Some problems in Extremal Combinatorics

A Thesis

submitted in partial fulfillment of
the Requirements for the Degree of
Doctor of Philosophy

by

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Mathematics and its Applications,
Central European University,
Budapest, Hungary.
October, 2021.

Certificate

This is to certify that this dissertation entitled Some problems in Extremal Combinatorics towards the partial fulfilment of the Doctor of Philosophy (Phd) in Mathematics And it's Applications program at the Central European University, Budapest represents study/work carried out by Debarun Ghosh at Central European University under the supervision of Ervin Győri, Professor, Department of Mathematics, during the academic year 2017-2021.

Ervin Győri

This thesis is dedicated to Mom and Dad.

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as specified in the text. It is not substantially the same as any work that has already been submitted before for any degree or other qualification.

Debarun Ghosh

Acknowledgments

I would like to express my gratitude towards my supervisor Ervin Győri for his continuous support during my time as a student at CEU. Special thanks go to my collaborator and batch-mates, Addisu Paulos and Chuanqi Xiao. Together with Ervin Győri, they majorly contributed to my success as a researcher.

I also want to thank my other collaborators Judit Nagy-György, Nika Salia, Oliver Janzer, Oscar Zamora, and Ryan Martin.

Finally, I would like to thank my parents for their irreplaceable love and support. I would also like to thank my friends Anisha, Ash, Atharva, Dwipayan, Eqra, Prachi, Vaibhav, and countless others for supporting me wholeheartedly throughout these four years and my life in general.

Abstract

My thesis investigates various problems in the field of Extremal Combinatorics. We work on the classical Turán problem for graphs before branching out into "Generalized Turán problems". Specifically, we were interested in the growing field of Planar Turán problems. We also explore the natural generalization of Turán type problems in hypergraphs. In addition, we apply the concepts of Planar Extremal Graph Theory in determining the Wiener index of planar graphs.

The thesis contains six chapters. The first chapter introduces the field of Extremal Graph Theory and Turán type problems for graphs and hypergraphs. We delve into the basic notations and famous theorems, which laid the foundations of Extremal Graph Theory and its generalizations.

The Turán number of a graph H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex graph that does not have H as a subgraph. Erdős-Stone-Simonovits Theorem asserts that $\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2)$, where $\chi(H)$ is the chromatic number of H. If H is non-bipartite, this result is an asymptotic form of $\mathrm{ex}(n, H)$, but if H is bipartite, the order of magnitude of $\mathrm{ex}(n, H)$ is in general open. Even when the graph H is non-bipartite, it is still interesting to know the order of magnitude of lower terms for $\mathrm{ex}(n, H)$. Let TP_k be the triangular pyramid of k-layers. For $k \geq 1$, the chromatic number of TP_k is 3. In Chapter 2, we determine that $\mathrm{ex}(n, TP_3) = \frac{1}{4}n^2 + n + o(n)$ and pose a conjecture for $\mathrm{ex}(n, TP_4)$. These results are based on the paper "The Turán Number of the Triangular Pyramid of 3-Layers", co-authored by Győri, Paulos, Xiao and Zamora.

Now we look at problems in generalized Turán numbers. Let $\mathrm{ex}_{\mathcal{P}}(n, T, H)$ denote the maximum number of copies of T in an n-vertex planar graph which does not contain H as a subgraph. When $T = K_2$, $\mathrm{ex}_{\mathcal{P}}(n, T, H)$ is the well-studied function, the planar Turán number of H. It is denoted by $\mathrm{ex}_{\mathcal{P}}(n, H)$ and was initiated by Dowden (2016). Unfortunately, the case when the forbidden subgraph is a complete graph (i.e., the analog to Turán) and stars is fairly trivial. The next most natural type of graph to investigate is perhaps a cycle. Dowden obtained a sharp upper bound for both $\mathrm{ex}_{\mathcal{P}}(n, C_4)$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5)$. Later, Lan, Shi and Song continued this topic and proved that $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{18(n-2)}{7}$. In Chapter 3, we improve this result and give the following sharp upper bound: $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5}{2}n - 7$, for all $n \geq 18$. We also pose a conjecture on $\mathrm{ex}_{\mathcal{P}}(n, C_k)$, for $k \geq 7$. These results are based on the paper "Planar Turán number of the 6-cycle", co-authored by Győri, Martin, Paulos and Xiao.

Another generalization of the planar Turán number of stars are the double stars as the forbidden graph. Double stars are two adjacent vertices of degree m and n, respectively, and are denoted by $S_{m,n}$. It is easy to see that $\mathrm{ex}(n, S_{m,n}) = 3n - 6$, for $m \geq 2$ and $n \geq 6$. In Chapter 4, we determine the upper bounds for $\mathrm{ex}_{\mathcal{P}}(n, S_{2,2})$, $\mathrm{ex}_{\mathcal{P}}(n, S_{2,3})$, $\mathrm{ex}_{\mathcal{P}}(n, S_{2,4})$, $\mathrm{ex}_{\mathcal{P}}(n, S_{2,5})$, $\mathrm{ex}_{\mathcal{P}}(n, S_{3,3})$

and $\mathrm{ex}_{\mathcal{P}}(n, S_{3,4})$. Moreover, the bound for $\mathrm{ex}_{\mathcal{P}}(n, S_{2,2})$ is sharp. These results are based upon the ongoing paper "Planar Turán Number of Double Stars", co-authored by Győri, Paulos and Xiao.

Next, we consider a Turán type problem in the field of Hypergraphs. One of the first problems in Extremal Graph Theory was the maximum number of edges that a triangle-free graph can have. Mantel solved this, which built the foundations of what we know as Extremal Graph Theory. The natural progression was to ask the maximum number of edges in a k-book free graph. A k-book, denoted by B_k, is k triangles sharing a common edge. Given a graph G on n vertices and having $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges. Erdős conjectured in 1962 [35] that the size of the largest book in G is $\frac{n}{6}$ and this was proved soon after by Edwards (unpublished, see also Khadziivanov and Nikiforov [110] for an independent proof). In the early 2000s, Győri [69] solved the hypergraph analog of the maximum number of hyperedges in a triangle-free hypergraph. In a hypergraph, k-book denotes k Berge triangles sharing a common edge. Let $\mathrm{ex}_3(n, \mathcal{F})$ denote the maximum number of hyperedges in a Berge-$\mathcal{F}$-free 3-uniform hypergraph on n vertices. In Chapter 5, we prove $\mathrm{ex}_3(n, B_k) = \dfrac{n^2}{8}(1+o(1))$. These results are from the paper "Book free 3-Uniform Hypergraphs", co-authored by Győri, Nagy-György, Paulos, Xiao, and Zamora.

The Wiener index is the sum of the distances between all the pairs of vertices in a connected graph. The Wiener index of an n-vertex maximal planar graph was conjectured to be at most $\lfloor \frac{1}{18}(n^3 + 3n^2) \rfloor$. In Chapter 6, we prove this conjecture and determine the unique n-vertex maximal planar graph attaining this maximum. These results are from the paper "The maximum Wiener index of maximal planar graphs", co-authored by Győri, Paulos, Salia, and Zamora.

Contents

Chapter 1

Introduction

Notations and definitions:

All the graphs we consider in this chapter are simple and finite. Let G be a graph. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. We denote the degree of a vertex v by $d(v)$, the minimum degree in graph G by $\delta(G)$ and the maximum degree in graph G by $\Delta(G)$. Let the complete graph on r vertices or a *r-clique* be denoted by K_r. The *Chromatic number*, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertex set of G such that no two adjacent vertices share the same color. The *Turán number* of a graph H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex graph that does not contain H as a subgraph. Let $\mathrm{EX}(n, H)$ denote the set of extremal graphs, i.e., the set of all n-vertex, H-free graph G such that $e(G) = \mathrm{ex}(n, H)$. A *k-partite graph* is a graph whose vertices are or can be partitioned into k different independent sets. A *complete bipartite graph* is a bipartite (2-partite) graph (V_1, V_2, E) such that for every two vertices $v_1 \in V_1$ and $v_2 \in V_2$, v_1v_2 is an edge in E. A complete bipartite graph, with partitions of size $|V_1|= m$ and $|V_2|= n$, is denoted by $K_{m,n}$. A *subgraph* H of a graph G is a graph whose vertex set $V(H)$ is a subset of the vertex set $V(G)$, and whose edge set $E(H)$ is a subset of the edge set $E(G)$. A graph H is called a *topological minor* of the graph G if H can be formed from G by deleting edges and vertices and by contracting edges. The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets X_1 and X_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. A connected graph G is called a *Hamiltonian graph*, if there is a cycle that includes every vertex of G, and the cycle is called a *Hamiltonian cycle.*

Extremal Graph Theory is among the most natural fields in Graph Theory and filled with intriguing results. Questions such as how global graph parameters, such as the edge number, density, and chromatic number, influence the local substructures have been an ever-growing research field. For instance, consider an n-vertex graph G. How many edges do we have to give the graph G to make sure that, no matter the arrangement of the edges, G contains a K_r subgraph for some given r? Or at least a K_r minor? Will some sufficiently high average degree or chromatic number ensure that one of these substructures occurs?

Extremal Graph Theory problems can be broadly subdivided into two categories, as follows: If we look for the global conditions that ensure a graph G contains some given graph H as a minor (or topological minor), it will suffice to raise G above the value of some linear function of $|G|$, i.e., to make $\epsilon(G) := \frac{|E|}{|V|}$ large enough. On the other hand, if we ask what global assumptions imply the existence of some given graph H as a subgraph, it will not help to raise in-variants such as ϵ. As soon as H contains a cycle, there are graphs of arbitrarily large chromatic numbers not containing H as a subgraph. In fact, let H be non-bipartite, and f be the function such that $f(n)$ edges on n vertices force a H as subgraph. Since complete bipartite graphs can have $\frac{n^2}{4}$ edges, $f(n)$ must exceed $\frac{n^2}{4}$.

A systematic study of these type problems started after Turán found and characterized $\mathrm{EX}(n, K_{r+1})$. The case $r = 2$ was solved by Mantel in 1907.

Theorem 1.0.1 (Mantel [103])**.** *The maximum number of edges in an n-vertex triangle-free graph is $\left\lfloor \frac{n^2}{4} \right\rfloor$. Furthermore, the only triangle-free graph with $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$.*

The *Turán graph*, $T_r(n)$, is an n-vertex complete r-partite graph whose parts have as equal as possible sizes. Precisely speaking, the graph has $(n \bmod r)$ parts of size $\lceil n/r \rceil$ and $r - (n \bmod r)$ parts of size $\lfloor n/r \rfloor$. Denote $e(T_r(n))$ by $t_r(n)$. Turán proved the following fundamental result in Extremal Graph Theory:

Theorem 1.0.2 (Turán [122])**.** *For an n-vertex K_{r+1}-free graph G, $e(G) \le t_r(n)$. Equality holds if and only if G is the Turán graph $T_r(n)$, i.e., $\mathrm{ex}(n, K_{r+1}) = t_r(n)$ and $\mathrm{EX}(n, K_{r+1}) = T_r(n)$.*

In 1966, Erdős, Stone, and Simonovits determined the asymptotic value of $\mathrm{ex}(n, H)$ by its chromatic number only. In particular, for any non-bipartite graph H, their result gives an asymptotic value which is best up to the leading term.

Theorem 1.0.3 (Erdős, Stone, and Simonovits [43, 45])**.** *Let F be a non-bipartite graph. Then*

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2),$$

where $\chi(H)$ denotes the chromatic number of H.

But when $\chi(H) = 2$ we get that $\mathrm{ex}(n, H) = o(n^2)$ and wish to know exactly how "small" is this $o(n^2)$. The most natural classes of graphs that arise from this question are paths and complete bipartite graphs. One of the oldest problems is the question of determining $\mathrm{ex}(n, P_k)$ and was solved by Erdős and Gallai.

Theorem 1.0.4 (Erdős and Gallai [40])**.** *If G_n is a graph containing no P_k, $(k \geq 2)$, then $e(G_n) \leq \frac{(k-2)n}{2}$.*

The tightness of Theorem 1.0.4 is shown by the graph with $\frac{n}{k-1}$ disjoint K_{k-1}, where n is divisible by $k - 1$. We refer the reader to an excellent survey on related topics by Füredi and Simonovits [55].

If k is even, then there are nearly extremal graphs having a completely different structure. Namely, one can take a complete bipartite graph with partite sets A and B of sizes $|A| = \frac{k-2}{2}$ and $|B| = n - \frac{k-2}{2}$ and add all edges in A. Faudree and Schelp [49] proved that the extremal graph for P_k can indeed be obtained in this way for all n and k.

We proceed to explore the complete bipartite graphs, specifically $K_{2,2}$. The upper bound was provided by Erdős, Rényi, and Sós [42]. The following $K_{2,2}$-free construction, due to Erdős, Rényi [41] (and independently re-discovered by Brown [17]), allows us to show that $\mathrm{ex}(n, K_{2,2}) \approx \frac{1}{2}n^{3/2}$.

Theorem 1.0.5 (Erdős, Rényi, and Sós [42], Erdős, Rényi [41], Brown [17])**.** *The maximum number of edges in a $K_{2,2}$-free graph on n vertices is:*

$$\left(\frac{1}{2} - o(1)\right) n^{\frac{3}{4}} \leq \mathrm{ex}(n, K_{2,2}) \leq \frac{n}{4}(1 + \sqrt{4n-3}).$$

Construction: We first show that for every prime p, we can construct a $K_{2,2}$-free graph on $n = p^2 - 1$ vertices with $m \geq (p^2-1)(p-1)/2 = (\frac{1}{2} - o(1))n^{\frac{3}{2}}$ edges. So given a prime p, we define a graph on $p^2 - 1$ vertices where each vertex is a pair $(a, b) \in F_p \times F_p, (a, b) \neq (0, 0)$. We "connect" vertex (a, b) to vertex (x, y), if and only if $ax + by = 1$(over F_p). Assume $v = (a, b) \neq (0, 0)$. Then, it is easy to check that in all cases (i.e., if v is either $(a, 0), (0, b)$ or (a, b) with $a, b \neq 0$), that there are exactly p solutions to $ax + by = 1$. This means that we always have $d(v) \geq p - 1$ (we omit the possible solution satisfying $x = a, y = b$ since we do not allow loops), implying that $m \geq (p^2 - 1)(p - 1)/2$ as needed. To show that the graph is indeed $K_{2,2}$-free, take any $v = (a', b')$, $u = (a, b)$, $u \neq v$. Then the equations $ax + by = 1$ and $a'x + b'y = 1$ have at most one solution implying that u and v have at most one common neighbor, so the graph is indeed $K_{2,2}$-free.

There is also a result of Füredi extending this. He proved that for each t, $\text{ex}(n, K_{2,t+1}) \approx \frac{\sqrt{t}}{2} n^{3/2}$ [52]. The following construction is due to Brown and provides the lower bound $\text{ex}(n, K_{3,3}) \geq c_0 n^{5/3}$[17]. Roughly speaking, take a prime $p \equiv 3 (\text{mod } 4)$ and consider the graph on p^3 vertices whose vertex set is $\mathbb{Z}_p^3$, where (x, y, z) is joined to (a, b, c) if and only if $(a-x)^2+(b-y)^2+(c-z)^2 \equiv 1 (\text{mod } p)$. For any given (x, y, z), there will be on the order of p^2 elements (a, b, c) to which it is connected. There are, therefore, around $c' n^{5/3}$ edges in the graph. Moreover, the unit spheres around the three distinct points $(x, y, z), (x', y', z')$ and (x'', y'', z'') cannot meet in more than two points, so the graph does not contain a $K_{3,3}$. The result follows for all n by a similar argument to above.

The following generalization was proved by Kővári, Sós and Turán [121].

Theorem 1.0.6 (Kővári, Sós, Turán [121])**.** *For every $s \leq t$ we have $\text{ex}(n, K_{s,t}) \leq \left(\frac{1}{2} + o(1)\right)(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}}$, where, $o(1) \to 0$ when $n \to \infty$.*

Other than the constructions mentioned, there is also an impressive construction of Alon, Kollár, Rónyai and Szabó which shows that if $t \geq (s-1)! + 1$, the upper bound mentioned above is essentially sharp, that is, $\text{ex}(n, K_{s,t}) \geq c' n^{2-\frac{1}{c}}$ [2, 90].

The next natural step in understanding Turán numbers is to consider the extremal problem for other bipartite graphs. Cycles of even length are one of the most obvious choices. Let cycles of length k be denoted by C_k. Bondy-Simonovits proved the following upper bound for any general cycle of even length:

Theorem 1.0.7 (Bondy, Simonovits [15]). *For any natural number $k \geq 2$, there exists a constant c such that* $\mathrm{ex}(n, C_{2k}) \leq O(cn^{1+\frac{1}{k}})$.

For $k = 2, 3$ and 5, that is, for C_4, C_6 and C_{10}, this is known to be sharp. For cycles of length 4, we have already seen that $\mathrm{ex}(n, C_4) \approx \frac{1}{2}n^{\frac{3}{2}}$ by Theorem 1.0.5. Benson [7] proved the lower bounds of C_6 and C_{10}. Some other constructions achieving it were found by Wenger [126], Lazebnik and Ustimenko [96], Mellinger [105], and Mellinger and Mubayi [106]. We refer the reader to the paper [97] that presents new constructions as well as gives numerous references. One of the most intriguing questions of modern Graph Theory is whether this upper bound is sharp or not.

Next, we consider the extremal number for odd cycles. We already know, by Theorem 1.0.3, that $\mathrm{ex}(n, C_{2k+1}) \approx \frac{n^2}{4}$. The so-called stability approach proves that, for n sufficiently large, $\mathrm{ex}(n, C_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor$. The idea behind the stability approach is to show that a C_{2k+1}-free graph with roughly the maximal number of edges is approximately bipartite. Then one uses this approximate structural information to prove an exact result.

Theorem 1.0.8 (Erdős [38]). *For every natural number $k \geq 2$ and $\epsilon > 0$, there exists $\delta > 0$ and a natural number n_0 such that, if G is a C_{2k+1}-free graph on $n \geq n_0$ vertices with at least $(\frac{1}{4} - \delta)n^2$ edges, then G may be made bipartite by removing at most ϵn^2 edges.*

For more detailed values of $\mathrm{ex}(n, C_{2k+1})$, we refer the reader to the works of Bondy [14, 13], Woodall [128], and Bollobás [8] (pp. 147–156). For a recent presentation, see Dzido [30], who also considered the Turán number of wheels. One of the problems we worked on with the "stability approach" was determining the Turán number of Triangular pyramids. *Triangular pyramids* with k-layers are smaller triangles arranged in k layers to form a bigger triangle (see Figure 1.1) and are denoted by TP_k.

For $k \geq 1$, the chromatic number of TP_k is 3. Hence, by Theorem 1.0.3, we have $\mathrm{ex}(n, TP_k) = \frac{n^2}{4} + o(n^2)$. Yet, it remains interesting to determine the exact value of $\mathrm{ex}(n, TP_k)$. Recall, by Theorem 1.0.1, the maximum number of edges in an n-vertex triangle-free graph is $\left\lfloor \frac{n^2}{4} \right\rfloor$. Since TP_1 is a triangle, we have $\mathrm{ex}(n, TP_1) = \left\lfloor \frac{n^2}{4} \right\rfloor$. The graph TP_2 denotes the flattened tetrahedron. Liu [99] determined $\mathrm{ex}(n, TP_2)$ for sufficiently large values of n. Later, C. Xiao, G. O.H. Katona, J. Xiao, and O. Zamora [129] determined $\mathrm{ex}(n, TP_2)$ for small values of n. They showed that the

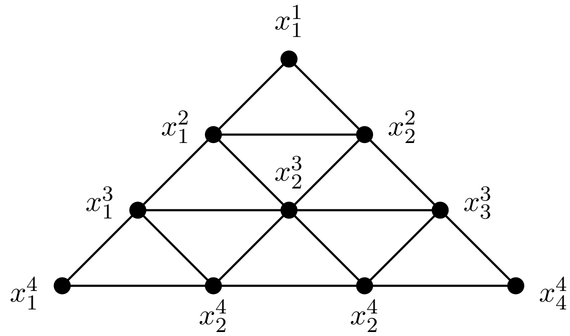


Figure 1.1: A Triangular Pyramid of 3 layers.

maximum number of edges in an n-vertex TP_2-free graph ($n \neq 5$) is,

$$\mathrm{ex}(n, TP_2) = \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n}{2} \right\rfloor, & n \not\equiv 2 \pmod 4, \\ \dfrac{n^2}{4} + \dfrac{n}{2} - 1, & n \equiv 2 \pmod 4. \end{cases}$$

We studied the Turán number for the Triangular Pyramid with three layers in [63]. The maximum number of edges a TP_3-free graph on n vertex can have is at most $\frac{1}{4}n^2+n+o(n)$. We refer the reader to the Chapter 2 for the details. The extremal constructions are interesting in their way. For example, if n is divisible by 6, we take a complete bipartite graph with each class containing $n/6$ vertices. Replace every vertex with a triangle and add the edges between the vertices of triangles in different classes, see Figure 1.2.

1.1 Generalized Turán Numbers

In this chapter, we briefly overview one of the many generalizations which have received an emerging significance in recent years, from the many notable extensions of the Turán function $\mathrm{ex}(n, F)$.

The first one officially carries the name of "Generalized Turán problems". Given graphs H and F, the generalized Turán number $\mathrm{ex}(n, H, F)$ is the maximum number of copies of H in an n-vertex F-free graph. Alon and Shikhelman [3] initiated the study of the function $\mathrm{ex}(n, H, F)$. Note that the case $H = K_2$ gives back the well-studied Turán function. Moon and Moser [107] studied the case when H is a triangle. A recent notable sharp result in this area is of Reiher [117],

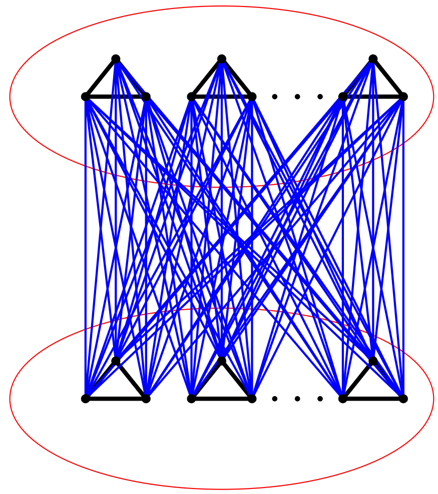


Figure 1.2: The extremal construction on n vertices with $\mathrm{ex}(n, TP_3)$ edges, when n is divisible by 6.

concerning the clique case $F = K_t$ for arbitrary $t \in \mathbb{Z}^+$, which answered the question of Lovász and Simonovits [101]. In general, similarly to the basic Turán-function, there is an essential difference in terms of the chromatic number of F. That is, we may expect asymptotically sharp results for $\chi(F) > 2$, while in the case $\chi(F) = 2$ even the exponent of the function can be unclear in certain domains of m. A few examples of $\mathrm{ex}(n, H, F)$, with $H \neq K_2$, were studied first by Zykov in [132] and independently by Erdős [36]. They determined $\mathrm{ex}(n, K_r, K_s)$ for all r and s. Later Győri, Pach and Simonovits [74] studied $\mathrm{ex}(n, H, K_s)$ for various graphs H when $s \geq 3$. A different example that has received considerable attention recently is $\mathrm{ex}(n, C_r, C_s)$ for various values of r and s. In 2008, Bollobás and Győri [11] showed that $\mathrm{ex}(n, C_3, C_5) = \Theta(n^{3/2})$, and this paper was the start of a more extensive study of this type of problems. Meanwhile, Győri and Li [73] obtained upper and lower bounds on $\mathrm{ex}(n, K_3, C_{2k+1})$, that were subsequently improved by Füredi and Ozkahya [54] and by Alon and Shikhelman [3]. One of the most famous problems in this field was the following conjecture by Erdős [37]:

Conjecture 1.1.1. *The number of cycles of length* 5 *in a triangle-free graph on* n*-vertices is at most* $(\frac{n}{5})^5$ *and the equality holds for the blown-up pentagon if* $5 \mid n$.

Hatami, Hladký, Král, Norine, and Razborov [81], and Grzesik [65] independently proved this conjecture. Recently, attention has been given to the problem of maximizing the number of induced copies of a fixed small graph H, see, for example, [48, 83, 116]. Morrison and Scott [108] determined the maximum possible number of induced cycles, without restriction on length, in an n-vertex graph. The maximal number of induced complete bipartite graphs and induced complete r-partite subgraphs have also been studied in [9, 12, 16]. The maximum number of induced C_5's has been elusive for a long time, and Balogh, Hu, Lidický and Pfender [5] finally solved it. We mention this result here since we worked on an extension in the case of planar Turán number, see Theorem 1.1.8.

Theorem 1.1.1 (Balogh, Hu, Lidický, Pfender [5]). *Let $C(n)$ denote the maximum number of induced copies of 5-cycles in graphs on n vertices. For n large enough, $C(n) = abcde + C(a) + C(b) + C(c) + C(d) + C(e)$, where $a + b + c + d + e = n$ and a, b, c, d, e are as equal as possible. Moreover, if n is a power of 5, the unique graph on n vertices maximizing the number of induced 5-cycles is an iterated blow-up of a 5-cycle.*

Another way to generalize the Turán problem is to study the so-called supersaturation problem (or Rademacher-Turán type problem). Erdős and Simonovits [44] were the first to investigate this systematically. At the same time, the pioneer result due to Rademacher (unpublished) revealed a particular case, see [35]. Here the aim, in general, is to determine the minimum number of subgraphs F in n-vertex graphs having m edges, in terms of m. This function was called the supersaturation function of F, and it takes a positive value exactly if $m > \mathrm{ex}(n, F)$. A sharp result in this case is only known for graphs $K_{2,t}$ due to the Nagy [111] and He, Ma and Yang [82], based on earlier work of Erdős and Simonovits [44], and the construction of Füredi [52]. Concerning the theory of supersaturation, we refer the reader to the surveys of Simonovits[119], Füredi–Simonovits [55] and Pikhurko and Yilma [112].

Now let us focus on a particular type of generalized Turán number of graphs. Let $f(n, H)$ be the maximum number of copies of H in an n-vertex planar graph. For a given graph H, let $\mathcal{F}$ be the collection of subdivisions of K_5 and $K_{3,3}$. It follows from Kuratowski's [92] theorem that $\mathrm{ex}(n, H, \mathcal{F})$ is equal to $f(n, H)$. In this sense, the problem of maximizing H copies in a planar graph is in some sense a special case of the problem of Alon and Shikelman. The simplest case of $H = P_2$, or in other words, the maximum number of edges in an n-vertex planar graph is $3n - 6$, and it follows from Euler's formula [47]. Hakimi and Schmeichel [79] initiated the study of this function for non-trivial cases, such as when H is a cycle. They determined the value of $f(n, C_3)$ and $f(n, C_4)$ precisely and showed that in general $f(n, C_k) = \Theta(n^{\lfloor k/2 \rfloor})$. Their result for $f(n, C_4)$ is as follows:

Theorem 1.1.2 (Hakimi, Schmeichel [79]). *For $n \geq 4$, $f(n, C_4) = \frac{1}{2}(n^2 + 3n - 22)$.*

Recently, Győri, Paulos, Salia, Tompkins, and Zamora [77] determined the exact answer for the 5-cycle. In the same paper, the order of magnitude is also given for $f(n, H)$ when H is a cycle of length more than 4.

Theorem 1.1.3 (Győri, Paulos, Salia, Tompkins, Zamora [77]). *For $n = 6$ and $n \geq 8$, $f(n, C_5) = 2n^2 - 10n + 12$. For $n = 5$ we have $f(n, C_5) = 6$, and for $n = 7$ we have $f(n, C_5) = 41$.*

It is natural to ask the value of $f(n, P_k)$, where P_k is a path of k vertices. Recall $f(n, P_2) = 3n-6$ if $n \geq 3$. Alon and Caro [1] determined the exact value of $f(n, H)$, where H is a complete bipartite graph in which the smaller class is of size 1 or 2. The previous result consequently determines the value of $f(n, P_3)$. They showed that

Theorem 1.1.4 (Alon, Caro [1]). *For $n \geq 4$, $f(n, P_3) = n^2 + 3n - 16$.*

Győri, Paulos, Salia, Tompkins, and Zamora in [76] determined the exact value of $f(n, P_4)$.

Theorem 1.1.5 (Győri, Paulos, Salia, Tompkins, Zamora [76]). *The maximum number of paths of length 4 in a planar graph on n vertices is as follows:*

$$f(n, P_4) = \begin{cases} 12, & \text{if } n = 4; \\ 147, & \text{if } n = 7; \\ 222, & \text{if } n = 8; \\ 7n^2 - 32n + 27, & \text{if } n = 5, 6 \text{ and } n \geq 9. \end{cases}$$

The order of magnitude of $f(n, H)$ when H is a fixed tree was determined in [75] and for general H (and in arbitrary surfaces) by Huynh, Joret and Wood [84] (see also [85] for results in general sparse settings). For a path on k vertices, we have $f(n, P_k) = \Theta(n^{\lfloor \frac{k-1}{2} \rfloor + 1})$. In a follow-up paper, we gave an asymptotic value of $f(n, P_5)$. This bound is asymptotically the best possible. Consider the maximal planar graph on n vertices containing two degree $n-1$ vertices as shown in Figure 1.3. It can be checked that this graph contains at least n^3 copies of P_5.

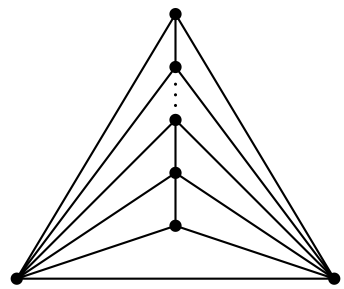


Figure 1.3: A graph on n vertices containing at least n^3 copies of P_5.

Theorem 1.1.6 (Ghosh, Győri, Martin, Paulos, Salia, Xiao, Zamora [58])**.** *The maximum number of paths of length* 5 *in an n vertex planar graph is $f(n, P_5) = n^3 + O(n^2)$.*

We also studied a variant of the above problem, where we are trying to maximize the number of *induced* copies of a certain fixed graph H in an n-vertex planar graph. Considering the growing interest in Turán number of induced graphs, see Theorem 1.1.1. In the planar case, the result of Hakimi and Schmeichel [79] investigated the case where H is a cycle. For $H = C_4$, note that the planar graph $K_{2,n-2}$ contains exactly $\frac{1}{2}(n^2 - 5n + 6)$ induced 4-cycles (see Figure 1.4(b)). It follows from this observation and Theorem 1.1.2 that the maximum number of induced 4-cycles in a planar graph with n vertices is $\frac{1}{2}n^2 + O(n)$. Induced 5-cycles are the next non-trivial case. We determined the maximum number of induced 5-cycles in a planar graph on n vertices, for n sufficiently large, exactly. To state the formula, we define the following function.

Definition 1.1.7. *For $n \geq 7$, let*

$$h(n) = \max\{k_1k_2 + k_2k_3 + k_3k_1 : k_1, k_2, k_3 \in \mathbb{N}, k_1 + k_2 + k_3 = n - 4\} + 2.$$

Clearly, the maximum is attained when k_1, k_2 and k_3 are as close as possible. In particular, $h(n) = n^2/3 + O(n)$.

Theorem 1.1.8 (Ghosh, Győri, Janzer, Paulos, Salia, Zamora [56])**.** *There exists a positive integer n_0 such that if $n \geq n_0$ and G is a planar graph on n vertices, then G contains at most $h(n)$ induced* 5*-cycles. Moreover, there exists a planar graph on n vertices which contains precisely $h(n)$ induced* 5*-cycles.*

Since the extremal graph has a rather complex structure, we present a simpler n-vertex planar

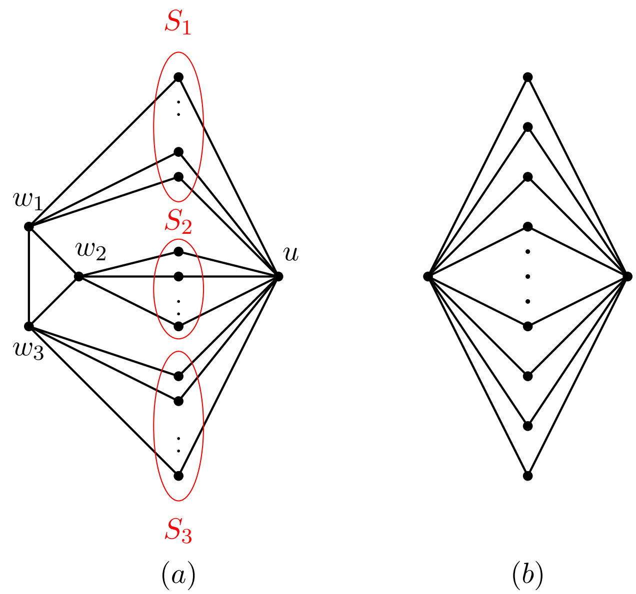


Figure 1.4: Planar graphs containing asymptotically the maximum number of induced 5-cycles and 4-cycles, respectively

graph which has $h(n)-2$ induced 5-cycles. Let S_1, S_2 and S_3 be pairwise disjoint sets of vertices such that $|S_1|+|S_2|+|S_3|= n-4$ and $|S_1|, |S_2|, |S_3|$ are as close as possible. We define an n-vertex planar graph G as follows. The vertex set of G is the three sets of vertices S_1, S_2 and S_3 together with four vertices, say w_1, w_2, w_3 and u. That is, $V(G) = S_1 \cup S_2 \cup S_3 \cup \{w_1, w_2, w_3, u\}$. We define the edges of G as $E(G) = \{w_1w_2, w_2w_3, w_3w_1\} \cup \{w_1v, vu|\ v \in S_1\} \cup \{w_2v, vu|\ v \in S_2\} \cup \{w_3v, vu|\ v \in S_3\}$ (see Figure 1.4 (a)). It can be checked that G contains exactly $|S_1||S_2|+|S_2||S_3|+|S_3||S_1|= h(n)-2$ induced C_5's.

Let us introduce a special case of the Generalized Turán numbers. Over the last decade, a large quantity of work has been carried out in the area of 'random' planar graphs (see, for example, [64], and [104]). However, there seem to be no known results on questions analogous to the Erdős-Stone Theorem, i.e., how many edges can an n-vertex planar graph have without containing a given smaller graph? In 2016, Dowden [28] initiated the study of these specific Turán-type problems. Note that this is the case of $f(n, H)$, where H is K_2. The planar Turán number of a graph H, $\mathrm{ex}_{\mathcal{P}}(n, H)$, is the maximum number of edges in a planar graph on n vertices which does not contain H as a subgraph.

Unfortunately, the case when the forbidden subgraph is a complete graph (i.e., the analog to

Turán) is fairly trivial. Since K_5 is not planar, the only meaningful cases to look at are K_3 and K_4; these are both straightforward. For the former, $K_{2,n-2}$ must be extremal (since all faces have size four when drawn in the plane), and so the extremal number of edges is $2n-4$. For the latter, it suffices to note that there exist planar triangulations not containing K_4 (e.g., take a cycle of length $n-2$ and then add two new vertices that are adjacent to all those in the cycle). Thus, the extremal number is $3n-6$. The planar Turán number, when the forbidden subgraph is a star, is also fairly trivial. The next most natural type of graph to investigate is perhaps a path. We refer the reader to [93] and [94], for extremal planar Turán number for paths of length $\{6, 7, 8, 9, 10, 11\}$. There are also various ways to extend the topic further. For example, one natural idea is to obtain several sufficient conditions on H which yield $\mathrm{ex}_{\mathcal{P}}(n, H) = 3n-6$ for all $n > |V(H)|$. The authors in [94] proved that $ex_{\mathcal{P}}(n, H) = 3n-6$ for all H with $n > |H|+2$ and either $\chi(H) = 4$ or $\chi(H) = 3$ and $\Delta(H) > 7$. They also completely determine $ex_{\mathcal{P}}(n, H)$ when H is a wheel or a star, and the case when H is a (t, r)-fan, that is, H is isomorphic to $K_1 + tK_{r-1}$, where $t > 2$ and $r > 3$ are integers.

The next most natural type of graph to investigate is perhaps a cycle. Dowden [28] obtained the tight bounds $\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq \frac{15(n-2)}{7}$, for all $n \geq 4$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$, for all $n \geq 11$. Later, Lan, Shi and Song [95] proved the following:

Theorem 1.1.9 (Lan, Shi, Song [95])**.** *Let θ_k denote the family of* Theta graphs *on $k \geq 4$ vertices, that is, graphs obtained from a cycle C_k by adding an edge joining two non-consecutive vertices.*

1. *For all $n \geq 4$, $\mathrm{ex}_{\mathcal{P}}(n, \theta_4) \leq \frac{12(n-2)}{5}$.*

2. *For all $n \geq 5$, $\mathrm{ex}_{\mathcal{P}}(n, \theta_5) \leq \frac{5(n-2)}{2}$.*

3. *For all $n \geq 7$, $\mathrm{ex}_{\mathcal{P}}(n, \theta_6) \leq \frac{18(n-2)}{7}$.*

Because of the bound for θ_6 in the same paper, they presented the following corollary for C_6: For all $n \geq 6$, $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{18(n-2)}{7}$, with equality when $n = 9$. The tight bound for $\mathrm{ex}_{\mathcal{P}}(n, C_6)$ was presented by Ghosh, Győri, Paulos, Xiao and Zamora in [57]. We proved that for $n \geq 18$ and an n-vertex C_6-free plane graph G, $e(G) \leq \frac{5}{2}n - 7$. In Chapter 3, we present this proof. In a follow-up paper, we improve the additive constant of the bound of $\mathrm{ex}_{\mathcal{P}}(n, \theta_6)$ given by Lan, Shi, Song. [95] and illustrate that our bound is sharp. We proved the following:

a, b	2	3	4	5	≥ 6
2	$2n-4$	$2n$	$(8/3)n$	$(20/7)n$	$3n-6$
3	$2n$	$(5/2)n-2$	$(20/7)n$	-	$3n-6$
4	$(8/3)n$	$(20/7)n$	-	$3n-6$	$3n-6$
5	$(20/7)n$	-	$3n-6$	$3n-6$	$3n-6$
≥ 6	$3n-6$	$3n-6$	$3n-6$	$3n-6$	$3n-6$

Table 1.1: Upper bounds for $\mathrm{ex}_{\mathcal{P}}(n, S_{a,b})$.

a, b	2	3	4	5	≥ 6
2	$2n-4$	$2n$	$(15/7)n$	$(5/2)n$	$3n-6$
3	$2n$	$(5/2)n-5$	$(9/4)n$	-	$3n-6$
4	$(15/7)n$	(9/4)n	-	$3n-6$	$3n-6$
5	$(5/2)n$	-	$3n-6$	$3n-6$	$3n-6$
≥ 6	$3n-6$	$3n-6$	$3n-6$	$3n-6$	$3n-6$

Table 1.2: Lower bounds for $\mathrm{ex}_{\mathcal{P}}(n, S_{a,b})$.

Theorem 1.1.10 (Ghosh, Győri, Paulos, Xiao, Zamora[62])**.** *Let G be a θ_6-free planar graph on n vertices. The maximum number of edges G can have is at most $\frac{18}{7}n - \frac{48}{7}$, for all $n \geq 14$. Equality holds when G is a 2-connected planar graph.*

Another generalization of the planar Turán number of stars might be double stars as the forbidden graph. *Double stars* are two adjacent vertices of degree m and n, respectively, and are denoted by $S_{m,n}$. It is easy to see that $\mathrm{ex}(n, S_{m,n}) = 3n - 6$, for $m \geq 2$ and $n \geq 6$. The other cases are non-trivial. The upper bounds were described by Ghosh, Győri, Paulos, and Xiao in [61]. They have some interesting extremal constructions, from disjoint copies of maximal planar graphs to Apollonian networks. For example, the extremal structure for $S_{3,3}$ is obtained by joining every vertex of the maximal matching on $n-2$ vertices with two vertices. For detailed proofs, we refer the reader to Chapter 4. The following Table 1.1 compiles the upper bounds.

1.2 Hypergraph Turán Type Problems

Definitions and Notations:

A *hypergraph* $H = (V; E)$ is a family E of distinct subsets of a finite set V. The members of E are called hyperedges and the elements of V are called vertices. A hypergraph is called r-*uniform* if each member of E has size r. A hypergraph $H = (V, E)$ is called *linear* if every two hyperedges have at most one vertex in common. For a family of forbidden r-uniform hypergraphs $\mathcal{F}$, the Turán number $\mathrm{ex}_r(n, \mathcal{F})$ denotes the maximum number of hyperedges in an r-uniform hypergraph on n vertices with no element of $\mathcal{F}$ as a sub-hypergraph. For convenience, whenever $\mathcal{F} = \{F\}$ consists of a single forbidden hypergraph, we write $\mathrm{ex}_r(n, F)$ instead of $\mathrm{ex}_r(n, \{F\})$. A *Berge cycle* of length k, denoted by Berge-C_k, is an alternating sequence of distinct vertices and distinct hyperedges of the form $v_1, h_1, v_2, h_2, \ldots, v_k, h_k$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, \ldots, k-1\}$ and $v_k v_1 \in h_k$.

It is natural to consider the Turán Problem in the setting of r-uniform hypergraphs (r-graphs for short). Surprisingly, while we have rather exact results for the graph Turán problem, already for 3-graphs the problem becomes much harder. Consider the "first" non-trivial case, when we forbid K_4^3, the complete 3-graph on 4 vertices. Let $\mathrm{ex}(n, K_4^3)$ denote the maximum number of edges a 3-graph can contain if it does not contain a copy of K_4^3. It is not hard to show that $\mathrm{ex}(n, K_4^3) \geq (\frac{5}{9} - o(1))\binom{n}{3}$. We do something very similar to what we did in the graph case. We take n vertices and partition them into 3 sets V_1, V_2, V_3 of almost equal size. We then take as edges all triples of vertices (x, y, z) if they are of the form $x \in V_1, y \in V_2, z \in V_3$, or of the form $x, y \in V_i, z \in V_{i+1}$, where $i \in [3]$ (and addition is modulo 3). Denote this graph by $T^{3,4}$. A well-known conjecture of Turán is that $\mathrm{ex}(n, K_4^3) \leq (\frac{5}{9} + o(1))\binom{n}{3}$. To date, the best known upper bound is $\mathrm{ex}(n, K_4^3) \leq 0.561\binom{n}{3}$ [115].

A probable explanation for the hardness of proving tight bounds for hypergraph Turán problems is the following: In the graph case, most proofs proved a tight upper bound along with the unique extremal graph (the Turán graph). As it turns out, the 3-graph $T^{3,4}$ we described above is not the unique K_4^3-free 3-graph with this many edges. In fact, for every n, there are exponentially many non-isomorphic 3-graphs that are K_4^3 -free and have the same number of edges as $T^{3,4}$. So, if indeed $\mathrm{ex}(n, K_4^3) = |E(T^{3,4})|$, then any proof would have to "avoid" proving that $T^{3,4}$ is the unique maximum.

A very natural and widely studied topic is Turán numbers of cycles in hypergraphs. Unlike graphs, there are several types of cycles in hypergraphs. Most studied of them are Berge cycles and linear cycles. The systematic study of Turán numbers of Berge cycles started with the investigation of Berge triangles by Győri [69], who proved that the maximum number of hyperedges in a Berge triangle-free 3-uniform hypergraph on n vertices is at most $\frac{n^2}{8}$. The construction for the lower bound is the following: Take 3 disjoint sets, $A = \{a_1, a_2, \dots, a_{\frac{n}{4}}\}, A' = \{a'_1, a'_2, \dots, a'_{\frac{n}{4}}\}$ and $B = \{b_1, b_2, \dots, b_{\frac{n}{2}}\}$. The hypergraph H, whose vertex set is $A \cup A' \cup B$ and the edge set is $\{a_i, a'_i, b_j \mid 1 \leq i \leq \frac{n}{4}, 1 \leq j \leq \frac{n}{2}\}$, is Berge triangle-free and has $\frac{n^2}{8}$ hyperedges. There is an informal but convenient way to see this construction. First, we take a complete bipartite graph, and then we make a copy of each vertex on one side. We create a triple corresponding to each edge of the original bipartite graph and assign hyperedges to these triples. It is worth noting that this type of hypergraph extension of a graph is quite common, and we will come across similarly obtained hypergraphs in Chapter 5.

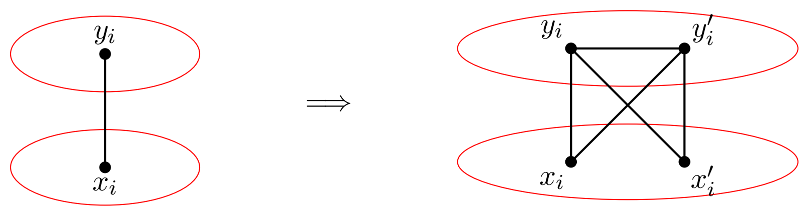


Figure 1.5: Replacing every graph edge x_iy_i in the bipartite graph with two hyperedges $x_iy_iy'_i$ and $y_iy'_ix'_i$

Recently, Füredi, Kostochka, and Luo [53] proved similar results for Berge cycles. Instead of forbidding Berge cycles of fixed length, they forbid all Berge cycles of length at least k. It continued with the study of Berge five cycles by Bollobás and Győri [11]. They showed that $n^{\frac{3}{2}}/3\sqrt{3} \leq \mathrm{ex}_3(n, C_5) \leq \sqrt{2}n^{\frac{3}{2}} + 4.5n$. Very recently, this estimate was considerably improved by Ergemlidze, Győri and Methuku [46]. Győri, Katona, and Lemons [70] proved the following analog of the Erdős-Gallai Theorem 1.0.4 for Berge paths.

Theorem 1.2.1 (Győri, Katona, Lemons [70])**.** *Fix $k > r + 1 > 3$ and let H be an r-uniform hypergraph containing no Berge path of length k. Then $e(H) \leq \frac{n}{k}\binom{k}{r}$. For the other case, fix $r \geq k > 2$. If H is an r-uniform hypergraph with no path of length k, then $e(H) \leq \frac{n(k-1)}{r+1}$.*

For other results, see [4, 86]. Győri and Lemons considered a more general question and estimated Turán number of Berge cycles of any given length.

Theorem 1.2.2 (Győri, Lemons [71],[72]). *For $r \geq 2$, we have* $\mathrm{ex}_r(n, C_{2l}) = O(n^{1+\frac{l}{l}})$. *For $r \geq 3$, we have* $\mathrm{ex}_r(n, C_{2l+1}) = O(n^{\frac{1+l}{l}})$.

The particular case of determining $ex_3^{lin}(n, C_3)$ is equivalent to the famous $(6, 3)$-problem, which is a special case of a general problem of Brown, Erdős, and Sós. The famous theorem of Ruzsa and Szemerédi states that there exists a constant $c > 0$ for which we have $n^{2-\frac{c}{\sqrt{\log n}}} < ex_3^{lin}(n, C_3) = o(n^2)$.

Recall that the maximum number of edges in a triangle-free graph is one of the classical results in Extremal Graph Theory, and Mantel proved it in 1907 [103]. The extremal problem for diamond-free graphs follows from this. If there are 2 triangles sitting on an edge in a graph, we call this a *diamond*. On the other hand, k triangles sitting on an edge is called a *k-book*, denoted by a B_k. Given a graph G on n vertices and having $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges. Mantel showed that G contains a triangle. Using Mantel's Theorem, we can also prove that this graph G also contains a diamond. Rademacher (unpublished and simplified later by Erdős in [39]) proved in the 1940s that the number of triangles in G is at least $\lfloor \frac{n}{2} \rfloor$. Erdős conjectured in 1962 [35] that the size of the largest book in G is $\frac{n}{6}$ and this was proved soon after by Edwards (unpublished [32], see also Khadziivanov and Nikiforov [110] for an independent proof). Both Rademacher's and Edwards' results are sharp. In the former, adding an edge to one part in the complete balanced bipartite graph (note that in G there is an edge contained in $\lfloor \frac{n}{2} \rfloor$ triangles) achieves the maximum. In the latter, every known extremal construction of G has $\Omega(n^3)$ triangles. For more details on book-free graphs, we refer the reader to the following articles [10], [114] and [131].

The hypergraph equivalent of diamonds and k-books is defined similarly, with 2-Berge triangles and k-Berge triangles sharing a common edge, respectively. We continue the work and determine the maximum number of hyperedges for a k-book free 3-uniform hypergraph in [59]. The main result is as follows: For a given $k \geq 2$ and a 3-uniform B_k-free hypergraph H on n vertices, $e(H) \leq \dfrac{n^2}{8}(1 + o(1))$. Recall that $\mathrm{ex}(n, C_3) = \mathrm{ex}(n, B_k) = \frac{n^2}{4}$ in graph setting. Győri [69] proved that the maximum number of hyperedges in a Berge triangle-free 3-uniform hypergraph on n vertices is at most $\frac{n^2}{8}$. So, there is an obvious parallel between the triangle-free and book-free graphs and hypergraphs, making the result much more intriguing. We refer the reader to Chapter 5 for the details.

1.3 Wiener Index

The Wiener index is named after Harry Wiener, who introduced it in 1947; at the time, Wiener called it the "path number" [127]. He was studying its correlations with boiling points of paraffin, taking into consideration its molecular structure. It is the summation of distances between all the unordered pair of the vertices of G. It is the oldest topological index related to molecular branching. Based on its success, many other topological indexes of chemical graphs, based on information in the distance matrix of the graph, have been developed subsequently to Wiener's work. Since undirected graphs, especially trees, are used to model molecules. The concept of Wiener index has been studied under different names such as the total status by [80], the total distance by Entringer, Jackson, and Snyder [34], and the transmission by Plesník [113] for various applications to topics including chemistry, communication, Sociometry, and the theory of social networks. Several survey papers [25, 26, 27, 88, 130] contain a great deal of knowledge on the Wiener index.

The average distance (or mean distance) $\mu(G)$ of the graph G is $\mu(G) = \frac{2W(G)}{n(n-1)}$. Networks with small mean distances are desirable due to their good properties. Thus, the average distance of a connected graph is at least 1 and can be realized only by a complete graph. On the other hand, the average distance of a connected graph of order n is at most $\frac{n+1}{3}$ and this bound is attained only by a path of order n, see [29, 34, 100]. Plesník [113] showed that this bound can be improved to $\frac{\left\lfloor \frac{n^2}{4} \right\rfloor}{(n-1)}$ for 2-connected or 2-edge-connected graphs of order n, and can be attained only by a cycle of order n. There is no closed or recursive formula to calculate their Wiener indices for most general classes of graphs. Thus, finding bounds on Wiener indices for a general class of graphs of a given order has been an attractive research topic.

Many sharp or asymptotically sharp bounds on $W(G)$ in terms of other graph parameters are known. Beezer, Riegsecker, and Smith [6] proved the following bound for graphs with minimum degree δ.

Theorem 1.3.1 (Beezer, Riegsecker, Smith [6]). *The Wiener index of a connected graph with n vertices, e edges and minimum degree δ satisfies* $\left\lfloor \frac{(n-1)n(n+1)-2e}{\delta+1} \right\rfloor$.

Sharp bounds on Wiener indices of k-connected graphs or k-edge-connected graphs of order n have been completely solved. The question in terms of finding a sharp upper bound on the average distances of k-connected (resp., k-edge-connected) graphs of a given order n for $k \geq 3$ was posed by Plesník [113] in 1984. In 2006, Gutman and Zhang [68] gave a sharp lower bound on Wiener indices of k-connected (resp., k-edge-connected) graphs of order n. Dankelmann, Mukwembi, and Swart [22, 21] established asymptotically sharp upper bounds on average distances of k-edge-connected graphs of a given order for $k \geq 3$. When $k \geq 3$ is odd, they [23] further showed that the upper bound on the average distances of k-connected n-vertex graphs, can be improved to $\mu(G) \leq \frac{n}{2k+1} + 30$. This bound is the best possible, apart from an additive constant. For other results involving graph parameters, for example, connectivity, edge-connectivity, and maximum degree see [51], respectively. For finding more details in the mathematical aspect of the Wiener index, see also results [24, 66, 87, 102, 89, 67, 109, 124, 123, 125, 20, 91].

One can study the Wiener index of the family of connected planar graphs. Since a path attains the above, it is natural to ask the same question for some family of planar graphs. For instance, the Wiener index of a maximal planar graph with n vertices, $n \geq 3$ has a sharp lower bound $(n-2)^2+2$. Any maximal planar graph such that the distance between any pair of vertices is at most 2 (for instance, a planar graph containing the n-vertex star) attains this bound. It is well-known [34] that the Wiener index of a tree on n vertices attains the minimum value $(n-1)^2$, when it is a star. On the other hand, it attains the maximum value $\frac{1}{6}(n^3-n)$ in case of a path. Marraki, Mohamed, and Abdelhafid [33] showed that a maximal planar graph minimizes the Wiener index of a planar map of order n and is equal to $(n-2)^2+2$. On the other hand, a path maximizes it and is equal to $\frac{1}{6}(n^3-n)$. Che and Collins [18], and independently Czabarka, Dankelmann, Olsen and Székely [19], gave a sharp upper bound of a particular class of maximal planar graphs known as Apollonian networks. An *Apollonian network* is an undirected graph formed by recursively subdividing a triangle into three smaller triangles. Starting from a single triangle, select a triangular face repeatedly and add a new vertex inside. Connect the new vertex to each vertex of the face containing it.

Theorem 1.3.2 (Che, Collins [18], Czabarka, Dankelmann, Olsen, Székely [19])**.** *Let G be an Apollonian network of order $n \geq 3$. Then $W(G)$ has a sharp upper bound*

$$W(G) \leq \left\lfloor \frac{1}{18}(n^3+3n^2) \right\rfloor = \begin{cases} \frac{1}{18}(n^3+3n^2), & \text{if } n \equiv 0 (mod\ 3); \\ \frac{1}{18}(n^3+3n^2-4), & \text{if } n \equiv 1 (mod\ 3); \\ \frac{1}{18}(n^3+3n^2-2), & \text{if } n \equiv 2 (mod\ 3). \end{cases}$$

The authors in [18] also conjectured that this bound also holds for every maximal planar graph. Dowden [19] showed that the conjectured bound holds asymptotically. In particular,

Theorem 1.3.3 (Dowden [19])**.** *For $k = 3, 4, 5$, there exists a constant C_k such that*

$$W(G) \leq \frac{1}{6k}n^3 + C_k n^{5/2},$$

for every k-connected maximal planar graph of order n.

In [60], we confirm the conjecture above. We refer the reader to Chapter 6 for the details. The authors in [18] also had a conjecture for Wiener index of quadrangulation graphs. Győri, Paulos, and Xiao [78] proved this conjecture recently. Their result is as follows:

Theorem 1.3.4 (Győri, Paulos, Xiao [78])**.** *Let G be a quadrangulation graph with $n \geq 4$ vertices. Then*

$$W(G) \leq \begin{cases} \frac{1}{12}n^3 + \frac{7}{6}n - 2, & \text{if } n \equiv 0 (mod\ 2); \\ \frac{1}{12}n^3 + \frac{11}{12}n - 1, & \text{if } n \equiv 1 (mod\ 2). \end{cases}$$

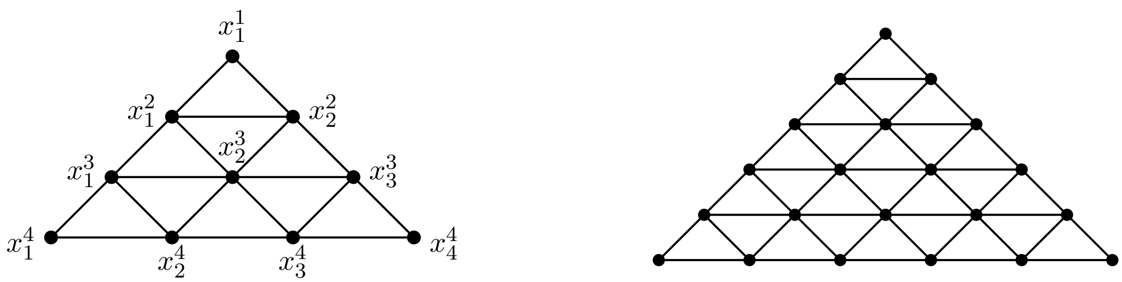
Chapter 2

Turán Number of TP_3

2.1 Introduction

All the graphs we consider in this chapter are simple and finite. Let G be a graph. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Let $e(G)$ and $v(G)$ denote the number of edges and vertices, respectively. We denote the degree of a vertex v by $d(v)$, the minimum degree in graph G by $\delta(G)$ and the maximum degree in graph G by $\Delta(G)$. The subgraph induced by $S \subseteq V(G)$, is denoted by $G[S]$. Moreover, $N(v)$ denotes the set of vertices in G adjacent to v. Let H be a subgraph of G and v be a vertex in H. Let $N_H(v)$ denote the set of vertices in H that are adjacent to v. Let $x_1, x_2, \dots, x_k$ be k vertices in H. The set of vertices in H which are adjacent to all these k vertices, $x_1, x_2, \dots, x_k$, is denoted by $N^*_H(x_1, x_2, \dots, x_k)$. We may omit the subscript in the notation whenever the underlying graph is clear. Let A and B be subsets of $V(G)$, then the number of edges between them is denoted by $e(A, B)$. We denote the cycle of length 6 by C_6 or a 6-cycle. A 7-wheel, denoted by W_7, is a 7-vertex graph containing a C_6 and a vertex adjacent to all the vertices of the 6-cycle. Recall that the Turán number of a graph H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex graph that does not contain H as a subgraph. Let $\mathrm{EX}(n, H)$ denote the set of extremal graphs, i.e., the set of all n-vertex, H-free graph G such that $e(G) = \mathrm{ex}(n, H)$.

Definition 2.1.1. *The Triangular Pyramid with k layers, denoted by TP_k, is defined as follows: Draw $k+1$ paths in layers such that the first layer is a 1-vertex path, the second layer is a 2-vertex*



(i) TP_3 (ii) TP_5

Figure 2.1: A Triangular Pyramid with 3 and 5 layers, respectively.

path, ..., and the $(k+1)^{st}$ layer is a $(k+1)$-vertex path. Label the vertices of the i^{th} layer's path from left to right as $x_1^i, x_2^i, \dots, x_i^i$, where $i \in \{1,2,3,\dots,k+1\}$. The vertex set of the graph TP_k is the set of all the vertices of the $(k+1)$ paths. The edge set contains all the edges of the paths. Additionally, for any two consecutive $(i-1)^{th}$ and i^{th} layer, $x_r^{i-1}x_r^i$ and $x_r^{i-1}x_{r+1}^i$ are in $E(TP_k)$, where $i \in \{1,2,\dots,k+1\}$ and $1 \le r \le i-1$ (see Figure 2.1).

For $k \geq 1$, the chromatic number of TP_k is 3. Hence, by Theorem 1.0.3, we have $\mathrm{ex}(n, TP_k) = \frac{n^2}{4} + o(n^2)$. Yet, it remains interesting to determine the exact value of $\mathrm{ex}(n, TP_k)$. The graph TP_1 is a triangle and by Mantel's Theorem, $\mathrm{ex}(n, TP_1) = \left\lfloor \frac{n^2}{4} \right\rfloor$. The graph TP_2 denotes the flattened tetrahedron. Liu [99] determined $\mathrm{ex}(n, TP_2)$ for sufficiently large values of n. Later, Katona, Xiao, Xiao, and Zamora [129] determined $\mathrm{ex}(n, TP_2)$ for small values of n.

Theorem 2.1.2 (Katona, Xiao, Xiao, Zamora [129])**.** *The maximum number of edges in an n-vertex TP_2-free graph ($n \neq 5$) is,*

$$\mathrm{ex}(n, TP_2) = \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n}{2} \right\rfloor, & n \not\equiv 2 \pmod 4, \\ \dfrac{n^2}{4} + \dfrac{n}{2} - 1, & n \equiv 2 \pmod 4. \end{cases}$$

In this chapter, we study the Turán number of TP_3, i.e., the Triangular Pyramid with three layers.

Theorem 2.1.3. *The maximum number of edges in an n-vertex TP_3-free graph is:*

$$\mathrm{ex}(n, TP_3) = \frac{1}{4}n^2 + n + o(n).$$

It can be checked that the constructions given in Figure 2.2, 2.3 and 2.4 are TP_3-free graphs containing $\frac{1}{4}n^2+n+1$, $\frac{1}{4}n^2+n+\frac{3}{4}$ and $\frac{1}{4}n^2+n$ edges, respectively. Thus, the bound in Theorem 2.2.3 is best up to the linear term for infinitely many values of n.

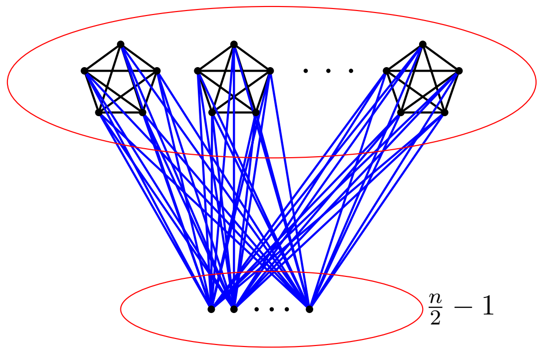


Figure 2.2: The extremal construction when n is even and $n \equiv 2 (\text{mod } 10)$.

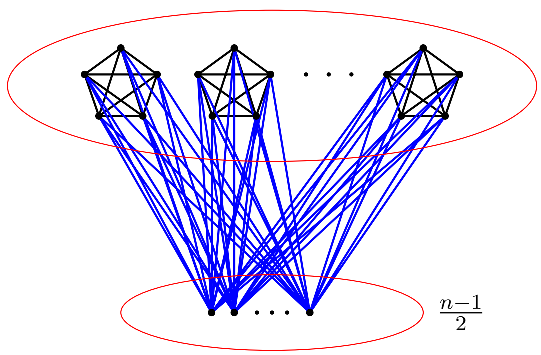


Figure 2.3: The extremal construction when n is odd and $n \equiv 1 (\text{mod } 10)$.

This chapter is structured as follows: In Section 2.2, the proof of the main theorem is presented. In Section 2.3, a conjecture on the exact extremal number of TP_3 and TP_4 is provided.

2.2 Proof of Theorem 2.1.3

We will be using the following classical stability result of Erdős and Simonovits:

Theorem 2.2.1 (Erdős, Simonovits [38]). *Let $k \geq 2$ and suppose that H is a graph with $\chi(H) = k+1$. If G is a H-free graph with $e(G) \geq t_k(n) - o(n^2)$, then G can be formed from $T_k(n)$ by adding and deleting $o(n^2)$ edges.*

Since $\chi(TP_3) = 3$, the above theorem can be restated as follows:

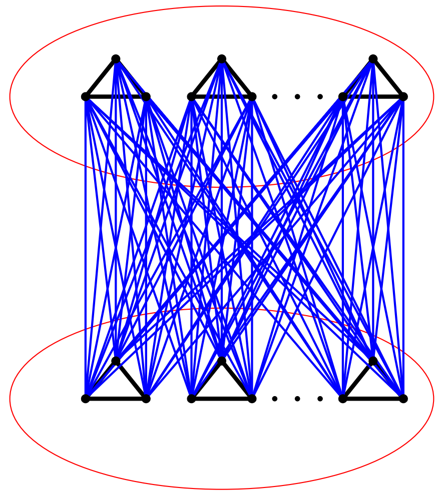


Figure 2.4: The extremal construction when n is divisible by 6.

Theorem 2.2.2. *For every $\gamma > 0$, there exists an $\epsilon > 0$ and $n_0(\gamma)$ such that for every TP_3-free graph G on n $(n > n_0(\gamma))$ vertices with $e(G) \geq \frac{n^2}{4} - \epsilon n^2$, we have*

$$|E(G)\Delta E(T_2(n))| \leq \gamma n^2.$$

We will prove the following version of Theorem 2.1.3.

Theorem 2.2.3. *For $\delta > 0$ and $n \geq \frac{5n_0(\delta)}{2\delta}$, the maximum number of edges in an n-vertex TP_3-free graph is* $\mathrm{ex}(n, TP_3) \leq \frac{n^2}{4} + (1+\delta)n$.

Given a δ, we define the following functions of δ: The $n_0(\delta)$ in Theorem 2.2.3 is coming from the Theorem 2.2.2. Let $\beta(\delta) \leq \min(\frac{1}{6} + \frac{2\delta}{3}, \frac{1}{8}, \frac{\delta}{9296}, \frac{n-2}{4n})$. On the other hand, $\gamma(\delta)$ satisfies the inequalities $\gamma < \min\{\frac{\delta(1-4\beta)}{7968}, \frac{\beta^2}{3\delta}, \frac{1}{24}, \frac{\beta^2}{24}\}$ and $4\beta\gamma < (\beta^2 - 16\gamma)(1-16\gamma)$. For brevity of the paper, we do not calculate these functions precisely.

We start by proving the following weaker version of Theorem 2.2.3:

Lemma 2.2.4. *Let G be a TP_3-free graph on n $(n \geq 10)$ vertices. Then $e(G) \leq \frac{n^2}{4} + \frac{7}{2}n$.*

Proof. Suppose $e(G) > \frac{n^2}{4} + \frac{7}{2}n$. The maximum number of edges in a 7-wheel free graph on n vertices is $\mathrm{ex}(n, W_7) = \lfloor \frac{n^2}{4} + \frac{n}{2} + 1 \rfloor$ [31], which is less than $\frac{n^2}{4} + \frac{7}{2}n$. So, we may assume that G contains a 7-wheel. We claim that each edge in G is contained in at least 8 triangles. Suppose not, i.e., there is an edge $xy \in E(G)$ such that $|N^*(x,y)| \leq 7$. In this case, the number of edges that are incident to either x or y is at most $n+6$. By the induction hypothesis,

$$e(G) \leq e(G - \{x,y\}) + (n+6) \leq \frac{(n-2)^2}{4} + \frac{7}{2}(n-2) + (n+6) = \frac{n^2}{4} + \frac{7}{2}n.$$

One can check that the statement also holds for small n.

Now consider a 7-wheel in G, containing the 6-cycle $x_1x_2x_3x_4x_5x_6x_1$ and the center vertex y. For any edge x_ix_j in the 6-cycle, it can be easily seen that there are at least 3 vertices in $V(G) \backslash \{x_1, x_2, \ldots, x_6, y\}$ which are adjacent to both x_i and x_j. Therefore, by the Pigeonhole principle, we can find three distinct vertices, say y_1, y_2 and y_3 which are in $N^*(x_1,x_2), N^*(x_3,x_4)$, and $N^*(x_5,x_6)$, respectively. This is a contradiction, as G does not contain a TP_3. □

Lemma 2.2.5. *Let $\delta > 0$ be given. Let G be a graph on n ($n \geq \frac{5n_0(\gamma)}{2\delta}$) vertices with $e(G) > \frac{n^2}{4} + (1+\delta)n$. Then either G contains a TP_3 as a subgraph or G contains a subgraph G_0 on n_0' vertices such that $e(G_0) > \frac{(n_0')^2}{4} + (1+\delta)n_0'$ with $d_{G_0}(x) > \left\lfloor \frac{n_0'}{2} + 1 \right\rfloor$, for all $x \in V(G_0)$ and any two adjacent vertices are incident to at least $n_0' + 2$ common vertices (so each edge is contained in at least three triangles).*

Proof. Let H be a subgraph of G. We call H a good subgraph if $e(H) > \frac{v(H)^2}{4} + (1+\delta)v(H)$. Additionally, for all $x \in V(H)$

$$d_{G_0} > \left\lfloor \frac{v(H)}{2} + 1 \right\rfloor, \tag{2.1}$$

and any two adjacent vertices in H are incident to at least $v(H)+2$ edges.

If every vertex in G satisfies the property (2.1) and any two adjacent vertices in G are incident to at least $v(G)+2$ edges (i.e., G itself is a good subgraph), then the lemma holds. Otherwise, we delete the vertex in G if it doesn't satisfy the degree condition in (2.1). If a vertex along with one of its neighbors has fewer than $v(G)+2$ edges incident to it, then we delete the vertex with the smaller degree. We repeat this step, say m times, till we get a subgraph H, satisfying the property (2.1) and any two adjacent vertices are incident to at least $v(H)+2$ edges.

We claim the following:

Claim 1. $e(H) \geq \frac{(n-m)^2}{4} + (1+\delta)(n-m) + \delta m.$

Proof. Suppose not. Let $e(H) < \frac{(n-m)^2}{4} + (1+\delta)(n-m) + \delta m$. We distinguish the following four cases based on the parity of n and m to complete the proof:

Case 1: n is odd. The sequence of the number of edges deleted in m steps from G, when m is even and m is odd, is as follows:

$$\left(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{n-m+3}{2}, \frac{n-m+3}{2}\right)$$

and

$$\left(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{n-m+4}{2}, \frac{n-m+4}{2}, \frac{n-m+2}{2}\right).$$

When m is even, the number of edges deleted after m steps is at least $\frac{m}{4}(2n-m+4)$. Hence,

$$\begin{aligned} e(G) \leq E(H) + \frac{m}{4}(2n-m+4) &< \left(\frac{(n-m)^2}{4} + (1+\delta)(n-m) + \delta m\right) + \frac{m}{4}(2n-m+4) \\ &= \frac{n^2}{4} + (1+\delta)n, \end{aligned}$$

which is a contradiction. Similarly, when m is odd, the number of edges deleted after m steps is at least $\frac{(m-1)}{4}(2n-m+5) + \frac{n-m+2}{2} = \frac{mn}{2} - \frac{m^2}{4} + m - \frac{1}{4}$. Hence,

$$\begin{aligned} e(G) \leq E(H) - \frac{m^2}{4} + \frac{mn}{2} + m - \frac{1}{4} &< \left(\frac{(n-m)^2}{4} + (1+\delta)(n-m) + \delta m\right) - \frac{m^2}{4} + \frac{mn}{2} + m - \frac{1}{4} \\ &= \frac{n^2}{4} + (1+\delta)n - \frac{1}{4}, \end{aligned}$$

which is again a contradiction.

Case 2: n is even. The sequence of the number of edges deleted in m steps from G, when m is odd and m is even, is as follows:

$$\left(\frac{n+2}{2}, \frac{n}{2}, \frac{n}{2}, \ldots, \frac{n-m+3}{2}, \frac{n-m+3}{2}\right)$$

and

$$\left(\frac{n+2}{2}, \frac{n}{2}, \frac{n}{2}, \ldots, \frac{n-m+4}{2}, \frac{n-m+4}{2}, \frac{n-m+2}{2}\right).$$

When m is odd, the number of edges deleted after m steps is at least $\frac{m-1}{4}(2n-m+3) + \frac{n+2}{2} = -\frac{m^2}{4} + \frac{mn}{2} + m + \frac{1}{4}$. Hence,

$$e(G) \leq E(H) - \frac{m^2}{4} + \frac{mn}{2} + m + \frac{1}{4} < \left(\frac{(n-m)^2}{4} + (1+\delta)(n-m) + \delta m\right) - \frac{m^2}{4} + \frac{mn}{2} + m + \frac{1}{4}$$
$$= \frac{n^2}{4} + (1+\delta)n + \frac{1}{4}.$$

Clearly, $e(G) \leq \frac{n^2}{4}+(1+\delta)n$. Otherwise, we get an integer between $\frac{n^2}{4}+(1+\delta)n$ and $\frac{n^2}{4}+(1+\delta)n+\frac{1}{4}$, which is not true. This contradicts the fact that $e(G) > \frac{n^2}{4} + (1+\delta)n$.

Similarly, when m is even, the number of edges deleted after m steps is at least $\frac{m-2}{4}(2n-m+4) + \frac{n+2}{2} + \frac{n-m+2}{2} = \frac{mn}{2} - \frac{m^2}{4} + m$. Hence,

$$e(G) \leq E(H) - \frac{m^2}{4} + \frac{mn}{2} + m < \left(\frac{(n-m)^2}{4} + (1+\delta)(n-m) + \delta m\right) - \frac{m^2}{4} + \frac{mn}{2} + m$$
$$= \frac{n^2}{4} + (1+\delta)n,$$

which is again a contradiction. □

If H contains a TP_3 as a subgraph, we are immediately done. Consider H is TP_3-free. By Lemma 2.2.4, $e(H) \leq \frac{(n-m)^2}{4} + \frac{7}{2}(n-m)$. Thus,

$$\frac{(n-m)^2}{4} + (1+\delta)(n-m) + \delta m \leq \frac{(n-m)^2}{4} + \frac{7}{2}(n-m).$$

Hence, $m \leq \frac{2.5-\delta}{2.5}n$. In other words $n-m \geq \frac{2\delta n}{5}$. The condition, $n \geq \frac{5n_0(\gamma)}{2\delta}$ implies $n-m \geq n_0(\gamma)$. Thus, we found the good subgraph H of G.

□

Definition 2.2.6. *Let a* 7*-wheel in* H *be with center* y *and the* 6*-cycle* $x_1x_2x_3x_4x_5x_1$*. We call the* 7*-wheel, a* ***sparse* 7*-wheel*,** *if* $x_ix_{i+2} \notin E(G)$ *for all* $i \in \{1, 2 \ldots, 6\}$ *(see Figure 2.5).*

We provide a sketch of the proof before going into the details. Suppose $e(G) \geq \frac{n^2}{4} + (1+\delta)n$, then one of the bipartitions must have more than $\frac{n}{2} + \frac{\delta n}{2}$ edges. We prove the following lemmas as stepping stones. Given a graph G on n vertices with at least $\frac{n^2}{4} + (1+\delta)n$ edges and containing a sparse 7-wheel as a subgraph, Lemma 2.2.7 proves that G contains a TP_3 as a subgraph. We take the biggest balanced bipartite splitting of G and prove in Lemma 2.2.9 that if one of the color

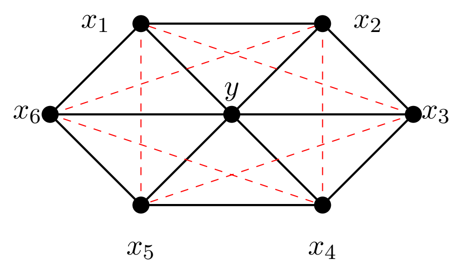


Figure 2.5: A sparse 7-wheel, the dashed red edges are not in G.

classes contains a vertex with large enough degree, then G contains a TP_3 as a subgraph. Lemma 2.2.12 shows that if one of the color classes contains more than $\frac{n}{2} + \frac{\delta n}{2}$ edges, then G contains a TP_3 as a subgraph.

Lemma 2.2.7. *Let G be a graph on n vertices, where $n \geq \frac{5n_0(\gamma)}{2\delta}$, and $e(G) > \frac{n^2}{4} + (1+\delta)n$. If G contains a sparse 7-wheel, then G contains a TP_3 as a subgraph.*

Proof. By Lemma 2.2.5, G contains a good subgraph H. In other words, for all $x \in V(H)$ we have

$$d(x) > \begin{cases} \frac{v(H)}{2} + 1, & 2 \mid v(H), \\ \frac{v(H)+1}{2}, & 2 \nmid v(H), \end{cases} \tag{2.2}$$

and any two adjacent vertices in H are incident to at least $v(H)+2$ edges (and so every edge is contained in at least three triangles).

Let a sparse 7-wheel in H be with center y and the 6-cycle $x_1x_2x_3x_4x_5x_6x_1$, as shown in Figure 2.5. Since H is a good subgraph, for each x_ix_{i+1}, $i \in \{1, 2, \ldots, 6\}$, $|N(x_i, x_{i+1})| \geq 3$. Moreover, for each x_ix_{i+1}, $i \in \{1, 2, \ldots, 6\}$, all the remaining four vertices of the cycle are not in $N(x_i, x_{i+1})$. Indeed, without loss of generality, consider the edge x_1x_2. The vertices x_3 and x_4 are not in $N(x_1, x_2)$, since the wheel is sparse. We can prove that the vertices x_6 and x_5 are not in $N(x_1, x_2)$ by a similar argument. Therefore, there exist at least two vertices in $V(H)\backslash\{x_1, x_2, \ldots, x_6, y\}$, which are in $N(x_i, x_{i+1})$. Take the matching x_1x_2, x_3x_4 and x_5x_6. If there are three distinct vertices in $V(H)\backslash\{x_1, x_2, \ldots, x_6, y\}$, which are in $N(x_1, x_2) \cup N(x_3, x_4) \cup N(x_5, x_6)$, then there is a TP_3 in H. Suppose not. Let z_1, z_2 and z_3 be the vertices in $V(H)\backslash\{x_1, x_2, \ldots, x_6, y\}$ such that $\{z_1, z_2, z_3\} \subset N(x_1, x_2) \cup N(x_3, x_4) \cup N(x_5, x_6)$. Since H is TP_3-free and $|N(x_1, x_2)|$, $N|(x_3, x_4)|$ and $|N(x_5, x_6)|$ are at least 3, it follows that each of the sets $N(x_1, x_2)$, $N(x_3, x_4)$ and $N(x_5, x_6)$ must contain at least two of the vertices in $\{z_1, z_2, z_3\}$. By Hall's Theorem, we get a distinct pairing of z_1, z_2, z_3 and $N(x_1, x_2), N(x_3, x_4)$ and $N(x_5, x_6)$ such that $z_i \in N(x_j, x_k)$, $i \in \{1, 2, 3\}$ and $(j, k) \in \{(1, 2), (3, 4), (5, 6)\}$, which is a contradiction to the fact that H does not contain a TP_3. Now we may assume that there are only two distinct vertices, say v_1 and v_2 in $V(H)\backslash\{x_1, x_2, \ldots, x_6, y\}$, such that $N(x_1, x_2, \ldots, x_6) = \{v, v_1, v_2\}$(see Figure 2.6).

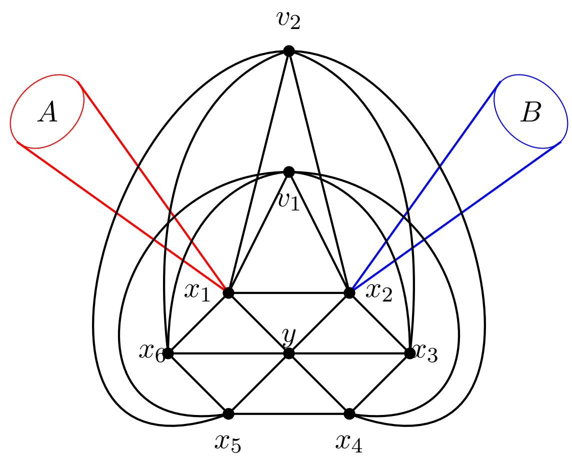


Figure 2.6: Structure of the subgraph of G with 2 common neighbors for each vertex on the cycle of the good wheel.

Consider the case when n is odd. Similarly, one can also solve when n is even. Let A and B be the set of vertices in $V(H)\backslash\{x_1, \ldots, x_6, y, v_1, v_2\}$ which are adjacent to x_1 and x_2, respectively (see Figure 2.6). Obviously, $A \cap B = \emptyset$. Otherwise, the graph contains a TP_3 as a subgraph. Thus, either $|A| \leq \frac{n-9}{2}$ or $|B| \leq \frac{n-9}{2}$. Without loss of generality, suppose $|A| \leq \frac{n-9}{2}$. If $|A| \leq \frac{n-11}{2}$, then $d(x_1) \leq |A| + 6 = \frac{n-11}{2} + 6 = \frac{n+1}{2}$, which is a contradiction.

So, assume $|A| = \frac{n-9}{2}$. In this case, we also have that $|B| = \frac{n-9}{2}$. We need the following claim to complete the proof of the lemma:

Claim 2. *Each vertex in A is adjacent to at least one other vertex in A.*

Proof. Suppose not. Let x be a vertex in A which is adjacent to no other vertex in A. The vertex x is not adjacent to x_2 and x_6, otherwise, H contains a TP_3 as a subgraph.

If x is adjacent to x_4, then x is not adjacent to both x_3 and x_5. Otherwise, the graph contains a TP_3 as a subgraph. In this case, the vertex x is possibly adjacent to y, v_1, v_2, and vertices in B. Thus, considering the vertex x_1 which is already adjacent to x, we get $d(x) \leq \frac{n-9}{2} + 5 = \frac{n+1}{2}$. This is a contradiction since H is a good subgraph.

Let x be adjacent to x_3. Then x cannot be adjacent to x_4. If x_5 is not adjacent to x, then $d(x) \leq \frac{n-9}{2} + 5 = \frac{n+1}{2}$, which is a contradiction. So, let x_5 be adjacent to x. If x is not adjacent to one of the vertices in $\{y, v_1, v_2\}$, then $d(x) \leq \frac{n-9}{2} + 5 = \frac{n+1}{2}$, which is a contradiction. Otherwise, consider the 7-wheel, with the 6-cycle $x_5yx_3v_1x_1v_2x_5$ (see the bold green cycle in Figure 2.7) and center x. Consider the matching x_5y, x_3v_1 and x_1v_2. Consider the vertices x_4, x_2, x_6, which are the common neighbors of the end vertices of the matching. These vertices along with the given 7-wheel form a TP_3, a contradiction. $\square$

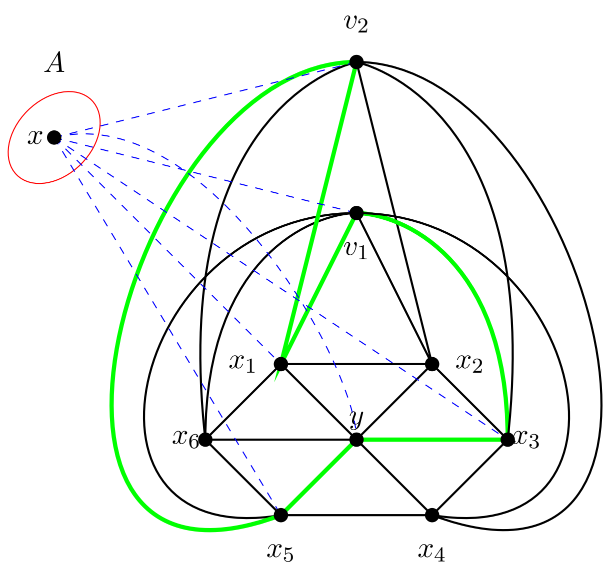


Figure 2.7: A graph containing a TP_3.

With the same argument, one can verify that the minimum degree of each vertex in B is at least 1. Now we finish the proof of Case 1. Consider the edge x_5x_6. Let A' and B' be the set of vertices in $V(H)\backslash\{x_1, \ldots, x_6, y, v_1, v_2\}$ which are adjacent to x_5 and x_6, respectively. For the same reason given above, $|A'| = |B'| = \frac{n-9}{2}$. Clearly $A' \cap B' = \emptyset$. Since $A \cap B' = \emptyset$ and $A' \cap B' = \emptyset$, i.e., $B' = B$ and $A' = A$. Thus, $|B' \cap B| = |A \cap A'| = \frac{n-9}{2}$.

Let $x \in A \cap A'$. We claim that x is not adjacent to y. Suppose on the contrary that x is adjacent to y. We can take the 7-wheel, with the 6-cycle $xx_1x_2x_3x_4x_5x$ and center y. By Claim 2, there is a vertex z in A which is adjacent to x. Since this vertex is adjacent to x_1, then taking the matching xx_1, x_2x_3 and x_4x_5 with common neighbors z, v_1 and v_2, respectively, we show that the graph contains a TP_3 as a subgraph. Hence, a contradiction.

Let $t \in B \cap B'$. We claim that t cannot be adjacent to y. Suppose on the contrary that t is adjacent to y. Consider the 7-wheel, with 6-cycle $tx_2x_3x_4x_5x_6t$ and center y. By Claim 2, t is adjacent to a vertex r in B. Since this vertex is adjacent to x_2, taking the matching tx_2, x_3x_4 and x_5x_6 with common neighbors r, v_1 and v_2, respectively, we show that H contains a TP_3 as a subgraph. Hence, a contradiction. Thus, we found that y is a vertex in H with constant degree, which is a contradiction to the fact that H is a good subgraph. $\square$

Remark 2.2.8. *For the rest of the write-up, we always work on this "good" subgraph, and to simplify notations, we denote it by G.*

Lemma 2.2.9. *Let G be a graph on n vertices, where $n \geq \frac{5n_0(\gamma)}{2\delta}$, and $e(G) \geq \frac{n^2}{4} + (1+\delta)n$. Let A and B be a partition of $V(G)$ with size as equal as possible and with $e(A, B)$ maximal. If A contains (or B contains) a vertex, say x, such that $d_A(x) \geq \beta n$ (or $d_B(x) \geq \beta n$), then G contains a TP_3 as a subgraph.*

Proof. Without loss of generality, suppose there exists a vertex $x \in A$ such that $d_A(x) \geq \beta n$. Note that, $e(G) > \frac{n^2}{4} - \epsilon n^2$, for any $\epsilon > 0$. Thus, by the stability theorem, $|E(G)\Delta E(T_{n,2})| \leq \gamma n^2$.

Let A_x be the graph induced by the vertices $N_A(x) \cup \{x\}$ in A. We have $e(A_x) \leq \gamma n^2$. The average degree of a vertex in A_x is

$$\bar{d}(A_x) \leq \frac{\sum\limits_{y \in V(A_x)} d_{A_x}(y)}{v(A_x)} \leq \frac{2e(A_x)}{v(A_x)} \leq \frac{2\gamma n^2}{\beta n} = \frac{2\gamma n}{\beta}.$$

Let $X = \{x \in V(A_x) \mid d_{A_x}(x) \geq \frac{4\gamma n}{\beta}\}$. The size of X is at most $\frac{2\gamma n^2}{\frac{4\gamma n}{\beta}} = \frac{\beta n}{2}$. Let $Y = V(A_x) - X$. Thus, $|Y| \geq \frac{\beta n}{2}$ and for each $y \in Y$, $d_{A_x}(y) < \frac{4\gamma n}{\beta}$. We color $G[Y]$ with $\frac{4\gamma n}{\beta}$ colors. The average size of the color class in $G[Y]$ is at least $\frac{\beta n/2}{4\gamma n/\beta} = \frac{\beta^2}{8\gamma} \geq 3$. Barring at most 2 vertices, each of these color classes can be divided into triples. Each of these triples form an induced $K_{1,3}$ with the vertex x as

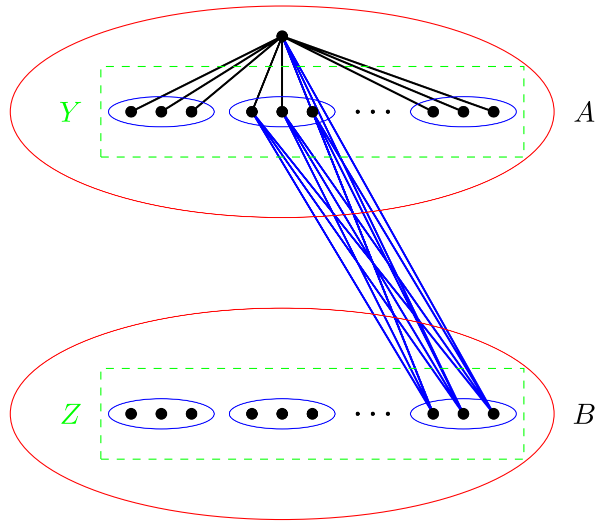


Figure 2.8: A sparse 7-wheel.

the center. Thus, we obtained at least $\frac{1}{3}\left(\frac{\beta n}{2} - \frac{2\cdot 4\gamma n}{\beta}\right)$ induced $K_{1,3}$'s in A_x(see Figure 2.8).

The graph induced by B, denoted by $G[B]$, also contains at most γn^2 edges. Thus, $\bar{d}(G_B) \leq 2\gamma n$. Delete vertices in B whose degree is at least $4\gamma n$. By similar reasoning as above, the number of vertices deleted is at most $\frac{n}{2}$. Let Z be the set of remaining vertices in B. We color $G[Z]$ with $4\gamma n$ colors. The average size of the color class in $G[Z]$ is at least $\frac{n/2}{4\gamma n} = \frac{1}{8\gamma} \geq 3$. Barring at most 2 vertices, each of these color classes can be divided into triples. This implies that we can find at least $\frac{1}{3}\left(\frac{n}{2} - 2\cdot 4\gamma n\right) = \frac{n}{3}\left(\frac{1}{2} - 8\gamma\right)$ induced triples in G_B (see Figure 2.8.)

If for each pair of induced $K_{1,3}$'s and induced triples obtained in A and B, respectively, there is a missing edge, then the number of missing edges is at least $\frac{n}{3}\left(\frac{\beta}{2} - \frac{8\gamma}{\beta}\right)\cdot\frac{n}{3}\left(\frac{1}{2} - 8\gamma\right)$. The following holds from the definition of β and γ:

$$\frac{n}{3}\left(\frac{\beta}{2} - \frac{8\gamma}{\beta}\right)\cdot\frac{n}{3}\left(\frac{1}{2} - 8\gamma\right) > \gamma n^2. \tag{2.3}$$

Thus, the number of missing edges is greater than γn^2, a contradiction. Hence, there must be an induced $K_{1,3}$ in A, which is joined completely to an induced triple of vertices in B. Therefore, we get a sparse 7-wheel and, by Lemma 2.2.7, G contains a TP_3 as a subgraph. $\square$

Definition 2.2.10. *The tree given in Figure 2.9 with* 3 *legs and one joint, is denoted as the spider graph.*

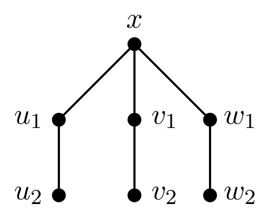


Figure 2.9: A spider graph with three legs and one joint.

Corollary 2.2.11. *Let G be a graph on n vertices, where $n \geq \frac{5n_0(\gamma)}{2\delta}$, and $e(G) > \frac{n^2}{4} + (1+\delta)n$. Let A and B be a partition of $V(G)$ with size as equal as possible and with $e(A, B)$ maximal. If A or B has a spider graph as a subgraph, then G contains a TP_3 as a subgraph.*

Proof. Let S denote the spider graph. Without loss of generality, suppose $S \subseteq G[A]$.

We consider the 4-vertex subsets of S, namely $\{x, u_1, u_2, v_1\}, \{x, v_1, v_2, w_1\}$ and $\{x, w_1, w_2, u_1\}$. Suppose we can find three vertices in B', namely y_1, y_2 and y_3, such that y_1 is connected to all the vertices in $\{x, u_1, u_2, v_1\}$, y_2 is connected to all the vertices in $\{x, v_1, v_2, w_1\}$ and y_3 is connected to all the vertices in $\{x, w_1, w_2, u_1\}$. We immediately find a TP_3. Thus, for every vertex $y \in B$ except for at most 2 vertices, y is not adjacent to at least one of the vertices in $\{x, u_1, u_2, v_1\}$. The average degree of vertices in $\{x, u_1, u_2, v_1\}$ to B is less than $\frac{3n}{8}$. So there exists a vertex $z \in \{x, u_1, u_2, v_1\}$, such that $d_B(z) < \frac{3n}{8}$. The minimum degree of the vertices in G is at least $\frac{n}{2}$, thus $d_A(z) > \frac{n}{8}$. Since $\beta \leq \frac{1}{8}$, we are done by Lemma 2.2.9. $\square$

We want to prove $\mathrm{ex}(n, TP_3) \leq \frac{1}{4}n^2 + (1+\delta)n$. Assume that there is a TP_3-free graph that has more than $\frac{1}{4}n^2 + (1+\delta)n$ edges. Then one of the bipartitions must have more than $\frac{n}{2} + \frac{\delta n}{2}$ edges. We prove in Lemma 2.2.12 that this is not possible.

Lemma 2.2.12. *Let G be a graph on n vertices, where $n \geq \frac{5n_0(\gamma)}{2\delta}$. Let A and B be a partition of $V(G)$ with size as equal as possible and with $e(A, B)$ maximal. Assume that neither A nor B contains a spider graph as a subgraph and the maximum degree of vertices inside each class is less than βn. If $e(A) \geq \frac{n}{2} + \frac{\delta n}{2}$, then G contains a TP_3 as a subgraph.*

Proof. We start by claiming the following:

Claim 3. *Given a graph G_k on k vertices, with $2k$ edges. We can find an independent set of vertices with size $\frac{3k}{55}$.*

Proof. Say we delete the vertices with degree greater than 10. Denote the remaining graph with G'_k. The number of vertices deleted is denoted by l. The number of edges deleted is at least $5l$. Since the number of edges in G_k is $2k$, $l \leq \frac{2k}{5}$. The number of vertices in G'_k is at least $\frac{3k}{5}$ and every vertex has degree at most 10. Start by choosing an arbitrary vertex $x \in G'_k$, delete its neighbors, and continue choosing another vertex in the graph $G'_k \setminus N(x)$. In each step, we delete at most 11 vertices. Thus, we can get an independent set of size $\frac{3k}{55}$ with this recursive procedure. □

We use this to prove the following claim:

Claim 4. *Let G be a graph on n vertices, where $n \geq \frac{5n_0(\gamma)}{2\delta}$. Let A and B be a partition of $V(G)$ with size as equal as possible and with $e(A, B)$ maximal. Assume that neither A nor B contains a spider graph as a subgraph and the maximum degree of vertices inside each class is less than βn. If $e(A) \geq \frac{n}{2} + \frac{\delta n}{2}$, then one of the following is true:*

1. *There are at least $\frac{\delta n}{664}$ vertex disjoint $K_{1,3}$'s in A.*
2. *There are stars of size at least 85 in A, such that partitioning each of their leaves into triples (1 or 2 vertices may be missed from each star), we have at least $\frac{\delta n}{664}$ triples forming an induced $K_{1,3}$ along with the center of the respective star.*

Proof. The degree sum of vertices in A is greater than or equal to $2(\frac{n}{2} + \delta\frac{n}{2})$. Since A has size at least $\lfloor\frac{n}{2}\rfloor$, we have vertices with degree more than 2.

Let v be a vertex in A such that $d_A(v) = \Delta_A$. Let A_v be the graph induced by the vertices $\{v\} \cup N_A(v)$. Note, A_v doesn't contain the spider graph as a subgraph. We consider the following cases:

Case 1: $\Delta \leq 83$. Let x_1, x_2 and x_3 be the vertices in $N_A(v)$. The vertices v, x_1, x_2, and x_3 form a $K_{1,3}$. Delete the set of vertices $\{v\} \cup N_A(v)$. We delete at most 332 edges in this step and since $e(A) \geq \delta\frac{n}{2}$, we can continue repeating this process by taking another vertex v', with $d_A(v')$ equal to the new maximum degree. Thus, the number of $K_{1,3}$'s we can find is at least $\frac{\delta n}{664}$.

Case 2: $\Delta \geq 84$. Denote the vertices in $N_A(v)$ with x_i. Note that we do not have 3 independent edges going out of $N_A(v)$ from x_i's, as we have a spider-free graph. Let x_1, x_2, and x_3 be the vertices with degree greater than 2. By Hall's Theorem, we immediately get 3 independent edges going from the set $N_A(v)$ to $A \setminus N_A(v)$. Thus, we have at most 2 vertices in the set $\{x_i\}$, who have degree greater than 2. Thus, the number of edges incident to $N_A(v)$ is at most $2(\Delta-1)+2(\Delta-2)+\Delta \leq 5\Delta$.

By the previous Claim 3, in the graph induced by the set of vertices x_i with 2Δ edges, we can find an independent set of size at least $\frac{3\Delta}{55}$. Hence, we can find at least $\frac{\Delta}{55}$ triples such that they form an induced $K_{1,3}$ with v being the center. Delete the set of vertices $\{v\} \cup N_A(v)$. We delete at most 5Δ edges in this step and since $e(A) \geq \delta \frac{n}{2}$, we can continue repeating this process by taking another vertex v', with $d_A(v')$ equal to the new maximum degree. Thus, the number of induced $K_{1,3}$s we can find is at least $\frac{\delta n}{550}$.

□

There are two possibilities to consider:

Case 1: Half of the triples in A lie in disjoint $K_{1,3}$'s. Consider a vertex $x \in B$. We know that the maximum degree of x inside B is less than βn. Thus, x has at most βn non-neighbors in A. There are at least $\frac{\delta n}{1328} - \beta n$ triples in disjoint $K_{1,3}$'s, such that all four of the vertices in the $K_{1,3}$ are adjacent to x. Recall that every vertex in G has degree strictly greater than $\lfloor \frac{n}{2} + 1 \rfloor$. Since the partitions A and B are with size as equal as possible and with $e(A, B)$ maximal, every vertex in B has degree at least 1 in B. The maximum degree of vertices inside each class is less than βn and $\frac{n}{2} \geq 2\beta n + 1$, thus we can find at least 3 independent edges in B. Let $y_1 z_1, y_2 z_2$ and $y_3 z_3$ denote any three independent edges in B. For each of these 6 vertices, we can find at least $\frac{\delta n}{1328} - \beta n$ triples in disjoint $K_{1,3}$'s, such that the vertices of the $K_{1,3}$ are joined completely to the given vertex. By the definition of β, we have:

$$\frac{\delta n}{1328} - \beta n \geq \frac{6}{7} \cdot \frac{\delta n}{1328}. \tag{2.4}$$

Thus, each of the vertices y_i (similarly z_i), for $i \in \{1, 2, 3\}$, is completely connected to all the vertices of at least $\frac{6}{7}$ triples of disjoint $K_{1,3}$'s in A. By the Pigeonhole principle, we have a common triple, such that these 3 independent edges are connected to it completely. Denote the vertices of this triple as x_1, x_2 and x_3. The vertices x_1, y_1, x_2, y_2, x_3, and y_3 along with the vertex x as the center form a 7-wheel. The triangles $x_1y_1z_1, x_2y_2z_2$, and $x_3y_3z_3$ along with the 7-wheel, form a TP_3.

Case 2: Half of the triples in A lie in induced $K_{1,3}$s with the center vertex having degree at least 84. Let $x \in A$, such that $d_A(x) \geq 84$. The maximum degree of x in A is at most βn. Hence, x can have at most βn non-neighbors in B. Delete these vertices in B and denote the graph remaining by B'. We know that $\Delta(B') \leq \beta n$. Hence, we can color it with βn colors. Barring at most 2 vertices, each of these color classes can be divided into triples. We can choose $\frac{\frac{n}{2}-2\beta n}{3}$ independent triples in B'. Each of these triples must have a missing edge to the root vertices in the $K_{1,3}$ chosen in A. Otherwise, we have a sparse 7-wheel. By Lemma 2.2.7, G contains a TP_3 as a subgraph, a contradiction. Thus, the total number of missing edges is at least $\frac{\delta n}{1328} \cdot \frac{\frac{n}{2}-2\beta n}{3}$. However, $\frac{\delta n}{1328} \cdot \frac{\frac{n}{2}-2\beta n}{3} > \gamma n^2$. Thus, the number of missing edges is greater than γn^2, which is a contradiction. Hence, we can find a sparse 7-wheel in G. By Lemma 2.2.7, G contains a TP_3 as a subgraph, a contradiction. □

2.3 Concluding remarks and Conjectures

Following the two constructions given in Figure 2.2 and Figure 2.3, we pose the following conjecture concerning $\text{ex}(n, TP_3)$:

Conjecture 2.3.1.

$$ex(n, TP_3) \leq \begin{cases} \frac{1}{4}n^2 + n + 1, & \textit{if } n \textit{ is even,} \\ \frac{1}{4}n^2 + n + \frac{3}{4}, & \textit{otherwise.} \end{cases}$$

We also pose the following conjecture related to $\text{ex}(n, TP_4)$.

Conjecture 2.3.2. *For n sufficiently large, $ex(n, TP_4) = \frac{n^2}{4} + \Theta(n^{4/3})$.*

To show the lower bound, we consider an n-vertex graph G obtained from a complete bipartite graph with color classes as equal as possible and adding a bipartite C_6-free graph with $cn^{4/3}$ edges in one of the color classes. Thus, $e(G) \geq \frac{n^2}{4} + O(n^{4/3})$. We need the following claim to show that G does not contain a TP_4:

Claim 5. *Every 2-coloring of the TP_4 such that color 1 is independent, contains either a C_3 or a C_6 in color 2.*

Proof. Consider a 2-coloring c of a TP_4 such that color 1 is independent. We want to show that there is either a C_3 or a C_6 in color 2. Suppose there is no such C_3. Then one of the vertices of the triangle $x_1x_2x_3$ (see Figure 2.10) is in color 1. Without loss of generality, let the color of x_1 be 1. Since c is a 2-coloring with the property that color 1 is independent, then all the 6 neighboring vertices of x_1 must be of color 2. Therefore, we obtain a C_6 with color 2 and this completes the proof. □

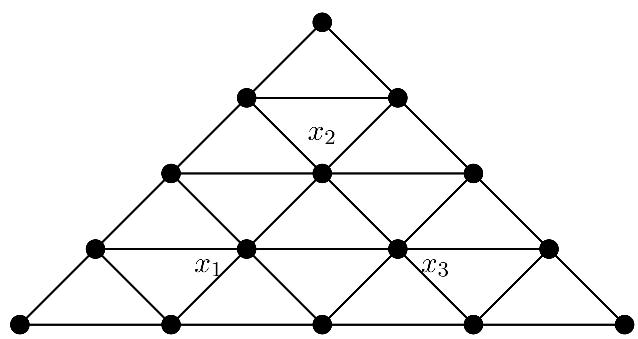


Figure 2.10: A Triangular Pyramid with 4 layers.

The following lemma is a consequence of Claim 5:

Lemma 2.3.1. *If G is a graph obtained from a complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ (with color class 1 and 2) by adding a bipartite, C_6-free graph to the color class 2, then G is a TP_4-free graph.*

Hence, the lower bound of Conjecture 2.3.2 holds.

Chapter 3

Planar Turán Number of the 6-Cycle

3.1 Introduction and Main Results

All the graphs we consider in this chapter are simple and finite. Let G be a graph. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Let $e(G)$ and $v(G)$ denote the number of edges and vertices, respectively. The minimum degree of G is denoted by $\delta(G)$, whereas the maximum degree of G is denoted by $\Delta(G)$. For a vertex v in G, the neighborhood of v, denoted by $N_G(v)$, is the set of all vertices in G which are adjacent to v. We denote the degree of v by $d_G(v) = |N_G(v)|$. We may avoid the subscripts if the underlying graph is clear. The number of components of G is denoted by $c(G)$. For the sake of simplicity, we may use the term k-cycle to mean a cycle of length k and k-face to mean a face bounded by a k-cycle. A k-path is a path with k edges. The *Turán number* of a graph H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex graph that does not contain H as a subgraph. Let $\mathrm{EX}(n, H)$ denote the set of extremal graphs, i.e., the set of all n-vertex, H-free graph G such that $e(G) = \mathrm{ex}(n, H)$.

In 2016, Dowden [28] initiated the study of Turán-type problems when host graphs are planar. The planar Turán number of a graph H, $\mathrm{ex}_{\mathcal{P}}(n, H)$, is the maximum number of edges in a planar graph on n vertices which does not contain H as a subgraph. Dowden obtained the tight bounds $\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq \frac{15(n-2)}{7}$, for all $n \geq 4$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$, for all $n \geq 11$. Later, Lan, Shi and Song [95] obtained the following bounds:

Theorem 3.1.1 (Lan, Shi, Song [95]). *Let θ_k denote the family of* Theta graphs *on $k \geq 4$ vertices,* *that is, graphs obtained from a cycle C_k by adding an edge joining two non-consecutive vertices.*

1. *For all* $n \geq 4$, $\mathrm{ex}_{\mathcal{P}}(n, \theta_4) \leq \frac{12(n-2)}{5}$.
2. *For all* $n \geq 5$, $\mathrm{ex}_{\mathcal{P}}(n, \theta_5) \leq \frac{5(n-2)}{2}$.
3. *For all* $n \geq 7$, $\mathrm{ex}_{\mathcal{P}}(n, \theta_6) \leq \frac{18(n-2)}{7}$.

They also demonstrated that their bounds for Θ_4 and Θ_5 are tight. They presented the following corollary, based on the bound for Θ_6:

Corollary 3.1.2 (Lan, Shi, Song [95])**.**

$$\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{18(n-2)}{7}$$

for all $n \geq 6$*, with equality when* $n = 9$.

In this chapter, we present a tight bound for $\mathrm{ex}_{\mathcal{P}}(n, C_6)$. The main ingredient of the result is as follows:

Theorem 3.1.3. *Let G be a 2-connected, C_6-free plane graph on $n \geq 6$ vertices with $\delta(G) \geq 3$. Then $e(G) \leq \frac{5}{2}n - 7$.*

We use Theorem 3.1.3, which considers only 2-connected graphs with no degree 2 (or 1) vertices and order at least 6, to prove the following result:

Theorem 3.1.4. *Let G be a C_6-free plane graph on $n \geq 18$ vertices. Then*

$$e(G) \leq \frac{5}{2}n - 7.$$

Indeed, there are C_6-free graphs on 17 vertices with 36 edges, see Figure 3.1.

We show that, for large graphs, Theorem 3.1.4 is tight.

Theorem 3.1.5. *For every $n \equiv 2 (mod\ 5)$, there exists a C_6-free plane graph G with $v(G) = \frac{18n+14}{5}$ and $e(G) = 9n$. Thus, $e(G) = \frac{5}{2}v(G) - 7$.*

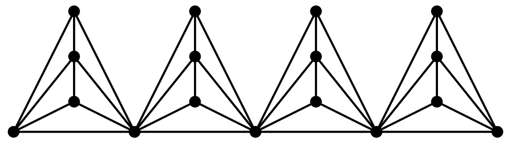


Figure 3.1: A graph G on 17 vertices, such that $e(G) > (5/2)v(G) - 7$.

The chapter is structured as follows: In Section 3.2, we present the construction of the extremal graph. In Section 3.3, we provide the definitions for the triangular blocks. In Section 3.4 and Section 3.5, we present the proofs of Theorem 3.1.3 and Theorem 3.1.4, respectively. In Section 3.6, we present conjectures on generalizing the proof for cycles of higher order. In Section 3.7, the tables give a summary of the results from Lemmas 3.4.2, 3.4.4, and 3.4.6.

3.2 Proof of Theorem 3.1.5: Extremal Graph Construction

Let G_0 be an n ($n \equiv 7 \pmod{10}$)-vertex plane graph, such that every face has length 7 and the degree of every vertex is either 2 or 3. Given such a G_0, we can construct G, where G is a C_6-free plane graph with $v(G) = \frac{18n+14}{5}$ and $e(G) = 9n$. We then give a construction for such a G_0, when $n \equiv 7 \pmod{10}$. In Remark 3.2.1, we summarize the construction for $n \equiv 2 \pmod{10}$ and will not show the details. Using Euler's formula, we have $e(G_0) = \frac{7(n-2)}{5}$ and the number of degree 2 and degree 3 vertices in G_0 is $\frac{n+28}{5}$ and $\frac{4n-28}{5}$, respectively.

Given G_0, we construct an intermediate graph G' by step (1):

(1) Add halving vertices to each edge of G_0 and join the pair of halving vertices with distance 2, see an example in Figure 3.2. Let G' denote this new graph. Thus, $v(G') = v(G_0) + e(G_0) = \frac{12n-14}{5}$ and the set of degree 2 vertices in G' is the set of degree 2 vertices in G_0. The same holds for the degree 3 vertices. Hence, the number of degree 2 and degree 3 vertices in G' equals the number of degree 2 and degree 3 vertices in G_0, respectively.

To obtain G, we apply the following steps (2) and (3) on the degree 2 and degree 3 vertices in G'.

(2) For each degree 2 vertex v in G', let $N(v) = \{v_1, v_2\}$, and so v_1vv_2 forms an induced triangle in G'. Fix v_1 and v_2. Replace v_1vv_2 with a K_5^- by adding the vertices v_1', v_2' to $V(G')$ and

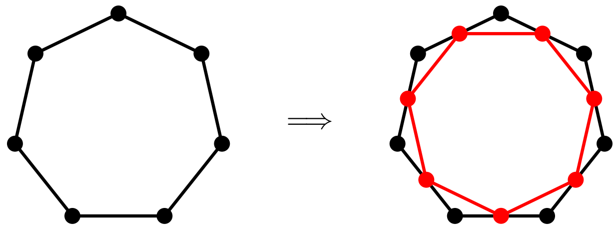


Figure 3.2: Adding a halving vertex to each edge of G_0.

the edges $v_1'v$, $v_1'v_2'$, $v_1'v_1$, $v_1'v_2$, $v_2'v_1$, $v_2'v_2$ to $E(G')$, see Figure 3.3.

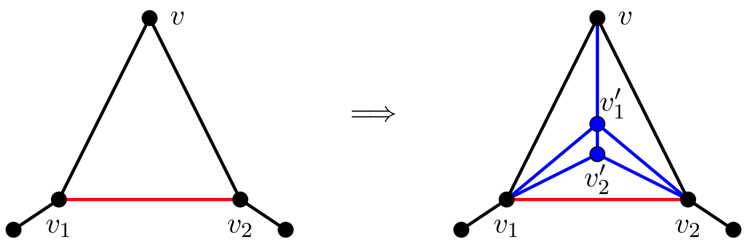


Figure 3.3: Replacing a degree 2 vertex of G_0 with a K_5^-.

(3) For each degree 3 vertex v in G', such that $N(v) = \{v_1, v_2, v_3\}$, the set of vertices $\{v, v_1, v_2, v_3\}$ forms an induced K_4 in G'. Fix v_1, v_2, and v_3. Replace this K_4 with a K_5^- by adding a new vertex v' to $V(G')$ and the edges $v'v$, $v'v_1$, $v'v_2$ to $E(G')$, see Figure 3.4.

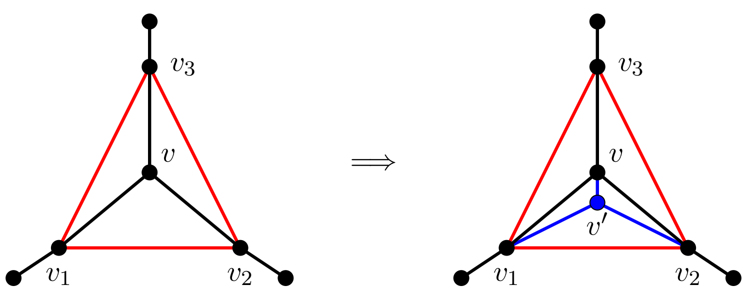


Figure 3.4: Replacing a degree 3 vertex of G_0 with a K_5^-.

We present a construction for such a G_0. For each integer $k \geq 0$ and $n = 10k + 7$, denote it by G_0^k. Let v_i^t and v_i^b $(1 \leq i \leq k+1)$ be a subset of the top and bottom vertices of the heptagonal grids with 3 layers and k columns, respectively (see the red vertices in Figure 3.5). Let v be the extra vertex in G_0^k but not in the heptagonal grid. We join $v_1^t v$, vv_1^b and $v_i^t v_i^b$ $(2 \leq i \leq k+1)$. Clearly, G_0^k

is a plane graph on $(10k+7)$-vertices and each face of G_0^k is a 7-face. Obviously $e\left(G_0^k\right) = 14k+7$, and the number of degree 2 and 3 vertices are $2k+7 = \frac{n+28}{5}$ and $8k = \frac{4n-28}{5}$, respectively.

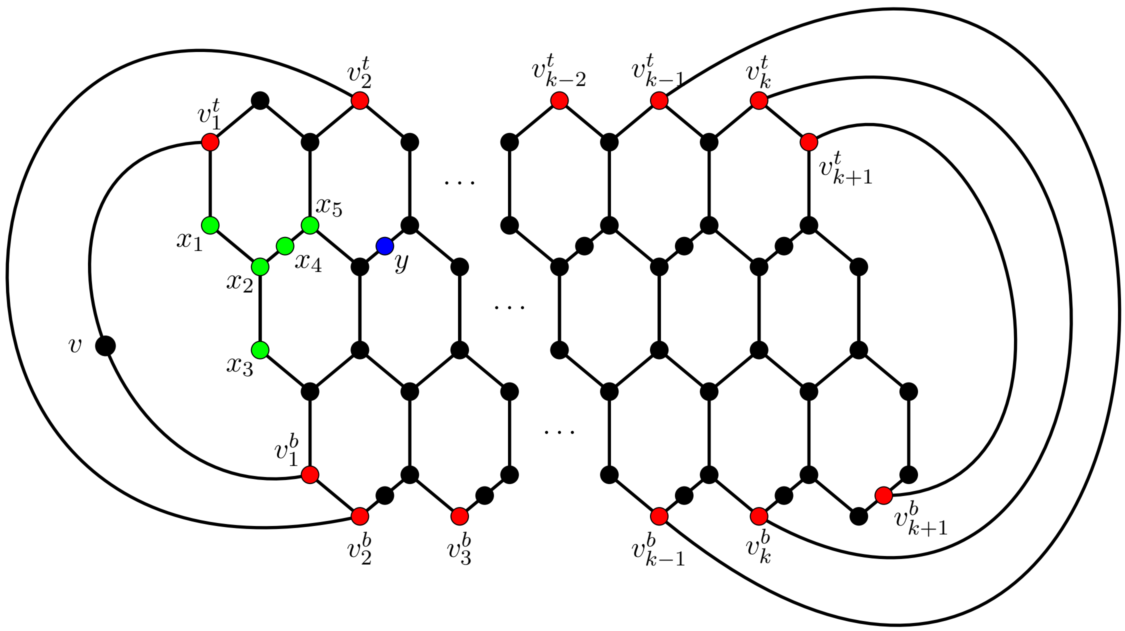


Figure 3.5: The graph G_0^k, $k \geq 1$, in which each face has length 7. The graph H_0^k (see Remark 3.2.1) is obtained by deleting $x_1, \ldots, x_5$ and adding the edge $v_1^t y$.

After applying steps (1), (2), and (3) on G_0^k, we get G. It is easy to verify that G is a C_6-free plane graph with

$$\begin{aligned} v(G) &= v(G_0^k) + e(G_0^k) + 2(2k+7) + 8k = (10k+7) + (14k+7) + 12k + 14 \qquad = 36k+28, \\ e(G) &= 9v(G_0^k) = 90k+63. \end{aligned}$$

Thus, $e(G) = \frac{5}{2}v(G) - 7$.

Remark 3.2.1. *In fact, for $k \geq 1$ and $n = 10k+2$, there exists a graph H_0^k which is obtained from G_0^k by deleting the vertices (colored green in Figure 3.5) x_1, x_2, x_3, x_4, x_5, and adding the edge $v_1^t y$. Clearly, H_0^k is a $10k+2$-vertex plane graph such that all faces have length 7. Moreover, $e(H_0^k) = 14k$, and the number of degree 2 and degree 3 vertices are $2k+6 = \frac{n+28}{5}$ and $8k-4 = \frac{4n-28}{5}$, respectively. After applying steps (1), (2), and (3) to H_0^k, we get a C_6-free plane graph H, with $e(H) = (5/2)v(H) - 7$.*

Thus, for any $n \equiv 2 (mod\ 5)$ ($n \geq 7$), we have graphs on n vertices such that each face is a 7-gon, and we get a C_6-free plane graph on n' vertices with $(5/2)n' - 7$ edges for $n' \equiv 10 (mod\ 18)$ if $n' \geq 28$.

3.3 Definitions and Preliminaries

We give some necessary definitions and preliminary results.

Definition 3.3.1. *Let G be a plane graph and $e \in E(G)$. If e is not in a 3-face of G, then we call it a **trivial triangular block**. Otherwise, we recursively construct a **triangular block** in the following way: Start with H as a subgraph of G, such that $E(H) = \{e\}$.*

(1) Add the other edges of the 3-face containing e to $E(H)$.

(2) Take $e' \in E(H)$ and search for a bounded 3-face containing e'. Add the other edge(s) in this bounded 3-face to $E(H)$.

(3) Repeat step (2) till we cannot find a bounded 3-face for any edge in $E(H)$.

Let $B(e)$ denote the triangular block obtained from e as the starting edge.

Let G be a plane graph. We have the following three observations:

(i) If H is a non-trivial triangular block and $e_1, e_2 \in E(H)$, then $B(e_1) = B(e_2) = H$.

(ii) Any two triangular blocks of G are edge disjoint.

(iii) Every triangular block of G contains at most 5 vertices.

Let $\mathcal{B}$ be the family of triangular blocks of G. From observation (ii) above, we have

$$e(G) = \sum_{B \in \mathcal{B}} e(B),$$

where $e(G)$ and $e(B)$ are the number of edges of G and B, respectively.

Definition 3.3.2. *A triangulated graph is a graph in which for every cycle of length $k > 3$, there is an edge joining two nonconsecutive vertices.*

Lemma 3.3.3. *If B is a triangular block with the unbounded region being a 3-face, then B is a triangulation graph.*

Proof. Suppose B is not a triangulation graph. Since each triangular block of G contains at most 5 vertices, we may assume that B contains a bounded 4-face or bounded 5-face.

If B contains a bounded 4-face, namely $v_1v_2v_3v_4$. By the definition of a triangular block, each edge of B is contained in a bounded 3-face. If $v(B) = 4$, $E(B)$ contains either v_2v_4 or v_1v_3. Without loss of generality, suppose $v_2v_4 \in E(B)$ and $v_2v_1v_4$ be the bounded 3-face, then the edges v_2v_3 and v_3v_4 are not in any bounded 3-face, which contradicts the fact that B is a triangular block. Let $v(B) = 5$. Without loss of generality, suppose $v_1v_2v_5$ forms a bounded 3-face in B. Since v_1v_4 should be contained in a bounded 3-face, then either $v_1v_3 \in E(B)$ or $v_4v_5 \in E(B)$. Suppose $v_1v_3 \in E(B)$. Similarly, v_2v_3 is contained in a bounded 3-face if the edge $v_3v_5 \in E(B)$. Hence, $v_1v_3v_5$ forms an unbounded 3-face, see Figure 3.6(a). However, B is not a triangular block in this case, a contradiction. Consider the case, when $v_4v_5 \in E(B)$ and $v_1v_4v_5$ is the resulting bounded 3-face. Since v_2v_3 should be contained in a bounded 3-face, either $v_2v_4 \in E(B)$ or $v_3v_5 \in E(B)$. If $v_2v_4 \in E(B)$, then B is not a triangular block. If $v_3v_5 \in E(B)$ and $v_2v_3v_5$ is the resulting bounded 3-face, clearly v_3v_4 is not contained in a bounded 3-face,see Figure 3.6(b). This contradicts the fact that B is a triangular block.

If B contains a bounded 5-face, namely $v_1v_2v_3v_4v_5$. Each edge of B is contained in a bounded 3-face. Without loss of generality, suppose that $v_1v_4, v_2v_4 \in E(B)$. Therefore, v_1v_2 is not contained in a bounded 3-face, which contradicts the fact that B is a triangular block. □

We describe all the possible triangular blocks in G based on the number of vertices of the block. For $k \in \{2, 3, 4, 5\}$, we denote the triangular blocks on k vertices as B_k.

Triangular blocks on 5 vertices.

There are four types of triangular blocks on 5 vertices, see Figure 3.7. Notice that $B_{5,a}$ is a K_5^-.

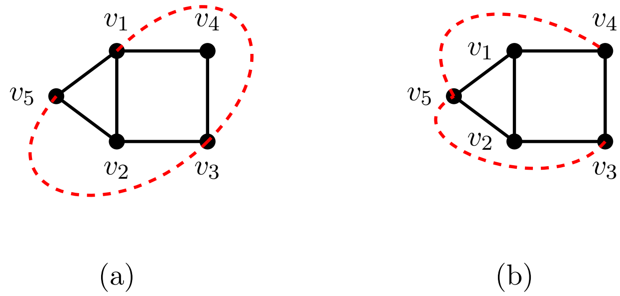


Figure 3.6: The triangular block B on 4 vertices containing a bounded 4 face.

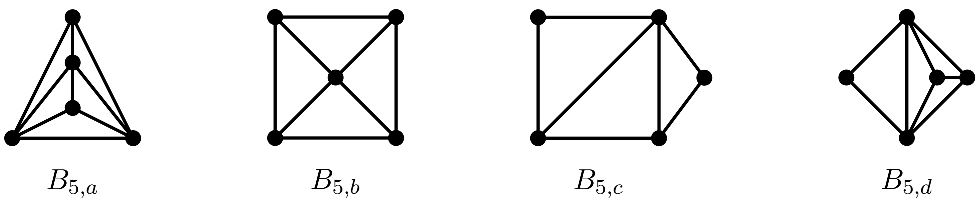


Figure 3.7: The triangular blocks on 5 vertices.

Triangular blocks on 4, 3, and 2 vertices.

There are two types of triangular blocks on 4 vertices, see Figure 3.8. Observe that $B_{4,a}$ is a K_4. The 3-vertex and 2-vertex triangular blocks are simply K_3 and K_2 (the trivial triangular block), respectively.

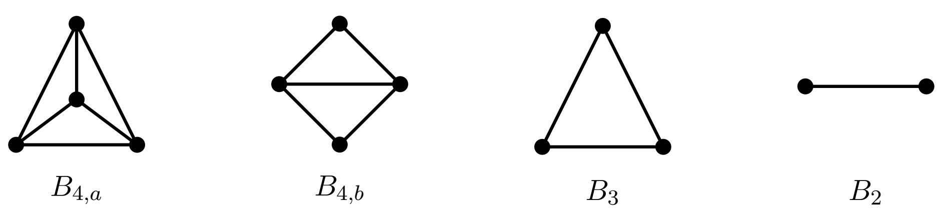


Figure 3.8: The triangular blocks on 4, 3, and 2 vertices.

Definition 3.3.4. *Let G be a plane graph.*

(i) A vertex v in G is called a ***junction vertex*** *if it is in at least two distinct triangular blocks of G.*

(ii) Let B be a triangular block in G. An edge of B is called an ***exterior edge*** *if it is on a boundary of a non-triangular face of G. Otherwise, we call it an* ***interior edge****. An end*

vertex of an exterior edge is called an ***exterior vertex****. We denote the set of all exterior and interior edges of B by $Ext(B)$ and $Int(B)$, respectively. Let $e \in Ext(B)$, a non-triangular face of G with e on its boundary is called the* ***exterior face*** *of e.*

Notice that, an exterior edge of a non-trivial triangular block has one exterior face exactly. On the other hand, if G is a 2-connected plane graph, then every trivial triangular block has two exterior faces.

Definition 3.3.5. *For a non-trivial triangular block B of a plane graph G, we call a path $P = v_1v_2v_3 \dots v_k$ an* ***exterior path*** *of B, if the following holds:*

(i) The vertices v_1 and v_k are junction vertices,

(ii) The edges v_iv_{i+1} are exterior edges of B, for $i \in \{1, 2, \dots, k-1\}$, and

(iii) The vertices v_j are not a junction vertex, for $j \in \{2, 3, \dots, k-1\}$.

The corresponding face in G, where P is on the boundary, is called the *exterior face* of P.

All graphs discussed from now on are C_6-free plane graphs. We define the **contribution** of a vertex to the number of vertices of a triangular block.

Definition 3.3.6. *Let G be a plane graph, B be a triangular block in G and $v \in V(B)$. The contribution of v to the vertex number of B is denoted by $n_B(v)$, and is defined as*

$$n_B(v) = \frac{1}{\#\ \textit{triangular blocks in } G \textit{ containing } v}.$$

We define the contribution of B to the number of vertices of G as $n(B) = \sum_{v \in V(B)} n_B(v)$.

Obviously, $v(G) = \sum_{B \in \mathcal{B}} n(B)$, where $v(G)$ is the number of vertices in G and $\mathcal{B}$ is the family of all the triangular blocks of G.

Let $B_{K_5^-}$ be a triangular block of G isomorphic to a $B_{5,a}$ with exterior vertices v_1, v_2, v_3, where v_1 and v_3 are the junction vertices, see Figure 3.9 for an example. Let F be a face in G such that $V(F)$ contains all exterior vertices $v_{1,1}, \dots, v_{1,m}, v_{2,1}, \dots, v_{2,m}, v_{3,1}, \dots, v_{3,m}$ of m $(m \geq 1)$ copies of

$B_{K_5^-}$, such that $v_{1,i}, v_{2,i}, v_{3,i}$ are the exterior vertices of the i-th $B_{K_5^-}$ and $v_{1,i}$, $v_{3,i}$ $(1 \leq i \leq m)$ are the junction vertices. Let C_F denote the cycle associated with the face F. We alter $E(C_F)$ in the following way:

$$\begin{aligned} E(C'_F) :=& E(C_F) - \{v_{2,1}, v_{1,1}\} - \{v_{2,1}, v_{3,1}\} - \cdots - \{v_{2,m}v_{1,m}\} - \{v_{2,m}, v_{3,m}\} \\ & \cup \{v_{1,1}v_{3,1}\} \cup \ldots \cup \{v_{1,m}, v_{3,m}\}. \end{aligned}$$

Thus, $|E(C'_F)|= |E(C_F)|-m$. For example, in Figure 3.9, $|E(C_F)|= 11$ but $|E(C'_F)|= 9$. If G is a C_6-free plane graph, we have $|E(C'_F)|> 6$.

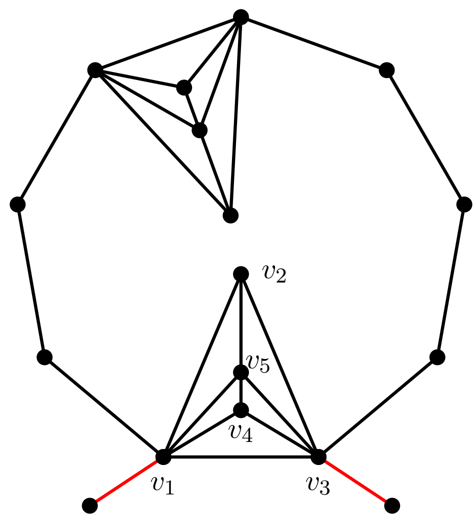


Figure 3.9: An example of a face containing all the exterior vertices of at least one $B_{K_5^-}$.

Now, we can define the **contribution** of an "*edge*" to the number of faces of a C_6-free plane graph G.

Definition 3.3.7. *Let F be an exterior face of G and $C_F := \{e_1, e_2, \ldots, e_k\}$ be the cycle associated with F. The contribution of an exterior edge e to the face number of the exterior face F, is denoted by $f_F(e)$ and is defined as follows:*

(i) For $1 \leq i \leq k-1$, if e_i and e_{i+1} are the adjacent exterior edges of a $B_{K_5^-}$, then $f_F(e_i) + f_F(e_{i+1}) = \dfrac{1}{|C'_F|}$. For the e_j's on the exterior face F (which do not lie as an exterior edge pair of a $B_{K_5^-}$), we define $f_F(e_j) = \dfrac{1}{|C'_F|}$.

(ii) Otherwise, $f_F(e) = \dfrac{1}{|C_F|}$.

Note that $\sum\limits_{e \in E(F)} f_F(e) = 1$. For a triangular block B, the total face contribution of B, denoted by $f(B)$, is defined as $f(B) = (\#\text{ interior faces of } B) + \sum\limits_{e \in Ext(B)} f_F(e)$, where F is the exterior face of B with respect to e. Obviously, $f(G) = \sum\limits_{B \in \mathcal{B}} f(B)$, where $f(G)$ is the number of faces of G.

3.4 Proof of Theorem 3.1.3

We begin by outlining our proof. Let f, n, and e be the number of faces, vertices, and edges of G, respectively. Let $\mathcal{B}$ be the family of all triangular blocks of G.

The main target of the proof is to show that

$$7f + 2n - 5e \leq 0. \tag{3.1}$$

Once we prove Equation (3.1), then by using Euler's Formula, we can finish the proof of Theorem 3.1.3. To prove (3.1), we show the existence of a partition $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ of $\mathcal{B}$ such that $7 \sum\limits_{B \in \mathcal{P}_i} f(B) + 2 \sum\limits_{B \in \mathcal{P}_i} n(B) - 5 \sum\limits_{B \in \mathcal{P}_i} e(B) \leq 0$, for all $i \in \{1, 2, 3 \ldots, m\}$. Since $f = \sum\limits_{B \in \mathcal{B}} f(B)$, $n = \sum\limits_{B \in \mathcal{B}} n(B)$ and $e = \sum\limits_{B \in \mathcal{B}} e(B)$ we have

$$\begin{aligned} 7f + 2n - 5e &= 7 \sum_{i}^{m} \sum_{B \in \mathcal{P}_i} f(B) + 2 \sum_{i}^{m} \sum_{B \in \mathcal{P}_i} n(B) - 5 \sum_{i}^{m} \sum_{B \in \mathcal{P}_i} e(B) \\ &= \sum_{i}^{m} \left(7 \sum_{B \in \mathcal{P}_i} f(B) + 2 \sum_{B \in \mathcal{P}_i} n(B) - 5 \sum_{B \in \mathcal{P}_i} e(B) \right) \leq 0. \end{aligned}$$

The following proposition is used throughout the chapter:

Proposition 3.4.1. *Let G be a 2-connected, C_6-free plane graph on n ($n \geq 6$) vertices with $\delta(G) \geq 3$.*

(i) If B is a non-trivial triangular block (that is, not B_2), then none of the exterior faces can have length 5.

(ii) If B is in $\{B_{5,a}, B_{5,b}\}$, then none of the exterior faces can have length 4.

(iii) If B is in $\{B_{5,d}, B_{4,a}, B_{4,b}\}$ and an exterior face of B has length 4, then that 4-face must share a 2-path with B (shown in blue in Figures 3.14 and 3.16). The other edges of the 4-face must be in trivial triangular blocks. If $B_{5,c}$ has an exterior face of B with length 4, then that 4-face must share a 3-path with $B_{5,c}$. The other edge of the 4-face must be a trivial triangular block.

(iv) No two 4-faces can be adjacent to each other.

Proof. (i) Observe that for every non-trivial triangular block B and every pair of vertices on the exterior of B, there is a 2-path (recall a 2-path has two edges) between them, using edges of B. That is, if u and v are vertices on the exterior of B, then there is a vertex $w \in V(B)$ such that uwv is a path. Thus, if an exterior face of B is a 5-face, the edge lying in B can be replaced with the 2-path, resulting in a 6-cycle.

(ii) Similar to (i). If $B \in \{B_{5,a}, B_{5,b}\}$ and it has an exterior 4-face, then either B shares an edge with the exterior face, or it shares a 2-path with the exterior face. If B shares an edge with the exterior face, then B has a 3-path between the vertices of the shared edge, which forms a 6-cycle with the remaining 3-path of the 4-face. If B shares a 2-path with the exterior face, then B has a 4-path which forms a 6-cycle with the remaining 2-path in the exterior face.

(iii) If $B \in \{B_{5,d}, B_{4,b}, B_{5,c}\}$, then any pair of consecutive exterior vertices has a path of length 3 between them. For $B_{5,d}$ (see Figure 3.14), we see that there is a path of length 4 between v_2 and v_4. Thus, the only way a 4-face can be adjacent to B is via a 2-path with end vertices v_1 and v_3. Since there is no vertex of degree 2, the path must be $v_1v_4v_3$. For $B_{4,b}$ (see Figure 3.16) since B cannot have a vertex of degree 2, the 4-face and B cannot share the path $v_2v_1v_4$ or the path $v_2v_3v_4$. Thus, the only paths that can share a boundary with a 4-face are $v_1v_4v_3$ and $v_1v_2v_3$. For $B_{4,a}$, it can be easily checked that the only path that can share a boundary with a 4-face is $v_1v_4v_3$. For $B_{5,c}$, any pair of consecutive exterior vertices has a path of length 4 between them. Thus, the only possibility that $B_{5,c}$ has an exterior 4-face is when $v_1v_5v_4v_3v_1$ forms a 4-face.

As to the other blocks that form edges of such a 4-face, consider the case when $B \in \{B_{5,d}, B_{4,a}, B_{4,b}\}$. Figure 3.10 shows that if, say v_1u is in a non-trivial triangular block,

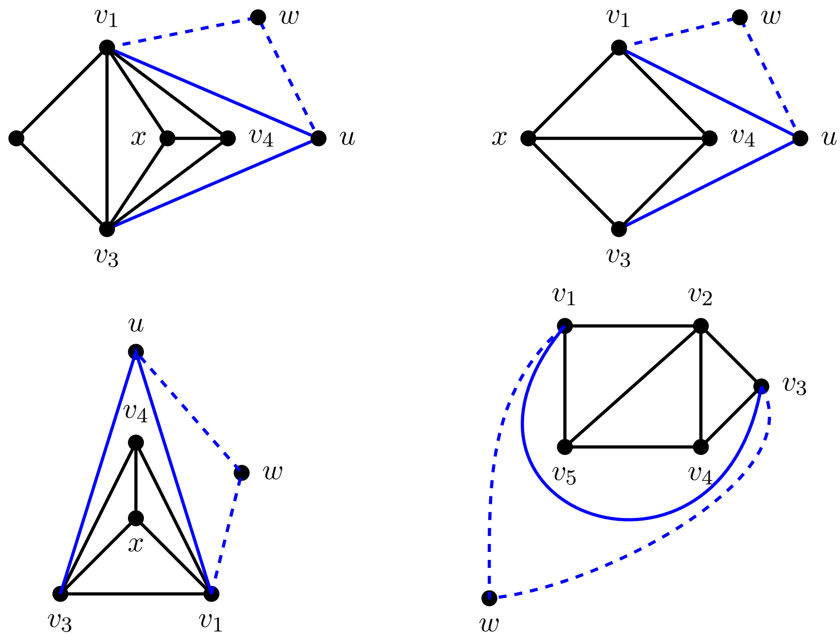


Figure 3.10: The blocks defined by blue edges must be trivial.

then there exists a vertex w in that block such that $wv_1xv_4v_3uw$ forms a 6-cycle, a contradiction. When $B = B_{5,c}$, Figure 3.10 shows that if v_1v_3 is in a non-trivial triangular block, then there exists a vertex w in that block, such that $wv_1v_5v_4v_2v_3w$ forms a 6-cycle, a contradiction.

(iv) If two 4-faces share an edge, then there is a 6-cycle formed by deleting that edge. If two 4-faces share a 2-path, then the midpoint of that path is a vertex of degree 2 in G. In both cases, we get a contradiction.

□

Lemma 3.4.2 verifies $7f(B) + 2n(B) - 5e(B) \leq 0$ for most blocks B. As for the exceptions, Lemmas 3.4.3, 3.4.4, 3.4.5, and 3.4.6 will give bounds for $7f(B) + 2n(B) - 5e(B)$ for $B_{5,c}$, $B_{5,d}$, $B_{4,a}$, and $B_{4,b}$, respectively. See Tables 3.2, 3.3, 3.4, and 3.5 in Section 3.7.

Lemma 3.4.2. *Let G be a 2-connected, C_6-free plane graph on n ($n \geq 6$) vertices with $\delta(G) \geq 3$. If B is a triangular block in G such that $B \notin \{B_{5,c}, B_{5,d}, B_{4,a}, B_{4,b}\}$, then $7f(B)+2n(B)-5e(B) \leq 0$.*

Proof. We separate the proof into several cases.

Case 1: B is $B_{5,a}$. Let v_1, v_2 and v_3 be the exterior vertices of the K_5^-. At least two of them must be junction vertices, otherwise G contains a cut vertex. We consider the following possibilities:

(a) Let B be $B_{5,a}$ with 3 junction vertices (see Figure 3.11(a)). By Proposition 3.4.1, every exterior edge in B is contained in an exterior face with length at least 7. Thus, $f(B) = (\#\text{ interior faces of } B) + \sum_{e \in Ext(B)} f_F(e) \leq 5 + 3/7$. Moreover, every junction vertex is contained in at least 2 triangular blocks, so we have $n(B) \leq 2 + 3/2$. With $e(B) = 9$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 0$.

(b) Let B be $B_{5,a}$ with 2 junction vertices, say v_2 and v_3 (see Figure 3.11(b)). Let F and F_1 be the exterior faces of the exterior edge v_2v_3 and exterior path $v_2v_1v_3$ of the triangular block, respectively. Notice that v_1v_2 and v_2v_3 are the adjacent exterior edges in the same face F_1, hence $|C(F_1)| \geq 8$. By Definition 3.3.7, we have $f_{F_1}(v_1v_2) + f_{F_1}(v_1v_3) \leq 1/7$. Since there can be no C_6, one can see that regardless of the configuration of the $B_{K_5^-}$, we have $f_F(v_2v_3) \leq 1/7$. Thus, $f(B) \leq 5 + 2/7$. Moreover, since v_2 and v_3 are contained in at least 2 triangular blocks, we have $n(B) \leq 3 + 2/2$. With $e(B) = 9$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 0$.

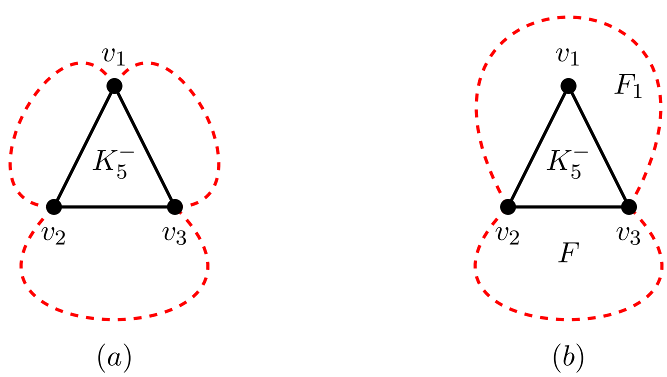


Figure 3.11: A $B_{5,a}$ triangular block with 3 and 2 junction vertices, respectively.

Case 2: B is $B_{5,b}$. When B is $B_{5,b}$, there are 4 faces inside the triangular block and each face incident to this triangular block has length at least 7. So, $f(B) \leq 4 + 4/7$. Since there is no cut-vertex, this triangular block must have at least two junction vertices, hence $n(B) \leq 3 + 2/2$. With $e(B) = 8$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 0$, as seen in Table 3.3.

Case 3: B is B_3. Let v_1, v_2 and v_3 be the exterior vertices of the triangular block B. Each of these three vertices must be a junction vertex, since there is no degree 2 vertex in G. Thus, the vertices are contained in at least 2 triangular blocks. We consider the following possibilities:

(a) Let the three exterior vertices be contained in exactly 2 triangular blocks. By Proposition 3.4.1(i), the length of each exterior face is either 4 or at least 7. We want to show that at most one exterior face has length 4. If not, then let v_1 be a vertex that is in two such faces. Consider the triangular block incident to B at v_1, call it B'. By Proposition 3.4.1(ii), B' is not in $\{B_{5,a}, B_{5,b}\}$.

If B' is in $\{B_{5,d}, B_{4,b}, B_3\}$, then the triangular block has vertices ℓ_2, ℓ_3, each adjacent to v_1 and the length 4 faces consist of $\{v_1, \ell_2, m_2, v_2\}$ and $\{v_1, \ell_3, m_3, v_3\}$. Either ℓ_2 is adjacent to ℓ_3 or there is a ℓ' distinct from v_1 that is adjacent to both ℓ_2 and ℓ_3. In the first case, $\ell_2 m_2 v_2 v_3 m_3 \ell_3 \ell_2$ is a 6-cycle (see Figure 3.12(a)). When ℓ' is distinct from m_2 and m_3, $\ell' \ell_2 m_2 v_2 v_1 \ell_3 \ell'$ is a 6-cycle. When ℓ' is m_2 or m_3, $\ell'(= m_2) v_2 v_1 v_3 m_3 \ell_3 \ell'(= m_2)$ or $\ell'(= m_3) v_3 v_1 v_2 m_2 \ell_2 \ell'(= m_3)$ is a 6-cycle, see Figure 3.12(b).

If B' is B_2, then the trivial triangular block is $\{v_1, \ell\}$, in which case $\{\ell, m_2, v_2, v_1, v_3, m_3\}$ is a C_6, see Figure 3.12(c). Thus, we may conclude that if each of the three exterior vertices are in exactly 2 triangular blocks, then $f(B) \leq 1 + 2/7 + 1/4$ and $n(B) \leq 3/2$. With $e(B) = 3$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -5/4$.

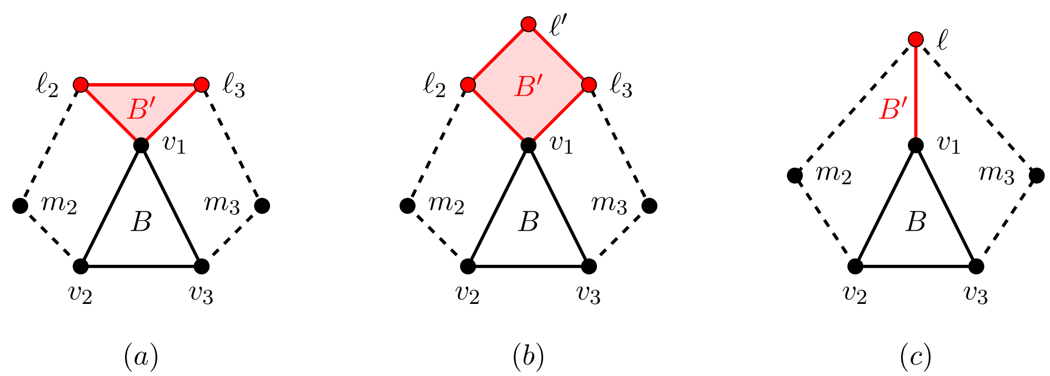


Figure 3.12: A B_3 triangular block, B which is incident to two 4-faces.

(b) Let at least one exterior vertex be contained in at least 3 triangular blocks and the others be contained in at least 2 triangular blocks. In this case, we have $f(B) \leq 1 + 3/4$ and $n(B) \leq 2/2 + 1/3$. With $e(B) = 3$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -1/12$.

Case 4**:** B **is** B_2**.** Recall, there is no vertex of degree 2. Thus, if an end vertex is in exactly two triangular blocks, then the other one cannot be a B_2. We consider the following possibilities:

(a) Let each end vertex be contained in exactly 2 triangular blocks. Since neither of the triangular blocks that are incident to B can be trivial, they cannot be incident to a face of length 5 by Proposition 3.4.1(i). Thus, B cannot be incident to a face of length 5. Moreover, the two faces incident to B cannot both be of length 4, again by Proposition 3.4.1(iv). Hence, $f(B) \leq 1/4+1/7$. Clearly $n(B) \leq 2/2$ and with $e(B) = 1$, we obtain $7f(B)+2n(B)-5e(B) \leq -1/4$.

(b) Let one end vertex be contained in exactly 2 triangular blocks and the other end vertex be contained in at least 3 triangular blocks. This is like case (a) in that neither of the faces can have length 5 and they cannot both have length 4. The only difference is that $n(B) \leq 1/2+1/3$ and so $7f(B) + 2n(B) - 5e(B) \leq -7/12$.

(c) Let each end vertex be contained in at least 3 triangular blocks. The two faces cannot both be of length 4 by Proposition 3.4.1(iv). Hence, $f(B) \leq 1/4 + 1/5$ and $n(B) \leq 2/3$. With $e(B) = 1$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -31/60$.

□

Lemma 3.4.3. *Let G be a 2-connected, C_6-free plane graph on n $(n \geq 6)$ vertices with $\delta(G) \geq 3$. If B is $B_{5,c}$, then $7f(B) + 2n(B) - 5e(B) \leq 7/12$. Moreover, $7f(B) + 2n(B) - 5e(B) \leq 0$ unless B shares a 3-path with a 4-face.*

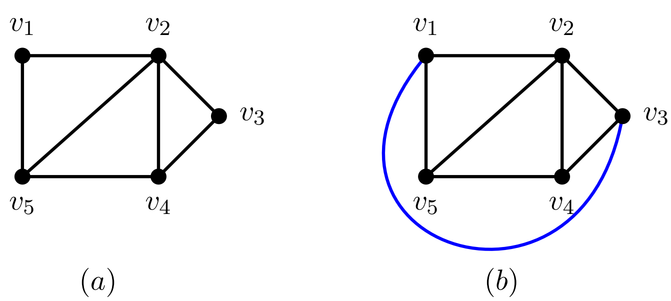


Figure 3.13: A $B_{5,c}$ triangular block and a $B_{5,c}$ with a 4-face incident to it.

Proof. Let B be $B_{5,c}$ with vertices v_1, v_2, v_3, v_4, and v_5, as shown in Figure 3.13(a). By Proposition 3.4.1(i), no exterior face of B can have length 5. By Proposition 3.4.1(iii), if there exists an exterior face of B that has length 4, this 4-face must be $v_1v_5v_4v_3v_1$. Since there is no vertex of degree 2 in G, v_1 and v_3 are junction vertices. We consider the following cases:

(a) None of the exterior faces of $B_{5,c}$ are of length 4. Thus, each exterior face has length at least 7. Hence, $f(B) \leq 3+5/7$ and $n \leq 3+2/2$. With $e = 7$, we obtain $7f(B)+2n(B)-5e(B) \leq -1$.

(b) The vertices $v_1v_5v_4v_3v_1$ form an exterior face of length 4. Since G contains no cut vertex, at least 2 of the vertices v_1, v_2 and v_3 are contained in one more triangular block. Hence, $f(B) \leq 3 + 3/4 + 2/7$ and $n(B) \leq 2 + 1 + 1/3 + 1/3$. With $e(B) = 7$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 7/12$.

□

Lemma 3.4.4. *Let G be a 2-connected, C_6-free plane graph on n ($n \geq 6$) vertices with $\delta(G) \geq 3$. If B is $B_{5,d}$, then $7f(B) + 2n(B) - 5e(B) \leq 1/2$. Moreover, $7f(B) + 2n(B) - 5e(B) \leq 0$ unless B shares a 2-path with a 4-face.*

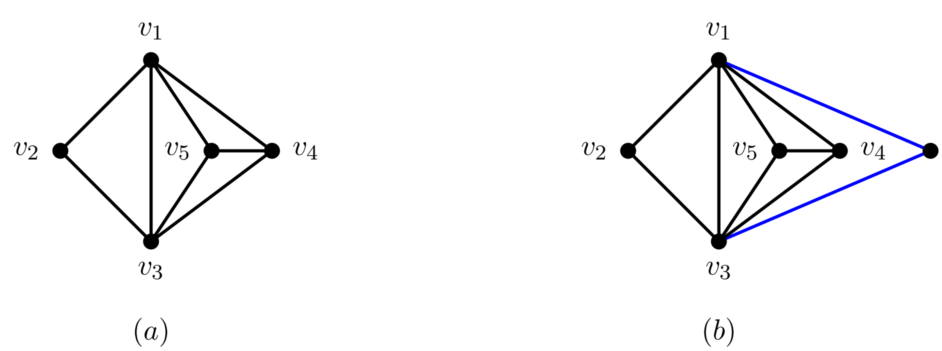


Figure 3.14: A $B_{5,d}$ triangular block and a $B_{5,d}$ with a 4-face incident to it.

Proof. Let B be $B_{5,d}$ with vertices v_1, v_2, v_3, v_4, and v_5, as shown in Figure 3.14(a). By Proposition 3.4.1(i), no exterior face of B can have length 5. By Proposition 3.4.1(iii), if there is an exterior face of B that has length 4, this 4-face must contain the path $v_1v_4v_3$. Since there is no vertex of degree 2, v_2 is a junction vertex. There is at least one other junction vertex, since G has no cut-vertex. We consider the following cases:

(a) Let v_4 be a junction vertex. This prevents an exterior face of length 4. Thus, each exterior face has length at least 7. Hence, $f(B) \leq 4 + 4/7$ and $n(B) \leq 3 + 2/2$. With $e(B) = 8$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 0$.

(b) Suppose v_4 fails to be a junction vertex and exactly one of the vertices v_1, v_3 is a junction vertex. Without loss of generality, let it be v_3. In this case, each exterior face has length at least 7. (In fact, it can be shown that the length of the exterior face containing the path $v_2v_1v_4v_3$ is at least 9. These yields $f(B) \leq 4+1/7+3/9$ and $7f(B)+2n(B)-5e(B) \leq -2/3$. However, this precision is unnecessary.) Again, $f(B) \leq 4 + 4/7$ and $n(B) \leq 3 + 2/2$. With $e(B) = 8$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 0$.

(c) Suppose v_4 fails to be a junction vertex and both the vertices v_1 and v_3 are junction vertices. Here, either the exterior path $v_1v_4v_3$ is part of an exterior face of length at least 4, or each edge must be in a face of length at least 7. If each exterior face is of length at least 7, then $f(B) \leq 4+4/7$, otherwise $f(B) \leq 4+2/4+2/7$. In both cases, $n(B) \leq 2+3/2$ and $e(B) = 8$. Hence, we obtain $7f(B)+2n(B)-5e(B) \leq -1$ in the first case and $7f(B)+2n(B)-5e(B) \leq 1/2$ in the case where B is incident to a 4-face.

□

Lemma 3.4.5. *Let G be a 2-connected, C_6-free plane graph on n ($n \geq 6$) vertices with $\delta(G) \geq 3$. If B is $B_{4,a}$, then $7f(B) + 2n(B) - 5e(B) \leq 3/2$. Moreover, $7f(B) + 2n(B) - 5e(B) \leq 0$ unless B shares a 2-path with any 4-face.*

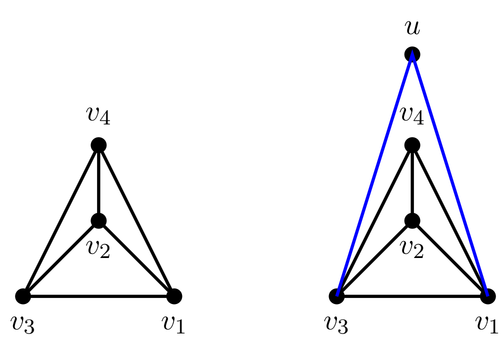


Figure 3.15: A $B_{5,a}$ triangular block and a $B_{5,a}$ with a 4-face incident to it.

Proof. Let B be $B_{4,a}$ with vertices v_1, v_2, v_3, and v_4, as shown in Figure 3.15(a). By Proposition 3.4.1(i), no exterior face of B can have length 5. If there exists an exterior face of B that has length 4, it is easy to verify that this exterior 4-face contains 2 exterior edges of B. Since G has no cut-vertex, there are at least two junction vertices in B. We consider the following cases:

(a) None of the exterior faces of $B_{4,a}$ has length 4. Thus, each exterior face has length at least 7. Hence, $f(B) \leq 3 + 3/7$ and $n \leq 2 + 2/2$. With $e = 6$, we obtain $7f(B) + 2n(B) - 5e(B) \leq 0$.

(b) Without loss of generality, suppose $v_1v_4v_3$ is part of an exterior face of length 4. Hence, $f(B) \leq 3+2/4+1/7$ and $n(B) \leq 2+2/2$. With $e(B) = 6$, we obtain $7f(B)+2n(B)-5e(B) \leq 3/2$.

□

Lemma 3.4.6. *Let G be a 2-connected, C_6-free plane graph on n ($n \geq 6$) vertices with $\delta(G) \geq 3$. If B is $B_{4,b}$, then $7f(B) + 2n(B) - 5e(B) \leq 4/3$. Moreover, $7f(B) + 2n(B) - 5e(B) \leq 1/6$ if B shares a 2-path with exactly one 4-face and $7f(B) + 2n(B) - 5e(B) \leq 0$ if B fails to share a 2-path with any 4-face.*

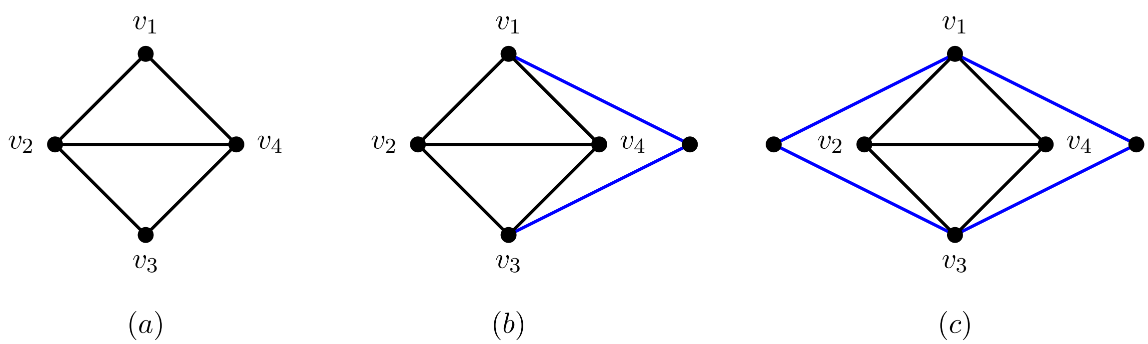


Figure 3.16: A $B_{4,b}$ triangular block and a $B_{4,b}$ with a 4-face incident to it.

Proof. Let B be $B_{4,b}$ with vertices v_1, v_2, v_3, and v_4, as shown in Figure 3.16(a). By Proposition 3.4.1(i), no exterior face of B can have length 5. Suppose there is an exterior face of B that has length 4. Since G is a C_6-free graph and contains no vertices of degree 2, v_1 and v_3 must be the junction vertices. We consider the following cases:

(a) Let either v_2 or v_4 be a junction vertex. Without loss of generality, let it be v_2. All the exterior faces have length at least 7 except for the possibility that the path $v_1v_4v_3$ may form two sides of a 4-face. Hence, $f(B) \leq 2 + 2/4 + 2/7$ and $n(B) \leq 1 + 3/2$. With $e(B) = 5$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -1/2$.

(b) Let neither v_2 nor v_4 be a junction vertex. Hence, there are two exterior faces: One that shares the exterior path $v_1v_4v_3$ and the other shares the exterior path $v_1v_2v_3$. Each exterior face has length either 4 or at least 7. We consider several subcases:

(i) If both faces are of length at least 7, then $f(B) \leq 2 + 4/7$, and $n(B) \leq 2 + 2/2$. With $e(B) = 5$, we obtain $7f(B) + 2n(B) - 5e(B) \leq -1$.

(ii) If only one of the exterior faces is of length 4, then $f(B) \leq 2 + 2/7 + 2/4$. Moreover, at least one of the vertices between v_1 and v_3 must be a junction vertex with more than two triangular blocks, otherwise either $v(G) = 5$ or the vertex incident to two blue edges in Figure 3.16(b) is a cut-vertex. Hence, $n(B) \leq 2 + 1/3 + 1/2$ and $e(B) = 5$, we have $7f(B) + 2n(B) - 5e(B) \leq 1/6$.

(iii) Both the exterior faces are of length 4. Thus, $f(B) \leq 2 + 4/4$. By Proposition 3.4.1(iii), the blocks represented by the blue edges in Figure 3.16(c) are trivial. Hence, $n(B) \leq 2 + 2/3$. With $e(B) = 5$, we get $7f(B) + 2n(B) - 5e(B) \leq 4/3$.

□

The last step in the proof of Theorem 3.1.3 is Lemma 3.4.7, which collects the blocks into parts of a partition such that the blocks B for which $7f(B) + 2n(B) - 5e(B)$ is positive, are balanced with those for which it is negative.

Lemma 3.4.7. *Let G be a 2-connected, C_6-free plane graph on n ($n \geq 6$) vertices with $\delta(G) \geq 3$. Then the triangular blocks of G can be partitioned into sets, $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ such that $7\sum_{B \in \mathcal{P}_i} f(B) + 2\sum_{B \in \mathcal{P}_i} n(B) - 5\sum_{B \in \mathcal{P}_i} e(B) \leq 0$ for all $i \in [m]$.*

Proof. As it can be seen from Tables 3.2, 3.3, 3.4, and 3.5 in Section 3.7, there are 5 possible cases where $7f(B)+2n(B)-5e(B)$ assumes a positive value. We deal with each of these blocks as follows:

(1) Let B be a $B_{5,c}$ triangular block as described in the proof of Lemma 3.4.3(b), see Figure 3.13(b). By Proposition 3.4.1(iii), the edge v_1v_3 is a trivial triangular block. Denote this triangular block by B'. One of the exterior faces of B' has length 4, whereas by Proposition 3.4.1(iv), the other has length at least 5. If it has length 5, then either (i) one of the edges

v_1v_2 or v_2v_3 is contained in this 5-face or (ii) none of the edges v_1v_2 and v_2v_3 are contained in it. For case (i), $v_2v_5v_4v_3$ or $v_2v_4v_5v_1$ would complete it to a 6-cycle. For case (ii), $v_1v_2v_3$ would complete it to a 6-cycle. Hence, the other exterior face of B' has length at least 7.

Thus, $f(B') \leq 1/4+1/7$ and $n(B') \leq 1/2+1/2$. With $e(B') = 1$, we obtain $7f(B')+2n(B')-5e(B') \leq -7/12$. Define $\mathcal{P}' = \{B, B'\}$. Thus, $7 \sum_{B^* \in \mathcal{P}'} f(B^*) + 2 \sum_{B^* \in \mathcal{P}'} n(B^*) - 5 \sum_{B^* \in \mathcal{P}'} e(B^*) \leq 1/2 + 2(-7/12) = -2/3$.

Therefore, for each triangular block in G as described in Lemma 3.4.3(b), it belongs to a set $\mathcal{P}'$ of three triangular blocks such that $7 \sum_{B^* \in \mathcal{P}'} f(B^*) + 2 \sum_{B^* \in \mathcal{P}'} n(B^*) - 5 \sum_{B^* \in \mathcal{P}'} e(B^*) \leq 0$. Denote such sets as $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{m_1}$ if they exist.

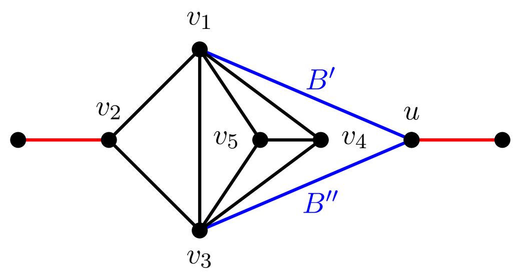


Figure 3.17: Structure of a $B_{5,d}$ if it is incident to a 4-face, as in Lemma 3.4.7. The triangular blocks B' and B'' are trivial.

(2) Let B be a $B_{5,d}$ triangular block as described in the proof of Lemma 3.4.4(c), see Figure 3.17. By Proposition 3.4.1(iii), the edges v_1u and v_3u are trivial triangular blocks. Denote these triangular blocks as B' and B''. Consider B'. One of the exterior faces of B' has length 4, whereas by Proposition 3.4.1(iv), the other has length at least 5. It must have length at least 7. Otherwise, if it had length 5, then the path v_1v_3u would complete it to a 6-cycle. Thus, $f(B') \leq 1/4+1/7$. Since the vertex u cannot be of degree 2, then this vertex is shared in at least three triangular blocks. Thus, $n(B') \leq 1/2 + 1/3$. With $e(B') = 1$, we obtain $7f(B') + 2n(B') - 5e(B') \leq -7/12$ and similarly $7f(B'') + 2n(B'') - 5e(B'') \leq -7/12$. Define $\mathcal{P}'' = \{B, B', B''\}$. Thus, $7 \sum_{B^* \in \mathcal{P}''} f(B^*) + 2 \sum_{B^* \in \mathcal{P}''} n(B^*) - 5 \sum_{B^* \in \mathcal{P}''} e(B^*) \leq 1/2 + 2(-7/12) = -2/3$.

Therefore, for each triangular block in G as described in Lemma 3.4.4(c), it belongs to a set $\mathcal{P}''$ of three triangular blocks such that $7 \sum_{B^* \in \mathcal{P}''} f(B^*) + 2 \sum_{B^* \in \mathcal{P}''} n(B^*) - 5 \sum_{B^* \in \mathcal{P}''} e(B^*) \leq 0$. Denote such sets as $\mathcal{P}_{m_1+1}, \mathcal{P}_{m_1+2}, \ldots, \mathcal{P}_{m_2}$ if they exist.

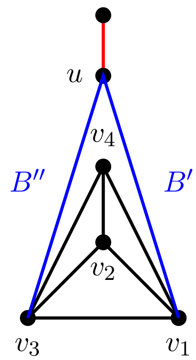


Figure 3.18: Structure of a $B_{4,a}$ triangular block if it is incident to a 4-face, as in Lemma 3.4.7. The triangular blocks $B'$and B'' are all trivial.

(3) Let B be a $B_{4,a}$ triangular block as described in the proof of Lemma 3.4.5(b), see Figure 3.18. By Proposition 3.4.1(iii), the edges v_1u_1 and v_3u_1 are trivial triangular blocks. Denote them as B' and B'', respectively. Consider B'. One of the exterior faces of B' has length 4 and by Proposition 3.4.1(iv), the other has length at least 5. It is easy to check that if another face of B' has length 5, then no matter whether this 5-face contains v_3 or not, we can find a copy of C_6 in G. Hence, $f(B') \leq 1/4 + 1/7$. Similarly, for B''.

Since the vertex u cannot be of degree 2, u is contained in at least 3 triangular blocks. Since G does not contain a cut vertex, at least one of v_1 or v_3 is contained in one more triangular block. To get the upper bound of $n(B) + n(B') + n(B'')$, we may assume that v_1 is contained in at least 3 triangular blocks and v_3 is contained in at least 2 triangular blocks. Thus, $n(B') \leq 1/3 + 1/3$. With $e(B') = 1$, we obtain $7f(B') + 2n(B') - 5e(B') \leq -11/12$. Similarly, for B'', we have $f(B'') \leq 1/4 + 1/7$. But $n(B'') \leq 1/3 + 1/2$, with $e(B'') = 1$, we obtain $7f(B'') + 2n(B'') - 5e(B'') \leq -7/12$. Define $\mathcal{P}''' = \{B, B', B''\}$. Thus, $7 \sum\limits_{B^* \in \mathcal{P}'''} f(B^*) + 2 \sum\limits_{B^* \in \mathcal{P}'''} n(B^*) - 5 \sum\limits_{B^* \in \mathcal{P}'''} e(B^*) \leq 3/2 + (-7/12) + (-11/12) = 0$.
Therefore, for each triangular block in G as described in Lemma 3.4.5(b), it belongs to a set $\mathcal{P}'''$ of three triangular blocks such that $7 \sum\limits_{B^* \in \mathcal{P}'''} f(B^*) + 2 \sum\limits_{B^* \in \mathcal{P}'''} n(B^*) - 5 \sum\limits_{B^* \in \mathcal{P}'''} e(B^*) \leq 0$.
Denote such sets as $\mathcal{P}_{m_2+1}, \mathcal{P}_{m_2+2}, \ldots, \mathcal{P}_{m_3}$ if they exist.

(4) Let B be a $B_{4,b}$ triangular block as described in the proof of Lemma 3.4.6(b)(ii), See Figure 3.19(a).

By Proposition 3.4.1(iii), the edges v_1u_1 and v_3u_1 are trivial triangular blocks. Denote them as B' and B'', respectively. Consider B'. One of the exterior faces of B' has length 4 and

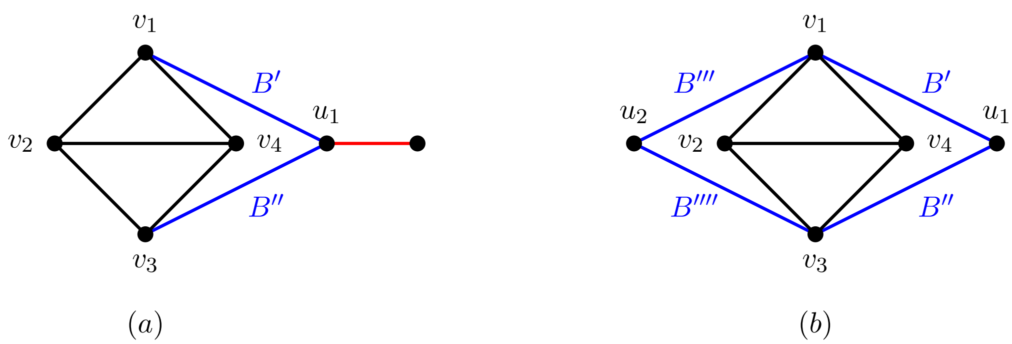


Figure 3.19: Structure of a $B_{4,b}$ triangular block if it is incident to a 4-face, as in Lemma 3.4.7. The triangular blocks B', B'', B''', and B'''' are all trivial.

by Proposition 3.4.1(iv), the other has length at least 5. Thus, $f(B') \leq 1/4 + 1/5$. Since the vertex u_1 cannot be of degree 2, then this vertex is shared in at least three triangular blocks. Thus, $n(B') \leq 1/2 + 1/3$. With $e(B') = 1$, we obtain $7f(B') + 2n(B') - 5e(B') \leq -11/60$. Similarly, $7f(B'') + 2n(B'') - 5e(B'') \leq -11/60$. Define $\mathcal{P}'''' = \{B, B', B''\}$. Thus, $7 \sum_{B^* \in \mathcal{P}''''} f(B^*) + 2 \sum_{B^* \in \mathcal{P}''''} n(B^*) - 5 \sum_{B^* \in \mathcal{P}''''} e(B^*) \leq 1/6 + 2(-11/60) = -1/5$.
Therefore, for each triangular block in G as described in Lemma 3.4.6(b)(ii), it belongs to a set $\mathcal{P}''''$ of three triangular blocks such that $7 \sum_{B^* \in \mathcal{P}''''} f(B^*) + 2 \sum_{B^* \in \mathcal{P}''''} n(B^*) - 5 \sum_{B^* \in \mathcal{P}''''} e(B^*) \leq 0$. Denote such sets as $\mathcal{P}_{m_3+1}, \mathcal{P}_{m_3+2}, \ldots, \mathcal{P}_{m_4}$ if they exist.

(5) Let B be a $B_{4,b}$ triangular block as described in the proof of Lemma 3.4.6(b)(iii), see Figure 3.19(b). By Proposition 3.4.1(iii), the edges v_1u_1, v_3u_1, v_1u_2, and v_3u_2 are trivial triangular blocks. Denote them as B', B'', B''' and B'''', respectively. Consider B'. One of the exterior faces of B' has length 4, whereas the other has length at least 5. Thus, $f(B') \leq 1/4 + 1/5$. Since the vertex u_1 cannot be of degree 2, then this vertex is shared in at least three triangular blocks. Clearly, v_1 is in at least three triangular blocks. Thus, $n(B') \leq 2/3$. With $e(B') = 1$, we obtain $7f(B') + 2n(B') - 5e(B') \leq -31/60$ and the same inequalities hold for B'', B''', and B''''.

Define $\mathcal{P}''''' = \{B, B', B'', B''', B''''\}$. Thus, $7 \sum_{B^* \in \mathcal{P}'''''} f(B^*) + 2 \sum_{B^* \in \mathcal{P}'''''} n(B^*) - 5 \sum_{B^* \in \mathcal{P}'''''} e(B^*) \leq 4/3 + 4(-31/60) = -11/15$.

Therefore, for each triangular block in G as described in Lemma 3.4.6(b)(iii), it belongs to a set $\mathcal{P}'''''$ of four triangular blocks such that $7 \sum_{B^* \in \mathcal{P}'''''} f(B^*) + 2 \sum_{B^* \in \mathcal{P}'''''} n(B^*) - 5 \sum_{B^* \in \mathcal{P}'''''} e(B^*) \leq 0$. Denote such sets as $\mathcal{P}_{m_4+1}, \mathcal{P}_{m_4+2}, \ldots, \mathcal{P}_{m_5}$ if they exist.

Now define $\mathcal{P}_{m_5+1} = \mathcal{B} - \bigcup_{i=1}^{m_5} \mathcal{P}_i$, where $\mathcal{B}$ is the set of all blocks of G. Clearly, for each block $B \in \mathcal{P}_{m_5+1}$, $7f(B)+2n(B)-5e(B) \le 0$. Thus, $7\sum_{B\in\mathcal{P}_{m_5+1}} f(B)+2\sum_{B\in\mathcal{P}_{m_5+1}} n(B)-5\sum_{B\in\mathcal{P}_{m_5+1}} e(B) \le 0$. Putting $m := m_5 + 1$ we got the partition $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ of $\mathcal{B}$ meeting the condition of the lemma. □

3.5 Proof of Theorem 3.1.4

Let G be a C_6-free plane graph. We will show that either $5v(G) - 2e(G) \ge 14$ or $v(G) \le 17$.

If we recursively delete a vertex x with degree at most two from G, then

$$\begin{aligned} 5v(G-x) - 2e(G-x) &= 5(v(G)-1) - 2(e(G) - \deg(x)) \\ &= 5v(G) - 2e(G) - 5 + 2\deg(x) \\ &\le 5v(G) - 2e(G) - 1. \end{aligned}$$

So, if the procedure ends with an empty graph, then $e(G) \le 2n - 3 \le \frac{5}{2}n - 7$, when $n \ge 8$. If not, the graph G has an induced subgraph G' with $\delta(G') \ge 3$ and

$$5v(G) - 2e(G) \ge 5v(G') - 2e(G') + \left(v(G) - v(G')\right). \tag{3.2}$$

In line with usual graph theoretic terminology, we call a maximal 2-connected subgraph a **block**. Let $\mathcal{B}'$ denote the set of blocks of G' with the i^{th} block having n_i vertices and e_i edges. Let b be the total number of blocks of G'. Specifically, let b_2, b_3, b_4, and b_5 denote the number of blocks of size 2, 3, 4, and 5, respectively. Let b_6 denote the number of blocks of size at least 6. Then we have $b = b_6 + b_5 + b_4 + b_3 + b_2$ and, using Table 3.1:

$$\begin{aligned} 5v(G') - 2e(G') &= 5\left(\sum_{i=1}^{b} n_i - (b-1)\right) - 2\sum_{i=1}^{b} e_i \\ &= \sum_{i=1}^{b} (5n_i - 2e_i - 5) + 5 \\ &\ge 9b_6 + 2b_5 + 3b_4 + 4b_3 + 3b_2 + 5. \end{aligned} \tag{3.3}$$

	min of $5n-2e-5$		
$n \geq 6$	$14-5 \ \geq$	9	Theorem 3.1.3
$n=5$	$5(5)-2(9)-5 \ \geq$	2	$B_{5,a}$, Figure 3.7
$n=4$	$5(4)-2(6)-5 \ \geq$	3	$B_{4,a}$, Figure 3.8
$n=3$	$5(3)-2(3)-5 \ \geq$	4	B_3, Figure 3.8
$n=2$	$5(2)-2(2)-5 \ \geq$	3	B_2, Figure 3.8

Table 3.1: Estimates of $5n-2e-5$ for various block sizes.

Combining (3.2) and (3.3), we obtain

$$5v(G)-2e(G) \geq 9b_6+2b_5+3b_4+4b_3+3b_2+5+\left(v(G)-v(G')\right). \tag{3.4}$$

If $b_6 \geq 1$, then the right-hand side of (3.4) is at least 14, as desired.

So, let us assume that $b_6=0$ and $b=b_5+b_4+b_3+b_2$. Furthermore,

$$\begin{aligned} v(G') &= 5b_5+4b_4+3b_3+2b_2-(b-1) \\ &= 4b_5+3b_4+2b_3+b_2+1. \end{aligned} \tag{3.5}$$

So, substituting $2b_5$ from (3.5) into (3.4), we have

$$\begin{aligned} 5v(G)-2e(G) &\geq 2b_5+3b_4+4b_3+3b_2+5+\left(v(G)-v(G')\right) \\ &= \left(\frac{1}{2}v(G')-\frac{3}{2}b_4-b_3-\frac{1}{2}b_2-\frac{1}{2}\right)+3b_4+4b_3+3b_2+5+\left(v(G)-v(G')\right) \\ &= v(G)-\frac{1}{2}v(G')+\frac{3}{2}b_4+3b_3+\frac{5}{2}b_2+\frac{9}{2} \\ &\geq \frac{1}{2}v(G)+\frac{9}{2}, \end{aligned}$$

which is strictly larger than 13 if $v(G) \geq 18$. Since $5v(G)-2e(G)$ is an integer, it is at least 14 and this completes the proof of Theorem 3.1.4.

Remark 3.5.1. *Observe that for $n \geq 17$, the only graphs on n vertices with e edges such that $e > (5/2)n-7$ have blocks of order 5 or less and by (3.4), there are at most 4 such triangular blocks. A bit of analysis shows that the maximum number of edges is achieved when the number of blocks of order 5 is as large as possible.*

3.6 Concluding remarks and Conclusions

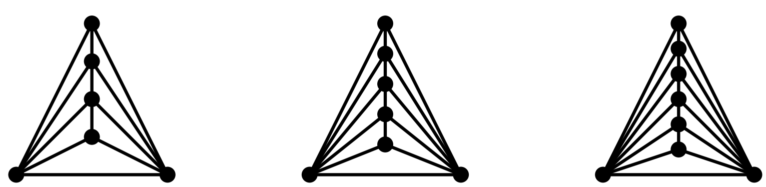
We note that the proof of Theorem 3.1.3, particularly Lemma 3.4.7, can be rephrased in terms of a discharging argument.

We believe that our construction in Theorem 3.1.5 can be generalized to prove $\mathrm{ex}_{\mathcal{P}}(n, C_\ell)$ for n sufficiently large. That is, we construct a plane graph G_0 such that all the faces are of length $\ell+1$ and all the vertices have degree either 2 or 3. (As above, n belongs to a congruence class.)

If such a G_0 exists, then the number of degree 2 and degree 3 vertices are $\frac{(\ell-5)n+4(\ell+1)}{\ell-1}$ and $\frac{4(n-\ell-1)}{\ell-1}$, respectively. We could then apply steps like (1), (2), and (3) in the proof of Theorem 3.1.5 in that we add halving vertices and insert a graph $B_{\ell-1}$ in Figure 3.20 (or another maximal planar graph of $\ell-1$ vertices) in place of the vertices of degree 2 and 3. For the resulting graph G,

$$\begin{aligned} v(G) &= v(G_0) + e(G_0) + (\ell-4)\frac{(\ell-5)n+4(\ell+1)}{\ell-1} + (\ell-5)\frac{4(n-\ell-1)}{\ell-1} \\ &= n + \frac{\ell+1}{\ell-1}(n-2) + \frac{(\ell^2-5\ell)n+2(\ell+1)}{\ell-1} \\ &= \frac{\ell^2-3\ell}{\ell-1}n + \frac{2(\ell+1)}{\ell}, \\ \text{and } e(G) &= (3\ell-9)v(G_0) = (3\ell-9)n. \end{aligned}$$

Therefore, $e(G) = \frac{3(\ell-1)}{\ell}v(G) - \frac{6(\ell+1)}{\ell}$. We conjecture that this is the maximum number of edges in a C_ℓ-free planar graph – at least if ℓ is small.



B_6 B_7 B_8

Figure 3.20: $B_{\ell-1}$ is used in the construction of a C_ℓ-free graph.

Conjecture 3.6.1. *Let G be C_ℓ-free plane graph ($10 \geq \ell \geq 7$) on n vertices, then there exists an integer $N_0 > 0$, such that when $n \geq N_0$, $e(G) \leq \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}$.*

Remark 3.6.1. *The conjecture is different if $\ell \geq 11$, since instead of $B_{\ell-1}$ we can use a bigger graph which is not Hamiltonian, or if ℓ is even bigger, then we can take an appropriate maximal* *planar graph not containing a cycle C_ℓ. Even more, in these cases, the proof technique is not just more complicated, but some other difficulties may rise too. Here, we do not want to go into details.*

3.7 Tables

The following tables give a summary of the results from Lemmas 3.4.2, 3.4.4, and 3.4.6.

A red edge incident to a vertex of a triangular block indicates the corresponding vertex is a junction vertex. Moreover, if a vertex has only one red edge, the vertex is shared in at least two triangular blocks. Whereas, if a vertex has two red edges, the vertex is shared in at least three blocks.

A pair of blue edges indicates the boundary of a 4-face.

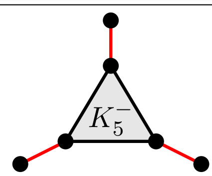
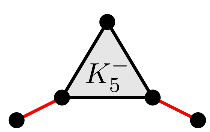
Case	B	Diagram	$f(B) \leq$	$n(B) \leq$	$e(B) =$	$7f + 2n - 5e \leq$
Lemma 3.4.2 1(a)	$B_{5,a}$		$5 + \frac{3}{7}$	$2 + \frac{3}{2}$	9	0
Lemma 3.4.2 1(b)	$B_{5,a}$		$5 + \frac{2}{7}$	$3 + \frac{2}{2}$	9	0

Table 3.2: The $B_{5,a}$ blocks in G and the estimation of $7f(B) + 2n(B) - 5e(B)$.

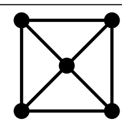
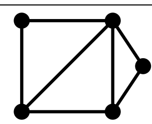
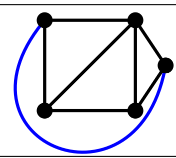
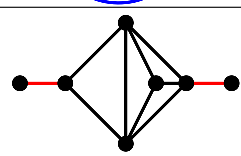
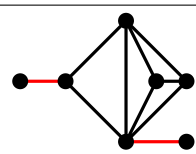
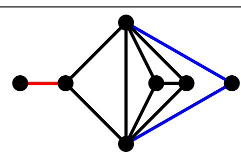
Case	B	Diagram	$f(B) \leq$	$n(B) \leq$	$e(B) =$	$7f + 2n - 5e \leq$
Lemma 3.4.2 2	$B_{5,b}$		$4 + \frac{4}{7}$	$3 + \frac{2}{2}$	8	0
Lemma 3.4.3 (a)	$B_{5,c}$		$3 + \frac{5}{7}$	$3 + \frac{2}{2}$	7	-1
Lemma 3.4.3 (b)	$B_{5,c}$		$3 + \frac{3}{4} + \frac{2}{7}$	$2 + 1 + \frac{1}{3} + \frac{1}{3}$	7	$\frac{7}{12}$ $\star$
Lemma 3.4.4 (a)	$B_{5,d}$		$4 + \frac{4}{7}$	$3 + \frac{2}{2}$	8	0
Lemma 3.4.4 (b)	$B_{5,d}$		$4 + \frac{4}{7}$	$3 + \frac{2}{2}$	8	0
Lemma 3.4.4 (c)	$B_{5,d}$		$4 + \frac{2}{4} + \frac{2}{7}$	$2 + \frac{3}{2}$	8	$\frac{1}{2}$ $\star$

Table 3.3: The possible B_5 blocks that are not a K_5^- in G and the estimation of $7f(B) + 2n(B) - 5e(B)$.

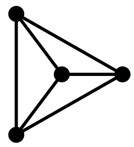
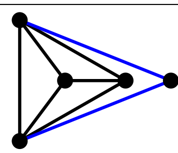
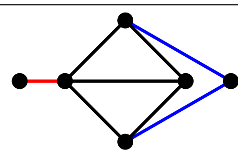
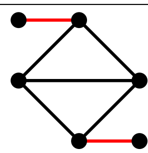
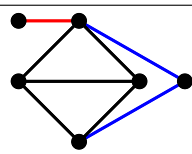
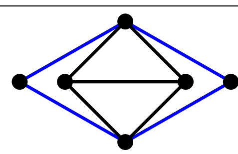
Case	B	Diagram	$f(B) \leq$	$n(B) \leq$	$e(B) =$	$7f + 2n - 5e \leq$
Lemma 3.4.5 (a)	$B_{4,a}$		$3 + \frac{3}{7}$	$2 + \frac{2}{2}$	6	0
Lemma 3.4.5 (b)	$B_{4,a}$		$3 + \frac{2}{4} + \frac{1}{7}$	$2 + \frac{2}{2}$	6	$\frac{3}{2}$ $\star$
Lemma 3.4.6 (a)	$B_{4,b}$		$2 + \frac{2}{4} + \frac{2}{7}$	$1 + \frac{3}{2}$	5	$-\frac{1}{2}$
Lemma 3.4.6 (b)(i)	$B_{4,b}$		$2 + \frac{4}{7}$	$2 + \frac{2}{2}$	5	-1
Lemma 3.4.6 (b)(ii)	$B_{4,b}$		$2 + \frac{2}{4} + \frac{2}{7}$	$2 + \frac{1}{3} + \frac{1}{2}$	5	$\frac{1}{6}$ $\star$
Lemma 3.4.6 (b)(iii)	$B_{4,b}$		$2 + \frac{2}{4} + \frac{2}{4}$	$2 + \frac{2}{3}$	5	$\frac{4}{3}$ $\star$

Table 3.4: All possible B_4 blocks in G and the estimate of $7f(B) + 2n(B) - 5e(B)$.

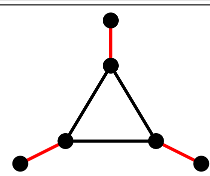
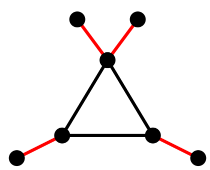
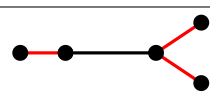
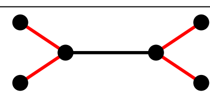
Case	B	Diagram	$f(B) \leq$	$n(B) \leq$	$e(B) =$	$7f + 2n - 5e \leq$
Lemma 3.4.6 3(a)	B_3		$1 + \frac{2}{7} + \frac{1}{4}$	$\frac{3}{2}$	3	$-\frac{5}{4}$
Lemma 3.4.6 3(b)	B_3		$1 + \frac{3}{4}$	$\frac{2}{2} + \frac{1}{3}$	3	$-\frac{1}{12}$
Lemma 3.4.6 4(a)	B_2		$\frac{1}{4} + \frac{1}{7}$	$\frac{2}{2}$	1	$-\frac{1}{4}$
Lemma 3.4.6 4(b)	B_2		$\frac{1}{4} + \frac{1}{7}$	$\frac{1}{2} + \frac{1}{3}$	1	$-\frac{7}{12}$
Lemma 3.4.6 4(c)	B_2		$\frac{1}{4} + \frac{1}{5}$	$\frac{2}{3}$	1	$-\frac{31}{60}$

Table 3.5: All possible B_3 and B_2 blocks in G and the estimate of $7f(B) + 2n(B) - 5e(B)$.

Chapter 4

Planar Turán Number of Double Stars

4.1 Introduction

All the graphs we consider in this chapter are simple and finite. Let G be a graph. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Let $e(G)$ and $v(G)$ denote the number of edges and vertices, respectively. We denote the degree of a vertex v by $d(v)$, the minimum degree in graph G by $\delta(G)$ and the maximum degree in graph G by $\Delta(G)$. The subgraph induced by $S \subseteq V(G)$, is denoted by $G[S]$. Moreover, $N(v)$ denotes the set of vertices in G adjacent to v. Let A and B be disjoint subsets of $V(G)$. Let $e(A, B)$ denote the number of edges in G, that joins a vertex in A and a vertex in B. An m-n edge is an edge such that the end vertices of the edge are with degree m and n. The *Turán number* of a graph H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex graph that does not contain H as a subgraph. Let $\mathrm{EX}(n, H)$ denote the set of extremal graphs, i.e., the set of all n-vertex, H-free graph G such that $e(G) = \mathrm{ex}(n, H)$. The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets X_1 and X_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

Recall that one of the famous problems in Extremal Graph Theory is determining the number of edges in an n-vertex graph to force a certain graph structure. The well-known result of Turán (Theorem 1.0.2) gives the maximum number of edges in an n-vertex graph, containing no complete graph of a given order. The result of Erdős, Stone and Simonovits (Theorem 1.0.3) asymptotically determines $\mathrm{ex}(n, F)$ for all non-bipartite graphs F. In the last decade, the area of 'random' planar graphs has received considerable attention. However, there seem to be no known results on questions analogous to the Erdős-Stone Theorem, i.e., how many edges can a planar graph on n vertices have without containing a given smaller graph? In 2016, Dowden [28] initiated the study of these specific Turán-type problems. The planar Turán number of a graph F, denoted by $\mathrm{ex}_{\mathcal{P}}(n, F)$, is the maximum number of edges in a planar graph on n vertices containing no F as a subgraph.

The analog to Turán's theorem in the case of planar graphs is fairly trivial. Since K_5 is not planar, there are only two meaningful cases. For the K_3, the extremal number of edges is $2n-4$, and the extremal graph is $K_{2,n-2}$ (since all faces have size four when drawn in the plane). Note that there exist planar triangulations not containing K_4 (e.g., take a cycle of length $n-2$ and then add two new vertices that are adjacent to all those in the cycle). Thus, the extremal number in the case of K_4 is $3n-6$. The planar Turán number when the forbidden subgraph is a star is also fairly trivial. The authors in [94] proved that $\mathrm{ex}_{\mathcal{P}}(n, H) = 3n-6$ for all H with $n > |H|+2$ and either $\chi(H) = 4$ or $\chi(H) = 3$ and $\Delta(H) > 7$. They also completely determine $\mathrm{ex}_{\mathcal{P}}(n, H)$ when H is a wheel or a star, and the case when H is a (t, r)-fan, that is, H is isomorphic to $K_1 + tK_{r-1}$, where $t > 2$ and $r > 3$ are integers. The next most natural type of graph to investigate is perhaps a path. For extremal planar Turán number for paths of length $\{6, 7, 8, 9, 10, 11\}$, we refer the reader to [93] and [94]. The next natural extension of the topic is considering double stars as the forbidden graph.

Definition 4.1.1. *An* $(m$,$n)$-double star*, denoted by $S_{m,n}$, is the graph obtained by taking an edge, say xy, and joining one of its end vertices, say x, with m vertices and the other end vertex, y, with n vertices which are different from the m vertices. The edge xy is called the* backbone *of the double star. The vertices adjacent to an end vertex of the backbone are called the* leaf-sets *of the double star. Figure 4.1 shows an m-n double star such that the backbone is xy and the leaf-sets are $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$, respectively.*

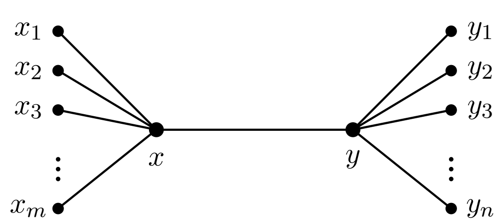


Figure 4.1: An (m, n)-double star.

In this chapter, we address the planar Turán number of double stars. The main results can be summarized as follows:

Theorem 4.1.2. *Estimates on the planar Turán number of double stars $S_{m,n}$ for given values of $\{m,n\}$ are as follows:*

(i) For any $n \geq 16$, $\mathrm{ex}_{\mathcal{P}}(n, S_{2,2}) = 2n - 4$.

(ii) For any $n \geq 1$, $\mathrm{ex}_{\mathcal{P}}(n, S_{2,3}) = 2n$.

(iii) For any $n \geq 1$, $\frac{15}{7}n \leq \mathrm{ex}_{\mathcal{P}}(n, S_{2,4}) \leq \frac{8}{3}n$.

(iv) For $n \geq 1$, $\frac{5}{2}n \leq \mathrm{ex}_{\mathcal{P}}(n, S_{2,5}) \leq \frac{20}{7}n$.

(v) For $n \geq 3$, $\frac{5}{2}n - 5 \leq \mathrm{ex}_{\mathcal{P}}(n, S_{3,3}) \leq \frac{5}{2}n - 2$.

(vi) For $n \geq 1$, $\frac{9}{4}n \leq \mathrm{ex}_{\mathcal{P}}(n, S_{3,4}) \leq \frac{20}{7}n$.

The chapter is structured as follows: Each section proves parts of the previous Theorem one by one.

4.2 Planar Turán number of $S_{2,2}$

We start by proving the following weaker bounds:

Lemma 4.2.1. *Let G be an $S_{2,2}$-free plane graph on n ($n \neq 5$) vertices, then $e(G) \leq 2n - 2$.*

Proof. Suppose that G contains 6 vertices. There are only two 6-vertex maximal planar graphs M_1 and M_2 as shown in Figure 4.2. It can be checked that M_1^- and M_2^- both contain an $S_{2,2}$. Thus, $e(G) \leq 10 = 2n - 2$, when $n = 6$. Now let G be an $S_{2,2}$-free planar graph on 7 vertices. If G contains a vertex of degree at most 2, we are done by induction. Moreover, there is no vertex of degree at least 5 in G. Suppose there is a vertex $x \in V(G)$ such that $d(x) = k$, where $k \geq 5$. In this case, each vertex in $N(x)$ must be of degree at most 2. Otherwise, it is easy to find an $S_{2,2}$ in G, which is a contradiction.

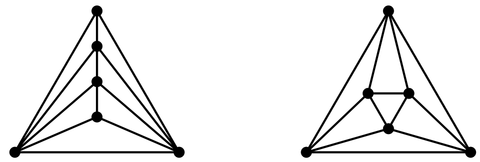


Figure 4.2: The two 6-vertex maximal planar graphs.

Assume that $\Delta(G) \leq 4$. Let the number of vertices in G with degree at most 3 be k. Hence, G contains at least $n-k$ vertices of degree 4, which implies

$$e(G) \leq \frac{4(n-k)+3k}{2} = 2n - \frac{k}{2}.$$

If there are at least 4 vertices of degree at most 3, then $e(G) \leq 2n - \frac{4}{2} = 2n-2$. Let v be a degree 4 vertex in G. If each vertex in $N(v)$ is of degree at most 3, then $e(G) \leq 2n-2$. So, there is a vertex $u \in N(v)$, such that uv is a 4-4 edge in G. Since G is an $S_{2,2}$-free plane graph, uv must be contained in 3 triangles. Let $N(u) \cap N(v) = \{x_1, x_2, x_3\}$ and $S = V(G) \backslash \{u, v, x_1, x_2, x_3\}$. Observe that no vertex in S is adjacent to a vertex in $\{x_1, x_2, x_3\}$. Deleting the vertices $\{u, v, x_1, x_2, x_3\}$. We deleted at most $3 \cdot 5 - 6 = 9$ edges. Hence, $e(G) = e(G - \{u, v, x_1, x_2, x_3\}) + 9 \leq 2(n-5) - 2 + 9 \leq 2n-2$. Hence, we are done by induction. $\square$

Lemma 4.2.2. *Let G be an $S_{2,2}$-free plane graph on n $(n \geq 8)$ vertices. If there is a vertex with degree at least 5, then $e(G) \leq 2n-4$.*

Proof. Let $x \in V(G)$ such that $d(x) = k \geq 5$. Let $N(x) = \{x_1, x_2, x_3, \ldots, x_k\}$, and S be the set of vertices in $V(G) \backslash N(x)$. Each vertex in $N(x)$ is adjacent to at most one other vertex in $N(x)$. Otherwise, it is easy to show that G contains an $S_{2,2}$. Similarly, for an edge $x_i x_j$, where $x_i, x_j \in N(x)$, there is no vertex in S which is adjacent to either x_i or x_j. Thus, the number of edges joining a vertex in $N(x)$ and a vertex in S is at most k. If $|S| \neq 5$, using Lemma 4.2.1, $e(G[S]) \leq 2(n-k-1) - 2$. Therefore $e(G) \leq 2(n-k-1) - 2 + k + k = 2n-4$, and we are done. Let $|S| = 5$. Let the graph induced by S be a K_5^-. Since G is an $S_{2,2}$-free plane graph, no vertex in $N(x)$ is adjacent to any vertex in S. This implies that $e(G[s]) = 9$ and $e(G[N(x)]) \leq \frac{k}{2}$. Hence, $e(G) \leq k + \frac{k}{2} + 9$. Since $n = k+6$, we have $e(G) \leq (n-6) + \frac{n-6}{2} + 9 = \frac{3n}{2} \leq 2n-4$ for $n \geq 8$. $\square$

Proof of Theorem 4.1.2(i). The lower bound is attained by considering the graph $K_{2,n-2}$, which is $S_{2,2}$-free and contains $2n-4$ edges. From Lemma 4.2.2, we may assume that $\Delta(G) \leq 4$. Let k be the number of vertices in G whose degree is at most 3. Then $e(G) \leq \frac{4(n-k)+3k}{2} = 2n - \frac{k}{2}$. If $k \geq 8$, then we are done. We may assume that the number of degree 4 vertices in G is at least 9, since $n \geq 16$. We start by proving the following claims:

Claim 6. *There is no degree* 4 *vertex in* G *such that all its neighbors are of degree at most* 3.

Proof. Suppose not. Let x be a degree 4 vertex in G, such that $d(y) \leq 3$ for all $y \in N(x)$. It is easy to check that for each $y \in N(x)$, if y is adjacent to any vertex in $V(G)\backslash(\{x\} \cup N(x))$, then $d(y) = 2$. Otherwise, G contains an $S_{2,2}$. Therefore, using Lemma 4.2.1, $e(G) = e(G[S]) + 4 + 4 \leq 2(n-5) - 2 + 8 = 2n - 4$. Assume that for each $y \in N(x)$, y is not adjacent to any vertex in $V(G)\backslash(\{x\} \cup N(x))$. Then $e(G[x \cup N(x)]) \leq 8$. Therefore, using Lemma 4.2.1, $e(G) = e(G[S]) + 8 \leq 2(n-5) - 2 + 8 = 2n - 4$. □

Claim 7. *The number of* 4-4 *edges in a matching in* G *is at least* 3.

Proof. Suppose not. Let the number of 4-4 edges in a matching in G be 2. Denote the 4-4 edges in the matching by uv and xy. Each of the edges is contained in 3 triangles. Let $N(u) \cap N(v) = \{u_1, u_2, u_3\}$ and $N(x) \cap N(y) = \{x_1, x_2, x_3\}$. Since G is an $S_{2,2}$-free plane graph, no vertex in $\{u_1, u_2, u_3\}$ is adjacent to a vertex in $V(G)\backslash\{x, y, x_1, x_2, x_3\}$ and no vertex in $\{x_1, x_2, x_3\}$ is adjacent to a vertex in $V(G)\backslash\{u, v, u_1, u_2, u_3\}$. Moreover, at least two vertices in $\{x_1, x_2, x_3\}$ and in $\{u_1, u_2, u_3\}$ are of degree at most three. The number of degree 4 vertices in G is at least 9. Thus, there is a degree 4 vertex, say z, in $V(G)\backslash\{x, y, u, v, x_1, x_2, x_3, u_1, u_2, u_3\}$ such that $d(t) \leq 3$ for every $t \in N(z)$. This contradicts Claim 6. A similar argument can be given, if we assume that the number of 4-4 edges in a matching in G is 1. □

From now on, suppose that the number of 4-4 edges in a matching in G is at least 3. We distinguish the following two cases:

Case 1: The number of 4-4 edges in a matching in G is at least 4. Denote the 4-4 edges in the matching in G by x_1x_2, x_3x_4, x_5x_6 and x_7x_8, respectively. Recall that each edge is contained in 3 triangles. Moreover, at least two vertices in $N(x_i) \cap N(x_{i+1})$ are of degree at most

3, for each $i \in \{1,3,5,7\}$. Thus, we have at least 8 vertices in G whose degree is at most 3. Hence, we are done in this case.

Case 2**: The number of** 4-4 **edges in a matching in** G **is** 3**.** Denote the 4-4 edges in the matching in G by x_1x_2, x_3x_4 and x_5x_6, respectively. At least two vertices in $N(x_i) \cap N(x_{i+1})$ are of degree at most 3, for $i \in \{1,3,5\}$. This implies that G contains at least 6 vertices whose degree is at most 3. Moreover, if a vertex in $N(x_i) \cap N(x_{i+1})$ is of degree at most 2, for some $i \in \{1,3,5\}$, then the remaining two vertices are of degree at most 3. Observe that in this case, $e(G) \leq \frac{4(n-7)+3\cdot 6+2}{2} = 2n-4$ and we are done.

So, we assume exactly two vertices in $N(x_i) \cap N(x_{i+1})$ are of degree 3, for each $i \in \{1,3,5\}$. In this case, the remaining vertex in $N(x_i) \cap N(x_{i+1})$ is of degree 4. Thus, the vertices $\{x_i, x_{i+1}\} \cup (N(x_i) \cap N(x_{i+1}))$, for each $i \in \{1,3,5\}$, induce a K_5^-, and it is a component in G. If $n = 16$, then there is an isolated vertex. In this case, $e(G) = 27 < 2n-4$. If $n > 17$, it is easy to find 2 more vertices of degree at most 3 since the number of 4-4 edges in a matching in G is 3. Thus, there are at least 8 degree 3 vertices in G. This completes the proof of Theorem 4.1.2(i). □

4.3 Planar Turán number of $S_{2,3}$

Proof of theorem 4.1.2(ii). Let G be an $S_{2,3}$-free plane graph on n vertices. Since the graph $S_{2,3}$ contains 7 vertices, a maximal planar graph with $n \leq 6$ vertices does not contain an $S_{2,4}$. Thus, the lower bound is attained by considering disjoint copies of the maximum planar graphs on 6 vertices, i.e., M_1 or M_2 (see Figure 4.2. If the maximum degree in G is at most 4, then $e(G) \leq 2n$. Now we separate the rest of the proof into 2 cases:

Case 1**: There exists a vertex** $v \in V(G)$**, such that** $d(v) \geq 6$**.** It is easy to check that for each $u \in N(v)$, $d(u) \leq 2$, otherwise, we find a copy of $S_{2,3}$ in G. Delete a vertex $u \in N(v)$, then the number of deleted edges is at most 2. By the induction hypothesis, we get $e(G - \{u\}) \leq 2(n-1)$. Hence, $e(G) = e(G - \{u\}) + d(u) \leq 2(n-1) + 2 = 2n$.

Case 2**: There exists a vertex** $v \in V(G)$**, such that** $d(v) = 5$**.** It is easy to check that for each $u \in N(v)$, if u is adjacent to any vertex in $V(G) \backslash (\{v\} \cup N(v))$, then $d(u) = 2$. Otherwise, G

contains an $S_{2,3}$. As in Case 1, we are done by induction. Assume that for each $u \in N(v)$, u is not adjacent to any vertex in $V(G)\backslash(\{v\}\cup N(v))$. Then $e(G[v\cup N(v)]) \leq 3\cdot 6-6 = 12$. By the induction hypothesis, $e(G-\{v\cup N(v)\}) \leq 2(n-6)$. Therefore, $e(G) = e(G-\{v\cup N(v)\})+e(G[v\cup N(v)]) \leq 2n$. □

4.4 Planar Turán number of $S_{2,4}$

Let G be an $S_{2,4}$-free plane graph on n vertices. Since $S_{2,4}$ contains 8 vertices, a maximal planar graph with $n \leq 7$ vertices, does not contain an $S_{2,4}$. Let $7|n$. Consider the plane graph consisting of $\frac{n}{7}$ disjoint copies of maximal planar graphs on 7 vertices. This graph does not contain an $S_{2,4}$. Hence, $\mathrm{ex}_{\mathcal{P}}(n, S_{2,4}) \geq \frac{15}{7}n$.

Claim 8. *Let G be an $S_{2,4}$ on n $(1 \leq n \leq 18)$ vertices. The number of edges in G is at most $\frac{8}{3}n$.*

Proof. Recall that, an n-vertex maximal planar graph contains $3n-6$ edges. Since $3n-6 \leq \frac{8}{3}n$ for $n \leq 18$, $e(G) \leq \frac{8}{3}n$ holds for all n, $1 \leq n \leq 18$. □

Lemma 4.4.1. *If G contains a vertex of degree greater than or equal to 7, then $e(G) \leq \frac{8}{3}n$.*

Proof. Let $v \in V(G)$, such that $d(v) \geq 7$. It is easy to check that for each $u \in N(v)$, $d(u) \leq 2$, otherwise, we find a copy of $S_{2,4}$ in G. Delete a vertex $u \in N(v)$, then the number of deleted edges is at most 2. By the induction hypothesis, we get $e(G-\{u\}) \leq \frac{8}{3}(n-1)$. Hence, $e(G) = e(G-\{u\})+d(u) \leq \frac{8}{3}(n-1)+2 \leq \frac{8}{3}n$. □

Lemma 4.4.2. *If G contains a vertex of degree 6, then $e(G) \leq \frac{8}{3}n$.*

Proof. Let $v \in V(G)$, such that $d(v) = 6$ and let $H = N(v)\cup\{v\}$. It is easy to check that for each $u \in N(v)$, if u is adjacent to any vertex in $V(G)\backslash H$, then $d(u) = 2$. Otherwise, G contains an $S_{2,4}$. We are done by induction in this case. Assume that for each $u \in N(v)$, u is not adjacent to any vertex in $V(G)\backslash H$. Then $e(G[H]) \leq 3\cdot 7-6 = 15$. Hence, $e(G) = e(G-H)+e(G[H]) \leq \frac{8}{3}(n-7)+15 \leq \frac{8}{3}n$. We are done by induction. □

Proof of Theorem 4.1.2(iii). If G contains a vertex of degree at least 6, we are done by Lemmas 4.4.1 and 4.4.2. Hence, we can assume that $\Delta(G) \leq 5$.

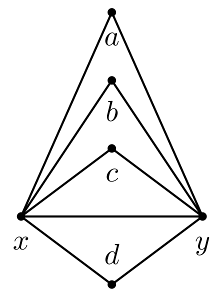


Figure 4.3: The graph G has a $5-5$ edge xy, with 4 triangles sitting on the edge xy.

Claim 9. *If G contains a $5-5$ edge, then $e(G) \leq \frac{8}{3}n$.*

Proof. Let $xy \in E(G)$ be a $5-5$ edge. There are at least 3 triangles sitting on the edge xy, otherwise G contains an $S_{2,4}$. We subdivide the cases based on the number of triangles sitting on the edge xy.

1. **There are 4 triangles sitting on the edge xy.** Let a, b, c and d be the vertices in G which are adjacent to both x and y (see Figure 4.3). Let $S_1 = \{a, b, c, d\}$ and $H = \{x, y, a, b, c, d\}$. Delete the vertices in H. The vertices in S_1 can have at most one neighbor in $V(G)\backslash H$ each and can form a path of length 3 in S_1. Hence, the number of edges deleted is at most $9+4+3 = 16$. Using the induction hypothesis, $e(G) \leq e(G-H)+16 \leq \frac{8}{3}(n-6)+16 = \frac{8n}{3}$.

2. **There are 3 triangles sitting on the edge xy.** Let a, b and c be the vertices in G which are adjacent to both x and y. Let d be the vertex adjacent to x but not adjacent to y, and e be the vertex adjacent to y but not adjacent to x. Let $S_1 = \{a, b, c\}$ and $H = \{x, y, d, e\} \cup S_1$. Delete the vertices in H. The vertices d and e can have at most one neighbor in $V(G)\backslash H$ each. We distinguish the cases as follows:

   (a) **The vertices d and e have no neighbors in S_1**, see Figure 4.4(i). The vertices in S_1 can have at most one neighbor in $V(G)\backslash H$ each and can form a path of length 2 in S_1. If the vertices d and e are adjacent, they cannot have a neighbor in $V(G)\backslash H$. Otherwise, we have an $S_{2,4}$ with dx (or ey) as the backbone. Thus, the number of edges deleted is at most $9+2+3+2 = 16$.

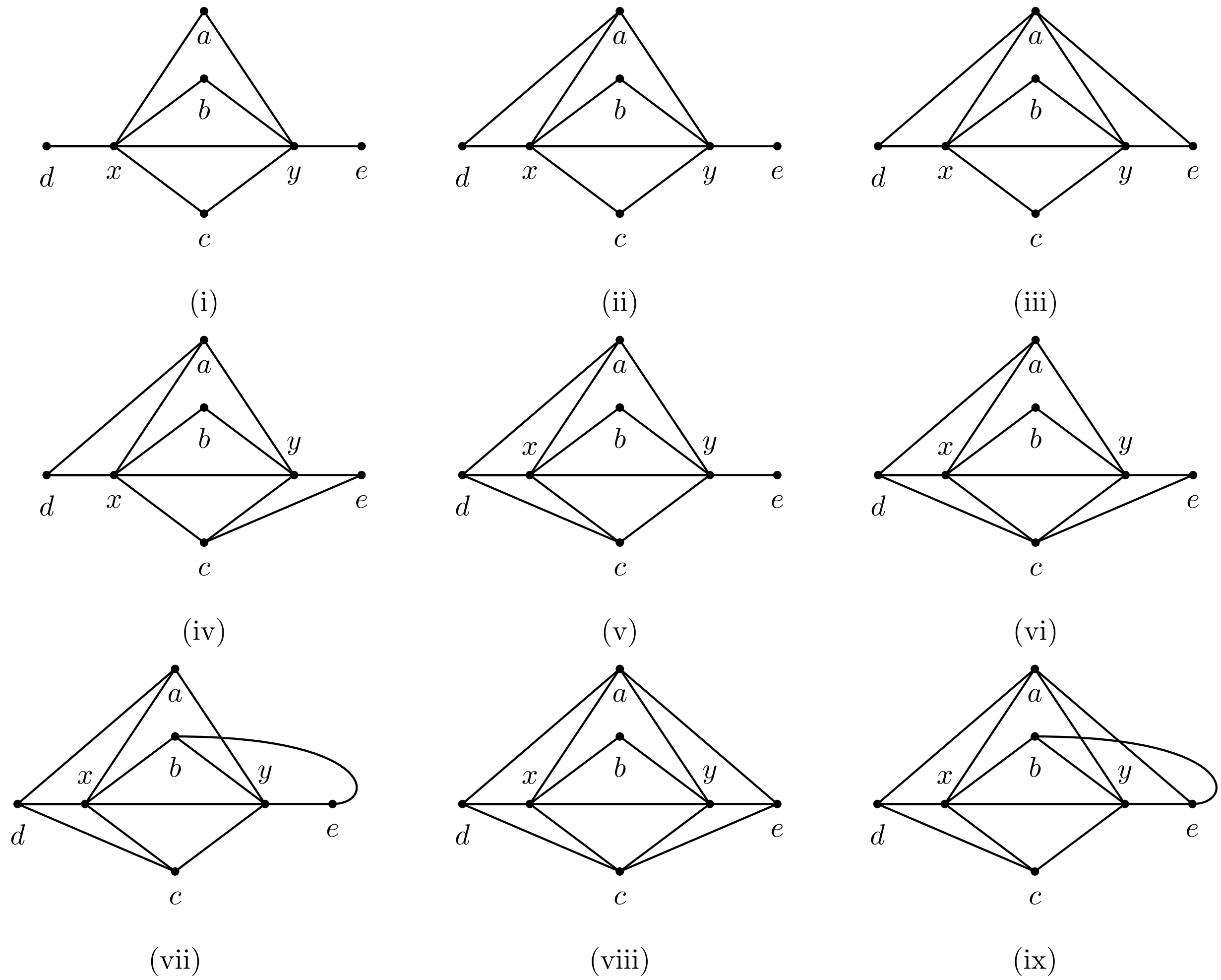


Figure 4.4: The graph G has a $5-5$ edge xy, with 3 triangles sitting on the edge xy.

(i) The vertices d and e have no neighbors in S_1.

(ii) The vertex d has one neighbor in S_1, and e has none.

(iii) The vertices d and e have one common neighbor in S_1.

(iv) The vertices d and e have one distinct neighbor in S_1.

(v) The vertex d has two neighbors in S_1, while e has none.

(vi) The vertex d is the neighbor of a and c, and while e is the neighbor of c.

(vii) The vertex d is the neighbor of a and c, while e is the neighbor of b.

(viii) The vertices d and e are neighbors of a and c both.

(ix) The vertex d is the neighbor of a and c, while e is the neighbor of a and b.

(b) **One of the vertices d or e has one neighbor in S_1, and the other has none.** Without loss of generality, suppose a is the neighbor of d, see Figure 4.4(ii). Note that a cannot have a neighbor in $V(G)\backslash H$. Otherwise, we have an $S_{2,4}$ with ay as the backbone. If the vertices d and e are adjacent, they cannot have a neighbor in $V(G)\backslash H$. Similarly, as before, the vertices b and c can have at most one neighbor in $V(G)\backslash H$ each and the vertices $\{a, b, c\}$ can form a path of length 2 in S_1. Thus, the number of edges deleted is at most $10 + 2 + 2 + 2 = 16$.

(c) **The vertices d and e have one neighbor in S_1.** There are two possibilities. In the first case, without loss of generality, suppose a is the common neighbor of d and e, see Figure 4.4(iii). Similarly, as before, a cannot have a neighbor in $V(G)\backslash H$. If the vertices d and e are adjacent, they cannot have a neighbor in $V(G)\backslash H$. The vertices b and c can have at most one neighbor in $V(G)\backslash H$ each, and the vertices $\{a, b, c\}$ can form a path of length 2 in S_1. Thus, the number of edges deleted is at most $11 + 2 + 2 + 2 = 17$.

Without loss of generality, suppose a is the neighbor of d while c is the neighbor of e, see Figure 4.4(iv). Similarly, as before, the vertices a and c cannot have a neighbor in $V(G)\backslash H$. The vertex b can have at most one neighbor in $V(G)\backslash H$ and the vertices $\{a, b, c\}$ can form a path of length 2 in S_1. Thus, the number of edges deleted is at most $11 + 2 + 1 + 2 = 16$. (If the vertices d and e are adjacent, there can only be a path of length 1 inside S_1. Again, this precision is unnecessary. We skip this in the following cases also.)

(d) **One of the vertices d or e has two neighbors in S_1, while the other has none.** Without loss of generality, suppose d is the neighbor of a and c, see Figure 4.4(v). Similarly, as before, the vertices a and c cannot have a neighbor in $V(G)\backslash H$. The vertex b can have at most one neighbor in $V(G)\backslash H$ and the vertices $\{a, b, c\}$ can form a path of length 2 in S_1. If the vertices d and e are adjacent, they cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $11 + 2 + 1 + 2 = 16$.

(e) **One of the vertices d or e has two neighbors in S_1, while the other has one neighbor.** There are two possibilities. In the first case, without loss of generality, suppose d is the neighbor of a and c, and e is the neighbor of c (see Figure 4.4(vi)). Similarly, as before, the vertices a and c cannot have a neighbor in $V(G)\backslash H$. The vertex b can have at most one neighbor in $V(G)\backslash H$ and the vertices $\{a, b, c\}$ can form a path of length 2 in S_1. If the vertices d and e are adjacent, they cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $12+2+1+2=17$.

In the other case, without loss of generality, assume that d is the neighbor of a and c, and e is the neighbor of b (see Figure 4.4(vii)). The vertices a, b and c cannot have a neighbor in $V(G)\backslash H$, but they can form a path of length 2 in S_1. Thus, the number of edges deleted is at most $12+2+2=16$.

(f) **Both the vertices d and e have two neighbors in S_1.** There are two possibilities. In the first case, without loss of generality, suppose d and e are the neighbors of a and c both (see Figure 4.4(viii)). Similarly, as before, the vertices a and c cannot have a neighbor in $V(G)\backslash H$. The vertex b can have one neighbor in $V(G)\backslash H$. The vertices $\{a, b, c\}$ can form a path of length 2 in S_1. If the vertices d and e are adjacent, they cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $13+1+2+2=18$.

On the other hand, without loss of generality, assume d is the neighbor of a and c, while e is the neighbor of a and b (see Figure 4.4(ix)). The vertices a, b and c cannot have a neighbor in $V(G)\backslash H$, but they can form a path of length 2 in S_1. Thus, the number of edges deleted is at most $13+2+2=17$.

Using the induction hypothesis,

$$e(G) \leq e(G-H)+18 \leq \frac{8}{3}(n-7)+18=\frac{8}{3}n.$$

This completes the proof. $\square$

Take $x, y \in V(G)$. By the previous claims, if $d(x)+d(y) \geq 10$, we are done by induction. Assume that $d(x)+d(y) \leq 9$. Summing it over all the edge pairs in G, we have $9e \geq \sum_{xy \in E(G)}(d(x)+d(y)) = \sum_{x \in V(G)}(d(x))^2 \geq n\overline{d}^2 = n(\frac{2e}{n})^2$, where $\overline{d}$ is the average degree in G. This gives us $e \leq \frac{9}{4}n \leq \frac{8}{3}n$ for $n \geq 1$. $\square$

4.5 Planar Turán number of $S_{2,5}$

Let G be an $S_{2,5}$-free plane graph on n vertices. Let $12|n$. Consider the plane graph consisting of $\frac{n}{12}$ disjoint copies of 5-regular maximal planar graphs with 12 vertices, see Figure 4.5. This graph does not contain an $S_{2,5}$, since it is a $5-$regular graph. Hence, $\mathrm{ex}_{\mathcal{P}}(n, S_{2,4}) \geq \frac{5}{2}n$.

Claim 10. *Let G be an $S_{2,5}$ on n $(1 \leq n \leq 42)$ vertices. The number of edges in G is at most $\frac{20}{7}n$.*

Proof. Recall that, an n-vertex maximal planar graph contains $3n-6$ edges. Since $3n-6 \leq \frac{20}{7}n$ for $n \leq 42$, $e(G) \leq \frac{20}{7}n$ holds for all n, $1 \leq n \leq 42$. □

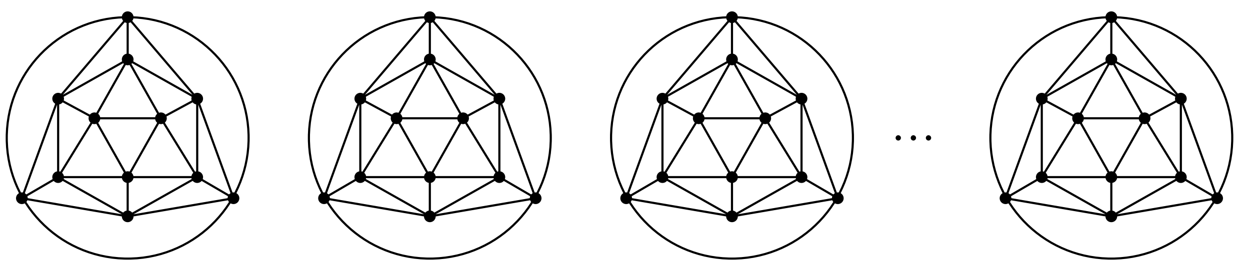


Figure 4.5: $(n/12)$-disjoint copies of 5-regular maximal planar graphs with 12 vertices.

Lemma 4.5.1. *If G contains a vertex v of degree greater than or equal to 8, then $e(G) \leq \frac{20}{7}n$.*

Proof. Let $v \in V(G)$, such that $d(v) \geq 8$. It is easy to check that for each $u \in N(v)$, $d(u) \leq 2$. Otherwise, we find a copy of $S_{2,5}$ in G. Delete a vertex $u \in N(v)$, then the number of deleted edges is at most 2. By the induction hypothesis, we get $e(G - \{u\}) \leq \frac{20}{7}(n-1)$. Hence, $e(G) = e(G - \{u\}) + d(u) \leq \frac{20}{7}(n-1) + 2 \leq \frac{20}{7}n$. □

Lemma 4.5.2. *If G contains a vertex v of degree equal to 7, then $e(G) \leq \frac{20}{7}n$.*

Proof. Let $v \in V(G)$, such that $d(v) = 7$ and let $H = N(v) \cup \{v\}$. It is easy to check that for each $u \in N(v)$, if u is adjacent to any vertex in $V(G)\backslash H$, then $d(u) = 2$. Otherwise, G contains an $S_{2,5}$. We are done by induction in this case. In the other case, assume that for each $u \in N(v)$, u is not adjacent to any vertex in $V(G)\backslash H$. Then $e(G[H]) \leq 3 \cdot 8 - 6 = 18$. Hence, $e(G) = e(G-H) + e(G[H]) \leq \frac{20}{7}(n-8) + 18 = \frac{20}{7}n$. We are done by induction. □

Proof of Theorem 4.1.2(iv). If G contains a vertex of degree at least 7, we are done by Lemmas 4.5.1 and 4.5.2. Hence, we can assume $\Delta(G) \leq 6$.

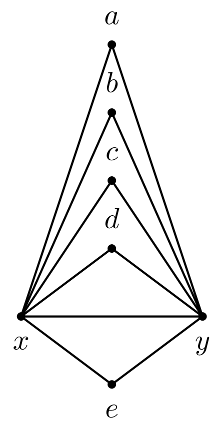


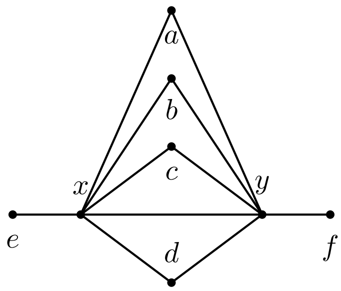
Figure 4.6: The graph G has a $6-6$ edge xy, with 5 triangles sitting on the edge xy.

Claim 11. *If G contains a $6-6$ edge, then $e(G) \leq \frac{20}{7}n$.*

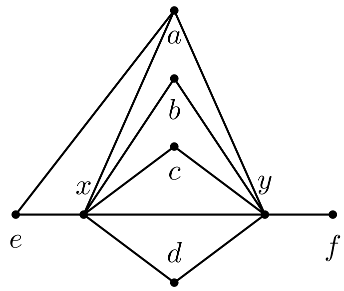
Proof. Let $xy \in E(G)$ be a 6-6 edge. There are at least 4 triangles sitting on the edge xy, otherwise G contains an $S_{2,5}$. We subdivide the cases based on the number of triangles sitting on the edge xy.

1. **There are 5 triangles sitting on the edge xy.** Let a, b, c, d and e be the vertices in G which are adjacent to both x and y (see Figure 4.6). Let $S_1 = \{a, b, c, d, e\}$ and $H = \{x, y, a, b, c, d, e\}$. Delete the vertices in H. The vertices in S_1 can have at most one neighbor in $V(G)\backslash H$ each and can form a path of length 4 in S_1. Hence, the number of edges deleted is at most $11 + 5 + 4 = 20$. Using the induction hypothesis, $e(G) \leq e(G - H) + 20 \leq \frac{20}{7}(n-7) + 20 = \frac{20}{7}n$.

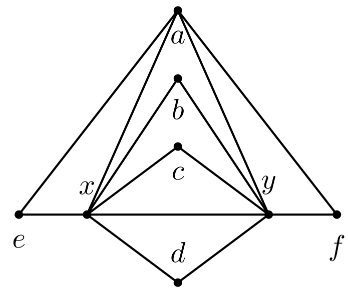
2. **There are 4 triangles sitting on the edge xy.** Let a, b, c and d be the vertices in G which are adjacent to both x and y. Let e be the vertex adjacent to x but not adjacent to y, and f be the vertex adjacent to y but not adjacent to x. Let $S_1 = \{a, b, c, d\}$ and $H = \{x, y\} \cup S_1 \cup \{e, f\}$. Delete the vertices in H. The vertices e and f can have at most one neighbor in $V(G)\backslash H$ each. We distinguish the cases based on the neighbors of e and f as follows:



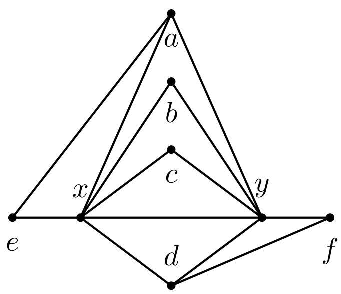
(i)



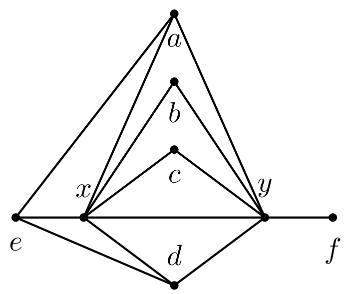
(ii)



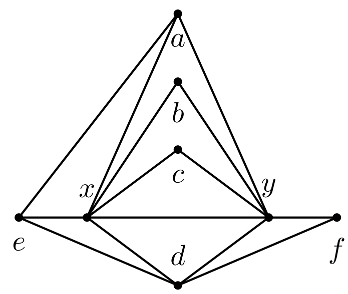
(iii)



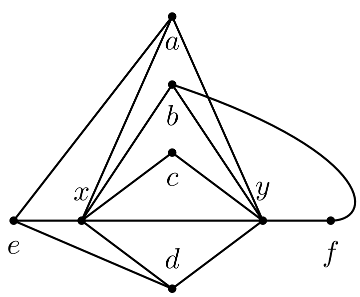
(iv)



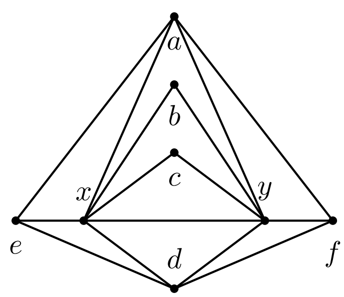
(v)



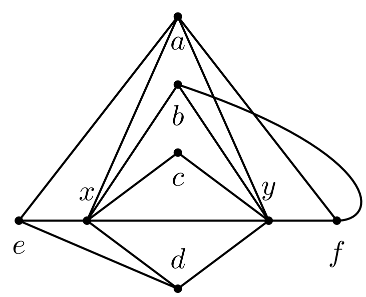
(vi)



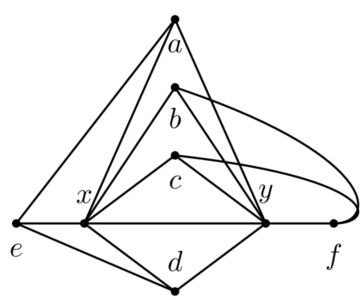
(vii)



$(viii)$



(ix)



(x)

Figure 4.7: The graph G has a $6-6$ edge xy, with 4 triangles sitting on the edge xy.

(i) the vertices e and f have no neighbors in S_1.

(ii) The vertex e has one neighbor in S_1 and f has none.

(iii) The vertices e and f have one common neighbor in S_1.

(iv) The vertices e and f have one distinct neighbor in S_1.

(v) The vertex e has two neighbors in S_1 and f has none.

(vi) The vertex e is the neighbor of a and d, and f is the neighbor of d.

(vii) The vertex e is the neighbor of a and d, and f is the neighbor of b.

(viii) The vertices e and f are neighbors of both a and d.

(ix) The vertex e is the neighbor of a and d, while f is the neighbor of a and b.

(x) The vertex e is the neighbor of a and d, while f is the neighbor of b and c.

(a) **The vertices e and f have no neighbors in S_1**, see Figure 4.7(i). The vertices in S_1 can have at most one neighbor in $V(G)\backslash H$ each and can form a path of length 3 in S_1. If the vertices e and f are adjacent, then they cannot have a neighbor in $V(G)\backslash H$. Otherwise, we have an $S_{2,5}$ with ex (or fy) as the backbone. Thus, the number of edges deleted is at most $11+3+4+2=20$.

(b) **One of the vertices e or f has one neighbor in S_1, and the other has none.** Without loss of generality, suppose a and e are adjacent (see Figure 4.7(ii)), then a cannot have a neighbor in $V(G)\backslash H$. Otherwise, we have an $S_{2,5}$ with ay as the backbone. If the vertices e and f are adjacent, then they cannot have a neighbor in $V(G)\backslash H$. Similarly, as before, the vertices b, c and d can have at most one neighbor in $V(G)\backslash H$ each and the vertices $\{a, b, c, d\}$ can form a path of length 3 in S_1. Thus, the number of edges deleted is at most $12+3+3+2=20$.

(c) **The vertices e and f have one neighbor in S_1.** There are two possibilities. In the first case, without loss of generality, suppose a is the common neighbor of e and f, see Figure 4.7(iii). Similarly, as before, a cannot have a neighbor in $V(G)\backslash H$. If the vertices e and f are adjacent, they cannot have a neighbor in $V(G)\backslash H$. The vertices b, c and d can have at most one neighbor in $V(G)\backslash H$ each, and the vertices $\{a, b, c, d\}$ can form a path of length 3 in S_1. Thus, the number of edges deleted is at most $13+3+3+2=21$. Without loss of generality, suppose a is the neighbor of e while d is the neighbor of f, see Figure 4.7(iv). Similarly, as before, the vertices a and d cannot have a neighbor in $V(G)\backslash H$. The vertex b and c can have at most one neighbor in $V(G)\backslash H$ each, and the vertices $\{a,b,c,d\}$ can form a path of length 3 in S_1. If the vertices e and f are adjacent, then they cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $13+3+2+2=20$. (In fact, it can be shown that if the vertices w and f are adjacent, there can only be a 2-path inside S_1. However, this precision is unnecessary. We skip this in the following cases also.)

(d) **One of the vertices e or f has two neighbors in S_1, while the other has none.** Without loss of generality, suppose e is the neighbor of a and d, see Figure 4.7(v). Similarly, as before, the vertices a and d cannot have a neighbor in $V(G)\backslash H$. The vertices b and c can have at most one neighbor in $V(G)\backslash H$ each, and the vertices $\{a,b,c,d\}$ can form a path of length 3 in S_1. If the vertices e and f are adjacent, then they cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $13+3+2+2=20$.

(e) **One of the vertices e or f has two neighbors in S_1, while the other has one neighbor.** There are two possibilities. In the first case, without loss of generality, suppose e is the neighbor of a and d, and f is the neighbor of d (see Figure 4.7(vi)). Similarly, as before, the vertices a and d cannot have a neighbor in $V(G)\backslash H$. The vertices b and c can have at most one neighbor in $V(G)\backslash H$ each, and the vertices $\{a,b,c,d\}$ can form a path of length 3 in S_1. If the vertices e and f are adjacent, then they cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $14+3+2+2=21$. In the other case, without loss of generality, assume that e is the neighbor of a and d, and f is the neighbor of b (see Figure 4.7(vii)). The vertices a, b and d cannot have a neighbor in $V(G)\backslash H$, but they along with c can form a path of length 3 in S_1. The vertex c can have at most one neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $14+3+1+2=20$.

(f) **Both the vertices e and f have two neighbors in S_1.** There are three possibilities. In the first case, without loss of generality, suppose e and f are neighbors of both a and d, see Figure 4.7(viii). Similarly, as before, the vertices a and d cannot have a neighbor in $V(G)\backslash H$. The vertices b and c can have at most one neighbor in $V(G)\backslash H$ each. The vertices $\{a, b, c, d\}$ can form a path of length 3 in S_1. If the vertices e and f are adjacent, then they cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $15 + 3 + 2 + 2 = 22$.

On the other hand, without loss of generality, assume e is the neighbor of both a and d, while f is the neighbor of a and b (see Figure 4.7(ix)). The vertices a, b and d cannot have a neighbor in $V(G)\backslash H$, but they along with c can form a path of length 3 in S_1. The vertex c can have at most one neighbor in $V(G)\backslash H$ each. Thus, the number of edges deleted is at most $15 + 3 + 1 + 2 = 21$.

In the last case, without loss of generality, assume e is the neighbor of a and d both, while f is the neighbor of b and c (see Figure 4.7(x)). The vertices a, b, c and d cannot have a neighbor in $V(G)\backslash H$, but they can form a path of length 3 in S_1. Thus, the number of edges deleted is at most $15 + 3 + 2 = 20$.

Using the induction hypothesis,

$$e(G) \leq e(G - H) + 22 \leq \frac{20}{7}(n - 8) + 22 = \frac{20}{7}n - \frac{6}{7}.$$

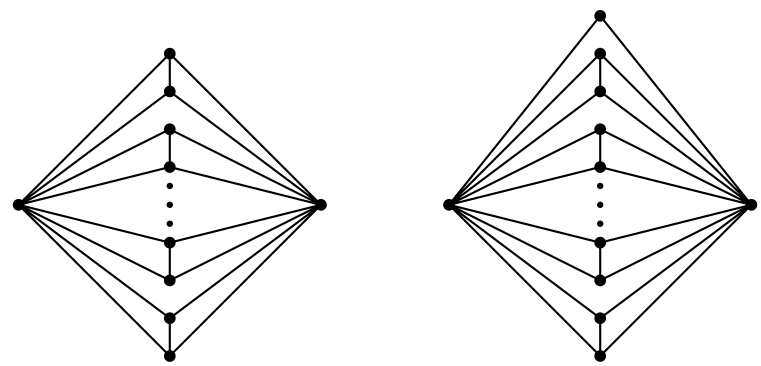
This completes the proof. $\square$

Consider $x, y \in V(G)$. By the previous claims, if $d(x) + d(y) \geq 12$, we are done by induction. Assume that $d(x) + d(y) \leq 11$. Summing it over all the edge pairs in G, we have $11e(G) \geq \sum_{xy\in E(G)} (d(x) + d(y)) = \sum_{x\in V(G)}(d(x))^2 \geq n\overline{d}^2 = n(\frac{2e(G)}{n})^2$, where $\overline{d}$ is the average degree in G. This gives us $e(G) \leq \frac{11}{4}n \leq \frac{20}{7}n$, for $n \geq 1$. $\square$

4.6 Planar Turán number of $S_{3,3}$

We show that for infinitely many integer values of n, we can construct an n-vertex $S_{3,3}$-free plane graph G_n with $\frac{5}{2}n - 5$ edges. This is to verify that the bound we have is best up to the linear term. Consider a plane graph G_n which is obtained by joining every vertex of the maximal matching on $n - 2$ vertices with two vertices. Constructions of G_n when n is even or odd is shown in Figure

4.8. Each edge in G_n has a degree 3 end vertex. Thus, G_n is an $S_{3,3}$-free planar graph. Moreover, $e(G_n) = \lfloor \frac{5}{2}n \rfloor - 5$.



(a) n is even. (b) n is odd.

Figure 4.8: Extremal Constructions for the lower bound of planar Turán number of $S_{3,3}$.

Claim 12. *Let G be an $S_{3,3}$ on $n(1 \leq n \leq 8)$ vertices. The number of edges in G is at most $\frac{5}{2}n - 2$.*

Proof. Recall that, an n-vertex maximal planar graph contains $3n-6$ edges. Since $3n-6 \leq \frac{5}{2}n-2$ for $n \leq 8$, $e(G) \leq \frac{5}{2}n - 2$ holds for all n, $1 \leq n \leq 8$. □

Let u be a vertex in G with degree at most 2. By the induction hypothesis, we get $e(G-\{u\}) \leq \frac{20}{7}(n-1)$. Hence, $e(G) = e(G-\{u\}) + d(u) \leq \frac{5}{2}(n-1) - 2 + 2 < \frac{5}{2}n - 2$. Similarly, if we have a 3-3 edge in G, we can finish the proof by induction. From now on, we may assume that G contains no vertex of degree at most 2 and no 3-3 edge. The following claims deal with the different cases of degree pairs in G:

Claim 13. *No vertex in G with a degree at least 7 is adjacent to a vertex of degree at least 4.*

Proof. Suppose not. Let xy be an edge in G such that $d(x) \geq 7$ and $d(y) \geq 4$. Obviously, there are three vertices in $V(G)\backslash\{x\}$, say y_1, y_2 and y_3, which are adjacent to y. Since $|N(x)\backslash\{y\}| \geq 6$, there are three vertices x_1, x_2 and x_3, not in $\{y, y_1, y_2, y_3\}$ which are adjacent to x. This implies we got an $S_{3,3}$ in G with backbone xy and leaf-sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$, respectively, which is a contradiction. This completes the proof of Claim 13. □

Claim 14. *If there is a 6-6 edge in G, then $e(G) \leq \frac{5}{2}n - 2$.*

Proof. Let $xy \in E(G)$ be a 6-6 edge. Since G is an $S_{3,3}$-free plane graph, xy must be contained in 5 triangles, see Figure 4.9. Let a, b, c, d and e be the vertices in G which are adjacent to both x

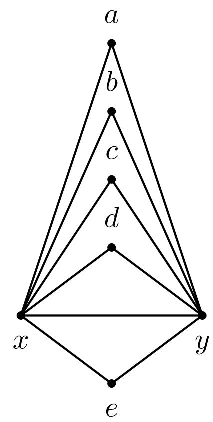


Figure 4.9: The graph G has a $6-6$ edge xy.

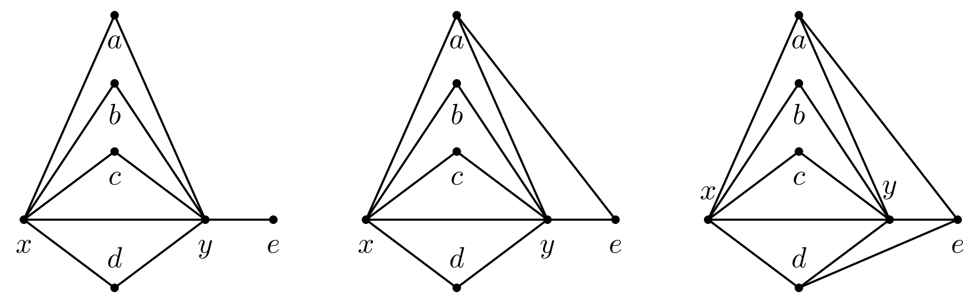
and y. Let $S_1 = \{a, b, c, d, e\}$ and $H = \{x, y, a, b, c, d, e\}$. Delete the vertices in H. Suppose a has two neighbors in $V(G)\backslash H$. We immediately get an $S_{3,3}$ with xa or ya as the backbone. Thus, any vertex in the set S_1 can have at most 1 neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, we deleted at most $11 + 5 = 16$ edges. Assume that there is an edge between the vertices in S_1, say ab. If a (or b) has a neighbor in $V(G)\backslash H$, xa (or xb) is the backbone of an $S_{3,3}$. Similarly, for the other edges in S_1, both the vertices cannot have a neighbor in $V(G)\backslash H$. Thus, if there is an edge joining any two vertices in S_1, the number of edges deleted is at most $11+4 = 15$. Using the induction hypothesis,

$$e(G) \leq e(G-H) + 16 \leq \frac{5}{2}(n-7) - 2 + 16 = \frac{5}{2}n - 3.5 < \frac{5}{2}n - 2.$$

This completes the proof. □

Claim 15. *If there is a* 5-6 *edge in* G*, then* $e(G) \leq \frac{5}{2}n - 2$.

Proof. Let xy be a 5-6 edge in G. Since G is an $S_{3,3}$-free plane graph, xy must be contained in 4 triangles, see Figure 4.10. Let a, b, c and d be the vertices in G which are adjacent to both x and y. Let e be the vertex adjacent to y but not adjacent to x. Let $S_1 = \{a, b, c, d\}$ and $H = \{x, y, a, b, c, d, e\}$. Delete the vertices in H. Any vertex in S_1 can have at most 1 neighbor in $V(G)\backslash H$. If there is an edge joining any two vertices in S_1, say ab. Similarly, as before, the vertices



(i) (ii) (iii)

Figure 4.10: The graph G has a $5-6$ edge xy.

(i) The vertex e has no neighbors in S_1.

(ii) The vertex e has one neighbor in S_1.

(iii) The vertex e has two neighbors in S_1.

a and b cannot have a neighbor in $V(G)\backslash H$. We further distinguish the cases based on the number of edges from e as follows:

1. **The vertex e has no neighbors in** S_1, see Figure 4.10(i). Clearly, the vertex e can have at most 2 neighbors in $V(G)\backslash H$. If there are no edges between the vertices in S_1, we deleted at most $10+2+4=16$ edges. If there is an edge joining any two vertices in S_1, the number of edges deleted is at most $10+2+3=15$.

2. **The vertex e has one neighbor in S_1.** Without loss of generality, suppose a is the neighbor of e, see Figure 4.10(ii). The vertex e can have at most one neighbor in $V(G)\backslash H$. If the vertex a has a neighbor in $V(G)\backslash H$, we have an $S_{3,3}$ with xa as the backbone. If there are no edges between the vertices in S_1, we have deleted at most $11+3+1=15$ edges. If there is an edge joining any two vertices in S_1, the number of edges deleted is at most $11+3+1=15$.

3. **The vertex e has two neighbors in S_1.** Without loss of generality, suppose e is the neighbor of a and d, see Figure 4.10(iii). The vertex e cannot have a neighbor in $V(G)\backslash H$, otherwise we have an $S_{3,3}$ with ye as the backbone. If either a or d has a neighbor in $V(G)\backslash H$, we have an $S_{3,3}$ with xa or xd as the backbone, respectively. Suppose there is no edge between the vertices in S_1. The total number of edges deleted is at most $12+2=14$. Suppose a and

b are adjacent, then b cannot have a neighbor in $V(G)\backslash H$. Similarly, for the other edges in S_1 except ad, which is still possible without any extra restrictions. If there is an edge joining any two vertices in S_1, the total number of edges deleted is at most $12+3=15$.

Thus, by induction

$$e(G) = e(G-H) + 16 \leq \frac{5}{2}(n-7) - 2 + 16 < \frac{5}{2}n - 2,$$

and we are done. □

Claim 16. *If there is a* 4-6 *edge in* G*, then* $e(G) \leq \frac{5}{2}n - 2$.

Proof. Let xy be a $4-6$ edge in G. Since G is an $S_{3,3}$-free plane graph, xy must be contained in 3 triangles, see Figure 4.11. Let a, b and c be the vertices in G which are adjacent to both x and y. Let d and e be the vertices adjacent to y but not to x. Let $S_1 = \{a, b, c\}$ and $H = \{x, y, a, b, c, d, e\}$. Delete the vertices in H. The vertices d and e can have at most two neighbors in $V(G)\backslash H$ each. The vertices in S_1 can have at most one neighbor in $V(G)\backslash H$ each. If there is an edge joining any two vertices in S_1, say ab. Similarly, as before, the vertices a and b cannot have a neighbor in $V(G)\backslash H$. We distinguish the cases based on the neighbors of d and e as follows:

1. **The vertices d and e have no neighbors in S_1**, see Figure 4.11(i). If the vertices d and e are adjacent, they can have at most one neighbor in $V(G)\backslash H$ each. Otherwise, we have an $S_{3,3}$ with dy (or ey) as the backbone. If there are no edges between the vertices in S_1, the number of edges deleted is at most $9+3+4=16$. If there is an edge joining any two vertices in S_1, the number of edges deleted is at most $9+2+4=15$.

2. **One of the vertices d or e has one neighbor in S_1, while the other has none.** Without loss of generality, suppose the vertices a and d are adjacent (see Figure 4.11(ii)). The vertex a cannot have a neighbor in $V(G)\backslash H$, and d can have at most one neighbor in $V(G)\backslash H$. If the vertices d and e are adjacent, then d cannot have a neighbor in $V(G)\backslash H$ and e can have at most one neighbor in $V(G)\backslash H$. Otherwise, we have an $S_{3,3}$ with dy (or ey) as the backbone. If there are no edges between the vertices in S_1, the number of edges deleted

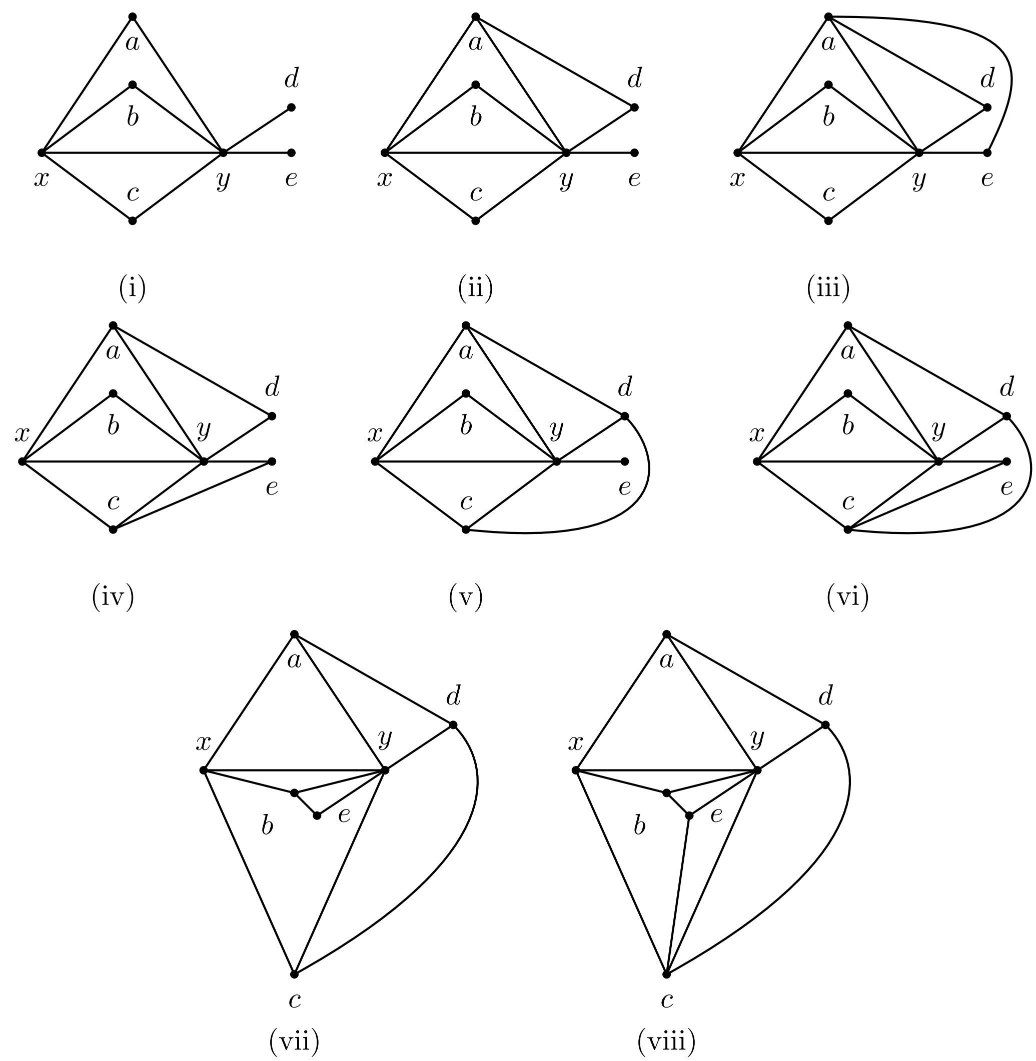


Figure 4.11: The graph G has a $4-6$ edge xy.

(i) The vertices d and e have no neighbors in S_1.

(ii) The vertex d has one neighbor in S_1, and f has none.

(iii) Both the vertices d and e have one common neighbor in S_1.

(iv) The vertices d and e have one distinct neighbor in S_1

(v) The vertex d has two neighbors in S_1, while e has none.

(vi) The vertex d is the neighbor of a and c, while e is the neighbor of c.

(vii) The vertex d is the neighbor of a and c, while e is the neighbor of b.

(viii) Both the vertices d and e have two neighbors in S_1.

is at most $10 + 2 + 3 = 15$. If there is an edge joining any two vertices in S_1, the number of edges deleted is at most $10 + 2 + 3 = 15$.

3. **Both the vertices d and e have one neighbor in S_1.** There are two possibilities. In the first case, without loss of generality, suppose a is the common neighbor of d and e (see Figure 4.11(iii)). Similarly, as before, a cannot have a neighbor in $V(G)\backslash H$, and d and e can have at most one neighbor in $V(G)\backslash H$ each. If the vertices d and e are adjacent, they cannot have a neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $11 + 2 + 2 = 15$. If there is an edge joining any two vertices in S_1, the number of edges deleted is at most $11 + 2 + 2 = 15$.

   Without loss of generality, suppose a is the neighbor of d while c is the neighbor of e (see Figure 4.11(iv)). Similarly, as before, the vertices a and c cannot have a neighbor in $V(G)\backslash H$, and d and e can have at most one neighbor in $V(G)\backslash H$ each. If the vertices d and e are adjacent, they cannot have a neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $11 + 1 + 2 = 14$. If a and b are adjacent, then b does not have a neighbor in $V(G)\backslash H$. Similarly, for the edge bc. The vertices a and c can be adjacent without any constraints. Thus, if there is an edge joining any two vertices in S_1, the number of edges deleted is at most $11 + 2 + 2 = 15$.

4. **One of the vertices d or e has two neighbors in S_1, while the other has none.** Without loss of generality, suppose d is the neighbor of a and c, while e has no neighbors in S_1 (see Figure 4.11(v)). The vertex d cannot have a neighbor in $V(G)\backslash H$. Similarly, as before, the vertices a and c cannot have a neighbor in $V(G)\backslash H$, whereas e can have at most two neighbors in $V(G)\backslash H$. If d and e are adjacent, then e can have at most one neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $11 + 1 + 2 = 14$. If a and b are adjacent, then b does not have a neighbor in $V(G)\backslash H$. Similarly, for the edge bc. The vertices a and c can be adjacent without any constraints. Thus, if there is an edge joining any two vertices in S_1, the number of edges deleted is at most $11 + 2 + 2 = 15$.

5. **One of the vertices d or e has two neighbors in S_1, while the other has one.** There are two possibilities. In the first case, without loss of generality, suppose d is the neighbor of a and c, while e is the neighbor of c (see Figure 4.11(vi)). Similarly, as before, the vertices

a, c and d cannot have a neighbor in $V(G)\backslash H$, whereas e can have at most one neighbor in $V(G)\backslash H$. If d and e are adjacent, then e cannot have a neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $12+1+1=14$. In the other case, the vertices a and c can be adjacent without any constraints. Thus, if there is an edge joining any two vertices in S_1, the number of edges deleted is at most $12+2+1=15$.

Without loss of generality, suppose d is the neighbor of a and c, while e is the neighbor of b (see Figure 4.11(vii)). Similarly, as before, the vertices a, b, c and d cannot have a neighbor in $V(G)\backslash H$, whereas e can have at most one neighbor in $V(G)\backslash H$. If d and e are adjacent, then e cannot have a neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $12+1=13$. In the other case, the vertices a, b and c can be adjacent without any constraints. Hence, the number of edges deleted is at most $12+2+1=15$.

6. **Both the vertices d and e have two neighbors in S_1.** Without loss of generality, suppose d is the neighbor of a and c, while e is the neighbor of b and c (see Figure 4.11(viii)). Similarly, as before, the vertices a, b, c, d and e cannot have a neighbor in $V(G)\backslash H$. The vertex c can be adjacent to a and b without any constraints. Hence, the number of edges deleted is at most $13+2=15$.

Thus, by induction

$$e(G) = e(G-H) + 16 \leq \frac{5}{2}(n-7) - 2 + 16 \leq \frac{5}{2}n - 2,$$

and we are done. $\square$

The following lemma completes the proof of the Theorem 4.1.2(v):

Lemma 4.6.1. *Let G be an $S_{3,3}$-free plane graph on n vertices, then $e(G) \leq \frac{5}{2}n - 2$.*

Proof. Let $A = \{x \in V(G) \mid d(x) = 3\}$, $B = \{x \in V(G) \mid d(x) = 4 \text{ or } d(y) = 5\}$ and $C = \{x \in V(G) \mid d(x) \geq 6\}$.

Since there is no 3-3 edge, the vertices in A are independent. From Claims 15 and 16, there is no edge between the sets B and C. Moreover, C is independent by Claim 14. The distribution of the edges in G is shown in Figure 4.12.

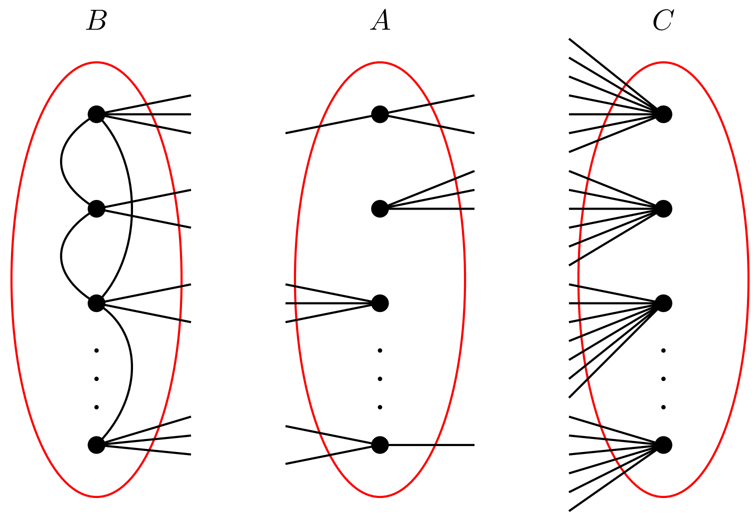


Figure 4.12: A graph showing the distribution of edges in G.

Let $|A|=a$, $|B|=b$ and $|C|=c$. Let x be the number of edges between the sets A and B, i.e., $e(A,B)=x$. Since the maximum degree in B is 5, we have $2e(G[B])=\sum_{v\in B} d(v)-x$ which implies $e(G)\leq \frac{5b-x}{2}$. Each vertex in A has degree 3 and the vertices in A are independent, hence $e(A,C)=3a-x$. Thus, the number of edges in G is

$$e(G)=e(G[B])+e(A,B)+e(A,C)\leq \frac{5b-x}{2}+x+3a-x=\frac{5}{2}a+\frac{5}{2}b+\frac{a-x}{2}.$$

On the other hand, since G is a plane graph and the graph induced by the vertices in A and the vertices in C is bipartite, $e(A,C)=3a-x\leq 2(a+c)-4$. This implies that $\frac{a-x}{2}\leq c-2=\frac{5}{2}c-\frac{3}{2}c-2$ for all $c\geq 0$. Therefore, using the inequality in (4.1), we get

$$e(G)\leq \frac{5}{2}a+\frac{5}{2}b+\frac{5}{2}c-\left(\frac{3}{2}c+2\right)=\frac{5}{2}(a+b+c)-\left(\frac{3}{2}c+2\right)\leq \frac{5}{2}n-2. \tag{4.1}$$

The last inequality in (4.1) holds (and hence Lemma 4.6.1) if $a\neq 0$ and $c\neq 0$. To finish the proof, we distinguish the following two cases:

Case 1: $a\neq 0$ and $c=0$. Observe that $e(G)\leq \frac{5(n-a)+3a}{2}=\frac{5}{2}n-a$. If $a\geq 2$, then we are done. Thus, $a=1$. Let the number of degree 4 vertices in G be k. Hence, $e(G)=\frac{5(n-k-1)+4k+3}{2}=\frac{5}{2}n-\frac{k}{2}-1$. If $k\geq 2$, then we are done.

Let the number of degree 4 vertices in G be at most 1. Let $A = \{x\}$ and $N(x) = \{x_1, x_2, x_3\}$. Let $d(x_i) = 5$, for every $i \in \{1, 2, 3\}$. Considering that there is at most one degree 4 vertex in G, it is easy to find a degree 5 vertex in $\cup_{i=1}^{3} N(x_i)$, such that all its 5 neighboring vertices are of degree 5. Moreover, the same property holds if one vertex in $N(x)$ is of degree 4. Let v be a degree 5 vertex in G, such that all its 5 neighboring vertices are of degree 5. Let $N(v) = \{x_1, x_2, x_3, x_4, x_5\}$, such that a plane drawing of G results in a clockwise alignment of the vertices x_1, x_2, x_3, x_4, x_5 around v. Since G is an $S_{3,3}$-free plane graph, every 5-5 edge in G must be contained in at least 3 triangles. Thus, an edge x_1v must be contained in at least 3 triangles. This implies, x_1 must be adjacent to at least one vertex in $\{x_3, x_4\}$. Without loss of generality, assume x_1 and x_3 are adjacent. Then the 5-5 edge x_2v is contained in at most 2 triangles, which results in an $S_{3,3}$ in G with x_2v as the backbone.

Case 2: $a = 0$**.** Let the number of degree 4 vertices in G be k. Thus, $e(G) = \frac{5(n-k)+4k}{2} = \frac{5}{2}n - \frac{k}{2}$. If the number of degree 4 vertices is at least 4, then $e(G) \leq \frac{5}{2}n - 2$ and we are done. Now assume that the number of degree 4 vertices in G is at most 3. Notice that, $v(G) \geq 8$. Otherwise, taking any maximal planar graph on n vertices, it can be checked that $3n - 6 < \frac{5}{2}n - 2$.

Let v be a degree 5 vertex in G, and $N(v) = \{x_1, x_2, x_3, x_4, x_5\}$. At least two vertices in $N(v)$ must be of degree 5. Otherwise, the number of degree 4 vertices is at least 4 and we are done. Let the plane drawing of G result in a clockwise alignment of the vertices x_1, x_2, x_3, x_4, x_5 around v. There are exactly 2 vertices in $N(v)$, which are of degree 5. Indeed, suppose that the number of degree 5 vertices is at least 3. We can assume that for some $i \in [5]$, $d(x_i) = d(x_{i+1}) = 5$. Without loss of generality, assume that these vertices are x_1 and x_2. Since x_1v and x_2v are 5-5 edges, they must be contained in at least 3 triangles. Thus, both x_1 and x_2 must be adjacent to x_4. On the other hand, it is easy to see that x_3v and x_5v are 4-5 edges. Thus, the vertex x_3 must be adjacent to x_2 and x_4. Similarly, the vertex x_5 must be adjacent to x_1 and x_4. Since $d(x_3) = 4$, there must be a vertex x_6, such that $x_3x_6 \in E(G)$. If $d(x_6) = 5$, then x_6 must be adjacent to x_4. This is impossible, as $d(x_4) = 5$. Hence, $d(x_6) = 4$. Similarly, we have another vertex x_7 adjacent to x_5 and $d(x_7) = 4$. This is a contradiction, as we found 4 vertices of degree 4, namely x_3, x_5, x_6 and x_7.

Thus, we can assume that only two vertices in $N(v)$ are of degree 5. Moreover, the vertices are not consecutive with respect to the alignment in the clockwise direction. Without loss of generality, assume that the vertices are x_2 and x_4. It can be checked that the vertices x_2 and x_4 are adjacent. Since x_1v and x_5v are 4-5 edges, then the edges x_1x_2, x_1x_5 and x_5x_4 are in G. Since $d(x_3) = 4$, there must exist a vertex x_6 adjacent to the vertex x_3. If this vertex is of degree 4, then it is a contradiction as we found 4 vertices of degree 4, namely x_1, x_3, x_5 and x_6. Hence, $d(x_6) = 5$ and the edges x_2x_6 and x_4x_6 are in G. Since $d(x_1)$ is 4, there must exist a vertex x_7 such that $x_1x_7 \in E(G)$. If $d(x_7)$ is 5, then x_7 is adjacent to x_2 and $d(x_2) \geq 6$, which is a contradiction. Hence, x_7 must be a vertex of degree 4. This is a contradiction, as we found 4 vertices of degree 4, namely x_1, x_3, x_5 and x_7. This completes the proof of Claim 4.6.1, and subsequently the proof of Theorem 4.1.2(v). □

4.7 Planar Turán number of $S_{3,4}$

Proof of the Theorem 4.1.2(vi). Let G be an n-vertex $S_{3,4}$-free plane graph. Since $S_{3,4}$ contains 9 vertices, a maximal planar graph with $n \leq 8$ vertices, does not contain an $S_{3,4}$. Let $8|n$. Consider the plane graph consisting of $\frac{n}{8}$ disjoint copies of maximal planar graphs on 8 vertices. This graph does not contain an $S_{3,4}$. Hence, $\mathrm{ex}_{\mathcal{P}}(n, S_{3,4}) \geq \frac{9}{4}n$.

Claim 17. *Let G be an $S_{3,4}$ on n $(1 \leq n \leq 42)$ vertices. The number of edges in G is at most $\frac{20}{7}n$.*

Proof. Recall that, an n-vertex maximal planar graph contains $3n - 6$ edges. Since $3n - 6 \leq \frac{20}{7}n$ for $n \leq 42$, $e(G) \leq \frac{20}{7}n$ holds for all n, $1 \leq n \leq 42$. □

The following claims deal with the different cases of degree pairs in G:

Claim 18. *No vertex in G with a degree at least 8 is adjacent to a vertex of degree at least 4.*

Proof. Suppose not. Let xy be an edge in G such that $d(x) \geq 8$ and $d(y) \geq 4$. Obviously, there are three vertices in $V(G)\backslash\{x\}$, say y_1, y_2 and y_3, which are adjacent to y. Since $|N(x)\backslash\{y\}| \geq 7$, there are four vertices x_1, x_2, x_3 and x_4, not in $\{y, y_1, y_2, y_3\}$ which are adjacent to x. This implies we got an $S_{3,4}$ in G with backbone xy and leaf-sets $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3\}$, respectively, which is a contradiction. This completes the proof. □

Claim 19. *If there is a 7-7 edge in G, then $e(G) \leq \frac{20}{7}n$.*

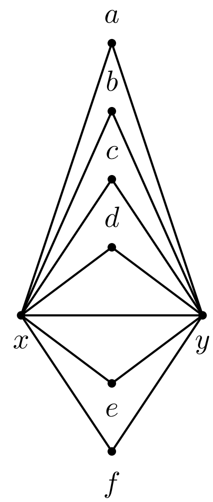


Figure 4.13: The graph G has a $7-7$ edge xy.

Proof. Let $xy \in E(G)$ be a 7-7 edge. Since G is an $S_{3,4}$-free plane graph, xy must be contained in 6 triangles. Let a, b, c, d, e and f be the vertices in G which are adjacent to both x and y, see Figure 4.13(i). Let $S_1 = \{a, b, c, d, e, f\}$ and $H = \{x, y, a, b, c, d, e, f\}$. Delete the vertices in H. Assume a has two neighbors in $V(G)\backslash H$. We immediately get an $S_{3,4}$ with xa or ya as the backbone. Thus, any vertex in the set S_1 can have at most 1 neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, we deleted at most $13 + 6 = 19$ edges. Assume that there is an edge between the vertices in S_1, say ab. If a (or b) has a neighbor in $V(G)\backslash H$, then xa (or xb) is the backbone of an $S_{3,3}$. Similarly, for the other edges in S_1, both the vertices cannot have a neighbor in $V(G)\backslash H$. Thus, if there is an edge joining any two vertices in S_1, the number of edges deleted is at most $13 + 5 = 18$. By the induction hypothesis, we get $e(G - H) \leq \frac{20}{7}(n - 8)$. Hence, $e(G) = e(G - H) + 19 \leq \frac{20}{7}(n - 8) + 19 \leq \frac{20}{7}n$. ☐

Claim 20. *If there is a* 6-7 *edge in* G*, then* $e(G) \leq \frac{20}{7}n$.

Proof. Let $xy \in E(G)$ be a 6-7 edge. Since G is an $S_{3,4}$-free plane graph, xy must be contained in 5 triangles. Let a, b, c, d and e be the vertices in G which are adjacent to both x and y. Let f be the vertex adjacent to y but not to x, see Figure 4.14. Let $S_1 = \{a, b, c, d, e\}$ and $H = \{x, y, a, b, c, d, e, f\}$. Delete the vertices in H. Any vertex in S_1 can have at most 1 neighbor in $V(G)\backslash H$. If there is an edge joining any two vertices in S_1, say ab. Similarly, as before, the vertices

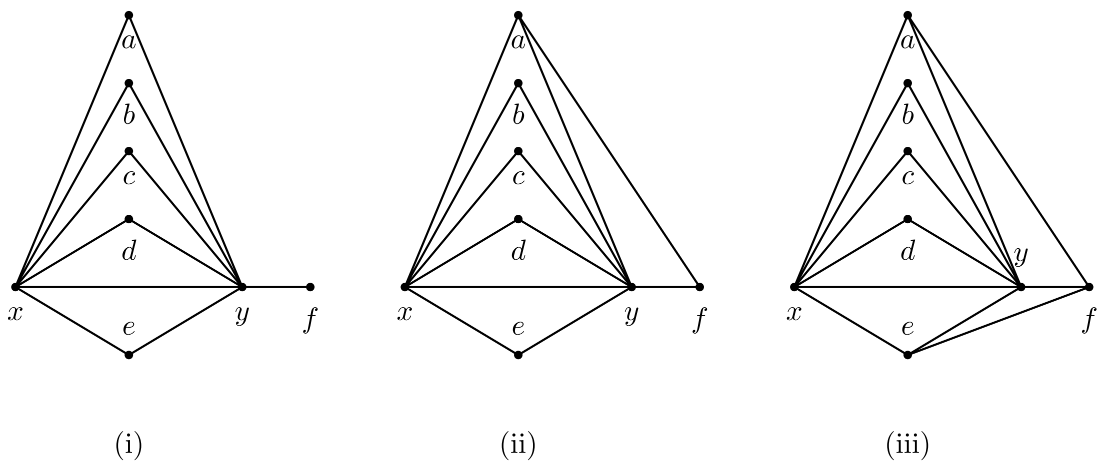


Figure 4.14: The graph G has a $6-7$ edge xy.

(i) The vertex f has no neighbors in S_1.

(ii) The vertex f has one neighbor in S_1.

(iii) The vertex f has two neighbors in S_1.

a and b cannot have a neighbor in $V(G)\backslash H$. We further distinguish the cases based on the number of edges from f as follows:

1. **The vertex f has no neighbors in** S_1, see Figure 4.14(i). Clearly, the vertex f can have at most 2 neighbors in $V(G)\backslash H$. If there are no edges between the vertices in S_1, we deleted at most $12+2+5=19$ edges. If there is an edge joining any two vertices in S_1, the number of edges deleted is at most $12+2+4=18$.

2. **The vertex f has one neighbor in S_1.** Without loss of generality, suppose a is the neighbor of f, see Figure 4.14(ii). The vertex f can have at most one neighbor in $V(G)\backslash H$. If the vertex a has a neighbor in $V(G)\backslash H$, we have an $S_{3,4}$ with xa as the backbone. If there are no edges between the vertices in S_1, we have deleted at most $13+4+1=18$ edges. If there is an edge joining any two vertices in S_1, the number of edges deleted is at most $13+4+1=18$.

3. **The vertex f has two neighbors in S_1.** Without loss of generality, suppose f is the neighbor of a and e, see Figure 4.14(iii). The vertex f cannot have a neighbor in $V(G)\backslash H$, otherwise we have an $S_{3,4}$ with yf as the backbone. If either a or e has a neighbor in $V(G)\backslash H$, we have an $S_{3,4}$ with xa or xe as the backbone, respectively. Suppose there is no edge between

the vertices in S_1. The total number of edges deleted is at most $14+3=17$. Suppose a and b are adjacent, then b cannot have a neighbor in $V(G)\backslash H$. Similarly, for the other edges in S_1 except ad, which is still possible without any extra restrictions. If there is an edge joining any two vertices in S_1, the total number of edges deleted is at most $14+4=18$.

Thus, by induction

$$e(G) = e(G-H)+19 \leq \frac{20}{7}(n-8)+19 < \frac{20}{7}n,$$

and we are done. □

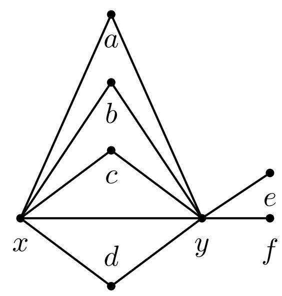
Claim 21. *If there is a* 5-7 *edge in* G*, then* $e(G) \leq \frac{20}{7}n$.

Proof. Let xy be a $5-7$ edge in G. Since G is an $S_{3,4}$-free plane graph, xy must be contained in 4 triangles. Let a, b, c and d be the vertices in G which are adjacent to both x and y. Let e and f be the vertices adjacent to y but not to x, see Figure 4.15. Let $S_1 = \{a, b, c, d\}$ and $H = \{x, y, a, b, c, d, e, f\}$. Delete the vertices in H.

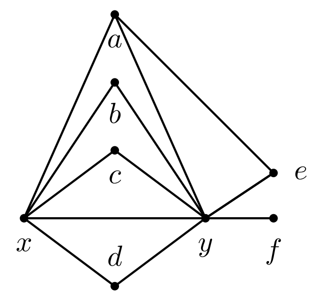
The vertices e and f can have at most two neighbors in $V(G)\backslash H$ each. The vertices in S_1 can have at most one neighbor in $V(G)\backslash H$ each. If there is an edge joining any two vertices in S_1, say ab. Similarly, as before, the vertices a and b cannot have a neighbor in $V(G)\backslash H$. We distinguish the cases based on the neighbors of e and f as follows:

1. **The vertices e and f have no neighbors in S_1**, see Figure 4.15(i)). If the vertices e and f are adjacent, they can have at most one neighbor in $V(G)\backslash H$ each. Otherwise, we have an $S_{3,4}$ with ey (or fy) as the backbone. If there are no edges between the vertices in S_1, the number of edges deleted is at most $11+4+4=19$. If there is an edge joining any two vertices in S_1, the number of edges deleted is at most $11+3+4=18$.

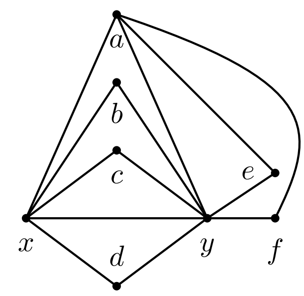
2. **One of the vertices e or f has one neighbor in S_1, while the other has none.** Without loss of generality, suppose e and a are adjacent (see Figure 4.15(ii)). The vertex a cannot have a neighbor in $V(G)\backslash H$, and e can have at most one neighbor in $V(G)\backslash H$. If the vertices e and f are adjacent, then e cannot have a neighbor in $V(G)\backslash H$ and f can have at most one neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number



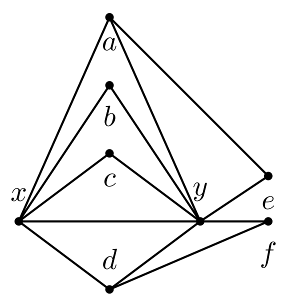
(i)



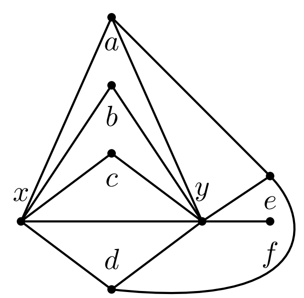
(ii)



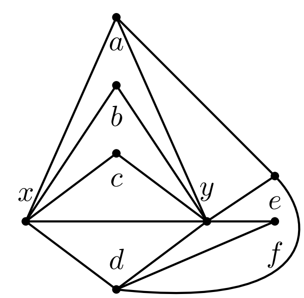
(iii)



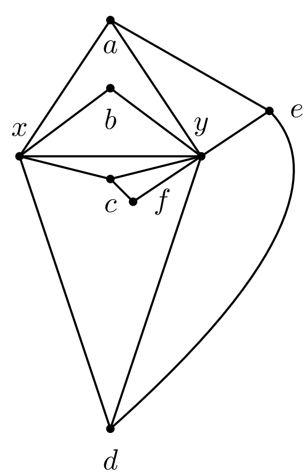
(iv)



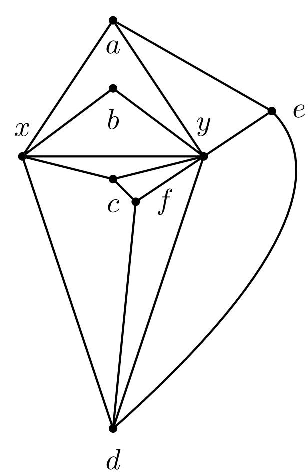
(v)



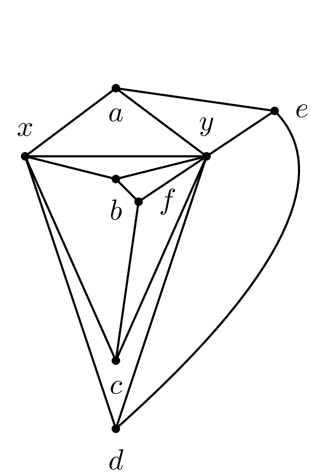
(vi)



(vii)



(viii)



(ix)

Figure 4.15: The graph G has a $5-7$ edge xy.

(i) The vertices e and f have no neighbors in S_1.

(ii) The vertex e has one neighbor in S_1, and f has none.

(iii) Both the vertices e and f have one common neighbor in S_1.

(iv) The vertices e and f have one neighbor in S_1 and they are distinct.

(v) The vertex e has two neighbors in S_1, while f has none.

(vi) The vertex e is the neighbor of a and d, while f is the neighbor of d.

(vii) The vertex e is the neighbor of a and d, while f is the neighbor of c.

(viii) The vertex e is the neighbor of a and d, while f is the neighbor of c and d.

(ix) The vertex e is the neighbor of a and d, while f is the neighbor of b and c.

of edges deleted is at most $12+3+3=18$. If there is an edge joining any two vertices in S_1, the number of edges deleted is at most $12+3+3=18$.

3. **Both the vertices e and f have one neighbor in S_1.** There are two possibilities. In the first case, without loss of generality, suppose a is the common neighbor of e and f (see Figure 4.15(iii)). Similarly, as before, the vertex a cannot have a neighbor in $V(G)\backslash H$, and e and f can have at most one neighbor in $V(G)\backslash H$ each. If the vertices e and f are adjacent, e and f have no neighbors in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $13+3+2=18$. If there is an edge joining any two vertices in S_1, the number of edges deleted is at most $13+3+2=18$.

   Without loss of generality, suppose a is the neighbor of e and d is the neighbor of f (see Figure 4.15(iv)). Similarly, as before, the vertices a and d cannot have a neighbor in $V(G)\backslash H$, and e and f can have at most one neighbor in $V(G)\backslash H$ each. If the vertices e and f are adjacent, then they have no neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $13+2+2=17$. If a and b are adjacent, then b does not have a neighbor in $V(G)\backslash H$. Similarly, for the other edges in S_1, except ad. The vertices a and d can be adjacent without any constraints. Thus, the number of edges deleted is at most $13+3+2=18$.

4. **One of the vertices e or f has two neighbors in S_1, while the other has none.** Without loss of generality, suppose e is the neighbor of a and d, while f has no neighbors in S_1 (see Figure 4.15(v)). The vertex e cannot have a neighbor in $V(G)\backslash H$. Similarly, as before, the vertices a and d cannot have a neighbor in $V(G)\backslash H$, whereas f can have at most two neighbors in $V(G)\backslash H$. If e and f are adjacent, then f can have at most one neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $13 + 2 + 2 = 17$. If a and b are adjacent, then b does not have a neighbor in $V(G)\backslash H$. Similarly, for the other edges in S_1, except ad. The vertices a and d can be adjacent without any constraints. Hence, the number of edges deleted is at most $13 + 3 + 2 = 18$.

5. **One of the vertices e or f has two neighbors in S_1, while the other has one.** There are two possibilities. In the first case, without loss of generality, suppose e is the neighbor of a and d, while f is the neighbor of d (see Figure 4.15(vi)). Similarly, as before, the vertices a, d and e cannot have a neighbor in $V(G)\backslash H$, whereas f can have at most one neighbor in $V(G)\backslash H$. If e and f are adjacent, then f cannot have a neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $14 + 2 + 1 = 17$. In the other case, the vertices a and d can be adjacent without any constraints. Hence, the number of edges deleted is at most $14 + 3 + 1 = 18$.

   Without loss of generality, suppose e is the neighbor of a and d, while f is the neighbor of c (see Figure 4.15(vii)). Similarly, as before, the vertices a, c, d and e cannot have a neighbor in $V(G)\backslash H$, whereas f can have at most one neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $14 + 1 + 1 = 16$. In the other case, the vertices a, c and d can be adjacent without any constraints. Hence, the number of edges deleted is at most $14 + 3 + 1 = 18$.

6. **Both the vertices e and f have two neighbors in S_1.** There are two possibilities. In the first case, without loss of generality, suppose e is the neighbor of a and d, while f is the neighbor of c and d (see Figure 4.15(viii)). Similarly, as before, the vertices a, c, d, e and f cannot have a neighbor in $V(G)\backslash H$. If there are no edges between the vertices in S_1, the number of edges deleted is at most $15 + 1 = 16$. In the other case, the vertices a, c and d can be adjacent without any constraints. Thus, the number of edges deleted is $15 + 3 = 18$.

   Without loss of generality, suppose e is the neighbor of a and d, while f is the neighbor of

b and c (see Figure 4.15(ix)). Similarly, as before, the vertices a, b, c, d, e and f cannot have a neighbor in $V(G)\backslash H$. The vertices a, b, c and d can be a path of length 3 without any constraints. In this case, the number of edges deleted is $15 + 3 = 18$.

Thus, by induction

$$e(G) = e(G - H) + 19 \leq \frac{20}{7}(n - 8) + 19 < \frac{20}{7}n,$$

and we are done. □

Claim 22. *If there is a* 6-6 *edge in* G*, then* $e(G) \leq \frac{20}{7}n$.

Proof. Let $xy \in E(G)$ be a 6-6 edge. There are at least 4 triangles sitting on the edge xy, otherwise G contains an $S_{3,4}$. We subdivide the cases based on the number of triangles sitting on the edge xy.

1. **There are** 5 **triangles sitting on the edge** xy**.** Let a, b, c, d and e be the vertices in G which are adjacent to both x and y, see Figure 4.6. Let $S_1 = \{a, b, c, d, e\}$, and $H = S_1 \cup \{x, y\}$. Delete the vertices in H. The vertices in S_1 can have at most one neighbor in $V(G)\backslash H$ each and can form a path of length 4 within S_1. Hence, the number of edges deleted is $11 + 5 + 4 = 20$. By the induction hypothesis, we get $e(G - H) \leq \frac{20}{7}(n - 7)$. Hence, $e(G) = e(G - H) + 20 \leq \frac{20}{7}(n - 7) + 20 \leq \frac{20}{7}n$.

2. **There are** 4 **triangles sitting on the edge** xy**.** Let a, b, c and d be the vertices in G which are adjacent to both x and y. Let e be the vertex adjacent to x but not adjacent to y, and f be adjacent to y but not adjacent to x. Let $S_1 = \{a, b, c, d\}$ and $H = \{x, y\} \cup S_1 \cup \{e, f\}$. Delete the vertices in H. The vertices e and f can have at most two neighbors in $V(G)\backslash H$ each. We distinguish the cases based on the neighbors of e and f as follows:

   (a) **The vertices** e **and** f **have no neighbors in** S_1, see Figure 4.7(i)). The vertices in S_1 can have at most one neighbor in $V(G)\backslash H$ each and can form a path of length 3 in S_1. If the vertices e and f are adjacent, then both can have at most one neighbor in $V(G)\backslash H$. Otherwise, we have an $S_{3,4}$ with ex (or fy) as the backbone. Thus, the number of edges deleted is at most $11 + 3 + 4 + 4 = 22$.

(b) **One of the vertices e or f has one neighbor in S_1, while the other has none.** Without loss of generality, assume that e and a are adjacent, see Figure 4.7(ii). The vertex a cannot have a neighbor in $V(G)\backslash H$, and e can have at most one neighbor in $V(G)\backslash H$. Otherwise, if a has a neighbor in $V(G)\backslash H$, ya is the backbone of an $S_{3,4}$. If the vertices e and f are adjacent, then the vertex e cannot have a neighbor in $V(G)\backslash H$, and f can have at most one neighbor in $V(G)\backslash H$. Otherwise, we have an $S_{3,4}$ with ex (or fy) as the backbone. Similarly, as before, the vertices b, c and d can have at most one neighbor in $V(G)\backslash H$ each and the vertices $\{a, b, c, d\}$ can form a path of length 3 in S_1. Thus, the number of edges deleted is at most $12 + 3 + 3 + 3 = 21$.

(c) **The vertices e and f have one neighbor in S_1.** There are two possibilities. In the first case, without loss of generality, suppose a is the common neighbor of e and f, see Figure 4.7(iii). Similarly, as before, a cannot have a neighbor in $V(G)\backslash H$, and e and f can have at most one neighbor in $V(G)\backslash H$ each. The vertices b, c and d can have at most one neighbor in $V(G)\backslash H$ each, and the vertices $\{a, b, c, d\}$ can form a path of length 3 in S_1. If the vertices e and f are adjacent, then they cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $13 + 3 + 3 + 2 = 21$.

    Without loss of generality, suppose a is the neighbor of e, and d is the neighbor of f, see Figure 4.7(iv). Similarly, as before, the vertices a and d cannot have a neighbor in $V(G)\backslash H$, and e and f can have at most one neighbor in $V(G)\backslash H$ each. The vertices b and c can have at most one neighbor in $V(G)\backslash H$ each, and the vertices $\{a, b, c, d\}$ can form a path of length 3 in S_1. If the vertices e and f are adjacent, then they cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $13+3+2+2 = 20$. (In fact, it can be shown that if the vertices e and f are adjacent, there can only be a 2-path inside S_1. However, this precision is unnecessary. We skip this in the following cases also.)

(d) **One of the vertices e or f has two neighbors in S_1, while the other has none.** Without loss of generality, suppose e is the neighbor of a and d, see Figure 4.7(v). Similarly, as before, a and d cannot have a neighbor in $V(G)\backslash H$, and e at most one neighbor in $V(G)\backslash H$. The vertices b and c can have at most one neighbor in $V(G)\backslash H$ each, and the vertices $\{a, b, c, d\}$ can form a path of length 3 in S_1. If the vertices e and f are adjacent, then e cannot have a neighbor in $V(G)\backslash H$ and f can have at most one neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $13+3+2+3=21$.

(e) **One of the vertices e or f has two neighbors in S_1, while the other has one neighbor.** There are two possibilities. In the first case, without loss of generality, suppose e is the neighbor of a and d, and f is the neighbor of d (see Figure 4.7(vi)). Similarly, as before, the vertices a and d cannot have a neighbor in $V(G)\backslash H$. The vertices e and f can have at most one neighbor in $V(G)\backslash H$ each. On the other hand, the vertices b and c can have at most one neighbor in $V(G)\backslash H$ each and the vertices $\{a,b,c,d\}$ can form a path of length 3 in S_1. If the vertices e and f are adjacent, e and f cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $14+3+2+2=21$.

Without loss of generality, assume that e is the neighbor of a and d, and f is the neighbor of b (see Figure 4.7(vii)). The vertices a,b and d cannot have a neighbor in $V(G)\backslash H$, but they along with c can form a path of length 3 in S_1. The vertices c,e and f can have at most one neighbor in $V(G)\backslash H$ each. Thus, the number of edges deleted is at most $14+3+1+2=20$.

(f) **Both the vertices e and f have two neighbors in S_1.** There are three possibilities. In the first case, without loss of generality, suppose e and f are neighbors of a and d both, see Figure 4.7(viii). Similarly, as before, the vertices a and d cannot have a neighbor in $V(G)\backslash H$. The vertices e and f can have at most one neighbor in $V(G)\backslash H$ each. The vertices b and c can have at most one neighbor in $V(G)\backslash H$ each. The vertices $\{a,b,c,d\}$ can form a path of length 3 in S_1. If the vertices e and f are adjacent, e and f cannot have a neighbor in $V(G)\backslash H$. Thus, the number of edges deleted is at most $15+3+2+2=22$.

On the other hand, without loss of generality, assume e is the neighbor of a and d both, while f is the neighbor of a and b (see Figure 4.7(ix)). The vertices a,b and d cannot have a neighbor in $V(G)\backslash H$, but they along with c can form a path of length 3 in S_1. The vertices c,e and f can have at most one neighbor in $V(G)\backslash H$ each. Thus, the number of edges deleted is at most $15+3+1+2=21$.

In the last case, without loss of generality, assume e is the neighbor of a and d both, while f is the neighbor of b and c (see Figure 4.7(x)). The vertices a,b,c and d cannot have a neighbor in $V(G)\backslash H$, but they can form a path of length 3 in S_1. The vertices e and f can have at most one neighbor in $V(G)\backslash H$ each. Thus, the number of edges deleted is at most $15+3+2=20$.

By the induction hypothesis, we get $e(G-H) \leq \frac{20}{7}(n-8)$. Hence, $e(G) = e(G-H)+22 \leq \frac{20}{7}(n-8)+22 \leq \frac{20}{7}n$.

□

Take $x, y \in V(G)$. By the previous claims, if $d(x)+d(y) \geq 12$, we are done by induction. Assume that $d(x)+d(y) \leq 11$. Summing it over all the edge pairs in G, we have $11e \geq \sum_{xy \in E(G)} (d(x)+d(y)) = \sum_{x \in V(G)} (d(x))^2 \geq n\overline{d}^2 = n(\frac{2e}{n})^2$, where $\overline{d}$ is the average degree in G. This gives us $e \leq \frac{11}{4}n \leq \frac{20}{7}n$ for $n \geq 1$. □

4.8 Concluding remarks and Conclusions

Concerning the exact value of $\mathrm{ex}_{\mathcal{P}}(n, S_{3,3})$, we conjecture the following:

Conjecture 4.8.1.

$$\mathrm{ex}_{\mathcal{P}}(n, S_{3,3}) = \begin{cases} 3n-6, & \textit{if } 3 \leq n \leq 7, \\ 16, & \textit{if } n = 8, \\ 18, & \textit{if } n = 9, \\ \left\lfloor \frac{5}{2}n \right\rfloor - 5, & \textit{otherwise.} \end{cases}$$

Chapter 5

Book free 3-Uniform Hypergraphs

5.1 Introduction

All the graphs we consider in this chapter are simple and finite. Let G be a graph. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. If there are 2 triangles sitting on an edge in a graph, we call this a *diamond*. On the other hand, k triangles sitting on an edge is called a *k-book* and is denoted by B_k. Similarly, let H be a hypergraph. The vertex and edge set of H are denoted by $V(H)$ and $E(H)$, respectively. A hypergraph is called *r-uniform* if each hyperedge has size r. A hypergraph $H = (V, E)$ is called *linear* if every two hyperedges have at most one vertex in common. A Berge cycle of length k, denoted by Berge-C_k, is an alternating sequence of distinct vertices and distinct hyperedges of the form $v_1, h_1, v_2, h_2, \ldots, v_k, h_k$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, \ldots, k-1\}$ and $v_k v_1 \in h_k$. The hypergraph equivalent of k-book is defined similarly, with k-Berge triangles sharing a common edge.

The maximum number of edges in a triangle-free graph is one of the classical results in Extremal Graph Theory. Mantel [103] proved this in 1907. The extremal problem of book-free graphs follows from this. Given a graph G on n vertices and having $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges. Mantel showed that G contains a triangle. Rademacher (unpublished and simplified later by Erdős in [39]) proved in the 1940s that the number of triangles in G is at least $\left\lfloor \frac{n}{2} \right\rfloor$. Erdős conjectured in 1962 [35] that the size of the largest book in G is $\frac{n}{6}$ and this was proved soon after by Edwards (unpublished, see also Khadziivanov and Nikiforov [110] for an independent proof).

Theorem 5.1.1. *[Edwards [32], Khadziivanov and Nikiforov [110]] Every n-vertex graph with more than $\frac{n^2}{4}$ edges, contains an edge that is in at least $\frac{n}{6}$ triangles.*

Both Rademacher's and Edwards' results are sharp. In the former, the addition of an edge to one of the parts in the complete balanced bipartite graph (note that, in G there is an edge contained in $\lfloor \frac{n}{2} \rfloor$ triangles) achieves the maximum. In the latter, every known extremal construction of G has $\Omega(n^3)$ triangles. For more details on book-free graphs, we refer the reader to the following articles [10], [114] and [131]. We investigate the equivalent problem in the case of hypergraphs.

Given a family of hypergraphs $\mathcal{F}$, we say that a hypergraph H is Berge-$\mathcal{F}$-free if, for every $F \in \mathcal{F}$, the hypergraph H does not contain a Berge-F as a sub-hypergraph. The maximum possible number of hyperedges in a Berge-$\mathcal{F}$-free r-uniform hypergraph on n vertices is denoted by $\mathrm{ex}_r(n, \mathcal{F})$. When $\mathcal{F} = \{F\}$, then we simply write $\mathrm{ex}_r(n, F)$ instead of $\mathrm{ex}_r(n, \mathcal{F})$. The linear Turán number $\mathrm{ex}_r^{lin}(n, F)$ is the maximum number of hyperedges in a r-uniform linear hypergraph on n vertices, which does not contain F as a sub-hypergraph.

The systematic study of the Turán number of Berge cycles started with Lazebnik and Verstraëte [98], who studied the maximum number of hyperedges in a r-uniform hypergraph containing no Berge cycle of length less than five. Another result was the study of Berge triangles by Győri [69]. He proved that:

Theorem 5.1.2 (Győri [69])**.** *The maximum number of hyperedges in a Berge triangle-free* 3-*uniform hypergraph on n vertices is at most $\frac{n^2}{8}$.*

It continued with the study of Berge five cycles by Bollobás and Győri [11]. In [70], Győri, Katona, and Lemons proved the analog of the Erdős-Gallai Theorem for Berge paths. For other results, see [4, 86]. The particular case of determining $\mathrm{ex}_3^{lin}(n, C_3)$ is equivalent to the famous $(6, 3)$-problem, which is a special case of a general problem of Brown, Erdős, and Sós. The famous theorem of Ruzsa and Szemerédi states:

Theorem 5.1.3. *[Ruzsa and Szemerédi [118]] For any $\epsilon > 0$ and $n > n_0(\epsilon)$, we have*

$$n^{2-\frac{c}{\sqrt{\log n}}} < \mathrm{ex}_3^{lin}(n, C_3) = \epsilon n^2.$$

We determine the maximum number of hyperedges for a k-book free 3-uniform hypergraph. The main result is as follows:

Theorem 5.1.4. *For a given $k \geq 2$ and $\epsilon > 0$, a 3-uniform B_k-free hypergraph H on n vertices can have at most $\dfrac{n^2}{8} + \epsilon n^2$ edges, where $n > \max(n_1(\epsilon)\sqrt{(6k-9)(3k-3)}, 12k)$.*

The following example shows that this result is asymptotically sharp. Take a complete bipartite graph with color classes of size $\lceil \frac{n}{4} \rceil$ and $\lfloor \frac{n}{4} \rfloor$, respectively. Denote the vertices in each class with x_i and y_i, respectively. Construct a graph by doubling each vertex and replacing each edge with two triangles, as shown below (Figure 5.1). Every graph edge $x_i y_i$ is replaced by the two hyperedges $x_i y_i y_i'$ and $y_i y_i' x_i'$. The construction does not contain a Berge triangle. Hence, it does not contain a k-book. With this, the number of hyperedges is $2 \cdot \frac{n^2}{16} = \frac{n^2}{8}$.

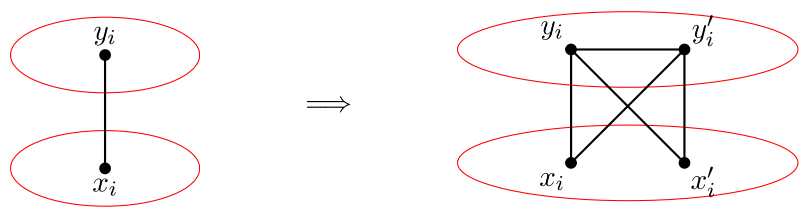


Figure 5.1: Replacing every graph edge $x_i y_i$ in the bipartite graph with two hyperedges $x_i y_i y_i'$ and $y_i y_i' x_i'$.

The chapter is structured as follows: In Section 5.2, we prove the main result of the chapter. In Section 5.3, we conjecture the tight bound.

5.2 Proof of Theorem 5.1.4

Let H be a B_k-free 3-uniform hypergraph. We are interested in the 2-shadow, i.e., let G be a graph with vertex set $V(H)$ and $E(G) = \{ab \mid \{a, b\} \subset e \in E(H)\}$. If an edge in G lies in more than one hyperedge in H, we color it blue. Otherwise, we color it red. We subdivide the hypergraph H into two sub-hypergraphs, namely H_1 and H_2, such that H_1 is the collection of hyperedges in H containing two or more red edges in G. On the other hand, H_2 is the collection of hyperedges in H, containing two or more blue edges in G. Note that, H is the disjoint union of H_1 and H_2.

Lemma 5.2.1. *Given $n > 12k$, the number of hyperedges in H_1 is at most $\frac{n^2}{8}$.*

Proof. Recall that each hyperedge in H_1 was replaced by two or more red edges in G. Let G_1 denote the subgraph of G, formed by the red colored edges. By the definition of red colored edges, we have $e(G_1) \geq 2e(H_1)$. If $e(G_1) \leq \frac{n^2}{4}$, we are done. Otherwise, assume $e(G_1) \geq \frac{n^2}{4} + 1$. By Theorem 5.1.1, we have a book of size $\frac{n}{6}$ in G_1. Denote the vertices of the $\frac{n}{6}$-book in G_1 with u, v and $\{x_i \mid 1 \leq i \leq \frac{n}{6}\}$, where the edge uv denotes the base of the book. Denote the third vertex of the hyperedge sitting on the edge uv by w. Let $V_{good} := \{x_i \mid 1 \leq i \leq \frac{n}{6}\}$ and $E_{good} := \emptyset$ denote the set of "good" vertices and "good" hyperedges, respectively.

Consider the triangle uvx_1. Denote the hyperedge sitting on the edge ux_1 and vx_1 by ux_1y_1 and vx_1z_1, respectively. We have a couple of cases to consider:

(i) Both the vertices y_1 and z_1 are distinct from $\{x_i \mid 1 \leq i \leq \frac{n}{6}\}$. In this case, add the hyperedges ux_1y_1 and vx_1z_1 to E_{good}.

(ii) The vertex $y_1 \notin \{x_i \mid 1 \leq i \leq \frac{n}{6}\}$, whereas the vertex $z_1 \in \{x_i \mid 1 \leq i \leq \frac{n}{6}\}$. In this case, add the hyperedges ux_1y_1 and vx_1z_1 to E_{good}. Also remove the vertex z_1 from V_{good}.

(iii) The vertex $y_1 \in \{x_i \mid 1 \leq i \leq \frac{n}{6}\}$, whereas the vertex $z_1 \notin \{x_i \mid 1 \leq i \leq \frac{n}{6}\}$. This is similar to the previous case, and we add the hyperedges ux_1y_1 and vx_1z_1 to E_{good}, and remove the vertex y_1 from V_{good}.

(iv) Both the vertices y_1 and z_1 belong to the set $\{x_i \mid 1 \leq i \leq \frac{n}{6}\}$. If $y_1 \neq z_1$, then remove the vertex x_1 from V_{good} and add the hyperedges ux_1y_1 and vx_1z_1 to E_{good}. Also add the hyperedges sitting on the edge vy_1 and uz_1 to E_{good}. When $y_1 = z_1$, we remove the vertex y_1 from V_{good} and add the hyperedges ux_1y_1 and vx_1z_1 to E_{good}.

Continue repeating this process for all the other vertices in V_{good}. Consider the set of hyperedges $E_{good} \cup \{uvw\}$. We delete at most one vertex from V_{good} in each step and since $\frac{n}{6} \geq 2k$, the above set of hyperedges forms a k-book. Hence, $e(G_1) \leq \frac{n^2}{4}$ and $e(H_1) \leq \frac{n^2}{8}$. □

Now let us work on the sub-hypergraph H_2.

Lemma 5.2.2. *A pair of vertices in H_2 is contained in at most $2k - 2$ hyperedges of H_2.*

Proof. Suppose u and v be the pair of vertices in H_2, which is contained in $2k-1$ hyperedges of H_2. The edge uv is colored blue. Denote the third vertex of each such hyperedge by x_i, where $1 \leq i \leq 2k-1$. Let $V_{good} := \{x_i \mid 1 \leq i \leq 2k-1\}$, $V_{bad} := \emptyset$ and $E_{good} := \emptyset$ denote the set of "good" vertices, "bad" vertices and "good" hyperedges, respectively. Consider the hyperedge uvx_1. At least one of the edges ux_1 or vx_1 is colored blue. Without loss of generality, assume ux_1 is colored blue, i.e., there is at least one more hyperedge sitting on the edge ux_1 other than ux_1v. Denote this hyperedge sitting on the ux_1 by ux_1y_1. If $y_1 \notin V_{good}$, add the hyperedges ux_1y_1 and ux_1v to E_{good}. If $y_1 \in V_{good}$, then remove the vertex y_1 from V_{good} and add it to V_{bad}. The hyperedges ux_1y_1 and ux_1v are added to E_{good}. Continue doing this for every other vertex in V_{good}. Let x_{bad} be a vertex in V_{bad}. Consider the set of hyperedges $E_{good} \cup \{uvx_{bad}\}$. Note, $|V_{good}| \geq k$ and in each step we add two hyperedges to E_{good}. Thus, $|E_{good}| \geq 2k$, and it is easy to see that the set $E_{good} \cup \{uvx_{bad}\}$ forms a k-book sitting on the edge uv. If V_{bad} is empty, then consider the set of hyperedges $E_{good} \cup \{uvx_{2k-1}\}$. Similarly, as above, $|E_{good}| \geq 2k$ and it is easy to see that the set $E_{good} \cup \{uvx_{2k-1}\}$ forms a k-book sitting on the edge uv. ☐

We now give an upper bound on the number of hyperedges in H_2.

Lemma 5.2.3. *For a given $\epsilon > 0$ and $n > n(\epsilon)\sqrt{(6k-9)(3k-3)}$, the number of hyperedges in H_2,i.e., $e(H_2)$, is at most ϵn^2.*

Proof. Take a hyperedge xyz in the sub-hypergraph H_2. By the previous Lemma 5.2.2, there are at most $2k-2$ hyperedges of H_2 sitting on each of the pairs of vertices xy, yz, and xz. Delete all such hyperedges barring xyz. We have deleted at most $6k-9$ hyperedges. Repeat this for every hyperedge left in H_2. Hence, the total number of hyperedges remaining is at least $\frac{e(H_2)}{6k-9}$. Denote the new hypergraph by H_2'. Any two hyperedges in H_2' have at most 1 vertex in common. In other words, H_2' is a linear 3-uniform hypergraph.

Consider a hyperedge abc in H_2'. Since H_2' is a B_k-free hypergraph, the number of Berge triangles sitting on the edge ab is at most $k-1$. Otherwise, the k-Berge triangles sitting on the edge ab along with the hyperedge abc form a k-book. Denote one such Berge triangle by abd. It is formed by the set of hyperedges $\{abc, ade, bdf\}$, where e and f are the end vertices of the hyperedges sitting on the edge ad and bd, respectively. We delete one of the hyperedges in $\{ade, bdf\}$. Continue this process for every Berge triangle sitting on the edge ab. Similarly, for the edges bc and ac. Repeat this for every hyperedge left in H_2'. Hence, the total number of hyperedges remaining is at least $\frac{e(H_2')}{3(k-1)}$. Denote the resulting hypergraph as $H_2^{''}$. By construction, $H_2^{''}$ is a linear triangle free 3-uniform hypergraph. From Theorem 5.1.3, for the given ϵ and $n_1(\epsilon) > n_0(\epsilon)$, the number of hyperedges in a 3-uniform triangle free linear hypergraph is at most $\epsilon n_1(\epsilon)^2$. Hence, $e(H_2^{''}) \leq \epsilon n_1(\epsilon)^2$ and $e(H_2) \leq \epsilon(n_1(\epsilon)\sqrt{(6k-9)(3k-3)})^2$. Since $n > n_1(\epsilon)\sqrt{(6k-9)(3k-3)}$, we are done. □

Proof of Theorem 5.1.4. By definition, $H = H_1 \cup H_2$ and they are non-intersecting. By Lemma 5.2.1 and 5.2.3, $e(H) \leq e(H_1) + e(H_2) \leq \frac{n^2}{8} + o(n^2)$. Hence, we are done. □

5.3 Concluding remarks and Conjectures

Recall that $\mathrm{ex}(n, C_3) = \mathrm{ex}(n, B_k) = \frac{n^2}{4}$ in graph setting. Győri [69] proved that the maximum number of hyperedges in a Berge triangle-free 3-uniform hypergraph on n vertices is at most $\frac{n^2}{8}$. Given the similarities, we conjecture the following:

Conjecture 5.3.1. *For a given $k \geq 2$ and a 3-uniform B_k-free hypergraph H on n vertices (n is large), $e(H) \leq \dfrac{n^2}{8}$.*

Chapter 6

The Maximum Wiener Index of Maximal Planar Graphs

6.1 Introduction

The Wiener Index was first introduced by H. Wiener [127] in 1947 while studying the correlations of the molecular structure with the boiling point of paraffin. It has become one of the most frequently used topological indices in chemistry. Since undirected graphs, especially trees, are used to model molecules. It has been used even in computer network representations and lattice hardware security enhancements.

Definition 6.1.1. *For a connected graph G, the Wiener index is the sum of distances between all the unordered pairs of vertices in the graph and is denoted by $W(G)$. That means*

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v),$$

where $d_G(u,v)$ denotes the distance from u to v i.e., the minimum length of a path from u to v in the graph G.

Many results on the Wiener index and closely related parameters such as the gross status [80], the distance of graphs [34], and the transmission [120] have been studied. The survey papers [25, 26, 27, 88, 130] are among many which accumulate a great deal of knowledge on the Wiener index.

Finding a sharp bound on the Wiener index for graphs under some constraints has been one of the research topics attracting many researchers. One of the most basic upper bounds for $W(G)$ is as follows:

Theorem 6.1.2. *[21, 113, 100] If G is a connected graph of order n, then,*

$$W(G) \leq \frac{(n-1)n(n+1)}{6}, \tag{6.1}$$

which is attained only by a path.

Many sharp or asymptotically sharp bounds on $W(G)$ in terms of other graph parameters are known, for instance, minimum degree [6, 20, 91], connectivity [23, 50], edge-connectivity [22, 21] and maximum degree [51]. More details of the mathematical aspect of Wiener index are covered in [24, 66, 87, 102, 89, 67, 109, 124, 123, 125].

One can study the Wiener index of the family of connected planar graphs. Since a path attains the bound given in Equation 6.1, it is natural to ask the same question for some family of planar graphs. For instance, the Wiener index of a maximal planar graph on n ($n \geq 3$) vertices has a sharp lower bound of $(n-2)^2 + 2$. Any maximal planar graph such that the distance between any pair of vertices is at most 2 attains this bound (for instance, a planar graph containing the n-vertex star).

Che and Collins [18], and independently Czabarka, Dankelmann, Olsen and Székely [19], gave a sharp upper bound of a particular class of maximal planar graphs known as *Apollonian networks.*

Definition 6.1.3. *An Apollonian network may be formed, starting from a single triangle embedded on the plane, by repeatedly selecting a triangular face of the embedding, adding a new vertex inside the face, and connecting the new vertex to each of the three vertices of the face.*

They showed that

Theorem 6.1.4 (Che, Collins [18] Czabarka, Dankelmann, Olsen, Székely [19])**.** *Let G be an*

Apollonian network of order $n \geq 3$. Then $W(G)$ has a sharp upper bound

$$W(G) \leq \left\lfloor \frac{1}{18}(n^3+3n^2) \right\rfloor = \begin{cases} \frac{1}{18}(n^3+3n^2), & \text{if } n \equiv 0 (mod\ \ 3); \\ \frac{1}{18}(n^3+3n^2-4), & \text{if } n \equiv 1 (mod\ \ 3); \\ \frac{1}{18}(n^3+3n^2-2), & \text{if } n \equiv 2 (mod\ \ 3). \end{cases}$$

It has been shown explicitly in [18] that the bound is attained for the maximal planar graphs T_n. We will give the construction of T_n in the next section, see Definition 6.2.2. The authors in [18] also conjectured that this bound also holds for every maximal planar graph. The authors in [19] showed the following result:

Theorem 6.1.5 (Czabarka, Dankelmann, Olsen, Székely [19]). *For $k \in \{3,4,5\}$, there exists a constant C_k such that*

$$W(G) \leq \frac{1}{6k}n^3 + C_k n^{5/2}$$

for every k-connected maximal planar graph of order n.

In this chapter, we confirm the above conjecture.

Theorem 6.1.6. *Let G be an $n \geq 6$ vertex maximal planar graph. Then*

$$W(G) \leq \left\lfloor \frac{1}{18}(n^3+3n^2) \right\rfloor = \begin{cases} \frac{1}{18}(n^3+3n^2), & \text{if } n \equiv 0 (mod\ \ 3); \\ \frac{1}{18}(n^3+3n^2-4), & \text{if } n \equiv 1 (mod\ \ 3); \\ \frac{1}{18}(n^3+3n^2-2), & \text{if } n \equiv 2 (mod\ \ 3). \end{cases}$$

Equality holds if and only if G is isomorphic to T_n for all $n \geq 9$.

The chapter is structured as follows: In Section 6.2, we have some notations and preliminaries. In Section 6.3, we prove the main result of the chapter. In Section 6.4, we provide some conjectures regarding the Wiener index of 4-connected planar graphs.

6.2 Notations and Preliminaries

A path in a graph is an alternating sequence of distinct vertices and edges, starting from a vertex and ending at a vertex, such that every edge is incident to neighboring vertices in the sequence. The length of the path is the number of edges in the given path. A cycle in a graph G is a non-zero length path from a vertex v to v itself. We use the standard function $d_G(v, u)$ to denote the length of the shortest path from the vertex v to the vertex u. Even more, we may define a function that denotes the distance from a vertex to a set of vertices. Let v be a vertex of G and $S \subseteq V(G)$, then $d_G(S, v) := min_{u \in S}\{d_G(u, v)\}$.

Definition 6.2.1. *For a vertex set $S \subset V(G)$, the status of S is defined as the sum of all distances from the vertices of the graph to the set S. It is denoted by $\sigma_G(S)$, thus*

$$\sigma_G(S) := \sum_{u \in V(G)} d_G(S, u).$$

For simplicity, we may omit the subscript G in the above functions if the underlying graph is obvious. With a slight abuse of notation, we use $\sigma_G(v) := \sigma_G(\{v\})$.

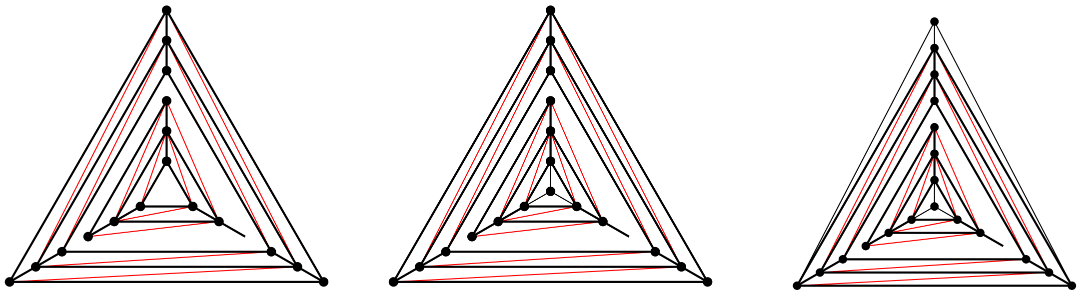
We have,

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} \sigma_G(v).$$

Here we use the definition from [18], for an Apollonian network T_n on n vertices. We will prove that T_n is the unique maximal planar graph that maximizes the Wiener index of the maximal planar graphs.

Definition 6.2.2. *([18]) The Apollonian network T_n is the maximal planar graph on $n \geq 3$ vertices, with the following structure, see Figure 6.1:*

If n is a multiple of 3, then the vertex set of T_n can be partitioned in three sets of the same size, $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_k\}$ and $C = \{c_1, c_2, \dots, c_k\}$. The edge set of T_n is the union of the following three sets: $E_1 = \bigcup_{i=1}^{k}\{(a_i, b_i), (b_i, c_i), (c_i, a_i)\}$ forming concentric triangles, $E_2 = \bigcup_{i=1}^{k-1}\{(a_i, b_{i+1}), (a_i, c_{i+1}), (b_i, c_{i+1})\}$ forming 'diagonal' edges, and $E_3 = \bigcup_{1}^{k-1}\{(a_i, a_{i+1}),$ $(b_i, b_{i+1}), (c_i, c_{i+1})\}$ forming paths in each vertex class, see Figure 6.1a.



(a) $3 \mid n$ (b) $3 \mid (n-1)$ (c) $3 \mid (n-2)$

Figure 6.1: Apollonian networks maximizing the Wiener index of maximal planar graphs [18].

If $3|(n-1)$, then T_n is the Apollonian network which may be obtained from T_{n-1} by adding a degree three vertex in the face a_1, b_1, c_1 or $a_{\frac{n-1}{3}}, b_{\frac{n-1}{3}}, c_{\frac{n-1}{3}}$, see Figure 6.1b. Note that both graphs are isomorphic.

If $3|(n-2)$, then T_n is the Apollonian network which may be obtained from T_{n-2} by adding a degree three vertex in each of the faces a_1, b_1, c_1 and $a_{\frac{n-1}{3}}, b_{\frac{n-1}{3}}, c_{\frac{n-1}{3}}$, see Figure 6.1c.

At first, we would like to recall some standard definitions. A connected graph G is said to be *s-vertex connected or simply s-connected* if it has more than s vertices and remains connected whenever fewer than s vertices are removed. Formally, let G be a graph and S be a subset of the vertices of G, $S \subseteq V(G)$. Then the induced subgraph $G[S]$ of G is a graph on the vertex set S and $E(G[S]) = \{e \in E(G) : e \subseteq S\}$.

Lemma 6.2.3. *Let G be an s-connected, maximal planar graph and S be a cut set of size s of G. Then $G[S]$ is Hamiltonian.*

Proof. Let us denote the vertices of S by $S = \{v_1, v_2, \ldots, v_s\}$. Let u and w be two distinct vertices, $\{u, w\} \in V(G) \setminus S$ such that any path from u to w contains at least one vertex from S. Since G is s-connected, by Menger's Theorem, there are s-pairwise internally vertex disjoint paths from u to w. Each of the paths intersects S in disjoint nonempty sets. Therefore, each of the paths contains one vertex from S exactly. Assume that in a particular planar embedding of G, those paths are ordered in the following way: One of the two regions determined by the cycle obtained from the two paths from u to w, containing v_{i_x} and $v_{i_{x+1}}$, has no vertex from S (where indices are modulo s), see Figure 6.2. From the maximality of the planar graph, there is a path from vertex u to vertex w

that does not contain a vertex from S, a contradiction. Thus, we must have the edges $\{v_{i_x}, v_{i_{x+1}}\}$. Therefore, we have a cycle of length s on the vertex set S, $v_{i_1}, v_{i_2}, \cdots, v_{i_s}, v_{i_1}$ in the given order. Hence, $G[S]$ is Hamiltonian. $\square$

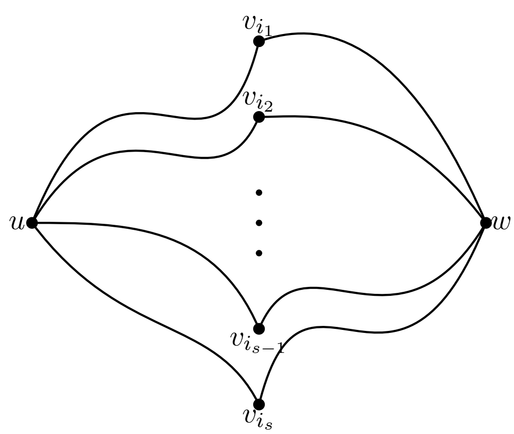


Figure 6.2: s-pairwise disjoint paths from the vertices u to w.

The following definition is particularly helpful for the proof of Theorem 6.1.6: Given a set $S \subseteq V$, the Breadth-First Search partition of V with root S, denoted by $\mathcal{P}_S^G$ (or simply $\mathcal{P}_S$ when the underlying graph is clear), is $\mathcal{P}_S = \{S_0, S_1, \dots\}$, where $S_0 = S$, and for $i \geq 1$, S_i is the set of vertices at distance exactly i from S. Formally $S_i = \{v \in V(G) : d_G(S, v) = i\}$. We refer to those sets as *levels* of $\mathcal{P}_s$. For example, S_1 is the *first level.* For the largest integer k, for which $S_k \neq \emptyset$, we refer to S_k as the *last level.* We refer to S_0 and the last level as *terminal levels.* Note that every level besides the terminal levels is a cut set of G. Let $\mathcal{P}_v$ denote the Breadth-First Search partition from $\{v\}$, i.e., the partition $\mathcal{P}_{\{v\}}$.

The following three lemmas play a critical role in proving Theorem 6.1.6.

Lemma 6.2.4. *Let G be an $n + s$ vertex graph and S, $S \subset V(G)$, be a set of vertices of size s. Suppose, each non-terminal level of $\mathcal{P}_S$ has size at least* 3*. Then we have*

$$\sigma(S) \leq \sigma_3(n) := \begin{cases} \frac{1}{6}(n^2 + 3n), & \textit{if } n \equiv 0 \ (mod\ 3); \\ \frac{1}{6}(n^2 + 3n + 2), & \textit{if } n \equiv 1, 2 \ (mod\ 3). \end{cases}$$

Proof. If $\mathcal{P}_S = \{S_0, S_1, \dots\}$, by definition, we have that $\sigma(S) = \sum_i i\,|S|$. Therefore

$$\begin{aligned}\sigma(S) &= |S_1| + 2\,|S_2| + 3\,|S_3| + \cdots \\ &\le 3\left(1 + 2 + \cdots + \left\lfloor \frac{n}{3} \right\rfloor\right) + \left(\left\lfloor \frac{n}{3} \right\rfloor + 1\right)\left(n - 3\left\lfloor \frac{n}{3} \right\rfloor\right) = \sigma_3(n).\end{aligned}$$

□

Lemma 6.2.5. *Let G be an $n+s$ vertex graph and S, $S \subset V(G)$, be a set of vertices of size s. Suppose, each non-terminal level of $\mathcal{P}_S$ has size at least 4. Then we have*

$$\sigma(S) \le \sigma_4(n) := \begin{cases} \frac{1}{8}(n^2 + 4n), & \text{if } n \equiv 0 \ (mod\ 4); \\ \frac{1}{8}(n^2 + 4n + 3), & \text{if } n \equiv 1, 3 \ (mod\ 4); \\ \frac{1}{8}(n^2 + 4n + 4), & \text{if } n \equiv 2 \ (mod\ 4). \end{cases}$$

Proof. We have

$$\sigma(S) \le 4\left(1 + 2 + \cdots + \left\lfloor \frac{n-1}{4} \right\rfloor\right) + \left(\left\lfloor \frac{n-1}{4} \right\rfloor + 1\right)\left(n - 1 - 4\left\lfloor \frac{n-1}{4} \right\rfloor\right) = \sigma_4(n)$$

□

Lemma 6.2.6. *Let G be an $n+s$ vertex graph and S, $S \subset V(G)$, be a set of vertices of size s. Suppose, each non-terminal level of $\mathcal{P}_S$ has size at least 5. Then we have*

$$\sigma(S) \le \sigma_5(n) := \begin{cases} \frac{1}{10}(n^2 + 5n), & \text{if } n \equiv 0 \ (mod\ 5); \\ \frac{1}{10}(n^2 + 5n + 4), & \text{if } n \equiv 1, 4 \ (mod\ 5); \\ \frac{1}{10}(n^2 + 5n + 6), & \text{if } n \equiv 2, 3 \ (mod\ 5). \end{cases}$$

Proof. We have

$$\sigma(S) \le 5\left(1 + 2 + \cdots + \left\lfloor \frac{n-1}{5} \right\rfloor\right) + \left(\left\lfloor \frac{n-1}{5} \right\rfloor + 1\right)\left(n - 1 - 5\left\lfloor \frac{n-1}{5} \right\rfloor\right) = \sigma_5(n)$$

□

6.3 Proof of Theorem 6.1.6

Proof. From [18], we know that the desired lower bound is attained by T_n. For the upper bound, we are going to prove Theorem 6.1.6 by induction on the number of vertices. In [18], the authors prove the upper bound for $n \le 10$ without computer aid. In [19], it is shown that the upper bound of Theorem 6.1.6 holds, for $6 \leq n \leq 18$ by using a computer program. It is also shown, by means of the computer program, that the upper bound is sharp for $6 \leq n \leq 18$ and the extremal graph is unique T_n for $9 \leq n \leq 18$.

In our proof, we use the computer-aided result of [19] only in Case 2.1 and for the uniqueness of the extremal graph. For the rest, the result in [18] is enough. When $n \leq 18$, unfortunately, we do not have a proof without the use of computers. So, we assume $n \geq 19$ from now on. Let G be a maximal planar graph of n vertices. Since G is a maximal planar graph, it is either 3, 4, or 5-connected. Notice that the result in [19] is much stronger asymptotically than ours if G is 4 or 5-connected.

Case 1: Let G be a 5-connected graph. For every fixed vertex $v \in V(G)$, consider $\mathcal{P}_v$. Since G is 5-connected, and each of the non-terminal levels of $\mathcal{P}_v$ is a cut set, we have that each non-terminal level has size at least 5. Therefore, from Lemma 6.2.6, we have,

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} \sigma(v) \leq \frac{n}{2} \sigma_5(n-1) \leq \frac{n}{20}(n^2 + 3n + 2) < \left\lfloor \frac{1}{18}(n^3 + 3n^2) \right\rfloor,$$

for all $n \geq 4$. Therefore, we have the desired result if G is 5-connected.

Case 2: Let G be a graph which is 4-connected, but not 5-connected. G contains a cut set of size 4, which induces a cycle of length four, by Lemma 6.2.3. Let us denote the vertices of this cut set as v_1, v_2, v_3 and v_4, forming the cycle in this given order. The cut-set divides the plane into two regions. We call them the inner and the outer region, respectively. Let us denote the number of vertices in the inner region by x. Without loss of generality, assume that x is minimal as possible but greater than one. Obviously $x \leq \frac{n-4}{2}$ or $x = n - 5$. From here on, we deal with several subcases depending on the value of x.

Case 2.1: Let $x \geq 4$ and $x \neq n - 5$. Let us consider the subgraph of G, say G', obtained by deleting all vertices from the outer region of the cycle v_1, v_2, v_3, v_4 in G. The graph G' is not maximal since the outer face is a 4-cycle. The graph G is 4-connected; therefore, it does not contain either $\{v_1, v_3\}$ or $\{v_2, v_4\}$. Consequently, we may add any of them to G' to obtain a maximal planar graph. Adding an edge decreases the Wiener index of G'. In the following paragraph, we prove that adding one of the edges decreases the Wiener index of G' by at most $\frac{x^2}{16}$.

Let $A_i = \{v \in V(G') | d(v, v_i) < d(v, v_j), \forall j \in \{1, 2, 3, 4\} \setminus \{i\}\}$, for $i \in \{1, 2, 3, 4\}$. Let A be the subset of the vertices of G' not contained in any of the A_i's. So A, A_1, A_2, A_3, A_4 is a partition of the vertices of G'. It is simple to observe that, if adding an edge $\{v_i, v_{i+2}\}$, for $i \in \{1, 2\}$, decreases the distance between a pair of vertices, then these vertices are from A_i and A_{i+2}. If there is a vertex u which has at least three neighbors from the cut set, without loss of generality say v_1, v_2, v_3, then $A_2 = \emptyset$, since G is 4-connected. Therefore, we are done if such vertex a vertex exists. Otherwise, for each pair $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_1\}$, there is a distinct vertex which is adjacent to both vertices of the pair. Hence, the size of A is at least 4. The size of the vertex set $\cup_{i=1}^{4} A_i$, is at most x. By the AM-GM inequality, we have that one of $|A_1| \cdot |A_3|$ or $|A_2| \cdot |A_4|$ is at most $\frac{x^2}{16}$. Therefore, we can choose one of the edges $\{v_1, v_3\}$ or $\{v_2, v_4\}$, such that after adding that edge to the graph G', the Wiener index of the graph decreases by at most $\frac{x^2}{16}$. Let us denote the maximal planar graph obtained by adding this edge to G' by G_{x+4}.

Similarly, the maximal planar graph obtained from G, by deleting all the vertices in the inner region and adding the diagonal, is denoted by G_{n-x}. This decreases the Wiener index by at most $\frac{(n-x-4)^2}{16}$.

Consider the graph G_{n-x} and a subset of its vertices $S = \{v_1, v_2, v_3, v_4\}$. Since the graph G is 4-connected, each non-terminal level of $\mathcal{P}_S^{G_{n-x}}$ has at least 4 vertices. Therefore, we get that $\sigma_{G_{n-x}}(S) \le \sigma_4(n - x - 4) = \frac{(n-x-2)^2}{8}$, from Lemma 6.2.5.

Recall that G' is the graph obtained from G by deleting the vertices from the outer region. For each $i \in \{1, 2, 3, 4\}$, consider the BFS partition $\mathcal{P}_{v_i}^{G'}$. Note that, $x \ge 4$, G is 4-connected, and by minimality of x, we have that every non-terminal level of $\mathcal{P}_{v_i}^{G'}$ has at least 5 vertices, except for two cases. The first level may contain only four vertices, and the penultimate level may also contain four vertices, with the last level having exactly one vertex. The status of the vertex v_i is maximized, if the first and the penultimate level contain four vertices each, the last level contains only one vertex and every other level contains exactly five vertices.

To simplify calculations of the status of the vertex v_i, we may hang a new temporary vertex on the root, and we may bring a vertex from the last level to the previous level. These modifications do not change the status of the vertex, but it increases the number of vertices. Now we may apply Lemma 6.2.6 for this BFS partition, considering that the number of vertices in all levels is 5 exactly.

Therefore, we have $\sigma_{G'}(v_i) \leq \frac{(x+4)^2+5(x+4)}{10}$. Observe that this status contains distances, from v_i to the other vertices in the cut set, which equals four. This is a uniform upper bound for the status of each of the vertices from the cut set.

Finally, we may upper bound the Wiener index of G in the following way:

$$\begin{aligned}W(G) \leq W(G_{n-x}) + \frac{(n-x-4)^2}{16} + W(G_{x+4}) + \frac{x^2}{16} - 8 \\ + x \cdot \sigma_{G_{n-x}}(\{v_1, v_2, v_3, v_4\}) + (n-x-4) \cdot (\sigma_{G'}(v_1) - 4).\end{aligned}$$

In the first line, we upper bound all the distances between pairs of vertices on the cut set and the outer region, and between pairs of vertices on the cut set and the inner region. We subtract 8 since distances between the pairs from the cut set were double counted. In the second line, we upper bound all the distances from the outer region to the inner region. These distances are split in two, distances from the outer region to the cut set and from a fixed vertex of the cycle to the inner region. Without loss of generality, let the fixed vertex of the cycle be v_1.

We are going to prove that $W(G) \leq \frac{1}{18}(n^3+3n^2)-1$. We need to prove the following inequality:

$$\begin{aligned}\frac{1}{18}(n^3+3n^2) - 1 \geq \frac{1}{18}((n-x)^3 + 3(n-x)^2) + \frac{(n-x-4)^2}{16} \\ + \frac{1}{18}((x+4)^3 + 3(x+4)^2) + \frac{x^2}{16} - 8 \\ + x \cdot \frac{(n-x-2)^2}{8} + (n-x-4) \cdot (\frac{(x+4)^2+5(x+4)}{10} - 4).\end{aligned}$$

After simplifications, we get

$$\frac{82}{45} - \frac{9n}{10} + \frac{n^2}{16} + \frac{x}{5} + \frac{41nx}{120} - \frac{n^2x}{24} - \frac{3x^2}{40} + \frac{nx^2}{60} + \frac{x^3}{40} \leq 0. \tag{6.2}$$

We know that $4 \leq x \leq \frac{n-4}{2}$ and if we set $x = 4$, we get $2176 + 528n - 75n^2 \leq 0$ which holds for all $n \geq 10$. The derivative of the right-hand side of the inequality is negative for all $\{x \mid 4 \leq x \leq \frac{n-4}{2}\}$. Thus, the inequality holds for all these values of x. Differentiating the LHS of the Inequality (6.2), with respect to x, we get

$$\begin{aligned}&\frac{\delta}{\delta x}\left(\frac{82}{45} - \frac{9n}{10} + \frac{n^2}{16} + \frac{x}{5} + \frac{41nx}{120} - \frac{n^2x}{24} - \frac{3x^2}{40} + \frac{nx^2}{60} + \frac{x^3}{40}\right) \\ &= \frac{1}{5} + \frac{41n}{120} - \frac{n^2}{24} - \frac{3x}{20} + \frac{nx}{30} + \frac{3x^2}{40}.\end{aligned} \tag{6.3}$$

If we set $x = 4$ in Equation 6.3, we get $\frac{1}{120}(96+57n-5n^2)$, which is negative for all $n \geq 13$. If we set $x = \frac{n-4}{2}$ in Equation 6.3, we get $\frac{1}{160}(-n^2+8n+128)$, which is negative for all $n \geq 17$. Therefore, Equation 6.3 is negative in the whole interval. Since $n \geq 19$, we have $W(G) \leq \frac{1}{18}(n^3 + 3n^2) - 1$, and this subcase is settled.

Case 2.2 : **Let** $2 \leq x \leq 3$**.** By the minimality of x and maximality of G, we have $x = 2$. Let G_{n-2} denote the maximal planar graph obtained from G by deleting these two vertices from the inner region and adding an edge which decreases the Wiener index by at most $\frac{(n-6)^2}{16}$. Such an edge exists, as in the previous case.

Since G is 4-connected, for each vertex v, $v \in V(G)$, each level of $\mathcal{P}_v^G$ contains at least 4 vertices, except the last one possibly. Therefore, status of both vertices inside can be bounded by $\sigma_5(n) = \frac{1}{8}((n-1)^2 + 4(n-1) + 4)$. This bound comes from Lemma 6.2.5. Finally, we have

$$\begin{aligned}
W(G) &\leq W(G_{n-2}) + \frac{(n-6)^2}{16} + \frac{2}{8}((n-1)^2 + 4(n-1) + 4) - 1 \\
&\leq \frac{1}{18}((n-2)^3 + 3(n-2)^2) + \frac{(n-6)^2}{16} + \frac{2}{8}((n-1)^2 + 4(n-1) + 4) - 1 \\
&= \frac{1}{18}n^3 + \frac{7}{48}n^2 - \frac{1}{4}n + \frac{49}{18} - 1 \leq \frac{1}{18}(n^3 + 3n^2) - 1.
\end{aligned} \tag{6.4}$$

The last inequality holds for all $n \geq 9$, so we have the desired result in this subcase.

Case 2.3**: Let** $x = n - 5$**.** We have a vertex outside the cut set. Let G_{n-1} denote the maximal planar graph, obtained from G by deleting the vertex from the outer region and adding an edge which decreases the Wiener index by at most $\frac{(n-5)^2}{16}$.

By the choice of x, we have that for the vertex outside the cut set v, each level of $\mathcal{P}_v^G$ contains at least 5 vertices, except the first one which contains only 4 vertices. The penultimate level may also contain 4 vertices followed by a vertex in the last level. The status of the vertex v is maximized, if the first and the penultimate level contain four vertices each, the last level contains only one vertex and every other level contains exactly five vertices.

Therefore, status of v can be bounded by $\frac{1}{10}(n^2 + 5n)$. This bound comes from Lemma 6.2.6. Finally, we have

$$
\begin{aligned}
W(G) &\leq W(G_{n-1}) + \frac{(n-5)^2}{16} + \frac{1}{10}(n^2+5n) \\
&\leq \frac{1}{18}((n-1)^3 + 3(n-1)^2) + \frac{(n-5)^2}{16} + \frac{1}{10}(n^2+5n) \\
&= \frac{1}{18}n^3 + \frac{13}{80}n^2 + \frac{7}{24}n - \frac{241}{144} \leq \frac{1}{18}(n^3+3n^2) - 1.
\end{aligned} \tag{6.5}
$$

The last inequality holds for all $n \geq 9$, so we have the desired result in this subcase.

Case 3**: Let** G **be a graph which is** 3**-connected and not** 4**-connected.** Since G is not 4-connected, and it is a maximal planar graph, it must have a cut set of size 3, say $\{v_1, v_2, v_3\}$. This induces a triangle from the Lemma 6.2.3. Without loss of generality, let us assume that the number of vertices in the inner smaller region of the cut set is as minimal as possible, say x.

Case 3.1 : **Let** $x \leq 2$**.** By the minimality of x, we have $x = 1$. Let us denote this vertex as v. Let G_{n-1} be a maximal planar graph obtained from G by deleting the vertex v. From the Lemma 6.2.4, we have $\sigma_G(v) \leq \frac{1}{6}(n^2+n) - \frac{1}{3}\mathbb{1}_{3|(n-1)}$, where $\mathbb{1}_{3|(n-1)}$ equals one if 3 divides $n-1$ and zero otherwise. Finally, we have,

$$
\begin{aligned}
W(G) &\leq W(G_{n-1}) + \sigma_G(v) \\
&\leq \frac{1}{18}((n-1)^3 + 3(n-1)^2) - \frac{1}{9}\mathbb{1}_{3|n} - \frac{2}{9}\mathbb{1}_{3|(n-2)} \\
&+ \frac{1}{6}(n^2+n) - \frac{1}{3}\mathbb{1}_{3|(n-1)} = \frac{n^3}{18} + \frac{n^2}{6} + \frac{1}{9} - \frac{1}{9}\mathbb{1}_{3|(n)} - \frac{2}{9}\mathbb{1}_{3|(n-2)} - \frac{1}{3}\mathbb{1}_{3|(n-1)} \\
&\leq \left\lfloor \frac{1}{18}(n^3+3n^2) \right\rfloor.
\end{aligned} \tag{6.6}
$$

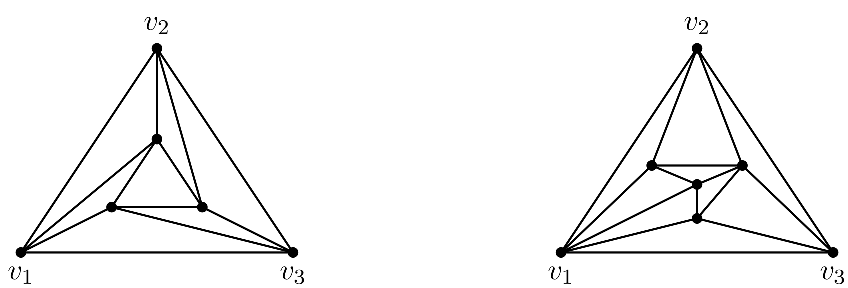
In this case, the equality holds if and only if the graph obtained after deleting the vertex v is T_{n-1}. We can observe that, if we add the vertex v to the graph T_{n-1}, the choice that maximizes the status of the vertex v, $\sigma_G(v) = \frac{1}{6}(n^2+n) - \frac{1}{3}\mathbb{1}_{3|(n-1)}$, is only when we add the vertex v in one of the two faces which gets us the graph T_n. Hence, we have the desired upper bound of the Wiener index and equality holds if and only if $G = T_n$.

Case 3.2 : **Let** $x = 3$**.** Let us denote the vertices in the inner region as x_1, x_2 and x_3. From the minimality of x and maximality of the plane graph G, the structure of G in the inner region is well-defined (see Figure 6.3a). If we remove these three inner vertices, the graph we get is denoted by G_{n-3} and is still maximal. Hence, we may use the induction hypothesis for the graph

G_{n-3}. Consider the graph G_{n-3} and a vertex set $S = \{v_1, v_2, v_3\}$. Each level of $\mathcal{P}_S^{G_{n-3}}$ has at least three vertices, except the terminal one. Therefore, we may apply Lemma 6.2.4, then we have $\sigma_{G_{n-3}}(\{v_1, v_2, v_3\}) \leq \frac{1}{6}((n-6)^2 + 3(n-6) + 2)$. To estimate the distances from the vertices in the outer region to the vertices in the inner region, we do the following: We first estimate the distances from the outer region to the cut set and from the fixed vertex on the cut set to all x_i. The distances from the vertices in the outer region to the set $\{v_1, v_2, v_3\}$, is $\sigma_{G_{n-3}}(\{v_1, v_2, v_3\})$. The sum of distances from v_i to the vertices $\{x_1, x_2, x_3\}$ is 4. If we take a vertex in the outer region that has at least two neighbors on the cut set, then for this vertex we need to count 3 for the distances from the cut set to the vertices $\{x_1, x_2, x_3\}$. Since we have at least two such vertices, all the cross distances can be bounded by $3\sigma_{G_{n-3}}(\{v_1, v_2, v_3\}) + 4(n-5) + 6$. Thus,

$$\begin{aligned} W(G) &\leq W(G_{n-3}) + W(K_3) + 3\sigma_{G_{n-3}}(\{v_1, v_2, v_3\}) + 4n - 14 \\ &\leq \frac{1}{18}((n-3)^3 + 3(n-3)^2) + \frac{1}{2}((n-6)^2 + 3(n-6) + 2) + 4n - 11 \\ &< \left\lfloor \frac{1}{18}(n^3 + 3n^2) \right\rfloor. \end{aligned} \tag{6.7}$$

Therefore, this case is also settled.



(a) $x = 3$.

(b) $x = 4$.

Figure 6.3: The unique inner regions for the 3-connected case when $x = 3$ and $x = 4$.

Case 3.3: **Let** $x = 4$**.** By the minimality of x and the maximality of the planar graph G, there is only one configuration of the inner region (see Figure 6.3b). Let G_{n-4} be the maximal planar graph on the $n-4$ vertices, obtained from G by deleting the four inner vertices. We will apply the induction hypothesis to upper bound the sum of distances between all pairs of vertices from $V(G_{n-4})$ in G. By applying Lemma 6.2.4 for G_{n-4} and $S = \{v_1, v_2, v_3\}$, we get $\sigma_{G_{n-4}}(\{v_1, v_2, v_3\}) \leq \frac{1}{6}((n-4-3)^2 + (n-4-3) + 2)$. The sum of the distances between the four inner vertices is 7. The sum of the distances from each v_i to all the vertices inside is at most six.

By a similar argument as in the previous case we have,

$$
\begin{aligned}
W(G) &\leq \frac{1}{18}((n-4)^3+3(n-4)^2)+7+\frac{4}{6}((n-7)^2+(n-7)+2)+6(n-4) \\
&< \left\lfloor \frac{1}{18}(n^3+3n^2) \right\rfloor .
\end{aligned}
\tag{6.8}
$$

Therefore, this case is also settled.

Case 3.4**: Let** $x = 5$**.** By the minimality of x and the maximality of the planar graph G, there are three configurations of the inner region, see Figure 6.4. Consider a maximal planar graph on the $n-5$ vertices, say G_{n-5}, obtained from G by deleting 5 vertices from the inner region. We will apply the induction hypothesis to bound the sum of the distances between the vertices of $V(G_{n-5})$ in the graph G. By applying Lemma 6.2.4 for G_{n-5} and $S = \{v_1, v_2, v_3\}$, we get $\sigma_{G_{n-5}}(\{v_1, v_2, v_3\}) \leq \frac{1}{6}((n-8)^2+(n-8)+2)$. The sum of the distances between the five inner vertices is at most 13. The sum of the distances from v_i to all the vertices inside is at most 8. We have,

$$
\begin{aligned}
W(G) &\leq \frac{1}{18}((n-5)^3+3(n-5)^2)+13+\frac{5}{6}((n-8)^2+(n-8)+2)+8(n-5) \\
&< \left\lfloor \frac{1}{18}(n^3+3n^2) \right\rfloor .
\end{aligned}
\tag{6.9}
$$

Therefore, this case is also settled.

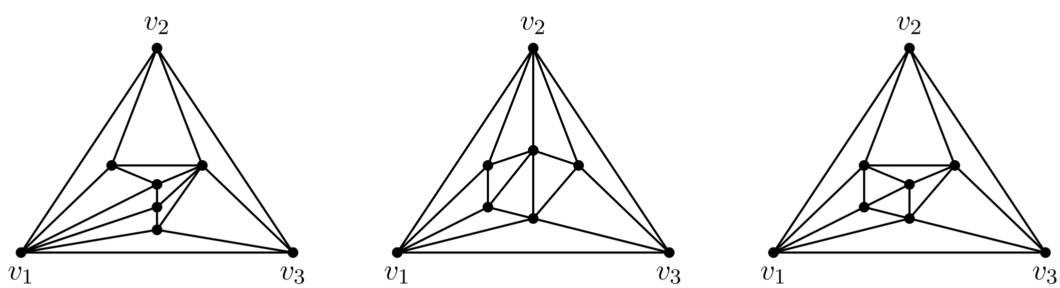


Figure 6.4: The unique inner regions for the 3-connected case when $x = 5$.

Case 3.5**: Let** $x \geq 6$**.** First, we settle for the case for $x \geq 7$. Consider the maximal planar graph on $n-x$ vertices, say G_{n-x}, which is obtained from G by deleting those x vertices from the inner region of the cut set $\{v_1, v_2, v_3\}$. Consider the maximal planar graph on $x+3$ vertices, say G_{x+3}, which is obtained from G, by deleting all $n-x-3$ vertices from the outer region of the cut set $\{v_1, v_2, v_3\}$. We know by induction that $W(G_{x+3}) \leq \frac{1}{18}((x+3)^3+3(x+3)^2)$. There are at least two vertices from the cut set $\{v_1, v_2, v_3\}$, such that each of them has at least two neighbors in the outer region of the cut set. Without loss of generality, we may assume they are v_1 and v_2. Hence, if we consider $\mathcal{P}_{v_1}^{G_{n-x}}$ and $\mathcal{P}_{v_2}^{G_{n-x}}$, we will have 4 vertices in the first level and at least three in the following levels until the last one. Therefore, we have $\sigma_{G_{n-x}}(v_1) \leq \sigma_3(n-x-2)+1 \leq \frac{1}{6}((n-x-2)^2+3(n-x-2)+8)$ from Lemma 6.2.4 and same for v_2. Now let us consider $\mathcal{P}_{\{v_1,v_2\}}^{G_{x+3}}$, from minimality of x, each non-terminal level of the $\mathcal{P}_{\{v_1,v_2\}}^{G_{x+3}}$ contains at least 4 vertices. Therefore, by applying Lemma 6.2.5, we get $\sigma_{G_{x+3}}(\{v_1,v_2\}) \leq \frac{1}{8}(x^2+6x+9)$. We have,

$$\begin{aligned} W(G) \leq & (W(G_{x+3}) + W(G_{n-x}) - 3) + (n-x-3)(\sigma_{G_{x+3}}(\{v_1,v_2\}) - 1) \\ & + x\Bigg(\max\Big\{\sigma_{G_{n-x}}(v_1), \sigma_{G_{n-x}}(v_2)\Big\} - 2\Bigg). \end{aligned} \tag{6.10}$$

The first term of the sum is an upper bound for the sum of all distances which does not cross the cut set. The second and the third terms upper bounds all the cross distances in the following way: we may split this sum into two parts. For each crossing pair, from inside to the set $\{v_1, v_2\}$. Secondly from v_i, $i \in \{1,2\}$ to the vertex outside. Therefore, applying estimates, we get

$$\begin{aligned} \frac{1}{18}(n^3+3n^2) - 1 \geq & \frac{1}{18}((x+3)^3 + 3(x+3)^2) + \frac{1}{18}((n-x)^3 + 3(n-x)^2) - 3 \\ & + \frac{(n-x-3)(x^2+6x+1)}{8} + \frac{x((n-x-2)^2+3(n-x-2)-4)}{6}. \end{aligned} \tag{6.11}$$

After simplification we have

$$-x^3 + x^2(n+3) + x(21-6n) - (15+3n) \geq 0, \tag{6.12}$$

where

$$\frac{\delta}{\delta x}\Bigg(-x^3 + x^2(n+3) + x(21-6n) - (15+3n)\Bigg) = -3x^2 + (2n+6)x + 21 - 6n.$$

The derivative is positive when $x \in [7, \frac{n}{2}]$. Hence, since the inequality (6.12) holds for $x = 7$, it also holds for all x, $x \in [7, \frac{n}{2}]$. Therefore, if $x \geq 7$, we have the desired result.

Finally, if $x = 6$, then distances from v_1 and v_2 to all vertices inside is 9 instead of $\frac{73}{8}$ as it was used in the Inequality 6.11. Thus, we get an improvement of Inequality (6.11), which shows that $W(G) < \lfloor \frac{1}{18}(n^3+3n^2) \rfloor$ even for $x = 6$. Therefore, we have settled the 3-connected case too.

□

6.4 Concluding remarks and Conjectures

The maximal planar graph T_n maximizes the Wiener index and is unique, by Theorem 6.1.6. Clearly T_n is not 4-connected. One may ask for the maximum Wiener index for the family of 4-connected and 5-connected maximal planar graphs. In [19], asymptotic results were proved for both cases. Moreover, based on their constructions, they conjecture sharp bounds for both 4-connected and 5-connected maximal planar graphs. Their conjectures are the following:

Conjecture 6.4.1. *Let G be an $n \geq 6$ vertex maximal 4-connected planar graph. Then*

$$W(G) \leq \begin{cases} \frac{1}{24}n^3 + \frac{1}{4}n^2 + \frac{1}{3}n - 2, & \textit{if } n \equiv 0, 2 \ (\textit{mod } 4); \\ \frac{1}{24}n^3 + \frac{1}{4}n^2 + \frac{5}{24}n - \frac{3}{2}, & \textit{if } n \equiv 1 \ (\textit{mod } 4); \\ \frac{1}{24}n^3 + \frac{1}{4}n^2 + \frac{5}{24}n - 1, & \textit{if } n \equiv 3 \ (\textit{mod } 4); \end{cases}$$

Conjecture 6.4.2. *Let G be an $n \geq 12$ vertex maximal 4-connected planar graph. Then*

$$W(G) \leq \begin{cases} \frac{1}{30}n^3 + \frac{3}{10}n^2 - \frac{23}{15}n + 32, & \textit{if } n \equiv 0 \ (\textit{mod } 5); \\ \frac{1}{30}n^3 + \frac{3}{10}n^2 - \frac{23}{15}n + \frac{156}{5}, & \textit{if } n \equiv 1 \ (\textit{mod } 5); \\ \frac{1}{30}n^3 + \frac{3}{10}n^2 - \frac{23}{15}n + \frac{168}{5}, & \textit{if } n \equiv 2 \ (\textit{mod } 5); \\ \frac{1}{30}n^3 + \frac{3}{10}n^2 - \frac{23}{15}n + 31, & \textit{if } n \equiv 3 \ (\textit{mod } 5); \\ \frac{1}{30}n^3 + \frac{3}{10}n^2 - \frac{23}{15}n + \frac{161}{5}, & \textit{if } n \equiv 4 \ (\textit{mod } 5); \end{cases}$$

Chapter 7

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