A Generalization of the Selberg Trace Formula

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List of Symbols

K	a totally real extension of $\mathbb{Q}$ of degree two 3
$K^\times$	the multiplicative group of K 3
d_K	a positive square-free integer determined uniquely by $K \cong \mathbb{Q}(\sqrt{d_K})$ 3
$d(K)$	the discriminant of K 3
$\mathcal{O}_K$	the ring of integers in K 3
$\mathcal{O}_K^\times$	the group of units in $\mathcal{O}_K$ 3
$\mathbb{H}$	the complex upper half-plane 3
Γ_K	the Hilbert modular group for the field K 4
$\hat{K}$	the set of cusps for Γ_K 4
ε	the fundamental unit of $\mathcal{O}_K$ 4
X_k	coordinates at the cusp 6
Y_0	coordinates at the cusp 6
Y_1	coordinates at the cusp 6
Γ_∞	the stabilizer of the point ∞ 7
F_∞	the fundamental domain for Γ_∞ 7
F	the fundamental domain for Γ_K 8
$L^2(\Gamma_K \setminus \mathbb{H}^2)$	the Hilbert space of square integrable automorphic functions on $\mathbb{H}^2$ 38
μ	the product measure on $\mathbb{H}^2$ induced by the usual measure on $\mathbb{H}$ 38
L_K	the lattice in $\mathbb{R}^2$ that consists of the points (α, α'), where $\alpha \in \mathcal{O}_K$ 38
L_K^*	the dual lattice of L_K 38
ω	$= \sqrt{d(K)}$, used in the definition of the dual lattice 39
$a_l(y)$	the Fourier coefficient of an automorphic form belonging to the lattice element l 39
c_l	for a lattice element $l \in L_K^*$: the constant coefficient in $a_l(y)$ 39
c_l	for an algebraic integer $l \in \mathcal{O}_K$: the constant coefficient in $a_{(l\omega^{-1},(l\omega^{-1})')}(y)$ 40
η	the coefficient of $y_1^{s_1} y_2^{s_2}$ in $a_0(y)$, in Chapter 3 it belongs to the fixed automorphic form u 39
ϕ	the coefficient of $y_1^{1-s_1} y_2^{1-s_2}$ in $a_0(y)$, in Chapter 3 it belongs to the fixed automorphic form u 39
a_l	for an $l \in \mathcal{O}_K$, $a_l = \lvert c_l\rvert^2 + \lvert c_{\varepsilon l}\rvert^2 + \lvert c_{-l}\rvert^2 + \lvert c_{-\varepsilon l}\rvert^2$ 45
$E(z, s, m)$	Eisenstein series 50
$\lambda_m(z)$	Grössencharacter-type exponential sum 50

$\phi(s, m)$	the zeroth Fourier coefficient of the Eisenstein series $E(z, s, m)$ 51
$\phi_l(s, m)$	the Fourier coefficient of the Eisenstein series $E(z, s, m)$ belonging to the lattice element $l \in L_K^* \setminus 0$ 51
$\zeta_K(s, m)$	a Hecke L-function 51
$E_A(z, s, m)$	truncated Eisenstein series 52
ψ	a function in $C^\infty(\mathbb{R}^2)$ used in the definition of a point-pair invariant kernel 55
$k(z, w)$	a point-pair invariant kernel function 55
$K(z, w)$	an automorphic kernel function 56
$Q(w_1, w_2)$	a transform of the function ψ 56
$g(u_1, u_2)$	a transform of the function ψ 56
$h(r_1, r_2)$	a transform of the function ψ 56
$u_j(z)$	a member of a complete orthonormal system of automorphic forms for the discrete spectrum of Γ_K 57
$s_k^{(j)}$	$\lambda_k^{(j)} = s_k^{(j)}(1 - s_k^{(j)})$ where $\lambda_k^{(j)}$ is an eigenvalue of an u_j 57
$r_k^{(j)}$	defined by $s_k^{(j)} = \frac{1}{2} + ir_k^{(j)}$ 57
ϕ_j	the constant coefficient in the zeroth Fourier coefficient of an u_j (zero if u_j is a cusp form) 57
$c_l^{(j)}$	the Fourier coefficient of an u_j belonging to the lattice element l 57
F_A	the central part of the fundamental domain F 61
m_u	the integer given by the eigenvalues of an automorphic form u 75
$\{\gamma\}$	the conjugacy class of $\gamma \in \Gamma_K$ 76
$C(\gamma)$	the centralizer of $\gamma \in \Gamma_K$ 76
T_γ^A	the truncated trace for the conjugacy class of γ 76
T_γ	the trace for the conjugacy class of γ 76
m_γ	for a totally elliptic $\gamma \in \Gamma_K$: the order of the centralizer $C(\gamma)$ 77
z_γ	for a totally elliptic $\gamma \in \Gamma_K$: the fixed point of γ in $\mathbb{H}^2$ 77
$\theta(\gamma^{(k)})$	for an elliptic component $\gamma^{(k)}$: the angle determined by the totally elliptic component with fixed point i that $\gamma^{(k)}$ is conjugate to 78
$N(\gamma^{(k)})$	for a hyperbolic component $\gamma^{(k)}$: the norm of (the conjugacy class of) $\gamma^{(k)}$ 80
P_γ	for a totally hyperbolic $\gamma \in \Gamma_K$: a parallelogram used in the description of the fundamental domain for a conjugate of $C(\gamma)$ 81
$\gamma_{m,\alpha}$	a representative of a hyperbolic-parabolic conjugacy class in Γ_K 84
Λ	for a hyperbolic-parabolic conjugacy class represented by $\gamma_{m,\alpha}$: a generator of the ideal $(\varepsilon^m - \varepsilon^{-m}, \alpha)$ 85
E	for a hyperbolic-parabolic conjugacy class represented by $\gamma_{m,\alpha}$: $E = E_m = \varepsilon^m - \varepsilon^{-m}$ 86
k	for a hyperbolic-parabolic conjugacy class represented by $\gamma_{m,\alpha}$: the positive integer determined by the generator of $C(\gamma_{m,\alpha})$ 86

Introduction

The main goal of this work is to develop a generalization of the Selberg trace formula for the manifold $\Gamma_K \setminus \mathbb{H}^2$, where Γ_K is the Hilbert modular group for a totally real quadratic field K (here $\mathbb{H}$ denotes the complex upper half-plane). This generalization was made by András Biró in [1] for $\Gamma \setminus \mathbb{H}$, where Γ is a finite volume Fuchsian group. We follow his proof closely but widely lean on the book [5] as well where the Selberg trace formula is worked out in detail for finite volume irreducible discrete subgroups of $PSL(2,\mathbb{R})^n$ $(n \geq 2)$. For simplicity, here we restrict ourselves to the case $n = 2$ and we will also assume that K has class number 1.

In Chapter 1 we begin by introducing the Hilbert modular group for a totally real quadratic field K. We give a classification of its elements and also describe its fundamental domain. In the special case $K = \mathbb{Q}(\sqrt{5})$ an alternative fundamental domain was given in [7]. In the second half of Chapter 1 we prove a sharp lower bound for it that was conjectured in [4]. This proof has a numerical flavour and in fact it was partly done by computer. However, the applied algorithms are simple and the argument is formalized so that it can basically be checked without computer. The omitted computational steps are reduced to the comparison of the magnitude of some numbers.

In Chapter 2 we continue the preparation for the proof of the trace formula. It describes a relation between the geometry of the manifold $\Gamma_K \setminus \mathbb{H}^2$ and the spectrum of its invariant differential operators whose algebra is generated by the Laplacians. For the description of the spectrum we need to investigate the automorphic forms that are Γ_K-invariant eigenfunctions of the Laplace operators. To describe the continuous spectrum we introduce the Eisenstein series and shortly list its basic properties that were proved in [5]. Also, some parts of the proof of the generalized trace formula in [1] rely on estimates proved in [11] related to the spectrum. In the second part of the chapter we give the two dimensional analogue of these results.

Chapter 3 contains the proof of the trace formula. As usual, we define an automorphic kernel function $K(z,w)$ in terms of Γ_K so that the eigenfunctions of integral operator defined by it are the same as the eigenfunctions of the Laplacians. Then we evaluate the trace

$$\int_F K(z,z)u(z)\, d\mu(z),$$

where F is the fundamental domain of Γ_K and following [1] we also include the factor $u(z)$ that is an eigenfunction of the Laplacians (in fact we *replace* the eigenfunction 1 by u). The measure

$$d\mu(z) = (y_1 y_2)^{-2}\, dx_1\, dy_1\, dx_2\, dy_2$$

is the product measure on $\mathbb{H}^2$ induced by the usual measure on $\mathbb{H}$. Note that this integral does not necessarily converge, hence we need to "cut" it at some "height" in general. Then, dividing Γ_K into conjugacy classes we compute the geometric trace, and after that we apply the spectral theorem for a different computation of the integral above obtaining the trace formula that is stated in Theorem 3.3.1.

The methods and computations that follow are undoubtedly quite involved and require the usage of a large set of notations. Note that we use many notations that are very common in the literature while almost every specific notation can be found in the List of Symbols (hopefully making the whole work easier to read).

Chapter 1

The Hilbert modular group

In this introductory chapter we define and examine a basic object that shows up consistently throughout this whole work: the Hilbert modular group. Being a discrete subgroup of $PSL(2,\mathbb{R})^n$ for some $n \geq 2$ it is a multidimensional analogue of the modular group $PSL(2,\mathbb{Z})$. The action of $PSL(2,\mathbb{R})$ on the upper half-plane $\mathbb{H}$ also provides a coordinate-wise action of $PSL(2,\mathbb{R})^n$ on the product space $\mathbb{H}^n$.

For simplicity we restrict ourselves to the case $n = 2$ and define the Hilbert modular group for a totally real extension of $\mathbb{Q}$ with degree 2 and class number 1. This makes it possible for us to give a quite explicit description of the fundamental domain in section 2. In fact we describe two different fundamental domains: a general one is taken from [15] while for the field $K = \mathbb{Q}(\sqrt{5})$ a different domain is defined in [7]. The reason for this is that in the second part of the chapter we make a detour and prove a sharp bound for the latter domain that was conjectured in [4]. The proof that is given in section 4 is a more or less straightforward, though at some points tedious analysis of certain functions. However, the investigation of the extreme values leads us to the identification of the totally elliptic elements of the Hilbert modular group which is done before the proof in section 3. As a byproduct, we easily derive Theorem 1.3.4 that will be useful in Chapter 3.

1.1 Definition and basic properties

In what follows, let $\mathbb{Q} \leq K$ denote a totally real number field of degree 2. Then K is isomorphic to $\mathbb{Q}(\sqrt{d_K})$ where d_K is a positive square-free integer uniquely determined by the field K. The notation d_K will be fixed. The discriminant of K will be denoted by $d(K)$, i.e. $d(K) = 4d_K$ if $d_K \not\equiv 1 \mod 4$ and $d(K) = d_K$ if $d_K \equiv 1 \mod 4$. Let $K^{(1)}$, $K^{(2)}$ be the two different embeddings of K into $\mathbb{R}$. If $a \in K$, then let $a^{(k)}$ denote the embedding of a into $K^{(k)}$. Moreover, let $\mathcal{O}_K$ be the ring of integers in K and

$$PSL(2,\mathcal{O}_K) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R}) : \, a,b,c,d \in \mathcal{O}_K \right\} / \{\pm 1\}.$$

The group of units in $\mathcal{O}_K$ will be denoted by $\mathcal{O}_K^\times$, while $K^\times$ is the multiplicative group of K.

Let $\mathbb{H}$ denote the complex upper half-plane. The group $PSL(2,\mathbb{R})$ acts on $\mathbb{H}$ in the usual way, if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{R})$ and $z \in \mathbb{H}$, then

$$\gamma z = \frac{az+b}{cz+d}. \tag{1.1}$$

The group $PSL(2,\mathbb{R})^2$ acts then coordinate-wise on $\mathbb{H}^2$. For the elements $z \in \mathbb{H}^2$ we will often use the notation $z = (z_1, z_2)$ where $z_k = x_k + iy_k \in \mathbb{H}$ with $x_k, y_k \in \mathbb{R}$ $(k = 1,2)$. So if $\gamma = (\gamma_1, \gamma_2) \in PSL(2,\mathbb{R})^2$ and $z \in \mathbb{H}^2$, then $\gamma z = (\gamma_1 z_1, \gamma_2 z_2)$.

Definition 1.1.1. Let K be a number field as above and let us define the group $\Gamma_K \leq PSL(2,\mathbb{R})^2$ as follows:

$$\Gamma_K = \left\{ \left(\begin{bmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{bmatrix}, \begin{bmatrix} a^{(2)} & b^{(2)} \\ c^{(2)} & d^{(2)} \end{bmatrix} \right) : \begin{bmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{bmatrix} \in PSL(2, \mathcal{O}_{K^{(1)}}) \right\}.$$

This group is called the *Hilbert modular group* for the field K.

It is known that Γ_K is a discrete subgroup of $PSL(2,\mathbb{R})^2$ which acts discontinuously on $\mathbb{H}^2$. The elements of Γ_K can and often will be represented by the first coordinates of the pairs in the definition above, i.e. by a two by two matrix of determinant 1. If $\sigma \in PSL(2,\mathbb{R})$, then $[\sigma]$ denotes a matrix which represents σ. In addition, the conjugate of an element $a \in K$ will often be denoted by a', the norm of this element is denoted by $N(a) = aa'$, while $\operatorname{tr} a = a + a'$ denotes the trace.

We introduce the set $\hat{K} = K \cup \{\infty\}$ together with the extended operations of K which satisfy the following:

$$a + \infty = \infty \quad \text{for any number } a \in K,$$

$$a \cdot \infty = \infty, \quad \frac{a}{0} = \infty, \quad \frac{a}{\infty} = 0 \qquad \text{for any number } 0 \neq a \in K,$$

$$\infty \cdot \infty = \infty.$$

The expressions $\infty \pm \infty$, $0 \cdot \infty$ and ∞/∞ remain undefined. The elements of $\hat{K}$ are called *cusps* and Γ_K acts on them, this action is defined by the action of the first coordinates of the elements of Γ_K as in (1.1). The cusps $\lambda, \mu \in \hat{K}$ are equivalent if $\mu = \gamma\lambda$ for some $\gamma \in \Gamma_K$, this is denoted by $\lambda \sim \mu$. The number of the equivalence classes of the cusps is the class number of K (see Proposition 20 on page 188 in [15]). In the following we assume that this class number is one, i.e. $\mathcal{O}_K$ is a principal ideal domain. This means that the action of Γ_K on $\hat{K}$ is transitive. We mention that W. Narkiewicz proved in [13] that all real quadratic number fields with class number one are Euclidean except for at most two fields. Moreover, any exception would contradict the Generalized Riemann hypothesis by the theorem of P. J. Weinberger, who showed in [17] that the Generalized Riemann hypothesis implies that every real quadratic number field with class number one is Euclidean. Still we do not restrict ourselves to Euclidean rings because of the two possible exceptions. But since the Euclidean case $K = \mathbb{Q}(\sqrt{5})$ will be discussed later in detail, we give here a simple statement about the structure of Γ_K when $\mathcal{O}_K$ is Euclidean. The proof is just an easy generalization of an analogous claim in [7] and will be omitted.

Proposition 1.1.1. *Let K be a quadratic field extension such that $\mathcal{O}_K$ is Euclidean and let β_1, β_2 be an integral basis in $\mathcal{O}_K$. Then Γ_K is generated by the elements*

$$S_1 = \begin{bmatrix} 1 & \beta_1 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & \beta_2 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix},$$

where ε is the fundamental unit of $\mathcal{O}_K$.

The notation ε will also be fixed in the following and it denotes fundamental unit of $\mathcal{O}_K$, i.e. the generator of the unit group $\mathcal{O}_K^\times$ (modulo the roots of unity) uniquely determined by the property $\varepsilon > 1$. Note that in the special case when $K = \mathbb{Q}(\sqrt{5})$ the number of the generator matrices can be reduced as in this case $\varepsilon = \frac{1+\sqrt{5}}{2}$, and with the notation of the previous proposition if we set $S = S_1$ with $\beta_1 = 1$, then

$$S_\varepsilon := \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix} = S^{-1}USU^{-1}.$$

Moreover, the set $\{1, \varepsilon\}$ is an integral basis in $\mathcal{O}_K$, so $\Gamma_K = \langle S, T, U \rangle$.

In the following we categorise the elements of Γ_K. We recall that an element $\gamma \in PSL(2, \mathbb{R})$ is called elliptic, parabolic or hyperbolic if $|\operatorname{tr}\gamma| < 2$, $|\operatorname{tr}\gamma| = 2$ or $|\operatorname{tr}\gamma| > 2$, respectively. An element of Γ_K is called *totally elliptic* or *totally parabolic*, if both of its components are elliptic or parabolic, respectively. If there are elements of different types among the components, then this element is called *mixed*. Note that if one component of an element is parabolic, then so is the other since in this case the (rational) trace remains unchanged. Hence a mixed element consists of an elliptic and a hyperbolic component.

Before we turn to the case when every component is hyperbolic we examine the fixed points of the elements. A totally elliptic element has a single fixed point $x \in \mathbb{H}^2$. Since Γ_K acts discontinuously on $\mathbb{H}^2$, x has a neighborhood U such that the set $\{\gamma \in \Gamma_K : \gamma U \cap U \neq \emptyset\}$ is finite. This means that a totally elliptic element must be of finite order. A totally parabolic element fixes a single point in $(\mathbb{R} \cup \{\infty\})^2$. An element of the form $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ where $\alpha \in \mathcal{O}_K$ is parabolic and fixes the point (∞, ∞). The coordinates of a parabolic fixed point different from (∞, ∞) can be expressed from the elements of the matrices via addition, multiplication and the inverse operations, so these points are of the form $(\alpha, \alpha') \in K^2$. Recall that Γ_K acts transitively on $\hat{K}$. In fact, every number in K can be expressed as a fraction a/c, where $a, c \in \mathcal{O}_K$, $(a, c) = 1$, hence finding an element which takes (∞, ∞) to $(a/c, a'/c') \in K^2$ is equivalent to finding a solution of the equation $ad - bc = 1$ which is possible because a and c are coprime. It follows that the parabolic fixed points are (∞, ∞) and the points $(\alpha, \alpha') \in K^2$.

A mixed element fixes two points in $\mathbb{H} \times (\mathbb{R} \cup \{\infty\})$ or in $(\mathbb{R} \cup \{\infty\}) \times \mathbb{H}$. If every component of $\gamma \in \Gamma_K$ is hyperbolic, then γ fixes $2^2 = 4$ points in $(\mathbb{R} \cup \{\infty\})^2$. The element γ is called *hyperbolic-parabolic* if there is a point among its fixed points that is also fixed by a (totally) parabolic element. Otherwise it is called a *totally hyperbolic* element. Finding the fixed point of a component of an element is equivalent to solving the equation

$$\frac{az + b}{cz + d} = z \iff cz^2 + (d - a)z - b = 0.$$

The solutions are in K if and only if the discriminant of the quadratic polynomial above is a square of an element of K. In this case the same is true for the polynomial $c'z^2 + (d' - a')z - b'$, and its roots will be the conjugates of the roots of the previous polynomial. It follows that an element with two hyperbolic components is hyperbolic-parabolic if and only if any of its components has a fixed point in K. It also follows that a hyperbolic component of a mixed element fixes no points in K.

Example 1.1.2. We have already seen an example of a totally parabolic element. It is easy to construct totally elliptic elements from those in $PSL(2, \mathbb{Z})$. For a mixed element we set

$K = \mathbb{Q}(\sqrt{2})$, here the fundamental unit in $\mathcal{O}_K$ is $\varepsilon = 1 + \sqrt{2}$. Let us consider

$$\left(\begin{bmatrix} \varepsilon & \varepsilon \\ 2 & \varepsilon \end{bmatrix}, \begin{bmatrix} \varepsilon' & \varepsilon' \\ 2 & \varepsilon' \end{bmatrix} \right) \in \Gamma_K.$$

Since $|2\varepsilon| > 2$ and $|2\varepsilon'| < 2$ this is a mixed element with fixed points $(\pm\sqrt{\varepsilon/2}, i\sqrt{-\varepsilon'/2})$. Note that $\sqrt{\varepsilon/2}$ is not in K since it is the root of the polynomial

$$(2x^2 - 1)^2 - 2 = 4x^4 - 4x^2 - 1$$

which is irreducible over $\mathbb{Q}$ (this can easily be seen by determining the decomposition of it into irreducible components over $\mathbb{R}$).

Since ε is a unit in $\mathcal{O}_K$, the element

$$\left(\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}, \begin{bmatrix} \varepsilon' & 0 \\ 0 & \varepsilon'^{-1} \end{bmatrix} \right)$$

is also in Γ_K and its fixed points are (∞, ∞), $(0, 0)$, $(\infty, 0)$ and $(0, \infty)$, hence this is a hyperbolic-parabolic element.

1.2 The fundamental domain

In this section we describe the fundamental domain of Γ_K. It is given in [15] in the general situation when $\mathbb{Q} \leq K$ is a totally real number field of arbitrary (finite) degree and of arbitrary class number. In our situation its description becomes simpler since every cusp is equivalent to ∞. The fundamental domain was constructed in a different way in [7] for the field $K = \mathbb{Q}(\sqrt{5})$. This will also be presented and we will prove a bound for it later that was conjectured in [4].

First we introduce the coordinates at the cusp ∞. We will see that the action of the stabilizer of ∞ can be given in a simple way in terms of them. It is also possible to define the "distance" of a point of $\mathbb{H}^2$ from a cusp which is a useful notion for the determination of the fundamental domain. At this point we fix an integral basis $\{\alpha_1, \alpha_2\}$ in $\mathcal{O}_K$. In fact we choose $\alpha_1 = 1$ and we set $\alpha_2 = \sqrt{d_K}$ if $d_K \not\equiv 1 \bmod 4$ while $\alpha_2 = \frac{1+\sqrt{d_K}}{2}$ if $d_K \equiv 1 \bmod 4$. For a point $z \in \mathbb{H}^2$ we define the coordinates X_k $(k = 1, 2)$ by the system of linear equations

$$\begin{aligned} \alpha_1 X_1 + \alpha_2 X_2 &= x_1, \\ \alpha_1' X_1 + \alpha_2' X_2 &= x_2, \end{aligned} \tag{1.2}$$

furthermore, let

$$Y_0 = y_1 y_2 \qquad \text{and} \qquad Y_1 = \frac{1}{4 \log \varepsilon} \log \frac{y_1}{y_2}.$$

The equations in (1.2) can be written in the form $AX = x$ where

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_1' & \alpha_2' \end{bmatrix}, \tag{1.3}$$

and then $X = A^{-1}x$, that is

$$X_1 = \frac{\alpha_2' x_1 - \alpha_2 x_2}{\alpha_1 \alpha_2' - \alpha_1' \alpha_2} \qquad \text{and} \qquad X_2 = \frac{-\alpha_1' x_1 + \alpha_1 x_2}{\alpha_1 \alpha_2' - \alpha_1' \alpha_2}.$$

In particular, if $d_K \not\equiv 1 \bmod 4$, then

$$X_1 = \frac{x_1 + x_2}{2}, \quad X_2 = \frac{x_1 - x_2}{2\sqrt{d_K}}.$$

On the other hand, if $d_K \equiv 1 \bmod 4$, then

$$X_1 = \frac{(\sqrt{d_K} - 1)x_1 + (\sqrt{d_K} + 1)x_2}{2\sqrt{d_K}}, \quad X_2 = \frac{x_1 - x_2}{\sqrt{d_K}}.$$

We may write $X_k = X_k(z)$ for $k = 1, 2$ and $Y_j = Y_j(z)$ for $j = 0, 1$ to indicate the point z that the coordinates belong to.

Next we examine the action of the stabilizer of the cusp ∞ on these coordinates. This stabilizer is denoted by Γ_∞ and is given by

$$\Gamma_\infty = \left\{ \left(\begin{bmatrix} u & \alpha \\ 0 & u^{-1} \end{bmatrix}, \begin{bmatrix} u' & \alpha' \\ 0 & u'^{-1} \end{bmatrix} \right) : u \in \mathcal{O}_K^\times, \, \alpha \in \mathcal{O}_K \right\} / \{\pm 1\}.$$

Here $u = \pm\varepsilon^l$ where $l \in \mathbb{Z}$ so each element $\gamma \in \Gamma_\infty$ is represented by a matrix of the form

$$\gamma = \begin{bmatrix} \varepsilon^l & \alpha \\ 0 & \varepsilon^{-l} \end{bmatrix}. \tag{1.4}$$

The action of such an element does not change the coordinate Y_0, i.e. $Y_0(\gamma z) = Y_0(z)$, moreover, $Y_1(\gamma z) = Y_1(z) + l$. Let us write $\alpha = m\alpha_1 + n\alpha_2$ where $m, n \in \mathbb{Z}$. If $l = 0$, then

$$X_1(\gamma z) = \frac{\alpha_2'(x_1 + \alpha) - \alpha_2(x_2 + \alpha')}{\alpha_1\alpha_2' - \alpha_1'\alpha_2} = X_1(z) + \frac{\alpha\alpha_2' - \alpha'\alpha_2}{\alpha_1\alpha_2' - \alpha_1'\alpha_2} = X_1(z) + m, \tag{1.5}$$

and similarly

$$X_2(\gamma z) = \frac{-\alpha_1'(x_1 + \alpha) + \alpha_1(x_2 + \alpha')}{\alpha_1\alpha_2' - \alpha_1'\alpha_2} = X_2(z) + \frac{\alpha_1\alpha' - \alpha_1'\alpha}{\alpha_1\alpha_2' - \alpha_1'\alpha_2} = X_2(z) + n. \tag{1.6}$$

As the numbers l, n, m can be chosen independently, it follows that every Γ_∞-orbit has a point in the set

$$F_\infty = \left\{ z \in \mathbb{H}^2 : -\frac{1}{2} \le Y_1 < \frac{1}{2}; -\frac{1}{2} \le X_1, X_2 < \frac{1}{2} \right\}. \tag{1.7}$$

Furthermore, if two points of F_∞ are on the same orbit, i.e. $z = \gamma w$ for some $z, w \in F_\infty$ and $\gamma \in \Gamma_\infty$, then it follows from the transformation rule of Y_1 and from $-\frac{1}{2} \le Y_1(z), Y_1(w) < \frac{1}{2}$ that in the matrix representation (1.4) of γ the exponent l must be 0. Then by (1.5) and (1.6) we get that $n = m = 0$ (as $-\frac{1}{2} \le X_k(z), X_k(w) < \frac{1}{2}$ for $k = 1, 2$).

Definition 1.2.1. Let the group G act on the topological space $\mathcal{X}$. A *fundamental domain for* G is a set $\mathcal{F} \subset \mathcal{X}$ that contains exactly one point from each G-orbit.

In view of the previous definition we have already proved the following

Proposition 1.2.1. *The set F_∞ defined in (1.7) is a fundamental domain for Γ_∞ in $\mathbb{H}^2$. If $d_k \not\equiv 1$ mod 4, then*

$$F_\infty = \{z \in \mathbb{H}^2 : \varepsilon^{-2} \le y_1/y_2 < \varepsilon^2, \, -1 \le x_1 + x_2 < 1; \, -\sqrt{d_K} \le x_1 - x_2 < \sqrt{d_K}\},$$

and if $d_K \equiv 1$ *mod* 4, *then*

$$F_\infty = \left\{ z \in \mathbb{H}^2 : \varepsilon^{-2} \le \frac{y_1}{y_2} < \varepsilon^2, \quad \begin{array}{c} -\sqrt{d_K} \le (\sqrt{d_K}-1)x_1 + (\sqrt{d_K}+1)x_2 < \sqrt{d_K}, \\ -\sqrt{d_K}/2 \le x_1 - x_2 < \sqrt{d_K}/2 \end{array} \right\}.$$

Note that we will often be a little unprecise about the notion of fundamental domain. Namely, we may call a set a fundamental domain if it contains more than one point from some orbits once it differs only in a measure zero set (with respect to the product measure obtained from the usual measure on $\mathbb{H}$, discussed in more detail in the next chapter) from a set that satisfies the requirements of the previous definition. This will not affect our results but usually simplifies the constructions. Let us remark though that the set F_∞ is a fundamental domain of Γ_∞ in the strict sense (i.e. in the sense of Definition 1.2.1).

Now we clarify the expression "distance from a cusp". Obviously the hyperbolic metric is not useful for our purposes as it would give infinite distance. Instead, we say that a point z is close to the cusp ∞ if $Y_0(z)$ is big, or equivalently, if $1/Y_0(z)$ is small. For an arbitrary cusp $\lambda \in \hat{K}$ there is a an element $\gamma_\lambda \in \Gamma_K$ such that $\gamma_\lambda \infty = \lambda$. We define the distance of a point $z \in \mathbb{H}^2$ from the cusp λ by

$$\Delta(z, \lambda) = Y_0(\gamma_\lambda^{-1} z)^{-\frac{1}{2}}.$$

First of all we note that Δ is well-defined. Indeed, if for the elements $\gamma_\lambda, \tilde{\gamma}_\lambda \in \Gamma_K$ we have $\gamma_\lambda \infty = \lambda = \tilde{\gamma}_\lambda \infty$, then $\tilde{\gamma}_\lambda^{-1} \gamma_\lambda \in \Gamma_\infty$ and hence

$$Y_0(\gamma_\lambda^{-1} z) = Y_0((\tilde{\gamma}_\lambda^{-1}\gamma_\lambda)(\gamma_\lambda^{-1} z)) = Y_0(\tilde{\gamma}_\lambda^{-1} z).$$

We mention that the exact value of the exponent $-\frac{1}{2}$ in the definition of Δ is irrelevant from our point of view, but this way we get same notion that is defined in section III.2 of [15], where the following is proved:

Theorem 1.2.2. *A fundamental domain of* Γ_K *is given by the set*

$$F = \{z \in F_\infty : \Delta(z, \infty) \le \Delta(z, \lambda) \text{ for every cusp } \lambda \in \hat{K}\}.$$

From now on the notation F is fixed for this fundamental domain (at least if it refers to a subset of $\mathbb{H}^2$). The previous theorem and the definition of Δ gives immediately that

$$F = \{z \in F_\infty : Y_0(\gamma z) \le Y_0(z) \text{ for every } \gamma \in \Gamma_K\}.$$

By the coordinate-wise application of the one dimensional formula for the imaginary part of a point σw where $\sigma \in PSL(2, \mathbb{R})$ and $w \in \mathbb{H}$ we get for a $z \in \mathbb{H}^2$ and a $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_K$ the important relation

$$Y_0(\gamma z) = \frac{Y_0(z)}{|cz_1 + d|^2 \, |c' z_2 + d'|^2}$$

which will be applied many times without referring to it. One easily derives the following:

Lemma 1.2.3. $\{z \in F_\infty : 1 \le Y_0(z)\} \subset F$.

Proof. Let us fix a $z \in F_\infty$ for which $Y_0(z) \ge 1$ holds. We have to show that $Y_0(\gamma z) \le Y_0(z)$ for every $\gamma \in \Gamma_K$. Equality holds if $\gamma \in \Gamma_\infty$. Otherwise $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathcal{O}_K$ and

$c \neq 0$. Then

$$Y_0(\gamma z) = \frac{Y_0(z)}{|cz_1+d|^2\,|c'z_2+d'|^2} = \frac{Y_0(z)}{[(cx_1+d)^2+(cy_1)^2][(c'x_2+d')^2+(c'y_2)^2]}$$
$$\leq \frac{Y_0(z)}{(cy_1)^2(c'y_2)^2} = \frac{1}{N(c)^2 Y_0(z)}.$$

As $N(c)^2$ is a positive integer and $Y_0(z) \geq 1$ we get that $Y_0(\gamma z) \leq 1 \leq Y_0(z)$ and we are done. $\square$

We mention another basic result regarding the fundamental domain. It is shown on page 200 in [15] that there exists a constant C_K depending on K such that for every $z \in \mathbb{H}^2$ there is a cusp λ such that $\Delta(z,\lambda) < C_K$. In fact $C_K = 2\sqrt{d(K)}$ can be chosen. If $z \in F$ and λ is a cusp such that $\Delta(z,\lambda) < C_K$, then $\Delta(z,\infty) \leq \Delta(z,\lambda) < C_K$. That is, one gets

Lemma 1.2.4. *For every $z \in F$ we have $Y_0(z) > \frac{1}{4d(K)}$.*

We make use of the following result later. It is a generalization of Lemma 2.10 in [11].

Lemma 1.2.5. *Let $z \in \mathbb{H}^2$ and $Y > 0$. We have*

$$\#\{\gamma \in \Gamma_\infty \setminus \Gamma_K : Y_0(\gamma z) > Y\} < 1 + \frac{\left(1+4\varepsilon\sqrt{d(K)}\right)^2}{Y^2}.$$

Proof. We may assume $z \in F$. Every coset different from the trivial one is represented by a matrix $M_{c,d} = \begin{bmatrix} * & * \\ c & d \end{bmatrix} \in SL(2,\mathcal{O}_K)$ where $c \neq 0$. The matrices $M_{c,d}$ and $M_{c',d'}$ represent the same coset if and only if $(c',d') = (\pm\varepsilon^l c, \pm\varepsilon^l d)$ for some $l \in \mathbb{Z}$. So every nontrivial coset can be represented uniquely by an ideal $0 \neq (c) \lhd \mathcal{O}_K$ and a fraction d/c, where $d \in \mathcal{O}_K$.

Let $\gamma \in \Gamma_\infty \setminus \Gamma$ be a nontrivial element with $Y_0(\gamma z) > Y$, then we have $c \neq 0$. Since $z \in F$,

$$Y_0(z) \geq Y_0(\gamma z) = \frac{Y_0(z)}{\left|c^{(1)}z_1+d^{(1)}\right|^2\left|c^{(2)}z_2+d^{(2)}\right|^2}$$

holds and hence $\left|c^{(1)}z_1+d^{(1)}\right|\left|c^{(2)}z_2+d^{(2)}\right| \geq 1$ follows.

Since $Y_0(\gamma z) > Y$, this implies $Y_0(z) > Y$ and $|N(c)| \leq Y_0(z)^{-\frac{1}{2}}Y^{-\frac{1}{2}}$. Note that since $z \in F$, we have $y_k \geq \varepsilon^{-1}\sqrt{y_1y_2} > (2\varepsilon\sqrt{d(K)})^{-1}$ for $k = 1,2$ by the previous lemma. Hence

$$\frac{(c^{(1)}x_1+d^{(1)})^2(c^{(2)})^2}{4\varepsilon^2 d(K)} < (c^{(1)}x_1+d^{(1)})^2(c^{(2)})^2y_2^2 \leq \frac{Y_0(z)}{Y},$$

that is,

$$\left|x_1+\frac{d^{(1)}}{c^{(1)}}\right| < \frac{2\varepsilon\sqrt{d(K)}Y_0(z)^{\frac{1}{2}}}{|N(c)|\,Y^{\frac{1}{2}}}.$$

The same bound holds for $\left|x_2+d^{(2)}/c^{(2)}\right|$.

Now assume that $\alpha = d/c$ and $\alpha' = d'/c'$ are different numbers, where $c, c', d, d' \in \mathcal{O}_K$. Then

$$\begin{aligned}\left|\alpha^{(1)} - \alpha'^{(1)}\right| + \left|(\alpha^{(2)} - \alpha'^{(2)})\right| &\geq 2\sqrt{|\alpha^{(1)} - \alpha'^{(1)}|\,|\alpha^{(2)} - \alpha'^{(2)}|} \\ &= 2\sqrt{\left|\frac{d^{(1)}}{c^{(1)}} - \frac{d'^{(1)}}{c'^{(1)}}\right| \cdot \left|\frac{d^{(2)}}{c^{(2)}} - \frac{d'^{(2)}}{c'^{(2)}}\right|} \\ &= 2\sqrt{\frac{|N(c'd - cd')|}{|N(c)N(c')|}} \\ &\geq \frac{2}{\sqrt{|N(c)N(c')|}}.\end{aligned}$$

This means that once the absolute value of the norm of c is fixed, then the number of the possibilities for d/c is at most

$$\left(1 + \frac{4\varepsilon\sqrt{d(K)}Y_0(z)^{\frac{1}{2}}}{|N(c)|\,Y^{1/2}} \cdot |N(c)|\right)^2 < \frac{\left(1 + 4\varepsilon\sqrt{d(K)}\right)^2 Y_0(z)}{Y}$$

since $Y_0(z) > Y$. Summing over $|N(c)|$ gives the bound

$$\frac{\left(1 + 4\varepsilon\sqrt{d(K)}\right)^2 Y_0(z)^{\frac{1}{2}}}{Y^{\frac{3}{2}}} \leq \frac{\left(1 + 4\varepsilon\sqrt{d(K)}\right)^2}{Y^2}$$

since $Y_0(z)^{\frac{1}{2}} \leq Y_0(z)^{\frac{1}{2}}\,|N(c)| \leq Y^{-\frac{1}{2}}$. Adding 1 to take account of Γ_∞ we get the claim. □

Now we turn to the special case $K = \mathbb{Q}(\sqrt{5})$ and shortly describe the fundamental domain given in [7]. Recall that $\Gamma_{\mathbb{Q}(\sqrt{5})}$ is generated by the elements S, T and U, where

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}.$$

We define three particular sets in $\mathbb{H}^2$. The first one is

$$\mathcal{U} = \{z \in \mathbb{H}^2 : \varepsilon^{-2} \leq y_2/y_1 < \varepsilon^2\}, \tag{1.8}$$

this is clearly a fundamental domain for the subgroup generated by U. Note that unlike in the general case we use here the quotient y_2/y_1 instead of its reciprocal to follow the notations of [7]. The subgroup generated by T is just a group of order 2 with the fundamental domain

$$\mathcal{T} = \{z \in \mathbb{H}^2 : |z_1 z_2| \geq 1\}. \tag{1.9}$$

Next we construct a fundamental domain for the subgroup N consisting of the totally parabolic elements of the form $\begin{bmatrix} 1 & \nu \\ 0 & 1 \end{bmatrix}$ where $\nu \in \mathcal{O}_K$. The action of an element of this form on the point $z \in \mathbb{H}^2$ does not change the values y_1 and y_2. So for some fixed $s_1, s_2 > 0$ N acts on the set $H_{s_1,s_2} = \{z \in \mathbb{H}^2 : y_1 = s_1, y_2 = s_2\}$. This set is homeomorphic to $\mathbb{R}^2$ and each N-orbit is a lattice in it. From a fixed orbit we choose exactly one point z such that the function $|z_1 z_2|$

restricted to that orbit takes its minimal value at z. This is possible since every orbit is a discrete subset of H_{s_1,s_2}. Choosing one point this way from every orbit we obtain the set $\mathcal{S}_{s_1,s_2}$. Finally, let

$$\mathcal{S} = \bigcup_{s_1,s_2>0} \mathcal{S}_{s_1,s_2}, \tag{1.10}$$

this is obviously a fundamental domain for N.

After this preparation we can formulate the following result (for a proof see [7]):

Theorem 1.2.6. *The set $\mathcal{F} = \mathcal{U} \cap \mathcal{T} \cap \mathcal{S}$ is a fundamental domain for $\Gamma_{\mathbb{Q}(\sqrt{5})}$.*

Besides this the following lemma was proved in [7]:

Lemma 1.2.7. *If $z \in \mathcal{T} \cap \mathcal{S}$, then $y_1 y_2 > 0.54$.*

The sharp bound here is $\sqrt{5}/4$ and we will prove this in section 1.4. This is also a lower bound then in the fundamental domain $\mathcal{U} \cap \mathcal{T} \cap \mathcal{S}$ and the minimum is taken at elliptic fixed points, i.e. at points fixed by totally elliptic elements of $\Gamma_{\mathbb{Q}(\sqrt{5})}$. Before the proof we investigate the totally elliptic conjugacy classes of Γ_K in the next section.

1.3 Totally elliptic elements

The main purpose of this section is to show that there are only finitely many totally elliptic conjugacy classes in Γ_K and to list all of these classes in $\Gamma_{\mathbb{Q}(\sqrt{5})}$. A totally elliptic element of Γ_K can be represented by a matrix $A \in SL(2, \mathcal{O}_K)$ which has finite order. Then all of the eigenvalues of A are roots of unity, and these are also the roots of the characteristic polynomial of A. This polynomial has coefficients in $\mathcal{O}_K$, so the degree of its roots over $\mathbb{Q}$ is at most 4. The degree of an nth primitive root of unity over $\mathbb{Q}$ is $\varphi(n)$ where $\varphi(n)$ is the number of integers m satisfying $1 \le m \le n$ with $(n, m) = 1$. In this case $\varphi(n)$ is at most 4 and it is easy to see that the possible values of n are $1, 2, 3, 4, 5, 6, 8, 10$ or 12. The corresponding minimal polynomials over $\mathbb{Q}$ are the nth cyclotomic polynomials:

$$\begin{aligned}
\Phi_1(x) &= x - 1, \\
\Phi_2(x) &= x + 1, \\
\Phi_3(x) &= x^2 + x + 1, \\
\Phi_4(x) &= x^2 + 1, \\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 = \left(x^2 - \tfrac{\sqrt{5}-1}{2}x + 1\right)\left(x^2 + \tfrac{\sqrt{5}+1}{2}x + 1\right), \\
\Phi_6(x) &= x^2 - x + 1, \\
\Phi_8(x) &= x^4 + 1 = \left(x^2 - \sqrt{2}x + 1\right)\left(x^2 + \sqrt{2}x + 1\right), \\
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 = \left(x^2 - \tfrac{\sqrt{5}+1}{2}x + 1\right)\left(x^2 + \tfrac{\sqrt{5}-1}{2}x + 1\right), \\
\Phi_{12}(x) &= x^4 - x^2 + 1 = \left(x^2 - \sqrt{3}x + 1\right)\left(x^2 + \sqrt{3}x + 1\right).
\end{aligned} \tag{1.11}$$

Now either both of the eigenvalues of A are real or they are both non-real and conjugate to each other. In the first case there are 3 possibilities for the set of the eigenvalues: $\{1\}$, $\{-1\}$ or $\{1, -1\}$. However, for the first two sets the trace of the matrix would be ± 2 which is impossible because A represents a totally elliptic element, whereas in the third case the determinant would be -1 contradicting $A \in SL(2, \mathcal{O}_K)$.

So the eigenvalues of A are non-real complex numbers, let us denote them by λ and $\overline{\lambda}$. They are the roots of the characteristic polynomial $k(x)$ of A which has real coefficients, hence $k(x)$ is irreducible in $\mathbb{R}[x]$ and divides the minimal polynomial of λ over $\mathbb{Q}$ in $\mathbb{R}[x]$. As $\mathbb{R}[x]$ is a unique factorization domain and $k(x)$ is a monic polynomial, it must coincide with one of the quadratic polynomials listed in (1.11). Moreover, both of the eigenvalues have the same order and this is also the order of the matrix A. Now we have proved the following:

Lemma 1.3.1. *Let $A \in SL(2, \mathcal{O}_K)$ represent a totally elliptic element of Γ_K. Then A has order $3, 4, 5, 6, 8, 10$ or 12 and the characteristic polynomial of A coincides with one of the monic quadratic polynomials that divide $\Phi_{\operatorname{ord} A}$.*

Corollary 1.3.2. *Let $\alpha \in \Gamma_K$ be a totally elliptic element which is represented by the matrix $A \in SL(2, \mathcal{O}_K)$. Then $\operatorname{ord} \alpha = \operatorname{ord} A$ if $\operatorname{ord} A$ is odd and $\operatorname{ord} \alpha = \operatorname{ord} A/2$ if $\operatorname{ord} A$ is even. Hence a totally elliptic element of Γ_K has order $2, 3, 4, 5$ or 6.*

Proof. Assume that the order of a totally elliptic element α is n and let $A \in SL(2, \mathcal{O}_K)$ be a matrix which represents this element. Then $A^n = \pm I$ and hence $A^{2n} = I$. It follows that $\operatorname{ord} A \mid 2n$. On the other hand $\alpha^{\operatorname{ord} A}$ is represented by $A^{\operatorname{ord} A} = I$, so $\alpha^{\operatorname{ord} A} = 1$, which means that $n \mid \operatorname{ord} A \mid 2n$. So $n = \operatorname{ord} A$ or $n = \operatorname{ord} A/2$. If $\operatorname{ord} A$ is odd then the first equality must hold. If $\operatorname{ord} A = 2k$ is even, then $A^{2k} = (A^k)^2 = I$. Now the equation $B^2 = I$ has only two solutions in $SL(2, \mathbb{R})$ since B is the root of its characteristic polynomial and hence

$$B^2 - \operatorname{tr} B \cdot B + \det B \cdot I = 2I - \operatorname{tr} B \cdot B = 0,$$

so $B = cI$ for some $c \in \mathbb{R}$ such that $B^2 = c^2 I = I$ and then $c = \pm 1$. But then $A^k = -I$, and we get that $\alpha^k = 1$, i.e. $\operatorname{ord} \alpha = \operatorname{ord} A/2$. Finally, from the previous lemma we get the possible values of $\operatorname{ord} \alpha$. □

Corollary 1.3.3. *Let $\alpha \in \Gamma_K$ be a totally elliptic element. If $\operatorname{ord} \alpha = 4$, then $K = \mathbb{Q}(\sqrt{2})$. If $\operatorname{ord} \alpha = 5$, then $K = \mathbb{Q}(\sqrt{5})$. Finally, if $\operatorname{ord} \alpha = 6$, then $K = \mathbb{Q}(\sqrt{3})$.*

Proof. Let $A \in SL(2, \mathcal{O}_K)$ a matrix which represents the element α. If $\operatorname{ord} \alpha = 4$, then by the previous corollary $\operatorname{ord} A = 8$, and from Lemma 1.3.1 we get that the characteristic polynomial $k(x)$ of A divides $\Phi_8(x)$. But each of its quadratic divisors has a coefficient from $\mathbb{Q}(\sqrt{2})$, and so $\mathbb{Q}(\sqrt{2}) \subset K$, and since $[K : \mathbb{Q}] = 2$ we have in fact $K = \mathbb{Q}(\sqrt{2})$.

If $\operatorname{ord} \alpha = 5$, then $\operatorname{ord} A = 5$ or $\operatorname{ord} A = 10$, and if $\operatorname{ord} \alpha = 6$, then $\operatorname{ord} A = 12$. Now as before, we see that in the first case $K = \mathbb{Q}(\sqrt{5})$ and in the second case $K = \mathbb{Q}(\sqrt{3})$. □

Next we calculate the fixed point of an elliptic element. Such an element is represented by a matrix

$$A = \begin{bmatrix} a & c^{-1}(a(\operatorname{tr} A - a) - 1) \\ c & \operatorname{tr} A - a \end{bmatrix} = \begin{bmatrix} a & -c^{-1}k(a) \\ c & \operatorname{tr} A - a \end{bmatrix}, \tag{1.12}$$

where $k(x)$ is the characteristic polynomial of A. Note that c cannot be zero because ∞ is not fixed by an elliptic element. If $z \in \mathbb{H}^2$ is the fixed point of A then

$$\frac{az_1 - c^{-1}k(a)}{cz_1 + \operatorname{tr} A - a} = z_1 \iff cz_1^2 + (\operatorname{tr} A - 2a)z_1 + c^{-1}k(a) = 0,$$

that is,

$$z_1 = \frac{2a - \operatorname{tr} A \pm \sqrt{(\operatorname{tr} A - 2a)^2 - 4k(a)}}{2c} = \frac{2a - \operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4}}{2c}.$$

As the imaginary part of z_1 is positive, we have in fact

$$\begin{aligned} z_1 &= \frac{2a - \operatorname{tr} A + \operatorname{sgn}(c)\sqrt{(\operatorname{tr} A)^2 - 4}}{2c}, \\ x_1 &= \frac{2a - \operatorname{tr} A}{2c}, \qquad y_1 = \frac{\sqrt{4 - (\operatorname{tr} A)^2}}{2\,|c|}, \end{aligned} \tag{1.13}$$

where we use the principal square root function, i.e. if $z = re^{i\varphi}$ with $r \geq 0$ and $-\pi < \varphi \leq \pi$, then $\sqrt{z} = \sqrt{r}e^{i\varphi/2}$. Similarly,

$$x_2 = \frac{2a' - (\operatorname{tr} A)'}{2c'}, \qquad y_2 = \frac{\sqrt{4 - (\operatorname{tr} A)'^2}}{2\,|c'|}. \tag{1.14}$$

Every totally elliptic conjugacy class has an element which has a fixed point in F. If A represents such an element, then by Lemma 1.2.4 we get that

$$y_1 y_2 = \frac{\sqrt{(4 - (\operatorname{tr} A)^2)(4 - (\operatorname{tr} A)'^2)}}{4\,|N(c)|} > \frac{1}{4d(K)}$$

and hence $1 \leq |N(c)| < 4d(K)$. Moreover, since $\varepsilon^{-2} \leq y_1/y_2 < \varepsilon^2$, the quotient $|c'/c|$ is also bounded from above and from below. Consequently, c and c' are bounded so we have only finitely many choices for c. Finally, the coordinates x_1 and x_2 are bounded too (since $z \in F$) and then so are a and a'. This means that we have finitely many possible values for a and we obtain

Theorem 1.3.4. *The number of the totally elliptic conjugacy classes in Γ_K is finite.*

Notice that in fact we described above an algorithm for finding all the totally elliptic conjugacy classes. Now we apply basically this method to list all such classes in the case $K = \mathbb{Q}(\sqrt{5})$, but instead of F we work with the fundamental domain $\mathcal{F}$ defined in Theorem 1.2.6. The field K is fixed for the rest of this chapter and to follow the computations below one may make use of the following table which contains the exact values of some powers of the fundamental unit and some relations between them:

$$\boxed{\begin{gathered} \varepsilon = \frac{1+\sqrt{5}}{2}, \quad \varepsilon^{-1} = \frac{\sqrt{5}-1}{2}, \quad \varepsilon^2 = \frac{3+\sqrt{5}}{2}, \quad \varepsilon^{-2} = \frac{3-\sqrt{5}}{2}, \\ \varepsilon^3 = 2+\sqrt{5}, \quad \varepsilon^{-3} = \sqrt{5}-2, \quad \varepsilon^4 = \frac{7+3\sqrt{5}}{2}, \quad \varepsilon^{-4} = \frac{7-3\sqrt{5}}{2}, \\ \varepsilon + \varepsilon^{-1} = \sqrt{5}, \qquad \varepsilon - \varepsilon^{-1} = 1. \end{gathered}}$$

There is an element in every conjugacy class which has a fixed point in the fundamental domain $\mathcal{F}$. Let then the matrix A represent a totally elliptic element with fixed point $z \in \mathcal{F}$. We use the same notations as in (1.12). By Lemma 1.2.7 we have

$$y_1 y_2 = \frac{\sqrt{(4 - (\operatorname{tr} A)^2)(4 - (\operatorname{tr} A)'^2)}}{4\,|N(c)|} > 0.54,$$

that is, $1 \leq |N(c)| < 1/0.54 < 2$ which means that the norm of c is ± 1, i.e. c is a unit. Then $c = \pm\varepsilon^k$ must hold for some $k \in \mathbb{Z}$, and since $z \in \mathcal{U}$ (where the set $\mathcal{U}$ is defined in (1.8)) we obtain

$$\varepsilon^{-2} \le \frac{y_2}{y_1} = \varepsilon^{2k}\sqrt{\frac{4-(\operatorname{tr} A)'^2}{4-(\operatorname{tr} A)^2}} < \varepsilon^2.$$

From Corollary 1.3.2, 1.3.3 and Lemma 1.3.1 we conclude that the possible values of $\operatorname{tr} A$ are 0, ± 1 and $\pm\varepsilon^{\pm 1}$. If $\operatorname{tr} A = 0$ or $\operatorname{tr} A = \pm 1$ then $k = 0$ or $k = -1$ follows. If $\operatorname{tr} A = \pm\varepsilon$ then

$$\sqrt{\frac{4-(\operatorname{tr} A)'^2}{4-(\operatorname{tr} A)^2}} = \sqrt{\frac{4-\varepsilon^{-2}}{4-\varepsilon^2}} = \sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}} = \sqrt{\frac{30+10\sqrt{5}}{20}} = \sqrt{\frac{3+\sqrt{5}}{2}} = \varepsilon,$$

and hence $k = 0$ or -1. Similarly, if $\operatorname{tr} A = \pm\varepsilon^{-1}$ then k is 0 or 1.

If one works with the set F it is easy to give exact bounds for x_1 and x_2. Instead of this we use the definition of the set $\mathcal{S}$ to determine the possible values of a. Since A^{-1} fixes z too, we have

$$y_1y_2 = Y_0(z) = Y_0(A^{-1}z) = \frac{Y_0(z)}{|-cz_1+a|^2\,|-c'z_2+a'|^2} = \frac{y_1y_2}{|z_1-a/c|^2\,|z_2-a'/c'|^2}$$

as c is a unit. Then $|(z_1-a/c)(z_2-a'/c')| = 1$ follows but since $z \in \mathcal{S}\cap\mathcal{T}$ and $a/c \in \mathcal{O}_K$ we get

$$1 = |(z_1-a/c)(z_2-a'/c')| \ge |z_1z_2| \ge 1,$$

and hence $|z_1z_2|^2 = 1$. In more detail

$$\begin{aligned}|z_1z_2|^2 &= \left[\frac{(2a-\operatorname{tr} A)^2}{4c^2}+\frac{4-(\operatorname{tr} A)^2}{4c^2}\right]\left[\frac{(2a'-(\operatorname{tr} A)')^2}{4c'^2}+\frac{4-(\operatorname{tr} A)'^2}{4c'^2}\right]\\ &= (a^2-(\operatorname{tr} A)a+1)(a'-(\operatorname{tr} A)'a'+1) = 1,\end{aligned}$$

hence $a^2-(\operatorname{tr} A)a+1$ is a unit. This means that for some $n\in\mathbb{Z}$

$$a^2-(\operatorname{tr} A)a+1 = \pm\varepsilon^n \iff a^2-(\operatorname{tr} A)a+1\mp\varepsilon^n = 0$$

and

$$a'^2-(\operatorname{tr} A)'a'+1\mp(-1)^n\varepsilon^{-n} = 0.$$

As a and a' are real roots of some quadratic polynomials the discriminant of these polynomials must be non-negative. Thus

$$(\operatorname{tr} A)^2-4\pm 4\varepsilon^n \ge 0 \qquad\text{and}\qquad (\operatorname{tr} A)'^2-4\pm(-1)^n4\varepsilon^{-n}\ge 0.$$

As $(\operatorname{tr} A)^2-4<0$ and $(\operatorname{tr} A)'^2-4<0$ we get that $a^2-(\operatorname{tr} A)a+1$ is positive and n is even. In other words, $n = 2m$ for some $m\in\mathbb{Z}$ and

$$a^2-(\operatorname{tr} A)a+1-\varepsilon^{2m} = 0.$$

So the possible values of a are

$$a = \frac{\operatorname{tr} A \pm\sqrt{(\operatorname{tr} A)^2-4(1-\varepsilon^{2m})}}{2}.$$

We also have

$$\varepsilon^{2m} \geq \frac{4-(\operatorname{tr} A)^2}{4} \qquad \text{and} \qquad \varepsilon^{-2m} \geq \frac{4-(\operatorname{tr} A)'^2}{4}.$$

If $\operatorname{tr} A$ is 0 or ± 1, then $m = 0$. In case of $\operatorname{tr} A = \pm\varepsilon$ we have

$$\varepsilon^{2m} \geq \frac{4-\varepsilon^2}{4} = \frac{5-\sqrt{5}}{8} > -2+\sqrt{5} = \varepsilon^{-3},$$

$$\varepsilon^{-2m} \geq \frac{4-\varepsilon^{-2}}{4} = \frac{5+\sqrt{5}}{8} > \frac{-4+4\sqrt{5}}{8} = \varepsilon^{-1},$$

so $2m > -3$ and $1 > 2m$, and hence $m = 0$ or $m = -1$. In the latter case we write down the details of the computation of a:

$$(\operatorname{tr} A)^2 - 4(1-\varepsilon^{-2}) = \varepsilon^2 - 4 + 4\varepsilon^{-2} = (\varepsilon - 2\varepsilon^{-1})^2,$$

and hence for $\operatorname{tr} A = \varepsilon$ we get $a = \frac{\varepsilon \pm (\varepsilon - 2\varepsilon^{-1})}{2}$, i.e. $a = 1$ or $a = \varepsilon^{-1}$ while for $\operatorname{tr} A = -\varepsilon$ we obtain $a = -1$ or $a = -\varepsilon^{-1}$.

Finally, if $\operatorname{tr} A = \pm\varepsilon^{-1}$, then

$$\varepsilon^{2m} \geq \frac{4-\varepsilon^{-2}}{4} > \varepsilon^{-1}, \qquad \varepsilon^{-2m} \geq \frac{4-\varepsilon^{2}}{4} > \varepsilon^{-3},$$

hence $2m > -1$ and $3 > 2m$, and $m = 0$ or $m = 1$ follows.

From all this we conclude the following:

Theorem 1.3.5. *Let $\alpha \in \Gamma_{\mathbb{Q}(\sqrt{5})}$ be a totally elliptic element with a fixed point $z \in \mathcal{F}$, and let*

$$A = \begin{bmatrix} a & c^{-1}(a(\operatorname{tr} A - a) - 1) \\ c & \operatorname{tr} A - a \end{bmatrix}$$

be the unique matrix in $SL(2, \mathcal{O}_K)$ representing α for which $a \geq 0$ and if $a = 0$ then $c > 0$. Then the possible values of $\operatorname{tr} A$ are 0, ± 1 and $\pm\varepsilon^{\pm 1}$.

If $\operatorname{tr} A = 0$, then $a = 0$ and $c = 1$ or $c = \varepsilon^{-1}$, so A is one of the following matrices:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\varepsilon \\ \varepsilon^{-1} & 0 \end{bmatrix}.$$

If $\operatorname{tr} A = 1$, then the possibilities for the pair (a, c) are $(0, 1)$, $(0, \varepsilon^{-1})$, $(1, 1)$, $(1, -1)$, $(1, \varepsilon^{-1})$ and $(1, -\varepsilon^{-1})$, and A is one of the following matrices:

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -\varepsilon \\ \varepsilon^{-1} & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -\varepsilon \\ \varepsilon^{-1} & 0 \end{bmatrix}, \begin{bmatrix} 1 & \varepsilon \\ -\varepsilon^{-1} & 0 \end{bmatrix}.$$

If $\operatorname{tr} A = -1$, then $a = 0$ and $c = 1$ or $c = \varepsilon^{-1}$, so A is one of the following matrices:

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -\varepsilon \\ \varepsilon^{-1} & -1 \end{bmatrix}.$$

If $\operatorname{tr} A = \varepsilon$, then $a = 0$, 1, ε or ε^{-1}, while $c = \pm 1$ or $\pm\varepsilon^{-1}$. Hence the possibilities for the pair (a, c) are $(0, 1)$, $(0, \varepsilon^{-1})$, $(1, \pm 1)$, $(1, \pm\varepsilon^{-1})$, $(\varepsilon, \pm 1)$, $(\varepsilon, \pm\varepsilon^{-1})$, $(\varepsilon^{-1}, \pm 1)$, $(\varepsilon^{-1}, \pm\varepsilon^{-1})$.

If $\operatorname{tr} A = -\varepsilon$, then the possibilities for the pair (a, c) are $(0, 1)$ and $(0, \varepsilon^{-1})$.

If $\operatorname{tr} A = \varepsilon^{-1}$*, then* $a = 0$*,* ε *or* ε^{-1} *and* $c = \pm 1$ *or* $c = \pm\varepsilon$*, so the possibilities for the pair* (a, c) *are* $(0, 1)$*,* $(0, \varepsilon)$*,* $(\varepsilon, \pm 1)$*,* $(\varepsilon, \pm\varepsilon)$*,* $(\varepsilon^{-1}, \pm 1)$ *and* $(\varepsilon^{-1}, \pm\varepsilon)$*.*

Finally, if $\operatorname{tr} A = -\varepsilon^{-1}$*, then* $a = 0$ *or* 1 *and* $c = \pm 1$ *or* $\pm\varepsilon$*, so the possibilities for the pair* (a, c) *are* $(0, 1)$*,* $(0, \varepsilon)$*,* $(1, \pm 1)$ *and* $(1, \pm\varepsilon)$*.*

The fixed point of the elements listed above are given by (1.13) and (1.14).

We add some remarks here. First note that we did not prove that all the element listed in the theorem have fixed point in $\mathcal{F}$, and it is in fact not true. It follows from our argument that all those fixed points are in the set $\mathcal{U} \cap \mathcal{T}$ but this does not hold for $\mathcal{S}$. For example, the elements $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ have the fixed points $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$ and $\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i, \frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$. These points are on the same N-orbit, so only at most one of them is contained in the set $\mathcal{S}$.

Furthermore, we did not show that all the matrices listed in the theorem represent elements from different conjugacy classes, and this is also not true. For example, we have

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

and this last matrix represents the same element as $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$.

What we *can* say is that every totally elliptic conjugacy classes of $\Gamma_{\mathbb{Q}(\sqrt{5})}$ is represented in the list above at least once. It is not difficult to identify the conjugate elements in this list, but it is still tiresome work and we will not do it here. At this point we only mention that for two matrices A and B which represent conjugate elements we have $|\operatorname{tr} A| = |\operatorname{tr} B|$ because matrix conjugation does not change the trace. Also, if two different elements of $\alpha, \beta \in \Gamma_K$ have the same fixed point, then they cannot be conjugate. For if $\gamma^{-1}\alpha\gamma = \beta$, then γ has the same fixed point as α and β, and it is easy to see that in this case they commute and hence $\alpha = \beta$ follows, which is a contradiction.

1.4 An estimate on the fundamental domain of $\Gamma_{\mathbb{Q}(\sqrt{5})}$

In this section the field $K = \mathbb{Q}(\sqrt{5})$ is fixed and we recall the definitions of the sets $\mathcal{U}$, $\mathcal{T}$ and $\mathcal{S}$ in (1.8), (1.9) and (1.10). Also, we remind that by Theorem 1.2.6 the set $\mathcal{F} = \mathcal{U} \cap \mathcal{T} \cap \mathcal{S}$ is a fundamental domain for $\Gamma_{\mathbb{Q}(\sqrt{5})}$. In the following we prove the following claim conjectured in [4].

Theorem 1.4.1. *If* $z \in \mathcal{S} \cap \mathcal{T}$*, then* $y_1 y_2 \geq \sqrt{5}/4$*.*

We are going to estimate the function

$$f_{y_1,y_2}(x_1, x_2) = x_1^2 x_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + y_1^2 y_2^2 = |z_1 z_2|^2$$

from above on the set $\mathcal{S}_{s_1,s_2} \cap \mathcal{T}$ where $s_1, s_2 > 0$. For this end we will estimate on the set

$$P_a^{s_1,s_2} = \left\{ z \in \mathbb{H}^2 : y_1 = s_1,\ y_2 = s_2, \quad \begin{array}{c} -\dfrac{\sqrt{5}}{2} \leq x_1 - x_2 < \dfrac{\sqrt{5}}{2}, \\ -1 \leq (1+a)x_1 + (1-a)x_2 < 1 \end{array} \right\},$$

where $a \in \mathbb{R}$ is a parameter. This is a parallelogram on the plane $\{z \in \mathbb{H}^2 : y_1 = s_1,\ y_2 = s_2\}$ symmetric to the origin. By the definition of $\mathcal{S}_{s_1,s_2}$ every upper bound on the latter set is clearly an upper bound on the former, since if $z \in \mathcal{S}_{s_1,s_2} \cap \mathcal{T}$, then for some $\nu \in \mathcal{O}_K$ we have $(z_1+\nu, z_2+\nu') \in P_a^{s_1,s_2}$. The parameter a will always be chosen so that $1+a>0$ and $1-a>0$ will hold. To simplify notation we may write P_a instead of $P_a^{s_1,s_2}$.

We will use the notations $b = y_1 y_2$ and $c = y_2/y_1$, then

$$f_{y_1,y_1}(x_1,x_2) = x_1^2 x_2^2 + (x_1^2 c + x_2^2 c^{-1})b + b^2.$$

The outline of the proof is the following: we always choose the parameter a so that the function $f_{y_1,y_2}(x_1,x_2) - b^2$ takes its maximum on P_a at a certain vertex. Let us denote this maximum temporarily by $g(a,b,c)$, we will estimate it from above by different expressions depending on c. To obtain the assertion of Theorem 1.4.1 we will use an inequality of the form

$$g(a,b,c) \le \alpha + \beta b,$$

where $\alpha, \beta \in \mathbb{R}$ are suitable numbers. So if $z \in \mathcal{S} \cap \mathcal{T}$, then because of the definition of $\mathcal{T}$ we have $|z_1 z_2| \ge 1$, on the other hand $(z_1+\nu, z_2+\nu') \in P_a$ holds for some $\nu \in \mathcal{O}_K$, then by the definition of $\mathcal{S}$ we get

$$1 \le |z_1 z_2|^2 \le |(z_1+\nu)(z_2+\nu')|^2 = f_{y_1,y_2}(x_1+\nu, x_2+\nu') \le g(a,b,c) + b^2 \le \alpha + \beta b + b^2.$$

But then

$$0 \le \alpha - 1 + \beta b + b^2 \tag{1.15}$$

holds. The roots of the quadratic polynomial on the right hand side are

$$\frac{-\beta \pm \sqrt{\beta^2 - 4(\alpha-1)}}{2}.$$

By Lemma 1.2.7 $b > 1/2$, so it is enough to choose α and β so that these roots are real, i.e.

$$\beta^2 \ge 4(\alpha - 1), \tag{1.16}$$

and the smaller root is less than $1/2$, then (1.15) can hold only if b is greater than or equal to the bigger root. Once the latter one is at least $\sqrt{5}/4$ we get the claim of the theorem. Thus, it will be sufficient to check that these conditions are fulfilled. If (1.16) holds and β is positive, then the smaller root is negative so it is smaller than $1/2$, while the last condition can be formalized in the following way:

$$\frac{-\beta + \sqrt{\beta^2 - 4(\alpha-1)}}{2} \ge \frac{\sqrt{5}}{4},$$

$$4\beta^2 - 16(\alpha - 1) \ge (\sqrt{5} + 2\beta)^2 = 5 + 4\sqrt{5}\beta + 4\beta^2,$$

that is,

$$R(\alpha,\beta) = 11 - 16\alpha - 4\sqrt{5}\beta \ge 0. \tag{1.17}$$

Note that (1.17) implies (1.16) so it is enough to check this latter inequality once $\beta > 0$.

First we show that it is enough to prove the statement if $c \in [\varepsilon^{-1}, \varepsilon]$. To this end consider the map $T_n : \mathbb{H}^2 \to \mathbb{H}^2$, $(z_1, z_2) \mapsto (\varepsilon^n x_1 + i\varepsilon^n y_1, (\varepsilon^n)' x_2 + i\varepsilon^{-n} y_2)$. As $(\varepsilon^n)' = (-1)^{-n}\varepsilon^{-n}$ this takes the set $P_a^{s_1,s_2}$ to

$$T_nP_a^{s_1,s_2} = \left\{ z \in \mathbb{H}^2 : \begin{array}{c} y_1 = \varepsilon^n s_1 \\ y_2 = \varepsilon^{-n} s_2 \end{array}, \quad \begin{array}{c} -\dfrac{\sqrt{5}}{2} \le \varepsilon^{-n}x_1 - (-1)^n\varepsilon^n x_2 < \dfrac{\sqrt{5}}{2} \\ -1 \le (1+a)\varepsilon^{-n}x_1 + (1-a)(-1)^n\varepsilon^n x_2 < 1 \end{array} \right\}.$$

As before, if $z \in \mathbb{H}^2$, $y_1 = \varepsilon^n s_1$ and $y_2 = \varepsilon^{-n}s_2$, then there is an integer $\nu \in \mathcal{O}_K$ such that $(z_1+\nu, z_2+\nu') \in T_nP_a^{s_1,s_2}$. Indeed, for any $\nu \in \mathcal{O}_K$ we have

$$\varepsilon^{-n}(x_1+\nu) - (-1)^n\varepsilon^n(x_2+\nu') = \varepsilon^{-n}x_1 - (-1)^n\varepsilon^n x_2 + \varepsilon^{-n}\nu - (\varepsilon^{-n}\nu)'. \tag{1.18}$$

If $\varepsilon^{-n}\nu = A + B\varepsilon$ where $A, B \in \mathbb{Z}$, then $\varepsilon^{-n}\nu - \varepsilon^{-n}\nu = B\sqrt{5}$ and hence the expression in (1.18) can be shifted into the interval $[-\sqrt{5}/2, \sqrt{5}/2)$ by choosing B properly. Similarly

$$(1+a)\varepsilon^{-n}(x_1+\nu) + (1-a)(-1)^n\varepsilon^n(x_2+\nu') =$$

$$= (1+a)\varepsilon^{-n}x_1 + (1-a)(-1)^n\varepsilon^n x_2 + (1+a)\varepsilon^{-n}\nu + (1-a)(\varepsilon^{-n}\nu)' =$$

$$= (1+a)\varepsilon^{-n}x_1 + (1-a)(-1)^n\varepsilon^n x_2 + (1+a)(A+B\varepsilon) + (1-a)(A+B\varepsilon') =$$

$$= (1+a)\varepsilon^{-n}x_1 + (1-a)(-1)^n\varepsilon^n x_2 + 2A + B + aB\sqrt{5},$$

so this value can be shifted into the interval $[-1,1)$ by choosing A independently from B. Moreover,

$$|z_1z_2|^2 = |(T_nz)_1(T_nz)_2|^2 \tag{1.19}$$

holds for any $n \in \mathbb{Z}$.

Let $z \in \mathcal{S} \cap \mathcal{T}$ an arbitrary point, then $c \in [\varepsilon^{2k-1}, \varepsilon^{2k+1}]$ for some $k \in \mathbb{Z}$. There is a $\nu \in \mathcal{O}_K$ such that $(z_1+\nu, z_2+\nu') \in T_{-k}P_a^{\varepsilon^k y_1, \varepsilon^{-k}y_2}$, and if $N(z) = N(z_1,z_2) = |z_1z_2|^2$, then by (1.19) and $z \in \mathcal{S} \cap \mathcal{T}$ we get

$$1 \le |z_1z_2|^2 \le |(z_1+\nu)(z_2+\nu')|^2 = N(z_1+\nu, z_2+\nu') = N(T_k(z_1+\nu, z_2+\nu')).$$

As $T_k(z_1+\nu, z_2+\nu') \in P_a^{\varepsilon^k y_1, \varepsilon^{-k}y_2}$ and the map T_k does not change the value y_1y_2, it is enough to estimate on this parallelogram. In other words, we can and will assume from now on that $c \in [\varepsilon^{-1}, \varepsilon]$.

1.4.1 Proof in the neighborhood of the endpoints of $[\varepsilon^{-1}, \varepsilon]$

The function f takes its maximum on the boundary of the parallelogram P_a, since at every local minimum or maximum in the interior of P_a the partial derivatives must vanish:

$$\partial_1 f_{y_1,y_2}(x_1,x_2) = 2x_1x_2^2 + 2x_1y_2^2 = 0,$$

$$\partial_2 f_{y_1,y_2}(x_1,x_2) = 2x_1^2x_2 + 2x_2y_1^2 = 0.$$

As y_1 and y_2 are positive, this implies that $x_1 = x_2 = 0$, and at this point f clearly takes its minimum. Moreover, since $f_{y_1,y_2}(x_1,x_2) = f_{y_1,y_2}(-x_1,-x_2)$, it is enough to estimate on the lines $x_1 = x_2 - \frac{\sqrt{5}}{2}$ and $x_1 = -\frac{1}{1+a} - \frac{1-a}{1+a}x_2$ between the vertices of the parallelogram. Here f depends on only one variable, say $x_2 =: x$, and we also omit the constant term for now, that is, we are looking for the maximum of the functions

$$g_{y_1,y_2}(x) = f_{y_1,y_2}\left(x - \frac{\sqrt{5}}{2}, x\right) - y_1^2 y_2^2$$

and

$$h_{y_1,y_2}(x) = f_{y_1,y_2}\left(-\frac{1}{1+a} - \frac{1-a}{1+a}x, x\right) - y_1^2 y_2^2$$

on some closed intervals. For the first function $g_{y_1,y_2}(x)$ the endpoints of this is interval come from the equations

$$x - \frac{\sqrt{5}}{2} = -\frac{1}{1+a} - \frac{1-a}{1+a}x; \quad x - \frac{\sqrt{5}}{2} = \frac{1}{1+a} - \frac{1-a}{1+a}x, \tag{1.20}$$

which means that $x \in \left[\frac{\sqrt{5}(1+a)-2}{4}, \frac{\sqrt{5}(1+a)+2}{4}\right]$, while in the case of $h_{y_1,y_2}(x)$ the first equation in (1.20) and

$$x_2 + \frac{\sqrt{5}}{2} = -\frac{1}{1+a} - \frac{1-a}{1+a}x_2$$

gives the interval $x \in \left[\frac{-\sqrt{5}(1+a)-2}{4}, \frac{\sqrt{5}(1+a)-2}{4}\right]$. We are going to show that for an appropriate choice of the parameter a these functions take their maximum at an endpoint of these intervals. But in this case it follows from $f_{y_1,y_2}(x_1, x_2) = f_{y_1,y_2}(-x_1, -x_2)$ that it is enough to consider the maximum of $g_{y_1,y_2}(x)$ (because two of the three vertices of the parallelogram that come into question are symmetrical to the origin). We remind that a will always be chosen from the interval $(-1, 1)$. The proof of the following propositions will be postponed to Section 1.4.4:

Proposition 1.4.2. *The function $g_{y_1,y_2}(x)$ restricted to the interval $\left[\frac{\sqrt{5}(1+a)-2}{4}, \frac{\sqrt{5}(1+a)+2}{4}\right]$ takes its maximum at an endpoint of the interval for any $a \in (-1, 1)$.*

To obtain an analogous result for $h_{y_1,y_2}(x)$ we must be careful by the choice of a.

Proposition 1.4.3. *Let us define the function*

$$H_a(c) := \left(\frac{1-a}{1+a}\right)^2 c + \frac{1}{c} - \frac{1}{(1+a)^2}.$$

If $H_a(c) \geq 0$, then the function $h_{y_1,y_2}(x)$ restricted to the interval $\left[\frac{-\sqrt{5}(1+a)-2}{4}, \frac{\sqrt{5}(1+a)-2}{4}\right]$ takes its maximum at an endpoint of the interval.

The parameter a will be a function of $c \in [\varepsilon^{-1}, \varepsilon]$. We choose different functions on different subintervals of $[\varepsilon^{-1}, \varepsilon]$, a constant function will do on the middle intervals, while we have to be more precautious by the endpoints where we will use linear functions. For these we will always check the condition $H_a(c) \geq 0$. We set $a = p(c - \varepsilon) + \frac{1}{\sqrt{5}}$ on the interval $[\varepsilon - \delta, \varepsilon]$ for some $p > 0$ and $\delta > 0$ such that $H_a(c) \geq 0$ holds. Similarly, we set $a = p'(c - \varepsilon^{-1}) - \frac{1}{\sqrt{5}}$ on the interval $c \in [\varepsilon^{-1}, \varepsilon^{-1} + \eta]$ for some $p' > 0$ and $\eta > 0$. While a detailed analysis will be made in the former case, we simply choose $p' = 1$ in the latter which makes the computations less tedious and fortunately works. Let us explain first the case of the right endpoint. Again, the following proposition is proved in Section 1.4.4 like some others in this section below which are given here without proof.

Proposition 1.4.4. *If $c \in [1, \varepsilon]$ and $p \in [0.24, 0.66]$, then for $a = p(c-\varepsilon) + \frac{1}{\sqrt{5}}$ we have $a \in (-1; 1)$ and $H_a(c) \geq 0$.*

For further simplification in the case of function $g_{y_1,y_2}(x)$ we substitute $w = x - \sqrt{5}/4$ and omit the indices y_1 and y_2. This means that once we $H_a(c) \geq 0$ holds we consider the function

$$g(w) = \left(w^2 - \frac{5}{16}\right)^2 + \left[\left(w - \frac{\sqrt{5}}{4}\right)^2 c + \left(w + \frac{\sqrt{5}}{4}\right)^2 c^{-1}\right] b$$

at the points $w = \frac{a\sqrt{5}-2}{4}$ and $w = \frac{a\sqrt{5}+2}{4}$ (for the detailed computation see the proof of Proposition 1.4.2). Putting these values in the place of w we get the expressions

$$g_1(a,b,c) = \left(\left(\frac{\sqrt{5}a+2}{4}\right)^2 - \frac{5}{16}\right)^2 + \left[\left(\frac{\sqrt{5}a+2}{4} - \frac{\sqrt{5}}{4}\right)^2 c + \left(\frac{\sqrt{5}a+2}{4} + \frac{\sqrt{5}}{4}\right)^2 c^{-1}\right] b,$$

$$g_2(a,b,c) = \left(\left(\frac{\sqrt{5}a-2}{4}\right)^2 - \frac{5}{16}\right)^2 + \left[\left(\frac{\sqrt{5}a-2}{4} - \frac{\sqrt{5}}{4}\right)^2 c + \left(\frac{\sqrt{5}a-2}{4} + \frac{\sqrt{5}}{4}\right)^2 c^{-1}\right] b.$$

Now we compute their difference $g_1(a,b,c) - g_2(a,b,c)$:

$$\left(\left(\frac{\sqrt{5}a+2}{4}\right)^2 - \frac{5}{16}\right)^2 - \left(\left(\frac{\sqrt{5}a-2}{4}\right)^2 - \frac{5}{16}\right)^2 +$$
$$+ \left\{\left[\left(\frac{\sqrt{5}a+2}{4} - \frac{\sqrt{5}}{4}\right)^2 - \left(\frac{\sqrt{5}a-2}{4} - \frac{\sqrt{5}}{4}\right)^2\right] c + \right.$$
$$\left.\left[\left(\frac{\sqrt{5}a+2}{4} + \frac{\sqrt{5}}{4}\right)^2 - \left(\frac{\sqrt{5}a-2}{4} + \frac{\sqrt{5}}{4}\right)^2\right] c^{-1}\right\} b.$$

The first term is

$$\left[\left(\frac{\sqrt{5}a+2}{4}\right)^2 - \left(\frac{\sqrt{5}a-2}{4}\right)^2\right]\left[\left(\frac{\sqrt{5}a+2}{4}\right)^2 + \left(\frac{\sqrt{5}a-2}{4}\right)^2 - \frac{10}{16}\right] =$$
$$= \frac{\sqrt{5}a}{2} \cdot \frac{(2(5a^2+4) - 10)}{16} = \frac{\sqrt{5}a(5a^2-1)}{16},$$

while the second one is

$$\left[\left(\frac{\sqrt{5}a - \sqrt{5}}{2}\right) c + \left(\frac{\sqrt{5}a + \sqrt{5}}{2}\right) c^{-1}\right] b = \frac{\sqrt{5}b}{2}[(a-1)c + (a+1)c^{-1}].$$

Let us introduce the notation

$$\Delta_{a,b,c} = g_1(a,b,c) - g_2(a,b,c) = \frac{\sqrt{5}a(5a^2-1)}{16} + \frac{\sqrt{5}b}{2}[(a-1)c + (a+1)c^{-1}].$$

For every subintervals of $[\varepsilon^{-1},\varepsilon]$ that we consider the parameter a will be always set so that the sign of $\Delta_{a,b,c}$ does not change on that interval. Once $\Delta_{a,b,c} \geq 0$ we need to estimate the value $g_1(a,b,c)$ on that particular interval while in the other case we work with $g_2(a,b,c)$. Note that for a fixed $a \in (-1,1)$ and $b > 0.54$ $\Delta_{a,b,c}$ is a decreasing function of c on the interval $[\varepsilon^{-1},\varepsilon]$.

Let us continue the analysis of the case when c is in the neighborhood of ε. Recall that the parameter $a = p(c-\varepsilon) + \frac{1}{\sqrt{5}}$ is chosen for some $p \in [0.24, 0.66]$ that we specify now. We have already seen that in this case $1 \pm a > 0$ and $H_a(c) \geq 0$ hold, and we will choose p such that $\Delta_{a,b,c}$ will be non-negative. We have the following:

Proposition 1.4.5. *If $c \in [1,\varepsilon]$ and $a = p(c-\varepsilon) + 1/\sqrt{5}$ where $p = 0.9/\sqrt{5}$, then $\Delta_{a,b,c} \geq 0$.*

This means that in a neighborhood of ε with the choice $a = p(c-\varepsilon)+1/\sqrt{5}$ where $p = 0.9/\sqrt{5}$ the maximum value of the function $f_{y_1,y_2}(x_1,x_2) - b^2$ on P_a is $g_1(a,b,c)$. Substituting the value of a in $g_1(a,b,c)$ and using the notation $q = \sqrt{5}p$ we get that $g_1(a,b,c)$ is

$$\left(\left(\frac{q(c-\varepsilon)+3}{4}\right)^2 - \frac{5}{16}\right)^2 + \left[\left(\frac{q(c-\varepsilon)+3}{4} - \frac{\sqrt{5}}{4}\right)^2 c + \left(\frac{q(c-\varepsilon)+3}{4} + \frac{\sqrt{5}}{4}\right)^2 c^{-1}\right] b.$$

This expression can be seen as a function of c with a fixed parameter b, let us denote its value by $g_1(b,c)$.

Without loss of generality we can assume in the following that $b < 0.56$ (otherwise the claim of the theorem holds). We will show that on some interval $[\varepsilon-\delta,\varepsilon]$ this function is strictly increasing. For this it suffices if its derivative is positive and this is true for some $\delta > 0$ small enough:

Proposition 1.4.6. *If $c \in [1.48,\varepsilon]$ and $q = 0.9$, then the derivative of $g_1(b,c)$ (with respect to c) is positive.*

We are now in the position to finish the first part of the proof. Since $g_1(b,c)$ is strictly increasing on $[1.48,\varepsilon]$ we simply estimate it on this interval by the value $g_1(b,\varepsilon)$:

$$g_1(b,c) \leq \frac{1}{16} + \left[\frac{\varepsilon^{-4}}{4}\varepsilon + \frac{\varepsilon^4}{4}\varepsilon^{-1}\right] b = \frac{1}{16} + \frac{b}{4}(\varepsilon^3+\varepsilon^{-3}) = \frac{1}{16} + \frac{\sqrt{5}}{2}b.$$

It remains to check the inequality (1.17) for $\alpha = \frac{1}{16}$ and $\beta = \frac{\sqrt{5}}{2}$. We have $R(\alpha,\beta) = 0$ so (1.17) holds and the theorem is proved in the case $c \in [1.48,\varepsilon]$. We have also proved that equality can only hold for $c = \varepsilon$.

Now we turn to the case when c is near to the other endpoint of the interval. As we mentioned before we chose the parameter $a = c - \varepsilon^{-1} - \frac{1}{\sqrt{5}}$ if $c \in [\varepsilon^{-1}, \varepsilon^{-1}+\delta]$ for some small positive δ specified later. Then we have the following:

Proposition 1.4.7. *If $c \in [\varepsilon^{-1};1]$ and $a = c - \varepsilon^{-1} - \frac{1}{\sqrt{5}}$, then $a \in (-1;1)$ and $H_a(c) \geq 0$ hold.*

Proposition 1.4.8. *If $c \in [\varepsilon^{-1},1]$ and $a = c - \varepsilon^{-1} - \frac{1}{\sqrt{5}}$, then $\Delta_{a,b,c} \leq 0$.*

This means that in a neighborhood of ε^{-1} with the choice of $a = c - \varepsilon^{-1} - 1/\sqrt{5}$ the maximum value of the function $f_{y_1,y_2}(x_1, x_2) - b^2$ on P_a is $g_2(a, b, c)$. If we substitute the value of a in $g_2(a, b, c)$ then we get that this maximum is

$$\left(\left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4}\right)^2 - \frac{5}{16}\right)^2$$

$$+\left[\left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4} - \frac{\sqrt{5}}{4}\right)^2 c + \left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4} + \frac{\sqrt{5}}{4}\right)^2 c^{-1}\right] b.$$

Let us denote this expression by $g_2(b, c)$, for a fixed b it is a function of c.

Proposition 1.4.9. *For a fixed* $0 < b < 0.56$ *the function* $g_2(b, c)$ *is strictly decreasing on the interval* $[\varepsilon^{-1}; 0.68]$.

It follows from this that

$$g_2(b, c) \leq g_2(b, \varepsilon^{-1}) = \frac{1}{16} + \frac{b}{4}(\varepsilon^4\varepsilon^{-1} + \varepsilon^{-4}\varepsilon) = \frac{1}{16} + \frac{\sqrt{5}}{2}b.$$

Then we get the same way as before that the theorem holds for $c \in [\varepsilon^{-1}; 0.68]$ and equality can hold only if $c = \varepsilon^{-1}$.

1.4.2 Proof on the middle intervals

In this section we prove the theorem in the case $c \in [0.68, 1.48]$. We divide this interval into subintervals and we fix the constant a on each of them. We will always check the conditions $a \in (-1, 1)$ and $H_a(c) \geq 0$. First we analyze $H_a(c)$ as a function of c. Its derivative is

$$H_a'(c) = \left(\frac{1-a}{1+a}\right)^2 - \frac{1}{c^2}.$$

If its sign is constant on an interval, then it is enough to check the sign of $H_a(c)$ at an endpoint to obtain this value for the whole interval. Similarly, to estimate g_1 or g_2 it will be sufficient to do this at the endpoints once their derivative has a constant sign on a subinterval. These derivatives are

$$g_1'(a, b, c) = \left[\left(\frac{\sqrt{5}a+2}{4} - \frac{\sqrt{5}}{4}\right)^2 - \left(\frac{\sqrt{5}a+2}{4} + \frac{\sqrt{5}}{4}\right)^2 c^{-2}\right] b,$$

$$g_2'(a, b, c) = \left[\left(\frac{\sqrt{5}a-2}{4} - \frac{\sqrt{5}}{4}\right)^2 - \left(\frac{\sqrt{5}a-2}{4} + \frac{\sqrt{5}}{4}\right)^2 c^{-2}\right] b.$$

As a first example we consider an interval $[1, 1+\delta)$ and set $a = 0$. As $H_a'(c)$ is strictly increasing and $H_0'(1) = 0$ it is enough to check the condition $H_a(c) \geq 0$ at the left endpoint. Since $H_0(1) = 1$, we get that $H_a(c) \geq 0$ holds if $c \in [1, 1+\delta]$. Let us examine the function $\Delta_{a,b,c}$.

Its derivative is

$$\Delta'_{a,b,c} = \frac{\sqrt{5}b}{2}[(a-1)-(a+1)c^{-2}],$$

which is also a strictly increasing function (for $c > 0$). Since

$$a - 1 - (a+1)\varepsilon^{-2} < -(a+1)\varepsilon^{-2} < 0,$$

the function $\Delta_{a,b,c}$ is strictly decreasing on $[\varepsilon^{-1}, \varepsilon]$ for every $a \in (-1, 1)$. Hence to show that $\Delta_{a,b,c} \leq 0$ on an interval it is enough to check this at the left endpoint. This is true for $a = 0$ and $c = 1$ since $\Delta_{0,b,1} = 0$.

It follows that the value $g_2(a, b, c)$ is an upper bound for the function $f_{y_1,y_2}(x_1, x_2) - b^2$ if $c \in [1, 1 + \delta]$. The derivative of $g_2(a, b, c)$ is again increasing (as a function of c) for every $a \in (-1, 1)$ and positive for $a = 0$ and $c = 1$, hence $g_2(0, b, c)$ is strictly increasing on $[1, 1 + \delta]$ and can be estimated from above by its value at $c_0 = 1 + \delta$. Now if $z \in \mathcal{S} \cap \mathcal{T}$ with $c \in [1, c_0]$, then

$$1 \leq |z_1 z_2|^2 \leq g_2(0, b, c_0) + b^2,$$

i.e. $0 \leq -1 + g_2(0, b, c_0) + b^2$. It is enough then if this quadratic polynomial r has real roots and the smaller one is less than $1/2$ (since $b > 1/2$) while the other one is bigger than $\sqrt{5}/4$. This is true for $c_0 = 1.08$ so with the choice $a = 0$ for the subinterval $[1, 1.08)$ the theorem is proved here. Note that on these subintervals b turns out to be strictly bigger than $\sqrt{5}/4$.

In the next step we increase a as much as possible, that is as long as $H_a(c_0) \geq 0$, $\Delta_{a,b,c_0} \leq 0$ and $g'_2(a, b, c_0) > 0$ hold. For the estimate of Δ_{a,b,c_0} we examine the sign of $(a-1)c + (a+1)c^{-1}$. This value is non-positive if and only if $(a-1)c^2 + a + 1 \leq 0$, that is (since $a - 1 < 0$)

$$c^2 \geq (1 + a)/(1 - a). \tag{1.21}$$

Then since $b > 1/2$, we have

$$\Delta_{a,b,c} \leq \frac{\sqrt{5}a(5a^2 - 1)}{16} + \frac{\sqrt{5}}{4}[(a-1)c + (a+1)c^{-1}] = D(a, c).$$

We will choose a such that the function $D(a, c)$ is non-positive. We have to take care also of the condition (1.21), so a should satisfy the inequality $a < (c_0^2 - 1)/(c_0^2 + 1)$. For practical reasons (i.e. to make this proof readable) we chose numbers that we can write down easily, we typically round down to 2 decimal places. This may increase the number of steps of the proof but not significantly. The value a that we get this way will be denoted by a_1. Like in the first step we increase c_0 after this as much as we can to get c_1 and the proof of the theorem for $c \in [c_0, c_1)$. The same procedure provides then the values $a_2, a_3, \ldots$ and $c_2, c_3, \ldots$ until we have $c_n > 1.48$ for some $n \in \mathbb{N}$, in which case we stop. We summarize this algorithm in the following:

1. Set $a_0 = 0$ (then $H'_{a_0}(c) \geq 0$ for $c \in [1, \varepsilon]$, $H_{a_0}(1) \geq 0$, $\Delta_{a_0,b,1} \leq 0$, $g'_2(a_0, b, 1) > 0$) and $n = 0$.

2. Choose the maximal c_n such that at most the first two digits of c_n after the decimal separator are non-zero and the smaller root of the polynomial $-1 + g_2(a_n, b, c_n) + b^2$ is less that $1/2$, while the bigger one is greater than $\sqrt{5}/4$.

3. If $c_n > 1.48$, then stop.

4. $n \to n + 1$

5. Choose the maximal a_n so that at most the first two digits of a_n after the decimal separator are non-zero, furthermore $H'_{a_n}(c_{n-1}) \geq 0$, $H_{a_n}(c_{n-1}) \geq 0$, $a_n \leq \frac{c_{n-1}^2-1}{c_{n-1}^2+1}$, $D(a_n, c_{n-1}) \leq 0$ and $g'(a_n, b, c_{n-1}) > 0$ hold. Continue with step 2.

The algorithm above gives the following values:

$$\begin{aligned}
a_1 &= 0.07, & c_1 &= 1.15,\\
a_2 &= 0.13, & c_2 &= 1.21,\\
a_3 &= 0.18, & c_3 &= 1.27,\\
a_4 &= 0.23, & c_4 &= 1.32,\\
a_5 &= 0.27, & c_5 &= 1.37,\\
a_6 &= 0.3, & c_6 &= 1.41,\\
a_7 &= 0.33, & c_7 &= 1.44,\\
a_8 &= 0.34, & c_8 &= 1.46,\\
a_9 &= 0.36, & c_9 &= 1.488.
\end{aligned}$$

This makes the proof complete if $c \in [1, \varepsilon]$.

Now we examine the other half of the interval and prove the assertion on a subinterval $[c_{-1}, 1)$. As before we require $H'_a(c_{-1}) \geq 0$ and $H_a(c_{-1}) \geq 0$ but now $\Delta_{a,b,c} \geq 0$ will be expected, so it will be checked at the right endpoint. We will need the condition

$$c^2 \leq (1+a)/(1-a).$$

Once this is fulfilled we get

$$\Delta_{a,b,c} \geq \frac{\sqrt{5}a(5a^2-1)}{16} + \frac{\sqrt{5}}{4}[(a-1)c + (a+1)c^{-1}],$$

so it is enough to show that the right hand side is non-negative at the right endpoint. In accordance with this we work with the function $g_1(a,b,c)$, its derivative is increasing for every $a \in (-1,1)$. We check that this derivative is negative at 1 (at the right endpoint) and so we can estimate by $g_1(a,b,c_{-1})$ (by the value at the left endpoint). Hence for $z \in \mathcal{S} \cap \mathcal{T}$ we have

$$1 \leq |z_1 z_2|^2 \leq g_1(a,b,c_{-1}) + b^2,$$

so $0 \leq -1 + g_1(a,b,c_{-1}) + b^2$. We choose c_{-1} so that the smaller root of the quadratic polynomial on the right hand side is smaller than $1/2$, while the bigger one is greater than $\sqrt{5}/4$.

We begin with $a = 0$ and looking for c_{-1}. We have already seen that $\Delta_{0,b,1} = 0$. Now $g_1(0,b,1) < 0$ also holds, and for $c_{-1} = 0.92$ the other conditions are fulfilled. Then we decrease a as much as we can so that the conditions $c_{-1} \leq (1+a)/(1-a)$, $\Delta_{a,b,c_{-1}} \geq 0$ and $g_1(a,b,c_{-1}) < 0$ hold. We get the value $a_{-1} = -0.08$ and continue with searching for the next left endpoit c_{-2}. We repeat these steps until we get that $c_{-n} < 0.68$. This way we obtain

$$\begin{aligned}
a_0 &= 0 & c_{-1} &= 0.92,\\
a_{-1} &= -0.08 & c_{-2} &= 0.86,\\
a_{-2} &= -0.14 & c_{-3} &= 0.82,\\
a_{-3} &= -0.19 & c_{-4} &= 0.78,\\
a_{-4} &= -0.24 & c_{-5} &= 0.75,
\end{aligned}$$

$$\begin{aligned} a_{-5} &= -0.27 & c_{-6} &= 0.73, \\ a_{-6} &= -0.3 & c_{-7} &= 0.71, \\ a_{-7} &= -0.32 & c_{-8} &= 0.7, \\ a_{-8} &= -0.342 & c_{-9} &= 0.682, \\ a_{-9} &= -0.365 & c_{-10} &= 0.668. \end{aligned}$$

Hence the assertion follows for $c \in [\varepsilon^{-1}, \varepsilon]$ and the proof of the theorem is complete (with the postponed computations in Section 1.4.4).

1.4.3 The case of equality

It is clear from Theorem 1.4.1 that $y_1 y_2 \geq \sqrt{5}/4$ for every point $z \in \mathcal{F}$ as it is a subset of $\mathcal{S} \cap \mathcal{T}$. In this section we shortly analyse the case when equality holds in the inequality above. Since $z \in \mathcal{F}$ we have $\varepsilon^{-2} \leq y_2/y_1 < \varepsilon^2$ and we have seen in the proof above that equality can hold only if $y_2/y_1 = \varepsilon^{\pm 1}$. If $y_2/y_1 = \varepsilon$, then

$$y_1 = \sqrt{(y_1 y_2)(y_1/y_2)} = \sqrt{\frac{\sqrt{5}}{4} \cdot \frac{\sqrt{5}-1}{2}} = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}},$$

$$y_2 = \sqrt{(y_1 y_2)(y_2/y_1)} = \sqrt{\frac{\sqrt{5}}{4} \cdot \frac{1+\sqrt{5}}{2}} = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}.$$

Following our argument above we see that for some $\nu \in \mathcal{O}_K$ the point $(z_1 + \nu, z_2 + \nu')$ is in $P_{1/\sqrt{5}}$. As before, we have

$$1 \leq |z_1 z_2|^2 \leq |(z_1 + \nu)(z_2 + \nu')| \leq g_1(\sqrt{5}/4, \varepsilon) + \frac{5}{16} = \frac{1}{16} + \frac{5}{8} + \frac{5}{16} = 1$$

This forces these values to be equal. That is, the point z can be translated to a vertex of the parallelogram $P_{1/\sqrt{5}}$ and it is clear from the proof that this vertex is one of the following points:

$$\left(\frac{\varepsilon^{-2}}{2} + i\frac{1}{2}\sqrt{1+\varepsilon^{-2}}, \frac{\varepsilon^{2}}{2} + i\frac{1}{2}\sqrt{1+\varepsilon^{2}}\right) \quad \text{or} \quad \left(-\frac{\varepsilon^{-2}}{2} + i\frac{1}{2}\sqrt{1+\varepsilon^{-2}}, -\frac{\varepsilon^{2}}{2} + i\frac{1}{2}\sqrt{1+\varepsilon^{2}}\right).$$

By (1.13) and (1.14) we get that these are the fixed points of totally elliptic elements represented by the matrices

$$A_1 = \begin{bmatrix} \varepsilon^{-1} & 1-\varepsilon^{-1} \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} \varepsilon^{-1} & \varepsilon^{-1}-1 \\ 1 & 1 \end{bmatrix}.$$

If $S_\nu = \begin{bmatrix} 1 & \nu \\ 0 & 1 \end{bmatrix}$, then z is the fixed point of a totally elliptic element $\alpha \in \Gamma_{\mathbb{Q}(\sqrt{5})}$ represented by a matrix $S_\nu^{-1} A_1 S_\nu$ or $S_\nu^{-1} A_2 S_\nu$ and hence it is an elliptic fixed point in $\mathcal{F}$. If α is represented by the unique matrix

$$A = \begin{bmatrix} a & c^{-1}(a(\operatorname{tr} A - a) - 1) \\ c & \operatorname{tr} A - a \end{bmatrix} \in SL(2, \mathcal{O}_K)$$

with the property $a \geq 0$ and $c > 0$ once $a = 0$, then $A = \pm S_\nu^{-1} A_1 S_\nu$ or $A = \pm S_\nu^{-1} A_2 S_\nu$ and hence $\operatorname{tr} A = \pm\varepsilon$. Furthermore, A must be contained in the finite list in Theorem 1.3.5.

We get the same way in the case when $y_2/y_1 = \varepsilon^{-1}$ that z is fixed by a totally elliptic element α such that $\operatorname{tr}[\alpha] = \pm\varepsilon^{-1}$. This element is also listed in Theorem 1.3.5. Moreover, every element with trace $\pm\varepsilon^{\pm 1}$ that is listed in this theorem has a fixed point for which $y_1 y_2 = \sqrt{5}/4$ holds. We have

Corollary 1.4.10. *Let $z \in \mathcal{F}$, then $y_1 y_2 \geq \sqrt{5}/4$. Equality holds if and only if z is fixed by a totally elliptic element $\alpha \in \Gamma_{\mathbb{Q}(\sqrt{5})}$ with $|tr[\alpha]| = \varepsilon^{\pm 1}$. There are only finitely many points in $\mathcal{F}$ with this property.*

1.4.4 Proof of some propositions

In the following we give the missing proofs of some propositions stated in Section 1.4.1:

Proof of Proposition 1.4.2. We consider the function

$$g_{y_1,y_2}(x) = \left(x - \frac{\sqrt{5}}{2}\right)^2 x^2 + \left(x - \frac{\sqrt{5}}{2}\right)^2 y_2^2 + x^2 y_1^2$$

$$= \left(x - \frac{\sqrt{5}}{2}\right)^2 x^2 + \left[\left(x - \frac{\sqrt{5}}{2}\right)^2 c + x^2 c^{-1}\right] b,$$

on the interval $\left[\frac{\sqrt{5}(1+a)-2}{4}, \frac{\sqrt{5}(1+a)+2}{4}\right]$, where $c = y_2/y_1$ and $b = y_1 y_2$. To simplify notations we omit the indices y_1 and y_2 in g_{y_1,y_2}. Furthermore, to make the computation easier we use the substitution $w = x - \sqrt{5}/4$ and we are looking for the maximum of the function $\tilde{g}(w) = g(w + \sqrt{5}/4) = g(x)$ on the interval $\left[\frac{a\sqrt{5}-2}{4}, \frac{a\sqrt{5}+2}{4}\right]$. This function is given by the formula

$$\tilde{g}(w) = \left(w - \frac{\sqrt{5}}{4}\right)^2 \left(w + \frac{\sqrt{5}}{4}\right)^2 + \left[\left(w - \frac{\sqrt{5}}{4}\right)^2 c + \left(w + \frac{\sqrt{5}}{4}\right)^2 c^{-1}\right] b$$

$$= \left(w^2 - \frac{5}{16}\right)^2 + \left[\left(w - \frac{\sqrt{5}}{4}\right)^2 c + \left(w + \frac{\sqrt{5}}{4}\right)^2 c^{-1}\right] b.$$

Its derivative is

$$\tilde{g}'(w) = 4w\left(w^2 - \frac{5}{16}\right) + 2b\left[\left(w - \frac{\sqrt{5}}{4}\right) c + \left(w + \frac{\sqrt{5}}{4}\right) c^{-1}\right]$$

$$= 4w^3 + \left(2b(c + c^{-1}) - \frac{5}{4}\right) w + \frac{b\sqrt{5}(c^{-1} - c)}{2}.$$

Recall that since our initial point z is in the set $\mathcal{S}_{y_1,y_2} \cap \mathcal{T}$ we have $b > 0.54$ by Lemma 1.2.7. By this and the inequality $c + c^{-1} \geq 2$ we get that the coefficient of w above is positive and hence the derivative of $\tilde{g}$ is strictly increasing on $\mathbb{R}$. Thus, it takes the value 0 only once, so $\tilde{g}$ has only one local extremum, and this must be a local minimum since $\lim_{w\to\pm\infty}\tilde{g}(w)=\infty$. This means that independently of y_1, y_2 and the choice of a the function $\tilde{g}$ and then also g takes their maximum on the intervals above at one of the endpoints. □

Proof of Proposition 1.4.3. We consider the function

$$h_{y_1,y_2}(x)=h(x)=\left(\frac{1-a}{1+a}x+\frac{1}{1+a}\right)^2x^2+\left(\frac{1-a}{1+a}x+\frac{1}{1+a}\right)^2y_2^2+x^2y_1^2$$

$$=\left(\frac{1-a}{1+a}\right)^2\left(x+\frac{1}{1-a}\right)^2x^2+\left[\left(\frac{1-a}{1+a}\right)^2\left(x+\frac{1}{1-a}\right)^2c+x^2c^{-1}\right]b$$

on the interval $\left[\frac{-\sqrt{5}(1+a)-2}{4},\frac{\sqrt{5}(1+a)-2}{4}\right]$. Let us introduce the notations $\alpha=(1-a)/(1+a)$, $\beta=1/(2(1-a))$ and $u=x+\beta$. Then

$$\tilde{h}(u)=h(u-\beta)=h(x)=\alpha^2(u+\beta)^2(u-\beta)^2+\left[\alpha^2(u+\beta)^2c+(u-\beta)^2c^{-1}\right]b$$

$$=\alpha^2(u^2-\beta^2)^2+\left[\alpha^2(u+\beta)^2c+(u-\beta)^2c^{-1}\right]b.$$

Now

$$\tilde{h}'(u)=4\alpha^2u(u^2-\beta^2)+2\alpha^2(u+\beta)bc+2(u-\beta)bc^{-1}$$

$$=4\alpha^2u^3+[2b(\alpha^2c+c^{-1})-4\alpha^2\beta^2]u+2\beta b(\alpha^2c-c^{-1}).$$

Here the coefficient of u^3 is positive, and as $\alpha\beta=\frac{1}{2(1+a)}$ the coefficient of u is

$$2\left(\frac{1-a}{1+a}\right)^2bc+2bc^{-1}-\frac{1}{(1+a)^2}\geq\left(\frac{1-a}{1+a}\right)^2c+\frac{1}{c}-\frac{1}{(1+a)^2}=H_a(c)$$

since $b>1/2$. Now as in the proof of Proposition 1.4.2 we can see the statement is true when $H_a(c)\geq 0$. □

Proof of Proposition 1.4.4. We set $a=p(c-\varepsilon)+\frac{1}{\sqrt{5}}$ for some parameter p. To fulfill the condition $H_a(c)\geq 0$ it is enough if we have

$$\frac{1}{c}-\frac{1}{(1+p(c-\varepsilon)+\frac{1}{\sqrt{5}})^2}\geq 0. \tag{1.22}$$

Of course we have to choose p such that $1\pm a>0$ holds (and then the denominator above does not vanish). Then (1.22) is equivalent to:

$$0\leq\left(1+p(c-\varepsilon)+\frac{1}{\sqrt{5}}\right)^2-c=\left(\frac{5+\sqrt{5}}{5}+p(c-\varepsilon)\right)^2-c$$

$$= \left(\frac{5+\sqrt{5}}{5}\right)^2 + p^2(c^2 - 2\varepsilon c + \varepsilon^2) + 2\left(\frac{5+\sqrt{5}}{5}\right) p(c-\varepsilon) - c$$

$$= p^2c^2 + \left(2\left(\frac{5+\sqrt{5}}{5}\right)p - 2\varepsilon p^2 - 1\right)c + p^2\varepsilon^2 - 2\left(\frac{5+\sqrt{5}}{5}\right)p\varepsilon + \left(\frac{5+\sqrt{5}}{5}\right)^2$$

$$= p^2c^2 + \left(2\left(\frac{5+\sqrt{5}}{5}\right)p - 2\varepsilon p^2 - 1\right)c + \left(p\varepsilon - \left(\frac{5+\sqrt{5}}{5}\right)\right)^2.$$

To fulfill this condition it is sufficient if the discriminant of this last quadratic polynomial is negative:

$$0 > \left[2p\left(\left(\frac{5+\sqrt{5}}{5}\right) - \varepsilon p\right) - 1\right]^2 - 4p^2\left(p\varepsilon - \left(\frac{5+\sqrt{5}}{5}\right)\right)^2$$

$$= 1 - 4p\left(\left(\frac{5+\sqrt{5}}{5}\right) - \varepsilon p\right) = 4\varepsilon p^2 - 4\left(\frac{5+\sqrt{5}}{5}\right)p + 1.$$

The roots of the latter quadratic polynomial are

$$p_{1,2} = \frac{4\left(\frac{5+\sqrt{5}}{5}\right) \pm \sqrt{16\left(\frac{5+\sqrt{5}}{5}\right)^2 - 16\varepsilon}}{8\varepsilon} = \frac{\left(\frac{5+\sqrt{5}}{5}\right) \pm \sqrt{\frac{6+2\sqrt{5}}{5} - \frac{1+\sqrt{5}}{2}}}{2\varepsilon}$$

$$= \frac{\left(\frac{5+\sqrt{5}}{5}\right) \pm \sqrt{\frac{7-\sqrt{5}}{10}}}{\sqrt{5}+1} = \frac{\frac{(5+\sqrt{5})(\sqrt{5}-1)}{5} \pm \sqrt{\frac{(7-\sqrt{5})(6-2\sqrt{5})}{10}}}{4}$$

$$= \frac{\frac{4\sqrt{5}}{5} \pm \sqrt{\frac{52-20\sqrt{5}}{10}}}{4} = \frac{1}{\sqrt{5}} \pm \frac{\sqrt{2}}{4}\sqrt{\frac{13-5\sqrt{5}}{5}}.$$

Every p between these roots is good, in particular we can choose any value in the interval $[0.24, 0.66]$. It remains to check that $1 \pm \left(p(c-\varepsilon) + \frac{1}{\sqrt{5}}\right) > 0$. If $1 \le c \le \varepsilon$, then

$$1 + p(c-\varepsilon) + \frac{1}{\sqrt{5}} \ge 1 + (1-\varepsilon) + \frac{1}{\sqrt{5}} > 2 - \frac{1+\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2} > 0$$

and

$$1 - p(c-\varepsilon) - \frac{1}{\sqrt{5}} \ge 1 - \frac{1}{\sqrt{5}} > 0,$$

so the assertion is proved. □

Proof of Proposition 1.4.5. It is clearly enough to see that

$$\sqrt{5}a(5a^2-1) + 8b[(\sqrt{5}a - \sqrt{5})c + (\sqrt{5}a + \sqrt{5})c^{-1}] \ge 0.$$

If $q=\sqrt{5}p$, then $\sqrt{5}a=q(c-\varepsilon)+1$ and the previous inequality can be written as

$$(q(c-\varepsilon)+1)q(c-\varepsilon)(q(c-\varepsilon)+2)+8b[(q(c-\varepsilon)+1-\sqrt{5})c+(q(c-\varepsilon)+1+\sqrt{5})c^{-1}]\geq 0.$$

Multiplying by c and using the substitution $t=\varepsilon-c$ we get that the left hand side is

$$f(t)=-qt(-qt+1)(-qt+2)(\varepsilon-t)+8b[(-qt+1-\sqrt{5})(\varepsilon-t)^2-qt+1+\sqrt{5}].$$

We are going to show that for an appropriate choice of q the inequality $f(t)\geq 0$ holds if $t\in[0,\varepsilon^{-1}]$. First we prove the inequality

$$\varphi(t)=(-qt+1-\sqrt{5})(\varepsilon-t)^2-qt+1+\sqrt{5}\geq 0$$

for any $t\in[0,\varepsilon^{-1}]$ and some q specified later. We rewrite $\varphi(t)$ in a different form:

$$\begin{aligned}\varphi(t)&=(-2\varepsilon^{-1}-qt)(\varepsilon^2-2\varepsilon t+t^2)-qt+2\varepsilon\\&=-2\varepsilon-\varepsilon^2qt+4t+2\varepsilon qt^2-2\varepsilon^{-1}t^2-qt^3-qt+2\varepsilon\\&=-qt^3+2(\varepsilon q-\varepsilon^{-1})t^2+(4-q(\varepsilon^2+1))t\\&=-t(qt^2+2(\varepsilon^{-1}-\varepsilon q)t+q(\varepsilon^2+1)-4).\end{aligned}$$

Hence it is enough to show that $qt^2+2(\varepsilon^{-1}-\varepsilon q)t+q(\varepsilon^2+1)-4\leq 0$. The roots of this polynomial are

$$\begin{aligned}t_{1,2}&=\frac{2(\varepsilon q-\varepsilon^{-1})\pm\sqrt{4(\varepsilon q-\varepsilon^{-1})^2-4q(q(\varepsilon^2+1)-4)}}{2q}\\&=\frac{\varepsilon q-\varepsilon^{-1}\pm\sqrt{\varepsilon^2q^2-2q+\varepsilon^{-2}-q^2(\varepsilon^2+1)+4q}}{q}\\&=\frac{\varepsilon q-\varepsilon^{-1}\pm\sqrt{-q^2+2q+\varepsilon^{-2}}}{q}.\end{aligned}$$

We want to choose a $q>0$ such the discriminant $-q^2+2q+\varepsilon^{-2}$ is positive, one of the roots above is non-positive and the other one is greater than ε^{-1}. Clearly, for such a q the inequality $\varphi(t)\geq 0$ will hold on the interval $[0,\varepsilon^{-1}]$. Let us begin with the condition

$$-q^2+2q+\varepsilon^{-2}>0\Longleftrightarrow q^2-2q-\varepsilon^{-2}<0, \tag{1.23}$$

the roots of the latter polynomial are $1\pm\sqrt{1+\varepsilon^{-2}}$ hence (1.23) holds for $0.53<q<1.48$ (and so for every p that comes into question by the earlier results). We also need

$$\varepsilon q-\varepsilon^{-1}\leq\sqrt{-q^2+2q+\varepsilon^{-2}},$$

and because of the assumption $0 < \varepsilon^{-2} < 0.58 < q$ both sides are non-negative, and then this is equivalent to

$$\varepsilon^2 q^2 - 2q + \varepsilon^{-2} \le -q^2 + 2q + \varepsilon^{-2},$$

$$(\varepsilon^2+1)q^2 - 4q = q((\varepsilon^2+1)q - 4) \le 0,$$

that is, to $q \le 4/(\varepsilon^2+1)$. Finally we need the following:

$$\frac{\varepsilon q - \varepsilon^{-1} + \sqrt{-q^2+2q+\varepsilon^{-2}}}{q} \ge \varepsilon^{-1},$$

$$(\varepsilon - \varepsilon^{-1})q - \varepsilon^{-1} + \sqrt{-q^2+2q+\varepsilon^{-2}} = q - \varepsilon^{-1} + \sqrt{-q^2+2q+\varepsilon^{-2}} \ge 0.$$

If $q > \varepsilon^{-1}$, then this holds.

So far we have seen that $\varphi(t) \ge 0$ if $0.3 \le p \le 0.49$, and since $b > 1/2$ we have

$$f(t) \ge -qt(-qt+1)(-qt+2)(\varepsilon - t) - 4t(qt^2 + 2(\varepsilon^{-1} - \varepsilon q)t + q(\varepsilon^2+1) - 4) = \tilde{f}(t).$$

We will show that for a certain q the following holds for $t \in (0, \varepsilon^{-1}]$:

$$\tilde{f}(t)/t = F(t) = q(t-\varepsilon)(qt-1)(qt-2) - 4(qt^2 + 2(\varepsilon^{-1} - \varepsilon q)t + q(\varepsilon^2+1) - 4) \ge 0.$$

We do this as follows:

$$\begin{aligned} F(t) &= (qt - \varepsilon q)(q^2t^2 - 3qt + 2) - 4qt^2 + 8(\varepsilon q - \varepsilon^{-1})t + 16 - 4q(\varepsilon^2+1) \\ &= q^3t^3 - \varepsilon q^3 t^2 - 3q^2t^2 + 3\varepsilon q^2 t + 2qt - 2\varepsilon q - 4qt^2 + 8(\varepsilon q - \varepsilon^{-1})t + 16 - 4q(\varepsilon^2+1) \\ &= q^3t^3 - (\varepsilon q^3 + 3q^2 + 4q)t^2 + (3\varepsilon q^2 + 2q + 8(\varepsilon q - \varepsilon^{-1}))t + 16 - 4q(\varepsilon^2+1) - 2\varepsilon q. \end{aligned}$$

Now

$$F(0) = 16 - 4q(\varepsilon^2+1) - 2\varepsilon q = 16 - (2(5+\sqrt{5}) + 1 + \sqrt{5})q = 16 - (11 + 3\sqrt{5})q \ge 0$$

holds if and only if

$$q \le \frac{16}{11 + 3\sqrt{5}},$$

and this is true when $q \le 0.9$.

Moreover,

$$F'(t) = 3q^3t^2 - 2q(\varepsilon q^2 + 3q + 4)t + 3\varepsilon q^2 + 2q + 8(\varepsilon q - \varepsilon^{-1}).$$

We show that for $q = 0.9$ this is positive on $[0, \varepsilon^{-1}]$, this clearly completes the proof. It is enough to see that $F'(1) > 0$ but $F'(q^{-1}) < 0$, since F' is a quadratic polynomial with positive leading coefficient. Indeed,

$$\begin{aligned} F'(q^{-1}) &= 3q - 2(\varepsilon q^2 + 3q + 4) + 3\varepsilon q^2 + 2q + 8(\varepsilon q - \varepsilon^{-1}) \\ &= 3q - 2\varepsilon q^2 - 6q - 8 + 3\varepsilon q^2 + 2q + 8(\varepsilon q - \varepsilon^{-1}) \\ &= \varepsilon q^2 - q + 8(\varepsilon q - \varepsilon^{-1} - 1) = \varepsilon q^2 - q + 8\varepsilon(q-1) \end{aligned}$$

$$= q(\varepsilon q - 1) + 8\varepsilon(q-1) < 0.9 \cdot (\varepsilon - 1) - 0.8 \cdot \varepsilon = 0.1 \cdot \varepsilon - 0.9 < 0,$$

but

$$\begin{aligned} F'(1) &= 3q^3 - 2q(\varepsilon q^2 + 3q + 4) + 3\varepsilon q^2 + 2q + 8(\varepsilon q - \varepsilon^{-1}) \\ &= (3-2\varepsilon)q^3 + (3\varepsilon - 6)q^2 - 6q + 8(\varepsilon q - \varepsilon^{-1}) > 0, \end{aligned}$$

what can be checked easily. □

Proof of Proposition 1.4.6. We set $q = 0.9$ but to make the proof readable we use the letter q instead of the value. The derivative of $g_1(b,c)$ is

$$\begin{aligned} g_1'(b,c) = 4 &\left(\left(\frac{q(c-\varepsilon)+3}{4} \right)^2 - \frac{5}{16} \right) \left(\frac{q(c-\varepsilon)+3}{4} \right) \cdot \frac{q}{4} + \\ &+ b \left[2 \left(\frac{q(c-\varepsilon)+3}{4} - \frac{\sqrt{5}}{4} \right) \cdot \frac{qc}{4} + \left(\frac{q(c-\varepsilon)+3}{4} - \frac{\sqrt{5}}{4} \right)^2 \right. \\ &\left. +2 \left(\frac{q(c-\varepsilon)+3}{4} + \frac{\sqrt{5}}{4} \right) \cdot \frac{q}{4c} - \left(\frac{q(c-\varepsilon)+3}{4} + \frac{\sqrt{5}}{4} \right)^2 \cdot \frac{1}{c^2} \right]. \end{aligned}$$

We would like to show that this is positive on an interval $[1+r, \varepsilon]$ for some $r \geq 0$. Multiplying the expression by $16c^2$ does not change its sign:

$$\begin{aligned} 16c^2 g_1'(b,c) = &\left((q(c-\varepsilon)+3)^2 - 5 \right) \left(\frac{q(c-\varepsilon)+3}{4} \right) qc^2 + \\ &+ b \left[2 \left(q(c-\varepsilon) + 2\varepsilon^{-2} \right) qc^3 + \left(q(c-\varepsilon) + 2\varepsilon^{-2} \right)^2 c^2 \right. \\ &\left. +2 \left(q(c-\varepsilon) + 2\varepsilon^2 \right) qc - \left(q(c-\varepsilon) + 2\varepsilon^2 \right)^2 \right]. \end{aligned}$$

This is a polynomial in c of degree 5. We define

$$A_1(c) = \left((q(c-\varepsilon)+3)^2 - 5 \right) \left(\frac{q(c-\varepsilon)+3}{4} \right) qc^2$$

and

$$\begin{aligned} B_1(c) = &\, 2 \left(q(c-\varepsilon) + 2\varepsilon^{-2} \right) qc^3 + \left(q(c-\varepsilon) + 2\varepsilon^{-2} \right)^2 c^2 \\ &+ 2 \left(q(c-\varepsilon) + 2\varepsilon^2 \right) qc - \left(q(c-\varepsilon) + 2\varepsilon^2 \right)^2 \end{aligned}$$

such that $16c^2 g_1'(b,c) = A_1(c) + bB_1(c)$. Since $1 \leq c \leq \varepsilon$ we have

$$\begin{aligned} B_1(c) \leq &\, 2 \left(q(c-\varepsilon) + 2\varepsilon^{-2} \right) q\varepsilon^3 + \left(q(c-\varepsilon) + 2\varepsilon^{-2} \right)^2 \varepsilon^2 \\ &+ 2 \left(q(c-\varepsilon) + 2\varepsilon^2 \right) q\varepsilon - \left(q(c-\varepsilon) + 2\varepsilon^2 \right)^2. \end{aligned}$$

This is a quadratic polynomial of c but to simplify the computation a little bit we substitute $t = q(c-\varepsilon)$. Then $c \in [1, \varepsilon] \Leftrightarrow t \in [-q\varepsilon^{-1}, 0]$ and the previous expression can be written as

$$\begin{aligned}
\gamma_1(t) &= 2\left(t + 2\varepsilon^{-2}\right) q\varepsilon^3 + \left(t + 2\varepsilon^{-2}\right)^2 \varepsilon^2 + 2\left(t + 2\varepsilon^2\right) q\varepsilon - \left(t + 2\varepsilon^2\right)^2 \\
&= 2q\varepsilon^3 t + 4q\varepsilon + (t^2 + 4\varepsilon^{-2}t + 4\varepsilon^{-4})\varepsilon^2 + 2q\varepsilon t + 4q\varepsilon^3 - t^2 - 4\varepsilon^2 t - 4\varepsilon^4 \\
&= (\varepsilon^2 - 1)t^2 + (2q\varepsilon^3 + 4 + 2q\varepsilon - 4\varepsilon^2)t + 4q(\varepsilon^3 + \varepsilon) + 4\varepsilon^{-2} - 4\varepsilon^4 \\
&= \varepsilon t^2 + (2q\varepsilon(\varepsilon^2+1) - 4\varepsilon)t + 4q\varepsilon(\varepsilon^2+1) + 4\varepsilon^{-2} - 4\varepsilon^4.
\end{aligned}$$

Now

$$\frac{\gamma_1(0)}{4\varepsilon} = q(\varepsilon^2+1) + \varepsilon^{-3} - \varepsilon^3 = \frac{q(5+\sqrt{5})}{2} - 4 < \frac{\sqrt{5}-3}{2} < 0,$$

while

$$\begin{aligned}
\gamma_1(-q\varepsilon^{-1}) &= q^2\varepsilon^{-1} + q(4 - 2q(\varepsilon^2+1)) + 4\varepsilon\left(\frac{q(5+\sqrt{5})}{2} - 4\right) \\
&= q^2\varepsilon^{-1} + 4q - q^2(5+\sqrt{5}) + 2\varepsilon q(5+\sqrt{5}) - 16\varepsilon \\
&= q(q\varepsilon^{-1} + 4) + q(5+\sqrt{5})(2\varepsilon - q) - 16\varepsilon \\
&< \varepsilon^{-1} + 4 + (5+\sqrt{5})(0.1+\sqrt{5}) - 8(1+\sqrt{5}) = 1 - 2.4\sqrt{5} < 0,
\end{aligned}$$

so $\gamma_1(t)$ is negative on the whole interval $[-q\varepsilon^{-1}, 0]$. It follows that $B_1(c) < 0$ for $c \in [1, \varepsilon]$, and then

$$16c^2 g_1'(b,c) = A_1(c) + bB_1(c) > A_1(c) + 0.56B_1(c).$$

Hence it is enough to show that $A_1(c) + 0.56B_1(c) > 0$ for $c \in [1.48, \varepsilon]$. To avoid the work with complicated algebraic expressions we do this by the following way. We show that the polynomial $F_1(c) = A_1(c) + 0.56B_1(c) + 8.001$ has 5 roots, and therefore if x_0 is the biggest root, then F_1 must be strictly increasing on the interval $[x_0, \infty)$. We do all this by giving pairs c_1, c_2 of real numbers such that $c_1 < c_2$ and the sign of $F_1(c_1)$ and $F_1(c_2)$ is different. One checks easily (e.g. by a computer) that

$$F_1(-14) < 0, \quad F_1(-13) > 0, \quad F_1(-0.1) < 0, \quad F_1(0) > 0, \quad F_1(0.1) < 0, \qquad F_1(0.6) > 0.$$

Thus the function $A_1(c) + 0.56B_1(c)$ is strictly increasing for $c \geq 0.6$. On the other hand, for $c = 1.48$ its value is positive and hence the same is true for $c \geq 1.48$. □

Proof of Proposition 1.4.7. First we check that $1 \pm a > 0$:

$$\begin{aligned}
1 + c - \varepsilon^{-1} - \frac{1}{\sqrt{5}} &\geq 1 - \frac{1}{\sqrt{5}} > 0, \\
1 - c + \varepsilon^{-1} + \frac{1}{\sqrt{5}} &\geq \varepsilon^{-1} + \frac{1}{\sqrt{5}} > 0.
\end{aligned}$$

In the case when c is near to the endpoint ε one may omit a term in $H_a(c)$ to simplify the computations but now we would lose too much this way. Instead, we work with the function $H_a(c)$ and show that

$$\left(\frac{1-a}{1+a}\right)^2 c + \frac{1}{c} - \frac{1}{(1+a)^2} \geq 0$$

if $c \in [\varepsilon^{-1}, 1]$. Multiplying by $(1+a)^2 c$ we get

$$(1-a)^2c^2 + (1+a)^2 - c = \left(1 - c + \varepsilon^{-1} + \frac{1}{\sqrt{5}}\right)^2 c^2 + \left(1 + c - \varepsilon^{-1} - \frac{1}{\sqrt{5}}\right)^2 - c$$

$$\geq \left(\varepsilon^{-1} + \frac{1}{\sqrt{5}}\right)^2 c^2 - c + \left(1 - \frac{1}{\sqrt{5}}\right)^2.$$

The discriminant of this latter quadratic polynomial is

$$1 - 4\left(\varepsilon^{-1} + \frac{1}{\sqrt{5}}(1-\varepsilon^{-1}) - \frac{1}{5}\right)^2 = 1 - 4\left(\varepsilon^{-1} + \frac{1}{\sqrt{5}}\varepsilon^{-2} - \frac{1}{5}\right)^2 \approx -0.387,$$

so the polynomial does not have real roots. Since its leading coefficient is positive it takes positive values for every c, i.e. $H_a(c) \geq 0$ holds. $\square$

Proof of Proposition 1.4.8. It is enough to show that

$$\sqrt{5}a(5a^2-1) + 8b[(\sqrt{5}a - \sqrt{5})c + (\sqrt{5}a + \sqrt{5})c^{-1}] \leq 0,$$

and as $a = c - \varepsilon^{-1} - \frac{1}{\sqrt{5}}$, the left hand side above is

$$(\sqrt{5}(c-\varepsilon^{-1}) - 1)\sqrt{5}(c-\varepsilon^{-1})(\sqrt{5}q(c-\varepsilon^{-1}) - 2) +$$
$$+ 8b[(\sqrt{5}(c-\varepsilon^{-1}) - 1 - \sqrt{5})c + (\sqrt{5}(c-\varepsilon^{-1}) - 1 + \sqrt{5})c^{-1}].$$

Multiplying by c and substituting $t = c - \varepsilon^{-1}$ we get

$$f(t) = \sqrt{5}t(\sqrt{5}t - 1)(\sqrt{5}t - 2)(t + \varepsilon^{-1}) +$$
$$+ 8b[(\sqrt{5}t - 1 - \sqrt{5})(t+\varepsilon^{-1})^2 + \sqrt{5}t - 1 + \sqrt{5}].$$

We show that $f(t) \leq 0$ once $t \in [0, 1-\varepsilon^{-1}]$. Note that $1 - \varepsilon^{-1} = \varepsilon^{-2}$. First we check that

$$\varphi(t) := (\sqrt{5}t - 1 - \sqrt{5})(t + \varepsilon^{-1})^2 + \sqrt{5}t - 1 + \sqrt{5} \leq 0 \tag{1.24}$$

if $t \in [0, \varepsilon^{-2}]$. The function $\varphi(t)$ is a polynomial function of degree 3:

$$\varphi(t) = (\sqrt{5}t - 2\varepsilon)(t^2 + 2\varepsilon^{-1}t + \varepsilon^{-2}) + \sqrt{5}t + 2\varepsilon^{-1}$$
$$= \sqrt{5}t^3 + 2\varepsilon^{-1}\sqrt{5}t^2 + \varepsilon^{-2}\sqrt{5}t - 2\varepsilon t^2 - 4t - 2\varepsilon^{-1} + \sqrt{5}t + 2\varepsilon^{-1}$$
$$= \left[\sqrt{5}t^2 + 2(\varepsilon^{-1}\sqrt{5} - \varepsilon)t + (\sqrt{5}(\varepsilon^{-2}+1) - 4)\right] t = \tilde{\varphi}(t)t$$

with $\tilde{\varphi}(t) = \sqrt{5}t^2 + 2(\varepsilon^{-1}\sqrt{5} - \varepsilon)t + (\sqrt{5}(\varepsilon^{-2}+1) - 4)$. Now $\varphi(0) = 0$ and on the interval $(0, \varepsilon^{-2}]$ the sign of φ is the same as the sign of $\tilde{\varphi}$. One checks that $\tilde{\varphi}(0) < 0$ and

$$\varphi(\varepsilon^{-2}) = \varphi(1 - \varepsilon^{-1}) = 2\sqrt{5}\varepsilon^{-2} - 2 < 0,$$

so $\tilde{\varphi}(t)$ is negative for $t \in (0, \varepsilon^{-2}]$ and then so is $\varphi(t)$, hence (1.24) is proved.

As $b > 0.5$ we have

$$f(t) \leq \sqrt{5}t(\sqrt{5}t - 1)(\sqrt{5}t - 2)(t + \varepsilon^{-1}) + 4\varphi(t) =: \tilde{f}(t).$$

Since $\tilde{f}(0) = 0$, it is enough to show that $\tilde{f}'$ is negative on the interval $[0, \varepsilon^{-2}]$. Now the first term of $\tilde{f}$ is

$$\begin{aligned}\psi(t) &= \sqrt{5}t(\sqrt{5}t - 1)(\sqrt{5}t - 2)(t + \varepsilon^{-1}) \\ &= (\sqrt{5}t^2 + \varepsilon^{-1}\sqrt{5}t)(5t^2 - 3\sqrt{5}t + 2) \\ &= 5\sqrt{5}t^4 + (5\sqrt{5}\varepsilon^{-1} - 15)t^3 + (2\sqrt{5} - 15\varepsilon^{-1})t^2 + 2\sqrt{5}\varepsilon^{-1}t,\end{aligned}$$

and

$$\psi'(t) = 20\sqrt{5}t^3 + 15(\sqrt{5}\varepsilon^{-1} - 3)t^2 + 2(2\sqrt{5} - 15\varepsilon^{-1})t + 2\sqrt{5}\varepsilon^{-1},$$

while

$$4\varphi'(t) = 12\sqrt{5}t^2 + 16(\sqrt{5}\varepsilon^{-1} - \varepsilon)t + 4(\sqrt{5}(\varepsilon^{-2} + 1) - 4),$$

so $\tilde{f}'(t) = \psi'(t) + 4\varphi'(t)$. One may check that

$$\begin{aligned}0.862 &\approx \tilde{f}'(-0.5) > 0, \\ -0.875 &\approx \tilde{f}'(0) < 0, \\ -3.118 &\approx \tilde{f}'(\varepsilon^{-2}) < 0,\end{aligned}$$

hence the assertion follows (because $\tilde{f}'$ is a polynomial function of degree 3 with positive leading coefficient). □

Proof of Proposition 1.4.9. It is enough to see that the derivative of $g_2(b, c)$ is negative. This derivative is

$$\begin{aligned}g_2'(b,c) = 4\Bigg(&\left(\left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4}\right)^2 - \frac{5}{16}\right)\left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4}\right)\cdot\frac{\sqrt{5}}{4} + \\ &+ b\left[2\left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4} - \frac{\sqrt{5}}{4}\right)\cdot\frac{\sqrt{5}c}{4} + \left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4} - \frac{\sqrt{5}}{4}\right)^2\right. \\ &\left. +2\left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4} + \frac{\sqrt{5}}{4}\right)\cdot\frac{\sqrt{5}}{4c} - \left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4} + \frac{\sqrt{5}}{4}\right)^2\cdot\frac{1}{c^2}\right].\end{aligned}$$

We will show that this function is negative on the interval $[\varepsilon^{-1}, \varepsilon^{-1}+r]$ for some $r>0$. we multiply by $16c^2$, this does not change the sign:

$$16c^2 g_2'(b,c) = \left(\left(\sqrt{5}(c-\varepsilon^{-1})-3\right)^2 - 5\right)\left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4}\right)\sqrt{5}c^2 +$$

$$+ b\left[2\left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^2\right)\sqrt{5}c^3 + \left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^2\right)^2 c^2\right.$$

$$\left. +2\left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^{-2}\right)\sqrt{5}c - \left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^{-2}\right)^2\right].$$

Like in an earlier proof, we define

$$A_2(c) = \left(\left(\sqrt{5}(c-\varepsilon^{-1})-3\right)^2 - 5\right)\left(\frac{\sqrt{5}(c-\varepsilon^{-1})-3}{4}\right)\sqrt{5}c^2,$$

$$B_2(c) = 2\left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^2\right)\sqrt{5}c^3 + \left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^2\right)^2 c^2$$

$$+ 2\left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^{-2}\right)\sqrt{5}c - \left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^{-2}\right)^2.$$

From now on we assume that $c\in[\varepsilon^{-1}, 0.68]$, so $c-\varepsilon^{-1} < 0.68-\varepsilon^{-1} < 0.062$, and then

$$\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^2 < 0, \qquad \sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^{-2} < 0,$$

therefore

$$B_2(c) \geq 2\left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^2\right)\sqrt{5}\cdot 0.68^3 + \left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^2\right)^2 \varepsilon^{-2}$$

$$+ 2\left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^{-2}\right)\sqrt{5}\cdot 0.68 - \left(\sqrt{5}(c-\varepsilon^{-1})-2\varepsilon^{-2}\right)^2.$$

This lower bound is a quadratic polynomial of c. We substitute $t=\sqrt{5}(c-\varepsilon^{-1})$, then $c\in[\varepsilon^{-1}, 0.68] \Leftrightarrow t\in[0, \sqrt{5}(0.68-\varepsilon^{-1})]$ and the latter expression can be written as

$$\gamma_2(t) = 2\left(t-2\varepsilon^2\right)\sqrt{5}\cdot 0.68^3 + \left(t-2\varepsilon^2\right)^2\varepsilon^{-2} + 2\left(t-2\varepsilon^{-2}\right)\sqrt{5}\cdot 0.68 - \left(t-2\varepsilon^{-2}\right)^2$$

$$= 2\cdot 0.68^3\sqrt{5}t - 4\cdot 0.68^3\sqrt{5}\varepsilon^2 + (t^2 - 4\varepsilon^2 t + 4\varepsilon^4)\varepsilon^{-2}$$

$$+ 1.36\sqrt{5}t - 2.72\sqrt{5}\varepsilon^{-2} - t^2 + 4\varepsilon^{-2}t - 4\varepsilon^{-4}$$

$$= -\varepsilon^{-1}t^2 + (2\cdot 0.68^3\sqrt{5} - 4 + 1.36\sqrt{5} + 4\varepsilon^{-2})t$$

$$+ (4 - 4\cdot 0.68^3\sqrt{5})\varepsilon^2 - 2.72\sqrt{5}\varepsilon^{-2} - 4\varepsilon^{-4}.$$

One may check that one of the roots of $\gamma_2(t)$ is greater than 3 and the other one is negative, and since the leading coefficient is also negative we get that $\gamma(t)$ is positive on the interval $[0, \sqrt{5}(0.68 - \varepsilon^{-1})]$. It follows that $B_2(c) > 0$ on $[\varepsilon^{-1}, 0.68]$ and hence

$$16c^2 g_2'(b,c) < A_2(c) + 0.56 B_2(c).$$

To see that this is negative we consider the function $F_2(c) = A_2(c) + 0.56B_2(c) + 2.5$. This is a polynomial of degree 5 with positive leading coefficient, and we have that

$$F_2(-0.4) < 0, \quad F_2(-0.3) > 0, \quad F_2(0) < 0, \quad F_2(0.2) > 0,$$
$$F_2(1.2) > 0, \quad F_2(1.3) < 0, \quad F_2(3) > 0.$$

This implies that F_2 has a root x_1 in $[0, 0.2]$ and another one in $[1.2, 1.3]$ denoted by x_2. Furthermore, F_2 is positive on (x_1, x_2), where it has exactly one local maximum taken at the point x_m, so F_2 is increasing on $[x_1, x_m]$, while it is decreasing on $[x_m, x_2]$. As $F_2(0.2) < F_2(0.7) < F_2(0.8)$ we get that $x_m > 0.7$ and hence F_2 is increasing on the interval $[0.2, 0.7]$, and so is $A_2(c) + 0.56B_2(c)$. Moreover, $A_2(0.7) + 0.56B(0.7) < 0$, therefore $g_2'(b,c) < 0$ on the interval $[\varepsilon^{-1}, 0.68]$. □

Chapter 2

Automorphic forms

In this chapter we introduce the notion of automorphic functions on $\mathbb{H}^2$ with respect to the group Γ_K. These are complex valued functions that are invariant under the action of Γ_K.

In the first section we examine the Fourier expansion of some special automoprhic fuctions, the so-called automorphic forms. These are smooth eigenfunctions of the Laplace operators, and also, certain restrictions on their growth are made. We derive basic estimates for them with special emphasis on square-integrable forms.

The Eisenstein series are introduced in the second section. These special examples of automophic forms have an essential role in the spectral decomposition of square-integrable automorphic functions (see Theorem 2.2.10). This section is basically a short summary of Chapter II of the book [5], though minor complements are added.

After that we define the automorphic kernel functions that are crucial in the remaining part of this work. In the last two sections further preparation is made for the next chapter: Proposition 7.2 and the formula (8.27) of [11] are generalized and some related results are given.

We remark that the results and estimates that are obtained in this chapter are not necessarily the best possible ones. Still, they provide sufficient tools for the work in Chapter 3.

2.1 Fourier expansion of automorphic forms

A function $f : \mathbb{H}^2 \to \mathbb{C}$ is called an *automorphic function* with respect to the Hilbert modular group Γ_K if it is invariant under the action of Γ_K, that is, $f(\gamma z) = f(z)$ holds for every $z \in \mathbb{H}^2$ and $\gamma \in \Gamma_K$.

A linear operator L acting on a vector space of functions $f : \mathbb{H}^2 \to \mathbb{C}$ is said to be *invariant* if it commutes with the action of the group $PSL(2, \mathbb{R})^2$, that is, if $L(f(\sigma z)) = (Lf)(\sigma z)$ holds for every $\sigma \in PSL(2, \mathbb{R})^2$. The *Laplace operators*

$$\Delta_k = y_k^2 \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} \right), \qquad (k = 1, 2)$$

act on the space of smooth automorhpic functions and they form a generating system in the algebra of the invariant differential operators on this space. An *automorphic form* u is a smooth automorphic function which is an eigenfunction of the Laplacians, that is, for which the equations

$$(\Delta_k + \lambda_k)u = 0$$

hold with some $\lambda_k \in \mathbb{C}$ $(k = 1, 2)$. Let $\mathcal{B}(\Gamma_K \setminus \mathbb{H}^2)$ denote the space of bounded smooth automorphic functions, and

$$\mathcal{D}(\Gamma_K \setminus \mathbb{H}^2) := \{f \in \mathcal{B}(\Gamma_K \setminus \mathbb{H}^2) : \Delta_k f \in \mathcal{B}(\Gamma_K \setminus \mathbb{H}^2),\ k = 1, 2\}.$$

Then $\mathcal{D}(\Gamma_K \setminus \mathbb{H}^2)$ is dense in $L^2(\Gamma_K \setminus \mathbb{H}^2)$, i.e. in the Hilbert space of the automorphic functions that are of square-integrable on F with respect to the measure

$$d\mu(z) = (y_1 y_2)^{-2}\, dx_1\, dy_1\, dx_2\, dy_2,$$

which is the product measure on $\mathbb{H}^2$ obtained from the usual measure $y^{-2} dx\, dy$ on the hyperbolic plane $\mathbb{H}$ derived from the Poincaré differential $ds = y^{-1}\, |dz|$.

The Laplacians are symmetric operators on $\mathcal{D}(\Gamma_K \setminus \mathbb{H}^2)$ and $-\Delta_k$ is non-negative. Hence by Friedrichs' theorem they have a unique self-adjoint extension to $L^2(\Gamma_K \setminus \mathbb{H}^2)$. It follows also that the *Laplace eigenvalues* $\lambda_k = s_k(1 - s_k)$ of an automorphic form $u \in \mathcal{D}(\Gamma_K \setminus \mathbb{H}^2)$ are real and non-negative. Therefore, either $s_k = \frac{1}{2} + ir_k$ for an $r_k \in \mathbb{R}$ or $0 \le s_k \le 1$ $(k = 1, 2)$.

If u is an automorphic form, then it is invariant under the action of the translation operator $T_\alpha u = u(z_1 + \alpha, z_2 + \alpha')$ for any $\alpha \in \mathcal{O}_K$. It is well known that the set

$$L_K := \{(\alpha, \alpha') : \alpha \in \mathcal{O}_K\} \subset \mathbb{R}^2$$

is a discrete additive subgroup of $\mathbb{R}^2$ of rank 2, i.e. a lattice. So for any fixed $y_1, y_2 > 0$ the function $u_{y_1,y_2} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $u_{y_1,y_2}(x_1, x_2) := u(x_1 + iy_1, x_2 + iy_2)$ is a smooth function which is invariant under translations by the elements of L_K, hence it has the Fourier expansion

$$u(z) = \sum_{l \in L_K^*} \phi(y, l) e^{2\pi i <l,x>},$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $L_K^* = \{v \in \mathbb{R}^2 : \langle v, w \rangle \in \mathbb{Z}$ for any $w \in L_K\}$ is the *dual lattice* of L_K. The elements of L_K^* can be given in terms of L_K:

Proposition 2.1.1. *If $\Lambda = A(\mathbb{Z}^n) \subset \mathbb{R}^n$ is a lattice, where $A \in GL(\mathbb{R}^n)$, then its dual lattice is given by $\Lambda^* = (A^{-1})^T(\mathbb{Z}^n)$.*

Proof. If $v \in \Lambda' = (A^{-1})^T(\mathbb{Z}^n)$, then $v = (A^{-1})^T u_1$ for some $u_1 \in \mathbb{Z}^n$. Similarly, for a $w \in \Lambda$ one has $w = Au_2$ for some $u_2 \in \mathbb{Z}^n$, hence

$$\langle v, w \rangle = \left\langle (A^{-1})^T u_1, Au_2 \right\rangle = \left\langle u_1, A^{-1} A u_2 \right\rangle = \langle u_1, u_2 \rangle \in \mathbb{Z},$$

so $\Lambda' \subset \Lambda^*$.

On the other hand, assume that $v \in \Lambda^*$. If $v' = A^T v$ and $u \in \mathbb{Z}^n$ arbitrary, then

$$\langle v', u \rangle = \left\langle v', A^{-1} A u \right\rangle = \langle v, Au \rangle \in \mathbb{Z},$$

hence $v' \in \mathbb{Z}^n$ must hold and then $v = (A^{-1})^T v' \in \Lambda'$. □

Since $L_K = A(\mathbb{Z}^2)$, where A is defined in (1.3) on page 6, we can give the elements of L_K^* explicitly. They are of the form

$$(l_1, l_2) = \frac{1}{\sqrt{d(K)}}(n - m\beta', -n + m\beta),$$

where $n, m \in \mathbb{Z}$ and $\beta = \sqrt{d_K}$ if $d_K \equiv 2, 3 \pmod 4$, and $\beta = \frac{1+\sqrt{d_K}}{2}$ if $d_K \equiv 1 \pmod 4$. In other words,

$$L_K^* = \{((\omega^{-1}l), (\omega^{-1}l)') : l \in \mathcal{O}_K\}, \tag{2.1}$$

where $\omega = 2\sqrt{d_K}$ if $d_K \equiv 2, 3 \pmod 4$, and $\omega = \sqrt{d_K}$ if $d_K \equiv 1 \pmod 4$.

For a number $\alpha \in K$ and a lattice $\Lambda \subset \mathbb{R}^2$ we define

$$\alpha\Lambda = \{(\alpha l_1, \alpha' l_2) : (l_1, l_2) \in \Lambda\}.$$

From (2.1) we infer that $uL_K^* = L_K^*$ for any $u \in \mathcal{O}_K^\times$.

The Fourier coefficients of an automorphic form can be expressed by means of the modified Bessel function of the second kind, denoted by $K_\nu(z)$ (see Theorem 5.1 in [16]):

Theorem 2.1.2. *Let u be an automorphic form with Lapalace eigenvalues $s_k(1-s_k)$ which satisfies the growth condition $u(z) = o(e^{2\pi y_k})$ as $y_k \to \infty$ ($k = 1, 2$). Then u admits a Fourier expansion of the form*

$$u(z) = \sum_{l \in L_K^*} a_l(y) e^{2\pi i <l,x>}, \tag{2.2}$$

where

$$a_l(y) = c_l \sqrt{y_1 y_2} K_{s_1 - 1/2}(2\pi |l_1| y_1) K_{s_2 - 1/2}(2\pi |l_2| y_2)$$

for $l \neq 0$, while $a_0(y)$ is the linear combination of $y_1^{s_1} y_2^{s_2}$, $y_1^{s_1} y_2^{1-s_2}$, $y_1^{1-s_1} y_2^{s_2}$ and $y_1^{1-s_1} y_2^{1-s_2}$, except for the case $s_1 = s_2 = \frac{1}{2}$, when $a_0(y)$ is the linear combination of $(y_1 y_2)^{\frac{1}{2}}$ and $\log(y_1 y_2)(y_1 y_2)^{\frac{1}{2}}$.

In the following we always assume that an automorphic form u satisfies the growth condition of Theorem 2.1.2 and hence admits the Fourier expansion (2.2). Since u is invariant under the action of the element $\rho = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}$, we have

$$\begin{aligned} u(z) = u(\rho z) &= a_0(\varepsilon^2 y_1, \varepsilon^{-2} y_2) + \sum_{l \in L_K^* \setminus 0} a_l(\varepsilon^2 y_1, \varepsilon^{-2} y_2) e^{2\pi i <(l_1, l_2), (\varepsilon^2 x_1, \varepsilon^{-2} x_2)>} \\ &= a_0(\varepsilon^2 y_1, \varepsilon^{-2} y_2) + \sum_{l \in L_K^* \setminus 0} c_l \sqrt{y_1 y_2} K_{s_1 - 1/2}(2\pi \left|\varepsilon^2 l_1\right| y_1) K_{s_2 - 1/2}(2\pi \left|\varepsilon^{-2} l_2\right| y_2) \times \\ &\qquad \times e^{2\pi i <(\varepsilon^2 l_1, \varepsilon^{-2} l_2), (x_1, x_2)>}. \end{aligned}$$

As $\varepsilon^2 L_K^* = L_K^*$ and the Fourier coefficients are determined uniquely, we get

Proposition 2.1.3. *Let u be an automorphic form with Fourier expansion (2.2). Then $c_{\varepsilon^2 l} = c_l$ for every $l \in L_K^* \setminus 0$ (where $\varepsilon^2 l = (\varepsilon^2 l_1, \varepsilon^{-2} l_2)$).*

We also have that $a_0(y_1, y_2) = a_0(\varepsilon^2 y_1, \varepsilon^{-2} y_2)$. One may deduce from this by a calculation that if $s_1 \neq \frac{1}{2} \neq s_2$ holds, then only two of the four terms mentioned in the theorem can occur in $a_0(y)$ except for in at most countably many possible cases. In the following we choose the notation so that

$$\boxed{\operatorname{Re} s_1, \operatorname{Re} s_2 \geq \frac{1}{2} \text{ is assumed}}$$

unless it is told otherwise, and in the important cases $a_0(y) = \eta y_1^{s_1} y_2^{s_2} + \phi y_1^{1-s_1} y_2^{1-s_2}$ will always hold for some $\eta, \phi \in \mathbb{C}$, hence from now on we disregard other cases. The notations η and ϕ will be fixed and (at least if they refer to complex numbers) they will always denote the coefficients in $a_0(y)$.

Assume that $a_0(y)$ is nonzero. Then $a_0(y_1, y_2) = a_0(\varepsilon^2 y_1, \varepsilon^{-2} y_2)$ gives

$$\eta\varepsilon^{2(s_1-s_2)} y_1^{s_1} y_2^{s_2} + \phi\varepsilon^{2(s_2-s_1)} y_1^{1-s_1} y_2^{1-s_2} = \eta y_1^{s_1} y_2^{s_2} + \phi y_1^{1-s_1} y_2^{1-s_2}. \tag{2.3}$$

We handle the case when $\eta \neq 0$, the case when $\phi \neq 0$ is similar. We multiply (2.3) by $y_1^{s_1-1} y_2^{s_2-1}$ to get

$$\eta\varepsilon^{2(s_1-s_2)} y_1^{2s_1-1} y_2^{2s_2-1} + \phi\varepsilon^{2(s_2-s_1)} = \eta y_1^{2s_1-1} y_2^{2s_2-1} + \phi,$$

$$(\varepsilon^{2(s_1-s_2)} - 1)\eta y_1^{2s_1-1} y_2^{2s_2-1} = \phi(1 - \varepsilon^{2(s_2-s_1)}).$$

The right hand side of the last equation is constant hence so is the left hand side. If at least one of s_1 and s_2 is not $1/2$ then $\varepsilon^{2(s_1-s_2)} = 1$ must hold. Also, if $s_1 = s_2 = 1/2$, then $\varepsilon^{2(s_1-s_2)} = 1$ holds anyway. We have

Proposition 2.1.4. *Let u be an automorphic form with Lapalace eigenvalues $s_k(1-s_k)$ for $k = 1, 2$. If the zeroth Fourier coefficient of u is nonzero, then $\log\varepsilon(s_1 - s_2) \equiv 0$ modulo πi. In other words,*

$$(s_1, s_2) = \left(s + \frac{\pi i m}{2\log\varepsilon}, s - \frac{\pi i m}{2\log\varepsilon}\right),$$

for $s = \frac{s_1+s_2}{2}$ and some $m \in \mathbb{Z}$.

The dominant term of the Fourier expansion is the zeroth coefficient. To estimate the remaining part we need the asymptotic behavior of the Bessel function $K_\nu(y)$. It is known that

$$K_\nu(y) = \left(\frac{\pi}{2y}\right)^{\frac{1}{2}} e^{-y} \left(1 + O\left(\frac{1+|\nu|^2}{y}\right)\right) \tag{2.4}$$

for $y > 1 + |\nu|^2$ (see formula (B.36) in [11]). First we use this to derive an upper bound for the Fourier coeffitients of an automorphic form u. By (2.1) and Proposition 2.1.3 we have

$$u(z) - a_0(y) = \sum_{0 \neq (l) \lhd \mathcal{O}_K} \left(c_l S_l(z) + c_{\varepsilon l} S_{\varepsilon l}(z) + c_{-l} S_{-l}(z) + c_{-\varepsilon l} S_{-\varepsilon l}(z)\right), \tag{2.5}$$

where for an $l \in \mathcal{O}_K$ we set $c_l = c_{(l\omega^{-1}, (l\omega^{-1})')}$ and $S_l(z)$ is

$$\sum_{k=-\infty}^{\infty} \sqrt{y_1 y_2} K_{s_1-\frac{1}{2}}(2\pi\varepsilon^{2k} \left|l\omega^{-1}\right| y_1) K_{s_2-\frac{1}{2}}(2\pi\varepsilon^{-2k} \left|(l\omega^{-1})'\right| y_2) e^{2\pi i(\varepsilon^{2k} l\omega^{-1} x_1 + \varepsilon^{-2k}(l\omega^{-1})' x_2)} \tag{2.6}$$

for every $0 \neq l \in \mathcal{O}_K$. As we sum over ideals, we may choose l so that $\varepsilon^{-2} \leq |l| / |l'| < \varepsilon^2$ holds. As the Fourier series converges absolutely, we may drop some terms in (2.6) and keep only those ones where $k = 0$, which gives that the sum

$$\sum_{\substack{0 \neq (l) \lhd \mathcal{O}_K \\ \varepsilon^{-2} \leq |l|/|l'| < \varepsilon^2}} |c_l| \sqrt{y_1 y_2} \left| K_{s_1-\frac{1}{2}}(2\pi \left|l\omega^{-1}\right| y_1) K_{s_2-\frac{1}{2}}(2\pi \left|(l\omega^{-1})'\right| y_2) \right|$$

converges for every $y_1, y_2 > 0$, hence the sequence of the terms tends to 0 if $|N(l)| \to \infty$.

Let us fix a small $\delta > 0$ and set $y_1 = y_2 = \frac{\delta\sqrt{d(K)}}{4\pi\varepsilon}$. As

$$\varepsilon^{-1}\sqrt{|N(l)|} \le |l|, |l'| < \varepsilon\sqrt{|N(l)|} \tag{2.7}$$

by our choice, (2.4) gives that if the absolute value of the norm of l is big enough, then

$$\sqrt{y_1 y_2}\left|K_{s_1-\frac{1}{2}}(2\pi\left|l\omega^{-1}\right| y_1)K_{s_2-\frac{1}{2}}(2\pi\left|(l\omega^{-1})'\right| y_2)\right| \gg e^{-\frac{\delta}{2\varepsilon}|l|}e^{-\frac{\delta}{2\varepsilon}|l'|} > e^{-\delta\sqrt{|N(l)|}},$$

therefore

$$c_l \ll e^{\delta\sqrt{|N(l)|}} \tag{2.8}$$

holds, where the implied constant depends on u. One can see this for $c_{\varepsilon l}$, c_{-l} and $c_{-\varepsilon l}$ similarly.

We will need an upper bound for $K_\nu(y)$ also in those cases when y is small. The following statement was formulated in this form by András Biró.

Lemma 2.1.5. *Assume that $\frac{1}{2} \le \operatorname{Re} s \le B$ for some constant $B > 0$. Then there are constants $C > 0$ and $d > 0$ such that*

$$K_{s-\frac{1}{2}}(y) \ll e^{-dy}$$

holds whenever $y > C\,|s|$, and the implied constant depends on B and C. On the other hand, we have

$$K_{s-\frac{1}{2}}(y) \ll \left(\frac{|s|}{y}\right)^{\operatorname{Re} s+\frac{1}{2}} e^{-\frac{\pi}{2}|s|}$$

for any $y > 0$, where the implied constant depends on B.

The first estimate follows easily from the the integral representation

$$K_\nu(z) = \pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})^{-1}\left(\frac{z}{2}\right)^\nu \int_1^\infty (t^2-1)^{\nu-1/2}e^{-tz}\,dt$$

while for the second estimate one can use the representation

$$K_\nu(z) = \pi^{-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})\left(\frac{z}{2}\right)^{-\nu} \int_0^\infty (t^2+1)^{-\nu-1/2}\cos(tz)\,dt$$

and integration by parts. Note that these formulae hold if $\operatorname{Re} z > 0$ and $\operatorname{Re}\nu > -1/2$.

Proposition 2.1.6. *Let u be an automorphic form with Laplace eigenvalues $s_k(1-s_k)$ that satisfies the growth condition $u(z) = o(e^{2\pi y_k})$ for $k = 1, 2$. If y_2 is bounded from below by a positive costant, then $u(z) - a_0(y) = O(y_1^{-\operatorname{Re} s_1})$ as $y_1 \to \infty$ (where $a_0(y)$ is the zeroth Fourier coefficient of u). The implied constant depends on the field K, the function u and the lower bound on y_2. An analogous statement holds if we replace the roles of y_1 and y_2.*

Proof. We only prove the first statement, the second one is similar. Assume that y_2 is bounded from below by B. We use (2.5) and hence first estimate $S_l(z)$ (defined in (2.6)).

As above, we may assume that $\varepsilon^{-2} \le |l|/|l'| < \varepsilon^2$ and hence (2.7) hold. Now

$$2\pi\varepsilon^{-2k}\left|(l\omega^{-1})'\right| y_2 \ge \frac{\varepsilon^{-1}\sqrt{N(l)}}{\sqrt{d(K)}}B\varepsilon^{-2k} \ge \frac{\varepsilon^{-1}B}{\sqrt{d(K)}}\varepsilon^{-2k} \ge C|s_2|$$

holds if

$$\varepsilon^{2k} \le \frac{B}{\varepsilon C|s_2|\sqrt{d(K)}}.$$

Here C is the constant from the previous lemma. On the other hand, if this latter inequality does not hold, then

$$2\pi\varepsilon^{2k}|l\omega^{-1}|y_1 > \frac{\varepsilon^{-1}\sqrt{N(l)}By_1}{\varepsilon C|s_2|d(K)} \ge \frac{By_1}{\varepsilon^2 C|s_2|d(K)} \ge C|s_1|$$

is true once

$$y_1 > \frac{\varepsilon^2 C^2|s_1 s_2|d(K)}{B}.$$

So let us set

$$N := \log\left(\frac{B}{\varepsilon C|s_2|\sqrt{d(K)}}\right)/(2\log\varepsilon),$$

and if $k \ge N$, then the previous lemma gives

$$K_{s_1-\frac{1}{2}}(2\pi\left|l\omega^{-1}\right|\varepsilon^{2k}y_1)K_{s_2-\frac{1}{2}}(2\pi\left|(l\omega^{-1})'\right|\varepsilon^{-2k}y_2) \ll$$
$$\ll e^{-d\varepsilon^{2k}|l|y_1}\left(\frac{\varepsilon^{2k}}{|l'|y_2}\right)^{\operatorname{Re}s_2+\frac{1}{2}} \ll e^{-d'\varepsilon^{2k}\sqrt{|N(l)|}y_1}\left(\frac{\varepsilon^{2k}}{y_2}\right)^{\operatorname{Re}s_2+\frac{1}{2}}$$

where the implied constant depends on u, while if $k < N$, then

$$K_{s_1-\frac{1}{2}}(2\pi\left|l\omega^{-1}\right|\varepsilon^{2k}y_1)K_{s_2-\frac{1}{2}}(2\pi\left|(l\omega^{-1})'\right|\varepsilon^{-2k}y_2) \ll$$
$$\ll \left(\frac{\varepsilon^{-2k}}{|l|y_1}\right)^{\operatorname{Re}s_1+\frac{1}{2}} e^{-d\varepsilon^{-2k}|l'|y_2} \ll \left(\frac{\varepsilon^{-2k}}{y_1}\right)^{\operatorname{Re}s_1+\frac{1}{2}} e^{-d'\varepsilon^{-2k}\sqrt{|N(l)|}y_2}.$$

As $\sqrt{N(l)}y_1$ is bounded from below by a positive constant, the exponential factor absorbs the power of ε^{2k} in the first estimate. The analogous claim holds in the second estimate as well since $\sqrt{N(l)}y_2$ is also bounded from below. We obtain that

$$S_l(z) \ll \sum_{k\ge N} y_1^{\frac{1}{2}} y_2^{-\operatorname{Re}s_2} e^{-d''\varepsilon^{2k}\sqrt{|N(l)|}y_1} + \sum_{k<N} y_1^{-\operatorname{Re}s_1} y_2^{\frac{1}{2}} e^{-d''\varepsilon^{-2k}\sqrt{|N(l)|}y_2}$$

Now we use that if $k \ge N$, then $\varepsilon^{2k}\sqrt{N(l)}$ is bounded from below while if $k < N$, then $\varepsilon^{-2k}\sqrt{|N(l)|}$ is bounded from below, and we can estimate from above by

$$B^{-\operatorname{Re}s_2}\sum_{k\ge N} e^{-d_3\varepsilon^{2k}\sqrt{|N(l)|}y_1} + y_1^{-\operatorname{Re}s_1}\sum_{k<N} e^{-d''\varepsilon^{-2k}\sqrt{|N(l)|}y_2}.$$

We rewrite the first sum as $\sum_{k=0}^{\infty} e^{-d_3\varepsilon^{2\lceil N\rceil}\varepsilon^{2k}\sqrt{|N(l)|}y_1}$. Since $k \ll \varepsilon^{2k}$ for a positive k, we obtain

$$\sum_{k=0}^{\infty} e^{-d_3\varepsilon^{2\lceil N\rceil}\varepsilon^{2k}\sqrt{|N(l)|}y_1} \ll e^{-d_4\varepsilon^{2\lceil N\rceil}\sqrt{|N(l)|}y_1}\sum_{k=0}^{\infty}(e^{-d_4\varepsilon^{2\lceil N\rceil}\sqrt{|N(l)|}y_1})^k$$
$$= e^{-d_4\varepsilon^{2\lceil N\rceil}\sqrt{|N(l)|}y_1}\frac{1}{1-e^{-d_4\varepsilon^{2\lceil N\rceil}\sqrt{|N(l)|}y_1}} \ll e^{-d_5\sqrt{|N(l)|}y_1}$$

since $\varepsilon^{2\lceil N\rceil}$ and $\sqrt{|N(l)|}y_1$ is bounded from below by a positive constant.

Similarly,

$$\sum_{k<N} e^{-d''\varepsilon^{-2k}\sqrt{|N(l)|}y_2} = \sum_{k<\lceil N\rceil} e^{-d''\varepsilon^{-2k}\sqrt{|N(l)|}y_2} = \sum_{k=1}^{\infty} e^{-d''\varepsilon^{2\lceil N\rceil}\varepsilon^{2k}\sqrt{|N(l)|}y_2}$$
$$\ll e^{-d_6\varepsilon^{2\lceil N\rceil}\sqrt{|N(l)|}y_2} \frac{1}{1-e^{-d_4\varepsilon^{2\lceil N\rceil}\sqrt{|N(l)|}y_1}} \ll e^{-d_7\sqrt{|N(l)|}},$$

and we summarize this in

$$S_l(z) \ll e^{-d\sqrt{|N(l)|}y_1} + y_1^{-\operatorname{Re}s_1}e^{-d'\sqrt{|N(l)|}} \ll y_1^{-\operatorname{Re}s_1}e^{-c\sqrt{|N(l)|}}. \tag{2.9}$$

We have seen in (2.8) that $c_l \ll e^{\delta\sqrt{|N(l)|}}$ holds for every $\delta > 0$, hence the right hand side of (2.9) is in fact an upper bound for $c_l S_l(z)$ (with different constants of course). The same argument applies for the terms $c_{\varepsilon l}S_{\varepsilon l}(z)$, $c_{-l}S_{-l}(z)$ and $c_{-\varepsilon l}S_{-\varepsilon l}(z)$ in (2.5), hence

$$u(z) - a_0(y) \ll y_1^{-\operatorname{Re}s_1} \sum_{0\neq(l)\triangleleft\mathcal{O}_K} e^{-c\sqrt{|N(l)|}} = y_1^{-\operatorname{Re}s_1}\sum_{n=1}^{\infty} a_n e^{-c\sqrt{n}}$$

where a_n is the number of ideals (l) with $N((l)) = |N(l)| = n$. This number can be expressed as a sum $\sum_{b|n}\chi_{d(K)}(b)$, where $\chi_{d(K)}$ is a quadratic character modulo $d(K)$ (see [12], Section 9.3 and 9.5). This means that $|a_n| \leq \tau(n)$, where $\tau(n)$ denotes the number of the divisors of n. It is known that $\tau(n) \ll_\delta n^\delta$ for any $\delta > 0$ (see e.g. [8], Chapter XVIII) and hence

$$u(z) - a_0(y) \ll y_1^{-\operatorname{Re}s_1}\sum_{n=1}^{\infty} n^\delta e^{-c\sqrt{n}} \ll y_1^{-\operatorname{Re}s_1}\sum_{n=1}^{\infty} e^{-c'\sqrt{n}} \ll y_1^{-\operatorname{Re}s_1}$$

if $y_1 > M$ for some $M > 0$. The proof shows that the constant M and the implied constant in the estimate above depends on the field K, u and the lower bound on y_2. □

It follows from the previous proposition that once an automorphic form satisfies the growth condition $u(z) = o(e^{2\pi y_k})$ it will be automatically of polynomial growth. We remark that by following the proof of Lemma 2.8.6 in [3] one easily gets a better upper bound for the Fourier coefficients than the trivial bound (2.8). Namely, if $u(z) = O(y_k^\alpha)$ for some $\alpha \geq 0$ and $k = 1, 2$, then $c_l = O(|N(l)|^{\frac{\alpha+1}{2}})$ for every $l \in \mathcal{O}_K \setminus \{0\}$ (where the constant depends on the field K and u).

We will investigate now the automorphic forms in the space $L^2(\Gamma_K \setminus \mathbb{H}^2)$. Since they are square integrable on the fundamental domain F, it follows that if their zeroth Fourier coefficient is nonzero, then it must be of the form $\phi y_1^{1-s_1}y_2^{1-s_2}$, where $\frac{1}{2} < s_1, s_2 \leq 1$. Now Proposition 2.1.4 gives that $s_1 = s_2$ must hold in this case. The automorphic forms with vanishing zeroth coefficient are called *cusp forms*. It is known that $\operatorname{Re}s_k = \frac{1}{2}$ for a cusp form. To a further analysis we need the following

Lemma 2.1.7. *If $l \in L_K^*$, then*

$$\iint\limits_{-\frac{1}{2}\leq X_1,X_2<\frac{1}{2}} e^{2\pi i<l,x>}\,dx_1\,dx_2 = \begin{cases} \sqrt{d(K)}, & \text{if } l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It can easily be checked by a computation. □

Now suppose that $u \in L^2(\Gamma_K \setminus \mathbb{H}^2)$ is a non-constant automorphic form with eigenvalues $s_k(1-s_k)$, where $\operatorname{Re} s_k \geq 1/2$ $(k=1,2)$. We give a two dimensional analogue of the argument in the proof of Theorem 3.2 in [11]. We are going to estimate u in terms of $Y_0 = y_1 y_2$ whenever $B_1 < \frac{y_1}{y_2} < B_2$ holds for some constants $0 < B_1 < 1$ and $1 < B_2$. Note that this implies $B_1 y_2 < y_1 < B_2 y_2$ and $B_2^{-1} y_1 < y_2 < B_1^{-1} y_1$. By the previous lemma we get

$$|a_0(y)|^2 + \sum_{l \in L_K^* \setminus 0} |a_l(y)|^2 = \frac{1}{\sqrt{d(K)}} \iint\limits_{-\frac{1}{2} \leq X_1, X_2 < \frac{1}{2}} |u(z)|^2 \, dx_1 \, dx_2, \tag{2.10}$$

where $a_0(y)$ and $a_l(y)$ are the Fourier coefficients of u given in Theorem 2.1.2. The formula above is in fact Parseval's identity. Now we integrate both sides of (2.10) over

$$P(A) = \{-\frac{1}{2} \leq Y_1 < \frac{1}{2}, \, Y_0 \geq A\}$$

with respect to the measure $\frac{dy_1 \, dy_2}{y_1^2 y_2^2} = 2 \log \varepsilon \, \frac{dY_0}{Y_0^2} \, dY_1$. Note that $y_1 = Y_0^{\frac{1}{2}} \varepsilon^{2Y_1}$ and $y_2 = Y_0^{\frac{1}{2}} \varepsilon^{-2Y_1}$, hence the integral of the left hand side is

$$2\,|\phi|^2 \log \varepsilon \frac{A^{1-(s_1+s_2)}}{s_1+s_2-1} +$$

$$+\, 2 \log \varepsilon \sum_{l \in L_K^* \setminus 0} |c_l|^2 \int\limits_A^\infty \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} Y_0 K^2_{s_1 - \frac{1}{2}}(2\pi \, |l_1| \, Y_0^{\frac{1}{2}} \varepsilon^{2Y_1}) K^2_{s_2 - \frac{1}{2}}(2\pi \, |l_2| \, Y_0^{\frac{1}{2}} \varepsilon^{-2Y_1}) \, dY_1 \, \frac{dY_0}{Y_0^2}. \tag{2.11}$$

Here the first term comes from the integral of the zeroth coefficient, and if u is a cusp form, then this is simply 0 since $\phi = 0$. Otherwise $s := s_1 = s_2$ and we simply integrate Y_0^{-2s}. We also used that $K_{s_k - \frac{1}{2}}(y)$ is real if $y \in \mathbb{R}^+$. This follows from the integral representation

$$K_\nu(z) = \int_0^\infty e^{-z \cosh t} \cosh(\nu t) \, dt$$

which holds for $\operatorname{Re} z > 0$ and $\operatorname{Re} \nu > -\frac{1}{2}$, taking into account that $s_k - \frac{1}{2}$ is either real or purely imaginary. Every orbit $\{\gamma z : \gamma \in \Gamma_K\}$ has at most $1 + C_K A^{-2}$ points in

$$P(A) \cap \{-1/2 \leq X_1, X_2 < 1/2\}$$

by Lemma 1.2.5, hence (2.11) is bounded from above by $d(K)^{-\frac{1}{2}} \|u\|^2 (1 + C_K A^{-2})$. Recall that for an $l \in \mathcal{O}_K \setminus 0$ the coefficient c_l was defined after (2.5) as the Fourier coefficient $c_{(l\omega^{-1}, (l\omega^{-1})')}$. We can rearrange the sum in (2.11) as we did in (2.5) to obtain

$$\sum_{0 \neq (l) \triangleleft \mathcal{O}_K} |c_l|^2 \Sigma_l + |c_{\varepsilon l}|^2 \Sigma_{\varepsilon l} + |c_{-l}|^2 \Sigma_{-l} + |c_{-\varepsilon l}|^2 \Sigma_{-\varepsilon l},$$

where

$$\Sigma_l = \sum_{k=-\infty}^{\infty} \int_A^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} Y_0 K^2_{s_1-\frac{1}{2}}(2\pi\varepsilon^{2k}\left|l\omega^{-1}\right| Y_0^{\frac{1}{2}}\varepsilon^{2Y_1}) K^2_{s_2-\frac{1}{2}}(2\pi\varepsilon^{-2k}\left|(l\omega^{-1})'\right| Y_0^{\frac{1}{2}}\varepsilon^{-2Y_1})\, dY_1\, \frac{dY_0}{Y_0^2}.$$

Substituting $Y_1' = Y_1 - k$ the inner integral adds up to an integral on $\mathbb{R}$, and in fact we get the same integral for Σ_l, $\Sigma_{\varepsilon l}$, Σ_{-l} and $\Sigma_{-\varepsilon l}$.

For an $l \in \mathcal{O}_K \setminus 0$ we set

$$a_l = |c_l|^2 + |c_{\varepsilon l}|^2 + |c_{-l}|^2 + |c_{-\varepsilon l}|^2 . \tag{2.12}$$

With this notation, the sum in (2.11) above equals 4 times

$$\sum_{0\neq(l)\lhd\mathcal{O}_K} a_l \int_A^{\infty} \int_{-\infty}^{\infty} Y_0 K^2_{s_1-\frac{1}{2}}(2\pi\left|l\omega^{-1}\right| Y_0^{\frac{1}{2}}\varepsilon^{2Y_1}) K^2_{s_2-\frac{1}{2}}(2\pi\left|(l\omega^{-1})'\right| Y_0^{\frac{1}{2}}\varepsilon^{-2Y_1})\, dY_1\, \frac{dY_0}{Y_0^2}.$$

Now we fix an $N > 0$ and omit the zeroth term and the terms with $|N(l)| > N$ from (2.11) to get a lower estimate. Then by the substitution $u_1 = 2\pi\left|l\omega^{-1}\right| y_1$ and $u_2 = 2\pi\left|(l\omega^{-1})'\right| y_2$ we obtain

$$\sum_{\substack{0\neq(l)\\ |N(l)|\leq N}} a_l \iint\limits_{u_1u_2\geq\frac{(2\pi)^2|N(l)|A}{d(K)}} K^2_{s_1-\frac{1}{2}}(u_1) K^2_{s_2-\frac{1}{2}}(u_2)\, \frac{du_1\, du_2}{u_1u_2} \ll \|u\|^2\left(1+\frac{C_K}{A^2}\right).$$

Moreover, we note that

$$\{(u_1,u_2)\in(\mathbb{R}^+)^2 : u_1\geq A_1,\, u_2\geq A_2,\, A_1A_2\geq A\} \subset \{(u_1,u_2)\in(\mathbb{R}^+)^2 : u_1u_2\geq A\}$$

and hence we infer

$$\sum_{\substack{0\neq(l)\\ |N(l)|\leq N}} a_l \int_{A_1}^{\infty} K^2_{s_1-\frac{1}{2}}(u_1)\, \frac{du_1}{u_1} \int_{A_2}^{\infty} K^2_{s_2-\frac{1}{2}}(u_2)\, \frac{du_2}{u_2} \ll \|u\|^2\left(1+\frac{C_K}{A^2}\right).$$

once $A_1A_2 \geq \frac{(2\pi)^2|N(l)|A}{d(K)}$. On the other hand, the following lower bound holds for the integral of $K_{s-\frac{1}{2}}(y)$ (see the proof of Theorem 3.2 in [11]):

$$\int_{|s|/2}^{\infty} K^2_{s-\frac{1}{2}}(y)\, \frac{dy}{y} \gg |s|^{-1}\, e^{-\pi|s|}. \tag{2.13}$$

Note that since $|s_k| \geq \frac{1}{2}$, we have $\frac{\sqrt{|s_k|}}{2\sqrt{2}} \leq \frac{|s_k|}{2}$, so choosing $A_k = \frac{\sqrt{|s_k|}}{2\sqrt{2}}$ with

$$A = \frac{d(K)\sqrt{|s_1s_2|}}{2(4\pi)^2N}$$

we conclude

$$\sum_{\substack{0\neq(l)\\|N(l)|\leq N}} a_l \ll \|u\|^2 \left(|s_1s_2| + N^2\right) e^{\pi(|s_1|+|s_2|)}, \tag{2.14}$$

where the implied constant depends on the field K.

Remark that if the zeroth term exists, then $s_1 = s_2 = s \in (1/2, 1]$, and choosing $A = 1$ we get $\phi \ll \|u\|(s - \frac{1}{2})^{\frac{1}{2}}$.

We will use (2.14) to estimate $u(z)$. First note that

$$|u(z) - a_0(y)| \leq \sum_{0\neq(l)\lhd\mathcal{O}_K} (|c_l||S_l(z)| + |c_{\varepsilon l}||S_{\varepsilon l}(z)| + |c_{-l}||S_{-l}(z)| + |c_{-\varepsilon l}||S_{-\varepsilon l}(z)|)$$

holds by (2.5). We only estimate the sum $\sum_{0\neq(l)} |c_l||S_l(z)|$, the other terms can be estimated similarly. That is, we work with the sum

$$\sum_{0\neq(l)} |c_l|\sqrt{y_1y_2} \sum_{k=-\infty}^{\infty} \left|K_{s_1-\frac{1}{2}}(2\pi\left|\varepsilon^{2k}l\omega^{-1}\right| y_1)\right| \left|K_{s_2-\frac{1}{2}}(2\pi\left|(\varepsilon^{2k}l\omega^{-1})'\right| y_2)\right|, \tag{2.15}$$

where we may choose l so that $\varepsilon^{-2} \leq \frac{|l|}{|l'|} < \varepsilon^2$ and hence $\varepsilon^{-1}\sqrt{|N(l)|} \leq |l|, |l'| \leq \varepsilon\sqrt{|N(l)|}$ hold.

First we assume that $L := \frac{\varepsilon C^2 d(K)}{B_1B_2^{-1}} \frac{|s_1s_2|^2}{y_1y_2} \geq 1$, where C is at least the constant denoted by the same letter in Lemma 2.1.5. This means that y_1y_2 is smaller than a constant times $|s_1s_2|^2$. We separate the sum in (2.15) and first handle the terms that belong to those ideals for which $|N(l)| > L$ holds. In this case, if $k \geq 0$ in the inner sum, then

$$2\pi\left|\varepsilon^{2k}l\omega^{-1}\right| y_1 \geq 2\varepsilon^{-1}\sqrt{|N(l)|}\sqrt{d(K)}^{-1}\sqrt{B_1}\sqrt{y_1y_2} \geq 2\sqrt{B_2}C\,|s_1s_2| \geq C\,|s_1|,$$

while if $k < 0$, then

$$2\pi\left|(\varepsilon^{2k}l\omega^{-1})'\right| y_2 \geq 2\varepsilon^{-1}\sqrt{|N(l)|}\sqrt{d(K)}^{-1}\sqrt{B_2^{-1}}\sqrt{y_1y_2} \geq 2\sqrt{B_1^{-1}}C\,|s_1s_2| \geq C\,|s_2|.$$

In the first case we use the following estimates (given by Lemma 2.1.5):

$$K_{s_1-\frac{1}{2}}(2\pi\left|\varepsilon^{2k}l\omega^{-1}\right| y_1) \ll e^{-\frac{2\pi\varepsilon^{2k}\sqrt{|N(l)|}d}{\varepsilon\sqrt{d(K)}}y_1} \leq e^{-D\varepsilon^{2k}\sqrt{|N(l)|y_1y_2}}$$

$$K_{s_2-\frac{1}{2}}(2\pi\left|(\varepsilon^{2k}l\omega^{-1})'\right| y_2) \ll \left(\frac{\varepsilon^{2k}\,|s_2|}{\sqrt{|N(l)|y_2}}\right)^{\operatorname{Re} s_2+\frac{1}{2}} e^{-\frac{\pi}{2}|s_2|} \ll \left(\frac{\varepsilon^{2k}}{\sqrt{|N(l)|\,y_1y_2}}\right)^{\operatorname{Re} s_2+\frac{1}{2}}$$

for some constants $d, D > 0$. We get an analogous bound for a negative k by switching the roles of the variables. Since $\frac{1}{2} \leq \operatorname{Re} s_2 \leq 1$ and therefore $|N(l)|\,y_1y_2$ is bounded from below by a positive constant, we estimate the sum over k's by

$$\sum_{k=0}^{\infty} \varepsilon^{2k} e^{-D\varepsilon^{2k}\sqrt{|N(l)|y_1y_2}} \ll \sum_{k=0}^{\infty} e^{-D'\varepsilon^{2k}\sqrt{|N(l)|y_1y_2}}.$$

Also, $k \ll \varepsilon^{2k}$ holds, hence we can bound the sum above by

$$e^{-D'\sqrt{|N(l)|y_1y_2}} + \sum_{k=1}^{\infty}\left(e^{-D'\sqrt{|N(l)|y_1y_2}}\right)^k = e^{-D'\sqrt{|N(l)|y_1y_2}}\left(1+\frac{1}{1-e^{-D'\sqrt{|N(l)|y_1y_2}}}\right)$$
$$\ll e^{-D'\sqrt{|N(l)|y_1y_2}},$$

again, since $\sqrt{|N(l)|\,y_1y_2}$ is bounded from below by a positive constant.

Now we turn to the case when $|N(l)| \le L$. Here the tails of the inner sum will be small again, but we make a different upper estimate for the central terms. By central terms, we mean the ones for which $\varepsilon^{\pm 4k}\,|N(l)| \le L$ (i.e. $|k| \le \frac{\log(L/|N(l)|)}{4\log\varepsilon}$) holds. Then the tails can be bounded by $e^{-c\sqrt{Ly_1y_2}}$ similarly as before. For the other terms we simply use the bounds

$$K_{s_1-\frac{1}{2}}(2\pi\left|\varepsilon^{2k}l\omega^{-1}\right|y_1) \ll \left(\frac{|s_1|}{\varepsilon^{2k}\sqrt{|N(l)|}y_1}\right)^{\mathrm{Re}\,s_1+\frac{1}{2}} e^{-\frac{\pi}{2}|s_1|},$$
$$K_{s_2-\frac{1}{2}}(2\pi\left|(\varepsilon^{2k}l\omega^{-1})'\right|y_2) \ll \left(\frac{\varepsilon^{2k}\,|s_2|}{\sqrt{|N(l)|}y_2}\right)^{\mathrm{Re}\,s_2+\frac{1}{2}} e^{-\frac{\pi}{2}|s_2|}.$$

As $\mathrm{Re}\,s := \mathrm{Re}\,s_1 = \mathrm{Re}\,s_2$, we infer

$$\sum_{k=-\infty}^{\infty}\left|K_{s_1-\frac{1}{2}}(2\pi\left|\varepsilon^{2k}l\omega^{-1}\right|y_1)\right|\left|K_{s_2-\frac{1}{2}}(2\pi\left|(\varepsilon^{2k}l\omega^{-1})'\right|y_2)\right| \ll$$
$$\ll \log\left(\frac{L}{|N(l)|}\right)\left(\frac{|s_1s_2|}{|N(l)|\,y_1y_2}\right)^{\mathrm{Re}\,s+\frac{1}{2}} e^{-\frac{\pi}{2}(|s_1|+|s_2|)} + e^{-c\sqrt{Ly_1y_2}}.$$

So far, we have that $\sum_{0\neq(l)}|c_l||S_l(z)|$ is bounded by

$$\sum_{0<|N(l)|\le L}|c_l|\sqrt{y_1y_2}\left[M_u(l,y_1,y_2)+e^{-c\sqrt{Ly_1y_2}}\right] + \sum_{|N(l)|>L}|c_l|\sqrt{y_1y_2}e^{-c\sqrt{|N(l)|y_1y_2}}, \tag{2.16}$$

where

$$M_u(l,y_1,y_2) = \log\left(\frac{L}{|N(l)|}\right)\left(\frac{|s_1s_2|}{|N(l)|\,y_1y_2}\right)^{\mathrm{Re}\,s+\frac{1}{2}} e^{-\frac{\pi}{2}(|s_1|+|s_2|)}.$$

We apply Cauchy's inequality for the first sum:

$$M^2 := \left(\sum_{0<|N(l)|\le L}|c_l|\sqrt{y_1y_2}M_u(l,y_1,y_2)\right)^2 \le \sum_{0<|N(l)|\le L}|c_l|^2\sum_{0<|N(l)|\le L}y_1y_2M_u(l,y_1,y_2)^2$$

As the number of the ideals with a given norm n is $O(|n|^\delta)$ for any $\delta > 0$ (see the proof of Proposition 2.1.6), we obtain

$$\sum_{0<|N(l)|\le L}y_1y_2M_u(l,y_1,y_2)^2 \ll (\log^2 L)\,|s_1s_2|^{2\mathrm{Re}\,s+1}\,(y_1y_2)^{-2\mathrm{Re}\,s}e^{-\pi(|s_1|+|s_2|)}\sum_{0<n\le L}n^{\delta-2\mathrm{Re}\,s-1}$$
$$\ll |s_1s_2|^{2\mathrm{Re}\,s+1}\,(y_1y_2)^{-2\mathrm{Re}\,s}e^{-\pi(|s_1|+|s_2|)}L^{\delta-2\mathrm{Re}\,s}.$$

On the other hand,

$$\sum_{0<|N(l)|\le L} |c_l|^2 \ll \sum_{0<|N(l)|\le L} a_l \ll \|u\|^2 e^{\pi(|s_1|+|s_2|)}(|s_1 s_2| + L^2),$$

by (2.14) and then

$$M^2 \ll \|u\|^2 \left(|s_1 s_2|^{2\mathrm{Re}\, s+2} (y_1 y_2)^{-2\mathrm{Re}\, s} L^{\delta-2\mathrm{Re}\, s} + |s_1 s_2|^{2\mathrm{Re}\, s+1} (y_1 y_2)^{-2\mathrm{Re}\, s} L^{2+\delta-2\mathrm{Re}\, s} \right)$$

As $L^{2+\delta-2\mathrm{Re}\, s} \ll |s_1 s_2|^{4-4\mathrm{Re}\, s+2\delta} (y_1 y_2)^{2\mathrm{Re}\, s-2-\delta}$, the second term is bounded by

$$|s_1 s_2|^{5-2\mathrm{Re}\, s+2\delta} (y_1 y_2)^{-2-\delta}.$$

while the first term can be bounded by

$$|s_1 s_2|^{2-2\mathrm{Re}\, s+2\delta} (y_1 y_2)^{-\delta}.$$

This can be expressed in terms of L:

$$M^2 \ll \|u\|^2 (|s_1 s_2|^{2-2\mathrm{Re}\, s} L^{\delta} + |s_1 s_2|^{1-2\mathrm{Re}\, s} L^{2+\delta}).$$

Note that for a cusp form this bound is $\|u\|^2(|s_1 s_2| L^{\delta} + L^{2+\delta})$, and this is a correct bound even if $s_1 = s_2$ are in the section $[1/2, 1]$.

Next, it follows from (2.14) that

$$\begin{aligned}\left(\sum_{0<|N(l)|\le L} |c_l| \right)^2 &\ll \|u\|^2 (|s_1 s_2| + L^2) e^{\pi(|s_1|+|s_2|)} \sum_{0<|N(l)|\le L} 1 \\ &\ll \|u\|^2 |s_1 s_2| L^{3+\delta} e^{\pi(|s_1|+|s_2|)},\end{aligned}$$

so

$$e^{-c\sqrt{L y_1 y_2}} \sqrt{y_1 y_2} \sum_{0<|N(l)|\le L} |c_l| \ll \|u\| \, |s_1 s_2|^{\frac{1}{2}} \sqrt{y_1 y_2} e^{-c\sqrt{L y_1 y_2}} e^{\frac{\pi}{2}(|s_1|+|s_2|)} L^{\frac{3}{2}+\delta}.$$

If the constant C in the definition of L is big enough, then the factor $e^{-c\sqrt{L y_1 y_2}} = e^{-c'|s_1 s_2|}$ absorbs the powers of $|s_1 s_2|$ and also the factor $e^{\frac{\pi}{2}(|s_1|+|s_2|)}$, hence we get the upper bound $\|u\|(y_1 y_2)^{-1-\delta}$ that is smaller than the bound for the central terms.

Finally, we investigate the sum

$$\sum_{|N(l)|>L} |c_l| \sqrt{y_1 y_2} e^{-c\sqrt{|N(l)| y_1 y_2}}. \tag{2.17}$$

We use the trivial bound (obtained from (2.14))

$$c_l \ll \|u\| \, |s_1 s_2|^{\frac{1}{2}} \, |N(l)| \, e^{\frac{\pi}{2}(|s_1|+|s_2|)},$$

and since $|N(l)|\, y_1y_2$ is bounded from below, we can estimate (2.17) by

$$\|u\|\, |s_1s_2|^{\frac{1}{2}}\, e^{\frac{\pi}{2}(|s_1|+|s_2|)} \sum_{|N(l)|>L} |N(l)|^{\frac{1}{2}}\, e^{-c'\sqrt{|N(l)|y_1y_2}} \ll$$

$$\ll \|u\|\, |s_1s_2|^{\frac{1}{2}}\, e^{\frac{\pi}{2}(|s_1|+|s_2|)} \sum_{n>L} n^{\frac{1}{2}+\delta} e^{-c'\sqrt{ny_1y_2}}$$

The function $x^{\frac{1}{2}+\delta} e^{-c'(y_1y_2)^{\frac{1}{2}}x^{\frac{1}{2}}}$ is decreasing if $x \geq \lfloor L \rfloor$ once the constant C in the definition of L is big enough (this can be seen by examining its derivative). Hence

$$\sum_{n>L} n^{\frac{1}{2}+\delta} e^{-c'\sqrt{ny_1y_2}} \leq \int_{\lfloor L \rfloor}^{\infty} x^{\frac{1}{2}+\delta} e^{-c'(y_1y_2)^{\frac{1}{2}}x^{\frac{1}{2}}}\, dx \leq \int_{L/2}^{\infty} x^{\frac{1}{2}+\delta} e^{-c'(y_1y_2)^{\frac{1}{2}}x^{\frac{1}{2}}}.$$

One may use integration by parts to show that this last integral is bounded from above by $e^{-c|s_1s_2|}(y_1y_2)^{-\frac{3}{2}-\delta}$. Again, if the constant C is big enough, then $e^{-c|s_1s_2|}$ absorbs the factor $e^{\frac{\pi}{2}(|s_1|+|s_2|)}$ and all the powers of $|s_1s_2|$, and we infer that (2.17) is bounded by $\|u\|(y_1y_2)^{-\frac{3}{2}-\delta}$ (and the implied constant depends only on the field K).

Now assume that y_1y_2 is bounded from below by a constant. This is the case when $L < 1$ and then only the terms in (2.17) occur on the right hand side of (2.16). Then the bound

$$\|u\|\, |s_1s_2|^{\frac{1}{2}}\, e^{\frac{\pi}{2}(|s_1|+|s_2|)} \sum_{n>L} n^{\frac{1}{2}+\delta} e^{-c\sqrt{ny_1y_2}}$$

is still valid, but now as $c\sqrt{y_1y_2}$ is bounded from below, we can estimate this sum by

$$\sum_{n>L} e^{-c'\sqrt{ny_1y_2}} \leq e^{-\frac{c'}{2}\sqrt{Ly_1y_2}} \sum_{n>L} e^{-\frac{c'}{2}\sqrt{ny_1y_2}} \leq e^{-C'|s_1s_2|} \sum_{n>0} e^{-\frac{c'}{2}\sqrt{ny_1y_2}},$$

and as before, $e^{-C'|s_1s_2|}$ absorbs the factor $|s_1s_2|^{\frac{1}{2}}\, e^{\frac{\pi}{2}(|s_1|+|s_2|)}$. As above, one can see that the last sum bounded by $e^{-d\sqrt{y_1y_2}}$ for some d. We have proved the following:

Theorem 2.1.8. *Let $u \in L^2(\Gamma_K \setminus \mathbb{H}^2)$ an automorphic form with eigenvalues $s_k(1-s_k)$, where $\operatorname{Re} s_k \geq \frac{1}{2}$ $(k = 1, 2)$, and let $a_0(y)$ be the zeroth Fourier coefficient of u. Assume that $z \in \mathbb{H}^2$ is a point for which $0 < B_1 < y_1/y_2 < B_2$ holds for some constants $0 < B_1 < 1$ and $B_2 > 1$. Then there is a constant C_K depending only on B_1, B_2 and the field K such that if $L := C_K \frac{|s_1s_2|^2}{y_1y_2} \geq 1$, then for any $\delta > 0$ we have*

$$u(z) - a_0(y) \ll \|u\| \left\{ (|s_1s_2|\, L^{\delta} + L^{2+\delta})^{\frac{1}{2}} + (y_1y_2)^{-\frac{3}{2}-\delta} \right\},$$

where the implied constant depends only on δ, B_1, B_2 and the field K. Moreover, if y_1y_2 is bounded from below by a constant, then the term $(y_1y_2)^{-\frac{3}{2}-\delta}$ in the estimate above can be omitted. Also, if $L < 1$, then

$$u(z) - a_0(y) \ll \|u\| e^{-d\sqrt{y_1y_2}}.$$

for some constant $d > 0$.

Observe that if $u \notin L^2(\Gamma_K \setminus \mathbb{H}^2)$ but satisfies the requirements of Theorem 2.1.2, then an analogue of the argument in the last paragraph above the theorem is still valid. The major difference in this case is that the estimate in (2.14) cannot be applied, but we may use the trivial bound in (2.8) to estimate the Fourier coefficients of u. We conclude that if $B_1 < y_1/y_2 < B_2$ holds and y_1y_2 is big enough, then $u(z) - a_0(y) \ll e^{-d\sqrt{y_1y_2}}$, where the implied constant and the lower bound on y_1y_2 depends on u, the field K, B_1 and B_2.

2.2 Eisenstein series

In this section we introduce an important family of automorphic forms, the *Eisenstein series*, and give a few basic results about them. Most of them will be stated without a proof, the details can be found for example in Chapter II of the book [5].

The function $y_k^{s_k}$ is an eigenfunction of Δ_k with eigenvalue $\lambda_k = s_k(1-s_k)$, and also, the Laplace operator commutes with the group action. Hence if $s_1, s_2 \in \mathbb{C}$ are such numbers for which the function $y_1^{s_1}y_2^{s_2}$ is invariant under the action of Γ_∞, then the sum

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} y_1(\gamma z)^{s_1} y_2(\gamma z)^{s_2}$$

(at least if it converges) is invariant under the action of Γ_K and also an eigenfunction of Δ_k, that is, an automorphic form.

As the translations does not change the function $y_1^{s_1}y_2^{s_2}$, it is invariant under the action of Γ_∞ if and only if $\varepsilon^{2(s_1-s_2)} = 1$. Then, as in the previous section, we must have

$$(s_1, s_2) = \left(s + \frac{\pi i m}{2\log\varepsilon}, s - \frac{\pi i m}{2\log\varepsilon}\right)$$

for $s = \frac{s_1+s_2}{2}$ and for some $m \in \mathbb{Z}$.

For an $s \in \mathbb{C}$ and $m \in \mathbb{Z}$ we define the *Eisenstein series* as follows:

$$\begin{aligned} E(z,s,m) :&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} y_1(\gamma z)^{s+\frac{\pi i m}{2\log\varepsilon}} y_2(\gamma z)^{s-\frac{\pi i m}{2\log\varepsilon}} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} Y_0(\gamma z)^s \left(\frac{y_1(\gamma z)}{y_2(\gamma z)}\right)^{\frac{\pi i m}{2\log\varepsilon}} \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} Y_0(\gamma z)^s e^{2\pi i m Y_1(\gamma z)} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} Y_0(\gamma z)^s \lambda_m(\gamma z), \end{aligned}$$

where $\lambda_m(z) := e^{2\pi i m Y_1(z)}$ (it is a so-called *Grössencharacter-type exponential sum*, see [5], Section II.1). Regarding the convergence properties of the series above, we have the following (see Proposition II.1.8 and Corollary II.1.9 in [5]):

Proposition 2.2.1. *The Eisenstein series $E(z,s,m)$ converges absolutely for* $\operatorname{Re} s > 1$ *and uniformly on the compact subsets of this half-plane.*

Proposition 2.2.2. *If $s \in \mathbb{C}$,* $\operatorname{Re} s > 1$*, then*

$$E(z,s,m) - Y_0(z)^s\lambda_m(z) = E(z,s,m) - y_1^{s_1}y_2^{s_2} \to 0$$

once $y_k \to \infty$ and the other y coordinate is fixed ($k = 1,2$).

Hence by Theorem 2.1.2 the function $E(z,s,m)$ admits the Fourier expansion

$$E(z,s,m) = \sum_{l\in L_K^*} a_l(y,s,m) e^{2\pi i <l,x>},$$

where for an $l \in L_K^* \setminus 0$ we have

$$a_l(y,s,m) = \phi_l(s,m)\sqrt{y_1y_2}K_{s_1-1/2}(2\pi\,|l_1|\,y_1)K_{s_2-1/2}(2\pi\,|l_2|\,y_2)$$

for some $\phi_l(s,m) \in \mathbb{C}$, while the last proposition also shows that

$$a_0(y,s,m) = y_1^{s_1}y_2^{s_2} + \phi(s,m)y_1^{1-s_1}y_2^{1-s_2} = Y_0(z)^s\lambda_m(z) + \phi(s,m)Y_0(z)^{1-s}\lambda_{-m}(z)$$

for some $\phi(s,m) \in \mathbb{C}$. That is,

$$\begin{aligned} E(z,s,m) =& Y_0(z)^s\lambda_m(z) + \phi(s,m)Y_0(z)^{1-s}\lambda_{-m}(z) + \\ &+ \sum_{l \in L_K^* \setminus 0} \phi_l(s,m)\sqrt{y_1y_2}K_{s_1-1/2}(2\pi\,|l_1|\,y_1)K_{s_2-1/2}(2\pi\,|l_2|\,y_2)e^{2\pi i<l,x>}. \end{aligned} \tag{2.18}$$

The functions $\phi(s,m)$ and $\phi_l(s,m)$ can be determined explicitly:

$$\begin{aligned} \phi(s,m) &= \frac{\pi}{\sqrt{d(K)}}\frac{\zeta_K(2s-1,-m)}{\zeta_K(2s,-m)}\frac{\Gamma(s_1-\frac{1}{2})\Gamma(s_2-\frac{1}{2})}{\Gamma(s_1)\Gamma(s_2)} \\ &= \frac{\pi}{\sqrt{d(K)}}\frac{\zeta_K(2s-1,-m)}{\zeta_K(2s,-m)}\frac{\Gamma(s+\frac{\pi i m}{2\log\varepsilon}-\frac{1}{2})\Gamma(s-\frac{\pi i m}{2\log\varepsilon}-\frac{1}{2})}{\Gamma(s+\frac{\pi i m}{2\log\varepsilon})\Gamma(s-\frac{\pi i m}{2\log\varepsilon})}, \end{aligned}$$

$$\begin{aligned} \phi_l(s,m) &= \frac{4\pi^{2s}}{\sqrt{d(K)}}\frac{\sigma_{1-2s,-m}(l)}{\zeta_K(2s,-m)}\frac{|l_1|^{s_1-\frac{1}{2}}\,|l_2|^{s_2-\frac{1}{2}}}{\Gamma(s_1)\Gamma(s_2)} \\ &= \frac{4\pi^{2s}}{\sqrt{d(K)}}\frac{\sigma_{1-2s,-m}(l)}{\zeta_K(2s,-m)}\frac{|l_1|^{s+\frac{\pi i m}{2\log\varepsilon}-\frac{1}{2}}\,|l_2|^{s-\frac{\pi i m}{2\log\varepsilon}-\frac{1}{2}}}{\Gamma(s+\frac{\pi i m}{2\log\varepsilon})\Gamma(s-\frac{\pi i m}{2\log\varepsilon})}, \end{aligned}$$

where

$$\zeta_K(s,m) = \sum_{0 \neq (\alpha) \lhd \mathcal{O}_K} \left|\frac{\alpha}{\alpha'}\right|^{\frac{\pi i m}{\log\varepsilon}} |N(\alpha)|^{-s}$$

is a *Hecke L-function* (see [9]) and

$$\sigma_{s,m}(l) = \sum_{(c)|(l)\mathcal{D}} \left|\frac{c}{c'}\right|^{\frac{\pi i m}{\log\varepsilon}} |N(c)|^s .$$

Here $\mathcal{D}$ denotes the *different* of K, i.e. the inverse of the fractional ideal

$$\mathcal{D}^{-1} = \{\alpha \in K :\ \mathrm{tr}\,(\alpha\mathcal{O}_K) \subset \mathbb{Z}\}.$$

Note that $\zeta_K(\overline{s},-m) = \overline{\zeta_K(s,m)}$, $\sigma_{\overline{s},-m}(l) = \overline{\sigma_{s,m}(l)}$. Moreover, since $\overline{s+\frac{\pi i m}{2\log\varepsilon}} = \overline{s} - \frac{\pi i m}{2\log\varepsilon}$ and $\Gamma(\overline{s}) = \overline{\Gamma(s)}$ hold, we obtain that $\overline{\phi(s,m)} = \phi(\overline{s},-m)$ and $\overline{\phi_l(s,m)} = \phi_l(\overline{s},-m)$. Similarly, using that $\overline{K_\nu(y)} = K_{\overline{\nu}}(y)$ holds for a positive y (where $K_\nu(y)$ is the Bessel function) and that the Fourier coefficient belonging to $-l$ is the same as the one belonging to l we conclude $\overline{E(z,s,m)} = E(z,\overline{s},-m)$.

The functions $E(z, s, m)$ and $\phi(s, m)$ are holomorphic on the half-plane $\operatorname{Re} s > 1$, they can be continued meromorphically to the whole complex plane and satisfy the following functional equation:

$$E(z, 1-s, -m) = \phi(1-s, -m)E(z, s, m).$$

Regarding the poles one can say the following (see Proposition II.6.1 in [5]):

Proposition 2.2.3. *The functions $E(z, s, m)$ and $\phi(s, m)$ has no poles on the half-plane $\operatorname{Re} s > \frac{1}{2}$ except for finitely many in $(1/2, 1]$ if $m = 0$.*

The functions $E(z, s, m)$ are not in $L^2(\Gamma_K \setminus \mathbb{H}^2)$, hence we define the *truncated Eisenstein series* for any $A > 0$ by

$$E_A(z, s, m) := \begin{cases} E(z, s, m) - Y_0(z)^s \lambda_m(z) - \phi(s, m) Y_0(z)^{1-s} \lambda_{-m}(z), & \text{if } Y_0(z) > A \\ E(z, s, m) & \text{otherwise.} \end{cases}$$

For these we have the following (see Theorem II.7.2 in [5]):

Theorem 2.2.4 (Maass-Selberg)**.** *Let $s, s' \in \mathbb{C}$, $m, m' \in \mathbb{Z}$, and assume that $(s, m) \neq (s', m')$ and $(s, m) + (s', m') \neq (1, 0)$. Then*

$$\int_F E_A(z, s, m) E_A(z, s', m') \, d\mu(z) =$$

$$= 2\sqrt{d(K)} \log \varepsilon \left[\delta_{m,-m'} \frac{A^{s+s'-1} - \phi(s, m)\phi(s', m') A^{1-s-s'}}{s + s' - 1} + \right.$$

$$\left. + \delta_{m,m'} \frac{A^{s-s'}\phi(s', m') - A^{s'-s}\phi(s, m)}{s - s'} \right].$$

A few corollaries can be derived from this:

Corollary 2.2.5. *If $\phi(s, m)$ is holomorphic at s, then $E(z, s, m)$ is also holomorphic at s.*

Corollary 2.2.6. *If $s = \frac{1}{2} + it$, then $|\phi(s, m)|^2 = \phi(s, m)\phi(\overline{s}, -m) = 1$. Therefore the function $E(z, s, m)$ is holomorphic on the line $s = \frac{1}{2} + it$.*

Corollary 2.2.7. *The exceptional poles of $E(z, s, 0)$ are simple.*

Corollary 2.2.8. *The function $\phi(s, m)$ is bounded in the half-plane $\operatorname{Re} s \geq \frac{1}{2}$ if s is bounded away from the real line.*

Now we use Theorem 2.2.4 to determine the value of the integral

$$\int_F |E_A(z, s, m)|^2 \, d\mu(z)$$

in the case when $s = \frac{1}{2} + ir$ for some $r \in \mathbb{R}$. Note that the theorem does not apply directly here since at least one of the requirements $(s, m) \neq (s', m')$ and $(s, m) + (s', m') \neq (1, 0)$ is not fulfilled. Assume that the Eisenstein series $E(z, s, m)$ is holomorphic in the strip $1/2 < \sigma \leq \sigma_0$.

If $\sigma + ir$ is in this strip and $m \neq 0$, then Theorem 2.2.4 gives

$$\begin{aligned}\int_F |E_A(z, \sigma + ir, m)|^2 \, d\mu(z) &= \int_F E_A(z, \sigma + ir, m) E_A(z, \sigma - ir, -m) \, d\mu(z) = \\ &= 2 \log \varepsilon \sqrt{d(K)} \frac{A^{2\sigma - 1} - \phi(\sigma + ir, m)\phi(\sigma - ir, -m) A^{1-2\sigma}}{2\sigma - 1} \\ &= 2 \log \varepsilon \sqrt{d(K)} \frac{A^{2\sigma - 1} - \phi(\sigma + ir, m)\overline{\phi(\sigma + ir, m)} A^{1-2\sigma}}{2\sigma - 1}\end{aligned}$$

in this strip.

The power series of $\phi(s, m)$ around $1/2 + ir$ is

$$\phi(1/2 + ir, m) + \phi'(1/2 + ir, m)(s - 1/2 - ir) + \sum_{j=2}^{\infty} \frac{\phi^{(j)}(\frac{1}{2} + ir, m)}{j!} (s - 1/2 - ir)^j,$$

and as $|\phi(s, m)| = 1$ on the line $s = 1/2 + ir$, we have for such an s that

$$\begin{aligned}\overline{\phi'(s, m)} &= \lim_{t \to 0} \overline{\frac{\phi(s + it, m) - \phi(s, m)}{it}} \\ &= \lim_{t \to 0} \frac{\phi(s + it, m)^{-1} - \phi(s, m)^{-1}}{-it} \\ &= \lim_{t \to 0} -\frac{\phi(s, m) - \phi(s + it, m)}{it\phi(s + it, m)\phi(s, m)} \\ &= \lim_{t \to 0} \frac{\phi(s + it, m) - \phi(s, m)}{it} \cdot \frac{1}{\phi(s + it, m)\phi(s, m)} \\ &= \frac{\phi'(s, m)}{\phi(s, m)^2}.\end{aligned}$$

Therefore,

$$\phi(\sigma + ir, m) = \phi(1/2 + ir, m) + \phi'(1/2 + ir, m)(\sigma - 1/2) + \sum_{j=2}^{\infty} \frac{\phi^{(j)}(\frac{1}{2} + ir, m)}{j!} (\sigma - 1/2)^j,$$

$$\begin{aligned}\overline{\phi(\sigma + ir, m)} &= \overline{\phi(1/2 + ir, m)} + \overline{\phi'(1/2 + ir, m)}(\sigma - 1/2) + \sum_{j=2}^{\infty} \frac{\overline{\phi^{(j)}(\frac{1}{2} + ir, m)}}{j!} (\sigma - 1/2)^j \\ &= \phi(1/2 + ir, m)^{-1} + \frac{\phi'(\frac{1}{2} + ir, m)}{\phi(\frac{1}{2} + ir, m)^2}(\sigma - 1/2) + \sum_{j=2}^{\infty} \frac{\overline{\phi^{(j)}(\frac{1}{2} + ir, m)}}{j!} (\sigma - 1/2)^j,\end{aligned}$$

and hence

$$\phi(\sigma + ir, m)\overline{\phi(\sigma + ir, m)} = 1 + 2\frac{\phi'(\frac{1}{2} + ir, m)}{\phi(\frac{1}{2} + ir, m)}(\sigma - 1/2) + \sum_{j=2}^{\infty} a(j, r, m)(\sigma - 1/2)^j$$

and

$$\frac{A^{2\sigma-1}-\phi(\sigma+ir,m)\overline{\phi(\sigma+ir,m)}A^{1-2\sigma}}{2\sigma-1}=$$

$$=\frac{A^{2\sigma-1}-A^{1-2\sigma}}{2\sigma-1}-\frac{\phi'(\frac{1}{2}+ir,m)}{\phi(\frac{1}{2}+ir,m)}A^{1-2\sigma}-\frac{A^{1-2\sigma}}{2}\sum_{j=2}^{\infty}a(j,r,m)(\sigma-1/2)^{j-1}.$$

We conclude

$$\int_F |E_A(z,1/2+ir,m)|^2\,d\mu(z)=\lim_{\sigma\to\frac{1}{2}}\int_F |E_A(z,\sigma+ir,m)|^2\,d\mu(z)$$
$$=2\log\varepsilon\sqrt{d(K)}\left[2\log A-\frac{\phi'(\frac{1}{2}+ir,m)}{\phi(\frac{1}{2}+ir,m)}\right]. \tag{2.19}$$

If $m=0$ and $r\neq 0$, then Theorem 2.2.4 gives

$$\int_F |E_A(z,\sigma+ir,0)|^2\,d\mu(z)=$$
$$=2\log\varepsilon\sqrt{d(K)}\left[\frac{A^{2\sigma-1}-\phi(\sigma+ir,0)\phi(\sigma-ir,0)A^{1-2\sigma}}{2\sigma-1}+\right.$$
$$\left.+\frac{\phi(\sigma-ir,0)A^{2ri}-\phi(\sigma+ir,0)A^{-2ri}}{2ri}\right],$$

hence, taking the limit as $\sigma\to 1/2$ we infer

$$\int_F |E_A(z,1/2+ir,0)|^2\,d\mu(z)=$$
$$=2\log\varepsilon\sqrt{d(K)}\left[2\log A-\frac{\phi'(\frac{1}{2}+ir,0)}{\phi(\frac{1}{2}+ir,0)}+\frac{\phi(1/2-ir,0)A^{2ri}-\phi(1/2+ir,0)A^{-2ri}}{2ri}\right]. \tag{2.20}$$

Finally, if $m=0$ and $r=0$, then

$$\int_F |E_A(z,1/2,0)|^2\,d\mu(z)=\lim_{r\to 0}\int_F |E_A(z,1/2+ir,0)|^2\,d\mu(z)=$$
$$=2\log\varepsilon\sqrt{d(K)}\left[2\log A-\frac{\phi'(\frac{1}{2},0)}{\phi(\frac{1}{2},0)}\right]$$
$$+2\log\varepsilon\sqrt{d(K)}\lim_{r\to 0}\left[\frac{\phi(1/2-ir,0)A^{2ri}-\phi(1/2+ir,0)A^{-2ri}}{2ri}\right]$$
$$=2\log\varepsilon\sqrt{d(K)}\left[2\log A-\frac{\phi'(\frac{1}{2},0)}{\phi(\frac{1}{2},0)}+2\phi(1/2,0)\log A-\phi'(1/2,0)\right].$$

We note that $\phi(1/2,0) = \overline{\phi(1/2,0)}$ and hence $\phi(1/2,0)^2 = 1$ hold. This means that $\phi(1/2,0) = \phi(1/2,0)^{-1} = \pm 1$ and we can write

$$\int_F |E_A(z,1/2,0)|^2 \, d\mu(z) = 2\log\varepsilon\sqrt{d(K)}(2\log A - \phi'(1/2,0))(1+\phi(1/2,0)). \tag{2.21}$$

We mention another important basic result, that will be useful for us also in a later section (see Theorem II.8.1 in [5]):

Theorem 2.2.9 (Plancherel formula). *Assume that $f(t), g(t) \in C_0^\infty(0,\infty)$, $m \in \mathbb{Z}$ and let us define*

$$F_m(z) = \int_0^\infty f(t) E\left(z, 1/2+it, m\right) \, dt, \qquad G_m(z) = \int_0^\infty g(t) E\left(z, 1/2+it, m\right) \, dt.$$

Then $F_m, G_m \in L^2(\Gamma_K \setminus \mathbb{H}^2)$ and we have

$$\frac{1}{4\pi \log\varepsilon\sqrt{d(K)}} \int_F F_m(z)\overline{G_m(z)} \, d\mu(z) = \int_0^\infty f(t)\overline{g(t)} \, dt.$$

Finally, we turn to the decomposition of $L^2(\Gamma_K \setminus \mathbb{H}^2)$ into subspaces which are invariant under the action of Δ_k. The map $f \mapsto F_m$ defined in the theorem above extends to an isometry of $L^2(0,\infty)$ into $L^2(\Gamma_K \setminus \mathbb{H}^2)$. Let $\mathcal{E}_m(\Gamma_K \setminus \mathbb{H}^2)$ denote the subspace in $L^2(\Gamma_K \setminus \mathbb{H}^2)$ generated by the images of $L^2(0,\infty)$ under the maps $f \mapsto F_m$ and $f \mapsto F_{-m}$. Then the $\mathcal{E}_m(\Gamma_K \setminus \mathbb{H}^2)$'s are invariant subspaces which are orthogonal to each other for different m's. Let us define $\mathcal{E}(\Gamma_K \setminus \mathbb{H}^2) = \oplus_{m=0}^\infty \mathcal{E}_m(\Gamma_K \setminus \mathbb{H}^2)$, moreover let us denote the subspace of cusp forms by $\mathcal{C}(\Gamma_K \setminus \mathbb{H}^2)$. The residues of $E(z,s,0)$ in $(1/2,1]$ are also automorphic forms which are in $L^2(\Gamma_K \setminus \mathbb{H}^2)$ and generate the finite dimensional subspace $\mathcal{R}(\Gamma_K \setminus \mathbb{H}^2)$. These subspaces are all invariant and we have the orthogonal decomposition

$$L^2(\Gamma_K \setminus \mathbb{H}^2) = \mathcal{C}(\Gamma_K \setminus \mathbb{H}^2) \oplus \mathcal{R}(\Gamma_K \setminus \mathbb{H}^2) \oplus \mathcal{E}(\Gamma_K \setminus \mathbb{H}^2).$$

Let $\{u_j : j \geq 0\}$ be a complete orthonormal system of L^2-eigenfunctions, which span $\mathcal{E}^\perp$. Then, together with the Eisenstein series they give the spectral decomposition of square-integrable functions (see Theorem II.9.8 in [5]):

Theorem 2.2.10. *If $f(z) \in L^2(\Gamma_K \setminus \mathbb{H}^2)$, then*

$$f(z) = \sum_{j=0}^\infty \langle f, u_j \rangle \, u_j(z) + \frac{1}{8\pi\log\varepsilon\sqrt{d(K)}} \sum_{m=-\infty}^\infty \int_{-\infty}^\infty \langle f, E\left(\cdot, 1/2+it, m\right)\rangle \, E(z, 1/2+it, m) \, dt.$$

2.3 The automorphic kernel

The so-called *automorphic kernel functions* will play a basic role in the following. In this section we define them and discuss some of their most important properties. For this definition we choose a function $\psi \in C^\infty(\mathbb{R}^2)$ and set

$$k_\psi(z,w) = k(z,w) = \psi\left(\frac{|z_1 - w_1|^2}{\operatorname{Im} z_1 \cdot \operatorname{Im} w_1}, \frac{|z_2 - w_2|^2}{\operatorname{Im} z_2 \cdot \operatorname{Im} w_2}\right)$$

for every $z, w \in \mathbb{H}^2$. The function $k(z,w)$ is a so-called *point-pair invariant kernel* which means that $k(z,w) = k(\sigma z, \sigma w)$ holds for every $z, w \in \mathbb{H}^2$ and $\sigma \in PSL(2,\mathbb{R})^2$ (this can easily be checked by a computation). Note that certain control over its growth will be required and this will be discussed later in this section.

A new kernel is given by the series

$$K(z,w) = \sum_{\gamma \in \Gamma_K} k(z, \gamma w). \tag{2.22}$$

This is clearly an automorphic function in every variable (at least if the series above converges absolutely), and hence called an *automorphic kernel*.

Now we define some transforms of ψ, they often occur in computations:

$$\begin{aligned} Q(w_1,w_2) &:= \int_{w_2}^{\infty}\int_{w_1}^{\infty} \frac{\psi(t_1,t_2)}{\sqrt{t_1-w_1}\sqrt{t_2-w_2}}\,dt_1\,dt_2, \\ g(u_1,u_2) &:= Q(e^{u_1}+e^{-u_1}-2, e^{u_2}+e^{-u_2}-2), \\ h(r_1,r_2) &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u_1,u_2)e^{i(r_1u_1+r_2u_2)}\,du_1\,du_2. \end{aligned} \tag{2.23}$$

These notations will be fixed in the following.

We often restrict ourselves to a compactly supported smooth function ψ. Then g is also a smooth function with compact support and hence h is rapidly decreasing. However, in many situations this condition is not essential. It is simpler to express the sufficient conditions in terms of h rather than k. Following [11], we will assume that h is even in every variable, holomorphic in the strip $|\mathrm{Im}\, r_k| \le \frac{1}{2} + \varepsilon$ in every variable and that

$$h(r_1,r_2) \ll (|r_1|+1)^{-2-\varepsilon}(|r_2|+1)^{-2-\varepsilon}.$$

Also, many times it is more convenient to choose g or h instead of ψ, therefore we frequently use the inverses of the transforms above, that are described in the following statement (see Proposition I.2.2 in [5]).

Proposition 2.3.1. *If h satisfies the above mentioned properties, then*

$$\begin{aligned} g(u_1,u_2) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(r_1,r_2)e^{-i(r_1u_1+r_2u_2)}\,dr_1\,dr_2, \\ Q(w_1,w_2) &= g\left(2\log\left(\sqrt{\frac{w_1}{4}+1}+\sqrt{\frac{w_1}{4}}\right), 2\log\left(\sqrt{\frac{w_2}{4}+1}+\sqrt{\frac{w_2}{4}}\right)\right), \\ \psi(t_1,t_2) &= \frac{1}{\pi^2}\int_{t_2}^{\infty}\int_{t_1}^{\infty} \frac{\frac{\partial^2 Q}{\partial w_1 \partial w_2}(w_1,w_2)}{\sqrt{w_1-t_1}\sqrt{w_2-t_2}}\,dw_1\,dw_2. \end{aligned}$$

We close this short section with an important claim that will be applied several times later (see Theorem 1.14 and Theorem 1.16 in [11] together with the remark after the proof of Theorem 1.16, and see also Lemma I.2.1 and Proposition I.2.2 in [5]):

Lemma 2.3.2. *If $h(r_1, r_2)$ satisfies the properties above, and $u : \mathbb{H}^2 \to \mathbb{C}$ is an eigenfunction of Δ_k with eigenvalue $\lambda_k = s_k(1-s_k)$, where $s_k = \frac{1}{2} + ir_k$ $(k = 1, 2)$, then*

$$\int_{\mathbb{H}^2} k(z,w)u(w)\,d\mu(w) = h(r_1, r_2)u(z).$$

2.4 An application of Bessel's inequality

In this section we generalize some results that are given in sections 7.2 and 10.2 of [11]. Let $\{u_j(z) : j \geq 0\}$ be a complete orthonormal system of automorphic forms for the discrete spectrum of Γ_K with eigenvalues $(\lambda_1^{(j)}, \lambda_2^{(j)})$, $\lambda_k^{(j)} = s_k^{(j)}(1 - s_k^{(j)})$, where $\operatorname{Re} s_k^{(j)} \geq \frac{1}{2}$ is assumed and we write $s_k^{(j)} = \frac{1}{2} + ir_k^{(j)}$ $(k = 1, 2)$. The Fourier expansion of u_j is

$$u_j(z) = \phi_j y_1^{1-s_1^{(j)}} y_2^{1-s_2^{(j)}} + \sum_{l \in L_K^* \setminus 0} c_l^{(j)} \sqrt{y_1 y_2} K_{s_1^{(j)}-\frac{1}{2}}(2\pi\,|l_1|\,y_1) K_{s_2^{(j)}-\frac{1}{2}}(2\pi\,|l_2|\,y_2) e^{2\pi i<x,l>}$$

where $\phi_j \neq 0$ only if u_j is not a cusp form.

We will need an analogue of the triangle inequality for the function

$$\rho(z,w) = \frac{|z-w|^2}{\operatorname{Im} z \cdot \operatorname{Im} w},$$

where $z, w \in \mathbb{H}$. One can express the hyperbolic distance function d by means of ρ:

$$d(z,w) = \cosh^{-1}\left(1 + \frac{\rho(z,w)}{2}\right).$$

As the triangle inequality holds for d, one derives easily that if $\rho(z,w) \leq \delta$ and $\rho(w,u) \leq \delta$, then

$$\rho(z,u) \leq \delta(4+\delta). \tag{2.24}$$

We fix a point $w \in \mathbb{H}^2$ and define the function

$$f(z) = K(z,w) = \sum_{\gamma \in \Gamma_K} k(z, \gamma w).$$

Since ψ has compact support, the same holds for f and hence $f \in L^2(\Gamma_K \setminus \mathbb{H}^2)$. Then by the spectral theorem and Lemma 2.3.2 we have $K(z,w) = \sum_j f_j(z) + \sum_{m \in \mathbb{Z}} E_m(z)$, where

$$f_j(z) = h(r_1^{(j)}, r_2^{(j)}) u_j(z) \overline{u_j(w)},$$

$$E_m(z) = \frac{1}{8\pi \log\varepsilon \sqrt{d(K)}} \int_{-\infty}^{\infty} h\left(r + \frac{\pi m}{2\log\varepsilon}, r - \frac{\pi m}{2\log\varepsilon}\right) \times$$
$$\times E\left(z, \frac{1}{2} + ir, m\right) \overline{E\left(w, \frac{1}{2} + ir, m\right)}\,dr.$$

In fact we do not need the spectral formula in this section, an approximation is sufficient. First, we have

$$\begin{aligned}\langle f, f_j\rangle &= \overline{h(r_1^{(j)}, r_2^{(j)})}u_j(w)\,\langle f, u_j\rangle = \overline{h(r_1^{(j)}, r_2^{(j)})}u_j(w)\int_{\mathbb{H}^2} k(z,w)\overline{u_j(z)}\,d\mu(z)\\ &= \left|h(r_1^{(j)}, r_2^{(j)})u_j(w)\right|^2 = \|f_j\|^2\end{aligned}$$

by Lemma 2.3.2. Now we define

$$G_m(z) = \frac{1}{4\pi\log\varepsilon\sqrt{d(K)}}\int_A^B h\left(r+\frac{\pi m}{2\log\varepsilon}, r-\frac{\pi m}{2\log\varepsilon}\right)E\left(z,\frac{1}{2}+ir,m\right)\overline{E\left(w,\frac{1}{2}+ir,m\right)}\,dr$$

for some numbers $0 < A < B < \infty$. We set

$$h(r) = \begin{cases}\dfrac{1}{4\pi\log\varepsilon\sqrt{d(K)}}h\left(r+\dfrac{\pi m}{2\log\varepsilon}, r-\dfrac{\pi m}{2\log\varepsilon}\right)\overline{E\left(w,\dfrac{1}{2}+ir,m\right)}, & \text{if } r\in[A,B],\\ 0 & \text{otherwise.}\end{cases}$$

Now applying Lemma 2.3.2 together with Theorem 2.2.9 for a smooth approximation of $h(r)$ and then taking limit we obtain

$$\begin{aligned}\langle f, G_m\rangle &= \int_F K(z,w)\overline{G_m(z)}\,d\mu(z)\\ &= \frac{1}{4\pi\log\varepsilon\sqrt{d(K)}}\int_A^B \left|h\left(r+\frac{\pi m}{2\log\varepsilon}, r-\frac{\pi m}{2\log\varepsilon}\right)E\left(w,\frac{1}{2}+ir,m\right)\right|^2 dr = \|G_m\|^2.\end{aligned}$$

Hence Bessel's inequality is applicable: $\sum_j \|f_j\|^2 + \sum_{m\in\mathbb{Z}}\|G_m\|^2 \le \|f\|^2$, that is,

$$\begin{aligned}&\sum_j \left|h(r_1^{(j)}, r_2^{(j)})u_j(w)\right|^2 +\\ &\quad+\sum_{m\in\mathbb{Z}}\frac{1}{8\pi\log\varepsilon\sqrt{d(K)}}\int_{-\infty}^{\infty}\left|h\left(r+\frac{\pi m}{2\log\varepsilon}, r-\frac{\pi m}{2\log\varepsilon}\right)E\left(w,\frac{1}{2}+ir,m\right)\right|^2 dr \qquad (2.25)\\ &\quad\le \int_F |K(z,w)|^2\,d\mu(z).\end{aligned}$$

We can drop the restriction $A \le r \le B$ in the second term since we integrate a non-negative function. Then by substituting $R = -r$, adding the terms belonging to m and $-m$ and finally, summing over $\mathbb{Z}$ and dividing by 2 we obtain the inequality above.

We apply this inequality for a compactly supported kernel ψ which is a smooth approximation of the characteristic function of the rectangle $[0,\delta_1]\times[0,\delta_2]$ for some small $\delta_1 > 0$ and $\delta_2 > 0$ satisfying $0\le\psi\le 1$ and $\psi|_{[0,\delta_1]\times[0,\delta_2]}\equiv 1$. We assume further that $\operatorname{supp}(\psi)\subset[-\eta_1,\delta_1+\eta_1]\times$

$[-\eta_2, \delta_2 + \eta_2]$ for some $\eta_1 > 0$, $\eta_2 > 0$. Then by Lemma 2.3.2 we get the integral representation

$$h(r_1, r_2) = \int_{\mathbb{H}^2} k((i,i), z) y_1^{s_1} y_2^{s_2}\, \mu(z)$$

for any $s_1, s_2 \in \mathbb{C}$, in particular, for $s_1 = s_2 = 0$ we obtain

$$h\left(\frac{i}{2}, \frac{i}{2}\right) = \int_{\mathbb{H}^2} k((i,i), z)\, d\mu(z).$$

If $\varrho_1 > 0$ and $\varrho_2 > 0$, then we have

$$\int\limits_{\substack{\rho(i,z_k) \leq \varrho_k \\ k=1,2}} 1\, d\mu(z) = \varrho_1 \varrho_2 \pi^2,$$

since it is the product of the areas of hyperbolic circles with hyperbolic radius $\cosh^{-1}(1 + \frac{\varrho_k}{2})$ $(k = 1, 2)$. It follows that $\delta_1 \delta_2 \pi^2 \leq h(i/2, i/2) \leq (\delta_1 + \eta_1)(\delta_2 + \eta_2)\pi^2$.

We would like to estimate the value of $h(r_1, r_2)$ where $\lambda_k = \frac{1}{4} + r_k^2$ $(k = 1, 2)$ and (λ_1, λ_2) is an eigenvalue vector of a u_j or an Eisenstein series. For this we estimate the distance of the numbers $h(r_1, r_2)$ and $h(i, i)$. We will handle this problem coordinate-wise, so assume that $\delta_k > 0$ and $\eta_k > 0$ are some small numbers and $0 < \delta_k + \eta_k < C$ for some constant $C > 0$. If $z_k \in \mathbb{H}$ and $\rho(i, z_k) \leq \delta_k + \eta_k$, then

$$\frac{|y_k - 1|^2}{y_k} \leq \frac{|z_k - i|^2}{\operatorname{Im} z_k} = \rho(i, z_k) \leq \delta_k + \eta_k,$$

and hence $y_k^2 - 2y_k + 1 \leq (\delta_k + \eta_k) y_k < C y_k$, that is, y_k lies between the values

$$\frac{2 + C \pm \sqrt{(2+C)^2 - 4}}{2} = \frac{2 + C \pm \sqrt{C(C+4)}}{2}.$$

Choosing a small enough C we can reach $|y_k - 1| \leq 2\sqrt{\delta_k + \eta_k}$.

If $|s| \leq 1$, then one can show easily using the power series of y_k^s that

$$|y_k^s - 1| \leq |s| \max \left\{ |y_k - 1|, \left| \frac{1}{y_k} - 1 \right| \right\} \leq 2\,|s|\,|y_k - 1|$$

holds (for a small enough C). However, we need a similar estimate also if $s = \frac{1}{2} + ir$ for some $r \in \mathbb{R}$. If $|s| \leq \frac{c}{|\log y_k|}$ for some constant $c > 0$, then

$$y_k^s - 1 = e^{s \log y_k} - 1 \ll |s|\,|\log y_k| \ll |s|\,|y_k - 1|,$$

where the implied constant depends on c and C. Finally, if $|s| \geq \frac{c}{|\log y_k|}$, then by the triangle inequality

$$|y_k^s - 1| \leq 1 + y_k^{\frac{1}{2}} \ll 1 \ll |s| \log y_k \ll |s|\,|y_k - 1|.$$

Now we restrict ourselves to the cases when the real part of s_1 and s_2 is $\frac{1}{2}$ or when both s_1 and s_2 are on the interval $[\frac{1}{2}, 1]$. If C is small enough and $\delta_k + \eta_k < C$, then we have

$$\begin{aligned}
|h(r_1, r_2) - h(i/2, i/2)| &\le |h(r_1, r_2) - h(r_1, i/2)| + |h(r_1, i/2) - h(i/2, i/2)| \\
&\le \int\limits_{\substack{\rho(i,z_k) \le \delta_k + \eta_k \\ k=1,2}} |y_1^{s_1} y_2^{s_2} - y_1^{s_1}| \, d\mu(z) + \int\limits_{\substack{\rho(i,z_k) \le \delta_k + \eta_k \\ k=1,2}} |y_1^{s_1} - 1| \, d\mu(z) \\
&\le \int\limits_{\substack{\rho(i,z_k) \le \delta_k + \eta_k \\ k=1,2}} 2 \, |y_2^{s_2} - 1| \, d\mu(z) + \int\limits_{\substack{\rho(i,z_k) \le \delta_k + \eta_k \\ k=1,2}} |y_1^{s_1} - 1| \, d\mu(z) \\
&\ll (|s_1| \sqrt{\delta_1 + \eta_1} + |s_2| \sqrt{\delta_2 + \eta_2})(\delta_1 + \eta_1)(\delta_2 + \eta_2) \\
&\ll (|s_1| \sqrt{\delta_1} + |s_2| \sqrt{\delta_2}) \delta_1 \delta_2
\end{aligned}$$

for $\eta_k = c\delta_k$ for example, where $0 < c < 1$ is a small constant, and if $|s_k| \ll \delta_k^{-\frac{1}{2}}$ with an appropriate implied constant, then

$$\frac{1}{2} \pi^2 \delta_1 \delta_2 \le h(r_1, r_2) \le 2\pi^2 \delta_1 \delta_2. \tag{2.26}$$

Now we estimate the L^2-norm of $K(z, w)$. First, we have

$$\begin{aligned}
\int_F |K(z,w)|^2 \, d\mu(z) &= \sum_{\gamma, \gamma' \in \Gamma_K} \int_F k(\gamma' z, w) k(\gamma' z, \gamma w) \, d\mu(z) \\
&= \sum_{\gamma \in \Gamma_K} \int_{\mathbb{H}^2} k(z, w) k(z, \gamma w) \, d\mu(z).
\end{aligned}$$

If $k(z, w)k(z, \gamma w) \neq 0$, then $\rho(z_k, w_k) \le 2\delta_k$ and $\rho(z_k, \gamma^{(k)} w_k) \le 2\delta_k$ hold and then we get $\rho(w_k, \gamma^{(k)} w_k) \le 4\delta_k(2 + \delta_k)$ by (2.24) ($k = 1, 2$). Setting

$$N_{\delta_1, \delta_2}(w) = \#\{\gamma \in \Gamma_K : \ \rho(w_k, \gamma^{(k)} w_k) \le 4\delta_k(2 + \delta_k), \ k = 1, 2\},$$

we obtain

$$\int_F |K(z,w)|^2 \, d\mu(z) \le N_{\delta_1, \delta_2}(w) \int_{\mathbb{H}^2} k((i,i), z) \, d\mu(z) \le N_{\delta_1, \delta_2}(w) (2\pi)^2 \delta_1 \delta_2.$$

If we restrict the summation and the integration in (2.25) to the points $s_k^{(j)} = \frac{1}{2} + ir_k^{(j)}$ and $\frac{1}{2} + ir + (-1)^{k-1} \frac{im}{2 \log \varepsilon}$ with absolute value less than a constant times $\delta_k^{-\frac{1}{2}}$ for $k = 1, 2$ and apply the lower estimate in (2.26), then we infer

$$\sum_j{}' |u_j(z)|^2 + \sum_{m \in \mathbb{Z}} \frac{1}{8\pi \log \varepsilon \sqrt{d(K)}} \int{}' \left| E\left(z, \frac{1}{2} + ir, m\right) \right|^2 dr \ll \delta_1^{-1} \delta_2^{-1} N_{\delta_1, \delta_2}(z),$$

where $'$ denotes our restriction. If $T_1 > 0$ and $T_2 > 0$ are big enough, then we can choose $\delta_k = (cT_k)^{-2}$ with some constant c to obtain

$$\sum_{\substack{|s_k^{(j)}|\leq T_k \\ k=1,2}} |u_j(z)|^2 + \sum_{m\in\mathbb{Z}} \frac{1}{8\pi \log\varepsilon \sqrt{d(K)}} \int' \left| E\left(z, \frac{1}{2}+ir, m\right)\right|^2 dr \ll \tag{2.27}$$

$$\ll T_1^2 T_2^2 N_{(cT_1)^{-2},(cT_2)^{-2}}(z),$$

where for a fixed m we integrate over those r's for which $\left|\frac{1}{2} + ir + (-1)^{k-1}\frac{im}{2\log\varepsilon}\right| \leq T_k$ holds.

We proceed by estimating $N_{\delta_1,\delta_2}(z)$ for some special points. The following set occurs many times later: for an $A > 0$ we define

$$F_A := \{z \in F : Y_0(z) = y_1 y_2 \leq A\}.$$

This is the "central part" of the fundamental domain, i.e. the points that are "close to the cusp" are omitted from F.

Lemma 2.4.1. *There is a constant C_K depending on the field K such that if $z \in F_A$ and $0 < \delta_1, \delta_2 < C_K$, then $N_{\delta_1,\delta_2}(z) \ll 1 + A(\sqrt{\eta_1} + \sqrt{\eta_2})^2$, where $\eta_k = 4\delta_k(2+\delta_k)$ for $k = 1, 2$ and the implied constant depends only on K.*

Proof. Assume that $z \in F_A$. If $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in N_{\delta_1,\delta_2}(z)$ then we have $\rho(z_k, \gamma^{(k)} z_k) \leq \eta_k$ for $k = 1, 2$, and this is equivalent to

$$\left|\gamma^{(k)} z_k - z_k\right|^2 \leq \eta_k \cdot \operatorname{Im} z_k \cdot \operatorname{Im} \gamma^{(k)} z_k. \tag{2.28}$$

If $c = 0$, then $a^{(k)} = \varepsilon^l$ for an $l \in \mathbb{Z}$. Then

$$\operatorname{Im}^2(\gamma^{(k)} z_k - z_k) = (\varepsilon^{2l} - 1)^2 y_k^2 \leq \left|\gamma^{(k)} z_k - z_k\right|^2 \leq \eta_k \varepsilon^{2l} y_k^2,$$

that is, $(\varepsilon^l - \varepsilon^{-l})^2 \leq \eta_k$. If δ_k is small enough, then so is η_k and this latter inequality holds only for $l = 0$, and then $a = d = 1$. It follows that

$$\left|\gamma^{(k)} z_k - z_k\right|^2 = \left|b^{(k)}\right|^2 \leq \eta_k y_k^2 \leq \eta_k \varepsilon^2 y_1 y_2 \leq \eta_k \varepsilon^2 A,$$

so $|b^{(k)}| \leq \varepsilon\sqrt{\eta_k A}$. It follows that there are at most $1 + c_K A(\sqrt{\eta_1} + \sqrt{\eta_2})^2$ possibilities for b, where c_K is a constant depending only on K.

Now assume that $c \neq 0$. Then (2.28) gives

$$\left(\frac{y_k}{|c^{(k)} z_k + d^{(k)}|^2} - y_k\right)^2 \leq \left|\gamma^{(k)} z_k - z_k\right|^2 \leq \eta_k \frac{y_k^2}{|c^{(k)} z_k + d^{(k)}|^2},$$

that is, $(B_k - B_k^{-1})^2 \leq \eta_k$ for $B_k = \left|c^{(k)} z_k + d^{(k)}\right|$. From this we obtain

$$\eta_k + 2 \geq B_k^2 + B_k^{-2} \geq B_k^2 = \left|c^{(k)} z_k + d^{(k)}\right|^2 \geq (c^{(k)})^2 y_k^2 > \frac{(c^{(k)})^2}{4\varepsilon^2 d(K)}$$

by Lemma 1.2.4. If $\eta_1, \eta_2 < 1$, then $|c^{(1)}|$ and $|c^{(2)}|$ are bounded by a constant depending only on K, so there are only finitely many possible values for c. For a fixed c we have

$$(c^{(k)}x_k + d^{(k)})^2 \leq \left|c^{(k)}z_k + d^{(k)}\right|^2 \leq \eta_k + 2,$$

and since $|x_k|$ is bounded, d can be chosen only from a finite set. So there are only finitely many possibilities for the pair (c, d). Finally, (2.28) gives

$$\left|a^{(k)}z_k + b^{(k)} - z_k(c^{(k)}z_k + d^{(k)})\right|^2 \leq \eta_k y_k^2.$$

Since $\operatorname{Im}(z_k(c^{(k)}z_k + d^{(k)})) = 2c^{(k)}x_k y_k + d^{(k)}y_k$, we obtain $(a^{(k)} - (2c^{(k)}x_k + d^{(k)}))^2 \leq \eta_k$, and since $(2c^{(k)}x_k + d^{(k)})$ is bounded, we get that so is $a^{(1)}$ and $a^{(2)}$. This completes the proof of the lemma. □

From (2.27) and the lemma above we infer

Theorem 2.4.2. *If $A > 0$, $z \in F_A$ and $T_1, T_2 > 0$ are big enough, then we have*

$$\sum_{\substack{|s_k^{(j)}| \leq T_k \\ k=1,2}} |u_j(z)|^2 + \sum_{m \in \mathbb{Z}} \frac{1}{8\pi \log \varepsilon \sqrt{d(K)}} \int' \left| E\left(z, \frac{1}{2} + ir, m\right)\right|^2 dr \ll$$

$$\ll T_1^2 T_2^2 + A(T_1^2 + T_2^2 + T_1 T_2), \qquad (2.29)$$

where for a fixed m we integrate over those r's for which $\left|\frac{1}{2} + ir + (-1)^{k-1}\frac{im}{2\log\varepsilon}\right| \leq T_k$ *holds.*

Next we omit the second sum from the estimate above, set $A = cT_1T_2$ for some big enough constant $c > 0$ and integrate both sides on F_A:

$$\sum_{\substack{|s_k^{(j)}| \leq T_k \\ k=1,2}} \int_{F_A} |u_j(z)|^2 \, d\mu(z) \ll T_1^2 T_2^2 + T_1^3 T_2 + T_1 T_2^3.$$

Also, we define $F_A^B := \{z \in F : A \leq Y_0(z) \leq B\}$. If we choose $A = cT_1T_2$, $B = cT_1^2T_2^2$, and integrate both sides on F_A^B, then we obtain

$$\sum_{\substack{|s_k^{(j)}| \leq T_k \\ k=1,2}} \int_{F_A^B} |u_j(z)|^2 \, d\mu(z) \ll (T_1^2T_2^2 + T_1^2T_2^2(T_1^2 + T_2^2 + T_1T_2)) \int_{F_A^B} 1 d\mu(z)$$

$$\ll \frac{1}{A}(T_1^2T_2^2 + T_1^2T_2^2(T_1^2 + T_2^2 + T_1T_2)) \ll T_1^2T_2^2 + T_1^3T_2 + T_1T_2^3.$$

Summing these two inequalities we get the following on the left hand side:

$$\sum_{\substack{|s_k^{(j)}| \leq T_k \\ k=1,2}} \int_{F_B} |u_j(z)|^2 \, d\mu(z) = \sum_{\substack{|s_k^{(j)}| \leq T_k \\ k=1,2}} 1 - \int_{F \setminus F_B} |u_j(z)|^2 \, d\mu(z).$$

For a big enough c we infer by Theorem 2.1.8 that

$$\int_{F\setminus F_B} |u_j(z)|^2 \, d\mu(z) \ll |\phi_j|^2 \int\limits_{cT_1^2T_2^2}^{\infty} Y_0^{1-s}\frac{dY_0}{Y_0^2} + \int\limits_{cT_1^2T_2^2}^{\infty} e^{-d\sqrt{Y_0}}\frac{dY_0}{Y_0^2}$$

$$\ll \delta_{\phi_j\neq 0}\frac{(cT_1^2T_2^2)^{-s}}{|s|} + e^{-d\sqrt{c}T_1T_2} \ll \frac{1}{\sqrt{c}} + e^{-d\sqrt{c}},$$

where s is defined to be $s_1 = s_2$ when $\phi_j \neq 0$, and the implied constant depends only on the field K. Hence if c is big enough, then the integral above is uniformly small for every j. We can summarize this as

$$\#\{j : |s_1^{(j)}| \leq T_1, |s_2^{(j)}| \leq T_2\} \ll T_1^2T_2^2 + T_1^3T_2 + T_1T_2^3. \tag{2.30}$$

As the last result of this section we prove a bound for the logarithmic derivative of the zeroth Fourier coefficient of the Eisenstein series:

Theorem 2.4.3. *If $T_1, T_2 > 0$ are big enough, then*

$$\sum_{m\in\mathbb{Z}} \int' \frac{-\phi'(\frac{1}{2}+ir, m)}{\phi(\frac{1}{2}+ir, m)}\, dr \ll T_1^2T_2^2 + \sqrt{T_1T_2}(T_1^2 + T_2^2 + T_1T_2),$$

where for a fixed m we integrate over the r's for which $\left|\frac{1}{2} + ir + (-1)^{k-1}\frac{\pi i m}{2\log\varepsilon}\right| \leq T_k$ holds.

Proof. First we fix some big numbers $T_1, T_2 > 0$ and for a fixed $z \in F$ we set $A = y_1y_2$ and integrate both sides of (2.29) over $\{-\frac{1}{2} \leq X_1, X_2 < \frac{1}{2}\}$ to infer

$$\sum_{\substack{|s_k^{(j)}|\leq T_k\\ k=1,2}} \sum_{l\in L_K^*\setminus 0} |c_l^{(j)}|^2 y_1y_2 K^2_{s_1^{(j)}-\frac{1}{2}}(2\pi\,|l_1|\,y_1) K^2_{s_2^{(j)}-\frac{1}{2}}(2\pi\,|l_2|\,y_2)+$$

$$+\sum_{m\in\mathbb{Z}} \int' \sum_{l\in L_K^*\setminus 0} |\phi_l(1/2+ir,m)|^2\, y_1y_2 K^2_{ir+\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_1|\,y_1) K^2_{ir-\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_2|\,y_2)\, dr \tag{2.31}$$

$$\ll T_1^2T_2^2 + y_1y_2(T_1^2 + T_2^2 + T_1T_2),$$

where for a fixed m we integrate over those r's for which $\left|\frac{1}{2} + ir + (-1)^{k-1}\frac{im}{2\log\varepsilon}\right| \leq T_k$ holds.

We use the bound (see [11] p. 141)

$$\int_T^\infty |K_{s-\frac{1}{2}}(2\pi y)|^2 \frac{dy}{y} \ll |s| \int_{T/2}^\infty |K_{s-\frac{1}{2}}(2\pi y)|^2 \frac{dy}{y^2}$$

and estimate as follows:

$$
\begin{aligned}
&\int_{F\setminus F_{\sqrt{T_1T_2}}} \left|E_{\sqrt{T_1T_2}}(z,s,m)\right|^2 \, d\mu(z) = \\
&\qquad = \sum_{l\in L_K^*\setminus 0} |\phi_l(s,m)|^2 \iint\limits_{\substack{\sqrt{T_1T_2}\le Y_0 \\ -\frac{1}{2}\le Y_1<\frac{1}{2}}} |K_{s-\frac{1}{2}+\frac{\pi im}{2\log\varepsilon}}(2\pi\,|l_1|\,y_1)|^2 |K_{s-\frac{1}{2}-\frac{\pi im}{2\log\varepsilon}}(2\pi\,|l_2|\,y_2)|^2 \frac{dy_1\,dy_2}{y_1y_2} \\
&\qquad \le \sum_{l\in L_K^*\setminus 0} |\phi_l(s,m)|^2 \iint\limits_{\substack{\sqrt{T_k}\le y_k \\ k=1,2}} |K_{s-\frac{1}{2}+\frac{\pi im}{2\log\varepsilon}}(2\pi\,|l_1|\,y_1)|^2 |K_{s-\frac{1}{2}-\frac{\pi im}{2\log\varepsilon}}(2\pi\,|l_2|\,y_2)|^2 \frac{dy_1\,dy_2}{y_1y_2} \\
&\qquad \ll \left|s+\frac{\pi im}{2\log\varepsilon}\right|\left|s-\frac{\pi im}{2\log\varepsilon}\right| \sum_{l\in L_K^*\setminus 0} |\phi_l(s,m)|^2 \cdot \\
&\qquad\qquad \cdot \iint\limits_{\substack{\sqrt{T_k}/2\le y_k \\ k=1,2}} |K_{s-\frac{1}{2}+\frac{\pi im}{2\log\varepsilon}}(2\pi\,|l_1|\,y_1)|^2 |K_{s-\frac{1}{2}-\frac{\pi im}{2\log\varepsilon}}(2\pi\,|l_2|\,y_2)|^2 \frac{dy_1\,dy_2}{y_1^2y_2^2},
\end{aligned}
$$

hence, using this and the estimate (2.31) we get that

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}} {\int}' \int_{F\setminus F_{\sqrt{T_1T_2}}} &\left|E_{\sqrt{T_1T_2}}(z,1/2+ir,m)\right|^2 \, d\mu(z)\,dr \ll \\
&\ll \iint\limits_{\substack{\sqrt{T_k}/2\le y_k \\ k=1,2}} T_1^3T_2^3 + y_1y_2T_1T_2(T_1^2+T_2^2+T_1T_2)\frac{dy_1\,dy_2}{y_1^3y_2^3} \\
&\ll T_1^2T_2^2 + \sqrt{T_1T_2}(T_1^2+T_2^2+T_1T_2)
\end{aligned}
$$

Now we apply Theorem 2.4.2 with $A=\sqrt{T_1T_2}$. Omitting the first sum on the left hand side of (2.29) and integrate the remaining terms over $F_{\sqrt{T_1T_2}}$ we infer

$$
\sum_{m\in\mathbb{Z}} {\int}' \int_{F_{\sqrt{T_1T_2}}} \left|E\left(z,\frac{1}{2}+ir,m\right)\right|^2 d\mu(z)\,dr \ll T_1^2T_2^2 + \sqrt{T_1T_2}(T_1^2+T_2^2+T_1T_2).
$$

Note that here we also have to estimate the integral of the zeroth coefficients, hence we show that

$$
\sum_{m\in\mathbb{Z}} {\int}' \log T_1T_2\,dr \ll (T_1^2+T_2^2)\log T_1T_2, \tag{2.32}
$$

where the integration is restricted to those points for which

$$
\left|\frac{1}{2}+ir+(-1)^{k-1}\frac{im}{2\log\varepsilon}\right| \le T_k
$$

holds ($k=1,2$). The inequalities above are equivalent to

$$\left|\frac{1}{2}+ir+(-1)^{k-1}\frac{im}{2\log\varepsilon}\right|^2=\frac{1}{4}+\left(r+(-1)^{k-1}\frac{m}{2\log\varepsilon}\right)^2\le T_k^2,$$

and adding the two inequalities we obtain

$$\frac{1}{2}+2r^2+\frac{m^2}{2\log^2\varepsilon}\le T_1^2+T_2^2.$$

That is, both $|r|$ and $|m|$ are bounded by a constant times $\sqrt{T_1^2+T_2^2}$ and our claim follows.

Now

$$\sum_{m\in\mathbb{Z}}\int'\int_F\left|E_{\sqrt{T_1T_2}}\left(z,\frac{1}{2}+ir,m\right)\right|^2\,d\mu(z)\,dr=$$

$$=\sum_{m\in\mathbb{Z}}\int'\int_{F_{\sqrt{T_1T_2}}}\left|E\left(z,\frac{1}{2}+ir,m\right)\right|^2\,d\mu(z)\,dr+$$

$$+\sum_{m\in\mathbb{Z}}\int'\int_{F\setminus F_{\sqrt{T_1T_2}}}\left|E_{\sqrt{T_1T_2}}\left(z,\frac{1}{2}+ir,m\right)\right|^2\,d\mu(z)\,dr,$$

hence

$$\sum_{m\in\mathbb{Z}}\int'\int_F\left|E_{\sqrt{T_1T_2}}\left(z,\frac{1}{2}+ir,m\right)\right|^2\,d\mu(z)\,dr\ll T_1^2T_2^2+\sqrt{T_1T_2}(T_1^2+T_2^2+T_1T_2)$$

follows. By (2.19) on page 54

$$\int_F\left|E_{\sqrt{T_1T_2}}(z,1/2+ir,m)\right|^2\,d\mu(z)=2\log\varepsilon\sqrt{d(K)}\left[\log T_1T_2-\frac{\phi'(\frac{1}{2}+ir,m)}{\phi(\frac{1}{2}+ir,m)}\right]$$

holds if $m\ne 0$, while for $m=0$ we have

$$\int_F\left|E_{\sqrt{T_1T_2}}(z,1/2+ir,0)\right|^2\,d\mu(z)=$$
$$=2\log\varepsilon\sqrt{d(K)}\left[\log T_1T_2-\frac{\phi'(\frac{1}{2}+ir,0)}{\phi(\frac{1}{2}+ir,0)}+\frac{\phi(1/2-ir,0)(T_1T_2)^{ri}-\phi(1/2+ir,0)(T_1T_2)^{-ri}}{2ri}\right]$$

if $r\ne 0$ and

$$\int_F\left|E_{\sqrt{T_1T_2}}(z,1/2,0)\right|^2\,d\mu(z)=2\log\varepsilon\sqrt{d(K)}(\log T_1T_2-\phi'(1/2,0))(1+\phi(1/2,0))$$

(see (2.20) and (2.21)). That is,

$$\sum_{m\in\mathbb{Z}} \int' \int_F \left|E_{\sqrt{T_1T_2}}(z, 1/2+ir, m)\right|^2 \, d\mu(z) =$$

$$= 2\log\varepsilon\sqrt{d(K)} \sum_{m\in\mathbb{Z}} \int' \left[\log T_1T_2 - \frac{\phi'(\frac{1}{2}+ir, m)}{\phi(\frac{1}{2}+ir, m)}\right] dr +$$

$$+ 2\log\varepsilon\sqrt{d(K)} \int\limits_{\left|\frac{1}{2}+ir\right| \le \min(T_1,T_2)} \frac{\phi(1/2-ir,0)(T_1T_2)^{ri} - \phi(1/2+ir,0)(T_1T_2)^{-ri}}{2ri} \, dr.$$

We define $T = \min(T_1, T_2)$. Since $\left|\frac{1}{2}+ir\right| \le T$ holds if and only if $|r| \le \sqrt{T^2-1/4}$, we need to estimate

$$\int\limits_{\sqrt{T^2-\frac{1}{4}}}^{\sqrt{T^2-\frac{1}{4}}} \frac{\phi(1/2-ir,0)(T_1T_2)^{ri} - \phi(1/2+ir,0)(T_1T_2)^{-ri}}{2ri} \, dr. \tag{2.33}$$

As $|\phi(1/2+ir)|^2 = 1$ by Corollary 2.2.6, we have for every $0 < c < \sqrt{T^2-1/4}$ that

$$\int\limits_{c}^{\sqrt{T^2-\frac{1}{4}}} \frac{\phi(1/2-ir,0)(T_1T_2)^{ri} - \phi(1/2+ir,0)(T_1T_2)^{-ri}}{2ri} \, dr \ll \log(T/c) \ll \log(T_1T_2/c),$$

hence it remains to estimate the integral around 0 and to choose an appropriate c. As

$$\int\limits_{-c}^{c} \frac{\phi(\frac{1}{2}-ir,0)(T_1T_2)^{ri} - \phi(\frac{1}{2}+ir,0)(T_1T_2)^{-ri}}{2ri} \, dr =$$

$$= \int\limits_{-c}^{c} \frac{\phi(\frac{1}{2}-ir,0)(T_1T_2)^{ri} - \phi(\frac{1}{2}-ir,0)(T_1T_2)^{-ri}}{2ri} \, dr$$

$$+ \int\limits_{-c}^{c} \frac{\phi(\frac{1}{2}-ir,0)(T_1T_2)^{-ri} - \phi(\frac{1}{2}+ir,0)(T_1T_2)^{-ri}}{2ri} \, dr$$

$$= \int\limits_{-c}^{c} \phi(\frac{1}{2}-ir,0)\frac{(T_1T_2)^{ri} - (T_1T_2)^{-ri}}{2ri} \, dr + \int\limits_{-c}^{c} (T_1T_2)^{-ri}\frac{\phi(\frac{1}{2}-ir,0) - \phi(\frac{1}{2}+ir,0)}{2ri} \, dr.$$

If $c < 1$, then the fraction in the second integral is bounded by a constant depending on the field K, so the second integral is $O(c)$. Similarly, the fraction in the first integral is bounded by a constant times $\log T_1T_2$ around zero, hence by choosing $c = 1/\log T_1T_2$ the second integral is $O(1)$ and we conclude that (2.33) is bounded by a constant times $\log T_1T_2$ and together with (2.32) the theorem follows. □

2.5 A spectral mean-value estimate

In this section we derive a mean-value estimate for the Fourier coefficients of automorphic forms and Eisenstein series. Our main goal is to generalize formula (8.27) in [11]. For this we first fix a complete orthonormal system of automorphic forms for the discrete spectrum of Γ_K like in the previous section, let us denote this set by $\{u_j : j \geq 0\}$. We also pick an appropriate kernel function ψ and apply the spectral theorem for the automorphic kernel K. In fact we choose the function g (that is defined in (2.23) on page 56) instead of ψ such that $g(u_1, u_2) = g_0(u_1)g_0(u_2)$ holds for some $g_0 : \mathbb{R} \to \mathbb{R}$ that is positive and decreasing on $\mathbb{R}^+$ making ψ non-negative and real by Proposition 2.3.1.

Like it was mentioned in the previous section, by Theorem 2.2.10 and Lemma 2.3.2 the function $K(\cdot, z')$ has the spectral decomposition

$$K(z,z') = \sum_j h(r_1^{(j)}, r_2^{(j)}) u_j(z)\overline{u_j(z')} + \frac{1}{8\pi\sqrt{d(K)}\log\varepsilon} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} h\left(r + \frac{\pi m}{2\log\varepsilon}, r - \frac{\pi m}{2\log\varepsilon}\right) \times E\left(z, \frac{1}{2} + ir, m\right) \overline{E\left(z', \frac{1}{2} + ir, m\right)} \, dr.$$

We write $C_K = 8\pi\sqrt{d(K)}\log\varepsilon$, $z_k = x_k + iy_k$ and $z'_k = x'_k + iy'_k$ for $k = 1, 2$, and then for every $l \in L_K^*$ we have

$$\frac{1}{d(K)} \iint\limits_{-\frac{1}{2} \leq X_1(z), X_2(z) < \frac{1}{2}} \iint\limits_{-\frac{1}{2} \leq X_1(z'), X_2(z') < \frac{1}{2}} e^{-2\pi i <l,x>} e^{2\pi i <l,x'>} K(z,z') \, dx'_1 \, dx'_2 \, dx_1 \, dx_2 = \sum_j h(r_1^{(j)}, r_2^{(j)}) |c_l^{(j)}|^2 \prod_{k=1,2} \sqrt{y_k y'_k} K_{s_k^{(j)} - \frac{1}{2}}(2\pi |l_k|) y_k) \overline{K_{s_k^{(j)} - \frac{1}{2}}(2\pi |l_k|) y'_k)} + \frac{1}{C_K} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} h\left(r + \frac{\pi m}{2\log\varepsilon}, r - \frac{\pi m}{2\log\varepsilon}\right) \left|\phi_l\left(\frac{1}{2} + ir, m\right)\right|^2 \times \prod_{k=1,2} \sqrt{y_k y'_k} K_{ir + (-1)^{k-1}\frac{\pi i m}{2\log\varepsilon}}(2\pi |l_k|) y_k) \overline{K_{ir + (-1)^{k-1}\frac{\pi i m}{2\log\varepsilon}}(2\pi |l_k|) y'_k)} \, dr. \tag{2.34}$$

On the other hand, as the function ψ is non-negative, so is the kernel function K and the absolute value of the left hand side above is less than

$$\frac{1}{\sqrt{d(K)}} \iint\limits_{-\frac{1}{2} \leq X_1(z), X_2(z) < \frac{1}{2}} H(z, y') \, dx_1 \, dx_2,$$

where

$$H(z,y') = \frac{1}{\sqrt{d(K)}} \iint\limits_{-\frac{1}{2}\le X_1(z'),X_2(z')<\frac{1}{2}} K(z,z')\,dx_1'\,x_2'$$

$$= \frac{1}{\sqrt{d(K)}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} \sum_{k=-\infty}^{\infty} \sum_{\nu\in\mathcal{O}_K} \iint\limits_{-\frac{1}{2}\le X_1(z'),X_2(z')<\frac{1}{2}} k((n(\nu)\rho_k\gamma)z,z')\,dx_1'\,dx_2',$$

and

$$\rho_k = \begin{bmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{bmatrix}, \qquad n(\nu) = \begin{bmatrix} 1 & \nu \\ 0 & 1 \end{bmatrix}.$$

We obtain

$$H(z,y') = \frac{1}{\sqrt{d(K)}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} \sum_{k=-\infty}^{\infty} \iint\limits_{\mathbb{R}^2} k((\rho_k\gamma)z,z')\,dx_1'\,dx_2', \tag{2.35}$$

and for further estimates we first compute the integral of the kernel $k(z,z')$ with respect to the arguments x_1' and x_2' (and then we may substitute anything in the place of z later):

$$\iint\limits_{\mathbb{R}^2} k(z,z')\,dx_1'\,dx_2' = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \psi\left(\frac{(x_1-x_1')^2+(y_1-y_1')^2}{y_1y_1'}, \frac{(x_2-x_2')^2+(y_2-y_2')^2}{y_2y_2'}\right) dx_1'\,dx_2'$$

$$= 4\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \psi\left(\frac{(x_1')^2+(y_1-y_1')^2}{y_1y_1'}, \frac{(x_2')^2+(y_2-y_2')^2}{y_2y_2'}\right) dx_1'\,dx_2'$$

$$= \sqrt{y_1y_1'y_2y_2'} \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \frac{\psi(u_1+\frac{y_1}{y_1'}+\frac{y_1'}{y_1}-2, u_2+\frac{y_2}{y_2'}+\frac{y_2'}{y_2}-2)}{\sqrt{u_1u_2}}\,du_1\,du_2$$

$$= \sqrt{y_1y_1'y_2y_2'}g(\log(y_1/y_1'),\log(y_2/y_2')).$$

The last equality above is obtained by (2.23).

We make the choice $g_0(u) = \frac{1}{\sqrt{\pi}}Te^{-u^2T^2}$ for some $T\ge 1$, and then

$$g(u_1,u_2) = \frac{1}{\pi}T^2e^{-(u_1^2+u_2^2)T^2} = \frac{1}{\pi}T^2e^{-\frac{(u_1+u_2)^2T^2}{2}}e^{-\frac{(u_1-u_2)^2T^2}{2}},$$

so that

$$\sqrt{y_1y_2y_1'y_2'}\,g(\log(y_1/y_1'),\log(y_2/y_2')) =$$

$$= \frac{\sqrt{y_1y_2y_1'y_2'}T^2}{\pi}e^{-\log^2\left(\frac{y_1y_2}{y_1'y_2'}\right)T^2/2}e^{-\log^2\left(\frac{y_1y_2'}{y_1'y_2}\right)T^2/2}$$

$$= \frac{\sqrt{Y_0(z)Y_0(z')}T^2}{\pi}e^{-\log^2(Y_0(z)/Y_0(z'))T^2/2}e^{-(4\log\varepsilon)^2(Y_1(z)-Y_1(z'))^2T^2/2}.$$

We write this in (2.35) and express $H(z,y')$ as a function of z, $Y_0' = y_1'y_2'$ and $Y_1' = \frac{1}{4\log\varepsilon}\log\frac{y_1'}{y_2'}$:

$$H(z,Y_0',Y_1') = \frac{T^2}{\pi\sqrt{d(K)}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K}\sqrt{Y_0(\gamma z)Y_0'}e^{-\log^2(Y_0(\gamma z)/Y_0')T^2/2}\times$$
$$\times\sum_{k=-\infty}^{\infty}e^{-(4\log\varepsilon)^2(Y_1(\rho_k\gamma z)-Y_1')^2T^2/2}.$$

As $Y_1(\rho_k\gamma z) = Y_1(\gamma z)+k$, we can apply the Poisson summation formula (in the form as it is written in (1.9) in [10]) to estimate the inner sum:

$$\sum_{k=-\infty}^{\infty}e^{-8(T\log\varepsilon)^2(k+Y_1'-Y_1(\gamma z))^2} = \frac{\sqrt{\pi}}{2\sqrt{2}T\log\varepsilon}\sum_{k=-\infty}^{\infty}e^{-\frac{\pi^2k^2}{8(T\log\varepsilon)^2}}e^{2\pi ik(Y_1'-Y_1(\gamma z))},$$

and this last sum can be estimated by

$$\frac{\sqrt{\pi}}{2\sqrt{2}T\log\varepsilon}\sum_{k=-\infty}^{\infty}e^{-\frac{\pi^2k^2}{8(T\log\varepsilon)^2}} = \sum_{k=-\infty}^{\infty}e^{-8(T\log\varepsilon)^2k^2} \le \sum_{k=-\infty}^{\infty}e^{-8(\log\varepsilon)^2|k|} = \frac{2}{1-e^{-8(\log\varepsilon)^2}}-1,$$

where we used the Poisson summation again after the (first) estimate. That is,

$$H(z,Y_0',Y_1') \ll T\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K}\sqrt{Y_0(\gamma z)Y_0'}\,g_0\left(\frac{\log(Y_0(\gamma z)/Y_0')}{\sqrt{2}}\right), \tag{2.36}$$

where the implied constant depends only on the field K. We set $f(y) = (yY_0')^{\frac{1}{2}}g_0\left(\frac{\log(y/Y_0')}{\sqrt{2}}\right)$ for a positive real number y. The contribution of the identity element in the previous sum is $f(Y_0) = \sqrt{Y_0Y_0'}\,g_0(\log(Y_0/Y_0')/\sqrt{2})$. We will use Lemma 1.2.5 to estimate the remaining part of the sum. Recall that for a $\gamma\in\Gamma_\infty\setminus\Gamma_K$ different from the identity we have $Y_0(\gamma z)\le Y_0(z)^{-1}$.

Examining the derivative of f one infers that f is increasing on the interval $(0,e^{\frac{1}{2T^2}}Y_0']$ and decreasing on $[e^{\frac{1}{2T^2}}Y_0',\infty)$. We first handle the case when $Y_0^{-1}(z)\le e^{\frac{1}{2T^2}}Y_0'$, then f is increasing on the interval $(0,Y_0(z)^{-1}]$. Now we partition the sum in (2.36) as follows:

$$\sum_{\mathrm{id}\ne\gamma\in\Gamma_\infty\backslash\Gamma_K}f(Y_0(\gamma z)) = \sum_{n=1}^{\infty}\sum_{\substack{\mathrm{id}\ne\gamma\in\Gamma_\infty\backslash\Gamma_K\\ \frac{Y_0(z)^{-1}}{n+1}<Y_0(\gamma z)\le\frac{Y_0(z)^{-1}}{n}}}f(Y_0(\gamma z))$$
$$\le\sum_{n=1}^{\infty}a_n(z,Y_0(z)^{-1})f\left(\frac{Y_0(z)^{-1}}{n}\right),$$

where

$$a_n(z,Y_0(z)^{-1}) = \#\left\{\mathrm{id}\ne\gamma\in\Gamma_\infty\setminus\Gamma_K:\ \frac{Y_0(z)^{-1}}{n+1}<Y_0(\gamma z)\le\frac{Y_0(z)^{-1}}{n}\right\}$$

for every $n\in\mathbb{N}^+$. Note that $a_n(z,Y_0(z)^{-1})$ is finite by Lemma 1.2.5. Let $A_z(x,Y_0(z)^{-1}) = \sum_{1\le n\le x}a_n(z,Y_0(z)^{-1})$, then $A_z(x,Y_0(z)^{-1})\ll x^2Y_0(z)^2$ by Lemma 1.2.5, and by partial summation we obtain

$$\sum_{n=1}^{\infty} a_n(z, Y_0(z)^{-1}) f(Y_0(z)^{-1}/n) \ll \int_0^{\infty} \frac{|f'(y)|}{y^2}\, dy.$$

Now we turn to the case when $Y_0(z)^{-1} > e^{\frac{1}{2T^2}} Y_0'$ so that $Y_0(\gamma z)$ may be in the interval $(e^{\frac{1}{2T^2}} Y_0', Y_0(z)^{-1}]$. There is an $N \in \mathbb{N}$ such that $M := N e^{\frac{1}{2T^2}} Y_0' \geq Y_0(z)^{-1}$, so we partition the sum in (2.36) the following way:

$$\begin{aligned}\sum_{\mathrm{id}\neq\gamma\in\Gamma_\infty\backslash\Gamma_K} f(Y_0(\gamma z)) &= \sum_{n=1}^{\infty} \sum_{\substack{\mathrm{id}\neq\gamma\in\Gamma_\infty\backslash\Gamma_K \\ \frac{M}{n+1} < Y_0(\gamma z) \leq \frac{M}{n}}} f(Y_0(\gamma z)) \\ &\leq \sum_{n=1}^{\infty} a_n(z, M) f(M/n) + \sum_{n=1}^{\infty} a_n(z, M) f(M/(n+1)),\end{aligned}$$

since f takes its maximum on every interval $[M/(n+1), M/n]$ at one of the endpoints. From here we can continue as in the previous case and obtain that the elements different from the identity contribute in (2.36) all together at most a constant times

$$\int_0^{\infty} y^{-2} |f'(y)|\, dy \leq \frac{1}{Y_0'} \int_0^{\infty} (\sqrt{2} g_0(u) - 2g_0'(u)) \cosh\left(\frac{3\sqrt{2}}{2} u\right) du,$$

where we get the last estimate by a straightforward computation. First we estimate the second term. Note that $\cosh u \leq e^u$ for any $u \geq 0$, hence

$$\begin{aligned}\int_0^{\infty} -2g_0'(u) \cosh\left(\frac{3\sqrt{2}}{2} u\right) du &\leq \int_0^{\infty} -2g_0'(u) e^{3u}\, du = \frac{4T^3}{\sqrt{\pi}} \int_0^{\infty} u e^{-u^2T^2 + 3u}\, du \\ &= \frac{4T}{\sqrt{\pi}} e^{\frac{9}{4T^2}} \int_{-\frac{3}{2T}}^{\infty} \left(t + \frac{3}{2T}\right) e^{-t^2}\, dt \ll T,\end{aligned}$$

where the implied constant is absolute. Turning to the first integral, we remark that the Fourier transform $\hat{g_0}$ of g_0 is $e^{-(2\pi)^2 t^2/(4T^2)}$, so

$$\int_0^{\infty} g_0(u)(e^{cu} + e^{-cu})\, du = \int_{-\infty}^{\infty} g_0(u) e^{cu} = \hat{g_0}\left(\frac{ic}{2\pi}\right) = e^{\frac{c^2}{4T^2}} \ll 1.$$

Therefore, we have

$$H(z, Y_0', Y_1') \ll T\sqrt{Y_0(z) Y_0'}\, g_0\left(\frac{\log(Y_0(z)/Y_0')}{\sqrt{2}}\right) + \frac{T^2}{Y_0'}.$$

Choosing $y_1 = y_1'$ and $y_2 = y_2'$ in (2.34) we infer

$$\sum_j h(r_1^{(j)}, r_2^{(j)})|c_l^{(j)}|^2 y_1 y_2 K^2_{s_1^{(j)}-\frac{1}{2}}(2\pi\,|l_1|\,y_1) K^2_{s_2^{(j)}-\frac{1}{2}}(2\pi\,|l_2|\,y_2)+$$

$$+\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(r+\frac{\pi m}{2\log\varepsilon}, r-\frac{\pi m}{2\log\varepsilon}\right)\left|\phi_l\left(\frac{1}{2}+ir,m\right)\right|^2 \times$$

$$\times\, y_1 y_2 K^2_{ir+\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_1|\,y_1) K^2_{ir-\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_2|\,y_2)\,dr \ll T^2\left(y_1 y_2 + \frac{1}{y_1 y_2}\right).$$

As

$$h(r_1, r_2) = \int_{-\infty}^{\infty} g_0(u_1) e^{i r_1 u_1}\,du_1 \int_{-\infty}^{\infty} g_0(u_2) e^{i r_2 u_2}\,du_2 = \hat{g}_0\left(\frac{r_1}{2\pi}\right)\hat{g}_0\left(\frac{r_2}{2\pi}\right) = e^{-\frac{r_1^2+r_2^2}{4T^2}} \gg 1$$

once $|r_1|, |r_2| \ll T$, we obtain

$$\sum_{|s_1^{(j)}|,|s_2^{(j)}|\le T} |c_l^{(j)}|^2 y_1 y_2 K^2_{s_1^{(j)}-\frac{1}{2}}(2\pi\,|l_1|\,y_1) K^2_{s_2^{(j)}-\frac{1}{2}}(2\pi\,|l_2|\,y_2)+$$

$$+\sum_{m\in\mathbb{Z}} \int' \left|\phi_l\left(\frac{1}{2}+ir,m\right)\right|^2 y_1 y_2 K^2_{ir+\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_1|\,y_1) K^2_{ir-\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_2|\,y_2)\,dr$$

$$\ll T^2\left(y_1 y_2 + \frac{1}{y_1 y_2}\right),$$

where for a fixed m we integrate over those r's for which $\left|\frac{1}{2} + ri \pm \frac{\pi i m}{2\log\varepsilon}\right| \le T$ holds. Note that the implied constant depends on the field K.

To separate the coordinates we assume that $T_1 \le T$, $T_2 \le T$ and then the previous estimate implies

$$\sum_{\frac{T_1}{2}\le|s_1^{(j)}|\le T_1,\ \frac{T_2}{2}\le|s_2^{(j)}|\le T_2} |c_l^{(j)}|^2 y_1 y_2 K^2_{s_1^{(j)}-\frac{1}{2}}(2\pi\,|l_1|\,y_1) K^2_{s_2^{(j)}-\frac{1}{2}}(2\pi\,|l_2|\,y_2)+$$

$$+\sum_{m\in\mathbb{Z}} \int' \left|\phi_l\left(\frac{1}{2}+ir,m\right)\right|^2 y_1 y_2 K^2_{ir+\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_1|\,y_1) K^2_{ir-\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_2|\,y_2)\,dr$$

$$\ll T^2\left(y_1 y_2 + \frac{1}{y_1 y_2}\right),$$

where for a fixed integer m we integrate over the r's for which $\frac{T_1}{2} \le \left|\frac{1}{2} + ir + \frac{\pi i m}{2\log\varepsilon}\right| \le T_1$ and $\frac{T_2}{2} \le \left|\frac{1}{2} + ir - \frac{\pi i m}{2\log\varepsilon}\right| \le T_2$ hold. Next we integrate on the square

$$\left[\frac{c_1 T_1}{|l_1|}, \frac{d_1 T_1}{|l_1|}\right] \times \left[\frac{c_2 T_2}{|l_2|}, \frac{d_2 T_2}{|l_2|}\right]$$

for some positive real constants c_1, d_1, c_2, d_2 w.r.t. the measure $\frac{dy_1\,dy_2}{y_1^2 y_2^2}$:

$$\sum_{\frac{T_1}{2}\le|s_1^{(j)}|\le T_1,\ \frac{T_2}{2}\le|s_2^{(j)}|\le T_2} |c_l^{(j)}|^2 \int\limits_{\frac{c_1T_1}{|l_1|}}^{\frac{d_1T_1}{|l_1|}} K^2_{s_1^{(j)}-\frac12}(2\pi\,|l_1|\,y_1)\frac{dy_1}{y_1} \int\limits_{\frac{c_2T_2}{|l_2|}}^{\frac{d_2T_2}{|l_2|}} K^2_{s_2^{(j)}-\frac12}(2\pi\,|l_2|\,y_2)\frac{dy_2}{y_2}+$$

$$+\sum_{m\in\mathbb{Z}}\int{}' \left|\phi_l\left(\frac12+ir,m\right)\right|^2 \int\limits_{\frac{c_1T_1}{|l_1|}}^{\frac{d_1T_1}{|l_1|}} K^2_{ir+\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_1|\,y_1)\frac{dy_1}{y_1} \int\limits_{\frac{c_2T_2}{|l_2|}}^{\frac{d_2T_2}{|l_2|}} K^2_{ir-\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_2|\,y_2)\frac{dy_2}{y_2}\,dr$$

$$\ll T^2\left[\int\limits_{\frac{c_1T_1}{|l_1|}}^{\frac{d_1T_1}{|l_1|}}\frac{dy_1}{y_1}\int\limits_{\frac{c_2T_2}{|l_2|}}^{\frac{d_2T_2}{|l_2|}}\frac{dy_2}{y_2}+\int\limits_{\frac{c_1T_1}{|l_1|}}^{\frac{d_1T_1}{|l_1|}}\frac{dy_1}{y_1^3}\int\limits_{\frac{c_2T_2}{|l_2|}}^{\frac{d_2T_2}{|l_2|}}\frac{dy_2}{y_2^3}\right]\ll_{c_1,c_2,d_1,d_2} T^2+\frac{T^2\,|N(l)|^2}{T_1^2T_2^2}.$$

Substituting $u_k = 2\pi\,|l_k|\,y_k$ for $k=1,2$, the left hand side becomes

$$\sum_{\frac{T_1}{2}\le|s_1^{(j)}|\le T_1,\ \frac{T_2}{2}\le|s_2^{(j)}|\le T_2} |c_l^{(j)}|^2 \int\limits_{2\pi c_1T_1}^{2\pi d_1T_1} K^2_{s_1^{(j)}-\frac12}(u_1)\frac{du_1}{u_1} \int\limits_{2\pi c_2T_2}^{2\pi d_2T_2} K^2_{s_2^{(j)}-\frac12}(u_2)\frac{du_2}{u_2}+$$

$$+\sum_{m\in\mathbb{Z}}\int{}' \left|\phi_l\left(\frac12+ir,m\right)\right|^2 \int\limits_{2\pi c_1T_1}^{2\pi d_1T_1} K^2_{ir+\frac{\pi i m}{2\log\varepsilon}}(u_1)\frac{du_1}{u_1} \int\limits_{2\pi c_2T_2}^{2\pi d_2T_2} K^2_{ir-\frac{\pi i m}{2\log\varepsilon}}(u_2)\frac{du_2}{u_2}\,dr$$

Since in the first sum we have $|s_k^{(j)}|\ge\frac{T_k}{2}$ $(k=1,2)$, we infer that

$$\int\limits_{2\pi c_kT_k}^{2\pi d_kT_k} K^2_{s_k^{(j)}-\frac12}(u_k)\frac{du_k}{u_k}\ge\int\limits_{|s_k^{(j)}|/2}^{2\pi d_kT_k} K^2_{s_k^{(j)}-\frac12}(u_k)\frac{du_k}{u_k}$$

holds if $c_k\le 1/(8\pi)$. Also, for $s_k(r,m)=\frac12+ir+(-1)^{k-1}\frac{\pi i m}{2\log\varepsilon}$ we have

$$\int\limits_{2\pi c_kT_k}^{2\pi d_kT_k} K^2_{s_k(r,m)-\frac12}(u_k)\frac{du_k}{u_k}\ge\int\limits_{|s_k(r,m)|/2}^{2\pi d_kT_k} K^2_{s_k(r,m)-\frac12}(u_k)\frac{du_k}{u_k}.$$

As $T_k\ge|s_k|\,,|s_k(r,m)|$, we have by Lemma 2.1.5 and by the estimate (2.13) on page 45 that if d_k is big enough, then

$$\int\limits_{|s_k^{(j)}|/2}^{2\pi d_kT_k} K^2_{s_k^{(j)}-\frac12}(u_k)\frac{du_k}{u_k}=\int\limits_{|s_k^{(j)}|/2}^{\infty} K^2_{s_k^{(j)}-\frac12}(u_k)\frac{du_k}{u_k}-\int\limits_{2\pi d_kT_k}^{\infty} K^2_{s_k^{(j)}-\frac12}(u_k)\frac{du_k}{u_k}\gg|s_k^{(j)}|^{-1}e^{-\pi|s_k^{(j)}|},$$

and an analogous statement holds with $s_k(r,m)$ instead of $s_k^{(j)}$. Using this and also the condition $|s_k|, |s_k(r,m)| \le T_k$ we conclude

$$\sum_{\substack{\frac{T_k}{2} \le |s_k^{(j)}| \le T_k \\ k=1,2}} |c_l^{(j)}|^2 e^{-\pi(|s_1^{(j)}|+|s_2^{(j)}|)} + \sum_{m\in\mathbb{Z}} \int' \left|\phi_l\left(\frac{1}{2}+ir,m\right)\right|^2 e^{-\pi(s_1(r,m)+s_2(r,m))}\,dr \tag{2.37}$$

$$\ll T_1T_2T^2 + \frac{T^2\,|N(l)|^2}{T_1T_2},$$

where the estimate holds once T, T_1 and T_2 are bounded from below by a positive constant and $T_1, T_2 \le T$. We recall that the second sum and the integration is restricted to those m's and r's for which $\frac{T_k}{2} \le |s_k(r,m)| \le T_k$ holds, where

$$s_k(r,m) = \frac{1}{2} + ir + (-1)^{k-1}\frac{\pi i m}{2\log\varepsilon}.$$

The estimate (2.37) is the one that will be needed in the next chapter, but we derive a statement from this that resembles formula (8.27) in [11]. Let us denote the left hand side above by $\Sigma(T_1,T_2)$, then taking the integral (for a fixed m) over those r's for which $|s_k(r,m)| \le T$ holds we obtain

$$\sum_{|s_1^{(j)}|\le T,\,|s_2^{(j)}|\le T} |c_l^{(j)}|^2 e^{-\pi(|s_1^{(j)}|+|s_2^{(j)}|)} + \sum_{m\in\mathbb{Z}} \int' \left|\phi_l\left(\frac{1}{2}+ir,m\right)\right|^2 e^{-\pi(s_1(r,m)+s_2(r,m))}\,dr$$

$$= \sum_{k_1=0}^{\log_2(2T)} \sum_{k_2=0}^{\log_2(2T)} \Sigma(2^{-k_1}T, 2^{-k_2}T) \ll \sum_{k_1=0}^{\lfloor\log_2(2T)\rfloor} \sum_{k_2=0}^{\lfloor\log_2(2T)\rfloor} 2^{-(k_1+k_2)}T^4 + 2^{k_1+k_2}N(l)^2$$

$$\ll T^4 + T^2N(l)^2.$$

Chapter 3

A generalization of the Selberg trace formula

In this chapter we give a two dimensional version of the generalized Selberg trace formula worked out in [1]. This was obtained by computing the integral

$$\mathrm{Tr}K = \int_F K(z,z)u(z)\frac{dx\,dy}{y^2}$$

in two different ways ("geometrically" and "spectrally"). Here $y^{-2}\,dx\,dy$ is the usual measure on $\mathbb{H}$, $K(z,w)$ is an automorphic kernel function and u is a fixed automorphic form with respect to a finite volume Fuchsian group Γ with fundamental domain $F \subset \mathbb{H}$.

For the two dimensional trace formula we fix an automorphic form u that satisfies the growth condition $o(e^{2\pi y_k})$ for $k = 1, 2$. Then by Theorem 2.1.2 and Proposition 2.1.6 it is in fact of polynomial growth. Its eigenvalues are denoted by $\lambda_k = s_k(1-s_k)$, and we assume for simplicity that $\frac{1}{2} \le \mathrm{Re}\, s_k < 1$ holds ($k = 1, 2$). If u is not a cusp form, then its zeroth Fourier coefficient is $\eta y_1^{s_1} y_2^{s_2} + \phi y_1^{1-s_1} y_2^{1-s_2}$ for some $\eta, \phi \in \mathbb{C}$ and by Proposition 2.1.4 we have

$$(s_1, s_2) = \left(s + \frac{\pi i m_u}{2\log\varepsilon}, s - \frac{\pi i m_u}{2\log\varepsilon}\right), \tag{3.1}$$

for $s = \frac{s_1+s_2}{2}$ and some $m_u \in \mathbb{Z}$. The notations η, ϕ, s and m_u will be fixed throughout this chapter. Further assumptions will be made on the function u in Section 3.2.

We also fix an automorphic kernel function defined in (2.22). To this end we have to choose a function $\psi \in C^\infty(\mathbb{R}^2)$ and from now on it will be assumed to be compactly supported. We will evaluate the integral

$$\mathrm{Tr}_u K = \int_F K(z,z)u(z)\,d\mu(z)$$

in two different ways. As to that, we have to be careful here since this integral does not necessarily converge. Therefore, instead of this we work with the expression

$$\mathrm{Tr}_u^A K := \int_{F_A} K(z,z)u(z)\,d\mu(z), \tag{3.2}$$

where

$$F_A = \{z \in F : Y_0(z) \le A\}$$

for any $A > 0$. That is, we remove the part that is "close to the cusp ∞" from the fundamental domain of Γ_K.

Most of the methods in this chapter are generalizations of the ones used in [1] and [5]. But the main arc of our argument is in fact very common. The so-called geometric trace is obtained by the evaluation of (3.2) by partitioning Γ_K into conjugacy classes which results in integrals over fundamental domains of centralizers whose structures are convenient for computations. After that, we make use of the spectral theorem to obtain a different evaluation of the trace and infer the trace formula by comparing the two results.

The trace formula is given in Theorem 3.3.1 that clearly resembles Theorem 1 in [1]. Among others the Hecke L-functions appears in the result (in fact on both sides: as the contribution of the totally parabolic classes on the geometric side and in the zeroth coefficient of the Eisenstein series on the spectral side) in the same way as the zeta function does in the one dimensional case.

3.1 The geometric trace

In the following we compute $\mathrm{Tr}_u^A K$ by partitioning Γ_K into conjugacy classes. For an element $\gamma \in \Gamma_K$ we denote the conjugacy class of γ by $\{\gamma\}$. This way we get

$$\mathrm{Tr}_u^A K = \sum_{\{\gamma\}} \sum_{\sigma \in \{\gamma\}} \int_{F_A} k(z, \sigma z) u(z)\, d\mu(z).$$

where F is the fundamental domain of Γ_K.

The conjugacy class of the identity element consists only of itself, and the term that belongs to it is a constant multiple of the integral

$$\int_{F_A} u(z)\, d\mu(z).$$

This integral converges as $A \to \infty$ and the limit is zero since the Laplacians are symmetric operators and the eigenvalues of 1 and u are different.

From now on we assume that $\gamma \in \Gamma_K$ is not the identity. Since $\sigma_1^{-1}\gamma\sigma_1 = \sigma_2^{-1}\gamma\sigma_2$ if and only if $\sigma_2\sigma_1^{-1} \in C(\gamma)$, where $C(\gamma)$ is the centralizer of γ, this is equivalent to $\sigma_2 \in C(\gamma)\sigma_1$ and we get that

$$T_\gamma^A := \sum_{\sigma \in \{\gamma\}} \int_{F_A} k(z, \sigma z) u(z)\, d\mu(z) = \sum_{\sigma \in C(\gamma)\backslash\Gamma_K} \int_{F_A} k(z, \sigma^{-1}\gamma\sigma z) u(z)\, d\mu(z).$$

As $k(\varrho z, \varrho w) = k(z, w)$ for every $\varrho \in PSL(2, \mathbb{R})^2$ and u is invariant under the action of Γ_K, this last sum is

$$\sum_{\sigma \in C(\gamma)\backslash\Gamma_K} \int_{F_A} k(\sigma z, \gamma\sigma z) u(\sigma z)\, d\mu(z) = \int_{C(\gamma)\backslash H_A} k(z, \gamma z) u(z)\, d\mu(z),$$

where $H_A = \cup_{\gamma \in \Gamma_K} F_A$. Now for every $\varrho \in PSL(2, \mathbb{R})^2$ this is

$$\int_{\varrho^{-1}(C(\gamma)\backslash H_A)} k(\varrho z, \gamma\varrho z) u(\varrho z)\, d\mu(z) = \int_{(\varrho^{-1}C(\gamma)\varrho)\backslash\varrho^{-1}H_A} k(z, \varrho^{-1}\gamma\varrho z) u(\varrho z)\, d\mu(z)$$

since the measure μ and the function k are $PSL(2,\mathbb{R})^2$-invariant. So far this holds for every $id \neq \gamma \in \Gamma_K$. If γ is totally elliptic, totally hyperbolic or a mixed element then $T_\gamma = \lim_{A\to\infty} T_\gamma^A$ exists and

$$T_\gamma = \int_{(\varrho^{-1}C(\gamma)\varrho)\backslash\mathbb{H}^2} k(z, \varrho^{-1}\gamma\varrho z)u(\varrho z)\, d\mu(z) \tag{3.3}$$

holds. The existence of the limit follows easily from the absolute convergence of the final results of the next three sections which justifies the correctness of the prior computations. Note that $(\varrho^{-1}C(\gamma)\varrho)\backslash\mathbb{H}^2$ is nothing else but the fundamental domain of the group $\varrho^{-1}C(\gamma)\varrho$. We proceed by calculating T_γ^A or in the three cases mentioned above the limit T_γ.

3.1.1 Contribution of totally elliptic elements

Let $\gamma \in \Gamma_K$ be a totally elliptic element with the elliptic fixed point $z_\gamma \in \mathbb{H}^2$. Then the centraizer $C(\gamma)$ consists of the elements in Γ_K which fix the point z_γ (see [14], p. 37) and the stabilizer Γ_{z_γ} of z_γ in Γ_K is a finite cyclic group (see Remark 2.14 in [6]). Let us denote by m_γ the order of $C(\gamma)$. Every elliptic element in $PSL(2,\mathbb{R})$ is conjugate to an element of the form $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, hence the generator γ_0 of $C(\gamma)$ can be chosen so that

$$\gamma_0 \sim \gamma_0' = \left(\begin{bmatrix} \cos\dfrac{\pi}{m_\gamma} & \sin\dfrac{\pi}{m_\gamma} \\ -\sin\dfrac{\pi}{m_\gamma} & \cos\dfrac{\pi}{m_\gamma} \end{bmatrix}, \begin{bmatrix} \cos\dfrac{k_2\pi}{m_\gamma} & \sin\dfrac{k_2\pi}{m_\gamma} \\ -\sin\dfrac{k_2\pi}{m_\gamma} & \cos\dfrac{k_2\pi}{m_\gamma} \end{bmatrix} \right),$$

where the sign $\sim$ means that the two elements are conjugate by an element $\varrho \in PSL(2,\mathbb{R})^2$, and $k_2 \in \mathbb{Z}$ with $\gcd(k_2, m_\gamma) = 1$. Let us write $\gamma' = \varrho^{-1}\gamma\varrho$. To compute T_γ we give the fundamental domain $F_{C(\gamma')}$ of $C(\gamma') = \langle\gamma_0'\rangle \leq \varrho^{-1}\Gamma_K\varrho$. Since the first coordinate of γ_0' is a rotation around the point $i \in \mathbb{H}$ by the angle $2\pi/m_\gamma$ every $C(\gamma')$-orbit has exactly one point in the set $F_0 \times \mathbb{H}$, where $F_0 \subset \mathbb{H}$ is a sector enclosed by two half-lines with endpoint i and angle $2\pi/m_\gamma$. Note that in fact both coordinates are rotations around i which means that ϱ takes the point (i,i) to the fixed point of γ, namely z_γ. Now by (3.3) we have

$$T_\gamma = \int_{F_{C(\gamma')}} k(z, \gamma' z)u(\varrho z)\, d\mu(z) = \frac{1}{m_\gamma} \int_{\mathbb{H}^2} k(z, \gamma' z)u(\varrho z)\, d\mu(z),$$

where we used the $PSL(2,\mathbb{R})^2$-invariance of the function k and the measure μ, the Γ_K-invariance of u and that γ' and γ_0' commute. As $z = (z_1, z_2)$ one can write

$$\int_{\mathbb{H}^2} k(z, \gamma' z)u(\varrho z)\, d\mu(z) = \int_{\mathbb{H}} \int_{\mathbb{H}} k(z, \gamma' z)u(\varrho z)\, d\mu(z_1)\, d\mu(z_2), \tag{3.4}$$

where $\mu(z_k)$ denotes the measure $y_k^{-2} dx_k\, dy_k$. In the inner integral above the coordinate z_2 is fixed. Then the function $u(\varrho z)$ is a function of z_1 and it is the eigenfunction of the Laplace operator Δ_1 (because the operator commutes with the group action), furthermore, the value of $k(z, \gamma' z)$ depends only on the hyperbolic distance of z_1 and $\gamma'^{(1)} z_1$. To simplify the notation later we now write $u(\varrho z) = u_{z_2}(z_1)$ and $k(z, w) = k_{z_2,w_2}(z_1, w_1)$. Furthermore, as γ' is fixed we can simply write $k(z,\gamma' z) = k_{z_2}(z_1,\gamma'^{(1)}z_1)$. With this notation the inner integral is

$$T_{z_2} = \int_{\mathbb{H}} k(z,\gamma' z)u(\varrho z)\,d\mu(z_1) = \int_{\mathbb{H}} k_{z_2}(z_1,\gamma'^{(1)}z_1)u_{z_2}(z_1)\,d\mu(z_1).$$

For an $\alpha \in \mathbb{R}$ we introduce the notation

$$R(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \tag{3.5}$$

moreover, for a vector $\varphi = (\varphi_1,\varphi_2) \in \mathbb{R}^2$ we define $R(\varphi) = (R(\varphi_1),R(\varphi_2)) \in PSL(2,\mathbb{R})^2$. Then the elements of $C(\gamma')$ are of the form $R(\varphi)$ and similarly $\gamma' = R(\theta)$ for some $\theta = (\theta_1,\theta_2)$ where $\theta_k \in [0,\pi)$. Since $\gamma' \in C(\gamma')$ we have in fact $\theta_k = l_k\pi/m_\gamma$ for some integer $0 < l_k < m_\gamma$ $(k = 1,2)$. Note that all the elements $R(\varphi)$ have the same fixed point, namely (i,i), and hence they commute with each other (see [14], p. 36). It follows from this that θ is determined by γ, i.e. it is independent of the choice of ϱ (at least if both coordinates are in the interval $[0,\pi)$). Indeed, if $\varrho^{-1}\gamma\varrho = R(\theta)$ and $\sigma^{-1}\gamma\sigma = R(\theta')$ then

$$(\varrho^{-1}\sigma)^{-1}R(\theta)(\varrho^{-1}\sigma) = \sigma^{-1}\varrho R(\theta)\varrho^{-1}\sigma = \sigma^{-1}\gamma\sigma = R(\theta'),$$

so $R(\theta)$ and $R(\theta')$ are conjugate. But then $\varrho^{-1}\sigma$ fixes the point (i,i) and hence it is of the form $R(\varphi)$ and commutes with $R(\theta)$. Consequently $R(\theta) = R(\theta')$ and then $\theta = \theta'$. From now on we write $\theta(\gamma^{(k)})$ instead of θ_k and $\theta(\gamma) = (\theta(\gamma^{(1)}),\theta(\gamma^{(2)}))$ instead of θ.

Next we use geodesic polar coordinates (see [11], section 1.3), i.e. we make the substitution $z_1 = R(\varphi_1)e^{-r_1}i$ where $r_1 \in (0,\infty)$ is the hyperbolic distance of i and z_1 and $\varphi_1 \in [0,\pi)$. Then we have

$$d\mu(z_1) = (2\sinh r_1)\,dr_1\,d\varphi_1$$

and the integral above is

$$T_{z_2} = \int_0^\infty \int_0^\pi k_{z_2}(R(\varphi_1)e^{-r_1}i, R(\theta(\gamma^{(1)}))R(\varphi_1)e^{-r_1}i))u_{z_2}(R(\varphi_1)e^{-r_1}i)2\sinh r_1\,d\varphi_1 dr_1.$$

As the elements $R(\theta(\gamma^{(1)}))$ and $R(\varphi_1)$ commute and k_{z_2} depends only on the hyperbolic distance of the variables we get that

$$T_{z_2} = \int_0^\infty k_{z_2}(e^{-r_1}i, R(\theta(\gamma^{(1)}))e^{-r_1}i))\left(\int_0^\pi u_{z_2}(R(\varphi_1)e^{-r_1}i)\,d\varphi_1\right)(2\sinh r_1)\,dr_1.$$

We recall that

$$k_{z_2,w_2}(z_1,w_1) = \psi\left(\rho(z_1,w_1),\rho(z_2,w_2)\right) =: \psi_{z_2,w_2}(\rho(z_1,w_1)),$$

where

$$\rho(z_k,w_k) = \frac{|z_k - w_k|^2}{\operatorname{Im} z_k \operatorname{Im} w_k}$$

for $k = 1,2$. One gets by a computation that

$$\rho(z_k, R(\theta(\gamma^{(k)}))z_k) = \frac{|z_k^2+1|^2\sin^2\theta(\gamma^{(k)})}{y_k^2}$$

where $z_k = x_k + iy_k$ and then

$$\rho(e^{-r_k} i, R(\theta(\gamma^{(k)}))e^{-r_k} i) = \frac{|-e^{-2r_k}+1|^2 \sin^2\theta(\gamma^{(k)})}{e^{-2r_k}} = (2\sinh r_k)^2 \sin^2\theta(\gamma^{(k)}).$$

This gives that

$$T_{z_2} = \int_0^\infty \psi_{z_2}((2\sinh r_1)^2 \sin^2\theta(\gamma^{(1)})) \left(\int_0^\pi u_{z_2}(R(\varphi_1)e^{-r_1} i)\, d\varphi_1\right)(2\sinh r_1)\, dr_1,$$

where ψ_{z_2} is just an abbreviation for $\psi_{z_2,\gamma'^{(2)}z_2}$. Let us define

$$G_{z_2}(z_1) = \frac{1}{\pi}\int_0^\pi u_{z_2}(R(\varphi_1)z_1)\, d\varphi_1.$$

By Lemma 1.10 in [11] the value of G_{z_2} depends only on the hyperbolic distance of z_1 and i. Moreover, G_{z_2} is the eigenfunction of the operator Δ_1 with eigenvalue λ_1, where λ_1 is the first coordinate of the eigenvalue vector of u. Now by Lemma 1.12 of [11] this function is unique up to a constant factor. Furthermore

$$\Delta_1 = \frac{\partial^2}{\partial r_1^2} + \frac{\cosh r_1}{\sinh r_1}\frac{\partial}{\partial r_1} + \frac{1}{4\sinh^2 r_1}\frac{\partial^2}{\partial \varphi_1^2},$$

so with the notation $G_{z_2}(z_1) = G_{z_2}(r_1, \varphi_1)$ we have

$$\frac{\partial^2}{\partial r_1^2} G_{z_2}(r_1,\varphi_1) + \frac{\cosh r_1}{\sinh r_1}\frac{\partial}{\partial r_1} G_{z_2}(r_1,\varphi_1) = \lambda_1 G_{z_2}(r_1,\varphi_1).$$

Let $g_{\lambda_1}(r) : [0,\infty) \to \mathbb{C}$ be the unique solution of the differential equation

$$g''(r) + \frac{\cosh r}{\sinh r} g'(r) = \lambda_1 g(r) \tag{3.6}$$

satisfying the initial condition $g(0) = 1$, then

$$G_{z_2}(z_1) = g_{\lambda_1}(r_1) u_{z_2}(i) = g_{\lambda_1}(r_1) u(\varrho^{(1)} i, \varrho^{(2)} z_2).$$

That is,

$$T_{z_2} = 2\pi u(\varrho^{(1)} i, \varrho^{(2)} z_2) \int_0^\infty \psi_{z_2}((2\sinh r_1)^2 \sin^2\theta(\gamma^{(1)})) g_{\lambda_1}(r_1) \sinh r_1\, dr_1.$$

Substituting this in (3.4) and interchanging the integrals we get

$$T_\gamma = \frac{2\pi}{m_\gamma}\int_0^\infty T_{r_1} g_{\lambda_1}(r_1) \sinh r_1\, dr_1,$$

where

$$T_{r_1} = \int_{\mathbb{H}} \psi((2\sinh r_1)^2 \sin^2\theta(\gamma^{(1)}), \rho(z_2, \gamma'^{(2)} z_2)) u(\varrho^{(1)} i, \varrho^{(2)} z_2)\, d\mu(z_2).$$

Evaluating T_{r_1} the same way as above we get that

$$T_\gamma = \frac{(2\pi)^2}{m_\gamma} u(z_\gamma) \int_0^\infty \int_0^\infty \psi(S(r_1, \theta(\gamma^{(1)})), S(r_2, \theta(\gamma^{(2)}))) \left(\prod_{k=1,2} g_{\lambda_k}(r_k) \sinh r_k \right) dr_1 dr_2,$$

where $S(r, \theta) = (2\sinh r)^2 \sin^2 \theta$. Finally we recall that by Theorem 1.3.4 the contribution of the elliptic conjugacy classes in the formula is a finite sum of T_γ's.

3.1.2 Contribution of totally hyperbolic elements

Let $\gamma \in \Gamma_K$ be a totally hyperbolic element. Then by Theorem I.5.7 in [5] the centralizer $C(\gamma)$ of γ is a free abelian group of rank 2. The element γ is conjugate in $PSL(2, \mathbb{R})^2$ to an element of the form

$$\nu = \left(\begin{bmatrix} N(\gamma^{(1)})^{1/2} & 0 \\ 0 & N(\gamma^{(1)})^{-1/2} \end{bmatrix}, \begin{bmatrix} N(\gamma^{(2)})^{1/2} & 0 \\ 0 & N(\gamma^{(2)})^{-1/2} \end{bmatrix} \right)$$

where $N(\gamma^{(k)}) > 1$. Note that $N(\gamma^{(k)})$ is called the *norm* of $\gamma^{(k)}$ and it is determined uniquely by $\gamma^{(k)}$ since if

$$\sigma^{-1} \alpha \sigma = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \qquad \text{and} \qquad \varrho^{-1} \alpha \varrho = \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}$$

hold for some elements in $PSL(2, \mathbb{R})$ and $a, b > 1$ is true as well, then

$$[\varrho^{-1}][\sigma] \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} [\sigma^{-1}][\varrho] = [\varrho][\alpha][\varrho^{-1}] = \pm \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix},$$

so for some real numbers x, y, z, w we have

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = \begin{bmatrix} ax & a^{-1}y \\ az & a^{-1}w \end{bmatrix} = \pm \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \pm \begin{bmatrix} bx & by \\ b^{-1}z & b^{-1}w \end{bmatrix}.$$

If the equation holds with negative sign, then $x = y = z = w = 0$, and this is impossible. Hence we have positive sign above, and if $x \neq 0$ then $a = b$. Otherwise $z \neq 0$ and then $a = b^{-1}$ which is also impossible since $a, b > 1$.

If $\nu = \varrho^{-1} \gamma \varrho$, then the centralizer $C(\nu) \leq \varrho^{-1} \Gamma_K \varrho$ is $\varrho^{-1} C(\gamma) \varrho$ and it is generated by the elements $\nu_i = \varrho^{-1} \gamma_i \varrho$ for $i = 1, 2$ where γ_1 and γ_2 are the generators of $C(\gamma)$. As the γ_i's have the same fixed points as γ this is true also for the conjugates and therefore

$$\nu_i = \left(\begin{bmatrix} N(\gamma_i^{(1)})^{1/2} & 0 \\ 0 & N(\gamma_i^{(1)})^{-1/2} \end{bmatrix}, \begin{bmatrix} N(\gamma_i^{(2)})^{1/2} & 0 \\ 0 & N(\gamma_i^{(2)})^{-1/2} \end{bmatrix} \right)$$

for $i = 1, 2$. Note that $N(\gamma_i^{(1)}) > 1$ can be reached by changing the generator to its inverse if necessary (but then $N(\gamma_i^{(2)}) > 1$ may not be assured).

For every $z = (z_1, z_2)$ we have

$$\left| \nu_i^{(k)} z_k \right| = N(\gamma_i^{(k)}) \left| z_k \right|, \quad \arg \nu_i^{(k)} z_k = \arg z_k,$$

and from this one easily gets the following

Proposition 3.1.1. *The fundamental domain $F_{C(\nu)} = F_{\varrho^{-1}C(\gamma)\varrho}$ for the group $C(\nu) = \varrho^{-1}C(\gamma)\varrho$ is given by*

$$(\log|z_1|, \log|z_2|) \in P_\gamma, \qquad (\arg z_1, \arg z_2) \in (0, \pi),$$

where

$$P_\gamma = \{s(\log N(\gamma_1^{(1)}), \log N(\gamma_1^{(2)})) + t(\log N(\gamma_2^{(1)}), \log N(\gamma_2^{(2)})) : 0 \le s, t < 1\} \subset \mathbb{R}^2.$$

Now from (3.3) we have

$$T_\gamma = \int_{F_{C(\nu)}} k(z, \nu z) u(\varrho z)\, d\mu(z).$$

We change to polar coordinates, i.e. make the substitution $z_k = r_k e^{i(\pi/2+\vartheta_k)}$ where $r_k \in (0, \infty)$ and $\vartheta_k \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $(k = 1, 2)$. We obtain by a computation that

$$\rho(z_k, \nu^{(k)} z_k) = \frac{N(\gamma^{(k)}) + N(\gamma^{(k)})^{-1} - 2}{\cos^2\vartheta_k},$$

and since

$$\frac{dx_k\, dy_k}{y_k^2} = \frac{dr_k\, d\vartheta_k}{r_k \cos^2\vartheta_k},$$

the integral T_γ is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi\left(\frac{N(\gamma^{(1)}) + N(\gamma^{(1)})^{-1} - 2}{\cos^2\vartheta_1}, \frac{N(\gamma^{(2)}) + N(\gamma^{(2)})^{-1} - 2}{\cos^2\vartheta_2}\right) \frac{F(e^{i(\frac{\pi}{2}+\vartheta_1)}, e^{i(\frac{\pi}{2}+\vartheta_2)})\, d\vartheta_1\, d\vartheta_2}{\cos^2\vartheta_1 \cos^2\vartheta_2},$$

where

$$F(z) = \int_{(\log r_1, \log r_2) \in P_\gamma} u(\varrho^{(1)}(r_1 z_1), \varrho^{(2)}(r_2 z_2)) \frac{dr_1\, dr_2}{r_1 r_2} \tag{3.7}$$

for any $z \in \mathbb{H}^2$.

Lemma 3.1.2. *The function F defined in* (3.7) *is invariant under coordinate-wise scalar multiplication, i.e. $F(R_1 z_1, R_2 z_2) = F(z_1, z_2)$ for every $R_1, R_2 \in (0, \infty)$ and $z_1, z_2 \in \mathbb{H}$.*

Proof. By the definition of the function F we have

$$F(R_1 z_1, R_2 z_2) = \int_{(\log r_1, \log r_2) \in P_\gamma} u(\varrho^{(1)}(r_1 R_1 z_1), \varrho^{(2)}(r_2 R_2 z_2)) \frac{dr_1\, dr_2}{r_1 r_2}$$

$$= \int_{(\log r_1, \log r_2) \in P_\gamma + (\log R_1, \log R_2)} u(\varrho^{(1)}(r_1 z_1), \varrho^{(2)}(r_2 z_2)) \frac{dr_1\, dr_2}{r_1 r_2}$$

We can divide the parallelogram $P_\gamma + (\log R_1, \log R_2)$ into at most four disjoint parts such that each part is entirely contained in a parallelogram of the form $P_\gamma + k\underline{n}_1 + m\underline{n}_2$, where $\underline{n}_i = (\log N(\gamma_i^{(1)}), \log N(\gamma_i^{(2)}))$ $(i = 1, 2)$ and $k, m \in \mathbb{Z}$, i.e. in a translated image of P_γ by a lattice point of the lattice generated by $\underline{n}_1$ and $\underline{n}_2$. Since u is invariant under the action of Γ_K, it follows that

$$u(\varrho\nu_i z) = u(\varrho(\varrho^{-1}\gamma_j\varrho)z) = u(\varrho z)$$

for $i = 1, 2$. Using this equality one can translate back the above mentioned parts into P_γ in the last integral by the same type of substitutions that we did above. Then the translated parts make up P_γ and the assertion follows. $\square$

Using the notation $z_k = r_k e^{i(\frac{\pi}{2}+\vartheta_k)}$ $(k = 1, 2)$, the previous lemma gives that F depends only on the vector $(\vartheta_1, \vartheta_2)$. Since u is the eigenfunction of Δ_k with eigenvalues λ_k and this operator commutes with the group action, we infer that $F(z)$ is also an eigenfunction of the Laplacians with the same eigenvalues. As

$$\Delta_k = (r_k \cos\vartheta_k)^2 \left(\frac{\partial^2}{\partial r_k^2} + r_k^{-1}\frac{\partial}{\partial r_k} + r_k^{-2}\frac{\partial^2}{\partial \vartheta_k^2} \right),$$

we obtain the differential equations

$$\frac{\partial^2 F}{\partial \vartheta_k^2}(\vartheta_1, \vartheta_2) = \frac{\lambda_k}{\cos^2\vartheta_k} F(\vartheta_1, \vartheta_2) \qquad (\vartheta_k \in (-\pi/2, \pi/2),\ k = 1, 2). \tag{3.8}$$

Let $f_{\lambda_k}(\vartheta)$ be the unique solution of the differential equation

$$F''(\vartheta) = \frac{\lambda_k}{\cos^2\vartheta} F(\vartheta) \qquad (\vartheta \in (-\pi/2, \pi/2)) \tag{3.9}$$

with the initial condition $f_{\lambda_k}(0) = 1$ and $f'_{\lambda_k}(0) = 0$, and $\tilde{f}_{\lambda_k}(\vartheta)$ the one with $\tilde{f}_{\lambda_k}(0) = 0$ and $\tilde{f}'_{\lambda_k}(0) = 1$. Note that $f_{\lambda_k}(-\vartheta)$ satisfies (3.9) and the initial conditions of $f_{\lambda_k}(\vartheta)$ and hence they agree, i.e. f_{λ_k} is an even function. Similarly, $\tilde{f}_{\lambda_k}$ is an odd function.

Now by (3.8)

$$F(\vartheta_1, \vartheta_2) = F(0, \vartheta_2) f_{\lambda_1}(\vartheta_1) + \frac{\partial F}{\partial \vartheta_1}(0, \vartheta_2)\tilde{f}_{\lambda_1}(\vartheta_1)$$

holds for every fixed ϑ_2, and using the notation

$$\psi_\gamma(\vartheta_1, \vartheta_2) := \psi\left(\frac{N(\gamma^{(1)}) + N(\gamma^{(1)})^{-1} - 2}{\cos^2\vartheta_1}, \frac{N(\gamma^{(2)}) + N(\gamma^{(2)})^{-1} - 2}{\cos^2\vartheta_2} \right)$$

we get that

$$T_\gamma = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_\gamma(\vartheta_1, \vartheta_2) \left(F(0, \vartheta_2) f_{\lambda_1}(\vartheta_1) + \frac{\partial F}{\partial \vartheta_1}(0, \vartheta_2)\tilde{f}_{\lambda_1}(\vartheta_1) \right) \frac{d\vartheta_1}{\cos^2\vartheta_1}\frac{d\vartheta_2}{\cos^2\vartheta_2}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_\gamma(\vartheta_1, \vartheta_2) F(0, \vartheta_2) f_{\lambda_1}(\vartheta_1) \frac{d\vartheta_1}{\cos^2\vartheta_1}\frac{d\vartheta_2}{\cos^2\vartheta_2}$$

because $\psi_\gamma(\cdot, \vartheta_2)$ and $\cos^{-2}$ are even and hence $\psi_\gamma(\cdot, \vartheta_2)\cos^{-2}\tilde{f}_{\lambda_1}$ is an odd function. Similarly,

$$F(0, \vartheta_2) = F(0, 0) f_{\lambda_2}(\vartheta_2) + \frac{\partial F}{\partial \vartheta_2}(0, 0)\tilde{f}_{\lambda_2}(\vartheta_2)$$

and using also that f_{λ_1} is an even function we have

$$T_\gamma = F(0,0) \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_\gamma(\vartheta_1, \vartheta_2) f_{\lambda_1}(\vartheta_1) f_{\lambda_2}(\vartheta_2) \frac{d\vartheta_1\, d\vartheta_2}{\cos^2\vartheta_1 \cos^2\vartheta_2},$$

where

$$F(0,0) = \int\limits_{(\log r_1, \log r_2) \in P_\gamma} u(\varrho^{(1)}(r_1 i), \varrho^{(2)}(r_2 i)) \frac{dr_1\, dr_2}{r_1 r_2}.$$

Note that since ψ has compact support and

$$\frac{N(\gamma^{(k)}) + N(\gamma^{(k)})^{-1} - 2}{\cos^2\vartheta_k} \geq N(\gamma^{(k)}) + N(\gamma^{(k)})^{-1} - 2 = \left|\mathrm{tr}\,[\gamma^{(k)}]\right| - 2$$

for $k = 1, 2$, we get $T_\gamma = 0$ once $t_k := \left|\mathrm{tr}\,[\gamma^{(k)}]\right|$ is big enough for some k. But $t_k \in \mathcal{O}_K$ and $t_2 = t_1'$ so there are only finitely many possible pairs (t_1, t_2) for which $T_\gamma \neq 0$. Moreover, for every $t_1 \in \mathcal{O}_K$ there are only finitely many totally hyperbolic conjugacy classes with trace t_1 (see [5], Proposition I.7.1 and the paragraph after Definition I.7.2) and hence $T_\gamma = 0$ for all but finitely many classes.

3.1.3 Contribution of mixed elements

For a mixed element we can apply the methods of the previous two sections and most of the computations will be omitted. Let $\gamma \in \Gamma_K$ be a mixed element, without loss of generality we may assume that its first coordinate is hyperbolic and the second one is elliptic. Then by Theorem I.5.7 of [5] the centralizer $C(\gamma)$ of γ is a free abelian group of rank 1 generated by an element γ_0 of the same type (since the fixed points of γ and γ_0 are the same) and hence for some $\varrho \in PSL(2,\mathbb{R})^2$ we have

$$\gamma' := \varrho^{-1}\gamma\varrho = \left(\begin{bmatrix} (N(\gamma^{(1)}))^{\frac{1}{2}} & 0 \\ 0 & (N(\gamma^{(1)}))^{-\frac{1}{2}} \end{bmatrix}, \begin{bmatrix} \cos\theta(\gamma^{(2)}) & \sin\theta(\gamma^{(2)}) \\ -\sin\theta(\gamma^{(2)}) & \cos\theta(\gamma^{(2)}) \end{bmatrix} \right)$$

and

$$\gamma_0' := \varrho^{-1}\gamma_0\varrho = \left(\begin{bmatrix} (N(\gamma_0^{(1)}))^{\frac{1}{2}} & 0 \\ 0 & (N(\gamma_0^{(1)}))^{-\frac{1}{2}} \end{bmatrix}, \begin{bmatrix} \cos\theta(\gamma_0^{(2)}) & \sin\theta(\gamma_0^{(2)}) \\ -\sin\theta(\gamma_0^{(2)}) & \cos\theta(\gamma_0^{(2)}) \end{bmatrix} \right),$$

and the centralizer of γ' in $\varrho^{-1}\Gamma_K\varrho$ is $\varrho^{-1}C(\gamma)\varrho$.

Now we determine the fundamental domain of $\varrho^{-1}C(\gamma)\varrho$. If $z \in \mathbb{H}^2$, then

$$\gamma_0' z = (N(\gamma_0^{(1)}) z_1, R(\theta(\gamma_0^{(2)})) z_2),$$

where $R(\theta_0) \in PSL(2,\mathbb{R})$ is defined in (3.5). Since γ_0 is the generator of $\varrho^{-1}C(\gamma)\varrho$ it follows immediately that the fundamental domain of this group is

$$F_{\varrho^{-1}C(\gamma)\varrho} = \{z \in \mathbb{H}^2 : \log|z_1| \in [0, \log N(\gamma_0^{(1)}))\}.$$

So then

$$T_\gamma = \int\limits_{F_{\varrho^{-1}C(\gamma)\varrho}} k(z,\gamma' z)u(\varrho z)\,d\mu(z) = \int\limits_{\log|z_1|\in[0,\log N(\gamma_0^{(1)}))} \int\limits_{\mathbb{H}} k(z,\gamma' z)u(\sigma z)\,d\mu(z_2)\,d\mu(z_1).$$

Like in Section 3.1.1 we get that this is

$$2\pi \int\limits_0^\infty \int\limits_{\log|z_1|\in[0,\log N(\gamma_0^{(1)}))} \psi(\rho(z_1,\gamma'^{(1)}z_1), S(r_2,\theta(\gamma^{(2)})))u(\varrho^{(1)}z_1,\varrho^{(2)}i)\,d\mu(z_1)\ g_{\lambda_2}(r_2)\sinh r_2\,dr_2.$$

where $S(r,\theta) = (2\sinh r)^2\sin^2\theta$ and $g_{\lambda_2}(r) : [0,\infty) \to \mathbb{C}$ is the unique solution of the differential equation (3.6) satisfying the initial condition $g(0) = 1$.

From here we can continue as in the previous section to obtain that T_γ is

$$2\pi \int\limits_1^{N(\gamma_0^{(1)})} u(\varrho^{(1)}(r_1 i),\varrho^{(2)}i)\frac{dr_1}{r_1}\cdot$$

$$\cdot\int\limits_0^\infty \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi(N(\gamma^{(1)},\vartheta_1), S(r_2,\theta(\gamma^{(2)})))\frac{f_{\lambda_1}(\vartheta_1)\,d\vartheta_1}{\cos^2\vartheta_1} g_{\lambda_2}(r_2)\sinh r_2\,dr_2,$$

where

$$N(\gamma^{(1)},\vartheta_1) = \frac{N(\gamma^{(1)}) + N(\gamma^{(1)})^{-1} - 2}{\cos^2\vartheta_1},$$

and $f_{\lambda_1}(\vartheta) : (-\frac{\pi}{2},\frac{\pi}{2}) \to \mathbb{C}$ is the unique solution of the differential equation (3.9) satisfying $f_{\lambda_j}(0) = 1$ and $f'_{\lambda_j}(0) = 0$. Finally, as in the previous section one can see that $T_\gamma \neq 0$ holds for only finitely many mixed classes.

3.1.4 Contribution of hyperbolic-parabolic elements

We continue with the identification of the hyperbolic-parabolic conjugacy classes. Every element of this type is conjugate in Γ_K to an element of the form

$$\gamma_{m,\alpha} := \begin{bmatrix} \varepsilon^m & \alpha \\ 0 & \varepsilon^{-m} \end{bmatrix}, \tag{3.10}$$

where $m \in \mathbb{Z}\setminus\{0\}$ and $\alpha \in \mathcal{O}_K$. If two such elements $\gamma_{m,\alpha}$ and $\gamma_{n,\beta}$ are conjugate in Γ_K, then there exists an element $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_K$ such that

$$\begin{bmatrix} \varepsilon^m & \alpha \\ 0 & \varepsilon^{-m} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \varepsilon^n & \beta \\ 0 & \varepsilon^{-n} \end{bmatrix},$$

that is

$$\begin{bmatrix} \varepsilon^m a + \alpha c & \varepsilon^m b + \alpha d \\ \varepsilon^{-m}c & \varepsilon^{-m}d \end{bmatrix} = \begin{bmatrix} a\varepsilon^n & a\beta + b\varepsilon^{-n} \\ c\varepsilon^n & c\beta + d\varepsilon^{-n} \end{bmatrix}$$

The equations $\varepsilon^{-m}c = c\varepsilon^n$ and $\varepsilon^m a + \alpha c = a\varepsilon^n$ give that if $c = 0$, then $m = n$, while if $c \neq 0$, then $n = -m$.

Next we show that for every $m \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathcal{O}_K$ the element $\gamma_{m,\alpha}$ is conjugate to a $\gamma_{-m,\beta}$ for some $\beta \in \mathcal{O}_K$. Suppose that $\tau\gamma_{m,\alpha}\tau^{-1} = \gamma_{-m,\beta}$. If q is the real fixed point of $\gamma_{m,\alpha}$, then the conjugate fixes $\tau\infty$ and also τq, so one of these two points must be ∞. Now if $\tau\infty = \infty$, then $\tau = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, which is impossible since then the conjugate would be of the form $\gamma_{m,\beta}$ (as we have seen in the previous paragraph). It follows that τ takes q to ∞. This point can be written explicitly:

$$\frac{\varepsilon^m q + \alpha}{\varepsilon^{-m}} = q \Longleftrightarrow q = \frac{-\alpha}{\varepsilon^m - \varepsilon^{-m}}.$$

The group Γ_K acts transitively on $K \cup \{\infty\}$, so there is a $\tau = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_K$ that takes $\frac{-\alpha}{\varepsilon^m - \varepsilon^{-m}}$ to ∞ and then

$$d = \frac{c\alpha}{\varepsilon^m - \varepsilon^{-m}}.$$

Moreover, the determinant of the matrix is 1, i.e.

$$\frac{ac\alpha}{\varepsilon^m - \varepsilon^{-m}} - bc = 1,$$

and from this we infer

$$c = \frac{\varepsilon^m - \varepsilon^{-m}}{a\alpha - b(\varepsilon^m - \varepsilon^{-m})}, \qquad d = \frac{\alpha}{a\alpha - b(\varepsilon^m - \varepsilon^{-m})}.$$

These two numbers are coprime in $\mathcal{O}_K$ and hence $\Lambda = a\alpha - b(\varepsilon^m - \varepsilon^{-m})$ is the generator of the ideal $(\varepsilon^m - \varepsilon^{-m}, \alpha)$ so it is determined up to a unit factor. This means that the positive integer $N(\Lambda)^2$ is independent of the choice of τ. As $\tau\gamma_{m,\alpha}\tau^{-1}$ fixes the point ∞ it must be of the form $\gamma_{n,\beta}$ for some $n \in \mathbb{Z} \setminus \{0\}$ and $\beta \in \mathcal{O}_K$. We have already seen that $n = \pm m$ holds in this case and since $c \neq 0$ we must have in fact $n = -m$. But $\tau\gamma_{m,\alpha}\tau^{-1}$ fixes also the point $\tau\infty = \frac{a\Lambda}{\varepsilon^m - \varepsilon^{-m}}$ and hence

$$\tau\gamma_{m,\alpha}\tau^{-1} = \begin{bmatrix} \varepsilon^{-m} & a\Lambda \\ 0 & \varepsilon^m \end{bmatrix}.$$

Now assume that the elements $\gamma_{m,\alpha}$ and $\gamma_{m,\beta}$ are conjugate to each other. Then for some $l \in \mathbb{Z}$ and $a \in \mathcal{O}_K$

$$\begin{aligned} \begin{bmatrix} \varepsilon^m & \beta \\ 0 & \varepsilon^{-m} \end{bmatrix} &= \begin{bmatrix} \varepsilon^{-l} & -a \\ 0 & \varepsilon^l \end{bmatrix} \begin{bmatrix} \varepsilon^m & \alpha \\ 0 & \varepsilon^{-m} \end{bmatrix} \begin{bmatrix} \varepsilon^l & a \\ 0 & \varepsilon^{-l} \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon^{m-l} & \varepsilon^{-l}\alpha - \varepsilon^{-m}a \\ 0 & \varepsilon^{l-m} \end{bmatrix} \begin{bmatrix} \varepsilon^l & a \\ 0 & \varepsilon^{-l} \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon^m & \varepsilon^{m-l}a + \varepsilon^{-2l}\alpha - \varepsilon^{-m-l}a \\ 0 & \varepsilon^{-m} \end{bmatrix}, \end{aligned}$$

that is,

$$\beta = \varepsilon^{-l}a(\varepsilon^m - \varepsilon^{-m}) + \varepsilon^{-2l}\alpha.$$

This means that β is congruent to $\varepsilon^{-2l}\alpha$ for some $l \in \mathbb{Z}$ modulo $\varepsilon^m - \varepsilon^{-m}$. On the other hand, choosing an appropriate a and l and conjugating by $\gamma_{l,a}$ we get that $\gamma_{m,\alpha}$ is conjugate to $\gamma_{m,\beta}$ for any element β of the cosets of the principal ideal $(\varepsilon^m - \varepsilon^{-m})$ represented by a number $\varepsilon^{2l}\alpha$ for some $l \in \mathbb{Z}$. The number of these cosets is finite (in fact at most $|N(\varepsilon^m - \varepsilon^{-m})|$) and we can summarize all this in the following statement:

Proposition 3.1.3. *Every hyperbolic-parabolic element is conjugate in Γ_K to an element of the form $\gamma_{m,\alpha}$ for some $m \in \mathbb{N}^+$ and $\alpha \in \mathcal{O}_K$. Moreover, for a fixed $m \in \mathbb{N}^+$, the number of the conjugacy classes represented by an element $\gamma_{m,\alpha}$ is finite.*

We also have the following result (see §20 in [14]):

Proposition 3.1.4. *The centralizer $C(\gamma_{m,\alpha})$ of the element $\gamma_{m,\alpha}$ is a cyclic group generated by an element*

$$\gamma_k(m,\alpha) = \gamma_k := \begin{bmatrix} \varepsilon^k & \alpha\frac{\varepsilon^k - \varepsilon^{-k}}{\varepsilon^m - \varepsilon^{-m}} \\ 0 & \varepsilon^{-k} \end{bmatrix},$$

where $k \in \mathbb{Z} \setminus \{0\}$.

One can prove by induction that $\gamma_k^n = \gamma_{nk}$ for any $n \in \mathbb{Z}$. As $\gamma_{m,\alpha}$ is in the centralizer, we have that $\gamma_{m,\alpha} = \gamma_k^n = \gamma_{nk}$ for some $n \in \mathbb{Z}$ which means that $nk = m$, i.e. $k \mid m$. If follows that k can be chosen as the smallest positive divisor of m for which $\frac{\alpha(\varepsilon^k - \varepsilon^{-k})}{\varepsilon^m - \varepsilon^{-m}}$ is an algebraic integer.

We now describe a fundamental domain $F_{C(\gamma_{\alpha,m})}$ of the centralizer $C(\gamma_{\alpha,m}) = \langle \gamma_k \rangle$. Let C_k denote the cyclic group generated by

$$\rho_k := \begin{bmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{bmatrix}. \tag{3.11}$$

We fix the notation $E = E_m = \varepsilon^m - \varepsilon^{-m}$. Then $C(\gamma_{\alpha,m}) = \sigma^{-1} C_k \sigma$ where $\sigma = \begin{bmatrix} 1 & \frac{\alpha}{E} \\ 0 & 1 \end{bmatrix}$ (because $\gamma_k = \sigma^{-1}\rho_k\sigma$) and hence if F_{C_k} is a fundamental domain for C_k, then

$$F_{C(\gamma_{\alpha,m})} = \sigma^{-1} F_{C_k} = F_{C_k} - \left(\frac{\alpha}{E}, \frac{\alpha'}{E'}\right)$$

is a fundamental domain for $C(\gamma_{\alpha,m})$.

As in the case of the totally hyperbolic elements we use polar coordinates. That is, for a point $z = (z_1, z_2) \in \mathbb{H}^2$ we write $z_j = r_j(\sin\vartheta_j + i \cdot \cos\vartheta_j)$ where $r_j \in \mathbb{R}^+$ and $-\frac{\pi}{2} \le \vartheta_j < \frac{\pi}{2}$ $(j = 1, 2)$. Now the fundamental domain F_{C_k} is given by

$$-\frac{\pi}{2} \le \vartheta_1, \vartheta_2 < \frac{\pi}{2}, \qquad 1 \le r_1 < \varepsilon^{2k}, \quad r_2 \in \mathbb{R}^+.$$

We consider the integral

$$T_\gamma^A = \sum_{\sigma \in C(\gamma_{m,\alpha}) \backslash \Gamma_K} \int_{\sigma F_A} k(z, \gamma_{m,\alpha} z) u(z)\, d\mu(z). \tag{3.12}$$

The union of the sets σF_A above makes up the fundamental domain of the centralizer $C(\gamma_{m,\alpha})$ except for the images of the part $F \setminus F_A =: F_A^*$. For some cosets it is unnecessary to omit the images of F_A^*. To see this we separate the cosets in the sum that contain elements that take the point ∞ to a fixed point of $\gamma_{m,\alpha}$. Note that in this case $\sigma\infty$ is the same for every element σ of the coset. If σ leaves ∞ fixed, then the part σF_A^* is the same as

$$\{z \in F_{C(\gamma_{m,\alpha})} : \sigma^{-1}z \in F,\, Y_0(\sigma^{-1}z) \geq A\}.$$

But as $\sigma\infty = \infty$, the values $Y_0(\sigma^{-1}z)$ and $Y_0(z)$ are the same.

If $\sigma\infty = q$, then

$$z \in \sigma F_A^* \iff z \in F_{C(\gamma_{m,\alpha})},\, \sigma^{-1}z \in F,\, Y_0(\sigma^{-1}z) \geq A. \tag{3.13}$$

As σ^{-1} takes q to ∞ it is of the form $\begin{bmatrix} a & b \\ \frac{E}{\Lambda} & \frac{\alpha}{\Lambda} \end{bmatrix}$ and then

$$Y_0(\sigma^{-1}z) = \frac{N(\Lambda^2)Y_0(z)}{|Ez_1+\alpha|^2\,|E'z_2+\alpha'|^2},$$

hence $Y_0(\sigma^{-1}z)$ is the same for every σ wich takes ∞ to q. So we can fix an element τ^{-1} with this condition and write the last inequality in (3.13) in the form $Y_0(\tau z) \geq A$.

Before we turn to the remaining cosets we compute the value $k(z, \gamma_{m,\alpha}z)$. Substituting the definitions we get

$$\begin{aligned}
k(z,\gamma_{m,\alpha}z) &= \psi\left(\frac{\left|z_1-\gamma_{m,\alpha}^{(1)}z_1\right|^2}{y_1(z)y_1(\gamma_{m,\alpha}z)}, \frac{\left|z_2-\gamma_{m,\alpha}^{(2)}z_2\right|^2}{y_2(z)y_2(\gamma_{m,\alpha}z)}\right) \\
&= \psi\left(\frac{\left|z_1-\frac{\varepsilon^m z_1+\alpha}{\varepsilon^{-m}}\right|^2}{\varepsilon^{2m}y_1(z)^2}, \frac{\left|z_2-\frac{(\varepsilon')^m z_2+\alpha'}{(\varepsilon')^{-m}}\right|^2}{(\varepsilon')^{2m}y_2(z)^2}\right) \\
&= \psi\left(\frac{|Ez_1+\alpha|^2}{y_1(z)^2}, \frac{|E'z_2+\alpha'|^2}{y_2(z)^2}\right) = \psi\left(\frac{(Ex_1+\alpha)^2}{y_1^2}+E^2, \frac{(E'x_2+\alpha')^2}{y_2^2}+E^2\right) \\
&= \psi\left(\frac{|z_1-q|^2}{E^{-2}y_1^2}, \frac{|z_2-q'|^2}{E^{-2}y_2^2}\right)
\end{aligned}$$

using that $q = -\alpha/E$ and $E^2 = (E')^2$ (since $\varepsilon' = \pm\varepsilon^{-1}$). Now assume that σ is from a coset such that $\sigma\infty \neq \infty, q$. Writing $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $c \neq 0$ we have

$$Y_0(\sigma z) = \frac{Y_0(z)}{|cz_1+d|^2\,|c'z_2+d'|^2},$$

and then

$$\frac{|\sigma z_1-q|^2}{E^{-2}y_1(\sigma z)^2}\cdot\frac{|\sigma z_2-q'|^2}{E^{-2}y_2(\sigma z)^2} = \frac{E^4}{Y_0(\sigma z)^2}\cdot\left|\sigma^{(1)}z_1+\frac{\alpha}{E}\right|^2\cdot\left|\sigma^{(2)}z_2+\frac{\alpha'}{E'}\right|^2$$

$$= \frac{E^4}{Y_0(\sigma z)^2} \cdot \left| \frac{az_1 + b}{cz_1 + d} + \frac{\alpha}{E} \right|^2 \cdot \left| \frac{a'z_2 + b'}{c'z_2 + d'} + \frac{\alpha'}{E'} \right|^2$$

$$\geq \frac{E^4}{Y_0(\sigma z)^2} \cdot \frac{(Ea + \alpha c)^2 y_1^2}{E^2 \left| cz_1 + d \right|^2} \cdot \frac{(E'a' + \alpha' c')^2 y_2^2}{E^2 \left| c'z_2 + d' \right|^2}$$

$$= \frac{N(Ea + \alpha c)^2 Y_0(z)^2}{Y_0(\sigma z)^2 \left| cz_1 + d \right|^2 \left| c'z_2 + d' \right|^2}$$

$$= N(Ea + \alpha c)^2 \left| cz_1 + d \right|^2 \left| c'z_2 + d' \right|^2$$

$$\geq N(Ea + \alpha c)^2 N(c)^2 Y_0(z)^2.$$

The numbers $N(c)^2$ and $N(Ea + \alpha c)^2$ are positive integers since $\sigma\infty \neq \infty, q$. From this we see that if $Y_0(z)$ is big enough, then

$$\psi \left(\frac{\left| \sigma z_1 - q \right|^2}{E^{-2} y_1(\sigma z)^2}, \frac{\left| \sigma z_2 - q' \right|^2}{E^{-2} y_2(\sigma z)^2} \right) = 0$$

as ψ has compact support. It follows that for a big enough A we can write σF instead of σF_A on the right hand side of (3.12) once $\sigma\infty \neq \infty, q$. Hence we have to integrate over

$$S_A = \{ z \in F_{C(\gamma_{m,\alpha})} : Y_0(z) \leq A \text{ and } Y_0(\tau z) \leq A \}.$$

First we make the substitutions $x_1 \mapsto x_1 - \alpha/E$ and $x_2 \mapsto x_2 - \alpha'/E'$ to get

$$\int_{S_A} k(z, \gamma z) u(z) \, d\mu(z) =$$

$$= \int_{S_A} \psi \left(\frac{(Ex_1 + \alpha)^2}{y_1^2} + E^2, \frac{(E'x_2 + \alpha')^2}{y_2^2} + E^2 \right) u(z) \frac{dx_1 \, dx_2 \, dy_1 \, dy_2}{y_1^2 y_2^2}$$

$$= \int_{S'_A} \psi \left(\frac{E^2(x_1^2 + y_1^2)}{y_1^2}, \frac{E^2(x_2^2 + y_2^2)}{y_2^2} \right) u \left(z_1 - \frac{\alpha}{E}, z_2 - \frac{\alpha'}{E'} \right) \frac{dx_1 \, dx_2 \, dy_1 \, dy_2}{y_1^2 y_2^2},$$

where

$$S'_A = \left\{ z \in F_{C(\gamma_{m,\alpha})} + \left(\frac{\alpha}{E}, \frac{\alpha'}{E'} \right) : Y_0 \left(z - \left(\frac{\alpha}{E}, \frac{\alpha'}{E'} \right) \right) \leq A, \, Y_0 \left(\tau \left(z - \left(\frac{\alpha}{E}, \frac{\alpha'}{E'} \right) \right) \right) \leq A \right\}$$

$$= \left\{ z \in F_{C_k} : y_1 y_2 \leq A, \, \frac{N(\Lambda)^2 y_1 y_2}{E^4 \left| z_1 z_2 \right|^2} \leq A \right\}.$$

As $y_1 y_2 = r_1 r_2 \cos\vartheta_1 \cos\vartheta_2$ the conditions above can be written in the following form:

$$A_{r_1,\vartheta_1,\vartheta_2} := \frac{N(\Lambda)^2 \cos\vartheta_1 \cos\vartheta_2}{E^4 A r_1} \leq r_2 \leq A^{r_1,\vartheta_1,\vartheta_2} := \frac{A}{r_1 \cos\vartheta_1 \cos\vartheta_2}.$$

Since $\dfrac{dx\,dy}{y^2} = \dfrac{dr\,d\vartheta}{r\cos^2\vartheta}$, after changing variables we get that

$$T_\gamma^A = \int_{S_A} k(z,\gamma z)u(z)\,d\mu(z) = \int\limits_{-\pi/2}^{\pi/2}\int\limits_{-\pi/2}^{\pi/2} \psi\left(\frac{E^2}{\cos^2\vartheta_1}, \frac{E^2}{\cos^2\vartheta_2}\right) I_u(A,\vartheta_1,\vartheta_2)\frac{d\vartheta_1\,d\vartheta_2}{\cos^2\vartheta_1\cos^2\vartheta_2}$$

where

$$I_u(A,\vartheta_1,\vartheta_2) = \int\limits_1^{\varepsilon^{2k}}\int\limits_{A_{r_1,\vartheta_1,\vartheta_2}}^{A^{r_1,\vartheta_1,\vartheta_2}} u\left(r_1e^{i(\frac{\pi}{2}+\vartheta_1)} - \frac{\alpha}{E}, r_2e^{i(\frac{\pi}{2}+\vartheta_2)} - \frac{\alpha'}{E'}\right)\frac{dr_2}{r_2}\frac{dr_1}{r_1}.$$

Note that if $|m|$ is big then so is $E^2 = (\varepsilon^m - \varepsilon^{-m})^2$ and hence

$$\psi\left(\frac{E^2}{\cos^2\vartheta_1}, \frac{E^2}{\cos^2\vartheta_2}\right) = 0.$$

It follows that the sum over the hyperbolic-parabolic conjugacy classes is in fact finite.

We divide $I_u(A,\vartheta_1,\vartheta_2)$ into two parts:

$$\int\limits_1^{\varepsilon^{2k}}\int\limits_{A_{r_1,\vartheta_1,\vartheta_2}}^{A^{r_1,\vartheta_1,\vartheta_2}} u\left(r_1e^{i(\frac{\pi}{2}+\vartheta_1)} - \frac{\alpha}{E}, r_2e^{i(\frac{\pi}{2}+\vartheta_2)} - \frac{\alpha'}{E'}\right)\frac{dr_2}{r_2}\frac{dr_1}{r_1} =$$

$$= \int\limits_1^{\varepsilon^{2k}}\int\limits_1^{A^{r_1,\vartheta_1,\vartheta_2}} u\left(r_1e^{i(\frac{\pi}{2}+\vartheta_1)} - \frac{\alpha}{E}, r_2e^{i(\frac{\pi}{2}+\vartheta_2)} - \frac{\alpha'}{E'}\right)\frac{dr_2}{r_2}\frac{dr_1}{r_1}$$

$$+ \int\limits_1^{\varepsilon^{2k}}\int\limits_{A_{r_1,\vartheta_1,\vartheta_2}}^{1} u\left(r_1e^{i(\frac{\pi}{2}+\vartheta_1)} - \frac{\alpha}{E}, r_2e^{i(\frac{\pi}{2}+\vartheta_2)} - \frac{\alpha'}{E'}\right)\frac{dr_2}{r_2}\frac{dr_1}{r_1}.$$

Since u is invariant under the action of Γ_K the second part equals

$$\int\limits_1^{\varepsilon^{2k}}\int\limits_{A_{r_1,\vartheta_1,\vartheta_2}}^{1} u\left(\tau\left(r_1e^{i(\frac{\pi}{2}+\vartheta_1)} - \frac{\alpha}{E}\right), \tau'\left(r_2e^{i(\frac{\pi}{2}+\vartheta_2)} - \frac{\alpha'}{E'}\right)\right)\frac{dr_2}{r_2}\frac{dr_1}{r_1}.$$

As

$$\tau\left(z - \frac{\alpha}{E}\right) = \frac{az - \frac{a\alpha}{E} + b}{\frac{E}{\Lambda}(z - \frac{\alpha}{E}) + \frac{\alpha}{\Lambda}} = \Lambda\frac{az - \frac{a\alpha}{E} + b}{Ez} = L + \frac{M}{z}$$

where $L = \frac{a\Lambda}{E}$ and $M = \frac{\Lambda}{E}\left(b - \frac{a\alpha}{E}\right) = -\frac{\Lambda^2}{E^2}$ we get after the substitutions $r_1 \mapsto \frac{1}{r_1}$ and $r_2 \mapsto \frac{1}{r_2}$ that the latter double integral is

$$\int\limits_{\varepsilon^{-2k}}^{1}\int\limits_1^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} u\left(-Mr_1e^{i(\frac{\pi}{2}-\vartheta_1)} + L, -M'r_2e^{i(\frac{\pi}{2}-\vartheta_2)} + L'\right)\frac{dr_2}{r_2}\frac{dr_1}{r_1} =$$

$$= \int_{\varepsilon^{-2k}}^{1} \int_{1}^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} u\left(\tau\left[T(r_1 e^{i(\frac{\pi}{2}-\vartheta_1)}) - \frac{\alpha}{E}\right], \tau'\left[T(r_2 e^{i(\frac{\pi}{2}-\vartheta_2)}) - \frac{\alpha'}{E'}\right]\right) \frac{dr_2}{r_2}\frac{dr_1}{r_1}$$

$$= \int_{\varepsilon^{-2k}}^{1} \int_{1}^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} u\left(T(r_1 e^{i(\frac{\pi}{2}-\vartheta_1)}) - \frac{\alpha}{E}, T(r_2 e^{i(\frac{\pi}{2}-\vartheta_2)}) - \frac{\alpha'}{E'}\right) \frac{dr_2}{r_2}\frac{dr_1}{r_1},$$

where T is the transformation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and

$$\hat{A}_{r_1,\vartheta_1,\vartheta_2} = \frac{E^4 A}{r_1 N(\Lambda)^2 \cos\vartheta_1 \cos\vartheta_2}.$$

Let us define

$$U_1(z) = u(z_1 - \alpha/E, z_2 - \alpha'/E') \tag{3.14}$$

and

$$U_2(z) = u(-Mz + L, -M'z + L') = U_1(Tz). \tag{3.15}$$

With this notation we have

$$I_u(A, \vartheta_1, \vartheta_2) = \int_{1}^{\varepsilon^{2k}} \int_{1}^{A^{r_1,\vartheta_1,\vartheta_2}} U_1(r_1 e^{i(\frac{\pi}{2}+\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}+\vartheta_2)}) \frac{dr_2}{r_2}\frac{dr_1}{r_1}$$

$$+ \int_{\varepsilon^{-2k}}^{1} \int_{1}^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} U_2(r_1 e^{i(\frac{\pi}{2}-\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}-\vartheta_2)}) \frac{dr_2}{r_2}\frac{dr_1}{r_1}.$$

Note that if u is a cusp form, then $u(z) = O(y_2^{-\frac{1}{2}})$ by Proposition 2.1.6 once $y_2 \to \infty$ and y_1 is bounded from below. Hence in this case the integrals above converge (note that $\cos\vartheta_j \geq \delta > 0$ for $j = 1, 2$ since ψ has compact support).

Now we handle the case when u is not a cusp form. We are going to subtract the main terms of U_1 and U_2 to get convergent integrals. By main term we mean the zeroth coefficient of the Fourier series. Recall that the zeroth coefficient u is $\eta y_1^{s_1} y_2^{s_2} + \phi y_1^{1-s_1} y_2^{1-s_2}$ where $\eta, \phi \in \mathbb{C}$ and at least one of them is non-zero. Also, by (3.1) we have

$$(s_1, s_2) = \left(s + \frac{\pi i m_u}{2\log\varepsilon}, s - \frac{\pi i m_u}{2\log\varepsilon}\right)$$

for some $s \in \mathbb{C}$ and $m_u \in \mathbb{Z}$. Hence by Proposition 2.1.6 the function

$$\overline{U}_1(z) = U_1(z) - (\eta y_1^{s_1} y_2^{s_2} + \phi y_1^{1-s_1} y_2^{1-s_2}) = U_1(z) - m_1(z) \tag{3.16}$$

is $O(y_2^{-\operatorname{Re} s_2})$ as $y_2 \to \infty$ and y_1 is bounded from below. Similarly, the main term of $U_2(z)$ is

$$m_2(z) = \eta \frac{|N(\Lambda)|^{2s} \lambda_{m_u}(\Lambda^2)}{|E|^{4s}} y_1^{s_1} y_2^{s_2} + \phi \frac{|N(\Lambda)|^{2(1-s)} \lambda_{-m_u}(\Lambda^2)}{|E|^{4(1-s)}} y_1^{1-s_1} y_2^{1-s_2},$$

where $\lambda_m(\alpha)= |\alpha/\alpha'|^{\frac{\pi i m}{2\log\varepsilon}}$ for any $\alpha \in K^\times$ and $m \in \mathbb{Z}$. It is a so-called *Grössencharacter* that will occur again in the next section. Hence

$$\overline{U}_2(z) = U_2(z) - m_2(z) \tag{3.17}$$

is also $O(y_2^{-\operatorname{Re} s_2})$ as $y_2 \to \infty$ and y_1 is bounded from below. Note that the numbers $\lambda_{\pm m_u}(\Lambda^2)$ and $N(\Lambda)^2$ depend only on the ideal (Λ).

We write

$$I_u(A,\vartheta_1,\vartheta_2) = I_u^1(A,\vartheta_1,\vartheta_2) + I_u^2(A,\vartheta_1,\vartheta_2),$$

where

$$I_u^1(A,\vartheta_1,\vartheta_2) = \int_1^{\varepsilon^{2k}} \int_1^{A^{r_1,\vartheta_1,\vartheta_2}} \overline{U}_1\left(r_1 e^{i(\frac{\pi}{2}+\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}+\vartheta_2)}\right) \frac{dr_2}{r_2}\frac{dr_1}{r_1}$$

$$+ \int_{\varepsilon^{-2k}}^{1} \int_1^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} \overline{U}_2\left(r_1 e^{i(\frac{\pi}{2}-\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}-\vartheta_2)}\right) \frac{dr_2}{r_2}\frac{dr_1}{r_1}$$

and

$$I_u^2(A,\vartheta_1,\vartheta_2) = \int_1^{\varepsilon^{2k}} \int_1^{A^{r_1,\vartheta_1,\vartheta_2}} m_1(r_1 e^{i(\frac{\pi}{2}+\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}+\vartheta_2)}) \frac{dr_2}{r_2}\frac{dr_1}{r_1}$$

$$+ \int_{\varepsilon^{-2k}}^{1} \int_1^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} m_2(r_1 e^{i(\frac{\pi}{2}-\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}-\vartheta_2)}) \frac{dr_2}{r_2}\frac{dr_1}{r_1}.$$

By the estimates on $\overline{U}_1(z)$ and $\overline{U}_2(z)$ the function $I_u^1(A,\vartheta_1,\vartheta_2)$ converges as $A \to \infty$ and hence we can write $I_u^1(A,\vartheta_1,\vartheta_2) = I_u^\gamma(\vartheta_1,\vartheta_2) + o(1)$ where

$$I_u^\gamma(\vartheta_1,\vartheta_2) = \int_1^{\varepsilon^{2k}} \int_1^{\infty} \overline{U}_1\left(r_1 e^{i(\frac{\pi}{2}+\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}+\vartheta_2)}\right) \frac{dr_2}{r_2}\frac{dr_1}{r_1}$$

$$+ \int_{\varepsilon^{-2k}}^{1} \int_1^{\infty} \overline{U}_2\left(r_1 e^{i(\frac{\pi}{2}-\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}-\vartheta_2)}\right) \frac{dr_2}{r_2}\frac{dr_1}{r_1}. \tag{3.18}$$

We continue with the calculation of the term $I_u^2(A,\vartheta_1,\vartheta_2)$. The first term is

$$\int_1^{\varepsilon^{2k}} \int_1^{A^{r_1,\vartheta_1,\vartheta_2}} \eta(r_1\cos\vartheta_1)^{s_1}(r_2\cos\vartheta_2)^{s_2} + \phi(r_1\cos\vartheta_1)^{1-s_1}(r_2\cos\vartheta_2)^{1-s_2}\frac{dr_2}{r_2}\frac{dr_1}{r_1} =$$

$$= \eta(\cos\vartheta_1)^{s_1}(\cos\vartheta_2)^{s_2} \int_1^{\varepsilon^{2k}} \left(\int_1^{A^{r_1,\vartheta_1,\vartheta_2}} r_2^{s_2-1}\, dr_2 \right) r_1^{s_1-1}\, dr_1$$

$$+ \phi(\cos\vartheta_1)^{1-s_1}(\cos\vartheta_2)^{1-s_2} \int_1^{\varepsilon^{2k}} \left(\int_1^{A^{r_1,\vartheta_1,\vartheta_2}} r_2^{-s_2}\, dr_2 \right) r_1^{-s_1}\, dr_1.$$

We compute the first inner integral on the right hand side:

$$\int_1^{A^{r_1,\vartheta_1,\vartheta_2}} r_2^{s_2-1}\, dr_2 = \frac{1}{s_2}\left[(A^{r_1,\vartheta_1,\vartheta_2})^{s_2} - 1\right] = \frac{1}{s_2}\left[\frac{A^{s_2}}{r_1^{s_2}(\cos\vartheta_1\cos\vartheta_2)^{s_2}} - 1\right],$$

and hence

$$\int_1^{\varepsilon^{2k}} \left(\int_1^{A^{r_1,\vartheta_1,\vartheta_2}} r_2^{s_2-1}\, dr_2 \right) r_1^{s_1-1}\, dr_1 = \frac{1}{s_2}\left[\frac{A^{s_2}}{(\cos\vartheta_1\cos\vartheta_2)^{s_2}} \int_1^{\varepsilon^{2k}} r_1^{s_1-s_2-1}\, dr_1 - \int_1^{\varepsilon^{2k}} r_1^{s_1-1}\, dr_1\right].$$

But once $s_1 \neq s_2$ we have that

$$\int_1^{\varepsilon^{2k}} r_1^{s_1-s_2-1}\, dr_1 = \frac{1}{s_1-s_2}(\varepsilon^{2k(s_1-s_2)} - 1) = 0$$

by (3.1), that is,

$$\int_1^{\varepsilon^{2k}} \left(\int_1^{A^{r_1,\vartheta_1,\vartheta_2}} r_2^{s_2-1}\, dr_2 \right) r_1^{s_1-1}\, dr_1 = \frac{1-\varepsilon^{2ks_1}}{s_1 s_2} = \frac{1-\mathrm{sg}(km_u)\varepsilon^{2ks}}{s^2 + \frac{\pi^2 m_u^2}{4\log^2\varepsilon}}.$$

Similarly,

$$\int_1^{A^{r_1,\vartheta_1,\vartheta_2}} r_2^{-s_2}\, dr_2 = \frac{1}{1-s_2}\left[(A^{r_1,\vartheta_1,\vartheta_2})^{1-s_2} - 1\right] = \frac{1}{1-s_2}\left[\frac{A^{1-s_2}}{r_1^{1-s_2}(\cos\vartheta_1\cos\vartheta_2)^{1-s_2}} - 1\right],$$

$$\int_1^{\varepsilon^{2k}} \left(\int_1^{A^{r_1,\vartheta_1,\vartheta_2}} r_2^{-s_2}\, dr_2 \right) r_1^{-s_1}\, dr_1 = \frac{1}{1-s_2}\left[\frac{A^{1-s_2}}{(\cos\vartheta_1\cos\vartheta_2)^{1-s_2}} \int_1^{\varepsilon^{2k}} r_1^{s_2-s_1-1}\, dr_1 - \int_1^{\varepsilon^{2k}} r_1^{-s_1}\, dr_1\right].$$

Again, if $s_1 \neq s_2$, then

$$\int_1^{\varepsilon^{2k}} r_1^{s_2-s_1-1}\, dr_1 = \frac{1}{s_2-s_1}(\varepsilon^{2k(s_2-s_1)} - 1) = 0,$$

$$\int_1^{\varepsilon^{2k}} \left(\int_1^{A^{r_1,\vartheta_1,\vartheta_2}} r_2^{-s_2}\, dr_2 \right) r_1^{-s_1}\, dr_1 = \frac{1-\varepsilon^{2k(1-s_1)}}{(1-s_1)(1-s_2)} = \frac{1-\mathrm{sg}(km_u)\varepsilon^{2k(1-s)}}{(1-s)^2+\frac{\pi^2 m_u^2}{4\log^2\varepsilon}}.$$

On the other hand, if $s_1 = s_2 = s$, then

$$\int_1^{\varepsilon^{2k}} r_1^{s_2-s_1-1}\, dr_1 = \int_1^{\varepsilon^{2k}} r_1^{s_1-s_2-1}\, dr_1 = \int_1^{\varepsilon^{2k}} r_1^{-1}\, dr_1 = 2k\log\varepsilon.$$

Summarizing all this we get

$$\int_1^{\varepsilon^{2k}} \int_1^{A^{r_1,\vartheta_1,\vartheta_2}} m_1(r_1 e^{i(\frac{\pi}{2}+\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}+\vartheta_2)}) \frac{dr_2}{r_2}\frac{dr_1}{r_1} =$$

$$= \delta_{s_1=s_2=s} 2k\log\varepsilon \left[\frac{\eta A^s}{s} + \frac{\phi A^{1-s}}{1-s} \right]$$

$$+ \frac{\eta(\cos\vartheta_1)^{s_1}(\cos\vartheta_2)^{s_2}(1-\mathrm{sg}(km_u)\varepsilon^{2ks})}{s^2+\frac{\pi^2 m_u^2}{4\log^2\varepsilon}}$$

$$+ \frac{\phi(\cos\vartheta_1)^{1-s_1}(\cos\vartheta_2)^{1-s_2}(1-\mathrm{sg}(km_u)\varepsilon^{2k(1-s)})}{(1-s)^2+\frac{\pi^2 m_u^2}{4\log^2\varepsilon}}.$$

Now we turn to the second double integral in $I_u^2(A,\vartheta_1,\vartheta_2)$:

$$\int_{\varepsilon^{-2k}}^{1} \int_1^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} m_2(r_1 e^{i(\frac{\pi}{2}-\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}-\vartheta_2)}) \frac{dr_2}{r_2}\frac{dr_1}{r_1}.$$

Substituting the definition of m_2 one gets

$$\frac{\eta\,|N(\Lambda)|^{2s}\lambda_{m_u}(\Lambda^2)}{E^{4s}}(\cos\vartheta_1)^{s_1}(\cos\vartheta_2)^{s_2} \int_{\varepsilon^{-2k}}^{1} \left(\int_1^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} r_2^{s_2-1}\, dr_2 \right) r_1^{s_1-1}\, dr_1 +$$

$$+ \frac{\phi\,|N(\Lambda)|^{2(1-s)}\lambda_{-m_u}(\Lambda^2)}{|E|^{4(1-s)}}(\cos\vartheta_1)^{1-s_1}(\cos\vartheta_2)^{1-s_2} \int_{\varepsilon^{-2k}}^{1} \left(\int_1^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} r_2^{-s_2}\, dr_2 \right) r_1^{-s_1}\, dr_1.$$

As before, we have

$$\int_1^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} r_2^{s_2-1}\, dr_2 = \frac{1}{s_2}\left[\frac{|E|^{4s_2} A^{s_2}}{r_1^{s_2}\,|N(\Lambda)|^{2s_2}(\cos\vartheta_1\cos\vartheta_2)^{s_2}} - 1 \right]$$

and

$$\int_{1}^{\hat{A}_{r_1,\vartheta_1,\vartheta_2}} r_2^{-s_2}\,dr_2 = \frac{1}{1-s_2}\left[\frac{|E|^{4(1-s_2)}\,A^{1-s_2}}{r_1^{1-s_2}\,|N(\Lambda)|^{2(1-s_2)}\,(\cos\vartheta_1\cos\vartheta_2)^{1-s_2}} - 1\right].$$

This gives the same way as before that the term that depends on A is

$$\delta_{s_1=s_2=s} 2k\log\varepsilon\left[\frac{\eta A^s}{s} + \frac{\phi A^{1-s}}{1-s}\right].$$

The constant term comes from

$$-\frac{1}{s_2}\int_{\varepsilon^{-2k}}^{1} r_1^{s_1-1}\,dr_1 = \frac{1}{s_1 s_2}(\varepsilon^{-2ks_1}-1) = \frac{\mathrm{sg}(km_u)\varepsilon^{-2ks}-1}{s^2+\frac{\pi^2 m_u^2}{4\log^2\varepsilon}}$$

and

$$-\frac{1}{1-s_2}\int_{\varepsilon^{-2k}}^{1} r_1^{-s_1}\,dr_1 = \frac{1}{(1-s_1)(1-s_2)}(\varepsilon^{-2k(1-s_1)}-1) = \frac{\mathrm{sg}(km_u)\varepsilon^{-2k(1-s)}-1}{(1-s)^2+\frac{\pi^2 m_u^2}{4\log^2\varepsilon}}.$$

We summarizing this in the following

Lemma 3.1.5. *If u is not a cusp form, then*

$$I_u^2(A,\vartheta_1,\vartheta_2) = \delta_{m_u} 4k\log\varepsilon\left[\frac{\eta A^s}{s}+\frac{\phi A^{1-s}}{1-s}\right] + \eta C_{\gamma,\vartheta_1,\vartheta_2}(s,m_u) + \phi C_{\gamma,\vartheta_1,\vartheta_2}(1-s,-m_u),$$

where

$$C_{\gamma,\vartheta_1,\vartheta_2}(s,m_u) = (\cos\vartheta_1)^{s_1}(\cos\vartheta_2)^{s_2} C_\gamma(s,m_u),$$

$$C_\gamma(s,m_u) = C_{\gamma_{m,\alpha}}(s,m_u) = \frac{(1-M_\gamma(s,m_u) - sg(km_u)(\varepsilon^{2ks} - M_\gamma(s,m_u)\varepsilon^{-2ks}))}{s^2+\frac{\pi^2 m_u^2}{4\log\varepsilon}},$$

and

$$M_\gamma(s,m_u) = \frac{|N(\Lambda)|^{2s}\,\lambda_{m_u}(\Lambda^2)}{|E|^{4s}}.$$

Notice that if u is a cusp form, then $\overline{U}_j = U_j$ and we have

Proposition 3.1.6. *If u is a cusp form, then*

$$T_\gamma^A = \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} \psi\left(\frac{E^2}{\cos^2\vartheta_1},\frac{E^2}{\cos^2\vartheta_2}\right) F(e^{i(\frac{\pi}{2}+\vartheta_1)}, e^{i(\frac{\pi}{2}+\vartheta_2)})\frac{d\vartheta_1\,d\vartheta_2}{\cos^2\vartheta_1\cos^2\vartheta_2} + o(1)$$

as $A\to\infty$, where

$$F(z_1,z_2) = \int_{1}^{\varepsilon^{2k}}\int_{0}^{\infty} U_1(r_1 z_1, r_2 z_2)\,\frac{dr_2}{r_2}\frac{dr_1}{r_1}.$$

The function $U_1(z) = u(z_1 - \alpha/E, z_2 - \alpha'/E')$ is not invariant under the action of Γ_K but it is still invariant under the action of ρ_k (defined in (3.11) on page 86), because

$$\begin{aligned}
U_1(\rho_k z) &= u\left(\varepsilon^{2k} z_1 - \frac{\alpha}{E}, \varepsilon^{-2k} z_2 - \frac{\alpha'}{E'}\right) \\
&= u\left(\varepsilon^{2k} z_1 - \varepsilon^{2k}\frac{\alpha}{E} + \varepsilon^{2k}\frac{\alpha}{E} - \frac{\alpha}{E}, \varepsilon^{-2k} z_2 - \varepsilon^{-2k}\frac{\alpha'}{E'} + \varepsilon^{-2k}\frac{\alpha'}{E'} - \frac{\alpha'}{E'}\right) \\
&= u\left(\varepsilon^{2k}\left(z_1 - \frac{\alpha}{E}\right) + \varepsilon^{k}\frac{\alpha}{E}(\varepsilon^{k} - \varepsilon^{-k}), \varepsilon'^{2k}\left(z_2 - \frac{\alpha'}{E'}\right) + \varepsilon'^{k}\frac{\alpha'}{E'}\left(\varepsilon'^{k} - \varepsilon'^{-k}\right)\right) \\
&= u\left(z_1 - \frac{\alpha}{E}, z_2 - \frac{\alpha'}{E'}\right)
\end{aligned}$$

since u is invariant under the action of γ_k (defined in Proposition 3.1.4). As in Section 3.1.2 we get that $F(R_1 z_1, R_2 z_2) = F(z_1, z_2)$ for any $R_1, R_2 \in (0, \infty)$, i.e. $F(z_1, z_2)$ is a function of ϑ_1 and ϑ_2 only and we can write $F(\vartheta_1, \vartheta_2)$. Also, since the Laplace operators Δ_j commute with the action of $PSL(2,\mathbb{R})^2$ this function satisfies the equation (3.8). Proceeding as in Section 3.1.2 we obtain

$$T_\gamma^A = F(0,0) \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \psi\left(\frac{E^2}{\cos^2\vartheta_1}, \frac{E^2}{\cos^2\vartheta_2}\right) f_{\lambda_1}(\vartheta_1) f_{\lambda_2}(\vartheta_2) \frac{d\vartheta_1\, d\vartheta_2}{\cos^2\vartheta_1 \cos^2\vartheta_2} + o(1),$$

where f_{λ_j} are the solutions of the equation (3.8) with $f_{\lambda_j}(0) = 1$ and $f'_{\lambda_j}(0) = 0$ and

$$F(0,0) = \int_1^{\varepsilon^{2k}} \int_0^{\infty} u\left(r_1 i - \frac{\alpha}{E}, r_2 i - \frac{\alpha'}{E'}\right) \frac{dr_2}{r_2}\frac{dr_1}{r_1}.$$

Next we turn to the case when u is not a cusp form. Note that by Lemma 3.1.5 we get a main term (i.e. a term that does not converge as $A \to \infty$) only in the case $m_u = 0$. To give the contribution of a class we write

$$\psi_\gamma(\vartheta_1, \vartheta_2) = \psi\left(\frac{E^2}{\cos^2\vartheta_1}, \frac{E^2}{\cos^2\vartheta_2}\right),$$

and then

$$T_\gamma^A = \delta_{m_u} 4k \log\varepsilon \left[\frac{\eta A^s}{s} + \frac{\phi A^{1-s}}{1-s}\right] \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \psi_\gamma(\vartheta_1, \vartheta_2) \frac{d\vartheta_1\, d\vartheta_2}{\cos^2\vartheta_1 \cos^2\vartheta_2}$$

$$+ \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \psi_\gamma(\vartheta_1, \vartheta_2) C_u^\gamma(\vartheta_1, \vartheta_2) \frac{d\vartheta_1\, d\vartheta_2}{\cos^2\vartheta_1 \cos^2\vartheta_2} + o(1)$$

where $C_u^\gamma(\vartheta_1,\vartheta_2)$ is

$$\eta(\cos\vartheta_1)^{s_1}(\cos\vartheta_2)^{s_2}C_\gamma(s,m_u)+\phi(\cos\vartheta_1)^{1-s_1}(\cos\vartheta_2)^{1-s_2}C_\gamma(1-s,-m_u)+I_u^\gamma(\vartheta_1,\vartheta_2).$$

Here $I_u^\gamma(\vartheta_1,\vartheta_2)$ was defined in (3.18) on page 91 while $C_\gamma(s,m_u)$ was introduced in Lemma 3.1.5. We compute the coefficient of the main term:

$$\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\psi_\gamma(\vartheta_1,\vartheta_2)\frac{d\vartheta_1\,d\vartheta_2}{\cos^2\vartheta_1\cos^2\vartheta_2}=4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\psi\left(\frac{E^2}{\cos^2\vartheta_1},\frac{E^2}{\cos^2\vartheta_2}\right)\frac{d\vartheta_1\,d\vartheta_2}{\cos^2\vartheta_1\cos^2\vartheta_2}$$

$$=\frac{1}{E^2}\int_{E^2}^{\infty}\int_{E^2}^{\infty}\frac{\psi(u_1,u_2)}{\sqrt{u_1-E^2}\sqrt{u_2-E^2}}\,du_1\,du_2$$

$$=\frac{1}{E^2}\,g(\log\varepsilon^{2m},\log\varepsilon^{-2m}).$$

Note that $E^2=|N(E)|$ and by Proposition III.3.3 in [5] we have the following for a fixed m:

$$\sum_{\{\gamma_{m,\alpha}\}\text{ hyperbolic-parabolic}}\frac{k}{|N(E)|}=1.$$

So if we gather the main terms that belong to the classes of $\gamma_{m,\alpha}$ for a fixed m, then we obtain

$$4g(\log\varepsilon^{2m},\log\varepsilon^{-2m})\log\varepsilon\left[\frac{\eta A^s}{s}+\frac{\phi A^{1-s}}{1-s}\right].$$

Note that $g(\log\varepsilon^{2m},\log\varepsilon^{-2m})=g(\log\varepsilon^{2|m|},\log\varepsilon^{-2|m|})$ for every $m\in\mathbb{Z}$ and hence instead of summing over the positive integers we may sum over the non-zero integers and divide by 2. Hence the hyperbolic-parabolic conjugacy classes give the main term

$$2\log\varepsilon\left[\frac{\eta A^s}{s}+\frac{\phi A^{1-s}}{1-s}\right]\sum_{m\in\mathbb{Z}\setminus\{0\}}g(\log\varepsilon^{2m},\log\varepsilon^{-2m}).$$

We elaborate also on the constant terms. Let us consider the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\psi_\gamma(\vartheta_1,\vartheta_2)(\cos\vartheta_1)^{s_1}(\cos\vartheta_2)^{s_2}\frac{d\vartheta_1\,d\vartheta_2}{\cos^2\vartheta_1\cos^2\vartheta_2}=$$

$$=4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\psi\left(\frac{E^2}{\cos^2\vartheta_1},\frac{E^2}{\cos^2\vartheta_2}\right)(\cos\vartheta_1)^{s_1}(\cos\vartheta_2)^{s_2}\frac{d\vartheta_1\,d\vartheta_2}{\cos^2\vartheta_1\cos^2\vartheta_2}.$$

As before, we make the substitution $u_j = E^2/\cos^2\vartheta_j$, so the previous integral is

$$\frac{1}{|E|^{2-s_1-s_2}}\int_{E^2}^{\infty}\int_{E^2}^{\infty}\frac{\psi(u_1,u_2)u_1^{-\frac{s_1}{2}}u_2^{-\frac{s_2}{2}}}{\sqrt{u_1-E^2}\sqrt{u_2-E^2}}\,du_1\,du_2.$$

In the following we omit the constant multiplier $|E|^{s_1+s_2-2}$ and calculate just the integral above. Using Proposition 2.3.1 we get that it is

$$\frac{1}{\pi^2}\int_{E^2}^{\infty}\int_{E^2}^{\infty}\int_{u_2}^{\infty}\int_{u_1}^{\infty}\frac{\frac{\partial^2 Q}{\partial w_1\partial w_2}(w_1,w_2)}{\sqrt{w_1-u_1}\sqrt{w_2-u_2}}\,dw_1\,dw_2\frac{u_1^{-\frac{s_1}{2}}u_2^{-\frac{s_2}{2}}}{\sqrt{u_1-E^2}\sqrt{u_2-E^2}}\,du_1\,du_2.$$

Now we interchange the order of integration to continue:

$$\frac{1}{\pi^2}\int_{E^2}^{\infty}\int_{E^2}^{\infty}\frac{\partial^2 Q}{\partial w_1\partial w_2}(w_1,w_2)\int_{E^2}^{w_2}\int_{E^2}^{w_1}\frac{u_1^{-\frac{s_1}{2}}u_2^{-\frac{s_2}{2}}}{\sqrt{w_1-u_1}\sqrt{u_1-E^2}\sqrt{w_2-u_2}\sqrt{u_2-E^2}}\,du_1\,du_2\,dw_1\,dw_2 =$$

$$= \frac{1}{\pi^2}\int_{0}^{\infty}\int_{0}^{\infty}\frac{\partial^2 Q}{\partial w_1\partial w_2}(w_1+E^2,w_2+E^2)\int_{0}^{w_2}\int_{0}^{w_1}\frac{(u_1+E^2)^{-\frac{s_1}{2}}(u_2+E^2)^{-\frac{s_2}{2}}}{\sqrt{w_1-u_1}\sqrt{u_1}\sqrt{w_2-u_2}\sqrt{u_2}}\,du_1\,du_2\,dw_1\,dw_2$$

$$= \frac{1}{\pi^2}\int_{0}^{\infty}\int_{0}^{\infty}\frac{\partial^2 Q}{\partial w_1\partial w_2}(w_1+E^2,w_2+E^2)\int_{0}^{1}\int_{0}^{1}\frac{(w_1u_1+E^2)^{-\frac{s_1}{2}}(w_2u_2+E^2)^{-\frac{s_2}{2}}}{\sqrt{1-u_1}\sqrt{u_1}\sqrt{1-u_2}\sqrt{u_2}}\,du_1\,du_2\,dw_1\,dw_2.$$

Then integration by parts with respect to the variable w_2 gives

$$-\frac{(E^2)^{-\frac{s_2}{2}}}{\pi^2}\int_{0}^{\infty}\frac{\partial Q}{\partial w_1}(w_1+E^2,E^2)\int_{0}^{1}\frac{(w_1u_1+E^2)^{-\frac{s_1}{2}}}{\sqrt{1-u_1}\sqrt{u_1}}\,du_1\int_{0}^{1}\frac{1}{\sqrt{1-u_2}\sqrt{u_2}}\,du_2\,dw_1$$

$$+\frac{s_2}{2\pi^2}\int_{0}^{\infty}\int_{0}^{\infty}\frac{\partial Q}{\partial w_1}(w_1+E^2,w_2+E^2)\times$$

$$\times\int_{0}^{1}\frac{(w_1u_1+E^2)^{-\frac{s_1}{2}}}{\sqrt{1-u_1}\sqrt{u_1}}du_1\int_{0}^{1}\frac{\sqrt{u_2}(w_2u_2+E^2)^{-\frac{s_2}{2}-1}}{\sqrt{1-u_2}}\,du_2\,dw_1\,dw_2.$$

Note that

$$\int_{0}^{1}\frac{1}{\sqrt{1-t}\sqrt{t}}\,dt = \pi$$

and hence the first term above is

$$-\frac{(E^2)^{-\frac{s_2}{2}}}{\pi}\int_{0}^{\infty}\frac{\partial Q}{\partial w_1}(w_1+E^2,E^2)\int_{0}^{1}\frac{(w_1u_1+E^2)^{-\frac{s_1}{2}}}{\sqrt{1-u_1}\sqrt{u_1}}\,du_1\,dw_1.$$

Again, integration by parts gives that this is

$$(E^2)^{-\frac{s_1+s_2}{2}} Q(E^2,E^2) - \frac{s_1(E^2)^{-\frac{s_2}{2}}}{2\pi} \int_0^\infty Q(w_1+E^2,E^2) \int_0^1 \frac{\sqrt{u_1}(w_1u_1+E^2)^{-\frac{s_1}{2}-1}}{\sqrt{1-u_1}}\, du_1\, dw_1 =$$

$$= (E^2)^{-\frac{s_1+s_2}{2}} Q(E^2,E^2) - (E^2)^{-\frac{s_2}{2}} \int_{E^2}^\infty Q(w_1,E^2)\tilde{F}_{s_1,m}(w_1)\, dw_1,$$

where

$$\tilde{F}_{s,m}(w) = \frac{s}{2\pi} \int_0^1 ((w-E^2)t+E^2)^{-\frac{s}{2}-1} \frac{\sqrt{t}}{\sqrt{1-t}}\, dt$$

We get similarly that the remaining terms are

$$-(E^2)^{-\frac{s_1}{2}} \int_{E^2}^\infty Q(E^2,w_2)\tilde{F}_{s_2,m}(w_2)\, dw_2$$

and

$$\int_{E^2}^\infty \int_{E^2}^\infty Q(w_1,w_2)\tilde{F}_{s_1,m}(w_1)\tilde{F}_{s_2,m}(w_2)\, dw_1\, dw_2.$$

Lastly, we substitute $w_j = e^{x_j}+e^{-x_j}-2$ and define

$$F_{s,m}(x) = \frac{(e^x-e^{-x})s}{2\pi} \int_0^1 ((e^x+e^{-x}-2-E^2)t+E^2)^{-\frac{s}{2}-1} \frac{\sqrt{t}}{\sqrt{1-t}}\, dt$$

to obtain

$$\Xi_{s_1,s_2}(m) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_\gamma(\vartheta_1,\vartheta_2)(\cos\vartheta_1)^{s_1}(\cos\vartheta_2)^{s_2} \frac{d\vartheta_1\, d\vartheta_2}{\cos^2\vartheta_1 \cos^2\vartheta_2} =$$

$$= \frac{1}{|E|^2} g(\log\varepsilon^{2m},\log\varepsilon^{2m}) - \frac{1}{|E|^{2-s_1}} \int_{\log\varepsilon^{2m}}^\infty g(x_1,\log\varepsilon^{2m})F_{s_1,m}(x_1)\, dx_1$$

$$- \frac{1}{|E|^{2-s_2}} \int_{\log\varepsilon^{2m}}^\infty g(\log\varepsilon^{2m},x_2)F_{s_2,m}(x_2)\, dx_2$$

$$+ \frac{1}{|E|^{2-2s}} \int_{\log\varepsilon^{2m}}^\infty \int_{\log\varepsilon^{2m}}^\infty g(x_1,x_2)F_{s_1,m}(x_1)F_{s_2,m}(x_2)\, dx_1\, dx_2.$$

Putting all these together we get

Proposition 3.1.7. *If u is not a cusp form, then the contribution of the hyperbolic-parabolic conjugacy classes in the trace is*

$$\delta_{m_u} 2\log\varepsilon \left[\frac{\eta A^s}{s} + \frac{\phi A^{1-s}}{1-s}\right] \sum_{m\in\mathbb{Z}\setminus\{0\}} g(\log\varepsilon^{2m}, \log\varepsilon^{-2m}) +$$

$$+\eta \sum_{m\in\mathbb{N}^+} \Xi_{s_1,s_2}(m) \sum_{\{\gamma_{m,\alpha}\}} C_{\gamma_{m,\alpha}}(s, m_u)$$

$$+\phi \sum_{m\in\mathbb{N}^+} \Xi_{1-s_1,1-s_2}(m) \sum_{\{\gamma_{m,\alpha}\}} C_{\gamma_{m,\alpha}}(1-s, -m_u)$$

$$+\sum_{m\in\mathbb{N}^+} \sum_{\{\gamma_{m,\alpha}\}} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \psi\left(\frac{E^2}{\cos^2\vartheta_1}, \frac{E^2}{\cos^2\vartheta_2}\right) I_u^{\gamma_{m,\alpha}}(\vartheta_1,\vartheta_2) \frac{d\vartheta_1\, d\vartheta_2}{\cos^2\vartheta_1 \cos^2\vartheta_2} + o(1).$$

Here $\gamma_{m,\alpha}$ is defined in (3.10)*, $E = \varepsilon^m - \varepsilon^{-m}$, $\Xi_{s_1,s_2}(m)$ is defined above the proposition, $C_{\gamma_{m,\alpha}}(s, m_u)$ is defined in Lemma* 3.1.5 *and*

$$I_u^{\gamma_{m,\alpha}}(\vartheta_1,\vartheta_2) = \int_1^{\varepsilon^{2k}} \int_1^{\infty} \overline{U}_1\left(r_1 e^{i(\frac{\pi}{2}+\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}+\vartheta_2)}\right) \frac{dr_2}{r_2}\frac{dr_1}{r_1}$$

$$+ \int_{\varepsilon^{-2k}}^{1} \int_1^{\infty} \overline{U}_2\left(r_1 e^{i(\frac{\pi}{2}-\vartheta_1)}, r_2 e^{i(\frac{\pi}{2}-\vartheta_2)}\right) \frac{dr_2}{r_2}\frac{dr_1}{r_1},$$

where k is the positive integer defined uniquely by Proposition 3.1.4 *and the functions $\overline{U}_1$ and $\overline{U}_2$ are defined by* (3.14)*,* (3.15)*,* (3.16) *and* (3.17) *above.*

3.1.5 Contribution of totally parabolic elements

Now we turn to the conjugacy classes of totally parabolic elements. Every element of this type is conjugate in Γ_K to an element of the form

$$\gamma_\alpha := \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \tag{3.19}$$

where $0 \neq \alpha \in \mathcal{O}_K$. Two such elements γ_α and γ_β are conjugate if and only if

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$$

for some $a, b, c, d \in \mathcal{O}_K$, that is,

$$\begin{bmatrix} a+\alpha c & b+\alpha d \\ c & d \end{bmatrix} = \begin{bmatrix} a & \beta a + b \\ c & \beta c + d \end{bmatrix}.$$

We obtain that $a + \alpha c = a$. This gives $c = 0$, which means that $d = a^{-1}$, hence $a \in \mathcal{O}_K^\times$. From $b + \alpha d = \beta a + b$ we have $\alpha = a^2\beta$. We obtained the following:

Proposition 3.1.8. *The representatives of the conjugacy classes of toltally parabolic elements are given by the elements γ_α defined in* (3.19)*, where $\alpha \in \mathcal{O}_K/(\mathcal{O}_K^\times)^2$.*

We compute the centralizer of these elements:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if and only if

$$\begin{bmatrix} a & \alpha a + b \\ c & \alpha c + d \end{bmatrix} = \begin{bmatrix} a + \alpha c & b + \alpha d \\ c & d \end{bmatrix}.$$

From $a = a + \alpha c$ we have $c = 0$ (since $\alpha \neq 0$), from $\alpha a + b = b + \alpha d$ we have $a = d$. So we have the following

Proposition 3.1.9. *The centralizer $C(\gamma_\alpha)$ of the totally parabolic element γ_α is*

$$\left\{ \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} : \beta \in \mathcal{O}_K \right\}.$$

We obtain immediately from (1.5) and (1.6) on page 7 that the fundamental domain of the centralizer $C(\gamma_\alpha)$ is

$$F_{C(\gamma_\alpha)} = \left\{ z \in \mathbb{H}^2 : -\frac{1}{2} \le X_1(z) < \frac{1}{2}, -\frac{1}{2} \le X_2(z) < \frac{1}{2} \right\}.$$

We consider the sum

$$\sum_{0 \neq \alpha \in \mathcal{O}_K/(\mathcal{O}_K^\times)^2} \sum_{\sigma \in C(\gamma_\alpha)\backslash\Gamma_K} \int_{\sigma F_A} k(z, \gamma_\alpha z) u(z)\, d\mu(z), \tag{3.20}$$

and proceed as in the previous section. The union of the sets σF_A above makes up the set $F_{C(\gamma_\alpha)}$ except for the images of the remainder part $F \setminus F_A =: F_A^*$. If σ leaves ∞ fixed, then so does every element in its coset, and the part σF_A^* is the same as

$$\{z \in F_{C(\gamma_\alpha)} : \sigma^{-1} z \in F, Y_0(\sigma^{-1}z) \ge A\}.$$

But since $\sigma\infty = \infty$, the values $Y_0(\sigma^{-1}z)$ and $Y_0(z)$ are the same.

Now assume that $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ does not fix the cusp ∞, so $c \neq 0$. If $z \in F_A^*$, then

$$Y_0(\sigma z) \le \frac{1}{N(c)^2 Y_0(z)} \le \frac{1}{A},$$

and since

$$k(z, \gamma_\alpha z) = \psi\left(\frac{|z_1 - (z_1 - \alpha)|^2}{y_1^2}, \frac{|z_2 - (z_2 - \alpha')|^2}{y_2^2} \right) = \psi\left(\frac{\alpha^2}{y_1^2}, \frac{\alpha'^2}{y_2^2} \right),$$

this is zero for σz if $z \in F_A^*$ and A is big enough (because ψ has compact support). So in this case we can integrate over σF instead of σF_A and (3.20) becomes

$$\sum_{0\neq\alpha\in\mathcal{O}_K/(\mathcal{O}_K^\times)^2} \int_{z\in F_{C(\gamma_\alpha)}, Y_0(z)\leq A} \psi\left(\frac{\alpha^2}{y_1^2}, \frac{\alpha'^2}{y_2^2}\right) u(z)\, d\mu(z) =$$

$$= \sum_{0\neq\alpha\in\mathcal{O}_K/(\mathcal{O}_K^\times)^2} \iint_{-\frac{1}{2}\leq X_1, X_2<\frac{1}{2}} \iint_{Y_0\leq A} \psi\left(\frac{\alpha^2}{y_1^2}, \frac{\alpha'^2}{y_2^2}\right) u(z) \frac{dy_1\, dy_2}{y_1^2 y_2^2}\, dx_1\, dx_2.$$

The function u has the Fourier expansion

$$u(z) = \eta y_1^{s_1} y_2^{s_2} + \phi y_1^{1-s_1} y_2^{1-s_2} + \sum_{l\in L_K^*\backslash 0} a_l(y) e^{2\pi i<l,x>}$$

and since for every $0 \neq l \in L_K^*$ we have

$$\iint_{-\frac{1}{2}\leq X_1, X_2<\frac{1}{2}} e^{2\pi i<l,x>}\, dx_1\, dx_2 = 0,$$

by Lemma 2.1.7, which also gives that

$$\iint_{-\frac{1}{2}\leq X_1, X_2<\frac{1}{2}} 1\, dx_1\, dx_2 = \sqrt{d(K)},$$

we can write the sum above in the following way:

$$\sqrt{d(K)} \sum_{0\neq\alpha\in\mathcal{O}_K/(\mathcal{O}_K^\times)^2} \iint_{Y_0\leq A} \psi\left(\frac{\alpha^2}{y_1^2}, \frac{\alpha'^2}{y_2^2}\right) (\eta y_1^{s_1} y_2^{s_2} + \phi y_1^{1-s_1} y_2^{1-s_2}) \frac{dy_1\, dy_2}{y_1^2 y_2^2}.$$

We substitute $u_k = \left|\alpha^{(k)}\right| / y_k$ $(k = 1, 2)$ to obtain

$$\frac{\sqrt{d(K)}}{|N(\alpha)|} \sum_{0\neq\alpha\in\mathcal{O}_K/(\mathcal{O}_K^\times)^2} \iint_{\substack{0<u_1,u_2<\infty \\ |\alpha_1\alpha_2|/A\leq u_1 u_2}} \psi\left(u_1^2, u_2^2\right) \times$$

$$\times \left[\eta \left|\frac{\alpha}{\alpha'}\right|^{\frac{\pi i m_u}{2\log\varepsilon}} \frac{|N(\alpha)|^s}{u_1^{s_1} u_2^{s_2}} + \phi \left|\frac{\alpha'}{\alpha}\right|^{\frac{\pi i m_u}{2\log\varepsilon}} \frac{|N(\alpha)|^{1-s}}{u_1^{1-s_1} u_2^{1-s_2}}\right] du_1\, du_2$$

since by (3.1) $s_1 = s + \frac{\pi i m_u}{2\log\varepsilon}$ and $s_2 = s - \frac{\pi i m_u}{2\log\varepsilon}$ for some $s \in \mathbb{C}$ and $m_u \in \mathbb{Z}$. Hence we have to examine two terms:

$$\eta\sqrt{d(K)} \iint_{0<u_1,u_2<\infty} \psi\left(u_1^2, u_2^2\right) u_1^{-s_1} u_2^{-s_2} \sum_{\substack{0\neq\alpha\in\mathcal{O}_K/(\mathcal{O}_K^\times)^2 \\ |N(\alpha)|\leq A u_1 u_2}} \frac{\lambda_{m_u}(\alpha)}{|N(\alpha)|^{1-s}}\, du_1\, du_2 \tag{3.21}$$

and

$$\phi\sqrt{d(K)} \iint_{0<u_1,u_2<\infty} \psi\left(u_1^2,u_2^2\right) u_1^{s_1-1}u_2^{s_2-1} \sum_{\substack{0\neq\alpha\in\mathcal{O}_K/(\mathcal{O}_K^\times)^2 \\ |N(\alpha)|\leq Au_1u_2}} \frac{\lambda_{-m_u}(\alpha)}{|N(\alpha)|^s}\, du_1\, du_2. \tag{3.22}$$

Here $\lambda_m(\alpha)=|\alpha/\alpha'|^{\frac{\pi i m}{2\log\varepsilon}}$ is the Grössencharacter that occurred also in the previous section.

To follow the usual notation s will denote a complex variable for a little while (instead of the fixed parameter of u). We consider the function

$$Z_K(s,m)=\sum_{0\neq\alpha\in\mathcal{O}_K/(\mathcal{O}_K^\times)^2}\frac{\lambda_m(\alpha)}{|N(\alpha)|^s},$$

it clearly converges absolutely for $\operatorname{Re}s>1$. We rewrite this sum in the following way:

$$Z_K(s,m)=\sum_{0\neq(\alpha)\lhd\mathcal{O}_K}\left(\frac{\lambda_m(\alpha)}{|N(\alpha)|^s}+\frac{\lambda_m(-\alpha)}{|N(-\alpha)|^s}+\frac{\lambda_m(\varepsilon\alpha)}{|N(\varepsilon\alpha)|^s}+\frac{\lambda_m(-\varepsilon\alpha)}{|N(-\varepsilon\alpha)|^s}\right).$$

The first two and the second two terms are equal and $|N(\alpha)|=|N(\varepsilon\alpha)|$, so let us calculate the third numerator:

$$\lambda_m(\varepsilon\alpha)=\left|\frac{\varepsilon\alpha}{\varepsilon'\alpha'}\right|^{\frac{\pi i m}{2\log\varepsilon}}=(\varepsilon^2)^{\frac{\pi i m}{2\log\varepsilon}}\lambda_m(\alpha)=e^{\pi i m}\lambda_m(\alpha).$$

Hence $Z_K(s,m)=0$ for an odd integer m. Note that if m_u is odd, then the terms in the partial sums in (3.21) and (3.22) also cancel each other and therefore the contribution of the totally parabolic conjugacy classes is zero. In the following we assume that m denotes an even integer. Then $Z_K(s,m)=4\zeta_K(s,m)$, where

$$\zeta_K(s,m)=\sum_{0\neq(\alpha)\lhd\mathcal{O}_K}\frac{\lambda_m(\alpha)}{|N(\alpha)|^s} \tag{3.23}$$

is a Hecke L-function. It is entire if $m\neq 0$. For $m=0$ it is the Dedekind zeta function, which has a simple pole at $s=1$ with residue $2\log\varepsilon/\sqrt{d(K)}$ and it is holomorphic elsewhere (see [9]).

We apply Theorem 5.2 and Corollary 5.3 in [12]. For this purpose we write

$$\zeta_K(s,m)=\sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

where

$$a_n=\sum_{\substack{(\alpha)\lhd\mathcal{O}_K \\ |N(\alpha)|=n}}\lambda_m(\alpha).$$

We estimate the sum a_n by estimating the number of the ideals of norm n. We have already mentioned in the proof of Proposition 2.1.6 that this is bounded by n^δ for every $\delta>0$, so we have $a_n \ll_\delta n^\delta$. Now if $0 < \operatorname{Re} s < 1$ and $\sigma_0 > 1 - \operatorname{Re} s$, then we have

$$\sum_{n \le A}{}' \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \zeta_K(S + s, m) \frac{A^S}{S} \, dS + R,$$

where $\sum'$ indicates that if A is an integer, then the last term is to be counted with half weight, further

$$R \ll \sum_{\substack{A/2 < n < 2A \\ n \ne A}} |a_n| \, n^{-\operatorname{Re} s} \min\left(1, \frac{A}{T\,|A - n|}\right) + \frac{4^{\sigma_0} + A^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0 + \operatorname{Re} s}}.$$

Let us fix a small number $0 < \varepsilon_0 < \operatorname{Re} s$, set the value $\sigma_0 = 1 - \operatorname{Re} s + \varepsilon_0 + \delta$ for some $\delta > 0$ and estimate the second term on the right hand side. Since $a_n \ll n^{\varepsilon_0}$ we have

$$\frac{4^{\sigma_0} + A^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0 + \operatorname{Re} s}} \ll \frac{A^{1 - \operatorname{Re} s + \varepsilon_0 + \delta}}{T} \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} \ll \frac{A^{1 - \operatorname{Re} s + \varepsilon_0} A^{\delta}}{T\delta}.$$

Now if we choose $\delta = \frac{1}{\log A}$, then the right hand side is

$$\ll \frac{A^{1 - \operatorname{Re} s + \varepsilon_0} \log A}{T},$$

where the implied constant depends on ε_0. If we set $T = A^{1 - \operatorname{Re} s + \nu}$ for some $\varepsilon_0 < \nu < \operatorname{Re} s$, then this term is $o(1)$ when $A \to \infty$.

Now we turn to the first error term

$$\sum_{\substack{A/2 < n < 2A \\ n \ne A}} |a_n| \, n^{-\operatorname{Re} s} \min\left(1, \frac{A}{T\,|A - n|}\right) = \sum_{\substack{A/2 < n < 2A \\ n \ne A}} |a_n| \, n^{-\operatorname{Re} s} \min\left(1, \frac{A^{\operatorname{Re} s - \nu}}{|A - n|}\right).$$

We divide this sum into two parts, the first is where $|A - n| < A^{\operatorname{Re} s - \nu}$, here we get $|a_n| \, n^{-\operatorname{Re} s}$, and the second part is where $|A - n| \ge A^{\operatorname{Re} s - \nu}$ for which we get $|a_n| \, n^{-\operatorname{Re} s} \frac{A^{\operatorname{Re} s - \nu}}{|A - n|}$, so this error term is

$$\begin{aligned}
&\ll A^{\varepsilon_0 - \operatorname{Re} s} A^{\operatorname{Re} s - \nu} + A^{\operatorname{Re} s - \nu} A^{\varepsilon_0 - \operatorname{Re} s} \sum_{|A - n| \ge A^{\operatorname{Re} s - \nu}} \frac{1}{|A - n|} \\
&= A^{\varepsilon_0 - \nu} \left[1 + \sum_{|A - n| \ge A^{\operatorname{Re} s - \nu}} \frac{1}{|A - n|} \right] \\
&\ll A^{\varepsilon_0 - \nu} \log A = o(1),
\end{aligned}$$

since we have $0 < \varepsilon_0 < \nu < \operatorname{Re} s$. Note that the implied constant depends on ε_0, ν and $\operatorname{Re} s$.

Now we return to the integral (3.21). Rewriting the partial sum as above we get

$$4\eta\sqrt{d(K)}\int_0^\infty\int_0^\infty \psi\left(u_1^2,u_2^2\right)u_1^{-s_1}u_2^{-s_2}\times$$

$$\times\left(\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\zeta_K(1-s+S,m_u)\frac{(u_1u_2A)^S}{S}\,dS\right)du_1\,du_2+o(1),$$

where $\sigma_0=1-1+\operatorname{Re}s+\varepsilon_0+\delta=\operatorname{Re}s+\varepsilon_0+\frac{1}{\log A}$ for some $0<\varepsilon_0<1-\operatorname{Re}s$. Note that since ψ has compact support and hence u_1 and u_2 are bounded from above, one can see by a closer studying of the arguments above that T can be chosen independently of them, i.e. $T=A^{1-1+\operatorname{Re}s+\nu}=A^{\operatorname{Re}s+\nu}$ works for some $\varepsilon_0<\nu<1-\operatorname{Re}s$. Now we interchange the order of integration to get

$$\frac{4\eta\sqrt{d(K)}}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}F(S)\zeta_K(1-s+S,m_u)\frac{A^S}{S}\,dS$$

where

$$F(S)=\int_0^\infty\int_0^\infty \psi\left(u_1^2,u_2^2\right)u_1^{S-s_1}u_2^{S-s_2}\,du_1\,du_2. \tag{3.24}$$

If $G(S)=F(S)\zeta_K(1-s+S,m_u)\frac{A^S}{S}$ and $\sigma_1<0$, then by the residue theorem

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}G(S)\,dS=&\zeta_K(1-s,m_u)F(0)+\delta_{m_u=0}\cdot\frac{2\log\varepsilon}{\sqrt{d(K)}}\cdot\frac{A^s}{s}F(s)\\ &-\frac{1}{2\pi i}\left(\int_{\sigma_0+iT}^{\sigma_1+iT}G(S)\,dS+\int_{\sigma_1+iT}^{\sigma_1-iT}G(S)\,dS+\int_{\sigma_1-iT}^{\sigma_0-iT}G(S)\,dS\right).\end{aligned}\tag{3.25}$$

To get the error terms we apply partial integration in $F(S)$ with respect to u_2, and here we assume also that $\operatorname{Re}S>\operatorname{Re}s-1$:

$$\int_0^\infty\int_0^\infty \psi\left(u_1^2,u_2^2\right)u_1^{S-s_1}u_2^{S-s_2}\,du_1\,du_2=\int_0^\infty\left(\int_0^\infty\psi\left(u_1^2,u_2^2\right)u_2^{1-s_2}u_2^{S-1}\,du_2\right)u_1^{S-s_1}\,du_1.$$

The inner integral is

$$-\frac{1}{S}\int_0^\infty\left[2\frac{\partial\psi}{\partial u_2}\left(u_1^2,u_2^2\right)u_2^{2-s_2}+(1-s_2)\psi\left(u_1^2,u_2^2\right)u_2^{-s_2}\right]u_2^S\,du_2=\frac{1}{S}\int_0^\infty H_{s_2}(u_1,u_2)u_2^{S-s_2}\,du_2,$$

where

$$H_s(u_1,u_2)=-2u_2^2\frac{\partial\psi}{\partial u_2}\left(u_1^2,u_2^2\right)-(1-s)\psi(u_1^2,u_2^2).$$

That is,

$$G(S) = \zeta_K(1-s+S, m_u)\frac{A^S}{S^2}\int_0^\infty\int_0^\infty H_{s_2}(u_1,u_2)u_1^{S-s_1}u_2^{S-s_2}\,du_1\,du_2.$$

We estimate the integrals on the right hand side of (3.25). First we examine the horizontal segments:

$$\int_0^\infty\int_0^\infty H_{s_2}(u_1,u_2)u_1^{-s_1}u_2^{-s_2}\left(\frac{1}{2\pi i}\int_{\sigma_0\pm iT}^{\sigma_1\pm iT}\zeta_K(1-s+S,m_u)\frac{(u_1u_2A)^S}{S^2}\,dS\right)du_1\,du_2.$$

We will set $\sigma_1 = \operatorname{Re} s - 1 + \delta_1$ for some small $\delta_1 > 0$ and estimate the inner integral by the convexity bound:

$$\zeta_K(\sigma+it,m) \ll |t|^{1-\sigma+\delta}$$

as $|t| \to \infty$ for any $\delta > 0$ and $0 \le \sigma \le 1$ (see e.g. [16]) [1]. So if $\operatorname{Re} S = \sigma$, then on the horizontal lines we have

$$\begin{aligned}\zeta_K(1-s+S,m_u)\frac{(u_1u_2A)^S}{S^2} &\ll \frac{A^\sigma}{T^2}\cdot T^{1-(1-\operatorname{Re} s)-\sigma+\delta} = A^\sigma T^{-2-\sigma+\operatorname{Re} s+\delta}\\ &= A^{\sigma+(\operatorname{Re} s+\nu)(-2+\operatorname{Re} s-\sigma+\delta)} = A^{\sigma(1-\operatorname{Re} s-\nu)+(\operatorname{Re} s+\nu)(-2+\operatorname{Re} s+\delta)},\end{aligned}$$

since u_1, u_2 and σ are bounded. We will show that the exponent of A is negative if $\delta > 0$ is chosen properly, so this last bound is $o(1)$. As $\nu < 1 - \operatorname{Re} s$ we increase this exponent by increasing σ:

$$\begin{aligned}A^{\sigma(1-\operatorname{Re} s-\nu)+(\operatorname{Re} s+\nu)(-2+\operatorname{Re} s+\delta)} &\le A^{\sigma_0(1-\operatorname{Re} s-\nu)+(\operatorname{Re} s+\nu)(-2+\operatorname{Re} s+\delta)}\\ &= A^{(\operatorname{Re} s+\varepsilon_0+\frac{1}{\log A})(1-\operatorname{Re} s-\nu)+(\operatorname{Re} s+\nu)(-2+\operatorname{Re} s+\delta)}\\ &\ll A^{(\operatorname{Re} s+\varepsilon_0)(1-\operatorname{Re} s-\nu)+(\operatorname{Re} s+\nu)(-2+\operatorname{Re} s+\delta)}\end{aligned}$$

Now this last exponent is negative if and only if

$$\begin{aligned}\delta(\operatorname{Re} s+\nu) &< (2-\operatorname{Re} s)(\operatorname{Re} s+\nu) - (\operatorname{Re} s+\varepsilon_0)(1-\operatorname{Re} s-\nu)\\ &= \operatorname{Re} s + 2\nu - \varepsilon_0 + \varepsilon_0(\operatorname{Re} s+\nu).\end{aligned}$$

The right hand side is positive since $\varepsilon_0 < \nu$, so this inequality holds for some $\delta > 0$ small enough. Then the integrals on the horizontal segments are $o(1)$.

On the vertical line we have

$$\zeta_K(1-s+S,m_u)\frac{(u_1u_2A)^S}{S^2} \ll \frac{A^{\sigma_1}}{(1+|t|)^2}\cdot|t|^{1-(1-\operatorname{Re} s+\sigma_1)+\delta} \ll A^{\operatorname{Re} s-1+\delta_1}\,|t|^{-1+\delta+\delta_1}$$

[1] Note that in [16] K is assumed to be of narrow class number one. However, by the derivation of the convexity bound this condition is used there only through the functional equation of $\zeta_K(s,m)$ which is proved in full generality in [9]. Hence we do not have to make this restriction.

and hence

$$\int_{\sigma_1+iT}^{\sigma_1-iT} G(S)\,dS \ll A^{\mathrm{Re}\,s-1+\delta_1}T^{\delta+\delta_1}+A^{\mathrm{Re}\,s-1+\delta_1}=A^{\mathrm{Re}\,s-1+\delta_1}A^{(\delta+\delta_1)(\mathrm{Re}\,s+\nu)}+A^{\mathrm{Re}\,s-1+\delta_1}$$

Now if we choose δ and δ_1 so that $(\delta+\delta_1)(\mathrm{Re}\,s+\nu)<1-\mathrm{Re}\,s-\delta_1=-\sigma_1$ holds, then we get that this term is also $o(1)$ and (3.21) is

$$4\eta\sqrt{d(K)}\zeta_K(1-s,m_u)F(0)+\delta_{m_u=0}\cdot 8\eta\log\varepsilon\cdot\frac{A^s}{s}F(s)+o(1),$$

where F is defined in (3.24). We remark for the completeness that on the half-plane $\sigma>1$ the convexity bound is not applicable. However, a similar polynomial bound is obtained in terms of $|t|$ in this region for example in [2] and one may apply integration by parts above more than once if necessary to get the result above.

One can show the same way that (3.22) is

$$4\phi\sqrt{d(K)}\zeta_K(s,-m_u)\tilde F(0)+\delta_{m_u=0}\cdot 8\phi\log\varepsilon\cdot\frac{A^{1-s}}{1-s}\tilde F(1-s)+o(1),$$

where

$$\tilde F(S)=\int_0^\infty\int_0^\infty \psi\left(u_1^2,u_2^2\right)u_1^{S+s_1-1}u_2^{S+s_2-1}\,du_1\,du_2.$$

It remains to evaluate F and $\tilde F$ at some points. If $m_u=0$, then

$$F(s)=\tilde F(1-s)=\int_0^\infty\int_0^\infty \psi(u_1^2,u_2^2)\,du_1\,du_2=\frac14\int_0^\infty\int_0^\infty\frac{\psi(u_1,u_2)}{\sqrt{u_1u_2}}\,du_1\,du_2=\frac14 g(0,0).$$

For an arbitrary even m_u we have

$$F(0)=\int_0^\infty\int_0^\infty \psi(u_1^2,u_2^2)u_1^{-s_1}u_2^{-s_2}\,du_1\,du_2=\frac14\int_0^\infty\int_0^\infty\frac{\psi(u_1,u_2)}{\sqrt{u_1u_2}}u_1^{-\frac{s_1}{2}}u_2^{-\frac{s_2}{2}}\,du_1\,du_2.$$

As in the previous section, one can use Proposition 2.3.1 to see that this double integral is

$$\frac{1}{4\pi^2}\int_0^\infty\int_0^\infty\frac{\partial^2Q}{\partial w_1\partial w_2}(w_1,w_2)\int_0^{w_1}\frac{u_1^{-\frac{s_1}{2}}}{\sqrt{w_1-u_1}\sqrt{u_1}}\,du_1\int_0^{w_2}\frac{u_2^{-\frac{s_2}{2}}}{\sqrt{w_2-u_2}\sqrt{u_2}}\,du_2\,dw_1\,dw_2.$$

We have

$$\int_0^w\frac{u^{-\frac{\alpha}{2}}}{\sqrt{w-u}\sqrt{u}}\,du=w^{-\frac{\alpha}{2}}B\left(\frac{1-\alpha}{2},\frac12\right)=\frac{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac12)}{\Gamma(\frac{1-\alpha}{2}+\frac12)}=w^{-\frac{\alpha}{2}}2^{-\alpha}\frac{\Gamma(\frac{1-\alpha}{2})^2}{\Gamma(1-\alpha)}$$

for any $w > 0$ and $\alpha \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < 1$, where B is the beta function and we used the following relations:

$$\Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

Then

$$F(0) = \frac{2^{-(s_1+s_2)}}{4\pi^2} \frac{\Gamma(\frac{1-s_1}{2})^2\Gamma(\frac{1-s_2}{2})^2}{\Gamma(1-s_1)\Gamma(1-s_2)} \int_0^\infty \int_0^\infty \frac{\partial^2 Q}{\partial w_1 \partial w_2}(w_1, w_2) w_1^{-\frac{s_1}{2}} w_2^{-\frac{s_2}{2}} \, dw_1 \, dw_2.$$

Now we substitute $w_j = e^{x_j} + e^{-x_j} - 2$ to express this in terms of the function g. The integral above becomes

$$\int_0^\infty \int_0^\infty \frac{\partial^2 g}{\partial x_1 \partial x_2}(x_1, x_2)(e^{x_1} + e^{-x_1} - 2)^{-\frac{s_1}{2}} (e^{x_2} + e^{-x_2} - 2)^{-\frac{s_2}{2}} \, dx_1 \, dx_2$$

Now

$$\frac{\partial^2 g}{\partial x_1 \partial x_2}(x_1, x_2) = -\frac{1}{(2\pi)^2} \int_0^\infty \int_0^\infty h(r_1, r_2) r_1 r_2 e^{-i(r_1 x_1 + r_2 x_2)} \, dr_1 \, dr_2,$$

and

$$\int_0^\infty (e^x + e^{-x} - 2)^{-\alpha} e^{-irx} \, dx = \int_0^1 \left(y^{-\frac{1}{2}} - y^{\frac{1}{2}}\right)^{-2\alpha} y^{ir-1} \, dy = \int_0^1 (1-y)^{-2\alpha} \, y^{\alpha+ir-1} \, dy$$

$$= B(\alpha + ir, 1 - 2\alpha) = \frac{\Gamma(\alpha + ir)\Gamma(1-2\alpha)}{\Gamma(1 - \alpha + ir)}.$$

We obtain that $F(0)$ is

$$-\left(\frac{2^{-(s_1+s_2)}}{4\pi^2}\right)^2 \Gamma\left(\frac{1-s_1}{2}\right)^2 \Gamma\left(\frac{1-s_2}{2}\right)^2 \int_0^\infty \int_0^\infty h(r_1, r_2) r_1 r_2 \frac{\Gamma(\frac{s_1}{2} + ir)\Gamma(\frac{s_2}{2} + ir)}{\Gamma(\frac{2-s_1}{2} + ir)\Gamma(\frac{2-s_2}{2} + ir)} \, dr_1 \, dr_2.$$

Similarly, $\tilde{F}(0)$ is

$$-\left(\frac{2^{-(2-s_1-s_2)}}{4\pi^2}\right)^2 \Gamma\left(\frac{s_1}{2}\right)^2 \Gamma\left(\frac{s_2}{2}\right)^2 \int_0^\infty \int_0^\infty h(r_1, r_2) r_1 r_2 \frac{\Gamma(\frac{1-s_1}{2} + ir)\Gamma(\frac{1-s_2}{2} + ir)}{\Gamma(\frac{s_1-1}{2} + ir)\Gamma(\frac{s_2-1}{2} + ir)} \, dr_1 \, dr_2,$$

and we have

Proposition 3.1.10. *Assume that u is not a cusp form, the main term of its Fourier expansion is $\eta y_1^{s_1} y_2^{s_2} + \phi y_1^{1-s_1} y_2^{1-s_2}$ and the numbers $s \in \mathbb{C}$ and m_u are defined in* (3.1) *on page* 75*. Then the contribution of the totally parabolic conjugacy classes is*

$$\sum_{\{\gamma\} \text{ totally parabolic}} \sum_{\sigma \in \{\gamma\}} \int_{F_A} k(z, \sigma z) u(z) \, d\mu(z) =$$

$$\delta_{m_u=0} 2\log\varepsilon \left[\frac{\eta A^s}{s} + \frac{\phi A^{1-s}}{1-s}\right] g(0,0)$$

$$- \delta_{m_u\equiv 0\,(2)}\eta\sqrt{d(K)}\left(\frac{2^{-(s_1+s_2)}}{2\pi^2}\right)^2 \zeta_K(1-s,m_u)\Gamma\left(\frac{1-s_1}{2}\right)^2\Gamma\left(\frac{1-s_2}{2}\right)^2\times$$

$$\times\int_0^\infty\int_0^\infty h(r_1,r_2)r_1r_2\frac{\Gamma(\frac{s_1}{2}+ir)\Gamma(\frac{s_2}{2}+ir)}{\Gamma(\frac{2-s_1}{2}+ir)\Gamma(\frac{2-s_2}{2}+ir)}\,dr_1\,dr_2$$

$$- \delta_{m_u\equiv 0\,(2)}\phi\sqrt{d(K)}\left(\frac{2^{-(2-s_1-s_2)}}{2\pi^2}\right)^2 \zeta_K(s,-m_u)\Gamma\left(\frac{s_1}{2}\right)^2\Gamma\left(\frac{s_2}{2}\right)^2\times$$

$$\times\int_0^\infty\int_0^\infty h(r_1,r_2)r_1r_2\frac{\Gamma(\frac{1-s_1}{2}+ir)\Gamma(\frac{1-s_2}{2}+ir)}{\Gamma(\frac{s_1-1}{2}+ir)\Gamma(\frac{s_2-1}{2}+ir)}\,dr_1\,dr_2+o(1),$$

where $\zeta_K(s,m)$ *is given in* (3.23).

3.2 Evaluation of the spectral part

As in Section 2.4 let $\{u_j(z) : j\geq 0\}$ be a complete orthonormal system of automorphic forms for the discrete spectrum of Γ_K with eigenvalue $(\lambda_1^{(j)},\lambda_2^{(j)})$, $\lambda_k^{(j)}=s_k^{(j)}(1-s_k^{(j)})$, where $\operatorname{Re}s_k^{(j)}\geq\frac{1}{2}$ and $s_k^{(j)}=\frac{1}{2}+ir_k^{(j)}$, hence $\lambda_k^{(j)}=\frac{1}{4}+(r_k^{(j)})^2$ $(k=1,2)$. Recall that the Fourier expansion of u_j is

$$u_j(z)=\phi_j y_1^{1-s_1^{(j)}}y_2^{1-s_2^{(j)}}+\sum_{l\in L_K^*\setminus 0} c_l^{(j)}\sqrt{y_1y_2}K_{s_1^{(j)}-\frac{1}{2}}(2\pi\,|l_1|\,y_1)K_{s_2^{(j)}-\frac{1}{2}}(2\pi\,|l_2|\,y_2)e^{2\pi i<x,l>}.$$

The Fourier expansion of the Eisenstein series is given in (2.18) on page 51. If $\phi_j\neq 0$ for some $j>0$, then u_j is a constant multiple of a residue of an Eisenstein series. Note that the Eisenstein series $E(z,s,m)$ has no poles unless $m=0$. In this case the poles in the half-plane $\operatorname{Re}s\geq\frac{1}{2}$ are in fact on the section $(1/2,1]$ and they are simple. Furthermore, every pole of $E(z,s,0)$ is also a pole of $\phi(s,0)$. This function has only finitely many poles in $\operatorname{Re}s>1/2$ and all of them are in $(1/2,1]$ (see Section 2.2). Let us denote this finite set by L and the residue of $\phi(s,0)$ at some $s_l\in L$ by R_{s_l}. Hence if $\phi_j\neq 0$, then $(s_1^{(j)},s_2^{(j)})=(s_l,s_l)$ for some $s_l\in L$. Recall that the Fourier expansion of u is

$$u(z)=\eta y_1^{s_1}y_2^{s_2}+\phi y_1^{1-s_1}y_2^{1-s_2}+\sum_{l\in L_K^*\setminus 0} c_l\sqrt{y_1y_2}K_{s_1-\frac{1}{2}}(2\pi\,|l_1|\,y_1)K_{s_2-\frac{1}{2}}(2\pi\,|l_2|\,y_2)e^{2\pi i<x,l>},\quad(3.26)$$

and if at least one of η and ϕ is non-zero, then

$$(s_1,s_2)=\left(s+\frac{\pi i m_u}{2\log\varepsilon},s-\frac{\pi i m_u}{2\log\varepsilon}\right)$$

holds for some $s\in\mathbb{C}$ and $m_u\in\mathbb{Z}$. In this case we also assume for simplicity that $\frac{1}{2}\leq\operatorname{Re}s<1$ and, in addition, that $\frac{s+1}{2},\frac{2-s}{2}\notin L$.

We are going to evaluate the truncated trace

$$\mathrm{Tr}_u^A K = \int_{F_A} K(z,z)u(z)\,d\mu(z)$$

using the spectral theorem which is applicable since ψ is assumed to be compactly supported (see Theorem 2.2.10):

$$\mathrm{Tr}_u^A K = \sum_j h(r_1^{(j)},r_2^{(j)}) I_u^A(u_j) + \\ + \frac{1}{8\pi\sqrt{d(K)}\log\varepsilon} \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} h\left(r + \frac{\pi m}{2\log\varepsilon}, r - \frac{\pi m}{2\log\varepsilon}\right) I_u^A(r,m)\,dr,$$

where

$$I_u^A(u_j) = \int_{F_A} |u_j(z)|^2\, u(z)\,d\mu(z),$$

and

$$I_u^A(r,m) = \int_{F_A} E\left(z, \frac{1}{2}+ir, m\right) E\left(z, \frac{1}{2}-ir, -m\right) u(z)\,d\mu(z)$$
$$= \int_{F_A} \left|E\left(z, \frac{1}{2}+ir, m\right)\right|^2 u(z)\,d\mu(z).$$

We define $F^1 = \{z \in F_A : Y_0(z) < 1\}$, then by Lemma 1.2.3 we have $F_A = F^1 \cup F_1^A$, where $F_1^A = \{z \in F_\infty : 1 \le Y_0(z) \le A\}$. Moreover, the set $\overline{F^1} \subset \mathbb{H}^2$ is compact by Lemma 1.2.4. Now we divide the integrals above into two parts: $I_u^A(u_j) = I^1(u_j) + I_1^A(u_j)$ and $I_u^A(r,m) = I^1(r,m) + I_1^A(r,m)$, where

$$I^1(u_j) = \int_{F^1} |u_j(z)|^2\, u(z)\,d\mu(z), \qquad I_1^A(u_j) = \int_{F_1^A} |u_j(z)|^2\, u(z)\,d\mu(z),$$

$$I^1(r,m) = \int_{F^1} \left|E\left(z, \frac{1}{2}+ir, m\right)\right|^2 u(z)\,d\mu(z),$$

$$I_1^A(r,m) = \int_{F_1^A} \left|E\left(z, \frac{1}{2}+ir, m\right)\right|^2 u(z)\,d\mu(z).$$

Let us examine first the integral $I_1^A(r,m)$. Recall that

$$F_1^A = \left\{z \in \mathbb{H}^2 : 1 \le Y_0 \le A,\ -\frac{1}{2} \le Y_1 < \frac{1}{2};\ -\frac{1}{2} \le X_1, X_2 < \frac{1}{2}\right\},$$

and that we denote by $a_0(y)$ the zeroth Fourier coefficient $\eta y_1^{s_1} y_2^{s_2} + \phi y_1^{1-s_1} y_2^{1-s_2}$ of the function u. Also, let $a_0(z, \frac{1}{2}+ir, m)$ be the zeroth Fourier coefficient of $E(z, \frac{1}{2}+ir, m)$ and we define

$\overline{E}(z, \frac{1}{2} + ir, m) = E(z, \frac{1}{2} + ir, m) - a_0(z, \frac{1}{2} + ir, m)$. Then by Lemma 2.1.7 we have

$$\int_{F_1^A} \left| E\left(z, \frac{1}{2} + ir, m\right) \right|^2 a_0(y)\, d\mu(z) =$$

$$= \int_{F_1^A} \left| a_0\left(z, \frac{1}{2} + ir, m\right) \right|^2 a_0(y)\, d\mu(z) + \int_{F_1^A} \left| \overline{E}\left(z, \frac{1}{2} + ir, m\right) \right|^2 a_0(y)\, d\mu(z). \tag{3.27}$$

Note that these integrals are non-zero only if u is not a cusp form. We calculate the first term above in that case. Substituting

$$a_0\left(z, \frac{1}{2} + ir, m\right) = (y_1 y_2)^{\frac{1}{2}+ir} \lambda_m(z) + \phi\left(\frac{1}{2} + ir, m\right) (y_1 y_2)^{\frac{1}{2}-ir} \lambda_{-m}(z)$$

and using Lemma 2.1.7 we get that it is

$$\begin{aligned}\sqrt{d(K)} \iint\limits_{\substack{1 \le Y_0 \le Y \\ -\frac{1}{2} \le Y_1 < \frac{1}{2}}} & \left[(y_1 y_2)^{\frac{1}{2}+ir} \lambda_m(z) + \phi\left(\frac{1}{2} + ir, m\right) (y_1 y_2)^{\frac{1}{2}-ir} \lambda_{-m}(z) \right] \\ & \cdot \left[(y_1 y_2)^{\frac{1}{2}-ir} \lambda_{-m}(z) + \phi\left(\frac{1}{2} - ir, -m\right) (y_1 y_2)^{\frac{1}{2}+ir} \lambda_m(z) \right] \\ & \cdot \left[\eta (y_1 y_2)^s \lambda_{m_u}(z) + \phi (y_1 y_2)^{1-s} \lambda_{-m_u}(z) \right] \frac{dy_1\, dy_2}{y_1^2 y_2^2}.\end{aligned}$$

As an abbreviation we write $\phi_{r,m} = \phi(\frac{1}{2} + ir, m)$. Since $\frac{dy_1\, dy_2}{y_1^2 y_2^2} = 2 \log \varepsilon \frac{dY_0}{Y_0^2}\, dY_1$, the previous integral becomes

$$\begin{aligned}2 \log \varepsilon \sqrt{d(K)} \int_1^A \int_{-\frac{1}{2}}^{\frac{1}{2}} & \left[Y_0^{\frac{1}{2}+ir} e^{2\pi i m Y_1} + \phi_{r,m} Y_0^{\frac{1}{2}-ir} e^{-2\pi i m Y_1} \right] \\ & \cdot \left[Y_0^{\frac{1}{2}-ir} e^{-2\pi i m Y_1} + \phi_{-r,-m} Y_0^{\frac{1}{2}+ir} e^{2\pi i m Y_1} \right] \\ & \cdot \left[\eta Y_0^s e^{2\pi i m_u Y_1} + \phi Y_0^{1-s} e^{-2\pi i m_u Y_1} \right] \frac{dY_1\, dY_0}{Y_0^2}.\end{aligned} \tag{3.28}$$

Since $\phi_{r,m} \phi_{-r,-m} = |\phi_{r,m}|^2 = 1$ by Corollary 2.2.6 and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i n Y_1}\, dY_1 = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases}$$

we get that (3.28) is

$$\begin{aligned}2 \log \varepsilon \sqrt{d(K)} \int_1^A & 2\delta_{m_u=0} (\eta Y_0^{s-1} + \phi Y_0^{-s}) + \phi_{-r,-m} (\delta_{m_u=-2m} \eta Y_0^{s-1+2ir} + \delta_{m_u=2m} \phi Y_0^{-s+2ir}) \\ & + \phi_{r,m} (\delta_{m_u=2m} \eta Y_0^{s-1-2ir} + \delta_{m_u=-2m} \phi Y_0^{-s-2ir})\, dY_0,\end{aligned}$$

that is,

$$2\log\varepsilon\sqrt{d(K)}\left[2\delta_{m_u=0}\left(\eta\frac{A^s-1}{s}+\phi\frac{A^{1-s}-1}{1-s}\right)\right.$$
$$+\phi_{-r,-m}\left(\delta_{m_u=-2m}\eta\frac{A^{s+2ir}-1}{s+2ir}+\delta_{m_u=2m}\phi\frac{A^{1-s+2ir}-1}{1-s+2ir}\right)$$
$$\left.+\phi_{r,m}\left(\delta_{m_u=2m}\eta\frac{A^{s-2ir}-1}{s-2ir}+\delta_{m_u=-2m}\phi\frac{A^{1-s-2ir}-1}{1-s-2ir}\right)\right].$$

Now we set

$$\Psi(A,u,r,m)=2\log\varepsilon\sqrt{d(K)}\left[2\delta_{m_u=0}\left(\eta\frac{A^s}{s}+\phi\frac{A^{1-s}}{1-s}\right)\right.$$
$$+\phi(1/2-ir,-m)\left(\delta_{m_u=-2m}\eta\frac{A^{s+2ir}}{s+2ir}+\delta_{m_u=2m}\phi\frac{A^{1-s+2ir}}{1-s+2ir}\right)$$
$$\left.+\phi(1/2+ir,m)\left(\delta_{m_u=2m}\eta\frac{A^{s-2ir}}{s-2ir}+\delta_{m_u=-2m}\phi\frac{A^{1-s-2ir}}{1-s-2ir}\right)\right].$$

and calculate

$$\frac{1}{8\pi\log\varepsilon\sqrt{d(K)}}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}h\left(r+\frac{\pi m}{2\log\varepsilon},r-\frac{\pi m}{2\log\varepsilon}\right)\Psi(A,u,r,m)\,dr=$$

$$=\frac{\delta_{m_u=0}}{2\pi}\left(\eta\frac{A^s}{s}+\phi\frac{A^{1-s}}{1-s}\right)\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}h\left(r+\frac{\pi m}{2\log\varepsilon},r-\frac{\pi m}{2\log\varepsilon}\right)dr$$

$$+\frac{\delta_{m_u\equiv 0\,(2)}}{4\pi}(H_1^u(A)+H_2^u(A)),$$

where

$$H_1^u(A)=\int_{-\infty}^{\infty}h\left(r+\frac{\pi m_u}{4\log\varepsilon},r-\frac{\pi m_u}{4\log\varepsilon}\right)\left(\eta\phi_{r,\frac{m_u}{2}}\frac{A^{s-2ir}}{s-2ir}+\phi\phi_{-r,-\frac{m_u}{2}}\frac{A^{1-s+2ir}}{1-s+2ir}\right)dr \tag{3.29}$$

and

$$H_2^u(A)=\int_{-\infty}^{\infty}h\left(r-\frac{\pi m_u}{4\log\varepsilon},r+\frac{\pi m_u}{4\log\varepsilon}\right)\left(\eta\phi_{-r,\frac{m_u}{2}}\frac{A^{s+2ir}}{s+2ir}+\phi\phi_{r,-\frac{m_u}{2}}\frac{A^{1-s-2ir}}{1-s-2ir}\right)dr.$$

Note that in fact $H_1^u(A)=H_2^u(A)$ since h is even in every variable. By the Poisson summation formula

$$\int_{-\infty}^{\infty}\sum_{m\in\mathbb{Z}}h\left(r+\frac{\pi m}{2\log\varepsilon},r-\frac{\pi m}{2\log\varepsilon}\right)dr=\int_{-\infty}^{\infty}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}h\left(r+\frac{\pi x}{2\log\varepsilon},r-\frac{\pi x}{2\log\varepsilon}\right)e^{-2\pi imx}\,dx\,dr.$$

Substituting $r_1 = r + \frac{\pi x}{2\log\varepsilon}$ and $r_2 = r - \frac{\pi x}{2\log\varepsilon}$ we have $x = \frac{\log\varepsilon}{\pi}(r_1 - r_2)$ and $dx\,dr = \frac{\log\varepsilon}{\pi}\,dr_1\,dr_2$:

$$\frac{\log\varepsilon}{\pi}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(r_1,r_2)e^{-2im\log\varepsilon(r_1-r_2)}\,dr_1\,dr_2 = 4\pi\log\varepsilon\sum_{m\in\mathbb{Z}} g(2m\log\varepsilon, 2m\log\varepsilon)$$

using that g is even in every variable.

Next we examine $H_1^u(A)$. We replace the line of integration to $\operatorname{Im} r = -\operatorname{Re} s/2 - \delta$ for the first term and to $\operatorname{Im} r = \frac{1-\operatorname{Re} s}{2} + \delta$ for the second one in (3.29) and use the residue theorem to get that this is

$$\begin{aligned}
&\pi\eta h\left(i\frac{s}{2} - \frac{\pi m_u}{4\log\varepsilon}, i\frac{s}{2} + \frac{\pi m_u}{4\log\varepsilon}\right)\phi\left(\frac{1+s}{2}, \frac{m_u}{2}\right) + \\
&+\pi\phi h\left(i\frac{1-s}{2} + \frac{\pi m_u}{4\log\varepsilon}, i\frac{1-s}{2} - \frac{\pi m_u}{4\log\varepsilon}\right)\phi\left(\frac{2-s}{2}, -\frac{m_u}{2}\right) \\
&-\delta_{m_u=0}2\pi\eta\sum_{\substack{\frac{1}{2}<s_l<\frac{\operatorname{Re} s+1}{2}\\ s_l\in L}} h\left(i\left(s_l-\frac{1}{2}\right), i\left(s_l-\frac{1}{2}\right)\right)\frac{A^{1+s-2s_l}}{1+s-2s_l}R_{s_l} \\
&-\delta_{m_u=0}2\pi\phi\sum_{\substack{\frac{1}{2}<s_l<\frac{2-\operatorname{Re} s}{2}\\ s_l\in L}} h\left(i\left(s_l-\frac{1}{2}\right), i\left(s_l-\frac{1}{2}\right)\right)\frac{A^{2-s-2s_l}}{2-s-2s_l}R_{s_l} + O(A^{-\delta}).
\end{aligned}$$

Here we used that $\frac{1+s}{2}, \frac{2-s}{2} \notin L$, the rapid decay of h and that $\phi(S,m)$ is bounded in $\operatorname{Re} S \geq \frac{1}{2}$ once S is bounded away from the real line (see Corollary 2.2.8). We summarize this in

Proposition 3.2.1.

$$\begin{aligned}
&\frac{1}{8\pi\sqrt{d(K)}\log\varepsilon}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty} h\left(r+\frac{\pi m}{2\log\varepsilon}, r-\frac{\pi m}{2\log\varepsilon}\right)\Psi(A,u,r,m)\,dr = \\
&= \delta_{m_u=0}\left[2\log\varepsilon\left(\eta\frac{A^s}{s} + \phi\frac{A^{1-s}}{1-s}\right)\sum_{m\in\mathbb{Z}} g(2m\log\varepsilon, 2m\log\varepsilon) + S_1 + S_2\right] + \\
&\qquad + \delta_{m_u\equiv 0\,(2)}C_{u,K}^{\psi} + O(A^{-\delta})
\end{aligned}$$

where

$$\begin{aligned}
C_{u,K}^{\psi} =& \pi\eta h\left(i\frac{s}{2} - \frac{\pi m_u}{4\log\varepsilon}, i\frac{s}{2} + \frac{\pi m_u}{4\log\varepsilon}\right)\phi\left(\frac{1+s}{2}, \frac{m_u}{2}\right) + \\
&+\pi\phi h\left(i\frac{1-s}{2} + \frac{\pi m_u}{4\log\varepsilon}, i\frac{1-s}{2} - \frac{\pi m_u}{4\log\varepsilon}\right)\phi\left(\frac{2-s}{2}, -\frac{m_u}{2}\right)
\end{aligned}$$

and

$$S_1 = -\eta \sum_{\substack{\frac{1}{2} < s_l \leq \frac{\mathrm{Re}\, s+1}{2} \\ s_l \in L}} h(r_l, r_l) \frac{A^{1+s-2s_l}}{1+s-2s_l} R_{s_l},$$

$$S_2 = -\phi \sum_{\substack{\frac{1}{2} < s_l \leq \frac{2-\mathrm{Re}\, s}{2} \\ s_l \in L}} h(r_l, r_l) \frac{A^{2-s-2s_l}}{2-s-2s_l} R_{s_l}$$

with $r_l = -i\left(s_l - \frac{1}{2}\right)$.

Note that we changed the sign of the arguments of the even function h in S_1 and S_2. This way the notation r_l resembles the previous notations, i.e. $s_l = \frac{1}{2} + ir_l$ holds.

Similarly, we evaluate the integral

$$\int_{F_1^A} |\phi_j|^2 \, y_1^{2(1-\mathrm{Re}\, s_1^{(j)})} y_2^{2(1-\mathrm{Re}\, s_2^{(j)})} a_0(y) \, d\mu(z),$$

here $\phi_j y_1^{1-s_1^{(j)}} y_2^{1-s_2^{(j)}}$ is the zeroth term of the Fourier expansion of some u_j which is not a cusp form and hence $\phi_j \neq 0$. Note that in this case $s_1^{(j)} = s_2^{(j)} = s^{(j)} = s_l \in L$, and we write ϕ_l instead of ϕ_j (as the u_j's are independent, at most one function belongs to an element of L, if there is no such function, we simply set $\phi_l = 0$). As before, we get that this is

$$2\log\varepsilon\sqrt{d(K)}\, |\phi_l|^2 \int_1^A \int_{-\frac{1}{2}}^{\frac{1}{2}} Y_0^{2(1-s_l)} \left[\eta Y_0^s e^{2\pi i m_u Y_1} + \phi Y_0^{1-s}(z) e^{-2\pi i m_u Y_1}\right] dY_1 \frac{dY_0}{Y_0^2} =$$

$$= \delta_{m_u=0} 2\log\varepsilon\sqrt{d(K)}\, |\phi_l|^2 \left[\eta \frac{A^{1+s-2s_l}-1}{1+s-2s_l} + \phi \frac{A^{2-s-2s_l}-1}{2-s-2s_l}\right].$$

We continue now with only those terms that depend on A:

$$\delta_{m_u=0} 2\log\varepsilon\sqrt{d(K)}\, |\phi_l|^2 \left[\eta \frac{A^{1+s-2s_l}}{1+s-2s_l} + \phi \frac{A^{2-s-2s_l}}{2-s-2s_l}\right].$$

Recall that L is a finite set, morover, if $s_l > (1+\mathrm{Re}\, s)/2$, then the first term above is $O(A^{-\delta})$ and the same is true for the second term once $s_l > (2-\mathrm{Re}\, s)/2$. Multiplying by $h(r_l, r_l)$, summing over the elements of L and adding R_1 and R_2 we get $\delta_{m_u=0}[\Sigma_{u,h}^A + \tilde{\Sigma}_{u,h}^A + O(A^{-\delta})]$, where

$$\Sigma_{u,h}^A = \sum_{\substack{\frac{1}{2} < s_l \leq \frac{1+\mathrm{Re}\, s}{2} \\ s_l \in L}} \eta h(r_l, r_l) \frac{A^{1+s-2s_l}}{1+s-2s_l} \left(2\log\varepsilon\sqrt{d(K)}\, |\phi_l|^2 - R_{s_l}\right),$$

$$\tilde{\Sigma}_{u,h}^A = \sum_{\substack{\frac{1}{2} < s_l \leq \frac{2-\mathrm{Re}\, s}{2} \\ s_l \in L}} \phi h(r_l, r_l) \frac{A^{2-s-2s_l}}{2-s-2s_l} \left(2\log\varepsilon\sqrt{d(K)}\, |\phi_l|^2 - R_{s_l}\right).$$

We proceed as follows. First, let us define

$$\tilde{I}_u^A(r,m) = I_u^A(r,m) - \Psi(A,u,r,m),$$

$$\tilde{I}_u^A(u_j) = I_u^A(u_j) - \delta_{m_u=0} 2\log\varepsilon\sqrt{d(K)}\,|\phi_j|^2 \left[\eta \frac{A^{1+s-2s_j}}{1+s-2s_j} + \phi \frac{A^{2-s-2s_j}}{2-s-2s_j}\right].$$

We will show that

$$\sum_{\substack{|s_k^{(j)}|<T_k \\ k=1,2}} \tilde{I}_u^A(u_j) + \sum_{m\in Z} \int' \tilde{I}_u^A(r,m)\,dr \ll p(T_1,T_2), \tag{3.30}$$

where p is a polynomial (and the implied constant does not depend on A) and for a fixed m we restrict the integration to those r's for which $\left|\frac{1}{2} + ir + (-1)^{k-1}\frac{im}{2\log\varepsilon}\right| < T_k$ holds. Then the limits

$$I_u(u_j) = \lim_{A\to\infty} \tilde{I}_u^A(u_j), \qquad I_u(r,m) = \lim_{A\to\infty} \tilde{I}_u^A(r,m)$$

exist and

$$\sum_{\substack{|s_k^{(j)}|<T_k \\ k=1,2}} I_u(u_j) + \sum_{m\in Z} \int' I_u(r,m)\,dr \ll p(T_1,T_2) \tag{3.31}$$

holds as well. We get by the rapid decay of the function h that the expression

$$\sum_j h(r_1^{(j)}, r_2^{(j)}) I_u(u_j) + \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} h\left(r + \frac{\pi m}{2\log\varepsilon}, r - \frac{\pi m}{2\log\varepsilon}\right) I_u(r,m)\,dr$$

is finite and it is the limit of

$$\sum_j h(r_1^{(j)}, r_2^{(j)}) \tilde{I}_u^A(u_j) + \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} h\left(r + \frac{\pi m}{2\log\varepsilon}, r - \frac{\pi m}{2\log\varepsilon}\right) \tilde{I}_u^A(r,m)\,dr.$$

It follows that the terms that tend to infinity as $A\to\infty$ are $\Sigma_{u,h}^A$, $\tilde{\Sigma}_{u,h}^A$ and

$$2\log\varepsilon\left(\eta\frac{A^s}{s} + \phi\frac{A^{1-s}}{1-s}\right)\sum_{m\in\mathbb{Z}} g(2m\log\varepsilon, 2m\log\varepsilon), \tag{3.32}$$

and these terms occur only if $m_u = 0$. T They also appeared in the geometric trace as well, in fact in two parts. The totally parabolic conjugacy classes give the part where $m=0$ in the last sum, while the main term that comes from hyperbolic-parabolic classes constitute the remaining part (see Proposition 3.1.7 and Proposition 3.1.10). Observe that we got no other terms in the previous sections that tend to infinity as $A\to\infty$. This implies that $\Sigma_{u,h}^A + \tilde{\Sigma}_{u,h}^A$ must be identically zero. Note that since s can be chosen freely, it is easy to see that in fact $2\log\varepsilon\sqrt{d(K)}\,|\phi_l|^2 - R_{s_l} = 0$ holds for every $s_l\in L$. Then, subtracting (3.32) from both sides the remaining terms are equal and give the trace formula, that is stated in Theorem 3.3.1 in the next section, where we summarize the results of the whole chapter.

It remains to show (3.30). The contribution of the integrals

$$\int_{F_1^A} |\phi_j|^2 \, y_1^{2(1-\operatorname{Re} s_1^{(j)})} y_2^{2(1-\operatorname{Re} s_2^{(j)})} a_0(y)\, d\mu(z) \qquad \text{and} \qquad \int_{F_1^A} \left| a_0\left(z, \frac{1}{2}+ir, m\right)\right|^2 a_0(y)\, d\mu(z)$$

on the left hand side of (3.30) is bounded by

$$\sum_{\substack{\frac{1}{2}<s_l\leq\frac{1+\operatorname{Re} s}{2}\\ s_l\in L}} \frac{1}{|1+s-2s_l|} + \sum_{\substack{\frac{1}{2}<s_l\leq\frac{2-\operatorname{Re} s}{2}\\ s_l\in L}} \frac{1}{|2-s-2s_l|} \ll 1$$

and

$$\sum_{m\in\mathbb{Z}} \int' \frac{1}{|s|} + \frac{1}{|1-s|} + \frac{1}{|s+2ir|} + \frac{1}{|1-s+2ir|}\frac{1}{|s-2ir|} + \frac{1}{|1-s-2ir|}\, dr,$$

respectively. As in the proof of Theorem 2.4.3 one can see easily that the latter sum is $O(T_1^2+T_2^2)$.

Next we consider the expression

$$\sum_{\substack{|s_k^{(j)}|<T_k\\ k=1,2}} I^1(u_j) + \sum_{m\in\mathbb{Z}} \int' I^1(r,m)\, dr. \tag{3.33}$$

Recall that

$$I^1(u_j) = \int_{F^1} |u_j(z)|^2\, u(z)\, d\mu(z), \qquad I^1(r,m) = \int_{F^1} \left|E\left(z, \frac{1}{2}+ir, m\right)\right|^2 u(z)\, d\mu(z),$$

where $F^1 = \{z \in F_A : Y_0(z) < 1\}$. Since $u(z)$ is bounded on F^1, we can use Theorem 2.4.2 to bound (3.33) by

$$\sum_{\substack{|s_k^{(j)}|<T_k\\ k=1,2}} \int_{F^1} |u_j(z)|^2 \, d\mu(z) + \sum_{m\in\mathbb{Z}} \int' \int_{F^1} \left|E\left(z, \frac{1}{2}+ir, m\right)\right|^2 d\mu(z)\, dr \ll T_1^2 T_2^2.$$

We continue by estimating

$$\sum_{\substack{|s_k^{(j)}|<T_k\\ k=1,2}} \int_{F_1^A} |u_j(z)|^2\, \overline{u}(z)\, d\mu(z) + \sum_{m\in\mathbb{Z}} \int' \int_{F_1^A} \left|E\left(z, \frac{1}{2}+ir, m\right)\right|^2 \overline{u}(z)\, d\mu(z),$$

where $\overline{u}(z) = u(z) - a_0(y)$. By the remark after Theorem 2.1.8 we have $\overline{u}(z) \ll e^{-d\sqrt{y_1 y_2}}$ once $Y_0 \gg T_1^2 T_2^2$. Using this and that the norm of u_j is 1 for any j, we immediately obtain that

$$\sum_{\substack{|s_k^{(j)}|<T_k \\ k=1,2}} \int_{F_1^A} |u_j(z)|^2 \, \overline{u}(z) \, d\mu(z) \ll \sum_{\substack{|s_k^{(j)}|<T_k \\ k=1,2}} 1 \ll T_1^2 T_2^2 + T_1^3 T_2 + T_1 T_2^3$$

by (2.30). We also get a polynomial bound for the second sum using the bound for $\overline{u}(z)$ and the following

Lemma 3.2.2. *For any big enough $A > 0$ and $T_1, T_2 > 0$ we have*

$$\sum_{m \in \mathbb{Z}} \int' \int_{F_A} |E(z, 1/2 + ir, m)|^2 \, d\mu(z) \, dr \ll T_1^2 T_2^2 + \sqrt{T_1 T_2}(T_1^2 + T_2^2 + T_1 T_2) + (T_1^2 + T_2^2) \log A,$$

where for a fixed m we integrate over the r's for which $\left|\frac{1}{2} + ir + (-1)^{k-1} \frac{\pi i m}{2 \log \varepsilon}\right| \leq T_k$ holds ($k = 1, 2$).

Proof. First, notice that

$$\begin{aligned}\sum_{m \in \mathbb{Z}} \int' \int_{F_A} |E(z, 1/2 + ir, m)|^2 \, d\mu(z) \, dr &= \sum_{m \in \mathbb{Z}} \int' \int_{F_A} |E_A(z, 1/2 + ir, m)|^2 \, d\mu(z) \, dr \\ &\leq \sum_{m \in \mathbb{Z}} \int' \int_{F} |E_A(z, 1/2 + ir, m)|^2 \, d\mu(z) \, dr,\end{aligned}$$

hence it is enough to estimate the last expression. As in the proof of Theorem 2.4.3 we can write

$$\begin{aligned}\sum_{m \in \mathbb{Z}} \int' \int_{F} |E_A(z, 1/2 + ir, m)|^2 \, d\mu(z) &= 2 \log \varepsilon \sqrt{d(K)} \sum_{m \in \mathbb{Z}} \int' \left[2 \log A - \frac{\phi'(\frac{1}{2} + ir, m)}{\phi(\frac{1}{2} + ir, m)} \right] dr + \\ &+ 2 \log \varepsilon \sqrt{d(K)} \int\limits_{\left|\frac{1}{2} + ir\right| \leq \min(T_1, T_2)} \frac{\phi(1/2 - ir, 0) A^{2ri} - \phi(1/2 + ir, 0) A^{-2ri}}{2ri} \, dr.\end{aligned} \tag{3.34}$$

Following that proof we also obtain that

$$\sum_{m \in \mathbb{Z}} \int' 2 \log A \ll (T_1^2 + T_2^2) \log A,$$

and that the last integral in (3.34) is bounded by $\log T_1 T_2 + \log \log A$. Together with the statement of Theorem 2.4.3 the lemma follows. □

Finally, we handle the sum

$$\sum_{\substack{|s_k^{(j)}|<T_k\\k=1,2}} \int_{F_1^A} |\overline{u}_j(z)|^2\, a_0(y)\, d\mu(z) + \sum_{m\in\mathbb{Z}} \int' \int_{F_1^A} \left|\overline{E}\left(z,\frac{1}{2}+ir,m\right)\right|^2 a_0(y)\, d\mu(z)\, dr,$$

where $\overline{u}_j(z)=u_j(z)-\phi_j y_1^{1-s_1^{(j)}} y_2^{1-s_2^{(j)}}$ for every $j\geq 0$ and $\overline{E}(z,\frac{1}{2}+ir,m)$ was defined similarly before (3.27) on page 110.

By Parseval's identity we have

$$\int_{F_1^A} |\overline{u}_j(z)|^2\, a_0(y)\, d\mu(z) =$$

$$= \sqrt{d(K)} \iint\limits_{\substack{1\leq Y_0<A\\ -\frac{1}{2}\leq Y_1<\frac{1}{2}}} a_0(y) \sum_{l\in L_K^*\setminus 0} |c_l^{(j)}|^2 K^2_{s_1^{(j)}-\frac{1}{2}}(2\pi\,|l_1|\,y_1) K^2_{s_2^{(j)}-\frac{1}{2}}(2\pi\,|l_2|\,y_2)\frac{dy_1\,dy_2}{y_1y_2}$$

and

$$\int_{F_1^A} \left|\overline{E}\left(z,\frac{1}{2}+ir,m\right)\right|^2 a_0(y)\, d\mu(z) =$$

$$\sqrt{d(K)} \iint\limits_{\substack{1\leq Y_0<A\\ -\frac{1}{2}\leq Y_1<\frac{1}{2}}} a_0(y) \sum_{l\in L_K^*\setminus 0} |\phi_l(1/2+ir,m)|^2 K^2_{ir+\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_1|\,y_1) K^2_{ir-\frac{\pi i m}{2\log\varepsilon}}(2\pi\,|l_2|\,y_2)\frac{dy_1\,dy_2}{y_1y_2}.$$

Note that if u is a cusp form, then these values are simply zero because $a_0(y)=0$. Otherwise $\operatorname{Re} s_1=\operatorname{Re} s_2 =: \operatorname{Re} s$ and $a_0(y)$ can be estimated by $|\eta|\,(y_1y_2)^{\operatorname{Re} s}+|\phi|\,(y_1y_2)^{1-\operatorname{Re} s}$.

Before stating the last lemma that finishes the proof of the trace formula we make a technical remark. To derive (3.31) from (3.30) we may use dyadic summation and set $T_k=T/2^{a_k}$ for some integers a_k. Hence it is enough to prove the following

Lemma 3.2.3. *For any $T\gg 1$, $\delta>0$ and for any integers $0\leq a_1,a_2\leq\lceil\log_2 T\rceil$ we have*

$$\Sigma_1 := \sum_{\substack{\frac{T}{2^{a_k+1}}\leq|s_k^{(j)}|<\frac{T}{2^{a_k}}\\k=1,2}} \int_{F_1^A} |\overline{u}_j(z)|^2\, a_0(y)\, d\mu(z) \ll \frac{T^{4+\delta}}{2^{a_1+a_2}}$$

and

$$\Sigma_2 := \sum_{m\in\mathbb{Z}} \int' \int_{F_1^A} \left|\overline{E}\left(z,\frac{1}{2}+ir,m\right)\right|^2 a_0(y)\, d\mu(z)\, dr \ll \frac{T^{4+\delta}}{2^{a_1+a_2}},$$

where we integrate over those points for which $\frac{T}{2^{a_k+1}}\leq\left|\frac{1}{2}+ir+(-1)^{k-1}\frac{im}{2\log\varepsilon}\right|<\frac{T}{2^{a_k}}$ holds ($k=1,2$). The implied constant depends on δ and the field K, but not on A.

Proof. First note that if u_j is not a cusp form, then for a big enough Y_0 we have $\overline{u}_j(z) \ll e^{-d\sqrt{y_1 y_2}}$ by Theorem 2.1.8, and as $a_0(y)$ is of polynomial growth, the integral

$$\int_{F_1^A} |\overline{u}_j(z)|^2 \, a_0(y) \, d\mu(z)$$

converges absolutely as $A \to \infty$. So the contribution of the finitely many terms belonging to these u_j's is $O(1)$. Hence we can assume that $\operatorname{Re} s_k^{(j)} = \frac{1}{2}$ holds.

We prove the statement for Σ_1, the other estimate follows similarly. As we have already remarked above we can estimate Σ_1 by

$$\sum_{\substack{\frac{T}{2^{a_k+1}} \le |s_k^{(j)}| < \frac{T}{2^{a_k}} \\ k=1,2}} \iint_{\substack{1 \le Y_0 < \infty \\ -\frac{1}{2} \le Y_1 < \frac{1}{2}}} (y_1 y_2)^{\operatorname{Re} S} \sum_{l \in L_K^* \setminus 0} |c_l^{(j)}|^2 K^2_{s_1^{(j)}-\frac{1}{2}}(2\pi \, |l_1| \, y_1) K^2_{s_2^{(j)}-\frac{1}{2}}(2\pi \, |l_2| \, y_2) \frac{dy_1 \, dy_2}{y_1 y_2}$$

where $0 < \operatorname{Re} S := \max(\operatorname{Re} s, 1 - \operatorname{Re} s) < 1$. If we collect the terms $\varepsilon^k l$ in the inner sum, this becomes

$$\sum_{\substack{\frac{T}{2^{a_k+1}} \le |s_k^{(j)}| < \frac{T}{2^{a_k}} \\ k=1,2}} \sum_{0 \ne (l)} a_l^{(j)} \iint_{1 \le Y_0 < \infty} (y_1 y_2)^{\operatorname{Re} S} K^2_{s_1^{(j)}-\frac{1}{2}}(2\pi \, |l/\omega| \, y_1) K^2_{s_2^{(j)}-\frac{1}{2}}(2\pi \, |(l/\omega)'| \, y_2) \frac{dy_1 \, dy_2}{y_1 y_2}, \tag{3.35}$$

where the inner sum runs over the non-zero ideals of $\mathcal{O}_K$ and $a_l^{(j)} = |c_l|^2 + |c_{\varepsilon l}|^2 + |c_{-l}|^2 + |c_{-\varepsilon l}|^2$ as a_l was defined in (2.12) on page 45. For simplicity we omit the last three terms from $a_l^{(j)}$, but an analogous proof works for those as well.

We divide the inner sum in (3.35) into three parts. To this end we fix a small $\delta > 0$ and set $N = 3/\delta$ and $\Delta_{1,2} = 2^{|a_1 - a_2|/N}$. Let us note that $\Delta_{1,2} \ge 1$ holds. We denote by $\Sigma_1^{(1)}$ the part of the (double) sum above where $|N(l)| \le \frac{c\Delta_{1,2} T^2}{2^{a_1+a_2}}$ holds for some positive constant c. Here we estimate the integral from above simply by extending it to $\mathbb{H}^2$. Substituting $u_k = 2\pi \left| (l\omega)^{(k)} \right| y_k$ in the integral we obtain

$$(4\pi^2 |N(\omega^{-1})|)^{-\operatorname{Re} S} \, |N(l)|^{-\operatorname{Re} S} \int_0^\infty \int_0^\infty (u_1 u_2)^{\operatorname{Re} S} K^2_{s_1^{(j)}-\frac{1}{2}}(u_1) K^2_{s_2^{(j)}-\frac{1}{2}}(u_2) \frac{du_1 \, du_2}{u_1 u_2}.$$

Now

$$\int_0^\infty \int_0^\infty (u_1 u_2)^{\operatorname{Re} S} K^2_{s_1^{(j)}-\frac{1}{2}}(u_1) K^2_{s_2^{(j)}-\frac{1}{2}}(u_2) \frac{du_1 \, du_2}{u_1 u_2} = \prod_{k=1,2} \int_0^\infty u^{\operatorname{Re} S - 1} K^2_{s_k^{(j)}-\frac{1}{2}}(u) \, du, \tag{3.36}$$

and since $\operatorname{Re} s_1^{(j)} = \operatorname{Re} s_2^{(j)} = \frac{1}{2}$, we get

$$\int_0^\infty u^{\operatorname{Re} S - 1} K^2_{s_k^{(j)}-\frac{1}{2}}(u) \, du = 2^{\operatorname{Re} S - 3} \Gamma(\operatorname{Re} S)^{-1} \Gamma\left(\frac{\operatorname{Re} S}{2}\right)^2 \Gamma\left(\frac{\operatorname{Re} S}{2} + i r_k^{(j)}\right) \Gamma\left(\frac{\operatorname{Re} S}{2} - i r_k^{(j)}\right)$$

(see the formula above (B.37) in [11]). Hence if $|r_k^{(j)}|\geq 1$ for $k=1,2$, then

$$\int\limits_0^\infty u^{\operatorname{Re}S-1}K^2_{s_k^{(j)}-\frac12}(u)\,du \ll \left|\Gamma\left(\frac{\operatorname{Re}S}{2}+ir_k^{(j)}\right)\Gamma\left(\frac{\operatorname{Re}S}{2}-ir_k^{(j)}\right)\right| \ll e^{-\pi|r_k^{(j)}|}|r_k^{(j)}|^{\operatorname{Re}S-1}$$

by Stirling's formula. Since $|r_k^{(j)}|$ is bounded away from zero, we also have

$$|r_k^{(j)}|\geq \frac{|s_k^{(j)}|}{2}\geq \frac{T}{2^{a_k+2}}.$$

On the other hand, if $|r_k^{(j)}|\leq 1$, then $\Gamma(\operatorname{Re}S/2\pm ir_k^{(j)})$ is bounded by a constant (that depends on u). As in this case $\frac12\leq |s_k^{(j)}|\leq\sqrt2$ also holds, we get that the left hand side of (3.36) is bounded by

$$e^{-\pi(|s_1^{(j)}|+|s_2^{(j)}|)}\left(\frac{T^2}{2^{a_1+a_2}}\right)^{\operatorname{Re}S-1}.$$

So using the estimate (2.37) on page 73 we estimate $\Sigma_1^{(1)}$ by

$$\begin{aligned}
&\left(\frac{T^2}{2^{a_1+a_2}}\right)^{\operatorname{Re}S-1}\sum_{0<|N(l)|\leq\frac{c\Delta_{1,2}T^2}{2^{a_1+a_2}}}|N(l)|^{-\operatorname{Re}S}\sum_{\substack{\frac{T}{2^{a_k+1}}\leq|s_k^{(j)}|<\frac{T}{2^{a_k}}\\ k=1,2}}|c_l^{(j)}|^2e^{-\pi(|s_1^{(j)}|+|s_2^{(j)}|)}\\
&\ll\left(\frac{T^2}{2^{a_1+a_2}}\right)^{\operatorname{Re}S-1}\sum_{0<|N(l)|\leq\frac{c\Delta_{1,2}T^2}{2^{a_1+a_2}}}\left(|N(l)|^{-\operatorname{Re}S}\frac{T^4}{2^{a_1+a_2}}+2^{a_1+a_2}|N(l)|^{2-\operatorname{Re}S}\right)\\
&=\sum_{0<|N(l)|\leq\frac{c\Delta_{1,2}T^2}{2^{a_1+a_2}}}\left(|N(l)|^{-\operatorname{Re}S}\frac{T^{2+2\operatorname{Re}S}}{2^{(a_1+a_2)\operatorname{Re}S}}+|N(l)|^{2-\operatorname{Re}S}\frac{T^{2\operatorname{Re}S-2}}{2^{(a_1+a_2)(\operatorname{Re}S-2)}}\right)\\
&\ll\frac{T^{2+2\operatorname{Re}S}}{2^{(a_1+a_2)\operatorname{Re}S}}\sum_{0<n\leq\frac{c\Delta_{1,2}T^2}{2^{a_1+a_2}}}n^{-\operatorname{Re}S+\delta}+\frac{T^{2\operatorname{Re}S-2}}{2^{(a_1+a_2)(\operatorname{Re}S-2)}}\sum_{0<n\leq\frac{c\Delta_{1,2}T^2}{2^{a_1+a_2}}}n^{2-\operatorname{Re}S+\delta}\\
&\ll\Delta_{1,2}^3\cdot\frac{T^{4+2\delta}}{2^{(a_1+a_2)(1+\delta)}}\leq\frac{T^{4+2\delta}}{2^{a_1+a_2}}.
\end{aligned}$$

If $|N(l)|=|ll'|>c\Delta_{1,2}T^2/2^{a_1+a_2}$, then we estimate in a different way. Let us note first that for a fixed $l\in\mathcal{O}_K$ we have

$$\iint\limits_{1\leq Y_0<\infty}(y_1y_2)^{\operatorname{Re}S}K^2_{s_1^{(j)}-\frac12}(2\pi\,|l/\omega|\,y_1)K^2_{s_2^{(j)}-\frac12}(2\pi\,|(l/\omega)'|\,y_2)\frac{dy_1\,dy_2}{y_1y_2}=$$

$$=\sum_{k=-\infty}^{\infty}\iint\limits_{\substack{1\leq Y_0<\infty\\ -\frac12\leq Y_1<\frac12}}(y_1y_2)^{\operatorname{Re}S}K^2_{s_1^{(j)}-\frac12}(2\pi\left|\varepsilon^kl/\omega\right|y_1)K^2_{s_2^{(j)}-\frac12}(2\pi\left|(\varepsilon^kl/\omega)'\right|y_2)\frac{dy_1\,dy_2}{y_1y_2}. \quad (3.37)$$

We estimate the Bessel functions using Lemma 2.1.5. If

$$2\pi \left|\varepsilon^k l/\omega\right| y_1 > C|s_1^{(j)}|, \tag{3.38}$$

where C is the constant in the lemma, then (using the condition $\varepsilon^{-2} \leq y_1/y_2 < \varepsilon^2$) we infer

$$K^2_{s_1^{(j)}-\frac{1}{2}}(2\pi \left|\varepsilon^k l/\omega\right| y_1) K^2_{s_2^{(j)}-\frac{1}{2}}(2\pi \left|(\varepsilon^k l/\omega)'\right| y_2) \ll e^{-d\varepsilon^k|l|\sqrt{y_1y_2}} \left(\frac{\varepsilon^k |s_2^{(j)}|}{|l'|\sqrt{y_1y_2}}\right)^{2\operatorname{Re} s_2^{(j)}+1} e^{-\pi|s_2^{(j)}|}.$$

On the other hand, if

$$2\pi \left|(\varepsilon^k l/\omega)'\right| y_2 > C|s_2^{(j)}|, \tag{3.39}$$

then we get the bound

$$e^{-d\varepsilon^{-k}|l'|\sqrt{y_1y_2}} \left(\frac{\varepsilon^{-k} |s_1^{(j)}|}{|l|\sqrt{y_1y_2}}\right)^{2\operatorname{Re} s_1^{(j)}+1} e^{-\pi|s_1^{(j)}|}.$$

Recall that $\operatorname{Re} s_1^{(j)} = \operatorname{Re} s_2^{(j)} = \frac{1}{2}$.

As we sum over ideals we can choose the generator l. We will make this choice so that (3.38) will hold for any non-negative k while (3.39) will hold for any negative k. Then, for a non-negative k the integral in (3.37) can be bounded by

$$\begin{aligned}
&|s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|} \iint\limits_{\substack{1\leq Y_0<\infty \\ -\frac{1}{2}\leq Y_1<\frac{1}{2}}} \varepsilon^{2k} \left|l'\right|^{-2} (y_1y_2)^{\operatorname{Re} S} e^{-d\varepsilon^k|l|\sqrt{y_1y_2}} \frac{dy_1\,dy_2}{y_1^2y_2^2} = \\
&\qquad = 2\log\varepsilon |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|} \iint\limits_{\substack{1\leq Y_0<\infty \\ -\frac{1}{2}\leq Y_1<\frac{1}{2}}} \varepsilon^{2k} \left|l'\right|^{-2} Y_0^{\operatorname{Re} S} e^{-d\varepsilon^k|l|\sqrt{Y_0}} \frac{dY_0\,dY_1}{Y_0^2} \\
&\qquad = 2\log\varepsilon |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|} \int_1^\infty \varepsilon^{2k} \left|l'\right|^{-2} Y_0^{\operatorname{Re} S-2} e^{-d\varepsilon^k|l|\sqrt{Y_0}}\,dY_0 \\
&\qquad = |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|} \frac{4\varepsilon^k \log\varepsilon}{d\,|l|\,|l'|^2} \int_1^\infty Y_0^{\operatorname{Re} S-\frac{3}{2}} \left(\frac{1}{2} d\varepsilon^k\,|l|\,Y_0^{-\frac{1}{2}} e^{-d\varepsilon^k|l|\sqrt{Y_0}}\right) dY_0 \\
&\qquad \ll |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|} \frac{\varepsilon^k}{|l|\,|l'|^2} \int_1^\infty \frac{1}{2} d\varepsilon^k\,|l|\,Y_0^{-\frac{1}{2}} e^{-d\varepsilon^k|l|\sqrt{Y_0}}\,dY_0 \\
&\qquad = |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|} \frac{\varepsilon^k\,|l|}{|N(l)|^2} e^{-d\varepsilon^k|l|}.
\end{aligned}$$

Similarly, for a negative k we get the estimate

$$|s_1^{(j)}|^2 e^{-\pi|s_1^{(j)}|} \frac{\varepsilon^{-k}\,|l'|}{|N(l)|^2} e^{-d\varepsilon^{-k}|l'|}.$$

Now we handle the part of (3.35) where $\frac{c\Delta_{1,2}T^2}{2^{a_1+a_2}} < |N(l)| \leq \frac{c^2\Delta_{1,2}^2T^4}{2^{2(a_1+a_2)}}$ holds. Let us denote this part by $\Sigma_1^{(2)}$. Here we choose l so that $\left|\varepsilon^k l\right| > \sqrt{c\Delta_{1,2}}T/2^{a_1}$ holds for any $k \geq 0$, while $\left|\varepsilon^k l\right| \leq \sqrt{c\Delta_{1,2}}T/2^{a_1}$ for any $k < 0$. Then for any $k < 0$ we have $\left|(\varepsilon^k l)'\right| > \sqrt{c\Delta_{1,2}}T/2^{a_2}$ because of the assumption on $|N(l)|$. As y_1 and y_2 are bounded from below in the integrals in (3.37) and $\Delta_{1,2} \geq 1$, we get that if c is big enough, then $2\pi\left|\varepsilon^k l/\omega\right| y_1 > CT/2^{a_1} \geq C|s_1^{(j)}|$ for any $k \geq 0$, while $2\pi\left|(\varepsilon^k l/\omega)'\right| y_2 > CT/2^{a_2} \geq C|s_2^{(j)}|$ holds for any $k < 0$, where C is the constant in Lemma 2.1.5. Note that the choice of c depends only on the field K. This means that the estimates above apply for the integral on the right hand side of (3.37). As $T/2^{a_1}$ and $T/2^{a_2}$ are bounded from below by a positive constant, so are $\left|\varepsilon^k l\right|$ for a $k \geq 0$ and $\left|(\varepsilon^k l)'\right|$ for a $k < 0$. Hence the referred integral can be bounded by

$$|s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|}\frac{\varepsilon^k\,|l|}{|N(l)|^2}e^{-d\varepsilon^k|l|} \ll |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|}\frac{1}{|N(l)|^2}e^{-d'\varepsilon^k|l|}$$

for a non-negative k and by

$$|s_1^{(j)}|^2 e^{-\pi|s_1^{(j)}|}\frac{\varepsilon^{-k}\,|l'|}{|N(l)|^2}e^{-d\varepsilon^{-k}|l'|} \ll |s_1^{(j)}|^2 e^{-\pi|s_1^{(j)}|}\frac{1}{|N(l)|^2}e^{-d'\varepsilon^{-k}|l'|}$$

for a negative k.

Summing over k we obtain

$$\frac{|s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|}}{|N(l)|^2}e^{-d'|l|} + \frac{|s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|}}{|N(l)|^2}\sum_{k=1}^{\infty} e^{-d'\varepsilon^k|l|} + \frac{|s_1^{(j)}|^2 e^{-\pi|s_1^{(j)}|}}{|N(l)|^2}\sum_{k=1}^{\infty} e^{-d'\varepsilon^k|l'|}.$$

As $k \ll \varepsilon^k$ for a positive k, we can bound the sums above by

$$\sum_{k=1}^{\infty}(e^{-d'|l|})^k = \frac{e^{-d'|l|}}{1-e^{-d'|l|}} \ll e^{-d'|l|} \quad \text{and} \quad \sum_{k=1}^{\infty}(e^{-d'|l'|})^k = \frac{e^{-d'|l'|}}{1-e^{-d'|l'|}} \ll e^{-d'|l'|}.$$

Hence we have the following bound for Σ_2:

$$\sum_{\frac{c\Delta_{1,2}T^2}{2^{a_1+a_2}} < |N(l)| \leq \frac{c^2\Delta_{1,2}^2T^4}{2^{2(a_1+a_2)}}} \frac{1}{|N(l)|^2} \sum_{\substack{\frac{T}{2^{a_k+1}} \leq |s_k^{(j)}| < \frac{T}{2^{a_k}} \\ k=1,2}} |c_l^{(j)}|^2 \left(|s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|}e^{-d'|l|} + |s_1^{(j)}|^2 e^{-\pi|s_1^{(j)}|}e^{-d'|l'|}\right).$$

If the constant $c > 0$ is big enough, then so is $c' > 0$, hence the inner sum above can be bounded by

$$\sum_{\substack{\frac{T}{2^{a_k+1}} \leq |s_k^{(j)}| < \frac{T}{2^{a_k}} \\ k=1,2}} |c_l^{(j)}|^2 e^{-\pi(|s_2^{(j)}|+|s_2^{(j)}|)}\left(|s_2^{(j)}|e^{-c_2\sqrt{\Delta_{1,2}}T/2^{a_1}} + |s_1^{(j)}|e^{-c_2\sqrt{\Delta_{1,2}}T/2^{a_2}}\right).$$

Now if $a_1 \geq a_2$, then

$$\frac{\Delta_{1,2}^{1/2}T}{2^{a_1}} \gg \left(\frac{2^{(a_1-a_2)}T^2}{2^{2a_1}}\right)^{\frac{1}{2N}} = \left(\frac{T^2}{2^{a_1+a_2}}\right)^{\frac{1}{2N}} \gg |s_2^{(j)}|^{\frac{1}{2N}}$$

while

$$\frac{\Delta_{1,2}^{1/2}T}{2^{a_2}} \geq \left(\frac{T^2}{2^{2a_2}}\right)^{\frac{1}{2}} \gg \left(\frac{T^2}{2^{a_1+a_2}}\right)^{\frac{1}{2}} \gg |s_1^{(j)}|^{\frac{1}{2}}.$$

Hence the exponential factors above in the parentheses absorb the polynomial ones. A similar argument applies if $a_2 \geq a_1$, and by (2.37) we obtain

$$\Sigma_1^{(2)} \ll \sum_{\frac{c\Delta_{1,2}T^2}{2^{a_1+a_2}}<|N(l)|\leq\frac{c^2\Delta_{1,2}^2T^4}{2^{2(a_1+a_2)}}} \frac{1}{|N(l)|^2} \sum_{\substack{\frac{T}{2^{a_k+1}}\leq|s_k^{(j)}|<\frac{T}{2^{a_k}} \\ k=1,2}} |c_l^{(j)}|^2 e^{-\pi(|s_2^{(j)}|+|s_2^{(j)}|)}$$

$$\ll \frac{T^4}{2^{a_1+a_2}} + 2^{a_1+a_2} \sum_{\frac{c\Delta_{1,2}T^2}{2^{a_1+a_2}}<|N(l)|\leq\frac{c^2\Delta_{1,2}^2T^4}{2^{2(a_1+a_2)}}} 1 \ll \frac{T^4}{2^{a_1+a_2}} + 2^{a_1+a_2}\frac{\Delta_{1,2}^{2+2\delta}T^{4+4\delta}}{2^{2(a_1+a_2)(1+\delta)}} \ll \frac{T^{4+4\delta}}{2^{a_1+a_2}}.$$

Finally, if even $|N(l)| > c^2\Delta_{1,2}^2T^4/2^{2(a_1+a_2)}$ is true, then we denote this part of (3.35) by $\Sigma_1^{(3)}$ and for every ideal we choose l uniquely so that $\varepsilon^{-2} \leq l/l' < \varepsilon^2$ holds. This means that

$$\left|\varepsilon^k l\right| \geq \varepsilon^{-1}\varepsilon^k\sqrt{|N(l)|} \gg c\Delta_{1,2}T^2/2^{a_1+a_2} \gg cT/2^{a_1}$$

once $k \geq 0$, while

$$\left|(\varepsilon^k l)'\right| \geq \varepsilon^{-1}\varepsilon^{-k}\sqrt{|N(l)|} \gg c\Delta_{1,2}T^2/2^{a_1+a_2} \gg cT/2^{a_2}$$

if $k < 0$, that is, (3.38) holds for any non-negative k and (3.39) holds for any negative k. As before, we get the bound

$$\begin{aligned}
|s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|}\frac{\varepsilon^k\,|l|}{|N(l)|^2}e^{-d\varepsilon^k|l|} &\ll |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|}\frac{\varepsilon^k\sqrt{|N(l)|}}{|N(l)|^2}e^{-d\varepsilon^k\sqrt{|N(l)|}} \\
&\leq |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|}\frac{(\varepsilon^k\sqrt{|N(l)|})^5}{|N(l)|^4}e^{-d\varepsilon^k\sqrt{|N(l)|}} \\
&\ll |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|}\frac{1}{|N(l)|^4}e^{-d'\varepsilon^k\sqrt{|N(l)|}}
\end{aligned}$$

for the integral on the right hand side of (3.37) if $k \geq 0$, and

$$|s_1^{(j)}|^2 e^{-\pi|s_1^{(j)}|}\frac{\varepsilon^{-k}\,|l'|}{|N(l)|^2}e^{-d\varepsilon^{-k}|l'|} \ll |s_1^{(j)}|^2 e^{-\pi|s_1^{(j)}|}\frac{1}{|N(l)|^4}e^{-d'\varepsilon^{-k}\sqrt{|N(l)|}}$$

if $k < 0$.

Summing over k as above we obtain

$$(|s_1^{(j)}|^2 e^{-\pi|s_1^{(j)}|} + |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|})\frac{1}{|N(l)|^4}e^{-d'\sqrt{|N(l)|}} \ll$$

$$\ll \frac{1}{|N(l)|^4}(|s_1^{(j)}|^2 e^{-\pi|s_1^{(j)}|} + |s_2^{(j)}|^2 e^{-\pi|s_2^{(j)}|})e^{-c'T^2/2^{a_1+a_2}} \ll \frac{e^{-\pi(|s_1^{(j)}|+|s_2^{(j)}|)}}{|N(l)|^4},$$

and hence

$$\Sigma_1^{(3)} \ll \sum_{|N(l)|>\frac{c^2\Delta_{1,2}^2 T^4}{2^{2(a_1+a_2)}}} \frac{1}{|N(l)|^4} \sum_{\substack{\frac{T}{2^{a_k+1}} \le |s_k^{(j)}| < \frac{T}{2^{a_k}} \\ k=1,2}} |c_l^{(j)}| e^{-\pi(|s_1^{(j)}|+|s_2^{(j)}|)}$$

$$\ll \frac{T^4}{2^{a_1+a_2}+2^{a_1+a_2}} \ll \frac{T^4}{2^{a_1+a_2}} + T^2 \ll \frac{T^4}{2^{a_1+a_2}}$$

by (2.37). As we mentioned at the beginning of the proof, we get the upper bound for Σ_2 in the same way. This completes the proof of the trace formula. □

3.3 The trace formula

In this section we summarize the results of the previous sections of this chapter. First of all, we repeat some of the important notations and definitions, though many of them will not be detailed here but can be found in the List of Symbols. We fix an automorphic form u that satisfies the growth condition $o(e^{2\pi y_k})$ for $k = 1, 2$ and hence admits the Fourier expansion (3.26) specified on page 108. Its eigenvaules are denoted by $\lambda_k = s_k(1 - s_k)$ and we assume for simplicity that $\frac{1}{2} \le \operatorname{Re} s_k < 1$ $(k = 1, 2)$. If u is not a cusp form, then the pair (s_1, s_2) has a special form given in (3.1) on page 75. In the latter case we also make an assumption on s (defined in (3.1)), namely we require that $\frac{s+1}{2}, \frac{2-s}{2} \notin L$ holds, where the finite set L is defined in the paragraph above (3.26).

We also fix a function $\psi \in C_0^\infty(\mathbb{R}^2)$ and define a point-pair invariant kernel

$$k_\psi(z, w) = k(z, w) = \psi\left(\frac{|z_1 - w_1|^2}{\operatorname{Im} z_1 \cdot \operatorname{Im} w_1}, \frac{|z_2 - w_2|^2}{\operatorname{Im} z_2 \cdot \operatorname{Im} w_2}\right)$$

for every $z, w \in \mathbb{H}^2$, for which $k(z, w) = k(\sigma z, \sigma w)$ holds for every $z, w \in \mathbb{H}^2$ and for any $\sigma \in PSL(2, \mathbb{R})^2$. The automorphic kernel $K(z, w)$ is defined by

$$K(z, w) = \sum_{\gamma \in \Gamma_K} k(z, \gamma w).$$

This is an automorphic function in both variables.

The main result of this chapter is obtained by the comparison of the results of two different kind of evaluation of the truncated trace

$$\operatorname{Tr}_u^A K := \int_{F_A} K(z, z) u(z)\, d\mu(z),$$

where $A > 0$ an arbitrary large enough number. The following terms come from the so-called geometric trace (calculated in Section 3.1):

$$\Sigma_{\text{ell}} := \sum_{\substack{\{\gamma\} \\ \gamma \in \Gamma_K \text{ totally} \\ \text{elliptic}}} T_\gamma^e, \qquad \Sigma_{\text{hyp}} := \sum_{\substack{\{\gamma\} \\ \gamma \in \Gamma_K \text{ totally} \\ \text{hyperbolic}}} T_\gamma^h, \qquad \Sigma_{\text{mix}} := \sum_{\substack{\{\gamma\} \\ \gamma \in \Gamma_K \text{ mixed}}} T_\gamma^m,$$

where the terms above are defined as follows:

$$T_\gamma^e = \frac{(2\pi)^2}{m_\gamma} u(z_\gamma) \int_0^\infty \int_0^\infty \psi(S(r_1, \theta(\gamma^{(1)})), S(r_2, \theta(\gamma^{(2)}))) \left(\prod_{k=1,2} g_{\lambda_k}(r_k) \sinh r_k \right) dr_1 dr_2,$$

where m_γ is the order of the centralizer of γ, z_γ is the fixed point of γ, $\theta(\gamma^{(k)})$ is defined below (3.5) on page 78, $S(r,\theta) = (2\sinh r)^2 \sin^2\theta$ and the function $g_{\lambda_k}(r) : [0,\infty) \to \mathbb{C}$ is the unique solution of the differential equation

$$g''(r) + \frac{\cosh r}{\sinh r} g'(r) = \lambda_k g(r)$$

satisfying the initial condition $g(0) = 1$,

$$T_\gamma^h = F(0,0) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi(N(\gamma^{(1)}, \vartheta_1), N(\gamma^{(2)}, \vartheta_2)) f_{\lambda_1}(\vartheta_1) f_{\lambda_2}(\vartheta_2) \frac{d\vartheta_1\, d\vartheta_2}{\cos^2\vartheta_1 \cos^2\vartheta_2},$$

here

$$N(\gamma^{(k)}, \vartheta_k) = \frac{N(\gamma^{(k)}) + N(\gamma^{(k)})^{-1} - 2}{\cos^2\vartheta_k},$$

where $N(\gamma^{(k)})$ is the norm of $\gamma^{(k)}$,

$$F(0,0) = \int_{(\log r_1, \log r_2) \in P_\gamma} u(\varrho^{(1)}(r_1 i), \varrho^{(2)}(r_2 i)) \frac{dr_1\, dr_2}{r_1 r_2}$$

(here $\varrho \in PSL(2,\mathbb{R})^2$ is an element for which $\varrho^{-1}\gamma\varrho$ is diagonal, see secion 3.1.2, and the set P_γ is defined in Proposition 3.1.1) and $f_{\lambda_k}(\vartheta)$ is the unique solution of the differential equation

$$F''(\vartheta) = \frac{\lambda_k}{\cos^2\vartheta} F(\vartheta) \qquad (\vartheta \in (-\pi/2, \pi/2))$$

with the initial condition $F(0) = 1$ and $F'(0) = 0$, while

$$T_\gamma^m = 2\pi \int_1^{N(\gamma_0^{(1)})} u(\varrho^{(1)}(r_1 i), \varrho^{(2)} i) \frac{dr_1}{r_1} \cdot$$
$$\cdot \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi(N(\gamma^{(1)}, \vartheta_1), S(r_2, \theta(\gamma^{(2)}))) \frac{f_{\lambda_1}(\vartheta_1)\, d\vartheta_1}{\cos^2\vartheta_1} g_{\lambda_2}(r_2) \sinh r_2\, dr_2,$$

where γ_0 is the generator of the centralizer $C(\gamma)$ and for the definition of ϱ see the beginning of section 3.1.3. This latter equatlity holds if the first component of the mixed element γ is hyperbolic and the second one is elliptic. In the other possible cases T_γ^m is simply obtained by interchanging the coordinates in the expressions above.

We denote the contribution of the parabolic conjugacy classes by Σ_{par}, it is given in Proposition 3.1.10. Recall that we defined the number s and the integer m_u in (3.1) on page 75 in the case when u is not a cusp form. Note that if u is a cusp form or if m_u is odd, then Σ_{par} is simply zero. Otherwise, we have

$$\begin{aligned}\Sigma_{\text{par}} =& -\delta_{m_u\equiv 0\,(2)}\eta\sqrt{d(K)}\left(\frac{2^{-(s_1+s_2)}}{2\pi^2}\right)^2\zeta_K(1-s,m_u)\Gamma\left(\frac{1-s_1}{2}\right)^2\Gamma\left(\frac{1-s_2}{2}\right)^2\times\\ &\times\int_0^\infty\int_0^\infty h(r_1,r_2)r_1r_2\frac{\Gamma(\frac{s_1}{2}+ir)\Gamma(\frac{s_2}{2}+ir)}{\Gamma(\frac{2-s_1}{2}+ir)\Gamma(\frac{2-s_2}{2}+ir)}\,dr_1\,dr_2\\ &-\delta_{m_u\equiv 0\,(2)}\phi\sqrt{d(K)}\left(\frac{2^{-(2-s_1-s_2)}}{2\pi^2}\right)^2\zeta_K(s,-m_u)\Gamma\left(\frac{s_1}{2}\right)^2\Gamma\left(\frac{s_2}{2}\right)^2\times\\ &\times\int_0^\infty\int_0^\infty h(r_1,r_2)r_1r_2\frac{\Gamma(\frac{1-s_1}{2}+ir)\Gamma(\frac{1-s_2}{2}+ir)}{\Gamma(\frac{s_1-1}{2}+ir)\Gamma(\frac{s_2-1}{2}+ir)}\,dr_1\,dr_2,\end{aligned}$$

where for an $S\in\mathbb{C}$ and $m\in\mathbb{Z}$ the function $\zeta_K(S,m)$ is the Hecke L-function with the Grössencharacter $\lambda_m(\alpha)$:

$$\zeta_K(S,m)=\sum_{0\neq(\alpha)\lhd\mathcal{O}_K}\frac{\lambda_m(\alpha)}{|N(\alpha)|^S},\qquad \lambda_m(\alpha)=\left|\frac{\alpha}{\alpha'}\right|^{\frac{\pi i m}{2\log\varepsilon}}.$$

The last component $\Sigma_{\text{h-p}}$ of the geometric trace comes from the hyperbolic-parabolic conjugacy classes. First of all, we recall that every hyperbolic-parabolic element is conjugate to an element $\gamma_{m,\alpha}$ that is given in (3.10) on page 84 ($m\in\mathbb{Z}\setminus\{0\}$, $\alpha\in\mathcal{O}_K$). By Proposition 3.1.3 there are only finitely many conjugacy classes for a given m. Also, for a fixed m and α we define k as the smallest positive divisor of m for which $\frac{\alpha(\varepsilon^k-\varepsilon^{-k})}{E}$ is an algebraic integer, where $E=\varepsilon^m-\varepsilon^{-m}$.

If u is a cusp form, then $\Sigma_{\text{h-p}}=\sum_{m,\alpha}T_{m,\alpha}$, where the terms $T_{m,\alpha}$ are similar to the terms T^h_γ above:

$$T_{m,\alpha}=F(m,\alpha)\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\psi\left(\frac{E^2}{\cos^2\vartheta_1},\frac{E^2}{\cos^2\vartheta_2}\right)f_{\lambda_1}(\vartheta_1)f_{\lambda_2}(\vartheta_2)\frac{d\vartheta_1\,d\vartheta_2}{\cos^2\vartheta_1\cos^2\vartheta_2},$$

where f_{λ_k} are the solutions of the equation (3.8) with $f_{\lambda_k}(0)=1$ and $f'_{\lambda_k}(0)=0$ and

$$F(m,\alpha)=\int_1^{\varepsilon^{2k}}\int_0^\infty u\left(r_1i-\frac{\alpha}{E},r_2i-\frac{\alpha'}{E'}\right)\frac{dr_2}{r_2}\frac{dr_1}{r_1}.$$

On the other hand, if u is not a cusp form, then the inner integral above is not convergent, and $\Sigma_{\text{h-p}}$ is given by Proposition 3.1.7:

$$\begin{aligned}\Sigma_{\text{h-p}}=&\eta\sum_{m\in\mathbb{N}^+}\Xi_{s_1,s_2}(m)\sum_{\{\gamma_{m,\alpha}\}}C_{\gamma_{m,\alpha}}(s,m_u)\\ &+\phi\sum_{m\in\mathbb{N}^+}\Xi_{1-s_1,1-s_2}(m)\sum_{\{\gamma_{m,\alpha}\}}C_{\gamma_{m,\alpha}}(1-s,-m_u)\end{aligned}$$

$$+\sum_{m\in\mathbb{N}^+}\sum_{\{\gamma_{m,\alpha}\}}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\psi\left(\frac{E^2}{\cos^2\vartheta_1},\frac{E^2}{\cos^2\vartheta_2}\right)I_u^{\gamma_{m,\alpha}}(\vartheta_1,\vartheta_2)\frac{d\vartheta_1\,d\vartheta_2}{\cos^2\vartheta_1\cos^2\vartheta_2},$$

where

$$\Xi_{s_1,s_2}(m):=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\psi\left(\frac{E^2}{\cos^2\vartheta_1},\frac{E^2}{\cos^2\vartheta_2}\right)(\cos\vartheta_1)^{s_1}(\cos\vartheta_2)^{s_2}\frac{d\vartheta_1\,d\vartheta_2}{\cos^2\vartheta_1\cos^2\vartheta_2}$$

(it is expressed in terms of the function g above Proposition 3.1.7), $C_{\gamma_{m,\alpha}}(s,m_u)$ is defined in Lemma 3.1.5 on page 94 and

$$I_u^{\gamma_{m,\alpha}}(\vartheta_1,\vartheta_2)=\int_1^{\varepsilon^{2k}}\int_1^{\infty}\overline{U}_1\left(r_1e^{i(\frac{\pi}{2}+\vartheta_1)},r_2e^{i(\frac{\pi}{2}+\vartheta_2)}\right)\frac{dr_2}{r_2}\frac{dr_1}{r_1}$$

$$+\int_{\varepsilon^{-2k}}^{1}\int_1^{\infty}\overline{U}_2\left(r_1e^{i(\frac{\pi}{2}-\vartheta_1)},r_2e^{i(\frac{\pi}{2}-\vartheta_2)}\right)\frac{dr_2}{r_2}\frac{dr_1}{r_1},$$

where k is given for a fixed $\gamma_{m,\alpha}$ in Proposition 3.1.4 and the functions $\overline{U}_1$ and $\overline{U}_2$ are defined by (3.14), (3.15), (3.16) and (3.17) on pages 90 and 90.

Now we turn to the spectral part and fix a complete orthonormal system of automorphic forms $\{u_j(z):\ j\geq 0\}$. Recall that the integrals $I_u^A(u_j)$ and $I_u^A(r,m)$ are defined by

$$I_u^A(u_j)=\int_{F_A}|u_j(z)|^2\,u(z)\,d\mu(z),\qquad I_u^A(r,m)=\int_{F_A}|E\,(z,1/2+ir,m)|^2\,u(z)\,d\mu(z).$$

Let us define

$$\begin{aligned}\Psi(A,u,r,m)=2\log\varepsilon\sqrt{d(K)}\Bigg[&2\delta_{m_u=0}\left(\eta\frac{A^s}{s}+\phi\frac{A^{1-s}}{1-s}\right)\\&+\phi(1/2-ir,-m)\left(\delta_{m_u=-2m}\eta\frac{A^{s+2ir}}{s+2ir}+\delta_{m_u=2m}\phi\frac{A^{1-s+2ir}}{1-s+2ir}\right)\\&+\phi(1/2+ir,m)\left(\delta_{m_u=2m}\eta\frac{A^{s-2ir}}{s-2ir}+\delta_{m_u=-2m}\phi\frac{A^{1-s-2ir}}{1-s-2ir}\right)\Bigg].\end{aligned}$$

and

$$\tilde{I}_u^A(r,m)=I_u^A(r,m)-\Psi(A,u,r,m),$$

$$\tilde{I}_u^A(u_j)=I_u^A(u_j)-\delta_{m_u=0}2\log\varepsilon\sqrt{d(K)}\,|\phi_j|^2\left[\eta\frac{A^{1+s-2s_j}}{1+s-2s_j}+\phi\frac{A^{2-s-2s_j}}{2-s-2s_j}\right],$$

where $\phi_j y_1^{1-s_1^{(j)}}y_2^{1-s_2^{(j)}}$ is the zeroth Fourier coefficient of u_j (and hence ϕ_j is non-zero only if u_j is not a cusp form). The results of the previous section show that the limits

$$I_u(r,m):=\lim_{A\to\infty}\tilde{I}_u^A(r,m),\qquad I_u(u_j):=\lim_{A\to\infty}\tilde{I}_u^A(u_j)$$

exist. Now we are ready to state our final result:

Theorem 3.3.1. *With the notations above we have*

$$\Sigma_{\mathrm{ell}}+\Sigma_{\mathrm{hyp}}+\Sigma_{\mathrm{mix}}+\Sigma_{\mathrm{par}}+\Sigma_{\mathrm{h-p}}=$$

$$=\delta_{m_u\equiv 0\,(2)}\frac{\eta}{2}h\left(i\frac{s}{2}-\frac{\pi m_u}{4\log\varepsilon},i\frac{s}{2}+\frac{\pi m_u}{4\log\varepsilon}\right)\phi\left(\frac{1+s}{2},\frac{m_u}{2}\right)$$

$$+\delta_{m_u\equiv 0\,(2)}\frac{\phi}{2}h\left(i\frac{1-s}{2}+\frac{\pi m_u}{4\log\varepsilon},i\frac{1-s}{2}-\frac{\pi m_u}{4\log\varepsilon}\right)\phi\left(\frac{2-s}{2},-\frac{m_u}{2}\right)$$

$$+\sum_j h(r_1^{(j)},r_2^{(j)})I_u(u_j)+$$

$$+\frac{1}{8\pi\sqrt{d(K)}\log\varepsilon}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}h\left(r+\frac{\pi m}{2\log\varepsilon},r-\frac{\pi m}{2\log\varepsilon}\right)I_u(r,m)\,dr.$$

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