From Poincaréan Intuition to Actual Infinity

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Submitted to

Department of Philosophy, Central European University

In partial fulfillment of the requirements for the degree of Master of Arts

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Vienna, Austria

2021

Abstract

This thesis focuses on Poincaré's philosophy of mathematics. Specifically, his theory of intuition as a foundation for mathematics and his ideas on actual infinity. My main goal is to present an analysis of Poincaré's philosophy as a whole, and connect his ideas within a Poincaréan framework. In this thesis, I deal with how he argues for mathematical intuition, and why he thinks that mathematics is synthetic a priori. In the second chapter, I present his views on transfinite cardinals, and show the underlying reasons of his rejection of actual infinity. In the third chapter, I interpret Poincaré's philosophy and show some possible reasons why he is dissatisfied with set theory and Cantor's *transfinite paradise.* At the end, I look at Cantor's argument for the uncountability of real numbers within a Poincaréan framework.

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Introduction

Poincaré is known for his rejection of the logicist program. Logicists like Zermelo, Cantor, Russell, Peano etc. claim that all true mathematical statements can be proven via logic alone, without depending on any "extra-logical" elements. In *Foundations of Arithmetic* and *Basic Laws of Arithmetic I* (1884), Frege posed five laws and claimed that one can derive all laws of arithmetic from these five logical axioms. However, his Basic Law 5 failed because it leads to Russell's Paradox. Following Frege, Zermelo and Russell defended the *axiom of infinity* which guarantees the existence of at least one infinite set, i.e. the set of natural numbers. The problem, however, is that the axiom of infinity cannot be proved via other axioms of ZFC Set Theory, and is therefore independent from the system. Such attempts were not as successful in reducing arithmetic to logic without any extra-logical elements involved. Poincaré was against the logicist tradition and thought that we need an extra-logical element. He argues that this extra-logical element is a form of intuition which cannot be found within the system and that mathematics is synthetic because of that reason. He claims in *Science and Hypothesis* that:

> No doubt we may refer back to axioms which are at the source of all these reasonings. If it is felt that they cannot be reduced to the principle of contradiction, if we decline to see in them any more than experimental facts which have no part or lot in mathematical necessity, there is still one resource left to us: we may class them among à priori synthetic views. (Poincaré, 1905, p. 2).

We may class them among a priori synthetic views because mathematical induction, or proof by recurrence, is at stake for every mathematical truth and the faculty of intuition provides us the ability of theoretically going ad infinitum which is necessary for proof by recurrence. It is also well-known that Poincaré was critical of the axioms of set-theory (Gray,1991), but more importantly for the purpose of this topic, of actual infinity. He states that:

> Now, as far as the second transfinite cardinal Aleph One, is concerned, I am not entirely convinced that it exists. One reaches it by considering the totality of ordinal numbers of the power Aleph Null; it is clear that this totality must be of a higher power. But the question arises whether it is self-contained, and therefore of whether we may speak of its power without contradiction. There is not in any case an actual infinite. (Poincaré, 1910).

His rejection of the actual infinity is mainly due to his theory of predicative and non-predicative definitions, and his argument that classifications concerning the elements of infinite collections must be predicative.

The overall aim of this project is to explain Poincaré's theory of intuition, his theory of predicativity and connect it with his rejection of actual infinity and his doubts on infinities bigger than Aleph Null. I wish to show what Poincaré means by intuition, how his views on such topics can be connected with his arguments about arithmetic intuition, and how can his views on such topics be connected with some other problems in mathematics, like paradoxes and antinomies. My primary sources for this project are his books *Science and Hypothesis* and *The Value of Science*, and his collections of articles *The Logic of Infinity* and *Mathematics and Logic*. In addition to Poincaré's own works, I use secondary literature to help me interpret Poincaré from different perspectives.

In Chapter One, I will explain what he means by intuition, show why he thinks that intuition is constitutive for arithmetic and explain its relation to mathematical induction. I also wish to show why Poincaré might think that mathematical induction is foundational for mathematics and why it cannot be reduced to "logical" elements and why we need it in the foundations. I will also show the ideas behind his argument that mathematics is synthetic a priori. In Chapter Two, I will present the theory of predicativity and show why he thinks that actual infinity does not exist. I will do so by explaining the difference between predicative and non-predicative definitions. I will also show the different presumptions between what Poincaré calls Cantorians and pragmatists, explain Cantor's diagonal argument and present Poincaré's argument on why the law of correspondence should also be predicative. In Chapter Three, I will connect Poincaré's ideas on intuition with his rejection of a completed infinity that is not formed via continuous succession. I will also consider whether Poincaré's views on intuition and infinity can illuminate some problems in contemporary philosophy of mathematics by illuminating some misconceptions. In this chapter, I also attempt to explain the reasons of Poincaré's dissatisfaction with axiomatic set theory, and the connection between that and the antinomies one faces in the study of infinity. At the end, I will reconsider Cantor's argument of the uncountability of the set of real numbers, and interpret it from a Poincaréan perspective.

Chapter 1: Poincaréan Intuition

My aim in this chapter is twofold. First, I wish to explain what Poincaré means by intuition when he claims that it is essential for mathematics. Second, I wish to show why he thinks that mathematics is synthetic a priori. These two claims are among the most critical parts of Poincaré's philosophy of mathematics. By explaining why he thinks so, I will pave the way for the rest of my project, i.e. to show why infinities bigger than the cardinality of natural numbers are not and cannot be constructed via intuition. I will also discuss Poincaré's circularity arguments on mathematical induction, and present my views on why he thinks that induction and intuition is essential for any part of mathematics.

1.1. Defining Poincaréan Intuition

Poincaré discusses the role of intuition in the foundation of mathematics in many of his writings. However, Poincaré's notion of mathematical intuition is fundamentally different from what is now called the "intuitionist program", and that is why he is sometimes called a pre-intuitionist or a semi-intuitionist. My aim in this chapter is to make clear what he means by *intuition* and to show intuition's role for him in the foundation of mathematics, since what he means by the term is not so obvious. Explaining what he means by "intuition" is necessary for my project, since I argue that it sets the ground for other parts of his philosophy of mathematics, including his rejection of actual infinity.

What is common to almost all of Poincaré's writings on philosophy of mathematics is his criticism of the logicist program, which claims that the arithmetic is reducible to logic and that number-theoretic truths are actually logical truths. His main claim is that intuition is crucial for mathematics, and some even interpret his claims in a stronger way— i.e. even if it had been successful in reducing arithmetic to logic, this would not show that mathematics does not require intuition (Folina, 1992). While talking about *intuition* and its role both in *Science and Hypothesis* and *Science and Its Value,* he does not make it clear what he means by the term. I wish to make it clearer.

According to Poincaré, we have several intuitions that we use in different sciences. He enumerates the types of intuition as:

> …first, the appeal to the senses and the imagination; next, generalization by induction, copied, so to speak, from the procedures of the experimental sciences; finally, we have the intuition of pure number, … which is able to create the real mathematical reasoning. (Poincaré, 1902, p. 20).

He argues that while the first two, appeal to the senses and the induction used in natural sciences, cannot give us certainty, intuition of pure number gives us the certainty that we have in mathematics. *Intuition of pure number*, as he calls it, is vague and its name might be misleading, since it is not about how we intuit the numbers, but more about the notion of continuity and repetition. For that, it is sometimes called "the intuition of indefinite iteration" (Folina, 1992). Intuition of pure number, or intuition of indefinite iteration, is closely connected for him with proof by recurrence (Poincaré, 1902) and mathematical induction. He gives a number of examples in the beginning of *Science and Hypothesis* of cases in which we use proof by recurrence. Then he gives a general definition as:

> The process is proof by recurrence. We first show that a theorem is true for n = 1; we then show that if it is true for n–1 it is true for n, and we conclude that it is true for all integers. (Poincaré, 1905, p. 11).

Proof by induction in mathematics is considered to be valid just like other proofs where every step should be justified. What is done, basically, is to show that the formula is true for the base case, e.g. n=0 or n=1, and assuming that the formula holds for an arbitrary number n=k, it holds for n=k+1. When this is done, the formula is said to be true for every natural number. Poincaré argues that:

> If we look carefully, we find this mode of reasoning at every step, either under the simple form which we have just given to it, or under a more or less modified form. It is therefore *mathematical reasoning par excellence,* and we must examine it closer. (Poincaré, 1905, p. 12).

Although the certainty of mathematical induction, or *proof by recurrence* as he calls it, is not to be debated, and it is actually the foundation of rigor in mathematics (Poincaré, 1902), this reasoning deserves more investigation. What is important here is that we are demonstrating the formula to be true for an arbitrary number n. It is not the case that we are trying every number separately to verify if the formula holds for them. What we are demonstrating is that if the formula is true for an arbitrary number, then it is also true for its successor. It is an arbitrary number because "n" stands for any number in the set, not a particular one. This set consists of numbers. This arbitrary number "n" can be substituted for any element in our set, and then we conceive that the formula holds for every number in the set. Because we imagine a number n and its successor n+1, and prove that the theorem stays true for the succession of these arbitrary numbers, we can say that the theorem or the formula holds for an infinite set, that is, the natural numbers. That is why Poincaré states that:

> To prove even the smallest theorem he [the mathematician] must use reasoning by recurrence, for that is the only instrument which enables us to pass from the finite to the infinite…it [reasoning by recurrence] contains, condensed, so to speak, in a single formula, an infinite number of syllogisms (Poincaré, 1902, p. 14).

It is this *passing from finite to the infinite* and that our ability conceive *an infinite number of syllogisms in a single formula* that gives mathematical induction its rigor and certainty.

For Poincaré, intuition of pure number is closely connected to proof by recurrence and mathematical induction. Having explained the importance of the reasoning by recurrence, we can look at why Poincaré considers this as an intuition. There are several interpretations of what intuition means for Poincaré. One, for example, is Goldfarb's interpretation. He claims that intuition is a *psychological* term. He argues that "If a mathematical proposition is convincing, that is, it seems self-evident to us, and the purported logical proofs of it are insufficient, then, tautologously, intuition in Poincare's sense is what is at work." (Goldfarb, 1988). It is well-known that Poincaré "seeks to defend intuition as an alternative foundation to logic for mathematics" (MacDougall, 2010, p. 139). Poincaré argues that we need proof by recurrence or mathematical induction at every step in the foundations of mathematics, and claims that these can be conceived via intuition alone. The reason why he argues this is that logic alone cannot justify or give us mathematical induction. Since he argues that there is an extra-logical element in mathematics which he calls intuition, he is against logicism, which claims that all mathematical truths are in fact truths of logic.

1.2. Why Is Mathematics Synthetic A Priori?

Although Poincaré is not considered as an intuitionist about mathematics, he is often referred as a *pre*-intuitionist or a *semi*-intuitionist. Poincaré was strictly against logicism. He claimed that the axioms which are at the heart of mathematical reasoning cannot be reduced to truths of logic, because they cannot be reduced to the principle of contradiction. He claimed that if mathematical views cannot be reduced to the principle of contradiction or to the truths of logic, “there is still one resource left to us: we may class them among a priori synthetic views.” (Poincaré, 1902, p. 2). Borrowing the terminology from Kant, Poincaré, along with him, claimed that mathematics is synthetic a priori. Although they both claim this, their reasons for that are quite different from each other. For Kant, space and time are a priori forms of intuition. Regarding the principle of mathematical induction, Folina explains/claims that:

> For Kant this principle would be synthetic because in considering an element together with its successor, we must employ our intuitions concerning succession. And these intuitions are not "analytic" because our intuitions concerning succession are not present in the concepts of number… but are only present in virtue of the a priori temporal form via which we understand the concepts. (Folina, 1992, p. 50).

Folina interprets Kant as he is emphasizing the role of intuition in our idea of succession while he is discussing the mathematical induction. It is known that in *Science and Hypothesis,* Poincaré adopts a more or less Kantian view about arithmetic; however, while sharing some similarities with the Kantian notion of intuition in general, Poincaré’s reasons for claiming that mathematics is synthetic a priori are not due to the reasons that Kant adopts in Folina’s interpretation. For Poincaré, mathematics is synthetic because in its foundation lies the principle of mathematical induction. And the principle of mathematical induction is synthetic because it is knowable only through our a priori arithmetic intuition. We *know* that a formula is true, when it holds for “0” and for S(n), or the successor of n, assuming that it holds for an arbitrary number n. Our ability to pass from finite to infinite, or consider n and S(n) as instantiations of the same concept, is a power of the mind. Regarding the principle of mathematical induction, Poincaré claims that:

> This rule, inaccessible to analytical proof and to experiment, is the exact type of the a priori synthetic intuition (Poincaré, 1905).

The claim that “mathematical induction is inaccessible to analytic proof and to experiment” needs further discussion, since Poincaré’s argument that mathematics is synthetic lies partially on this statement. He argues:

> We may readily pass from one enunciation to another, and thus give ourselves the illusion of having proved that reasoning by recurrence is legitimate. But we shall always be brought to a full stop—we shall always come to an indemonstrable axiom, which will at bottom be but the proposition we had to prove translated into another language. We cannot therefore escape the conclusion that the rule of reasoning by recurrence is irreducible to the principle of contradiction. Nor can the rule come to us from experiment. Experiment may teach us that the rule is true for the first ten or the first hundred numbers, for instance; it will not bring us to the indefinite series of numbers, but only to a more or less long, but always limited, portion of the series. (Poincaré, 1905, p. 15)

Intuition is the bridge that connects symbols and empirical world, and it is what makes mathematical truths meaningful. Although not only about mathematics, the below quotation shows the importance of the connection with the empirical reality for him:

> The eternal contemplation of its own navel is not the sole object of the science. It touches nature, and one day or other it will come into contact with it. Then it will be necessary to shake off purely verbal definitions and no longer to content ourselves with words. (Poincaré, 1905, p. 24)

Just like his argument on why science should not aim for solely coherent verbal definitions, mathematical statements should be meaningful too. This part is important in showing the ideas behind his rejection of a completed reality. Mathematics is meaningful and connected to the external world because it is synthetic; mathematical statements have certainty because they are a priori. If intuition is what constitutes the bridge between pure mathematics, or symbols and the empirical world, a mathematics that is not constituted via intuition would lack that connection.

1.3 Poincaré and Mathematical Induction

As is known by now, Poincaré argues that mathematics is synthetic a priori, because it involves a form of a priori intuition. This is what he calls arithmetic or mathematical intuition, and is closely connected with mathematical induction or proof by recurrence. Poincaré's main arguments for intuition are mainly circularity arguments against the logicist who claims that mathematics can be reconstructed without any "extra-logical" elements. These circularity arguments are not explicitly listed by Poincaré; however, we can find their traces in his writings. People claim that there are more than one circularity arguments in Poincaré's attack against logicism (Folina, 2006), I will interpret Poincaré's claims in light of modern number theory and set theory. According to Folina:

Circularity arguments make two fundamental claims:

1. In order to derive some non-trivial portion of mathematical theory, some mathematics must always be presupposed.

2. What is presupposed is not entirely arbitrary. At least some presupposed is intuitive, which means that it is built into the nature of the human (finite thinking) mind. (Folina, 2006, p.275).

Poincaré claims that:

> The principle of complete induction, they say, is not an assumption properly so called or a synthetic judgement a priori; it is simply the definition of whole number (Poincaré, 1905).

He claims that the logicist uses mathematical induction to define the number itself. What he means by that, as far as I understand it, is the mathematician needs to use mathematical induction to have the numbers, or even define the numbers. We can look at two ways of doing that. First is the famous *Von Neumann method* of constructing natural numbers in set theory.[1] Von Neumann ordinals basically goes like this:

Take 0 = { } as the empty set.

Define S(a) = a ∪ {a} for every set a, where S(a) is the *Successor* of a, and S is the *Successor Function.*

[1] Zermelo's method is similar to this in many ways considering my objective. Therefore, I will not mention about that.

Each natural number will be the set of all natural numbers less than it. The sets looks like this:

$0 = \{ \}$,

$1 = 0 \cup \{0\} = \{0\}$,

$2 = 1 \cup \{1\} = \{0, 1\}$,

$3 = 2 \cup \{2\} = \{0, 1, 2\}$,

$S(n) = n \cup \{n\} = \{0, 1, ..., n\}$ etc.

This way we can construct natural numbers in set theory. The problem, however, lies in its inductive nature. The mathematician needs to go ad infinitum to construct the natural numbers. However, the ability to go on, or the indefinite iteration would not be something that is inherent to the system for him. Therefore, if we accept that mathematical induction is intuitive, we also have to accept that we need intuition even to define numbers. Poincaré's emphasis on repetition, succession and induction can also be found in modern number theory, specifically in Peano axioms, by the repeated application of successor function. These axioms are one of the most accepted postulates in number theory, simply for their consistency in constructing natural numbers. In Peano's construction of natural numbers, the first axiom states that "1 is a natural number" and his postulate about the successor function states that "For every natural number n, S(n) is a natural number" (Peano, 1889). These two postulates show that natural numbers are closed under the successor function, i.e. the function always produces a natural number. When we put “1”, for example, to the successor function S(x), the result would be “2”, and it goes on this way. By repeating this process, we end up having the set of natural numbers. What I have said about Von Neumann method also applies to Peano axioms when we want to define the numbers this way.

Another circularity argument in Poincaré’s works, according to Folina (2006) is that a consistency proof is required for the induction without using the induction. Folina interprets Poincaré’s claims as this:

> If the new principles are disguised definitions of the new logical constants, then a consistency proof is required; but since the proof will need to presuppose the induction is true, any derivation of induction from the logical principles is circular. (ibid.)

To my interpretation, what she means by this is that the induction cannot be proved without using a form of inductive method. If we want to eliminate induction as an “extra-logical” element, we have to prove induction from the logical elements that we have, and these should not include induction. So what one is supposed to do is make a consistency proof for induction without using induction. However, it is not possible. One has to assume that induction is true to derive induction from logical principles. Therefore, proving induction would be circular.

In the next chapter I will explain why Poincaré rejects actual infinity and infinities bigger than Aleph Zero. I will do so by summarizing his theory of predicative and non-predicative definitions, and show the reasons why he rejects actual infinity. In connection to this, in the third chapter, I will argue that Cantorian infinities which have a bigger cardinality than natural numbers cannot be formed via arithmetic intuition. These two parts of his work, i.e. predicative definitions/classifications and intuition, are usually seen as distinct parts within

Poincaré's philosophy of mathematics. I argue that they are not, and that his ideas on intuition should lead him to reject actual infinity as well.

Chapter 2: Poincaré and Transfinite Cardinals

Poincaré starts his famous essay "The Logic of Infinity" by asking two important questions:

> 1. Do the ordinary rules of logic apply without change when we consider collections comprising an infinite number of objects?
>
> 2. Do the contradictions that mathematicians who specialize in the study of infinity encounter arise from the fact that the rules of logic have been incorrectly applied, or from the fact that these rules cease to be valid outside of their proper domain, i.e. the collections formed only of a finite number of objects? (Poincaré, 1909, p. 45)

After posing these questions, he argues "that the classification which is adopted be immutable" (ibid.) for the rules of logic to be valid. He claims that the antinomies arise from the violation of this condition, because mathematicians depend on a classification which is not immutable. Although these classifications look as if they are immutable, and people may claim that they are immutable, he claims that the classification which was relied on should be immutable in fact. In order to show why an infinite collection would not check for his immutability condition, he makes a distinction between predicative definitions/classifications and non-predicative ones.

2.1. Predicative & Non-Predicative Classifications

To show the distinction between predicative and non-predicative classifications Poincaré puts a case in front of us. He prompts us to ask a question:

> What is the smallest integer which cannot be defined by a sentence with fewer than one hundred French words? And furthermore does this number exist? (Poincaré, 1909, p. 46)

There are two possible answers to this question according to him. Here are the answers:

2. Yes, it exists. For with one hundred words we can constitute a finite number of sentences. Among these sentences, there will be ones with no meaning or that do not define an integer. But among the ones that they do, each sentence should be capable of defining at most one single integer. And the number of integers capable of being defined this way is limited. And among these integers, there would be one which is smaller than the others (because of the well-ordering). By this reasoning we can find the smallest integer if we list all the sentences that define an integer.

3. No, it does not exist. Because it implies a contradiction. If the number exists, it can be defined by a sentence with fewer than one hundred words, that sentence being “the smallest integer which cannot be defined by a sentence with fewer than one hundred words”. With this definition, it can be defined by a sentence with fewer than one hundred words. Hence, a contradiction.

According to Poincaré, this reasoning rests on a classification: Integers which can be defined with one hundred French words, and integers that cannot be defined so.

He argues that this way we claim that the classification is immutable. But it is not the case. He argues that:

> The classification can be conclusive only after we have reviewed all the sentences with fewer than one hundred words, when we have rejected those which have no meaning, and when we have definitively fixed the meaning of those which possess a meaning (Poincaré, 1909, p. 46).

The problem is, however, there will be some sentences which can have meaning only after we fix the classification. In sum, he says:

> The classification of the numbers can be fixed only after the selection of the sentences is completed, and this selection can be completed only after the classification is determined, so that neither the classification nor the selection can ever be terminated. (ibid.).

Such difficulties are even more apparent when we consider infinite collections. And this reasoning lies under Poincaré's rejection of the actual infinity. He argues in the same paragraph:

> There is no actual infinity, and when we speak of an infinite collection, we understand a collection to which we can add new elements unceasingly…For the classification could not properly be completed except when the list was ended; every time that new elements are added to the collection, this collection is modified; it is therefore possible to modify the relation of this collection with the elements already classified; and since it is in accordance with this relation that these elements have been arranged in this or that drawer, it can happen that, once this relation is modified, these elements will no longer be in the correct drawer and that it will be necessary to shift them. As long as there are new elements to be introduced, it is to be feared that the work may have to be begun all over again; for

> it will never happen that there will not be new elements to be introduced; the classification can therefore never be fixed. (Poincaré, 1909, p. 47)

By this reasoning Poincaré draws a distinction concerning the classification applicable to the elements of infinite collections: i.e. predicative and non-predicative classifications. Where the elements of a set ordered with predicative classification cannot be disordered when the new elements are introduced, this is not the case for non-predicative classifications: the items that non-predicative definitions pick out necessitate constant modification when new elements are introduced (Poincaré, 1909, p. 47).

Imagine that you are classifying integers into two: those that are greater than 10 and those that are less than 10. Adding new elements to the collection will not require modification in the classification. The first classification might be concerning the first 100 integers, and you might add 101 to the collection and this adding does not change where the first 100 integers stand in relation to our classification. This classification, according to Poincaré, is *predicative.* Whereas a classification of points in space might be different:

> Let us imagine that we want to classify the points in space and that we differentiate between those which can be defined in a finite number of words and those which cannot. Among the possible sentences there will be some which will refer to the entire collection, that is, to space or else to some portions of space. When we introduce new points in space, these sentences will change in meaning, they will no longer define the same point: or they will lose all meaning; or else they will acquire a new meaning although they did not have any previously. And then points which were not definable will become capable of being defined, others which were definable will cease to be definable. (Poincaré, 1909, p. 47).

This type of classification, according to Poincaré, is *non-predicative.* Because by the introduction of new integers or new points, the classification will change. But according to Poincaré, it is not enough that the classification is not changed for it to be predicative. He says, "from a certain point of view, we should not say that a classification is predicative in an absolute manner, but that it is predicative in relation to a method of definition." (Poincaré, 1909, p. 49). He means that by the method of definition, whether a classification is predicative or not might change. We know that he thinks a classification should be immutable, and elements of non-predicative definitions necessitate constant modification to be in accord with the method of classification. However, I believe Poincaré argues here that, although a classification might seem as predicative, it does not mean that it is predicative in an absolute manner. Rather, he means that the type of classification is relative to the method of definition. Overall, what he argues is that one should avoid non-predicative classifications and definitions, for the elements classified that way require constant modification. He also argues that impredicative definitions are viciously circular. I will discuss Poincaré's vicious circularity arguments in the next chapter related to his criticisms of axiomatic set theory.

2.2. Cantor's Diagonal Argument and the Law of Correspondence

Cantor uses his infamous diagonal method to show that the cardinality of real numbers is bigger than the cardinality of natural numbers, or integers. This applies not only to integers and real numbers, but to any set and its power set. I will begin by explaining Cantor's diagonal *proof*, and then explain Poincaré's argument that the law of correspondence that Cantor uses is not predicative.

Cantor's diagonal argument shows us that the integers and real numbers cannot be put to one-to-one correspondence. To have the same cardinality, two sets should be put to one-to-one correspondence, i.e. a bijection. Cantor shows that the integers and the real numbers cannot be put into one-to-one correspondence. When we map the integers with the real numbers, for each integer that is in one-to-one correspondence with real numbers, there will still be infinite number of elements left in the set of real numbers. From this, Cantor concludes that the infinity of real numbers is different than the infinity of integers. These two sets, i.e. real numbers and integers, have different cardinalities, and the cardinality of real numbers is bigger than the cardinality of integers. Both of these sets are considered infinite sets, and with this argument Cantor shows that there are different infinities with different cardinalities. The cardinality of integers, and natural numbers are called "Aleph-zero", and the cardinality of real numbers is called "the continuum". Poincaré discusses the law of correspondence that Cantor uses in the form of "one-to-one" correspondence. I explain this correspondence below.

In the second chapter of "The Logic of Infinity", Poincaré argues that the law of correspondence that Cantor uses to show that real numbers and integers have different cardinalities should be predicative as well. He argues that the law of correspondence between the points in space (continuum, real numbers, or whatever you might say) and the integers is not predicative. Suppose that you are comparing the set of integers to the points in space capable of being defined by a finite number of words. We establish a correspondence between them in this way:

> I shall list all possible sentences. I shall arrange them according to the number of words in them, placing in alphabetical order those which have the same number of words. I shall erase all those which have no meaning or which do not define any point, or which define a point already defined by one of the preceding sen-

> tences. To each point I shall have correspond the sentence which defines it, and the *number* which represents the position of this sentence in the revised list. (Poincaré, 1909, p. 50).

When you introduce new points to the set, i.e. the points in space, where the points correspond with the sentences that define some sentences which had no meaning before may acquire a meaning, and you would have to replace them in the list with the ones you erased them at first. This way, the sequence number of the other sentences will change. The correspondence will entirely change. It means that our law of correspondence in this case would not be predicative. Following his belief that the method of classification that we adopt should be predicative, Poincaré argues that it is necessary to modify the definition of cardinal numbers that are proposed by Cantor. He argues that we must specify the law of correspondence that we use to define cardinal numbers in a way that it would be predicative. (Poincaré, 1909, p. 50).

2.3. Presumptions of Cantorians & Pragmatists[2]

Poincaré considers two opposite tendencies when discussing on infinity. He claims that for some people infinity is derived from the finite. For these people infinity consists of possible finite things. In other words, the finite precedes the infinite. For the other mathematicians, infinity precedes the finite, the finite is "obtained by cutting out a small piece from infinity" (Poincaré, 1905b). He names the first group *pragmatists,* and the second group of people *Cantorians.* Poincaré gives this bit of an imaginary dialogue between the pragmatist and the Cantorian

[2] These terms, *Cantorians* and *pragmatists* are used by Poincaré himself in "Mathematics and Logic".

to show their attitudes towards Zermelo's transformation of space into a well-ordered set:

> Let us take Zermelo's theorem according to which space is capable of being transformed into a well-ordered set. The Cantorians will be charmed by the rigor, real or apparent, of the proof. The pragmatists will answer:
>
> — You say that you can transform the space into a well-ordered set. Well! Transform it!
>
> — It would take too long.
>
> — Then at least show us that someone with enough time and patience could execute the transformation.
>
> — No, we cannot, because the number of operations to be performed is infinite; it is even greater than aleph zero. (Poincaré, 1905b)

We get a hint at Poincaré's thoughts about Zermelo's argument that space is capable of being transformed into a well-ordered set. He, adopting the point of view of the pragmatist, implies that the Cantorian does not have a sufficient way of demonstrating the so called transformation. Cantorians give us the argument that with an infinite number of operations, the space can be transformed into a well-ordered set, and they say that the infinity mentioned here is bigger than the infinity of integers, or aleph-zero. The main reason for Cantorians to adopt this point of view is, according to Poincaré, that it is imaginable, or one can comprehend this. He states that "The pragmatists adopt the point of view of extension, and the Cantorians the point of view of comprehension" (Poincaré, 1905b, p. 67). This is where the presuppositions of Cantorians and pragmatists come into play. Poincaré argues that from the point of view of the comprehension, we begin by accepting the notion that there are pre-existing objects. These objects exist before the act of the mathematician, and we recognize these objects by labelling them and “arrange them in drawers” (Poincaré, 1905b). The drawers he mentions are the sets. Since the Cantorian thinks that mathematical objects exist before the act of the mathematician, the objects between 0 and 1, or the objects spanned within real numbers are infinite. This infinity is the second type of infinity, or the continuum, according to Cantor and his followers. However, pragmatists do not begin with the presumption of the independent reality of mathematical objects. If we adopt the point of view of extension, according to Poincaré, “a collection is formed by the successive addition of new members; we can construct new objects by combining old objects, then with these new objects construct newer ones, and if the collection is infinite, it is because there is no reason for stopping” (Poincaré, 1905b).

Poincaré argues that for the pragmatist there can only be objects which can be defined in a finite number of words, he says that “the possible definitions, which can be expressed in sentences, can always be numbered with ordinary numbers from one to infinity” (Poincaré, 1905b). This reasoning allows you to accept only one type of infinity, i.e. aleph-zero. The reason why is that it is only a potential infinity, that goes up by successive addition. You number the sentences that are made up of finite number of words, and this numeration goes from one to infinity. There would be a one-to-one correspondence between this and natural numbers. Hence, only aleph-zero. Then in the same paragraph he asks, “Why then do we say that the power of the continuum is not the power of the integers?”. Since there are no objects in the continuum that cannot be defined by a finite number of words, according to pragmatist, the power of the continuum will be same with the power of the integers. Hence, no actual infinity in the Cantorian sense for the pragmatist.

Another reason why the pragmatist denies the existence of objects which could not be defined in a finite number of words, according to Poincaré, is that they believe objects exist when they are conceived by the mind. As opposed to the Cantorian who accepts that mathematical objects pre-exist the mathematical activity, we can say that the pragmatist position is closer to the anti-realist about mathematical objects. Poincaré argues that pragmatists "believe that an object exists only when it is conceived by the mind and that an object could not be conceived by the mind independently of a being capable of thinking." (Poincaré, 1905b, p. 72). This mind capable of thinking is nothing other than the rational person, or something that resembles it, and it is a finite being. Since this is the case, for the pragmatist, "infinity can have no other meaning than the possibility of creating as many finite objects as we wish." (Poincaré, 1905b).

Chapter 3: Poincaré's Philosophy of Mathematics

In the final chapter, I would like to present my arguments on Poincaré's philosophy of mathematics, and to show that the two seemingly distinct branches of his philosophy is not distinct at all. Rather, I argue that Poincaré's views on intuition and his arguments on how intuition is foundational and necessary for mathematics should lead him to deny the actual infinity, the existence of the continuum, Aleph-One or bigger transfinite cardinals. Then, I will evaluate the potential reasons why Poincaré was dissatisfied with axiomatic set theory, and show the relation of the paradoxes with impredicative definitions. Lastly, I will elaborate on Cantor's argument that the real numbers are uncountable. I will do so in a Poincaréan framework.

3.1 Intuition and Actual Infinity

We have seen Poincaré's argument on the foundations of mathematics. Recall that according to Poincaré we need mathematical induction at every step in the mathematics, because it is foundational. He claims that mathematics itself does not provide us a basis for mathematical induction. Rather, the ability to continue is a power of the mind, so it can be conceived via what he calls "intuition" alone. In *Science and Hypothesis,* he claims that:

> "We may readily pass from one enunciation to another, and thus give ourselves the illusion of having proved that reasoning by recurrence is legitimate. But we shall always be brought to a full stop—we shall always come to an indemonstrable axiom, which will at bottom be but the proposition we had to prove translated into another language. We cannot therefore escape the conclusion that the rule of

> reasoning by recurrence is irreducible to the principle of contradiction. Nor can the rule come to us from experiment. Experiment may teach us that the rule is true for the first ten or the first hundred numbers, for instance; it will not bring us to the indefinite series of numbers, but only to a more or less long, but always limited, portion of the series." (Poincaré, 1905, p. 15)

He argues that we always come to a full stop, an indemonstrable axiom etc. when we try to prove reasoning by recurrence with the tools of logic. He argues that we need intuition for mathematics. This point brings us to the point of discussion that I want to have. His later arguments in his article "The Logic of Infinity" do not deal with his departing points about intuition. One might say that his aim is different in "The Logic of Infinity", since his main point in the article is the difference between predicative and non-predicative definitions. By introducing this distinction, he also brings up a new light to a paradox discussed by Russell, which is "the smallest integer which cannot be defined with fewer than one hundred words". He talks about how the elements ordered by non-predicative classifications need constant modification, and they always can be disordered. Then, he argues against non-predicative classifications. In the article, his famous passage about the actual infinity is this:

> There is no actual infinity, and when we speak of an infinite collection, we understand a collection to which we can add new elements unceasingly…For the classification could not properly be completed except when the list was ended; every time that new elements are added to the collection, this collection is modified; it is therefore possible to modify the relation of this collection with the elements already classified; and since it is in accordance with this relation that these elements have been arranged in this or that drawer, it can happen that, once this relation is modified, these elements will no longer be in the correct drawer and that it will be necessary to shift them. As long as there are new elements to be in-

troduced, it is to be feared that the work may have to be begun all over again; for it will never happen that there will not be new elements to be introduced; the classification can therefore never be fixed. (Poincaré, 1909, p. 47)

He clearly rejects actual infinity, on the grounds that every time new elements are added, the set is disordered, and it needs to be modified. Because elements ordered by non-predicative classifications necessitate constant modification; after the modification, an element might be "in the wrong drawer" and it might be necessary to shift the elements. He argues, the classification might never be fixed. So far we don't see any reasoning about his theory of intuition in his denial of actual infinity. However, the connection becomes more clear in his other late article "Mathematics and Logic". While discussing different approaches to infinity, he hints us that he is closer to the pragmatist, as discussed in 2.3. He argues one way to look at infinity, the pragmatist's way, is this:

> a collection is formed by the successive addition of new members; we can construct new objects by combining old objects, then with these new objects construct newer ones, and if the collection is infinite, it is because there is no reason for stopping (Poincaré, 1905b).

Infinity, according to the pragmatist, and according to Poincaré is closely connected with successive addition. By adding new members you construct a new object from the older ones. This might be the text-book definition of a potential infinity where infinity is not completed, and where it increases by successive addition. Now recall that Poincaréan intuition was about continuity and iteration. Intuition provides us the power to go on without stopping and this power does not belong to the system [logic] itself. From these remarks, we can see that the infinity constructed by successive addition is the one where the intuition is at stake. The infinity of natural numbers, or Aleph-Zero is constructed like this, whereas we come to the actual infinity by cutting out small pieces from an already existing object. We lose the continuity and iteration when we come to actual infinity. We might even say that from a Poincaréan point of view, actual infinity has no connection to intuition whatsoever.

3.2. Paradoxes and Set Theory

It is well known that Poincaré was critical about set theory, particularly modern set theory. He criticized logicists because he saw their attempt as unsuccessful, however, his main target in set theory was Zermelo's axiomatic set theory. It is to be debated whether he actually stated the famous words ascribed to him, the words where he brutally criticizes set theory, "Later generations will regard Mengenlehre (set theory) as a disease from which one has recovered". Some (Gray, 1991) argue that he did not say that, but in each case Poincaré was heavily dissatisfied with the axiomatization of set theory. Considering aforementioned views of Poincaré, it is more than normal for him to criticize ZFC set theory, where the existence of at least one infinite set is guaranteed with *axiom of infinity*. The problem is, however, not that ZFC guarantees the first infinite set, i.e. natural numbers, *power set axiom* guarantees that a set's power set has a bigger cardinality than the set itself. It is proven with, again, Cantor's diagonal argument. If the existence of the first infinite set is given with axiom of infinity, then with the power set axiom, we should say that the P(N) [the power set of N] has a bigger cardinality than N. Given Poincaré's criticism of Cantor's diagonal argument and the impredicativity of the law of correspondence, it is more than usual for him to be critical of set theory. Another point is the *axiom of choice*, where the well-ordering of any set is guaranteed with an axiom. If such is the case, then one can say that the points in space [continuum, actual infinity] can be well-ordered, which is problematic given Poincaré's views on the subject. It is problematic because the elements of a completed infinity, the set of real numbers etc., require constant modification even when the classification is fixed. If such is the case, it is going to be a problem to *well-order* the elements of such a collection. Recall that, Poincaré means a "collection which we can add new elements unceasingly" when he is discussing about infinity. If those elements are already there, it is not possible to well-order them without specifying what these elements are. The next issue I want to discuss is the paradoxes.

Paradoxes of the set theory were seen as inevitable by some, and as a sign that set theory was problematic by the other, including Poincaré. His answer to the paradoxes like Russell's was to exclude non-predicative definitions [definitions involving only a finite number of words] in his own terms. He barred the paradox that "the smallest integer which cannot be defined with fewer than one hundred words" this way (Gray, 1991). Poincaré attempted at solving that paradox by excluding non-predicative definitions, however, the kind of paradoxes that I would like to mention here are of a more specific nature. I would like to talk about the paradoxes including transfinite cardinals. Paradoxes like Burali-Forti, Zermelo-König and Richard's are maybe the ones where the inadequacy becomes most obvious. Poincaré mentions those antinomies and deals with them in his 1906 lectures (Ewald, 1996).

I don't have enough space to explain those paradoxes in a detailed way; however, these paradoxes, especially Richard's lead people to accept predicativism. This, in my opinion, is another reason why Poincaré was critical of set theory. Considering his insistence on excluding non-predicative definitions and allowing for only predicative ones, set theory should be a field where immutability of mathematics disappears since impredicative definitions are common in modern set theory.

One of Poincaré's most famous objections against impredicative definitions is that they are viciously circular. The circularity, in my opinion, comes from the fact that while we are trying to define an object with an impredicative definition, we already presuppose its existence. Poincaré was not alone at this project. Russell (1908) also argues against impredicative definitions and presents the *Vicious Circle Principle:*

> Whatever involves all of a collection must not be one of the collection.
>
> If, provided a certain collection has a total, it would have members only definable in terms of that total, then the said collection has no total. (Russell, 1908, p.225).

Paradoxes, according to Poincaré and Russell were the results of vicious circularity, caused by impredicative definitions. In some of them the circularity is more obvious, like Liar's Paradox, but in most of them it is more implicit. I believe that the paradoxes, or antinomies, consisting of transfinite numbers generally fall into the latter category. Most of them actually consist impredicative definitions and classifications implicitly. One should keep that in mind whenever they face an antinomy consisting of transfinite numbers, and impredicative definitions.

3.3. Cantor's Argument Revisited

We should recall Cantor's argument that the set of real numbers is uncountable to interpret it in a Poincaréan framework. For the purposes, I will reconstruct Cantor's proof:

First, take that N=1, 2, 3,… and X is a set of all real numbers between (0,1)

Cantor's theorem (1890) tells us that X is uncountable. To prove that he uses *reductio ad absurdum* with his famous diagonal method that I mentioned before.

Assume that X is countable. Then there is an enumeration of all real numbers in X.

Let

$x_1, x_2, x_3, \ldots$

be some arbitrary enumeration of all real numbers in X. Then, there is a bijection from the elements of X to all elements of N.

Applying his diagonal method, Cantor constructs an anti-diagonal which differs from every enumeration in the first application. Consequently, the given enumeration is not an enumeration of all real numbers from the set X.

Therefore, we face a contradiction.

Hence, the assumption "X is countable" is false.

The proof looks good at the first instance. However, Zenkin argues that Cantor has a hidden necessary condition in his argument. I think he is right. He argues that the hidden necessary condition is the actuality of X. His *corrected* version of Cantor's proof is:

If X is *actual,* then it is *uncountable.* (Zenkin, 2005, p. 7)

It looks like we come to the point where we were discussing from the beginning. Whether a completed, actual infinity exists. To check for Cantor's corrected proof, one has to show that X [actual infinity] exists, or is *actual*.

Poincaré argues that Cantorians fail to show the actuality of X. Then, Cantor's argument turn into a conditional which is "valid only within the framework of the Cantor's paradigm of the actualization of all infinite sets" (Zenkin, 2005, p.7). Cantor's proof, then, does not prove anything in a Poincaréan framework. To accept the proof, we should accept the actualization of *X* first. If we look at the proof with Poincaré's argument on actual infinity, how it is formed with a non-predicative classification, and how the construction of actual infinity breaks the ties with intuition, it becomes more than hard for one to accept the validity of Cantor's proof.

Poincaré argues that there are two ways to deal with infinity as I mentioned in 2.3. Where one group, Cantorians, see infinity as pre-existing and we come to smaller infinities by cutting out bits from the first one; the other group, pragmatists, believe that infinity is formed via successive addition and accept Aleph-Zero only. At this point, it almost becomes a matter of belief. You might accept Cantor's proof, even though it requires you to believe in actual infinity beforehand; or you might deny actual infinity, like Poincaré, and disregard Cantor's proof because it is a conditional. If you deny the existence of actual infinity [or X in the reconstructed proof], you do not have to accept that X is uncountable. The argument turns into two premises where there is not a conclusion:

1. If X is actual, then X is uncountable
2. X is not actual [from a Poincaréan perspective].

Conclusion

In the history of the philosophy of mathematics Poincaré is usually known with his rejection of logicism and his theory of predicativism. Being a great mathematician and a physicist, he is less known about his writings in philosophy. What I wanted to do in this project was to show that Poincaré was also a great philosopher. I mainly focused on his philosophy of mathematics and tried to present his arguments to the reader. While doing that I also tried to show my position and interpret him authentically. In the concluding chapter, I want to give a brief summary of what I did in this project and also why I did that.

I hope it is clear by now why Poincaré was insisting on the idea that intuition is foundational for mathematics. Intuition, for Poincaré, mainly is the power to go on, and to pass from the finite to the infinity. It is most obvious in the case of mathematical induction. We can conceive that a rule applies to an infinite set, because we can theoretically go ad infinitum and see that the formula is indeed true. Because induction is not reducible to any other logical principle [e.g. principle of contradiction], and because induction is at stake at every part of mathematics, Poincaré classifies mathematics as *a priori synthetic*. I showed why he thinks that in the first chapter and presented why I take him to argue that induction is not reducible to any other principle.

The first part of my thesis was about Poincaré's theory of intuition (or Poincaréan intuition). In the second part I explained Poincaré's theory of predicativity and showed what he thinks about Cantor's argument and about the people who argue for the existence of transfinite numbers bigger than Aleph-Zero. He argues that we should eliminate non-predicative definitions and classifications from our system, because our criteria should be immutable when grouping elements of sets according to a criterion. He also argues that the law of correspondence that Cantor uses in his diagonal proof is non-predicative. Because of all these, Poincaré argues against the existence of actual infinity. In the final chapter, I looked at Poincaré's philosophy of mathematics as a whole. I argued that although seen as different theories (intuition, and predicativity) Poincaré's views on intuition should lead him to deny the actual infinity. Successive addition, continuity, and induction are key elements in Poincaré's philosophy, and cutting the ties with those would be cutting the ties with the intuition which should be at the core of mathematical reasoning. I also showed what Poincaré thinks about set theory and the inevitable paradoxes within set theory. I argued that Poincaré was dissatisfied with set theory because of the fact that it allows for non-predicative definitions, and those definitions are viciously circular. At the end I argued that, following Zenkin, Cantor's argument has a hidden premise which presupposes the actuality of actual infinity. Within a Poincaréan framework, this presupposition is problematic, and if you deny the actuality of actual infinity, you have every reason to be suspicious about Cantor's proof of the sets with bigger cardinalities.

There are important points in Poincaré's philosophy of mathematics. He argues that:

> In my view an object is only thinkable when it can be defined with a finite number of words. An object that is in this sense finitely definable, I shall for brevity call simply 'definable'. Accordingly, an undefinable object is also unthinkable. (Poincaré, 1910, p. 1072)

We should keep this in mind when doing mathematics. Poincaré has very strong reasons to argue that the objects should be definable with a finite number of words, and to eliminate the objects which cannot be definable that way. How do we define a completed, actual infinity? How do we define the objects, if they exist beforehand, within the actual infinity? Following this remark, I would like to quote Poincaré again to emphasize this point:

> Now, as far as the second transfinite cardinal X, is concerned, I am not entirely convinced that it exists. One reaches it by considering the totality of ordinal numbers of the power Aleph Null; it is clear that this totality must be of a higher power. But the question arises whether it is self-contained, and therefore of whether we may speak of its power without contradiction. There is not in any case an actual infinite.

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