Positive Hilbert Space Operators as Real Valued Functions

by

Andriamanankasina Ramanantoanina

Supervisor: Prof. Lajos Molnár (University of Szeged)

Central European University

June 2020, Budapest

A thesis submitted in fulfilment of the requirements for the degree of Practical Mathematics Specialization Program

Declaration of Authorship

I, Andriamanankasina Ramanantoanina, declare that this thesis entitled, *Positive Hilbert Space Operators as Real Valued Functions* and the work presented in it are my own.

I confirm that:

This work was done wholly or mainly while in candidature for a research degree at this University. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated. Where I have consulted the published work of others, this is always clearly attributed. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work. I have acknowledged all main sources of help.

Abstract

This essay is a short exploration of functional representations of positive Hilbert space operators. These are non-negative functions associated to positive operators in a way that that transfers order structures of operators into pointwise order on the functions. The orders of interest here are the usual Löwner order and the spectral order. The representations are built to behave well with the orders but they present interesting algebraic and topological properties like compatibility with mean operations and continuous functional calculus. We present here an application of this machinery to describe the spectral order isomorphism group of the set of selfadjoint operators on infinite dimensional Hilbert spaces.

Table of Contents

Chapter 1

Introduction

The theme of representation is very common in mathematics. The main idea is to embed one structure into an other in a way that preserves the property that one wants to study. The property is hopefully more transparent in the target structure. Here our object of study are operators on Hilbert spaces and orders among them. And we want to map the order structure into appropriately chosen space of real valued functions.

An operator T on a Hilbert space H is said to be positive when it is selfadjoint and its spectrum, which is contained in $\mathbb{R}$, is non-negative. There are two orders that are of interests for us on the positive operators the usual order, or the Löwner order, denoted by $\leq$, and the spectral order, or the Olson order, denoted by $\preccurlyeq$. We say that $A \leq B$ when $B - A$ is positive, and we say that $A \preccurlyeq B$ when $B^n - A^n$ is positive for any integer $n \geq 1$.

Why the spectral order? The usual order has been the object of many investigation and it still is. Many issues concerning this order are due to two things. First, the set of positive operators with the usual order is not a lattice, it has actually been proved by Kadison [4] that it is as far as it could be from being a lattice. Second, for a continuous increasing function, the operators $f(A)$ and $f(B)$ are well defined but there is something a bit counter intuitive with them which is that only for a very peculiar class of such function we have $f(A) \leq f(B)$ when $A \leq B$. The spectral order remedies these flaws of the usual order.

The idea of representing selfadjoint operators with real valued function is very natural. For example, a selfadjoint operator is totally determined by its quadratic form, moreover, order among selfadjoint operator is the same as the pointwise order among quadratic forms. In this essay, we consider four examples of functional representation of positive Hilbert space operators: wo of them turn the usual order between operators to the pointwise order between functions, another two turn the spectral order between operators to the same, pointwise order between functions. We investigate which algebraic operations those representations preserve. At the end we showcase two applications in the description of the spectral order isomorphism of the set of selfadjoint operators and the definition of natural metrics derived from these functions.

Chapter 2

Functional representations

Notations

In what follows, H is a Hilbert space and $\mathcal{B}(H)$ is the algebra of bounded operators over H. The inner product on H is denoted by $\langle .,. \rangle$ and let the generated norm be $\|.\|$. The unit sphere in H is denoted by $\mathcal{S}_H$.

The class of selfadjoint operators on H is denoted by $\mathcal{B}_{sa}(H)$ and the class of positive operators is denoted by $\mathcal{B}^+(H)$. The symbol $\mathcal{P}_1(H)$ stands for the set of all rank-one projections on H. For any $x, y \in H$, $x \otimes y$ denotes the rank at most one operator on H defined by $(x \otimes y)(z) = \langle z, y \rangle x$, $z \in H$. Clearly, the elements of $\mathcal{P}_1(H)$ are exactly the operators in $\mathcal{B}(H)$ which are of the form $x \otimes x$ with some vector $x \in \mathcal{S}_H$. We write $P_x = x \otimes x$ for any $x \in \mathcal{S}_H$.

An operator $A \in \mathcal{B}(H)$ is called positive if $\langle Ax, x \rangle \geq 0$ holds for all $x \in H$. Denote the set of all positive operators on H by $\mathcal{B}^+(H)$. For any operator $A \in \mathcal{B}(H)$, $\mathrm{rng}A$ stands for its range and $\ker A$ denotes its kernel.

2.1 Orders on positive operators

There are two important partial orders on the space $\mathcal{B}_{sa}(H)$. The usual Löwner order $\leq$ which is defined as follows. For any $A, B \in \mathcal{B}_{sa}(H)$ we write $A \leq B$ if and only if $B - A \in \mathcal{B}^+(H)$. The second one is the so-called spectral order $\preccurlyeq$ which was introduced by Olson in [13].

There are equivalent definitions for restricted classes of selfadjoint operators, but the original definition of Olson was formulated in terms of spectral measures. Let $A, B \in \mathcal{B}_{sa}(H)$ and denote by E_A, E_B the corresponding spectral measures defined on the Borel subsets of the real line. We write $A \preceq B$ if and only if $E_A(]-\infty, t]) \geq E_B(]-\infty, t])$ holds for every real number t. For positive operators, comparison in the spectral order is equivalent to comparison of all the integral powers in the usual order.

2.1.1 Theorem (Theorem 3 in [13])**.** *Given two positive operators $A, B \in \mathcal{B}^+(H)$, we have $A \preccurlyeq B$ if and only if $A^n \leq B^n$ for $n = 1, 2, \cdots$.*

There are several substantial differences among the properties of those two partial orders. As already mentioned, $\mathcal{B}_{sa}(H)$ with the usual order $\leq$ form what is called an anti-lattice (see the next theorem) while the spectral order $\preceq$ makes it a conditionally complete lattice.

2.1.2 Theorem (Theorem 6 in [4])**.** *A greatest lower bound with respect to the usual order for two selfadjoint operators $A, B \in \mathcal{B}_{sa}(H)$ exists if and only if A and B are comparable.*

2.1.3 Theorem (Theorem 1 in [13])**.** *Any nonempty collection of elements of $\mathcal{B}_{sa}(H)$ having an upper bound necessarily has a supremum.*

The second important big difference, we already mentioned in the introduction, is the following. A real function f is said to be operator monotone with respect to some given partial order $\mathcal{R}$ on $\mathcal{B}_{sa}(H)$ if, for any $A, B \in \mathcal{B}_{sa}(H)$, $A\mathcal{R}B$ implies $f(A)\mathcal{R}f(B)$. The operator monotone functions with respect to the usual order $\leq$ are very special, they have a well-known and deep theory essentially due to Löwner. On the other hand, every monotone increasing function is operator monotone with respect to the spectral order $\preceq$.

2.1.4 Theorem (Corollary 1 in [13])**.** *Given two positive operators $A, B \in \mathcal{B}_{sa}(H)$, and an interval $J \subset \mathbb{R}$ containing the spectrum of A and B. Then $A \preccurlyeq B$ if and only if $f(A) \preccurlyeq f(B)$ for any continuous increasing function f on J.*

2.2 Faithful functional representations

Here are the four non-negative real valued functions on the unit sphere in H, which we associate to an arbitrary positive operator. For any $A \in \mathcal{B}^+(H)$ we define

$$w(A, x) = \langle Ax, x\rangle, \quad x \in \mathcal{S}_H;$$

$$\lambda(A, x) = \sup\{t \geq 0 \,:\, tP_x \leq A\}, \quad x \in \mathcal{S}_H;$$

$$\nu(A, P_x) = \sup\{t \geq 0 \,:\, tP_x \preccurlyeq A\}, \quad x \in \mathcal{S}_H;$$

$$r(A, x) = \lim_n \|A^n x\|^{1/n}, \quad x \in \mathcal{S}_H.$$

The function $w(A, .)$ is the quadratic form corresponding to $A \in \mathcal{B}^+(H)$ restricted to $\mathcal{S}_H$, we call it the numerical range function of A. The function $\lambda(A, .)$ is called the strength function of A. The strength functions were initially defined by Busch and Gudder [2] for positive contractions on Hilbert space but their definition extends verbatim to positive operators. The investigation of strength functions by Busch and Gudder was prompted by a remark of Ludwig in his book *Foundations of Quantum Mechanics* [6], proof of Theorem 5.22. The spectral order analogue, $\nu(A, .)$, of the strength function of A is called the spectral strength function of A. It is introduced and studied in [11]. The fourth function, $r(A, .)$, is called the spectral radius function.

Notice that the sup in the definitions of $\lambda(A, x)$ and $\nu(A, x)$ can be replaced by max. The definition of $r(A, x)$ in terms of limit raises question about the existence of this limit. In general this limit does not necessarily exist, but recall that we are dealing with positive operators here. It is proved in [11] that this limit exists for any positive operator $A \in \mathcal{B}^+(H)$ and any unit vector $x \in \mathcal{S}_H$.

The first step in our investigation is to check that the representations we are dealing with here are faithful and they turn order among positive operators to pointwise order on the representing functions. Let us start with the usual order and the first two representations $A \longmapsto w(A,.)$ and $A \longmapsto \lambda(A,.)$. It is clear that for any $A, B \in \mathcal{B}^+(H)$ we have $A \leq B$ if and only if $w(A,x) \leq w(B,x)$ for all $A \in \mathcal{S}_H$. It was proved by Busch and Gudder in [2] that the same equivalence holds for the strength functions.

2.2.1 Theorem (Theorem 1 in [2])**.** *Given two positive operators $A, B \in \mathcal{B}^+(H)$, the following are equivalent:*

1. $\lambda(A,x) \leq \lambda(B,x)$, *for all* $x \in \mathcal{S}_H$.
2. $A \leq B$.

Bush and Gudder's proof was formulated for effects on a Hilbert space but it works for general positive operators. We then see that, not only the strength function determines the operator, but the embedding is actually order preserving.

The beauty of the above theorem is weakened by the fact that we do not know of any sufficient condition for a real valued function on the unit sphere to be the strength function of a positive operator. This being said, we point out that there is an explicit formula to compute $\lambda(A,x)$. The following formula for the strength function was proved in [2]. An alternative proof based on Douglas' factorization theorem is also presented in [11].

2.2.2 Proposition (Theorem 4 in [2], Theorem 6 in [11])**.** *Given a positive operator $A \in \mathcal{B}^+(H)$ and a unit vector $x \in \mathcal{S}_H$, the numbers $\lambda(A,x)$ are of the following form:*

1. $\lambda(A,x) = \left|\left|A^{-1/2}x\right|\right|^{-2}$, *if* $x \in \mathrm{rng}\left(A^{1/2}\right)$.
2. $\lambda(A,x) = 0$, *otherwise.*

In this formula, $A^{-1/2}$ is the possibly unbounded inverse of $A^{1/2}$ from the orthogonal complement of $\ker A^{1/2}$ onto $\mathrm{rng} A^{1/2}$. We obtain two corollaries of this proposition. These are characterisations of invertible operators and projection operators by means of their strength functions.

Note that if A is invertible then the strength function of A and the numerical range function of A satisfy the following relation:

$$\lambda(A,x) = \frac{1}{w(A^{-1},x)}, \quad x \in \mathcal{S}_H.$$

We clearly see that this is a continuous function of x on the unit sphere of H (with respect to the restriction of the topology of H onto its unit sphere). The converse is also true as remarked in [9].

2.2.3 Corollary (Remark 6 in [9])**.** *A positive operator $A \in \mathcal{B}^+(H)$ is invertible if and only if its strength function $\lambda(A,\cdot)$ is continuous.*

We also see from the above proposition that the strength function of a projection is a characteristic function of the unit sphere of the range space.

2.2.4 Corollary (Lemma 4 in [2])**.** *Let $A \in \mathcal{B}^+(H)$. Then A is a projection if and only if* $\mathrm{Im}(\lambda(P,\cdot)) = \{0,1\}$. *In this case, the range of the projection is given by*

$$\mathrm{rng}\,(A) = \mathrm{span}\,\{x \in H : \ \lambda(A,x) = 1\}\,.$$

Now we turn to the spectral order and the representations $A \longmapsto \nu(A,.)$ and $A \longmapsto r(A,.)$. One can check that they are faithful and turn spectral order among positive operators to pointwise order on the representing functions. First, notice how the spectral strength function and the local spectral radius function relate to the spectral measures as shown in the following proposition.

2.2.5 Proposition (Proposition 1 and Proposition 5 in [11]). *Given a positive operator $A \in \mathcal{B}^+(H)$ and a unit vector $x \in \mathcal{S}_H$, we have*

$$r(A,x) = \min\{t \geq 0 \,:\, P_x \leq E_A([0,t])\},$$

$$\nu(A,x) = \max\{t \geq 0 \,:\, P_x \leq E_A([t,\infty))$$

Using the above facts, it was also proved in [11] that the spectral strength functions and the local spectral radius functions characterise the spectral order.

2.2.6 Theorem (Proposition 3 and Proposition 5 in [11]). *Given two positive operators $A, B \in \mathcal{B}^+(H)$, the following are equivalent:*

1. *$r(A,x) \leq r(B,x)$, for all $x \in \mathcal{S}_H$;*
2. *$\nu(A,x) \leq \nu(B,x)$, for all $x \in \mathcal{S}_H$;*
3. *$A \preccurlyeq B$.*

We do not know of any sufficient condition for a non-negative real valued function on $\mathcal{S}_H$ to be the spectral strength function of some positive operator. However, an explicit formula to compute $\nu(A,x)$ is given in [11].

2.2.7 Theorem (Theorem 8 in [11]). *Given $A \in \mathcal{B}^+(H)$ and a unit vector $x \in \mathcal{S}_H$, we have that $\nu(A,x) > 0$ if and only there exists a closed invariant subspace M of A, containing x, on which A is invertible and in that case we have*

$$\nu(A,x) = \frac{1}{r(A\big|_M^{-1}, x)}.$$

2.3 Algebraic properties

The set of positive operators has two immediate operations: addition and multiplication by positive scalars. It turns out that the four functional representations (the numerical range function, the strength function, the spectral strength function and the local spectral radius function) we have seen in the previous section all behave well with the scalar multiplication.

It is clear that the numerical range function is additive. On the contrary, it is proved in [9] that the strength function is highly non-additive. In fact, the sum of two strength functions is again a strength function if and only if the underlying operators are linearly dependent.

2.3.1 Proposition (Proposition 2 in [9]). *Given $A, B \in \mathcal{B}^+(H)$, $\lambda(A,\cdot) + \lambda(B,\cdot) = \lambda(C,\cdot)$, for some positive operator $C \in \mathcal{B}^+(H)$ if and only if A, B are linearly dependent.*

There are other important operations on positive operators connected to the so called means. Here we are considering Kubo-Ando means [5]. The most fundamental means are defined as follows for positive invertible operators

1. the arithmetic mean: $A\nabla B = (A+B)/2$.
2. the geometric mean: $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$.
3. the harmonic mean: $A!B = 2(A^{-1}+B^{-1})^{-1}$.

These are defined for invertible operators and extend to $\mathcal{B}^+(H)$ by monotone decreasing strong convergence. So on one hand we have these means defined for positive operators, and on the other hand we have non-negative real valued functions so that mean operations are naturally defined. We want to know which of these operations are preserved by the representations. In the following we denote the arithmetic, geometric and harmonic means of functions with the corresponding symbol for the mean for positive operators.

Clearly, the embedding $A \longmapsto w(A,.)$ transfers arithmetic means on $\mathcal{B}^+(H)$ to arithmetic means on the numerical range functions. The non-additivity of the strength functions indicates that this is not the case for the embedding $A \longmapsto \lambda(A,.)$.

Concerning the geometric and the harmonic means, there are some examples which suggest that the embedding $A \longmapsto \lambda(A,.)$ also transfers these two means. In studying the preservers of geometric and harmonic means on $\mathcal{B}^+(H)$, Molnár [8, 7] proved the following formulas for the strength functions.

2.3.2 Proposition (Lemma 1 in [8] and Lemma 2 in [7])**.** *Given $A \in \mathcal{B}^+(H)$ and a unit vector $x \in \mathcal{S}_H$, we have*

$$A\#P_x = \sqrt{\lambda(A,x)}P_x, \quad \textit{and} \quad A!P_x = \frac{2\lambda(A,x)}{\lambda(A,x)+1}P_x.$$

We first see that $A\#P_x$ and $A!P_x$ are non zero if and only if $x \in \operatorname{rng}A^{1/2}$. Recalling that the strength function of the rank one projection P_x takes only the values 0 and 1, we can write these formulas as follows. For all $y \in \mathcal{S}_H$, we have

$$\lambda(A\#P_x,y) = \sqrt{\lambda(A,x)}\lambda(P_x,y) = \sqrt{\lambda(A,x)\lambda(P_x,y)} = \lambda(A,y)\#\lambda(P_x,y),$$

and

$$\lambda(A!P_x,\cdot) = \frac{2\lambda(A,x)}{\lambda(A,x)+1}\lambda(P_x,\cdot) = \lambda(A,\cdot)!\lambda(P_x,\cdot).$$

Here, for non negative real numbers, if $st=0$ then their harmonic mean $s!t$ is defined to be 0.

Here is one more evidence to the fact that the strength function representation should transfer the harmonic mean. If $A,B \in \mathcal{B}^{++}(H)$ are positive invertible operators, then their strength functions do not vanish, the harmonic mean is then well defined. The harmonic mean of the strength functions is equal to the strength function of the harmonic mean. Indeed, given such $A,B \in \mathcal{B}^{++}(H)$ and a unit vector $x \in \mathcal{S}_H$, we have $\lambda(A,x) = \langle A^{-1}x,x\rangle^{-1}$ and $\lambda(B,x) = \langle B^{-1}x,x\rangle^{-1}$. Computing the harmonic mean of these two functions, we get

$$\begin{aligned}\lambda(A,x)!\lambda(B,x) &= 2(\lambda(A,x)^{-1}+\lambda(B,x)^{-1})^{-1}\\ &= 2(\langle A^{-1}x,x\rangle+\langle B^{-1}x,x\rangle)^{-1}\\ &= 2(\langle (A^{-1}+B^{-1})x,x\rangle)^{-1}.\end{aligned}$$

Since the operator in the last line is also invertible, we have

$$\begin{aligned}\lambda(A,x)!\lambda(B,x) &= 2(\langle (A^{-1}+B^{-1})x,x\rangle)^{-1}\\ &= 2\lambda((A^{-1}+B^{-1})^{-1},x)\\ &= \lambda(2(A^{-1}+B^{-1})^{-1},x).\end{aligned}$$

Hence

$$\lambda(A,x)!\lambda(B,x)=\lambda(A!B,x). \tag{2.1}$$

Although this with the above proposition suggest that the embedding $A \longmapsto \lambda(A,.)$ should transfer the harmonic mean, it is highly non-trivial to see how this relation (2.1) holds for non invertible operators since there is even no explicit formula for the harmonic mean of two positive singular operators. In [11], we proved that this identity actually holds on the whole $\mathcal{B}^+(H)$.

2.3.3 Theorem ([11])**.** *Given $A,B\in\mathcal{B}^+(H)$, we have*

$$w(A\nabla B,x)=w(A,x)\ \nabla\ w(B,x),\quad x\in\mathcal{S}_H;$$

$$\lambda(A!B,x)=\lambda(A,x)\ !\ \lambda(B,x),\quad x\in\mathcal{S}_H.$$

But, surprisingly, there is no injective map from $\mathcal{B}^+(H)$ to the set of all bounded non-negative real valued functions on $\mathcal{S}_H$ which would transfer the geometric mean.

Now we turn to the spectral strength function and the local spectral radius function. The non linearity of the spectral order dismisses any possible compatibility of these functions with addition on $\mathcal{B}^+(H)$. Rather, we are more interested in their properties related to lattice operations. At first sight, the lattice operations (the sup $\vee$ and the inf $\wedge$) might appear unrelated to the operations above. We see in the following proposition that these are limits of arithmetic and harmonic means.

2.3.4 Proposition (Theorem 2 and Theorem 3 in [3])**.** *Given two positive operators $A,B\in\mathcal{B}^{++}(H)$, $A\vee B=\lim_n(A^n\nabla B^n)^{1/n}$, and $A\wedge B=\lim_n(A^n!B^n)^{1/n}$ where the limits are taken in the strong operator topology.*

It is then consistent that the local spectral radius and the spectral strength functions are compatible with these operations, as we proved in [11].

2.3.5 Theorem (Proposition 11 in [11])**.** *Given two positive operators $A,B\in\mathcal{B}^+(H)$, we have*

$$r(A\vee B,\cdot)=r(A,\cdot)\vee r(B,\cdot)\quad\text{and}\quad r(A\wedge B,\cdot)\le r(A,\cdot)\wedge r(B,\cdot).$$

$$\nu(A\vee B,\cdot)\ge\nu(A,\cdot)\vee\nu(B,\cdot)\quad\text{and}\quad \nu(A\wedge B,\cdot)=\nu(A,\cdot)\wedge\nu(B,\cdot),$$

Chapter 3

Applications

3.1 Metric defined by functional representations

Recall that we now have four functional representations of positive operators: $w(A,.), \lambda(A,.), r(A,.)$ and $\nu(A,.)$. One can prove that these are all bounded functions on $\mathcal{S}_H$. Indeed, given a positive operator $A \in \mathcal{B}^+(H)$, the following inequalities are proved in [11]:

$$\nu(A,x) \leq \lambda(A,x) \leq w(A,x) \leq r(A,x) \leq ||A||\,, \quad x \in \mathcal{S}_H.$$

The most natural metric on bounded real valued functions is the metric coming from the supremum norm. One can prove that the metric from the supremum norm of the functions $w(A,.), A \in \mathcal{B}^+(H)$ is exactly the metric of the operator norm on $\mathcal{B}^+(H)$

$$||A - B|| = \sup_{x \in \mathcal{S}_H} |w(A,x) - w(B,x)|,$$

and this is a complete metric on $\mathcal{B}^+(H)$.

The metric from the supremum norm of the functions $\lambda(A,.), A \in \mathcal{B}^+(H)$,

$$d_{BG}(A,B) = \sup_{x \in \mathcal{S}_H} |\lambda(A,x) - \lambda(B,x)|,$$

is called the Busch-Gudder metric, and the first investigation of the properties of this metric was done by Molnár in [9]. He proved several properties of the topology generated by this metric which shows that this is a rather peculiar metric. For example, the addition is not continuous in the Busch-Gudder distance and $\mathcal{B}^{++}(H)$ is not dense in $\mathcal{B}^+(H)$. We summarise in the following proposition some of Molnár's findings about the properties of the Busch-Gudder metric.

3.1.1 Proposition (Proposition 10 and Proposition 14 in [9])**.** *Let H be finite dimensional Hilbert space.*

1. *The topology of the norm distance is weaker than that of the Busch-Gudder metric on $\mathcal{B}^+(H)$, and they coincide on $\mathcal{B}^{++}(H)$.*

2. *$\mathcal{B}^+(H)$ is complete in the Busch-Gudder metric as well.*

It is definitely interesting to see that although this is a very natural distance, it has some unexpected pathological behaviors. Mainly we notice natural properties of the generated topology on non-singular elements and unexpected behavior on the singular ones. This is due to the fact that the strength function is very sensible to invertibility.

Surprisingly, despite the discrepancy in the behavior of the norm metric and the Busch-Gudder metric, their group of surjective isometries are the same as it is shown in the following theorem.

3.1.2 Theorem (Theorem 16 in [9])**.** *A surjective map $\phi : \mathcal{B}^+(H) \longrightarrow \mathcal{B}^+(H)$ preserves the norm distance or the Busch-Gudder distance if and only if it is of the form $\phi(A) = UAU^*$ for some unitary or anti-unitary operator U on H.*

The so called Thompson metric on $\mathcal{B}^{++}(H)$ is also an other important metric. The Thompson distance on $\mathcal{B}^{++}(H)$ is defined by

$$d_T(A, B) = \log \max\{M(A/B), M(B/A)\}, \quad A, B \in \mathcal{B}^{++}(H),$$

where $M(X/Y) = \inf\{t > 0 : X \leq tY\}$ for any $X, Y \in \mathcal{B}^{++}(H)$. But one can show that this metric can also be expressed in terms of the supremum norm derived from the functions $\lambda(A, .), A \in \mathcal{B}^{++}(H)$. For a positive real number t, the inequality $A \leq tB$ is equivalent to $w(A, .) \leq tw(B, .)$ and $\lambda(A, .) \leq t\lambda(B, .)$. From that, one can prove that the supremum norm of the functions $\log \lambda(A, .), A \in \mathcal{B}^{++}(H)$ or the functions $w(A, .), A \in \mathcal{B}^{++}(H)$ yield exactly the Thomspon metric

$$d_T(A, B) = \sup_{x \in \mathcal{S}_H} |\log \lambda(A, x) - \log \lambda(B, x)| = \sup_{x \in \mathcal{S}_H} |\log w(A, x) - \log w(B, x)|.$$

We have the following result as a particular case of Thompson's completeness result in Lemma 3 in [14].

3.1.3 Proposition. *The Thompson metric d_T is a complete metric on $\mathcal{B}^{++}(H)$.*

A spectral order variant of the Thompson metric can also be defined by changing the usual order $\leq$ in the definition of $M(X/Y)$ to the spectral order $\preccurlyeq$,

$$d_{sT}(A, B) = \log \max\{N(A/B), N(B/A)\}, \quad A, B \in \mathcal{B}^{++}(H),$$

where $N(X/Y) = \inf\{t > 0 : X \preccurlyeq tY\}$ for any $X, Y \in \mathcal{B}^{++}(H)$. We call it the spectral Thompson metric. Following a similar argument as above, since we know that $A \preccurlyeq tB$ is equivalent to $\nu(A, .) \leq t\nu(B, .)$ (resp. $r(A, .) \leq tr(B, .)$), we see that the spectral Thompson metric is also a supremum norm distance derived from the functions $\log \nu(A, .), A \in \mathcal{B}^{++}(H)$ or the functions $r(A, .), A \in \mathcal{B}^{++}(H)$,

$$d_{sT}(A, B) = \sup_{x \in \mathcal{S}_H} |\log \nu(A, x) - \log \nu(B, x)| = \sup_{x \in \mathcal{S}_H} |\log r(A, x) - \log r(B, x)|.$$

The following proposition was proved in

3.1.4 Proposition (Proposition 26 in [11])**.** *The spectral Thompson metric d_{sT} is a complete metric on $\mathcal{B}^{++}(H)$.*

Note that Thompson's completeness result relied on the linearity of the underlying order. So the completeness of the spectral Thompson metric is not a mere consequence of Thompson's original result.

3.2 Spectral order isomorphisms of $\mathcal{B}_{sa}(H)$

Let $\Omega \subset \mathcal{B}_{sa}(H)$. A spectral order isomorphism of Ω is a bijective map $\phi : \Omega \to \Omega$ such that

$$A \preccurlyeq B \Longleftrightarrow \phi(A) \preccurlyeq \phi(B), \quad A, B \in \Omega.$$

The structure of spectral order isomorphisms of set of selfadjoint operators was initiated by Molnár and Šemrl in [12]. They first gave a description of the spectral order isomorphisms of the effect algebra $[0, I]$ (this is the set of $A \in \mathcal{B}^+(H)$ such that $0 \preccurlyeq A \preccurlyeq I$). Then they used an elaborated scaling argument to extend their result to $\mathcal{B}_{sa}(H)$. The key point of their argument is to prove that any spectral order isomorphism of $\mathcal{B}_{sa}(H)$ preserves the scalar multiples of the identity operator. A proof that any spectral order isomorphism of $\mathcal{B}_{sa}(H)$ preserves the scalar multiples of the identity operator was provided in [12] when $3 \leq \dim(H) < \infty$, and in [1] for any Hilbert space H.

In this section, we first give a brief description of the spectral order isomorphisms of the effect algebra. Then we rederive the characterisation of scalar multiples of the identity from [1], using the functional representations.

The description of the spectral order isomorphisms of $[0.I]$ involves a family of highly non-trivial maps that we explain now. First of all, by a resolution of the identity we mean a function from the set of reals into the lattice of all projections on the Hilbert space H which is monotone increasing, right-continuous, for small enough real numbers it takes the value 0 and for large enough real numbers it takes the value I. It is well-known that there is a one-to-one correspondence between the compactly supported spectral measures on the Borel sets of $\mathbb{R}$ and the resolutions of the identity. In fact, every resolution of the identity is of the form $t \mapsto E(]-\infty, t])$ with a (uniquely determined) compactly supported spectral measure E on the Borel sets of $\mathbb{R}$. If $A \in \mathcal{B}_{sa}(H)$, then the resolution of the identity corresponding to E_A is what we have already called the spectral resolution of A.

Let $S : H \to H$ now be a bijective bounded linear or conjugate linear operator if H is infinite dimensional, or a bijective semilinear operator if H is finite dimensional. For any $A \in \mathcal{B}_{sa}(H)$ with spectral measure E_A, the map

$$t \longmapsto I - P_{S(\mathrm{rng} E_A(]t,\infty[))}, \quad t \in \mathbb{R}$$

is a resolution of the identity. Denote the corresponding spectral measure by E_A^S. Define

$$\psi_S(A) = \int_{-\infty}^{+\infty} t \, dE_A^S(]-\infty, t]), \quad A \in \mathcal{B}_{sa}(H).$$

It was proved in Proposition 1 in [12] that $\psi_S : \mathcal{B}_{sa}(H) \to \mathcal{B}_{sa}(H)$ is a spectral order isomorphism.

Since ψ_S fixes 0 and I, $\psi_S : [0, I] \to [0, I]$ is also a spectral order isomorphism. Notice first that if $f : [0, 1] \longrightarrow [0, 1]$ is bijective increasing then the map $A \longmapsto f(A)$ is a spectral order isomorphisms of $[0, I]$. These two maps make up all the spectral order isomorphisms of $[0, I]$, as it is shown in the following theorem.

3.2.1 Theorem (Theorem 3 in [12])**.** *Let H be a Hilbert space with* $\dim H \geq 3$ *and let* $\phi : [0, I] \longrightarrow [0, I]$ *be a spectral order isomorphism. Then there exists a bijective increasing function* $f : [0, 1] \longrightarrow [0, 1]$, *and an operator* $S : H \longrightarrow H$, *which is semilinear in the case where* $3 \leq \dim(H) < \infty$ *and bounded linear or conjugate linear in the case where H is infinite dimensional, such that*

$$\phi(A) = \psi_S(f(A)), \quad A \in [0, I].$$

Now to describe the spectral order isomorphisms of $\mathcal{B}_{sa}(H)$, $\mathcal{B}^+(H)$ and $\mathcal{B}^{++}(H)$, a similar argument as in the proof Theorem 4 in [12] can be applied to reduce the problem to the case of the effect algebra which we already know. The key point in the aforementioned proof is to prove that a spectral order isomorphisms on one of the sets $\mathcal{B}_{sa}(H)$, $\mathcal{B}^+(H)$ or $\mathcal{B}^{++}(H)$ preserves the scalar multiple of the identity. In [11], we give an order characterisation of the scalar operators using their local spectral radius functions.

3.2.2 Proposition (Lemma 22 in [11])**.** *Let $A \in \mathcal{B}^+(H)$. Then $A = aI$ holds with some real number $a \geq 0$ if and only if for every $B \in \mathcal{B}^+(H)$ and $x \in \mathcal{S}_H$ we have $r(A \wedge B, x) = r(A, x) \wedge r(B, x)$.*

As consequence of this, the following spectral order characterisation of scalar operators is obtained in [11]. This was originally proved in [1] for more general case.

3.2.3 Theorem (Theorem 23 in [11])**.** *Given an operator $T \in \mathcal{B}^+(H)$. We have T is a scalar multiple of the identity operatorif and only if $T \wedge (A \vee B) = (T \wedge A) \vee (T \wedge B)$ for any $A, B \in \mathcal{B}^+(H)$.*

Using the fact that translations by scalar multiples of the identity are spectral order isomorphisms which also preserve the scalar multiples of the identity, the above theorem can also be modified to give the same characterisation of the scalar multiples of the identity in $\mathcal{B}_{sa}(H)$ and in $\mathcal{B}^{++}(H)$. As a consequence of these characterisations, we see that any spectral order isomorphism of $\mathcal{B}_{sa}(H)$ or $\mathcal{B}^+(H)$ or $\mathcal{B}^{++}(H)$ preserves the scalar multiples of the identity. Then the description of the structure of the spectral order isomorphisms of these sets can be completed, as show in the theorem below.

3.2.4 Theorem. *Let ϕ be a spectral order isomorphism of $\mathcal{B}_{sa}(H)$ (resp. $\mathcal{B}^+(H), \mathcal{B}^{++}(H)$). Then there is a strictly increasing bijective function f on $]-\infty, +\infty[$ (resp. $[0, +\infty[$, $]0, +\infty[$) and an additive bijection $S : H \to H$ which is semilinear in the case where $3 \leq \dim(H) < \infty$ and bounded linear or conjugate linear in the case where H is infinite dimensional such that for all $A \in \mathcal{B}_{sa}(H)$ (resp.$\mathcal{B}^+(H), \mathcal{B}^{++}(H)$) we have*

$$\phi(A) = \psi_S(f(A)).$$

Proof. See [12, 1] for the case of $\mathcal{B}_{sa}(H), \mathcal{B}^+(H)$ and [11] for the case of $\mathcal{B}^{++}(H)$. □

Chapter 4

Outlook

In conclusion, we have seen some examples of functional representations of positive operators here which were initially constructed to be compatible with orders. They reflect different interactions of the order structures with other structure present in $\mathcal{B}^+(H)$. It is natural to ask whether these interactions characterise the representation as well. For instance, is the strength function the only representation of the positive operators into the positive real valued functions compatible with the harmonic mean? We could ask the same question about the spectral strength function, the local spectral radius function and the lattice operations.

We also have seen several interesting metrics derived from these representation in a natural way, namely the Thompson metric and the spectral Thompson metric. It would be an interesting "preserver problem" to describe the isometry group of these metrics.

One can take a different direction. Notice that the result discussed in this essay are all formulated in $\mathcal{B}(H)$, which is a type I von Neumann algebra. It would be interesting to see how much of these constructions could be carried out in more general von Neumann algebras.

Bibliography

[1] M. Bohata, *Spectral order isomorphisms and AW*-factors,* preprint, arXiv:1911.07491.

[2] P. Busch and S.P. Gudder, *Effects as functions on projective Hilbert space,* Lett. Math. Phys. **47** (1999), 329–337.

[3] M. Fujii, R. Nakamoto, and M. Nakamura. *Spectral order and operator means,* Scientiae Mathematicae **1** (1998), 149–152.

[4] R.V. Kadison, *Order properties of bounded self-adjoint operators,* Proc. Amer. Math. Soc. **2** (1951), 505–510.

[5] F. Kubo and T. Ando, *Means of positive linear operators,* Math. Ann. **246** (1980), 205-224.

[6] G. Ludwig, *Foundations of Quantum Mechanics. I.* Texts and Monographs in Physics. Springer-Verlag, New York, 1983. xii+426 pp.

[7] L. Molnár, *Maps preserving the harmonic mean or the parallel sum of positive operators,* Linear Algebra Appl. **430** (2009), 3058–3065.

[8] L. Molnár, *Maps preserving the geometric mean of positive operators,* Proc. Amer. Math. Soc. **137** (2009), 1763–1770.

[9] L. Molnár, *Busch-Gudder metric on the cone of positive semidefinite operators and its isometries,* Integr. Equ. Oper. Theory (2018) 90:20.

[10] L. Molnár and G. Nagy: *Spectral order automorphisms of the spaces of Hilbert space effects and observables: The 2-dimensional case*, Lett. Math. Phys. **106**, 535–544 (2016).

[11] L. Molnár and A. Ramanantoanina, *On functional representations of positive Hilbert space operators,* in preparation.

[12] L. Molnár and P. Šemrl, *Spectral order automorphisms of the spaces of Hilbert space effects and observables,* Lett. Math. Phys. **80** (2007), 239–255.

[13] M.P. Olson, *The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice,* Proc. Amer. Math. Soc. **28** (1971), 537–544.

[14] A.C. Thompson, *On certain contraction mappings in a partially ordered vector space,* Proc. Amer. Math. Soc. **14** (1963), 438–443.