Optimal long-term investment in illiquid markets when prices have negative memory

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Declaration of Authorship

I, Lornat Nagy, declare that this thesis entitled, Optimal long-term investment in illiquid markets when prices have negative memory and the work presented in it are my own. I confirm that: This work was done wholly or mainly while in candidature for a research degree at this University. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated. Where I have consulted the published work of others, this is always clearly attributed. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work. I have acknowledged all main sources of help.

Abstract

In a discrete-time financial market model with instantaneous price impact, we find an asymptotically optimal strategy for an investor maximizing her expected wealth. The asset price is assumed to follow a process with negative memory. We determine how the optimal growth rate depends on the impact parameter and on the covariance decay rate of the price.

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1 Introduction

Fractional Brownian motions (FBMs) with various Hurst parameters $H \in (0,1)$ have been enticing researchers of financial mathematics for a long time, since the appearance of [4]. In idealistic models of trading, however, FBMs do not provide admissible models since they generate arbitrage opportunities (for $H \neq 1/2$), see [5]. In the presence of market frictions arbitrage disappears and FBMs become interesting candidates for describing prices.

In markets with instantaneous price impact the first analysis of long-term investment has been carried out in [2]: the optimal growth rate of expected portfolio wealth has been found and an asymptotically optimal strategy has been exhibited. The robustness of such results was the next natural question: is the particular structure of FBMs needed for these conclusions? In [2] a larger class of Gaussian processes could also be treated where future increments are positively correlated to the past and the covariance structure is similar to that of FBMs with $H > 1/2$. The question of extending the case of FBMs with $H < 1/2$ remained open.

The current paper provides such an extension, more involved than in the positively correlated case. For simplicity, we stay in a discrete-time setting. We derive the same conclusions as [2] did in the case of FBMs with $H < 1/2$ but for a larger class of Gaussian processes.

2 Market model

Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a filtration $\mathscr{F}_t$, $t \in \mathbb{Z}$. Let $E[X]$ denote the expectation of a real-valued random variable X (when exists). Consider a financial market where the price of a risky asset follows a process S_t, $t \in \mathbb{N}$, adapted to $\mathscr{F}_t$, $t \in \mathbb{N}$.

We will present a model where trading takes place with a temporary, nonlinear price impact, along the lines of [3] but in discrete time. For some $T \in \mathbb{N}$ the class of *feasible strategies* up to terminal time T is defined as

$$\mathscr{S}(T) := \left\{ \phi = (\phi_t)_{t=0}^{T} : \phi \text{ is an } \mathbb{R}\text{-valued, adapted process} \right\}. \tag{1}$$

As we will see, ϕ_t represents the *change* in the investor's position in the given asset. Let $z = (z^0, z^1) \in \mathbb{R}^2$ be a deterministic initial endowment where z^0 is in cash and and z^1 is in the risky asset.

For a feasible strategy $\phi \in \mathscr{S}(T)$, at any time $t \geq 0$, the number of shares in the risky asset is equal to

$$\Phi_t := z^1 + \sum_{u=0}^{t} \phi_u \,. \tag{2}$$

We will shortly derive a similar formula for the cash position of the investor.

In classical, frictionless models of trading, cash at time $T+1$ equals

$$\sum_{u=1}^{T+1} \Phi_{u-1}(S_u - S_{u-1}). \tag{3}$$

Algebraic manipulation of (3) yields

$$\sum_{u=1}^{T+1} \Phi_{u-1}(S_u - S_{u-1}) = -\sum_{u=0}^{T} \phi_u S_u + S_{T+1} \sum_{u=0}^{T} \phi_u.$$

We assume that price impact is a superlinear power function of the "trading speed" ϕ so we augment the above with a term that implements the effect of friction:

$$-\sum_{u=0}^{T} \phi_u S_u + S_{T+1} \sum_{u=0}^{T} \phi_u - \sum_{u=0}^{t} \lambda |\phi_u|^\alpha$$

where we assume $\alpha > 1$ and $\lambda > 0$. We wish to utilize only those portfolios where the risky asset is liquidated by the end of the trading period so we define

$$\mathscr{G}(T) := \mathscr{S}(T) \cap \left\{ \phi : \Phi_T = \sum_{u=0}^{T} \phi_u = 0 \right\}.$$

We finally get that, for $\phi \in \mathscr{G}(T)$, the position in the riskless asset at time $T+1$ is given by

$$X_T(\phi) \quad := \quad z^0 - \sum_{u=0}^{T} \phi_u S_u - \sum_{u=0}^{T} \lambda |\phi_u|^\alpha. \tag{4}$$

For simplicity, we also assume $z^0 = z^1 = 0$ from now on, i.e. portfolios start from nothing.

To investigate the potential of realizing monetary profits, we focus on a risk-neutral objective: a linear utility function. Let $x_- := \max\{-x, 0\}$ for $x \in \mathbb{R}$. Define, for $T \in \mathbb{N}$,

$$\mathscr{A}(T) := \left\{ \phi \in \mathscr{G}(T) : E[(X_T(\phi))_-] < \infty \right\},$$

the class of strategies starting from a zero initial position in both assets and ending at time $T+1$ in a cash only position with expected value greater than $-\infty$. The value of the problem we will consider is thus

$$u(T) := \sup_{\phi \in \mathscr{A}(T)} E[X_T(\phi)].$$

The investors's objective is to find ϕ which, at least asymptotically as $T \to \infty$, achieves the same growth rate as $u(T)$.

3 Asymptotically optimal investment

First we introduce assumptions on the price process and its dependence structure.

Assumption 3.1. *Let Z_t, $t \in \mathbb{Z}$ be an adapted, real-valued, zero-mean stationary Gaussian process which will represent price increments. Let $r(t) := \mathrm{cov}(Z_0, Z_t)$, $t \in \mathbb{Z}$ denote its covariance function. We assume that there exists $T_0 > 0$ and $J_1, J_2 < 0$ such that for all $t \geq T_0$,*

$$J_1 t^{2H-2} \leq r(t) \leq J_2 t^{2H-2} \tag{5}$$

is satisfied for some parameter $H \in \left(0, \frac{1}{2}\right)$. Furthermore,

$$\sum_{t \in \mathbb{Z}} r(t) = 0. \tag{6}$$

Let us introduce the adapted price process defined by $S_0 = 0$ and $S_t = S_{t-1} + Z_t$, $t \geq 1$.

Remark 3.2. Properties (5) and (6) express that Z is a *process with negative memory*, see Definition 1.1.1 on page 1 of [1]. When Z_t, $t \in \mathbb{Z}$ are the increments of a FBM with Hurst parameter $H < 1/2$, then (5) is satisfied. This is the motivation for choosing H for parametrization (and not $2H-2$).

The next theorem is our main result: it provides the explicit form of an (asymptotically) optimal strategy and determines its expected asymptotic growth rate.

Theorem 3.3. *Let Assumption 3.1 be in force. If λ is small enough then*

(i) maximal expected profits satisfy

$$\limsup_{T \to \infty} \frac{u(T)}{T^{H\left(1+\frac{1}{\alpha-1}\right)+1}} < \infty; \tag{7}$$

(ii) the strategy

$$\phi_t(T, \alpha) := \begin{cases} -\mathrm{sgn}(S_t)|S_t|^{\frac{1}{\alpha-1}}, & 0 \leq t < T/2, \\ -\frac{1}{T/2} \sum_{s=0}^{T/2} \phi_s, & T/2 \leq t \leq T \end{cases} \tag{8}$$

satisfies

$$\liminf_{T \to \infty} \frac{E X_T(\phi(T, \alpha))}{T^{H\left(1+\frac{1}{\alpha-1}\right)+1}} > 0. \tag{9}$$

where T runs through multiples of 6 everywhere.

4 Proofs

4.1 General bounds for variance and covariance

First we make some useful preliminary observations. Using stationarity of the increments of the process S, we have

$$\begin{aligned}
\mathrm{var}(S_t) = \mathrm{cov}(S_t, S_t) &= \mathrm{cov}(\sum_{j=1}^{t} S_j - S_{j-1}, \sum_{i=1}^{t} S_i - S_{i-1}) \\
&= t \cdot \mathrm{var}(S_1 - S_0) + 2 \sum_{i=2}^{t} \sum_{j=1}^{i-1} \mathrm{cov}(S_j - S_{j-1}, S_i - S_{i-1}) \\
&= t \cdot \mathrm{var}(S_1 - S_0) + 2 \sum_{i=2}^{t} \sum_{j=1}^{i-1} \mathrm{cov}(S_1 - S_0, S_{i-j+1} - S_{i-j}) \\
&= t \cdot r(0) + 2 \sum_{i=2}^{t} \sum_{j=1}^{i-1} r(i-j).
\end{aligned} \tag{10}$$

Furthermore, for $s > t$ we similarly have

$$\mathrm{cov}(S_s - S_t, S_t) = \sum_{i=t+1}^{s} \sum_{j=1}^{t} r(i-j). \tag{11}$$

Observe also that we can write

$$r(0) = -2 \sum_{j=1}^{\infty} r(j). \tag{12}$$

Turning to the variances, we first obtain a convenient expression for them. Using (9) and (11), we have

$$\mathrm{var}(S_t) = -2t \sum_{j=1}^{t-1} r(j) - 2t \sum_{j=t}^{\infty} r(j) + 2 \sum_{i=2}^{t} \sum_{j=1}^{i-1} r(j),$$

and algebraic manipulation of the summation operation $\left(-2t \sum_{j=1}^{t-1} + 2 \sum_{i=2}^{t} \sum_{j=1}^{i-1}\right)$ yields

$$\begin{aligned}
&-2t \sum_{j=1}^{t-1} + 2 \sum_{i=2}^{t} \sum_{j=1}^{i-1} \\
&= -2t \left(\sum_{j=1}^{T_0-1} + \sum_{j=T_0}^{t-1} \right) + 2 \left(\sum_{i=2}^{T_0-1} + \sum_{i=T_0}^{t} \right) \sum_{j=1}^{i-1} \\
&= -2t \sum_{j=1}^{T_0-1} - 2t \sum_{j=T_0}^{t-1} + 2 \sum_{i=2}^{T_0-1} \sum_{j=1}^{i-1} + 2 \sum_{i=T_0}^{t} \sum_{j=1}^{i-1} \\
&= -2t \sum_{j=1}^{T_0-1} - 2t \sum_{j=T_0}^{t-1} + 2 \sum_{i=2}^{T_0-1} \sum_{j=1}^{i-1} + 2 \sum_{j=1}^{T_0-1} + 2 \sum_{i=T_0+1}^{t} \left(\sum_{j=1}^{T_0-1} + \sum_{j=T_0}^{i-1} \right) \\
&= -2t \sum_{j=1}^{T_0-1} - 2t \sum_{j=T_0}^{t-1} + 2 \sum_{i=2}^{T_0-1} \sum_{j=1}^{i-1} + 2 \sum_{j=1}^{T_0-1} + 2 \sum_{i=T_0+1}^{t} \sum_{j=1}^{T_0-1} + 2 \sum_{i=T_0+1}^{t} \sum_{j=T_0}^{i-1} \\
&= -2t \sum_{j=1}^{T_0-1} + 2 \sum_{i=2}^{T_0-1} \sum_{j=1}^{i-1} + 2 \sum_{j=1}^{T_0-1} + 2 \sum_{i=T_0+1}^{t} \sum_{j=1}^{T_0-1} - 2t \sum_{j=T_0}^{t-1} + 2 \sum_{i=T_0+1}^{t} \sum_{j=T_0}^{i-1},
\end{aligned}$$

where the last line is only a reordering of terms. Setting $C_1 = \sum_{j=1}^{T_0-1} r(j)$, $C_2 = \sum_{i=2}^{T_0-1}\sum_{j=1}^{i-1} r(j)$ and $C_3 = 2(C_2-(T_0-1)C_1)$, the above calculation gives

$$\begin{aligned}\operatorname{var}(S_t) &= -2tC_1+2C_2+2C_1+2(t-T_0)C_1+\left(-2t\sum_{j=t}^{\infty}-2t\sum_{j=T_0}^{t-1}+2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\right)r(j)\\ &= C_3+\left(-2t\sum_{j=t}^{\infty}-2t\sum_{j=T_0}^{t-1}+2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\right)r(j)\end{aligned} \tag{13}$$

Now we are ready to present three lemmas, providing a lower and an upper bound for the variance and an upper bound for the covariance.

Lemma 4.1. *There exist $T_1 \in \mathbb{N}$ and $B_1 > 0$ such that for all $t \geq T_1$ we have*

$$\operatorname{var}(S_t) \geq B_1 t^{2H}.$$

Proof. Using properties induced by the choice of T_0 in Assumption 3.1 first note that

$$\begin{aligned}&\left(-2t\sum_{j=T_0}^{t-1}+2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\right)r(j)\\ &\geq \left(-2t\sum_{j=T_0}^{t-1}+2(t-T_0)\sum_{j=T_0}^{t-1}\right)r(j)\\ &= -T_0\sum_{j=T_0}^{t-1} r(j) \geq 0.\end{aligned}$$

Also notice that

$$\begin{aligned}-2t\sum_{j=t}^{\infty} r(j) &\geq -2J_2t\sum_{j=t}^{\infty} j^{2H-2} \geq -2J_2t\int_t^{\infty} u^{2H-2}du\\ &= -2J_2t\frac{1}{2H-1}\left(-t^{2H-1}\right) = \frac{2J_2}{2H-1}t^{2H}.\end{aligned}$$

Using these and (**??**)

$$\operatorname{var}(S_t) \geq C_3 + \frac{2J_2}{2H-1}t^{2H}.$$

The threshold T_1 and the constant B_1 can be explicitly calculated in terms of the constants present in the above expression. This completes the proof. □

Lemma 4.2. *There exist $T_2 \in \mathbb{N}$ and $B_2 > 0$ such that for all $t \geq T_2$ we have*

$$\operatorname{var}(S_t) \leq B_2 t^{2H}.$$

Proof. First note that algebraic manipulation of the operation $\left(-2t\sum_{j=T_0}^{t-1}+2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\right)$

yields

$$
\begin{aligned}
&-2t\sum_{j=T_0}^{t-1}+2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}=-2(t-T_0+T_0)\sum_{j=T_0}^{t-1}+2\sum_{i=T_0}^{t-1}\sum_{j=T_0}^{i}\\
&=-2\sum_{i=T_0}^{t-1}\sum_{j=T_0}^{t-1}+2\sum_{i=T_0}^{t-1}\sum_{j=T_0}^{i}-2T_0\sum_{j=T_0}^{t-1}=-2\sum_{i=T_0}^{t-1}\left(\sum_{j=T_0}^{t-1}-\sum_{j=T_0}^{i}\right)-2T_0\sum_{j=T_0}^{t-1}\\
&=-2\sum_{i=T_0}^{t-1}\sum_{j=i+1}^{t-1}-2T_0\sum_{j=T_0}^{t-1}.
\end{aligned}
$$

By Assumption 3.1, this implies

$$
\begin{aligned}
&\left(-2t\sum_{j=T_0}^{t-1}+2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\right)r(j)\leq -2J_1\left(\sum_{i=T_0}^{t-1}\sum_{j=i+1}^{t-1}j^{2H-2}+T_0\sum_{j=T_0}^{t-1}j^{2H-2}\right)\\
&\leq -2J_1\left(\sum_{i=T_0}^{t-1}\int_i^{t-1}u^{2H-2}du+T_0\int_{T_0-1}^{t-1}u^{2H-2}du\right)\\
&=-\frac{2J_1}{2H-1}\left(\sum_{i=T_0}^{t-1}\left((t-1)^{2H-1}-i^{2H-1}\right)+T_0\left((t-1)^{2H-1}-(T_0-1)^{2H-1}\right)\right)\\
&=-\frac{2J_1}{2H-1}\left(t(t-1)^{2H-1}-\sum_{i=T_0}^{t-1}i^{2H-1}-T_0(T_0-1)^{2H-1}\right)\\
&\leq \frac{2J_1}{2H-1}\sum_{i=T_0}^{t-1}i^{2H-1}+\frac{2J_1}{2H-1}T_0(T_0-1)^{2H-1}\\
&\leq \frac{2J_1}{2H(2H-1)}((t-1)^{2H}-(T_0-1)^{2H})+\frac{2J_1}{2H-1}T_0(T_0-1)^{2H-1}\\
&\leq \frac{2J_1}{2H(2H-1)}t^{2H}+\frac{2J_1}{2H-1}T_0(T_0-1)^{2H-1}.
\end{aligned}
$$

To proceed observe that, using the asymptotics in Assumption 3.1, for $t>2$ we have

$$
\begin{aligned}
-2t\sum_{j=t}^{\infty}r(j)\leq -2J_1t\sum_{j=t}^{\infty}j^{2H-2}&\leq -2J_1t\int_{t-1}^{\infty}u^{2H-2}du\\
&=\frac{2J_1t}{2H-1}(t-1)^{2H-1}\leq\frac{2J_1t}{2H-1}(t-t/2)^{2H-1}\\
&=\frac{2^{2-2H}J_1}{2H-1}t^{2H}.
\end{aligned}
$$

These results yield for $t>\max(2,T_0)$, using again (**??**), that

$$
\operatorname{var}(S_t)\leq C_3+\left(\frac{2J_1}{2H(2H-1)}+\frac{2^{2-2H}J_1}{2H-1}\right)t^{2H}+\frac{2J_1}{2H-1}T_0(T_0-1)^{2H-1} \tag{14}
$$

The threshold T_2 and the constant B_2 could again be explicitly given. The proof is complete. □

We proceed with the lemma controlling the covariance $\operatorname{cov}(S_s-S_t,S_t)$.

Lemma 4.3. *There exist $T_3\in\mathbb{N}$ and $D_1,D_2>0$ such that*

$$
\operatorname{cov}(S_s-S_t,S_t)\leq D_1\ \textit{ for all }\ s>t>T_3.
$$

For a fixed $v > 1$, define

$$U(v) := J_2 (2H)^{-1} (2H-1)^{-1} \left(1 - \left(v^{2H} - (v-1)^{2H}\right)\right).$$

Then

$$\operatorname{cov}(S_s - S_t, S_t) \le D_2 - U(v) t^{2H} < 0 \textit{ holds for all } s > t > T_3 \textit{ satisfying } \frac{s}{t} > v.$$

There exists $K > 1$ and $T_4 \in \mathbb{N}$ such that

$$\operatorname{cov}(S_s - S_t, S_t) \le 0 \textit{ for all } s > t > T_4 \textit{ satisfying } s - t > K.$$

Proof. Let us set

$$C_4 = \sum_{j=-T_0+1}^{0} \sum_{i=1}^{1+T_0} r(i-j), \quad C_5 = J_2 \sum_{j=-T_0+1}^{0} \sum_{i=1}^{1+T_0} (i-j)^{2H-2},$$

and define $C_6 = C_4 - C_5$. Note that, for each $t \in \mathbb{N}$, $C_4 = \sum_{j=t-T_0+1}^{t} \sum_{i=t+1}^{t+1+T_0} r(i-j)$, and $C_5 = J_2 \sum_{j=t-T_0+1}^{t} \sum_{i=t+1}^{t+1+T_0} (i-j)^{2H-2}$. For $t > T_0$, we have

$$\begin{aligned}
\operatorname{cov}(S_s - S_t, S_t) &= \sum_{j=1}^{t} \sum_{i=t+1}^{s} r(i-j) \\
&\le C_6 + J_2 \sum_{j=1}^{t} \sum_{i=t+1}^{s} (i-j)^{2H-2} \le C_6 + J_2 \sum_{j=1}^{t} \int_{t+1-j}^{s+1-j} u^{2H-2} du \\
&\le C_6 + \frac{J_2}{2H-1} \sum_{j=1}^{t} \left((s+1-j)^{2H-1} - (t+1-j)^{2H-1}\right) \\
&= C_6 + \frac{J_2}{2H-1} \sum_{j=1}^{t} (s+1-j)^{2H-1} - \frac{J_2}{2H-1} \sum_{j=1}^{t} (t+1-j)^{2H-1} \qquad (15) \\
&\le C_6 + \frac{J_2}{2H-1} \int_{s-t}^{s} u^{2H-1} du - \frac{J_2}{2H-1} \int_{1}^{t+1} u^{2H-1} du \\
&= C_6 + \frac{J_2}{2H(2H-1)} \left(s^{2H} - (s-t)^{2H}\right) - \frac{J_2}{2H(2H-1)} \left((t+1)^{2H} - 1\right) \\
&= C_6 + \frac{J_2}{2H(2H-1)} \left(s^{2H} - (s-t)^{2H} - \left((t+1)^{2H} - 1\right)\right). \\
&=: C_6 + C_7 \left(s^{2H} - (s-t)^{2H} - \left((t+1)^{2H} - 1\right)\right).
\end{aligned}$$

Since $k \ge t+1$ the expression $C_7 \left(s^{2H} - (s-t)^{2H} - \left((t+1)^{2H} - 1\right)\right)$ is non-positive, which yields

$$\operatorname{cov}(S_s - S_t, S_t) \le C_6,$$

proving the first statement of the lemma. Now, for all $v > 1$ the property $\frac{s}{t} > v$ - together with the previous constraint of $t > T_0$ - further implies

$$\begin{aligned}
\operatorname{cov}(S_s - S_t, S_t) &\le C_6 + C_7 \left(s^{2H} - (s-t)^{2H} - \left((t+1)^{2H} - 1\right)\right) \\
&\le C_6 + C_7 \left((v^{2H} - (v-1)^{2H} - 1) t^{2H} + 1\right) \qquad (16) \\
&= C_6 + C_7 + C_7 (v^{2H} - (v-1)^{2H} - 1) t^{2H}.
\end{aligned}$$

Obviously, for large enough t the bound becomes strictly negative, proving the second statement. Now, assuming $s-t \geq K > 1$ beside $t > T_0$ we have

$$\begin{aligned} \operatorname{cov}(S_s - S_t, S_t) &\leq C_6 + C_7\left((t+K)^{2H} - K^{2H} - \left((t+1)^{2H} - 1\right)\right) \\ &= C_6 - C_7\left(K^{2H} - 1\right) + C_7\left((t+K)^{2H} - (t+1)^{2H}\right) \\ &\leq C_6 - C_7\left(K^{2H} - 1\right) + C_7 2HKt^{2H-1}. \end{aligned} \tag{17}$$

This shows that K can be chosen so large that $C_6 - C_7\left(K^{2H}-1\right) < 0$ and then, since $2H-1<0$, a threshold T_4 - depending on K - for the variable t can be specified so that

$$C_6 - C_7\left(K^{2H}-1\right) + C_7 2HKt^{2H-1} \leq 0$$

whenever t exceeds the threshold, proving the third statement, completing the proof of the lemma. □

4.2 Key estimates

Define

$$\rho(s,t) := \frac{\operatorname{cov}(S_s, S_t)}{\operatorname{var}(S_t)} = \frac{\operatorname{cov}(S_s - S_t, S_t)}{\operatorname{var}(S_t)} + 1, \ s \in \mathbb{N}, \ t \in \mathbb{N}\setminus\{0\}.$$

Lemma 4.4. *There exist $\bar{T} \in \mathbb{N}$ and constants $R>0$, $K>1$, $\eta \in (1/2,1)$ and $\varepsilon > 0$ such that*

(i) $\rho(s,t) < 1+R$, *for all* $t<s$;

(ii) $\rho(s,t) \leq 1$, *whenever* $\bar{T} < t < s$ *and* $s-t > K$;

(iii) *For all* $T \in \mathbb{N}$, $\rho(s,t) \leq 1-\varepsilon$, *whenever* $\bar{T} < t < \frac{T}{2} < \eta T < s$. *Furthermore, one can also guarantee* $T/2 + K < \eta T$ *in this case.*

Proof of Lemma 4.3. Let B_2, $U(\cdot)$, T_1, T_2, T_3, T_4, D_1, D_2 and K be as in Lemma 4.2 and Lemma **??**. Choose $T' > \max\{T_1, T_2, T_3\}$ so large that $\frac{D_2}{B_2}(T')^{-2H} - \frac{U(4/3)}{B_2} < 0$ and set $\eta := 2/3$. Lemma 4.2 and Lemma **??** now show that whenever $T' < t < T/2$ and $s \in (\eta T, T)$, we have

$$\frac{\operatorname{cov}(S_s - S_t, S_t)}{\operatorname{var}(S_t)} \leq \frac{D_2}{B_2} t^{-2H} - \frac{U(4/3)}{B_2} \leq \frac{D_2}{B_2}(T')^{-2H} - \frac{U(4/3)}{B_2}, \tag{18}$$

which yields $\rho(s,t) \leq 1-\varepsilon$, where $\varepsilon = -\frac{D_2}{B_2}(T')^{-2H} + \frac{U(4/3)}{B_2}$. Lemma **??** shows that $t > T_4$, ensures that $s-t > K$ implies $\rho(s,t) \leq 1$. Finally, set $\bar{T} = \max\{T', T_4, 3K\}$. It is clear – using (**??**) in the proof of Lemma **??** – that for fixed t, the function $(s,t) \mapsto \rho(s,t)$ is bounded. So let $D_1' = \max_{0<t<\bar{T}} \sup_{s\geq 0} \rho(s,t)$ and define $R = \max\{D_1, D_1'\} - 1$ It remains to guarantee $T/2 + K < \eta T$ but this follows since $\bar{T} < t < T/2$ implies $T > 6K$. The quantities η, $\bar{T}$, R, K and ε constructed above fulfill all the requirements. □

Proof of Theorem 3.3. First we determine the maximal expected growth rate of portfolios. Let us define

$$Q(T) = \sum_{t=0}^{T} E|S_t|^{\frac{\alpha}{\alpha-1}}.$$

Let $G(x) := \lambda|x|^{\alpha}$, $x \in \mathbb{R}$ and denote its Fenchel-Legendre conjugate

$$G^*(y) := \sup_{x\in\mathbb{R}}(xy - G(x)) = \frac{\alpha-1}{\alpha}\alpha^{\frac{1}{1-\alpha}}\lambda^{\frac{1}{1-\alpha}}|y|^{\frac{\alpha}{\alpha-1}}, \qquad y \in \mathbb{R}. \tag{19}$$

By definition of G^*, for all $\phi \in \mathscr{G}(T)$,

$$X_T(\phi) \le \sum_{t=0}^{T} G^*(-S_t) = C\sum_{t=0}^{T}|S_t|^{\alpha/(\alpha-1)}$$

for some $C > 0$ and hence

$$EX_T(\phi) \le CQ(T) < \infty. \tag{20}$$

Note that this bound is independent of ϕ. Using Lemma 4.2 it holds that

$$\begin{aligned} Q(T) &= C_{\frac{\alpha}{\alpha-1}}\sum_{t=0}^{T}\operatorname{var}(S_t)^{\frac{\alpha}{2(\alpha-1)}} \\ &\le C_{\frac{\alpha}{\alpha-1}}\sum_{t=0}^{T_2-1}\operatorname{var}(S_t)^{\frac{\alpha}{2(\alpha-1)}} + C_{\frac{\alpha}{\alpha-1}}B_2\sum_{t=T_2}^{T} t^{\frac{H\alpha}{(\alpha-1)}} \\ &\le C_{\frac{\alpha}{\alpha-1},T_2} + C_{\alpha,H,B_2}T^{H(1+\frac{1}{\alpha-1})+1}. \end{aligned} \tag{21}$$

Thus the maximal expected profit grows as $T^{H(1+\frac{1}{\alpha-1})+1}$ with the power of the horizon, this proves (**??**). With the strategy defined in (7), the dynamics takes the form

$$\begin{aligned} X_T(\phi) = &\sum_{t=0}^{T/2}|S_t|^{\frac{\alpha}{\alpha-1}} \\ &-\sum_{t=0}^{T/2}\lambda|S_t|^{\frac{\alpha}{\alpha-1}} \\ &-\frac{1}{T/2}\sum_{s=T/2+1}^{T}S_s\sum_{t=0}^{T/2}\operatorname{sgn}(S_t)|S_t|^{\frac{1}{\alpha-1}} \\ &-\frac{1}{T/2}\sum_{s=T/2+1}^{T}\lambda\left|\sum_{t=0}^{T/2}\operatorname{sgn}(S_t)|S_t|^{\frac{1}{\alpha-1}}\right|^{\alpha}. \end{aligned}$$

In the above expression let us denote the four terms by $I_1(T)$, $I_2(T)$, $I_3(T)$, $I_4(T)$, respectively, so that

$$X_T(\phi) = I_1(T) - I_2(T) - I_3(T) - I_4(T).$$

The upper bound constructed in (**??**) for $Q(T)$ right away gives us an upper estimate for $EI_1(T)$ as $EI_1(T) = Q(T/2)$. Using Lemma 4.1, we likewise present a lower estimate as

$$\begin{aligned} Q(T/2) = E[I_1] &= C_{\frac{\alpha}{\alpha-1}}\sum_{t=0}^{T/2}\operatorname{var}(S_t)^{\frac{\alpha}{2(\alpha-1)}} \\ &\ge C_{\frac{\alpha}{\alpha-1}}\sum_{t=0}^{T_1-1}\operatorname{var}(S_t)^{\frac{\alpha}{2(\alpha-1)}} + C_{\frac{\alpha}{\alpha-1}}B_1\sum_{t=T_1}^{T/2} t^{\frac{H\alpha}{\alpha-1}} \\ &\ge C_{\frac{\alpha}{\alpha-1},H,B_1,T_1} + C_{\frac{\alpha}{\alpha-1},H,B_1}T^{H(1+\frac{1}{\alpha-1})+1}, \end{aligned} \tag{22}$$

To treat the terms $I_2(T)$ and $I_4(T)$, note that with $\alpha > 1$ the function $x \mapsto |x|^\alpha$ is convex, thus applying Jensen's inequality

$$|EI_4(T)| \le E|I_2(T)| = \lambda E\left[\sum_{t=0}^{T/2} |S_t|^{\frac{\alpha}{\alpha-1}}\right] = \lambda \sum_{t=0}^{T/2} E|S_t|^{\frac{\alpha}{\alpha-1}} = \lambda E[I_1(T)] = \lambda Q(T/2). \tag{23}$$

Controlling term $I_3(T)$ is done via exploiting a specific property of Gaussian processes, namely that S_s for $s > t$ can be decomposed as $S_s = \rho(s,t)S_t + W_{s,t}$, where $W_{s,t}$ is independent of S_t and zero mean. With this, observe that

$$\begin{aligned} EI_3(T) &= \frac{1}{T/2} \sum_{s=T/2+1}^{T} \sum_{t=0}^{T/2} E[\rho(s,t) S_t \operatorname{sgn}(S_t) |S_t|^{\frac{1}{\alpha-1}}] \\ &= \frac{1}{T/2} \sum_{s=T/2+1}^{T} \sum_{t=0}^{T/2} E[\rho(s,t) |S_t|^{\frac{\alpha}{\alpha-1}}]. \end{aligned} \tag{24}$$

Let the constants $\bar{T}$, R, K, $\eta = 2/3$ and ε be as in Lemma 4.3, and decompose the double sum in (**??**) as

$$\sum_{s=T/2+1}^{T} \sum_{t=0}^{T/2} = \sum_{s=T/2+1}^{T} \sum_{t=0}^{\bar{T}} + \sum_{s=T/2+1}^{T/2+K} \sum_{t=\bar{T}}^{T/2} + \sum_{s=T/2+K}^{\eta T} \sum_{t=\bar{T}}^{T/2} + \sum_{s=\eta T}^{T} \sum_{t=\bar{T}}^{T/2}$$

Note that applying the upper bound developed in Lemma 4.3 to the double sum in (**??**), the summand no longer depends on the running variable of the outer sum. Denoting $C_{\bar{T}} := \sum_{t=0}^{\bar{T}} E|S_t|^{\frac{\alpha}{\alpha-1}}$, this implies that

$$\begin{aligned} EI_3(T) &\le \left(\sum_{t=0}^{T/2} + R\sum_{t=0}^{\bar{T}} + \frac{2RK}{T}\sum_{t=\bar{T}}^{T/2} - 2\varepsilon\left(1 - \frac{2}{3}\right)\sum_{t=\bar{T}}^{T/2}\right) E|S_t|^{\frac{\alpha}{\alpha-1}} \\ &= E[I_1(T)] + \left(R\sum_{t=0}^{\bar{T}} + \frac{2RK}{T}\sum_{t=\bar{T}}^{T/2} - \frac{2\varepsilon}{3}\sum_{t=\bar{T}}^{T/2}\right) E|S_t|^{\frac{\alpha}{\alpha-1}} \\ &= E[I_1(T)] + \left(R\sum_{t=0}^{\bar{T}} + \left(\frac{2\varepsilon}{3} - \frac{2RK}{T}\right)\sum_{t=0}^{\bar{T}-1} + \frac{2RK}{T}\sum_{t=0}^{T/2} - \frac{2\varepsilon}{3}\sum_{t=0}^{T/2}\right) E|S_t|^{\frac{\alpha}{\alpha-1}} \\ &= \left(1 - \frac{2\varepsilon}{3}\right) E[I_1(T)] + RC_{\bar{T}} + \left(\frac{2\varepsilon}{3} - \frac{2RK}{T}\right) C_{\bar{T}-1} + \frac{2RK}{T} E[I_1(T)], \end{aligned}$$

So we have

$$\begin{aligned} E[I_1(T)] - E[I_3(T)] &\ge \frac{2\varepsilon}{3} E[I_1(T)] - RC_{\bar{T}} - \left(\frac{2\varepsilon}{3} - \frac{2RK}{T}\right) C_{\bar{T}-1} - \frac{2RK}{T} E[I_1(T)] \\ &= \frac{2\varepsilon}{3} E[I_1(T)] - RC_{\bar{T}} - \frac{2\varepsilon}{3} C_{\bar{T}-1} + \frac{2RK}{T} C_{\bar{T}-1} - \frac{2RK}{T} E[I_1(T)]. \end{aligned}$$

The above, using (**??**), boils down to

$$X_T(\phi) \ge \frac{2\varepsilon}{3} Q(T/2) - RC_{\bar{T}} - \frac{2\varepsilon}{3} C_{\bar{T}-1} + \frac{2RK}{T} C_{\bar{T}-1} - \frac{2RK}{T} Q(T/2) - 2\lambda Q(T/2)$$

Using (**??**) and (**??**), with $\lambda < \varepsilon/3$, dividing through with $T^{H(1+\frac{1}{\alpha-1})+1}$ proves the statement in (8), and the proof of Theorem 3.3 is complete. □

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