Extremal problems for paths and cycles

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For my mother

It is difficult and often impossible to judge the value of a problem correctly in advance; for the final award depends upon the gain which science obtains from the problem. Nevertheless we can ask whether there are general criteria which mark a good mathematical problem. An old French mathematician said: "A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street." This clearness and ease of comprehension, here insisted on for a mathematical theory, I should still more demand for a mathematical problem if it is to be perfect; for what is clear and easily comprehended attracts, the complicated repels us.

David Hilbert, Mathematical Problems,
International Congress of Mathematicians,
Paris, 1900.

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Abstract

A classical result of Erdős and Gallai determines the maximum number of edges in a simple n vertex graph without a path of given length as a subgraph, i.e. they determined Turán number of paths. They also determined Turán number of a class of long cycles. In this dissertation, we extend those results for Hypergraphs. We follow one of the most general definitions of paths and cycles in hypergraphs. A Berge-path of length k in a hypergraph $\mathcal{H}$ is a sequence $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1}$ of distinct vertices and hyperedges with $v_{i+1} \in e_i, e_{i+1}$ for all $i \in [k]$. Berge-cycles are defined similarly. We study several generalizations of Erdős-Gallai theorem for hypergraphs forbidding Berge families of paths and cycles and some related problems.

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Notations, symbols

Functions

$\chi(G)$ Chromatic number of G.

$\mathrm{ex}(n,H)$ The maximum number of edges in an n-vertex H-free graph.

$\mathrm{ex}(n,H,F)$ $max\{\mathcal{N}(G,F) : G \subseteq K_n,\ H \not\subseteq G\}$

$\mathrm{ex}^c(n,T)$ Extremal function for vertex colored graphs.

$\mathrm{ex}^{conn}(n,H)$ The maximum number of edges in an n-vertex H-free connected graph.

$\mathcal{N}(G,F)$ The number of sub-graphs of G isomorphic to F

$\partial_k(\mathcal{H})$ k-shadow of a hypergraph $\mathcal{H}$, i.e. all k-sets contained in a hyperedge of $\mathcal{H}$.

$E(G)$ The edge set of graph / hypergraph G.

$e(G)$ The number of edges of graph / hypergraph G.

$f_s(n,k,a)$ $\binom{k-a}{s} + (n-k+a)\binom{a}{s-1}$.

$G_1 + G_2$ A graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{(v,u) : v \in V(G_1), u \in V(G_2)\}$.

$G_1 \cup G_2$ Disjoint union of graphs, i.e. it is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

$N(G,F)$ The number of sub-graphs of G isomorphic to F.

$t_r(n)$ The number of edges of Turán graph $T_r(n)$ i.e. $t_r(n) = e(T_r(n))$

$V(G)$ The vertex set of graph / hypergraph G.

$v(G)$ The number of vertices of graph / hypergraph G.

General

$[n]$	$\{1,2,3,\cdots,n\}$.
$\mathbb{1}_A(\cdot)$	Indicator function, $\mathbb{1}_A(x)=1$ if $x\in A$ and $\mathbb{1}_A(x)=0$ otherwise.

Graphs

$\mathcal{C}$	The family of all cycles, $\{C_3,C_4,\dots\}$.
$\mathcal{C}_{\geq k}$	The family of cycles of length at least k, $\{C_k,C_{k+1},\dots\}$.
$\mathcal{C}_{odd}$	The set of all cycles of odd length, i.e. $\{C_3,C_5,\dots\}$.
$\mathcal{S}_n^{(r)}$	The n-vertex r-uniform hypergraph with $n-r+1$ hyperedges all intersecting in the same $r-1$ set.
C_n	A cycle with n vertices.
$G_{n,k,s}$	$\left(K_{k-2s}\cup\overline{K}_{n-k+1}\right)+K_s$.
K_n	The complete graph with n vertices.
$K_{n,m}$	Complete bipartite graph, with partite sets of size n and m.
K_n^r	The complete r-uniform hypergraph on n vertices.
P_n	A path of length n, i.e. $n+1$-vertex path.
$T_r(n)$	Complete r-partite Turán graph with n-vertices.

Chapter 1

Introduction

> Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians
>
> *Extremal Graph Theory* [14], Béla Bollobás

Extremal combinatorics is a branch of discrete mathematics that studies the maximum or the minimum size of discrete structures under given constraints. For example, a classical question studied by Mantel 1907 asks, 'What is the maximum number of edges that a triangle-free graph can have?', see Theorem 1.1.4. Thus, extremal combinatorics can be thought of as solving certain optimization problems, and as such has many real-world applications. Extremal combinatorics, as one might guess, is about extreme behaviors therefore it fascinates a wide range of audiences. In particular, since we are all naturally fascinated by everyday objects that are extreme for various properties like speed, color, size, or time in much the same way extremal objects in mathematics are inherently fascinating. In addition to the natural beauty, extremal combinatorics is a useful tool for other fields of mathematics, even more, extremal combinatorial problems encourage elegant mathematics that uses a variety of techniques from different fields of mathematics.

In some sense, the first extremal result in graph theory was by Euler [40] in 1758, when he showed that the maximum number of edges in a planar graph is at most $3n-6$. After a couple of centuries in 1907 Mantel determined the maximum number of edges in a triangle-free graph. While working on a problem of number theory, Erdős maximized the number of edges in an n-vertex graph without 4-cycles in 1938 [29]. Surprisingly, after obtaining this result Erdős did not discover extremal graph theory as a research subject as he said. Extremal combinatorics as a research subject was not discovered until 1941 when Turán determined the maximum number of edges in an n-vertex graph without a complete graph of a given size, Theorem 1.1.5. This result is counted as the birth of extremal combinatorics, naturally, we refer to the extremal number of a graph as the Turán number of that graph. Soon after Turán's Theorem 1.1.5, Erdős and Stone in 1946, later Erdős and Simonovits in 1966 determined the asymptotic behavior of Turán function for all non-bipartite graphs. On one hand, it seems they have settled the majority of the problems but in the case of a bipartite graphs Turán function is hard to determine and in some cases, we do not even know the order of magnitude.

This phenomenon having multiple results without discovering the subject repeats with so-called Generalized Turán numbers also. If the Turán number of a graph counts the maximum number of edges under some constraints, the generalized Turán number counts the maximum number of given substructures under the same constraints. In particular, in 1949 Zykov [117] determined the maximum number of cliques of a given size in an n vertex graph without cliques of larger size, since this result was about linear complexes. The same result was done later by Erdős [24], independently. While Erdős was measuring how far are the triangle-free graphs from bipartite graphs he naturally asked a question 'What is the maximum number of pentagons in a triangle-free graph' [26]. This question was settled half a century later by two groups of mathematicians using flag algebras. In 1991, Győri, Pach, Simonovits [76], defined the generalized Turán number and obtained some results. In particular, they maximized copies of a bipartite graph with 1-factor in triangle-free graphs. While investigating pentagon-free 3-uniform hypergraphs Bollobás-Győri [15] initiated the study of the converse of the problem of Erdős. They asked the following question 'What is the maximum number of triangles in pentagon-free graphs'. Surprisingly this question is still open but we show a simple proof of an upper-bound in Chapter 3. After a decade Alon, Shikhelman [3] defined the function $\mathrm{ex}(n,H,F)$, and obtained some general results after which the generalized Turán number gained a wide range of interest.

Extremal questions for hypergraphs are even harder. Lately, numerous mathematicians are investigating the Turán number for hypergraphs. In here there are lots of questions and lots of open problems the majority of which are hard. The underlying theme of this work is investigating variants of the Turán extremal problem for Berge-hypergraphs. In this work, we determine extremal numbers of various kinds of paths and cycles in hypergraphs in the spirit of the Erdős-Gallai theorem. In some sections, we prove some results for graphs which are useful tools for getting results for hypergraphs.

1.1 The Turán number

Extremal Graph theory is a branch of discrete mathematics that studies relations between graph invariants. In particular, how global parameters, such as the number of edges, can influence local substructures. One of the most studied problems of this field is to determine the Turán number of a graph.

Definition 1.1.1. *The Turán number of a graph H,* $\mathrm{ex}(n,H)$*, is the maximum number of edges in a simple graph on n vertices which does not have H as a sub-graph. In particular,*

$$\mathrm{ex}(n,H) = max\left\{e(G) : G \subseteq K_n \text{ and } H \nsubseteq G\right\}.$$

For a given graph H, all n-vertex H-free graphs with $\mathrm{ex}(n,H)$ edges are *extremal graphs*. Let us denote a path of length n by P_n, and cycle of length n by C_n. Note that number of vertices of P_n is $n+1$.

Example 1.1.2. *It is easy to note that*

$$\mathrm{ex}(n,P_2) = \left\lfloor \frac{n}{2} \right\rfloor.$$

even more, for a fixed n the only extremal graph is a perfect matching if n is even and almost perfect matching otherwise.

This function naturally generalizes to a setting where rather than forbidding just a graph H but a class of graphs $\mathcal{H}$. In particular,

$$\mathrm{ex}(n,\mathcal{H}) = max\left\{e(G) : G \subseteq K_n \text{ and } H \nsubseteq G \text{ for all } H \in \mathcal{H}\right\}.$$

Let us denote class of all cycles by $\mathcal{C}$, i.e. $\mathcal{C} = \{C_3, C_4, \dots\}$. We give you another example from classical graph theory.

Example 1.1.3. *We know the maximum number of edges in an n-vertex graph without a cycle is $n-1$. Therefore we have another Turán-type result*

$$\mathrm{ex}(n,\mathcal{C}) = n-1.$$

In this case, extremal graphs are all n-vertex trees.

The basic theorem from extremal graph theory is Mantel's theorem from 1907.[1]

[1] For the reader, not familiar with extremal graph theory we recommend to check different proofs of Mantel's theorem, see [21].

Theorem 1.1.4 (Mantel [100])**.** *If a graph G is triangle-free, then it contains at most* $\left\lfloor \frac{n^2}{4} \right\rfloor$ *edges. Equality holds if and only if G is a complete balanced bipartite graph* $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Mantel's theorem in extremal graph theory language is $\mathrm{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$. The only extremal graph for fixed integer n, is a balanced bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. Later Mantel's theorem was generalized for all complete graphs by Turán in 1941. This result is counted as the birth of extremal combinatorics.

Theorem 1.1.5 (Turán [114])**.** *For fixed integers n and r, let Turán graph $T(n,r)$ be a balanced, n-vertex r-partite graph. Let us denote the number of edges in $T(n,r)$ by $t(n,r)$. Then we have*

$$\mathrm{ex}(n, K_{r+1}) = t(n,r) \leq \left(1 - \frac{1}{r}\right)\binom{n}{2}.$$

The only extremal graph is $T(n,r)$.

Since then researchers studied the Turán number of various graphs. Note that there are different stages while searching for $\mathrm{ex}(n, \cdot)$. In particular, for a fixed graph it may be difficult to find the exact Turán number, hence there are different relaxations of the problem. At first, we try to find the order of magnitude of $\mathrm{ex}(n, \cdot)$ as a function of n, then asymptotic, only after the exact value. Which is followed by characterizing all extremal graphs. Even more, the research may be continued afterward by asking saturation and stability-type questions, see Section 1.9. Note that there is a spectrum of other steps in between those steps. Like sometimes it is hard to get the exact result for every n but it is possible to get the exact result for infinitely many n. This spectrum of different problems, excited mathematicians of different tastes which made this subject widely popular.

Soon after Turán's Theorem 1.1.5, Erdős and Stone in 1946 proved a theorem, this was later strengthened by Erdős and Simonovits in 1966, Theorem 1.1.6. This theorem at first glance suggests that the majority of Turán problems are settled asymptotically.

Theorem 1.1.6 (Erdős, Stone, Simonovits [28, 33])**.**

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}.$$

Note that if H is not bipartite i.e. $\chi(H) > 2$ the this theorem determines asymptotic of $\mathrm{ex}(n, H)$. This was a motivation for Zarankiewicz to generalize Turán's problem in 1951 [116]. Before we state Zarankiewicz problem let us make a cosmetic change in the definition of the function $\mathrm{ex}(\cdot, \cdot)$. As we have seen $\mathrm{ex}(n, H)$ denotes the maximum number of edges in an n-vertex graph not containing H as a sub-graph. Therefore we are searching between all sub-graphs of K_n one with the maximum number of edges and no H. Naturally, one may generalize Turán function.

$$\text{ex}(F,H) = max\left\{e(G) : G \subseteq F \text{ and } H \nsubseteq G\right\}.$$

or even more, let $\mathcal{F}_n$ be a family of n-vertex graphs. Then we may ask the following variant of the Turán number

$$\text{ex}(\mathcal{F}_n,H) = max\left\{e(G) : G \in \mathcal{F}_n \text{ and } H \nsubseteq G\right\}.$$

For the reader, we specify $\mathcal{F}$ can be all n-vertex planar graphs, or all n-vertex connected graphs,[2] or all n-vertex bipartite graphs. In this settings $\text{ex}(n,H)$ is the same as $\text{ex}(K_n,H)$. Zarankiewicz proposed to study $\text{ex}(K_{n,n},\cdot)$ in 1951 [116].

1.2 The Turán number of degenerate (bipartite) graphs

As we have seen Erdős-Stone-Simonovits Theorem 1.1.6 settling Turán problem for all but non-bipartite graphs. Naturally, mathematicians started investigating Turán functen for bipartite graphs. Füredi and Simonovits [52] devote a one-hundred-page survey to this topic, therefore we refer to this survey all the readers who are deeply interested in this topic. In this section, we try to present some central theorems from this area. Some of those results will be applied later.

Let C_{odd} be the set of all cycles of odd length.

Theorem 1.2.1 (Kővári, Sós, Turán [92])**.**

$$\text{ex}(n,C_4 \cup \{C_{odd}\}) = \frac{1}{2\sqrt{2}}n^{\frac{3}{2}} + o(n^{\frac{3}{2}}).$$

Theorem 1.2.2 (Kővári, Sós, Turán [92])**.** *For all positive integers a,b, $a \le b$, we have*

$$\text{ex}(n,K_{a,b}) \le \frac{1}{2}\sqrt[a]{b-1}\,n^{2-\frac{1}{a}} + \frac{a-1}{2}n.$$

The proof of Theorem 1.2.2 is an intuitive double counting idea. On one hand they count the number of a-stars,[3] which is $\sum_{v\in V(G)}\binom{d(v)}{a}$. If G is $K_{a,b}$-free then the number of a-stars is at most $(b-1)\binom{V(G)}{a}$. Finally they apply Jansen's Inequality [85] to get the desired result.

[2] A graph is connected if there is a path between any two vertices of it.

[3] $K_{1,a}$.

Conjecture 1.2.3 (Kővári, Sós, Turán [92])**.** *The order of the upper bound in Theorem 1.2.2 is sharp.*

This conjecture holds for some special cases. For example we have

Theorem 1.2.4 (Erdős, Rényi, Sós [27], Brown [19])**.**

$$\mathrm{ex}(n, C_4) = \frac{1}{2} n^{3/2} + O(n^{\frac{3}{2}} - c).$$

The upper bound of this theorem comes from cherry[4] counting argument as before. A corresponding lower bound comes from finite geometry constructions.

Example 1.2.5 ([19, 27], see also [16, 52])**.** *Let q be a prime power. The vertices of our graph G are the equivalence classes of the non-zero triples $(a,b,c) \in GF(q)^3$, Where $GF(q)$ is a finite field. Two triples (a,b,c) and (x,y,z) are considered in the same equivalence class if $(a,b,c) = (\lambda x, \lambda y, \lambda z)$ for some non-zero element $\lambda \in GF(q)$. We have $v(G) = \frac{q^3-1}{q-1} = q^2+q+1$. The edge set of graph G is*

$$E(G) = \{((a,b,c),(x,y,z)) : ax+by+cz = 0 \text{ and } (a,b,c) \neq (x,y,z)\}$$

One can easily see that the graph G from Example 1.2.5 is C_4-free.

Theorem 1.2.6 (Füredi [44])**.** *Let $n = q^2+q+1$ where q is an integer such that $q \neq 1,7,9,11,13$, then*

$$\mathrm{ex}(n, C_4) \leq \frac{1}{2} q(q+1)^2.$$

Moreover, if q is a power of a prime, then

$$\mathrm{ex}(n, C_4) = \frac{1}{2} q(q+1)^2.$$

Here we would like to highlight our favorite conjecture of Erdős.

Conjecture 1.2.7 (Erdős)**.**

$$\mathrm{ex}(n, \{C_3, C_4\}) = \frac{1}{2\sqrt{2}} n^{\frac{3}{2}} + o(n^{\frac{3}{2}}).$$

On the other hand, there is a counter-conjecture from Allen, Keevash, Sudakov, and Verstraëte.

[4]Cherry is a bipartite graph $K_{1,2}$

Conjecture 1.2.8 (Allen, Keevash, Sudakov, Verstraëte [1])**.**

$$\limsup_{n\to\infty} \frac{\mathrm{ex}(n,\{C_3,C_4\})}{\mathrm{ex}(n,C_{odd}\cup\{C_4\})} > 1.$$

Erdős, Simonovits determined extremal number of $\{C_4,C_5\}$.

Theorem 1.2.9 (Erdős, Simonovits [34])**.**

$$\mathrm{ex}(n,\{C_5,C_4\}) = \frac{1}{2\sqrt{2}}n^{\frac{3}{2}} + o(n^{\frac{3}{2}}).$$

Allen, Keevash, Sudakov, Verstraëte [1] generalized Erdős, Simonovits theorem.

Theorem 1.2.10 (Allen, Keevash, Sudakov, Verstraëte [1])**.** *Let ℓ and t be integers, such that $\ell \geq 2$. Then we have*

$$\limsup_{n\to\infty} \frac{\mathrm{ex}(n,\{K_{2,t},C_{2\ell+1}\})}{\mathrm{ex}(n,\{K_{2,t},C_{odd}\}))}.$$

Here comes another mysterious open problems for cycles. A well-known result of Bondy-Simonovits [17] asserts that for all $\ell \geq 2$ we have $\mathrm{ex}(n,C_{2\ell}) = O(n^{1+1/\ell})$, however the order of magnitude is only known to be sharp in the cases $\ell = 2,3,5$.

In the following section, we concentrate on paths and long cycles.

1.3 Erdős-Gallai Theorem, paths and cycles

As we have seen Erdős-Stone-Simonovits Theorem 1.1.6 settling Turán problem for all but non-bipartite graphs. Erdős-Gallai determined the Turán number of paths and a class of long cycles in 1959.

Theorem 1.3.1 (Erdős, Gallai [31])**.** *For two integers n and ℓ,*

$$\mathrm{ex}(n,P_\ell) \leq \frac{(\ell-1)n}{2}.$$

The equality holds if and only if $\ell|n$ and extremal graph is the disjoint union of $\frac{n}{\ell}$ cliques of size ℓ, see Figure 1.1.

Strictly speaking, this theorem was a corollary of the following more general theorem. Let us denote a family of cycles of length at least ℓ by $\mathcal{C}_{\geq \ell}$.

Theorem 1.3.2 (Erdős, Gallai [31])**.** *For two integers n and ℓ,*

$$\mathrm{ex}(n,\mathcal{C}_{\geq\ell}) \leq \frac{(\ell-1)(n-1)}{2}.$$

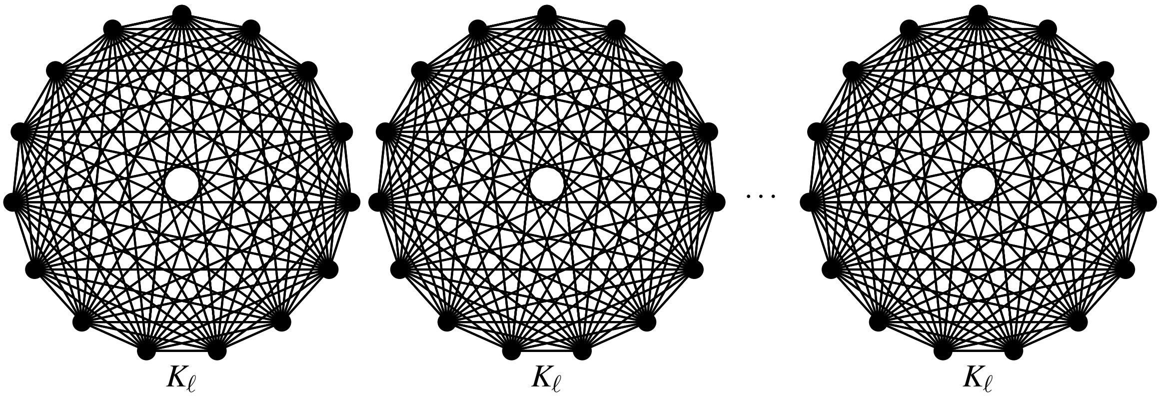

Fig. 1.1 The extremal graph of Theorem 1.3.1, $\mathrm{ex}(n, P_\ell)$.

The equality holds if and only if $\ell-2|n-1$ *and* G *is the union of* $\frac{n-1}{\ell-2}$ *disjoint cliques of size* $\ell-1$ *sharing a vertex in a tree-like structure* [5]*, see Figure 1.2.*

Let G be an n vertex P_ℓ-free graph. We construct an auxiliary graph G' from G, by adding a vertex v to G which is joined to all vertices of G. In particular, we have $V(G') = V(V) \cup \{v\}$ and $E(G') = E(G) \cup \{(v,u) : u \in V(G)\}$. Note that since G is P_ℓ-free, G' is $C_{\geq \ell+2}$-free. Hence from Theorem 1.3.2 we have the desired inequality

$$e(G) = e(G') - n \leq \frac{(\ell+2-1)(n+1-1)}{2} - n = \frac{(\ell-1)n}{2}.$$

Since we find the following trick is a useful tool we would like to show the proof of Theorem 1.3.1 using Theorem 1.3.2 as in [31].

Let G be an n vertex P_ℓ-free graph. We construct an auxiliary graph G' from G, by adding a vertex v to G which is joined to all vertices of G. In particular, we have $V(G') = V(V) \cup \{v\}$ and $E(G') = E(G) \cup \{(v,u) : u \in V(G)\}$. Note that since G is P_ℓ-free, G' is $C_{\geq \ell+2}$-free. Hence from Theorem 1.3.2 we have the desired inequality

$$e(G) = e(G') - n \leq \frac{(\ell+2-1)(n+1-1)}{2} - n = \frac{(\ell-1)n}{2}.$$

After finding the Turán number of P_ℓ, one may attempt to determine the Turán number of $\ell+1$-vertex trees. Surprisingly this problem remains still open despite of the interest around it.

[5] Every vertex-maximal two-connected component of G is isomorphic to $K_{\ell-1}$. A graph is two-connected if there are two internally disjoint paths between any two vertices.

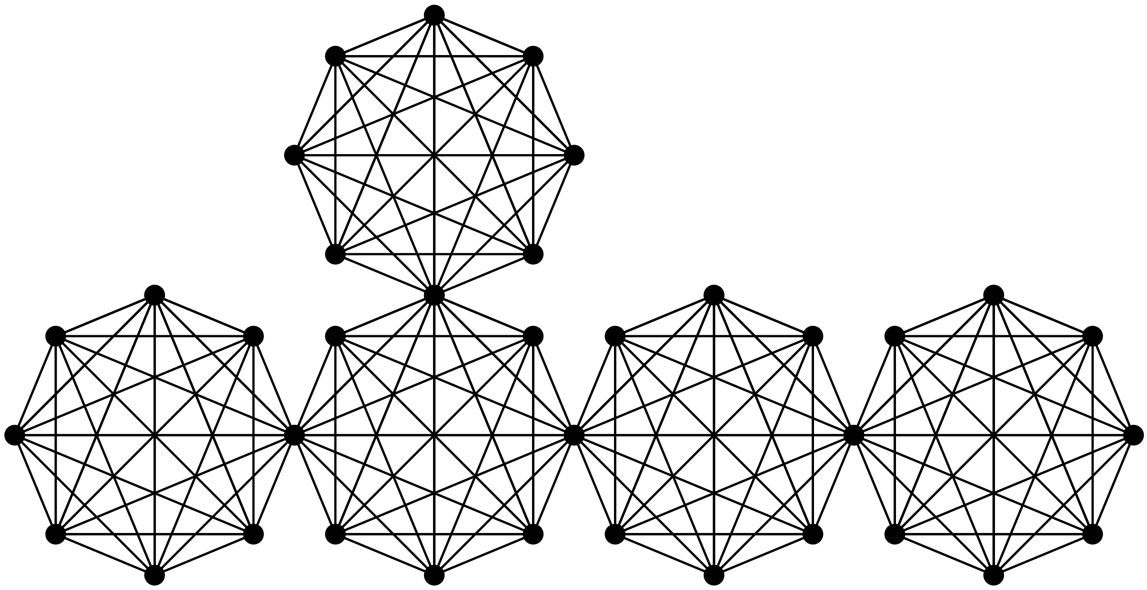

Fig. 1.2 The extremal graph of Theorem 1.3.2, $\mathrm{ex}(n,\mathcal{C}_{\geq\ell})$.

Conjecture 1.3.3 (Erdős, Sós [30]). *Let $T_{\ell+1}$ be an arbitrary $\ell+1$ vertex tree, then*

$$\mathrm{ex}(n,T_{\ell+1}) \leq \frac{(\ell-1)n}{2}.$$

A corresponding lower-bound comes from the same construction as in Theorem 1.3.1, see Figure 1.1. It is believed that this conjecture is true since there are some specific cases done. For the reader, interested in this subject we refer to the sixth chapter of the Survey [52].

Mordechai Lewin extended Erdős-Gallai Theorem 1.3.2 for directed cycles and obtained sharp results for all n.

Woodall extended Theorem 1.3.2, they obtained sharp results for every n.

Theorem 1.3.4 (Woodall [115]). *Let $0 \leq t$ and $0 \leq r < \ell-2$*

$$\mathrm{ex}(t(\ell-2)+r+1,\mathcal{C}_{\geq\ell}) = t\binom{l-1}{2} + \binom{r+1}{2}$$

A corresponding lower-bound comes from a graph containing t copies of $K_{\ell-1}$ and a K_{r+1} sharing a vertex. Similarly, as before one may get a corollary for the paths, this was already obtained by Faudree and Schelp including the classification of extremal graphs.

Theorem 1.3.5 (Faudree, Schelp [41]). *Let $0 \leq t$ and $0 \leq r < \ell$*

$$\mathrm{ex}(t\ell+r,P_\ell) = t\binom{\ell}{2} + \binom{r}{2}.$$

Extremal graphs are G_1 and $G_{2,j}$. Where

$$G_1 := \left(\bigcup_{i=1}^{t} K_\ell\right) \bigcup K_r.$$

When ℓ is odd, $t > 0$ and $r = \frac{n\pm1}{2}$ then

$$G_{2,j} := \bigcup_{i=1}^{j} K_\ell \bigcup \left(K_{\frac{\ell-1}{2}} + \overline{K}_{\frac{\ell+1}{2}+(t-j-1)\ell+r}\right)$$

for every j, $0 \leq j \leq k-1$.

Jackson [84], Gyárfás [65] obtained extremal numbers of paths in bipartite graphs. Another extension of Theorem 1.3.2 was suggested by Woodall [115]. They considered 2-connected graphs without long cycles and obtained partial results. Later in 1977,Kopylov settled the proposed conjecture in [86]. They determined extremal number for 2-connected n-vertex graphs without a cycle of length at least ℓ. This result with the already mentioned trick was implying result for paths too. Which was later also proved by Balister, Győri, Lehel, Schelp in [9], including Extremal graphs.

Let us denote the maximum number of edges in an n-vertex H-free connected graph by $\mathrm{ex}^{conn}(n,H)$.

Definition 1.3.6. *For $n \geq k \geq 2s$ let*

$$G_{n,k,s} := \left(K_{k-2s} \cup \overline{K}_{n-k+1}\right) + K_s.$$

See figure 1.3.

Theorem 1.3.7 (Kopylov [86], Balister, Győri, Lehel, Schelp [9])**.** *Let $n > \ell \geq 3$*

$$\mathrm{ex}^{conn}(n,P_\ell) = max\left\{e(G_{n,\ell,1}), e(G_{n,\ell,\lfloor\frac{\ell-1}{2}\rfloor})\right\}.$$

Extremal graphs are $G_{n,\ell,1}$ or $G_{n,\ell,\lfloor\frac{\ell-1}{2}\rfloor}$, see Definition 1.3.6.

Another variant of Erdős-Gallai problem is to determine the Turán number of paths for Erdős-Rényi random graphs. In particular, the random variable $\mathrm{ex}(G_{n,p},H)$, where $G_{n,p}$ is the Erdős–Rényi random graph, was introduced by Babai, Simonovits, Spencer [8], and by Frankl, Rödl [43]. For recent developments in this direction see [93, 12].

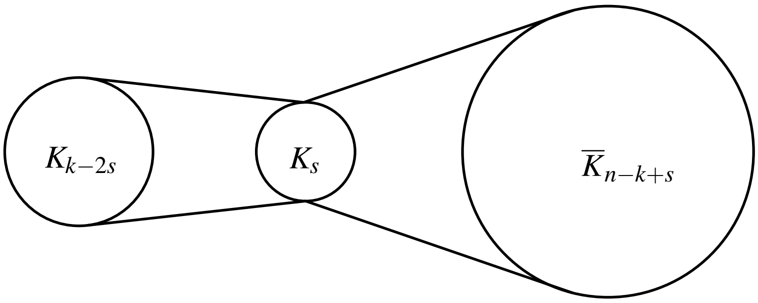

Fig. 1.3 The graph $G_{n,k,s}$, Definition 1.3.6.

1.4 Erdős-Gallai Theorem, for vertex colored graphs

In a relatively recent paper, Győri, Lemons [71] investigated the extremal number of hypergraphs avoiding Berge-cycles. To this end, they introduced a generalization of the theorem of Erdős-Gallai about paths. By a proper vertex coloring of a graph G, we mean a coloring of the vertices of G such that no two adjacent vertices are the same color. Győri, Lemons proved the following lemma.

Lemma 1.4.1 (Győri-Lemons [71])**.** *Let k be a positive integer and G be an n-vertex graph with a proper vertex coloring such that G contains no P_{2k+1} with endpoints of different colors, then*

$$|E(G)| \leq 2kn.$$

We show that the factor of 2 in Theorem 1.4.1 is not needed and, thus, recover the original upper bound from the Erdős-Gallai theorem. We also determine which graphs achieve this upper bound.

Theorem 1.4.2 (Salia, Tompkins, Zamora [106])**.** *Let $k \geq 0$ and G be an n-vertex graph with a proper vertex coloring such that G contains no P_{2k+1} with endpoints of different colors, then*

$$|E(G)| \leq kn,$$

and equality holds if and only if $2k+1$ divides n and G is the union of $\frac{n}{2k+1}$ disjoint cliques of size $2k+1$.

We believe that an analog of Theorem 1.4.1 should hold in the setting of trees. Recall that the extremal number $\mathrm{ex}(n,H)$ of a graph H is defined to be the largest number of edges an n-vertex graph may have if it does not contain H as a sub-graph.

We introduce a new variation of the extremal function $\mathrm{ex}(n,T)$ in the case of trees.

Definition 1.4.3. *Let* $\mathrm{ex}^c(n,T)$ *denote the maximum number of edges possible in an* n*-vertex graph* G *with a proper vertex coloring (using any number of colors), such that in every copy of* T *in* G *the leaves of* T *are all the same color.*

As for what we experienced for paths we have two different cases for trees. The first when the tree T has leaves from different color classes.

Theorem 1.4.4 (Salia, Tompkins, Zamora [106])**.** *Let* T *be a tree with* k *edges such that in the (unique) proper vertex* 2*-coloring of* T *all leaves are not in the same color, then* $\mathrm{ex}^c(n,T) \leq (k-1)n$*.*

The second when all leaves are from the same color class.

Theorem 1.4.5 (Salia, Tompkins, Zamora [106])**.** *Let* T *be a tree with* k *edges such that in the proper vertex* 2*-coloring of* T *all leaves are the same color, then* $\mathrm{ex}^c(n,T) = \left\lfloor \frac{n^2}{4} \right\rfloor$*, provided* n *is sufficiently large.*

We believe that a strengthening of Conjecture 1.3.3 should hold for trees whose 2-coloring yields two leaves of different colors.

Conjecture 1.4.6 (Salia, Tompkins, Zamora [106])**.** *Let* T *be a tree with* $k \geq 1$ *edges such that in the proper vertex* 2*-coloring of* T *all leaves are not the same color, then* $\mathrm{ex}^c(n,T) \leq \frac{(k-1)n}{2}$*.*

One would hope that Conjecture 1.4.6 could be deduced directly from Conjecture 1.3.3, but unfortunately, this does not seem to be the case. We take the first step towards Conjecture 1.4.6 by proving it in the case of double stars.

Theorem 1.4.7 (Salia, Tompkins, Zamora [106])**.** *For positive integers* a *and* b*, let* $S_{a,b}$ *denote the tree on* $a+b+2$ *vertices consisting of an edge* $\{u,v\}$ *where* $|N(u)\setminus v| = a$*,* $|N(v)\setminus u| = b$ *and* $N(u)\cap N(v) = \varnothing$ *(See Figure 2.2, left). We have* $\mathrm{ex}^c(n,S_{a,b}) \leq \frac{a+b}{2}n$*.*

Proofs of these theorems are in Chapter 2.

1.5 The generalized Turán number

Recall, for a fixed graph F, the classical Turán number $\mathrm{ex}(n,F)$ is defined to be the maximum number of edges possible in an n-vertex graph not containing F as a sub-graph. This function naturally generalizes to a setting where, rather than edges, we maximize the number of copies of a given graph H in an n-vertex F-free graph. Following Alon, Shikhelman [3] (see also [4]), we denote this more general function by $\mathrm{ex}(n,H,F)$. In particular, we have

$$\mathrm{ex}(n,F,H) = max\{\mathcal{N}(G,F) : G \subseteq K_n,\ H \not\subseteq G\}.$$

Where $\mathcal{N}(G,F)$ denotes the number of sub-graphs of G (not necessarily induced) isomorphic to F.

Problems of this type have a long history beginning with a result of Zykov [117] (and later independently Erdős [24]) who determined the value of $\mathrm{ex}(n,K_r,K_t)$ for any pair of cliques. While Erdős was measuring how far are the triangle-free graphs from bipartite graphs he naturally asked a question 'What is the maximum number of pentagons in a triangle-free graph' [26]. This question was settled half a century later by Hatami, Hladký, Král, Norine, Razborov [83] and independently by Grzesik [62], using flag algebras. In 1991, Győri, Pach, Simonovits [76], defined the generalized Turán number and obtained some results. In particular, they maximized copies of a bipartite graph with 1-factor in triangle-free graphs. While investigating pentagon-free 3-uniform hypergraphs Bollobás-Győri [15] initiated the study of the converse of the problem of Erdős. They asked the following question 'What is the maximum number of triangles in a pentagon-free graph'. Surprisingly this question is still open but we show a simple proof of an upper-bound in Chapter 3. After a decade Alon, Shikhelman [3] defined the function $\mathrm{ex}(n,H,F)$, and obtained some general results after which the generalized Turán number gained a wide range of interest.

Bollobás, Győri [15] showed that

Theorem 1.5.1 (Bollobás, Győri [15])**.**

$$\frac{(n)^{\frac{3}{2}}}{3\sqrt{3}}(1+o(1)) \leq \mathrm{ex}(n,C_3,C_5) \leq n^{3/2}(\frac{5}{4}+o(1)).$$

The lower-bound of Theorem 1.5.1 comes from the following construction.

Example 1.5.2. *There exists a balanced bipartite C_4-free extremal graph G' with $\frac{2n}{3}$ vertices and $\frac{1}{3\sqrt{3}}n^{\frac{3}{2}}$ edges, from Theorem 1.2.1. Let us denote partite sets of G' by $A=\{a_1,a_2,\dots,a_{\frac{n}{3}}\}$ and $B=\{b_1,b_2,\dots,b_{\frac{n}{3}}\}$. Let A' be a disjoint copy of A, $A'=\{a'_1,a'_2,\dots,a'_{\frac{n}{3}}\}$.*

Let G be a graph on n vertices, such that $3|n$. The vertex set of G is $A\cup A'\cup B$. The edge set of G is

$$E(G)=\{(a_i,a'_i): i\in[\frac{n}{3}]\}\cup E(G')\cup\{(a'_i,b_j):(a_i,b_j)\in E(G'),\ i,j\in[\frac{n}{3}]\}.$$

Theorem 1.5.1 was improved since then but it still stays open [3, 36, 39]. In [39], Ergemlidze and Methuku prove the best known upper-bound.

Theorem 1.5.3 (Ergemlidze, Győri, Methuku, Salia [36])**.**

$$\mathrm{ex}(n,C_3,C_5) \leq \frac{1}{2\sqrt{2}}(1+o(1))n^{3/2}.$$

See proof of this theorem in Section 3.2.

Luo determined the maximum number of cliques in a graph without long paths. Before we need to introduce a function $f_s(n,k,a)$.

$$f_s(n,k,a) = \binom{k-a}{s} + (n-k+a)\binom{a}{s-1}.$$

Theorem 1.5.4 (Luo [97]). *Let $n-1 \geq k \geq 4$. Let G be a connected n-vertex graph with no P_k, then the number of s-cliques in G is at most*

$$\max\{f_s(n,k,\lfloor (k-1)/2 \rfloor), f_s(n,k,1)\}.$$

As a corollary, she also showed

Corollary 1.5.5 (Luo). *Let $n \geq k \geq 3$. Assume that G is an n-vertex graph with no cycle of length k or more, then*

$$N_s(G) \leq \frac{n-1}{k-2}\binom{k-1}{s}.$$

Nowadays the generalized Turán number is in the center of attention of extremal combinatorialists, therefore there are many fresh results obtained in this subject [74, 77, 113, 59, 95, 75, 54, 95, 60, 79].

1.6 Berge-hypergraphs

A hypergraph $\mathcal{H}$ is a pair $V(\mathcal{H})$ and $E(\mathcal{H})$ where $V(\mathcal{H})$ is a vertex set and $E(\mathcal{H})$ is a hyperedge set. Similarly, as for graphs $v(\mathcal{H}) := |V(\mathcal{H})|$, $e(\mathcal{H}) := |E(\mathcal{H})|$. Where $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})}$. For a fixed set of positive integers R, a hypergraph $\mathcal{H}$ is R-uniform if the cardinality of each edge belongs to R, i.e. $\forall h \in \mathcal{H}$ we have $|h| \in R$. If $R = \{r\}$, then an R-uniform hypergraph is simply an r-uniform hypergraph. We say a hypergraph is $\mathcal{F}$-free if it does not contain a copy of any hypergraph from the family $\mathcal{F}$ as a sub-hypergraph.

Let $\mathcal{H}$ be a hypergraph. Then its *k-shadow*, denoted by $\partial_k\mathcal{H}$, is a hypergraph on the same set of vertices and the hyperedge set is the collection of all k-sets that lie in some hyperedge of $\mathcal{H}$. In particular, $E(\partial_k\mathcal{H}) = \{e : |e| = k \text{ and } e \subseteq h \in \mathcal{H}\}$

A hypergraph $\mathcal{H}$ is *connected* if $\partial_2(\mathcal{H})$ is a connected graph.

Incidence bipartite graph G of a hypergraph $\mathcal{H}$ is a bipartite graph with partite sets $V(\mathcal{H})$ and $E(\mathcal{H})$, i.e. $V(G) = V(\mathcal{H}) \cup E(\mathcal{H})$. Two vertices of G, v and h, $v \in V(\mathcal{H})$ and $h \in E(\mathcal{H})$, are joined by an edge in G if $v \in h$.

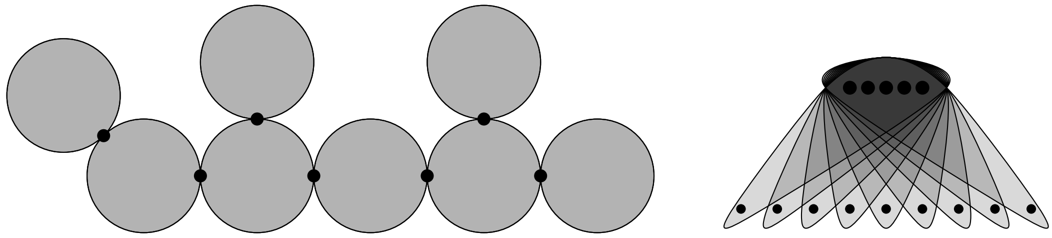

Fig. 1.4 A block tree and $\mathcal{S}_n^{(r)}$.

Let n,k,r be integers such that $k \le r$. Fix s, $s \in \{r, r+1\}$. An r-uniform hypergraph $\mathcal{H}$ is called an $(s,k-1)$-block tree if $\partial_2(\mathcal{H})$ is connected and every 2-connected block of $\partial_2(\mathcal{H})$ consists of s vertices which induce $k-1$ hyperedges in $\mathcal{H}$. An $(s,k-1)$-block tree contains no Berge-cycle of length at least k, because each of its blocks contain fewer than k hyperedges, see Figure 1.4.

We define the r-star, $\mathcal{S}_n^{(r)}$, as the n-vertex r-uniform hypergraph with vertex set $V(\mathcal{S}_n^{(r)}) = \{v_1, v_2, \dots, v_n\}$ and edge set

$$E(\mathcal{S}_n^{(r)}) = \{\{v_1, v_2, \dots, v_{r-1}, v_i\} : r \le i \le n\},$$

the set $\{v_1, v_2, \dots, v_{r-1}\}$ is called the center of the star. Since $\mathcal{S}_n^{(r)}$ has $r-1$ vertices of degree larger than 1, $\mathcal{S}_n^{(r)}$ contains no Berge-cycle of length at least r.

Definition 1.6.1. *For a set $S \subseteq V$, the* hyperedge neighborhood *of S in an r-uniform hypergraph $\mathcal{H}$ is the set*

$$N_{\mathcal{H}}(S) := \{h \in E(\mathcal{H}) | h \cap S \neq \emptyset\}$$

of hyperedges that are incident with at least one vertex of S.

There are no natural ways to generalize graph paths and cycles for hypergraphs. Hence there are numerous definitions for them. In this work, we follow the definition of Berge [13]. Berge-paths and Berge-cycles are some of the most general definitions of paths and cycles in hypergraphs. For example, there are linear-paths/cycles, loose-paths/cycles, and tight-paths/cycles all of which are specific examples of a Berge-path/cycles. All such paths/cycles coincide with the classical definition of path/cycle for graphs.

Definition 1.6.2. *A hypergraph BC_ℓ is a* Berge-cycle *of length ℓ if*

- $V(BC_\ell) \supseteq \{v_1, v_2, \dots, v_\ell\}$;
- $E(BC_\ell) = \{e_1, e_2, \dots, e_\ell\}$;

- $v_i, v_{i+1} \in e_i$ *for all* $i \in [\ell]$.[6]

The vertices $v_1, v_2, \ldots, v_\ell$ *are called defining vertices and hyperedges* $e_1, e_2, \ldots, e_\ell$ *defining hyperedges of the Berge-cycle.*

The class of all Berge-cycles of length ℓ *is denoted by* $\mathcal{BC}_\ell$ *and the class of all Berge-cycles of length at least* ℓ *by* $\mathcal{BC}_{\geq \ell}$.

Remark 1.6.3. *We have defined that a hypergraph* $\mathcal{H}$ *is* connected *if* $\partial_2(\mathcal{H})$ *is a connected graph. We can give an equivalent definition to this one.*

A hypergraph $\mathcal{H}$ *is* connected *if for all pairs of vertices v and u there is a Berge-path from v to u.*

For the completeness of this work we define Berge-paths.

Definition 1.6.4. *A hypergraph* BP_ℓ *is a* Berge-path *of length* ℓ *if*

- $V(BP_\ell) \supseteq \{v_1, v_2, \ldots, v_{\ell+1}\}$;
- $E(BP_\ell) = \{e_1, e_2, \ldots, e_\ell\}$;
- $v_i, v_{i+1} \in e_i$ *for all* $i \in [\ell]$.

The vertices $v_1, v_2, \ldots, v_{\ell+1}$ *are called defining vertices and hyperedges* $e_1, e_2, \ldots, e_\ell$ *defining hyperedges of the Berge-path.*

The class of all Berge-paths of length ℓ *is denoted by* $\mathcal{BP}_\ell$

Note that the number of vertices in $\mathcal{BP}_\ell$ is at least $\ell + 1$ but it can be larger. The index of path classically denotes the number of vertices in a path, but as we have seen it is not a case for hypergraphs therefore we decided to use the index to denote the length of the path.

Those definitions of Berge-paths and cycles naturally generalize to other graphs.

Definition 1.6.5. *A hypergraph BG is a* Berge-*G, for some fixed graph G if*

- *There is an injective function* $f_1 : V(G) \to V(BG)$;
- *There is a bijective function* $f_2 : E(G) \to E(BG)$;
- *If* $\{v_1, v_2\} \in E(G)$, *then* $\{f_1(v_1), f_1(v_2)\} \subseteq f_2(\{v_1, v_2\})$

The vertices from $f_1(V(G))$ *are called defining vertices and hyperedges from* $f_2(E(G))$ *defining hyperedges of the BG.*

The class of all Berge-G is denoted by $\mathcal{B}G$.

[6] indices are taken modulo ℓ

Recently there has been extensive study of various parameters of Berge-hypergraphs. For example, it is a popular subject to determine Ramsey number for Berge-hypergraphs. Ramsey theory is among the oldest and most intensely investigated topics in combinatorics. It began with the seminal result of Ramsey from 1930.

Theorem 1.6.6 (Ramsey [104])**.** *Let r,t and k be positive integers. Then there exists an integer N such that any coloring of the N-vertex r-uniform complete hypergraph with k colors contains a monochromatic copy of the t-vertex r-uniform complete hypergraph.*

Estimating the smallest value of such an integer N (the so-called Ramsey number) is a notoriously difficult problem and only weak bounds are known. Given the difficulty of this problem, many people began investigating variations of this problem where graphs other than the complete graphs are considered. An example of an early result in this direction due to Chvátal [20] asserts that the Ramsey number of a t-clique versus any m-vertex tree is precisely $N = 1+(m-1)(t-1)$. That is any red-blue coloring of the complete graph K_N yields a red K_t or a blue copy of a given m-vertex tree. Ramsey problems for a variety of hypergraphs and classes of hypergraphs have been considered (for a recent survey of such problems see [102]). The Ramsey problem for Berge-paths and cycles has received much attention. Of particular interest is a result of Gyárfás and Sárközy [66] showing that the 3-color Ramsey number of a 3-uniform Berge-cycle of length n is asymptotic to $\frac{5n}{4}$ (the 2-color case was settled exactly in [64]). Since this is not a subject of this work we refer the reader to the following manuscripts [7, 105, 55, 96, 53, 98].

1.7 The Turán number of Berge-hypergraphs

The Turán number naturally generalizes for hypergraphs.

Definition 1.7.1. *The Turán number of a family of R-uniform hypergraphs $\mathcal{F}$, denoted $\mathrm{ex}_R(n,\mathcal{F})$, is the maximum number of hyperedges in an n-vertex, R-uniform, simple-hypergraph which does not contain an isomorphic copy of $\mathcal{H}$ as a sub-hypergraph, for all $F \in \mathcal{F}$.*

The same question may be asked for multi-hypergraphs, we denote the Turán number for multi-hypergraphs by $\mathrm{ex}_R^{multi}(n,\mathcal{F})$. But this question is not always interesting.

Remark 1.7.2. *If every hypergraph in $\mathcal{F}$ has at least $r+1$ vertices, then $\mathrm{ex}_r^{multi}(n,\mathcal{F})$ is infinite, since a hypergraph on r vertices and multiple copies of the same hyperedge is $\mathcal{F}$-free.*

Gerbner, Palmer [58] obtained general bounds for the Turán number of r-uniform Berge-hypergraphs using the classical Turán number.

Theorem 1.7.3 (Gerbner, Palmer [58]). *For all integer $r \geq 2$ and a graph H we have*

$$\mathrm{ex}(n,F) \leq \mathrm{ex}_r(n,\mathcal{B}F) \leq \mathrm{ex}(n,K_r,F) + \mathrm{ex}(n,F).$$

In 2004 Győri extended mantels theorem for 3 and 4 uniform hypergraphs [67].

Theorem 1.7.4 (Győri [67]). *Let $\mathcal{H}$ be a $\mathcal{B}C_3$-free, n-vertex hypergraph. Then*

$$\sum_{h \in E(\mathcal{H})} (|h|-2) \leq \frac{n^2}{8}.$$

for all n, $n \geq 100$.

Example 1.7.5 (Győri [67]). *For an integer n multiple of 4, let $\mathcal{H}_4$ be a 4-uniform hypergraph. The vertex set of $\mathcal{H}_4$ is partitioned to four k-sets A, A', B, and B'. The edge set of $\mathcal{H}_4$ is $E(\mathcal{H}_4) = \left\{ \{a_i, a'_i, b_j, b'_j\} : 1 \leq i, j \leq \frac{n}{4} \right\}$*

Clearly $\mathcal{H}_4$ is $\mathcal{B}C_3$-free and $e(\mathcal{H}_4) = \frac{n^2}{16}$. Hence we have equality in Theorem 1.7.4 $\sum_{h \in E(\mathcal{H}_4)} (|h|-2) = 2e(\mathcal{H}_4) = \frac{n^2}{8}$.

Example 1.7.6 (Győri [67]). *For an integer n multiple of 4, let $\mathcal{H}_3$ be a 3-uniform hypergraph. The vertex set of $\mathcal{H}_3$ is partitioned to three sets A, A', and B of sizes $\frac{n}{4}$, $\frac{n}{4}$, and $\frac{n}{2}$ accordingly. The edge set of $\mathcal{H}_3$ is $E(\mathcal{H}_3) = \left\{ \{a_i, a'_i, b_j\} : 1 \leq i, j \leq \frac{n}{4} \right\}$*

Clearly $\mathcal{H}_3$ is $\mathcal{B}C_3$-free and $e(\mathcal{H}_3) = \frac{n^2}{8}$. Hence we have equality in Theorem 1.7.4 $\sum_{h \in E(\mathcal{H}_3)} (|h|-2) = 2e(\mathcal{H}_4) = \frac{n^2}{8}$.

Putting together Theorem 1.7.4 and Examples 1.7.5, 1.7.6 we have a corollary.

Corollary 1.7.7 (Győri [67]). *For all $n \geq 100$, such that $4|n$, we have*

$$\mathrm{ex}_3(n,\mathcal{B}C_3) = \frac{n^2}{8},$$

and

$$\mathrm{ex}_4(n,\mathcal{B}C_3) = \frac{n^2}{16}.$$

Later Győri, Lemons [71] gave an upper bound for the size of 3-uniform hypergraphs avoiding $\mathcal{B}C_{2k+1}$. In particular, they show

$$\mathrm{ex}_3(n,\mathcal{B}C_{2k+1}) \leq 4k^4 n^{1+\frac{1}{k}} + O(n).$$

They provide constructions showing that these bounds are best possible for $k \in \{1,2,3,5\}$ up to the constant factor. Győri, Lemons [72] extended their results for r-uniform hypergraphs and non-uniform hypergraphs as well.

The Turán number of C_4 was determined in Theorem 1.2.4.

$$\mathrm{ex}(n,C_4) = \frac{n^{3/2}}{2} + o(n^{3/2}).$$

Füredi, Özkahya [51] proved

$$\mathrm{ex}_3(n,\mathcal{B}C_4) \le (1+o(1))\frac{2}{3}n^{3/2}.$$

Recently we improved this result.

Theorem 1.7.8 (Ergemlidze, Győri, Methuku, Salia, Tompkins [37])**.**

$$\mathrm{ex}_3(n,\mathcal{B}C_4) \le (1+o(1))\frac{n^{3/2}}{\sqrt{10}}.$$

See proof in Section 3.3. On the other hand, we have a lower-bound.

Theorem 1.7.9 (Bollobás, Győri [15], see also [70])**.**

$$(1-o(1))\frac{n^{3/2}}{3\sqrt{3}} \le \mathrm{ex}_3(n,\mathcal{B}C_4).$$

This lower bound comes from the following example.

Example 1.7.10 (Bollobás, Győri [15])**.** *There exists a balanced bipartite C_4-free extremal graph G with $\frac{2n}{3}$ vertices and $\frac{1}{3\sqrt{3}}n^{\frac{3}{2}}$ edges, 1.2.1. Let us denote partite sets of G by $A = \{a_1,a_2,\dots,a_{\frac{n}{3}}\}$ and $B = \{b_1,b_2,\dots,b_{\frac{n}{3}}\}$. Let A' be a disjoint copy of A, $A' = \{a'_1,a'_2,\dots,a'_{\frac{n}{3}}\}$.*

Let $\mathcal{H}$ be a 3-uniform hyper-graph on n vertices, such that $3|n$. The vertex set of $\mathcal{H}$ is $E(\mathcal{H}) = A\cup A'\cup B$. The edge set of $\mathcal{H}$ is

$$E(\mathcal{H}) = \{(a_i,a'_i,b_j) : (a_i,b_j) \in E(G),\ i,j \in [\frac{n}{3}]\}.$$

Interestingly the example of Bollobás, Győri initially was for pentagon-free hypergraphs but it happened to be C_4-free too. Which is an interesting phenomenon. We know even and odd cycles behave extremely differently in the case of graphs. Here I would suggest an exciting conjecture from Győri.

Conjecture 1.7.11 (Győri [68])**.** *For all integer $k \ge 2$, we have*

$$\lim_{n\to\infty} \frac{\mathrm{ex}_3(n,\mathcal{B}C_{2k})}{\mathrm{ex}_3(n,\mathcal{B}C_{2k+1})} = 1.$$

This version of the conjecture is a modest version of the one we believe.

The Turán problem for Berge-cliques has been investigated heavily by several authors in [67, 99, 63]. For further results see [6, 103, 61] also.

1.8 Extensions of Erdős-Gallai Theorem

In the last decade, there has been extensive study of hypergraphs without long Berge-paths and cycles. But in contrast to graphs, the pattern shows that there are two distinct cases for hypergraphs. The first when the forbidden structure has more vertices than uniformity and the second when uniformity is 'larger' than the number of defining vertices in a forbidden structure. But in some cases, there are different constructions and situations when the number of defining vertices of a forbidden structure is close to uniformity.

1.8.1 Hypergraphs without long Berge-paths

We open this section with a theorem of Győri, Katona, Lemons[69].

Theorem 1.8.1 (Győri, Katona, Lemons [69])**.** *Let $r \geq k \geq 3$, then*

$$\mathrm{ex}(n, \mathcal{BP}_k) \leq \frac{(k-1)n}{r+1}.$$

Theorem 1.8.2 (Győri, Katona, Lemons [69])**.** *Let $k > r+1 > 3$, then*

$$\mathrm{ex}(n, \mathcal{BP}_k) \frac{n}{k} \binom{k}{r}.$$

In the remaining case when $k = r+1$ was solved later by Davoodi, Győri, Methuku, Tompkins [22], the extremal number matches the upper bound of Theorem 1.8.2.

Theorem 1.8.3 (Davoodi, Győri, Methuku, Tompkins [22])**.** *Let $r > 2$, then*

$$\mathrm{ex}(n, \mathcal{BP}_{r+1}) \leq \frac{n}{r+1} \binom{k}{r}.$$

Similarly, as in the graph case in Theorem 1.3.7, it makes sense to ask the same question requiring connectivity. We obtained the first partial result in this direction.

Theorem 1.8.4 (Győri, Methuku, Salia, Tompkins, Vizer [75])**.** *Let $\mathcal{H}_{n,k}$ be a largest r-uniform connected n-vertex hypergraph with no Berge-path of length k, then*

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{\left|E(\mathcal{H}_{n,k})\right|}{k^{r-1}n} = \frac{1}{2^{r-1}(r-1)!}.$$

See proof of this theorem in Chapter 5.

In the recent work of Füredi, Kostochka, Luo [49], they investigate 2-connected hypergraphs[7] and obtain many interesting results. To present their results, we start by introducing the following two functions,

$$f(n,k,r,a) := \binom{k-a}{\min\{r, \lfloor \frac{k-a}{2} \rfloor\}} + (n-k+a)\binom{a}{\min\{r-1, \lfloor a/2 \rfloor\}},$$

and

$$f^*(n,k,r,a) := \binom{k-a}{r} + (n-k+a)\binom{a}{r-1}.$$

Let us introduce another notion, a family of sets $\mathcal{F}$ is called a *Sperner family*, if for all $f \subset \mathcal{F}$, there is no $f' \subset \mathcal{F}$ such that $f' \not\subset f$.

Füredi, Kostochka, Luo [49] obtained a sharp upper bound for 2-connected Sperner families without a long Berge-cycle.

Theorem 1.8.5 (Füredi, Kostochka, Luo [49]). *Let $n \geq k \geq r \geq 3$. If $\mathcal{H}$ is an n-vertex Sperner 2-connected r-uniform hypergraph with no Berge-cycle of length at least k, then*

$$e(\mathcal{H}) \leq \max\{f(n,k,r,\lfloor (k-1)/2 \rfloor), f(n,k,r,2)\}.$$

They also obtained a sharp upper-bound for 2-connected r-uniform hypergraphs without a long Berge-cycle.

Theorem 1.8.6 (Füredi, Kostochka, Luo [49]). *Let $n \geq n_{k,r} \geq k \geq 4r \geq 12$. If $\mathcal{H}$ is an n-vertex 2-connected r-uniform hypergraph with no Berge-cycle of length k or longer, then*

$$e(\mathcal{H}) \leq f(n,k,r,\lfloor (k-1)/2 \rfloor) = f^*(n,k,r,\lfloor (k-1)/2 \rfloor).$$

The following two results are for 2-connected hypergraphs, the first when the hypergraph is a Sperner family and the second when the hypergraph is uniform.

Theorem 1.8.7 (Füredi, Kostochka, Luo [49]). *Let $n \geq k \geq r \geq 3$. If $\mathcal{H}$ is an n-vertex Sperner connected $\leq r$-uniform hypergraph with no Berge-path of length k, then*

$$e(\mathcal{H}) \leq \max\{f(n,k,r,\lfloor (k-1)/2 \rfloor), f(n,k,r,1)\}.$$

[7]A hypergraph is k-connected if it's incidence bipartite graph is k-connected.

Theorem 1.8.8 (Füredi, Kostochka, Luo [49]). *Let $n \geq n'_{k,r} \geq k \geq 4r \geq 12$. If $\mathcal{H}$ is an n-vertex connected r-uniform hypergraph with no Berge-path of length k, then*

$$e(\mathcal{H}) \leq f(n,k,r,\lfloor (k-1)/2 \rfloor) = f^*(n,k,r,\lfloor (k-1)/2 \rfloor).$$

Theorem 1.8.8 has the sharp upper bound. We extended Theorem for $k \geq 2r+13$ as well as we prove that there is only one hypergraph with the extremal number of hyperedges. To describe the extremal result and to introduce our contributions, we need the following definition that can be considered as an analog of Definition 1.3.6 for higher uniformity.

Definition 1.8.9. *For integers $n, a \geq 1$ and $b_1, \dots, b_t \geq 2$ with $n \geq 2a + \sum_{i=1}^t b_i$ let us denote by $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ the following r-uniform hypergraph.*

- *Let the vertex set of $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ be $A \cup L \cup \bigcup_{i=1}^t B_i$, where $A, B_1, B_2, \dots, B_t$ and L are pairwise disjoint sets of sizes $|A| = a$, $|B_i| = b_i$ $(i = 1,2,\dots,t)$ and $|L| = n - a - \sum_{i=1}^t b_i$.*
- *Let the hyperedges of $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ be*

$$\binom{A}{r} \cup \bigcup_{i=1}^{t} \binom{A \cup B_i}{r} \cup \left\{ \{c\} \cup A' : c \in L, A' \in \binom{A}{r-1} \right\}.$$

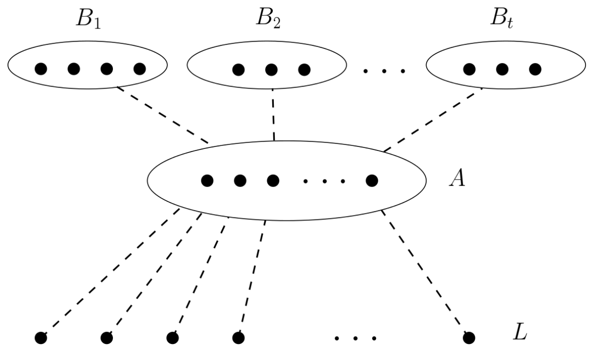

Fig. 1.5 The hypergraph $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$.

Observe that the number of hyperedges in $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is

$$\left(n - a - \sum_{i=1}^{t} b_i\right) \binom{a}{r-1} + \sum_{i=1}^{t} \binom{a+b_i}{r} - (t-1)\binom{a}{r}.$$

Note that, if $a \leq a'$ and $b_i \leq b'_i$ for all $i = 1,2,\dots,t$, then $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is a sub-hypergraph of $\mathcal{H}_{n,a',b'_1,b'_2,\dots,b'_t}$. Finally, the length of the longest path in $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is $2a - t + \sum_{i=1}^t b_i$ if $t \leq a+1$, and $a - 1 + \sum_{i=1}^{a+1} b_i$ if $t > a+1$ and the b_i's are in non-increasing order.

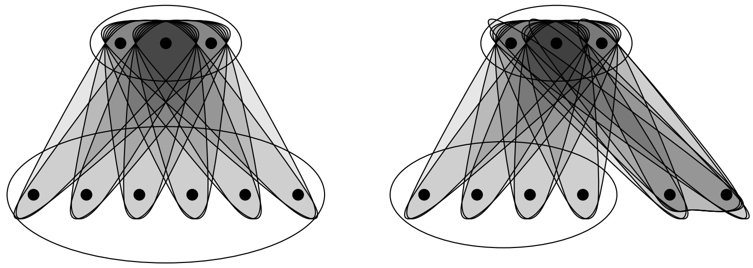

Fig. 1.6 The hypergraphs $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$ and $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$
.

With a slight abuse of notation, we define $\mathcal{H}^+_{n,a}$ to be a hypergraph obtained from $\mathcal{H}_{n,a}$ by adding an arbitrary hyperedge. Hyperedges containing at least $r-1$ vertices from A are already in $\mathcal{H}_{n,a}$, therefore there are $r-1$ pairwise different hypergraphs that we denote by $\mathcal{H}^+_{n,a}$ depending on the number of vertices from A in the extra hyperedge. Observe that the length of the longest path in $\mathcal{H}^+_{n,a}$ is one larger than in $\mathcal{H}_{n,a}$, in particular, if k is even, then $\mathcal{H}^+_{n,\lfloor\frac{k-1}{2}\rfloor}$ does not contain a Berge-path of length k.

Let us denote the maximum number of edges in an n-vertex H-free connected r-uniform hypergraph by $\mathrm{ex}^{conn}_r(n,H)$.

Theorem 1.8.10 (Győri, Salia , Zamora [80]). *For all integers k,r with $k \geq 2r+13 \geq 18$ there exists $n_{k,r}$ such that if $n > n_{k,r}$, then we have*

- $\mathrm{ex}^{conn}_r(n,\mathcal{BP}_k) = |\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}|$, *if k is odd, and*
- $\mathrm{ex}^{conn}_r(n,\mathcal{BP}_k) = |\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}|$, *if k is even.*

Depending on the parity of k, the unique extremal hypergraph is $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$ or $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$, (see Figure 1.6).

Later we extended this result to a stability result, since the proof is similar we will skip the proof of this theorem in this work. Instead, we show the idea of it here.

At first let us study the extremal hypergraphs $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$ and $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$, Figure 1.6. For fixed parity of k the extremal hypergraph is unique, let us describe the case when k is odd.[8] The hypergraph $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$ has two classes of vertices, one with degree $\Theta(n)$ and the others with constant.[9] It is easy to note that two vertices from the sparse family can not be neighbors in a Berge-path, since there is no hyperedge incident with both. Therefore the longest Berge-path and cycle of $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$ has length $k-1$. This special structure was an inspiration

[8] This case is technically easier than the other.
[9] As function of n.

to use the following argument for the desired result. Proof is by contradiction, let $\mathcal{H}$ be a connected r-uniform $\mathcal{BP}_k$-free hypergraph with $e(\mathcal{H}) \geq e(\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor})$. At first we found a sub-hypergraph with a large minimum degree, by a classical argument, removing low degree vertices. The second important step was to show that the hypergraph we got was still connected. Then by classical Dirac argument [23], we show that the hypergraph contains a Berge-cycle of length $k-1$ as the unique extremal hypergraph $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$. The rest is to show that there is a partitioning of vertices in two sets with the desired properties, which is deduced from the Berge-cycle of length $k-1$.

This result was followed by a stability result which we show in Section 1.9, and prove in Chapter 5.

1.8.2 Hypergraphs without long Berge-cycles

Similarly, as for the forbidden path case, the extremal hypergraphs when Berge-cycles of length at least k are forbidden are different in the cases when $k \geq r+2$ and $k \leq r+1$ with an exceptional third case when $k = r$ case. The latter has a surprisingly different extremal hypergraph. Füredi, Kostochka, Luo [47] provide sharp bounds and extremal constructions for infinitely many n, for $k \geq r+3 \geq 6$. Later in [48] they also determined the exact bounds and extremal constructions for all n, for the case $k \geq r+4$. Kostochka, Luo [90] determine a bound for $k \leq r-1$ which is sharp for infinitely many n. Ergemlidze, Győri, Methuku, Salia, Tompkins, Zamora [38] determine a bound in the cases where $k \in \{r+1, r+2\}$. The case when $k = r$ remained open. Both papers [90, 38] conjectured the maximum number of edges to be bounded by $\max\left\{\frac{(n-1)(r-1)}{r}, n-(r-1)\right\}$ (See Figure 1.4, the hypergraph $\mathcal{S}_n^{(r)}$ is on the right).

Theorem 1.8.11 (Füredi, Kostochka, Luo [48, 47])**.** *Let $r \geq 3$ and $k \geq r+3$, then*

$$\mathrm{ex}(n, \mathcal{BC}_{\geq k}) \leq \frac{n-1}{k-2}\binom{k-1}{r}.$$

Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{k-1}$ and $\mathcal{H}[D] = K_{k-1}^r$, (see Figure 1.4).

Theorem 1.8.12 (Győri, Methuku, Salia, Tompkins, Zamora [38])**.** *If $r \geq 3$ then*

$$\mathrm{ex}(n, \mathcal{BC}_{\geq r+2}) \leq \frac{(n-1)(r+1)}{r}.$$

Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D] = K_{r+1}^r$, (see Figure 1.4).

Theorem 1.8.13 (Kostochka, Luo [90]). *Let $k \geq 4, r \geq k+1$ and*

$$\mathrm{ex}(n, \mathcal{B}C_{\geq k}) \leq \frac{(k-1)(n-1)}{r}.$$

Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D]$ consists of $k-1$ hyperedges, (see Figure 1.4).

Theorem 1.8.14 (Győri, Methuku, Salia, Tompkins, Zamora [38]). *If $r \geq 3$ then*

$$\mathrm{ex}(n, \mathcal{B}C_{\geq r+1}) \leq n-1.$$

Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D]$ consists of r hyperedges, (see Figure 1.4).

Let us note that our proof method is very different from the proof of Füredi, Kostochka, Luo [47] [90]. One of our main ideas in proving the above two theorems is an unusual application of Hall's theorem to vertices of the hypergraph (instead of applying it to edges in the shadow of the hypergraph). This allows us to assign a distinct hyperedge to each vertex which is then very helpful in finding Berge-cycles. The notion of connectivity (i.e., the notion of "cut hyperedges") in Berge-hypergraphs has also been very important. These ideas may have further applications (see for e.g. [56]).

Note that Theorem 1.8.14 implies Theorem 1.8.3. In fact, it gives the following stronger form. We prove this implication in Chapter 4.

Theorem 1.8.15 (Győri, Methuku, Salia, Tompkins, Zamora [38]). *Fix $k = r+1 > 2$ and let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge-path of length k. Then, $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r} = n$. Moreover, equality holds if and only if each connected component D of $\partial_2(\mathcal{H})$ is K_{r+1}, and $\mathcal{H}[D] = K_{r+1}^r$.*

Theorem 1.8.16 (Győri, Lemons, Salia, Zamora [73]). *Let k, n and r be positive integers such that $3 \leq k < r$. Then*

$$\mathrm{ex}_r(n, \mathcal{B}C_{\geq k}) = \left\lfloor \frac{n-1}{r} \right\rfloor (k-1) + \mathbb{1}_{r\mathbb{N}^*}(n).$$

If $r|(n-1)$, then the only extremal n-vertex r-graphs are the $(r+1, k-1)$-block trees, (see Figure 1.4).

We note that as a corollary of Theorem 1.8.16, we obtain a slightly stronger version of Theorem 1.8.1.

Corollary 1.8.17 (Győri, Lemons, Salia, Zamora [73]). *Let k,n and r be positive integers with $3 \leq k \leq r$. Then*

$$\mathrm{ex}_r(n,\mathcal{BP}_k) = \left\lfloor \frac{n}{r+1} \right\rfloor (k-1) + \mathbb{1}_{(r+1)\mathbb{N}^*}(n+1).$$

Theorem 1.8.18 (Győri, Lemons, Salia, Zamora [73]). *Let $r > 2$ and n be positive integers. Then*

$$\mathrm{ex}_r(n,\mathcal{BC}_{\geq r}) = \max\left\{ \left\lfloor \frac{n-1}{r} \right\rfloor (r-1), n-r+1 \right\}.$$

When $n-r+1 > \frac{n-1}{r}(r-1)$ the only extremal graph is $\mathcal{S}_n^{(r)}$. When $\frac{n-1}{r}(r-1) > n-r+1$ and $r|(n-1)$ the only extremal graphs are the $(r+1,k-1)$-block trees, (see Figure 1.4).

Remark 1.8.19 (Győri, Lemons, Salia, Zamora [73]). *In particular, when $n \geq r(r-2)+2$, we have that $\mathrm{ex}_r(n,\mathcal{BC}_{\geq r}) = n-r+1$ and $\mathcal{S}_n^{(r)}$ is the only extremal hypergraph.*

Theorem 1.8.20 (Győri, Lemons, Salia, Zamora [73]). *Let k,n and r be positive integers such that $2 \leq k \leq r$. Then*

$$\mathrm{ex}_r^{multi}(n,\mathcal{BC}_{\geq k}) = \left\lfloor \frac{n-1}{r-1} \right\rfloor (k-1).$$

If $r-1|(n-1)$ then the only extremal graphs with n vertices are the $(r,k-1)$-block trees.

As a corollary of Theorem 1.8.20 we obtain a version of Theorem 1.8.1 with multiple hyperedges.

Corollary 1.8.21 (Győri, Lemons, Salia, Zamora [73]). *Let k,n and r be positive integers with $2 \leq k \leq r$ then*

$$\mathrm{ex}_r^{multi}(n,\mathcal{BP}_k) = \left\lfloor \frac{n}{r} \right\rfloor (k-1).$$

Kostochka and Luo obtain Theorem 1.8.13 from the incidence bipartite graph by investigating the structure of 2-connected bipartite graphs. Similarly, Jackson [84] gives an upper bound on the number of edges of a multi r-uniform hypergraph with no Berge-cycle of length at least r.

Theorem 1.8.22 (Jackson [84]). *Let G be a bipartite graph with bipartition A and B such that $|A| = n$ and every vertex in B has degree at least r. If $|B| > \left\lfloor \frac{n-1}{r-1} \right\rfloor (r-1)$, then G contains a cycle of length at least $2r$.*

We study the structure of r-uniform hypergraphs containing no Berge-cycles of length at least k, for all $3 \leq k \leq r$. By exploring the structure of the hypergraphs, instead of bipartite graphs, we can find the extremal number in the case when $k = r$, which also gives us a simple proof for Theorem 1.8.13. Furthermore, our method lets us determine the extremal number for every value of n in both simple r-uniform hypergraphs and multi r-uniform hypergraphs.

At the end of this section, we want to present one of our favorite conjectures.

Super-cyclic bipartite graphs

Kostochka, Lavrov, Luo, Zirlin [89, 91, 88] investigated pan-cyclic and super-pan-cyclic graphs and hypergraphs. Without getting into the details we would like to state a version of their conjecture here.

Conjecture 1.8.23 (Kostochka, Lavrov, Luo, Zirlin [88])**.** *Let G be a bipartite graph, with partite sets A and B. Let for every subset A' of A, $|A'| \geq 2$, the number of vertices incident with at least two vertices of A' is at least $|A'|$ in G. Then for every subset A' of A, $|A'| \geq 2$, there is a cycle $C_{A'}$ in G such that $V(C_{A'}) \cap A = A'$.*

Note that this conjecture has an equivalent variant for non-uniform Hypergraphs.

1.8.3 Hypergraphs without Berge-trees

The Turán number of certain kinds of trees in r-uniform hypergraphs has long been a major topic of research. For example, there is a notoriously difficult conjecture of Kalai [42] which is more general than the Erdős-Sós conjecture (see Conjecture 1.3.3). The trees which Kalai considers are generalizations of the notion of tight paths in hypergraphs. In another direction, Füredi [45] investigated linear trees, constructed by adding $r-2$ new vertices to every edge in a (graph) tree. In this setting, he proved asymptotic results for all uniformities at least 4. Whereas, the articles above considered classes of trees containing tight and linear paths, respectively, we will consider the setting of Berge-trees.

In the range when $k > r$, a number of results on forbidding Berge-trees were obtained by Gerbner, Methuku, Palmer [56]. In particular, they proved that if we assume the Erdős-Sós Conjecture 1.3.3 holds for a tree T with k edges and all of its sub-trees and also that $k > r+1$, we have $\mathrm{ex}_r(n, \mathcal{B}T) \leq \frac{n}{k}\binom{k}{r}$ (a construction matching this bound when k divides n is given by n/k disjoint copies of the complete r-uniform hypergraph on k vertices). In the present paper, we will consider the range $r > k$, where we prove some exact results.

Considering multi-hypergraphs, we prove the following.

Theorem 1.8.24 (Győri, Salia, Tompkins, Zamora [78])**.** *Let n,k,r be positive integers and let T be a k-edge tree, then for all $r \geq (k-1)(k-2)$,*

$$\mathrm{ex}_r^{multi}(n,\mathcal{B}T) \leq \frac{n(k-1)}{r}.$$

If $r > (k-1)(k-2)$ and T is not a star, equality holds if and only if r divides n and the extremal multi-hypergraph is $\frac{n}{r}$ disjoint hyperedges, each with multiplicity $k-1$. If T is a star equality holds only for all $(k-1)$-regular multi-hypergraphs.

We conjecture that Theorem 1.8.24 holds for the following wider set of parameters.

Conjecture 1.8.25 (Győri, Salia, Tompkins, Zamora [78])**.** *Let n,k,r be positive integers and let T be a k-edge tree, then for all $r \geq k+1$,*

$$\mathrm{ex}_r^{multi}(n,\mathcal{B}T) \leq \frac{n(k-1)}{r}.$$

For all trees T, where T is not a star, equality holds if and only if r divides n and the extremal multi-hypergraph is $\frac{n}{r}$ disjoint hyperedges each with multiplicity $k-1$.

The special case of Conjecture 1.8.25, when the forbidden tree is a path, was settled by Győri, Lemons, Salia, Zamora [73] (see the first corollary).

We now define a class of hypergraphs which we will need when we classify the extremal examples in our main result about simple hypergraphs, Theorem 1.8.28.

Definition 1.8.26. *An r-uniform hypergraph $\mathcal{H}$ is* two-sided *if $V(\mathcal{H})$ can be partitioned into a set X and pairwise disjoint sets A_i, $i = 1,2,\ldots,t$ (also disjoint from X) of size $r-1$, such that every hyperedge is of the form $\{x\} \cup A_i$ for some $x \in X$. We say that a two-sided r-uniform hypergraph is (a,b)-regular if every vertex of X has degree a and every vertex of $\bigcup_{i=1}^{t} A_i$ has degree b.*

Remark 1.8.27. *A two-sided r-uniform hypergraph can also be viewed as a graph obtained by taking a bipartite graph G with bipartite classes X and Y, and "blowing up" each vertex of Y to a set of size $r-1$, and replacing each edge $\{x,y\}$ by the r-hyperedge containing x together with the blown-up set for y.*

Theorem 1.8.28 (Győri, Salia, Tompkins, Zamora [78])**.** *Let n,k,r be positive integers and let T be a k-edge tree which is not a star, then for all $r \geq k(k-2)$,*

$$\mathrm{ex}_r(n,\mathcal{B}T) \leq \frac{n(k-1)}{r+1}.$$

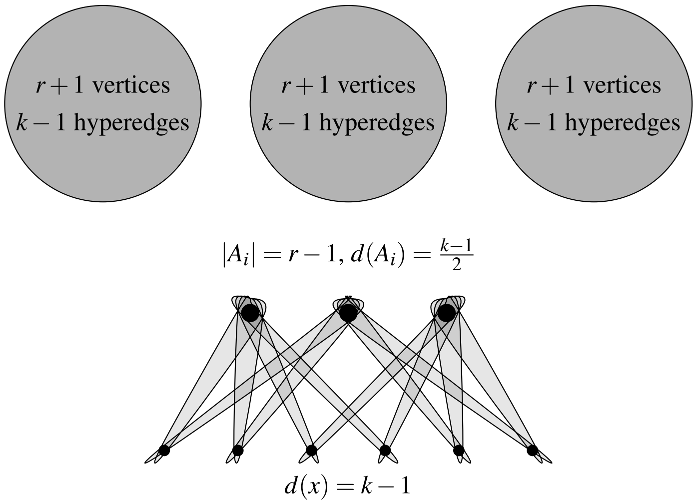

Fig. 1.7 An extremal graph, Theorem 1.8.28.

Equality holds if and only if $r+1$ divides n, and the extremal hypergraph is obtained from $\frac{n}{r+1}$ disjoint sets of size $r+1$, each containing $k-1$ hyperedges. Unless k is odd, and T is the balanced double star, where the balanced double star is the tree obtain from an edge by adding $\frac{k-1}{2}$ incident edges to each of the ends of the edge, in which case equality holds if and only if $r+1$ divides n and $\mathcal{H}$ is obtained from the disjoint union of sets of size $r+1$ containing $k-1$ hyperedges each and possibly a $(k-1, \frac{k-1}{2})$-regular two-sided r-uniform hypergraph (see Figure 1.7).

1.9 Stability

As we have seen, the Turán number depicts how a global parameter number of edges influence local substructures. In particular, for a given substructure, we determine the maximum number of edges. The graphs with the maximum number of edges without a given substructure are called Extremal graphs. Often such graphs are finite and unique. In other words, if we have ’many’ edges and no given substructure, our graph must be one from the given set of Extremal graphs. Naturally, arises another question are these properties stable? in particular, if we have ’nearly as many’ edges as in an extremal graph and no substructure are we ’close to’ a graph from the extremal family? One needs to define what do we mean in ’nearly as many’ and ’close to’. Those concepts are always dependent on the settings of the problem.

We open this section with a famous stability Theorem of Erdős, Simonovits [25, 108], which has many useful applications [2, 10, 107, 112].

Theorem 1.9.1 (Erdős, Simonovits [25, 108]). *Let G be an n vertex K_{r+1} graph with $t(n,r)+o(n^2)$ edges, then by changing $o(n^2)$ edges we can get $T(n,r)$ graph from G.*

While studying almost extremal graphs without K_{r+1}, Simonovits [109] observed that by replacing a vertex from one color class to another in a Turán graph $T(n,r)$ we lose a constant number of edges but the distance to the Turán graph is $\theta(n)$. Therefore one may ask a modified question- what is the minimum number of edges for a K_{r+1}-free graph to be an r-partite graph. This type of questions were investigated in a series of papers [5, 87, 11, 18, 82, 32].

Let us recall Theorem 1.3.7,

$$\mathrm{ex}^{conn}(n,P_\ell) = max\left\{e(G_{n,\ell,1}), e(G_{n,\ell,\lfloor\frac{\ell-1}{2}\rfloor})\right\}.$$

Extremal graphs for this problem were $G_{n,\ell,1}$ or $G_{n,\ell,\lfloor\frac{\ell-1}{2}\rfloor}$ depending how large was n. The stability version of these results was proved by Füredi, Kostochka, Verstraëte [50].

Theorem 1.9.2 (Füredi, Kostochka, Verstraëte [50]). *Let $t \geq 2$, $n \geq 3t-1$ and $k \in \{2t, 2t+1\}$. Suppose we have a n-vertex connected P_k-free graph G with more edges than $\left|G_{n+1,k+1,t-1}\right| - n$, where the graph $G_{n+1,k+1,t-1}$ is described in Definition 1.3.6. Then we have either*

- *$k = 2t$, $k \neq 6$ and G is a sub-graph of $G_{n,k,t-1}$, or*
- *$k = 2t+1$ or $k = 6$, and $G \setminus A$ is a star forest for $A \subseteq V(G)$ of size at most $t-1$.*

Our main result provides a stability version (and thus a strengthening) of Theorem 1.8.10 and also an extension of Theorem 1.9.2 for uniformity at least 3.

First we state it for hypergraphs with minimum degree at least 2, and then in full generality. In the proof, the hypergraphs $\mathcal{H}_{n,\frac{k-3}{2},3}$ and $\mathcal{H}_{n,\frac{k-3}{2},2,2}$ will play a crucial role in case k is odd, while if k is even, then the hypergraphs $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}$, $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3,2}$ and $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,2,2,2}$ will be of importance (Definition 1.8.9), note that all of them are n-vertex, maximal, $\mathcal{BP}_k$-free hypergraphs. In both cases, the hypergraph listed first contains the largest number of hyperedges. This number gives the lower bound in the following theorem.

Theorem 1.9.3 (Gerbner, Nagy, Patkós, Salia, Vizer ,[57]). *For any $\varepsilon > 0$ there exist integers $q = q_\varepsilon$ and $n_{k,r}$ such that if $r \geq 3$, $k \geq (2+\varepsilon)r+q$, $n \geq n_{k,r}$ and $\mathcal{H}$ is a connected n-vertex, r-uniform hypergraph with minimum degree at least 2, without a Berge-path of length k, then we have the following.*

- *If k is odd and $|\mathcal{H}| > |\mathcal{H}_{n,\frac{k-3}{2},3}| = (n-\frac{k+3}{2})\binom{\frac{k-3}{2}}{r-1} + \binom{\frac{k+3}{2}}{r}$, then $\mathcal{H}$ is a sub-hypergraph of $\mathcal{H}_{n,\frac{k-1}{2}}$.*

- *If k is even and* $|\mathcal{H}| > |\mathcal{H}_{n,\lfloor \frac{k-3}{2}\rfloor,4}| = (n - \lfloor \frac{k+5}{2} \rfloor)\binom{\lfloor \frac{k-3}{2}\rfloor}{r-1} + \binom{\lfloor \frac{k+5}{2}\rfloor}{r}$, *then* $\mathcal{H}$ *is a sub-hypergraph of* $\mathcal{H}_{n,\lfloor \frac{k-1}{2}\rfloor,2}$ *or* $\mathcal{H}^{+}_{n,\lfloor \frac{k-1}{2}\rfloor}$.

Let $\mathbb{H}'_{n',a,b_1,b_2,\dots,b_t}$ be the class of hypergraphs that can be obtained from $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ for some $n \le n'$ by adding hyperedges of the form $A'_j \cup D_j$, where the D_j's partition $[n'] \setminus [n]$, all D_j's are of size at least 2 and $A'_j \subseteq A$ for all j. Let us define $\mathbb{H}^{+}_{n',\lfloor \frac{k-1}{2}\rfloor}$ analogously.

Theorem 1.9.4 (Gerbner, Nagy, Patkós, Salia, Vizer ,[57]). *For any* $\varepsilon > 0$ *there exist integers* $q = q_\varepsilon$ *and* $n_{k,r}$ *such that if* $r \ge 3$, $k \ge (2+\varepsilon)r + q$, $n \ge n_{k,r}$ *and* $\mathcal{H}$ *is a connected n-vertex, r-uniform hypergraph without a Berge-path of length k, then we have the following.*

- *If k is odd and* $|\mathcal{H}| > |\mathcal{H}_{n,\frac{k-3}{2},3}|$, *then* $\mathcal{H}$ *is a sub-hypergraph of some* $\mathcal{H}' \in \mathbb{H}'_{n,\frac{k-1}{2}}$.
- *If k is even and* $|\mathcal{H}| > |\mathcal{H}_{n,\lfloor \frac{k-3}{2}\rfloor,4}|$, *then* $\mathcal{H}$ *is a sub-hypergraph of some* $\mathcal{H}' \in \mathbb{H}'_{n,\lfloor \frac{k-1}{2}\rfloor,2}$ *or* $\mathbb{H}^{+}_{n,\lfloor \frac{k-1}{2}\rfloor}$.

We prove Theorem 1.9.3 and 1.9.4 in Capter 5.

Chapter 2

Erdős-Gallai theorem, for vertex colored graphs

> Working with Paul Erdos was like taking a walk in the hills. Every time when I thought that we had achieved our goal and deserved a rest, Paul pointed to the top of another hill and off we would go.
>
> Fan Chung

In a relatively recent paper, Győri, Lemons [71] investigated the extremal number of hypergraphs avoiding Berge-cycles. To this end, they introduced a generalization of the theorem of Erdős-Gallai about paths. Győri, Lemons proved the following Lemma 1.4.1.

Lemma (Győri-Lemons [71])**.** *Let k be a positive integer and G be an n-vertex graph with a proper vertex coloring such that G contains no P_{2k+1} with endpoints of different colors, then*

$$|E(G)| \leq 2kn.$$

This was a useful tool for determining the Turán number of Hypergraphs avoiding Berge-cycles. One can see this lemma as a version of Theta-lemma of Bondy-Simonovits [17], which has many useful applications like in [111].

In this Chapter we show proofs of theorems raised in Section 1.4. In the following section at first we prove Theorem 1.4.2 for paths and in Section 2.2 we prove theorems for trees-Theorem 1.4.4, Theorem 1.4.5, and Theorem 1.4.7.

2.1 Forbidden paths in vertex colored graphs

We start with a proof of Theorem 1.4.2, which eliminates the factor of 2 in Theorem 1.4.1. Thus, we recover the original upper bound from the Erdős-Gallai theorem 1.3.1. We also determine the family of extremal graphs.

Theorem (Salia, Tompkins, Zamora [106])**.** *Let $k \geq 0$ and G be an n-vertex graph with a proper vertex coloring such that G contains no P_{2k+1} with endpoints of different colors, then*

$$|E(G)| \leq kn,$$

and equality holds if and only if $2k+1$ divides n and G is the union of $\frac{n}{2k+1}$ disjoint cliques of size $2k+1$.

Proof. By induction on the number of vertices, we may assume that G is connected and has minimum degree $\delta(G) \geq k$. Indeed, if $\delta(v) < k$ then

$$e(G) = e(G-v) + \delta(v) \leq k(n-1) + k - 1 < kn.$$

If G is C_ℓ-free for all $\ell \geq 2k+1$, then by Theorem 1.3.2 we have

$$|E(G)| \leq \frac{(n-1)2k}{2} < kn.$$

Thus, assume there is a cycle of length at least $2k+1$, and let C be the smallest such cycle with length ℓ. Let the vertices of C be $v_0, v_1, v_2, \ldots, v_{\ell-1}, v_0$, consecutively. Addition and subtraction in subscripts will always be taken modulo ℓ. We say that an edge e is *outgoing* if it has one vertex in $V(C)$ and the other in $V(G) \setminus V(C)$. We say a vertex $v \in V(C)$ is outgoing if it is contained in an outgoing edge.

We will consider cases based on the value of ℓ. Observe that $\ell = 2k+2$ is impossible since $v_0, v_1, \ldots, v_{2k+1}$ is a path of length $2k+1$ but v_0 and v_{2k+1} are adjacent, contradiction.

Case 1. Suppose $\ell \geq 2k+4$. Since we have chosen ℓ to be the length of the smallest C_ℓ with $\ell \geq 2k+1$, we have v_0 cannot be adjacent to any of $v_2, v_3, \ldots, v_{\ell-2k}$ nor any of $v_{2k}, v_{2k+1}, \ldots, v_{\ell-2}$, for otherwise we would have a shorter cycle of length at least $2k+1$. Also note that v_0 is adjacent to v_1 and $v_{\ell-1}$.

Observe that v_0 cannot have two consecutive neighbors in the ℓ-cycle. Indeed, if v_i and v_{i+1} are neighbors of v_0, then we have the following $(2k+1)$-paths starting at v_1: $v_1, v_2, \ldots, v_{2k+1}, v_{2k+2}$ and $v_1, v_2, \ldots, v_i, v_0, v_{i+1}, v_{i+2}, \ldots, v_{2k}, v_{2k+1}$. Thus, v_{2k+1} and v_{2k+2} would have to be colored the same, but this is impossible since they are neighbors.

If v_0 has a neighbor outside of C, say u_0, then we have two paths of length $2k+1$: $u_0, v_0, v_1, \ldots, v_{2k}$ and $v_{2k}, v_{2k-1}, \ldots, v_0, v_{\ell-1}$. It follows that u_0 and $v_{\ell-1}$ have the same color. Similarly, u_0 and v_1 have the same color. Thus, $v_{\ell-1}$ and v_1 also have the same color, and similarly, for every i such that v_i is outgoing, we can conclude v_{i-1} and v_{i+1} have the same color.

If $\ell = 2k+4$ and there is an outgoing vertex, say v_0, then v_1 and v_{2k+3} have the same color (from above), a contradiction since v_1 and v_{2k+2} also have the same color (they are endpoints of a length $2k+1$ path along the cycle C). If there is no outgoing vertex in $V(C)$, then C uses all vertices of the graph. Since no vertex of the cycle has two consecutive neighbors, it follows that each degree is bounded by $2 + \lceil \frac{2k-5}{2} \rceil \leq k$ and so the number of edges is at most $\frac{(2k+4)k}{2} = \frac{nk}{2} < nk$.

If $\ell \geq 2k+5$, we will show that v_0 has an outgoing edge from the ℓ-cycle C. Suppose not, then since v_0 does not have consecutive neighbors, it follows that v_0 has at most

$$2 + \left\lceil \frac{2k - (\ell - 2k + 1)}{2} \right\rceil \leq k - 1$$

neighbors, a contradiction. Thus, v_0 and similarly every other v_i has an outgoing neighbor, and it follows that for every i, the vertices v_i and v_{i+2} have the same color. Hence v_0 and v_{2k} have the same color, contradicting that v_0 and v_{2k+1} have the same color, since they are endpoints of a P_{2k+1}.

Case 2. Suppose $\ell = 2k+3$. For all $0 \leq i \leq \ell - 1$, $v_{i+2}, v_{i+1}, \ldots, v_{\ell-1}, v_0, \ldots, v_i$ is a path of length $2\ell + 1$, and so v_i and v_{i+2} have the same color. Thus, v_0 and v_{2k+2} have the same color, but they are adjacent, contradiction.

Case 3. Finally, suppose $\ell = 2k+1$. If no edge is outgoing, then we are done, since by connectivity the total number of edges in the graph is at most $\binom{2k+1}{2} = kn$. If indeed the total number of edges is kn, then G is a clique. This is the only case when equality holds. From here on, we will assume there is an outgoing edge.

Observe that if u is not a vertex of C, then u cannot have two consecutive neighbors in C, for otherwise we would have a cycle of length $2k+2$. Moreover, u cannot be connected to v_i and v_{i+3}, since there would be paths of length $2k+1$ from u to v_{i+1} and v_{i+2}. It follows that u can have at most $k-1$ neighbors in C and, thus, must have a neighbor outside C.

If there are some two consecutive non outgoing vertices in C, then we may take two such vertices v_i and v_{i+1}, for some index i, so that the next vertex v_{i+2} is outgoing. Suppose $\{v_{i+2}, u\}$ is an outgoing edge. By the previous observation, there is an edge $\{u, w\}$ where $w \notin C$. So we have a $2k+1$ length path from v_i to w, then v_{i+1} cannot have two consecutive neighbors from C, since that would also imply that there is also $2k+1$ length path from w to

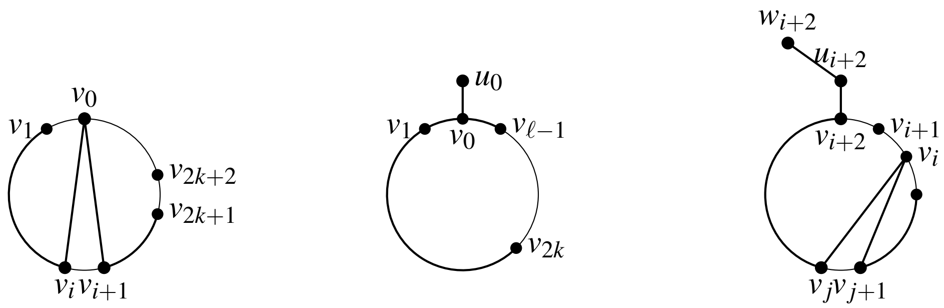

Fig. 2.1 Sketch of the proof of Theorem 1.4.2. The picture on the left is for Case 1, and the other pictures show Case 4.

v_{i-1}, similarly v_i cannot have two consecutive neighbors in C, hence v_i and v_{i+1} have degree at most k. By removing these two vertices, we remove $2k-1$ edges, and by the induction hypothesis the resulting graph has at most $k(n-2)$ edges. So $e(G) < kn$.

For every i, either v_{i+1} or v_{i+2} is an outgoing vertex. Hence the vertex v_i has either the same color as v_{i+2}, if v_{i+1} is an outgoing vertex, or the same color as v_{i+4}, if v_{i+2} is an outgoing vertex. Hence by repeatedly applying this argument we obtain that v_0 has the same color as v_{2k} or v_1, contradiction. $\square$

2.2 Forbidden trees in vertex colored graphs

Let us recall Theorem 1.4.4.

Theorem. *Let T be a tree with k edges such that in the (unique) proper vertex 2-coloring of T all leaves are not the same color, then* $\mathrm{ex}^c(n,T) \leq (k-1)n$.

Proof. There is a path of odd length in T with endpoints which are leaves. Let G be an n-vertex graph with more than $(k-1)n$ edges with a proper vertex coloring. We may find a subgraph G' of G with the average degree at least that of G and minimum degree greater than $k-1$. The proper coloring of G induces a proper coloring of G' and so applying Theorem 1.4.1 we may find a copy of $P_{2\ell+1}$ in G' with endpoints of distinct colors. We may now build up the rest of the tree with a greedy argument, as every degree in G' is at least k and T has $k+1$ vertices. Thus, we have found a copy of T in the graph G with leaves of at least two colors. $\square$

We recall Theorem 1.4.5.

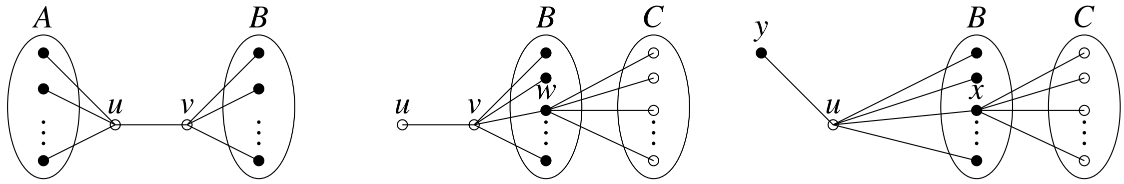

Fig. 2.2 The graph $S_{a,b}$.

Theorem. *Let T be a tree with k edges such that in the proper vertex 2-coloring of T all leaves are the same color, then $\mathrm{ex}^c(n,T) = \left\lfloor \frac{n^2}{4} \right\rfloor$, provided n is sufficiently large.*

Proof. The fact that all leaves are colored the same by a 2-coloring implies that all paths between a pair of leaves have even length. We add an edge e to T connecting an arbitrary pair of leaves and let G be the resulting graph. Since G has an odd cycle, its chromatic number is 3, and the deletion of e yields a 2-chromatic graph. It follows from a theorem of Simonovits [108] that if n is sufficiently large, the extremal number of G is precisely $\mathrm{ex}(n,G) = \left\lfloor \frac{n^2}{4} \right\rfloor$. Thus, in any n-vertex graph with more than $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges we have a copy T with two adjacent leaves, and so in any proper coloring of this graph we have a copy of T with leaves of at least 2 colors. It follows that $\mathrm{ex}^c(n,T) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$, and this bound is realized by the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. □

Remark 2.2.1. *The paths of even length P_{2k} are a special case of Theorem 1.4.5. Better bounds on n are known to exist. For example, the result of Füredi [46] on the extremal number of odd cycles implies that $n \geq 4k$ is sufficient.*

We believe that a strengthening of Conjecture 1.4.6 should hold for trees whose 2-coloring yields two leaves of different colors.

Let us recall and prove a special case of conjecture 1.4.6, Theorem 1.4.7.

Theorem. *For positive integers a and b, let $S_{a,b}$ denote the tree on $a+b+2$ vertices consisting of an edge $\{u,v\}$ where $|N(u)\setminus v| = a$, $|N(v)\setminus u| = b$ and $N(u)\cap N(v) = \varnothing$ (See Figure 2.2, left). We have $\mathrm{ex}^c(n,S_{a,b}) \leq \frac{a+b}{2}n$.*

Proof. Let G be a vertex colored graph with $|E(G)| > |V(G)|\,\frac{a+b}{2}$. Without loss of generality, suppose $a \leq b$. We may assume by induction that $\delta(G) > \frac{a+b}{2} \geq a$. Since $\mathrm{ex}(m,S_{a,b}) = m\frac{a+b}{2}$ (see, for example [101]), it follows that G contains a copy of $S_{a,b}$. Suppose this copy is defined by the edge $\{u,v\}$ together with the disjoint sets $A \subseteq N(u)$, $B \subseteq N(v)$ with $|A| = a$, $|B| = b$. Now, if there is more than one color in $A\cup B$, then we are done. So suppose the color of all vertices in $A\cup B$ is the same. Hence $A\cup B$ is an independent set.

If u is not adjacent to some $w \in B$ (See Figure 2.2, middle), since $|N(w)| \geq a+1$, we can pick $C \subseteq N(w) \setminus \{u,v\}$ of size a. So the edge $\{v,w\}$ together with the sets $B' = (B \cup \{u\}) \setminus \{w\}$ and C define a $S_{a,b}$, where the colors of all vertices in C are different from the colors of $B' \setminus \{u\}$.

If u is adjacent to all $w \in B$, then fix $x \in B$ (See Figure 2.2, right). Since $|N(x)| \geq a+1$, we can pick $C \subseteq N(x) \setminus \{u\}$ of size a. Let $y \in A$ and define $B' = (B \cup \{y\}) \setminus \{x\}$. Observe that $B' \subseteq N(u)$, and the edge $\{u,x\}$ together with the sets B' and C defines a $S_{a,b}$, where again the color of the vertices in C is different from the color of vertices in B'. $\square$

Chapter 3

Hypergraph girth problem

> It's like asking why is Ludwig van Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is.
>
> Paul Erdős

3.1 Connection

In this chapter, we prove an upper bound on the number of triangles in pentagon-free graphs Theorem 1.5.3 and an upper bound on the number of hyperedges in three uniform hypergraphs without Berge-cycle of length 4, Theorem 1.7.8. At first one may think these two problems are non-related, and it may seem an odd choice to put them together in this chapter. We try to justify our choice at the end of this chapter, in Section 3.4.

3.2 Pentagons vs. triangles

We open this section by recalling Theorem 1.5.3.

Theorem (Ergemlidze, Győri, Methuku, Salia [36])**.**

$$\mathrm{ex}(n, C_3, C_5) \leq \frac{1}{2\sqrt{2}}(1 + o(1))n^{3/2}.$$

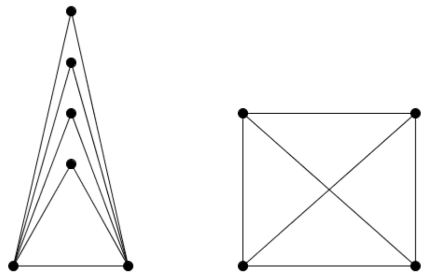

Fig. 3.1 An example of a crown-block and a K_4-block.

Proof. Let G be a C_5-free graph with the maximum possible number of triangles. We may assume that each edge of G is contained in a triangle because otherwise, we can delete it without changing the number of triangles. Two triangles T, T' are said to be in the same *block* if they either share an edge or if there is a sequence of triangles $T, T_1, T_2, \dots, T_s, T'$ where each triangle of this sequence shares an edge with the previous one (except the first one of course). It is easy to see that all the triangles in G are partitioned uniquely into blocks. Below we will characterize the blocks of G.

A block of the form $\{abc_1, abc_2, \dots, abc_k\}$ where $k \geq 1$, is called a *crown-block* (i.e., a collection of triangles containing the same edge) and a block consisting of all triangles contained in the complete graph K_4 is called a K_4*-block*. See Figure 3.1.

Claim 3.2.1. *Every block of G is either a crown-block or a K_4-block.*

Proof. If a block contains only one or two triangles, then it is easy to see that it is a crown-block. So we may assume that a block of G contains at least three triangles and let abc_1, abc_2 be two of them. We claim that if bc_1x or ac_1x is a triangle in G which is different from abc_1, then $x = c_2$. Indeed, if $x \neq c_2$, then the vertices a, x, c_1, b, c_2 contain a C_5, a contradiction. Similarly, if bc_2x or ac_2x is a triangle in G which is different from abc_2, then $x = c_1$.

Therefore, if ac_i or bc_i (for $i = 1, 2$) is contained in two triangles, then abc_1c_2 forms a K_4. However, then there is no triangle in G which shares an edge with this K_4 but is not contained in it, because otherwise, it is easy to find a C_5 in G, a contradiction. So in this case, the whole block consists only of a K_4, and we are done.

So we can assume that whenever abc_1, abc_2 are two triangles then the edges ac_1, bc_1, ac_2, bc_2 are each contained in exactly one triangle. Therefore, any other triangle which shares an edge with either abc_1 or abc_2 must contain ab. Let abc_3 be such a triangle. Then applying the same argument as before for the triangles abc_1, abc_3 one can conclude that the edges ac_3, bc_3 are contained in exactly one triangle and so, any other triangle of G which shares an edge with one of the triangles abc_1, abc_2, abc_3 must contain ab again. So by induction, it is easy to see that all of the triangles in this block must contain ab. Therefore, it is a crown-block, as needed. □

Notice that, by the maximality, blocks of G are edge-disjoint. We claim that there is no C_4 in G whose edges lie in different blocks of G.

Claim 3.2.2. *The edge set of every C_4 is contained in some block of G.*

Proof. Let $xyzw$ be a 4-cycle in G. Every edge of G is contained in a triangle. So in particular, let xyu be a triangle containing the edge xy. If $u \notin \{x,y,z,w\}$ then $uxwzy$ is a C_5, a contradiction. Therefore, $u = z$ or $u = w$. So either xyz and yzw or xyw and ywz are triangles of G. In both cases, the two triangles share an edge, so they belong to the same block. Hence, all four edges of $xyzw$ lie in the same block. □

We are now ready to prove the theorem. We want to select a C_4-free subgraph G_0 of G such that the number of edges in G_0 is the same as the number of triangles in G. By Claim 3.2.1 the edge set of every C_4 is contained in some block of G. To make sure the selected subgraph G_0 is C_4-free, it suffices to make sure the edges selected from each block of G do not contain a C_4, which is done as follows: From each crown-block $\{abc_1, abc_2, \ldots, abc_k\}$, we select the edges $ac_1, ac_2, \ldots, ac_k$ to be in G_0. From each K_4-block $abcd$ we select the edges ab, bc, ac, ad to be in G_0 (since every block is either a crown-block or a K_4-block by Claim 3.2.1, we have dealt with all blocks of G). Finally, notice that the number of selected edges in each block is exactly the number of triangles in that block. Moreover, since blocks are edge-disjoint, we never select the same edge twice. Therefore, since every triangle of G is contained in some block, the total number of triangles in G is the same as the number of edges in G_0. On the other hand, as G_0 is C_4-free and also C_5-free (as it is a subgraph of G), we can use Theorem 1.2.9, to show that the number of edges in it is at most $\frac{1}{2\sqrt{2}}(1+o(1))n^{3/2}$, completing the proof of Theorem 1.5.3.

□

3.3 3-uniform $\mathcal{BC}_4$-free hypergraphs

In this section we prove Theorem 1.7.8.

Theorem (Ergemlidze, Győri, Methuku, Salia, Tompkins [37])**.**

$$\mathrm{ex}_3(n, C_4) \leq (1+o(1))\frac{n^{3/2}}{\sqrt{10}}.$$

Proof. Let $\mathcal{H}$ be a 3-uniform hypergraph with no $\mathcal{BC}_4$ and no isolated vertices. A block $\mathcal{B}$ of a hypergraph $\mathcal{H}$ is defined to be a maximal subparagraph of $\mathcal{H}$ with the property that for any two edges $e, f \in E(\mathcal{B})$, there is a sequence of edges of $\mathcal{H}$, $e = e_1, e_2, \ldots, e_t = f$, such that $|e_i \cap e_{i+1}| = 2$ for all $1 \leq i \leq t-1$ and $V(\mathcal{B}) = \cup_{h \in E(\mathcal{B})} h$. It is easy to see that the blocks of $\mathcal{H}$ define a unique partition of $E(\mathcal{H})$.

For a block $\mathcal{B}$ and an edge $h \in E(\mathcal{B})$, we say h is a *leaf* if there exists $x \in h$ such that the only edge of $\mathcal{B}$ incident to x is h. It is simple to observe that the set of non-leaf edges of a block $\mathcal{B}$ is either the empty set, a single edge, or the edges of a complete hypergraph on 4-vertices minus an edge, $K_4^{(3)-}$. Even more, if the set of non-leaf edges of $\mathcal{B}$ is $E(K_4^{(3)-})$, then $\mathcal{B} = K_4^{(3)-}$. This implies that the set $B(\mathcal{H}) = \{\mathcal{B} \mid \mathcal{B} \text{ is a block in } \mathcal{H}\}$ of all blocks of $\mathcal{H}$, can be partitioned into the following types of blocks:

1. We say $\mathcal{B} \in B(\mathcal{H})$ is *type 1* if there exists an edge $e \in E(\mathcal{B})$ such that for all distinct $f_1, f_2 \in E(\mathcal{B})$, $f_1, f_2 \neq e$, we have $|e \cap f_i| = 2$, for $i = 1, 2$ and $f_1 \cap f_2 \subseteq e$.
2. We say $\mathcal{B} \in B(\mathcal{H})$ is *type 2* if $\mathcal{B} = K_4^{(3)-}$.

Define the 2-shadow of a hypergraph to be the graph on the same set of vertices whose edges are all pairs of vertices $\{x, y\}$ for which there exists an edge $e \in E(\mathcal{H})$ such that $\{x, y\} \subset e$. We denote the 2-shadow of a hypergraph $\mathcal{H}$ by $\partial\mathcal{H}$. The proof of Theorem 1.7.9 will proceed by estimating the number of 3-paths (3-vertex paths) in the 2-shadow of a Berge-C_4-free hypergraph in two different ways. Given a vertex v in a hypergraph $\mathcal{H}$, $d(v)$ denotes the classical hypergraph degree of v. In particular, $d(v) = |\{h \in E(\mathcal{H}) : v \in h\}|$. Let $d_s(v)$ be the (graph) degree of v in the 2-shadow of the hypergraph. In particular, $d_s(v) = |\{e \in E(\partial\mathcal{H}) : v \in e\}|$. Then, we define the *excess degree* of the vertex v to be $d_{ex}(v) = d_s(v) - d(v)$. Finally, we define the *block degree* $d_b(v)$ to be the total number of blocks containing an edge which contains v.

Notice that for every 4-cycle x_1, x_2, x_3, x_4, x_1 of $\partial\mathcal{H}$, there exists three distinct integers $1 \leq i < j < k \leq 4$ such that $\{x_i, x_j, x_k\} \in E(\mathcal{H})$, otherwise, $\mathcal{H}$ contains a copy of Berge-C_4. We call this edge a *representative edge* of this 4-cycle. Note that each 4-cycle of $\partial\mathcal{H}$ has either 1, 2 or 3 representative edges. Two edges of $\mathcal{H}$ sharing two vertices yield a C_4 in $\partial\mathcal{H}$. However these are not the only types of C_4's in $\partial\mathcal{H}$. We call a 4-cycle of $\partial\mathcal{H}$ *rare* if the induced subhypergraph of $\mathcal{H}$ on the vertices of cycle does not contain two edges sharing a diagonal pair of vertices of the 4-cycle. In the following claim, we show that the number of such cycles is small.

We define a particular type of 3-path of $\partial\mathcal{H}$. A 3-path, x_1, x_2, x_3, is called *good* if $\{x_1, x_2, x_3\} \notin E(\mathcal{H})$ and there is no $x \in V(\mathcal{H})$ such that x, x_1, x_2, x_3, x is a rare cycle of $\partial\mathcal{H}$.

Claim 3.3.1. *For any $a,b \in V(\mathcal{H})$, there are at most two good* 3*-paths in $\partial\mathcal{H}$ with end points a and b.*

Proof. Suppose, by contradiction, that there are three distinct vertices v_1, v_2, v_3 different from a and b such that a, v_i, b forms a good 3-path of $\partial\mathcal{H}$ for all integer $1 \le i \le 3$. It follows that there are three Berge-paths a, e_i, v_i, f_i, b, for all integer $1 \le i \le 3$ in $\mathcal{H}$. Note that those edges are not necessarily distinct. But we have $e_i \neq f_i$ and $e_i \neq f_j$, $i \neq j$, since $\{a, v_i\} \subset e_i$ and $\{b, v_j\} \subset f_j$ and $\mathcal{H}$ is 3-uniform. Note that if $e_2 = e_3$, then $e_2 = \{a, v_2, v_3\}$, hence $e_1 \neq e_2$. Similarly, we have either $f_1 \neq f_2$ or $f_1 \neq f_3$. We may assume, without loss of generality, that $e_1 \neq e_2, e_3$. It follows that either $a, e_1, v_1, f_1, b, f_2, v_2, e_2, a$ or $a, e_1, v_1, f_1, b, f_3, v_3, e_3, a$ is a Berge-C_4, a contradiction. □

Claim 3.3.2. *There are at most $6|E(\mathcal{H})|$ rare* 4*-cycles in $\partial\mathcal{H}$.*

Proof. We fix an edge $\{a,b,c\} \in E(\mathcal{H})$. It suffices to show that the edge $\{a,b,c\}$ is representative of at most 6 rare 4-cycles (that is, $\{a,b,c\}$ is contained in the vertex set of at most 6 rare 4-cycles). Suppose by contradiction that this is not true. Observe that there are three possible positions for a fixed vertex v among the vertices of a 4-cycle in $\partial\mathcal{H}$ with $\{a,b,c\}$. By the pigeonhole principle there are 3 distinct vertices v_1, v_2, v_3 different from a, b or c with the same position in the 4-cycle. Without loss of generality, we may assume they form a 4-cycle in the order v_i, a, c, b, v_i. Therefore from the definition of a rare 4-cycle, there are at least three good 3-paths in $\partial\mathcal{H}$ from a to b, a contradiction to Claim 3.3.1.

□

Using Claim 3.3.2, it is easy to see that the number of 3-paths in $\partial\mathcal{H}$ which are not good is at most $3|E(\mathcal{H})| + 3 \cdot 6|E(\mathcal{H})| = 21|E(\mathcal{H})|$. Here we use the fact that each rare 4-cycle induces an edge of $\mathcal{H}$.

By conditioning on the middle vertex of the 3-path, we have the following estimate on the number of 3-paths in $\partial\mathcal{H}$:

$$\#(\text{3-paths in } \partial\mathcal{H}) = \sum_{v \in V(\mathcal{H})} \binom{d_s(v)}{2} = \sum_{v \in V(\mathcal{H})} \binom{d(v) + d_{ex}(v)}{2}.$$

The following claim provides an upper bound on the number of good 3-paths in $\partial\mathcal{H}$.

Claim 3.3.3.

$$\#(\text{good 3-paths in } \partial\mathcal{H}) \le 2\binom{n}{2} - 4\sum_{v \in V(\mathcal{H})} \binom{d_b(v)}{2}.$$

Proof. Fix a vertex v and consider two adjacent edges $\{v, x_1, x_2\}$ and $\{v, y_1, y_2\}$ such that they belong to the different blocks; clearly the vertices v, x_1, x_2, y_1, y_2 are all distinct. We claim that there is at most one good 3-path, namely x_i, v, y_j, between x_i and y_j, for each $i, j \in \{1,2\}$. Suppose this is not the case, then without loss of generality, there exists $u \neq v$ such that x_1, u, y_1 is a good 3-path. By the definition of a good 3-path, there are two distinct edges $h_x, h_y \in \mathcal{H}$ such that $x_1, u \in h_x$ and $y_1, u \in h_y$. If $\{v, x_1, x_2\}$, $\{v, y_1, y_2\}$, h_x and h_y are all different edges, then clearly there is a Berge-4-cycle. Therefore either $\{v, x_1, x_2\} = h_x$ or $\{v, y_1, y_2\} = h_y$. Hence we have $u \in \{x_2, y_2\}$, without loss of generality we may assume $u = x_2$. Observe that the 4-cycle x_1, x_2, y_1, v of $\partial\mathcal{H}$ contains a good 3-path and so by definition the 4-cycle x_1, x_2, y_1, v is not a rare 4-cycle. Hence we have a contradiction to the statement that edges $\{v, x_1, x_2\}$ and $\{v, y_1, y_2\}$ belong to the different blocks. Concluding that there is at most one good path between x_i and y_j. So there are at least $4\sum_{v\in V(\mathcal{H})}\binom{d_b(v)}{2}$ pairs of vertices which have at most one good 3-path between them. From Claim 3.3.1, for each pair of vertices there are at most two good 3-paths in $\partial\mathcal{H}$. These observations complete the proof of Claim 3.3.3. □

Thus, since the number of 3-paths which are not good is at most $21\,|E(\mathcal{H})|$, we have

$$\sum_{v\in V(\mathcal{H})}\binom{d(v)+d_{ex}(v)}{2} = \#(\text{3-paths in } \partial\mathcal{H}) \leq 2\binom{n}{2} - 4\sum_{v\in V(\mathcal{H})}\binom{d_b(v)}{2} + 21\,|E(\mathcal{H})|. \tag{3.1}$$

Now, we will obtain estimates for $\sum_{v\in V(\mathcal{H})} d_{ex}(v)$ and $\sum_{v\in V(\mathcal{H})} d_b(v)$. For each block $\mathcal{B}$ and $v \in V(\mathcal{B})$, let $d_{ex}^{\mathcal{B}}(v)$ denote an excess degree of v inside the hypergraph $\mathcal{B}$. If $\mathcal{B}$ is type 1, then every vertex $v \in V(\mathcal{B})$ has $d_{ex}^{\mathcal{B}}(v) \geq 1$, so for type 1 blocks, $\sum_{v\in V(\mathcal{B})} d_{ex}^{\mathcal{B}}(v) \geq |V(\mathcal{B})|$. It is easy to see that for every block $\mathcal{B}$ we have $|V(\mathcal{B})| > |E(\mathcal{B})|$, so $\sum_{v\in V(\mathcal{B})} d_{ex}^{\mathcal{B}}(v) > |E(\mathcal{B})|$, for every type 1 block $\mathcal{B}$. If $\mathcal{B}$ is a type 2 block, then $\sum_{v\in V(\mathcal{B})} d_{ex}^{\mathcal{B}}(v) = 3 = |E(\mathcal{B})|$. Therefore,

$$\sum_{v\in V(\mathcal{B})} d_{ex}^{\mathcal{B}}(v) \geq |E(\mathcal{B})|$$

for every block $\mathcal{B}$ in $B(\mathcal{H})$. This together with the fact that the blocks define a partition of the edges $E(\mathcal{H})$ implies

$$\sum_{v\in V(\mathcal{H})} d_{ex}(v) = \sum_{\mathcal{B}\in B(\mathcal{H})}\sum_{v\in V(\mathcal{B})} d_{ex}^{\mathcal{B}}(v) \geq \sum_{\mathcal{B}\in B(\mathcal{H})} |E(\mathcal{B})| = |E(\mathcal{H})|. \tag{3.2}$$

On the other hand, a simple double counting argument yields

$$\sum_{v\in V(\mathcal{H})} d_b(v) = \sum_{\mathcal{B}\in B(\mathcal{H})} |V(\mathcal{B})|.$$

Therefore,

$$\sum_{v\in V(\mathcal{H})} d_b(v) = \sum_{\mathcal{B}\in B(\mathcal{H})} |V(\mathcal{B})| \geq \sum_{\mathcal{B}\in B(\mathcal{H})} |\mathcal{B}| = |E(\mathcal{H})|. \tag{3.3}$$

Now we will use the inequalities derived so far to get desired upper bound on $|E(\mathcal{H})|$. By (3.2),

$$4|E(\mathcal{H})| = 3|E(\mathcal{H})| + |E(\mathcal{H})| \leq \sum_{v\in V(\mathcal{H})} (d(v)+d_{ex}(v)).$$

Since $\binom{x}{2}$ is a convex function, by Jensen's inequality we have

$$\binom{\frac{1}{n}\sum_{v\in V(\mathcal{H})}(d(v)+d_{ex}(v))}{2} \leq \frac{1}{n}\sum_{v\in V(\mathcal{H})}\binom{d(v)+d_{ex}(v)}{2}.$$

Combining the above two inequalities we get

$$n\binom{\frac{4|E(\mathcal{H})|}{n}}{2} \leq \sum_{v\in V(\mathcal{H})}\binom{d(v)+d_{ex}(v)}{2}. \tag{3.4}$$

Similarly, by (3.3) and Jensen's inequality, we have

$$n\binom{\frac{|E(\mathcal{H})|}{n}}{2} \leq \sum_{v\in V(\mathcal{H})}\binom{d_b(v)}{2}. \tag{3.5}$$

Combining (3.1), (3.4) and (3.5) we obtain

$$n\binom{\frac{4|E(\mathcal{H})|}{n}}{2} + 4n\binom{\frac{|E(\mathcal{H})|}{n}}{2} \leq 2\binom{n}{2} + 21|E(\mathcal{H})|. \tag{3.6}$$

Rearranging (3.6) yields the desired bound,

$$|E(\mathcal{H})| \leq (1+o(1))\frac{n^{3/2}}{\sqrt{10}}. \qquad \square$$

3.4 Concluding remarks

We start with Lazebnik-Verstraëte theorem [94].

Theorem 3.4.1 (Lazebnik, Verstraëte [94])**.**

$$\mathrm{ex}_3(n,\{\mathcal{BC}_2,\mathcal{BC}_3,\mathcal{BC}_4\}) = \frac{1}{6}n^{3/2} + o(n^{3/2}).$$

Surprisingly it is not known if there is a similar theorem for 4-uniform hypergraph of girth six[1]. Let us state a bold conjecture.

Conjecture 3.4.2.

$$\mathrm{ex}_4(n,\{\mathcal{BC}_2,\mathcal{BC}_3,\mathcal{BC}_4,\mathcal{BC}_5\}) = o(n^{3/2}).$$

Even more, if one believes Conjecture 3.4.2 is false, one may try to confirm the following conjecture for 3-uniform hypergraphs.

Conjecture 3.4.3.

$$\mathrm{ex}_3(n,\{\mathcal{BC}_2,\mathcal{BC}_3,\mathcal{BC}_4,\mathcal{BC}_5\}) = \Theta(n^{3/2}).$$

Conjecture 3.4.3 is a strengthening of Theorem 3.4.1. Note that construction Lazebnik-Verstraëte in Theorem [94] is full of $\mathcal{BC}_5$'s.

Example 3.4.4 (Lazebnik, Verstraëte [94])**.** *Take Füredi graph G from Theorem 1.2.6. In particular, G is a C_4-free graph with a property that for every pair of vertices there is the unique vertex connected with both of them. Therefore every edge is in a unique triangle. Let $\mathcal{H}$ be a 3-uniform hypergraph on the vertex set $V(G)$ and a triple is a hyperedge if and only if it induces a triangle in G.*

From the unique property of G, that every edge is in a unique triangle, $\mathcal{H}$ is linear. Even more, since a Berge-triangle in $\mathcal{H}$ is a triangle in the shadow G, the same property implies that $\mathcal{H}$ is Berge-triangle-free. Similarly, $\mathcal{H}$ is $\mathcal{BC}_4$-free too.

The graph G is full of C_5's. Since G is C_4-free then every C_5 is an induced C_5 in G. Finally, since every edge is in a triangle, we have for every C_5 in G there is a $\mathcal{BC}_5$ in $\mathcal{H}$.

All of these problems are closely related with the problem to determine $\mathrm{ex}_3(n,\mathcal{BC}_4)$. Since we conjectured the lower bound is the correct asymptotic see Theorem 1.7.9, and the hypergraph in Example 1.7.10 is not only $\mathcal{BC}_4$-free but $\mathcal{BC}_3$ and $\mathcal{BC}_5$-free, we have the following relaxed conjecture.

Conjecture 3.4.5.

$$\mathrm{ex}_3(n,\{\mathcal{BC}_3,\mathcal{BC}_4,\mathcal{BC}_5\}) = (1+o(1))\frac{n^{3/2}}{3\sqrt{3}}.$$

Theorem 3.4.6 (Ergemlidze, Győri, Methuku [35])**.**

$$\mathrm{ex}_3(n,\{\mathcal{BC}_2,\mathcal{BC}_3,\mathcal{BC}_5\}) = \mathrm{ex}_3(n,\{\mathcal{BC}_2,\mathcal{BC}_5)\} = \frac{1}{3\sqrt{3}}n^{3/2}+O(n).$$

[1] $\{\mathcal{BC}_2,\mathcal{BC}_3,\mathcal{BC}_4,\mathcal{BC}_5\}$-free

Ergemlidze, Győri, Methuku provided the construction which has a beautiful geometric interpretation. This geometric construction was found by Mubayi-Solymosi.

Example 3.4.7 (Mubayi, Solymosi [110]). *For a fixed integer n, take a cube* $[n]^3$. *The axes parallel lines are* $\{(a_1,b_1,x) : x \in [n]\}$ *for all* $a_1,b_1 \in [n]$, $\{(a_2,x,c_2) : x \in [n]\}$ *for all* $a_2,c_2 \in [n]$, *and* $\{x,b_3,c_3) : x \in [n]\}$ *for all* $b_3,c_3 \in [n]$. *The vertices of the hypergraph* $\mathcal{H}$ *are those lines, hence* $e(\mathcal{H}) = 3n^2$. *Three vertices form a hyperedge of size three if and only if they meet in exactly one point-*

$$E(\mathcal{H}) = \{((a,b,\cdot),(a,\cdot,c),(\cdot,b,c)) : a,b,c \in [n]\}.$$

In other words, all vertices (a,b,c) *of the cube are hyperedges.*

It is easy to verify that, this hypergraph is linear, $\mathcal{BC}_3$ *and* $\mathcal{BC}_5$*-free but full of* $\mathcal{BC}_4$.

Finally, we give you reasoning why the problem of maximizing triangles in a C_5-free graph is closely related to these problems. Let G be a C_5-free graph. As we have seen in the proof of Theorem 1.5.3, triangles are distributed in G as crown blocks or K_4-blocks, see Figure 3.1. It is easy to see that if one constructs a 4-uniform hypergraph on the same set of vertices, taking 4-edges on vertices inducing K_4 in G we would get a 4-uniform girth-6 graph. Even more, by the conjectured lower-bound construction, Example 1.5.2, does not contain any copy of K_4. Therefore proving Conjecture 3.4.2 may be an important step towards proving conjectured upper bound of $\mathrm{ex}(n,C_3,C_5)$.

Chapter 4

The Structure of Hypergraphs avoiding long Berge-cycles

Simplicity is the final achievement. After one has played a vast quantity of notes and more notes, it is simplicity that emerges as the crowning reward of art.

Frederic Chopin

In this Chapter we consider a problem $\mathrm{ex}(n, \mathcal{BC}_{\geq k})$. Naturally, this problem is divided in three parts with different extremal values and constructions.

1. $k \geq r+2$;
2. $k = r$;
3. $k < r$ and $k = r+1$;

Recently, Füredi, Kostochka, Luo [48, 47] proved exact bounds for $k > r+2$. Kostochka, Luo [90] settled $k < r$ case. In the following section we prove cases $k = r+1$ and $k = r+2$, fully settling 1 and 3. After we prove $k = r$ case with a lemma which also slightly improves Kostochka, Luo [90] result in the case $k < r$.

4.1 Avoiding long Berge-cycles, cases $k = r+1$ and $k = r+2$

In this section we prove Theorem 1.8.14 and theorem 1.8.12. Let us recall those theorems.

Theorem (Győri, Methuku, Salia, Tompkins, Zamora [38]). *If $r \geq 3$ then*

$$\mathrm{ex}(n, \mathcal{B}C_{\geq r+1}) \leq n-1.$$

Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D]$ consists of r hyperedges, (see Figure 1.4).

Theorem (Győri, Methuku, Salia, Tompkins, Zamora [38]). *If $r \geq 3$ then*

$$\mathrm{ex}(n, \mathcal{B}C_{\geq r+2}) \leq \frac{(n-1)(r+1)}{r}.$$

Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D] = K^r_{r+1}$, (see Figure 1.4).

4.1.1 Basic Lemmas, used in Subsection 4.1.2 and 4.1.3

Lemma 4.1.1. *For any $r \geq 3$, if a set S of size $r+1$ contains r hyperedges of size r, then between any two vertices $u, v \in S$, there is a Berge-path of length r consisting of these hyperedges.*

Proof. Let $\mathcal{H}$ be the hypergraph consisting of r hyperedges on $r+1$ vertices. First notice that for any pair of vertices $x, y \in S$, the number of hyperedges $h \subset S$ such that $\{x, y\} \not\subset h$ is at most 2. (Indeed, there is at most one hyperedge that does not contain x and at most one hyperedge that does not contain y.) This means that every pair $x, y \in S$ is contained in some hyperedge, as there are at least 3 hyperedges contained in S. In other words, $\partial_2(\mathcal{H}) = K_{r+1}$.

Consider an arbitrary path $x_1x_2, \ldots, x_{r+1}$ of length r in the $\partial_2(\mathcal{H})$ connecting $u = x_1$ and $v = x_{r+1}$. We want to show that there are distinct hyperedges containing the pairs x_ix_{i+1} for each $1 \leq i \leq r$. To this end, we consider an auxiliary bipartite graph with pairs $\{x_1x_2, x_2x_3, \ldots, x_rx_{r+1}\}$ in one class and the r hyperedges $h \subset S$ in the other class, and a pair is connected to a hyperedge if it is contained in the hyperedge. We will show that Hall's condition holds [81]. As noted before, every pair is contained in a hyperedge. Given any two distinct pairs x_ix_{i+1} and x_jx_{j+1}, there is at most one hyperedge that does not contain either of them; i.e., at least $r-1$ hyperedges contain one of them. Thus we need $2 \leq r-1$ for Hall's condition to hold, but this is true as we assumed $r \geq 3$. Moreover, if we take any $3 \leq j \leq r$ distinct pairs, then every hyperedge contains one of them. Therefore, we need $j \leq r$, but this is true by assumption. This finishes the proof of the lemma. □

Lemma 4.1.2. *For any $r \geq 4$, if a set S of size $r+1$ contains $r-1$ hyperedges of size r, then between any two vertices $u, v \in S$, there is a Berge-path of length $r-1$ consisting of these hyperedges.*

Proof. The proof is similar to that of Lemma 4.1.1. Let $\mathcal{H}$ be the hypergraph consisting of $r-1$ hyperedges on $r+1$ vertices. First notice that for any pair of vertices $x, y \in S$, the number of hyperedges $h \subset S$ such that $\{x,y\} \not\subset h$ is at most 2. This means that every pair $x, y \in S$ is contained in some hyperedge, as there are at least $r-1 \geq 3$ hyperedges contained in S. In other words, $\partial_2(\mathcal{H}) = K_{r+1}$.

Consider an arbitrary path $x_1x_2\ldots x_r$ of length $r-1$ in the $\partial_2(\mathcal{H})$ connecting $u = x_1$ and $v = x_r$. We want to show that there are distinct hyperedges containing the pairs x_ix_{i+1} for each $1 \leq i \leq r-1$. To this end, we consider an auxiliary bipartite graph with pairs $\{x_1x_2, x_2x_3, \ldots, x_{r-1}x_r\}$ in one class and the $r-1$ hyperedges $h \subset S$ in the other class, and a pair is connected to a hyperedge if it is contained in the hyperedge. We show that Hall's condition holds: As noted before, every pair is contained in a hyperedge. Given any two distinct pairs x_ix_{i+1} and x_jx_{j+1}, there is at most one hyperedge that does not contain either of them; i.e., at least $r-2$ hyperedges contain one of them. Thus we need $2 \leq r-2$ for Hall's condition to hold, but this is true as we assumed $r \geq 4$. Moreover, if we take any $3 \leq j \leq r-1$ distinct pairs, then every hyperedge contains one of them. Therefore, we need $j \leq r-1$ for Hall's condition to hold, and this is true by assumption. This finishes the proof of the lemma. □

4.1.2 Extremal hypergraphs without $\mathcal{BC}_{\geq r+1}$

We recall Theorem 1.8.14.

Theorem (Győri, Methuku, Salia, Tompkins, Zamora [38])**.** *If $r \geq 3$ then*

$$\mathrm{ex}(n, \mathcal{BC}_{\geq r+1}) \leq n-1.$$

Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D]$ consists of r hyperedges, (see Figure 1.4).

Proof. We use induction on n. For the base cases, notice that the statement of the theorem is trivially true if $1 \leq n \leq r$. Moreover, if $n = r+1$, then $e(\mathcal{H}) \leq r$ because otherwise, $\mathcal{H} = K^r_{r+1}$ and then it is easy to see that there is a (Hamiltonian) Berge-cycle of length $r+1$ in $\mathcal{H}$, a contradiction. Therefore, $e(\mathcal{H}) \leq r = n-1$. Moreover, equality holds if and only if $\partial_2(\mathcal{H}) = K_{r+1}$ and $\mathcal{H}$ consists of r hyperedges.

We will show the statement is true for n assuming it is true for all smaller values. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices having no Berge-cycle of length $r+1$ or longer. We show that we may assume the following two properties hold for $\mathcal{H}$.

(1) For any set $S \subseteq V(\mathcal{H})$ with $|S| \leq |V(\mathcal{H})| - 1 = n-1$, the number of hyperedges of $\mathcal{H}$ incident to the vertices of S is at least $|S|$.

Indeed, suppose there is a set $S \subset V(\mathcal{H})$ (i.e., $|S| \leq |V(\mathcal{H})| - 1$) with fewer than $|S|$ hyperedges incident to the vertices of S. We delete the vertices of S from $\mathcal{H}$ to obtain a new hypergraph $\mathcal{H}'$ on $n-|S|$ vertices. By induction, $\mathcal{H}'$ contains at most $(n-|S|-1)$ hyperedges, so $\mathcal{H}$ contains less than $(n-1-|S|)+|S| = (n-1)$ hyperedges, as required.

(2) There is no cut-hyperedge in $\mathcal{H}$.

Indeed, if $h \in E(\mathcal{H})$ is a cut-hyperedge, then $\partial_2(\mathcal{H} \setminus \{h\})$ is not a connected graph, so there are disjoint non-empty sets V_1 and V_2 such that $V(\mathcal{H}) = V_1 \cup V_2$ and there are no edges of $\partial_2(\mathcal{H} \setminus \{h\})$ between V_1 and V_2. So the hypergraphs $\mathcal{H}[V_1]$ and $\mathcal{H}[V_2]$ do not contain a Berge-cycle of length $r+1$ or longer. Therefore, by induction, $e(\mathcal{H}[V_1]) \leq |V_1| - 1$ and $e(\mathcal{H}[V_2]) \leq |V_2| - 1$. In total, $e(\mathcal{H}) = e(\mathcal{H}[V_1]) + e(\mathcal{H}[V_2]) + 1 \leq (|V_1| + |V_2| - 2) + 1 = |V(\mathcal{H})| - 1$, as desired.

Moreover, we claim that the equality $e(\mathcal{H}) = |V(\mathcal{H})| - 1$ cannot hold in this case (i.e., if there is a cut-hyperedge). Indeed, if equality holds, then we must have $e(\mathcal{H}[V_1]) = |V_1| - 1$ and $e(\mathcal{H}[V_2]) = |V_2| - 1$. Notice that since $r \geq 3$, the hyperedge h either contains at least two vertices $x, y \in V_1$ or two vertices $x, y \in V_2$. Without loss of generality, assume the former is true. By induction, $\partial_2(\mathcal{H}[V_1])$ is connected and for every block D of $\partial_2(\mathcal{H}[V_1])$, we have $D = K_{r+1}$ and the subhypergraph induced by D consists of r hyperedges. So by Lemma 4.1.1, there is a Berge-path of length r (consisting of the r hyperedges induced by D) between any two vertices of a block D. Then it is easy to see that since $\partial_2(\mathcal{H}[V_1])$ is connected, there is a Berge-path $\mathcal{P}$ of length at least r between any two vertices of V_1. In particular, between x and y. Then $\mathcal{P}$ together with h forms a Berge-cycle of length $r+1$ in $\mathcal{H}$, a contradiction.

Consider an auxiliary bipartite graph B consisting of vertices of $\mathcal{H}$ in one class and hyperedges of $\mathcal{H}$ on the other class. Then property (1) shows that Hall's condition holds for all subsets of $V(\mathcal{H})$ of size up to $|V(\mathcal{H})| - 1$. Therefore, there is a matching in B that matches all the vertices in $V(\mathcal{H})$, except at most one vertex, say x. In other words, there exists an injection $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ such that for every $v \in V(\mathcal{H}) \setminus \{x\}$, we have $v \in f(v)$. Given an injection $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ with $v \in f(v)$, let $\mathcal{P}_f$ be a longest Berge-path of the form $v_1 f(v_1) v_2 f(v_2) \dots v_{l-1} f(v_{l-1}) v_l$ where for each $1 \le i \le l-1$, $v_{i+1} \in f(v_i)$. Moreover, among all injections $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ with $v \in f(v)$, suppose $\phi : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ is an injection for which the path $\mathcal{P}_\phi = v_1 \phi(v_1) v_2 \phi(v_2) \dots v_{l-1} \phi(v_{l-1}) v_l$ is a longest path.

Because of the way $\mathcal{P}_\phi$ was constructed, it is also clear that $x \notin \{v_1, v_2, \dots, v_{l-1}\}$. We consider two cases depending on whether v_l is equal to x or not.

Case 1: $v_l \neq x$**.** Our aim is to get a contradiction, and show that this case is impossible.

Claim 4.1.3. *If $v_l \neq x$, then $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \dots, v_l\}$.*

Proof. If $v_l \neq x$, then we claim $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \dots, v_l\}$. Indeed, if $\phi(v_l)$ contains a vertex $v_i \in \{v_1, v_2, \dots, v_{l-r}\}$, then the Berge-cycle $v_i \phi(v_i) v_{i+1} \phi(v_{i+1}) \dots v_l \phi(v_l) v_i$ is of length $r+1$ or longer, a contradiction. Moreover, if $\phi(v_l)$ contains a vertex $v \notin \{v_1, v_2, \dots, v_l\}$, then P_ϕ can be extended to a longer path $v_1 \phi(v_1) v_2 \phi(v_2), \dots, v_{l-1} \phi(v_{l-1}) v_l \phi(v_l) v$, a contradiction again, proving that $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \dots, v_l\}$. □

Fix some $i \in \{l-r+1, l-r+2, \dots, l-1\}$. Let us define a new injection $\psi : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v \notin \{x, v_1, v_2, \dots, v_l\}$, and for every $v \in \{v_1, v_2, \dots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_l)$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l \ge k \ge i+1$. Now consider the Berge-path $v_1 \phi(v_1) v_2 \phi(v_2) \dots v_i \phi(v_l) v_l \phi(v_{l-1}) \dots v_{i+2} \phi(v_{i+1}) v_{i+1}$ $= v_1 \psi(v_1) v_2 \psi(v_2) \dots v_i \psi(v_i) v_l \psi(v_l) \dots v_{i+2} \psi(v_{i+2}) v_{i+1}$. This path has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, $v_{i+1} \neq x$, so we can apply Claim 4.1.3 to conclude that $\psi(v_{i+1}) = \{v_{l-r+1}, v_{l-r+2}, \dots, v_l\} = \phi(v_i)$. But then $\phi(v_i) = \phi(v_l)$, a contradiction to the fact that ϕ was an injection.

Case 2: $v_l = x$**.**

Claim 4.1.4. $\phi(v_{l-1}) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_l\}$.

Proof. If $\phi(v_{l-1})$ contains a vertex $v \notin \{v_1, v_2, \dots, v_l\}$, then we consider the Berge-path $v_1 \phi(v_1) v_2\ \phi(v_2)\ , \dots,\ v_{l-1} \phi(v_{l-1}) v$. Since $v \neq x$, we get a contradiction by Case 1. Moreover, if $\phi(v_{l-1})$ contains a vertex v_i with $i \in \{1, 2, \dots, l-r-1\}$, then the Berge-cycle $v_i \phi(v_i) v_{i+1} \phi(v_{i+1}) \dots v_{l-1} \phi(v_{l-1}) v_i$ is of length $r+1$ or longer, a contradiction. This finishes the proof of the claim. □

By Claim 4.1.4, we know that $\phi(v_{l-1}) = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\} \setminus \{v_j\}$ for some j with $l-r \le j \le l-2$. In the rest of the proof we fix this j.

Claim 4.1.5. *For any $i \in \{l-r, l-r+1, \dots, l-1\} \setminus \{j\}$, we have*

$$\phi(v_i) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}.$$

Proof. When $i = l-1$, we know the statement is true by Claim 4.1.4.

Suppose $i \in \{l-r, l-r+1, \dots, l-2\} \setminus \{j\}$. Let us define a new injection $\psi : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v \notin \{v_1, v_2, \dots, v_l\}$, and for every $v \in \{v_1, v_2, \dots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_{l-1})$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l-1 \geq k \geq i+1$. Now consider the Berge-path $v_1\phi(v_1)v_2\phi(v_2) \dots v_i\phi(v_{l-1})v_{l-1}\phi(v_{l-2})\dots v_{i+1}$ $= v_1\psi(v_1)v_2\psi(v_2) \dots v_i\psi(v_i)v_{l-1}\psi(v_{l-1})\dots v_{i+1}$.
(Note that when $i = l-2$, the Berge-path is simply $v_1\phi(v_1)v_2\phi(v_2) \dots v_i\phi(v_{l-1})v_{l-1} = v_1\psi(v_1)v_2\ \psi(v_2)\dots v_i\psi(v_i)v_{l-1}$.)

If $\psi(v_{i+1})$ contains a vertex $v \notin \{v_1, v_2, \dots, v_l\}$, then the Berge-path $v_1\psi(v_1)v_2\psi(v_2) \dots v_i\psi(v_i)\ v_{l-1}\psi(v_{l-1})\dots v_{i+2}\psi(v_{i+2})v_{i+1}\psi(v_{i+1})v$ has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, since $v \neq x$, we get a contradiction by Case 1.

If $\psi(v_{i+1})$ contains a vertex $v_k \in \{v_1, v_2, \dots, v_{l-r-1}\}$ then one can see that the Berge-cycle $v_k\psi(v_k)v_{k+1}\psi(v_{k+1}) \dots v_{l-1}\psi(v_{l-1})v_k$ is of length $r+1$ or longer, a contradiction. Therefore, we have $\psi(v_{i+1}) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_l\}$. But we defined $\psi(v_{i+1}) = \phi(v_i)$, proving the claim. □

Note that Claim 4.1.5 shows that $r-1$ hyperedges of $\mathcal{H}$ are contained in a set $S := \{v_{l-r}, v_{l-r+1}, \dots,\ v_{l-1}, v_l\}$ of size $r+1$. The following claim shows that if we can find one more hyperedge of $\mathcal{H}$ contained in S, then S must induce a block of $\partial_2(\mathcal{H})$.

Claim 4.1.6. *Suppose $r \geq 3$. If a set S of size $r+1$ contains r hyperedges of $\mathcal{H}$ then it induces a block of $\partial_2(\mathcal{H})$.*

Proof. Since the set S contains at least 3 hyperedges every pair $x, y \in S$ is contained in some hyperedge. Thus $\partial_2(\mathcal{H}[S]) = K_{r+1}$. Consider a (maximal) block D of $\partial_2(\mathcal{H})$ containing S.

Suppose D contains a vertex $t \notin S$. Then since D is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between t and S, which are vertex-disjoint besides t. Let $V(P_1) \cap S = \{u\}$ and $V(P_2) \cap S = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge-path $\mathcal{P}$ between u and v.

On the other hand, by Lemma 4.1.1, there is a Berge-path $\mathcal{P}'$ of length r between u and v consisting of the r hyperedges contained in S. Note that $\mathcal{P}$ and $\mathcal{P}'$ do not share any hyperedges (indeed, each hyperedge of $\mathcal{P}$ contains a vertex not in S, while hyperedges of $\mathcal{P}'$ are contained in S). Therefore, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge-cycle of length $r+1$ or longer, a contradiction. Therefore, D contains no vertex outside S; thus S induces a block of $\partial_2(\mathcal{H})$, as required. □

We will use the above claim several times later. At this point, we need to distinguish the cases $r = 3$ and $r \geq 4$ since Lemma 4.1.2 only applies in the latter case.

The case $r \geq 4$

Since $r \geq 4$, by Claim 4.1.5 and Lemma 4.1.2 there is a Berge-path of length $r-1$ between any two vertices of $S = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$. This will allow us to show the following. (Recall that j is defined just before Claim 4.1.5.)

Claim 4.1.7. $\phi(v_j) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\} = S$.

Proof. Suppose for a contradiction that $\phi(v_j)$ contains a vertex $v \notin S$. The hyperedge $\phi(v_j)$ contains at least two vertices from S, namely v_j and v_{j+1}. By property (2), $\phi(v_j)$ is not a cut-hyperedge of $\mathcal{H}$. So after deleting $\phi(v_j)$ from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{\phi(v_j)\}$ is still connected – so there is a (shortest) Berge-path $\mathcal{Q}$ in $\mathcal{H} \setminus \{\phi(v_j)\}$ between v and a vertex $s \in S$ (note that the hyperedges of $\mathcal{Q}$ are not contained in S). The vertex s is different from either v_j or v_{j+1}, say $s \neq v_j$, without loss of generality. By Lemma 4.1.2, there is a Berge-path $\mathcal{Q}'$ of length $r-1$ between s and v_j (consisting of the hyperedges contained in S). Then $\mathcal{Q}, \mathcal{Q}'$ and $\phi(v_j)$ form a Berge-cycle of length at least $r+1$ in $\mathcal{H}$, a contradiction. $\square$

Claim 4.1.5 and Claim 4.1.7 together show that there are at least r hyperedges of $\mathcal{H}$ contained in S. If all $r+1$ subsets of S of size r are hyperedges of $\mathcal{H}$, then S induces K_{r+1}^r and it is easy to show that it contains a Berge-cycle of length $r+1$, a contradiction. This means S contains exactly r hyperedges of $\mathcal{H}$. Then by Claim 4.1.6, we know that S induces a block of $\partial_2(\mathcal{H})$.

Let $D_1, D_2, \dots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. Claim 4.1.6 shows that one of these blocks, say D_1, is induced by S. Let us contract the vertices of S to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \dots, D_p$. So $\mathcal{H}'$ does not contain any Berge-cycle of length $r+1$ or longer, also; moreover, $|V(\mathcal{H}')| = |V(\mathcal{H})| - r$ and $e(\mathcal{H}') = e(\mathcal{H}) - r$. By induction, we have $e(\mathcal{H}') \leq |V(\mathcal{H}')| - 1$. Therefore,

$$e(\mathcal{H}) = e(\mathcal{H}') + r \leq (|V(\mathcal{H}')| - 1) + r = (|V(\mathcal{H})| - r - 1) + r = |V(\mathcal{H})| - 1.$$

If $e(\mathcal{H}) = |V(\mathcal{H})| - 1$, then we must have $e(\mathcal{H}') = |V(\mathcal{H}')| - 1$ and S must contain exactly r hyperedges. Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected and for each block D_i (with $2 \leq i \leq p$) of $\partial_2(\mathcal{H}')$, $D_i = K_{r+1}$ and $\mathcal{H}'[D_i]$ contains exactly r hyperedges. This means that for every block D of $\partial_2(\mathcal{H})$, we have $D = K_{r+1}$ and $\mathcal{H}[D]$ contains exactly r hyperedges, completing the proof in the case $r \geq 4$.

The case $r = 3$

Recall that using Claim 4.1.5 we can find a set S of size 4 which contains 2 hyperedges of $\mathcal{H}$. Let $S = \{x, y, a, b\}$ and the two hyperedges be xab and yab. By property (2), xab is not a cut-hyperedge of $\mathcal{H}$. So after deleting xab from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{xab\}$ is still connected – so there is a (shortest) Berge-path $\mathcal{Q}$ between x and $\{y, a, b\}$. If $\mathcal{Q}$ is of length at least 2, then it is easy to see that $\mathcal{Q}$ together with yab and xab form a Berge-cycle of length at least 4, a contradiction. So $\mathcal{Q}$ consists of only one hyperedge, say h.

Our goal is to find a set of vertices that induces a block of $\partial_2(\mathcal{H})$ so that we can apply induction.

If $|h \cap \{y, a, b\}| = 2$ then h, xab, yab are 3 hyperedges of $\mathcal{H}$ contained in S, so by Claim 4.1.6, we can conclude that S induces a block of $\partial_2(\mathcal{H})$. (Notice that S contains exactly $|S| - 1 = 3$ hyperedges of $\mathcal{H}$, otherwise it is easy to find a Berge-cycle of length 4; this will be useful later.) So we can suppose $|h \cap \{y, a, b\}| = 1$. We consider two cases depending on whether $h \in \{xat, xbt\}$, or whether $h = xyt$ for some $t \notin S$.

Case 1. First suppose without loss of generality that $h = xat$ for some $t \notin S$. Consider the set $\mathcal{D}$ of all hyperedges of $\mathcal{H}$ containing the pairs xa, ab or xb and let D be the set of vertices spanned by them. For each pair of vertices $i, j \in \{x, a, b\}$, let $V_{ij} = \{v \mid ijv \in \mathcal{H}\} \setminus \{x, a, b\}$. We claim that the sets V_{xa}, V_{ab}, V_{xb} are pairwise disjoint. Suppose for the sake of contradiction that $t' \in V_{xa} \cap V_{ab}$. Then the hyperedges xat', abt', xab are contained in a set of 4 vertices $\{x, a, b, t'\}$. Thus by Claim 4.1.6, this set induces a block of $\partial_2(\mathcal{H})$ and we are done. Thus we can suppose $V_{xa} \cap V_{ab} = \emptyset$. Similarly, $V_{ab} \cap V_{xb} = \emptyset$ and $V_{xa} \cap V_{xb} = \emptyset$. This shows that $|D| = 3 + |V_{xa}| + |V_{xb}| + |V_{ab}|$. On the other hand, $\mathcal{D}$ consists of $1 + |V_{xa}| + |V_{xb}| + |V_{ab}|$ hyperedges, so $|\mathcal{D}| = |D| - 2$.

We will now show that D induces a block of $\partial_2(\mathcal{H})$.

Let D' be a (maximal) block of $\partial_2(\mathcal{H})$ containing D and suppose for the sake of a contradiction that it contains a vertex $p \notin D$. Then since D' is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between p and D, which are vertex-disjoint besides p. Let $V(P_1) \cap D = \{u\}$ and $V(P_2) \cap D = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge-path $\mathcal{P}$ between u and v. If $uv \notin \{xa, ab, xb\}$, then it is easy to see that there is a path $\mathcal{P}'$ of length 3 between u and v consisting of the hyperedges of $\mathcal{D}$. Then $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge-cycle of length at least 4 in $\mathcal{H}$, a contradiction. On the other hand, if $uv \in \{xa, ab, xb\}$, then $\mathcal{P}$ must contain at least two hyperedges of $\mathcal{H}$ because otherwise $\mathcal{P} = \{puv\}$ but then puv should have been in $\mathcal{D}$ (since by definition $\mathcal{D}$ must contain all the hyperedges of $\mathcal{H}$ containing the pair uv); moreover, it is easy to check that between u and

v there is a Berge-path $\mathcal{P}'$ of length 2 consisting of the hyperedges of $\mathcal{D}$. Then again, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge-cycle of length at least 4 in $\mathcal{H}$, a contradiction. Therefore, D' contains no vertex outside D; so D induces a block of $\partial_2(\mathcal{H})$ (which contains $|D|-2$ hyperedges of $\mathcal{H}$), as desired.

Case 2. Finally suppose $h=xyt$ for some $t \notin S$. Let $\mathcal{D}$ be the set of all hyperedges of $\mathcal{H}$ containing the pair xy plus the hyperedges xab and yab, and let D be the set of vertices spanned by the hyperedges of $\mathcal{D}$. Let $V_{xy}=\{v \mid xyv \in \mathcal{H}\}$. We claim that $a \notin V_{xy}$ and $b \notin V_{xy}$. Indeed suppose for the sake of a contradiction that $a \in V_{xy}$. Then the hyperedges xab, yab, xya are contained in a set of 4 vertices $\{x,y,a,b\}$. So by Claim 4.1.6, this set induces a block of $\partial_2(\mathcal{H})$, and we are done. So $a \notin V_{xy}$. Similarly, we can conclude $b \notin V_{xy}$. Therefore, $|D|=|V_{xy}|+4$. On the other hand, $|\mathcal{D}|=|V_{xy}|+2$, so $|\mathcal{D}|=|D|-2$.

We claim that D induces a block of $\partial_2(\mathcal{H})$. The proof is very similar to that of Case 1, but we still give it for completeness. Let D' be a (maximal) block of $\partial_2(\mathcal{H})$ containing D and suppose for the sake of a contradiction that it contains a vertex $p \notin D$. Then since D' is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between p and D, which are vertex-disjoint besides p. Let $V(P_1) \cap D=\{u\}$ and $V(P_2) \cap D=\{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge-path $\mathcal{P}$ between u and v.

If $uv \neq xy$, then it is easy to see that there is a path $\mathcal{P}'$ of length 3 or 4 between u and v consisting of the hyperedges of $\mathcal{D}$. (Indeed if $u,v \in V_{xy}$, then $\mathcal{P}'$ is of length 4, otherwise it is of length 3.) Then $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge-cycle of length at least 4 in $\mathcal{H}$, a contradiction. On the other hand, if $uv=xy$, then $\mathcal{P}$ must contain at least two hyperedges of $\mathcal{H}$ because otherwise $\mathcal{P}=\{puv\}$ but then puv should have been in $\mathcal{D}$ (since by definition $\mathcal{D}$ must contain all the hyperedges of $\mathcal{H}$ containing the pair uv); moreover, it is easy to check that between u and v there is a Berge-path $\mathcal{P}'$ of length 2 consisting of the hyperedges of $\mathcal{D}$. Then again, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge-cycle of length at least 4 in $\mathcal{H}$, a contradiction. Therefore, D' contains no vertex outside D; so D induces a block of $\partial_2(\mathcal{H})$ (and contains $|D|-2$ hyperedges of $\mathcal{H}$), as desired.

Let $D_1, D_2, \dots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. In Case 1 and Case 2 we showed that one of these blocks, (say) $D_1=D$ is such that $\mathcal{H}[D_1]$ contains $|D_1|-2$ hyperedges of $\mathcal{H}$, otherwise, D_1 is a set of 4 vertices such that $\mathcal{H}[D_1]$ contains exactly $|D_1|-1=3$ hyperedges of $\mathcal{H}$. In all these cases, note that $e(\mathcal{H}[D_1]) \leq |D_1|-1$.

Let us contract the vertices of D_1 to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \dots, D_p$. So $\mathcal{H}'$ does not contain any Berge-cycle of length 4 or longer, also; moreover, $|V(\mathcal{H}')|=$

$|V(\mathcal{H})|-|D_1|+1$ and $e(\mathcal{H}')=e(\mathcal{H})-e(\mathcal{H}[D_1])$. By induction, we have $e(\mathcal{H}')\le |V(\mathcal{H}')|-1$. Therefore,

$$\begin{aligned} e(\mathcal{H}) &= e(\mathcal{H}')+e(\mathcal{H}[D_1]) \le |V(\mathcal{H}')|-1+|D_1|-1 \\ &= (|V(\mathcal{H})|-|D_1|+1)-1+|D_1|-1 = |V(\mathcal{H})|-1. \end{aligned}$$

If $e(\mathcal{H})=|V(\mathcal{H})|-1$, then we must have $e(\mathcal{H}')=|V(\mathcal{H}')|-1$ and $\mathcal{H}[D_1]$ must contain exactly $|D_1|-1$ hyperedges. As noted before, this is only possible if D_1 has 4 vertices and induces exactly 3 hyperedges of $\mathcal{H}$. Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected and for each block D_i (with $2\le i\le p$) of $\partial_2(\mathcal{H}')$, $D_i=K_4$ and $\mathcal{H}'[D_i]$ contains exactly 3 hyperedges. This means for every block D of $\partial_2(\mathcal{H})$, we have $D=K_4$ and $\mathcal{H}[D]$ contains exactly 3 hyperedges of $\mathcal{H}$, completing the proof in the case $r=3$. □

Note that Theorem 1.8.14 easily implies Theorem 1.8.3. In fact, it gives the following stronger form.

Theorem 4.1.8. *Fix $k=r+1>2$ and let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge-path of length k. Then, $e(\mathcal{H})\le \frac{n}{k}\binom{k}{r}=n$. Moreover, equality holds if and only if each connected component D of $\partial_2(\mathcal{H})$ is K_{r+1}, and $\mathcal{H}[D]=K_{r+1}^r$.*

Proof. We proceed by induction on n. The base cases $n\le r+1$ are easy to check. Let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge-path of length $k=r+1$ such that $e(\mathcal{H})\ge n$. Then by Theorem 1.8.14, $\mathcal{H}$ contains a Berge-cycle $\mathcal{C}$ of length $r+1$ or longer. $\mathcal{C}$ must be of length exactly $r+1$, otherwise it would contain a Berge-path of length $r+1$. Let $v_1,\dots,v_{r+1}$ and $e_1,\dots,e_{r+1}$ be the vertices and edges of $\mathcal{C}$ where $\{v_i,v_{i+1}\}\subseteq e_i$ (indices are taken modulo $r+1$). For any i with $1\le i\le r+1$, if e_i contains a vertex $v\notin\{v_1,\dots,v_{r+1}\}$, then $v_{i+1}e_{i+1}v_{i+2}e_{i+2}\dots e_{i-1}v_ie_iv$ forms a Berge-path of length $r+1$ in $\mathcal{H}$, a contradiction. Therefore, all of the edges e_i (for $1\le i\le r+1$) are contained in the set $S:=\{v_1,\dots,v_{r+1}\}$. That is, $\mathcal{H}[S]=K_{r+1}^r$. It is easy to see that S forms a connected component in $\partial_2(\mathcal{H})$ because if any hyperedge h of $\mathcal{H}$ (with $h\notin\mathcal{C}$) contains a vertex of $\mathcal{C}$, then $\mathcal{C}$ can be extended to form a Berge-path of length $r+1$.

Let $S_1,S_2,\dots,S_t$ be the vertex sets of connected components of $\partial_2(\mathcal{H})$. As noted before, one of them, say S_1, is equal to S. We delete the vertices of S_1 from $\mathcal{H}$ to form a new hypergraph $\mathcal{H}'$; note that $|V(\mathcal{H}')|=|V(\mathcal{H})|-(r+1)$ and $|E(\mathcal{H}')|=|E(\mathcal{H})|-(r+1)$ and the connected components of $\partial_2(\mathcal{H}')$ are $S_2,\dots,S_t$. By induction $|E(\mathcal{H}')|\le|V(\mathcal{H}')|$. Thus $|E(\mathcal{H})|=|E(\mathcal{H}')|+(r+1)\le|V(\mathcal{H}')|+(r+1)=|V(\mathcal{H})|$. Moreover, if $|E(\mathcal{H})|=|V(\mathcal{H})|$, then $|E(\mathcal{H}')|=|V(\mathcal{H}')|$, so by the induction hypothesis each connected component S_i ($i\ge 2$) of $\partial_2(\mathcal{H}')$ is K_{r+1}, and $\mathcal{H}'[S_i]=K_{r+1}^r$, proving the theorem. □

4.1.3 Extremal hypergraphs without $\mathcal{BC}_{\geq r+2}$

We will prove the theorem by induction on n. For the base cases, note that if $1 \leq n \leq r$ then the statement of the theorem is trivially true. If $n = r+1$, the statement is true since there are at most $r+1$ hyperedges of size r on $r+1$ vertices. Moreover, equality holds if and only if $\mathcal{H} = K_{r+1}^r$.

We will show the statement is true for $n \geq r+2$ assuming it is true for all smaller values. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices having no Berge-cycle of length $r+2$ or longer. We show that we may assume the following two properties hold for $\mathcal{H}$.

(1) For any set $S \subseteq V(\mathcal{H})$ of vertices, the number of hyperedges of $\mathcal{H}$ incident to the vertices of S is at least $|S|$.

Indeed, suppose there is a set $S \subseteq V(\mathcal{H})$ with fewer than $|S|$ hyperedges incident to the vertices of S. If $|S| = n$ we immediately have the required bound on $e(\mathcal{H})$, so assume $n > |S|$. We can delete the vertices of S from $\mathcal{H}$ to obtain a new hypergraph $\mathcal{H}'$ on $n-|S|$ vertices. By induction, $\mathcal{H}'$ contains at most $\frac{r+1}{r}(n-|S|-1)$ hyperedges, so $\mathcal{H}$ contains less than $\frac{r+1}{r}(n-1-|S|)+|S| < \frac{r+1}{r}(n-1)$ hyperedges, as desired.

(2) There is no cut-hyperedge in $\mathcal{H}$.

Indeed, if $h \in E(\mathcal{H})$ is a cut-hyperedge, then $\partial_2(\mathcal{H} \setminus \{h\})$ is not a connected graph, so there are non-empty disjoint sets V_1 and V_2 such that $V(\mathcal{H}) = V_1 \cup V_2$, and there are no edges of $\partial_2(\mathcal{H} \setminus \{h\})$ between V_1 and V_2. So both hypergraphs $\mathcal{H}[V_1]$ and $\mathcal{H}[V_2]$ do not contain a Berge-cycle of length $r+2$ or longer. By induction, $e(\mathcal{H}[V_1]) \leq \frac{r+1}{r}(|V_1|-1)$ and $e(\mathcal{H}[V_2]) \leq \frac{r+1}{r}(|V_2|-1)$. In total, $e(\mathcal{H}) = e(\mathcal{H}[V_1]) + e(\mathcal{H}[V_2]) + 1 \leq \frac{r+1}{r}(|V_1| + |V_2|-2)+1 < \frac{r+1}{r}(|V(\mathcal{H})|-1)$, as desired.

Consider an auxiliary bipartite graph B consisting of vertices of $\mathcal{H}$ in one class and hyperedges of $\mathcal{H}$ in the other class, and the edges of B are defined as follows: $xh \in E(B)$ if and only if the vertex x is contained in the hyperedge h.

Then property (1) shows that Hall's condition holds in B. Therefore, there is a perfect matching in B. In other words, there exists an injection $f : V(\mathcal{H}) \to E(\mathcal{H})$ such that $v \in f(v)$.

Given an injection $f : V(\mathcal{H}) \to E(\mathcal{H})$ with $v \in f(v)$, let $\mathcal{P}_f$ be a longest Berge-path of the form $v_1 f(v_1) v_2 f(v_2) \ldots v_{l-1} f(v_{l-1}) v_l$ where for each $1 \leq i \leq l-1$, $v_{i+1} \in f(v_i)$. Moreover, among all injections $f : V(\mathcal{H}) \to E(\mathcal{H})$ with $v \in f(v)$, suppose $\phi : V(\mathcal{H}) \to E(\mathcal{H})$ is an injection for which the path $\mathcal{P}_\phi = v_1 \phi(v_1) v_2 \phi(v_2) \ldots v_{l-1} \phi(v_{l-1}) v_l$ is a longest path.

Claim 4.1.9. $\phi(v_l) \subset \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}$.

Proof. First notice that if $\phi(v_l)$ contains a vertex $v_i \in \{v_1, v_2, \ldots, v_{l-r-1}\}$, then the Berge-cycle $v_i\phi(v_i)v_{i+1}\phi(v_{i+1})\ldots v_l\phi(v_l)v_i$ is of length $r+2$ or longer, a contradiction. Moreover, if $\phi(v_l)$ contains a vertex $v \notin \{v_1, v_2, \ldots, v_l\}$, then $\mathcal{P}_\phi$ can be extended to a longer path $v_1\phi(v_1)v_2\phi(v_2)\ \ldots v_{l-1}\phi(v_{l-1})v_l\phi(v_l)v$, a contradiction. This completes the proof of the claim. □

By Claim 4.1.9, we know that $\phi(v_l) = \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\} \setminus \{v_j\}$ for some $l-r \leq j \leq l-1$.

Claim 4.1.10. *For any $i \in \{l-r, l-r+1, \ldots, l\} \setminus \{j\}$, we have*

$$\phi(v_i) \subset \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}.$$

Proof. When $i = l$, we know the statement is true. Suppose $i \in \{l-r, l-r+1, \ldots, l-1\} \setminus \{j\}$. Let us define a new injection $\psi : V(\mathcal{H}) \to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v \notin \{v_1, v_2, \ldots, v_l\}$, and for every $v \in \{v_1, v_2, \ldots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_l)$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l \geq k \geq i+1$.

Now consider the Berge-path

$$v_1\phi(v_1)v_2\phi(v_2)\ldots v_i\phi(v_l)v_l\phi(v_{l-1})\ldots v_{i+2}\phi(v_{i+1})v_{i+1},$$

equivalently $v_1\psi(v_1)v_2\psi(v_2)\ldots v_i\psi(v_i)v_l\psi(v_l)\ldots v_{i+2}\psi(v_{i+2})v_{i+1}$. This path has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, notice that the sets of last $r+1$ vertices of both paths are the same. Thus we can apply Claim 4.1.9 to conclude that $\phi(v_i) = \psi(v_{i+1}) \subset \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}$, as desired. □

Claim 4.1.10 shows that there are r hyperedges (each of size r) contained in the set $S := \{v_{l-r}, v_{l-r+1}\ , \ldots,\ v_{l-1}, v_l\}$ of size $r+1$. We will apply Lemma 4.1.1 to S.

Claim 4.1.11. *The set $S = \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}$ induces a block of $\partial_2(\mathcal{H})$.*

Proof. Since the set $S = \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}$ contains $r \geq 3$ hyperedges every pair $x, y \in S$ is contained in some hyperedge. Thus $\partial_2(\mathcal{H}[S]) = K_{r+1}$. Consider a (maximal) block D of $\partial_2(\mathcal{H})$ containing S.

Suppose D contains a vertex $t \notin S$. Then since D is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between t and S, which are vertex-disjoint besides t. Let $V(P_1) \cap S = \{u\}$ and $V(P_2) \cap S = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge-path $\mathcal{P}$ between u and v.

On the other hand, by Lemma 4.1.1, there is a Berge-path $\mathcal{P}'$ of length r between u and v consisting of the r hyperedges contained in S. Note that $\mathcal{P}$ and $\mathcal{P}'$ do not share any hyperedges (indeed, each hyperedge of $\mathcal{P}$ contains a vertex not in S, while hyperedges of $\mathcal{P}'$ are contained in S). Therefore, $\mathcal{P} \cup \mathcal{P}'$ forms a Berge-cycle of length $r+2$ or longer unless $\mathcal{P}$ consists of only one hyperedge, say h. Note that h contains a vertex $x \notin S$ and $u, v \in h$; moreover by property (2), h is not a cut-hyperedge of $\mathcal{H}$. So after deleting h from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{h\}$ is still connected – so there is a (shortest) Berge-path $\mathcal{Q}$ in $\mathcal{H} \setminus \{h\}$ between x and a vertex $s \in S$. (Note that the hyperedges of $\mathcal{Q}$ are not contained in S, and as it is a shortest Berge-path, both u and v do not appear among the "defining" vertices of $\mathcal{Q}$.) The vertex s is different from either u or v, say $s \neq u$ without loss of generality. By Lemma 4.1.1, there is a Berge-path $\mathcal{Q}'$ of length r between s and u (consisting of hyperedges contained in S). Then, $\mathcal{Q}, \mathcal{Q}'$ and h form a Berge-cycle of length at least $r+2$, a contradiction. Therefore, D contains no vertex outside S; thus S induces a block of $\partial_2(\mathcal{H})$, as required. □

Let $D_1, D_2, \ldots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. Claim 4.1.11 shows that one of these blocks, say D_1, is induced by S. Let us contract the vertices of S to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \ldots, D_p$. So $\mathcal{H}'$ does not contain any Berge-cycle of length $r+2$ or longer, also; moreover $|V(\mathcal{H}')| = |V(\mathcal{H})| - r$. Thus, by induction, we have $e(\mathcal{H}') \leq \frac{r+1}{r}(|V(\mathcal{H}')| - 1)$. Therefore,

$$e(\mathcal{H}) \leq \frac{r+1}{r}(|V(\mathcal{H}')| - 1) + (r+1) = \frac{r+1}{r}(|V(\mathcal{H})| - r - 1) + (r+1) = \frac{r+1}{r}(|V(\mathcal{H})| - 1).$$

Now if $e(\mathcal{H}) = \frac{r+1}{r}(|V(\mathcal{H})| - 1)$, then we must have $e(\mathcal{H}') = \frac{r+1}{r}(|V(\mathcal{H}')| - 1)$ and S must contain all $r+1$ subsets of size r (i.e., $\mathcal{H}[S] = \mathcal{H}[D_1] = K_{r+1}^r$). Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected and for each block D_i (with $2 \leq i \leq p$) of $\partial_2(\mathcal{H}')$, $D_i = K_{r+1}$ and $\mathcal{H}'[D_i] = K_{r+1}^r$. This means that for every block D of $\partial_2(\mathcal{H})$, we have $D = K_{r+1}$ and $\mathcal{H}[D] = K_{r+1}^r$, completing the proof.

4.1.4 A corollary

We note that Theorem 1.8.14 implies Theorem 1.8.3. In fact, it gives the following stronger form. Here we prove this implication.

Theorem 4.1.12. *Fix $k = r+1 > 2$ and let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge-path of length k. Then, $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r} = n$. Moreover, equality holds if and only if each connected component D of $\partial_2(\mathcal{H})$ is K_{r+1}, and $\mathcal{H}[D] = K_{r+1}^r$.*

Proof. We proceed by induction on n. The base cases $n \leq r+1$ are easy to check. Let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge-path of length $k = r+1$ such that $e(\mathcal{H}) \geq n$. Then by Theorem 1.8.14, $\mathcal{H}$ contains a Berge-cycle $\mathcal{C}$ of length $r+1$ or longer. $\mathcal{C}$ must be of length exactly $r+1$, otherwise it would contain a Berge-path of length $r+1$. Let $v_1, \ldots, v_{r+1}$ and $e_1, \ldots, e_{r+1}$ be the vertices and edges of $\mathcal{C}$ where $\{v_i, v_{i+1}\} \subseteq e_i$ (indices are taken modulo $r+1$). For any i with $1 \leq i \leq r+1$, if e_i contains a vertex $v \notin \{v_1, \ldots, v_{r+1}\}$, then $v_{i+1}e_{i+1}v_{i+2}e_{i+2}\ldots e_{i-1}v_ie_iv$ forms a Berge-path of length $r+1$ in $\mathcal{H}$, a contradiction. Therefore, all of the edges e_i (for $1 \leq i \leq r+1$) are contained in the set $S := \{v_1, \ldots, v_{r+1}\}$. That is, $\mathcal{H}[S] = K^r_{r+1}$. It is easy to see that S forms a connected component in $\partial_2(\mathcal{H})$ because if any hyperedge h of $\mathcal{H}$ (with $h \notin \mathcal{C}$) contains a vertex of $\mathcal{C}$, then $\mathcal{C}$ can be extended to form a Berge-path of length $r+1$.

Let $S_1, S_2, \ldots, S_t$ be the vertex sets of connected components of $\partial_2(\mathcal{H})$. As noted before, one of them, say S_1, is equal to S. We delete the vertices of S_1 from $\mathcal{H}$ to form a new hypergraph $\mathcal{H}'$; note that $|V(\mathcal{H}')| = |V(\mathcal{H})| - (r+1)$ and $|E(\mathcal{H}')| = |E(\mathcal{H})| - (r+1)$ and the connected components of $\partial_2(\mathcal{H}')$ are $S_2, \ldots, S_t$. By induction $|E(\mathcal{H}')| \leq |V(\mathcal{H}')|$. Thus $|E(\mathcal{H})| = |E(\mathcal{H}')| + (r+1) \leq |V(\mathcal{H}')| + (r+1) = |V(\mathcal{H})|$. Moreover, if $|E(\mathcal{H})| = |V(\mathcal{H})|$, then $|E(\mathcal{H}')| = |V(\mathcal{H}')|$, so by the induction hypothesis each connected component S_i ($i \geq 2$) of $\partial_2(\mathcal{H}')$ is K_{r+1}, and $\mathcal{H}'[S_i] = K^r_{r+1}$, proving the theorem. □

4.2 Hypergraphs with a circumference at most uniformity

In this Section we prove Theorem 1.8.16, Theorem 1.8.18 and Theorem 1.8.20 using a powerful Lemma 4.2.2.

We recall Theorem 1.8.16.

Theorem. *Let k, n and r be positive integers such that $3 \leq k < r$. Then*

$$\mathrm{ex}_r(n, \mathcal{BC}_{\geq k}) = \left\lfloor \frac{n-1}{r} \right\rfloor (k-1) + \mathbb{1}_{r\mathbb{N}^*}(n)$$

If $r|(n-1)$ the only extremal n-vertex r-graphs are the $(r+1, k-1)$-block trees.

We recall Theorem 1.8.18.

Theorem. *Let $r > 2$ and n be positive integers. Then*

$$\mathrm{ex}_r(n, \mathcal{BC}_{\geq r}) = \max\left\{ \left\lfloor \frac{n-1}{r} \right\rfloor (r-1), n-r+1 \right\}$$

When $n-r+1 > \frac{n-1}{r}(r-1)$ the only extremal graph is $\mathcal{S}_n^{(r)}$. When $\frac{n-1}{r}(r-1) > n-r+1$ and $r|(n-1)$ the only extremal graphs are the $(r+1,k-1)$-block trees.

Remark 4.2.1. *In particular when $n \geq r(r-2)+2$, we have that $\mathrm{ex}_r(n,\mathcal{BC}_{\geq r}) = n-r+1$ and $\mathcal{S}_n^{(r)}$ is the only extremal hypergraph.*

We recall Theorem 1.8.20.

Theorem. *Let k,n and r be positive integers such that $2 \leq k \leq r$.*

Then

$$\mathrm{ex}_r^{multi}(n,\mathcal{BC}_{\geq k}) = \left\lfloor \frac{n-1}{r-1} \right\rfloor (k-1)$$

If $r-1|(n-1)$ the only extremal graphs with n vertices are the $(r,k-1)$-block trees.

All three theorems have essentially the same proof since, these results follow from our Lemma 4.2.2, which to some extent lets us understand the structure of long Berge-cycle free hypergraphs.

Lemma 4.2.2. *Let r,k,n and m be positive integers, with $r \geq k \geq 3$, and let $\mathcal{H}$ be an n-vertex r-graph which is $\mathcal{BC}_{\geq k}$-free such that every hyperedge has multiplicity at most m. Then at least one of the following holds.*

i) There exists $S \subseteq V$ of size $r-1$ such that $|N_h(S)| \leq m$. Moreover, if $m < k-1$ there exists a set S of size $r-1$ such that $N_h(S)$ is $d \leq m$ copies of a hyperedge h and $S \subset h$.

ii) There exists $S \subseteq V$ of size r such that $|N_h(S)| \leq k-1$.

iii) $k = r$, $m < k-1$, and there exists $e \in E(\mathcal{H})$ such that after removing e from $\mathcal{H}$ the resulting r-graph can be decomposed in two r-graphs, $\mathcal{S}$ and $\mathcal{K}$ sharing one vertex, such that $\mathcal{S}$ is an r-star with at least $r-1$ edges, the shared vertex is in the center of $\mathcal{S}$, $e \cap V(\mathcal{S})$ is a subset of the center of $\mathcal{S}$ and $v(\mathcal{K}) \geq 2$.

In particular, since no hyperedge can have multiplicity larger than $k-1$, by setting $m = k-1$ we have that there exists a set S of size $r-1$ incident with at most $k-1$ edges.

In Subsection 4.2.1 we deduce Theorems 1.8.16, 1.8.18 and 1.8.20 from Lemma 4.2.2. We deduce corollaries of those theorems in the end of this chapter Subsection 4.2.3. | We give the proof of Lemma 4.2.2 in Subsection 4.2.2.

4.2.1 Proof of Theorem 1.8.16, 1.8.18, 1.8.20

To obtain the extremal constructions in Theorem 1.8.16, first we are going to show that in an $(r+1,k-1)$-block tree, for every pair of vertices there exists a Berge-path of length $k-1$ joining them. For this, we prove the following statement by induction.

Claim 4.2.3. *Let $r \geq 4$ and $\mathcal{H}_1$ a multi (not necessarily uniform) hypergraph such that $v(\mathcal{H}_1) = r+1 > e(\mathcal{H}_1) \geq 2$, and every hyperedge $h \in E(\mathcal{H}_1)$, $h \neq V(\mathcal{H}_1)$, has size at least r and multiplicity at most one. Then every pair of vertices of $\mathcal{H}_1$ are joined by a Berge-path of length $e(\mathcal{H}_1)$.*

Proof. The proof is by induction on r. The case where $r = 4$ is simple to check, as well as the case when $e(\mathcal{H}_1) = 2$, since every edge contains all but at most one vertex. So suppose $r > 4$ and $e(\mathcal{H}_1) > 2$. Let v, u be two distinct vertices, take any hyperedge h containing v, then choose $w \in \mathcal{H} \backslash \{v, u\}$, consider $\mathcal{H}_2$ obtain by removing v and h from $\mathcal{H}_1$ and by deleting v from the remaining hyperedges, then $\mathcal{H}_2$ satisfy the conditions of the claim, hence there exists a Berge-path of length $e(\mathcal{H}_2) = e(\mathcal{H}_1) - 1$ joining w and u, we can extend this path with h to be a Berge-path of length $e(\mathcal{H}_1)$ joining v and u. $\square$

Therefore we proved that, when $r \geq 4$, in an $(r+1,k-1)$-block tree for every pair of vertices from the same block there exists a Berge-path of length $k-1$ joining them, hence the statement trivially holds for every pair of vertices too, since an $(r+1,k-1)$-block tree is a connected hypergraph. The same is true for $(r,k-1)$-block trees.

Proof of Theorem 1.8.16. For the lower bound we can observe that an $(r+1,k-1)$-block tree on $ar+1$ vertices is a $\mathcal{BC}_{\geq k}$-free graph with $a(k-1)$ edges, for $n \in \{ar+1, ar+2, \dots, (a+1)r-1\}$ this proves the lower bound, if $n = (a+1)r$ add an extra edge containing $r-1$ new vertices to this construction and we will get a desired lower bound.

For the upper bound, let $\mathcal{H}$ is an r-uniform, n-vertex, hypergraph, without a Berge-cycle of length at least k. The proof is by induction on the number of vertices. The theorem trivially holds for $n \leq r$. So suppose $n > r$ and that the theorem holds for any graph with less than n vertices, by Lemma 4.2.2 there exists a set $S \subseteq V$ such that either $|S| = r-1$ and $|N_{\mathcal{H}}(S)| = 1$ or $|S| = r$ and $|N_{\mathcal{H}}(S)| = k-1$. Let $\mathcal{H}'$ be the graph induced by $V' = V \backslash S$. Then either

$$e(\mathcal{H}) \leq 1 + e(\mathcal{H}') \leq 1 + \left\lfloor \frac{n-r}{r} \right\rfloor (k-1) + \mathbb{1}_{r\mathbb{N}^*}(n-r+1) \leq \left\lfloor \frac{n-1}{r} \right\rfloor (k-1) + \cdot \mathbb{1}_{r\mathbb{N}^*}(n), \text{ or}$$

$$e(\mathcal{H}) \leq (k-1) + e(\mathcal{H}') \leq (k-1) + \left\lfloor \frac{n-(r-1)-1}{r} \right\rfloor (k-1) + \mathbb{1}_{r\mathbb{N}^*}(n-r)$$
$$= \left\lfloor \frac{n-1}{r} \right\rfloor (k-1) + \mathbb{1}_{r\mathbb{N}^*}(n).$$

From the above calculations equality holds, only when $|S| = r$, $|N_{\mathcal{H}}(S)| = k-1$ and $e(\mathcal{H}') = \left\lfloor \frac{n-r-1}{r} \right\rfloor (k-1)$ or $r|n$, $|S| = r-1$, $|N_{\mathcal{H}}(S)| = 1$ and $e(\mathcal{H}') = \left\lfloor \frac{n-r-1}{r} \right\rfloor (k-1)+1$. If $r|n-1$, we will prove that the only extremal hypergraph is an $(r+1,k-1)$-block tree. We have $|S| = r$ and by induction $\mathcal{H}'$ is an $(r+1,k-1)$-block tree. For any hyperedge h incident with S we have that $|h \cap V'| \leq 1$, otherwise, since any two vertices of V' are joined by a Berge-path of length at least $k-1$ in $\mathcal{H}'$, we have a Berge-cycle of length at least k in $\mathcal{H}$, a contradiction. If there exist two hyperedges h, h' incident with S such that, there exists two distinct vertices v, $v' \in V'$, such that $v \in h$ and $v' \in h'$ then both $h \backslash \{v\}$ and $h' \backslash \{v'\}$ have $r-1$ elements in S, then these hyperedges must intersect in a vertex x, $x \in S$. So v,h,x,h',v' together with a Berge-path of length at least $k-1$ joining v to v' in $\mathcal{H}'$ is a Berge-cycle of length at least $k+1$. Therefore every edge in $N_{\mathcal{H}}(S)$ is either S or intersects the same vertex v of V', hence $\mathcal{H}$ is an $(r+1,k-1)$-block tree. □

Remark 4.2.4. *If $\mathcal{H}$ is an n-vertex multi r-graph in which each edge has multiplicity at most $m \leq k-1$ and contains no Berge-cycle of length at least k, then Lemma 4.2.2 implies* $e(\mathcal{H}) \leq \max\{a(k-1)+bm : ar+b(r-1) < n\}$, *this holds for all $k \geq 2$.*

Proof of Theorem 1.8.20. This theorem follows by induction in the same way as Theorem 1.8.16 since we can always find a set S of size $r-1$ incident with at most $k-1$ edges. □

Proof of Theorem 1.8.18. The Theorem trivially holds for $n \leq r$. So suppose $n \geq r+1$ We will assume by induction that Theorem 1.8.18 holds for $n' < n$. Note that for $n' > 2$, we have that any n' vertex $\mathcal{BC}_{\geq r}$-free r-graph have at most $n'-2$ hyperedges, more over the only such r-graphs with precisely $n'-2$ hyperedges are $\mathcal{S}_{n'}^{(3)}$, when $r=3$, or an $(r+1,r-1)$-block when $n' = r+1$, both of which are connected.

Let $\mathcal{H}$ be an n-vertex $\mathcal{BC}_{\geq k}$-free r-graph, with maximum number of hyperedges. Applying Lemma 4.2.2, one of $i)$, $ii)$ or $iii)$ must hold.

Suppose $iii)$ holds in Lemma 4.2.2, and let $\mathcal{S}$ and $\mathcal{K}$ be the given decomposition of $\mathcal{H}$ after removing the hyperedge e. Let v be the only vertex in $V(\mathcal{S}) \cap V(\mathcal{K})$. If $e \cap V(\mathcal{S}) = \{v\}$, then we can in fact decompose $\mathcal{H}$ into $\mathcal{S}$ and $\mathcal{K}'$, where $\mathcal{K}'$ is obtain from $\mathcal{K}$ by adding e. By induction we have that $e(\mathcal{H}) \leq e(\mathcal{S}) + e(\mathcal{K}') \leq v(\mathcal{S}) - (r-1) + v(\mathcal{K}) - 2 < n-(r-1)$, a contradiction. Hence we have that $e \cap V(\mathcal{S}) \neq \{v\}$. Let $u \in e \cap V(\mathcal{S})$ and $w \in e \cap V(\mathcal{K})$, with both u,w different from v. If $v(\mathcal{K}) \geq 3$, then $e(\mathcal{H}) = e(\mathcal{S}) + e(\mathcal{K}) + 1 \leq v(\mathcal{S}) - (r-1) + v(\mathcal{K}) - 2 + 1 = n-(r-1)$, but equality is not possible, since by connectivity of $\mathcal{K}$ there is a Berge-path from w to v in $\mathcal{K}$ and we have a Berge-path of length $r-2$ in S from v to u, finally we can use the hyperedge e to connect u to w, we get a Berge-path of length at least k, a contradiction. So $\mathcal{H}$ has $n-(r-1)$ edges only if $v(\mathcal{K}) = 2$. In this case $e(\mathcal{K}) = 0$, and therefore e contains the center of S and the only vertex of $V(\mathcal{K})\backslash\{v\}$, hence $\mathcal{H}=\mathcal{S}_n^{(r)}$. Finally we have $e(\mathcal{H})\leq n-(r-1)$ and equality holds when $\mathcal{H}=\mathcal{S}_n^{(r)}$.

If $\left\lfloor\frac{n-1}{r}\right\rfloor(r-1)\geq n-r+1$ then either *iii*) in Lemma 4.2.2 holds and $e(\mathcal{H})\leq n-(r-1)$, or the proof of extremal number follows by induction in the similar way as Theorem 1.8.16.

Suppose $n-r+1>\left\lfloor\frac{n-1}{r}\right\rfloor(r-1)$. If *i*) holds in Lemma 4.2.2 then we have $e(\mathcal{H})<n-(r-1)$ since $n'-r+1\geq\left\lfloor\frac{n'-1}{r}\right\rfloor(r-1)$ for $n'=n-(r-1)$, a contradiction. If *ii*) holds in Lemma 4.2.2 then we have $e(\mathcal{H})<n-(r-1)$ since $n'-r+1\geq\left\lfloor\frac{n'-1}{r}\right\rfloor(r-1)$ for $n'=n-r$, which is also a contradiction. Therefore *iii*) holds in Lemma 4.2.2, hence $\mathcal{H}=\mathcal{S}_n^{(r)}$. □

4.2.2 Proof of Lemma 4.2.2

Definition 4.2.5. *A* semi-path *of length t in a hypergraph, is an alternating sequence of distinct hyperedges and vertices,* $e_1,v_1,e_2,v_2,\dots,e_t,v_t$ *(starting with a hyperedge and ending in a vertex) such that,* $v_1\in e_1$ *and* $v_{i-1},v_i\in e_i$*, for* $i=2,3,\dots t$.

Let $r\geq k\geq 3$ be fixed integers and let $\mathcal{H}$ be a $\mathcal{BC}_{\geq k}$-free multi r-graph, consider a semi-path $P=e_1,v_1,e_2,v_2,\dots,e_t,v_t$ of maximal length. Consider P' the semi-path $e_1,v_1,e_2,v_2,\dots,$ e_ℓ,v_ℓ obtained from the first ℓ vertices and hyperedges of P, where $\ell=\min\{k-1,t\}$, let $\mathcal{F}=\{e_1,e_2,\dots,e_\ell\}$ and $U=\{v_1,v_2,\dots,v_\ell\}$, the defining vertices and hyperedges of this path. Note that $|e_1\cap U|\leq k-1<r$, so $e_1\backslash U\neq\emptyset$.

First, we will show that any vertex from $e_1\cap U$ is the only incident with the defining hyperedges from $\mathcal{F}$.

Lemma 4.2.6. *Suppose* $w\in e_1\backslash U$*, then* $N_{\mathcal{H}}(w)\subseteq\mathcal{F}$*. Hence* $N_{\mathcal{H}}(e_1\backslash U)\subseteq\mathcal{F}$.

Proof. If w is incident with a hyperedge of P not in $\mathcal{F}$, then $t\geq k$. Let $j\geq k$ be the smallest index such that w is incident with e_j, then $v_1,e_2,v_2\dots,v_{j-1},e_j,w,e_1,v_1$ is a Berge-cycle of length at least k, a contradiction. If w is incident with an edge e not in the semi-path P, then e,w,P is a longer semi-path, a contradiction to the maximality of P.

For simplicity Lemma 4.2.6 was stated and proved for the maximal semi-path $\mathcal{P}$, but similarly, it holds for every maximal semi-path. Hence we may apply Lemma 4.2.6 for other maximal semi-paths.

□

For each defining vertex v_i, $v_i\in e_1\cap U$, we find another maximal semi-path by rearranging P, starting at e_i, without changing the set of the first ℓ vertices and hyperedges.

Lemma 4.2.7. *If for some i we have that* $v_i\in e_1\cap U$*, then* $N_{\mathcal{H}}(e_i\backslash U)\subseteq\mathcal{F}$.

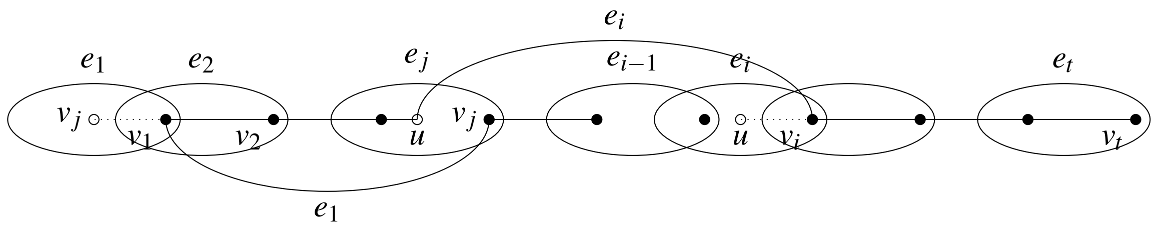

Fig. 4.1 Semi-path P_1 in the proof of Lemma 4.2.8

Proof. Consider the semi-path $e_i, v_{i-1}, e_{i-1}, v_{i-2}, \dots, e_2, v_1, e_1, v_i, e_{i+1}, v_{i+1}, \dots, e_t, v_t$, this semi-path has length t, so it is maximal, then $N_{\mathcal{H}}(e_i \backslash U) \subseteq \mathcal{F}$ follows from Lemma 4.2.6 for this path. □

Lemma 4.2.8. *If there are two vertices $v_i, v_j \in e_1 \cap U$, with $i > j$ such that $(e_i \cap e_j) \backslash U \neq \emptyset$, then $N_{\mathcal{H}}(v_{i-1}) \subseteq \mathcal{F}$ and $N_{\mathcal{H}}(v_j) \subseteq \mathcal{F}$.*

Proof. Fix a vertex $u \in (e_i \cap e_j) \backslash U$ and consider maximal length semi-paths P_1 and P_2, (see Figure 4.1).

$$P_1 = e_{i-1}, v_{i-2}, e_{i-2}, v_{i-3}, \dots, e_{j+1}, v_j, e_1 v_1, e_2, v_2, \dots, v_{j-1}, e_j, u, e_i, v_i, e_{i+1}, v_{i+1}, \dots, e_t, v_t,$$

$$P_2 = e_{j+1}, v_{j+1}, e_{j+2}, v_{j+2}, \dots, v_{i-1}, e_i, u, e_j v_{j-1}, e_{j-1}, v_{j-2}, \dots, e_2, v_1, e_1, v_i, e_{i+1}, \dots, e_t, v_t.$$

Applying Lemma 4.2.6 for P_1 and P_2, we get $N_{\mathcal{H}}(v_{i-1}) \subseteq \mathcal{F}$ and $N_{\mathcal{H}}(v_{j+1}) \subseteq \mathcal{F}$. □

Let $d \leq m$ be an integer such that $V(e_1) = V(e_2) = \dots = V(e_d) \neq V(e_{d+1})$.

Claim 4.2.9. *If $e_1 \cap U = \{v_1, v_2, v_3, \dots, v_d\}$ then either $e_1 \backslash \{v_d\}$ is incident with d, $d \leq m$, hyperedges or there exists a set S of size r such that $N_{\mathcal{H}}(S) \subseteq \mathcal{F}$. In particular if $e_1 \cap U = \{v_1, v_2, v_3, \dots, v_d\}$ then Lemma 4.2.2 holds.*

Proof. First note that the vertices $v_1, v_2, \dots, v_{d-1}$ can be exchanged with the vertices of $e_1 \backslash U \neq \emptyset$, hence from the Lemma 4.2.6, we have $N_h(e_1 \backslash \{v_d\}) \subseteq \mathcal{F}$. Suppose $w \in e_1 \backslash \{v_d\}$ is incident with a hyperedge e_j, $\ell \geq j > d$, we may assume $w \in e_1 \backslash U$, then the semi-path $P_1 = e_{j-1}, v_{j-2}, e_{j-2}, v_{j-3}, \dots \; \dots, e_2, v_1, e_1, w, e_j, v_j, \dots, e_t, v_t$ has maximal length. Since v_{j-1} is a non-defining vertex in the first hyperedge P_1, applying Lemma 4.2.6 to P_1, we have that $N_{\mathcal{H}}(v_{j-1}) \subseteq \mathcal{F}$, therefore the set $(v_1 \backslash \{v_d\}) \cup \{v_{j-1}\}$ is a set of r vertices incident with at most $k-1$ hyperedges from $\mathcal{F}$. Otherwise, if there is no such w then we have a set of $r-1$ vertices, $e_1 \backslash \{v_d\}$ incident with at most m hyperedges. □

From here we may assume that $|e_1 \cap U| > d$. Let $e_1 \cap U = \{v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_s}\}$, where $1 = i_0 < i_1 < i_2 < \cdots < i_s$, define recursively the sets $A_1 := e_1 \backslash U$ and for $j = 1, \ldots, s$, if $(e_{i_j} \backslash U) \cap A_j = \emptyset$, take $A_{j+1} := A_j \cup (e_{i_j} \backslash U)$, otherwise take $A_{j+1} := A_j \cup (e_{i_j} \backslash U) \cup \{v_{i_j-1}\}$. Note that the only possible defining vertices in A_j are $v_{i_1-1}, v_{i_2-1}, \ldots, v_{i_{j-1}-1}$, therefore v_{i_j-1} is not contained in A_j. Let us denote $A := A_{s+1}$. We have that $|A_j| < |A_{j+1}|$, for all $j \in \{1, 2, \ldots, s\}$, so $|A| \geq |A_1| + s \geq r - 1$, by Lemmas 4.2.6, 4.2.7 and 4.2.8, we have that $N_{\mathcal{H}}(A) \subseteq \mathcal{F}$. If $m \geq k-1$ then A is a set of at least $r-1$ vertices incident with at most m hyperedges, hence Lemma 4.2.2 holds. If $|A| \geq r$ then A is a set of at least r vertices incident with at most $k-1$ hyperedges, hence Lemma 4.2.2 holds. From here we may assume $m < k-1$ and $|A| = r-1$. Observe that $|A| = r-1$ is only possible if for every $i = 1, 2, \ldots, s$, $|A_{i+1}| = |A_i| + 1$. We will assume, without loss of generality, that among all possible semi-paths of maximal length, P is one for which $|e_1 \backslash U|$ is minimal. There are two cases:

Case 1: There exists an index $j \geq d$, such that A_j intersects $(e_{i_j} \backslash U)$, let j' be the first such index, then there is another index $d-1 \leq q < j'$ such that $(e_{i'_j} \backslash U) \cap (e_{i_q} \backslash U) \neq \emptyset$, and let u be an element in this intersection.

If $i_q < i_{j'} - 1$ then $v_{i_q} \notin A$ from the minimality of j', and by Lemma 4.2.8, $N_{\mathcal{H}}(v_{i_q}) \subseteq \mathcal{F}$ so $A \cup \{v_{i_q}\}$ is a set of vertices, of size r, incident with at most $k-1$ hyperedges, hence Lemma 4.2.2 holds.

If $d < i_q = i_{j'} - 1$ then by applying Lemma 4.2.6 to the maximal semi-path,

$$e_{i_q-1}, v_{i_q-2}, e_{i_q-2}, v_{i_q-3}, \ldots, v_2, e_2, v_1, e_1, v_{i_q}, e_{i_q}, u, e_{i_q+1}, v_{i_q+1}, e_{i_q+2}, \ldots, e_t, v_t$$

we get $N_{\mathcal{H}}(v_{i_q-1}) \subseteq \mathcal{F}$, since v_{i_q-1} is a non-defining vertex in the first hyperedge. Also we have $v_{i_q-1} \notin A$ from the minimality of j', hence $A \cup \{v_{i_q-1}\}$ is a set of r vertices, incident with at most $k-1$ hyperedges and therefore Lemma 4.2.2 holds.

Suppose $d = i_q = i_{j'} - 1$. Since $|A_{d+1}| = |A_d| + 1$, $A_{d+1} = A_d \cup (e_{i_d} \backslash U) \cup \{v_{i_d-1}\}$ and $v_{i_d-1} \notin A_d$, we have $(e_{d+1} \backslash U) \subseteq (e_1 \backslash U)$, otherwise A_{d+1} would have at least two new elements, but by the minimality of $|e_1 \backslash U|$, we have $(e_{d+1} \backslash U) = (e_1 \backslash U)$. Fix any vertex v_x, $v_x \in U \cap (e_{d+1} \backslash e_1)$. We need a similar lemma as Lemma 4.2.8.

Claim 4.2.10. *Suppose $v_j \in e_1$ is such that $(e_j \backslash U)$ intersects $(e_x \backslash U)$ then*

$$N_h(v_{\max\{j,x\}-1}) \subseteq \mathcal{F}.$$

We skip the proof of Claim 4.2.10, since it is similar to the proof of Lemma 4.2.8.

Let $(e_1 \cap U) \cup \{v_x\} = \{v_{j_0}, v_{j_1}, \dots, v_{j_{s+1}}\}$, where $1 = j_0 < j_1 < \cdots < j_{s+1}$, define recursively the following sets $B_1 = e_1 \backslash U$, and for $c = 1, 2, \dots, s+1$ let $B_{c+1} = B_c \cup (e_{v_{j_c}} \backslash U)$, if $B_c \cap (e_{v_{j_c}} \backslash U) = \emptyset$, otherwise take $B_{c+1} = B_c \cup \{v_{j_c - 1}\}$. Finally B_{s+2} has size at least r and is incident with at most $k-1$ hyperedges, therefore Lemma 4.2.2 holds.

Case 2: For every index $j \geq d$, A_j and $(e_{i_j} \backslash U)$ are disjoint. In this case, by construction, we have that $r - 1 = |A| = |e_1 \backslash U| + (d-1) + |e_{i_d} \backslash U| + \cdots + |e_{i_s} \backslash U|$, this implies that $\left|e_{i_j} \backslash U\right| = 1$ for every j, hence $|U| \geq r-1$, but since $k - 1 \geq |U|$, we have that $k = r$ and $|U| = r-1$. So there exists distinct vertices $u_d, u_{d+1}, \dots, u_{r-1}$ such that $e_i = U \cup \{u_i\}$ for each $i \in \{d, d+1, \dots, r-1\}$ and $A = \{v_1, v_2, \dots, v_{d-1}, u_d, u_{d+1}, \dots, u_{r-1}\}$.

If $d > 1$, take the maximal semi-path, obtained from P, by changing the vertex v_{r-2} with v_1 and the vertex v_1 with u_d, that is $e_1, u_d, e_2, v_2, \dots, v_{r-3}, e_{r-2}, v_1, e_{r-1}, v_{r-1}, \dots, v_t$. By Lemma 4.2.6, we have that $N_{\mathcal{H}}(v_{r-2}) \subseteq \mathcal{F}$. Therefore $A \cup \{v_{r-2}\}$ is a set of vertices of size r incident with at most $k-1$ hyperedges, thus Lemma 4.2.2 holds. Thus, we can suppose, that $d = 1$, and then each u_i is vertex of degree one.

We may also assume that the length of P is at least r, otherwise $N_{\mathcal{H}}(v_{r-1}) \subset \mathcal{F}$, hence $A \cup \{v_{r-1}\}$ is a vertex set of size r incident with at most $k-1$ hyperedges, therefore Lemma 4.2.2 holds.

Claim 4.2.11. *If there exists a hyperedge e, such that $e \neq e_r$ and $e \cap (U \backslash \{v_{r-1}\}) \neq \emptyset$ then the vertices in $e \backslash U$ are only incident with e.*

Proof. Suppose without loss of generality $v_1 \in e$, otherwise we can rearrange the path. If e is a hyperedge of semi-path P, then $e = e_j$ for some $j < r$, otherwise we have a Berge-cycle length at least k, a contradiction. If $e = e_j$, $j < r$ then we already deduced that Claim 4.2.11 holds. If e is a non-defining hyperedge of semi-path P, then consider P' obtain by replacing e_1 in P with e, from Lemma 4.2.6, a vertex in $v \in e \backslash U$ can only be incident with $e, e_2, e_3, \dots, e_{r-1}$, but if v is incident with one of these hyperedges from $e_2, e_3, \dots, e_{r-1}$ then $e, e_1, e_2, \dots, e_{r-1}$ together with the vertices $v, v_1, v_2, \dots, v_{r-1}$ in some order would be a Berge-cycle of length r, a contradiction. Finally we have $N_{\mathcal{H}}(e \backslash U) = \{e\}$, therefore Claim 4.2.11 holds. □

Let $h_1, h_2, \dots, h_p$ be the hyperedges incident with $U \backslash v_{r-1}$. If $|h_i \backslash U| \geq 2$, for some i, then $(e_1 \cup e_2 \cup \cdots \cup e_{r-2} \cup h_i) \backslash U$ is a set of size at least r incident with $k-1$ hyperedges, hence Lemma 4.2.2 holds. Otherwise we have $|h_i \backslash U| = 1$, so these hyperedges form an r-star $\mathcal{S}$ with $p \geq r-1$ hyperedges. Every hyperedge from $E(\mathcal{H}) \backslash \{e_r, h_1, h_2, \dots, h_p\}$

can only intersect $V(\mathcal{S})$ in v_{r-1}, by setting $\mathcal{K}$ to be the r-graph induce from $\mathcal{H}$ by the vertices $\{v_{r-1}\} \cup (V(\mathcal{H}) \backslash V(\mathcal{S}))$, we get the desired partition of $\mathcal{H}$ after removing the hyperedge e_r, therefore Lemma 4.2.2 holds.

4.2.3 Corollaries

We note that as a corollary of Theorem 1.8.16 we obtain a slightly stronger version of Theorem 1.8.1.

Let us recall Corollary 1.8.17.

Corollary. *Let k,n and r be positive integer with $3 \le k \le r$. Then*

$$\mathrm{ex}_r(n, \mathcal{BP}_k) = \left\lfloor \frac{n}{r+1} \right\rfloor (k-1) + \mathbb{1}_{(r+1)\mathbb{N}^*}(n+1)$$

Proof of Corollary 1.8.17. Let $\mathcal{H}$ be an n-vertex r-graph containing no Berge-path of length k. Define an $(r+1)$-graph $\mathcal{H}'$ by adding a new vertex v to the vertex set of $\mathcal{H}$ and extending every hyperedge of $\mathcal{H}$ with v.

If $\mathcal{H}'$ is $\mathcal{BC}_{\ge k}$-free, then from Theorem 1.8.16, we have

$$e(\mathcal{H}) = e(\mathcal{H}') \le \left\lfloor \frac{n+1-1}{r+1} \right\rfloor (k-1) + \mathbb{1}_{(r+1)\mathbb{N}^*}(n+1) = \left\lfloor \frac{n}{r} \right\rfloor (k-1) + \mathbb{1}_{(r+1)\mathbb{N}^*}(n+1).$$

Suppose $\mathcal{H}'$ contains a copy of a Berge-cycle $v_1, h_1, v_2 \dots, h_{\ell-1}, v_\ell, h_\ell, v_1$, of length ℓ, for some $\ell \ge k$. If v is one of the defining vertices, suppose without loss of generality $v = v_1$, and let $h_i' = h_i \backslash \{v\}$ for each $i = 1, 2 \dots \ell$ then $\left|(h_1' \cup h_k') \backslash \{v_2, \dots, v_k\}\right| \ge r+1-(k-1) \ge 2$ and that set intersects both h_1' and h_k' hyperedges. Therefore we can find two distinct vertices $u \in h_1'$ and $u' \in h_k'$ different from all v_i, $i \in \{1, 2, \dots, k\}$ then $u, h_1', v_2, h_2', v_3, \dots, h_{k-1}', v_k, h_k', u'$ is a Berge-path of length k in $\mathcal{H}$, a contradiction. If v is not one of the defining vertices, then a similar argument leads us to contradiction.

□

As a corollary of Theorem 1.8.20 we obtain a version of Theorem 1.8.1 with multiple hyperedges.

Let us recall Corollary 1.8.21.

Corollary. *Let k,n and r be positive integer with $2 \le k \le r$ then*

$$\mathrm{ex}_r^{multi}(n, \mathcal{BP}_k) = \left\lfloor \frac{n}{r} \right\rfloor (k-1).$$

Proof of Corollary 1.8.21. This follows in a similar way as the previous corollary, by constructing a $\mathcal{BC}_{\geq k}$-free r-multi-graph $\mathcal{H}'$.
Hence, by Theorem 1.8.20, $e(\mathcal{H}) = e(H') \leq \left\lfloor \frac{n+1-1}{r+1-1} \right\rfloor (k-1) = \left\lfloor \frac{n}{r} \right\rfloor (k-1)$. $\square$

Chapter 5

Connected Hypergraphs without long Berge-paths

> If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy.
>
> Alfréd Rényi

In this Chapter we prove the analogues of Theorem 1.3.7 for higher uniformity. Let us recall Theorem 1.3.7.

Theorem (Kopylov [86], Balister, Győri, Lehel, Schelp [9])**.** *Let $n > \ell \geq 3$*

$$\mathrm{ex}^{conn}(n, P_\ell) = max\left\{e(G_{n,\ell,1}), e(G_{n,\ell,\lfloor\frac{\ell-1}{2}\rfloor})\right\}.$$

Extremal graphs are $G_{n,\ell,1}$ or $G_{n,\ell,\lfloor\frac{\ell-1}{2}\rfloor}$, see Definition 1.3.6.

In the following section we prove Theorem 1.8.4.

Theorem (Győri, Methuku, Salia, Tompkins, Vizer [75])**.** *Let $\mathcal{H}'_{n,k}$ be a largest r-uniform connected n-vertex hypergraph with no Berge-path of length k, then*

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{\left|E(\mathcal{H}'_{n,k})\right|}{k^{r-1}n} = \frac{1}{2^{r-1}(r-1)!}.$$

We omit the proof of Theorem 1.8.10.

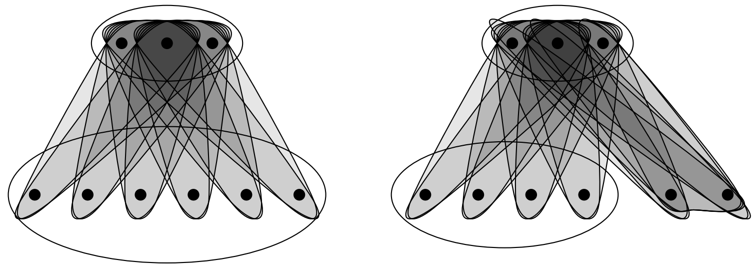

Fig. 5.1 The hypergraphs $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$ and $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$.

Theorem (Győri, Salia , Zamora [75])**.** *For all integers k,r with $k \geq 2r+13 \geq 18$ there exists $n_{k,r}$ such that if $n > n_{k,r}$, then we have*

- $\mathrm{ex}_r^{conn}(n,\mathcal{BP}_k) = |\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}|$, *if k is odd, and*
- $\mathrm{ex}_r^{conn}(n,\mathcal{BP}_k) = |\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}|$, *if k is even.*

Depending on the parity of k, the unique extremal hypergraph is $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$ or $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$, (see Figure 5.1).

Instead we prove a stability version of Theorem 1.8.10, in Section 5.2.

5.1 Proof of asymptotic

Here we start to prove Theorem 1.8.4. We will use the following simple corollary of Theorem 1.5.4.

Corollary 5.1.1. *Let G be a connected graph on n vertices with no P_k, then G has at most*

$$\frac{k^{r-1}n}{2^{r-1}(r-1)!}$$

r-cliques if $n \geq c_{k,r}$ for some constant $c_{k,r}$ depending only on k and r.

Proof. From Theorem 1.5.4, it follows that for large enough n, the number of r-cliques is at most

$$\left(n-\left\lfloor\frac{k-1}{2}\right\rfloor\right)\binom{\lfloor\frac{k-1}{2}\rfloor}{r-1}+\binom{\lfloor\frac{k-1}{2}\rfloor}{r}+\binom{\lfloor\frac{k-1}{2}\rfloor}{r-2}<n\binom{\frac{k}{2}}{r-1}. \qquad \square$$

Given an r-uniform hypergraph $\mathcal{H}$ we define the shadow graph of $\mathcal{H}$, denoted $\partial\mathcal{H}$ to be the graph on the same vertex set with edge set:

$$E(\partial\mathcal{H}) := \{\{x,y\} : \{x,y\} \subset e \in E(\mathcal{H})\}.$$

Definition 5.1.2. *If $r = 3$, then we call an edge $e \in E(\partial\mathcal{H})$* fat *if there are at least 2 distinct hyperedges h_1, h_2 with $e \subset h_1, h_2$. If $r > 3$, then we call an edge $e \in E(\partial\mathcal{H})$* fat *if there are at least k distinct hyperedges $h_1, h_2, \dots, h_k$ in $\mathcal{H}$ with $e \subset h_i$ for $1 \leq i \leq k$.*
We call an edge $e \in E(\partial\mathcal{H})$ thin *if it is not fat.*

Thus, the set $E(\partial\mathcal{H})$ decomposes into the set of fat edges and the set of thin edges. We will refer to the graph whose edges consist of all fat edges in $\partial\mathcal{H}$ as the *fat graph* and denote it by F.

Lemma 5.1.3. *There is no P_k in the fat graph F of the hypergraph $\mathcal{H}$.*

Proof. Suppose we have such a P_k with edges $e_1, e_2, \dots, e_k$. For $r = 3$, if a hyperedge contains two edges from the path, then it must contain consecutive edges e_i, e_{i+1}. Select hyperedges $h_1, h_2, \dots, h_k$ where $e_i \subset h_i$ in such a way that h_{i+1} is different from h_i for all $1 \leq i \leq k-1$, and these edges yield the required Berge-path.

Suppose now that $r > 3$, we will find a Berge-path of length k in $\mathcal{H}$, with a greedy argument. For e_1, select an arbitrary hyperedge h_1 containing it. Suppose we have found a distinct hyperedge h_i containing the fat edge e_i for all $1 \leq i < i^*$. Since the edge e_{i^*} is fat, there are at least k different hyperedges $h^1_{i^*}, h^2_{i^*}, \dots, h^k_{i^*}$ containing it. Select one of them, say $h^j_{i^*}$, which is not equal to any of $h_1, h_2, \dots, h_{i^*-1}$. Thus, we may find distinct hyperedges $h_1, h_2, \dots, h_k$ where $e_i \subset h_i$ for $1 \leq i \leq k$, and thus, we have a Berge-path of length k. □

We call a hyperedge $h \in E(\mathcal{H})$ *fat* if h contains no thin edge. Let $\mathcal{F}$ denote the hypergraph on the same set of vertices as $\mathcal{H}$ consisting of the fat hyperedges, then

Lemma 5.1.4. *If $r = 3$, then*

$$|E(\mathcal{H} \setminus \mathcal{F})| \leq \frac{(k-1)n}{2}.$$

If $r > 3$, then

$$|E(\mathcal{H} \setminus \mathcal{F})| \leq \frac{(k-1)^2 n}{2}.$$

Proof. Arbitrarily select a thin edge from each $h \in \mathcal{H} \setminus \mathcal{F}$. Let G be the graph consisting of the selected thin edges. We know that each edge in G was selected at most once if $r = 3$ and at most $k-1$ times in the $r > 3$. Thus, we have that $|\mathcal{H} \setminus \mathcal{F}| \leq |E(G)|$ for $r = 3$ and $|\mathcal{H} \setminus \mathcal{F}| \leq (k-1)\,|E(G)|$ for $r > 3$. Moreover, G is P_k-free since a P_k in G would imply a Berge P_k in $\mathcal{H}$ by considering any hyperedge from which each edge was selected. It follows by Theorem 1.3.1 that $|E(G)| \leq \frac{(k-1)n}{2}$, so $|\mathcal{H} \setminus \mathcal{F}| \leq \frac{(k-1)n}{2}$ if $r = 3$, and $|\mathcal{H} \setminus \mathcal{F}| \leq \frac{(k-1)^2 n}{2}$ if $r > 3$. □

Any hyperedge of $\mathcal{F}$ contains only fat edges, so it corresponds to a unique r-clique in F. This implies the following.

Observation 5.1.5. *The number of hyperedges in $E(\mathcal{F})$ is at most the number of r-cliques in the fat graph F.*

To this end we will upper bound the number of r-cliques in F, by making use of the following important lemma.

Lemma 5.1.6. *There are no two disjoint cycles of length at least $k/2+1$ in the fat graph F.*

Proof. Let C and D be two such cycles. By connectivity, there are vertices $v \in V(C)$ and $w \in V(D)$ and a Berge-path from v to w in $\mathcal{H}$ containing no additional vertices of C or D as defining vertices. This path can be extended using the hyperedges containing the edges of C and D to produce a Berge-path of length k in $\mathcal{H}$ (note that here we used that the edges of C and D are fat), a contradiction. $\square$

Assume that F has connected components $C_1, C_2, \dots, C_t$. Trivially,

$$N_r(F) = \sum_{i=1}^{t} N_r(C_i). \tag{5.1}$$

If $|V(C_i)| \le k/2$, then trivially

$$N_r(C_i) \le \binom{|V(C_i)|}{r} \le \frac{|V(C_i)|^r}{r!} \le \frac{k^{r-1}\,|V(C_i)|}{2^{r-1}(r-1)!}.$$

So we can assume $|V(C_i)| \ge k/2$. By Lemma 5.1.6, we have that for all but at most one i, C_i does not contain a cycle of length at least $k/2+1$. So by Corollary 1.5.5, for all but at most one i, say i_0, we have

$$N_r(C_i) \le \frac{|V(C_i)|-1}{k/2-2}\binom{k/2-1}{r} \le \frac{k^{r-1}\,|V(C_i)|}{2^{r-1}(r-1)!} + O(k^{r-2}).$$

If $|V(C_{i_0})| \ge c_{k,r}$, then by Lemma 5.1.3 and by Corollary 5.1.1 we have

$$N_r(C_{i_0}) \le \frac{k^{r-1}\,|V(C_i)|}{2^{r-1}(r-1)!}.$$

Otherwise, $N_r(C_{i_0}) \le \binom{|V(C_{i_0})|}{r} = o(n)$. Therefore, by (5.1), we have

$$N_r(F) = \sum_{i=1}^{t} N_r(C_i) \le$$

$$\leq \sum_{i=1}^{t}\left(\frac{k^{r-1}|V(C_i)|}{2^{r-1}(r-1)!}+O(k^{r-2})\right)+o(n)\leq \frac{k^{r-1}n}{2^{r-1}(r-1)!}+O(k^{r-2})n+o(n).$$

Therefore, by Observation 5.1.5,

$$|E(\mathcal{F})|\leq N_r(F)\leq \frac{k^{r-1}n}{2^{r-1}(r-1)!}+O(k^{r-2})n+o(n). \tag{5.2}$$

Since $|E(\mathcal{H})|=|E(\mathcal{H}\setminus\mathcal{F})|+|E(\mathcal{F})|$, adding up the upper bounds in (5.2) and Lemma 5.1.4, we obtain the desired upper bound on $|E(\mathcal{H})|$.

□

5.2 Proof of stability

Let us start with recalling Definition 1.8.9.

Definition 5.2.1. *For integers $n,a\geq 1$ and $b_1,\dots,b_t\geq 2$ with $n\geq 2a+\sum_{i=1}^t b_i$ let us denote by $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ the following r-uniform hypergraph.*

• *Let the vertex set of $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ be $A\cup L\cup\bigcup_{i=1}^t B_i$, where $A,B_1,B_2,\dots,B_t$ and L are pairwise disjoint sets of sizes $|A|=a$, $|B_i|=b_i$ $(i=1,2,\dots,t)$ and $|L|=n-a-\sum_{i=1}^t b_i$.*

• *Let the hyperedges of $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ be*

$$\binom{A}{r}\cup\bigcup_{i=1}^{t}\binom{A\cup B_i}{r}\cup\left\{\{c\}\cup A' : c\in L, A'\in\binom{A}{r-1}\right\}.$$

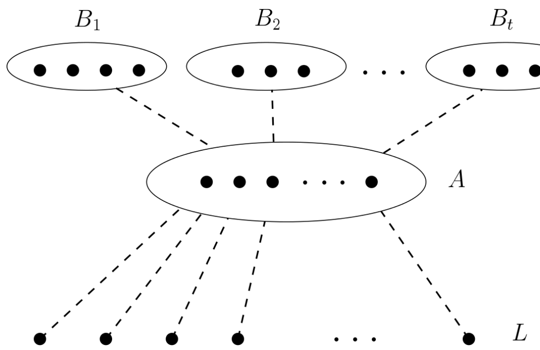

Fig. 5.2 The hypergraph $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$.

The result here provides a stability version of Theorem 1.8.10 and also an extension of Theorem 1.9.2 for uniformity at least 3.

First we state a theorem for hypergraphs with minimum degree at least 2, and then in full generality. In the proof, the hypergraphs $\mathcal{H}_{n,\frac{k-3}{2},3}$ and $\mathcal{H}_{n,\frac{k-3}{2},2,2}$ will play a crucial role in case k is odd, while if k is even, then the hypergraphs $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}$, $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3,2}$ and $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,2,2,2}$ will be of importance (Definition 1.8.9), note that all of them are n-vertex, maximal, $\mathcal{BP}_k$-free hypergraphs. In both cases, the hypergraph listed first contains the largest number of hyperedges. This number gives the lower bound in the following theorem. We recall Theorem 1.9.3.

Theorem (Gerbner, Nagy, Patkós, Salia, Vizer ,[57]). *For any $\varepsilon > 0$ there exist integers $q = q_\varepsilon$ and $n_{k,r}$ such that if $r \geq 3$, $k \geq (2+\varepsilon)r + q$, $n \geq n_{k,r}$ and $\mathcal{H}$ is a connected n-vertex, r-uniform hypergraph with minimum degree at least 2, without a Berge-path of length k, then we have the following.*

- *If k is odd and $|\mathcal{H}| > |\mathcal{H}_{n,\frac{k-3}{2},3}| = (n - \frac{k+3}{2})\binom{\frac{k-3}{2}}{r-1} + \binom{\frac{k+3}{2}}{r}$, then $\mathcal{H}$ is a sub-hypergraph of $\mathcal{H}_{n,\frac{k-1}{2}}$.*
- *If k is even and $|\mathcal{H}| > |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}| = (n - \lfloor\frac{k+5}{2}\rfloor)\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1} + \binom{\lfloor\frac{k+5}{2}\rfloor}{r}$, then $\mathcal{H}$ is a sub-hypergraph of $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$ or $\mathcal{H}^+_{n,\lfloor\frac{k-1}{2}\rfloor}$.*

Let $\mathbb{H}'_{n',a,b_1,b_2,\ldots,b_t}$ be the class of hypergraphs that can be obtained from $\mathcal{H}_{n,a,b_1,b_2,\ldots,b_t}$ for some $n \leq n'$ by adding hyperedges of the form $A'_j \cup D_j$, where the D_j's partition $[n'] \setminus [n]$, all D_j's are of size at least 2 and $A'_j \subseteq A$ for all j. Let us define $\mathbb{H}^+_{n',\lfloor\frac{k-1}{2}\rfloor}$ analogously. We recall Theorem 1.9.4.

Theorem 5.2.2 (Gerbner, Nagy, Patkós, Salia, Vizer ,[57]). *For any $\varepsilon > 0$ there exist integers $q = q_\varepsilon$ and $n_{k,r}$ such that if $r \geq 3$, $k \geq (2+\varepsilon)r + q$, $n \geq n_{k,r}$ and $\mathcal{H}$ is a connected n-vertex, r-uniform hypergraph without a Berge-path of length k, then we have the following.*

- *If k is odd and $|\mathcal{H}| > |\mathcal{H}_{n,\frac{k-3}{2},3}|$, then $\mathcal{H}$ is a sub-hypergraph of some $\mathcal{H}' \in \mathbb{H}'_{n,\frac{k-1}{2}}$.*
- *If k is even and $|\mathcal{H}| > |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}|$, then $\mathcal{H}$ is a sub-hypergraph of some $\mathcal{H}' \in \mathbb{H}'_{n,\lfloor\frac{k-1}{2}\rfloor,2}$ or $\mathbb{H}^+_{n,\lfloor\frac{k-1}{2}\rfloor}$.*

We start the proof of Theorem 1.9.3 with a technical lemma that will be crucial later.

Lemma 5.2.3. *Let $\mathcal{H}$ be a connected r-uniform hypergraph with minimum degree at least 2 and with longest Berge-path and Berge-cycle of length $\ell - 1$. Let C be a Berge-cycle of length $\ell - 1$ in $\mathcal{H}$, with defining vertices $V = \{v_1, v_2, \ldots, v_{\ell-1}\}$ and defining edges $\mathcal{E}(C) = \{e_1, e_2, \ldots, e_{\ell-1}\}$ with $v_i, v_{i+1} \in e_i$ (modulo $\ell - 1$). Then, we have* *(i) every hyperedge* $h \in \mathcal{H} \setminus C$ *contains at most one vertex from* $V(\mathcal{H}) \setminus V$.

(ii) If u, v *are not necessarily distinct vertices from* $V(\mathcal{H}) \setminus V$, *then there cannot exist distinct hyperedges* $h_1, h_2 \in \mathcal{H} \setminus \mathcal{C}$ *and an index* i *with* $v, v_i \in h_1$ *and* $u, v_{i+1} \in h_2$.

(iii) If there exists a vertex $v \in V(\mathcal{H}) \setminus V$ *and there exist different hyperedges* $h_1, h_2 \in \mathcal{H} \setminus \mathcal{C}$ *with* $v, v_{i-1} \in h_1$ *and* $v, v_{i+1} \in h_2$, *then there exists a cycle of length* $\ell - 1$ *not containing* v_i *as a defining vertex.*

Proof. We prove (i) by contradiction. Suppose $h \in \mathcal{H} \setminus C$ contains two vertices from $V(\mathcal{H}) \setminus V$. We distinguish two cases.

Case 1. Hyperedge h contains a vertex $u \notin V$ and a different vertex $v \in e_i \setminus V$ for some $i \leq \ell$. Then $v_{i+1}, e_{i+1}, v_{i+2}, \ldots, v_\ell, e_\ell, v_1, e_1, \ldots, v_i, e_i, v, h, u$ is a path of length ℓ, a contradiction.

Case 2. Hyperedge h contains two vertices u and v from $V(\mathcal{H}) \setminus V(C)$. We consider the hypergraph $\mathcal{H}'$ obtained from $\mathcal{H}$ by removing a hyperedge h.

Case 2.1. There is a Berge-path in $\mathcal{H}'$ from $\{v, u\}$ to the cycle C, in particular, to a defining vertex of C. Then let P be a shortest such path, let us assume P is from v to v_i. Without loss of generality we may suppose that P does not contain e_i as a defining hyperedge, (it is possible P contains e_{i-1} as a defining hyperedge). Then $u, h, v, P, v_i, e_i, v_{i+1}, \ldots, e_{i-2}, v_{i-1}$ is a Berge-path of length at least ℓ, contradicting the assumption that the longest path in $\mathcal{H}$ is of length $\ell - 1$.

Case 2.2. Suppose there is no Berge-path from the vertex v to the cycle C in $\mathcal{H}'$. However by connectivity of $\mathcal{H}$, there is a shortest path P from v to a defining vertex of C, say v_i and it does not use any defining hyperedge of C but possibly e_{i-1}. Also, h is not a hyperedge of P. There exists a hyperedge $h' \neq h$ containing v, as the minimum degree is at least 2 in $\mathcal{H}$. Note that h' is not a hyperedge of the path P, even more all vertices of h' different from v are not defining vertices of P or C. Fix a vertex $u' \in h' \setminus \{v\}$. Then $u', h', P, e_i, v_{i+1}, \ldots, e_{i-2}, v_{i-1}$ is a Berge-path of length at least ℓ, a contradiction.

To prove (ii), assume first that $u = v$. Then one could enlarge C by removing e_i and adding h_1, v, h_2 to obtain a longer cycle, a contradiction. Assume now $u \neq v$. Then removing e_i and adding h_1, v and h_2, u, one would obtain a path of length ℓ, a contradiction.

Finally to show (iii), we can replace e_{i-1}, v_i, e_i in C by h_1, v, h_2 to obtain the desired cycle. □

We say that an r-uniform hypergraph $\mathcal{H}$ has the *set degree condition*, if for any set X of vertices with $|X| \leq k/2$, we have $|E(X)| \geq |X| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$, i.e., the number of those hyperedges that are incident to some vertex in X is at least $|X|\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1}$. We first prove Theorem 1.9.3 for such hypergraphs.

Proof of Theorem 1.9.3 for hypergraphs having the set degree condition. Let a hypergraph $\mathcal{H}$ be an n-vertex $\mathcal{BP}_k$-free with the set degree condition. Also, assume $|\mathcal{H}|$ is as claimed in the statement of the theorem. However, for the most part of the proof, we will only use the set degree condition.

Claim 5.2.4. *Let P be a longest Berge-path in $\mathcal{H}$ with defining vertices $U = \{u_1,\dots,u_\ell\}$ and defining hyperedges $\mathcal{F} = \{f_1, f_2,\dots, f_{\ell-1}\}$ in this given order. Suppose P minimizes $x_1 + x_\ell$ among longest Berge-paths of $\mathcal{H}$, where x_i for $i \in [\ell]$, denotes the number of hyperedges in $\mathcal{F}$ incident to u_i. Then the sizes of $N_{\mathcal{H}\setminus\mathcal{F}}(u_1)$ and $N_{\mathcal{H}\setminus\mathcal{F}}(u_\ell)$ are at least $\lfloor\frac{k-3}{2}\rfloor$.*

Proof of Claim 5.2.4. Observe that the statement is trivially true for $r \geq 4$ and for arbitrary longest path, as by the set degree condition, there exist at least $\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1} - k + 1$ hyperedges in $\mathcal{H}\setminus\mathcal{F}$ incident to u_1. This is strictly greater than $\binom{\lfloor\frac{k-5}{2}\rfloor}{r-1}$ if $r \geq 4$ and $k \geq (2+\varepsilon)r + q$, for large enough q, hence $|N_{\mathcal{H}\setminus\mathcal{F}}(u_1)| > \frac{k-5}{2}$, finishing the proof for $r \geq 4$.

Thus we can assume that $r = 3$. Let P be a longest Berge-path in $\mathcal{H}$, minimizing $x_1 + x_\ell$. First we claim that if $u_1 \in f_i$ then $x_i \geq x_1$. Note that the Berge-path

$$u_i, f_{i-1}, u_{i-1}, f_{i-2}, u_{i-2},\dots, u_2, f_1, u_1, f_i, u_{i+1}, f_{i+1},\dots, u_\ell, f_{\ell-1}, u_\ell$$

is also a longest Berge-path, with the same set of defining vertices and defining hyperedges and endpoint x_ℓ, hence by the minimality of the sum $x_1 + x_\ell$, the number of hyperedges from $\mathcal{F}$ incident to u_i is at least x_1.

This means that if we consider all possible Berge-paths obtained from P by the way described above (including itself), then the number of pairs (u, f), where $u \in U$, $f \in \mathcal{F}$ and $u \in f$, is at least x_1^2. On the other hand, this number is upper bounded by $3|\mathcal{F}| = 3(\ell-1)$, hence we have $x_1^2 \leq 3(\ell-1) \leq 3(k-1)$, therefore $x_1 \leq \sqrt{3(k-1)}$. The same holds for the other end vertex u_ℓ and so for x_ℓ by symmetry.

Since the degree of u_1 is at least $\binom{\lfloor\frac{k-3}{2}\rfloor}{2}$, out of which at most $\sqrt{3(k-1)}$ of the hyperedges are defining hyperedges, the degree of u_1 in $\mathcal{H}\setminus\mathcal{F}$ is at least

$$\binom{\lfloor\frac{k-3}{2}\rfloor}{2} - \sqrt{3(k-1)} > \binom{\lfloor\frac{k-3}{2}\rfloor - 1}{2},$$

if $k \geq 21$. Thus $|N_{\mathcal{H}\setminus\mathcal{F}}(u_1)| \geq \lfloor\frac{k-3}{2}\rfloor$ and in the same way we have $|N_{\mathcal{H}\setminus\mathcal{F}}(u_l)| \geq \lfloor\frac{k-3}{2}\rfloor$. □

Claim 5.2.5. *Let $\ell - 1$ be the length of the longest Berge-path in $\mathcal{H}$. Then $\ell \geq k-3$ and $\mathcal{H}$ contains a Berge-cycle of length $\ell - 1$.*

Proof of Claim 5.2.5. Let $u_1, f_1, u_2, f_2, \ldots, u_{\ell-1}, f_{\ell-1}, u_\ell$ be a longest Berge-path given by Claim 5.2.4 with defining hyperedges $\mathcal{F} = \{f_1, f_2, \ldots, f_{\ell-1}\}$ and defining vertices $U = \{u_1, u_2, \ldots, u_\ell\}$.

For $\mathcal{E} \subseteq \mathcal{E}(\mathcal{H})$ and an integer j with $1 \leq j \leq \ell$, let $S_{j,\mathcal{E}}$ denote the set of indices of vertices in $U \cap N_{\mathcal{H} \setminus \mathcal{E}}(u_j)$, and we simply denote $S_{j,\mathcal{F}}$ by S_j. In particular S_j denotes the set of indices i such that there is a hyperedge of $\mathcal{H}$ that contains both u_i and u_j and is not a defining hyperedge of the path. For any set S of integers let $S^- := \{a : a > 0, a+1 \in S\}$, $S^{--} = (S^-)^-$. The operations $^+$ and $^{++}$ are defined analogously.

To start the proof, observe first that $\mathcal{H}$ cannot contain a Berge-cycle C of length ℓ. Indeed, the hyperedges of such a cycle contain at most $\ell(r-1)$ vertices. Therefore there is a vertex $v \in V(\mathcal{H}) \setminus V(C)$, then as $\mathcal{H}$ is connected, there exists a path from v to C and we obtain a path of length at least ℓ, contradicting our assumption on the length of the longest path.

If $\ell \in S_1$ or equivalently $1 \in S_\ell$, then a hyperedge showing this, together with $\mathcal{F}$ forms a Berge-cycle of length ℓ in $\mathcal{H}$. So we can assume $S_1, S_\ell \subseteq \{2, \ldots, \ell-1\}$ and so $S_1^- \subseteq \{1, 2, \ldots, \ell-1\}$.

If $S_1^- \cap S_\ell \neq \emptyset$ (or symmetrically $S_1 \cap S_\ell^+ \neq \emptyset$), then $\mathcal{H}$ contains a Berge-cycle of length ℓ. Indeed, if $i \in S_1^- \cap S_\ell$, then there are hyperedges e and e' in $\mathcal{H} \setminus \mathcal{F}$ with $u_1, u_i \in e$ and $u_\ell, u_{i-1} \in e'$. Then

$$u_{i-1}, f_{i-2}, u_{i-2}, \ldots, f_2, u_2, f_1, u_1, e, u_i, f_{i+1}, u_{i+2}, \ldots, f_{\ell-1}, u_\ell, e'$$

is a Berge-cycle of length ℓ. (Note that e and e' are distinct hyperedges as $\ell \notin S_1$.) Note that by Claim 5.2.4, we have $|S_\ell|, |S_1^-| \geq \left\lfloor \frac{k-3}{2} \right\rfloor$. So to avoid $S_1^- \cap S_\ell \neq \emptyset$, we have $\ell \geq k-3$.

The exact same argument shows that if $S_1^{--} \cap S_\ell \neq \emptyset$ or symmetrically $S_1 \cap S_\ell^{++} \neq \emptyset$, then $\mathcal{H}$ contains a Berge-cycle of length $\ell - 1$ and we are done in this case.

For two indices $x < y \in S_\ell$, let us introduce the relation $x \sim y$ if $S_1 \cap (x, y] = \emptyset$. Clearly, $\sim$ is an equivalence relation. Assume S_ℓ has m_1 equivalence classes. Also, we say that a maximal subset of consecutive integers in S_ℓ is an interval of S_ℓ. As $S_\ell^+ \cap S_1 = \emptyset$ by the above, elements of the same interval belong to the same equivalence class. Let m_2 be the number of intervals in S_ℓ. If $\mathcal{H}$ does not contain cycles of length ℓ and $\ell - 1$, then for the maximal element z of each equivalence class, we have that $z+1, z+2 \notin S_1$ and so by the definition of equivalence classes $z+1, z+2 \notin S_\ell$. Moreover, if an element z belongs both to S_1 and S_ℓ, then z is the smallest element of an equivalence class. Also if z is the largest element of an interval that is not the rightmost interval in an equivalence class, then $z+1 \notin S_1 \cup S_\ell$. These observations show that $2\lfloor\frac{k-3}{2}\rfloor+m_1-2+(m_2-m_1)\leq \ell-2$ holds. As $\ell\leq k$, we must have $m_2\leq 4$.

Similarly as in the proof of Claim 5.2.4 we can see that for any $j\in S_1^-$, the vertex u_j is the endpoint of a longest path $\mathcal{F}_j$ with other end vertex u_ℓ and with defining vertex set U. Observe that the neighborhood S_ℓ of u_ℓ with respect to the non-defining hyperedges of $\mathcal{F}$ and $\mathcal{F}_j$ is the same, as the single hyperedge $h\in\mathcal{F}_j\setminus\mathcal{F}$ contains u_1 and therefore cannot contain u_ℓ without creating a cycle of length ℓ. Therefore $S_{j,\mathcal{F}_j}\subseteq U$ and similarly as above if $[(S_\ell^-\cup S_\ell^{--})\cap(S_\ell^+\cup S_\ell^{++})]\cap S_{j,\mathcal{F}_j}\neq\emptyset$, then $\mathcal{H}$ contains a Berge-cycle of length ℓ or $\ell-1$.

Let $S^*:=(S_\ell^-\cup S_\ell^{--})\cap(S_\ell^+\cup S_\ell^{++})$ then $|S^*|\geq|S_\ell|-2m_2\geq\lfloor\frac{k-3}{2}\rfloor-8$. Let $U_{S_1^-}:=\{u_i: i\in S_1^-\}$ and consider $E(U_{S_1^-})$. Observe that all but one of the defining hyperedges of $\mathcal{F}_i$ are in $\mathcal{F}$, thus there are at most $|\mathcal{F}|+|U_{S_1^-}|\leq k-1+|S_1^-|$ hyperedges altogether in $E(U_{S_1^-})$ that are defining hyperedges of $\mathcal{F}$ or an $\mathcal{F}_i$. From above, all other hyperedges in $E(U_{S_1^-})$ are completely in $U\setminus S^*$, thus we have

$$E(U_{S_1^-})\subseteq\binom{U\setminus S^*}{r}\cup\mathcal{F}\cup\bigcup_{x\in S_1^-}\mathcal{F}_x.$$

By the set degree condition and the above, we must have

$$|S_1^-|\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1}\leq|E(U_{S_1^-})|\leq\binom{k-\lfloor\frac{k-3}{2}\rfloor+8}{r}+k-1+|S_1^-|. \tag{5.3}$$

Using $\lfloor\frac{k-3}{2}\rfloor\leq|S_1^-|$, $\binom{a}{r}=\frac{a}{r}\binom{a-1}{r-1}$ and $\frac{\binom{a+1}{r-1}}{\binom{a}{r-1}}=\frac{a+1}{a-r+2}\leq\frac{a}{a-r}$, and writing $k=\alpha r$ we have

$$\begin{aligned}\binom{k-\lfloor\frac{k-3}{2}\rfloor+8}{r}&=\frac{k-\lfloor\frac{k-3}{2}\rfloor+8}{r}\binom{k-\lfloor\frac{k-3}{2}\rfloor+7}{r-1}\\&\leq\left(\frac{k/2+9}{r}\right)\binom{\lfloor\frac{k+17}{2}\rfloor}{r-1}=\left(\frac{\alpha}{2}+9/r\right)\binom{\lfloor\frac{k+17}{2}\rfloor}{r-1}\\&\leq\left(\frac{\alpha}{2}+9/r\right)\left(\frac{\alpha}{\alpha-2}\right)^{10}\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1}.\end{aligned}\tag{5.4}$$

Therefore (5.3), (5.4) and $k-1+|S_1^-|\leq 2k=2\alpha r$ implies $\alpha r/2-2\leq(\frac{\alpha}{2}+9/r)(\frac{\alpha}{\alpha-2})^{10}+2\alpha$. This shows that for any $\varepsilon>0$, there is an r_0 such that if $r>r_0$, then $\alpha<2+\varepsilon$ must hold, a contradiction. For the finitely many smaller values of r, the above inequality gives an upper bound β_r for $\alpha=k/r$, which might be larger than $2+\varepsilon$. In that case we can choose $q_\varepsilon:=\max_{r\leq r_0}\beta_r r$. Then we have $k>q_\varepsilon\geq\alpha r=k$, a contradiction. $\square$

Note that the cycle C given by Claim 5.2.5 is a longest Berge-cycle in $\mathcal{H}$ and let its defining vertices and defining hyperedges be $V := \{u_1, u_2, \dots, u_{\ell-1}\}$ and $E(C) := \{e_1, e_2, \dots, e_{\ell-1}\}$, respectively, with $u_i, u_{i+1} \in e_i$. We have ℓ is either $k-3,\ k-2,\ k-1$ or k by Claim 5.2.5. Let us call u_{i-1} and u_{i+1} the *neighbors of u_i on C*.

5.2.1 Preliminary technical claims

By Lemma 5.2.3 (i), for any vertex $w \in V(\mathcal{H}) \setminus V$ we have $N_{\mathcal{H}\setminus C}(w) \subseteq V$. For any vertex $w \in V(\mathcal{H}) \setminus V$, we partition $N_{\mathcal{H}\setminus C}(w)$ into two parts the following way: let M_w denote the set of vertices $v \in V$ such that there exists exactly one hyperedge in $\mathcal{H} \setminus C$ containing both w and v, and let D_w denote the set of those vertices $v \in V$ for which there exist at least 2 hyperedges in $\mathcal{H} \setminus C$ containing both v and w.

Claim 5.2.6. *For any w and w' with $w, w' \in V(\mathcal{H}) \setminus V$ and not necessarily distinct, we have*

(i) If $u_j \in N_{\mathcal{H}\setminus C}(w)$, $u_{j+1} \in N_{\mathcal{H}\setminus C}(w')$, then $w = w'$, $u_j, u_{j+1} \in M_w$ and there exists a non-defining hyperedge h with $w, u_j, u_{j+1} \in h$.

(ii) If $u_j \in N_{\mathcal{H}\setminus C}(w)$, $u_{j+2} \in D_w$, then there exists a cycle C' of length $\ell - 1$ in $\mathcal{H}$ such that the defining vertices of C' are those of C but u_{j+1} replaced by w.

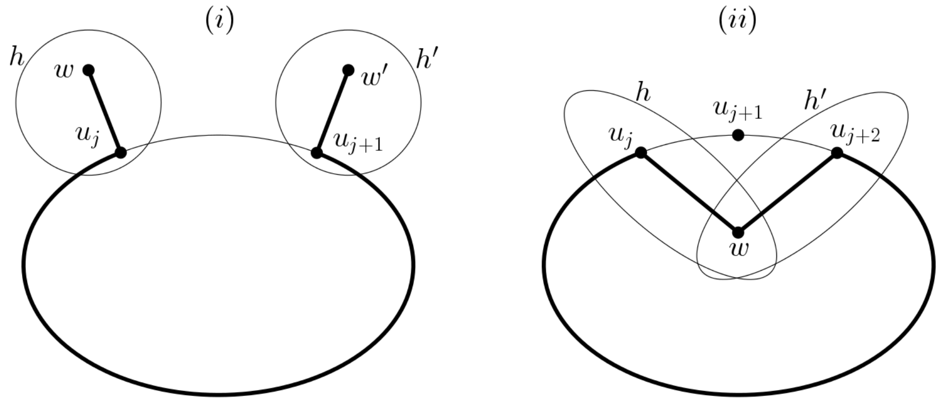

Fig. 5.3 Sketch of the proof of Claim 5.2.6.

Proof. Let $u_j \in N_{\mathcal{H}\setminus C}(w)$, $u_{j+1} \in N_{\mathcal{H}\setminus C}(w')$. If $w \neq w'$, then for the hyperedges $h, h' \in \mathcal{H} \setminus C$ with $u_j, w \in h$ and $u_{j+1}, w' \in h'$, we have $h \neq h'$, from Lemma 5.2.3 (i). But then

$$w', h', u_{j+1}, e_{j+1}, u_{j+2}, \dots, u_{\ell-1}, e_{\ell-1}, u_1, e_1, \dots, u_j, h, w$$

is a Berge-path of length ℓ, see Figure 5.3, a contradiction. So $w = w'$, and if there exist $h \neq h'$ with $u_j, w \in h$ and $u_{j+1}, w \in h'$, then the Berge-path presented above is in fact a Berge-cycle that is longer than C, a contradiction. This proves (i).

For the second part of the claim, observe that if $u_j \in N_{\mathcal{H}\setminus C}(w)$ and $u_{j+2} \in D_w$, then there exist two distinct hyperedges $h, h' \in \mathcal{H} \setminus C$ such that $u_j, w \in h$ and $u_{j+2}, w \in h'$, so in C we can replace e_j, u_{j+1}, e_{j+1} by h, w, h' to obtain desired cycle C', see Figure 5.3. $\square$

Claim 5.2.7. *Suppose $u_{i-1}, u_{i+1}, u_j \in D_w$ are three distinct vertices for some $w \in V(\mathcal{H}) \setminus V$ and let $w^* \in V(\mathcal{H}) \setminus V$ be a vertex distinct from w. Then we have the following.*

(i) There is no hyperedge $h \in \mathcal{H} \setminus C$ with $u_i, u_{j-1} \in h$ nor with $u_i, u_{j+1} \in h$.

(ii) If $u_{j+2} \in N_{\mathcal{H}\setminus C}(w)$, then e_{i-1}, e_i do not contain u_{j+1}.

(iii) Hyperedges e_{i-1} and e_i are not incident with the vertices w, w^*.

(iv) Suppose $u_{t+1} \in D_{w^*}$ or $u_{t-1} \in D_{w^*}$ for some $t \neq i$. Then there is no $h \in \mathcal{H} \setminus C$ incident to u_i and u_t.

(v) The hyperedges e_{j-1}, e_j are not incident with u_i.

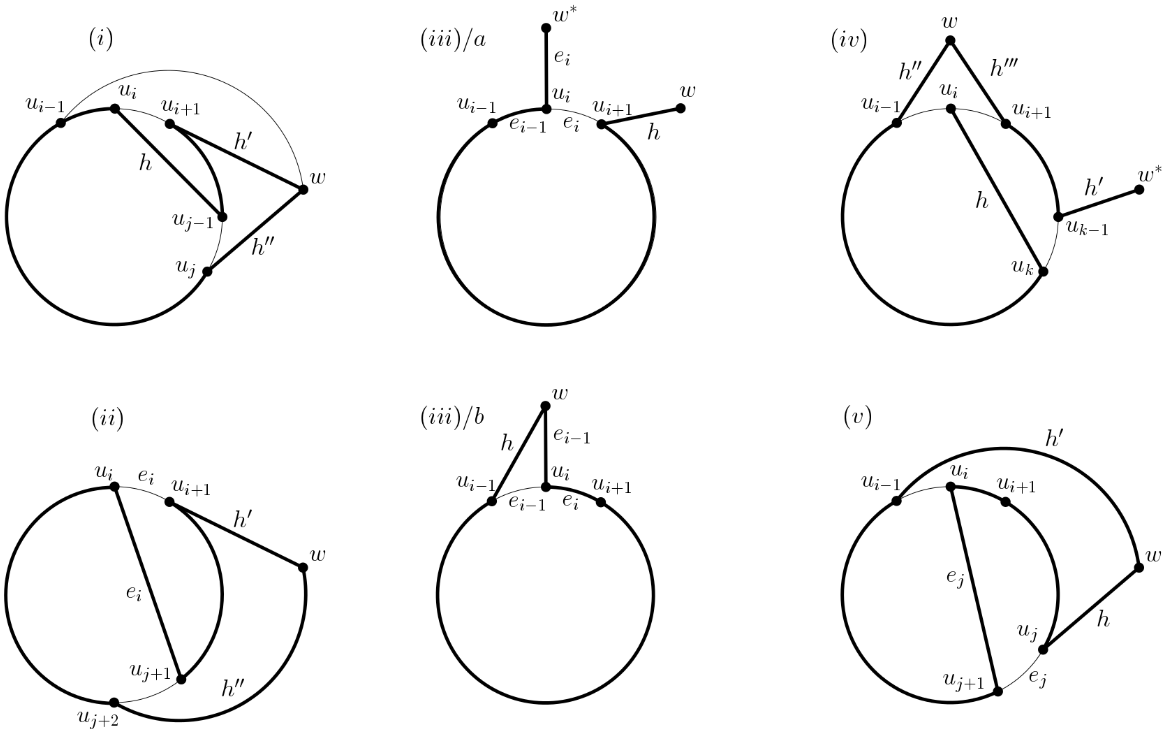

Fig. 5.4 Sketch of the proof of Claim 5.2.7.

Proof. We start with the proof of (i), see Figure 5.4 (i). Suppose by contradiction that $u_i, u_{j-1} \in h \in \mathcal{H} \setminus C$. Then by Claim 5.2.6 (i), we have $w \notin h$ (as otherwise $u_i, u_{i-1} \in M_w$, contradicting $u_{i-1} \in D_w$). Furthermore, as $u_{i+1}, u_j \in D_w$, there exist two distinct hyperedges $h', h'' \in \mathcal{H} \setminus C$ with $u_{i+1}, w \in h'$ and $u_j, w \in h''$. Using the fact that u_{j-1} and u_{i+1} are different vertices as there can not be neighboring vertices in D_w by Lemma 5.2.3 (ii), we have that

$$u_{i-1}, e_{i-1}, u_i, h, u_{j-1}, e_{j-2}, u_{j-2}, \ldots, u_{i+1}, h', w, h'', u_j, e_j, u_{j+1}, \ldots, e_{i-2}$$

is a Berge-cycle longer than C, a contradiction. Similarly we can extend the cycle C if $u_i, u_{j+1} \in h \in \mathcal{H} \setminus C$. This proves (i).

To show (ii) see Figure 5.4 (ii), it is enough to get a contradiction if e_i contains u_{j+1}, since the other case e_{i-1} contains u_{j+1} is symmetric. We have two non-defining distinct hyperedges, a hyperedge h'' incident to w and u_{j+2} and a hyperedge h' incident to w and u_{i+1} as $u_{i+1} \in D_w$. Then

$$u_i, e_i, u_{j+1}, e_j, u_j, e_{j-1}, \ldots, e_{i+1}, u_{i+1}, h', w, h'', u_{j+2}, e_{j+2}, \ldots, u_{i-2}, e_{i-2}, u_{i-1}, e_{i-1}$$

is a Berge-cycle longer than C, a contradiction.

To show statement (iii), suppose first $w^* \in e_i$. Then for a non-defining hyperedge h incident to w and u_{i+1}, we have that $w^*, e_i, u_i, e_{i-1}, u_{i-1}, \ldots, u_{i+1}, h, w$ is a path of length ℓ - a contradiction. If $w^* \in e_{i-1}$, then similarly, for a non-defining hyperedge h incident to w and u_{i-1}, we have that $w^*, e_{i-1}, u_i, e_i, u_{i+1}, \ldots, e_{i-2}, u_{i-1}, h, w$ is a path of length ℓ - a contradiction. If $w \in e_{i-1}$, then we have a contradiction since there exists a cycle longer than C, which is obtained from C by exchanging the edge e_{i-1} with h, w, e_{i-1}, where h is a non-defining hyperedge incident to w and u_{i-1}. Similarly we get a contradiction if $w \in e_i$.

To prove (iv) by a contradiction, suppose that we have a non-defining hyperedge h of C incident to u_i and u_t. Assume without loss of generality that $u_{t-1} \in D_{w^*}$ since the other case is symmetrical. Then there exists a non-defining hyperedge h' different from h, incident to u_{t-1} and w^*. Also there are two distinct non-defining hyperedges h'', h''' with $w, u_{i-1} \in h''$ and $w, u_{i+1} \in h'''$. At first note that hyperedge h is distinct from h'' and h''' by Claim 5.2.6 (i). From Lemma 5.2.3 (i) we have that hyperedges h'' and h''' distinct from h'. Finally we have a contradiction since the following is a Berge-path of length ℓ

$$w^*, h', u_{t-1}, e_{t-2}, \cdots u_{i+1}, h''', w, h'', u_{i-1}, e_{i-2}, \cdots, u_{t+1}, e_t, u_t, h, u_i.$$

To prove (v) suppose by a contradiction that e_j contains u_i. There are distinct non-defining hyperedges h, h' with $w, u_j \in h$ and $w, u_{i-1} \in h'$. Then

$$u_{j+1}, e_j, u_i, e_i, u_{i+1}, e_{i+1}, \ldots, e_{j-1}, u_j, h, w, h', u_{i-1}, e_{i-2}, u_{i-2}, \ldots, e_{j+2}, u_{j+2}, e_{j+1}$$

is a Berge-cycle of length longer than C. This contradiction proves (v). The proof for the case $u_i \in e_{j-1}$ is analogous. □

By Claim 5.2.6 and the set degree condition

$$\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1} \leq |M_w| + \binom{\min\left\{\left\lfloor\frac{\ell-1-|M_w|}{2}\right\rfloor, |D_w|\right\}}{r-1}$$

must hold for all $w \in V(\mathcal{H}) \setminus V(C)$. At first we observe that $|M_w| \leq 3$ as otherwise $\ell - 1 - |M_w| \leq k-5$. Therefore, we have $|D_w| \geq \lfloor\frac{k-3}{2}\rfloor$, if $k \geq 11$.

We say that a vertex $u_i \in V$ is *replaceable by* w, if $u_{i-1}, u_{i+1} \in D_w$, and we denote by R_w the set of vertices that are replaceable by w. A vertex is called *replaceable*, if it is replaceable by w for some $w \in V(\mathcal{H}) \setminus V$. For a replaceable vertex w', we define $D_{w'}$ and $M_{w'}$ as for vertices in $V(\mathcal{H}) \setminus V$.

For a vertex $w \in V(\mathcal{H}) \setminus V$ let us call a maximal set I of consecutive defining vertices of C in $V \setminus D_w$ a *missing interval for* w (or just missing intervals, if w is clear from the context), if its size is at least two. Let $I_1, I_2, \dots, I_s$ be the missing intervals of C for w and let us denote by $\overline{I_1}, \overline{I_2}, \dots, \overline{I_s}$ the same intervals without the terminal vertices (it is possible that $\overline{I_j} = \emptyset$). We have $\sum_{i=1}^{s}(|I_i| - 1) = \ell - 1 - 2|D_w|$. In particular, as $|D_w| \geq \left\lfloor\frac{k-3}{2}\right\rfloor$ by the set degree condition and Lemma 5.2.3 (i), we have $s \leq 3$, if k is even and $s \leq 2$, if k is odd. Let us consider a hyperedge $e_j \in C$ such that u_j or u_{j+1} is from a missing interval. The number of such hyperedges is $\sum_{i=1}^{s}(|I_i| + 1)$, which is at most 9, if k is even, and at most 6, if k is odd. Our next technical claim is about missing intervals.

Claim 5.2.8. *Suppose that $u_i, u_{i+1}, \dots, u_{i+t}$ form a missing interval for some $w \in V(\mathcal{H}) \setminus V$. Then*

(i) e_{i-1} and e_{i+t} do not contain vertices $w^* \in V(\mathcal{H}) \setminus V$; and

(ii) if $u_{i-1} \in D_{w'}$ (resp. $u_{i+t+1} \in D_{w'}$) for some $w' \neq w$, then e_{i-1} (resp. e_{i+t}) does not contain a vertex from R_w.

Proof. To prove (i) observe that there exists a Berge-path starting with the vertex w, a non-defining hyperedge h, the vertex u_{i-1}, going around C with defining vertices and hyperedges and finishing with a vertex u_i. Such h exists since u_{i-1} does not belong to the missing interval, so $u_{i-1} \in D_w$. Note that we did not use a hyperedge e_{i-1} which contains w^*. If $w = w^*$, then e_{i-1} closes a Berge-cycle longer than C, a contradiction, while if $w \neq w^*$, then finishing with e_{i-1}, w^* we obtain a Berge-path of length ℓ, a contradiction. This contradiction proves (i). Similar argument shows the statement for the hyperedge e_{i+t}.

We omit the proof of part (ii) since the same argument will provide the desired result after changing a replaceable vertex with w. □

Here we will show that $|D_{w^*}| \geq \left\lfloor \frac{k-3}{2} \right\rfloor$ holds even for vertices $w^* \in V(C) \setminus V$, therefore we have $|D_{w'}| \geq \left\lfloor \frac{k-3}{2} \right\rfloor$ for all $w' \in V(\mathcal{H}) \setminus V$.

By Claim 5.2.7 (iii) and Claim 5.2.8 (i), if $w^* \in V(C) \setminus V$ and $u_i \in D_w$, then $w^* \notin e_{i-1}, e_i$. Therefore the number of defining hyperedges that may contain w^* is at most 3. So Claim 5.2.6 and the set degree condition implies

$$\binom{\lfloor \frac{k-3}{2} \rfloor}{r-1} \leq 3 + |M_{w^*}| + \binom{\min\left\{ \left\lfloor \frac{\ell-1-|M_{w^*}|}{2} \right\rfloor, |D_{w^*}| \right\}}{r-1}.$$

Just as for $w \in V(\mathcal{H}) \setminus V(C)$, in two steps we obtain $|D_w| \geq \lfloor \frac{k-3}{2} \rfloor$ for k large enough.

Before continuing with a give possible embeddings of $\mathcal{H}$ into some $\mathcal{H}_{n,a,b_1,\ldots,b_s}$ let us state a last technical claim that will be used several times. Let us recall that a terminal vertex v is a vertex of a missing interval that is adjacent to a vertex from D_w.

Claim 5.2.9. *Suppose $D_w = D_{w'}$ for some $w' \in V(\mathcal{H}) \setminus V$ with $w' \neq w$.*

(i) There does not exist $h \in \mathcal{H} \setminus C$ such that h contains terminal vertices of two distinct missing intervals of w.

(ii) If $\{u_i, u_{i+1}, u_{i+2}\}$ and $\{u_j, u_{j+1}\}$ form missing intervals of w and there exists $h \in \mathcal{H} \setminus C$ with $u_{i+1}, u_j \in h$ or $u_{i+1}, u_{j+1} \in h$, then there does not exist $h' \in \mathcal{H} \setminus C$, with $u_i, u_{i+2} \in h'$.

Proof. We prove (i) by contradiction. Suppose $\{u_i, u_{i+1}, \ldots, u_{i+t}\}$ and $\{u_j, u_{j+1}, \ldots, u_{j+z}\}$ are two distinct missing intervals of w.

• Suppose first $u_i, u_{j+z} \in h \in \mathcal{H} \setminus C$. We have $u_{i+t+1}, u_{i-1}, u_{j+z+1} \in D_w$, therefore there are three different hyperedges h_w, h'_w and $h_{w'}$, such that h_w is incident to w and u_{i+t+1}, h'_w is incident to w and u_{i-1} and $h_{w'}$ is incident to u_{j+z+1} and w'. Note that all those hyperedges are different from h by Claim 5.2.6 (i). Then we have a contradiction since the following Berge-path is of length ℓ, as it contains all the $\ell-1$ defining vertices of C and w and w':

$$u_{i+t}, \ldots, u_i, h, u_{j+z}, e_{j+z-1}, \ldots, u_{i+t+1}, h_w, w, h'_w, u_{i-1}, e_{i-2}, \ldots, u_{j+z+1}, h_{w'}, w'.$$

• If $u_{i+t}, u_{j+z} \in h \in \mathcal{H} \setminus C$, then the Berge-path of length ℓ (using similar ideas as in the previous bullet) is

$$u_i, \ldots, u_{i+t}, h, u_{j+z}, e_{j+z-1}, \ldots, u_{i+t+1}, h_w, w, h'_w, u_{i-1}, e_{i-2}, \ldots, u_{j+z+1}, h_{w'}, w',$$

and we are done with the proof of (i).

In (ii) we can assume that $u_{i+1}, u_{j+1} \in h$ holds since the case $u_{i+1}, u_j \in h$ is identical. The proof of this part is similar, at first we observe from part (i) that we have $h \neq h'$. Then the following Berge-path of length ℓ gives us a contradiction:

$$u_i, h', u_{i+2}, e_{i+1}, u_{i+1}, h, u_{j+1}, e_j, u_j, e_{j-1}, \dots, u_{i+3}, h_w, w, h'_w, u_{i-1}, e_{i-2}, \dots, u_{j+2}, h_{w'}, w'.$$

□

5.2.2 Possible embeddings of $\mathcal{H}$ypergraph

Now we are in the situation to be able to give possible embeddings of $\mathcal{H}$ into some $\mathcal{H}_{n,a,b_1,\dots,b_s}$. In this subsection, we gather all the information that we know about these embeddings so far and in the next subsection, we analyze further the different cases to finish the proof.

Let us fix $w \in V(\mathcal{H}) \setminus V$ with D_w of maximum size and let $\mathcal{H}^*$ denote the subhypergraph of $\mathcal{H}$ that we obtain by removing those defining hyperedges e_i of C for which at least one of u_i or u_{i+1} is a vertex of a missing interval for w. By the above, $|\mathcal{H}| \leq |\mathcal{H}^*| + 9$.

If we are in a case when for all $w' \in V(\mathcal{H}) \setminus V$ we have $D_{w'} \subseteq D_w$, then let $A = D_w$, $B_i = I_i$ for $i = 1, 2, \dots, s$ and $L = V(\mathcal{H}) \setminus (D_w \cup_{i=1}^{s} I_i)$. Let us summarize the findings of the technical claims and enumerate the types of different hyperedges in $\mathcal{H} \setminus \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$ in this scenario.

Summary 5.2.10. *If $h \in \mathcal{H} \setminus \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$ is not a defining hyperedge of C (i.e., $h \in \mathcal{H} \setminus C$), then*

1. *either there exists $v \in (V(\mathcal{H}) \setminus V) \cup R_w$ such that $h \setminus \{v\} \subseteq D_w \cup \bigcup_{i=1}^{s} \overline{I_i}$ and $h \cap \bigcup_{i=1}^{s} \overline{I_i} \neq \emptyset$; We refer to these hyperedges as type 1 hyperedges in what follows.*

2. *$h \subseteq V \setminus R_w$ and h contains vertices from at least two distinct missing intervals. We refer to these hyperedges as type 2 hyperedges in what follows.*

If $e_i \in \mathcal{H} \setminus \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$ is a defining hyperedge of C, then

3. *either $e_i \in \mathcal{H} \setminus \mathcal{H}^*$; or*

4. *u_i or u_{i+1} belongs to R_w, $e_i \setminus \{u_i, u_{i+1}\} \subseteq D_w \cup \bigcup_{i=1}^{s} I_i$ and $e_i \cap \bigcup_{i=1}^{s} I_i \neq \emptyset$.*

Proof. Suppose first that h is not a defining hyperedge of C and h contains a vertex $v \in (V(\mathcal{H}) \setminus V) \cup R_w$. We claim that h cannot contain any $v' \in V(\mathcal{H}) \setminus V$ with $v' \neq v$. Indeed, if $v \notin V$, then it follows from Lemma 5.2.3 (i). If $v \in R_w$ and $v' = w$, then w can be inserted to obtain a longer cycle than C, while if $w \neq v'$, then using h, the defining vertices and hyperedges of C one can create a Berge-path of length ℓ from v' to w.

We also claim that h cannot contain a neighbor of a vertex in D_w on C. Indeed, if $v \notin V$, then it follows from Lemma 5.2.3 (ii) and (iii). If $v \in R_w$, then it follows from Claim 5.2.7 (i).

Therefore, h cannot contain other vertices of R_w, nor terminal vertices of missing intervals. This gives possibility 1.

Otherwise if $h \in \mathcal{H} \setminus \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$ is not a defining hyperedge of C, then we must have $h \subseteq V \setminus R_w$. As all hyperedges in $\binom{A \cup I_j}{r}$ belong to $\mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$, there must exist two distinct missing intervals meeting h. This gives possibility 2.

Let $e_i \in \mathcal{H} \setminus \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$ be a defining hyperedge of C. If at least one of u_i or u_{i+1} belongs to a missing interval, then $e_i \in \mathcal{H} \setminus \mathcal{H}^*$ by definition of $\mathcal{H}^*$. This gives possibility 3. Note that we have more information on some of these hyperedges by Claim 5.2.8.

Otherwise u_i or u_{i+1} belongs to R_w. By Claim 5.2.7 (ii), e_i does not contain any other vertex from R_w, and by Claim 5.2.7 (iii) e_i cannot contain any vertex from $V(\mathcal{H}) \setminus V$. This gives us possibility 4. Even more, if the unique element of $e_i \cap R_w$ is also replaceable by some $w' \neq w$, then e_i cannot contain w either. $\square$

If we are in a case when we have vertices $w, w' \in V(\mathcal{H}) \setminus V$ with $D_w \not\subseteq D_{w'}$ and $D_{w'} \not\subseteq D_w$, then as $\lfloor \frac{k-3}{2} \rfloor \leq |D_w|, |D_{w'}|$, we will have $\lfloor \frac{k-1}{2} \rfloor \leq |D_w \cup D_{w'}|$. Since the elements of $D_w \cup D_{w'}$ cannot be neighbors on C by Claim 5.2.6 (i) and $|C| \leq k-1$, we must have $|D_w \cup D_{w'}| = \lfloor \frac{k-1}{2} \rfloor$.

If $|C| = 2\lfloor \frac{k-3}{2} \rfloor + 2$, then we will embed $\mathcal{H}$ to $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}$, with $A = D_w \cup D_{w'}$ and all the other vertices are going to L.

If $|C| = 2\lfloor \frac{k-3}{2} \rfloor + 3$, then we will embed $\mathcal{H}^*$ to $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ with $A = D_w \cup D_{w'}$, the unique missing interval goes to B_1 and all the remaining vertices are going to L.

Summary 5.2.11. *If for $w, w' \in V(\mathcal{H}) \setminus V$ we have $D_w \not\subseteq D_{w'}$ and $D_{w'} \not\subseteq D_w$, then*

1. *there is no hyperedge $h \in \mathcal{H} \setminus C$ with $h \in \mathcal{H} \setminus \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}$ or $h \in \mathcal{H} \setminus \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ depending on whether $|V| = 2\lfloor \frac{k-3}{2} \rfloor + 2$ or $|V| = 2\lfloor \frac{k-3}{2} \rfloor + 3$; and*

2. *if $u_{i-1}, u_{i+1} \in D_w \cup D_{w'}$, then $e_{i-1} \setminus \{u_i\}, e_i \setminus \{u_i\} \subseteq D_w \cup D_{w'} \cup I$, where I is the unique possible interval u_j, u_{j+1} of size two disjoint with $D_w \cup D_{w'}$. Furthermore, if u_i is replaceable by either w or w', then $e_{i-1} \setminus \{u_i\}, e_i \setminus \{u_i\} \subseteq D_w \cup D_{w'}$.*

Proof. Note that every $u \in V \setminus (D_w \cup D_{w'})$ has a neighbor on C in $D_w \cup D_{w'}$. Therefore, if $v \in h \in \mathcal{H} \setminus C$ with $v \in V(\mathcal{H}) \setminus V$, then Claim 5.2.6 (i) yields $h \setminus \{v\} \subseteq D_w \cup D_{w'}$. So we only have to consider hyperedges $h \subset V$. If u_i is replaceable by either w or w' and $u_i \in h \in \mathcal{H} \setminus C$, then Claim 5.2.7 (i) and (iv) yield $h \setminus \{u_i\} \subseteq D_w \cup D_{w'}$. Finally, if u_j, u_{j+1} form the unique interval of $V \setminus (D_w \cup D_{w'})$, and u_i is neither replaceable by w nor by w', then one of u_{i-1}, u_{i+1} belong to D_w, the other to $D_{w'}$. Suppose that $u_i, u_j \in h \in \mathcal{H} \setminus C$, the other case $u_i, u_{j+1} \in h \in \mathcal{H} \setminus C$ is symmetric. Then $u_{j-1} \in D_{w^*}$ and $u_{i-1} \in D_{w^{**}}$ for some $w^*, w^{**} \in \{w, w'\}$. Therefore

$$w^*, h', u_{j-1}, e_{j-2}, \ldots, u_{i+1}, e_i, u_i, h, u_j, e_j, u_{j+1}, \ldots, e_{i-2}, u_{i-1}, h'', w^{**}$$

is either a cycle (if $w^* = w^{**}$) or a path (if $w^* \neq w^{**}$) of length k. Such distinct hyperedges h', h'' exist from the definition of $D_{w^*}, D_{w^{**}}$ as well as they are different from the hyperedge h since $h \subset V$. This settles part 1.

For part 2, let us consider defining hyperedges e_{i-1}, e_i of C with $u_{i-1}, u_{i+1} \in D_w \cup D_{w'}$. Observe first that all but at most one of the u_i's are replaceable either by w or by w'. If u_i is indeed replaceable by w or by w', then Claim 5.2.7 (iii) yields $e_{i-1} \setminus \{u_i\}, e_i \setminus \{u_i\} \subseteq D_w \cup D_{w'}$. For the at most one exception u_i, we have that one of u_{i-1}, u_{i+1} is in D_w, the other one is in $D_{w'}$ and by Claim 5.2.7 (v) we are done. □

5.2.3 Case-by-case analysis

We finish the proof with a case-by-case analysis according to the length of the longest Berge-cycle C and subcases will be defined according to the size of D_w. Let us remind the reader that the length of the cycle C, $\ell - 1$, might take the values $2\left\lfloor \frac{k-3}{2} \right\rfloor$, $2\left\lfloor \frac{k-3}{2} \right\rfloor + 1$, $2\left\lfloor \frac{k-3}{2} \right\rfloor + 2$ or $2\left\lfloor \frac{k-3}{2} \right\rfloor + 3$, and in the last case k is even. In each case we will use the summaries from the previous subsection.

Case I $\ell - 1 = 2\left\lfloor \frac{k-3}{2} \right\rfloor$.

As $|D_w| \geq \left\lfloor \frac{k-3}{2} \right\rfloor$, then by Claim 5.2.6 (i), D_w must consist of every second vertex of V, so there are no missing intervals. Summary 5.2.10 implies $\mathcal{H} \subseteq \mathcal{H}_{n, \left\lfloor \frac{k-3}{2} \right\rfloor}$ thus

$$|\mathcal{H}| \leq |\mathcal{H}_{n, \left\lfloor \frac{k-3}{2} \right\rfloor}| < |\mathcal{H}_{n, \left\lfloor \frac{k-3}{2} \right\rfloor, 3}|,$$

which contradicts the assumption on $|\mathcal{H}|$.

Case II $\ell - 1 = 2\left\lfloor \frac{k-3}{2} \right\rfloor + 1$.

$|D_w| \geq \left\lfloor \frac{k-3}{2} \right\rfloor$ and Claim 5.2.6 (i) imply that, after a possible relabelling we have $D_w = \{u_1, u_4, \ldots, u_{2\left\lfloor \frac{k-3}{2} \right\rfloor}\}$ and thus $\{u_2, u_3\}$ is the only missing interval for w, and all other vertices in $V \setminus D_w$ are in R_w. As all vertices in $V \setminus D_w$ are neighbors to some vertex in D_w, by Summary 5.2.10, all hyperedges in $\mathcal{H} \setminus C$ belong to $\mathcal{H}_{n, \left\lfloor \frac{k-3}{2} \right\rfloor, 2}$.

To consider the defining hyperedges of C, let us analyze those that contain an $u_i \in R_w$. Observe that by Claim 5.2.6 (i) a vertex in D_w cannot be a neighbor on C of a vertex in $D_{w'}$ for some $w' \in V(\mathcal{H}) \setminus V$, so $D_w = D_{w'}$ for any two $w, w' \in V(\mathcal{H}) \setminus V$. Hence we have that

e_{i-1} and e_i cannot contain any of u_2 and u_3 by Claim 5.2.6 (i) applied to the cycle C' we obtain by Claim 5.2.6 (ii). Therefore by Summary 5.2.10 we have that $\mathcal{H}^* \subseteq \mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,2}$, thus

$$|\mathcal{H}| \leq |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,2}| + 3 < |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}|,$$

which contradicts the assumption on $|\mathcal{H}|$.

CASE III $\ell - 1 = 2\left\lfloor\frac{k-3}{2}\right\rfloor + 2$.

The three subcases below cover this case.

CASE III/A There exists $w \in V(\mathcal{H}) \setminus V$ with $|D_w| = \left\lfloor\frac{k-3}{2}\right\rfloor + 1$.

Then there is no missing interval for w, and so $V \setminus D_w \subseteq R_w$, so by Summary 5.2.10 we have $\mathcal{H} = \mathcal{H}^* \subseteq \mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$.

CASE III/B There exists $w \in V(\mathcal{H}) \setminus V$, for which there are two missing intervals, $\{u_i, u_{i+1}\}$ and $\{u_j, u_{j+1}\}$.

Note that there is no type 1 hyperedge of $\mathcal{H} \setminus C$, as each vertex of the missing intervals is terminal. Observe that all the vertices in $V \setminus D_w$ have neighbors in D_w, therefore the fact that $|D_{w'}| \geq \left\lfloor\frac{k-3}{2}\right\rfloor$, together with $|D_w| = \left\lfloor\frac{k-3}{2}\right\rfloor$ and Claim 5.2.6 (i) imply $D_w = D_{w'}$ for all $w, w' \in V(\mathcal{H}) \setminus V$. This enables us to conclude that

- by Claim 5.2.9 (i), there is no hyperedge $h \in \mathcal{H} \setminus C$ of type 2; and
- by Claim 5.2.7 (v), if $u_l \in R_w$, then e_{l-1}, e_l do not contain vertices of missing intervals.

So by Summary 5.2.10 we have $\mathcal{H}^* \subseteq \mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,2,2}$ and thus

$$|\mathcal{H}| \leq |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,2,2}| + 6 < |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}|,$$

contradicting the assumption on $|\mathcal{H}|$.

CASE III/C For all $w \in V(\mathcal{H}) \setminus V$, there is only one missing interval containing three vertices $\{u_{i(w)}, u_{i(w)+1}, u_{i(w)+2}\}$.

If there exist two vertices $w, w' \in V(\mathcal{H}) \setminus V$ with $i(w) \neq i(w')$, then $D_w \cup D_{w'}$ must contain every second vertex of C. So by Summary 5.2.11, we have $\mathcal{H} \subseteq \mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$ as claimed by the theorem.

So we can assume that the D_ws are the same and without loss of generality suppose that for every $w \in V(\mathcal{H}) \setminus V$, the missing interval is $\{u_1, u_2, u_3\}$. Moreover, as every replaceable vertex u_i is replaceable by any $w \in V(\mathcal{H}) \setminus V$, replaceable vertices and defining hyperedges e_{i-1}, e_i behave as vertices in $V(\mathcal{H}) \setminus V$ and hyperedges in $\mathcal{H} \setminus C$. By Summary 5.2.10 and the above, we have to deal with type 1 hyperedges of $\mathcal{H} \setminus C$ and $\mathcal{H} \setminus \mathcal{H}^* = \{e_{2\lfloor\frac{k-3}{2}\rfloor+2}, e_1, e_2, e_3\}$.

• At first suppose that there exists a type 1 hyperedge of $\mathcal{H}\setminus C$, i.e., $h\in\mathcal{H}\setminus C$ with $v,u_2\in h$ for some $v\in(V(\mathcal{H})\setminus V)\cup R_w$. Without loss of generality we may assume $v\in V(\mathcal{H})\setminus V$. Then we claim that there is no hyperedge $h'\in\mathcal{H}$ with $u_1,u_3\in h'$. Suppose by a contradiction that such h' exists, then observe that $h'\neq h$, as otherwise we would have $v,u_1,u_3\in h'$ that is not possible by Summary 5.2.10. Also, either $h'\notin\{e_1,e_3\}$ or $h'\notin\{e_{2\lfloor\frac{k-3}{2}\rfloor+2},e_2\}$, so we may assume $h'\notin\{e_1,e_3\}$ without loss of generality. Since $u_{2\lfloor\frac{k-3}{2}\rfloor+2}\in D_v$, there is a hyperedge h'' different from the hyperedges h and h', incident to the vertices v and $u_{2\lfloor\frac{k-3}{2}\rfloor+2}$. We have a contradiction since the following is a longer Berge-cycle than C, containing all defining vertices of C and v:

$$v,h,u_2,e_1,u_1,h',u_3,e_3,u_4,\cdots,u_{2\lfloor\frac{k-3}{2}\rfloor+2},h''.$$

As no hyperedge contains both u_1 and u_3, we obtained $\mathcal{H}\subseteq\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor}$ in this case.

• Suppose next that there is no type 1 hyperedge of $\mathcal{H}\setminus C$, i.e., by Summary 5.2.10, we have $\mathcal{H}^*\subseteq\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}$. Observe that $e_{2\lfloor\frac{k-3}{2}\rfloor+2}$ and e_3 do not contain vertices from $(V(\mathcal{H})\setminus V)\cup R_w$ by Claim 5.2.8 (i) and (ii). If the same holds for e_1,e_2, then $\mathcal{H}\subseteq\mathcal{H}_{n,2\lfloor\frac{k-3}{2}\rfloor,3}$ contradicting the assumption $|\mathcal{H}|>|\mathcal{H}_{n,2\lfloor\frac{k-3}{2}\rfloor,3}|$. So we can assume that e_2 contains a vertex $v\in(V(\mathcal{H})\setminus V)\cup R_w$. Then we claim that there is no hyperedge $h\in\mathcal{H}\setminus C$ with $u_1,u_3\in h$. In here we get a contradiction as in the previous settings with a longer Berge-cycle, therefore we omit the proof. We obtained the following contradiction

$$|\mathcal{H}|\leq 2+|\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}|-\binom{\lfloor\frac{k-3}{2}\rfloor}{r-2}<|\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}|.$$

CASE IV $\ell-1=2\left\lfloor\frac{k-3}{2}\right\rfloor+3$.

Note that in this case k is even and the length of C is $k-1$. We again distinguish several subcases.

CASE IV/A $|D_w|=\left\lfloor\frac{k-1}{2}\right\rfloor$.

Then as D_w does not contain neighboring vertices on C, after relabelling, we can suppose that we have $D_w=\{u_1,u_4,u_6,\dots,u_{k-2}\}$. So there is one missing interval $\{u_2,u_3\}$, therefore there does not exist a type 1 or type 2 hyperedge $h\in\mathcal{H}\setminus C$. If $u_i\in R_w$, then by Claim 5.2.7 (iii) e_{i-1} and e_i do not contain vertices from $V(\mathcal{H})\setminus V$. We claim that e_{i-1} and e_i do not contain vertices from the missing interval $\{u_2,u_3\}$. Indeed, if there exists $w^*\neq w$ with $u_1\in N_{\mathcal{H}\setminus C}(w^*)$ and $u_2\in e_i$ or e_{i-1}, then the following is a Berge-path of length k:

$$u_i,e_i\ (\text{or } e_{i-1}),u_2,e_2,u_3,\dots,u_{i-1},h,w,h',u_{i+1},e_{i+1},\dots,u_{k-1},e_{k-1},u_1,h'',w^*.$$

Here h and h' exist and are distinct as u_i is in R_w and h'' exists by the choice of w^*.

Similarly, if there exists $w^{**} \neq w$ with $u_4 \in N_{\mathcal{H}\setminus C}(w^*)$, then e_{i-1}, e_i cannot contain u_3. As all D_{w^*} is of size at least $\lfloor \frac{k-3}{2} \rfloor$, the only cases when we are not yet done is when $u_1 \notin N_{\mathcal{H}\setminus C}(w^*)$ and $D_{w^*} = \{u_4, u_6, \dots, u_{2\lfloor \frac{k-3}{2} \rfloor + 2}\}$ or $u_4 \notin N_{\mathcal{H}\setminus C}(w^*)$ and $D_{w^*} = \{u_6, u_8, \dots, u_{2\lfloor \frac{k-3}{2} \rfloor + 2}, u_1\}$. By symmetry, we can assume the first. But then any replaceable u_i but $u_{2\lfloor \frac{k-3}{2} \rfloor + 3}$ can be replaced with some $w^* \neq w$, and the above arguments applied to the new cycle C' show that any $u_i \in h \in \mathcal{H} \setminus C'$ (in particular, it applies to e_i and e_{i-1}!) cannot contain u_3, and by Summary 5.2.10, we already know that e_{i-1}, e_i cannot contain $u_{2\lfloor \frac{k-3}{2} \rfloor + 3}$. Therefore setting $A = D_w \setminus \{u_1\}$, $B_1 = \{u_{2\lfloor \frac{k-3}{2} \rfloor + 3}, u_1, u_2, u_3\}$ we have that $\mathcal{H}$ is a subfamily of $\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}$ apart from $e_{2\lfloor \frac{k-3}{2} \rfloor + 2}, e_{2\lfloor \frac{k-3}{2} \rfloor + 3}, e_1, e_2, e_3$ and the hyperedges containing both w and u_1. On the other hand, there cannot exist $h \in \mathcal{H} \setminus C$ with $u_{2\lfloor \frac{k-3}{2} \rfloor + 3}, u_2 \in h$ nor with $u_{2\lfloor \frac{k-3}{2} \rfloor + 3}, u_3 \in h$ as in the former case

$$w, h', u_1, e_{2\lfloor \frac{k-3}{2} \rfloor + 3}, u_{2\lfloor \frac{k-3}{2} \rfloor + 3}, h, u_2, e_2, \dots, u_{2\lfloor \frac{k-3}{2} \rfloor + 2}, h'', w^*,$$

while in the latter case

$$w, h', u_1, e_1, u_2, e_2, u_3, h, u_{2\lfloor \frac{k-3}{2} \rfloor + 3}, e_{2\lfloor \frac{k-3}{2} \rfloor + 2}, u_{2\lfloor \frac{k-3}{2} \rfloor + 2}, \dots, e_4, u_4, h'', w^*$$

is a Berge-path of length k. So we have

$$|\mathcal{H}| \leq |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}| + 5 + \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} - 2\binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|,$$

contradicting the assumption on $|\mathcal{H}|$. So we obtained that e_{i-1}, e_i cannot contain u_2, u_3 and thus so far by Summary 5.2.10 we have $\mathcal{H}^* \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$.

Now let us concentrate on the hyperedges in $\mathcal{H} \setminus \mathcal{H}^*$. So $\{u_2, u_3\}$ is the unique missing interval (all other vertices of $V \setminus D_w$ are in R_w), and thus $\mathcal{H} \setminus \mathcal{H}^*$ contains three hyperedges: e_1, e_2 and e_3. Observe that by Claim 5.2.8 (i), e_1 and e_3 do not contain any $w' \in V(\mathcal{H}) \setminus V$. By Claim 5.2.7 (v), e_1 and e_3 do not contain any vertex in R_w.

• If e_2 does not contain any vertex in $R_w \cup (V(\mathcal{H}) \setminus V)$, then we are done, since $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$.

• If e_2 does contain a vertex from $R_w \cup (V(\mathcal{H}) \setminus V)$, then there does not exist any other hyperedge h that contains both u_2 and u_3. Indeed, if e_2 contained w, then w could be inserted in between u_2 and u_3 in the Berge-cycle C to form a longer cycle than C, a contradiction. If e_2 contains some $w' \neq w$ from $V(\mathcal{H}) \setminus V$, then we can reach a contradiction as before: we would find a Berge-path of length k starting with w', e_2, u_2, h, u_3, then going through C and ending with u_1, h', w as $u_1 \in D_w$.

Finally, if e_2 contains a replaceable u_i, then at least one of u_1, u_4 belongs to $D_{w'}$ for some $w' \in V(\mathcal{H}) \setminus V$ with $w' \neq w$, since $D_{w'} \subseteq D_w$ from Claim 5.2.6 (i) and $|D_w \setminus D_{w'}| \leq 1$. By symmetry, we may assume that $u_1 \in D_{w'}$. Then we have a contradiction since the following Berge-path has length k. The Berge-path is $u_i, e_2, u_2, h, u_3, u_4, \ldots$ that goes around the cycle C, replaces u_i by w and finishes with $u_1, h_{w'}, w'$, such $h_{w'}$ exists from the definition of $D_{w'}$. Therefore, if e_2 does contain a vertex from $R_w \cup (V(\mathcal{H}) \setminus V)$, then there does not exist any other hyperedge h that contains both u_2 and u_3. Hence, $\mathcal{H} \subseteq \mathcal{H}^+_{n,\lfloor \frac{k-1}{2} \rfloor}$ with $A = D_w$, $L = V(\mathcal{H}) \setminus D_w$ and e_2 being the unique hyperedge of $\mathcal{H}^+_{n,\lfloor \frac{k-1}{2} \rfloor}$ that contains less than $r-1$ vertices of A.

CASE IV/B For all $w' \in V(\mathcal{H}) \setminus V$, we have $|D'_w| = |D_w| = \lfloor \frac{k-3}{2} \rfloor$.

As the length of C is $k-1$, k is even and vertices of D_w are not neighbors on C, we have at most three missing intervals. If there are three missing intervals, then each of them contains two vertices. If there are two missing intervals, then they contain two and three vertices and if there is only one missing interval, then it contains 4 vertices. According to this structure, we are going to consider the following three subcases.

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CASE IV/B/1 There exists $w \in V(\mathcal{H}) \setminus V$ with $V \setminus D_w$ containing 3 intervals of length 2.

Observe that as all the missing intervals are of size 2, we do not have type 1 hyperedges $h \in \mathcal{H} \setminus C$. As all vertices in $V \setminus D_w$ have neighbors in D_w, we obtain that for any $w' \in V(\mathcal{H}) \setminus V$ we have $D_w = D_{w'}$. So Claim 5.2.9 (i) implies that there does not exist any type 2 hyperedges $h \in \mathcal{H} \setminus C$. Finally, Claim 5.2.7 (v) implies that defining hyperedges of C, apart from those in $\mathcal{H} \setminus \mathcal{H}^*$, are in $\mathcal{H}_{n,\lfloor \frac{k-3}{2} \rfloor,2,2,2}$. So we obtained a contradiction as

$$|\mathcal{H}| \leq 9 + |\mathcal{H}_{n,\lfloor \frac{k-3}{2} \rfloor,2,2,2}| < |\mathcal{H}_{n,\lfloor \frac{k-3}{2} \rfloor,4}|.$$

CASE IV/B/2 For all $w \in V(\mathcal{H}) \setminus V$, the number of missing intervals is at most 2 and there exist $w, w' \in V(\mathcal{H}) \setminus V$ with $D_w \neq D_{w'}$.

By relabeling, we can assume that $\{u_2, u_3\}$ forms the unique missing interval for both w and w', i.e., the unique interval of length more than 1 in $V \setminus (D_w \cup D_{w'})$. According to Summary 5.2.11, if every $u_i \notin D_w \cup D_{w'} \cup \{u_2, u_3\}$ is replaceable, then we have $\mathcal{H} \setminus \{e_1, e_2, e_3\} \subseteq \mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor,2}$, while if there is $u_i \in V \setminus (D_w \cup D_{w'})$ $(i \neq 2,3)$ that is not in $R_w \cup R_{w'}$, then we know $e_{i-1} \setminus \{u_i\}, e_i \setminus \{u_i\} \subseteq D_w \cup D_{w'} \cup \{u_2, u_3\}$.

• At first we suppose that there exists a $u \in D_w \cup D_{w'}$ such that $|\{w^* \in (V(\mathcal{H}) \setminus V) \cup R_w \cup R_{w'} : u \in N_{\mathcal{H}\setminus C}(w^*)\}| = 1$. In that case the unique w^* must be either w or w', say w. Consider the hypergraph $\mathcal{H} \setminus \mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}$ with $\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}$ having $A = D_w \cup D_{w'} \setminus \{u\} = D_{w'}$ and $B_1 = \{u, u_2, u_3\}$. Then, by Summary 5.2.11, the hyperedges left are incident with the vertex u, thus the number of hyperedges is at most $\binom{\lfloor\frac{k-3}{2}\rfloor}{r-2} + 5$. Here the first term is an upper bound for those hyperedges that are incident with both u and w, while the second term is 5 for $\{e_{i-1}, e_i, e_1, e_2, e_3\}$. So we have a contradiction as

$$|\mathcal{H}| \leq \binom{\lfloor\frac{k-3}{2}\rfloor}{r-2} + 5 + |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}| < |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}|.$$

• Suppose now that for all $u \in D_w \cup D_{w'}$,

$$|\{w^* \in (V(\mathcal{H}) \setminus V) \cup R_w \cup R_{w'} : u \in N_{\mathcal{H}\setminus C}(w^*)\}| \geq 2.$$

At first we show that $u_2, u_3 \notin e_{i-1}, e_i$ if $u_i \in V \setminus (D_w \cup D_{w'})$ ($i \neq 2, 3$). This holds by Summary 5.2.11, if u_i is replaceable by either w or w'. Therefore without loss of generality we may assume $u_{i+1} \in D_w \setminus D_{w'}$ and $u_{i-1} \in D_{w'} \setminus D_w$. Note that $D_w = (D_w \cup D_{w'}) \setminus \{u_{i-1}\}$ and $D_{w'} = (D_w \cup D_{w'}) \setminus \{u_{i+1}\}$. Because of symmetry, it is enough to show a contradiction only if $u_2 \in e_i$, the three remaining cases are similar to this one. The following is a Berge-path of length k

$$u_i, e_i, u_2, e_2, u_3, e_3, \ldots, u_{i-1}, h, w', h', u_1, e_{k-1}, u_{k-1}, e_{k-2} \ldots, e_{i+1}, u_{i+1}, h'', w,$$

a contradiction. The hyperedges h, h', h'' can be chosen distinct as $u_1, u_{i-1} \in D_{w'}$ and $u_{i+1} \in D_w$ and by Lemma 5.2.3 (i), $h^* \in \mathcal{H} \setminus C$ cannot contain distinct vertices from outside V.

By Claim 5.2.8 (i) and (ii), e_1 and e_3 are not incident with vertices in $V(\mathcal{H}) \setminus V$ or in $R_w \cup R_{w'}$. Even more, they are not incident with u_i either, since otherwise if $u_i \in e_1$, the following path is of length k, a contradiction:

$$u_i, e_1, u_2, e_2, u_3, e_3, \ldots, u_{i-1}, h, w', h', u_1, e_{k-1}, u_{k-1}, \ldots, e_{i+1}, u_{i+1}, h'', w.$$

An analogous argument shows $u_i \notin e_3$.

Finally, if e_2 contains any vertex from $V(\mathcal{H}) \setminus (D_w \cup D_{w'})$, then similarly to previous cases a hyperedge $e_2 \neq h \in \mathcal{H}$ containing both u_2, u_3 would lead to a Berge-path of length k. So if no such hyperedge h exists, then $\mathcal{H} \subseteq \mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$. Otherwise, we have $\mathcal{H} \subseteq \mathcal{H}^+_{n,\lfloor\frac{k-1}{2}\rfloor}$. Both possibilities are as claimed by the theorem.

Case IV/B/3 For all $w' \in V(\mathcal{H}) \setminus V \cup R_w$, the number of missing intervals is at most 2 and $D_w = D_{w'}$.

As $D_w = D_{w'}$ for all $w, w' \in V(\mathcal{H}) \setminus V$, it follows that we do not have to distinguish between vertices in $V(\mathcal{H}) \setminus V$ and vertices in R_w. Also, anything that we prove for hyperedges $h \in \mathcal{H} \setminus C$ is valid for all e_i, e_{i-1} if $u_i \in R_w$, by Claim 5.2.6 (ii).

Case IV/B/3/1 Let us consider first the case when for every $v \in V(\mathcal{H}) \setminus V \cup R_w$, the missing intervals for v are $\{u_2, u_3, u_4\}$ and $\{u_i, u_{i+1}\}$ for some $6 \leq i \leq k-2$, after possible relabeling. By Summary 5.2.10 and Claim 5.2.9 (i), we need to consider the 7 hyperedges in $\mathcal{H} \setminus \mathcal{H}^*$, the hyperedges in $\mathcal{H} \setminus C$ containing u_3, u_i or u_3, u_{i+1} and the hyperedges in $\mathcal{H} \setminus C$ containing u_3 and some $v \in V(\mathcal{H}) \setminus V \cup R_w$.

- If there are no hyperedges in $\mathcal{H} \setminus C$ containing u_3, u_i or u_3, u_{i+1} or u_3 and some $v \in V(\mathcal{H}) \setminus V \cup R_w$, then $\mathcal{H}^* \subseteq \mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2}$, with embedding $A = D_w$, $B_1 = \{u_2, u_3, u_4\}$, $B_2 = \{u_i, u_{i+1}\}$ and
$$|\mathcal{H}| \leq |\mathcal{H}^*| + 7 \leq |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2}| + 7 < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|,$$
contradicting the assumption on $|\mathcal{H}|$.

- If there are no hyperedges in $\mathcal{H} \setminus C$ containing u_3 and some $v \in V(\mathcal{H}) \setminus V \cup R_w$, but there exist a hyperedge $h \in \mathcal{H} \setminus C$ containing u_3, u_i or u_3, u_{i+1}, then by Claim 5.2.9 (ii), there is no hyperedge containing both u_2 and u_4. In particular, with embedding $A = D_w$, $B_1 = \{u_2, u_3, u_4\}$, $B_2 = \{u_i, u_{i+1}\}$ we have $|\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2} \setminus \mathcal{H}| \geq \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2}$. Also, by Summary 5.2.10, the hypergraph $\mathcal{H} \setminus \mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2}$ may contain the 7 hyperedges of $\mathcal{H} \setminus \mathcal{H}^*$ and at most $2\binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} + \binom{\lfloor \frac{k-3}{2} \rfloor}{r-3}$ hyperedges containing u_i or/and u_{i+1} and u_3. So we have
$$|\mathcal{H}| \leq |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2}| + 7 + 2\binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} + \binom{\lfloor \frac{k-3}{2} \rfloor}{r-3} - \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|,$$
which contradicts the assumption on $|\mathcal{H}|$.

- Suppose that there is a hyperedge $h \in \mathcal{H} \setminus C$ containing u_3 and some $v \in V(\mathcal{H}) \setminus V \cup R_w$. There is no $h' \in \mathcal{H} \setminus C$ incident with u_2 and u_4. Indeed, otherwise
$$v, u_3, e_2, u_2, h', u_4, e_4, \ldots, u_1, h_w, w$$
is a Berge-path of length k, a contradiction.

By the above, Summary 5.2.10 and Claim 5.2.9 (i), we have that $\mathcal{H}^* \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ with embedding $A = D_w \cup \{u_3\}$, $B_1 = \{u_i, u_{i+1}\}$. Even more, since $D_v = D_w \ni u_1, u_5$, by Lemma 5.2.3 (iii) there exist cycles C_2, C_4 with v replacing u_2 and u_4, respectively. Observe that the set D_{w^*} does not change when we apply these changes from C to C_2 and C to C_4. In C_2,

e_1, e_2 are not defining hyperedges, while in C_4, e_3, e_4 are not defining hyperedges. Therefore, applying Lemma 5.2.3 (ii), we obtain that e_1, e_2 do not contain u_4, u_i, u_{i+1} and e_3, e_4 do not contain u_2, u_i, u_{i+1}. Hence hyperedges e_1, e_2, e_3, e_4 are also from $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$ by Summary 5.2.10. By Claim 5.2.8 (i) and Claim 5.2.9 (i), we have that the hyperedges e_{i-1} and e_{i+1} are also from $\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$. Finally, if e_i does not contain any vertex from $(V(\mathcal{H})\setminus V)\cup R_w\cup\{u_3\}$, then we have $\mathcal{H}\subseteq\mathcal{H}_{n,\lfloor\frac{k-1}{2}\rfloor,2}$. Otherwise, as in Case IV/A, one can see that there does not exist $h\neq e_i$ with $u_i, u_{i+1}\in h$ and thus $\mathcal{H}=\mathcal{H}^{+}_{n,\lfloor\frac{k-1}{2}\rfloor}$ with $A = D_w\cup\{u_3\}$ and e_i being the unique hyperedge with less than $r-1$ elements in A.

CASE IV/B/3/2 For all $v\in V(\mathcal{H})\setminus V\cup R_w$, after possible relabelling the only missing interval consists of $\{u_2, u_3, u_4, u_5\}$.

By Summary 5.2.10, we need to handle hyperedges e_1, e_2, e_3, e_4, e_5 and those $h\in\mathcal{H}\setminus C$ that contain a $v\in V(\mathcal{H})\setminus V\cup R_w$ and u_3 and/or u_4.

• If there are no such hyperedges and $e_1, e_2, e_3, e_4, e_5\subseteq D_w\cup\{u_2, u_3, u_4, u_5\}$, then $\mathcal{H}\subseteq\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}$ contradicting the assumption on $|\mathcal{H}|$.

• Suppose next there is no $h\in\mathcal{H}\setminus C$ with a vertex from $V(\mathcal{H})\setminus V\cup R_w$ containing u_3 or u_4, but some e_i ($i = 1,2,3,4,5$) does contain a vertex from outside V. By Claim 5.2.8 (i), it is neither e_1 nor e_5. If e_i contains a vertex v from outside V, then there cannot exist $h\in\mathcal{H}\setminus C$ with $u_2, u_{i+1}\in h$, as then

$$v, e_i, u_i, e_{i-1}, \dots, u_2, h, u_{i+1}, e_{i+1}, u_{i+2}, e_{i+2}, \dots, u_{k-1}, e_{k-1}, u_1, h'$$

is a Berge-cycle of length k. For the existence of h' we used $D_v = D_w\ni u_1$. Therefore we have

$$|\mathcal{H}|\leq 3+|\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}|-\binom{\lfloor\frac{k-3}{2}\rfloor}{r-2}<|\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}|,$$

contradicting the assumption on $|\mathcal{H}|$.

• If there exists a hyperedge $h\in\mathcal{H}\setminus C$ incident with some vertex $v\in V(\mathcal{H})\setminus V\cup R_w$ and u_3, then there is no $h'\neq h$, $h'\in\mathcal{H}\setminus C$ incident with some vertex from $V(\mathcal{H})\setminus V\cup R_w$ and u_4, by Claim 5.2.6 (i). Even more, there is no $h''\in\mathcal{H}\setminus C$ with $u_2, u_4\in h''$. The argument is the same as if e_3 contained v from the previous bullet. Similarly one can get that there is no hyperedge $h''\in\mathcal{H}\setminus C$ with $u_2, u_5\in h''$.

Observe that there should exist at least two distinct $v_1, v_2\in V(\mathcal{H})\setminus V\cup R_w$ for which hyperedges h_{v_1}, h_{v_2} with $v_1, u_3\in h_{v_1}$ and $v_2, u_3\in h_{v_2}$ exist. Indeed, otherwise using that there is no non-defining edge incident to u_2, u_4, we have

$$|\mathcal{H}|\leq 5+1+|\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}|-\binom{\lfloor\frac{k-3}{2}\rfloor}{r-2}<|\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}|.$$

We will show that either $\mathcal{H} \subseteq \mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor,2}$ or $\mathcal{H} \subseteq \mathcal{H}^{+}_{n,\lfloor \frac{k-1}{2} \rfloor}$ with $A = D_w \cup \{u_3\}$ and $B_1 = \{u_4, u_5\}$. Let w^* denote an arbitrary vertex $w^* \in V(\mathcal{H}) \setminus V$ with $w^* \neq v_1, v_2$. We will use that $u_1, u_6 \in D_w = D_{w^*} = D_{v_1} = D_{v_2}$ thus there exists a hyperedge that is not a defining hyperedge of C and is different from h_{v_1} and h_{v_2}, containing either u_1 or u_6 together with v_1 or v_2 or w^*.

We need to prove that $u_4, u_5 \notin e_1, e_2$ and $u_2 \notin e_3, e_5$. In each of the cases we present a Berge-path of length k below, which is a contradiction.

If $u_4 \in e_1$, then the path is

$$v_1, h_{v_1}, u_3, e_2, u_2, e_1, u_4, e_4, u_5, \dots, u_{k-1}, e_{k-1}, u_1, h, w^*.$$

If $u_4 \in e_2$, then the path is

$$u_2, e_2, u_4, e_4, u_5, e_5, \dots, u_{k-1}, e_{k-1}, u_1, h, v_1, h_{v_1}, u_3, h_{v_2}, v_2.$$

If $u_5 \in e_1$ or e_2 , then the path is

$$u_2, e_1 \text{ or } e_2, u_5, e_4, u_4, e_3, u_3, h_{v_1}, v_1, h, u_6, e_6, \dots, u_{k-1}, e_{k-1}, u_1, h', w^*.$$

If $u_2 \in e_3$, then the path is

$$v_1, h_{v_1}, u_3, e_2, u_2, e_3, u_4, e_4, u_5, \dots, u_{k-1}, e_{k-1}, u_1, h, w^*.$$

If $u_2 \in e_5$, then the path is

$$u_2, e_5, u_5, e_4, u_4, e_3, u_3, h_{v_1}, v_1, h, u_6, e_6, \dots, u_{k-1}, e_{k-1}, u_1, h', w^*.$$

From here, one can conclude to $\mathcal{H} \subseteq \mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor,2}$ or $\mathcal{H} \subseteq \mathcal{H}^{+}_{n,\lfloor \frac{k-1}{2} \rfloor}$ as in Case IV/A, depending on whether $e_4 \subseteq D_w \cup \{u_3, u_4, u_5\}$ or not.

The above case-by-case analysis concludes the proof of Theorem 1.9.3 under the set degree condition, i.e., for any set X of vertices with $|X| \leq k/2$ the number of hyperedges incident with some vertex in X, $|E(X)|$, is at least $|X|\binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$. □

Let $n'_{k,r}$ denote the threshold such that the statement of Theorem 6 holds for hypergraphs with the set degree condition if $n \geq n'_{k,r}$. We are now ready to prove the general statements.

Proof of Theorem 1.9.3 and Theorem 1.9.4. Let $\mathcal{H}$ be a connected n-vertex r-uniform hypergraph without a Berge-path of length k, and suppose that if k is odd, then

$$|\mathcal{H}| > |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,3}| = \left(n - \frac{k+3}{2}\right)\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1} + \binom{\lfloor\frac{k+3}{2}\rfloor}{r},$$

while if k is even, then

$$|\mathcal{H}| > |\mathcal{H}_{n,\lfloor\frac{k-3}{2}\rfloor,4}| = \left(n - \left\lfloor\frac{k+5}{2}\right\rfloor\right)\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1} + \binom{\lfloor\frac{k+5}{2}\rfloor}{r}.$$

We obtain a subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ using a standard greedy process: as long as there exists a set S of vertices with $|S| \le k/2$ such that $|E(S)| < |S|\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1}$, we remove S from $\mathcal{H}$ and all hyperedges in $E(S)$. Let $\mathcal{H}'$ denote the subhypergraph at the end of this process.

Claim 5.2.12. *There exists a threshold $n''_{k,r}$, such that if $|V(\mathcal{H})| \ge n''_{k,r}$, then $\mathcal{H}'$ is connected and contains at least $n'_{k,r}$ vertices.*

Proof. To see that $\mathcal{H}'$ is connected, observe that every component of $\mathcal{H}'$ possesses the set degree condition. Therefore Claim 5.2.5 yields that every component contains a cycle of length at least $k-4$. Therefore, as $\mathcal{H}$ is connected, $\mathcal{H}$ contains a Berge-path with at least $2k-8$ vertices from two different components of $\mathcal{H}'$, a contradiction as $k \ge 9$.

Suppose to the contrary that $\mathcal{H}'$ has less than $n'_{k,r}$ vertices. Observe that, by definition of the process, $|\mathcal{E}(\mathcal{H}')| - |V(\mathcal{H}')|\binom{\lfloor\frac{k-3}{2}\rfloor}{r-1}$ strictly increases at every removal of some set X of at most k vertices. Therefore if $n > n'_{k,r} + k\binom{n'_{k,r}}{r} = n''_{k,r}$ and $|V(\mathcal{H}')| < n'_{k,r}$, then at the end we would have more hyperedges than those in the complete r-uniform hypergraph on $|v(\mathcal{H}')|$ vertices, a contradiction. □

By Claim 5.2.12 and the statement for hypergraphs with the set degree property, we know that $\mathcal{H}'$ has $n_1 \ge n'_{k,r}$ vertices, and $\mathcal{H}' \subseteq \mathcal{H}_{n_1,\lfloor\frac{k-1}{2}\rfloor}$ if k is odd, and $\mathcal{H}' \subseteq \mathcal{H}_{n_1,\lfloor\frac{k-1}{2}\rfloor,2}$ or $\mathcal{H}^+_{n_1,\lfloor\frac{k-1}{2}\rfloor}$ if k is even. Then for any hyperedge $h \in \mathcal{E}(\mathcal{H}) \setminus \mathcal{E}(\mathcal{H}')$ that contain at least one vertex from $V(\mathcal{H}) \setminus V(\mathcal{H}')$ with degree at least two, we can apply Lemma 5.2.3 (i) to obtain that all such h must meet the A of $\mathcal{H}'$ in $r-1$ vertices. This shows that if the minimum degree of $\mathcal{H}$ is at least 2, then $\mathcal{H} \subseteq \mathcal{H}_{n_2,\lfloor\frac{k-1}{2}\rfloor}$ if k is odd, and $\mathcal{H} \subseteq \mathcal{H}_{n_2,\lfloor\frac{k-1}{2}\rfloor,2}$ or $\mathcal{H} \subseteq \mathcal{H}^+_{n_2,\lfloor\frac{k-1}{2}\rfloor}$ if k is even, where $n_2 \le n$ is the number of vertices that are contained in a hyperedge of $\mathcal{H}$ that is either in $\mathcal{H}'$ or has a vertex in $V(\mathcal{H}) \setminus V(\mathcal{H}')$ with degree at least 2. This finishes the proof of Theorem 1.9.3.

Finally, consider the hyperedges that contain the remaining $n-n_2$ vertices. As all these vertices are of degree 1, they are partitioned by these edges. For such a hyperedge h let

D_h denote the subset of such vertices. Observe that for such a hyperedge h, we have that $h \setminus D_h \subseteq A$. Indeed if $v \in h \setminus (D_h \cup A)$, then there exists a cycle C of length $k-1$ in $\mathcal{H}'$ not containing v. Thus there is a path of length at least k starting at an arbitrary $d \in D_h$, continuing with h, v, and having $k-1$ more vertices as it goes around C with defining hyperedges and vertices. This contradicts Claim 5.2.5 and finishes the proof of Theorem 1.9.4. $\square$

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