, $c^{l'_1+l'_2}\circ s^{l'_1,l'_2}=c^{l'_1}\otimes c^{l'_2}$ in $\mathrm{Pic}^{l'_1+l'_2}$. In the next discussions we replace $c^{l'}$ by the composition

$$\widetilde{c}^{l'}:\mathrm{ECa}^{l'}(Z)\xrightarrow{c^{l'}}\mathrm{Pic}^{l'}(Z)\xrightarrow{\mathcal{O}_Z(-l')}\mathrm{Pic}^0(Z),$$

where the second map is the multiplication by the natural line bundle $\mathcal{O}_Z(-l')$. Since $\mathcal{O}_Z(l'_1+l'_2)=\mathcal{O}_Z(l'_1)\otimes\mathcal{O}_Z(l'_2)$ we also have $\widetilde{c}^{l'_1+l'_2}\circ s^{l'_1,l'_2}=\widetilde{c}^{l'_1}\otimes\widetilde{c}^{l'_2}$ in Pic^0. After identification of Pic^0 with (the additive) $H^1(\mathcal{O}_Z)$, this reads as $\widetilde{c}^{l'_1+l'_2}\circ s^{l'_1,l'_2}=\widetilde{c}^{l'_1}+\widetilde{c}^{l'_2}$ in $H^1(\mathcal{O}_Z)$. The advantage of this new map is that it collects all the images of the effective Cartier divisors in a single vector space $H^1(\mathcal{O}_Z)$. Lemma 3.4.1.1 and the construction imply

$$\mathrm{im}(\widetilde{c}^{l'_1}) + \mathrm{im}(\widetilde{c}^{l'_2}) \subset \mathrm{im}(\widetilde{c}^{l'_1+l'_2}) \subset \overline{\mathrm{im}(\widetilde{c}^{l'_1}) + \mathrm{im}(\widetilde{c}^{l'_2})}. \tag{3.4.1.2}$$

Definition 3.4.1.3. For any $l' \in -\mathcal{S}'$ let $A_Z(l')$ (if there is no confusion, $A(l')$) be the smallest dimensional affine subspace of $H^1(\mathcal{O}_Z)$ which contains $\mathrm{im}(\widetilde{c}^{l'})$. Let $V_Z(l')$ be the parallel vector subspace of $H^1(\mathcal{O}_Z)$, the translation of $A_Z(l')$ to the origin.

Remark 3.4.1.4. From this definition follows that $\dim V_Z(l')$ is greater than or equal to the dimension of the Zariski tangent space at any $\mathcal{L} \in \overline{\mathrm{im}(c^{l'}(Z))}$; in particular, $\dim V_Z(l')) \geq \dim \mathrm{im}(c^{l'}(Z))$. Hence, by (3.3.5.2) one also has $\dim V_Z(l') \geq h^1(\mathcal{O}_Z) - h^1(Z, \mathcal{L})$ for any $\mathcal{L} \in \mathrm{im}(c^{l'}(Z))$.

Example 3.4.1.5. In general, $\mathrm{im}(\widetilde{c}^{l'}) \subsetneq A_Z(l')$; take e.g. the first case of Example 3.1.4.2, when $\dim \mathrm{im}(c^{l'}) = 1$ and $A_Z(l') = \mathbb{C}^2$. (The fact that $A_Z(l') = \mathbb{C}^2$ can be deduced in the following way as well. $c^{nl'}$ is dominant for $n \gg 1$, hence $A_Z(nl') = \mathbb{C}^2$. But $V_Z(l') = V_Z(nl')$, see e.g. the next Lemma.)

Using (3.4.1.2) one obtains the following properties of the spaces $\{A_Z(l')\}_{l'}$ of $H^1(\mathcal{O}_Z)$:

Lemma 3.4.1.6.

(a) $A_Z(l'_1 + l'_2) = A_Z(l'_1) + A_Z(l'_2) := \{a_1 + a_2 \,:\, a_i \in A_Z(l'_i\}$*; in particular,* $V_Z(l'_1) \subset V_Z(l'_2)$ *whenever* $l'_1 \leq l'_2$ *and* $V_Z(nl') = V_Z(l')$ *for any* $n \geq 1$.

(b) For any $-l' = \sum_v a_v E^*_v \in \mathcal{S}'$ *let the* E^**–support of* l' *be* $I(l') := \{v \,:\, a_v \neq 0\}$*. Then* $V_Z(l')$ *depends only on* $I(l')$.

E.g., if $I(l') = \mathcal{V}$*, then* $c^{nl'}$ *is dominant for any* $n \gg 1$ *(use Theorem 3.2.1.1(3).) Hence,* $V_Z(l') = V_Z(nl') = H^1(\mathcal{O}_Z)$.

Proof. (b) $V_Z(l') \subset V_Z(l' + nE^*_v) \subset V_Z(l') + V_Z(nE^*_v) \subset V_Z(l') + V_Z(E^*_v) \subset V_Z(l')$ for $v \in I(l')$. □

Definition 3.4.1.7. (a) 3.4.1.6*(b)* motivates to use the notation $V_Z(I)$ for $V_Z(l')$ whenever $I = I(l')$.

Hence Lemma 3.4.1.6(a) reads as $V_Z(I_1 \cup I_2) = V_Z(I_1) + V_Z(I_2)$.

(b) If $Z_2 \geq Z_1$, then the restriction (cf. 3.1.1) satisfies $r(V_{Z_2}(l')) = V_{Z_1}(l')$, hence $\dim V_{Z_2}(l') \geq \dim V_{Z_1}(l')$ and the pair $V_Z(l') \subset H^1(\mathcal{O}_Z)$ stabilizes as Z increases. Set $\big(V_{\widetilde{X}}(l') \subset H^1(\mathcal{O}_{\widetilde{X}})\big)$ for $\lim_{\leftarrow} \big(V_Z(l') \subset H^1(\mathcal{O}_Z)\big)$ and $\big(V_{\widetilde{X}}(I) \subset H^1(\mathcal{O}_{\widetilde{X}})\big) := \lim_{\leftarrow} \big(V_Z(I) \subset H^1(\mathcal{O}_Z)\big)$.

Remark 3.4.1.8. The multiplicative structure — that is, the general properties what must be satisfied by $\widetilde{c}^{nl'}$ for a certain $n > 1$ — imposes strong hidden properties for the original map $\widetilde{c}^{l'} : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^0(Z) = H^1(\mathcal{O}_Z)$ as well. Let us exemplify this via the following case. Assume e.g. that $Z \geq E$ and $\mathrm{ECa}^{l'}(Z)$ is 1–dimensional. Then $\mathrm{ECa}^{l'}(Z)$ can be identified with some $E_v^{reg} := E_v \setminus \cup_{w \neq v} E_w$. Therefore, the symmetric product $\mathrm{ECa}^{l'}(Z)^{\times n}/\mathfrak{S}_n$ (where $\mathfrak{S}_n$ is the permutation group of n letters) embeds as a Zariski open set into $\mathrm{ECa}^{nl'}(Z)$. Hence, by Lemma 3.1.1.8, the generic fibers of the restriction of $\widetilde{c}^{nl'}$ ($\mathrm{ECa}^{l'}(Z)^{\times n}/\mathfrak{S}_n \to H^1(\mathcal{O}_Z)$, $[D_1, \cdots, D_n] \mapsto \sum_i \widetilde{c}^{l'}(D_i)$) must be irreducible. This fact imposes serious restrictions for the original map $\widetilde{c}^{l'}$ as well.

E.g., $\mathbb{C} \to \mathbb{C}^2$, $t \mapsto (t, t^4)$ cannot be birational equivalent with a certain $\widetilde{c}^{l'}$. Indeed, its 'double', $\mathbb{C}^{\times 2}/\mathfrak{S}_2 \to \mathbb{C}^2$, $(t,s) \mapsto (t+s, t^4+s^4)$, rewritten in terms of elementary symmetric functions reads as $\mathbb{C}^2 \to \mathbb{C}^2$, $(\sigma_1, \sigma_2) \mapsto (\sigma_1, \sigma_1^4 - 4\sigma_2\sigma_1^2 + 2\sigma_2^2)$, which has non–irreducible generic fibers.

By the next theorem, $V_Z(l') = H^1(\mathcal{O}_Z)$ if and only if $c^{nl'}$ is dominant for $n \gg 1$; and in 3.4.3 we will characterize those cases when $V_Z(l') = 0$. But besides these two limit situations the construction provides a rather complex *linear subspace arrangement* $\{V_Z(l')\}_{l'}$, which, in general, contains deep analytic information about (X, o).

Theorem 3.4.1.9. *Fix $l' \in -\mathcal{S}'$ and $Z > 0$ as above. Then for $n \gg 1$ the following facts hold.*

(a) The image of $\widetilde{c}^{nl'}$ *is the affine subspace* $A_Z(nl')$ *of* $H^1(\mathcal{O}_Z)$ *(a translated of* $A_Z(l')$*).*

(b) All the (non–empty) fibers of $\widetilde{c}^{nl'}$ *have the same dimension.*

In particular, for any $\mathcal{L} \in \mathrm{Pic}^{nl'}(Z)$ *without fixed components (and* $n \gg 1$*) one has*

$$h^1(Z,\mathcal{L}) = h^1(\mathcal{O}_Z) - \dim V_Z(l') = \mathrm{codim}\big(V_Z(l') \subset H^1(\mathcal{O}_Z)\big). \tag{3.4.1.10}$$

(c) Let $I \subset \mathcal{V}$ *be the* E^**–support of* l'*. Decompose* Z *as* $Z|_I + Z|_{\mathcal{V}\setminus I}$ *according to the supports* I *and* $\mathcal{V} \setminus I$*. Then for all* $\mathcal{L} \in \mathrm{Pic}^{nl'}(Z)$ *without fixed components (and* $n \gg 1$*)* $h^1(Z,\mathcal{L})$ *depends only on the* E^**–support* I *of* l'*:*

$$h^1(Z,\mathcal{L}) = h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}}). \tag{3.4.1.11}$$

Hence, by (3.4.1.10),

$$\dim V_Z(I) = h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}}). \tag{3.4.1.12}$$

In particular, if $(\widetilde{X}/E_{\mathcal{V}\setminus I}, o_{\mathcal{V}\setminus I})$ *denotes the multi–germ (the disjoint union of singularities) obtained by contracting the connected components of* $E_{\mathcal{V}\setminus I}$ *in* $\widetilde{X}$*, then for* $Z \gg 0$ *we obtain*

$$\dim V_Z(I) = p_g(X,o) - p_g(\widetilde{X}/E_{\mathcal{V}\setminus I}, o_{\mathcal{V}\setminus I}). \tag{3.4.1.13}$$

Therefore, $V_Z(I) = H^1(\mathcal{O}_Z) = \mathbb{C}^{p_g(X,o)}$*, if and only if* $\Gamma \setminus I$ *is a disjoint union of rational graphs.*

(d) With the notations of (c), $V_Z(I) = \ker(H^1(\mathcal{O}_Z) \to H^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}}))$*.*

(e) Any $\mathcal{L} \in \mathrm{Pic}^{nl'}(Z)$ *without fixed components is generated by global sections.*

Remark 3.4.1.14. (a) In (3.4.1.10) $h^1(Z,\mathcal{L}) > -\chi(Z,\mathcal{L})$ (since $h^0(Z,\mathcal{L})>0$), which gives a topological lower bound for $\mathrm{codim}\big(V_Z(l')\subset H^1(\mathcal{O}_Z)\big)$.

(b) (3.4.1.13) generalizes the 'p_g–additivity formula' of Okuma [O08], which was proved for splice quotient singularities, for details see 3.7.2. Note that the present formula is valid for any singularity.

(c) Part *(a)* of Theorem 3.4.1.9 is equivalent (by a similar argument as the proof of Lemma 3.4.1.6*(b)*) by the following statement: (a$'$) If $-l'=\sum_{v\in I} a_vE_v^*$ with $a_v\gg 0$ (but no other relations between them), then the image of $\widetilde{c}^{l'}$ is an affine subspace, a translated of $V_Z(I)$.

(d) Parts *(b)–(c)* of Theorem 3.4.1.9 imply that $\mathrm{im}(c^{nl'})$ (for $n\gg 1$) is closed and consists of only one h^1–strata: $\mathrm{im}(c^{nl'})=W_{nl',h^1(\mathcal{O}_Z)-\dim V_Z(I)}$.

Proof of Theorem 3.4.1.9. *(a)* Write $A(l')$ as $a+V(l')$ for some $a\in A(l')$. Then by (3.4.1.2) $\mathrm{im}(\widetilde{c}^{nl'})\subset na+V(l')$. We have to show that for $n\gg 0$ we have equality $\mathrm{im}(\widetilde{c}^{nl'})=na+V(l')$.

We choose smooth points $x_1,\dots,x_k$ in $\mathrm{im}(\widetilde{c}^{l'})$ such that the tangent spaces $T_{x_i}\mathrm{im}(\widetilde{c}^{l'})$, translated to the origin, generate $V(l')$. Then taking Zariski neighborhoods U_i of x_i in $\mathrm{im}(\widetilde{c}^{l'})$, we notice that $\sum_i(-x_i+U_i)$ contains a Zariski open set of $V(l')$. But $\sum_i(-x_i+U_i)\subset\sum_i(-x_i+\mathrm{im}(\widetilde{c}^{l'}))\subset -\sum_i x_i+\mathrm{im}(\widetilde{c}^{kl'})\subset V(l')$, hence $-\sum_i x_i+\mathrm{im}(\widetilde{c}^{kl'})$ contains a Zariski open subset of $V(l')$. On the other hand, if U is a Zariski open set of a vector space V, then $U+U=V$. This shows that $\mathrm{im}(\widetilde{c}^{2kl'})$ is an affine space associated with $V(l')$.

(b) If we replace l' by some multiple if it, by part *(a)* we can assume that $\widetilde{c}^{l'}:\mathrm{ECa}^{l'}(Z)\to H^1(\mathcal{O}_Z)$ has image $A(l')$. Consider the following diagram (for some $m\in\mathbb{Z}_{>0}$ which will be determined later):

$$\begin{array}{ccc}
(\,\mathrm{ECa}^{l'}(Z)\,)^m & \xrightarrow{s} & \mathrm{ECa}^{ml'}(Z)\\
\oplus\,\widetilde{c}^{l'}\downarrow & & \downarrow\widetilde{c}^{ml'}\\
(\,A(l')\,)^m & \xrightarrow{\Sigma} & A(ml')
\end{array}$$

Fix any $x \in A(ml')$. Since $\oplus\, \widetilde{c}^{l'}$ and Σ are surjective, the fiber $(\widetilde{c}^{ml'})^{-1}(x)$ intersects $\mathrm{im}(s)$ at some point p. Since the source and target spaces of s are smooth of the same dimension, by Open Mapping Theorem (see e.g. [GR70, p. 107]) there exists an (analytic) open neighbourhood U of p (hence intersecting the fiber) contained in $\mathrm{im}(s)$. Hence, using also the quasi–finiteness of s, $\dim(\widetilde{c}^{ml'})^{-1}(x) = \dim(\widetilde{c}^{ml'} \circ s)^{-1}(x) = \dim(\Sigma \circ \oplus\, \widetilde{c}^{l'})^{-1}(x)$. Thus, if $\mathbf{x} = (x_1, \dots, x_m)$ are the coordinates in $(\, A(l')\,)^m$, then we have to analyse the set $(\oplus\, \widetilde{c}^{l'})^{-1}\{\mathbf{x} : \sum_i x_i = x\}$ for any fixed x.

In $A(l')$ there is a Zariski open subset U, with the following two properties:

(i) for any $y \in U$, the fiber $(\widetilde{c}^{l'})^{-1}(y)$ has the minimal possible dimension, namely $\dim \mathrm{ECa}^{l'}(Z) - \dim A(l') = (l', Z) - d(l')$;

(ii) if $F := A(l') \setminus U$ is its complement, then $\dim(\widetilde{c}^{l'})^{-1}(F) < \dim \mathrm{ECa}^{l'}(Z) = (l', Z)$.

We stratify $H_x := \{\mathbf{x} : \sum_i x_i = x\}$ with the sets $\mathcal{F}_k := \{\mathbf{x} \in H_x \,:\, \#\{i : x_i \in F\} = k\}$, where $0 \le k \le m$. Set also $E\mathcal{F}_k := (\oplus\, \widetilde{c}^{l'})^{-1}(\mathcal{F}_k)$.

Then $\mathcal{F}_0$ is a non–empty open set of H_x of dimension $(m-1)d(l')$, hence $\dim E\mathcal{F}_0 = (m-1)d(l') + m((l', Z) - d(l')) = (ml', Z) - d(l')$. Next we estimate the dimensions of the other strata as well.

First, we consider the case $1 \le k < m$. Then $\mathcal{F}$ is covered by several components according to the position of $I = \{i_1, \dots, i_k\}$ indexing those x_i which belong to F. Fix suxh a component $\mathcal{F}_{k,I}$, and write $(\oplus\, \widetilde{c}^{l'})^{-1}(\mathcal{F}_{k,I}) = E\mathcal{F}_{k,I}$. We consider the projection $pr_I : \mathcal{F}_{k,I} \to \sqcap_I F$, $\mathbf{x} \mapsto (x_{i_1}, \dots, x_{i_k})$, and the lifted one $Epr_I : E\mathcal{F}_{k,I} \to \sqcap_I (\widetilde{c}^{l'})^{-1}(F)$. Note that Epr_I is an injection and its target has dimension $\le k((l', Z) - 1)$. Furthermore, the fibers of Epr_I have dimension $(m-k-1)d(l') + (m-k)((l', Z) - d(l')) = (m-k)(l', Z) - d(l')$. Hence, $\dim E\mathcal{F}_{k,I} \le (m-k)(l', Z) - d(l') + k((l', Z) - 1) = (ml', Z) - d(l') - k$.

The case $k = m$ is slightly different. Using the injection $\mathcal{F}_m \to \sqcap_m (\widetilde{c}^{l'})^{-1}(F)$ we get 'only' $\dim E\mathcal{F}_m \le m((l', Z) - 1)$. Therefore, if $m \ge d(l')$ then we get $\dim E\mathcal{F}_m \le$

$\dim E\mathcal{F}_0$. Hence, finally, $\dim(\widetilde{c}^{ml'})^{-1}(x) = \dim E\mathcal{F}_0 = \dim \mathrm{ECa}^{ml'}(Z) - \dim A(ml')$.

For (3.4.1.10) use part *(b)* and Lemma 3.3.5.1.

(c) For any $n \gg 1$ and $\mathcal{L} \in \mathrm{im}(c^{nl'})$ (3.4.1.10) gives $h^1(Z,\mathcal{L}) = h^1(\mathcal{O}_Z) - d_Z(l')$. By Grauert–Riemenschneider vanishing theorem $h^1(Z|_I, \mathcal{L}(-Z|_{\mathcal{V}\setminus I})) = 0$, hence $h^1(Z,\mathcal{L}) = h^1(Z|_{\mathcal{V}\setminus I}, \mathcal{L})$. If $\mathcal{L}$ is associated with certain effective divisor $D \in \mathrm{ECa}^{nl'}(Z)$ (as the image of $c^{nl'}$), then $\mathcal{L}|_{Z|_{\mathcal{V}\setminus I}}$ is associated with the restriction of this divisor to $Z|_{\mathcal{V}\setminus I}$. But this restriction has an empty support, hence $\mathcal{L}|_{Z|_{\mathcal{V}\setminus I}}$ is the trivial bundle over $Z|_{\mathcal{V}\setminus I}$.

(d) Since the restriction of any element of $\mathrm{ECa}^{nl'}(Z)$ to $Z|_{\mathcal{V}\setminus I}$ is the empty divisor, the image of the composition $\mathrm{ECa}^{nl'}(Z) \to \mathrm{ECa}^0(Z|_{\mathcal{V}\setminus I}) \to \mathrm{Pic}^0(Z|_{\mathcal{V}\setminus I})$ is the trivial bundle (that is, the zero element of $\mathrm{Pic}^0(Z_{\mathcal{V}\setminus I})$). Therefore, $\mathrm{im}(c^{nl'}) \subset \ker(H^1(\mathcal{O}_Z) \to H^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}}))$. Since they have the same dimension (cf. 3.4.1.12) they must agree.

(e) Let n be so large that $\mathrm{im}(\widetilde{c}^{nl'}) = A_Z(nl')$ is an affine subspace. We claim that any $\mathcal{L} \in \mathrm{im}(\widetilde{c}^{2nl'}) = A_Z(2nl')$ is generated by global sections. Indeed, fix such a bundle and one of its sections $s \in H^0(Z,\mathcal{L})$ whose divisor is an element of $\mathrm{ECa}^{2nl'}(Z)$, whose support with reduced structure is $\mathbf{p} := \{p_1, \ldots, p_k\} \subset E$. Let $\mathrm{ECa}^{nl'}_p(Z)$ be the subspace of $\mathrm{ECa}^{nl'}(Z)$ consisting of divisors supported in the complement of $\mathbf{p}$. This is a Zariski open set of $\mathrm{ECa}^{nl'}(Z)$, hence $c(\mathrm{ECa}^{nl'}_p(Z))$ contains a Zariski open set U in $A_Z(nl')$. Then $U + U = A_Z(2nl')$, hence $\mathcal{L}$ admits a section whose divisor has support off $\mathbf{p}$. $\square$

3.4.2 Cohomological reinterpretations of $V_Z(l')$.

Fix $\mathcal{L} \in \mathrm{im}(c^{nl'})$ ($n \gg 1$), $D \in (c^{nl'})^{-1}(\mathcal{L})$, and $s \in H^0(Z,\mathcal{L})$ without fixed components. Then, as in the situation of 3.1.2 one has the cohomological long exact sequence $H^0(Z,\mathcal{L}) \xrightarrow{R_{\mathcal{L}}} \mathcal{O}_D \xrightarrow{\delta} H^1(\mathcal{O}_Z) \to H^1(Z,\mathcal{L}) \to 0$ from (3.1.2.1). Then by Theorem 3.4.1.9, $\mathrm{im}(c^{nl'}) = A(nl')$. Therefore, $\mathrm{im}(T_D c^{nl'}) \subset T_{\mathcal{L}} A(nl')$. But, by Lemma 3.1.1.8, $\dim \mathrm{im} T_D c^{nl'} = \dim \mathrm{ECa}^{nl'}(Z) - \dim \mathrm{im}(c^{nl'})^{-1}(\mathcal{L}) = h^1(\mathcal{O}_Z) - h^1(Z,\mathcal{L}) =$

$\dim T_{\mathcal{L}}A(nl') = \dim V_Z(l')$. Hence, $\mathrm{im}(T_D\widetilde{c}^{nl'}) = V_Z(l')$. As $\mathrm{im}(T_D\widetilde{c}^{nl'}) = \mathrm{im}\delta$ (cf. Prop. 3.1.2.2) for $V_Z(l')$ we get two other cohomological reinterpretations. Either it is the Artin algebra $\mathcal{O}_D/\mathrm{im}(R_{\mathcal{L}})$, as a vector space, identified as the image of O_D into $H^1(\mathcal{O}_Z)$, or it is also the kernel of $H^1(\times s) : H^1(\mathcal{O}_Z) \to H^1(Z, \mathcal{L})$.

In other words, for $n \gg 1$, the image of $\mathcal{O}_D \to H^1(\mathcal{O}_Z)$ is independent of the choice of D, while the kernel of $H^1(\times s) : H^1(\mathcal{O}_Z) \to H^1(Z, \mathcal{L})$ is independent of the choice of s. Furthermore, they are equal, and in fact this subspace of $H^1(\mathcal{O}_Z)$ depends only on the E^*–support I of l', and it equals $V_Z(I)$.

There is a parallel analogous discussion for $\widetilde{X}$ (instead of Z) as well (in that case the reduced structure of D is Stein, hence $h^1(\mathcal{O}_D) = 0$ again).

3.4.3 Example. Characterization of the cases $\dim \mathrm{im}(c) = 0$

Fix $l' \in -\mathcal{S}'$ with E^*–support $I \subset \mathcal{V}$ and $Z > 0$ as above. Using (3.1.1.9) and (3.4.1.12) one proves that the following facts are equivalent (for an additional equivalent property see also Example 3.6.1.4):

(i) $\mathrm{im}(c^{l'})$ is a point (or, $V_Z(l') = 0$);

(ii) there exists $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ without fixed components such that $h^1(Z, \mathcal{L}) = h^1(Z)$;

(iii) any $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ without fixed components satisfies $h^1(Z, \mathcal{L}) = h^1(Z)$;

(iv) all line bundles $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ without fixed components are isomorphic to each other;

(v) $h^1(\mathcal{O}_Z) = h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}})$.

Let us define $\mathcal{S}'_{pt}$ as $\{-l' \in \mathcal{S}' : \mathrm{im}(c^{l'}) \text{ is a point}\} \subset \mathcal{S}'$, this is the set of Chern classes satisfying the above equivalent conditions. Using (3.4.1.2) we obtain that $\mathcal{S}'_{pt}$ is a semigroup.

Part (v) via Proposition 3.3.4.1 reads as follows:

$$\mathcal{S}'_{pt} = \mathbb{Z}_{\geq 0}\langle\, E^*_v \mid E_v \not\subset |Z_{coh}(Z, \mathcal{O}_Z)|\,\rangle. \tag{3.4.3.1}$$

Note that (in contrast with $\mathcal{S}'_{dom}$) $\mathcal{S}'_{pt}$ is not topological. Indeed, take e.g. the graph from Example 3.1.4.1, $-l' := Z_{min} = E^*_v$ (where v is the (-2)–vertex adjacent with the (-7) vertex), and set $Z = Z_K$. Then, if $p_g(X, o) = 2$ (that is, (X, o) is Gorenstein) then $Z_{coh}(Z, \mathcal{O}_Z) = Z$, and $\mathcal{S}'_{pt} = \{0\}$. If $p_g(X, o) = 1$, then $Z_{coh}(Z, \mathcal{O}_Z)$ is the minimally elliptic cycle, and $\mathcal{S}'_{pt} = \mathbb{Z}\langle E^*_v\rangle$.

In [OWY14, OWY15a, OWY15b] a cycle $l \in \mathcal{S}' \cap L$ is called p_g–cycle if $\mathcal{O}_{\widetilde{X}}(-l) \in \mathrm{Pic}(\widetilde{X})$ has no fixed components, and $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-l)) = p_g$. Note that this in our language means that $-l \in \mathcal{S}'_{pt}$ for $Z \gg 0$. Our results generalizes several statements of [loc.cit.] for arbitrary bundles $\mathcal{L}$ without fixed components (replacing $\mathcal{O}_{\widetilde{X}}(-l)$) and arbitrary $\dim \mathrm{im}(c^{l'})$.

This particular case and several similar classical results valid for bundles of type $\mathcal{O}(l')$ motivate to investigate the position of the natural line bundles with respect to $\mathrm{im}(c^{l'})$ (i.e., whether $\mathcal{O}(l')$ has fixed components or no). This is the subject of section 3.7.

3.5 The Abel map via differential forms

3.5.1 Review of Laufer Duality [La72], [La77, p. 1281]

Following Laufer, we identify the dual space $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^*$ with the space of global holomorphic 2-forms on $\widetilde{X}\setminus E$ up to the subspace of those forms which can be extended holomorphically over $\widetilde{X}$.

For this, use first Serre duality $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^* \simeq H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}})$. Then, in the exact sequence

$$0 \to H^0_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}) \to H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^1(\widetilde{X}, \Omega^2_{\widetilde{X}})$$

$H^0_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) = H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^* = 0$ by dimension argument, while $H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}) = 0$ by the Grauert–Riemenschneider vanishing. Hence,

$$H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^* \simeq H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) \simeq H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}). \tag{3.5.1.1}$$

The second isomorphism can be realized as follows. Fix a small tubular neighbourhood $N \subset \widetilde{X}$ of E such that its closure is compact in $\widetilde{X}$. Take any $\omega \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$, and extend the restriction $\omega|_{\widetilde{X}\setminus N}$ to a $C^\infty(2,0)$–form $\tilde{\omega}$ on $\widetilde{X}$. Then $\bar{\partial}\tilde{\omega}$ is a compactly supported $C^\infty(2,1)$–form, $\bar{\partial}\bar{\partial}\tilde{\omega} = 0$, hence $\bar{\partial}\tilde{\omega}$ determines a class in $H^1_c(\widetilde{X}, \Omega^2)$. If $\tilde{\omega}$ is a holomorphic extension then $\bar{\partial}\tilde{\omega} = 0$. Next, let λ be a $C^\infty(0,1)$ form in $\widetilde{X}$. Then the duality $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \otimes H^1_c(\widetilde{X}, \Omega^2) \to \mathbb{C}$ is the perfect pairing

$$\langle [\lambda], [\bar{\partial}\tilde{\omega}] \rangle = \int_{\widetilde{X}} \lambda \wedge \bar{\partial}\tilde{\omega}.$$

Assume that the class $[\lambda] \in H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is realized by a *Čech cocyle* $\lambda_{ij} \in \mathcal{O}(U_i \cap U_j)$, where $\{U_i\}_i$ is an open cover of E, $U_i \cap U_j \cap U_k = \emptyset$, and each connected component of the intersections $U_i \cap U_j$ is either a coordinate bidisc $B = \{|u| < 2\epsilon,\ |v| < 2\epsilon\}$ with coordinates (u, v), such that $E \cap B \subset \{uv = 0\}$, or a punctured coordinate bidisc $B = \{\epsilon/2 < |v| < 2\epsilon,\ |u| < 2\epsilon\}$ with coordinates (u, v), such that $E \cap B = \{u = 0\}$. Then λ is obtained as follows: one finds C^∞ functions λ_i on U_i such that $\lambda_i - \lambda_j = \lambda_{ij}$ on $U_i \cap U_j$, and one sets λ as $\bar{\partial}\lambda_i$ on U_i. Then, by Stokes theorem

$$\langle [\lambda], [\bar{\partial}\tilde{\omega}] \rangle = \sum_B \int_{|u|=\epsilon,\ |v|=\epsilon} \lambda_{ij}\omega. \tag{3.5.1.2}$$

By Stokes theorem, if ω has no pole along E in B, then the B–contribution in the above sum is zero.

3.5.1.3. Above $H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}})$ can be replaced by $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))$ for a large cycle Z (e.g. for $Z \geq \lfloor Z_K \rfloor$). Indeed, for any cycle $Z > 0$ from the exacts sequence of sheaves $0 \to \Omega^2_{\widetilde{X}} \to \Omega^2_{\widetilde{X}}(Z) \to \mathcal{O}_Z(Z + K_{\widetilde{X}}) \to 0$ and from the vanishing $h^1(\Omega^2_{\widetilde{X}}) = 0$ and Serre duality one has

$$H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) = H^0(\mathcal{O}_Z(Z+K)) \simeq H^1(\mathcal{O}_Z)^*. \qquad (3.5.1.4)$$

Since $H^1(\mathcal{O}_Z) \simeq H^1(\mathcal{O}_{\widetilde{X}})$ for $Z \geq \lfloor Z_K \rfloor$, the natural inclusion

$$H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \hookrightarrow H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}}) \qquad (3.5.1.5)$$

is an isomorphism.

3.5.1.6. The above duality, via the isomorphism $\exp : H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to c_1^{-1}(0) \subset H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}}) = \mathrm{Pic}(\widetilde{X})$, can be transported as follows. Consider the following situation. We fix a smooth point p on E, a local bidisc $B \ni p$ with local coordinates (u, v) such that $B \cap E = \{u = 0\}$. We assume that a certain form $\omega \in H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))$ has local equation $\omega = \sum_{i\in\mathbb{Z}, j\geq 0} a_{i,j}u^iv^j du \wedge dv$ in B.

In the same time, we fix a divisor $\widetilde{D}$ on $\widetilde{X}$, whose local equation in B is v^n, $n \geq 1$. Let $\widetilde{D}_t$ be another divisor, which is the same as $\widetilde{D}$ in the complement of B and in B its local equation is $(v + tu^{o-1})^n$, where $o \geq 1$ and $t \in \mathbb{C}$ (with $|t| \ll 1$ whenever $o = 1$).

Next we will provide three type of formulae.

The first one is the composition of several maps. Note that the pairing $\langle \cdot, [\bar{\partial}\tilde{\omega}]\rangle$ (abridged as $\langle \cdot, \omega\rangle$) produces a map $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to \mathbb{C}$. Then we identify $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ with $\mathrm{Pic}^0(\widetilde{X})$ by the exponential map. Then we consider the composition $t \mapsto \widetilde{D}_t -$

$\widetilde{D} \mapsto \mathcal{O}_{\widetilde{X}}(\widetilde{D}_t - \widetilde{D}) \mapsto \exp^{-1}\mathcal{O}_{\widetilde{X}}(\widetilde{D}_t - \widetilde{D}) \mapsto \langle \exp^{-1}\mathcal{O}_{\widetilde{X}}(\widetilde{D}_t - \widetilde{D}), \omega\rangle$. The first formula makes this composition explicit. This restricted to any cycle $Z \gg 0$ can be reinterpreted as ω–coordinate of the Abel map restricted to the path $t \mapsto D_t := \widetilde{D}_t|_Z$ (and shifted by the image of $D := \widetilde{D}|_Z$).

The second formula determines the tangent application of the above composition (in this way it determines the ω–coordinate of the tangent application of the Abel map restricted to D_t).

In the third formula we replace the path D_t by a complete neighborhood of D in $\mathrm{ECa}(Z)$.

Note that if we consider — instead of a single form ω — a complete set of representatives of a basis of $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$, then we get by the above three constructions the restriction of the Abel map to the path D_t, the tangent map of this restriction, and in the third case the 'complete' Abel map defined in some neighbourhood of D.

3.5.2 The Abel map restricted to D_t

The first two cases start with the explicit computation of $\langle \exp^{-1}\mathcal{O}_{\widetilde{X}}(\widetilde{D}_t - \widetilde{D}), \omega\rangle$, as follows. $\widetilde{D}_t - \widetilde{D}$ is the divisor $\widetilde{D}' = \mathrm{div}((v + tu^{o-1})/v)^n$, supported in $B = \{|u|, |v| < \epsilon\}$. We can fix ϵ such that the support of $\widetilde{D}'$ is in $\{|v| < \epsilon/2\}$, and set $B^* := \{\epsilon/2 < |v| < \epsilon, |u| < \epsilon\}$. Using the trivialization of $\mathcal{O}(\widetilde{D}')$ in $\widetilde{X} \setminus \{|v| \le \epsilon/2\}$ and the realization $\mathcal{O}(\widetilde{D}')$ on B, we get that $\mathcal{O}(\widetilde{D}')$ can be represented by the cocycle $g = ((v + tu^{o-1})/v)^n \in \mathcal{O}^*(B^*)$. Therefore, $\log((v + tu^{o-1})/v)^n = n\log(1 + tu^{o-1}/v)$ is a cocycle in B^* representing its lifting into $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. This paired with ω gives:

$$\langle\langle \widetilde{D}_t, \omega\rangle\rangle := \langle \exp^{-1}\mathcal{O}_{\widetilde{X}}(\widetilde{D}_t - \widetilde{D}), \omega\rangle = n\int_{|u|=\epsilon,\ |v|=\epsilon} \log(1 + t\frac{u^{o-1}}{v}) \cdot \sum_{i\in\mathbb{Z}, j\ge 0} a_{i,j}u^i v^j du \wedge dv. \tag{3.5.2.1}$$

If $\omega_1, \dots, \omega_{p_g}$ are representatives of a basis for $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$, and $Z \gg 0$, then

$$\widetilde{D}_t \mapsto (\langle\langle \widetilde{D}_t, \omega_1\rangle\rangle, \dots, \langle\langle \widetilde{D}_t, \omega_{p_g}\rangle\rangle) \tag{3.5.2.2}$$

is the restriction of the Abel map to $\widetilde{D}_t$ (associated with Z, and shifted by the image of $\widetilde{D}$).

At the level of tangent application on has the formula for $(T_{\widetilde{c}(D)}\omega) \circ T_D\widetilde{c})(\frac{d}{dt}D_t|_{t=0})$:

$$\frac{d}{dt}\Big|_{t=0}\Big[n\int_{|u|=\epsilon,\ |v|=\epsilon} \log(1+t\frac{u^{o-1}}{v}) \cdot \sum_{i\in\mathbb{Z}, j\geq 0} a_{i,j}u^iv^j du\wedge dv\Big] = \lambda\cdot a_{-o,0} \quad (\lambda \in \mathbb{C}^*). \tag{3.5.2.3}$$

If ω has no pole along the divisor $\{u=0\}$ then $\langle \exp^{-1}\mathcal{O}_{\widetilde{X}}(\widetilde{D}_t - \widetilde{D}), \omega\rangle = 0$ for any path $\widetilde{D}_t$.

Definition 3.5.2.4. Consider the above situation in the bidisc B: $B\cap E = \{u=0\}$, $\widetilde{D}$ has local equation v (i.e. $n=1$), and $\omega = \sum_{i\in\mathbb{Z}, j\geq 0} a_{i,j}u^iv^j du\wedge dv$. Then we introduce the *Leray residue* of ω/du along $\{v=0\}$ as the 1–form (with possible poles at $\widetilde{D}\cap E$) defined by $(\omega/dv)|_{v=0} = \sum_i a_{i,0}u^i du$. We denote it by $\mathrm{Res}_D(\omega)$.

Note that the right hand side of (3.5.2.3) tests exactly the pole part of the Leray residue $\mathrm{Res}_D(\omega)$.

3.5.3 The Abel map

Assume as above that in the ball B the divisor $\widetilde{D}$ is given by $v=0$ (i.e. $n=1$), and its 'perturbation' $\widetilde{D}(c)$ is given by $v = c_0 + c_1u + c_2u^2 + \cdots$ with $|c_0| \ll \epsilon$. Furthermore, assume that the form ω in B has the form $(f(v)/u^{\ell+1})du\wedge dv$, where $f\in\mathcal{O}(B)$ and $\ell \geq 0$. (Note that the Laurent expansion in variable u of any differential form is a sum of such terms.)

Our aim is the computation of $\langle\langle \widetilde{D}(c), \omega\rangle\rangle$.

If $\{p_i\}_{\geq 1}$ (resp. $\{h_i\}_{i\geq 1}$) denote the power sum (resp. complete) symmetric polynomials (functions) then (cf. [Mac95, p. 23])

$$p_1 u + p_2 u^2/2 + p_3 u^3/3 + \cdots = \log(1 + h_1 u + h_2 u^2 + \cdots). \tag{3.5.3.1}$$

Furthermore, by [Mac95, p. 28], for $n \geq 1$,

$$(-1)^{n+1} p_n = \begin{pmatrix} h_1 & 1 & 0 & \dots & 0 \\ 2h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ nh_n & h_{n-1} & h_{n-2} & \dots & h_1 \end{pmatrix} \tag{3.5.3.2}$$

We rewrite (3.5.3.1) as $\log(A) + p_1 u + p_2 u^2/2 + \cdots = \log(A + h_1 Au + h_2 Au^2 + \cdots)$ and we make the substitution $A = (v - c_0)/v$, $h_1 A = -c_1/v$, $h_2 A = -c_2/v$, etc., and we obtain

$$\log\big(1 - \frac{c_0 + c_1 u + c_2 u^2 + \cdots}{v}\big) = \log\big(1 - \frac{c_0}{v}\big) + \delta_1(c)u + \delta_2(c)u^2 + \cdots, \tag{3.5.3.3}$$

where for $n \geq 1$

$$\delta_n(c) = \sum_{i=1}^{n} \frac{\delta_{n,i}(c)}{(v - c_0)^i} = \frac{-1}{n} \begin{pmatrix} \frac{c_1}{v-c_0} & -1 & 0 & \dots & 0 \\ \frac{2c_2}{v-c_0} & \frac{c_1}{v-c_0} & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{nc_n}{v-c_0} & \frac{c_{n-1}}{v-c_0} & \frac{c_{n-2}}{v-c_0} & \dots & \frac{c_1}{v-c_0} \end{pmatrix}. \tag{3.5.3.4}$$

Note that $\delta_{n,i}$ are certain universal polynomials in variables $c_1, \dots, c_n$. Then $\langle\langle \widetilde{D}(c), \omega \rangle\rangle$ equals

$$\int_{|u|=\epsilon,\ |v|=\epsilon} \log\Big(1 - \frac{c_0 + c_1 u + \cdots}{v}\Big) \cdot \frac{f(v)}{u^{\ell+1}}\, du \wedge dv = \sum_{i=1}^{\ell} \frac{\delta_{\ell,i}(c)}{(i-1)!} \cdot \frac{d^{i-1} f}{dv^{i-1}}(c_0). \tag{3.5.3.5}$$

3.5.4 Reduction to an arbitrary $Z > 0$.

Consider the above perfect pairing $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \otimes H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}}) \to \mathbb{C}$ given via integration of class representatives. In $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ let A be the image of the $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z))$, hence $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/A = H^1(\mathcal{O}_Z)$. On the other hand, in $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}})$ consider the subspace $B := H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$ of dimension $h^1(\mathcal{O}_Z)$ (cf. (3.5.1.4). Since $\langle A, B\rangle = 0$, the pairing factorizes to a perfect pairing $H^1(\mathcal{O}_Z) \otimes H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \to \mathbb{C}$. It can be described by the very same integral form of the corresponding class representatives.

Moreover, if $\widetilde{D}_t$ is an 1–parameter family of divisors as in 3.5.1.6, representing an element in $H^1(\mathcal{O}_Z)$ (via the surjection $H^1(\mathcal{O}_{\widetilde{X}}) \to H^1(\mathcal{O}_Z)$), and ω is a representative of a class $[\omega] \in H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$, then the expression of the pairing $H^1(\mathcal{O}_Z) \otimes H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \to \mathbb{C}$, $\langle \exp^{-1} \mathcal{O}_Z(\widetilde{D}_t - \widetilde{D}), [\omega]\rangle$, can be represented by the very same formula (3.5.2.1) (as in the case $Z \gg 0$). Furthermore, all other formulae of subsections 3.5.2 and 3.5.3 also have their extended versions. E.g., (3.5.2.3) gives $T_{\widetilde{c}(D)}(\omega) \circ T_D\widetilde{c}^{l'}(Z))(\frac{d}{dt}D_t|_{t=0})$, and (3.5.3.5) is the $[\omega]$–coordinate of the Abel map $\mathrm{ECa}^{l'}(Z) \to H^1(\mathcal{O}_Z)$.

3.6 The 'stable' arrangement $\{V_{\widetilde{X}}(I)\}_{I\subset\mathcal{V}}$ and differential forms

3.6.1 The arrangement $\{\Omega_{\widetilde{X}}(I)\}_I$ of forms and its duality with $\{V_{\widetilde{X}}(I)\}_I$

Definition 3.6.1.1. Let $\Omega_{\widetilde{X}}(I)$ (or, $\Omega(I)$) be the subspace of $H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ generated by differential forms $\omega \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$, which have no poles along $E_I \setminus \cup_{v\notin I} E_v$.

As in Theorem 3.4.1.9*(c)*, let $(\widetilde{X}/E_{\mathcal{V}\setminus I}, o_{\mathcal{V}\setminus I})$ denote the multi–germ obtained by contracting the connected components of $E_{\mathcal{V}\setminus I}$ in $\widetilde{X}$. Let $\widetilde{X}(\mathcal{V}\setminus I)$ be a small neighbourhood of $E_{\mathcal{V}\setminus I}$ in $\widetilde{X}$, which is the inverse image by ϕ of a small Stein neighbourhood of $(\widetilde{X}/E_{\mathcal{V}\setminus I}, o_{\mathcal{V}\setminus I})$.

Proposition 3.6.1.2. *(a)* $\dim \Omega(I) = p_g(\widetilde{X}/E_{\mathcal{V}\setminus I}, o_{\mathcal{V}\setminus I})$.

(b) Set $\overline{\Omega}(\emptyset) := H^0(\widetilde{X}(\mathcal{V}\setminus I)\setminus E_{\mathcal{V}\setminus I}, \Omega^2_{\widetilde{X}(\mathcal{V}\setminus I)})/H^0(\widetilde{X}(\mathcal{V}\setminus I), \Omega^2_{\widetilde{X}(\mathcal{V}\setminus I)})$*. Then linear map* $\rho : \Omega(I) \to \overline{\Omega}(\emptyset)$*, induced by restriction, is an isomorphism.*

(c) Fix $I \subset \mathcal{V}$ *as above and set* $J \subset \mathcal{V}$ *with* $J \cap I = \emptyset$*. Let* $\overline{\Omega}(J)$ *be the subspace of* $\overline{\Omega}(\emptyset)$ *generated by forms from* $H^0(\widetilde{X}(\mathcal{V}\setminus I)\setminus E_{\mathcal{V}\setminus I}, \Omega^2_{\widetilde{X}(\mathcal{V}\setminus I)})$ *without pole along* E_J*. Then the restriction of* ρ *to* $\Omega(J)\cap\Omega(I)$ *induces an isomorphism* $\Omega(J)\cap\Omega(I) \to \overline{\Omega}(J)$*.*

In particular, for any I*, the subspace arrangement* $\{\overline{\Omega}(J)\}_{J\cap I=\emptyset}$ *of the multi-germ* $(\widetilde{X}/E_{\mathcal{V}\setminus I}, o_{\mathcal{V}\setminus I})$ *and resolution* $\widetilde{X}(\mathcal{V}\setminus I)$ *can be recovered from the arrangement* $\{\Omega(M)\}_M$ *via* $\{\Omega(I)\cap\Omega(J)\}_{J\cap I=\emptyset}$*.*

Proof. *(a)* Fix $Z = \sum_{v\in\mathcal{V}\setminus I} n_v E_v$ with all $n_v \gg 0$. By (3.5.1.4) $\dim \Omega(I) = \dim\ H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}$
$h^1(\mathcal{O}_Z)$, which equals $p_g(\widetilde{X}/E_{\mathcal{V}\setminus I}, o_{\mathcal{V}\setminus I})$ by formal function theorem.

(b) If $[\omega] \in \ker(\rho)$, then ω has no pole along E_I (since $[\omega] \in \Omega(I)$), and has no pole along $E_{\mathcal{V}\setminus I}$ either (since $\rho[\omega] = 0$). Hence $[\omega] = 0$, and ρ is injective. Since by *(a)* the dimension of the source and the target is the same, ρ is an isomorphism.

(c) By *(b)*, for any $\bar{\omega} \in \overline{\Omega}(J)$ there exists $\omega \in \Omega(I)$ with $\rho(\omega) = \bar{\omega}$. Note that ω is necessarily in $\Omega(I\cap J)$, hence $\Omega(J)\cap\Omega(I) \to \overline{\Omega}(J)$ is onto. □

The next result shows that the linear subspace arrangement $\{V_{\widetilde{X}}(I)\}_I$ of $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ (cf. 3.4.1.7) is dual to the linear subspace arrangement $\{\Omega_{\widetilde{X}}(I)\}_I$ of $\Omega_{\widetilde{X}}(\emptyset) = H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$.

Theorem 3.6.1.3. *Via duality (3.5.1.1) one has* $V_{\widetilde{X}}(I)^* = \Omega_{\widetilde{X}}(I)$*.*

Proof. We fix a cycle $Z \gg 0$ for which $V_Z(I) = V_{\widetilde{X}}(I)$. Choose $l' = -\sum_{v\in I} a_v E^*_v$ such that each a_v is so large that $\mathrm{im}(c^{l'})$ is an affine space, cf. Theorem 3.4.1.9. Then, any element $\mathcal{L}$ of $V_Z(I)$ has the form $\mathcal{O}_Z(D_1 - D_2)$, with both D_1, $D_2 \in \mathrm{ECa}^{l'}(Z)$. Lift $\{D_i\}_{i=1,2}$ to effective divisors $\{D'_1\}_{i=1,2}$ in $\widetilde{X}$. Since they do not intersect $E_{\mathcal{V}\setminus I}$, the class $[\lambda]$ of $\mathcal{O}_{\widetilde{X}}(D'_1 - D'_2)$ in $\mathrm{Pic}^0(\widetilde{X})$ can be represented by a Čech cocycles $\{\lambda_{ij}\}$, which in a neighbourhood of $E_{\mathcal{V}\setminus I}$ are all zero. Therefore, if ω is a form which has no pole along E_I, $\langle[\lambda], [\omega]\rangle = 0$ by (3.5.1.2). That is, $\langle V_{\widetilde{X}}(I), \Omega(I)\rangle = 0$, or $V_{\widetilde{X}}(I) \subset \Omega(I)^*$. Since by (3.4.1.12) and Proposition 3.6.1.2*(a)* one has $\dim V_{\widetilde{X}}(I) = p_g - \dim \Omega(I)$, we get $V_{\widetilde{X}}(I) = \Omega(I)^*$. □

Example 3.6.1.4. (Continuation of Example 3.4.3) Fix $l' \in -\mathcal{S}'$ with E^*–support $I \subset \mathcal{V}$ as in 3.4.3, and choose $Z \gg 0$. Then

$$\mathrm{im}(c^{l'}) \text{ is a point } \Leftrightarrow V_{\widetilde{X}}(I) = 0 \Leftrightarrow \Omega_{\widetilde{X}}(I) = \Omega_{\widetilde{X}}(\emptyset).$$

3.6.2 Convexity property of $\Omega(\{v\})$'s

Clearly, the subspace arrangement has the properties $\Omega(\emptyset) \simeq \mathbb{C}^{p_g}$, and $\Omega(I \cup J) = \Omega(I) \cap \Omega(J)$. In this subsection we establish an interesting additional structure property of the arrangement. It is the analytical analogue of topological convexity property [LNN14, Prop. 4.4.1].

For simplicity write $\Omega_v := \Omega(\{v\})$ for $v \in \mathcal{V}$, and define

$$\Pi(I) := \begin{cases} \emptyset & \text{if } I = \emptyset \\ \sum_{v\in I} \Omega_v & \text{if } I \neq \emptyset. \end{cases}$$

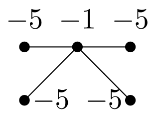
Proposition 3.6.2.1. *For any $I \subset \mathcal{V}$ let Γ_I be the smallest connected subtree of Γ whose set of vertices $\overline{I}$ contains I. Then $\Pi(J) = \Pi(I)$ for any $I \subset J \subset \overline{I}$.*

Proof. By induction, it is enough to consider the case $J = I \cup \{u\}$, such that u is on the geodesic path connecting v, w with $v, w \in I$. Moreover, it is enough to show that $\Omega_u \subset \Omega_v + \Omega_w$. Write the connected components of $\Gamma \setminus u$ as $\cup_{k=0}^{s}\Gamma_k$, and set

$I_k := \mathcal{V}(\Gamma_k)$. Assume that $w \in I_0$.

Choose an arbitrary $\omega \in \Omega_u$ and consider its restriction $\omega|_{\widetilde{X}(I_0)}$ in $\overline{\Omega}(\emptyset) := H^0(\Omega^2(\widetilde{X}(I_0) \setminus E_{I_0}))/H^0(\Omega^2(\widetilde{X}(I_0)))$. By Proposition 3.6.1.2(b) $\Omega(\mathcal{V} \setminus I_0) \to \overline{\Omega}(\emptyset)$ is bijective, hence there exists $\omega_v \in \Omega(\mathcal{V} \setminus I_0)$ such that $\omega_v|_{\widetilde{X}(I_0)} = \omega|_{\widetilde{X}(I_0)}$. But $\Omega_v \supset \Omega(\mathcal{V} \setminus I_0)$, hence $\omega_v \in \Omega_v$. On the other hand, $(\omega - \omega_v)|_{\widetilde{X}(I_0)} = 0$, hence $\omega_w := \omega - \omega_v \in \Omega_w$. Thus $\omega = \omega_v + \omega_w \in \Omega_v + \Omega_w$. $\square$

Example 3.6.2.2. Consider the weighted homogeneous isolated hypersurface singularity $(X, o) = \{x^4 + y^4 + z^5 = 0\} \subset (\mathbb{C}^3, 0)$. One verifies that $p_g = 4$ (use either [Pi77]). We consider the minimal good resolution, whose graphs is



If ω is the Gorenstein form, then ω, $z\omega$, $x\omega$ and $y\omega$ generate $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}})$. The pole orders along the central curve E_0 are 7, 3, 2, 2. Let v_i $(1 \leq i \leq 4)$ be the end–vertices. Then for fixed i, $\mathcal{V} \setminus \{v_i\}$ represents a minimally elliptic singularity. Hence $\Omega_{v_i} \simeq \mathbb{C}$ by (3.4.1.12) and Theorem 3.6.1.3. If ξ_i are the roots of $\xi^4 + 1 = 0$, then $(x + \xi_i y)\omega$ generates Ω_{v_i}, hence $\sum_{i=1}^4 \Omega_{v_i} \simeq \mathbb{C}^2 = \langle x\omega, y\omega \rangle$.

In particular, the linear subspace arrangement $\{\Omega_v\}_v$ in $\mathbb{C}^{p_g} = \mathbb{C}^4$ is not generic at all. Furthermore, $\Omega_{v_0} = 0$ hence 3.6.2.1 can also be exemplified on this concrete example.

3.6.3 Reduction to an arbitrary $Z > 0$.

The duality from Theorem 3.6.1.3, valid for $\widetilde{X}$ (or, for any $Z \gg 0$) can be generalized for any $Z \geq E$ as follows. For the definition of $V_Z(I)$ see Definitions 3.4.1.3 and 3.4.1.7. In parallel, define $\Omega_Z(I)$ as the subspace $H^0(\Omega^2_{\widetilde{X}}(Z|_{\mathcal{V}\setminus I}))/H^0(\Omega^2_{\widetilde{X}})$ in $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$. By (3.5.1.4) $\dim H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) = h^1(\mathcal{O}_Z)$, while $\dim \Omega_Z(I) = h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}})$. But, by pairing (similarly as in the proof of Theorem 3.6.1.3)

$V_Z(I) \subset \Omega_Z(I)^*$. Furthermore, by (3.4.1.12), $\dim V_Z(I) = \dim \Omega_Z(I)^*$. Hence

$$V_Z(I) = \Omega_Z(I)^*. \tag{3.6.3.1}$$

3.7 The 'stable' dimensions $\{\dim(V_Z(I))\}_I$ and natural line bundles

3.7.1 Natural line bundles and the image of the Abel map

Recall that the *saturation* in $\mathcal{S}'$ of a submonoid $\mathcal{M} \subset \mathcal{S}'$ is the submonoid $\overline{\mathcal{M}} := \{l' \in \mathcal{S}' : \exists\, n \geq 1 \text{ with } nl' \in \mathcal{M}\}$.

Let us fix some cycle $Z \geq E$. Recall that $\mathrm{ECa}^{-l'}(Z) \neq \emptyset$ if and only if $l' \in \mathcal{S}'$. For $l' \in \mathcal{S}'$ regarding the mutual position of the natural line bundle $\mathcal{O}_Z(-l')$ with respect to the image of $c^{-l'} : \mathrm{ECa}^{-l'}(Z) \to \mathrm{Pic}^{-l'}(Z)$ we can consider three cases.

(a) $\mathcal{O}_Z(-l') \in \mathrm{im}(c^{-l'})$, *or, equivalently,* $0 \in \mathrm{im}(\widetilde{c}^{-l'})$. The set of cycles l' satisfying this property is denoted by $\mathcal{S}'_{im}$. Clearly $0 \in \mathcal{S}'_{im}$ and by the first paragraphs of 3.4.1 it is a sub-monoids of $\mathcal{S}'$. (In the literature, this monoid — defined for bundles over $Z \gg 0$, or over $\widetilde{X}$ —, is called the *analytic monoid* of (X, o), in contrast with the *topological monoid* $\mathcal{S}'$, since it indexes the restrictions to E of the divisors of different holomorhic sections of the natural line bundles of $\widetilde{X}$, or divisors of fuctions of the universal abelian covering of (X, o), cf. [N99b].)

(b) $\mathcal{O}_Z(-nl') \in \mathrm{im}(c^{-nl'})$, or $0 \in \mathrm{im}(\widetilde{c}^{-nl'})$, for $n \gg 1$. The cycles l' satisfying this property are indexed by $\overline{\mathcal{S}'_{im}}$.

(c) $l' \in \mathcal{S}' \setminus \overline{\mathcal{S}'_{im}}$.

Example 3.7.1.1. In general, $\mathcal{S}'_{im} \subsetneq \overline{\mathcal{S}'_{im}}$. E.g. in Example 3.1.4.3, $\mathcal{O}_Z(-Z_{min}) \notin \mathrm{im}(c)$, however $\mathcal{O}_Z(-2Z_{min}) \in \mathrm{im}(c)$. Furthermore, in general, $\overline{\mathcal{S}'_{im}} \subsetneq \mathcal{S}'$ either. Indeed, take e.g. a situation when $\mathrm{im}(c^{-l'})$ is a point different than $\mathcal{O}_Z(-l')$. Then

$\mathcal{O}_Z(-nl') \notin \mathrm{im}(c^{-nl'})$ for $n \geq 1$, hence $nl' \notin \mathcal{S}'_{im}$ for $n \geq 1$. In such cases $\mathcal{S}' \setminus \overline{\mathcal{S}'_{im}}$ is even infinite. For a concrete example see the last case of 3.1.4.1.

Lemma 3.7.1.2. *Let $Z \geq E$ be an arbitrary cycle as above.*

(a) Fix $l' \in -\mathcal{S}'$ as above, and assume that $n \geq 1$ satisfies the next assumptions:

(i) $\mathrm{im}(\widetilde{c}^{nl'}) = A(nl')$ (automatically satisfied if n is sufficiently large, cf. Theorem 3.4.1.9),

(ii) $0 \in \mathrm{im}(\widetilde{c}^{nl'})$.

Then $0 \in A(l')$ and $\mathrm{im}(\widetilde{c}^{ml'}) = A(l')$ for any $m \geq n$.

(b) $\overline{\mathcal{S}'_{im}} = \mathcal{S}'$ if and only if $\mathcal{S}' \setminus \mathcal{S}'_{im}$ is finite.

Proof. (a) Since $0 \in A(nl')$, by Theorem 3.4.1.9*(a)* necessarily $A(kl') = A(l') = V(l')$ for any $k \geq 1$. Fix $\mathcal{L} \in \mathrm{im}(\widetilde{c}^{kl'})$. Then, $\mathcal{L} \in A(kl')$ and by (3.4.1.2) and Lemma 3.4.1.6, $A(l') = A(l') + \mathcal{L} \subset \mathrm{im}(\widetilde{c}^{nl'}) + \mathrm{im}(\widetilde{c}^{kl'}) \subset \mathrm{im}(\widetilde{c}^{(n+k)l'}) \subset A((n+k)l') = A(l')$. Part (b) follows from (a). $\square$

In the remaining part of this subsection we will work with line bundles defined over $Z \gg 0$.

Definition 3.7.1.3. (a) Following Neumann and Wahl [NW10], we say that (X, o) and its resolution ϕ satisfy the *End Curve Condition* (ECC) if $E^*_v \in \mathcal{S}'_{im}$ for any *end vertex* $v \in \mathcal{V}$ (i.e. for $\delta_v = 1$).

(b) We say that (X, o) and its resolution ϕ satisfies the *Weak End Curve Condition* (WECC) if $E^*_v \in \overline{\mathcal{S}'_{im}}$ for any end vertex $v \in \mathcal{V}$.

If we restrict ourselves to singularities with rational homology sphere links, by End Curve Theorem [NW10] (see also [O10]) singularities which satisfy ECC are exactly the splice quotient singularities of Neumann and Wahl [NW05]. The WECC terminology is new in the literature, however its necessity and importance appeared in many private discussions of the second author with T. Okuma in the last decade. The main question regarding singularities satisfying WECC is how can one generalize the results valid for splice quotient singularities to this larger family. The present article shows that e.g. the p_g–additivity formula of Okuma extends. Indeed, the general additivity formula (3.4.1.12) provides an additivity with correction term $\dim V_{\widetilde{X}}(I)$. Furthermore, as we will see in the next discussions, the correction term $\dim V_{\widetilde{X}}(I)$ has different reinterpretations in terms of certain Hilbert polynomials or Poincaré series (similarly as in the splice quotient case) whenever WECC is satisfied.

Proposition 3.7.1.4. *(a)* (**Convexity property of** $\overline{\mathcal{S}'_{im}}$) *Fix $u, v \in \mathcal{V}$, $u \neq v$. If E^*_u, $E^*_v \in \overline{\mathcal{S}'_{im}}$ then for any vertex w on the geodesic path in the graph connecting u and v one has $E^*_w \in \overline{\mathcal{S}'_{im}}$ too.*

(b) (X, o) satisfies WECC if and only if $\overline{\mathcal{S}'_{im}} = \mathcal{S}'$.

Proof. Fix integers n_u, n_v, n_w sufficiently large such that (i) $n_uE^*_u$, $n_vE^*_v$, $n_wE^*_w$ belong to L, (ii) the E_w–multiplicities of these three cycles are equal, and (iii) $n_uE^*_u$ and $n_vE^*_v$ belong to $\mathcal{S}'_{im}$. Set $l := n_uE^*_u - n_wE^*_w$, and let the connected components of $\Gamma \setminus w$ be $\cup_i\Gamma_i$. We distinguish Γ_{i_0}, which contains u. Then l is supported on $\cup_i\Gamma_i$. Since $(l, E_z) = 0$ for any $z \in \mathcal{V}(\cup_{i\neq i_0}\Gamma_i)$, $l|\Gamma_i = 0$ for all $i \neq i_0$. Since $(l, E_z) \leq 0$ for any $z \in \mathcal{V}(\Gamma_{i_0})$, and $(l, E_u) < 0$, all the entries of $l|\Gamma_{i_0}$ are strict positive. We have similar property for $n_vE^*_v - n_wE^*_w$ too. Hence $\min\{n_uE^*_u, n_vE^*_v\} = n_wE^*_w$. Since, by assumption there exist functions f_u and f_v, which can be regarded as sections of $\mathcal{O}(-n_uE^*_u)$ and $\mathcal{O}(-n_vE^*_v)$ without fixed components, the generic linear combination $af_u + bf_v$ is a section of $\mathcal{O}(-n_wE^*_w)$ without fixed components. For *(b)* use part *(a)* and the fact that Γ is a tree. $\square$

3.7.2 Different reinterpretations of $\dim(V_{\widetilde{X}}(l'))$ when $l' \in \overline{\mathcal{S}'_{im}}$.

In the sequel we apply the results of the previous section (e.g. Theorem 3.4.1.9) for natural line bundles. This will also include the 'classical' cases $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(-l)$, where l

is an effective integral cycle. In order to do this we will need additional assumptions of type $\mathcal{L} \in \mathrm{im}(c^{nl'})$.

We fix the following setup. We consider line bundles over $\widetilde{X}$, or over $Z \gg 0$. We write $V_{\widetilde{X}}(l')$ for the stabilized $V_Z(l')$ with $Z \gg 0$. We fix $l' \in \mathcal{S}'$ from $\overline{\mathcal{S}'_{im}}$, this means that there exists $n \gg 1$ such that $\mathcal{O}(-nl')$ admits sections without fixed components. Let $o \in \mathbb{Z}_{>0}$ be the order of $[l']$ in L'/L. We also write $ol' = l \in L$. Note that $V_{\widetilde{X}}(l') = V_{\widetilde{X}}(ol')$, cf. Lemma 3.4.1.6.

3.7.3 $\dim(V_{\widetilde{X}}(l'))$ as a coefficient of a Hilbert polynomial

Consider the situation of subsection 3.7.2. For $n \gg 1$ from the exact sequence of sheaves $0 \to \mathcal{O}_{\widetilde{X}}(-nl) \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{nl} \to 0$, we get

$$\dim H^0(\mathcal{O})/H^0(\mathcal{O}(-nl)) = \chi(nl) - h^1(\mathcal{O}(-nl)) + p_g(X, o),$$

which combined with Theorem 3.4.1.9 gives

$$\dim H^0(\mathcal{O})/H^0(\mathcal{O}(-nl)) = \chi(nl) + \dim V_{\widetilde{X}}(l). \tag{3.7.3.1}$$

This already shows that $V_{\widetilde{X}}(l)$ is the free term of the Hilbert polynomial associated with $n \mapsto \dim H^0(\mathcal{O})/H^0(\mathcal{O}(-nl))$. This fact can be reorganized even more. Note that by Theorem 3.4.1.9*(d)* $\mathcal{O}(-nl)$ is generated by global sections for all $n \geq n_0$ for some n_0. Therefore, if we denote the ideal $H^0(\widetilde{X}, \mathcal{O}(-n_0 l)) \subset \mathcal{O}_{X,o}$ by $\mathcal{J}$, then the integral closure of its powers satisfy $\overline{\mathcal{J}^m} = H^0(\widetilde{X}, \mathcal{O}(-mn_0 l))$ [Li69]. In particular, $\dim(\mathcal{O}_{X,o}/\overline{\mathcal{J}^m}) = \chi(mn_0 l) + \dim V_{\widetilde{X}}(l)$.

Recall that there exist integral coefficients $\overline{e}_i(\mathcal{J})$ (where $i = 1, 2, 3$) such that $\dim(\mathcal{O}_{X,o}/\overline{\mathcal{J}^m}) = \overline{e}_0(\mathcal{J})\binom{m+1}{2} - \overline{e}_1(\mathcal{J})\binom{m}{1} + \overline{e}_2(\mathcal{J})$ for $m \gg 1$. Here, the polynomial from the right hand side is called the *normal Hilbert polynomial of $\mathcal{J}$*. One verifies that $\overline{e}_2(\mathcal{J})$ is independent of the choice of n_0. Then, the two identities combined provide $\dim V_{\widetilde{X}}(l) = \overline{e}_2(\mathcal{J})$.

If in our general identities from Theorem 3.4.1.9 we insert $\overline{e}_2(\mathcal{J})$ for $\dim V_{\widetilde{X}}(l)$, then we recover e.g. the results from [OWY15a, §3]; or the additivity statement from [O08, Cor. 4.5].

3.7.4 $\dim(V_{\widetilde{X}}(l'))$ in terms of the multivariable series $P_{h=0}(\mathbf{t})$.

Assume again that $l' \in \overline{\mathcal{S}'_{im}}$, and let I be the E^*_v–support of l', that is, $l' = \sum_{v\in I} a_v E^*_v$ with $a_v \in \mathbb{Z}_{>0}$. Then with the notations of 3.7.2, for n sufficiently large $\mathcal{O}(-nol')$ has no fixed components and $h^1(\widetilde{X}, \mathcal{O}(-nl)) = p_g - \dim V_{\widetilde{X}}(I)$. This combined with (2.1.4.5) gives that for cycles of type nl ($n \gg 1$)

$$\sum_{\tilde{l}\in L,\ \tilde{l}\ngeq nl} p_{\mathcal{O}(-\tilde{l})} = \chi(nl) + \dim V_{\widetilde{X}}(I); \tag{3.7.4.1}$$

that is, the counting function $nl \mapsto \sum_{\tilde{l}\in L, \tilde{l}\ngeq nl} p_{\mathcal{O}(-\tilde{l})}$ of the coefficients of $P_{h=0}(\mathbf{t})$ is (for $n \gg 1$) the multivariable quadratic polynomial $\chi(nl) + \dim V_{\widetilde{X}}(I)$ in nl, whose free term is exactly $\dim V_{\widetilde{X}}(I)$.

The above counting function can be simplified even more: we will reduce the variables of P_0 to the variables indexed by I. For this we define the projection (along the E–coordinates) $\pi_{\mathcal{I}} : \mathbb{R}\langle E_v\rangle_{v\in\mathcal{V}} \to \mathbb{R}\langle E_v\rangle_{v\in\mathcal{I}}$, denoted also as $x \mapsto x|_{\mathcal{I}}$, by $\sum_{v\in\mathcal{V}} l_v E_v \mapsto \sum_{v\in\mathcal{I}} l_v E_v$.

For further motivations and topological analogues of the next statements see also [LNN14] (where $Z(\mathbf{t})$ plays the role of $P(\mathbf{t})$).

Lemma 3.7.4.2. *Assume that $l' = \sum_{v\in I} a_v E^*_v$ with $a_v > 0$, and $l'' \in \mathcal{S}'$ too. Then $l'' \geq l'$ if and only if $l''|_I \geq l'|_I$.*

Proof. We prove the $\Leftarrow$ part. Write $l'' - l'$ as $x + y$, where x (resp y) is supported on E_I (resp. on $E_{\mathcal{V}\setminus I}$). By assumption, $x \geq 0$. For any $u \in \mathcal{V} \setminus I$ one has $0 \geq (l'', E_u) =$

$(l', E_u) + (x, E_u) + (y, E_u)$. But $(l', E_u) = 0$ and $(x, E_u) \geq 0$. Hence $(y, E_u) \leq 0$ for any u in the support of y. Since $(\ ,\ )$ is negative definite, $y \geq 0$. $\square$

According to the π_I projection, we also define the series $P_{I,h}(\mathbf{t}_I)$ (for any $h \in H$), in variables $\{t_v\}_{v\in I}$ by $P_{I,h}(\mathbf{t}_I) := P_h(\mathbf{t})|_{t_v=1, v\notin I}$.

Note that the series $P_{I,0}(\mathbf{t}_I)$ has the form $\sum_{l_I\in\pi_I(\mathcal{S}'\cap L)} p_I(l_I)\mathbf{t}_I^{l_I}$. By Lemma 3.7.4.2 one has

$$\sum_{\tilde{l}\in L,\ \tilde{l}\not\geq nl} p_{\mathcal{O}(-\tilde{l})} = \sum_{l_I\in\pi_I(L),\ l_I\not\geq nl|_I} p_I(l_I).$$

Therefore, for $n \gg 1$, one also has that the counting function of the coefficients of the reduced series $P_{I,0}$ provides the same expression

$$\sum_{l_I\in\pi_I(L),\ l_I\not\geq nl|_I} p_I(l_I) = \chi(nl) + \dim V_{\widetilde{X}}(I). \tag{3.7.4.3}$$

(Note that if the E^*–support of nl is I, then $nl|_I$ determines uniquely nl.)

E.g., if $I = \{v\}$ (under the assumption $E_v^* \in \overline{\mathcal{S}'_{im}}$), $P_{I,0} = \sum_{m\geq 0} p_v(m)t_v^m$ has only one variable, and $\sum_{m\geq nl|_v} p_v(m) = \chi(nl) + \dim V_{\widetilde{X}}(I)$ for $n \gg 1$.

Theorem 3.7.4.4. *Assume that (X, o) is a splice quotient singularity associated with the graph Γ (or, equivalently, $\phi : \widetilde{X} \to X$ satisfies the ECC, cf. Definition 3.7.1.3). Then for any I the dimension $\dim V_{\widetilde{X}}(I)$ is topological, computable from Γ.*

Proof. For splice quotient singularities $P(\mathbf{t})$ equals the topological series $Z(\mathbf{t})$, cf. [N12]. Hence, in (3.7.4.1) the left hand side can be replaced by the corresponding sum of the coefficients of $Z(\mathbf{t})$. $\square$

Remark 3.7.4.5. Let us denote the Seiberg–Witten invariant of the link $M(\Gamma)$, associate with the canonical *spin*^{c}–structure of $M(\Gamma)$ with $\mathfrak{sw}_{can}(M(\Gamma))$, and the corresponding normalized Seiberg–Witten invariant by $\overline{\mathfrak{sw}}_{can}(M(\Gamma)) := \mathfrak{sw}_{can}(M(\Gamma)) + (Z_K^2 + |\mathcal{V}(\Gamma)|)/8$, see e.g. [LNN14]. Recall also that in the splice quotient case

$P(\mathbf{t}) = Z(\mathbf{t})$ (cf. [N12]). Therefore, if we replace in (3.7.4.3) $P(\mathbf{t})$ by $Z(\mathbf{t})$, in the terminology of [LNN14] (3.7.4.3) reads as follows: $\dim V_{\widetilde{X}}(I)$ is the periodic constant of the I–reduction $Z_{I,0}(\mathbf{t}_I)$ of $Z_0(\mathbf{t})$, and by Theorem 3.1.1 of [LNN14] it equals $-\overline{\mathfrak{sw}}_{can}(M(\Gamma)) + \overline{\mathfrak{sw}}_{can}(M(\Gamma \setminus I))$.

3.7.5 The equivariant version of 3.7.4.

Note that the identity (†) $h^1(\widetilde{X}, \mathcal{O}(-nl')) = p_g - \dim V_{\widetilde{X}}(I)$ holds uniformly for any $n \gg 1$, though $[nl'] \in H$ might have different H–classes. Such stability usually cannot be proved via cohomology exact sequence of type $0 \to \mathcal{L}(-l) \to \mathcal{L} \to \mathcal{L}|_l \to 0$, $l \in L_{>0}$ (since in this situation $c_1(\mathcal{L}(-l)) - c_1(\mathcal{L}) \in L$), or by eigenspace decomposition of some sheaf associated with the universal abelian cover (X_{ab}, o). Maybe one should emphasize that in the above identity (†) the contribution p_g comes from the dimension of $\mathrm{Pic}^{l'}$, which is independent of the class $[l'] \in H$, and not from the $p_g(X_{ab}, o)_h$ for $h = 0$.

Now, if we apply (2.1.4.5) for (†) for different classes we obtain the following fact. Let us fix, as above $l' \in \overline{\mathcal{S}'_{im}}$ with E^*–support I, and let us fix also some $k \in \mathbb{Z}_{\geq 0}$, $h := [kl'] \in H$, and write $kl' = r_h + l_k$ for some $l_k \in L$. Let o be the order of $[l']$ in H as above. Then from (2.1.4.5) one has

$$h^1(\mathcal{O}(-r_h - l_k - nol')) = -\sum_{a\in L,\, a\ngeq 0} p_{\mathcal{O}(-r_h-l_k-nol')} + p_g(X_{ab}, o)_h + \chi(l_k + nol') - (l_k + nol', r_h).$$

or, for any k and any $n \gg 1$,

$$\sum_{a\in L,\, a\ngeq 0} p_{\mathcal{O}(-r_h-l_k-nol')} = \chi(l_k + nol') - (l_k + nol', r_h) + p_g(X_{ab}, o)_h - p_g + \dim V_{\widetilde{X}}(I).$$

Hence $\dim V_{\widetilde{X}}(I)$ connects the asymptotic behaviour of *different* h–components of $P(\mathbf{t})$ of the form $h = [kl']$, $k \in \mathbb{Z}$.

3.8 The 'non–stable' $\dim \mathrm{im}(c^{l'})$ and differential forms.

3.8.1 Stabilization of the image

The first theorem of this section is a generalization of that statement of section 3.6, which says that for $Z \geq E$ the dual of the vector subspace $V_Z(nl') \subset H^1(\mathcal{O}_Z)$, the 'stable image affine subspace' $\mathrm{im}(\widetilde{c}^{nl'}) = A_Z(nl')$ ($n \gg 1$) shifted to the origin, agrees with the subspace of forms $\Omega_Z(I)$, where I is the E^*–support of l' (see Theorem 3.6.1.3 and subsection 3.6.3). $V_Z(nl')$ can also be interpreted (up to a shift) as the tangent space at any $\mathcal{L} \in A_Z(nl')$ of $A_Z(nl')$. Hence, $\mathcal{L} + V_Z(nl')$ is the intersection of all the kernels of linear maps $T_{\mathcal{L}}\omega$, where $\omega \in \Omega_Z(I)$ (that is, for all ω without pole along those E_v's which support the divisors from $\mathrm{ECa}^{nl'}(Z)$). For the explicit description of the duality see 3.5.1.

The new setup is the following. Consider a divisor $D \in \mathrm{ECa}^{l'}(Z)$, which is a union of (l', E) disjoint divisors $\{D_i\}_i$, each of them $\mathcal{O}_Z$–reduction of divisors $\{\widetilde{D}_i\}_i$ from $\mathrm{ECa}^{l'}(\widetilde{X})$ intersecting E transversally. Set $\widetilde{D} = \cup_i \widetilde{D}_i$ and $\mathcal{L} := \widetilde{c}^{l'}(D) \in H^1(Z, \mathcal{O}_Z)$. Set also $Z = \sum_v m_v E_v$.

We introduce a subsheaf $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}$ of $\Omega^2_{\widetilde{X}}(Z)$ consisting of those forms ω which have the property that the residue $\mathrm{Res}_{\widetilde{D}_i}(\omega)$ has no poles along $\widetilde{D}_i$ for all i. This means that the restrictions of $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}$ and $\Omega^2_{\widetilde{X}}(Z)$ on the complement of the support of $\widetilde{D}$ coincide, however along $\widetilde{D}$ is satisfies the following requirement. If $p = E \cap \widetilde{D}_i = E_{v_i} \cap \widetilde{D}_i$ has local coordinates (u, v) with $\{u = 0\} = E$ and $\widetilde{D}_i$ with local equation v, then a local section of $\Omega^2_{\widetilde{X}}(Z)$ near p has the form $\omega = \sum_{i \geq -m_{v_i}, j \geq 0} a_{i,j} u^i v^j du \wedge dv$. Then the residue $\mathrm{Res}_{\widetilde{D}_i}(\omega)$ is $(\omega/dv)|_{v=0} = \sum_i a_{i,0} u^i du$, hence the pole–vanishing reads as $a_{i,0} = 0$ for all $i < 0$. Note that $\Omega^2_{\widetilde{X}}(Z - \widetilde{D})$ and the sheaf of regular forms $\Omega^2_{\widetilde{X}}$ are subsheaves of $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}$.

Theorem 3.8.1.1. *In the above situation one has the following facts.*

(a) The sheaves $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}} / \Omega^2_{\widetilde{X}}$ and $\mathcal{O}_Z(K_{\widetilde{X}} + Z - D)$ are isomorphic.

(b) $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}})/H^0(\widetilde{X},\Omega^2_{\widetilde{X}}) \simeq H^1(Z,\mathcal{L})^*$. *(The left hand side can be regarded as a subspace of* $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}}) \simeq H^1(Z,\mathcal{O}_Z)^*$.*)*

(c) The image $T_D\widetilde{c}^{l'}(T_D\mathrm{ECa}^{l'}(Z))$ *of the tangent map* $T_D\widetilde{c}^{l'}$ *at* D *of* $\widetilde{c}^{l'} : \mathrm{ECa}^{l'}(Z) \to H^1(Z,\mathcal{O}_Z) = H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})$ *is the intersection of kernels of linear maps* $T_{\mathcal{L}}\omega : T_{\mathcal{L}}H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}) \to \mathbb{C}$, *where* $\omega \in H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}})$.

Proof. *(a)* Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & \Omega^2_{\widetilde{X}}(-\widetilde{D}) & \longrightarrow & \Omega^2_{\widetilde{X}}(Z-\widetilde{D}) & \longrightarrow & \mathcal{O}_Z(K_{\widetilde{X}}+Z-D) & \to & 0 \\ & & \big\downarrow \alpha & & \big\downarrow \beta & & \big\downarrow \gamma & & \\ 0 & \to & \Omega^2_{\widetilde{X}} & \longrightarrow & \Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}} & \longrightarrow & \Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}/\Omega^2_{\widetilde{X}} & \to & 0 \end{array}$$

Above α and β are the natural inclusions. We claim that their cokernels are isomorphic. Indeed, with the notation $M_{i,j} = u^iv^j du\wedge dv$ one has $\mathrm{coker}(\alpha) = \{\sum_{j\geq 0, i\geq 0} a_{i,j}M_{i,j}\}/\{\sum_{j\geq 1,i\geq 0} a_{i,j}M_{i,j}\}$ and $\mathrm{coker}(\beta) = \{\sum_{j\geq 0, i\geq -m_{v_i}} a_{i,j}M_{i,j} \,|\, a_{i<0,0} = 0\}/\{\sum_{j\geq 1, i\in\mathbb{Z}} a_{i,j}M_{i,j}\}$. Hence γ is an isomorphism.

(b) Since $H^1(\widetilde{X},\Omega^2_{\widetilde{X}}) = 0$, by part *(a)* we have $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}})/H^0(\widetilde{X},\Omega^2_{\widetilde{X}}) = H^0(\mathcal{O}_Z(K_{\widetilde{X}}+Z-D))$. But, this last one equals $H^1(Z,\mathcal{O}_Z(D))^*$ by Serre duality.

(c) We prove the statement in the case $(l',E)=1$, the general case follows similarly. Hence, set $l' = -E_v^*$ for some vertex $v\in\mathcal{V}$, that is, $\widetilde{D}$ is a transversal cut at the point p of the exceptional divisor E_v. Consider local coordinates (u,v) around p as above. Recall that the local equation of D is v. Let $\{\widetilde{D}_t\}_{t\in\mathbb{C},\,|t|\ll 1}$ be a path in $\mathrm{ECa}^{l'}$ at D whose local equation is $v+tu^{o-1}$ for some $o\geq 1$.

Consider also an arbitrary form $\omega \in H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))$ (with local equation as above). Then (the class of) ω is in the dual space of the image $T_D\widetilde{c}^{l'}(T_D\mathrm{ECa}^{l'}(Z))$ if and only if $(T_{\mathcal{L}}\omega)(T_D\widetilde{c}^{l'}(\delta)) = 0$ for all tangent vectors δ, the tangent vectors of paths of type D_t at D. But $T_{\mathcal{L}}\omega(T_D\widetilde{c}^{l'}(\delta)) = \lambda\cdot a_{-o,0}$ $(\lambda\neq 0)$ by 3.5.2.3. Therefore, the dual space of forms is exactly the class of forms from $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}})$.

In fact, one also sees that the dimensions of these two spaces $\mathrm{im}(T_D\widetilde{c})$ and $\cap_\omega T_{\mathcal{L}}\omega$

agree. Indeed, $\dim \mathrm{im}(T_D\widetilde{c}) = h^1(\mathcal{O}_Z) - h^1(Z,\mathcal{L})$ by (3.1.1.9). But, $\dim \cap_\omega T_{\mathcal{L}}\omega$ is the same by *(b)*. $\square$

Corollary 3.8.1.2. *Assume that* $\{\omega_1,\dots,\omega_h\}$ *form a basis of* $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$. *Then*

$$H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}})/H^0(\widetilde{X},\Omega^2_{\widetilde{X}}) = \{(a_1,\dots,a_h)\in\mathbb{C}^h : \mathrm{Res}_{\widetilde{D}_i}(\textstyle\sum_\alpha a_\alpha\omega_\alpha) \text{ has no pole along } \widetilde{D}_i \text{ for all } i\}.$$

Hence, by Theorem 3.8.1.1, the dimension of the right hand side is $h^1(Z,\mathcal{L})$, *and the number of independent relations between* $(a_1,\dots,a_h)$, $h^1(\mathcal{O}_Z)-h^1(Z,\mathcal{L})$, *is the dimension of* $\mathrm{im}T_Dc^{l'}(T_D\mathrm{ECa}^{l'}(Z))$.

In particular, $\dim(\mathrm{im}(c^{l'}(Z)))$ *is the number of independent relations for* $\{\widetilde{D}_i\}_i$ *generic.*

3.8.1.3. The above theorem can be applied rather directly in several situations, when we can provide a bases for $H^1(Z,\mathcal{O}_Z)^* = H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$, and verify directly for certain (or for all) divisors D the above pole–vanishing property. In the next subsections we provide such applications.

3.8.2 The Gorenstein case.

Assume that (X,o) is Gorenstein, fix a resolution $\widetilde{X}\to X$ as above, and let $\omega_0\in H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z_K))$ be the pullback of the Gorenstein form, well defined up to a non–zero constant. Its pole is Z_K, the (anti)canonical cycle. Since $\Omega^2_{\widetilde{X}}=\mathcal{O}_{\widetilde{X}}(-Z_K)$, $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z_K))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$ is isomorphic with $H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-Z_K))$, hence if we fix a basis of $H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-Z_K))$ consisting of classes of functions $\{f_1,\dots,f_{p_g}\}\subset H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}})$ with divisors $\mathrm{div}_E f_\alpha \not\geq Z_K$ then in $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$ the classes of forms $\{f_1\omega_0,\dots,f_{p_g}\omega_0\}$ form a basis.

Therefore, for any fixed $I \subset \mathcal{V}$,

$$\Omega(I) = \{(a_1, \dots, a_{p_g}) \in \mathbb{C}^{p_g} : m_{E_v}(\textstyle\sum_\alpha a_\alpha f_\alpha) \geq m_{E_v}(Z_K) \ \text{ for any } v \in I, \tag{3.8.2.1}$$

where $m_{E_v}(\cdot)$ denotes the coefficient of a cycle along E_v.

By Theorem 3.4.1.9 $\dim \Omega(I) = h^1(\widetilde{X}, \mathcal{L})$ for any $\mathcal{L}$ with $c_1(\mathcal{L}) = nl'$ with $n \gg 1$ and where $I := \{E^*\text{–support of } l'\}$. Furthermore, the number of independent relations between $(a_1, \dots, a_{p_g})$, $p_g - \dim \Omega(I)$, is the dimension of the *stable* $\mathrm{im}(c^{nl'})$ $(n \gg 1)$.

According to Theorem 3.8.1.1, these facts have the following generalizations. Set $\widetilde{D} = \cup_i \widetilde{D}_i$ be a divisor as in 3.8.1: each $\widetilde{D}_i$ is a transversal cut intersecting $E_{v(i)}$. Let $\gamma_i : (\mathbb{C}, 0) \to (\widetilde{D}_i, \widetilde{D}_i \cap E_{v(i)})$, $t \mapsto \gamma_i(t)$, be a parametrization (local diffeomorphism). Set $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(D)$ and $c_1(\mathcal{L}) = l'$.

Theorem 3.8.2.2. *With the above notations one has*

$$H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) = \{(a_1, \dots, a_{p_g}) \in \mathbb{C}^{p_g} : \mathrm{ord}_t(\textstyle\sum_\alpha a_\alpha f_\alpha \circ \gamma_i) \geq m_{E_{v(i)}}(Z_K) \ \textit{for all } i$$

Similarly as in Corollary 3.8.1.2, the dimension of the right hand side is $h^1(\widetilde{X}, \mathcal{L})$, and the number of independent relations between $(a_1, \dots, a_{p_g})$, $p_g - h^1(\widetilde{X}, \mathcal{L})$, is the dimension of $\mathrm{im} T_D c^{l'}(T_D \mathrm{ECa}^{l'}(Z))$ *($Z \gg 0$), and* $\dim(\mathrm{im}(c^{l'}))$ *is the number of independent relations for $\{\widetilde{D}_i\}_i$ generic.*

We will apply this theorem in section 3.9 for superisolated (hypersurface, hence Gorenstein) germs.

3.9 Superisolated singularities

3.9.1 The setup.

We will exemplify the Gorenstein case on a special family of isolated hypersurface singularities. The family of superisolated singularities creates a bridge between the theory of projective plane curves and the theory of surface singularities. This bridge will be present in the next discussions as well. For details and results regarding such germs see e.g. [Lu87, LNM05].

Assume that (X, o) is a hypersurface superisolated singularity. This means that (X, o) is a hypersurface singularity $\{F(x_1, x_2, x_3) = 0\}$, where the homogeneous terms $F_d + F_{d+1} + \cdots$ of F satisfy the following properties: $\{F_d = 0\}$ is reduced and it defines in $\mathbb{CP}^2$ an irreducible rational cuspidal curve C; furthermore, the intersection $\{F_{d+1} = 0\} \cap \mathrm{Sing}\{F_d = 0\}$ in $\mathbb{CP}^2$ is empty. The restrictions regarding F_d implies that the link of (X, o) is a rational homology sphere (this fact motivates partly the presence of these restrictions). With F_d fixed, all the possible choices for $\{F_i\}_{i>d}$ define an equisingular family of singularities with fixed topology and fixed $p_g = d(d-1)(d-2)/6$. For simplicity, here we will take for F_{d+1} the $(d+1)^{th}$–power of some linear function and $F_i = 0$ for $i > d+1$. Moreover, by linear change of variables, we can assume $F_{d+1} = -x_3^{d+1}$. (Note that in our treatment the analytic type of the singularity plays a crucial role, hence, by the choice $F_{d+1} = -x_3^{d+1}$ we restrict ourselves to a special analytic family. We do this since in this case the presentation of the next subsections are more transparent. However, it would be interesting to analyse the stability/non-stability of the Abel map in the whole equisingular family when we vary $F_i,\ i \geq d+1$.)

If we blow up the origin of $\mathbb{C}^3$ then the strict transform X' of X is already smooth (this property is responsible for the name 'superisolated') — hence a minimal resolution of X —, the exceptional curve $C' \subset X'$ is irreducible and it can be identified with C [Lu87]. Hence, resolving the plane curve singularities of C' we get a minimal good resolution of X; for the precise resolution graph see e.g. [Lu87, LNM05]. In the minimal (or the minimal good) resolution the exceptional curve corresponding to C' will be denoted by E_0.

In the chart $x_1 = uw,\ x_2 = vw,\ x_3 = w$ of the blow up the total transform has equation $w^d(w - F_d(u,v,1)) = 0$, $X' = \{w = F_d(u,v,1)\}$, $C' = \{w = F_d(u,v,1) = 0\}$.

We wish to discuss the Abel map associated with several choices of l' and Z.

3.9.2 The case $l' = -kE_0^*$ $(k \geq 1)$, $Z = Z_K$ (and generic divisor on $\mathrm{ECa}^{l'}(Z)$).

In this case a generic point D of $\mathrm{ECa}^{l'}(Z)$ consists of k transversal cuts of E_0 at generic points. In order to determine $\dim \mathrm{im}(c^{l'})$, which equals $\dim \mathrm{im} T_D\widetilde{c}^{l'}(T_D\mathrm{ECa}^{l'}(Z))$, we will apply Theorem 3.8.2.2. Hence, we need to analyse the restriction of forms on the components of the divisor D. Note that Theorem 3.8.2.2 automatically provides $h^1(Z_K, \mathcal{O}(D))$ too. Furthermore, by Grauert–Riemenschneider vanishing $h^1(\widetilde{X}, \mathcal{O}(\widetilde{D} - Z_K)) = 0$, one also has $h^1(Z_K, \mathcal{O}(D)) = h^1(\widetilde{X}, \mathcal{O}(\widetilde{D}))$.

Since the first blow up already creates the exceptional divisor $C' = E_0$, all the computation can be done in this minimal resolution $\phi : X' \to X$, and we can even assume that D is in the chart considered above. First, we find $\{f_\alpha\}_{\alpha=1}^{p_g}$ such that $\{f_1\omega_0, \cdots, f_{p_g}\omega_0\}$ induces a basis in $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$. Notice that the pull-back of any monomial $\mathbf{x}^{\mathbf{m}} = x_1^{m_1}x_2^{m_2}x_3^{m_3}$ has vanishing order $\deg(\mathbf{x}^{\mathbf{m}}) = \sum_i m_i = |\mathbf{m}|$ along E_0. Moreover, the multiplicity of Z_K along C' is $d-2$. Since the number of monomials of degree strict less than $d-2$ is $p_g = d(d-1)(d-2)/6$, the set $\{\mathbf{x}^{\mathbf{m}} : \deg(\mathbf{x}^{\mathbf{m}}) \leq d-3\}$ serve as a basis for $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_K))$.

Next, we consider parametrizations of each component $\{\widetilde{D}_i\}_{i=1}^k$ (the liftings of the divisors $\{D_i\}_i$), $t \mapsto \gamma_i(t) = (u_i(t), v_i(t), w_i(t)) \subset X'$. In fact, we can start with a parametrization $t \mapsto (u_i(t), v_i(t))$ of a transversal cut of $\{F_d(u,v,1) = 0\} \subset \mathbb{C}^2$ at some smooth point. Then we lift it to X' by setting $w_i(t) := f(u_i(t), v_i(t), 1)$. The tranversality implies that $w_i(t)$ has the form $c_1 t + c_2 t^2 + \cdots$ with $c_1 \neq 0$, hence after a reparametrization with $t' := w_i(t)$, we can assume that $w_i(t) = t$.

We denote the point $(u_i(0), v_i(0)) \in \{F_d(u,v,1) = 0\} \subset \mathbb{C}^2$ by p_i. We abridge $(u,v)^{\mathbf{m}}(p_i) := u_i(0)^{m_1} v_i(0)^{m_2}$. Then, the restriction of a monomial $\mathbf{x}^{\mathbf{m}}$ to $\widetilde{D}_i$ is

$$u(t)^{m_1} v(t)^{m_2} t^{|\mathbf{m}|} = t^{|\mathbf{m}|}\big((u,v)^{\mathbf{m}}(p_i) + H_{\mathbf{m}}(t)\big),$$

where $H_{\mathbf{m}}(t)$ denotes the 'higher order terms' with $H_{\mathbf{m}}(0) = 0$. Hence, by Theorem 3.8.2.2,

$$h^1(Z_K, \mathcal{O}(D)) = \dim \left\{ (a_{\mathbf{m}})_{\mathbf{m}} \in \mathbb{C}^{p_g} : \sum_{\mathbf{m}} a_{\mathbf{m}} \cdot \frac{(u,v)^{\mathbf{m}}(p_i) + H_{\mathbf{m}}}{t^{d-2-|\mathbf{m}|}} \text{ has no pole for all } i \right\}.$$

Expanding the sum into its Laurent series in t, and separating the coefficients of $\{t^{-d+2+j}\}_{0 \le j \le d-3}$, we get for each D_i a linear system with $d-2$ equations for the variable $(a_{\mathbf{m}})_{\mathbf{m}}$. We need to determine the rank of the corresponding matrix. This matrix has a natural block decomposition, a block is indexed by j and the set $\mathbf{m}$ with fixed $|\mathbf{m}|$. We prefer to order the rows by t^{-d+2}, $t^{-d+3}, \ldots, t^{-1}$.

E.g., for fixed D_i, the first row has its first entry 1 (corresponding to the block t^{-d+2} and $|\mathbf{m}| = 0$) and all other entries zero. The second row has some entry in the first place, the second block corresponding to t^{-d+3} and $|\mathbf{m}| = 1$) has three entries, namely $u(p_i), v(p_i), 1$ (which are the evaluations of the degree ≤ 1 (u,v)–monomials at p_i), and the blocks corresponding to $|\mathbf{m}| > 1$ are zero. More generally, above the diagonal all the blocks are zero, the diagonal block indexed by t^{-d+2+j} and $|\mathbf{m}| = j$ contains the evaluation of the (u,v)–monomials of degree $\le j$ at p_i.

E.g., if $k = 1$, then the matrix has $d-2$ rows and p_g columns, and each diagonal block contains one entry 1, hence its rank of the linear system is $d-2$. In particular,

$\dim \operatorname{im}(c^{-E_0^*}) = d-2$.

For $k \geq 2$, we have to put together all the linear equations corresponding to all D_i. A block indexed by t^{-d+2+j} and $|\mathbf{m}| = j'$ will have k rows. Again, all the blocks above the diagonal are zero. On the other hand, the rank of the diagonal block indexed by t^{-d+2+j} and $|\mathbf{m}| = j$ is as large as possible, it is $\min\{k, \binom{j+2}{2}\}$. Indeed, its rows consists of the evaluation of (u,v)–monomials of degree $\leq j$ at points p_i: since the points p_i are generic they impose independent conditions on the corresponding (homogeneous) linear system (in variable (x_1, x_2, x_3)) of degree j. Hence, the rank of the matrix is $\sum_{j=0}^{d-3} \min\{k, \binom{j+2}{2}\}$.

Theorem 3.9.2.1. *For any $k \geq 1$ the dimension of $\operatorname{im}(c^{-kE_0^*})$ is $\sum_{j=0}^{d-3} \min\{k, \binom{j+2}{2}\}$. The first value of k when $c^{-kE_0^*}$ is dominant is $k = \binom{d-1}{2}$. $\operatorname{im}(c^{-kE_0^*})$ has codimension 1 for $k = \binom{d-1}{2} - 1$.*

Accordingly, for a generic $\mathcal{L} \in \operatorname{im}(c^{-kE_0^*})$, $h^1(Z_K, \mathcal{L}) = p_g - \dim(\operatorname{im}(c^{-kE_0^*}))$.

3.9.3 The case $l' = -kE_0^*$ $(k \geq 1)$, $Z = Z_K$ (and special divisor on $\mathrm{ECa}^{l'}(Z)$).

In the previous subsection we considered *generic* points $\mathcal{P} := \{p_1, \ldots, p_k\}$ on C, in particular, for all j $(0 \leq j \leq d-3)$ they imposed independent conditions on the linear system $\mathcal{O}_{\mathbb{P}^2}(j)$ (or, on the (u,v)–monomials of degree $\leq j$). However, taking special points they might fail to impose independent conditions on some $\mathcal{O}_{\mathbb{P}^2}(j)$. The discussion will show that $\operatorname{im}(c^{l'})$ has several (rather complicated) h^1–stratification, (some of them) imposed by special divisors.

Here we will indicate such possibilities; nevertheless, for simplicity we will restrict ourselves only to certain cases when only one block degenerates and the rang of the total linear system is determined again by the diagonal blocks. Even under this restriction we find the situation extremely rich, since it accumulates the classical plane curve geometry. However, the reader is invited to work out cases when the global rank depends on certain entries from the sub–diagonal blocks as well, covering even more sophisticated h^1–strata.

Recall that in the diagonal block of $(t^{-d+2+j}, |\mathbf{m}| = j)$ we test if $\mathcal{P}$ impose independent conditions on $\mathcal{O}_{\mathbb{P}^2}(j)$ or not. In the sequel we will assume that there exits exactly one j, say j_0, when $\mathcal{P}$ fails to impose independent conditions. Clearly $j_0 > 0$. Furthermore, we will also assume that $\binom{j_0+1}{2} \le k \le \binom{j_0+3}{2}$. This means that in all the diagonal blocks with $j < j_0$ the number k of rows is greater than or equal to the number $\binom{j+2}{2}$ of columns, hence the j–blocks has rank $\binom{j+2}{2}$. Symmetrically, in all the j–diagonal blocks with $j > j_0$ the number k or rows is $\le$ than the number $\binom{j+2}{2}$ of columns, hence the rank is k. Therefore, if the j_0–block is degenerated with rank $\min\{k, \binom{j_0+2}{2}\} - \Delta$ for some $\Delta > 0$, then independently of the sub–diagonal entries, the rank of the matrix of the system is $\sum_{j=0}^{d-3} \min\{k, \binom{j+2}{2}\} - \Delta$. In particular, $h^1(Z_K, \mathcal{O}(D))$ increases by Δ compared with the generic situation of 3.9.2.

Let us list some cases when such a degeneration can occur. Take e.g. $j_0 = 1$ and $k = 3$ and $\{p_1, p_2, p_3\}$ are collinear. For $j_0 = 2$ we give two possibilities: either $k = 4$ and the four points are collinear, or $k = 6$ and the six points are contained in a conic.

We recall here two classical theorems of plane curve geometry, which can be used to produce similar examples; for more see the article [EGH96] and the citations therein.

(a) [EGH96, Prop. 1] For $j_0 \ge 1$ and $k \le 2j_0 + 2$ the points $\mathcal{P}$ fail to impose independent conditions on $\mathcal{O}_{\mathbb{P}^2}(j_0)$ if and only if either j_0+2 points of $\mathcal{P}$ are collinear or $k = 2j_0 + 2$ and $\mathcal{P}$ is contained in a conic.

(b) [EGH96, Th. Cayley-Bacharach4] Assume that $\mathcal{P}$ consists of $k = e \cdot f$ poinst which are the intersection points of two curves of degree e and f. Then if a plane curve of degree $j_0 = e + f - 3$ contains all but one point of $\mathcal{P}$ then it contains all of $\mathcal{P}$.

Chapter 4

Invariants of generic normal surface singularities

In this chapter we wish to define what we mean by a generic analytic structure corresponding to a fixed resolution graph $\mathcal{T}$ relying mostly on the results of Laufer about local deformation spaces of normal surface singularities.

Next we compute some analytic invariants of this generic analytic structure, like it's geometric genus and analytic PoincarĂŠ series.

4.1 Generic analytic structures on normal surface singularities

4.1.1 The setup

We fix a topological type of a normal surface singularity. This means that we fix either the C^∞ oriented diffeomorphism type of the link, or, equivalently, one of the dual graphs of a good resolution (all of them are equivalent up to blowing up/down rational (-1)–vertices). We assume that the link is a rational homology sphere, that is, the graph is a tree of rational vertices.

Any such topological type might support several analytic structures. The moduli space of the possible analytic structures is not described yet in the literature, hence we cannot rely on it. In particular, the 'generic analytic structure', as a 'generic' point of this moduli space, in this way is not well–defined. However, in order to run/prove the concrete properties regarding generic analytic structures, instead of such theoretical definition it would be even much better to consider a definition based on a list of stability properties under certain concrete deformations (whose validity could be expected for the 'generic' analytic structure in the presence of a classification space). Hence, for us in this note, a generic analytic structure will be a structure, which will satisfy such stability properties. In order to define them it is convenient to fix a resolution graph Γ and treat deformation of analytic structures supported on resolution spaces having dual graph Γ.

The type of stability we wish to have is the following. The topological type (or, the graph Γ) determines a lower bound for the possible values of the geometric genus (which usually depends on the analytic type). Let $\mathrm{MIN}(\Gamma)$ be the unique optimal bound, that is, $\mathrm{MIN}(\Gamma) \leq p_g(X,o)$ for any singularity (X,o) which admits Γ as a resolution graph, and $\mathrm{MIN}(\Gamma) = p_g(X,o)$ for some (X,o). Then one of the requirements for the 'generic analytic structure' (X_{gen},o) is that $p_g(X_{gen},o) = \mathrm{MIN}(\Gamma)$. (In the body of the paper $\mathrm{MIN}(\Gamma)$ will be determined explicitly.) However, we will need several similar stability requirements involving other line bundles as well (besides the trivial one, which provides p_g). For their definition we need a preparation.

4.1.2 The '0–generic analytic structure'

We wish to define when is the analytic structure of a fiber Z_q ($q \in Q$) of a deformation 'generic'. We proceed in two steps. The '0–genericity' is the first one (corresponding to the Chern class $l' = 0$), which will be defined in this subsection.

It is rather advantageous to set a definition, which is compatible with respect to all the restrictions $\mathcal{O}_Z \to \mathcal{O}_{Z'}$. In order to do this, let us fix the coefficients $\tilde{r} = \{\tilde{r}_v\}_v$ so large that for them Theorem 2.2.1.1 is valid. In this way basically we fix a resolution $(\widetilde{X}, E)$ and some large infinitesimal neighbourhood $Z(\tilde{r})$ associated with it. Moreover, let us also fix a *complete* deformation $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ whose fibers have the topological type of $\Gamma(\tilde{r})$. Next, we consider all the other coefficient sets $r := \{r_v\}_v$ such that $0 \le r_v \le \tilde{r}_v$ for all v, not all $r_v = 0$. Such a choice, by restriction as in 2.2.1.5, automatically provides a deformation $\lambda(r) : \mathcal{Z}(r) \to Q$. Then set

$$\Delta(0,r) := \{q \in Q \,:\, h^i(Z(r)_q, \mathcal{O}_{Z(r)_q}) \text{ is not constant in a neighbourhood of } q \text{ for some } i\}. \tag{4.1.2.1}$$

Then $\Delta(0,r)$ is a closed (reduced) proper subspace of Q, see [Ri74, Ri76] (one can use also an argument similar to Lemma 4.1.4.1 written for $l' = 0$). Define $\Delta^0(\tilde{r}) := \cup_{r_v \le \tilde{r}_v} \Delta(0,r)$. Then $\Delta^0(\tilde{r})$ is also closed and $\Delta^0(\tilde{r}) \neq Q$.

Definition 4.1.2.2. We say that the fiber $Z(\tilde{r})_q$ of $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ is 0–generic if $q \in Q \setminus \Delta^0(\tilde{r})$.

Next, we wish to generalize this definition for all Chern classes $l' \in L'$, or, for all ‘natural line bundles’, as generalizations of the trivial bundle corresponding to $l' = 0$.

4.1.3 The universal family of natural line bundles

Next, we wish to extend the definition of the line bundles $\mathcal{O}_Z(l')$ to the total space of a deformation (at least locally, over small balls in the complement of $\Delta^0(\tilde{r})$).

We fix some $Z = Z(\tilde{r})$ with all $\tilde{r}_v \gg 0$, supported on E, such that Theorem 2.2.1.1 is valid (similarly as in 4.1.2). Fix also some $Y \subset Z$, and a complete deformation $\lambda : \mathcal{Z}(\tilde{r}) \to Q$ of (Z, Y) as in Definition 2.2.1.2 such that all the fibers have the same fixed topological type $\Gamma(\tilde{r})$. We consider the discriminant $\Delta^0(\tilde{r}) \subset Q$, and we fix some $q_0 \in Q \setminus \Delta^0(\tilde{r})$, and a small ball U, $q_0 \in U \subset Q \setminus \Delta^0(\tilde{r})$. Above U the topologically trivial family of irreducible exceptional curves form the irreducible divisors $\{\mathcal{E}_v\}_v$, such that $\mathcal{E}_v$ above any point $q \in U$ is the corresponding irreducible exceptional curve $E_{v,q}$ of $\widetilde{X}_q$. With the notations of the previous paragraph, if nl' has the form $\sum_v n_v E_v$ write $\mathrm{div}_\lambda(nl') := \sum_v n_v \mathcal{E}_v$ for the corresponding divisor in $\lambda^{-1}(U)$. Since U is contractible, one has $H^2(\lambda^{-1}(U), \mathbb{Z}) = L'$ and $H^1(\lambda^{-1}(U), \mathbb{Z}) = 0$, hence the exponential exact sequence on $\lambda^{-1}(U)$ gives

$$0 \to \mathrm{Pic}^0(\lambda^{-1}(U)) \longrightarrow \mathrm{Pic}(\lambda^{-1}(U)) \xrightarrow{c_1} L' \to H^2(\lambda^{-1}(U), \mathcal{O}_{\lambda^{-1}(U)}). \qquad (4.1.3.1)$$

Lemma 4.1.3.2. *$H^2(\lambda^{-1}(U), \mathcal{O}_{\lambda^{-1}(U)}) = 0$ and the first Chern class morphism c_1 in (4.1.3.1) is onto.*

Proof. We use the Leray spectral sequence. Recall, see e.g. EGA III.2 §7, or [Os], that if $q \mapsto h^i(Z(\tilde{r})_q, \mathcal{O}_{Z(\tilde{r})_q})$ is constant over some open set U (and all i) then $R^i\lambda(\tilde{r})_*\mathcal{O}_{\mathcal{Z}(\tilde{r})}$ is locally free over U and $R^i\lambda(\tilde{r})_*\mathcal{O}_{\mathcal{Z}(\tilde{r})} \otimes_{\mathcal{O}_U} \mathbb{C}(q) \to H^i(Z(\tilde{r})_q, \mathcal{O}_{Z(\tilde{r})_q})$ is an isomorphism for $q \in U$.

Hence, since $R^i\lambda_*\mathcal{O}_{\lambda^{-1}(U)}$ is locally free, $H^i(U, R^{2-i}\lambda_*\mathcal{O}_{\lambda^{-1}(U)}) = 0$ for $i > 0$. On the other hand, $R^2\lambda_*\mathcal{O}_{\lambda^{-1}(U)} = 0$ since $R^2\lambda_*\mathcal{O}_{\lambda^{-1}(U)} \otimes_{\mathcal{O}_U} \mathbb{C}(q) \to H^2(Z(\tilde{r})_q, \mathcal{O}_{Z(\tilde{r})_q})$ is an isomorphism and $H^2(Z(\tilde{r})_q, \mathcal{O}_{Z(\tilde{r})_q}) = 0$ by dimension argument. $\square$

Then, if in the above construction of the split of c_1 in (2.1.4.1) we replace $\widetilde{X}$ by $\lambda^{-1}(U)$ and $\mathrm{div}(nl')$ by $\mathrm{div}_\lambda(nl')$, we get the following statement.

Lemma 4.1.3.3. *For any $l' \in L'$ there exists a divisor $D_\lambda(l')$ in $\lambda^{-1}(U)$ such that one has a linear equivalence $nD_\lambda(l') \sim \mathrm{div}_\lambda(nl')$ in $\lambda^{-1}(U)$ and $c_1(\mathcal{O}_{\lambda^{-1}(U)}(D_\lambda(l'))) = l'$. Furthermore, $D_\lambda(l')$ is unique up to linear equivalence, hence $l' \mapsto \mathcal{O}_{\lambda^{-1}(U)}(D_\lambda(l'))$ is a split of (4.1.3.1) which extends the natural split $L \ni \sum_v m_v E_v \mapsto \mathcal{O}_{\lambda^{-1}(U)}(\sum_v m_v \mathcal{E}_v)$ over L. Since $\mathrm{Pic}^0(\lambda^{-1}(U)) = H^1(\lambda^{-1}(U), \mathcal{O}_{\lambda^{-1}(U)})$ is torsion free, there exists a unique split over L' with this extension property.*

Let us summarize what we obtained: For any $q_0 \in Q \setminus \Delta^0(\tilde{r})$, and small ball U with $q_0 \in U \subset Q \setminus \Delta^0(\tilde{r})$, we have defined for each $l' \in L'$ a line bundle $\mathcal{O}_{\lambda^{-1}(U)}(D_\lambda(l'))$ in

$\mathrm{Pic}(\lambda^{-1}(U))$, such that its restriction to each fiber $Z(\tilde{r})_q$ is the line bundle $\mathcal{O}_{Z(\tilde{r})_q}(l')$. Let us denote it by $\mathcal{O}_{\lambda^{-1}(U)}(l')$.

4.1.4 The semicontinuity of $q \mapsto h^1(Z_q, \mathcal{O}_{Z_q}(l'))$

We fix a complete deformation $\lambda : \mathcal{Z}(\tilde{r}) \to Q$, and we consider the set of multiplicities $r_v \leq \tilde{r}_v$, not all zero, as in 4.1.2. Then, for each r, we have a restricted deformation $\lambda(r) : \mathcal{Z}(r) \to Q$ of $Z(r)$ as in 4.1.3.

Lemma 4.1.4.1. *For any restricted natural line bundle the map $q \mapsto h^i(Z(r)_q, \mathcal{O}_{Z(r)_q}(l'))$ is semicontinuous over $Q \setminus \Delta^0(\tilde{r})$, for $i = 0, 1$.*

(Note that if each $r_v > 1$ then the intersection form on $\Gamma(r)$ is well–defined. In particular, the semicontinuities of h^0 and h^1 are equivalent, since $h^0 - h^1 = (Z(r), l') + \chi(Z(r))$ by Riemann–Roch.)

Proof. We fix a small ball U in $Q \setminus \Delta^0(\tilde{r})$ as in subsection 4.1.3, and we run $q \in U$.

Let us denote (as above) the exceptional curves in the fiber $\lambda(r)^{-1}(q)$ by $\{E_{v,q}\}_v$, hence the cycle $Z(r)_q$ is $\sum_v r_v E_{v,q}$. Then one has the short exact sequence of sheaves

$$0 \to \mathcal{O}_{Z(r)_q} \otimes \mathcal{O}_{\lambda^{-1}(U)}(l') \to \oplus_v \mathcal{O}_{r_v E_{v,q}} \otimes \mathcal{O}_{\lambda^{-1}(U)}(l') \to \oplus_{(v,w)} \mathbb{C}\{x, y\}/(x^{r_v} y^{r_w}) \to 0,$$

where the sum in the last term runs over the edges (v, w) of $\Gamma(r)$. This gives the Mayer–Vietoris exact sequence

$$0 \to H^0(Z(r)_q, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{Z(r)_q}) \to \oplus_v H^0(r_v E_{v,q}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_v E_{v,q}}) \xrightarrow{\delta} \oplus_{(v,w)} \mathbb{C}\{x, y\}/(x^{r_v} y^{r_w}) \to .$$

Next, we analyse the vector space $H^0(r_v E_{v,q}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_v E_{v,q}})$ for any v. Let us fix an arbitrary $q_0 \in U$. Note that a singularity with a resolution consisting only one rational irreducible divisor is taut, see [La73b], hence the analytic family $\{Z(\tilde{r})_q\}_q$ restricted to $\{r_v E_{v,q}\}_v$ over a small neighbourhood $U' \subset U$ of q_0 can be trivialized.

Furthermore, $\mathrm{Pic}^0(r_vE_{v,q}) = 0$, hence the line bundle $\mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_vE_{v,q}}$ is uniquely determined topologically by l' and r. Hence, $\mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_vE_{v,q}}$ also can be trivialised over a small U'. In particular, by these trivializations, $H^0(r_vE_{v,q}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_vE_{v,q}})$ can be replaced by the fixed $H^0(r_vE_{v,q_0}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_vE_{v,q_0}})$, and the q–dependence is codified in the restriction morphism δ. Hence, there exists a morphism

$$\oplus_v\, H^0(r_vE_{v,q_0}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_vE_{v,q_0}}) \xrightarrow{\delta(q)} \oplus_{(v,w)}\mathbb{C}\{x,y\}/(x^{r_v}y^{r_w}) \tag{4.1.4.2}$$

whose kernel is $H^0(Z(r)_q, \mathcal{O}_{Z(r)_q}(l'))$. Since the rank of $\delta(q)$ is semicontinuous, the statement follows for h^0. But $h^1(Z(r)_q, \mathcal{O}_{Z(r)_q}(l')) = \dim\mathrm{coker}(\delta(q)) + h^1(r_vE_{v,q}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_vE_{v,q}})$, and the second term in this last sum is also topological and constant (by the same argument as above), hence semicontinuity for h^1 follows as well. $\square$

4.1.5 The 'generic analytic structure'

Now we are ready to give the definition of the 'generic structure'. Let us fix a *complete* deformation $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ as in 4.1.2 (with $\tilde{r}_v$ large) whose fibers have the topological type of $\Gamma(\tilde{r})$. Similarly as there, we consider all the other coefficient sets $r := \{r_v\}_v$ such that $r_v \leq \tilde{r}_v$ for all v, not all zero, and the induced deformations $\lambda(r) : \mathcal{Z}(r) \to Q$. Then for any $l' \in L'$ consider

$$\mathrm{MIN}(l', r) := \min_{q\in Q\setminus\Delta^0(\tilde{r})} \{h^1(Z(r)_q, \mathcal{O}_{Z(r)_q}(l'))\} \tag{4.1.5.1}$$

and

$$\Delta(l', r) := \text{closure of } \{q \in Q \setminus \Delta^0(\tilde{r}) \,:\, h^1(Z(r)_q, \mathcal{O}_{Z(r)_q}(l')) > \mathrm{MIN}(l', r)\}. \tag{4.1.5.2}$$

Then $\Delta(l', r)$ is a closed (reduced) proper subspace of Q (for this use e.g. an argument as in the proof of Lemma 4.1.4.1, or [Ri74, Ri76]). Then set the countable union of closed proper subspaces $\Delta(\tilde r) := (\cup_{l'\in L'}\ \cup_{r_v\leq \tilde r_v}\Delta(l',r))\cup\Delta^0(\tilde r)$. Clearly, $\Delta(\tilde r)\subsetneqq Q$.

Definition 4.1.5.3. (a) For a fixed $\Gamma(\tilde r)$ and for any complete deformation $\lambda(\tilde r):\mathcal{Z}(\tilde r)\to Q$ (with all $\tilde r_v\gg 0$) we say that the fiber $Z(\tilde r)_q$ of $\lambda(\tilde r):\mathcal{Z}(\tilde r)\to Q$ is generic if $q\in Q\setminus\Delta(\tilde r)$.

(b) Consider a singularity (X,o) and one of its resolutions $\widetilde X$ with dual graph Γ. We say that the analytic type on $\widetilde X$ is generic if there exists $\tilde r\gg 0$, and a complete deformation $\lambda(\tilde r):\mathcal{Z}(\tilde r)\to Q$ with fibers of topological type $\Gamma(\tilde r)$, and $q\in Q\setminus\Delta(\tilde r)$ such that $\lambda(\tilde r)^{-1}(q)=\mathcal{O}_{\widetilde X}|_{\sum_v \tilde r_v E_v}$.

Remark 4.1.5.4. (a) Fix any 1–dimensional space Z with fixed topology $\Gamma(\tilde r)$ with all $\tilde r_v\gg 0$. Then in any complete deformation λ of Z there exists a generic structure arbitrary close to Z.

(b) Though the above construction does not automatically imply that $Q\setminus\Delta(\tilde r)$ is open, for any $q_0\in Q\setminus\Delta(\tilde r)$ and for any *finite* set $FL'\subset L'$ there exists a small neighbourhood U of q_0 such that $h^1(\mathcal{O}_{Z(r)_q},\mathcal{O}_{Z(r)_q}(l'))=\mathrm{MIN}(l',r)$ for any r (as above), $l'\in FL'$, and $q\in U$.

(c) Fix a complete deformation $\lambda:\mathcal{Z}(\tilde r)\to Q$ of some (Z,Y) with some fixed $\tilde r_v\gg 0$ as above. Then, by Theorem 2.2.1.1*(b)* for any $q\in Q$ the fiber $Z(\tilde r)_q$ determines uniquely a holomorphic neighborhood $\widetilde X_q$ of E. (Some $\{\tilde r_v\}_v$ very large works uniformly for all fibers, since a convenient $\{\tilde r_v\}_v$ can be chosen topologically.) Furthermore, $h^1(\widetilde X_q,\mathcal{O}_{\widetilde X_q})$ can be recovered from λ as $h^1(Z(\tilde r)_q,\mathcal{O}_{Z(\tilde r)_q})$ by the formal function theorem. This is the geometric genus of the singularity (X_q,o) obtained by contracting E in this $\widetilde X_q$. Since $\Delta(0,\tilde r)=\{q\in Q\,:\,p_g(X_q,o)=\mathrm{MIN}(\Gamma)\}$ is part of the discriminant $\Delta(\tilde r)$ (and it is closed), for any 'generic' $q\in Q\setminus\Delta(\tilde r)$ there is a ball $q\in U\subset Q\setminus\Delta(0,\tilde r)$ such that λ simultaneously blows down to a flat family $\mathcal{X}\to U$. This follows from [Ri74, Ri76, Wa76] by the constancy of Γ and p_g.

4.1.6 Extension of sections.

Consider a complete deformation $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ as above, and *let* $Z(\tilde{r})_q$ *be a generic fiber* as in Definition 4.1.5.3. Let U be a small neighbourhood of q such that $U \subset Q \setminus \Delta^0(\tilde{r})$. For any $l' \in L'$ fixed consider the universal family of line bundles $\mathcal{O}_{\lambda^{-1}(U)}(D_\lambda(l'))$ constructed in subsection 4.1.3. Fix also some $r := \{r_v\}_v$ $(0 \leq r_v \leq \tilde{r}_v$ for all v, not all $r_v = 0$, as above). Assume that $\mathcal{O}_{Z(r)_q}(l') = \mathcal{O}_{\lambda^{-1}(U)}(D_\lambda(l'))|_{Z(r)_q}$ admits a global section $s \in H^0(Z(r)_q, \mathcal{O}_{Z(r)_q}(l'))$ without fixed components.

Lemma 4.1.6.1. *After decreasing U if it necessary, the following facts hold:*

(a) the section s has an extension $\mathfrak{s} \in H^0(\lambda(r)^{-1}(U), \mathcal{O}_{\lambda(r)^{-1}(U)}(D_\lambda(l'))$ with $\mathfrak{s}_q = s$.

(b) $\mathfrak{s}_{q'}$ $(q' \in U,\ q' \neq q)$ has no fixed components either.

Proof. (a) Since $Z(\tilde{r})_q$ is generic, q does not sit in the union of the discriminant spaces considered in 4.1.5. In that subsection we considered all the discriminants associated with all the Chern classes and the 'r–tower', hence, in particular, we had countably many discriminant obstructions. By assumption, q is not contained in any of these. In this proof we have to concentrate on the Chern class l' and the tower level $Z(r)$, hence only one discriminant. In particular, $q \in Q$ has a small neighbourhood which does not intersect it. Therefore, decreasing the representative of (Q, q) we get the stability of the corresponding h^1–cohomology sheaves. Furthermore, λ is proper, $\mathcal{O}_{\lambda(r)^{-1}(U)}(D_\lambda(l'))$ is coherent, and $q' \mapsto h^1(Z(r)_{q'}, \mathcal{O}_{Z(r)_{q'}}(l'))$ is constant. Hence by EGA III.2 §7 (or, see e.g. [Os]), $R^0\lambda_*(\mathcal{O}_{\lambda(r)^{-1}(U)}(D_\lambda(l')))$ is locally free and $R^0\lambda_*(\mathcal{O}_{\lambda(r)^{-1}(U)}(D_\lambda(l'))) \otimes_{\mathcal{O}_{(Q,q)}} \mathbb{C}(q) \to H^0(Z(r)_q, \mathcal{O}_{Z(r)_q}(l'))$ is an isomorphism. $\square$

4.2 A special 1–parameter deformation.

4.2.1 The construction of the deformation

Next, we describe a special 1–parameter deformation of a fixed resolution of a normal surface singularity (X, o), what will play a crucial role in the proof of the main Theorem 4.3.1.1.

We choose any good resolution $\phi : (\widetilde{X}, E) \to (X, o)$, and write $\cup_v E_v = E = \phi^{-1}(o)$ as above. Since each E_v is rational, a small tubular neighborhood of E_v in $\widetilde{X}$ can be identified with the disc-bundle associated with the total space $T(e_v)$ of $\mathcal{O}_{\mathbb{P}^1}(e_v)$, where $e_v = E_v^2$. (We will abridge $e := e_v$.) Recall that $T(e)$ is obtained by gluing $\mathbb{C}_{u_0} \times \mathbb{C}_{v_0}$ with $\mathbb{C}_{u_1} \times \mathbb{C}_{v_1}$ via identification $\mathbb{C}^*_{u_0} \times \mathbb{C}_{v_0} \sim \mathbb{C}^*_{u_1} \times \mathbb{C}_{v_1}$, $u_1 = u_0^{-1}$, $v_1 = v_0 u_0^{-e}$, where $\mathbb{C}_w$ is the affine line with coordinate w, and $\mathbb{C}^*_w = \mathbb{C}_w \setminus \{0\}$.

Next, fix any curve E_w of $\phi^{-1}(o)$ and also a *generic* point $P_w \in E_w$. There exists an identification of the tubular neighbourhood of E_w via $T(e)$ such that $u_1 = v_1 = 0$ is P_w. By blowing up $P_w \in \widetilde{X}$ we get a second resolution $\psi : \widetilde{X}' \to \widetilde{X}$; the strict transforms of $\{E_v\}$'s will be denoted by E'_v, and the new exceptional (-1) curve by E_{new}. If we contract $E'_w \cup E_{new}$ we get a cyclic quotient singularity, which is taut, hence the tubular neighbourhood of $E'_w \cup E_{new}$ can be identified with the tubular neighbourhood of the union of the zero sections in $T(e-1) \cup T(-1)$. Here we represent $T(e-1)$ as the gluing of $\mathbb{C}_{u'_0} \times \mathbb{C}_{v'_0}$ with $\mathbb{C}_{u'_1} \times \mathbb{C}_{v'_1}$ by $u'_1 = u_0'^{-1}$, $v'_1 = v'_0 u_0'^{-e+1}$. Similarly, $T(-1)$ as $\mathbb{C}_\beta \times \mathbb{C}_\alpha$ with $\mathbb{C}_\delta \times \mathbb{C}_\gamma$ by $\delta = \beta^{-1}$, $\gamma = \alpha\beta$. Then $T(e-1)$ and $T(-1)$ are glued along $\mathbb{C}_{u'_1} \times \mathbb{C}_{v'_1} \sim \mathbb{C}_\beta \times \mathbb{C}_\alpha$ by $u'_1 = \alpha$, $v'_1 = \beta$ providing a neighborhood of $E'_w \cup E_{new}$ in $\widetilde{X}'$. Then the neighbourhood $\widetilde{X}'$ of $\cup_v E'_v \cup E_{new}$ will be modified by the following 1–parameter family of spaces: the neighbourhood of $\cup_v E'_v$ will stay unmodified, however $T(-1)$, the neighbourhood of E_{new} will be glued along $\mathbb{C}_{u'_1} \times \mathbb{C}_{v'_1} \sim \mathbb{C}_\beta \times \mathbb{C}_\alpha$ by $u'_1 + t = \alpha$, $v'_1 = \beta$, where $t \in (\mathbb{C}, 0)$ is a small holomorphic parameter. The smooth complex surface obtained in this way will be denoted by $\widetilde{X}'_t$, and the ‘moved’ (-1)–curve in $\widetilde{X}'_t$ by $E_{new,t}$. If we blow down $E_{new,t}$ we obtain the surface $\widetilde{X}_t$.

By construction, the family of spaces $\{\widetilde{X}'_t\}_{t\in(\mathbb{C},0)}$ form a smooth 3–fold $\widetilde{\mathcal{X}}'$, together with a flat map $\lambda' : (\widetilde{\mathcal{X}}', \widetilde{X}') \to (\mathbb{C}, 0)$, a C^∞ trivial fibration, such that $\lambda'^{-1}(t) = \widetilde{X}'_t$. Similarly, the family $\{\widetilde{X}_t\}_{t\in(\mathbb{C},0)}$ form a smooth 3–fold $\widetilde{\mathcal{X}}$, together with a flat map $\lambda : (\widetilde{\mathcal{X}}, \widetilde{X}) \to (\mathbb{C}, 0)$, a C^∞ trivial fibration, such that $\lambda^{-1}(t) = \widetilde{X}_t$.

Remark 4.2.1.1. Such a deformation $\lambda : (\widetilde{\mathcal{X}}, \widetilde{X}) \to (\mathbb{C}, 0)$, reduced to some $\Gamma(\tilde{r})$, say with $\tilde{r} \gg 0$, is always the pullback of a complete deformation of $\mathcal{O}_{\widetilde{X}}|Z(\tilde{r})$. Hence, if $\widetilde{X}$ is generic, then the base point q_0 corresponding to the fiber $\mathcal{O}_{\widetilde{X}}|Z(\tilde{r})$ is in $Q \setminus \Delta(\tilde{r})$. Since for such q_0 there is a ball $q \in U \subset Q \setminus \Delta(0, \tilde{r})$ such that λ simultaneously blows down to a flat family $\mathcal{X} \to U$ (cf. 4.1.5.4(c)), the deformation $\lambda : (\widetilde{\mathcal{X}}, \widetilde{X}) \to (\mathbb{C}, 0)$ also blows down to a deformation $\mathcal{X} \to (\mathbb{C}, 0)$ of (X, o). In fact, λ is a weak simultaneous resolution of the (topological constant) deformation $\mathcal{X} \to (\mathbb{C}, 0)$, cf. [La83, KSB88]. The point is that along the deformation λ automatically we will have the h^1–stabilities for *any* other finitely many restricted natural line bundles as well, cf. Remark 4.1.5.4(b) (that is, for the very same $\widetilde{X}$ and its deformation λ, the finitely many Chern classes — whose h^1–stability we wish — can be chosen arbitrarily, depending on the geometrical situation we treat).

4.3 The cohomology of restricted natural line bundles

4.3.1 The setup

We fix a normal surface singularity (X, o) and one of its good resolutions $\widetilde{X}$ with exceptional divisor E and dual graph Γ. For any integral effective cycle $Z = Z(r)$ whose support $|Z|$ is included in E (not necessarily the same as E) write $\mathcal{V}(|Z|)$ for the set of vertices $\{v: E_v \subset |Z|\}$ and $\mathcal{S}'(|Z|) \subset L'(|Z|)$ for the Lipman cone associated with the induced lattice $L(|Z|)$. As above, for any $l' \in L'$ we denote the restriction of the natural line bundle $\mathcal{O}_{\widetilde{X}}(l')$ to Z by $\mathcal{O}_Z(l')$. Denote also by $\tilde{l}$ the cohomological restriction $R(l')$ of $l' \in L'$ to $L'(|Z|)$. Recall also that for any $-\tilde{l} \in \mathcal{S}'(|Z|)$ one has the Abel map $c^{\tilde{l}} : \mathrm{ECa}^{\tilde{l}}(Z) \to \mathrm{Pic}^{\tilde{l}}(Z)$.

Theorem 4.3.1.1. *Assume that $\widetilde{X}$ is generic in the sense of Definition 4.1.5.3. Fix also some $Z = Z(r)$ as above. Choose $l' = \sum_{v \in \mathcal{V}} l'_v E_v \in L'$ such that $l'_v < 0$ for any $v \in \mathcal{V}(|Z|)$. Then the following facts hold.*

(I) Assume additionally that $-\tilde{l} \in \mathcal{S}'(|Z|) \setminus \{0\}$. Then the following facts are equivalent:

(a) $\mathcal{O}_Z(l') \in \mathrm{im}(c^{\tilde{l}})$, that is, $H^0(Z, \mathcal{O}_Z(l'))_{reg} \neq \emptyset$;

(b) $c^{\tilde{l}}$ is dominant, or equivalently, for a generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{\tilde{l}}(Z)$ one has $\mathcal{L}_{gen} \in \mathrm{im}(c^{\tilde{l}})$ (that is, $H^0(Z, \mathcal{L}_{gen})_{reg} \neq \emptyset$).

(c) $\mathcal{O}_Z(l') \in \mathrm{im}(c^{\tilde{l}})$, and for any $D \in (c^{\tilde{l}})^{-1}(\mathcal{O}_Z(l'))$ the tangent map $T_D c^{\tilde{l}} : T_D \mathrm{ECa}^{\tilde{l}}(Z) \to T_{\mathcal{O}_Z(l')} \mathrm{Pic}^{\tilde{l}}(Z)$ is surjective.

(II) $h^i(Z, \mathcal{O}_Z(l')) = h^i(Z, \mathcal{L}_{gen})$ for a generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{\tilde{l}}(Z)$ and $i = 0, 1$.

(For a remark regarding the assumptions of the theorem see 4.4.1.1(c).)

Remark 4.3.1.2. The theorem shows that if we fix $\Gamma(r)$ then the restrictions of natural line bundles of generic singularities cohomologically behave similarly as the generic line bundles. This is the main guiding principle of the present article. This principle, in general, can be formulated as follows. Fix some invariant associated with line bundles of resolutions with fixed graph and fixed Chern class. Then one expects that the invariant evaluated on the restricted natural line bundle in the context of the generic singularity agrees with the value of the invariant evaluated on the generic bundle with the same topological data (associated with an arbitrary fixed analytic type).

Note that by [NN18, Theorem 5.3.1] the cohomology of the generic line bundles depends only on the combinatorics of Γ (for the formula see e.g. the introduction or (4.4.1.2)).

4.3.1.3. Starting the proof of Theorem 4.3.1.1. We use double induction over the cardinality of the subset $\mathcal{V}(|Z|) \subset \mathcal{V}$ and $\sum_v r_v$.

If $|\mathcal{V}(|Z|)| = 1$ then $\mathrm{Pic}^0(Z) = 0$ and all line bundles with the same Chern class are isomorphic, hence all the statements are trivially true for any Z and any l'. Hence let us fix some virtual support $|Z|$ and assume that all the statements are valid for any cycle with support smaller than $|Z|$ and for any l' with the corresponding restrictions.

Next, we run induction over $\sum_{v\in\mathcal{V}(|Z|)} r_v$. Assume that $r_v \leq 1$ for all v. Then $\mathrm{Pic}^0(Z) = 0$ again and both (I) and (II) hold. Hence, we assume that (I) and (II) hold for all cycles with $\sum_v r_v < N$ (and any l' with the required restrictions) and we consider some $Z = Z(r)$ with $\sum_v r_v = N$.

4.3.1.4. The first part of the proof of Theorem 4.3.1.1(I). First we verify the 'easy' implications.

$(c) \Rightarrow (b)$ Since $\mathrm{ECa}^{\tilde{l}}(Z)$ is smooth (cf. [NN18, Th. 3.1.10]), by local submersion theorem, if $T_D c^{\tilde{l}}$ is surjective then the germ $c^{\tilde{l}} : (\mathrm{ECa}^{\tilde{l}}(Z), D) \to (\mathrm{Pic}^{\tilde{l}}(Z), \mathcal{O}_Z(l'))$ is surjective too. Since $c^{\tilde{l}}$ is an algebraic morphism and its image contains a small analytic ball of top dimension, $c^{\tilde{l}}$ is dominant.

$(b) \Rightarrow (a)$ Since $H^0(Z, \mathcal{L}_{gen})_{reg} \neq \emptyset$, one has $h^0(Z, \mathcal{L}_{gen}) \neq 0$, hence by the semi-continuity of $\mathcal{L} \mapsto h^0(Z, \mathcal{L})$ (cf. [NN18, Lemma 5.2.1]) $h^0(Z, \mathcal{O}_Z(l')) \neq 0$ too. Next, assume that $h^0(Z, \mathcal{O}_Z(l'))_{reg} = \emptyset$, that is, there exists $v \in \mathcal{V}(|Z|)$ such that $h^0(Z, \mathcal{O}_Z(l')) = h^0(Z - E_v, \mathcal{O}_Z(l')(-E_v))$. Note that $\mathcal{O}_Z(l')(-E_v)|_{Z-E_v}$ is also a restricted natural line bundle, it is $\mathcal{O}_{Z-E_v}(l' - E_v)$. Furthermore, from $l'_u < 0$ for $u \in \mathcal{V}(|Z|)$ we obtain $(l' - E_v)_u < 0$ too. Therefore, by the inductive step (part II) $h^0(Z - E_v, \mathcal{O}_Z(l' - E_v)) = h^0(Z - E_v, \mathcal{L}_{gen}(-E_v))$ and by the assumption

$h^0(Z-E_v, \mathcal{L}_{gen}(-E_v)) < h^0(Z, \mathcal{L}_{gen})$. Thus $h^0(Z, \mathcal{O}_Z(l')) < h^0(Z, \mathcal{L}_{gen})$, a fact, which contradicts the semicontinuity of $\mathcal{L} \mapsto h^0(Z, \mathcal{L})$.

The proof of $(a) \Rightarrow (c)$ in (I) is much harder and longer, and it is the core of the present theorem.

4.3.2 The proof of $(a) \Rightarrow (c)$ in short

The detailed proof is presented in 4.3.3; in this subsection we summarize the main steps in order to help the reading of the complete proof, though in this way inevitably some repetitions will occur. (Since the idea of the proof – based on the construction of the 1–parameter family – is quite fruitful, it will be used several times in forthcoming manuscripts as well, hence in the future work we will refer to these paragraphs as the basic prototype.)

First we identify $\mathrm{Pic}^{\tilde{l}}(Z)$ with $\mathrm{Pic}^0(Z)$ by $\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{O}_Z(-l')$, and $\mathrm{Pic}^0(Z)$ with $H^1(Z, \mathcal{O}_Z)$, and we replace $c^{\tilde{l}}(Z)$ with $\widetilde{c}^{l'}(Z) : \mathrm{ECa}^{\tilde{l}}(Z) \to H^1(\mathcal{O}_Z)$. Therefore, we wish to show that for any $D \in (\widetilde{c}^{l'})^{-1}(0)$ the tangent map $T_D\widetilde{c}^{l'} : T_D\mathrm{ECa}^{\tilde{l}}(Z) \to T_0H^1(\mathcal{O}_Z)$ is surjective.

Assume that this is not happening. Then there exists a linear functional $\varsigma \in H^1(\mathcal{O}_Z)^*$, $\varsigma \neq 0$, such that $\varsigma|_{\mathrm{im}(T_D\widetilde{c}^{l'})} = 0$. This lifts to a nonzero functional $\widetilde{\varsigma}$ of $H^1(\mathcal{O}_{\widetilde{X}})$, which necessarily has the form $\widetilde{\varsigma} = \langle \cdot, [\widetilde{\omega}] \rangle$ for some $\widetilde{\omega} \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$, which necessarily must have a pole along some E_w. Using [NN18] one shows that in fact we can choose $E_w \subset |Z|$. Next, we modify $\widetilde{X}$ by a sequence of blow ups. First we blow up $\widetilde{X}$ at generic point of E_w creating the new exceptional divisor F_1, then we blow up a generic point of F_1 creating F_2, etc. The sequence of n such blow ups will be denoted by $b_n : \widetilde{X}_n \to \widetilde{X}$, which has exceptional divisors $\cup_{i=1}^n F_i$. We define ς_n by the composition $H^1(\mathcal{O}_{b_n^*(Z)}) \to H^1(\mathcal{O}_Z) \xrightarrow{\varsigma} \mathbb{C}$ (where the first arrow is an isomorphism by Leray spectral sequence); and similarly we set $\widetilde{\varsigma}_n$ associated with some $\widetilde{Z} \gg 0$ (instead of Z). Note that $\widetilde{\varsigma}_n \circ \widetilde{c}^{-F_n^*}(b_n^*(\widetilde{Z}))$ corresponds to an integration of the 2–form $b_n^*(\widetilde{\omega})$ paired with divisors supported on F_n. Since the pole order along F_n of $b_n^*(\widetilde{\omega})$ decreases by one after each blow up, after some steps n it will have no pole along F_n, hence $\varsigma_n \circ \widetilde{c}^{-F_n^*}(b_n^*(Z)) : \mathrm{ECa}^{-F_n^*}(b_n^*(Z)) \to H^1(\mathcal{O}_{b_n^*(Z)}) \to \mathbb{C}$ is constant. Let k be the smallest integer such that this map is constant. Then $b_k^*(\widetilde{\omega})$ has a pole of order one along F_{k-1}.

Next, let $U \subset \widetilde{X}_k$ be a small tubular neighbourhood of the exceptional curve $E_U := E \cup (\cup_{i=1}^{k-1} F_i)$. Let Γ_U be the dual graph of E_U. One considers the homological projection $\pi_U : L(\Gamma) \to L(\Gamma_U)$ and the cohomological restriction $R_U : L'(\Gamma) \to L'(\Gamma_U)$ (dual to the natural homological injection of cycles). Then first one identifies the germs in the corresponding spaces of effective Cartier divisors $(\mathrm{ECa}^{\tilde{l}}(Z), D) \simeq (\mathrm{ECa}^{b_k^*(\tilde{l})}(b_k^*(Z)), D) \simeq (\mathrm{ECa}^{R_U(b_k^*(\tilde{l}))}(\pi_U(b_k^*(Z))), D)$, then one shows that $(\mathrm{ECa}^{\tilde{l}}(Z), D) \xrightarrow{\widetilde{c}^{l'}} H^1(\mathcal{O}_Z) \xrightarrow{\varsigma} \mathbb{C}$ factorizes through $(\mathrm{ECa}^{R_U(b_k^*(\tilde{l}))}(\pi_U(b_k^*(Z))), D) \xrightarrow{\widetilde{c}^{R_U b_k^*(l')}} H^1(\mathcal{O}_{\pi_U(b_k^*(Z))}) \xrightarrow{\varsigma_k^U} \mathbb{C}$. This, and the choice of ς show that

$$(\dagger) \qquad \varsigma_k^U \circ T_D(\widetilde{c}^{R_U(b_k^*(l'))}(\pi_U(b_k^*(Z))) = 0.$$

Now we continue with the key construction of the proof. Using the exceptional divisors F_{k-1} and F_k we construct the 1–parameter family of deformation $\{\widetilde{X}_{k,t}\}_t$ of $\widetilde{X}_k$ (by mowing the intersection point of $F_{k,t}$ along F_{k-1}), as in section 4.2. In this deformation one considers the universal family of natural line bundles. Since in the central fiber D is the divisor of a section of the corresponding natural line bundle, and along the deformation the cohomology groups of the bundles are stable (here we use the genericity), by Lemma 4.1.6.1 this extends to a family of sections. In this way we construct a path in $\mathrm{ECa}^{R_U(b_k^*(\tilde{l}))}(\pi_U(b_k^*(Z)))$ at D, $t \mapsto \gamma(t)$ (or, $\{D_t\}_t$ with $D_0 = D$). By the choice of ς and $(\dagger)$ and the chain rule, $\varsigma \circ \widetilde{c} \circ \gamma$ must have zero derivative at $t = 0$. This is valid even for any common multiple of the divisors $\{D_t\}_t$. On the other hand, this derivative can be computed differently by Laufer integration. Indeed, by taking a convenient multiple, the corresponding powers of the members of the family of natural line bundles restricted on U have the form $\mathcal{O}_{\pi_U(b_k^*(Z))}(\sum_v Nl'_vE_v + \ell\sum_{i=1}^{k-1}F_i + \ell F_{k,t})$ with $\ell \neq 0$. Here $\ell F_{k,t} \cap F_{k-1}$ is moving divisor along F_{k-1}. It paired with the differential form of pole one by Laufer pairing has a non-trivial linear part, cf. (3.5.2.2). Hence its derivative at $t = 0$ is nonzero, a fact which contradicts the previous statement.

4.3.3 The detailed proof of $(a) \Rightarrow (c)$

Fix any $l^* \in L'$ and write $\bar{l} \in L'(|Z|)$ for its restriction. Then there is a canonical identification of $\mathrm{Pic}^{\bar{l}}(Z)$ with $\mathrm{Pic}^0(Z)$ by $\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{O}_Z(-l^*)$. Also, $\mathrm{Pic}^0(Z)$ identifies with $H^1(Z, \mathcal{O}_Z)$ by the inverse of the exponential map such that $\mathcal{O}_Z$ is identified with 0. In particular, $c^{\bar{l}}(Z) : \mathrm{ECa}^{\bar{l}}(Z) \to \mathrm{Pic}^{\bar{l}}(Z)$ can be identified with its composition with the above two maps, namely with $\widetilde{c}^{l^*}(Z) : \mathrm{ECa}^{\bar{l}}(Z) \to H^1(\mathcal{O}_Z)$. In the sequel l^* will stay either for l' or for different cycles of type E_u^* with $E_u \in |Z|$. In this latter case, the restriction of $E_u^* \in L'$ is $E_u^*(|Z|)$, where this second dual is considered in $L'(|Z|)$. We use sometimes the same notation E_u^* for both of them, from the context will be clear which one is considered.

Therefore, the wished statement $(a) \Rightarrow (c)$ transforms into the following: If $D \in (\widetilde{c}^{l'})^{-1}(0)$ then the tangent map $T_D\widetilde{c}^{l'} : T_D\mathrm{ECa}^{\bar{l}}(Z) \to T_0H^1(\mathcal{O}_Z)$ is surjective (under the assumptions of part (I)).

Assume that this is not the case for some D. Then there exists a linear functional $\varsigma \in H^1(\mathcal{O}_Z)^*$, $\varsigma \neq 0$, such that $\varsigma|_{\mathrm{im}(T_D\widetilde{c}^{l'})} = 0$. During the proof we fix such a $D \in (\widetilde{c}^{l'})^{-1}(0)$ and ς.

First, we concentrate on ς.

Lemma 4.3.3.1. *For any $\varsigma \in H^1(\mathcal{O}_Z)^*$, $\varsigma \neq 0$, there exists $E_w \subset |Z|$ such that $\varsigma \circ \widetilde{c}^{-E_w^*} : \mathrm{ECa}^{-E_w^*}(Z) \to \mathbb{C}$ is not constant.*

Proof. Let $\widetilde{Z}=\sum_v \tilde{r}_v E_v$ be a large cycle with all $\tilde{r}_v \gg 0$ ($v\in\mathcal{V}$) so that $h^1(\mathcal{O}_{\widetilde{Z}})=h^1(\mathcal{O}_{\widetilde{X}})$. Define $\widetilde{\varsigma}$ by the composition $H^1(\mathcal{O}_{\widetilde{Z}})\xrightarrow{\rho} H^1(\mathcal{O}_Z)\xrightarrow{\varsigma}\mathbb{C}$. Since ρ is onto, $\widetilde{\varsigma}\neq 0$ too. Recall that any functional on $H^1(\mathcal{O}_{\widetilde{X}})$ has the form $\widetilde{\varsigma}=\langle\cdot,[\widetilde{\omega}]\rangle$, cf. (3.5.1.2), for some $\widetilde{\omega}\in H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})$. Since $\widetilde{\varsigma}\neq 0$ the form necessarily must have a pole along some E_w. By combination of Theorems 6.1.9(d) and 8.1.3 of [NN18] we know that the kernel of ρ is dual with the subspace of forms which have no pole along $|Z|$. Therefore, $\widetilde{\omega}$ must have a pole along some $E_w\subset|Z|$. Since $\mathrm{ECa}^{-E^*_w}(Z)$ is the space of effective Cartier divisors of $\widetilde{X}$ (up to the equation of Z), which intersect (transversally) only E_w, again by local nature of the integration formula, $\widetilde{\varsigma}\circ\widetilde{c}^{-E^*_w}(\widetilde{Z}):\mathrm{ECa}^{-E^*_w}(\widetilde{Z})\to\mathbb{C}$ is nonconstant, cf. (3.5.2.2). But $\varsigma\circ\widetilde{c}^{-E^*_w}(Z)$ composed with $R:\mathrm{ECa}^{-E^*_w}(\widetilde{Z})\to\mathrm{ECa}^{-E^*_w}(Z)$ is exactly this map $\widetilde{\varsigma}\circ\widetilde{c}^{-E^*_w}(\widetilde{Z})$. Since R is surjective (cf. [NN18, Theorem 3.1.10]), $\varsigma\circ\widetilde{c}^{-E^*_w}(Z)$ is nonconstant too. $\square$

4.3.3.2. Let Z, ς and $E_w\subset|Z|$ be as in Lemma 4.3.3.1, and $\widetilde{\omega}$ as in its proof, $\widetilde{\varsigma}=\langle\cdot,[\widetilde{\omega}]\rangle$. We wish to modify the resolution $\widetilde{X}$ (and the space Z) dictated by a certain property of $\widetilde{\omega}$. For this we blow up $\widetilde{X}$ at generic point of E_w creating the new exceptional divisor F_1, then we blow up a generic point of F_1 creating the new exceptional divisor F_2, etc. The sequence of n such blow ups will be denoted by $b_n:\widetilde{X}_n\to\widetilde{X}$, which has exceptional divisors $\cup_{i=1}^n F_i$. Note also that $H^1(\mathcal{O}_{b_n^*(Z)})\to H^1(\mathcal{O}_Z)$ is an isomorphism (use Leray spectral sequence). We define ς_n by the composition $H^1(\mathcal{O}_{b_n^*(Z)})\to H^1(\mathcal{O}_Z)\xrightarrow{\varsigma}\mathbb{C}$.

Lemma 4.3.3.3. *For n sufficiently large the next morphism is constant:*

$$\varsigma_n\circ\widetilde{c}^{-F_n^*}(b_n^*(Z)):\mathrm{ECa}^{-F_n^*}(b_n^*(Z))\to H^1(\mathcal{O}_{b_n^*(Z)})\to\mathbb{C}. \tag{4.3.3.4}$$

Proof. Consider $\widetilde{Z}$ and the notations of the proof of Lemma 4.3.3.1, and the composition $\widetilde{\varsigma}_n\circ\widetilde{c}^{-F_n^*}(b_n^*(\widetilde{Z}))$, similar to (4.3.3.4), but with $\widetilde{Z}$ instead of Z. This for any n

gives the diagram

$$\begin{array}{ccccc} \mathrm{ECa}^{-F_n^*}(b_n^*(\widetilde{Z})) & \xrightarrow{\widetilde{c}^{-F_n^*}} & H^1(\mathcal{O}_{b_n^*(\widetilde{Z})}) & \xrightarrow{\widetilde{\varsigma}_n} & \mathbb{C} \\ \downarrow\!\!\!\downarrow R_n & & \downarrow\!\!\!\downarrow & & \downarrow \simeq \\ \mathrm{ECa}^{-F_n^*}(b_n^*(Z)) & \xrightarrow{\widetilde{c}^{-F_n^*}} & H^1(\mathcal{O}_{b_n^*(Z)}) & \xrightarrow{\varsigma_n} & \mathbb{C} \end{array} \tag{4.3.3.5}$$

Note that $\widetilde{\varsigma}_n \circ \widetilde{c}^{-F_n^*}(b_n^*(\widetilde{Z}))$ corresponds to an integration of the 2–form $b_n^*(\widetilde{\omega})$ paired with a divisor supported on F_n. Since the pole order along F_n of $b_n^*(\widetilde{\omega})$ decreases by one after each blow up, after some steps n it will have no pole along F_n, hence $\widetilde{\varsigma}_n \circ \widetilde{c}^{-F_n^*}(b_n^*(\widetilde{Z})) = \varsigma_n \circ \widetilde{c}^{-F_n^*}(b_n^*(Z)) \circ R_n$ is constant. Since R_n is surjective (see e.g. [NN18, Theorem 3.1.10]), the statement follows. □

4.3.3.6. In the sequel, let $k \geq 1$ be the smallest integer such that $\varsigma_k \circ \widetilde{c}^{-F_k^*}(b_k^*(Z))$ is constant. Consider again $\widetilde{Z}$ as in the proof of Lemmas 4.3.3.1 and 4.3.3.3. The functionals ς_{k-1}, ς_k, $\widetilde{\varsigma}_{k-1}$ and $\widetilde{\varsigma}_k$ (as in 4.3.3.2 and (4.3.3.5)) form the following commutative diagram:

$$\begin{tikzcd} H^1(\mathcal{O}_{b_k^*(\widetilde{Z})}) \arrow[r, "\simeq"] \arrow[d, two heads] \arrow[rr, bend left, "\widetilde{\varsigma}_k"] & H^1(\mathcal{O}_{b_{k-1}^*(\widetilde{Z})}) \arrow[r, "\varsigma_{k-1}"] \arrow[d, two heads] & \mathbb{C} \arrow[d, "\simeq"] \\ H^1(\mathcal{O}_{b_k^*(Z)}) \arrow[r, "\simeq"] \arrow[rr, bend right, "\varsigma_k"'] & H^1(\mathcal{O}_{b_{k-1}^*(Z)}) \arrow[r, "\varsigma_{k-1}"] & \mathbb{C} \end{tikzcd} \tag{4.3.3.7}$$

By the choice of k and by the diagrams (4.3.3.5)–(4.3.3.7) $\widetilde{\varsigma}_{k-1} \circ \widetilde{c}^{-F_{k-1}^*}(b_k^*(\widetilde{Z}))$ is nonconstant, while $\widetilde{\varsigma}_k \circ \widetilde{c}^{-F_k^*}(b_k^*(\widetilde{Z}))$ is constant. Therefore, $b_k^*(\widetilde{\omega})$ has a pole of order one along F_{k-1}. In particular, the maps $\mathrm{ECa}^{-F_{k-1}^*}(b_k^*(V)) \to H^1(\mathcal{O}_{b_k^*(V)}) \to \mathbb{C}$ (where V is either $\widetilde{Z}$ or Z) depend only on the reduced structure of $b_k^*(V)$ along F_{k-1}, and they all can be identified with the map represented by Laufer's integration pairing.

4.3.3.8. In Lemma 4.3.3.3 and in the discussion from 4.3.3.6 one can replace in $\mathrm{ECa}^{-F_{k-1}^*}$ and in $\mathrm{ECa}^{-F_k^*}$ the cycles F_{k-1}^* and F_k^* by any multiple of them: NF_{k-1}^* and NF_k^* respectively, for any $N \in \mathbb{Z}_{>0}$. Indeed, the space of divisors has a natural 'additive' structure, namely a dominant map $s^{l'_1,l'_2}(V):\mathrm{ECa}^{l'_1}(V)\times \mathrm{ECa}^{l'_2}(V)\to \mathrm{ECa}^{l'_1+l'_2}(V)$ which satisfies $\widetilde{c}^{l'_1+l'_2}\circ s^{l'_1,l'_2}=\widetilde{c}^{l'_1}+\widetilde{c}^{l'_2}$. Therefore, if for $n=k-1$ or $n=k$ the image $\mathrm{im}(\widetilde{c}^{-F^*_n})$ belongs to an affine subspace A of $H^1(\mathcal{O}_{b^*_n(Z)})$, then $\mathrm{im}(\widetilde{c}^{-NF^*_n})$ belongs to $NA:=A+\cdots+A$ too. In particular, $\varsigma_{k-1}\circ\widetilde{c}^{-NF^*_{k-1}}(b^*_k(Z))$ is nonconstant, while $\varsigma_k\circ\widetilde{c}^{-NF^*_k}(b^*_k(Z))$ is constant. (Compare also with the ℓ–dependence in (3.5.2.1).) Furthermore, the discussion from 4.3.3.6 can be repeated for any N, the composed maps depend only on the reduced structure of $b^*_k(Z)$, hence Z can be replaced by any large $\widetilde{Z}$, in which case the composition can be computed by Laufer's integration duality formula.

This shows that one has a factorization (where $V=\widetilde{Z}$ or Z, and $\varsigma_{V,k}=\widetilde{\varsigma}_k$ or ς_k respectively)

$$\begin{tikzcd}
\mathrm{ECa}^{-NF^*_{k-1}}(b^*_k(V)) \arrow[r, "\widetilde{c}^{-NF^*_{k-1}}"] \arrow[d] & H^1(\mathcal{O}_{b^*_k(V)}) \arrow[r, "\varsigma_{V,k}"] & \mathbb{C} \\
\mathrm{ECa}^{-NF^*_{k-1}}(F_{k-1}) \arrow[ur, bend right] & &
\end{tikzcd} \tag{4.3.3.9}$$

Though in (4.3.3.9) this factorization through $\mathrm{ECa}^{-NF^*_{k-1}}(F_{k-1})$ exists (and it is nonconstant), a factorization through $\mathrm{ECa}^{-NF^*_{k-1}}(F_{k-1})\to H^1(\mathcal{O}_{F_{k-1}})$ definitely does not exists (because, e.g., $H^1(\mathcal{O}_{F_{k-1}})=0$). On the other hand, a factorization through a non-trivial quotient of $H^1(\mathcal{O}_{b^*_k(V)})=H^1(\mathcal{O}_V)$ do exists, a fact which will be crucial later. This is what we explain next.

4.3.3.10. In the space of resolution $\widetilde{X}_k$ let $U\subset\widetilde{X}_k$ be a small tubular neighbourhood of the exceptional curve $E_U:=E\cup(\cup_{i=1}^{k-1}F_i)$. Let Γ_U be the dual graph of E_U. (Note that contracting E_U in U provides a singularity with different topological type than Γ, one of its dual graphs is Γ_U.) One can restrict sheaves/bundles from $\widetilde{X}_k$ to U. At cycle level one has the homological projection $\pi_U(\sum_v n_vE_v+\sum_{i=1}^k m_iF_i):=\sum_v n_vE_v+\sum_{i=1}^{k-1}m_iF_i$. One also has the cohomological restriction $R_U:L'(\Gamma)\to L'(\Gamma_U)$ (dual to the natural homological injection of cycles); e.g. the restriction

$R_U(F^*_{k-1})$ of F^*_{k-1} is the antidual rational cycle $F^*_{k-1}(\Gamma_U)$ associated with F_{k-1} in the lattice of Γ_U. Then, for both $V = \widetilde{Z}$ or Z, one has the natural injection (which, for $V = \widetilde{Z}$ and Z fit in a commutative diagram): $\mathrm{ECa}^{-NF^*_{k-1}}(b^*_k(V))$ is a Zariski open set in $\mathrm{ECa}^{-NR_U(F^*_{k-1})}(\pi_U(b^*_k(V)))$. Indeed, both of them depend only on the multiplicity m_{k-1} of F_{k-1} in $b^*_k(V)$ and $\pi_U(b^*_k(V))$ (which are equal), the second set contains divisors up to the equation of $m_{k-1}F_{k-1}$ supported on $F_{k-1}\setminus F_{k-2}$ with total multiplicity N, while in the first set consists of those divisors of the second set whose support does not contain $F_{k-1}\cap F_k$.

On the other hand, the natural epimorphism $\rho_V : H^1(\mathcal{O}_{b^*_k(V)}) \to H^1(\mathcal{O}_{\pi_U(b^*_k(V))})$ usually is not a monomorphism. However, one has the following fact.

Lemma 4.3.3.11. $\varsigma_{V,k} : H^1(\mathcal{O}_{b^*_k(V)}) \to \mathbb{C}$ *factors through* $\rho_V : H^1(\mathcal{O}_{b^*_k(V)}) \to H^1(\mathcal{O}_{\pi_U(b^*_k(V))})$.

Proof. First, we concentrate on the map $\widetilde{c}^{-F^*_k} : \mathrm{ECa}^{-F^*_k}(b^*_k(V)) \to H^1(\mathcal{O}_{b^*_k(V)})$. Let A be the smallest affine subspace of $H^1(\mathcal{O}_{b^*_k(V)})$ which contains $\mathrm{im}(\widetilde{c}^{-F^*_k})$, and let A_0 be the parallel linear subspace of the same dimension. As above, we denote the sum $A+\cdots+A$ (m times) by mA, clearly all of these affine subspaces have the same dimension, and are parallel to each other. Next, consider also the 'multiples' $\widetilde{c}^{-mF^*_k} : \mathrm{ECa}^{-mF^*_k}(b^*_k(V)) \to H^1(\mathcal{O}_{b^*_k(V)})$ (cf. [NN18, §6], or see 4.3.3.8). Therefore, $\mathrm{im}(\widetilde{c}^{-mF^*_k}) \subset mA$, and in fact, by [NN18, Theorem 6.1.9], for $m \gg 0$, they agree. Furthermore, by the same theorem, $A_0 = \ker(\rho_V)$.

By the choice of k, $\varsigma_{V,k}$ restricted on the image of $\widetilde{c}^{-F^*_k}$ is constant, which means that $\varsigma_{V,k}|A$ is constant, or $A_0 \subset \ker(\varsigma_{V,k})$. Hence $\ker(\rho_V) \subset \ker(\varsigma_{V,k})$, and $\varsigma^U_{V,k}$ with $\varsigma^U_{V,k} \circ \rho_V = \varsigma_{V,k}$ exists. $\square$

This lemma has the following geometric interpretation. If $\varsigma_{V,k} = \langle \cdot, [b^*_k\widetilde{\omega}]\rangle$ (at the level of V or $\widetilde{X}_k$), then $\varsigma^U_{V,k} = \langle \cdot, [b^*_k\widetilde{\omega}|_U]\rangle$ at the level of U. The form $b^*_k\widetilde{\omega}|_U$ again has order one along F_{k-1} and all the local integration formulas along E_U are the same.

4.3.3.12. Next, we concentrate on the divisor $D \in \mathrm{ECa}^{\tilde{l}}(Z)$ and on the line bundle $\mathcal{O}_Z(l') = \mathcal{O}_Z(D)$. As the center of blow up of b_1 is generic on E_w, we can assume that it is not in the support of D. This guarantees that the divisor D lifts canonically into any of the spaces $\mathrm{ECa}^{b_k^*(\tilde{l})}(b_k^*(Z))$ (still denoted by D), and the germs $(\mathrm{ECa}^{\tilde{l}}(Z), D)$ and $(\mathrm{ECa}^{b_k^*(\tilde{l})}(b_k^*(Z)), D)$ are canonically isomorphic.

Furthermore, this germ is preserved under the restriction to U (see also the argument from 4.3.3.10), hence all these facts together with the existence of factorization from Lemma 4.3.3.11 can be inserted in the following commutative diagram:

$$\begin{array}{ccccc} (\mathrm{ECa}^{\tilde{l}}(Z), D) & \xrightarrow{\tilde{c}^{l'}} & H^1(\mathcal{O}_Z) & \xrightarrow{\varsigma} & \mathbb{C} \\ \uparrow\simeq & & b_n' \uparrow\simeq & & \uparrow\simeq \\ (\mathrm{ECa}^{b_k^*(\tilde{l})}(b_k^*(Z)), D) & \xrightarrow{\tilde{c}^{b_k^*(l')}} & H^1(\mathcal{O}_{b_k^*(Z)}) & \xrightarrow{\varsigma_k} & \mathbb{C} \\ \downarrow\simeq & & \rho_Z \downarrow & & \downarrow\simeq \\ (\mathrm{ECa}^{R_U(b_k^*(\tilde{l}))}(\pi_U(b_k^*(Z))), D) & \xrightarrow{\tilde{c}^{R_U b_k^*(l')}} & H^1(\mathcal{O}_{\pi_U(b_k^*(Z))}) & \xrightarrow{\varsigma_k^U} & \mathbb{C} \end{array} \tag{4.3.3.13}$$

This diagram shows that $\varsigma_k \circ T_D(\tilde{c}^{b_k^*(l')}(b_k^*(Z))) = 0$ and also

$$\varsigma_k^U \circ T_D(\tilde{c}^{R_U(b_k^*(l'))}(\pi_U(b_k^*(Z))) = 0. \tag{4.3.3.14}$$

4.3.3.15. On $b_k^*(Z)$ now we have the pullback line bundle $b_k^*(\mathcal{O}_Z(l')) = b_k^*(\mathcal{O}_Z(D)) = \mathcal{O}_{b_k^*(Z)}(D)$.

Lemma 4.3.3.16. $b_k^*(\mathcal{O}_{\widetilde{X}}(l')) = \mathcal{O}_{\widetilde{X}_k}(b_k^*(l'))$, *that is, the pullback of the natural line bundle* $\mathcal{O}_{\widetilde{X}}(l')$ *is the natural line bundle associated with the Chern class* $b_k^*(l')$. *Therefore,* $b_k^*(\mathcal{O}_Z(l')) = \mathcal{O}_{\widetilde{X}_k}(b_k^*(l')|_{b_k^*(Z)})$ *(which will be denoted by* $\mathcal{O}_{b_k^*(Z)}(b_k^*(l'))$*).*

Proof. A bundle is natural if one of its power has the form $\mathcal{O}(l)$ for some integral cycle l. In this case the Chern classes of the two bundles agree. Furthermore, if nl' is integral for certain $n \in \mathbb{Z}_{>0}$, then $b_k^*(\mathcal{O}_{\widetilde{X}}(l')^{\otimes n}) = \mathcal{O}_{\widetilde{X}_k}(b_k^*(nl'))$, hence $b_k^*(\mathcal{O}_{\widetilde{X}}(l'))$ is natural with Chern class $b_k^*(l')$. $\square$

After all these preparations, we start with the key construction of the proof.

We will construct a path in $\mathrm{ECa}^{R_U(b_k^*(\tilde{l}))}(\pi_U(b_k^*(Z)))$ at D, $t \mapsto \gamma(t)$ (or, $\{D_t\}_t$ with $D_0 = D$) with the following properties. Firstly, by the choice of ς and (4.3.3.14) $\varsigma \circ \widetilde{c} \circ \gamma$ must have zero derivative at $t=0$. On the other hand, we will compute by integration explicitly $\varsigma \circ \widetilde{c} \circ \gamma$ and we will show that its linear part is nontrivial, hence its derivative at $t=0$ is nonzero, a fact which leads to a contradiction.

The local path of divisors will be constructed via a deformation, based on section 4.2.

4.3.3.17. A special deformation of the analytic structure of $\mathcal{O}_{\widetilde{X}_k}$.

Let $(\widetilde{X}_k, E \cup \cup_{i=1}^k F_i)$ be the resolution as in 4.3.3.2, with the choice of k as in 4.3.3.6. Here we concentrate on the exceptional components F_{k-1} and F_k, where F_k is obtained by blowing up a generic point P. (If $k=1$ then $F_{k-1} = E_w$.) Then for the pair (F_{k-1}, F_k) we apply the construction of section 4.2, that is, we move F_k and its intersection point with F_{k-1} locally along F_{k-1}. In this way we obtain a 1–parameter family of deformations of the resolution $\widetilde{X}_k$, denoted by $\lambda_k : (\widetilde{\mathcal{X}}_k, \widetilde{X}_k) \to (\mathbb{C}, 0)$, with fibers $\widetilde{X}_{k,t}$. In $\widetilde{X}_{k,t}$ the exceptional curve has components $E \cup \cup_{i=1}^{k-1} F_i \cup F_{k,t}$. If we blow down the F–type curves in $\widetilde{X}_{k,t}$ we get a resolution $\widetilde{X}_t$, they form a family $(\widetilde{\mathcal{X}}, \widetilde{X})$. If we contract all the exceptional curves we get a family of singularities $\{(X_t, o)\}_t$. Since the analytic structure we started with is generic, the geometric genus $h^1(\mathcal{O}_{\widetilde{X}_{k,t}})$ stays constant and the deformation blows down to a deformation $(\mathcal{X}, X) \to (\mathbb{C}, 0)$ with fibers X_t (cf. 4.2). We denote the contraction $\widetilde{\mathcal{X}}_k \to \widetilde{\mathcal{X}}$ by the same symbol b_k.

We assume that the base space of λ is so small that the universal map $(\mathbb{C}, 0) \to Q$ to the base space of a complete deformation omits the discriminant $\Delta(\tilde{r})$; this fact is guaranteed by the choice of the generic structure of the singularity.

Therefore, for the very same $l' \in L'$ (which provides the bundle $\mathcal{O}_Z(l')$) we can consider the universal line bundles constructed in Lemma 4.1.3.3, namely $\mathcal{O}_{\widetilde{\mathcal{X}}_k}(b_k^*(l')) \in \mathrm{Pic}(\widetilde{\mathcal{X}}_k)$ and $\mathcal{O}_{\widetilde{\mathcal{X}}}(l') \in \mathrm{Pic}(\widetilde{\mathcal{X}})$. By similar argument as in Lemma 4.3.3.16 we have $b_k^*(\mathcal{O}_{\widetilde{\mathcal{X}}}(l')) = \mathcal{O}_{\widetilde{\mathcal{X}}_k}(b_k^*(l'))$. The restriction to the fibers of the deformations are the natural line bundles of the fibers.

Corresponding to the irreducible exceptional curves $\{E_v\}_v$ and $\{F_i\}_{i=1}^k$ in $\widetilde{X}_k$ we have the irreducible exceptional surfaces $\{\mathcal{E}_v\}_v$ and $\{\mathcal{F}_i\}_{i=1}^k$ in $\widetilde{\mathcal{X}}_k$. (Here $(\mathcal{F}_n)_t = F_n$ for $n < k$ but $(\mathcal{F}_k)_t = F_{k,t}$.) If $Z = \sum_v r_v E_v$ then $b_k^*(Z) = \sum_v r_v E_v + r_w \sum_{i=1}^k F_i$. Let we set $b_k^*(\mathcal{Z}) = \sum_{v \in \mathcal{V}} r_v \mathcal{E}_v + r_w \sum_{i=1}^k \mathcal{F}_i$. Then we restrict $\mathcal{O}_{\widetilde{\mathcal{X}}_k}(b_k^*(l'))$ to $b_k^*(\mathcal{Z})$ and we get $\mathcal{O}_{b_k^*(\mathcal{Z})}(b_k^*(l')) \in \mathrm{Pic}(b_k^*(\mathcal{Z}))$.

Let $\lambda : b_k^*(\mathcal{Z}) \to (\mathbb{C}, 0)$ be the projection of the deformation. The central fiber is $\mathcal{O}_{b_k^*(Z)}(b_k^*(l'))$. In particular, over $t = 0$ the bundle $\mathcal{O}_{b_k^*(Z)}(b_k^*(l'))$ has a global section s whose divisor is D (by the definition of D from 4.3.3 and identification (4.3.3.13)). Then Lemma 4.1.6.1 implies the following fact.

Lemma 4.3.3.18. *There exists an extension $\mathfrak{s} \in H^0(b_k^*(\mathcal{Z}), \mathcal{O}_{b_k^*(\mathcal{Z})}(b_k^*(l')))$ of $s \in H^0(b_k^*(Z), \mathcal{O}_{b_k^*(Z)}(b_k^*(l')))$ such that $\mathfrak{s}_0 = s$. Furthermore, $\mathfrak{s}_t$ has no fixed component either.*

Let D_t be the restriction of the divisor of $\mathfrak{s}$ to the fiber over t.

Since the support of $D = D_0$ is disjoint with the center of b_1, the same is true for each D_t (for $|t| \ll 1$). Hence, in this way we get a path germ γ with $\gamma(t) \in \mathcal{O}_{b_k^*(\mathcal{Z})_t}(D_t) = \mathcal{O}_{b_{k,t}^*(Z)}(D_t) = \mathcal{O}_{b_{k,t}^*(Z)}(b_{k,t}^*(l'))$, where $b_{k,t}$ is the contraction/blow up $\widetilde{X}_{k,t} \to \widetilde{X}_t$.

Note also that in the cycles $b_{k,t}^*(Z)$ the curve $F_{k,t}$ (with its stable multiplicity) is 'moving' along the deformation, the other components with their multiplicities are stable, and the divisors D_t are supported by this stable part (but they might move). More precisely, by the construction from 4.3.3.17 we obtain that $\pi_U(b_k^*(\mathcal{Z})_t)$ is t–independent, and it equals $\pi_U(b_k^*(Z))$. (It is worth to mention that $\pi_U(b_k^*(Z))$ is not the same as $b_{k-1}^*(Z)$, they differ even topologically at Euler number level.)

Then, by the choice of ς and D and the chain rule (compare also with (4.3.3.13) and (4.3.3.14):

$$\frac{d}{dt}\Big|_{t=0}(\varsigma_k\circ \widetilde{c}^{b^*_{k,t}(l')}(b^*_{k,t}(Z))(\gamma(t)) = \frac{d}{dt}\Big|_{t=0}(\varsigma_k^U\circ \widetilde{c}^{R_U(b^*_{k,t}(l'))}(\pi_U(b^*_{k,t}(Z)))(\gamma(t))$$
$$=T_D(\varsigma_k^U\circ \widetilde{c}^{R_U(b^*_{k}(l'))}(\pi_U(b^*_{k}(Z)))(\frac{d\gamma}{dt}\Big|_{t=0}) = \varsigma_k^U\circ T_D(\widetilde{c}^{R_U(b^*_{k}(l'))}(\pi_U(b^*_{k}(Z)))(\frac{d\gamma}{dt}\Big|_{t=0}) = 0. \tag{4.3.3.19}$$

The same is valid if we replace the family D_t by any of its multiple $N\cdot D_t$.

4.3.3.20. Let us summarize what we have. On each $b^*_{k,t}(Z)$ we can consider the restricted natural line bundle $\mathcal{O}_{b^*_{k,t}(Z)}(b^*_{k,t}(l'))$. Then, if we take its restriction to U, namely $\mathcal{O}_{b^*_{k,t}(Z)}(b^*_{k,t}(l'))|_U\in \mathrm{Pic}(\pi_U(b^*_k(Z)))$ and we shift it back with the natural line bundle $\mathcal{O}_{\pi_U(b^*_k(Z))}(R_U(b^*_k(l')))^{-1}$ we get a path in $\mathrm{Pic}^0(\pi_U(b^*_k(Z))) = H^1(\mathcal{O}_{\pi_U(b^*_k(Z))})$, whose differential at $t=0$ is in the kernel of ς_k^U.

Now, let us compute these objects directly, in fact, for a certain N–multiple of the corresponding bundles. Let N be an integer so that $Nl' = \sum_v Nl'_vE_v$ is an integral cycle and write $\ell := Nl'_w$. Then, $Nb^*_k(l') = \sum_v Nl'_vE_v + \ell\sum_{i=1}^k F_i$. Furthermore, $(\mathcal{O}_{b^*_{k,t}(Z)}(b^*_{k,t}(l')))^N$, being natural with integral Chern class, should equal $\mathcal{O}_{b^*_{k,t}(Z)}(\sum_v Nl'_vE_v + \ell\sum_{i=1}^k F_{i,t})$ and its restriction to U is $\mathcal{O}_{\pi_U(b^*_k(Z))}(\sum_v Nl'_vE_v + \ell\sum_{i=1}^{k-1} F_i+\ell F_{k,t})$. By the same reason, $\mathcal{O}_{\pi_U(b^*_k(Z))}(R_U(b^*_k(l')))^{-N}$ is $\mathcal{O}_{\pi_U(b^*_k(Z))}(\sum_v Nl'_vE_v+\ell\sum_{i=1}^k F_i)$. Hence, the N–multiple of the path is $\mathcal{O}_{\pi_U(b^*_k(Z))}(\ell(P_t-P))$, where $P_t = F_{k,t}\cap F_{k-1}$, $P=F_k\cap F_{k-1}$ as above. By assumption on l'_w we have $\ell\neq 0$.

That is, $\mathcal{O}_{\pi_U(b^*_k(Z))}(\ell P_t-\ell P)$ is a path in $H^1(\mathcal{O}_{\pi_U(b^*_k(Z))})$ and (4.3.3.19) reads as

$$\frac{d}{dt}\Big|_{t=0}(\varsigma_k^U(\mathcal{O}_{\pi_U(b^*_k(Z))}(\ell P_t-\ell P)) = 0. \tag{4.3.3.21}$$

Next we compute the left hand side of (4.3.3.21) in a different way.

By Lemma 4.3.3.11 (and comment after it) $\varsigma_k^U = \langle\cdot,[b^*_k\widetilde{\omega}|_U\rangle$, and the form $b^*_k\widetilde{\omega}|_U$ has a pole of order one along F_{k-1}. Moreover, P is a generic point of F_{k-1} and in a local neighborhood B of P in local coordinates (u,v) one has $F_{k-1}\cap B=\{u=0\}$,

$P_t = \{v + t = 0\}$. Hence (3.5.2.2) with $o = 1$ reads as

$$\varsigma_k^U(\mathcal{O}_{\pi_U(b_k^*(Z))}(\ell P_t - \ell P)) = t\ell c + \{\text{higher order terms}\} \quad (c \in \mathbb{C}^*), \tag{4.3.3.22}$$

whose derivative at $t = 0$ is non–zero. This contradicts (4.3.3.21).

4.3.4 The proof of part (II)

Note that the equalities for $i = 0$ and $i = 1$ are equivalent by Riemann–Roch. We will prove (II) in three steps.

4.3.4.1. The proof of part (II), case 1. Assume that $l'_v < 0$ for any $v \in \mathcal{V}(|Z|)$ and $-\tilde{l} \in \mathcal{S}'(|Z|) \setminus \{0\}$.

Then part (I) — already proved — can be applied.

First assume that the equivalent assumptions *(a)-(b)-(c)* of (I) are satisfied. Then by [NN18, Th. 4.1.1] $h^1(Z, \mathcal{L}_{gen}) = 0$. Hence we have to show that $h^1(Z, \mathcal{O}_Z(l')) = 0$ too. Choose an element $s \in H^0(Z, \mathcal{O}_Z(l'))_{reg}$ with divisor D and consider the exact sequence of sheaves $0 \to \mathcal{O}_Z \xrightarrow{\times s} \mathcal{O}_Z(l') \to \mathcal{O}_D(D) \to 0$ (where the second morphism is multiplication by s).

Then one has the cohomology exact sequence

$$H^0(Z, \mathcal{O}_Z(l')) \to \mathcal{O}_D(D) \xrightarrow{\delta} H^1(\mathcal{O}_Z) \to H^1(Z, \mathcal{O}_Z(l')) \to 0.$$

Then δ can be identified with $T_D(c^{\tilde{l}})$ (see [NN18, Prop. 3.2.2], or [Mu66, p. 164], [Kl05, Remark 5.18], [Kl13, §5]). Since $T_D(c^{\tilde{l}})$ is onto by (I)(c), $h^1(Z, \mathcal{O}_Z(l')) = 0$ follows.

Next, assume that the equivalent assumptions of (I) are not satisfied. That is, $H^0(Z, \mathcal{O}_Z(l'))_{reg} = H^0(Z, \mathcal{L}_{gen})_{reg} = \emptyset$. These facts read as $h^0(Z, \mathcal{O}_Z(l')) = \max_v\{h^0(Z - E_v, \mathcal{O}_Z(l' - E_v))\}$ and $h^0(Z, \mathcal{L}_{gen}) = \max_v\{h^0(Z - E_v, \mathcal{L}_{gen}(-E_v))\}$. But, by induction (applied for part (II) similarly as in the proof of case $(b) \Rightarrow (c)$ in 4.3.1.4, see also 4.3.1.3) $\max_v\{h^0(Z-E_v, \mathcal{O}_Z(l'-E_v))\} = \max_v\{h^0(Z-E_v, \mathcal{L}_{gen}(-E_v))\}$, hence $h^0(Z, \mathcal{O}_Z(l')) = h^0(Z, \mathcal{L}_{gen})$ follows too.

4.3.4.2. The proof of part (II), case 2. Assume that $l'_v < 0$ for any $v \in \mathcal{V}(|Z|)$ and $\tilde{l} = 0$. (If this happens then necessarily $|Z| < E$. Recall also that $\mathcal{O}_Z(l')$ is the restriction of the natural line bundle $\mathcal{O}_{\widetilde{X}}(l')$ to Z.)

If $h^1(\mathcal{O}_Z) = 0$ then $\mathcal{L}_{gen} = \mathcal{O}_Z(l')$, hence the statement follows. If $h^0(\mathcal{O}_Z(l')) = 0$ then by the semicontinuity of $\mathcal{L} \mapsto h^0(Z, \mathcal{L})$ (cf. [NN18, Lemma 5.2.1]) $h^0(\mathcal{L}_{gen}) = 0$ too.

In the sequel we assume that $h^1(\mathcal{O}_Z) \neq 0$ and $h^0(\mathcal{O}_Z(l')) \neq 0$.

Assume that $H^0(Z, \mathcal{O}_Z(l'))_{reg} \neq \emptyset$, that is, $\mathcal{O}_Z(l')$ has a section without fixed components. But, then by Chern class computation, this section has no zeros, hence $\mathcal{O}_Z(l') = \mathcal{O}_Z$, see also (3.1.1.5).

We claim that this identity $\mathcal{O}_Z(l') = \mathcal{O}_Z$ cannot happen for generic (X, o).

The argument runs similarly as the proof of $(a) \Rightarrow (c)$ in (I).

Since $h^1(\mathcal{O}_Z) \neq 0$ we can choose a nonzero functional $\omega \in H^1(\mathcal{O})^*$ for which we can repeat the arguments from 4.3.3. In particular, there exists $E_w \subset |Z|$ which satisfies Lemma 4.3.3.1, we can consider the sequence of blow ups as in 4.3.3.2, and we can choose k as in 4.3.3.6. Finally we consider the deformation of singularities as in 4.3.3.17. In this way we get a family of restricted line bundles $\mathcal{O}_{b^*_{k,t}(Z)}(b^*_{k,t}(l'))$, so that for $t = 0$ the corresponding bundle is the trivial one. We wish to show that for generic t the corresponding term cannot be the trivial bundle. Indeed, as in (4.3.3.22) we get that $t \mapsto \mathcal{O}_{b^*_{k,t}(Z)}(b^*_{k,t}(l'))|_U \in \mathrm{Pic}(\pi_U(b^*_k(Z)))$ is not constant. This implies that the path $t \mapsto b^*_{k,t}(\mathcal{O}_Z(l')) = \mathcal{O}_{b^*_{k,t}(Z)}(b^*_{k,t}(l'))$ cannot give for all t the trivial bundle either since otherwise its restriction to $\pi_U(b^*_k(Z))$ would be constant (since the restriction of the structures sheaf is the t–independent constant structure sheave). In particular, for generic t we have $\mathcal{O}_{Z_t}(l') \neq \mathcal{O}_{Z_t}$.

However, we can prove that in this situation necessarily $h^1(\mathcal{O}_{Z_t}(l')) < h^1(\mathcal{O}_{Z_t})$ for generic t (though the Chern classes agree), hence $t = 0$ is a jumping discriminant point of $l' \mapsto h^1(\mathcal{O}_{Z_t}(l'))$, a fact which contradict the genericity.

Indeed, since $\mathcal{O}_{Z_t}(l') \neq \mathcal{O}_{Z_t}$ for generic t (and $H^1(\mathcal{O}_{Z_t})$ is constant nonzero), $\mathcal{O}_{Z_t}(l')$ must have fix components (use $c_1(\mathcal{O}_{Z_t}(l')) = 0$ and (3.1.1.5)). Let $E_u \in |Z|$ be a fix component. Then $H^0(Z_t, \mathcal{O}_{Z_t}) \to H^0(E_u, \mathcal{O}_{Z_t}) = \mathbb{C}$ is surjective, while $H^0(Z_t, \mathcal{O}_{Z_t}(l')) \to H^0(E_u, \mathcal{O}_{Z_t}(l')) = \mathbb{C}$ is zero. Since their kernels have the same h^0 by the inductive step, $h^0(\mathcal{O}_{Z_t}(l')) < h^0(\mathcal{O}_{Z_t})$, hence the inequality follows by Riemann–Roch. This proves the claim.

After this discussion we can assume that $h^1(\mathcal{O}_Z) \neq 0$, $h^0(\mathcal{O}_Z(l')) \neq 0$, but $H^0(Z, \mathcal{O}_Z(l'))_{reg} = \emptyset$. By (3.1.1.5) $\mathcal{L}_{gen} \neq \mathcal{O}_Z$ (since $\mathrm{Pic}^0(\mathcal{O}_Z) \neq 0$), hence $H^0(Z, \mathcal{L}_{gen})_{reg} = \emptyset$ too. Then we proceed as in the last paragraph of 4.3.4.1, induction shows that $h^0(Z, \mathcal{O}_Z(l')) = h^0(Z, \mathcal{L}_{gen})$.

4.3.4.3. The proof of part (II), case 3. Finally, assume that $l'_v < 0$ for all $v \in \mathcal{V}(|Z|)$, and $-\tilde{l} \notin \mathcal{S}'(|Z|)$. Then there exists E_v in the support of Z such that $(l', E_v) = (\tilde{l}, E_v) < 0$. Hence for any $\mathcal{L} \in \mathrm{Pic}^{\tilde{l}}(Z)$ the exact sequence $0 \to \mathcal{L}(-E_v)|_{Z-E_v} \to \mathcal{L} \to \mathcal{L}|_{E_v} \to 0$ and vanishing $H^0(\mathcal{L}|_{E_v}) = 0$ give $h^0(Z - E_v, \mathcal{L}(-E_v)) = h^0(Z, \mathcal{L})$. By this step we replaced the Chern class $\tilde{l}$ by $\tilde{l} - E_v$. After finitely many such steps we necessarily get a new Chern class in the corresponding Lipman cone (see e.g. [N07, Prop. 4.3.3]). Hence, in this way we reduced this third case to the first two cases.

4.4 Applications. Analytic invariants

4.4.1 The start

In this section we will fix a resolution graph Γ (hence, the lattice L associated with it as well), and we treat singularities (X, o), together with their resolution $\widetilde{X}$ whose dual graph is Γ. The goal is to list some consequences of Theorem 4.3.1.1: hence we will assume that $\widetilde{X}$ is generic, and we will provide combinatorial expressions for several analytic invariants in terms of L. We will use the notations from the setup of 4.3.1.

The first group of results provides topological formulae for the **cohomology of** certain **natural line bundles** over an arbitrary $Z>0$.

Remark 4.4.1.1. (a) By [NN18, Theorem 5.3.1] for any $l'\in L'$ and $\mathcal{L}_{gen}$ generic in $\mathrm{Pic}^{R(l')}(Z)$

$$h^1(Z,\mathcal{L}_{gen})=\chi(-l')-\min_{0\leq l\leq Z, l\in L}\{\chi(-l'+l)\}. \tag{4.4.1.2}$$

In particular, if $l'=\sum_{v\in\mathcal{V}}l'_vE_v\in L'$ satisfies $l'_v<0$ for any $v\in\mathcal{V}(|Z|)$ and $\widetilde{X}$ is generic then Theorem 4.3.1.1 gives the following topological characterization for the cohomology of $\mathcal{O}_Z(l')$

$$h^1(Z,\mathcal{O}_Z(l'))=\chi(-l')-\min_{0\leq l\leq Z, l\in L}\{\chi(-l'+l)\}. \tag{4.4.1.3}$$

This will be extended in Theorem 4.4.1.5 for a larger family of l'–values.

(b) Note that the identity $h^1(Z,\mathcal{O}_Z(l'))=h^1(Z,\mathcal{L}_{gen})$ (hence (4.4.1.3) too) is not valid for any l' (that is, without some negativity condition regarding the coefficients of l'). Indeed, assume e.g. that $|Z|=E$ and all the coefficients of Z are very large, and $l'=0$. Then using the quadratic form of χ one has $\min_{0\leq l\leq Z, l\in L}\{\chi(l)\}=\min_{l\in L_{\geq 0}}\{\chi(l)\}$, hence $h^1(Z,\mathcal{L}_{gen})=-\min_{l\in L_{\geq 0}}\{\chi(l)\}$ by (4.4.1.2). But $h^1(Z,\mathcal{O}_Z)=1-\min_{l\in L_{\geq 0}}\{\chi(l)\}$ whenever (X,o) is not rational, see Corollary 4.4.2.4.

(c) Recall that if $-l'\in\mathcal{S}'\setminus\{0\}$ then all the coefficients l'_v of l' are strict negative. However, if the support of $|Z|$ is strict smaller than E, then $-R(l')\in\mathcal{S}'(|Z|)\setminus\{0\}$ does not necessarily imply that $l'_v<0$ for $v\in\mathcal{V}(|Z|)$. (Take e.g. $Z=E_v$ a (-2)–curve, choose E_u an adjacent vertex with it and set $l'=E_v+3E_u$. Then $-R(l')\in\mathcal{S}'(E_v)\setminus\{0\}$ however $l'_v=1$.)

4.4.1.4. The setup for generalization. We construct the following 'Laufer type computation sequence' (see e.g. [La72] or [N07, Prop. 4.3.3]). We start with a class $l' \in L'$ and an effective cycle Z with $|Z| \subset E$. Let $\tilde{l} \in L'(|Z|)$ be the restriction of l' as in Theorem 4.3.1.1.

Assume that $-\tilde{l} \notin \mathcal{S}'(|Z|)$. Then there exists $E_w \subset |Z|$ so that $(l', E_w) < 0$. Then, for both line bundles $\mathcal{L} = \mathcal{L}_{gen}$ and $\mathcal{L} = \mathcal{O}_Z(l')$ of $\mathrm{Pic}^{\tilde{l}}(Z)$ one can consider the exact sequence $0 \to \mathcal{L}(-E_w)|_{Z-E_w} \to \mathcal{L} \to \mathcal{L}|_{E_w} \to 0$, hence $h^0(\mathcal{L}(-E_w)|_{Z-E_w}) = h^0(\mathcal{L})$. Hence whenever $h^0(\mathcal{O}_Z(l'-E_w)|_{Z-E_w}) = h^0(\mathcal{L}_{gen}(-E_w)|_{Z-E_w})$ one also has $h^0(\mathcal{O}_Z(l')) = h^0(\mathcal{L}_{gen})$.

Let us construct the following sequence of pairs $(l'_k, Z_k)_{k=0}^t$. By definition, $(l'_0, Z_0) = (l', Z)$ the objects we started with. If $-\tilde{l} = -R(l') \notin \mathcal{S}'(|Z|)$, then define $(l'_1, Z_1) := (l' - E_w, Z - E_w)$ for some $E_w \subset |Z|$ with $(E_w, l') < 0$. If $-\tilde{l}_1 := -R(l'_1) \notin \mathcal{S}'(|Z_1|)$ we repeat the procedure, otherwise we stop. After finitely many steps necessarily $-\tilde{l}_t := -R(l'_t) \in \mathcal{S}'(|Z_t|)$ (here $Z_t = 0$ is also possible). (The choice of the sequence is not unique, however by similar argument as in [La72] or [N07, Prop. 4.3.3]) one can show that the last term (l'_t, Z_t) of the sequence is independent of all the choices: it is the unique $(l' - D, Z - D)$ with D minimal such that $Z \geq D \geq 0$, $D \in L$, and $-(l' - D) \in \mathcal{S}'(|Z - D|)$.)

Theorem 4.4.1.5. *Assume that $\widetilde{X}$ is generic with fixed dual graph Γ, and we choose an effective cycle Z and $l' \in L'$. Assume that the last term (l'_t, Z_t) of the Laufer type computation sequence $\{(l'_k, Z_k)\}_{k=0}^t$ has the following property: if $l'_t = \sum_v l'_{t,v} E_v$, then $l'_{t,v} < 0$ for any $v \in \mathcal{V}(|Z_t|)$. Then $h^i(Z, \mathcal{O}_Z(l')) = h^i(Z, \mathcal{L}_{gen})$ for a generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{\tilde{l}}(Z)$ ($i = 0, 1$), i.e. (4.4.1.3) holds.*

Proof. Use Theorem 4.3.1.1(II) and the discussion from 4.4.1.4. □

Example 4.4.1.6. Let $\widetilde{X}$ be generic, Z an effective cycle and $l' \in L'$. Assume that $l'_v \leq 0$ for all $v \in \mathcal{V}(|Z|)$ and for any connected component Z_{con} of Z there exists $v \in \mathcal{V}$

adjacent with Z_{con} with $l'_v < 0$. (The adjacent condition is $|Z_{con}| \cap E_v \neq \emptyset$.) Then the conditions from Theorem 4.4.1.5 are satisfied, hence $h^i(Z, \mathcal{O}_Z(l')) = h^i(Z, \mathcal{L}_{gen})$ and (4.4.1.3) holds.

Indeed, first note that if for some vertex with $l'_v = 0$ one has $(l', E_v) \geq 0$ then $l'_u = 0$ for all adjacent vertices u of v. Hence, $(l', E_v) \geq 0$ for all vertices v with $l'_v = 0$ contradicts the assumption. That is, there exists $v \in \mathcal{V}(|Z|)$ so that $l'_v = 0$ and $(l', E_v) < 0$.

Then we construct the computation sequence as follows. At the first part of the computation sequence, at step (l'_k, Z_k) we choose $E_{w(k)}$ so that $E_{w(k)} \subset |Z_k|$, the $E_{w(k)}$–coefficient of l'_k is zero, and $(E_{w(k)}, l'_k) < 0$. After finitely many such steps we arrive to the situation when along the support of $Z_{k'}$ all the coefficients of $l'_{k'}$ will be strict negative. Then we can continue the algorithm arbitrarily.

Corollary 4.4.1.7. *If $\widetilde{X}$ is generic with dual graph Γ and $|Z|$ is connected then*

$$h^1(\mathcal{O}_Z) = 1 - \min_{0<l\leq Z, l\in L} \{\chi(l)\} = 1 - \min_{|Z|\leq l\leq Z, l\in L} \{\chi(l)\}. \tag{4.4.1.8}$$

Proof. For $D = |Z|$ or $D = E_v$ for any $E_v \subset |Z|$ one has

$$0 \to H^0(Z-D, \mathcal{O}_Z(-D)) \to H^0(\mathcal{O}_Z) \xrightarrow{\delta} H^0(\mathcal{O}_D) \to H^1(Z-D, \mathcal{O}_Z(-D)) \xrightarrow{\iota} H^1(\mathcal{O}_Z) \to 0. \tag{4.4.1.9}$$

Since δ is onto ι is an isomorphism. But for $h^1(Z - D, \mathcal{O}_Z(-D))$ Example 4.4.1.6 and (4.4.1.3) hold. □

4.4.2 The cohomology of natural line bundles over $\widetilde{X}$.

Next we apply the results of the previous subsection for a cycle Z with all its coefficients very large. Recall that by Artin's Criterion $p_g = 0$ (that is, (X, o) is rational) if and only if $\min_{l\in L_{>0}}\{\chi(l)\} = 1$ [A62, A66]. Furthermore, for any singularity

$\min_{l\in L_{\geq 0}}\{\chi(l)\} = \min_{l\in L}\{\chi(l)\}$, see e.g. [N07, Prop. 4.3.3].

Corollary 4.4.2.1.

$$p_g(X,o) = 1 - \min_{l\in L_{>0}}\{\chi(l)\} = -\min_{l\in L}\{\chi(l)\} + \begin{cases} 1 & \textit{if } (X,o) \textit{ is not rational,} \\ 0 & \textit{else.} \end{cases} \tag{4.4.2.2}$$

Proof. For the first identity use (4.4.1.8), for the second one use Artin's Criterion for rationality. □

Remark 4.4.2.3. (a) For *any* non–rational analytic structure (X,o) one has $p_g(X,o) \geq 1 - \min_{l\in L}\{\chi(l)\}$ [Wa70, NO17]. The above corollary shows that this topological bound in fact is optimal.

(b) If (X,o) is elliptic then $\min_{l\in L_{>0}}\{\chi(l)\} = 0$. Hence, if the analytic structure is generic then $p_g = 1 - \min_{l\in L_{>0}}\{\chi(l)\} = 1$. This was proved by Laufer in [La77].

Corollary 4.4.2.4. *Assume that $\widetilde{X}$ is generic with dual graph Γ. Choose any $l' \in L'$ and consider $\mathcal{O}_{\widetilde{X}}(l')$, the natural line bundle on $\widetilde{X}$. Then*

$$h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(l')) = \chi(-l') - \min_{l\in L_{\geq 0}}\{\chi(-l'+l)\} + \epsilon(l'), \tag{4.4.2.5}$$

where

$$\epsilon(l') = \begin{cases} 1 & \textit{if } \ l' \in L,\ l' \geq 0, \textit{ and } (X,o) \textit{ is not rational,} \\ 0 & \textit{else.} \end{cases}$$

Proof. For any effective cycle Z (with $|Z| = E$) and $l' \in L'$ let us write $\Delta(Z,l') := h^1(Z, \mathcal{O}_Z(l')) - \chi(-l') + \min_{0\leq l\leq Z, l\in L}\{\chi(-l'+l)\}$. In order to compute $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(l'))$ let us fix some Z with all its coefficients very large. Then, if we start with the pair (l', Z), the Laufer sequence from 4.4.1.4 ends with some (l'_t, Z_t) with $Z_t \geq E$ (still with large coefficients), and $-l'_t \in \mathcal{S}'$. We claim that $\Delta(Z_k, l'_k)$ is constant along the computation sequence. Indeed, from the cohomological exact sequence used in

4.4.1.4 (for $k = 0$) $h^1(Z, \mathcal{O}(l')) = h^1(Z - E_w, \mathcal{O}(l' - E_w)) - 1 - (E_w, l')$. Then, we compare $\min_{0\leq l\leq Z} \chi(-l'+l)$ and $\min_{0\leq l\leq Z-E_w} \chi(-l'+E_w+l)$. Since for any $x \geq 0$ with $E_w \notin |x|$ we have $\chi(-l'+E_w+x) \leq \chi(-l'+x)$, these two minima agree. Hence the claim follows.

Now, for the pair (l'_t, Z_t), with $-l'_t \in \mathcal{S}'$, we distinguish two cases. The case $l'_t = 0$ occurs exactly when $l' \in L_{\geq 0}$ (because l'_t is the largest element of $(-\mathcal{S}')\cap(l'-L_{\geq 0})$, cf. [N07, Prop. 4.3.3]). In this case $\Delta(Z_t, l'_t)$ can be computed from (4.4.2.2). Or, $l'_t \neq 0$. In this case all the coefficients of l'_t are strict negative (use e.g. Remark 4.4.1.1(c)), and $\Delta(Z_t, l'_t) = 0$ by (4.4.1.3). $\square$

Example 4.4.2.6. For any $h \in H$ define $k_h := K + 2r_h$ and

$$\chi_{k_h}(x) := -(x, x+k_h)/2 = \chi(x) - (x, r_h) = \chi(x+r_h) - \chi(r_h).$$

It is known (use e.g. the algorithm from [N07, Prop. 4.3.3]) that for any $h \in H$ one has $\min_{l\in L_{\geq 0}} \chi(r_h + l) = \min_{l\in L} \chi(r_h + l)$. Therefore, for $h \neq 0$ one has

$$h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)) = \chi(r_h) - \min_{l\in L} \chi(r_h + l) = -\min_{l\in L}\{\chi_{k_h}(l)\} = -\min_{l\in L_{\geq 0}}\{\chi_{k_h}(l)\}. \tag{4.4.2.7}$$

Remark 4.4.2.8. (a) Let (X_{ab}, o) be the **universal abelian covering** of (X, o). Then

$$p_g(X_{ab}, 0) = \sum_{h\in H} h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)),$$

see e.g. [N07]. Hence $p_g(X_{ab}, 0)$ is topologically (and explicitly) computable by (4.4.2.2) and (4.4.2.7).

(b) For a conjectural identity which connects $\min_{l\in L} \chi(r_h + l)$ with the Heegaard Floer d–invariant associated with the link of the singularity and the *spin*^{c}–structure attached to the characteristic element k_h see [N08b, §5.2].

4.4.3 The cohomological cycle of $\widetilde{X}$

For any non–rational germ and fixed resolution the set $\{Z \in L_{>0} : h^1(\mathcal{O}_Z) = p_g(X,o)\}$ has a unique minimal element Z_{coh}, called the cohomological cycle. It also satisfies the next property: $h^1(\mathcal{O}_Z) < p_g$ for any $Z \not\geq Z_{coh}$, $Z > 0$ (see e.g. [Re97, 4.8]).

In parallel, let us mention the following topological statement. For any fixed non–rational resolution graph, $\mathcal{M} := \{Z \in L_{>0} : \chi(Z) = \min_{l\in L} \chi(l)\}$ has a unique minimal and a unique maximal element. Indeed, if $l_1, l_2 \in \mathcal{M}$, then for $m := \min\{l_1, l_2\}$ and $M := \max\{l_1, l_2\}$ one has $\chi(M)+\chi(m) = \chi(l_1)+\chi(l_2)-(l_1-m, l_2-m) \leq 2\min\chi$, hence $\chi(m) = \chi(M) = \min\chi$. Hence, $M \in \mathcal{M}$ always, and $m \in \mathcal{M}$ whenever $m \neq 0$. However, if $m = 0$ then the germ is elliptic and $\mathcal{M}$ admits a minimal element, namely the minimally elliptic cycle [La77, N99, N99b].

Corollary 4.4.3.1. *Assume that $\widetilde{X}$ is generic with a non–rational dual graph Γ. Then the cohomological cycle $Z_{coh} := \min\{Z \in L_{>0} : h^1(\mathcal{O}_Z) = p_g(X,o)\}$, is $\min\{Z \in L_{>0} : \chi(Z) = \min_{l\in L} \chi(l)\}$.*

4.4.4 The cohomological cycle of a line bundle

For any $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ with $h^1(\widetilde{X}, \mathcal{L}) > 0$ the set $L_{\mathcal{L}} := \{l \in L_{>0} : h^1(l, \mathcal{L}) = h^1(\widetilde{X}, \mathcal{L})\}$ has a unique minimal element, denoted by $Z_{coh}(\mathcal{L})$, called the cohomological cycle of $\mathcal{L}$ (and of ϕ). Similarly, for any $Z > 0$ and $\mathcal{L} \in \mathrm{Pic}(Z)$ with $h^1(Z, \mathcal{L}) > 0$ the set $L_{Z,\mathcal{L}} := \{l \in L,\ 0 < l \leq Z : h^1(l, \mathcal{L}) = h^1(Z, \mathcal{L})\}$ has a unique minimal element, denoted by $Z_{coh}(Z, \mathcal{L})$, called the cohomological cycle of $(Z, \mathcal{L})$. (For detail see e.g. [NN18, 5.5].)

Corollary 4.4.4.1. *Assume that $\widetilde{X}$ is generic.*

(a) Fix any $l' \in L'$ with $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(l')) \neq 0$. Then the set

$$L_{l'} := \{l_{min} \in L_{\geq 0} \mid \chi(-l' + l_{min}) = \min_{l \in L_{\geq 0}} \chi(-l' + l)\}$$

has a unique minimal element $Z_{coh}(l')$, which coincides with the cohomological cycle of $\mathcal{O}_{\widetilde{X}}(l')$.

(b) For any $Z > 0$ and $l' \in L'$ with $h^1(Z, \mathcal{O}_{\widetilde{X}}(l')) \neq 0$ the set

$$L_{Z,l'} := \{l_{min} \in L,\ 0 \leq l_{min} \leq Z,\ \mid \chi(-l' + l_{min}) = \min_{0 \leq l \leq Z,\, l \in L} \chi(-l' + l)\}.$$

has a unique minimal element $Z_{coh}(Z, l')$, which coincides with the cohomological cycle of $\mathcal{O}_{\widetilde{X}}(l')|_Z$.

Remark 4.4.4.2. [NN18, 5.5] For any analytic structure (X, o) supported on the fixed topological type and for any resolution ϕ, fix l' such that for the generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{l'}(\widetilde{X})$ one has $h^1(\widetilde{X}, \mathcal{L}_{gen}) \neq 0$. Then the cohomology cycle of $\mathcal{L}_{gen}$ is $Z_{coh}(l')$ (independently of the analytic structure). Similarly, if $h^1(Z, \mathcal{L}_{gen}) \neq 0$ for the generic $\mathcal{L}_{gen} \in \mathrm{Pic}^{l'}(Z)$ then the cohomological cycle of the pair $(Z, \mathcal{L}_{gen})$ is $Z_{coh}(Z, l')$.

4.4.5 The Hilbert series

Fix $\widetilde{X}$ generic and let $H(\mathbf{t})$ be the multivariable (equivariant) Hilbert series associated with the divisorial filtration of the local algebra of the universal abelian covering of (X, o) associated with divisors supported on all irreducible exceptional divisors of $\widetilde{X}$; for details see e.g. [CDGZ04, CDGZ08, N12]. Write $H(\mathbf{t}) = \sum_{l' \in L'} \mathfrak{h}(l') \mathbf{t}^{l'}$. (Here if $l' = \sum_v l'_v E_v$ then $\mathbf{t}^{l'} = \prod_v t_v^{l'_v}$.) It is known that for any l' there exists a unique $s(l') \in \mathcal{S}'$ such that $s(l') - l' \in L_{\geq 0}$, and $s(l')$ is minimal with these properties. Furthermore, for any $l' \in L'$ one has $\mathfrak{h}(l') = \mathfrak{h}(s(l'))$. Hence it is enough to determine

$\mathfrak{h}(l')$ for the (closed) first quadrant (because $\mathcal{S}' \subset L'_{\geq 0}$).

Write l' as $r_h + l_0$ for some $l_0 \in L_{\geq 0}$ (and $h = [l']$). Recall that $\mathfrak{h}(l')$ is the dimension of $H^0(\mathcal{O}_{\widetilde{X}}(-r_h))/H^0(\mathcal{O}_{\widetilde{X}}(-l_0 - r_h))$, see e.g. [N12, (2.3.3)]. Therefore, for $l_0 = 0$ we get $\mathfrak{h}(r_h) = 0$.

Proposition 4.4.5.1. *Assume that $l' = r_h + l_0$ with $l_0 > 0$. Then for $h \neq 0$*

$$\mathfrak{h}(l') = \min_{l \in L_{\geq 0}} \{\chi(l'+l)\} - \min_{l \in L_{\geq 0}} \{\chi(r_h+l)\} = \min_{l \in L_{\geq 0}} \{\chi_{k_h}(l_0+l)\} - \min_{l \in L_{\geq 0}} \{\chi_{k_h}(l)\}. \quad (4.4.5.2)$$

For $h = 0$ (i.e. when $r_h = 0$ and $l' = l_0 > 0$)

$$\mathfrak{h}(l_0) = \min_{l \in L_{\geq 0}} \{\chi(l_0 + l)\} - \min_{l \in L_{\geq 0}} \{\chi(l)\} + \begin{cases} 1 & \textit{if } (X, o) \textit{ is not rational,} \\ 0 & \textit{else.} \end{cases} \quad (4.4.5.3)$$

Proof. Use the exact sequence $0 \to \mathcal{O}(-r_h - l_0) \to \mathcal{O}(-r_h) \to \mathcal{O}_{l_0}(-r_h) \to 0$ and Corollary 4.4.2.4. $\square$

Remark 4.4.5.4. Proposition 4.4.5.1 via (4.4.2.7) and Corollary 4.4.2.1 can be written h–uniformly:

$$\mathfrak{h}(r_h + l_0) = \min_{l \in L_{\geq 0}} \{\chi_{k_h}(l_0 + l)\} + h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)) \quad (\forall\ h \in H,\ l_0 \in L_{>0}).$$

4.4.6 The Poincaré series

Let $P(\mathbf{t})$ be the multivariable equivariant Poincaré series associated with (X, o) and its fixed resolution, cf. [CDGZ04, CDGZ08, N12]. It is defined as $P(\mathbf{t}) = -H(\mathbf{t}) \cdot \prod_{v \in \mathcal{V}}(1 - t_v^{-1})$. It is known that it is supported on $\mathcal{S}'$. Proposition 4.4.5.1 implies the following.

Corollary 4.4.6.1. *Write $P(\mathbf{t}) = \sum_{l' \in \mathcal{S}'} \mathfrak{p}(l')\mathbf{t}^{l'}$. Then $\mathfrak{p}(0) = 1$ and for $l' > 0$ one* *has*

$$\mathfrak{p}(l') = \sum_{I\subset\mathcal{V}} (-1)^{|I|+1} \min_{l\in L_{\geq 0}} \chi(l'+l+E_I).$$

4.4.7 The analytic semigroup

The analytic semigroup is defined as

$$\mathcal{S}'_{an} := \{l' \,:\, H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(l'))_{reg} \neq \emptyset\} = \{l' \,:\, \mathfrak{h}(l') < \mathfrak{h}(l'+E_v) \text{ for any } v\in\mathcal{V}\}.$$

Corollary 4.4.7.1. *If (X,o) is generic then $\mathcal{S}'_{an} = \{l' \,:\, \chi(l') < \chi(l'+l)$ for any $l\in L_{>0}\}\cup\{0\}$ and $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(l')) = 0$ for any $l'\in -\mathcal{S}'_{an}\setminus\{0\}$.*

Proof. Use Corollary 4.4.2.4 and Proposition 4.4.5.1. □

Remark 4.4.7.2. (a) This formula emphasizes once more the parallelism between generic line bundles (associated with an arbitrary analytic structure) and the natural line bundles associated with a generic analytic structure, cf. 4.3.1.2 and 4.4.4.2. To explain this in the present situation, consider first an arbitrary analytic structure, a resolution with fixed graph Γ, and an effective cycle $|Z|$ as usual. By [NN18, §4] the fact that the Abel map $c^{l'} : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$ is dominant is independent of the analytic structure, and it has a purely combinatorial description: $\chi(-l') < \chi(-l'+l)$ for any $l\in L$, $0<l\leq Z\}$. Assume that $Z\gg 0$ and $l'\neq 0$. Then a generic line bundle $\mathcal{L}_{gen}\in \mathrm{Pic}^{l'}(Z)$ is in $\mathrm{im}(c^{l'})$ if and only if $-l'\in \mathcal{S}'_{dom} := \{-l' \,:\, \chi(-l') < \chi(-l'+l)$ for any $l\in L_{>0}\}$. On the other hand, by Corollary 4.4.7.1, in the context of a generic analytic type, this happens exactly when the natural line $\mathcal{O}_Z(l')$ is in the image of $\mathrm{im}(c^{l'})$ (that is, $\mathcal{O}_Z(l')$ behaves as a generic line bundle). In particular, for generic $\widetilde{X}$, $\mathcal{S}'_{an} = \mathcal{S}'_{dom}\cup\{0\}$.

(b) In [NN18, §4] several combinatorial properties of $\mathcal{S}'_{dom}$ are listed.

(c) Corollary 4.4.7.1 can be compared with the definition of $\mathcal{S}' = \{l' \,:\, \chi(l') < \chi(l'+E_v)$ for any $v\in\mathcal{V}\}$.

4.4.7.3. $\mathcal{S}_{an} := \mathcal{S}'_{an} \cap L$ is the semigroup of divisors (restricted to E) of functions $\phi^*\mathcal{O}_{(X,o)}$. Let Z_{max} be the **maximal ideal cycle** (of S. S.-T. Yau [Y80]), that is, the divisorial part of $\phi^*(\mathfrak{m}_{(X,o)})$ (here $\mathfrak{m}_{(X,o)}$ is the maximal ideal of $\mathcal{O}_{(X,o)}$). It is the unique smallest nonzero element of $\mathcal{S}_{an}$.

Corollary 4.4.7.4. *Assume that $\widetilde{X}$ is generic with non–rational graph Γ. Then $\mathcal{M} = \{Z \in L_{>0} : \chi(Z) = \min_{l\in L}\chi(l)\}$ has a unique maximal element and $Z_{max} = \max \mathcal{M}$.*

Proof. For the first part see the second paragraph of 4.4.3. $\max \mathcal{M} \in \mathcal{S}_{an}$ by the right hand side of 4.4.7.1, but $\min \mathcal{S}_{an}$ cannot be smaller than $\max \mathcal{M}$ by the very same identity. □

Remark 4.4.7.5. Recall that the fundamental (or minimal, or Artin) cycle $Z_{min} := \min\{\mathcal{S}' \cap L_{>0}\}$ has the property $h^0(\mathcal{O}_{Z_{min}}) = 1$, hence $h^1(\mathcal{O}_{Z_{min}}) = 1 - \chi(Z_{min})$ (see e.g. [N99b]). For $\widetilde{X}$ generic and (X, o) non–rational any cycle $Z \in \mathcal{M}$ (in particular Z_{max} too) has this property. Indeed, $h^1(\mathcal{O}_Z) = 1 - \min_{0<l\leq Z}\chi(l) = 1 - \chi(Z)$, hence $h^0(\mathcal{O}_Z) = 1$ too.

Corollary 4.4.7.6. *For (X, o) generic one has $Z_{max} \geq Z_{coh}$. If additionally (X, o) is numerically Gorenstein then $Z_{coh} + Z_{max} = Z_K$.*

4.4.8 The $\mathcal{O}_{(X,o)}$–multiplication on $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$

Assume that $p_g > 0$. On $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ the $\mathcal{O}_{(X,o)}$–module multiplication transforms on the dual vector space $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^* = H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ into the multiplication of forms by functions. The filtration on $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ induced by the powers of the maximal ideal agrees with the filtration associated by the nilpotent operator determined by multiplication by a generic element of $\mathfrak{m}_{(X,o)}$. For details see e.g. [To86].

The poles of forms are bounded by Z_{coh}. Indeed, by the exact sequence $0 \to \Omega^2 \to \Omega^2(Z_{coh}) \to \mathcal{O}_{Z_{coh}}(Z_{coh} + K_{\widetilde{X}}) \to 0$ and from the vanishing $h^1(\Omega^2) = 0$ (and from Serre duality) we have $\dim H^0(\Omega^2(Z_{coh}))/H^0(\Omega^2) = h^0(\mathcal{O}_{Z_{coh}}(Z_{coh}+K_{\widetilde{X}})) = h^1(\mathcal{O}_{Z_{coh}}) = p_g$. Hence the subspace $H^0(\Omega^2(Z_{coh}))/H^0(\Omega^2) \subset H^0(\widetilde{X}\setminus E, \Omega^2)/H^0(\Omega^2)$ has codimension zero, hence the spaces agree.

Corollary 4.4.8.1. *If $\widetilde{X}$ is generic then $\mathfrak{m}_{(X,o)} \cdot H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$. In particular, the $\mathcal{O}_{(X,o)}$–module multiplication factorizes to the $\mathbb{C} = \mathcal{O}_{(X,o)}/\mathfrak{m}_{(X,o)}$–vector space structure.*

Proof. Since $Z_{max} \geq Z_{coh}$, cf. 4.4.7.6, $\mathfrak{m}_{(X,o)} \cdot H^0(\Omega^2(Z_{coh})) \subset H^0(\Omega^2(-Z_{max}+Z_{coh})) \subset H^0(\Omega^2)$. □

4.4.9 Generic $\mathbb{Q}$–Gorenstein singularities

Recall that a singularity (X,o) is Gorenstein if the anticanonical cycle Z_K is integral, and $\Omega^2_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ equals $\mathcal{O}_{\widetilde{X}}(-Z_K)$. Hence in this case $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ is natural. Recall, that more generally, a line bunlde $\mathcal{L}$ is natural if and only if one of its powers has the form $\mathcal{O}_{\widetilde{X}}(l)$ for some $l \in L$, or equivalently, if and only if its restriction $\mathcal{L}|_{\widetilde{X}\setminus E} \in \mathrm{Pic}(\widetilde{X}\setminus E) = \mathrm{Cl}(X,o)$ has finite order (that is, it is $\mathbb{Q}$–Cartier). In particular, (X,o) is $\mathbb{Q}$–Gorenstein if and only if $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ is a natural line bundle, which automatically should agree with $\mathcal{O}_{\widetilde{X}}(-Z_K)$.

Proposition 4.4.9.1. *If a $\mathbb{Q}$–Gorenstein singularity (X,o) admits a resolution $\widetilde{X}$ with generic analytic structure, then (X,o) is either rational of minimally elliptic.*

Proof. **Step 1.** Let us fix a resolution $\widetilde{X}$ of a normal surface singularity (X,o). We claim that if (X,o) is neither rational nor minimally elliptic then there exists an effective cycle $Z>0$, $|Z| \subset E$, with $Z \not\geq Z_K$ and with $h^1(\mathcal{O}_Z) > 0$.

Assume first that $\widetilde{X} = \widetilde{X}_{min}$ is a minimal resolution. Then $Z_K \geq 0$ (by adjunction formulae, see also [La87]). By vanishing $h^1(\mathcal{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor)) = 0$ we get that $h^1(\mathcal{O}_{\lfloor Z_K \rfloor}) = p_g$. Since (X,o) is not rational, necessarily $\lfloor Z_K \rfloor > 0$. Hence, if $\lfloor Z_K \rfloor < Z_K$ then $Z = \lfloor Z_K \rfloor$ works.

Assume that $\lfloor Z_K \rfloor = Z_K$. Then $Z_K \in L$ and $Z_K > 0$ (since $p_g > 0$) hence necessarily $Z_K \geq E$ (see [La87]). For any $v \in \mathcal{V}$ consider the exacts sequence $0 \to \mathcal{O}_{E_v}(-Z_K + E_v) \to \mathcal{O}_{Z_K} \to \mathcal{O}_{Z_K - E_v} \to 0$. If $h^1(\mathcal{O}_{Z_K - E_v}) > 0$ for some v then we take $Z = Z_K - E_v$. Otherwise, $h^1(\mathcal{O}_{Z_K - E_v}) = 0$ for every v. Since $h^1(\mathcal{O}_{E_v}(-Z_K + E_v)) = 1$ we get that $p_g = 1$ and $Z_K = Z_{coh}$. Then the geometric genus of the singularities obtained by contracting any $E \setminus E_v$ is rational, hence (X, o) is minimally elliptic (for details see [La77] or [Re97]).

Finally, let $\widetilde{X}$ be arbitrary and let $\pi : \widetilde{X} \to \widetilde{X}_{min}$ be the corresponding modification of the minimal one. Let $0 < Z < Z_K$ be the cycle obtained previously for $\widetilde{X}_{min}$. Then $\pi^*(Z)$ works in $\widetilde{X}$.

Step2. Fix the generic resolution $\widetilde{X}$. Assume that (X, o) is neither rational nor minimally elliptic. Chose a cycle Z as in Step 1. Using $0 \to \Omega^2_{\widetilde{X}} \to \Omega^2_{\widetilde{X}}(Z) \to \mathcal{O}_Z(Z + K_{\widetilde{X}}) \to 0$, we get that $h^1(\Omega^2_{\widetilde{X}}(Z)) = h^1(\mathcal{O}_Z(Z + K_{\widetilde{X}})) = h^0(\mathcal{O}_Z)$. Since (X, o) is $\mathbb{Q}$–Gorenstein, $\Omega^2_{\widetilde{X}}(Z) = \mathcal{O}_{\widetilde{X}}(Z - Z_K)$, hence $h^1(\mathcal{O}_{\widetilde{X}}(Z - Z_K)) = h^0(\mathcal{O}_Z) = \chi(Z) + h^1(\mathcal{O}_Z)$. Now we apply (4.4.2.5) and (4.4.1.8), and we get

$$\chi(Z_K - Z) - \min_{l \geq 0}\{\chi(Z_K - Z + l)\} = \chi(Z) + 1 - \min_{0 < l \leq Z}\{\chi(l)\}.$$

Since $\chi(D) = \chi(Z_K - D)$ this transforms into $-\min_{l \leq Z}\{\chi(l)\} = 1 - \min_{0<l\leq Z}\{\chi(l)\}$. Next we claim that $\min_{l \leq Z}\{\chi(l)\} = \min_{0 \leq l \leq Z}\{\chi(l)\}$. Indeed, if $l = l_+ - l_-$ with $l_+, l_- \geq 0$ and with different supports, then there exists $E_v \in |l_-|$ such that $(E_v, l_-) < 0$; then by a computation $\chi(l + E_v) \leq \chi(l)$. Hence inductively $\chi(l_+) \leq \chi(l)$. Therefore,

$$-\min_{0 \leq l \leq Z}\{\chi(l)\} = 1 - \min_{0 < l \leq Z}\{\chi(l)\}.$$

This means that $\min_{0 \leq l \leq Z}\{\chi(l)\}$ cannot be realized by an element $l > 0$, hence $0 = \chi(0) < \min_{0<l\leq Z}\{\chi(l)\}$. But this implies $h^1(\mathcal{O}_Z) = 0$ (see [NN18, Example 4.1.3]), a contradiction. □

Remark 4.4.9.2. Proposition 4.4.9.1 generalizes the following result of Laufer [La77, Th. 4.3] (with a different proof): if the generic analytic structure of a numerically Gorenstein topological type is Gorenstein then the topological type is either Klein or minimally elliptic. (Recall that the Klein — or ADE — singularities are exactly the Gorenstein rational singularities.)

Chapter 5

Dimensions of images of Abel maps

In this chapter we want to investigate the images of the Abel maps $c^{l'}(Z) : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$. Since the space of effective Cartier divisors is an irreducible algebraic variety, the clousure of the image of the Abel map is an irreducible affine subvariety of $\mathrm{Pic}^{l'}(Z)$, which is also a Brill-Noether strata.

In this chapter we want to investigate the dimension of these images, so the numbers $\dim(\mathrm{im}(c^{l'}(Z)))$, we calculate them explicitely from cohomology numbers of the base singularity $\widetilde{X}$ and we give combinatorial formulas for them in the case of generic singularities.

Let us first briefly summarise from the previous chapters the main definitions and statements what we will need in this chapter.

5.1 Prelinimaries

5.1.1 Review of some needed statements

5.1.1.1. The modified Abel map. Multiplication by $\mathcal{O}_Z(-l')$ gives an isomorphism of the affine spaces $\mathrm{Pic}^{l'}(Z) \to \mathrm{Pic}^0(Z)$. Furthermore, we identify (via the exponential exact sequence) $\mathrm{Pic}^0(Z)$ with the vector space $H^1(Z, \mathcal{O}_Z)$.

It is convenient to replace the Abel map $c^{l'}$ with the composition

$$\widetilde{c}^{l'} : \mathrm{ECa}^{l'}(Z) \xrightarrow{c^{l'}} \mathrm{Pic}^{l'}(Z) \xrightarrow{\mathcal{O}_Z(-l')} \mathrm{Pic}^0(Z) \xrightarrow{\simeq} H^1(\mathcal{O}_Z).$$

The advantage of this new set of maps is that all the images sit in the same vector space $H^1(\mathcal{O}_Z)$.

Consider the natural additive structure $s^{l'_1,l'_2}(Z) : \mathrm{ECa}^{l'_1}(Z)\times\mathrm{ECa}^{l'_2}(Z) \to \mathrm{ECa}^{l'_1+l'_2}(Z)$ ($l'_1, l'_2 \in -\mathcal{S}'$) provided by the sum of the divisors. One verifies (see e.g. [NN18, Lemma 6.1.1]) that $s^{l'_1,l'_2}(Z)$ is dominant and quasi–finite. There is a parallel multiplication $\mathrm{Pic}^{l'_1}(Z) \times \mathrm{Pic}^{l'_2}(Z) \to \mathrm{Pic}^{l'_1+l'_2}(Z)$, $(\mathcal{L}_1, \mathcal{L}_2) \mapsto \mathcal{L}_1 \otimes \mathcal{L}_2$, which satisfies $c^{l'_1+l'_2} \circ s^{l'_1,l'_2} = c^{l'_1} \otimes c^{l'_2}$ in $\mathrm{Pic}^{l'_1+l'_2}$. This, in the modified case, using $\mathcal{O}_Z(l'_1 + l'_2) = \mathcal{O}_Z(l'_1) \otimes \mathcal{O}_Z(l'_2)$, reads as $\widetilde{c}^{l'_1+l'_2} \circ s^{l'_1,l'_2} = \widetilde{c}^{l'_1} + \widetilde{c}^{l'_2}$ in $H^1(\mathcal{O}_Z)$.

Let's recall, that for any $l' \in -\mathcal{S}'$ $A_Z(l')$ is the smallest dimensional affine subspace of $H^1(\mathcal{O}_Z)$ which contains $\mathrm{im}(\widetilde{c}^{l'})$ and $V_Z(l')$ is the parallel vector subspace of $H^1(\mathcal{O}_Z)$, the translation of $A_Z(l')$ to the origin..

For any $I \subset \mathcal{V}$, $I \neq \emptyset$, let (X_I, o_I) be the multigerm $\widetilde{X}/_{\cup_{v\in I}E_v}$ at its singular points, obtained by contracting the connected components of $\cup_{v\in I}E_v$ in $\widetilde{X}$. If $I = \emptyset$ then by convention (X_I, o_I) is a smooth germ.

We had the following theorem before:

Theorem 5.1.1.2. *Assume that $Z \geq E$.*

(a) For any $-l' = \sum_v a_v E^*_v \in \mathcal{S}'$ let the E^*–support of l' be $I(l') := \{v : a_v \neq 0\}$. Then $V_Z(l')$ depends only on $I(l')$. (This motivates to write $V_Z(l')$ as $V_Z(I)$ where $I = I(l')$.)

(b) $V_Z(I_1 \cup I_2) = V_Z(I_1) + V_Z(I_2)$ and $A_Z(l'_1 + l'_2) = A_Z(l'_1) + A_Z(l'_2)$.

(c) $\dim V_Z(I) = h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}})$.

(d) If $\mathcal{L}^{im}_{gen}$ is a generic bundle of $\mathrm{im}(c^{l'})$ then $h^1(Z, \mathcal{L}^{im}_{gen}) = h^1(\mathcal{O}_Z) - \dim(\mathrm{im}(c^{l'}))$.

(e) For $n \gg 1$ one has $\mathrm{im}(\widetilde{c}^{nl'}) = A_Z(nl')$, and $h^1(Z, \mathcal{L}) = h^1(\mathcal{O}_Z) - \dim V_Z(I) =$

$h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}})$ *for any* $\mathcal{L} \in \mathrm{im}(c^{nl'})$.

5.1.1.3. The linear subspace arrangement $\{V_Z(I)\}_I \subset H^1(\mathcal{O}_Z)$ **and differential forms.** The arrangement $\{V_Z(I)\}_I$ transforms into a linear subspace arrangement of $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$ via the (Laufer) non–degenerate pairing $H^1(\mathcal{O}_Z)\otimes H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \to \mathbb{C}$ as follows. Let $\Omega_Z(I)$ be the subspace $H^0(\Omega^2_{\widetilde{X}}(Z|_{\mathcal{V}\setminus I}))/H^0(\Omega^2_{\widetilde{X}})$ in $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$, that is, the subspace generated by those forms which have no poles along generic points of any E_v, $v \in I$. Via Laufer duality we have $V_Z(I) = \Omega_Z(I)^{\perp} = \{x : \langle x, \Omega_Z(I)\rangle = 0\}$ for $Z \geq E$.

5.1.1.4. Furthermore, for any $l' \in -\mathcal{S}' \setminus \{0\}$ consider a divisor $D \in \mathrm{ECa}^{l'}(Z)$, which is a union of (l', E) disjoint divisors $\{D_i\}_i$, each of them $\mathcal{O}_Z$–reduction of reduced divisors $\{\widetilde{D}_i\}_i$ of $\widetilde{X}$ intersecting E transversally. Set $\widetilde{D} = \cup_i \widetilde{D}_i$ and $\mathcal{L} := \widetilde{c}^{l'}(D) \in H^1(\mathcal{O}_Z)$. Write also $Z = \sum_{v\in\mathcal{V}} r_v E_v$.

We introduced a subsheaf $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}$ of $\Omega^2_{\widetilde{X}}(Z)$ consisting of those forms ω which have the property that the residue $\mathrm{Res}_{\widetilde{D}_i}(\omega)$ has no poles along $\widetilde{D}_i$ for all i. This means that the restrictions of $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}$ and $\Omega^2_{\widetilde{X}}(Z)$ on the complement of the support of $\widetilde{D}$ coincide, however along $\widetilde{D}$ one has the following local picture. Introduce near $p = E \cap \widetilde{D}_i = E_{v_i} \cap \widetilde{D}_i$ local coordinates (u, v) such that $\{u = 0\} = E$ and $\widetilde{D}_i$ has local equation v. Then a local section of $\Omega^2_{\widetilde{X}}(Z)$ in this system has the form $\omega = \sum_{k\geq -r_{v_i}, j\geq 0} a_{k,j} u^k v^j du \wedge dv$. Then, by definition, the residue $\mathrm{Res}_{\widetilde{D}_i}(\omega)$ is $(\omega/dv)|_{v=0} = \sum_k a_{k,0} u^k du$, hence the pole–vanishing reads as $a_{k,0} = 0$ for all $k < 0$. Note that $\Omega^2_{\widetilde{X}}(Z - \widetilde{D})$ and the sheaf of regular forms $\Omega^2_{\widetilde{X}}$ are subsheaves of $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}$.

Set $\Omega_Z(D) := H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$. This can be regarded as a subspace of $H^1(\mathcal{O}_Z)^* = H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$.

Theorem 5.1.1.5. *In the above situation one has the following facts.*

(a) The sheaves $\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}}/\Omega^2_{\widetilde{X}}$ *and* $\mathcal{O}_Z(K_{\widetilde{X}} + Z - D)$ *are isomorphic.*

(b) $H^1(Z,\mathcal{L})^* \simeq \Omega_Z(D)$.

(c) The image $(T_D\widetilde{c})(T_D\mathrm{ECa}^{l'}(Z))$ *of the tangent map at* D *of* $\widetilde{c}:\mathrm{ECa}^{l'}(Z)\to H^1(\mathcal{O}_Z)$ *is the intersection of kernels of linear maps* $T_{\mathcal{L}}\omega: T_{\mathcal{L}}H^1(\mathcal{O}_Z)\to\mathbb{C}$, *where* $\omega\in H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z)^{\mathrm{regRes}_{\widetilde{D}}})$.

If I is the E^*–support of l' (that is, $\widetilde{D}$ intersects E exactly along $\cup_{v\in I}E_v$), then $\Omega_Z(I)\subset\Omega_Z(D)\subset H^1(\mathcal{O}_Z)^*$.

Dually, via Theorem 5.1.1.5*(c)* (and up to a linear translation of $\mathrm{im}(T_D\widetilde{c})$)

$$(T_D\widetilde{c})(T_D\mathrm{ECa}^{l'}(Z))=\Omega_Z(D)^\perp\subset\Omega_Z(I)^\perp=V_Z(I)\subset H^1(\mathcal{O}_Z). \tag{5.1.1.6}$$

Let us fix a point $p\in E$ and a local coordinate system (u,v) around p such that $E=\{u=0\}$, cf. 5.1.1.4. Fix also some $\omega\in H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))$ which has pole of order $o>0$ at the exceptional divisor in E containing p. We say that (the divisor of) ω has no support point at p if it can be represented locally as $(\varphi(u,v)/u^o)du\wedge dv$ with φ holomorphic and $\varphi(0,0)\neq 0$. The other points are the support points denoted by $\mathrm{supp}(\omega)$.

Lemma 5.1.1.7. *Fix* $\omega\in H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))$ *such that there exists a point* $p\in E_v$, *a local divisor* $\widetilde{D}_1$ *in* $\widetilde{X}$ *with the following properties: (a)* $\widetilde{D}_1$ *is part of certain* $\widetilde{D}=\widetilde{D}_1+\widetilde{D}_2$, *such that* $\widetilde{D}_1\cap E=\widetilde{D}_1\cap E_v=p\notin\widetilde{D}_2\cup\mathrm{supp}(\omega)$, *and (b)* $\widetilde{D}$ *is a lift of* $D\in\mathrm{ECa}^{l'}(Z)$, *and the class of* ω *in* $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$ *restricted on* $\mathrm{im}T_D\widetilde{c}^{l'}(Z)$ *is zero. Then* ω *has no pole along* E_v.

Proof. Assume that ω has a pole of order $o>0$ along E_v. Fix some local coordinated (u,v) at $p:=\widetilde{D}_1\cap E_v$ such that ω locally is $du\wedge dv/u^o$ and $\widetilde{D}_1$ is $\{g(u,v)=0\}$. A deformation $g_t(u,v)$ of g produces a tangent vector in $T_D\mathrm{ECa}^{l'}(Z)$ and the action of ω on it is given by (for details see [NN18, 7.2])

$$\frac{d}{dt}\Big|_{t=0}\int_{|u|=\epsilon,\,|v|=\epsilon}\log\frac{g_t(u,v)}{g(u,v)}\cdot\frac{du\wedge dv}{u^o}. \tag{5.1.1.8}$$

Hence if we realize a deformation g_t for which the expression from (5.1.1.8) is non–zero, we get a contradiction. Note that g necessarily has the form $cv^k + \sum_{n>k} c_n v^n + uh(u,v) = cv^k + h'$ for some $k \geq 1$, $c_n \in \mathbb{C}$ and $c \in \mathbb{C}^*$. Then set $g_t = c(v - tu^{o-1})^k + h'$. Then the t–coefficient of the integrant is $\frac{kdu\wedge dv}{uv} \cdot (1 - \frac{h'}{cv^k} + (\frac{h'}{cv^k})^2 - \cdots)$, hence (5.1.1.8) is non–zero. $\square$

Definition 5.1.1.9. Additionally to the linear subspace arrangement $\{\Omega_Z(I)\}_I \subset H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \simeq H^1(\mathcal{O}_Z)^*$ we consider a more subtle object, a filtration indexed by $l \in L$, $0 \leq l \leq Z$ as well, called the *multivariable divisorial filtration of forms*. Indeed, for any such l we define $\mathcal{G}_l := H^0(\Omega^2_{\widetilde{X}}(l))/H^0(\Omega^2_{\widetilde{X}}) \subset H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$, equivalent to $H^1(\mathcal{O}_l)^* \hookrightarrow H^1(\mathcal{O}_Z)^*$, dual to the natural epimorphisms $H^1(\mathcal{O}_Z) \twoheadrightarrow H^1(\mathcal{O}_l)$. In particular, $\mathcal{G}_l \simeq H^1(\mathcal{O}_l)^*$. $\mathcal{G}_l$ is generated by forms with pole $\leq l$. In particular, $\mathcal{G}_0 = 0$, $\mathcal{G}_Z$ is the total vector space, $\mathcal{G}_{l_1} \subset \mathcal{G}_{l_2}$ whenever $l_1 \leq l_2$, and $\mathcal{G}_{l_1} \cap \mathcal{G}_{l_2} = \mathcal{G}_{\min\{l_1,l_2\}}$.

Note that if $l = \sum_{v\notin I} r_v E_v$ and all $r_v \gg 0$ then $\mathcal{G}_{\min(l,Z)} = \Omega_Z(I)$.

5.2 The first algorithm for the computation of $\dim \mathrm{Im}(c^{l'}(Z))$

5.2.1 Preparation and the statement

We fix $Z \geq E$ and $l' \in -\mathcal{S}'$ as above.

Definition 5.2.1.1. For any $l' \in -\mathcal{S}'$ with E^*–support I ($\emptyset \subset I \subset \mathcal{V}$) we set the following notations: $e_Z(l') = e_Z(I) := \dim V_Z(l') = \dim V_Z(I)$ and $d_Z(l') := \dim \mathrm{im}(c^{l'}(Z))$.

From definitions and Propositions 5.1.1.2 and 5.1.1.6:

$$\begin{aligned} &d_Z(l') \leq e_Z(l') \\ &e_Z(I) = h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}}) = h^1(\mathcal{O}_Z) - \dim \Omega_Z(I). \end{aligned} \tag{5.2.1.2}$$

Usually $d_Z(l') \neq e_Z(l')$. Next statement provides a criterion for the validity of the equality.

Lemma 5.2.1.3. *Let $l' \in -S'$ with E^*–support I and $Z \geq E$. Assume that $\mathcal{L}$ is a regular value of $\widetilde{c}^{l'}$ in $\mathrm{im}(\widetilde{c}^{l'})$ such that for any $\omega \in H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))$ there exists a section $s \in H^0(\mathcal{L})_{reg}$ such that $\mathrm{div}(s) \cap \mathrm{supp}(\omega) = \emptyset$. (This is guaranteed e.g. if the bundle $\mathcal{L}$ has no base points.) Then $T_{\mathcal{L}}(\mathrm{im}\widetilde{c}^{l'}) = A_Z(l')$, hence $d_Z(l') = e_Z(l')$.*

Proof. Since $\mathcal{L}$ is a regular value, $\mathcal{L}$ is a smooth point of $\mathrm{im}(\widetilde{c}^{l'})$ and $T_{\mathcal{L}}\mathrm{im}(\widetilde{c}^{l'}) = \mathrm{im}(T_D\widetilde{c}^{l'})$ for *any* $D \in (\widetilde{c}^{l'})^{-1}(\mathcal{L})$. We have to prove that $T_{\mathcal{L}}\mathrm{im}(\widetilde{c}^{l'}) = A_Z(l')$; we prove the dual identity in the space of forms, namely, $(T_{\mathcal{L}}\mathrm{im}(\widetilde{c}^{l'})^{\perp} = \Omega_Z(I)$ (see (5.1.1.6)).

Assume the contrary, that is, $(T_{\mathcal{L}}\mathrm{im}(\widetilde{c}^{l'}))^{\perp} \neq \Omega_Z(I)$. Since $\Omega_Z(I) \subset (T_{\mathcal{L}}\mathrm{im}(\widetilde{c}^{l'}))^{\perp}$ (the duality integral on $\Omega_Z(I) \times T_{\mathcal{L}}\mathrm{im}(\widetilde{c}^{l'})$ is zero, cf. 5.1.1.6 we get, that there is a form $\omega \in (T_{\mathcal{L}}\mathrm{im}(\widetilde{c}^{l'}))^{\perp} \setminus \Omega_Z(I)$.

Next choose $D \in (\widetilde{c}^{l'})^{-1}(\mathcal{L})$ such that its lift $\widetilde{D}$ satisfies $\widetilde{D} \cap \mathrm{supp}(\omega) = \emptyset$. But $\omega \in (T_{\mathcal{L}}\mathrm{im}(\widetilde{c}^{l'}))^{\perp} = (\mathrm{im}(T_D\widetilde{c}^{l'}))^{\perp}$ and $\omega \notin \Omega_Z(I)$ contradict Lemma 5.1.1.7. $\square$

In this section we provide an algorithm, valid for any analytic structure, which determines $d_Z(l')$ in terms of a finite collection of invariants of type $e_Z(l')$, associated with a finite sequence of resolutions obtained via certain extra blowing ups from $\widetilde{X}$.

5.2.1.4. Preparation Fix some resolution $\widetilde{X}$ of (X, o) and $-l' = \sum_{v \in \mathcal{V}} a_v E^*_v \in \mathcal{S}' \setminus \{0\}$ (hence each $a_v \in \mathbb{Z}_{\geq 0}$). In the next construction we will consider a finite sequence of blowing ups starting from $\widetilde{X}$. In order to find a bound for the number of blowing ups recall that for any representative ω in $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ the order of pole of ω along some E_v is less than or equal to the E_v–multiplicity m_v of $\max\{0, \lfloor Z_K \rfloor\}$ (see 3.5.1 here). Then, for every $v \in \mathcal{V}$ with $a_v > 0$ we fix a_v generic points on E_v, say p_{v,k_v}, $1 \leq k_v \leq a_v$. Starting from each p_{v,k_v} we consider a sequence of blowing ups of length m_v: first we blow up p_{v,k_v} and we create the exceptional curve $F_{v,k_v,1}$, then we blow up a generic point of $F_{v,k_v,1}$ and we create $F_{v,k_v,2}$, and we do this all together m_v times. We proceed in this way with all points p_{v,k_v}, hence we get $\sum_v a_v$ chains of modifications. If $a_v m_v = 0$ we do no modification along E_v. A set of integers $\mathbf{s} = \{\mathbf{s}_{v,k_v}\}_{v\in\mathcal{V},\ 1\le k_v\le a_v}$ with $0 \le \mathbf{s}_{v,k} \le m_v$ provides an intermediate step of the tower: in the (v, k_v) tower we do exactly $\mathbf{s}_{v,k_v}$ blowing ups; $\mathbf{s}_{v,k_v} = 0$ means that we do not blow up p_{v,k_v} at all. (In the sequel, in order to avoid aggregation of indices, we simplify k_v into k.) Let us denote this modification by $\pi_{\mathbf{s}} : \widetilde{X}_{\mathbf{s}} \to \widetilde{X}$. In $\widetilde{X}_{\mathbf{s}}$ we find the exceptional curves $\cup_{v\in\mathcal{V}} E_v \cup \cup_{v,k} \cup_{1\le t\le \mathbf{s}_{v,k}} F_{v,k,t}$; we index the set of vertices as $\mathcal{V}_{\mathbf{s}} := \mathcal{V} \cup \cup_{v,k} \cup_{1\le t\le \mathbf{s}_{v,k}} \{w_{v,k,t}\}$. At each level $\mathbf{s}$ we set the next objects: $Z_{\mathbf{s}} := \pi^*_{\mathbf{s}}(Z)$, $I_{\mathbf{s}} := \cup_{v,k}\{w_{v,k,\mathbf{s}_{v,k}}\}$, $-l'_{\mathbf{s}} := \sum_{v,k} F^*_{v,k,\mathbf{s}_{v,k}}$ (in $L'_{\mathbf{s}}$, where $F_{v,k,0} = E_v$), $d_{\mathbf{s}} := \dim \operatorname{im} c^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})$ and $e_{\mathbf{s}} := e_{Z_{\mathbf{z}}}(I_{\mathbf{s}})$ (both considered in $\widetilde{X}_{\mathbf{s}}$).

By similar argument as in (5.2.1.2) one has again $d_{\mathbf{s}} \le e_{\mathbf{s}}$ for any $\mathbf{s}$.

From definitions, for $\mathbf{s} = \mathbf{0}$ one has $I_{\mathbf{0}} = |l'|$, $e_{\mathbf{0}} = e_Z(l')$ and $d_{\mathbf{0}} = d_Z(l')$.

There is a natural partial ordering on the set of $\mathbf{s}$–tuples. Some of the above invariants are constant with respect to $\mathbf{s}$, some of them are only monotonous. E.g., by Leray spectral sequence one has $h^1(\mathcal{O}_{Z_{\mathbf{s}}}) = h^1(\mathcal{O}_Z)$ for all $\mathbf{s}$. One the other hand,

$$\text{if } \mathbf{s}_1 \le \mathbf{s}_2 \text{ then } e_{\mathbf{s}_1} = h^1(\mathcal{O}_{Z_{\mathbf{s}_1}}) - \dim \Omega_{Z_{\mathbf{s}_1}}(I_{\mathbf{s}_1}) \ge h^1(\mathcal{O}_{Z_{\mathbf{s}_2}}) - \dim \Omega_{Z_{\mathbf{s}_2}}(I_{\mathbf{s}_2}) = e_{\mathbf{s}_2} \tag{5.2.1.5}$$

because $\Omega_{Z_{\mathbf{s}_1}}(I_{\mathbf{s}_1}) \subset \Omega_{Z_{\mathbf{s}_2}}(I_{\mathbf{s}_2})$. In fact, for any ω, the pole–order along $F_{v,k,\mathbf{s}_{v,k}+1}$ of its pullback is one less than the pole–order of ω along $F_{v,k,\mathbf{s}_{v,k}}$. Hence, for $\mathbf{s} = \mathbf{m}$ (that is, when $\mathbf{s}_{v,k} = m_v$ for all v and k, hence all the possible pole–orders along $I_{\mathbf{m}}$ automatically vanish) one has $\Omega_{Z_{\mathbf{m}}}(I_{\mathbf{m}}) = H^0(\widetilde{X}_{\mathbf{m}}, \Omega^2_{\widetilde{X}_{\mathbf{m}}}(Z_{\mathbf{m}}))/H^0(\Omega^2_{\widetilde{X}_{\mathbf{m}}})$. Hence $e_{\mathbf{m}} = 0$. In particular, necessarily $d_{\mathbf{m}} = 0$ too.

More generally, for any $\mathbf{s}$ and (v, k) let $\mathbf{s}^{v,k}$ denote that tuple which is obtained from $\mathbf{s}$ by increasing $\mathbf{s}_{v,k}$ by one. By the above discussion if no form has pole along $F_{v,k,\mathbf{s}}$ then $\Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}}) = \Omega_{Z_{\mathbf{s}^{v,k}}}(I_{\mathbf{s}^{v,k}})$, hence $e_{\mathbf{s}} = e_{\mathbf{s}^{v,k}}$. Furthermore, by Laufer duality under such condition $d_{\mathbf{s}} = d_{\mathbf{s}^{v,k}}$ as well.

Therefore, we can redefine $e_{\mathbf{s}}$ and $d_{\mathbf{s}}$ for tuples $\mathbf{s} = \{\mathbf{s}_{v,k}\}_{v,k}$ even for arbitrary $\mathbf{s}_{v,k} \geq 0$: $e_{\mathbf{s}} = e_{\min\{\mathbf{s},\mathbf{m}\}}$ and $d_{\mathbf{s}} = d_{\min\{\mathbf{s},\mathbf{m}\}}$ (and these values agree with the ones which might be obtained by the first original construction applied for larger chains of blow ups).

The next theorem relates the invariants $\{d_{\mathbf{s}}\}_{\mathbf{s}}$ and $\{e_{\mathbf{s}}\}_{\mathbf{s}}$.

Theorem 5.2.1.6. (First algorithm) *With the above notations the following facts hold.*

(1) $d_{\mathbf{s}} - d_{\mathbf{s}^{v,k}} \in \{0,1\}$.

(2) If for some fixed $\mathbf{s}$ *the numbers* $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ *are not the same, then* $d_{\mathbf{s}} = \max_{v,k}\{\, d_{\mathbf{s}^{v,k}}\}$. *In the case when all the numbers* $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ *are the same, then if this common value* $d_{\mathbf{s}^{v,k}}$ *equals* $e_{\mathbf{s}}$, *then* $d_{\mathbf{s}} = e_{\mathbf{s}} = d_{\mathbf{s}^{v,k}}$*; otherwise* $d_{\mathbf{s}} = d_{\mathbf{s}^{v,k}} + 1$.

The proof of Theorem 5.2.1.6 together with the proof of Theorem 5.3.1.2 (the 'Second algorithm') from the next section will be given in a more general context in section 5.6.

5.2.1.7. Theorem 5.2.1.6 is suitable to run a decreasing induction over the entries of $\mathbf{s}$ in order to determine $\{d_{\mathbf{s}}\}_{\mathbf{s}}$ from $\{e_{\mathbf{s}}\}_{\mathbf{s}}$. In fact we can obtain even a closed–form expression.

Corollary 5.2.1.8. *With the notations of Theorem 5.2.1.6 one has* $d_{\mathbf{s}} = \min_{\mathbf{s}\leq\widetilde{\mathbf{s}}\leq\mathbf{m}}\{|\widetilde{\mathbf{s}}-\mathbf{s}| + e_{\widetilde{\mathbf{s}}}\}$ *for any* $\mathbf{0} \leq \mathbf{s} \leq \mathbf{m}$. *(Here* $|\mathbf{s}| = \sum_{v,k} s_{v,k_v}$.*) In particular,*

$$d_Z(l') = d_{\mathbf{0}} = \min_{\mathbf{0}\leq\mathbf{s}\leq\mathbf{m}}\{|\mathbf{s}| + e_{\mathbf{s}}\}.$$

(By the end of 5.2.1 one also has $\min_{\mathbf{s}\leq\widetilde{\mathbf{s}}\leq\mathbf{m}}\{|\widetilde{\mathbf{s}}-\mathbf{s}| + e_{\widetilde{\mathbf{s}}}\} = \min_{\mathbf{s}\leq\widetilde{\mathbf{s}}}\{|\widetilde{\mathbf{s}}-\mathbf{s}| + e_{\widetilde{\mathbf{s}}}\}$ *and* $\min_{\mathbf{0}\leq\mathbf{s}\leq\mathbf{m}}\{|\mathbf{s}| + e_{\mathbf{s}}\} = \min_{\mathbf{0}\leq\mathbf{s}}\{|\mathbf{s}| + e_{\mathbf{s}}\}$.*)*

Proof. By Theorem 5.2.1.6*(1)* for any $\widetilde{\mathbf{s}} \geq \mathbf{s}$ one has $d_{\mathbf{s}} - d_{\widetilde{\mathbf{s}}} \leq |\widetilde{\mathbf{s}}-\mathbf{s}|$, and by (5.2.1.2) $d_{\widetilde{\mathbf{s}}} \leq e_{\widetilde{\mathbf{s}}}$. These two imply $d_{\mathbf{s}} \leq |\widetilde{\mathbf{s}}-\mathbf{s}| + e_{\widetilde{\mathbf{s}}}$, hence $d_{\mathbf{s}} \leq \min_{\mathbf{s}\leq\widetilde{\mathbf{s}}\leq\mathbf{m}}\{|\widetilde{\mathbf{s}}-\mathbf{s}| + e_{\widetilde{\mathbf{s}}}\}$.

Next we show that $d_{\mathbf{s}}$ in fact equals $|\widetilde{\mathbf{s}}-\mathbf{s}|+e_{\widetilde{\mathbf{s}}}$ for some $\widetilde{\mathbf{s}}$. The wished $\widetilde{\mathbf{s}}$ is the last term of the sequence $\{\mathbf{s}_i\}_{i=0}^t$ constructed as follows. Set $\mathbf{s}_0:=\mathbf{s}$. Then, assume that $\mathbf{s}_i$ is already constructed, and that there exists (v,k) such that $d_{\mathbf{s}_i}=d_{(\mathbf{s}_i)^{v,k}}+1$. Then set $\mathbf{s}_{i+1}:=(\mathbf{s}_i)^{v,k}$ (for one of the choices of such possible (v,k)). This inductive construction will stop after finitely many steps (since each $d_{\mathbf{s}}\geq 0$). But if $d_{\mathbf{s}_t}=d_{(\mathbf{s}_t)^{v,k}}$ for all (v,k), then by 5.2.1.6*(2)* $d_{\mathbf{s}_t}=e_{\mathbf{s}_t}$. Hence $e_{\mathbf{s}_t}=d_{\mathbf{s}_t}=d_{\mathbf{s}}-|\mathbf{s}_t-\mathbf{s}|$. $\square$

5.3 The second algorithm for the computation of $\dim \mathrm{Im}(c^{l'}(Z))$

5.3.1 Preparation and the algorithm

The algorithm from the previous section determines the dimensions of the Abel maps $d_Z(l')$ in terms of a finite collection of invariants of type $e_Z(l')$ associated with a finite sequence of resolutions obtained via certain extra blowing ups from $\widetilde{X}$. Though, in principle, $e_Z(l')$ is much simpler than $d_Z(l')$ (it is the 'stabilizer' of $d_Z(l')$), the algorithm is still slightly cumbersome, it is more theoretical, it is not easy to apply in concrete examples: one needs to know all the integers $\{e_{\mathbf{s}}\}_{\mathbf{s}}$, that is, cf. Proposition 5.1.1.2, all the integers $\{h^1(\mathcal{O}_{Z_{\mathbf{s}}|_{\mathcal{V}_{\mathbf{s}}\setminus I_{\mathbf{s}}}})\}_{\mathbf{s}}$ associated with the tower of blowing ups. (However, it is a necessary intermediate step in the proof of the new algorithm).

The new algorithm is considerably simpler, e.g. it can be formulated in terms of the resolution $\widetilde{X}$ (see also the comments below). It provides $d_Z(l')$ in terms of the filtration $\{\mathcal{G}_l\}_l$ of 2–forms.

As a starting point, consider the construction from 5.2.1. For any $\mathbf{s}$ define the cycle $l_{\mathbf{s}}\in L$ of $\widetilde{X}$ by

$$l_{\mathbf{s}}:=\min\Big\{\sum_{v\in\mathcal{V}}\min_{1\leq k_v\leq a_v}\{\mathbf{s}_{v,k_v}\}E_v,Z\Big\}\in L.$$

Set $\mathcal{G}_{\mathbf{s}} := \mathcal{G}_{l_{\mathbf{s}}}$ and $g_{\mathbf{s}} := \dim \mathcal{G}_{\mathbf{s}}$ as well. Note that (via pullback) there is an inclusion $\mathcal{G}_{\mathbf{s}} \subset \Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}})$. Indeed, if the pole order of certain ω along E_v is $\leq \mathbf{s}_{v,k_v}$ then its pullback along $F_{v,k_v,\mathbf{s}_{v,k_v}}$ has no pole. Hence $g_{\mathbf{s}} \leq \dim \Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}}) = h^1(\mathcal{O}_Z) - e_{\mathbf{s}}$ too (cf. (5.2.1.2)). In particular,

$$d_{\mathbf{s}} \leq e_{\mathbf{s}} \leq h^1(\mathcal{O}_Z) - g_{\mathbf{s}}. \tag{5.3.1.1}$$

However, in principle it can happen that for a certain ω with even higher pole than $l_{\mathbf{s}}$ its pullback is in $\Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}})$. E.g., if ω in some local coordinates (u, v) of an open set U is $vdu \wedge dv/u^o$ (and $U \cap E = \{u = 0\}$) then its pullback via blowing up (once) at $u = v = 0$ has pole order $o - 2$. This phenomenon can happen even if we blow up a generic point: imagine a family of forms ω_t with 'moving divisor', parametrized by t given by $(v - t)du \wedge dv/u^o$. Then, even if we blow up E at a generic point $u = v - t_0 = 0$, in the family $\{\omega_t\}_t$ there is a form ω_{t_0} whose pole along E_v is o while its pullback has pole $o - 2$. Hence the equality of subspaces $\mathcal{G}_{\mathbf{s}} \subset \Omega_{Z_{\mathbf{s}}}(I_{\mathbf{s}})$, or of the equality $e_{\mathbf{s}} = h^1(\mathcal{O}_Z) - g_{\mathbf{s}}$ in principle is subtle and it is hard to test.

Note also that the invariant $h^1(\mathcal{O}_Z) - g_{\mathbf{s}}$ conceptually (and technically) is much simpler than $e_{\mathbf{s}}$. E.g., it depends only on $v \mapsto \min_{k_v \leq a_v}\{\mathbf{s}_{v,k_v}\}$, and it can be described via a cycle of $\widetilde{X}$ (namely $l_{\mathbf{s}}$) instead of the geometry of the tower $\widetilde{X}_{\mathbf{s}}$. Nevertheless, via the next theorem, it still contains sufficient information to determine $d_{\mathbf{s}}$, in particular $d_Z(l')$. In order to emphasize the parallelism between the two algorithms we formulate them in a completely symmetric way (in particular, the first parts are completely identical).

Theorem 5.3.1.2. (Second algorithm) *With the above notations the following facts hold.*

(1) $d_{\mathbf{s}} - d_{\mathbf{s}^{v,k}} \in \{0, 1\}$.

(2) If for some fixed $\mathbf{s}$ *the numbers* $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ *are not the same, then* $d_{\mathbf{s}} = \max_{v,k}\{d_{\mathbf{s}^{v,k}}\}$. *In the case when all the numbers* $\{d_{\mathbf{s}^{v,k}}\}_{v,k}$ *are the same, then if this common value*

$d_{\mathbf{s}^{v,k}}$ *equals* $h^1(\mathcal{O}_Z)-g_{\mathbf{s}}$*, then* $d_{\mathbf{s}}=h^1(\mathcal{O}_Z)-g_{\mathbf{s}}=d_{\mathbf{s}^{v,k}}$*; otherwise* $d_{\mathbf{s}}=d_{\mathbf{s}^{v,k}}+1$.

For the proof see section 5.6.

Corollary 5.3.1.3. *With the notations of 5.3.1 and of Theorem 5.3.1.2, for* $l'\in -\mathcal{S}'$ *and* $Z\geq E$ *one has*

$$d_Z(l')=\min_{\mathbf{s}}\{\,|\mathbf{s}|+h^1(\mathcal{O}_Z)-g_{\mathbf{s}}\,\}. \tag{5.3.1.4}$$

The proof runs similarly as the proof of Corollary 5.2.1.8.

The formula (5.3.1.4) can be rewritten in a different flavour.

Corollary 5.3.1.5. *For* $l'\in -\mathcal{S}'$ *and* $Z\geq E$ *one has*

$$d_Z(l')=\min_{0\leq Z_1\leq Z}\{\,(l',Z_1)+h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})\,\}. \tag{5.3.1.6}$$

Proof. From 5.1.1.9 $g_{\mathbf{s}}=\dim\mathcal{G}_{\mathbf{s}}=h^1(\mathcal{O}_{l_{\mathbf{s}}})$ and also $|\mathbf{s}|\geq\sum_v a_v(l_{\mathbf{s}})_v=(l',l_{\mathbf{s}})$, and $0\leq l_{\mathbf{s}}\leq Z$, hence $\min_{\mathbf{s}}\{\,|\mathbf{s}|+h^1(\mathcal{O}_Z)-g_{\mathbf{s}}\,\}\geq\min_{0\leq Z_1\leq Z}\{\,(l',Z_1)+h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})\,\}$. The opposite inequality is also true since any such Z_1 can be represented as a certain $l_{\mathbf{s}}$ with $|\mathbf{s}|=(l',l_{\mathbf{s}})$. $\square$

Example 5.3.1.7. (1) **($c^{l'}(Z)$ constant)** For any $0\leq Z_1\leq Z$ one has $(l',Z_1)\geq 0$ and $h^1(\mathcal{O}_Z)\geq h^1(\mathcal{O}_{Z_1})$, hence $d_Z(l')=0$ happens exactly when there exists Z_1 with $(l',Z_1)+h^1(\mathcal{O}_Z)-h^1(\mathcal{O}_{Z_1})=0$, or, $(l',Z_1)=0$ and $h^1(\mathcal{O}_Z)=h^1(\mathcal{O}_{Z_1})$. This means that $Z_1\leq Z|_{\mathcal{V}\setminus I}$, where I is the E^*–support of l', a fact which (together with $h^1(\mathcal{O}_Z)=h^1(\mathcal{O}_{Z_1})$) implies $h^1(\mathcal{O}_Z)=h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}})$ too. Hence, $d_Z(l')=0$ if and only if $h^1(\mathcal{O}_Z)=h^1(\mathcal{O}_{Z|_{\mathcal{V}\setminus I}})$. This is exactly the statement of [NN18, 6.3(v)].

(2) $c^{l'}(Z)$ **is dominant** if and only if $d_Z(l')=h^1(\mathcal{O}_Z)$, hence, via (5.3.1.6), if and only if $h^1(\mathcal{O}_{Z_1})\leq(l',Z_1)$ for any $0\leq Z_1\leq Z$. This can be seen in a different way as follows. First, if $c^{l'}(Z)$ is dominant, then, for any $0<Z_1\leq Z$, $c^{l'}(Z_1)$ is dominant too, hence $(l',Z_1)=\dim(\mathrm{ECa}^{l'}(Z_1))\geq\dim(H^1(\mathcal{O}_{Z_1}))$. Conversely, if $(l',Z_1)\geq h^1(\mathcal{O}_{Z_1})$

and $Z_1 > 0$ then $(l', Z_1) - h^1(\mathcal{O}_{Z_1}) > -h^0(\mathcal{O}_{Z_1})$, that is, $\chi(-l') < \chi(-l' + Z_1)$, hence $c^{l'}(Z)$ is dominant.

(3) By (5.3.1.6) $\mathrm{im}(c^{l'}(Z))$ **is a hypersurface** if and only if $\min_{0\leq Z_1\leq Z}\{(l', Z_1) - h^1(\mathcal{O}_{Z_1})\} = -1$. Since $h^0(\mathcal{O}_{Z_1}) \geq 1$, this implies that $\chi(-l') = \min_{0\leq l\leq Z}\chi(-l' + l)$.

The converse statement is not true: take e.g. a Gorenstein elliptic singularity with length of elliptic sequence $m+1$. (For elliptic singularities consult [N99, NN19a, NNtop]. For more on the Abel map of elliptic singularities see [NN19a].) Set $Z \gg 0$ and $-l' = Z_{min}$, the fundamental (minimal) cycle. Then $\mathrm{im}(c^{l'}(Z)) = 1$ and $h^1(Z) = p_g = m + 1$. However, $\chi(Z_{min}) = \min_{0\leq l\leq Z}\chi(Z_{min} + l) = 0$. Therefore, if $m = 1$ then $\mathrm{im}(c^{l'})$ is a hypersurface, but for $m \geq 2$ it is not. It is instructive to consider with the same topological data (elliptic numerically Gorenstein singularity with $m \geq 1$, $Z \gg 0$, $-l' = Z_{min}$) the generic analytic structure. Then $p_g = 1$ (cf. [La77, NN18]) but $\mathrm{im}(c^{l'}(Z))$ is a point (this follows from part *(1)* too). Hence $\mathrm{im}(c^{l'}(Z))$ is a hypersurface for any $m \geq 1$. In particular, the property that $\mathrm{im}(c^{l'}(Z))$ is a hypersurface is not a topological property.

Example 5.3.1.8. (Superisolated singularities) Assume that (X, o) is a hypersurface superisolated singularity whose link is a rational homology sphere. More precisely, $(X, o) = \{F(x_1, x_2, x_3) = 0\}$, where the homogeneous terms F_i of F are as follows: $\{F_d = 0\}$ defines an irreducible rational cuspidal curve in $\mathbb{CP}^2$ and $\{F_{d+1} = 0\} \cap \mathrm{Sing}\{F_d = 0\}$ is empty in $\mathbb{CP}^2$. (For details see [Lu87, LNM05, NN18].) Consider the minimal good resolution and let E_0 be the irreducible exceptional curve corresponding to C (the exceptional curve of the first blow up of the maximal ideal). Assume that $l' = -kE_0^*$ for some $k \geq 1$ and $Z \geq Z_K$. For any $\mathbf{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3_{\geq 0}$ write $|\mathbf{m}| = \sum_i m_i$. Then by the discussion from [NN18, 11.2] one has the following facts: $p_g = d(d-1)(d-2)/6 = \#\{\mathbf{m} : |\mathbf{m}| \leq d-3\}$, this is exactly the cardinality of the set of forms of type $\mathbf{x}^{\mathbf{m}}\omega$, where ω is the Gorenstein form. The pole order of ω along E_0 is $d-2$, and the vanishing order of $\mathbf{x}^{\mathbf{m}}$ along E_0

is $|\mathbf{m}|$. $\{\mathbf{x^m}\omega\}_{\mathbf{m}}$ constitute a basis in $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$. Hence, for $0 \leq s \leq d-2$ one has $g_s = \dim \mathcal{G}_{sE_0} = \#\{\mathbf{m} : d-2-s \leq |\mathbf{m}| \leq d-3\}$ and $h^1(\mathcal{O}_Z) - g_s = \binom{d-s}{3}$. In particular,

$$d_Z(-kE_0^*) = \min_{0\leq s\leq d-2} \{ks + \tbinom{d-s}{3}\}.$$

In [NN18, 11.2] $d_Z(-kE_0^*)$ was computed in a different way as $\sum_{j=0}^{d-3} \min\{k, \binom{j+2}{2}\}$. The identification of the two numerical answers is left to the reader. (Use $\sum_{j=0}^{t} \binom{j+2}{2} = \binom{t+3}{3}$.)

Remark 5.3.1.9. (1) In Theorems 5.2.1.6 and 5.3.1.2 (and Corollaries 5.2.1.8 and 5.3.1.3 as well) the functions $\mathbf{s} \mapsto e_{\mathbf{s}}$ and $\mathbf{s} \mapsto h^1(\mathcal{O}_Z) - g_{\mathbf{s}}$ serve as 'test–functions': "if this common value $d_{\mathbf{s}^{v,k}}$ equals the test value, then $d_{\mathbf{s}} = d_{\mathbf{s}^{v,k}}$, otherwise $d_{\mathbf{s}} = d_{\mathbf{s}^{v,k}}+1$". Via this fact in mind, the second algorithm is rather surprising: the test function for each fixed v depends only on $\mathbf{s} \mapsto \min_{0\leq k_v\leq a_v} s_{v,k_v} = (l_{\mathbf{s}})_v$, hence does not depend on the number of integers $\{s_{v,k_v}\}_{0\leq k_v\leq a_v}$, or, on a_v. However, the final output, namely $d_{\mathbf{s}}$ (and the right hand side of (5.3.1.4) and the algorithm itself) do depend on l'. We encourage the reader to work out the algorithm for an example when $a_v \geq 2$ (say, for $-l' = 2E_v^*$).

(2) Notice that the formulas $\min_{\mathbf{s}}(|\mathbf{s}|+h^1(Z)-g_{\mathbf{s}})$ and $\min_{\mathbf{s}}(|\mathbf{s}|+e_{\mathbf{s}})$ can be defined without any restriction on the numbers $g_{\mathbf{s}}$ and $e_{\mathbf{s}}$, however in our case these numbers are restricted. For example we have $\min_{\mathbf{s}\geq\mathbf{s}_1}(|\mathbf{s}| - |\mathbf{s}_1| + h^1(Z) - g_{\mathbf{s}}) - \min_{\mathbf{s}\geq\mathbf{s}_1^{v,k}}(|\mathbf{s} \geq \mathbf{s}_1^{v,k}| + h^1(Z) - g_{\mathbf{s}}) \in \{0,1\}$ for all $v, k, \mathbf{s}_1$. Or, $g_{\mathbf{s}} \leq |\mathbf{s}|$ for all $\mathbf{s}$ if and only if $\chi(-l') < \chi(-l'+l)$ for all $Z \geq l > 0$ (cf. Example 5.3.1.7(2)).

(3) **(Bounds for** $\operatorname{codim}\operatorname{im} c^{l'}(Z)$**)** In some expression the codimension of $\operatorname{im}(c^{l'}(Z))$ appears more naturally. E.g., we have the following two general statements from [NN18, Prop. 5.6.1] (under the conditions of Corollary 5.3.1.5):

(a) $h^1(Z, \mathcal{L}) \geq \operatorname{codim}\operatorname{im}(c^{l'}(Z))$ for any $\mathcal{L} \in \operatorname{im}(c^{l'}(Z))$. Equality holds whenever $\mathcal{L}$ is generic in $\operatorname{im}(c^{l'}(Z))$.

(b) $\operatorname{codim} \operatorname{im} c^{l'}(Z) \geq \chi(-l') - \min_{0\leq l\leq Z} \chi(-l'+l)$, and this inequality is strict whenever $c^{l'}(Z)$ is not dominant. (This can be compared with the discussion from Example 5.3.1.7(3).)

Note that Corollary 5.3.1.5 reads as:

$$\operatorname{codim} \operatorname{im}(c^{l'}(Z)) = \max_{0\leq Z_1\leq Z} \{\, h^1(\mathcal{O}_{Z_1}) - (l', Z_1)\, \}. \tag{5.3.1.10}$$

5.3.1.11. Before we state the next theorem let us emphasise the obvious fact that for any $0 \leq Z_1 \leq Z$ the natural restriction (linear projection) $r : H^1(\mathcal{O}_Z) \to H^1(\mathcal{O}_{Z_1})$ is surjective, hence for any irreducible constructible subset $C_1 \subset H^1(\mathcal{O}_{Z_1})$ one has $\dim r^{-1}(C_1) - \dim C_1 = h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$.

However, though the restriction of r to $\operatorname{im}(c^{l'}(Z)) \to \operatorname{im}(c^{l'}(Z_1))$ is dominant, in general $\dim \operatorname{im}(c^{l'}(Z))$ can be smaller than $\dim r^{-1}(\operatorname{im}(c^{l'}(Z_1)))$.

5.3.1.12. It is instructive to see that certain extremal geometric phenomenons (indexed by effective cycles) are realized by the very same set of cycles.

Lemma 5.3.1.13. *The following three sets of cycles coincide (for fixed $Z \geq E$ and $l' \in -\mathcal{S}'$ as above):*

(I) the set of cycles Z_1 with $0 \leq Z_1 \leq Z$ realizing the minimality in (5.3.1.6), that is: $d_Z(l') = (l', Z_1) + h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$.

(II) the set of cycles Z_1 with $0 \leq Z_1 \leq Z$ such that (i) the map $\mathrm{ECa}^{l'}(Z) \to H^1(Z_1)$ is birational onto its image, and (ii) the generic fibres of the restriction of r, $r^{im} : \operatorname{im}(c^{l'}(Z)) \to \operatorname{im}(c^{l'}(Z_1))$, have dimension $h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$. (That is, the fibers of r^{im} have maximal possible dimension.)

(III) the set of cycles Z_1 with $0 \leq Z_1 \leq Z$ such that for the generic element $\mathcal{L}^{im}_{gen} \in \operatorname{im}(c^{l'}(Z))$ and arbitrary section $s \in H^0(Z_1, \mathcal{L}^{im}_{gen})_{reg}$ with divisor D (i) in the (analogue of the Mittag-Lefler sequence associated with the exact sequence $0 \to$

$\mathcal{O}_{Z_1} \xrightarrow{\times s} \mathcal{L}_{gen}^{im} \to \mathcal{O}_D \to 0$, *cf. [NN18, 3.2]),*

$$0 \to H^0(\mathcal{O}_{Z_1}) \xrightarrow{\times s} H^0(Z_1, \mathcal{L}_{gen}^{im}) \to \mathbb{C}^{(Z_1,l')} \xrightarrow{\delta} H^1(\mathcal{O}_{Z_1}) \to h^1(Z_1, \mathcal{L}_{gen}^{im}) \to 0$$

δ is injective, and (ii) $h^1(Z, \mathcal{L}_{gen}^{im}) = h^1(Z_1, \mathcal{L}_{gen}^{im})$.

Proof. For (I)⇒(II) use the following. First recall that $\dim \mathrm{ECa}^{l'}(Z') = (l', Z')$ for any effective cycle Z'. Next, from (5.3.1.6), there exists an effective cycle $Z_1 \leq Z$, such that $\dim \mathrm{im}(c^{l'}(Z)) = (l', Z_1) + h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$. But $\dim(\mathrm{im}(c^{l'}(Z_1))) \leq \dim \mathrm{ECa}^{l'}(Z_1) = (l', Z_1)$ and $\dim(\mathrm{im}(c^{l'}(Z))) - \dim(\mathrm{im}(c^{l'}(Z_1))) \leq h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$. Hence, necessarily we have equalities in both these inequalities. (I)⇐(II) is similar.

For (II)(i)⇔(III)(i) use the fact that δ is the tangent application $T_D \mathrm{im} c^{l'}(Z_1)$ at D, cf. [NN18, 3.2], and for (II)(ii)⇔(III)(ii) use Remark 5.3.1.9(3)(a). □

5.3.2 Structure theorem for the Abel map

The geometric interpretation from Lemma 5.3.1.13(II) has the following consequence.

Theorem 5.3.2.1. (Structure theorem) *Fix a resolution $\widetilde{X}$, a cycle $Z \geq E$ and a Chern class $l' \in -\mathcal{S}'$ as above.*

(a) There exists an effective cycle $Z_1 \leq Z$, such that: (i) the map $\mathrm{ECa}^{l'}(Z) \to H^1(Z_1)$ is birational onto its image, and (ii) the generic fibres of the restriction of r, $r^{im} : \mathrm{im}(c^{l'}(Z)) \to \mathrm{im}(c^{l'}(Z_1))$, have dimension $h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$. (Cf. Lemma 5.3.1.13(II).)

(b) In particular, for any such Z_1, the space $\mathrm{im}(c^{l'}(Z))$ is birationally equivalent with an affine fibration with affine fibers of dimension $h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_{Z_1})$ over $\mathrm{ECa}^{l'}(Z_1)$.

(c) The set of effective cycles Z_1 with property as in (a) has a unique minimal and a unique maximal element denoted by $C_{min}(Z, l')$ and $C_{max}(Z, l')$. Furthermore,

$C_{min}(Z,l')$ coincides with the cohomology cycle of the pair $(Z,\mathcal{L}^{im}_{gen})$ (the unique minimal element of the set $\{0\leq Z_1\leq Z\,:\, h^1(Z,\mathcal{L}^{im}_{gen})=h^1(Z_1,\mathcal{L}^{im}_{gen})\}$) for the generic $\mathcal{L}^{im}_{gen}\in \mathrm{im}(c^{l'}(Z))$.

Proof. *(a)* Use Lemma 5.3.1.13.

(c) Assume that two cycles Z_1 and Z_2 satisfy *(a)*. We claim that $Z':=\max\{Z_1,Z_2\}$ satisfies too.

First, for any cycle Z'' with $Z_1\leq Z''\leq Z$, if Z_1 satisfies *(a)(ii)* then Z'' satisfies too. This applies for Z' too. To prove *(a)(i)* for Z', let us denote by $\mathrm{ECa}^{l'}(Z'')_0\subset \mathrm{ECa}^{l'}(Z'')$ the set of divisors whose support is disjoint from the singular points of E. If $l'=\sum_v a_vE^*_v$ then $\mathrm{ECa}^{l'}(Z)_0=\prod_v \mathrm{ECa}^{a_vE^*_v}(Z)_0$. Using this fact one shows that the product $\mathrm{ECa}^{l'}(Z')\to \mathrm{ECa}^{l'}(Z_1)\times \mathrm{ECa}^{l'}(Z_2)$ of the two restrictions $\mathrm{ECa}^{l'}(Z')\to \mathrm{ECa}^{l'}(Z_j)$ $(j=1,2)$ is birational onto its image (BioIm). This composed with the product of the maps $\mathrm{ECa}^{l'}(Z_1)\to H^1(Z_1)$ and $\mathrm{ECa}^{l'}(Z_2)\to H^1(Z_2)$ (both BioIm) guarantees that $\mathrm{ECa}^{l'}(Z')\to H^1(Z_1)\times H^1(Z_2)$ is BioIm too. This map writes as the composition $\mathrm{ECa}^{l'}(Z')\to H^1(Z')\to H^1(Z_1)\times H^1(Z_2)$, hence the first term $\mathrm{ECa}^{l'}(Z')\to H^1(Z')$ should be BioIm. Hence the claim and the existence of $C_{max}(Z,l')$ follows.

In order to prove the existence of $C_{min}(Z,l')$, first we claim that the set of cycles Z^{ii}, which satisfy *(a)(ii)* has a unique minimal element Z^{ii}_{min}. This fact via Remark 5.3.1.9(3)(a) is equivalent with the existence of the (unique) cohomological cycle for the pair $(Z,\mathcal{L}^{im}_{gen})$. This was proved in [NN18, 5.5], see also [Re97, 4.8]. Next, we claim that the map $\mathrm{ECa}^{l'}(Z^{ii}_{min})\to H^1(Z^{ii}_{min})$ is BioIm as well. From the existence of the cycle $C_{max}(\cdot,l')$ (already proved above), applied for Z^{ii}_{min}, there exists a cycle $C_{max}(Z^{ii}_{min},l')\leq Z^{ii}_{min}$, which satisfies *(a)*. In particular, *(a)(ii)* is valid for the pair $C_{max}(Z^{ii}_{min},l')\leq Z^{ii}_{min}$. By the definition of Z^{ii}_{min} the condition *(a)(ii)* is valid for the pair $Z^{ii}_{min}\leq Z$ too. Hence, *(a)(ii)* is valid for the pair $C_{max}(Z^{ii}_{min},l')\leq Z$ as well. Therefore, by the definition of Z^{ii}_{min} necessarily $C_{max}(Z^{ii}_{min},l')=Z^{ii}_{min}$, hence Z^{ii}_{min}

satisfies *(a)*. $\square$

5.3.3 Example. The case of generic analytic structure

Let us fix the topological type of a good resolution of a normal surface singularity, and we assume that the analytic type on $\widetilde{X}$ is generic (in the sense of chapter 5., see [La73] as well). Recall that in such a situation, if $Z' = \sum n_v E_v$ is a non–zero effective cycle, whose support $|Z'| = \cup_{n_v \neq 0} E_v$ is connected, then by [NN18, Corollary 6.1.7] one has

$$h^1(\mathcal{O}_{Z'}) = 1 - \min_{|Z'| \leq l \leq Z',\ l \in L} \{\chi(l)\}.$$

Corollary 5.3.3.1. *Assume that $\widetilde{X}$ has a generic analytic type, $Z \geq E$ an integral cycle and $l' \in -S'$. For any $0 \leq Z_1 \leq Z$ write $E_{|Z_1|}$ for $\sum_{E_v \subset |Z_1|} E_v$. Then*

$$d_Z(l') = 1 - \min_{E \leq l \leq Z} \{\chi(l)\} + \min_{0 \leq Z_1 \leq Z} \big\{ (l', Z_1) + \min_{E_{|Z_1|} \leq l \leq Z_1} \{\chi(l)\} - \chi(E_{|Z_1|}) \big\}. \quad (5.3.3.2)$$

In particular, $d_Z(l') = \dim(\mathrm{im} c^{l'}(Z))$ is topological.

Let us concentrate again on the *codimension* $h^1(\mathcal{O}_Z) - d_Z(l')$ of $\mathrm{im}(c^{l'}(Z)) \subset \mathrm{Pic}^{l'}(Z)$ instead of the dimension. Then, (5.3.3.2) reads as

$$\operatorname{codim} \mathrm{im}(c^{l'}(Z)) = \max_{0 \leq Z_1 \leq Z} \big\{ -(l', Z_1) - \min_{E_{|Z_1|} \leq l \leq Z_1} \{\chi(l)\} + \chi(E_{|Z_1|}) \big\}. \quad (5.3.3.3)$$

This is a rather complicated combinatorial expression in terms of the intersection lattice L. The next lemma aims to simplify it.

Proposition 5.3.3.4. *Consider the assumptions of Corollary 5.3.3.1. Let Z_1 be minimal such that the maximum in (5.3.3.3) is realized for it. Then $\min_{E_{|Z_1|} \leq l \leq Z_1} \{\chi(l)\} = \chi(Z_1)$. In particular,*

$$\operatorname{codim} \mathrm{im}(c^{l'}(Z)) = \max_{0 \leq Z_1 \leq Z} \big\{ -(l', Z_1) - \chi(Z_1) + \chi(E_{|Z_1|}) \big\}. \quad (5.3.3.5)$$

The maximum at the right hand side is realized e.g. for the cohomology cycle of $\mathcal{L}^{im}_{gen} \in \mathrm{im}(c^{l'}(Z)) \subset \mathrm{Pic}^{l'}(Z)$. *Furthermore,*

$$h^1(Z,\mathcal{L}) \geq \max_{0\leq Z_1\leq Z}\big\{ -(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\big\} \tag{5.3.3.6}$$

for any $\mathcal{L}\in \mathrm{im}(c^{l'}(Z))$ *and equality holds for generic* $\mathcal{L}^{im}_{gen} \in \mathrm{im}(c^{l'}(Z))$.

Proof. Assume that the minimum $\min_{E_{|Z_1|}\leq l\leq Z_1}\{\chi(l)\} = \chi(Z_1)$ is realized by some l_1. Then $(l',Z_1)\geq (l',l_1)$ (since $l'\in -\mathcal{S}'$), $\min_{E_{|Z_1|}\leq l\leq Z_1}\{\chi(l)\} = \min_{E_{|l_1|}\leq l\leq l_1}\{\chi(l)\}$ and $\chi(E_{|Z_1|}) = \chi(E_{|l_1|})$ hence $-(l',Z_1)-\min_{E_{|Z_1|}\leq l\leq Z_1}\{\chi(l)\}+\chi(E_{|Z_1|}) \leq -(l',l_1) - \min_{E_{|l_1|}\leq l\leq l_1}\{\chi(l)\}+\chi(E_{|l_1|})$. Since the maximality in (5.3.3.3) is realized by Z_1, which is minimal with this property, necessarily $Z_1 = l_1$. Next,

$$\max_{0\leq Z_1\leq Z}\{-(l',Z_1)-\min_{E_{|Z_1|}\leq l\leq Z_1}\{\chi(l)\}+\chi(E_{|Z_1|})\} \geq \max_{0\leq Z_1\leq Z}\{-(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\}.$$

But the maximum at the left hand side is realized by a term from the right.

For the last statement use again Remark 5.3.1.9(3)(a). □

5.3.4 Application for an arbitrary structure

The identity (5.3.3.5), valid for a generic analytic structure of $\widetilde{X}$, extends to an optimal inequality valid for *any analytic structure.*

Theorem 5.3.4.1. *Consider an* arbitrary *normal surface singularity* (X,o), *its resolution* $\widetilde{X}$, $Z\geq E$ *and* $l'\in -\mathcal{S}'$. *Then* $\mathrm{codim}\,\mathrm{im}(c^{l'}(Z)) = h^1(Z,\mathcal{L}^{im}_{gen})$ *(cf. Remark 5.3.1.9(3)(a)) satisfies*

$$\mathrm{codim}\,\mathrm{im}(c^{l'}(Z)) \geq \max_{0\leq Z_1\leq Z}\big\{ -(l',Z_1)-\chi(Z_1)+\chi(E_{|Z_1|})\big\}. \tag{5.3.4.2}$$

In particular, for any $\mathcal{L} \in \mathrm{im}(c^{l'}(Z))$ *one also has (everything computed in* $\widetilde{X}$*)*

$$h^1(Z,\mathcal{L}) \geq h^1(Z,\mathcal{L}^{im}_{gen}) = \operatorname{codim}\mathrm{im}(c^{l'}(Z)) \geq \max_{0\leq Z_1\leq Z}\Big\{ -(l',Z_1) - \chi(Z_1) + \chi(E_{|Z_1|})\Big\}. \tag{5.3.4.3}$$

Note that the right hand side of (5.3.4.2) is a sharp topological lower bound for $\operatorname{codim}\mathrm{im}(c^{l'}(Z))$. The inequality (5.3.4.2) can also be interpreted as the semi-continuity statement

$$\operatorname{codim}\mathrm{im}(c^{l'}(Z))(\text{arbitrary analytic structure}) \geq \operatorname{codim}\mathrm{im}(c^{l'}(Z))(\text{generic analytic structure}).$$

Proof. Consider the identity (5.3.1.10) applied for an arbitrary $\widetilde{X}$ and for the generic $\widetilde{X}$, denoted by $\widetilde{X}_{gen}$. Then, by semi-continuity of $h^1(\mathcal{O}_{Z_1})$ with respect to the analytic structure as parameter space (see e.g. [NN18, 3.6]), for any fixed effective cycle $Z_1 > 0$, $h^1(\mathcal{O}_{Z_1})$ computed in $\widetilde{X}$ is greater than or equal to $h^1(\mathcal{O}_{Z_1})$ computed in $\widetilde{X}_{gen}$. Therefore, by (5.3.1.10) one has $\operatorname{codim}\mathrm{im}(c^{l'}(Z))(\text{in } \widetilde{X}) \geq \operatorname{codim}\mathrm{im}(c^{l'}(Z))(\text{in } \widetilde{X}_{gen})$. Then for $\widetilde{X}_{gen}$ apply (5.3.3.5). □

Remark 5.3.4.4. Certain upper bounds for $\{h^1(Z,\mathcal{L})\}_{\mathcal{L}\in \mathrm{Pic}^{l'}(Z)}$, valid for any analytic structure, were established in [NN18, Prop. 5.7.1] (see alo Remark 5.3.5.3). However, an optimal upper bound is not known (see [NO17] for a particular case). Large h^1–values are realized by special strata, whose existence and study is extremely hard.

5.3.5 The cohomology of $\mathcal{L}^{im}_{gen}(l)$

Assume that $Z \geq E$, $l' \in -\mathcal{S}'$ and let $\mathcal{L}^{im}_{gen}$ be a generic element of $\mathrm{im}(c^{l'}(Z))$. If the analytic structure of (X, o) is generic, then by Proposition 5.3.3.4 $h^1(Z,\mathcal{L}^{im}_{gen}) = t_Z(l')$, where $t_Z(l')$ is the topological expression from the right hand side of (5.3.3.5).

Our goal is to give a topological lower bound for $h^1(Z,\mathcal{L})$, where $\mathcal{L} := \mathcal{L}^{im}_{gen}(l) =$

$\mathcal{L}^{im}_{gen}\otimes\mathcal{O}(l)\in \mathrm{Pic}^{l'+l}(Z)$ whenever $l\in L_{>0}$. In this way we will control the generic element of the 'new' strata $\mathcal{O}(l)\otimes(\mathrm{im}(c^{l'}(Z)))$ of $\mathrm{Pic}^{l'+l}(Z)$, unreachable directly by the previous result. Our hidden goal is to construct in this way line bundles with 'high' h^1.

For simplicity we will assume that all the coefficients of Z are sufficiently large (even compared with l, hence the coefficients of $Z-l$ are large as well). The monomorphism of sheaves $\mathcal{L}^{im}_{gen}|_{Z-l}\hookrightarrow \mathcal{L}^{im}_{gen}(l)$ gives $h^0(Z-l,\mathcal{L}^{im}_{gen})\leq h^0(Z,\mathcal{L}^{im}_{gen}(l))$, hence

$$h^1(Z-l,\mathcal{L}^{im}_{gen})+\chi(Z-l,\mathcal{L}^{im}_{gen})\leq h^1(Z,\mathcal{L}^{im}_{gen}(l))+\chi(Z,\mathcal{L}^{im}_{gen}(l)).$$

By a computation regarding χ this transforms into

$$h^1(Z,\mathcal{L}^{im}_{gen}(l))\geq h^1(Z-l,\mathcal{L}^{im}_{gen})+\chi(-l'-l)-\chi(-l').$$

If $\widetilde{X}$ is generic and $Z,Z-l\gg 0$ then $h^1(Z-l,\mathcal{L}^{im}_{gen})=t_{Z-l}(l')=t_Z(l')$, hence

$$h^1(Z,\mathcal{L}^{im}_{gen}(l))\geq t_Z(l')-\chi(-l')+\chi(-l'-l). \tag{5.3.5.1}$$

E.g., with the choice $l=-l'\in\mathcal{S}'\cap L_{>0}$ we get that $\mathcal{L}^{im}_{gen}(-l')\in\mathrm{Pic}^0(Z)$ and

$$h^1(Z,\mathcal{L}^{im}_{gen}(-l'))\geq t_Z(l')-\chi(-l'). \tag{5.3.5.2}$$

Remark 5.3.5.3. By [NN18, Prop. 5.7.1] for $Z\gg 0$, $\mathcal{L}\in\mathrm{Pic}(Z)$ with $c_1(\mathcal{L})\in-\mathcal{S}'$ one has $h^1(Z,\mathcal{L})\leq p_g$ whenever either $H^0(Z,\mathcal{L})=0$ or $\mathcal{L}\in\mathrm{im}(c^{l'}(Z))$. For other line bundles a weaker bound is established (see [loc. cit.]), which does not guarantee $h^1(\mathcal{L})\leq p_g$. However, it is not so easy to find singularities and bundles with $h^1(\mathcal{L})>p_g$ in order to show that such cases indeed might appear. In the next 5.3.5.4 we provide such an example (with a recipe to find many others as well) based partly on

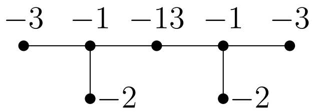
(5.3.5.2).

Example 5.3.5.4. Assume that we can construct a nonrational resolution graph which satisfies the following (combinatorial) properties, valid for certain $Z \gg 0$ and $l' \in -\mathcal{S}' \cap L$:

$$\begin{aligned} &(a) \quad t_Z(l') \geq \chi(-l') - \min_{l \geq 0} \chi(-l'+l) + 2, \ \text{and} \\ &(b) \quad -l' \leq \max \mathcal{M}, \ \text{where } \mathcal{M} := \{l \in L_{>0} : \chi(l) = \min \chi\}. \end{aligned} \tag{5.3.5.5}$$

Now, if we consider the generic analytic structure supported on this topological type, then $\min_{l\geq 0} \chi(-l'+l) \overset{(b)}{=} \min \chi = 1 - p_g$ (for the second identity use [NN18, Cor. 5.2.1]), hence $t_Z(l') - \chi(-l') \overset{(a)}{\geq} -1 + p_g + 2 = p_g + 1$. This combined with (5.3.5.2) gives $h^1(Z, \mathcal{L}^{im}_{gen}(-l')) > p_g$.

Next we show that (5.3.5.5) can be realized. Consider two copies Γ_1 and Γ_2 of the following graph



The wished graph Γ consists of Γ_1, Γ_2 and a new vertex v, which has two adjacent edges connecting v to the (-13)-vertices of Γ_1 and Γ_2. Let the decoration of v be $-b_v$ where $b_v \gg 0$. One verifies that the minimal cycle is $Z_{min} = (b_v - 2)E^*_v$, whose E_v–multiplicity is 1. We set $-l' := Z_{min}$. Since $\max \mathcal{M} \in \mathcal{S}_{an} \subset \mathcal{S}' \cap L$ (cf. [NN18, 5.7]) we get that $-l' = Z_{min} \leq \max \mathcal{M}$. One verifies that $\chi(Z_{min}) = -3$ (e.g. by Laufer's criterion), and also that $\min \chi = -5$ (realized e.g. for $2Z_{min} - E_v$). Therefore $\chi(-l') - \min_{l\geq 0} \chi(-l'+l) + 2 = -3 + 5 + 2 = 4$. On the other hand, the expression (under max) in (5.3.3.5) for $Z_1 = Z_{min}(\Gamma_1) + Z_{min}(\Gamma_2)$ supported on $\Gamma \setminus v$ is 4, hence $t_Z(l') \geq 4$.

5.4 Geometrical aspects behind Theorem 5.3.4.1

5.4.1 Reinterpretation of $t_Z(l')$

Let us discuss with more details the geometry behind the inequality (5.3.4.2). Along the discussion we will provide a second independent proof of it and we also provide several examples, which show its sharpness/weakness in several situations. Similar construction (with similar philosophy) will appear in forthcoming manuscripts on the subject as well. The construction of the present section shows also in a conceptual way how one can produce different sharp lower bounds for sheaf cohomologies (for another case see e.g. subsection 5.5.2).

We provide the new proof in several steps. First, we define a topological lower bound for $\operatorname{codim}\operatorname{im}(c^{l'}(Z))$, which (a priori) will have a more elaborated form then the right hand side $t_Z(l')$ of (5.3.4.2). Then via several steps we will simplify it and we show that in fact it is exactly $t_Z(l')$.

Definition 5.4.1.1. For any $Z > 0$ with $|Z|$ connected we define $D(Z, l')$ as 0 if $c^{l'}(Z)$ is dominant and 1 otherwise. Furthermore, set

$$T(Z, l') := \chi(-l') - \min_{0\leq l\leq Z, l\in L} \chi(-l' + l) + D(Z, l'). \tag{5.4.1.2}$$

By [NN18, Theorem 5.3.1] for any singularity (X, o), any resolution $\widetilde{X}$, any $Z > 0$ and $l' \in L'$, and for $\mathcal{L}_{gen}$ generic in $\operatorname{Pic}^{l'}(Z)$ one has

$$h^1(Z, \mathcal{L}_{gen}) = \chi(-l') - \min_{0\leq l\leq Z, l\in L} \chi(-l' + l). \tag{5.4.1.3}$$

By [NN18, Prop. 5.6.1], see also 5.3.1.9(3), for any $Z \geq E$ and for any $l' \in -\mathcal{S}'$, if $\mathcal{L}^{im}_{gen}$ is a generic element of $\operatorname{im}(c^{l'}(Z))$, then $h^1(Z, \mathcal{L}^{im}_{gen}) = \operatorname{codim}\operatorname{im}(c^{l'}(Z))$ satisfies

(the semicontinuity)

$$h^1(Z,\mathcal{L}^{im}_{gen})\geq \chi(-l')-\min_{0\leq l\leq Z, l\in L}\chi(-l'+l)+D(Z,l')=h^1(Z,\mathcal{L}_{gen})+D(Z,l')=T(Z,l'). \tag{5.4.1.4}$$

Remark 5.4.1.5. Assume that $Z>0$ is a nonzero cycle with connected support $|Z|$, but with $Z\not\geq E$. Then the statements from (5.4.1.4) remain valid for such Z once we replace l' by its restriction $R(l')$, where $R: L'\to L'(|Z|)$ is the natural cohomological operator dual to the natural homological inclusion $L(|Z|)\hookrightarrow L$. (For this apply the statement for the singularity supported on $|Z|$.) On the other hand, for $l\in L(|Z|)$ one has $\chi(-R(l'))-\chi(-R(l')+l)=-\chi(l)-(R(l'),l)_{L(|Z|)}=-\chi(l)-(l',l)=\chi(-l')-\chi(-l'+l)$. Hence, in fact, (5.4.1.4) remains valid in its original form for any such $Z>0$ with $|Z|$ connected.

Example 5.4.1.6. The difference $h^1(Z,\mathcal{L}^{im}_{gen})-h^1(Z,\mathcal{L}_{gen})$ can be arbitrary large. Indeed, let us start with a singularity with an arbitrary analytic structure, we fix a resolution $\widetilde{X}$ with dual graph Γ, and we distinguish a vertex, say v_0, associated with the irreducible divisor E_0. Let k ($k>0$) be the number of connected components of $\Gamma\setminus v_0$, and we assume that each of them is non–rational. Furthermore, we choose $Z\gg 0$, hence $h^1(\mathcal{O}_Z)=p_g$. Let $\widetilde{X}|_{\mathcal{V}\setminus v_0}$ be a small neighbourhood of $\cup_{v\neq v_0}E_v$, let $\{\widetilde{X}_i\}_{i=1}^k$ be its connected components, and set $p_{g,i}=h^1(\mathcal{O}_{\widetilde{X}_i})$ for the geometric genus of the singularities obtained from $\widetilde{X}_i$ by collapsing its exceptional curves. Write also $\Gamma\setminus v_0=\cup_i\Gamma_i$. We also assume that $-l'=nE_0^*$ with $n\gg 0$.

Since n is large, $\mathrm{im}(\widetilde{c}^{l'}(Z))=A_Z(l')$, hence $d_Z(l')=e_Z(l')=p_g-\sum_i p_{g,i}$, cf. [NN18, Th. 6.1.9] or Theorem 5.1.1.2 here. Hence, cf. (5.4.1.4), $\mathrm{codim}(\mathrm{im}\widetilde{c}^{l'}(Z))=h^1(\mathcal{O}_Z)-d_Z(l')=h^1(Z,\mathcal{L}^{im}_{gen})=\sum_i p_{g,i}$ (in particular, $\widetilde{c}^{l'}$ is not dominant).

Next we compute $h^1(Z,\mathcal{L}_{gen})=\chi(nE_0^*)-\min_{l\geq 0}\chi(nE_0^*+l)$. Write l as $l_0E_0+\widetilde{l}$, where $\widetilde{l}$ is supported on $\cup_{v\neq v_0}E_v$. Then $\chi(nE_0^*)-\chi(nE_0^*+l)=-\chi(l)-nl_0$. If $l_0=0$ then $-\chi(l)=-\chi(\widetilde{l})$, and its maximal value is $M:=\sum_i(-\min\chi(\Gamma_i))$. On the other hand, if $l_0 > 0$ then for $n > -M - \min\chi$ one has $-\chi(l) - l_0 n < M$. Hence $h^1(Z, \mathcal{L}_{gen}) = \chi(nE_0^*) - \min_{l\geq 0}\chi(nE_0^* + l) = \sum_i(-\min\chi(\Gamma_i))$.

Now, $p_{g,i} \geq 1 - \min\chi(\Gamma_i)$ (cf. [Wa70] or [NN18]), hence $h^1(Z, \mathcal{L}^{im}_{gen}) - h^1(Z, \mathcal{L}_{gen}) \geq k$.

5.4.1.7. We wish to estimate $h^1(Z, \mathcal{L}^{im}_{gen})$. Note that the estimate given by (5.4.1.4), that is, $h^1(Z, \mathcal{L}^{im}_{gen}) \geq T(Z, l')$, sometimes is week, see the previous example. However, surprisingly, if we replace Z by a smaller cycle $Z' \leq Z$, then we might get a better bound. More precisely, first note that if $\mathcal{L}^{im}_{gen}$ is a generic element of $\mathrm{im}(c^{l'}(Z))$, and $0 < Z' \leq Z$, then its restriction $r(\mathcal{L}^{im}_{gen})$ (via $r : \mathrm{Pic}^{l'}(Z) \to \mathrm{Pic}^{R(l')}(Z')$) is a generic element of $\mathrm{im}(c^{l'}(Z'))$. If Z' has more connected components, $Z' = \sum_i Z'_i$ (where each $|Z'_i|$ is connected and $|Z'_i| \cap |Z'_j| = \emptyset$ for $i \neq j$), then for each Z'_i we can apply (5.4.1.4). Therefore, we get

$$h^1(Z, \mathcal{L}^{im}_{gen}) \geq h^1(Z', r(\mathcal{L}^{im}_{gen})) = \sum_i h^1(Z'_i, r(\mathcal{L}^{im}_{gen})) \geq \sum_i T(Z'_i, l'). \tag{5.4.1.8}$$

Define

$$t(Z, l') := \max_{0<Z'\leq Z} \sum_i T(Z'_i, l') = \max_{0<Z'\leq Z} \Big(\sum_i (\chi(-l') - \min_{0\leq l_i \leq Z'_i} \chi(-l'+l_i) + D(Z'_i, l')) \Big). \tag{5.4.1.9}$$

(Here there is no need to restrict l', cf. Remark 5.4.1.5.) Hence (5.4.1.8) reads as

$$h^1(Z, \mathcal{L}^{im}_{gen}) \geq t(Z, l'). \tag{5.4.1.10}$$

In this estimate the point is the following: though $\sum_i(\chi(-l') - \min_{0\leq l_i\leq Z'_i}\chi(-l'+l_i) = \chi(-l') - \min_{0\leq l\leq Z'}\chi(-l'+l)$ is definitely not larger than $\chi(-l') - \min_{0\leq l\leq Z}\chi(-l'+l)$, the number of components of Z' might be large, and the sum of the 'non-dominant' contribution terms $\sum_i D(Z'_i, l')$ might increase the right hand side of (5.4.1.10) — compared with $T(Z, l')$ — drastically.

Example 5.4.1.11. (Continuation of Examle 5.4.1.6) The last computation of Example 5.4.1.6 shows that the maximum of $\chi(nE_0^*) - \min_{l\geq 0}\chi(nE_0^*+l)$ is obtained for $l_0 = 0$ and $T(Z,l') = 1 + \sum_i(-\min\chi(\Gamma_i))$. Hence, taking $Z' = \sum_i Z'_i$, each Z'_i supported on Γ_i and large, we get that the restriction of l' is zero and $\sum_i T(Z'_i,l') = \sum_i(1-\min\chi(\Gamma_i)) = T(Z,l') + k - 1$.

Summarized (also from Example 5.4.1.6), for any analytic type one has $\sum_i p_{g,i} = h^1(Z,\mathcal{L}^{im}_{gen}) \geq t(Z,l') \geq \sum_i T(Z'_i,l') = \sum_i(1-\min\chi(\Gamma_i))$. However, if $\widetilde{X}$ is generic then $p_{g,i} = 1-\min\chi(\Gamma_i)$ (cf. [NN18]), hence, all the inequalities transform into equalities. Hence, for generic analytic structure $h^1(Z,\mathcal{L}^{im}_{gen}) = t(Z,l')$, that is, (5.4.1.10) provides the optimal sharp topological lower bound.

Note also that both $t(Z,l')$ and $\sum_i(1-\min\chi(\Gamma_i))$ are topological, hence if they agree for $\widetilde{X}$ generic, then they are in fact equal. Since $p_{g,i}-1+\min\chi(\Gamma_i)$ for arbitrary analytic type can be considerably large, for *arbitrary* analytic types the inequality (5.4.1.10) can be rather week.

5.4.1.12. Our goal is to simplify the expression (5.4.1.9) of $t(Z,l')$.

First we analyse the set of cycles Z' for which the maximum in the right hand side of (5.4.1.9) can be realized. E.g., if $c^{l'}(Z)$ is dominant, then any $0 \leq Z' \leq Z$ realizes the maximum 0 (with all $l_i = 0$). (Indeed, use the fact that $D(Z_2,l') \geq D(Z_1,l')$ for $Z_2 \geq Z_1$ and $|Z_i|$ connected.)

In the next Lemmas 5.4.1.13 and 5.4.1.16 we will assume that $c^{l'}(Z)$ is not dominant.

Lemma 5.4.1.13. *(a) Assume that Z' is a minimal cycle (or a cycle with minimal number of connected components) among those cycles which realize the maximum in the right hand side of (5.4.1.9). Then $D(Z'_i,l') = 1$ for all i.*

(b) If $D(Z'_i,l') = 1$ then the minimal value $\min_{0\leq l_i\leq Z'_i}\chi(-l'+l_i)$ can be realized by $l_i > 0$.

Proof. *(a)* Otherwise, $c^{l'}(Z'_i)$ is dominant, so we have $\chi(-l')-\min_{0\leq l_i\leq Z'_i}\chi(-l'+l_i)=0$ (realized for $l_i=0$). Hence $T(Z'_i,l')=0$, that is, the right hand side of (5.4.1.9) is realized by $Z'-Z'_i$ too, contradicting the minimality of Z'. *(b)* If the wished minimum is realized by $l_i=0$, and *only* by $l_i=0$, then $c^{l'}(Z'_i)$ is dominant, contradicting $D(Z'_i,l')=1$. $\square$

Example 5.4.1.14. Though in Example 5.4.1.6 we have shown that $h^1(Z,\mathcal{L}^{im}_{gen})=t(Z,l')$ can be much larger than $T(Z,l')$ (that is, the maximizing Z' usually should be necessarily strict smaller than Z), in some cases $Z'=Z$ still works. Indeed, we claim that

if the E^*–support I of l' is included in the set of end vertices of Γ, then $t(Z,l')=T(Z,l')$.

Let Z' be a cycle for minimal number n of connected components $\{Z'_i\}_{i=1}^n$ for which the right hand side of (5.4.1.9) is realized. We claim that $n=1$. Indeed, by Lemma 5.4.1.13, each $D(Z'_i,l')=1$. Let l_i be a cycle which realizes $\chi(-l')-\min_{0\leq l\leq Z'_i}\chi(-l'+l)$. By Lemma 5.4.1.13 we can assume $l_i\neq 0$.

If $n>1$ then let Z_1 and Z_2 be two adjacent component, which means, that there is a vertex $u\in|Z'_1|$ and $v\in|Z'_2|$ and a (minimal) path $u_1=u,u_2,\cdots,u_t=v$, such that $u_2,\cdots,u_{t-1}\notin|Z'|$ and u_k and u_{k+1} are neighbours in the resolution graph. Moreover, define a new cycle by $Z'_{1,new}=Z'_1+Z'_2+\sum_{2\leq k\leq t-1}E_{u_k}$ and $Z'_{new}=Z'_{1,new}+\sum_{3\leq i\leq n}Z'_i$. Similarly, let us have a minimal path between $|l_1|$ and $|l_2|$: vertices $w_1,\cdots,w_l$, such that $w_1\in|l_1|$ and $w_l\in|l_2|$, $w_2,\cdots,w_{l-1}\notin|l_1|\cup|l_2|$ and w_k,w_{k+1} are neighbours in the resolution graph. Then define $l_{1,new}=l_1+l_2+\sum_{2\leq k\leq l-1}E_{w_k}$. The point is that the vertices $w_2,\cdots,w_{l-1}$ are not end vertices, in particular $(l',\sum_{2\leq k\leq l-1}E_{w_k})=0$.

Note also that $D(Z'_{1,new},l')=1$. Then a computation gives that

$$\chi(-l')-\chi(-l'+l_{1,new})+D(Z'_{1,new},l')\geq T(Z_1,l')+T(Z_2,l'), \tag{5.4.1.15}$$

or, $T(Z_{1,new}, l') \geq T(Z_1, l') + T(Z_2, l')$, contradicting the minimality of Z'. Hence necessarily $n = 1$.

On the other hand, if Z' is connected, then $T(Z', l') \leq T(Z, l')$, hence the maximal value in the right hand side of (5.4.1.10) is realized for Z as well (and maybe by several other smaller cycles too; here we minimalized $\#|Z'|$ by increasing Z').

The present example together with Examples 5.4.1.6 and 5.4.1.11 show that the structure of possible cycles Z' for which the maximality in (5.4.1.9) realizes can be rather subtle.

Lemma 5.4.1.16. *Assume that Z' is a minimal cycle among those cycles, which realizes the maximum in the right hand side of (5.4.1.9). Then the following facts hold:*

(a) $\min_{0\leq l_i\leq Z'_i}\chi(-l'+l_i)$ *is realized by* $l_i = Z'_i$.

(b) $\min_{0\leq l_i\leq Z'_i}\chi(l)$ *is realized by* $l_i = Z'_i$.

(c) $t(Z', l') = t(Z, l') = \sum_i \big(-(Z'_i, l') - \chi(Z'_i) + 1\big)$.

Proof. *(a)* For each Z'_i let l_i be minimal non–zero cycle (cf. Lemma 5.4.1.13) such that $M_i := \chi(-l') - \min_{0\leq l\leq Z'_i}\chi(-l'+l)$ is realized by l_i. Let $l_i = \cup_k l_{i,k}$ be its decomposition into cycles with $|l_{i,k}|$ connected and disjoint. Since $M_i = -\chi(l_i) - (l', l_i) \geq 0$, there exists k such that $\chi(-l') - \chi(-l' + l_{i,k}) = -\chi(l_{i,k}) - (l', l_{i,k}) \geq 0$, hence the Abel map $c^{l'}(l_{i,k})$ must be non–dominant. Thus (using also $D(Z'_i, l') = 1$ from Lemma 5.4.1.13*(a)*)

$$\sum_k T(l_{i,k}, l') \geq \chi(-l') - \chi(-l' + l_i) + 1 = T(Z'_i, l'). \tag{5.4.1.17}$$

In particular, by the minimality of Z'_i, $Z'_i = l_i$.

(b) By part *(a)* $\chi(Z'_i) + (Z'_i, l') \leq \chi(l_i) + (l_i, l')$ for any $0 \leq l_i \leq Z'_i$. But, since $l' \in -\mathcal{S}'$, $(Z'_i, l') \geq (l_i, l')$, hence $\chi(Z'_i) \leq \chi(l_i)$ for any $0 \leq l_i \leq Z'_i$. Part *(c)* follows from (5.4.1.9) and *(a)*. □

Recall that in 5.3.5 we defined $t_Z(l') := \max_{0\leq Z'\leq Z}\big\{ -(l',Z')-\chi(Z')+\chi(E_{|Z'|})\big\}$.

Corollary 5.4.1.18. $t(Z,l') = t_Z(l')$.

Proof. If $c^{l'}(Z)$ is dominant then both sides are zero. Otherwise, by Lemma 5.4.1.16(c) (with its notations) $t(Z,l') = \sum_i\big(-(Z_i',l')-\chi(Z_i')+1\big) \leq t_Z(l')$. On the other hand, let us fix some $Z' = \cup_i Z_i'$ for which the maximum in $t_Z(l')$ is realized. Then we can assume that each $c^{l'}(Z_i')$ is not dominant. Then $-(Z_i',l') - \chi(Z_i') + 1 = \chi(-l') - \chi(-l'+Z_i') + 1 \leq \chi(-l') - \min_{0\leq l_i\leq Z_i'}\chi(-l'+l_i) + D(Z_i',l')$. Hence $t_Z(l') \leq t(Z,l')$ too. $\square$

Remark 5.4.1.19. The second proof of Theorem 5.3.4.1 follows from (5.4.1.10) and Corolary 5.4.1.18.

5.5 The $\mathcal{L}_0$–projected Abel map

In this section we introduce a new object, a modification of the Picard group $\mathrm{Pic}(Z)$, which will play a key role in the cohomology computation of the shifted line bundles of type $\{\mathcal{L}_0\otimes\mathcal{L}\}_{\mathcal{L}\in \mathrm{im}(c^{l'}(Z))}$.

5.5.1 The $\mathcal{L}_0$–projected Picard group

Let (X,o) be a normal surface singularity. For simplicity we assume (as always in this thesis) that the link is a rational homology sphere. Let $\widetilde{X}$ be one of its good resolutions and $Z\geq E$ an effective cycle. Fix also $\mathcal{L}_0\in \mathrm{Pic}(Z)$ such that $H^0(Z,\mathcal{L}_0)_{reg}\neq\emptyset$.

Choose $s_0\in H^0(Z,\mathcal{L}_0)_{reg}$ arbitrarily, and write $\mathrm{div}(s_0) = D_0 \in \mathrm{ECa}^{l_0'}(Z)$, where $l_0' = c_1(\mathcal{L}_0) \in -\mathcal{S}'$. Motivated by the exponential exact sequence of sheaves $0\to \mathbb{Z}_Z \xrightarrow{i} \mathcal{O}_Z \to \mathcal{O}_Z^* \to 0$, we define $\mathcal{L}_0^* := \mathrm{coker}(\mathbb{Z}_Z \xrightarrow{i} \mathcal{O}_Z \xrightarrow{s_0} \mathcal{L}_0)$, where the second morphism is the multiplication by (restrictions of) s_0. Then we have the following commutative diagram of sheaves:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_Z & \xrightarrow{i} & \mathcal{O}_Z & \longrightarrow & \mathcal{O}_Z^* & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow s_0 & & \downarrow s_0^* & & \\
0 & \longrightarrow & \mathbb{Z}_Z & \longrightarrow & \mathcal{L}_0 & \longrightarrow & \mathcal{L}_0^* & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{D_0} & = & \mathcal{O}_{D_0} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where s_0^* is induced by s_0. At cohomological level we get the (identical/renamed) diagrams

$$
\begin{array}{ccccccc}
 & H^0(\mathcal{O}_{D_0}) & = & H^0(\mathcal{O}_{D_0}) & & & \\
 & \downarrow \delta^0 & & \downarrow \delta & & & \\
0 \to & H^1(\mathcal{O}_Z) & \to & H^1(\mathcal{O}_Z^*) & \xrightarrow{c_1} & L' & \to 0 \\
 & \downarrow s^0 & & \downarrow s & & \downarrow = & \\
0 \to & H^1(\mathcal{L}_0) & \to & H^1(\mathcal{L}_0^*) & \xrightarrow{c_1} & L' & \to 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & H^0(\mathcal{O}_{D_0}) & = & H^0(\mathcal{O}_{D_0}) & & & \\
 & \downarrow \delta^0 & & \downarrow \delta & & & \\
0 \to & \mathrm{Pic}^0(Z) & \to & \mathrm{Pic}(Z) & \xrightarrow{c_1} & L' & \to 0 \\
 & \downarrow s^0 & & \downarrow s & & \downarrow = & \\
0 \to & \mathrm{Pic}^0_{\mathcal{L}_0}(Z) & \to & \mathrm{Pic}_{\mathcal{L}_0}(Z) & \xrightarrow{c_1} & L' & \to 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}
$$

where we use the notation $\mathrm{Pic}_{\mathcal{L}_0}(Z) := H^1(Z, \mathcal{L}_0^*)$ — and call it *the $\mathcal{L}_0$–projected Picard group* —, and (its linearization) $\mathrm{Pic}^0_{\mathcal{L}_0}(Z) := H^1(Z, \mathcal{L}_0)$. Note that the classical first Chern class map c_1 factorizes to a well–defined map $c_1 : \mathrm{Pic}_{\mathcal{L}_0}(Z) \to L'$. Set also $\mathrm{Pic}^{l'}_{\mathcal{L}_0}(Z) := c_1^{-1}(l')$ for any $l' \in L'$; it is an affine space isomorphic to $\mathrm{Pic}^{l'}(Z)/\mathrm{im}(\delta)$ associated with the vector space $\mathrm{Pic}^0_{\mathcal{L}_0}(Z) = H^1(Z, \mathcal{L}_0) = H^1(\mathcal{O}_Z)/\mathrm{im}(\delta^0)$.

The corresponding vector spaces appear in the following exact sequences as well.

Let us take another line bundle $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ without fixed components, $s \in H^0(Z,\mathcal{L})_{reg}$ and $D := \mathrm{div}(s)$. Then one can take the exact sequences $0 \to \mathcal{O}_Z \xrightarrow{s} \mathcal{L} \to \mathcal{O}_D \to 0$ and $0 \to \mathcal{L}_0 \xrightarrow{s} \mathcal{L}_0 \otimes \mathcal{L} \to \mathcal{O}_D \to 0$. They induce (at cohomology, or 'tangent' vector space level) the following commutative diagram

$$\begin{array}{ccccccc}
 & & H^0(\mathcal{O}_{D_0}) & = & H^0(\mathcal{O}_{D_0}) & & \\
 & & \downarrow \delta^0 & & \downarrow & & \\
H^0(\mathcal{O}_D) & \xrightarrow{\delta^0_{\mathcal{L}}} & H^1(\mathcal{O}_Z) & \xrightarrow{s} & H^1(\mathcal{L}) & \to 0 \\
\Big\downarrow = & & \Big\downarrow s^0_{\mathcal{L}_0} & & \Big\downarrow & & \\
H^0(\mathcal{O}_D) & \xrightarrow{\bar\delta^0_{\mathcal{L}}} & H^1(\mathcal{L}_0) & \xrightarrow{s} & H^1(\mathcal{L}_0 \otimes \mathcal{L}) & \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

This is related with the Abel map $c^{l'}(Z) : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$ as follows. Recall from [NN18, 3.2.2] that the tangent linear map $T_D\, c^{l'}(Z) : T_D\, \mathrm{ECa}^{l'}(Z) \to T_{\mathcal{L}}\, \mathrm{Pic}^{l'}(Z)$ can be identified with $\delta^0_{\mathcal{L}} : H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_Z)$. Therefore, if $\mathcal{L} = \mathcal{L}^{im}_{gen}$ is a generic element of $\mathrm{im}(c^{l'}(Z))$ then $\mathrm{codim}\,\mathrm{im}(c^{l'}(Z)) = \dim\, H^1(\mathcal{O}_Z)/\mathrm{im}(\delta^0_{\mathcal{L}}) = h^1(Z,\mathcal{L})$. Similarly, consider the composition

$$c^{l'}_{\mathcal{L}_0}(Z) : \mathrm{ECa}^{l'}(Z) \xrightarrow{c^{l'}(Z)} \mathrm{Pic}^{l'}(Z) \xrightarrow{s^0_{\mathcal{L}_0}} \mathrm{Pic}^{l'}_{\mathcal{L}_0}(Z).$$

We call it *the $\mathcal{L}_0$–projection of the Abel map* $c^{l'}(Z)$. Using the previous paragraph we obtain that the tangent linear map $T_D\, c^{l'}_{\mathcal{L}_0}(Z) : T_D\, \mathrm{ECa}^{l'}(Z) \to T_{\mathcal{L}}\, \mathrm{Pic}^{l'}_{\mathcal{L}_0}(Z)$ can be identified with $\bar\delta^0_{\mathcal{L}} = s^0_{\mathcal{L}_0} \circ \delta^0_{\mathcal{L}} : H^0(\mathcal{O}_D) \to H^1(\mathcal{L}_0)$. Therefore, if $\mathcal{L}$ is a generic element of $\mathrm{im}(c^{l'}_{\mathcal{L}_0}(Z))$ (or, it is the image by $s_{\mathcal{L}_0}$ of a generic element $\mathcal{L}^{im}_{gen}$ of $\mathrm{im}(c^{l'}(Z))$) then

$$\mathrm{codim}\,\mathrm{im}(c^{l'}_{\mathcal{L}_0}(Z)) = \dim\, H^1(\mathcal{L}_0)/\mathrm{im}(\bar\delta^0_{\mathcal{L}}) = h^1(Z,\mathcal{L}_0 \otimes \mathcal{L}). \qquad (5.5.1.1)$$

This fact fully motivates the next point of view: if one wishes to study $h^1(Z, \mathcal{L}_0 \otimes \mathcal{L})$ with $\mathcal{L}_0$ fixed and $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ then — as a tool — the right Abel map is the $\mathcal{L}_0$–projected $c^{l'}_{\mathcal{L}_0}(Z)$.

5.5.2 The cohomology $h^1(Z, \mathcal{L}_0 \otimes \mathcal{L})$.

Using the exact sequence $H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_Z) \xrightarrow{s} H^1(Z, \mathcal{L}) \to 0$ and $h^0(\mathcal{O}_D) = (l', Z)$ we obtain the inequality $h^1(Z, \mathcal{L}) \geq h^1(\mathcal{O}_Z) - (l', Z)$. Usually it is not sharp, since $\delta^0_{\mathcal{L}}$ might not be injective. However, as in the prototype construction from section 5.4 (and even in its preceding sections), if we consider any $Z_1 \leq Z$ then we also have $h^1(Z, \mathcal{L}) \geq h^1(Z_1, \mathcal{L}) \geq h^1(\mathcal{O}_{Z_1}) - (l', Z_1)$, hence $h^1(Z, \mathcal{L}) \geq \max_{Z_1 \leq Z}\{h^1(\mathcal{O}_{Z_1}) - (l', Z_1)\}$, and, remarkably, this for the generic $\mathcal{L}^{im}_{gen} \in \mathrm{im}(c^{l'}(Z))$ is an equality (cf. (5.3.1.10)).

Similarly, using the exact sequence $H^0(\mathcal{O}_D) \to H^1(Z, \mathcal{L}_0) \xrightarrow{s} H^1(Z, \mathcal{L}_0 \otimes \mathcal{L}) \to 0$ we obtain $h^1(Z, \mathcal{L}_0 \otimes \mathcal{L}) \geq h^1(Z, \mathcal{L}_0) - (l', Z)$. Again, this usually is not sharp. However, by the same procedure,

$$h^1(Z, \mathcal{L}_0 \otimes \mathcal{L}) \geq \max_{0 \leq Z_1 \leq Z}\{h^1(Z_1, \mathcal{L}_0) - (l', Z_1)\}. \tag{5.5.2.1}$$

In the next section (cf. Corollary 5.6.2.4) we will prove that this is again an equality for the generic $\mathcal{L} = \mathcal{L}^{im}_{gen} \in \mathrm{im}(c^{l'}_{\mathcal{L}_0}(Z))$. (The above inequality (5.5.2.1) can be compared with (5.3.5.1) as well.)

5.5.3 Compatibility with Laufer duality and differential forms

Consider the perfect pairing $\langle\, , \rangle : H^1(\mathcal{O}_Z) \otimes H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \to \mathbb{C}$ from 3.5.1.3, see alo [NN18]. Once we fix $D_0 = \mathrm{div}(s_0)$ of certain $s_0 \in H^0(Z, \mathcal{L}_0)_{reg}$, we can define $\Omega_Z(D_0) := (\mathrm{im}(\delta^0_{\mathcal{L}_0}))^{\perp} \subset H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$. It is generated by forms which vanish on the image of the tangent map $T_{D_0} c^{l'_0}(Z)$, identified with $\delta^0_{\mathcal{L}_0}$, cf. 5.1.1.4 and

(5.1.1.6). The pairing $\langle\,,\rangle$ induces a perfect pairing $\langle\,,\rangle_{\mathcal{L}_0} : H^1(Z,\mathcal{L}_0)\otimes\Omega_Z(D_0)\to\mathbb{C}$, see also Theorem 5.1.1.5.

5.5.4 The $\mathcal{G}$–filtration of $\Omega_Z(D_0)=H^1(\mathcal{L}_0)^*$

Consider the situation and notations of Definition 5.1.1.9; in particular, $\mathcal{G}_l = H^0(\Omega^2_{\widetilde{X}}(l))/H^0(\Omega^2_{\widetilde{X}})$ for any $0<l\leq Z$. In the presence of $\mathcal{L}_0=\mathcal{O}_Z(D_0)$ as above, we have the subspace $\Omega_Z(D_0)=(\mathrm{im}\delta^0)^{\perp}\subset H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$, and the induced perfect pairing $\langle\,,\rangle_{\mathcal{L}_0} : H^1(Z,\mathcal{L}_0)\otimes\Omega_Z(D_0)\to\mathbb{C}$. Similarly, for any $0<l\leq Z$, we have the analogues data $\Omega_l(D_0)=(\mathrm{im}(\delta^0|_l))^{\perp}\subset H^0(\Omega^2_{\widetilde{X}}(l))/H^0(\Omega^2_{\widetilde{X}})$, and the induced perfect pairing $\langle\,,\rangle_{\mathcal{L}_0|_l} : H^1(l,\mathcal{L}_0)\otimes\Omega_l(D_0)\to\mathbb{C}$. One has the following inclusions inside $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$

$$\begin{array}{ccc}\Omega_l(D_0) & \longrightarrow & \Omega_Z(D_0)\\ \downarrow & & \downarrow\\ \mathcal{G}_l & \longrightarrow & H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})\end{array}$$

and, in fact, $\Omega_l(D_0)=\Omega_Z(D_0)\cap\mathcal{G}_l$. Hence $\{\Omega_l(D_0)\}_l=\{\Omega_Z(D_0)\cap\mathcal{G}_l\}_l$ filters $\Omega_Z(D_0)$. Moreover, by $\langle\,,\rangle_{\mathcal{L}_0|_l}$, one has $\dim\,\Omega_Z(D_0)\cap\mathcal{G}_l=\dim\,\Omega_l(D_0)=h^1(l,\mathcal{L}_0)$.

5.5.5 Dimensions/Notations

The dimension of $\mathrm{im}(c^{l'}_{\mathcal{L}_0}(Z))$ is denoted by $d_{\mathcal{L}_0,Z}(l')$.

If $A_Z(l')$ is the smallest affine space which contains $\mathrm{im}(c^{l'}(Z))$ in $\mathrm{Pic}^{l'}(Z)$, then $s_{\mathcal{L}_0}(A_Z(l'))$ is the smallest affine space which contains $\mathrm{im}(c^{l'}_{\mathcal{L}_0}(Z))$. We denote it by $A_{\mathcal{L}_0,Z}(l')$ and its dimension by $e_{\mathcal{L}_0,Z}(l')$. From definitions $d_{\mathcal{L}_0,Z}(l')\leq e_{\mathcal{L}_0,Z}(l')$.

In the next section we provide two algorithms for the computation of $d_{\mathcal{L}_0,Z}(l')$, the analogues of the algorithms from Theorems 5.2.1.6 and 5.3.1.2.

5.6 $\mathcal{L}_0$–projected versions of the algorithms

5.6.1 The setup

Let us fix (X, o), a good resolution $\widetilde{X}$, $Z \geq E$ and $l' \in -\mathcal{S}'$. We also fix a line bundle $\mathcal{L}_0$ as in section 5.5, whose notations we will adopt. In order to estimate $d_{\mathcal{L}_0,Z}(l')$ we proceed as in sections 5.2 and 5.3. In particular, we perform the modificatiosn $\pi_{\mathbf{s}} : \widetilde{X}_{\mathbf{s}} \to \widetilde{X}$, and we adopt the notations of 5.2.1 as well. By the generic choice of the centers of blow ups we can assume that they differ from the support of D_0. Notice that we have a natural identification between $H^1(\mathcal{O}_Z)$ and $H^1(\mathcal{O}_{Z_{\mathbf{s}}})$, and also between $H^1(\mathcal{O}^*_Z)$ and $H^1(\mathcal{O}^*_{Z_{\mathbf{s}}})$. Furthermore, we denote the divisor $\pi_{\mathbf{s}}^{-1}(D_0)$ on $\widetilde{X}_{\mathbf{s}}$ still by D_0 (basically unmodified), and the line bundle $\mathcal{O}_{Z_{\mathbf{s}}}(D_0)$ still by $\mathcal{L}_0$. Then we have the identification of $H^0(Z, \mathcal{O}_D)$ with $H^0(Z_{\mathbf{s}}, \mathcal{O}_D)$, and also $H^1(Z, \mathcal{L}_0) \simeq H^1(Z_{\mathbf{s}}, \mathcal{L}_0)$ and $H^1(Z, \mathcal{L}^*_0) \simeq H^1(Z_{\mathbf{s}}, \mathcal{L}^*_0)$ (hence identifications of the corresponding commutative diagrams from 5.5.1 as well). The subspace $\Omega_{Z_{\mathbf{s}}}(D_0)$ in $H^1(\mathcal{O}_{Z_{\mathbf{s}}})^* = H^1(\mathcal{O}_Z)^*$ is also 'stable' of dimension $h^1(Z, \mathcal{L}_0)$.

Write $d_{\mathcal{L}_0,\mathbf{s}}$ and $e_{\mathcal{L}_0,\mathbf{s}}$ the corresponding dimensions associated with $\widetilde{X}_{\mathbf{s}}$ defined as in 5.5.5. Then $d_{\mathcal{L}_0,\mathbf{s}} \leq e_{\mathcal{L}_0,\mathbf{s}}$. If $\mathbf{s} = \mathbf{0}$ then $d_{\mathcal{L}_0,\mathbf{0}} = d_{\mathcal{L}_0,Z}(l')$ and $e_{\mathcal{L}_0,\mathbf{0}} = e_{\mathcal{L}_0,Z}(l')$.

Theorem 5.6.1.1. *(1) $d_{\mathcal{L}_0,\mathbf{s}} - d_{\mathcal{L}_0,\mathbf{s}^{v,k}} \in \{0, 1\}$. Moreover, $d_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ if and only if for a generic point $\bar{\mathcal{L}} \in \operatorname{im}(c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}}))$ the set of divisors in $(c^{l'_{\mathbf{s}}}_{\mathcal{L}_0}(Z_{\mathbf{s}}))^{-1}(\bar{\mathcal{L}})$ do not have a base point on $F_{v,k,\mathbf{s}_{v,k}}$.*

(2) If for some fixed $\mathbf{s}$ the numbers $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}_{v,k}$ are not the same, then $d_{\mathcal{L}_0,\mathbf{s}} = \max_{v,k}\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$. In the case when all the numbers $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}_{v,k}$ are the same, then if this common value $d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ equals $e_{\mathcal{L}_0,\mathbf{s}}$, then $d_{\mathcal{L}_0,\mathbf{s}} = e_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$; otherwise $d_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}} + 1$.

Proof. (1) Assume first that either $s_{v,k} \geq 1$ or $a_v = 1$. Then divisors from $\mathrm{ECa}^{l'_{\mathbf{s}}}(Z_{\mathbf{s}})$ intersect $F_{v,k,\mathbf{s}_{v,k}}$ by multiplicity one, hence the intersection (supporting) point gives a map $q : \mathrm{ECa}^{l'_{\mathbf{s}}}(Z_{\mathbf{s}}) \to F_{v,k,\mathbf{s}_{v,k}}$, which is dominant. Moreover, $\mathrm{ECa}^{l'_{\mathbf{s}^{v,k}}}(Z_{\mathbf{s}^{v,k}})$ is birational with a generic fiber of q (the fiber over the point which was blown up), hence the first statement follows. Note also that $d_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ if and only if the generic fiber of the $\mathcal{L}_0$–projected Abel map $c^{l'_\mathbf{s}}_{\mathcal{L}_0}$ is not included in a q–fiber. This implies the second part of *(1)*.

If $\mathbf{s}_{v,k} = 0$ and $a_v > 1$ then write $l'_- := l'_\mathbf{s} - E^*_v$ and consider the 'addition map' $s : \mathrm{ECa}^{E^*_v}(Z_\mathbf{s}) \times \mathrm{ECa}^{l'_-}(Z_\mathbf{s}) \to \mathrm{ECa}^{l'_\mathbf{s}}(Z_\mathbf{s})$, which is dominant and quasifinite (cf. [NN18, Lemma 6.1.1]). Let $q : \mathrm{ECa}^{E^*_v}(Z_\mathbf{s}) \to E_v$ be given by the supporting point as before. Then if $q^{-1}(gen)$ is a generic fiber of q (above the point which was blown up), then the restriction of s to $q^{-1}(gen) \times \mathrm{ECa}^{l'_-}(Z_\mathbf{s})$ with target $\mathrm{ECa}^{l'_{\mathbf{s}^{v,k}}}(Z_{\mathbf{s}^{v,k}})$ is dominant and quasifinite. Hence the arguments can be repeated.

(2) First notice that if the numbers $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$ are not the same then from *(1)* we have $d_{\mathcal{L}_0,\mathbf{s}} \leq \min_{v,k} d_{\mathcal{L}_0,\mathbf{s}^{v,k}} + 1 \leq \max_{v,k} d_{\mathcal{L}_0,\mathbf{s}^{v,k}} \leq d_{\mathcal{L}_0,\mathbf{s}}$, hence $d_{\mathcal{L}_0,\mathbf{s}} = \max_{v,k} d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$.

Next, assume that the numbers $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$ are the same, say d.

If $d_{\mathcal{L}_0,\mathbf{s}} = d$ then part *(1)* reads as follows: $d_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ for all v and k if and only if for a generic $\bar{\mathcal{L}} \in \mathrm{im}(c^{l'_\mathbf{s}}_{\mathcal{L}_0}(Z_\mathbf{s}))$ the set of divisors in $(c^{l'_\mathbf{s}}_{\mathcal{L}_0}(Z_\mathbf{s}))^{-1}(\bar{\mathcal{L}})$ do not have a base point on any of the curves $\{F_{v,k,\mathbf{s}_{v,k}}\}_{v,k}$.

Let us choose a generic element $\bar{\mathcal{L}} \in \mathrm{im}(c^{l'_\mathbf{s}}_{\mathcal{L}_0}(Z_\mathbf{s}))$, which is in particular a regular value of $c^{l'_\mathbf{s}}_{\mathcal{L}_0}(Z_\mathbf{s})$ and the generic divisors in $\mathrm{ECa}^{l'_\mathbf{s}}(Z_\mathbf{s})$ mapped to $\bar{\mathcal{L}}$ are in fact generic divisors of $\mathrm{ECa}^{l'_\mathbf{s}}(Z_\mathbf{s})$ itself.

Next, take an element in $\Omega_{Z_\mathbf{s}}(D_0)$ (for details see 5.5.3) represented by a form ω, such that the class of ω vanish on $T_{\bar{\mathcal{L}}}\mathrm{im}(c^{l'_\mathbf{s}}_{\mathcal{L}_0}(Z_\mathbf{s}))$.

Then choose a generic D from $\mathrm{ECa}^{l'_\mathbf{s}}(Z_\mathbf{s})$, which is mapped to $\bar{\mathcal{L}}$ and which has no common points with the support of ω (we can even assume additionally that it is transversal and reduced). Then we apply the previous statements for $\bar{\mathcal{L}} := c^{l'_\mathbf{s}}_{\mathcal{L}_0}(Z_\mathbf{s})(D)$.

In particular, the class of ω vanish on $\mathrm{im}(T_D c^{l'_\mathbf{s}}_{\mathcal{L}_0}(Z_\mathbf{s}))$ so ω cannot have pole along any of the curves $\{F_{v,k,\mathbf{s}_{v,k}}\}_{v,k}$, that is, it belongs to $\Omega_{Z_\mathbf{s}}(I_\mathbf{s})$, cf. Theorem 5.1.1.5 and

Lemma 5.1.1.7. Hence $d_{\mathcal{L}_0,\mathbf{s}} = e_{\mathcal{L}_0,\mathbf{s}}$, cf. Lemma 5.2.1.3, and also $d = e_{\mathcal{L}_0,\mathbf{s}}$ too.

On the other hand if $d = e_{\mathcal{L}_0,\mathbf{s}}$, then from $d_{\mathcal{L}_0,\mathbf{s}^{v,k}} \le d_{\mathcal{L}_0,\mathbf{s}} \le e_{\mathcal{L}_0,\mathbf{s}}$ we get $d = d_{\mathcal{L}_0,\mathbf{s}}$. Hence $d_{\mathcal{L}_0,\mathbf{s}} = d$ if and only if $d = e_{\mathcal{L}_0,\mathbf{s}}$. Otherwise $d_{\mathcal{L}_0,\mathbf{s}}$ should be $d+1$ by *(1)*. □

5.6.2 Notations for the second algorithm

Consider the setup of 5.3.1 and combine it with the one from 5.6.1, where $\mathcal{L}_0$ enters in the picture. Accordingly, we have the following subspaces (inclusions):

$$\begin{array}{ccccccc}
\Omega_{Z_\mathbf{s}}(D_0)\cap\mathcal{G}_{l_\mathbf{s}} & \to & \Omega_{Z_\mathbf{s}}(D_0)\cap\Omega_{Z_\mathbf{s}}(I_\mathbf{s}) & \xrightarrow{j} & \Omega_{Z_\mathbf{s}}(D_0) & = & H^1(Z,\mathcal{L}_0)^* \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{G}_{l_\mathbf{s}} & \to & \Omega_{Z_\mathbf{s}}(I_\mathbf{s}) & \xrightarrow{i} & H^0(\Omega^2_{\widetilde{X}_\mathbf{s}}(Z_\mathbf{s}))/H^0(\Omega^2_{\widetilde{X}_\mathbf{s}}) & = & H^1(\mathcal{O}_Z)^*
\end{array}$$

The codimension of the inclusion i is $e_\mathbf{s}$ and the dimension of $\mathcal{G}_\mathbf{s}$ is $g_\mathbf{s}$ providing the inequality $e_\mathbf{s} \le h^1(\mathcal{O}_Z) - g_\mathbf{s}$. Similarly, the codimension of j is $e_{\mathcal{L}_0,\mathbf{s}}$ and the dimension of $\Omega_{Z_\mathbf{s}}(D_0)\cap\mathcal{G}_{l_\mathbf{s}}$ will be denoted by $g_{\mathcal{L}_0,\mathbf{s}}$ providing the inequality $e_{\mathcal{L}_0,\mathbf{s}} \le h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}}$. Hence

$$d_{\mathcal{L}_0,\mathbf{s}} \le e_{\mathcal{L}_0,\mathbf{s}} \le h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}}. \tag{5.6.2.1}$$

It is convenient to lift the $\mathbf{s}$–independent subspace $\Omega_{Z_\mathbf{s}}(D_0) = \Omega_Z(D_0)$ of $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$ as $\Omega_{\widetilde{X}}(D_0) := \pi^{-1}(\Omega_Z(D_0))$ by the projection $\pi : H^0(\Omega^2_{\widetilde{X}}(Z)) \to H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$.

Theorem 5.6.2.2. *(1)* $d_{\mathcal{L}_0,\mathbf{s}} - d_{\mathcal{L}_0,\mathbf{s}^{v,k}} \in \{0,1\}$.

(2) If for some fixed $\mathbf{s}$ *the numbers* $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}_{v,k}$ *are not the same, then* $d_{\mathcal{L}_0,\mathbf{s}} = \max_{v,k}\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$. *In the case when all the numbers* $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}_{v,k}$ *are the same, then if this common value* $d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ *equals* $h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}}$, *then* $d_{\mathcal{L}_0,\mathbf{s}} = h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$; *otherwise* $d_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}} + 1$.

Proof. Part *(1)* was already proved in Theorem 5.6.1.1. Regarding part *(2)*, if the numbers $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$ are not the same then we argue again as in the proof of Theorem

5.6.1.1.

Next, assume that the numbers $\{d_{\mathcal{L}_0,\mathbf{s}^{v,k}}\}$ are the same, say d. Via (5.6.2.1) and the first algorithm Theorem 5.6.1.1 we need to show that if $d = e_{\mathcal{L}_0,\mathbf{s}}$ then necessarily $d = h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}}$ as well. However, if $d = e_{\mathcal{L}_0,\mathbf{s}}$ then we have $e_{\mathcal{L}_0,\mathbf{s}} = d_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ for all (v,k), hence by (5.6.2.1) we get $e_{\mathcal{L}_0,\mathbf{s}} = d = d_{\mathcal{L}_0,\mathbf{s}^{v,k}} \leq e_{\mathcal{L}_0,\mathbf{s}^{v,k}}$. But $e_{\mathcal{L}_0,\mathbf{s}} \geq e_{\mathcal{L}_0,\mathbf{s}^{v,k}}$ by the combination of the argument from (5.2.1.5) and the diagram from 5.6.2. Hence, $d_{\mathcal{L}_0,\mathbf{s}^{v,k}} = e_{\mathcal{L}_0,\mathbf{s}}$ for all k and v implies $e_{\mathcal{L}_0,\mathbf{s}^{v,k}} = e_{\mathcal{L}_0,\mathbf{s}}$ for all v and k.

In particular, it is enough to verify the (stronger statement):

$$\text{if } e_{\mathcal{L}_0,\mathbf{s}^{v,k}} = e_{\mathcal{L}_0,\mathbf{s}} \text{ for all } v \text{ and } k \text{ then } e_{\mathcal{L}_0,\mathbf{s}} = h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}} \text{ as well.} \tag{5.6.2.3}$$

Assume that (5.6.2.3) is not true, that is, $e_{\mathcal{L}_0,\mathbf{s}^{v,k}} = e_{\mathcal{L}_0,\mathbf{s}}$ for all v and k, but $e_{\mathcal{L}_0,\mathbf{s}} < h^1(Z,\mathcal{L}_0) - g_{\mathcal{L}_0,\mathbf{s}}$. The last inequality via the diagram from 5.6.2 says that the inclusion $\Omega_{Z_\mathbf{s}}(D_0) \cap \mathcal{G}_{l_\mathbf{s}} \subset \Omega_{Z_\mathbf{s}}(D_0) \cap \Omega_{Z_\mathbf{s}}(I_\mathbf{s})$ is strict. This means, that there is a differential form $\omega \in \Omega_{\widetilde{X}}(D_0)$, with class $[\omega]$ in $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \subset H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$, such that ω does not have a pole along the exceptional divisor $F_{v,k,\mathbf{s}_{v,k}}$, however $[\omega] \notin \mathcal{G}_\mathbf{s}$. In particular, there exists a vertex $v \in |l'|$, such that the pole order of ω along E_v is larger than $(l_\mathbf{s})_v$. Notice that this also means $(l_\mathbf{s})_v = \min_{1\leq i\leq a_v} \mathbf{s}_{v,i} < Z_v$.

Let $1 \leq i \leq a_v$ be an integer such that $\mathbf{s}_{v,i} = (l_\mathbf{s})_v$ (abridged in the sequel by t) and we denote the order of vanishing of ω on an arbitrary exceptional divisor E_u by b_u, where u is an arbitrary vertex along the blowing up procedure. Next we focus on the string between v and $w_{v,i,\mathbf{s}_{v,i}}$ and we denote them by $v_0 = v, \ldots, v_t = w_{v,i,\mathbf{s}_{v,i}}$. Set $r := \min\{0 \leq s \leq t \,:\, b_{v_s} + t - s \geq 0\}$. Since for $s = t$ one has $b_{v_t} \geq 0$ (since ω has no pole along $F_{v,i,\mathbf{s}_{v,i}}$) r is well–defined. On the other hand we have $r \geq 1$. Indeed, $b_{v_0} + t < 0$, since pole order of ω along E_v is higher than $(l_\mathbf{s})_v = t$. Note that $b_{v_{r-1}} + t - r + 1 < 0$ and $b_{v_r} + t - r \geq 0$ imply $b_{v_r} - b_{v_{r-1}} \geq 2$ (†).

Let $\widetilde{X}'$ be that resolution obtained from $\widetilde{X}$, as an intermediate step of the tower between $\widetilde{X}$ and $\widetilde{X}_{\mathbf{s}}$, when in the (v,i) sequence of blow ups we do not proceed all $\mathbf{s}_{v,i}$ of them, but we create only the divisors $\{F_{v,i,k}\}_{k\leq r-1}$. Let $\mathcal{V}'$ be its vertex set and $\{E_u\}_{u\in\mathcal{V}'}$ its exceptional divisors. On $\widetilde{X}'$ consider the line bundle $\mathcal{L}:=\Omega^2_{\widetilde{X}'}(-\sum_{u\in\mathcal{V}'}b_uE_u)$. Since F_{v,i,v_r} was created by blowing up a *generic point* p of $E_{v_{r-1}}=F_{v,i,v_{r-1}}$, the existence of ω guarantees the existence of a section $s\in H^0(\widetilde{X}',\mathcal{L})$, which does not vanish along $E_{v_{r-1}}$ and it has multiplicity $m:=b_{v_r}-b_{v_{r-1}}-1$ at the generic point $p\in E_{v_{r1}}$. By (†) $m\geq 1$. By construction, ω (or s) belongs also to the subvectorspace $\Omega_{\widetilde{X}}(D_0)$ after certain identifications.

Now by the technical Lemma 5.6.3.1 (valid for general line bundles, and separated in section 5.6.3) for any $0\leq k<m$ and a generic point $p\in E_{v_{r-1}}$ there exists a section $s'\in H^0(\widetilde{X}',\mathcal{L})$, which does not vanish along the exceptional divisor $E_{v_{r-1}}$, and the divisor of s' has multiplicity k at p. We apply for $k=-(b_{v_{r-1}}+t-r+1)-1$. (Note that $0\leq k<m$.) The section s' gives a differential form $\omega'\in\Omega_{\widetilde{X}}(D_0)$, such that if we blow up $E_{v_{r-1}}$ in the generic point p and we denote the new exceptional divisor by $E_{v_{r,new}}$, then ω' has wanishing order $-(t-r+1)$ on $E_{v_{r,new}}$. This means, that if we blow up it in generic points $t-r+1$ times, then ω' has a pole on $E_{v_{t,new}}$, but has no pole on $E_{v_{t+1,new}}$. This means that $e_{\mathcal{L}_0,\mathbf{s}^{v,i}}\neq e_{\mathcal{L}_0,\mathbf{s}}$, which is a contradiction. □

The analogues of Corollaries 5.3.1.3 and 5.3.1.5 (with similar proofs) are:

Corollary 5.6.2.4. *For any $l'\in-\mathcal{S}'$, $Z\geq E$ and $\mathcal{L}_0$ with $H^0(Z,\mathcal{L}_0)_{reg}\neq\emptyset$ one has*

$$d_{\mathcal{L}_0,Z}(l')=\min_{\mathbf{s}}\{\,|s|+h^1(Z,\mathcal{L}_0)-g_{\mathcal{L}_0,\mathbf{s}}\,\}=\min_{0\leq Z_1\leq Z}\{\,(l',Z_1)+h^1(Z,\mathcal{L}_0)-h^1(Z_1,\mathcal{L}_0)\}.$$

This combined with (5.5.1.1) gives for a generic $\mathcal{L}^{im}_{gen}\in\mathrm{im}(c^{l'}(Z))$:

$$h^1(Z,\mathcal{L}_0\otimes\mathcal{L}^{im}_{gen})=\max_{0\leq Z_1\leq Z}\{\,h^1(Z_1,\mathcal{L}_0)-(l',Z_1)\}.$$

Example 5.6.2.5. This is a continuation of Example 5.3.1.8 (based on [NN18, §11]), whose notations and statements we will use. Assume that $Z \gg 0$ and $l' = -kE_0^*$ as in 5.3.1.8. Additionally we take a generic line bundle $\mathcal{L}_0$ with $c_1(\mathcal{L}_0) = l'_0 = -k_0E_0^*$, $k_0 \geq 0$, (hence $\widetilde{D}_0$ consists of k_0 generic irreducible cuts of E_0). Recall that $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}})$ admits a basis consisting of elements of type $\mathbf{x}^{\mathbf{m}}\omega$, where ω is the Gorenstein form and $0 \leq |\mathbf{m}| \leq d-3$. Each 'block' $\{|\mathbf{m}| = j\}$ $(0 \leq j \leq d-3)$ (which can be identified with $H^0(\mathbb{P}^2, \mathcal{O}(j))$) contributes with $\binom{j+2}{2}$ monomials. The k_0 generic divisors impose $\min\{k_0, \binom{j+2}{2}\}$ independent conditions (see [NN18, 11.2] for the explication), hence the block $\{|\mathbf{m}| = j\}$ $(0 \leq j \leq d-3)$ contributes into $\dim \Omega_Z(D_0) = h^1(\mathcal{L}_0)$ with $\binom{j+2}{2} - \min\{k_0, \binom{j+2}{2}\} = \max\{0, \binom{j+2}{2} - k_0\}$. In particular, $h^1(\mathcal{L}_0) = \sum_{j=0}^{d-3} \max\{0, \binom{j+2}{2} - k_0\}$ and $h^1(\mathcal{L}_0) - g_{\mathcal{L}_0,s} = \sum_{j=0}^{d-3-s} \max\{0, \binom{j+2}{2} - k_0\}$ $(0 \leq s \leq d-2)$. Therefore,

$$d_{\mathcal{L}_0,Z}(-kE_0^*) = \min_{0 \leq s \leq d-2} \Big\{ ks + \sum_{j=0}^{d-3-s} \max\Big\{0, \binom{j+2}{2} - k_0\Big\} \Big\}.$$

However, if $\mathcal{L}_0 = \mathcal{O}_Z(D_0)$ is not generic, then the points D_0 might fail to impose independent conditions on the corresponding linear systems, and the determination of the dimension of $\Omega_Z(D_0)$ can be harder. See [NN18, 11.3] for discussion, examples and connection with the Cayley–Bacharach type theorems (cf. [EGH96]). Those discussions with combined with the present section produces further examples for $d_{\mathcal{L}_0,Z}(l')$ whenever D_0 is special (and (X, o) is superisolated).

5.6.3 A technical lemma

The next lemma is used in the body of the article, however, it might have also an independent general interest.

Lemma 5.6.3.1. *Let $\widetilde{X}$ be an arbitrary resolution of a normal surface singularity $(X, 0)$. Let us fix an arbitrary line bundle $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ with $c_1(\mathcal{L}) = l' \in -S'$, an irreducible exceptional curve E_v, and an integer $m > 0$.*

Assume that there exists a sub-vectorspace $V \subset H^0(\widetilde{X}, \mathcal{L})$ *with the following property: for a generic point* $p \in E_v$ *there exists a section* $s \in V$ *such that* s *does not vanish along* E_v *and the multiplicity of the divisor of* s *at* $p \in E_v$ *is* m*. Then for any number* $0 \le k \le m$ *and a generic point* $p \in E_v$ *there exists a section* $s \in V$ *such that* s *does not vanish along* E_v *and the multiplicity of the divisor of* s *at* $p \in E_v$ *is* k*.*

Proof. By induction we need to prove the statement only for $k = m - 1$.

First we fix a very large integer $N \gg m$, and consider the restriction $r : H^0(\widetilde{X}, \mathcal{L}) \to H^0(NE_v, \mathcal{L})$. Then r induces a map from $H^0(\widetilde{X}, \mathcal{L})_{reg} := H^0(\widetilde{X}, \mathcal{L}) \setminus H^0(\widetilde{X}, \mathcal{L}(-E_v))$ to $H^0(NE_v, \mathcal{L})_{reg} := H^0(NE_v, \mathcal{L}) \setminus H^0((N-1)E_v, \mathcal{L}(-E_v))$. Denote its restriction $H^0(\widetilde{X}, \mathcal{L})_{reg} \cap V \to H^0(NE_v, \mathcal{L})_{reg} \cap r(V)$ by r_V. Consider also the natural map $\mathrm{div} : H^0(NE_v, \mathcal{L})_{reg} \to \mathrm{ECa}^{l'}(NE_v)$, and the composition map $\mathrm{div} \circ r_V = g : H^0(\widetilde{X}, \mathcal{L})_{reg} \cap V \to \mathrm{ECa}^{l'}(NE_v)$, which sends a section to its divisor restricted to the cycle NE_v.

Next, for any $p \in E_v^0 := E_v \setminus \cup_{u \neq v} E_u$ set $D_{m,p} \subset \mathrm{ECa}^{l'}(NE_v)$, the set of divisors with multiplicity m at p. (Since $N \gg m$ this notion is well–defined). Set also $D_m := \cup_p D_{m,p}$.

By the assumption, the image of g intersects $D_{m,p}$ for any generic p. Since D_m is constructible subvariety of $\mathrm{ECa}^{l'}(NE_v)$, $g^{-1}(D_m)$ is a nonempty constructible subset of $H^0(\widetilde{X}, \mathcal{L})_{reg} \cap V$. Define an analytic curve $h_0 : (-\epsilon, \epsilon) \to g^{-1}(D_m)$ such that its image is not a subset of some $g^{-1}(D_{m,p})$. Let us denote the zeros of the section $h_0(0)$ along E_v^0 by $\{p_1, \ldots, p_r\}$. Then there exists a small neighborhood U of one of the points p_i and a restriction of h_0 to some smaller $(-\epsilon', \epsilon')$, such that for any $t \in (-\epsilon', \epsilon')$ the restriction of $h_0(t)$ to U has a unique zero, say $p(t)$, and its multiplicity is m. Furthermore, $t \mapsto p(t)$, $(-\epsilon', \epsilon') \to U \cap E_v^0$ is not constant, hence taking further restrictions to some interval we can assume that $t \mapsto p(t)$ is locally invertible. Reparametrising h_0 by the inverse of this map, we obtain an analytic map $U \cap E_v^0 \to g^{-1}(D_m)$, $t \mapsto h(t)$ such that the restriction of the section $h(t)$ to some local chart U

has only one zero, namely t, and the multiplicity of the section at t is m. In some local coordinates (x,y) of U (with $U\cap E_v=\{y=0\}$) the equation of $h(t)$ has the form (modulo y^N)

$$h(t)=\sum_{j\geq 0,i\geq 0}(x-t)^jy^ic_{j,i}(t), \tag{5.6.3.2}$$

where by the multiplicity condition $c_{j,i}\equiv 0$, if $j+i<m$ and, there is a pair (j,i), such that $j+i=m$ and $c_{j,i}(t)\not\equiv 0$. Moreover, by the non–vanishing condition $y\not| h(t)$, or, $c_{j,0}(t)\not\equiv 0$ for some j.

We claim that there is a generic choice of $t_1,\ldots,t_r$ (for some large r) of t–values, and a convenient choice of the coefficients $\{\alpha_l\}_{l=1}^r$ such that $s:=\sum_{l=1}^r\alpha_lh(t_l)$ satisfies the requirements. Indeed, first we consider the Taylor expansion of $h(t)$ in variables (x,y) at a point $(x,y)=(q,0)$ with q generic (and modulo y^N as usual):

$$\sum_{j,i}(x-q+q-t)^jy^ic_{j,i}(t)=\sum_{j,i}\sum_{k=0}^{j}(x-q)^ky^i\binom{j}{k}(q-t)^{j-k}c_{j,i}(t).$$

The fact that s at $(q,0)$ has multiplicity $\geq m-1$ transforms into a linear system

$$\sum_{l=1}^r\alpha_l\Big(\sum_{j\geq k}\binom{j}{k}(q-t_l)^{j-k}c_{j,i}(t_l)\Big)=0$$

for any (k,i) with $k,i\geq 0$ and $k+i\leq m-2$. This linear system $LS(r,m-2)$ with unknowns $\{\alpha_l\}_{l=1}^r$ has matrix $M(r,m-2)$ of size $r\times m(m-1)/2$. If $r\gg m(m-1)/2$ then the system has a nontrivial solution. We need to show that for a generic choice of the solutions $\{\alpha_l\}_l$ the section s has multiplicity $m-1$ at q. Assume that this is not the case. Then the generic solution of the system $LS(r,m-2)$ is automatically solution of $LS(r,m-1)$ too (the last one defined similarly). This means that $\mathrm{rank}M(r,m-2)=\mathrm{rank}M(r,m-1)$ (†) for generic $\{t_l\}_l$.

The matrix $M(r,m-1)$ has m additional rows corresponding to the indexes (k,i) with $k,i\geq 0$ and $k+i=m-1$. Let us fix one of them, corresponding to the following choice.

Now let d be the minimal number, such that there exists j, i such that $i \leq m-1$, $j+i=d$ and $c_{j,i}(t)$ is not identically 0. Since by assumption (by non–vanishing of $h(t)$ along E_v) there exists certain $j \geq m$ with $c_{j,0} \not\equiv 0$, such a d exists. Fix i_0 such that $i_0 \leq m-1$, $j_0+i_0=d$ and $c_{j_0,i_0}(t) \not\equiv 0$.

Then, from the additional rows of $M(r, m-1)$ we chose the one indexed by $(m-1-i_0, i_0)$.

Consider the minor of $M(r, m-1)$ of size $m(m-1)/2+1$, whose last row is the row corresponding to $(m-1-i_0, i_0)$, and the other rows belong to $M(r, m-2)$, while the last column corresponds to the generic $t_r = t$. Then its determinant should be zero by (†). Expanded it by the last column gives

$$\sum_{j \geq m-1-i_0} \binom{j}{m-1-i_0} (q-t)^{j-m+1+i_0} c_{j,i_0}(t) = \sum_{k,i \geq 0; k+i \leq m-2} \beta_{k,i}(q) \cdot \sum_{j \geq k} \binom{j}{k} (q-t)^{j-k} c_{j,i}(t)$$

for some holomorphic functions $\beta_{k,i}(q)$. But such an identity cannot exist. Indeed, since $c_{j_0,i_0} \not\equiv 0$, but $c_{j,i_0} \equiv 0$ for any $j < j_0$, the vanishing order of $q-t$ at the left hand side is exactly $d-m+1$, while on the right hand side — since $j \geq d-i$ (otherwise $c_{j,i} \equiv 0$) and $k \leq m-2-i$ implies $j-k \geq d-m+2$ — we get vanishing order $\geq d-m+2$.

Finally we need to show that this generic s does not vanish along E_v. This follows from a similar argument as above, or one can proceed as follows. For any generic q consider a section s which has multiplicity $m-1$ at $(q, 0)$. If it vanishes along E_v then $s + h(q)$ does not vanish along E_v and it has multiplicity $m-1$ at $(q, 0)$. □

Remark 5.6.3.3. We claim that under the assumptions of Lemma 5.6.3.1 the following property also holds: *For any finite set $F \subset E_v$ there exists a section $s \in V$ such that s does not vanish along E_v, $\mathrm{div}(s) \cap F = \emptyset$, and at each each $p \in \mathrm{div}(s) \cap E_v$ the intersection is transversal.* Indeed, we can use first Lemma 5.6.3.1 for $k=1$ and then show that a generic combination of ‘moving’ sections of multiplicity one works.

Chapter 6

Gorenstein singularities

Let us have a numerically Gorenstein resolution graph Γ. From [PPP11] we know, that if Γ is numerically Gorenstein, then there is a Gorenstein surface singularity with resolution $\widetilde{X}$, whose dual is graph Γ.

However, the construction in [PPP11] is a very special analytic plumbing.

In this chapter we wish to describe a gluing construction, which for every numerically Gorenstein resolution graph Γ provides a Gorenstein singularity (X, o) with resolution has dual graph Γ. Furthermore, every Gorenstein singularity with resolution graph Γ can be given by this construction.

6.1 Existence of Gorenstein singularities supported on numerically Gorenstein resolution graphs

6.1.1 Preliminaries

In this chapter, for any resolution $\widetilde{X}$, we denote by $K = K_{\widetilde{X}}$ the canonical divisor of $\widetilde{X}$, that is, $\Omega^2_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(K)$.

We fix a numerically Gorenstein resolution graph Γ. This means that $Z_K \in L$. From [PPP11] we know, that there exists a Gorenstein surface singularity with resolution $\widetilde{X}$ and dual graph Γ. This means that there exists a differential form $\omega \in H^0(\Omega^2_{\widetilde{X}}(Z_K)) = H^0(\mathcal{O}_{\widetilde{X}}(K + Z_K))$, such that ω has a pole on the exceptional divisor of order Z_K and it does not vanish anywhere in $\widetilde{X} \setminus E$.

Lemma 6.1.1.1. *If we have a resolution, for which $E \leq Z_K \in L$, then the above Gorenstein property is equivalent with any of the following facts:*

(i) $h^1(Z_K, \mathcal{O}_{Z_K}) > h^1(Z_K - E_u, \mathcal{O}_{Z_K - E_u})$ for every $u \in \mathcal{V}$.

(ii) there exists a vertex $u \in \mathcal{V}$ such that $h^1(Z_K, \mathcal{O}_{Z_K}) > h^1(Z_K - E_u, \mathcal{O}_{Z_K - E_u})$.

Proof. *(i)* The classes of differential forms in $\frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K))}{H^0(\mathcal{O}_{\widetilde{X}}(K))}$, which have a pole of order Z_K are exactly the ones, which are not in $\frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K-E_u))}{H^0(\mathcal{O}_{\widetilde{X}}(K))}$ for every vertex $u \in \mathcal{V}$. On the other hand, by Laufer's duality, $\dim\left(\frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K))}{H^0(\mathcal{O}_{\widetilde{X}}(K))}\right) = h^1(Z_K, \mathcal{O}_{Z_K})$ and $\dim\left(\frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K-E_u))}{H^0(\mathcal{O}_{\widetilde{X}}(K))}\right) = h^1(Z_K - E_u, \mathcal{O}_{Z_K - E_u})$.

(ii) Notice that if $\frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K-E_u))}{H^0(\mathcal{O}_{\widetilde{X}}(K))} = \frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K))}{H^0(\mathcal{O}_{\widetilde{X}}(K))}$ for some vertex $u \in \mathcal{V}$, then $\frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K-E))}{H^0(\mathcal{O}_{\widetilde{X}}(K))} = \frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K))}{H^0(\mathcal{O}_{\widetilde{X}}(K))}$, because $c^1(\mathcal{O}_{\widetilde{X}}(K+Z_K-E_u)) = -E_u$, so the Laufer sequence which starts at E_u goes through E as well.

However $\frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K-E))}{H^0(\mathcal{O}_{\widetilde{X}}(K))} = \frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K))}{H^0(\mathcal{O}_{\widetilde{X}}(K))}$ means, that $\frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K-E_v))}{H^0(\mathcal{O}_{\widetilde{X}}(K))} = \frac{H^0(\mathcal{O}_{\widetilde{X}}(K+Z_K))}{H^0(\mathcal{O}_{\widetilde{X}}(K))}$ for every vertex $v \in \mathcal{V}$. □

6.1.1.2. Although [PPP11] guarantees the existence of a Gorenstein analytic structure, the construction was given by a very special analytic plumbing.

In this section we wish to describe a construction for every numerically Gorenstein resolution graph Γ, which gives a Gorenstein singularity with resolution graph Γ, and furthermore every Gorenstein singularity with resolution graph Γ can be given by this construction.

Although, very little is known about the moduli space of the possible Gorenstein analytic structures of a singularity corresponding to the numerically Gorenstein resolution graph Γ, or even about the possible analytic structures, the minimal value of the geometric genus of Gorenstein structures should correspond to a 'generic Gorenstein structure'.

If we consider all the possible analytic structures for the resolution graph Γ, then an appropriate definition of generic analytic structure was given in [NN18] (see Chapter 4 here), and it was showed that the minimal possible geometric genus is $1 - \min_{0<l\in L} \chi(l)$, which is the geometric genus of this generic analytic structure.

In this section we describe a way to construct generically a Gorenstein analytic structure, and we hope, that the p_g of this singularity is the least possible among all Gorenstein analytic singularities supported on Γ, and that we can compute them explicitly in a combinatorial way from the resolution graph Γ in the future.

First we prove a few lemmas, which will be useful in the following:

Lemma 6.1.1.3. *Let us have a numerically Gorenstein resolution graph Γ, such that $Z_K > E$ and let us have a vertex set $I \subset \mathcal{V}$ which consists of end vertices of the graph and $\mathcal{V} \setminus I \neq \emptyset$ and we have $(Z_K)_v = 1$ for every vertex $v \in I$.*

Let us have a singularity $\widetilde{X}$ with resolution graph Γ, and we denote a small tubular neighbourhood of $\cup_{v\in\mathcal{V}\setminus I} E_v$ by $\widetilde{X}_r$ (with dial graph Γ_r), and the restriction of the cycle Z_K to $L(\Gamma_r)$ by $(Z_K)_r$.

Then the singularity $\widetilde{X}$ is Gorenstein if and only if the line bundle $\mathcal{O}_{(Z_K)_r}(K+Z_K)$ is trivial on the cycle $(Z_K)_r$.

Proof. Assume first that $\widetilde{X}$ is Gorenstein. This happens if and only if $h^1(\mathcal{O}_{Z_K}) > h^1(\mathcal{O}_{Z_K-E_u})$ for every vertex $u \in \mathcal{V}$.

Notice, that $h^1(\mathcal{O}_{Z_K}) = h^0(\mathcal{O}_{Z_K}(K+Z_K))$ and $h^1(\mathcal{O}_{Z_K-E_u}) = h^0(\mathcal{O}_{Z_K-E_u}(K+Z_K-E_u))$, which means, that if the singularity is Gorenstein, then $h^0(\mathcal{O}_{Z_K}(K+Z_K)) > h^0(\mathcal{O}_{Z_K-E_u}(K+Z_K-E_u))$ for every vertex $u \in \mathcal{V}$, which means, that $H^0(\mathcal{O}_{Z_K}(K+Z_K))_{reg} \neq 0$.

On the other hand we know, that $c^1(\mathcal{O}_{Z_K}(K+Z_K)) = 0$, so the line bundle $\mathcal{O}_{Z_K}(K+Z_K)$ is trivial hence its restriction, the line bundle $\mathcal{O}_{(Z_K)_r}(K+Z_K)$, is trivial as well.

Next assume that $\mathcal{O}_{(Z_K)_r}(K+Z_K)$ is trivial, and let us fix a vertex $u \in \mathcal{V}\setminus I$ such that $(Z_K)_u > 1$. Then we have $h^0(\mathcal{O}_{(Z_K)_r}(K+Z_K)) > h^0(\mathcal{O}_{(Z_K)_r - E_u}(K+Z_K-E_u))$.

With the notation $E_I = \sum_{v\in I} E_v$ consider the following exact sequence:

$$0 \to H^0(\mathcal{O}_{Z_K-E}(-E)) \to H^0(\mathcal{O}_{Z_K}) \to H^0(\mathcal{O}_E) \to H^1(\mathcal{O}_{Z_K-E}(-E)) \to H^1(\mathcal{O}_{Z_K}) \to 0.$$

We know, that the map $H^0(\mathcal{O}_{Z_K}) \to H^0(\mathcal{O}_E)$ is surjective, hence we get that $h^1(\mathcal{O}_{Z_K-E}(-E)) = h^1(\mathcal{O}_{Z_K})$. On the other hand, the Laufer sequence starting from E_I goes through E, therefore $h^1(\mathcal{O}_{Z_K-E_I}(-E_I)) = h^1(\mathcal{O}_{Z_K-E}(-E)) + \chi(E_I) - \chi(E)$. This means that $h^1(\mathcal{O}_{Z_K-E_I}(-E_I)) = h^1(\mathcal{O}_{Z_K-E}(-E)) + |E_I| - 1 = h^1(\mathcal{O}_{Z_K}) + |E_I| - 1$, where $|E_I|$ denotes the number of connected components of E_I.

By duality $h^0(\mathcal{O}_{(Z_K)_r}(K+Z_K)) = h^1(\mathcal{O}_{Z_K}) + |E_I| - 1$.

Similarly we have the following exact sequence:

$$\begin{aligned} 0 \to H^0(\mathcal{O}_{Z_K-E_u-E}(-E)) \to H^0(\mathcal{O}_{Z_K-E_u}) \to H^0(\mathcal{O}_E) \\ \to H^1(\mathcal{O}_{Z_K-E_u-E}(-E)) \to H^1(\mathcal{O}_{Z_K-E_u}) \to 0. \end{aligned}$$

We know, that the map $H^0(\mathcal{O}_{Z_K-E_u}) \to H^0(\mathcal{O}_E)$ is surjective, hence $h^1(\mathcal{O}_{Z_K-E_u-E}(-E)) = h^1(\mathcal{O}_{Z_K-E_u})$. On the other hand the Laufer sequence starting from E_I to the Lipman cone goes through E, therefore $h^1(\mathcal{O}_{Z_K-E_u-E_I}(-E_I)) = h^1(\mathcal{O}_{Z_K-E_u-E}(-E)) + \chi(E_I) - \chi(E)$, which means, that $h^1(\mathcal{O}_{Z_K-E_u-E_I}(-E_I)) = h^1(\mathcal{O}_{Z_K-E_u-E}(-E)) + |E_I| - 1$.

This means that $h^0(\mathcal{O}_{(Z_K)_r-E_u}(K+Z_K-E_u)) = h^1(\mathcal{O}_{Z_K-E_u}) + |E_I| - 1$.

But we know that $h^0(\mathcal{O}_{(Z_K)_r}(K+Z_K)) > h^0(\mathcal{O}_{(Z_K)_r-E_u}(K+Z_K-E_u))$, which yields, that $h^1(\mathcal{O}_{Z_K}) > h^1(\mathcal{O}_{Z_K-E_u})$. In particular, by Lemma 6.1.1.1 $\widetilde{X}$ is Gorenstein, which proves the claim of the lemma completely. $\square$

6.1.1.4. In the main construction the next fact will be used several times. It follows from Theorem 3.4.1.9.

Lemma 6.1.1.5. *Let Γ be the dual graph of a resolution $\widetilde{X}$ and fix $\widetilde{X}_I$ (neighbourhood of $\cup_{v\in I}E_v$) associated with some $I \subset \mathcal{V}$. Assume that the E^*–support of a line bundle $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ is in $\mathcal{V} \setminus I$ (i.e. if $(c_1\mathcal{L}, E_v) \neq 0$ then $v \notin I$). Then for $N \gg 0$ the bundle $\mathcal{L}^N$ is in the image of $c^{Nc_1(\mathcal{L})}(Z)$ $(Z \gg 0)$ if and only if $\mathcal{L}|_{\widetilde{X}_I}$ is trivial.*

6.1.2 The construction

In the following we will describe our main construction, which provides for a minimal numerically Gorenstein resolution graph Γ a Gorenstein analytic structure.

In the following we can assume that the resolution graph Γ is not rational, because in the rational case the resolution graph is numerically Gorenstein if and only if it is an ADE graph, and in that case any analytic type is Gorenstein.

Hence, we start with a nonrational minimal numerically Gorenstein resolution graph Γ. Then automatically $Z_K > E$. Write $Z_K = \sum_{v\in\mathcal{V}} t_v \cdot E_v$, where $0 < t_v \in \mathbb{Z}$, and set $I := \{v \in \mathcal{V} \,|\, t_v > 1\}$. Furthermore, choose a very large integer N.

Whenever $v \in I$ we blow it up N times, then we get N new vertices, then we blow up each new vertex N times, then we get N^2 new vertices, and we repeat this procedure $t_v - 1$ times. We denote the new resolution graph by Γ_b with vertex set $\mathcal{V}_b$. For a vertex $v \in I$, $1 \leq i \leq t_v - 1$ we denote by $L_{v,i} \subset \Gamma_b$ the subset of new vertices constructed during the i-th iteration step of the blowing up procedure applied at the vertex v. Set also $L_{v,0} = \{v\}$.

Then $|L_{v,i}| = N^i$ and $(Z_K)_b = \sum_{v\in\mathcal{V}} t_v \cdot E_v + \sum_{v\in I, 1\leq i\leq t_v-1, u\in L_{v,i}} (t_v - i) \cdot E_u$.

For a vertex $u \in L_{v,i}$ we denote by $A_u \in \cup_{i+1\leq j\leq t_v-1} L_{v,j} \subset \mathcal{V}_b$ the set of vertices obtained by the blowing up procedure via blowing up (infinitesimally close) the vertex u some times. We have $|A_u| = \sum_{1\leq j\leq t_v-i-1} N^j$.

Lemma 6.1.2.1. *(1) Fix a vertex $u \in L_{v,i}$ (where $v \in I$ and $0 \leq i \leq t_v - 1$) and a subgraph $\Gamma' \subset \Gamma_b$ such that $\mathcal{V}(\Gamma') \cap A_u = \emptyset$ and $u \in \mathcal{V}(\Gamma')$.*

Assume that $\widetilde{X}'$ is a resolution of an arbitrary (analytic) singularity corresponding *to the resolution graph Γ'. Then the pole of any differential form from $H^0(\mathcal{O}_{Z'_K}(Z'_K+K'))$ on the exceptional divisor E_u is of order at most 1.*

(2) The subgraph $\Gamma_0 \subset \Gamma_b$ supported on the vertex set $\mathcal{V}$ is rational.

Proof. Part *(1)* follows from the following facts (under the assumption $Z_K > 0$)

(a) If $\widetilde{X}$ is a resolution with graph Γ then the pole–cycle of any form is $\leq \lfloor Z_K \rfloor$.

(b) If $\Gamma' \subset \Gamma$ is a full subgraph, then $Z_K(\Gamma') \leq Z_K(\Gamma)|_{\Gamma'}$.

(c) If Γ is a graph, and $u \in \mathcal{V}$, and $\Gamma_{u,e}$ is obtained from Γ by replacing the Euler decoration E_u^2 of Γ by some $e \ll 0$, then the E_u–coefficient of $Z_K(\Gamma_{u,e})$ is < 2.

(a) follows from the generalized Grauert–Riemenschneider vanishing and Laufer's duality $H^0(\mathcal{O}_{\widetilde{X}}(K+\lfloor Z_K \rfloor))/H^0(\mathcal{O}_{\widetilde{X}}(K)) = H^1(\mathcal{O}_{\widetilde{X}})^*$. For (b) use $(Z_K(\Gamma'), E_v) \geq (Z_K(\Gamma)|_{\Gamma'}, E_v)$ for any $v \in \mathcal{V}(\Gamma')$. Finally, for (c) decompose $\Gamma \setminus u$ into disjoint full subgraphs $\cup_k \Gamma_k$, and let $v_k \in \mathcal{V}(\Gamma_k)$ be adjacent to u in Γ. Set $E^*_{v_k}(\Gamma_k)$ be the dual in Γ_k, and write $E^* := \sum_k E^*_{v_k}(\Gamma_k)$ and $Z := \sum_k Z_K(\Gamma_k)$ too.

Then we claim that $Z_K(\Gamma_{u,e})$ has the form $Z(x) := Z + x(E^* + E_u)$ for certain $x \in \mathbb{Q}$. Indeed, $(Z(x), E_v) = (Z, E_v)$ for any $v \neq u$. Hence x is uniquely determined from $(Z(x), E_u) = e + 2$, that is, $(Z, E_u) + x(E^*, E_u) + xe = e + 2$. Here (Z, E_u) and (E^*, E_u) are e–independent. By a limit argument the coefficient x is < 2 if $e \ll 0$.

(2) In this case any pole of differential form is at most one, hence the cohomological cycle is $\leq E$. In particular, $p_g \leq h^1(\mathcal{O}_E) = 0$. □

6.1.2.2. Now we can describe the construction of a Gorenstein analytic type from the resolution graph Γ_b.

Denote by Γ_0 the full subgraph of vertices (strict transforms) $\mathcal{V}$ (as in Lemma 6.1.2.1), and let $\widetilde{X}_0$ be the corresponding resolution. By Lemma 6.1.2.1 it is rational, let is fix an arbitrary analytic structure on $\widetilde{X}_0$.

In the following we glue the tubular neighborhoods of the other exceptional divisors E_u, $u \in \mathcal{V}_b \setminus \mathcal{V}$, in $T := \max_{v \in I}\{t_v - 1\}$ steps.

In the i–th step ($1 \leq i \leq T$) we contract from the resolution space $\widetilde{X}_{i-1}$ the new space $\widetilde{X}_i$. For $0 \leq i \leq T$ the space $\widetilde{X}_i$ is associated with the full subgraph Γ_i with vertices $\mathcal{V}(\Gamma_i) := \mathcal{V} \cup \cup_{v \in I, 1 \leq j \leq \min\{t_v - 1, i\}} L_{v,j}$. The resolution space $\widetilde{X}_i$ is obtained from $\widetilde{X}_{i-1}$ by gluing the tubular neighborhoods of the divisors E_w with $w \in \Delta\mathcal{V}_i := \mathcal{V}(\Gamma_i) \setminus \mathcal{V}(\Gamma_{i-1}) = \cup_{v \in I, i \leq t_v - 1} L_{v,i}$ in a special way.

For any $1 \leq i \leq T$ set the line bundle $\mathcal{L}_i = \mathcal{O}_{\widetilde{X}_i}(-Z_K(\Gamma_b)|_{\Gamma_i} - K_b) \in \mathrm{Pic}^{c_1(\mathcal{L}_i)}(\widetilde{X}_i)$. (Here $Z_K(\Gamma_b)|_{\Gamma_i}$ denotes the homomological restriction: projection in E–coordinates to the lattice of Γ_i.) Note that $\Gamma_T = \Gamma_b$, $\widetilde{X}_T = \widetilde{X}_b$ and $\mathcal{L}_T = \mathcal{O}_{\widetilde{X}_b}(-Z_K(\Gamma_b) - K_b)$.

Also, for $1 \leq i \leq T$ let Γ'_i be the full subgraph with vertices $\cup_{v \in I, 1 \leq j \leq \min\{t_v - 1, i\}} L_{v,j-1}$ and $\widetilde{X}'_i$ the corresponding resolution space. Since $\Gamma'_i \subset \Gamma_i$ one also has $\widetilde{X}'_i \subset \widetilde{X}_i$.

By induction we wish to prove that the following condition: for any $1 \leq i \leq T$ the restricted line bundle $\mathcal{L}_i|_{\widetilde{X}'_i}$ is a trivial. Note that the E^*–support of $c_1(\mathcal{L}_i)$ is contained in $Ch_i := \cup_{v \in I, i \leq t_v - 2} L_{v,i}$. (I.e., if $(Z_K(\Gamma_b)|_{\Gamma_i} + K_b, E_w) \neq 0$ with $w \in \mathcal{V}(\Gamma_i)$ then $w \in Ch_i$; Ch_i are those vertices from $\mathcal{V}(\Gamma_i)$, which have adjacent vertices from $\mathcal{V}(\Gamma_b) \setminus \mathcal{V}(\Gamma_i)$.) In particular, the Chern class of the restriction $\mathcal{L}_i|_{\widetilde{X}'_i}$ is automatically trivial; here we impose the analytic triviality of the bundle.

Note that for $i = 1$ the graphs Γ'_1 is a subgraph of the rational Γ_0 hence $\mathcal{L}_I|_{\widetilde{X}'_1}$ is trivial.

Next, assume that $i \geq 2$ and $\mathcal{L}_{i-1} \in \mathrm{Pic}^{c_1(\mathcal{L}_{i-1})}(\widetilde{X}_{i-1})$ has the property that its restriction $\mathcal{L}_{i-1}|_{\widetilde{X}'_{i-1}}$ is trivial. Then we wish to glue the tubular neighborhoods of $\{E_w\}_{\Delta\mathcal{V}_i}$ in such a way that the property will be true at level i as well.

Write $\mathcal{L}^r_i = \mathcal{O}_{\widetilde{X}_i}(-Z_K(\Gamma_b)|_{\Gamma_{i-1}} - K_b) \in \mathrm{Pic}(\widetilde{X}_i)$, and $\mathcal{L}^\delta_i = \mathcal{O}_{\widetilde{X}_i}(-Z_K(\Gamma_b)|_{\Delta\mathcal{V}_i}) \in \mathrm{Pic}(\widetilde{X}_i)$. Then, clearly, $\mathcal{L}_i = \mathcal{L}^r_i \otimes \mathcal{L}^\delta_i$. Since $E_u \cap E_w = \emptyset$ whenever $u \in \Delta\mathcal{V}_i$ and $w \in \Gamma'_{i-1}$ we get that $\mathcal{L}^\delta_i|_{\widetilde{X}'_{i-1}}$ is trivial. Note also that $\mathcal{L}^r_i|_{\widetilde{X}'_{i-1}} = \mathcal{L}_{i-1}|_{\widetilde{X}'_{i-1}}$, which is trivial by the inductive step. In particular, $\mathcal{L}_i|_{\widetilde{X}'_{i-1}} = \mathcal{L}^r_i|_{\widetilde{X}'_{i-1}} = \mathcal{L}_{i-1}|_{\widetilde{X}'_{i-1}}$ is trivial. Hence we need to concentrate only on the extension of this triviality on the whole $\widetilde{X}'_i$. Note that $L_{v,i} \subset \Delta\mathcal{V}_i := \mathcal{V}(\Gamma_i) \setminus \mathcal{V}(\Gamma_{i-1})$ if and only if $L_{v,i-1} \subset \Delta\mathcal{V}'_i := \mathcal{V}(\Gamma'_i) \setminus \mathcal{V}(\Gamma'_{i-1})$.

Correspondingly, when we glue the E_w's from $\Delta\mathcal{V}_i$ we glue them along curves indexed by the vertices from $\Delta\mathcal{V}'_i$, and by convenient gluing we wish to guarantee the triviality of the bundle along the extra support $\Delta\mathcal{V}'_i$.

Consider again the bundle $\mathcal{G}_i := \mathcal{L}^r_i|_{\widetilde{X}'_i} \in \mathrm{Pic}(\widetilde{X}'_i)$. Since $\mathcal{G}_i = \mathcal{L}_{i-1}|_{\widetilde{X}'_i}$, it is independent on the gluing of the tubular neighbourhoods of $\{E_w\}_{w\in\Delta\mathcal{V}_i}$. Moreover, $c_1(\mathcal{G}_i)$ is E^*–supported on $\Delta\mathcal{V}'_i$, and $\mathcal{G}_i|_{\widetilde{X}'_{i-1}}$ is trivial. Therefore, by Lemma 6.1.1.5, $\mathcal{G}_i^M$ is in the image of the Abel map $c^{Mc_1(\mathcal{G}_i)}(Z_i)$ $(Z_i \in L(\Gamma'_i),\ Z_i \gg 0)$.

On the other hand, $\mathcal{L}^\delta_i|_{\widetilde{X}'_i} = \mathcal{O}_{\widetilde{X}'_i}(-Z_K(\Gamma_b)|_{\Delta\mathcal{V}_i})$ depends essentially on the gluing of the curves $\{E_w\}_{w\in\Delta\mathcal{V}_i}$. Furthermore, $c_1(\mathcal{L}^\delta_i|_{\widetilde{X}'_i}) = -c_1(\mathcal{G}_i)$. Since by construction N is very large, the image of the corresponding Abel map is stabilized, it is an affine space of dimension $h^1(\mathcal{O}_{\widetilde{X}'_i}) - h^1(\mathcal{O}_{\widetilde{X}'_{i-1}})$. Since the differential forms in $\widetilde{X}'_i$ have pole order at most one, and the coefficients of $Z_K(\Gamma_b)|_{\Delta\mathcal{V}_i}$ all along $\Delta\mathcal{V}_i$ are non–zero, the Abel map depends only on the intersection points of the effective divisors along $\{E_{w'}\}_{w'\in\Delta\mathcal{V}'_i}$. Hence, by moving the intersection points of the curves $\{E_w\}_{w\in\Delta\mathcal{V}_i}$ with $\{E_{w'}\}_{w'\in\Delta\mathcal{V}'_i}$ (and due to the choice $N \gg 0$) the bundle $(\mathcal{L}^\delta_i|_{\widetilde{X}'_i})^M$ can be any point of the $c^{Mc_1(\mathcal{G}_i)}$. In particular we can arrange that $(\mathcal{L}^\delta_i|_{\widetilde{X}'_i})^M \otimes (\mathcal{L}^r_i|_{\widetilde{X}'_i})^M$ is trivial. In particular, $\mathcal{L}_i|_{\widetilde{X}'_i}$ is trivial, what we wish to realize.

Therefore, at the end of the induction, we get that $\mathcal{L}_T|_{\widetilde{X}'_T}$ is trivial. But $\widetilde{X}'_T = \widetilde{X}_r$ in the notation of Lemma 6.1.1.3, hence by that Lemma $\widetilde{X}_b$ is Gorenstein. Then $\widetilde{X}$ is Gorenstein too.

6.1.2.3. In the following we wish to show that every Gorenstein singularity $\widetilde{X}$ supported on the resolution graph Γ can be given by this construction.

Indeed, consider the line bundle $\mathcal{L} = \mathcal{O}_{\widetilde{X}_r}(-Z_K(\Gamma_b) - K_b)$. If $\widetilde{X}$ (hence $\widetilde{X}_b$ too) is Gorenstein then it is trivial, cf. Lemma 6.1.1.3. Hence, its restriction to any $\widetilde{X}_i$ is trivial too.

Next, write $\mathcal{L}' = \mathcal{O}_{\widetilde{X}_r}(-Z_K(\Gamma_b)|_{\Gamma_i} - K_b)$ and $\mathcal{L}'' = \mathcal{O}_{\widetilde{X}_r}(-Z_K(\Gamma_b)|_{\Gamma_b\setminus\Gamma_i})$. Then $\mathcal{L} = \mathcal{L}' \otimes \mathcal{L}''$. Moreover, by support argument as above, $\mathcal{L}''|_{\widetilde{X}'_i}$ is trivial. Hence $\mathcal{L}'|_{\widetilde{X}'_i}$

should be trivial too. But $\mathcal{L}'|_{\widetilde{X}'_i} = \mathcal{L}_i|_{\widetilde{X}'_i}$, and the triviality of each $\mathcal{L}_i|_{\widetilde{X}'_i}$ characterizes our main construction.

This means exactly, that every Gorenstein singularity supported on the resolution graph Γ can be given by the construction described above.

Chapter 7

Further results

In this chapter we summarise a couple of further results from [N19a] and [N19b], where we do not prove most of the statements, we just wish to make a clear picture about the results.

7.1 Relatively generic structures on normal surface singularities

In this section we investigate a relative setup of generic structures on surface singularities, where we fix a given analytic type or line bundle on a smaller subgraph or more generally on a smaller cycle and we choose a relatively generic line bundle or analytic type on the large cycle and wish to compute it's invariants, like geometric genus or h^1 of natural line bundles.

The formulas yielding the answers to this questions are quite intresting on their own, however the real power of these results, that they give possibility for inductive proofs of problems regarding generic surface singularities.

The two main theorems will be the analouges of our main theorems about generic line bundles and invariants of generic normal surface singularities in the previous sections.

We consider a cycle $Z \geq E$ on a resolution $\widetilde{X}$, and a smaller cycle $Z_1 \leq Z$, where we denote $|Z_1| = \mathcal{V}_1$ and the subgraph corresponding to it by $\mathcal{T}_1$. We have the restriction map $r : \mathrm{Pic}(Z) \to \mathrm{Pic}(Z_1)$ and one has also the (cohomological) restriction operator $R_1 : L'(\mathcal{T}) \to L'_1 := L'(\mathcal{T}_1)$ (defined as $R_1(E^*_v(\mathcal{T})) = E^*_v(\mathcal{T}_1)$ if $v \in \mathcal{V}_1$, and $R_1(E^*_v(\mathcal{T})) = 0$ otherwise). For any $\mathcal{L} \in \mathrm{Pic}(Z)$ and any $l' \in L'(\mathcal{T})$ it satisfies

$$c_1(r(\mathcal{L})) = R_1(c_1(\mathcal{L})). \qquad (7.1.0.1)$$

In particular, we have the following commutative diagram as well:

$$\begin{array}{ccc} \mathrm{ECa}^{l'}(Z) & \xrightarrow{c^{l'}} & \mathrm{Pic}^{l'}(Z) \\ \downarrow_{\mathfrak{r}} & & \downarrow_{r} \\ \mathrm{ECa}^{R_1(l')}(Z_1) & \xrightarrow{c^{R_1(l')}} & \mathrm{Pic}^{R_1(l')}(Z_1) \end{array}$$

By the 'relative case' we mean that instead of the 'total' Abel map $c^{l'}$ (with $l' \in -\mathcal{S}'$ and $Z \geq E$) we study its restriction above a fixed fiber of r. That is, we fix some $\mathfrak{L} \in \mathrm{Pic}^{R_1(l')}(Z_1)$, and we study the restriction of $c^{l'}$ to $(r \circ c^{l'})^{-1}(\mathfrak{L}) \to r^{-1}(\mathfrak{L})$.

If we denote the subvariety $(r \circ c^{l'})^{-1}(\mathfrak{L}) = (c^{R_1(l')} \circ \mathfrak{r})^{-1}(\mathfrak{L}) \subset \mathrm{ECa}^{l'}(Z)$ by $\mathrm{ECa}^{l',\mathfrak{L}}$, then in the relative setup $\mathrm{ECa}^{l',\mathfrak{L}}$ plays the role of the space of effective Cartier divisors $\mathrm{ECa}^{l'}(Z)$ and we have the relative Abel map $\mathrm{ECa}^{l',\mathfrak{L}} \to r^{-1}(\mathfrak{L})$.

In the nonrelative case one of the crucial facts we use is that the space $\mathrm{ECa}^{l'}(Z)$ is a nice smooth algebraic variety, although at this point we don't know anything about the space $\mathrm{ECa}^{l',\mathfrak{L}}$.

To be able to control the behaviour of the space $\mathrm{ECa}^{l',\mathfrak{L}}$ we need some key properties of the map $\mathfrak{r}$, namely we have the following lemma:

Proposition 7.1.0.2. *(a) $\mathfrak{r}$ is a local submersion, that is, for any $D \in \mathrm{ECa}^{l'}(Z)$ and $D_1 := \mathfrak{r}(D)$, the tangent map $T_D\mathfrak{r}$ is surjective.*

(b) $\mathfrak{r}$ is dominant.

(c) any non–empty fiber of $\mathfrak{r}$ *is smooth of dimension* $(l', Z) - (l', Z_1) = (l', Z_2)$, *and it is irreducible.*

The main corrolary will be, that the space $\mathrm{ECa}^{l',\mathfrak{L}}$ is indeed a smooth irreducible algebraic variety:

Corollary 7.1.0.3. *Fix* $l' \in -\mathcal{S}'$, $Z \geq E$, $Z_1 \leq Z$ *and* $\mathfrak{L} \in \mathrm{Pic}^{R(l')}(Z_1)$. *Assume that* $\mathrm{ECa}^{l',\mathfrak{L}}$ *is nonempty. Then it is smooth of dimension* $h^1(Z_1, \mathfrak{L}) - h^1(Z_1, \mathcal{O}_{Z_1}) + (l', Z)$ *and it is irreducible.*

We can ivestigate also in the relative case the dominance property of the relative Abel map and it turns out, that again it depends just only on the analytic structre of the subsingularity $\widetilde{X}_1$ of $\widetilde{X}$ supported on $|Z_1|$, and the cuts D_v how we glue the excpetional divisors E_v, which have got a neighbour in $|Z_1|$.

Fix $l' \in -\mathcal{S}'$, $Z \geq E$, $Z_1 \leq Z$ and $\mathfrak{L} \in \mathrm{Pic}^{R_1(l')}(Z_1)$ as above.

Let's say that the pair $(l', \mathfrak{L})$ is *relative dominant* on the cycle Z if the closure of $r^{-1}(\mathfrak{L}) \cap \mathrm{Im}(c^{l'}(Z))$ in $r^{-1}(\mathfrak{L})$ is $r^{-1}(\mathfrak{L})$.

We prove the following theorem:

Theorem 7.1.0.4. *One has the following facts:*

(1) If $(l', \mathfrak{L})$ *is relative dominant on the cycle* Z, *then* $\mathrm{ECa}^{l',\mathfrak{L}}$ *is nonempty and* $h^1(Z, \mathcal{L}) = h^1(Z_1, \mathfrak{L})$ *for any generic* $\mathcal{L} \in r^{-1}(\mathfrak{L})$.

(2) $(l', \mathfrak{L})$ *is relative dominant on the cycle* Z, *if and only if for all* $0 < l \leq Z$, $l \in L$ *one has*

$$\chi(-l') - h^1(Z_1, \mathfrak{L}) < \chi(-l' + l) - h^1((Z - l)_1, \mathfrak{L}(-R_1(l))).$$

, where we denote $(Z - l)_1 = \min(Z - l, Z_1)$.

We will also state the analouge of our main theorem about cohomology numbers of generic line bundles in the relative setup:

Theorem 7.1.0.5. *Fix $l' \in -\mathcal{S}'$, $Z \geq E$, $Z_1 \leq Z$ and $\mathfrak{L} \in \mathrm{Pic}^{R_1(l')}(Z_1)$ as in Theorem 7.1.0.4. Then for any $\mathcal{L} \in r^{-1}(\mathfrak{L})$ one has*

$$h^1(Z,\mathcal{L}) \geq \chi(-l') - \min_{0\leq l\leq Z,\ l\in L}\{\,\chi(-l'+l) - h^1((Z-l)_1, \mathfrak{L}(-R_1(l)))\,\}, \quad \textit{or, equivalently,}$$
$$h^0(Z,\mathcal{L}) \geq \max_{0\leq l\leq Z, l\in L}\{\,\chi(Z-l,\mathcal{L}(-l)) + h^1((Z-l)_1, \mathfrak{L}(-R_1(l)))\,\}. \tag{7.1.0.6}$$

Furthermore, if $\mathcal{L}$ is generic in $r^{-1}(\mathfrak{L})$ then in both inequalities we have equalities.

7.2 Holes in possible values of h^1 of line bundles and geometric genus

In this section we summarise the results from [N19b] about the possible geometric genuses correspinging to a fixed topological type $\mathcal{T}$.

Let's have a resolution graph $\mathcal{T}$ and a corresponding normal surface singularity with resolution space $\widetilde{X}$, furthermore let's fix a Chern class l' and an effective cycle Z. Then our first main theorem states that the possible values of $h^1(Z,\mathcal{L})$, where $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$ form an interval of integers, more precisely we have:

Theorem 7.2.0.1. *Let's have an arbitrary resolution graph $\mathcal{T}$ and a corresponding singularity $\widetilde{X}$, an effective cycle Z and an abitrary chern class l'. Let's denote $k = \max_{\mathcal{L}\in \mathrm{Pic}^{l'}(Z)} h^1(Z,\mathcal{L})$ and let's have an arbitrary integer $\chi(-l') - \min_{0\leq l\leq Z}\chi(-l'+l) \leq r \leq k$, then there is a line bundle $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$, such that $h^1(Z,\mathcal{L}) = r$.*

Similarly let's have a resolution graph $\mathcal{T}$ and let's fix a Chern class l' and an effective cycle Z, such that if $l' = \sum_{v\in\mathcal{V}} b_v E_v$, then $b_v < 0$ for every vertex $v \in |Z|$. The second main theorem we prove states that the possible values of $h^1(\mathcal{O}_Z(l'))$ form an interval of integers if we consider any possible surface singularity with resolution graph $\mathcal{T}$ and $\mathcal{O}_Z(l')$ is the natural line bundle, more precisely we have:

Theorem 7.2.0.2. *Let $\mathcal{T}$ be an arbitrary resolution graph with vertex set $\mathcal{V}$ and let*

Z be an effective cycle on it, let's have furthermore a Chern class $l' \in L'$, such that $l' = \sum_{v\in\mathcal{V}} a_v E_v^*$. Let's write $l' = \sum_{v\in\mathcal{V}} b_v E_v$, and assume, that $b_v < 0$ if $v \in |Z|$.

Let's have a singularity $\widetilde{X}$ supported on $\mathcal{T}$, and let's look at the natural line bundle restricted to the cycle Z, $\mathcal{O}_Z(l')$. Suppose, that $k = h^1(\mathcal{O}_Z(l')) > \chi(-l') - \min_{0\leq l\leq Z}\chi(-l'+l)$, and let's have an arbitrary number $k > m \geq \chi(-l') - \min_{0\leq l\leq Z}\chi(-l'+l)$, then there is another singularity $\widetilde{X}'$ supported on the resolution graph $\mathcal{T}$, for which one has $m = h^1(\mathcal{O}_Z(l'))$.

As a corollary it yields that the possible values of the geometric genus $p_g(\widetilde{X})$ form an interval of integers.

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