Some log-Concavity in Geometry

PhD Thesis

Pavlos Kalantzopoulos

Supervisor:
Károly Böröczky

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTORAL PHILOSOPHY IN MATHEMATICS

Central European University

Budapest, Hungary, 2022

The author hereby declares that the dissertation contains no materials accepted for any other degrees in any other institution and that the dissertation contains no materials previously written or published by another person except where appropriate acknowledgment is made in the form of bibliographic reference.

September 1^{st} September, 2022

Pavlos Kalantzopoulos

Abstract

The results of this thesis belong to area of Geometric Functional Analysis. In particular we study some problems in Convex Geometry using geometrical, analytical and also algebraic tools. We deal with problems on volume concavity, extremizability and isoperimetric type inequality. It follows a summary of the thesis.

log-Brunn-Minkowski conjecture. We show (with Károly Böröczky) the well-known log-Brunn-Minkowski conjecture posed by Böröczky, Lutwak, Yang and Zhang [33], for convex bodies which are symmetric with respect to n-independent linear hyperplanes. In particular, under this high symmetry we show

$$|[h_K^{1-\lambda} h_L^{\lambda}]| \geq |K|^{1-\lambda}|L|^{\lambda},$$

where $[\cdot]$ stands for the Wulff shape and h_K, h_L are the support functions of K, L. Our results strengthen a previous result due to Saroglou [142] (see Bollobás-Leader [24], Uhrin [150], Cordero-Erausquin, Fradelizi, Maurey [60]), treating the unconditional symmetry. We also clarify its equality case and we discuss some consequences including the uniqueness for the solution of the logarithmic Minkowski problem for convex bodies under this symmetry.

Equality on Geometric Barhte's inequality. We characterize (with Károly Böröczky and Dongmeng Xi) the equality case of the geometric reverse Brascamp-Lieb or Barthe's inequality [15], that states the following: if E_i be some subspaces in $\mathbb{R}^n$ and $c_i > 0$ be some positive numbers that satisfy $\sum_{i=1}^k c_i P_{E_i} = I_n$, then for any non-negative integrable function $f_i : E_i \to [0, \infty)$, $i = 1, \ldots, k$, it holds

$$\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^{k} f_i(x_i)^{c_i}\, dx \geq \prod_{i=1}^{k} \left(\int_{E_i} f_i \right)^{c_i}.$$

Here, P_E stands for the orthogonal projection from $\mathbb{R}^n$ onto E and I_n the identity map. It turns out that the extremizers follow almost the same form with the extremizers in the Brascamp-Lieb inequality, found by Valdimarisson [152]. However, our argument is quite different from the one used by Valdimarisson [152].

j-Santaló conjecture. We introduce (with Christos Saroglou) a new family of sharp Santaló type conjectures, motivated by a recently work of Kolesnikov and Werner [97], and we prove them in some cases. For integers $1 \leq j \leq k$ denote s_j the elementary symmetric polynomial of k variables and degree j (see (2.47)). Fix a basis $\{e_m\}$ in $\mathbb{R}^n$ and denote B_j^n the ball of ℓ_j-norm. We study the following question: If $K_1, \ldots, K_k$ symmetric convex bodies in $\mathbb{R}^n$ that satisfy

$$\sum_{l=1}^{n} s_j(x_1(l), \ldots, x_k(l)) \leq \binom{k}{j}, \qquad \forall x_i \in K_i,\ i = 1, \ldots, k,$$

(where $x(l)$ is the l'th coordinate of a vector $x \in \mathbb{R}^n$), then does it hold

$$|K_1| \cdots |K_k| \leq |B_j^n|^k \ ?$$

We were able to prove it in some cases, including the case $j = 1$, $j = k$ and the unconditional case for all j's and we set up an equivalent functional form. Our results also strengthen one of the main results in [97], which corresponds to the case $j = 2$. All members of the family of our conjectured inequalities, excluding the exceptional case $j = 1$, can be interpreted as generalizations of the classical Blaschke-Santaló inequality, which corresponds to the case $j = k = 2$. Related, we discuss an analogue of a conjecture due to K. Ball [10] in the multi-entry setting and establish a connection to the j-Santaló conjecture.

Acknowledgments

I will always be grateful to Károly Böröczky, my supervisor, for the support and nurturing throughout the last 4 years. Károly introduced me to new mathematical areas and gave me the opportunity to attend conferences and meet many fellow mathematicians. His dedication and his encouragement in difficult times as well as our KV-meetings will always be in my memory.

I wish to express my deep gratitude to my collaborator and dear friend Christos Saroglou. I have learned a lot from him which reinforced my ability to deal with mathematical problems. I also gratefully acknowledge the hospitality that university of Ioannina offered me during the summer period of 2021.

The completion of my PhD thesis would not have been possible without the financial support provided by the Central European University as well as Alfréd Rényi Institute of Mathematics.

I wish to extend my special thanks to Souzanna Papadopoulou and Apostolos Giannopoulos for offering me their precious time and knowledge, including Stelios Chatzidakis for always being by my side.

Finally, I would like to express my deep feelings to Stergia Georgopoulou, my love, for her support and for showing me how to ''always look on the bright side of life".

Contents

Chapter 1

Notation and Preliminaries

This section shortly quote some fundamental elements from the area of Convex Geometry that we will need later, and in parallel settles thesis notation. For proofs and more information we refer to the Schneider's books [144].

Notation

We work in $\mathbb{R}^n$ endowed with an Euclidean structure $\|\cdot\|_2, \langle\cdot,\cdot\rangle$ and $|\cdot|$, $\mathcal{H}^k$ stands for volume (Lebesgue measure) and k-dimensional Hausdorff measure, respectively. We write $P_E M$ for the orthogonal projection of a set M in $\mathbb{R}^n$ into a linear subspace E. We denote with ∂B and relint$B$ the relative boundary and the relative interior with respect to the affine hull of a subset B in $\mathbb{R}^n$, respectively, and int$B$ the interior of B with respect to $\mathbb{R}^n$. For $p \geq 1$ we write B_p^n for the unit ball of the usual ℓ_p-norm and $S^{n-1} = \partial B_2^n$ stands for the unite sphere. Also, ellipsoid is any set of the form $\Phi(B_2^n)$ where $\Phi \in GL(n)$. The Minkowski sum of two sets A and B in $\mathbb{R}^n$ is given by

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and scalar multiplication by $\lambda A = \{\lambda a : a \in A\}$, $\lambda \in \mathbb{R}$. A set X in $\mathbb{R}^n$ is said to be invariant under $\Phi \in \mathrm{GL}(n)$ if $\Phi X = X$. A subset X of $\mathbb{R}^n$ is said to be origin symmetric (or simply symmetric) if X is invariant under $-I_n$. Moreover, X is said to be unconditional (with respect to a prefixed orthonormal basis $e_1, \dots, e_n$) if X is invariant under the the orthogonal reflections $\mathrm{Ref}_{e_1^\perp}, \dots, \mathrm{Ref}_{e_n^\perp}$.

Functions on convex sets

In this thesis a compact convex subset of $\mathbb{R}^n$ with non-empty interior is called convex body. The Minkowski functional or gauge function $\|\cdot\|_K : \mathbb{R}^n \to [0, \infty]$ of a convex body K that contains the origin in its interior, is given by

$$\|x\|_K := \min\{t > 0 : x \in tK\}.$$

If in addition K is a symmetric then $\|\cdot\|_K$ is a norm with unit ball K. For a compact convex set K in $\mathbb{R}^n$ its support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h_K(x) := \max\{\langle x, y\rangle : y \in K\}.$$

Note that h_K is 1-homogeneous which means that can be viewed as a function on the S^{n-1}. It can be checked that $h_K \leq h_L$ if and only if $K \subseteq L$, also $h_{P_E K}(u) = h_K(u)$ for any subspace E and $u \in E$ and $K \mapsto h_K$ is linear with respect to the Minkowski sum and scalar multiplication. Moreover, if a function $h : \mathbb{R}^n \to \mathbb{R}$ is sublinear and positively homogeneous, namely, $h(u+v) \leq h(u) + h(v)$ and $h(\lambda u) = \lambda h(u)$ for $u, v \in \mathbb{R}^n$ and $\lambda > 0$, then there exist a unique compact convex set K with $h = h_K$. For a compact star-shaped K (i.e. $\lambda x \in K$ for any $x \in K$ and $\lambda \in (0,1)$) its radial function $\rho_K : \mathbb{R}^n \setminus \{o\} \to \mathbb{R}$ is given by

$$\rho_K(x) := \max\{\lambda : x \in \lambda K\}.$$

Note that radial function is positively homogeneous of degree -1. The polar (or dual) set K° of a convex body K in $\mathbb{R}^n$ is given by

$$K^\circ := \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1,\ \forall y \in K\}.$$

From the definition it follows that $(\Phi K)^\circ = \Phi^{-t}(K^\circ)$ for $\Phi \in GL(n)$. We note that, polarity for symmetric convex bodies can be explained or equivalently defined by the relation $(B_X)^\circ = B_{X^*}$ where B_X is the unit ball of a Banach space norm X and X^* is the dual space. Böröczky, Schneider [35] have shown that, if a map T that goes from the class of convex bodies that contains the origin to itself, satisfy $K \subseteq L \Rightarrow T(K) \supseteq T(L)$ and $T(T(K)) = K$, then $T = SK^\circ$ for some self-adjoint linear transformation S. We also note that, for any convex body K that contains the origin in its interior $h_{K^o}(u) = \|u\|_K = \rho_K(u)^{-1}$, for any $u \in S^{n-1}$. Last, by the use of spherical coordinates, one can represent volume with respect to the above functions, see Schneider's book [144] pp 62-63.

Mixed volumes and the Brunn-Minkowski inequality

Minkowski's mixed volumes theorem asserts that, for any integer $k \geq 1$, any convex bodies $K_1, \ldots, K_k$ in $\mathbb{R}^n$ and positive real number $\lambda_1, \ldots, \lambda_k > 0$, one has

$$|\lambda_1 K_1 + \ldots + \lambda_k K_k| = \sum_{i_1,\ldots,i_n=1}^{k} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n}), \tag{1.1}$$

where the coefficients $V(K_1, \ldots, K_n)$, called mixed volumes. We note that, mixed volume are nonnegative, symmetric (namely, invariant under permutations of the K_i's), multilinear and continuous with respect to the Hausdorff metric. The Brunn-Minkowski inequality expose the $\frac{1}{n}$-concavity of volume $|\cdot| = V(\cdot, \ldots, \cdot)$, stating, for any convex bodies K and L in $\mathbb{R}^n$ it holds

$$|K + L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}, \tag{1.2}$$

with equality if and only if K and L are (positive) homothetic. Minkowski's first inequality asserts that, for any convex body K and L in $\mathbb{R}^n$ it holds

$$\frac{1}{n}\int_{S^{n-1}} h_L \, dS_K = V(L, K, \ldots, K) \geq |L|^{\frac{1}{n}} |K|^{\frac{n-1}{n}}. \tag{1.3}$$

with equality if and only if K and L are (positive) homothetic. The classic isoperimetric inequality and Urysohn (consequently isodiametric) inequality follows from Minkowski's first inequality, and in particular corresponds to the cases $L = B_2^n$ and $K = B_2^n$, respectively. Minkowski's second inequality, asserts that for any convex bodies K and L in $\mathbb{R}^n$ it holds

$$V(L, K, \ldots, K)^2 \geq V(L, L, K, \ldots, K) V(K, K, \ldots, K). \tag{1.4}$$

For the equality we refer to Shenfeld and Handel [146]. It is well known that inequalities (1.2), (1.3) and (1.4) are equivalent. Far strengthening of Minkowski's second inequality is the Alexandrov-Fenchel inequality, stating that, for any convex bodies $K, L, K_3, \ldots, K_{n-2}$ in $\mathbb{R}^n$ it holds

$$V(K, L, K_3, \ldots, K_{n-2})^2 \geq V(K, K, K_3 \ldots, K_{n-2}) V(L, L, K_3, \ldots, K_{n-2}). \tag{1.5}$$

For the equality cases (which is not completely known) we refer to the paper Shenfeld and Handel [147] dealing with the case of polytopes.

Smoothness - (Local mixed volumes)

A vector $u \in S^{n-1}$ is said to be exterior unit normal vector at a boundary point $x \in \partial K$ if $\langle x, u\rangle = h_K(u)$. The family of all exterior unit normal vectors at a point $x \in \partial K$ is denoted by $\nu_K(x)$. It is known that for $\mathcal{H}^{n-1}$ almost all $x \in \partial K$ the $\nu_K(x)$ consist from one vector and in this case x is called smooth boundary point. The set of smooth boundary points is denoted by $\partial' K$ and the $\nu_K : \partial' K \to S^{n-1}$ is known as Gauss map. A convex body K is called smooth (or of class C^1) if $\partial' K = \partial K$. The surface area measure S_K of a convex body K in $\mathbb{R}^n$, is a Borel measure on the sphere S^{n-1}, given for Borel sets $\omega \subseteq S^{n-1}$ by

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}\omega).$$

We note, S_K is translation invariant and S_K is barycentered at the origin, that means $\int_{S^{n-1}} u \, dS_K(u) = o$. A convex body K is said to be of class C^2_+ if h_K is twice differentiable on the sphere S^{n-1} and $\det(\nabla^2 h_K + h_K \,\mathrm{Id}) > 0$. The L_0-surface area measure or cone volume measure V_K of a convex body K with $o \in \mathrm{int}K$, is a Borel measure on the sphere S^{n-1}, given by

$$dV_K = \frac{1}{n}\, h_K \, dS_K.$$

It is known that, the surface area measure can be obtained as a coefficient from the polynomial representation of $|K + tB_2^n|$ in a local sense. More informations can be found in [144] refereed as Cristoffel problems. It is also known that, the cone volume measure can be captured from the same point of view. This place both previous measure under the light of local mixed volumes.

John (Löwner) position and isotropic meaures on the sphere

For a general reference for the material in this short section, see Artstein-Avidan, Giannopoulos, Milman [4, 5]. For any symmetric convex body K there exists a unique ellipsoid $\mathcal{E}$ of maximal (minimal) volume contained (cotaining) in K, known as John (Löwner) ellipsoid. Furthermore, we say that K is in John (Löwner) position if B_2^n is the maximal (minimal) volume ellipsoid. For $u \in S^{n-1}$ denote $u \otimes u$ the orthogonal projection in direction u; namely $u \otimes u(x) = \langle x, u\rangle u$, $x \in \mathbb{R}^n$. John's Theorem asserts that, for any symmetric convex body K that is in John (Löwner) position, there exist contact points $u_1, \ldots, u_k \in \partial K \cap S^{n-1}$ and positive real numbers $c_1, \ldots, c_k > 0$ such that

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = I_n. \tag{1.6}$$

A known consequence of the John Theorem is that, for any symmetric convex body K in $\mathbb{R}^n$ in John position, it holds $K \subseteq \sqrt{n} B_2^n$, and for any symmetric convex body K in $\mathbb{R}^n$ in Löwner position, it holds $\frac{1}{\sqrt{n}} B_2^n \subseteq K$. Note that there is always a linear image TK of a symmetric convex body K in $\mathbb{R}^n$ for $T \in \mathrm{GL}(n)$, such that TK is in John (Löwner) position.

A (not necessarily even) Borel measure μ on the sphere S^{n-1} is said to be isotropic if

$$\int_{S^{n-1}} u \otimes u \, d\mu(u) = I_n.$$

1.1 log-Concavity

The purpose of this section is to formulate a picture with some facts, conjectures and links among them related to concavity or log-concavity of certain functions in the area of Convex Geometry. The starting point is the known concavity of volume, stated for general measures, and then we discuss related elements as Santaló and Prékopa-Leilder inequalities keeping this general setting.

Let $\alpha \in [-\infty, \infty]$. A function $h : \Omega \to [0, \infty)$ on a convex subset $\Omega \subseteq \mathbb{R}^n$ is called α-concave if for any $x, y \in \Omega$ and $\lambda \in (0, 1)$ it holds

$$h((1-\lambda)x + \lambda y) \geq ((1-\lambda)h(x)^\alpha + \lambda h(y)^\alpha)^{\frac{1}{\alpha}}, \tag{1.7}$$

where the case $p = -\infty, 0, \infty$ are given in the limit sense; meaning that, the left hand side of (1.7) is greater than or equal to $\min\{h(x), h(y)\}$, $h(x)^{1-\lambda}h(y)^{\lambda}$, $\max\{h(x), h(y)\}$, respectively. It is simple to check that, if $\alpha_1 \le \alpha_2$ then α_2-concavity implies α_1-concavity. The case $p = 0$ called log-concavity and we note that, any functions of the form $h = e^{-w(x)}$ with w convex function is a log-concave function and visa versa. We note also that, product $f \cdot g$ and convolution $f * g$ are two closed operations of log-concavity. It is clear that the characteristic function 1_K on a convex set K is ∞-concave while a centered Gaussian function $e^{-\pi\langle Ax,x\rangle}$ where A be a positive definite map is log-concave. In the same spirit, a Borel measure μ in $\mathbb{R}^n$ is said to be α-concave, $\alpha \in [-\infty, \infty]$, if for any Borel sets K and L in $\mathbb{R}^n$ it holds

$$\mu((1-\lambda)K + \lambda L) \ge ((1-\lambda)\mu(K)^{\alpha} + \lambda\mu(L)^{\alpha})^{\frac{1}{\alpha}}. \tag{1.8}$$

As previously, the cases $+\infty, -\infty$ are interpreted in the limit sense as well as the 0-concavity or log-concavity which is of high interest, given by

$$\mu((1-\lambda)K + \lambda\, L) \ge \mu(K)^{1-\lambda}\mu(L)^{\lambda}. \tag{1.9}$$

We remark the followings:

- As α increases the condition is stronger; namely, if $\alpha_1 \le \alpha_2$ and μ is α_2-concave then μ is α_1-concave.

- By Borell [28], (or Brascamp, Lieb [39], Corollary 3.4) it holds that, if $\alpha \in [-\frac{1}{n}, \infty]$ and a density $d\mu/dx$ is α-concave then the measure μ is $\frac{1}{\frac{1}{\alpha}+n}$-concave. Consequently, log-concave densities are log-concave measure, uniform measure on convex set are $\frac{1}{n}$-concave.

- On the other hand, a log-concave measure μ on $\mathbb{R}^n$ whose support does not lie in a hyperplane has a log-concave density function according to Borell [28], Theorem 3.2.

Brunn-Minkowski inequality (1.2) appeared much earlier from the last two remarks and usually referred to the log-concavity of volume (with equality on translates) or to the $\frac{1}{n}$-concavity of the volume (with equality on homothetics) in the class of convex sets. Actually, in case of volume, a classical argument that uses its n-homogeneity shows that the $\frac{1}{n}$-concavity derive from log-concavity. This full concavity of volume has been studied for more than a century and form the core of various areas in fully non-liner partial differential equations, probability, statistics, information theory, additive combinatorics and convex geometry (see for example Trudinger, Wang [149], Tao, Vu [148], Schneider [144]). It was shown by Colesanti [55] that Brunn-Minkowski inequality receives a Poincaré type realization (see also Kolesnikov, Milman [94] for an analogue about Ehrhard inequlaity), while its equality case clarifies uniqueness of the surface area measure.

In the following two paragraphs we shortly analyze Hilbert's and Gromov's proofs of Brunn-Minkowksi inequality. Their methods inspired several later results, however we note that, one can retrieve the $\frac{1}{n}$-concavity of volume in the class of compact sets (not only convex sets), using elementary arguments; induction and arithmetic-geometric mean inequality.

Hilbert's proof of Minkowski's second inequality (1.4) (equivalent to Brunn-Minkowski), is based on interpretation of it as a reverse form of Cauchy-Schwarz inequality. He associated to any C^2_+ convex body K an (elliptic differential) operator $\mathcal{A}_K$ and a measure μ_K for which mixed volume admits the representation $V(L, M, K, \ldots, K) = \langle h_L, \mathcal{A}_K h_M\rangle_{L^2(\mu_K)}$. We refer to Shenfeld, Handel [145] for the exact construction and to Kolesnikov, Milman [95] for a slightly revised approach. This operator turns out to be self-adjoint with discrete spectrum. The remarkable fact is that $\mathcal{A}_K$ contains Minkowski's second inequality on its eigenvalue distribution; namely, inequality (1.4) holds if and only if the second largest eigenvalue of $\mathcal{A}_K$ is negative. A proof of the equivalence leads to a reverse form of Cauchy-Schwarz inequality of $\langle\cdot, \mathcal{A}_K\cdot\rangle_{L^2(\mu_K)}$ which is exactly Minkowski's second inequality. Hilbert's argument extended by Alexandrov (see Bonnesen, Fenchel [26]) providing a second proof of Aleksandrov-Fencel inequality (1.5). Recently, in the papers of Kolesnikov, Milman [95], Handel [81] the validity of the log-Brunn-Minkowski conjecture is studied under this approach, and we shortly discuss it in section 2.1.

Gromov observed that a measure transportation argument (see section 4.4.1) imply the log-concavity of volume. It is easy to check that the map $S(x) = (1-\lambda)x + \lambda T(x)$, $x \in \mathbb{R}^n$ where T push forward the measure $1_K\, dx$ to the measure $1_L\, dx$, provided that the bodies have the same volume, deduce $1_{(1-\lambda)K+\lambda L}(S(x)) \geq 1_K(x)$, $x \in \mathbb{R}^n$. Thus, multiplying the last inequality with the Jacobian $\det(DS(x))$ and then integrating it, one retrieve Brunn-Minkowski inequality by the log-concavity of $A \mapsto \det(A)$. Then, a classical rescaling argument finishes the proof. In other words, Gromov's idea was to reduce Brunn-Minkowksi inequality to its discriminant analogue. This method extended from MacCann [118] providing a second proof of Prékopa-Leindler inequality, and later Barthe [15] discovered the famous reverse Brascamp-Lieb (or Barthe's) inequality. The last, is related with the "best" log-concavity property that it holds for the determinant (see (2.28)) and we discuss it in section 2.2.

Turning to general measures, Gardner and Zvavitch [77] posed the question if any even log-concave measure is also $\frac{1}{n}$-concave on the class of symmetric convex bodies. In other words, if all α-concavities for $\alpha \in [0, \frac{1}{n}]$ are equivalent for even measures on symmetric sets. This conjectured is true in dimension two combining the work of Böröczky, Lutwak, Yang, Zhang [33], Saroglou [143], Livshyts, Marsiglietti, Nayar, Zvavitch [107]. Moreover the conjecture has been established locally near the Euclidean ball by Colesanti, Livshyts, Marsiglietti [57]. The most recent result is due to Cordero-Erausquin, Rotem [62] where they proved that any rotational invariant density $e^{-w(\|x\|_2)}$, where $w : (0,\infty) \to (-\infty,\infty]$ be non-increasing and $t \mapsto w(e^t)$ convex, is $\frac{1}{n}$-concave. Consequently, any density $e^{-\frac{1}{p}\|x\|_2^p}$ with $p > 0$ is $\frac{1}{n}$-concave. We note that the case $p = 2$ treated earlier by Eskenazis, Moschidis [68].

Meyer-Pajor [119, 120] elegantly proved that the classical Blaschke-Santaló inequality derives from the log-concavity of volume; namely, when $d\mu/dx = 1$ and K be a symmetric convex body in $\mathbb{R}^n$, then

$$\mu(K)\mu(K^\circ) \leq \mu(B_2^n)^2. \tag{1.10}$$

Cordero-Erasquin [59] posed the question whether inequality (1.10) holds for any even log-concave measure μ and any symmetric convex body K in $\mathbb{R}^n$. The same author in [59] proved that inequality (1.10) holds for certain class of measures and sets in $\mathbb{C}^n$ and noted that it holds for the Gaussian measure. Fradelizi-Meyer [75], applying the multiplicative Prékopa-Leindler inequality obtained (1.10) for unconditional log-concave measures and unconditional sets while also for any rotationally invariant log-concave measure and symmetric sets. Klartag [91], found that there exist a family of densities that are not all log-concave and satisfy (1.10) for symmetric convex bodies. In section 2.3 we discuss functional forms of the classical Blaschke-Santaló inequality which started by K. Ball [9].

We close this section with functional forms of log-concavity (1.9). Prékopa-Leindler inequality states that, if $\lambda \in (0,1)$ and $h, f, g : \mathbb{R}^n \to [0,\infty)$ be some functions that satisfy, for any $x, y \in \mathbb{R}^n$,

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda, \tag{1.11}$$

then, for any log-concave measure μ in $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} h\, d\mu \geq \left(\int_{\mathbb{R}^n} f\, d\mu\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\, d\mu\right)^{\lambda}. \tag{1.12}$$

The original Prékopa-Leindler inequality state the case where μ be the volume dx, however, volume case implies the same statement for any log-concave density. It is also clear that plug in appropriate characteristic functions one retrieve (1.9). Moreover, there is an equivalently and shorter way to state this result by taking h to be the smallest function that satisfies (1.11) (see Theorem 4.4.3). We note also that, the arithmetic mean in (1.11) can be replaced by the geometric mean (coordinatewise) when h, f, g defined on the orthant $\mathbb{R}_+^n$ (see Theorem 5.1.3 for $n = 1$). Borell [28] (see Marsiglietti [114] for a shorter proof) replaced geometric and arithmetic means that appear in (1.11) and (1.12) by general functions ϕ, Φ that follows some differentiable and homogeneity assumptions, and provided an equivalent formula to this generalized Prékopa-Leindler inequality correlating f, g, ϕ, Φ pointwise. We note that, a Prékopa-Leindler type inequality on Riemmannian manifolds has been established by Cordero-Erausquin, McCann, Schmuckenschlager [61], and recently Crasta, Fragalá [65] presented a new geometric mean

$f \star_\lambda g$ with integral equal to $(\int f)^{1-\lambda}(\int g)^\lambda$ for any $\lambda \in (0,1)$. It is easy to check that Prékopa-Leindler inequality and Hölder inequality can be expand inductively for several functions. This multi version admits far reaching generalization and we discuss it in section 2.2

Chapter 2

Presentation of the results

In this section we present our results which are essentially independent. In sections 3, 4, 5 we discuss the following three papers in the same order.

- K. J. Böröczky, P. Kalantzopoulos, *Log-Brunn-Minkowski inequality under symmetry*, Transactions of the American Mathematical Society, 375 (2022), 5987-6013.
- K. J. Böröczky, P. Kalantzopoulos, D. Xi, *About the case of equality in the Reverse Brascamp-Lieb inequality*, Preprint available at https://arxiv.org/pdf/2203.01428.pdf
- P. Kalantzopoulos, C. Saroglou, *On a j-Santaló Conjecture*, Preprint available at https://arxiv.org/pdf/2203.14815.pdf

2.1 L_p-Brunn-Minkowski theory

Central role in Brunn-Minkowksi theory have the mixed volumes inequalities. Despite the nature of these inequalities, which is completely geometric, they are linked with PDE and Operator theory. In the rapid developed L_p-Brunn-Minkowski theory, significant effort has been given to find and clarify L_p-analogues of the inequalities (1.2), (1.3), (1.4). The main motivation of this is the existence and uniqueness Minkowski problems for the so called L_p-surface area measure $S_{K,p}$, introduced by Lutwak [109]. For $p \in \mathbb{R}$ and a convex body K in $\mathbb{R}^n$ that $o \in \text{int}K$ (namely $h_K > 0$), the L_p-surface area measure $S_{K,p}$ is given by

$$dS_{K,p} := \frac{1}{n} h_K^{1-p} dS_K.$$

The existence L_p-Minkowski problem posed by Lutwak in [109] and asks, for fixed $p \in \mathbb{R}$, find necessary and sufficient conditions for a finite non-trivial Borel measure μ on the sphere S^{n-1}, so that there exist a convex body K with $o \in \text{int}K$ which

$$S_{K,p} = \mu. \tag{2.1}$$

Equation (2.1) can be viewed as a nonlinear partial differential equation on the sphere S^{n-1} (namely as a Monge-Ampère equation) in the class of C^2_+ convex bodies, with known the density of μ and unknown the density of $S_{K,p}$. In particular, the existence Minkowski problem in PDE sense asks, for a given function $f : S^{n-1} \to (0, \infty)$, solve

$$\frac{1}{n} h^{1-p} \det(\nabla^2 h + h\,\text{Id}) = f, \tag{2.2}$$

where h is the restriction of the support function of a convex body containing the origin to S^{n-1}, and $\nabla^2 h$ denotes the Hessian matrix of h with respect to a moving orthonormal frame on S^{n-1}. Alexandrov proposed a way to include non smooth solution in (2.2), the so called Monge-Ampère solutions in the

Alexandrov sense (we refer to Figalli [72] for more informations). The uniqueness L_p-Minkowski problem in a class of convex bodies $\mathcal{K}$, ask to clarify the following relation $\sim$ between two convex bodies K and L in the class $\mathcal{K}$,

$$S_{K,p} = S_{L,p} \Rightarrow K \sim L. \tag{2.3}$$

Let us quote some historical importance result, towards the previous two L_p-Minkowksi problems. First Minkowski [129] studied the existence problem (2.1) for the surface area measure, the case $p = 1$. The discrete case treated by him while the general by Alexandrov [1], Fenchel, Jessen [71]. Firey [73] established the so called L_p-Brunn-Minkowksi inequality, for $p \geq 1$, that is based on an extended Minkowski sum, while also treat some uniqueness result for $p = 0$ in [74]. Lutwak posed the L_p-Minkowski problem and studied the case $p \geq 1$ in [109]. A result for $p < 1$ appeared first by Chen, Huang [54]. The papers of Böröczky, Lutwak, Yang and Zhang [33, 34] approach the case $p = 0$ in a very inspired way, characterizing the even cone volume measures. This triggered several later results under the L_p-setting and we discuss some of them in the following sections. Our result clarify the relation in (2.3) for $p = 0$ and $\mathcal{K}$ the class of convex bodies with n-hyperplane symmetries.

L_p-sums and L_p-Problems

To state Firey [73] extension of the usual Minkowski's sum and scalar multiplication, we recall that, any sub-linear and positive homogeneous function h in $\mathbb{R}^n$ corresponds to a unique convex body K in $\mathbb{R}^n$, in which $h = h_K$. Thus, Minkowski's operations, can be viewed (or equivalently defined) as follows, denote with $K + L$ and αK the convex bodies with support function,

$$h_{K+L} = h_K + h_L \quad \text{and} \quad h_{\alpha K} = \alpha h_K.$$

Recall, the map $K \mapsto h_K$ is linear with respect to the initially Minkowski definition for sum and scalar multiplication. Firey [73] defined the so called L_p-sum and scalar multiplication attaching the L_p-norm as follows: for any $p \in [1, \infty]$, denote with $K +_p L$ and $\alpha \cdot K$ the convex bodies with support functions,

$$h_{K+_pL} := (h_K^p + h_L^p)^{\frac{1}{p}} \quad \text{and} \quad h_{\alpha\cdot K} := \alpha^{\frac{1}{p}} h_K. \tag{2.4}$$

For $p \in (1, \infty]$ the function $(h_K^p + h_L^p)^{\frac{1}{p}}$ is sub-additive by Minkowksi's inequality, while for $p < 1$ definition (2.4) fails. Böröczky, Lutwak, Yang and Zhang extended Firey's L_p-sum for $p < 1$ in [33]. To state it, the Wulff shape $[f]$ of a continues function $f : S^{n-1} \to (0, \infty)$ is the convex body given by

$$[f] := \left\{x \in \mathbb{R}^n : \langle x, u\rangle \leq f(u) \ \forall u \in S^{n-1}\right\}. \tag{2.5}$$

Readily, $K = [h_K]$, thus Firey L_p-combinations for $p \in [1, \infty]$ and bodies K and L that contain the origin in its interior, can be written (or equivalently defined) as

$$(1 - \lambda) \cdot K +_p \lambda \cdot L := [((1 - \lambda)h_K^p + \lambda h_L^p)^{\frac{1}{p}}]. \tag{2.6}$$

Böröczky, Lutwak, Yang and Zhang [33] defined the extended L_p-sum for any $p \in [-\infty, \infty]$ through (2.6). The case $p = 0$ is known as logarithmic convex combination and is given in the limit sense,

$$(1 - \lambda) \cdot K +_0 \lambda \cdot L := [h_K^{1-\lambda} h_L^{\lambda}].$$

It is easy to check that the support function of the body $[f]$ is less than or equal to f, namely $h_{[f]} \leq f$. The well known Aleksandrov's Lemma states that $h_{[f]}(u) = f(u)$, for any u belonging to the support of the surface area measure $S_{[f]}$. So, when $[f]$ has smooth boundary then necessarily $h_{[f]} = f$ and in turn f is a support function. We note, when $p \in [0, 1)$ the L_p-convex combination of two smooth convex bodies is not necessarily smooth.

We denote with $\mathcal{K}_o^n$ the set of all convex bodies in $\mathbb{R}^n$ containing the origin in its interior and with $\mathcal{K}_e^n$ the set of all origin symmetric convex bodies. Böröczky, Lutwak, Yang and Zhang [33] posed the problem: for fixed $p \in [-\infty, \infty]$, find a reasonably big class $\mathcal{K}'$ of convex bodies so that, for any $K, L \in \mathcal{K}'$ and $\lambda \in (0,1)$ it holds the so called L_p-Brunn-Minkowksi inequality,

$$|(1-\lambda)\cdot K +_p \lambda \cdot L| \geq \left((1-\lambda)|K|^{\frac{p}{n}} + \lambda |L|^{\frac{p}{n}}\right)^{\frac{n}{p}}; \tag{2.7}$$

which, as usual the cases $p = -\infty, 0, \infty$ are interpreted in the limit sense. Note that, a rescaling argument (see [33]) shows that for any $p \in (0, \infty]$ inequality (2.7) is equivalent to

$$|(1-\lambda)\cdot K +_p \lambda \cdot L| \geq |K|^{1-\lambda}|L|^{\lambda}, \tag{2.8}$$

and by the definition of L_p-sum, it holds the following monotonicity: if $-\infty \leq p \leq q \leq \infty$ then

$$(1-\lambda)\cdot K +_p \lambda \cdot L \subseteq (1-\lambda)\cdot K +_q \lambda \cdot L. \tag{2.9}$$

Firey [73] established (2.8) for $p \geq 1$ and convex bodies in $\mathcal{K}_o^n$. However, this inequality turns out to be a direct consequence of Brunn-Minkowksi inequality (case $p = 1$) and monotonicity (2.9). It is known that for the ranges $p \in [0,1)$ and $p < 0$ inequality (2.8) fails when is viewed as inequality on $\mathcal{K}_o^n$ and $\mathcal{K}_e^n$, respectively. Examples that shows this, is to consider two translated cubes and non-homothetic centered cubes, respectively. However, it was conjectures by Böröczky, Lutwak, Yang and Zhang [33] that for $p \in [0,1)$ and symmetric convex bodies K and L in $\mathbb{R}^n$, inequality (2.8) should hold, known as L_p-Brunn-Minkowski conjecture. The validity of that conjecture will significant strengthen the Brunn-Minkowski inequality under central symmetry and will have various consequences in the area of Convex Geometry. One of the most attractive problems corresponds to the case $p = 0$.

Conjecture 2.1.1 (log-Brunn-Minkowsi Conjecture). *For any symmetric convex bodies K and L in $\mathbb{R}^n$ and any $\lambda \in (0,1)$ it holds,*

$$|(1-\lambda)\cdot K +_0 \lambda \cdot L| \geq |K|^{1-\lambda}|L|^{\lambda}. \tag{2.10}$$

In addition, equality holds if and only if $K = K_1 + \ldots + K_m$ and $L = L_1 + \ldots + L_m$ for compact convex sets $K_1, \ldots, K_m, L_1, \ldots, L_m$ of dimension at least one where $\sum_{i=1}^m \dim K_i = n$ and K_i and L_i are dilates $i = 1, \ldots, m$.

Böröczky, Lutwak, Yang and Zhang [33] state the following related conjecture.

Conjecture 2.1.2 (Log-Minkowski conjecture). *For any symmetric convex bodies K and L in $\mathbb{R}^n$, we have*

$$\int_{S^{n-1}} \log \frac{h_L}{h_K}\, dV_K \geq \frac{|K|}{n} \log \frac{|L|}{|K|}, \tag{2.11}$$

with equality as in Conjecture 2.1.1.

The same authors in [33] proved that Conjecture 2.1.1 and Conjecture 2.1.2 are equivalent on the family of all symmetric convex bodies. Actually, it holds the following slightly general statement (see Proposition 3.5.2).

Lemma 2.1.3. *If $\mathcal{F}$ is a class of convex bodies containing the origin in their interor and $\mathcal{F}$ is closed under dilation and L_0-sum, then* (2.10) *for all $\lambda \in [0,1]$ and $K, L \in \mathcal{F}$ is equivalent with* (2.11) *for all $K, L \in \mathcal{F}$.*

Following again [33], the log-Minkowski conjecture (if true) characterize the even uniqueness Minkowski problem for the cone-volume measure $S_{K,0} = V_K$.

Conjecture 2.1.4 (Uniqueness for even V_K). *Within the class of symmetric convex bodies in $\mathbb{R}^n$, the equation $V_K = V_L$ implies that K and L are related as equality in Conjecture 2.1.1*

In the opposite direction, within the class of origin symmetric convex bodies with C_+^∞ boundary, uniqueness of the convex body with a prescribed cone volume measure implies the log-Minkowski conjecture according to Böröczky, Lutwak, Yang and Zhang [34]. Moreover the same authors in [33] proved that the three previous conjectures are true in the plane.

Theorem 2.1.5 (Böröczky, Lutwak, Yang, Zhang). *The log-Brunn-Minkowksi Conjecture 2.1.1 holds in dimension two.*

Local case for $p = 0$ and the L_p-triad

The first local result on the log-Brunn-Minkowski conjecture is due to Colesanti, Livshyts, Marsiglietti [57], concerning neighborhood of B_2^n. They proved that for any $R \in (0, \infty)$ and any even strictly positive $\phi \in C^2(S^{n-1})$ there exist $\alpha > 0$ such that the conjecture holds for any pair with support function in the class $\{R\phi^\epsilon : \epsilon \in (0, \alpha)\}$. This was strengthened by Kolesnikov and Milman [95] and we state here a particular case for simplicity.

Theorem 2.1.6 (Kolesnikov, Milman)**.** *For any $q \in [2, \infty]$ there exist $n_q \geq 2$ such that, for any $n \geq n_q$ the log-Brunn-Minkowski conjecture* (2.10) *holds for symmetric convex bodies in a C^2-neighborhood of B_q^n. Moreover, $n_2 = 2$.*

We briefly give some explanation for the last two Theorems. Böröczky, Lutwak, Yang and Zhang [33] established the equivalence between (2.10) and (2.11) and solved (2.11) in the plane. The key point concerning this equivalence, is that, inequality (2.10) deduce the log-concavity of $\lambda \mapsto |(1-\lambda) \cdot K +_0 \lambda \cdot L|$ and in turn when this concavity expressed under the first order derivative one obtains (2.11). This gives the idea that a second equivalent form of (2.10) can be deduced from the second order condition for this concavity. We note that, completely analogues equivalence has been obtained for the L_p-sum. The following Theorem states these two reformulations of the L_p-Brunn-Minkowski inequality, proven by the combine work of Böröczky, Lutwak, Yang and Zhang [33], Colesanti, Livshyts, Marsiglietti [57], Kolesnikov, Milman [95], Chen, Huang, Li, Liu [51], Putterman [136].

Theorem 2.1.7. *Let $p \in [0, 1]$. Withing the class symmetric convex bodies in $\mathbb{R}^n$ the following statements are equivalent.*

(i) The L_p-Brunn-Minkowski inequliaty (2.8) *holds.*

(ii) The so called L_p-Minkowski's inequality holds:

$$\left(\frac{1}{n} \int_{S^{n-1}} h_L^p h_K^{1-p} \, dS_k \right)^{\frac{1}{p}} \geq |L|^{\frac{1}{n}} |K|^{\frac{1}{p} - \frac{1}{n}},$$

where case $p = 0$ corresponds to the log-Minkowski inequality (2.11).

(iii) The so called local L_p-Brunn-Minkowski inequality holds

$$\frac{V(L, K \cdots, K)^2}{|K|} \geq \frac{n-1}{n-p} V(L, L, K, \ldots, K) + \frac{1-p}{n(n-p)} \int_{S^{n-1}} \frac{h_L^2}{h_K} \, dS_K. \tag{2.12}$$

where case $p = 0$ is the so called local log-Brunn-Minkowski inequality.

For $p = 1$ these inequalities coincides with inequalities (1.2), (1.3), (1.4), respectively. We note that the third statement, that reminiscent the Minkowski's second inequality (in fact strengthens it) proved equivalent with L_p-Brunn-Minkowksi inequality by Kolesnikov, Milman [95], but in a local sense. This was extended from local to global by Chen, Huang, Li, Liu [51] and Putterman [136] using different approaches, and build on Kolesnikov, Milman [95] they proved that all three inequalities in Theorem 2.1.7 hold for $p \in [1 - c/n^{\frac{3}{2}}, 1)$ where c be an absolute constant. Handel approach this problem from (2.12) (see related studies Shenfeld, Handel [146, 147]) and in [81] proved the following.

Theorem 2.1.8 (Handel)**.** *The log-Minkowksi inequality* (2.11) *holds if K is a zonoid.*

Interrelated Problems

Banaszcyk (see Latala [99]) asked if for any symmetric convex set K in $\mathbb{R}^n$ the function $t \mapsto \gamma(e^t K)$ is log-concave, where γ be the standard Gaussian measure on $\mathbb{R}^n$. This was confirmed by Cordero-Erasquin, Fradelizi and Maurey [60] applying a Poincare inequality, known as (B)-Theorem. The same authors asked if the function $t \mapsto \mu(e^t K)$ is log-concave where μ be any even log-concave measure and K a symmetric convex set, known as (B)-conjecture. Saroglou [142] proved that, the log-Brunn-Minkowksi inequality (if true) implies (B)-conjecture for the uniform measure on symmetric convex bodies. In addition he obtain an equivalent formulation of the log-Brunn-Minkowski conjecture as a (B)-type problem. To state it, let C_n the normalized cube and $\mathrm{diag}(s_1, \ldots, s_n)$ the diagonal transformation with diagonal entries $s_1, \ldots, s_n$.

Theorem 2.1.9 (Saroglou). *The log-Brunn-Minkowksi conjecture is equivalent with the following statement: for any symmetric convex body K in $\mathbb{R}^n$ and any $t_1, \ldots, t_n > 0$ the function*

$$s \mapsto |C_n \cap \mathrm{diag}(t_1^s, \ldots, t_n^s)K| \tag{2.13}$$

is log-concave.

Nayar-Tkocz [130] raised the following problem: for fixed $q \in [1, \infty]$ it holds that for any m-dimensional subspace H of $\mathbb{R}^n$ the function

$$(t_1, \ldots, t_n) \mapsto |\mathrm{diag}(e^{t_1}, \ldots, e^{t_n})B_q^n \cap H| \tag{2.14}$$

is log-concave (here $|\cdot|$ stands for the m-dimensional Lebesque measure). The same author were able to confirm it for $q = 1$ in [130] and they noticed that their problem for $q = \infty$ is equivalent with Saroglou's statement in (2.13). Under the same spirit of (B)-conjecture, Saroglou [143] observed that if one knows the validity of the log-Brunn-Minkowski conjecture then volume can be replaced by any even log-concave measure in (2.10). On the other hand, Livshyts, Marsiglietti, Nayar, Zvavitch [107], showed that the validity of the log-Brunn-Minkowski conjecture for a particular even log-concave measure μ (instead of volume) implies the $\frac{1}{n}$-concavity of μ. Combining last two, it follows that the log-Brunn-Minkowksi conjecture implies the Gardner-Zvavitch conjecture. Actually, Eskenazis, Moschidis [68] confirmed the Gardner-Zvavitch conjecture in the Gaussian case; namely, the Gaussian measure is $\frac{1}{n}$-concave on the class of symmetric convex bodies.

Case $p = 0$ under high Symmetry

Rotem [140] established the log-Brunn-Minkowksi conjecture for complex bodies $K, L \subseteq \mathbb{C}^n$, namely unit balls of norms in $\mathbb{C}^n$, using a general theorem about complex interpolation of Cordero–Erausquin [59]. Turning to the real setting, the classical coordinatewise product of two unconditional convex bodies K and L in $\mathbb{R}^n$ is given

$$K^{1-\lambda} \cdot L^{\lambda} = \{(\pm|x_1|^{1-\lambda}|y_1|^{\lambda}, \ldots, \pm|x_n|^{1-\lambda}|y_n|^{\lambda}) : (x_1, \ldots, x_n) \in K \text{ and } (y_1, \ldots, y_n) \in L\}.$$

As a strengthening of the log-Brunn-Minkowski inequality, it was shown that the volume of the above set is greater than or equals to the geometric mean of the volumes of K and L, by Bollobás, Leader [24], Uhrin [150] and Cordero-Erausquin, Fradelizi, Maurey [[60], Proposition 8], and the arguments are based on the multiplicative Prékopa-Leindler inequality Theorem 5.1.3. The equality case was characterized by Saroglou [142]. In the next Theorem, the set $K_1 \oplus \ldots \oplus K_m$ denotes the Minkowski sum of some compact convex sets $K_1, \ldots, K_m \subseteq \mathbb{R}^n$ if their affine hulls are pairwise orthogonal.

Theorem 2.1.10 (Bollobás-Leader, Uhrin, Saroglou). *If K and L are unconditional convex bodies in $\mathbb{R}^n$ with respect to the same orthonormal basis and $\lambda \in (0, 1)$, then*

$$|(1 - \lambda) \cdot K +_0 \lambda \cdot L| \geq |K|^{1-\lambda}|L|^{\lambda}. \tag{2.15}$$

Equality holds if and only if $K = K_1 \oplus \ldots \oplus K_m$ and $L = L_1 \oplus \ldots \oplus L_m$ for unconditional compact convex sets $K_1, \ldots, K_m, L_1, \ldots, L_m$ of dimension at least one where K_i and L_i are dilates, $i = 1, \ldots, m$.

Our main result strengthen the previous Theorem. To state it, a map $A \in \mathrm{GL}(n)$ is said to be linear reflection (see Davis [66], Humphreys [85], Vinberg [155]) if there is a hyperplane H for which

$$A^2 = \mathrm{Id}_n,\ A \neq \mathrm{Id}_n,\ A|_H = \mathrm{Id}_H. \tag{2.16}$$

A linear reflection satisfies $\det A = -1$ and there exists $u \in S^{n-1} \setminus H$ with $A(u) = -u$. Recall, a set X in $\mathbb{R}^n$ is invariant under $A \in GL(n)$ if $AX = X$. Our main result is the following.

Theorem 2.1.11 (Böröczky, K.)**.** *Let $\lambda \in (0,1)$. If $A_1, \dots, A_n$ are linear reflections such that $H_1 \cap \ldots \cap H_n = \{o\}$ holds for the associated hyperplanes $H_1, \dots, H_n$ and the convex bodies K and L are invariant under $A_1, \dots, A_n$, then*

$$|(1-\lambda) \cdot K +_0 \lambda \cdot L| \geq |K|^{1-\lambda}|L|^{\lambda}. \tag{2.17}$$

In addition, equality holds if and only if $K = K_1 + \ldots + K_m$ and $L = L_1 + \ldots + L_m$ for compact convex sets $K_1, \dots, K_m, L_1, \dots, L_m$ of dimension at least one and invariant under $A_1, \dots, A_n$ where $\sum_{i=1}^m \dim K_i = n$ and K_i and L_i are homothetic, $i = 1, \dots, m$.

Note that, a subset of $\mathbb{R}^n$ with this type of symmetry described above is not necessarily origin symmetric, the n-simplex is an example of that. Barthe, Fradelizi [20] and Barthe, Cordero-Erausquin [19] provided lower bound of the volume product and bounded the isotropic constant, respectively, under this type of symmetry. Very recently Crasta, Fragalá [65] reprove Theorem 2.1.11 introducing a new Prékopa-Leindler type inequality. In what follows, we list some consequences of Theorem 2.1.11, and for convenient, two convex bodies K and L that satisfy the assumption of Theorem 2.1.11 are said to be n-hyperplane symmetric with respect to the same linear reflections.

Symmetries of Regular polytopes. Theorem 2.1.11 settles the log-Brunn-Minkowski conjecture for convex bodies invariant under the symmetry group of a regular polytope.

Log-Minkowksi inequality. According to Böröczky, Lutwak, Yang and Zhang [33], the L_0-sum is covariant under linear tranformations (see (3.12)). Therefore, Theorem 2.1.11 and the method of [33] imply the log-Minkowski inequality under this symmetry (see section 3.5.1): Let K and L convex bodies in $\mathbb{R}^n$ which are n-hyperplane symmetric with respect to the same linear reflections, then

$$\int_{S^{n-1}} \log \frac{h_L}{h_K}\, dV_K \geq \frac{|K|}{n} \log \frac{|L|}{|K|} \tag{2.18}$$

with equality as in Theorem 2.1.11. We remark that inequality (2.18) is scaling invariant. So, it can be equivalently written for bodies of volume one, which in this case receives the form

$$\int_{S^{n-1}} \log h_L\, dV_K \geq \int_{S^{n-1}} \log h_K\, dV_K.$$

Uniqness of V_K. The log-Minkowski inequality (2.18) characterize the uniqueness of cone-volume measure according to Böröczky, Lutwak, Yang and Zhang [33]. We repeat this method in our case in Section 3.5.2, and we obtain that, if K and L be two convex bodies in $\mathbb{R}^n$ which are n-hyperplane symmetric with respect to the same linear reflections $A_1, \dots, A_n$, then the followings are equivalent:

(i) $V_K = V_L$

(ii) $V(K) = V(L)$ and $K = K_1 + \ldots + K_m$ and $L = L_1 + \ldots + L_m$ for compact convex sets $K_1, \dots, K_m, L_1, \dots, L_m$ of dimension at least one and invariant under $A_1, \dots, A_n$ where $\sum_{i=1}^m \dim K_i = n$ and K_i and L_i are homothetic, $i = 1, \dots, m$.

According to Chen, Li, Zhu [52], for general convex bodies, no analogue of the above equivalence can be expected. In particular, one can find two non-homothetic convex bodies with smooth boundary and the same cone-volume measure.

log-concavity with respect to the L_0-sum. Saroglou [[143], Theorem 3.1] proved that, the log-Brunn-Minkowski conjecture (2.10) implies that for any even log-concave measure and any symmetric convex bodies K and L in $\mathbb{R}^n$ and $\lambda \in (0,1)$, it holds

$$\mu((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq \mu(K)^{1-\lambda}\mu(L)^{\lambda}. \tag{2.19}$$

We confirm the above inequality for convex bodies K and L in $\mathbb{R}^n$ which are n-hyperplane symmetric with respect to the same linear reflections, in Theorem 3.5.3.

Gardner-Zvavitch inequality. Livshyts, Marsiglietti, Nayar, Zvavitch [[107], Proposition 1] showed that, if a Borel measure μ with a radial decreasing density f (namely, $f(tx) \geq f(x)$ for $x \in \mathbb{R}^n$ and $t \in [0,1]$) satisfies inequality (2.19) for K, L in a class $\mathcal{K}$, then for any $K, L \in \mathcal{K}$ and any $\lambda \in (0,1)$, it holds

$$\mu((1-\lambda)K + \lambda\, L)^{\frac{1}{n}} \geq (1-\lambda)\mu(K)^{\frac{1}{n}} + \lambda\, \mu(L)^{\frac{1}{n}}.$$

Thus, we immediately obtain the above inequality for a measure μ and convex bodies K, L which are n-hyperplane symmetric with respect to the same linear reflections.

Log-Minkowksi solution We note that for any convex body K, its centroid $\frac{1}{V(K)}\int_K x\,dx$ is invariant under any affine transformation which leaves K invariant. Therefore, Theorem 1.1 in Böröczky, Henk [29] and Theorem 1.4 in Bianchi, Böröczky, Colesanti, Yang [22] yield that the subspace concentration condition characterizes the cone volume measures of convex bodies with high symmetry: Let $G \subseteq O(n)$ be a group acting on S^{n-1} without fixed points, and let μ be a finite non-trivial Borel measure on S^{n-1} invariant under G. Then there exists a G invariant solution of the logarithmic Minkowski equation (namely, (2.1) for $p = 0$) in the Alexandrov sense if and only if μ satisfy the subspace concentration condition.

L_p-Minkowski problems

The purpose of this section is to state some known results about the existence and uniqueness L_p-Minkowksi problems that has been given by several groups of authors. On what follows, μ will always be a finite non-trivial Borel measure on the sphere S^{n-1} and $\operatorname{supp}\mu$ stands for the support of measure μ. A measure μ satisfy the subspace concentration condition (introduced by Böröczky, Lutwak, Yang and Zhang [34]) if, for any proper linear subspace L of $\mathbb{R}^n$, it holds

$$\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n}\mu(S^{n-1}), \tag{2.20}$$

with equality if and only if there exists a complementary subspace L' to L such that $\operatorname{supp}\mu \subseteq L \cup L'$. We say that μ is not concentrated on a set W if the support of μ is not contained in W. A subspace L is said to be essential with respect to μ if $L \cap \operatorname{supp}\mu$ is not concentrated on any closed hemisphere of $L \cap S^{n-1}$. A measure μ satisfy the essential subspace concentration condition if it satisfy the subspace concentration condition only for essential subspaces (with respect to μ). A great subsphere is a set of the form $S^{n-1} \cap L$, where L be a subspace of $\mathbb{R}^n$ codimension one, and hemisphere is clearly the half sphere. The following Theorem list some sufficient conditions for the L_p-surface area measure in the range $p > -n$ and $p \neq n$ and in the following paragraph we quote some necessary conditions.

Theorem 2.1.12 (L_p-Minkowski's Existence)**.** *For $p \in \mathbb{R}$ and finite non-trivial Borel measure μ on S^{n-1}, there exists a convex body K in $\mathbb{R}^n$ containing the origin such that $\mu = S_{K,p}$ if either of the following conditions hold.*

(i) $p > 1$ and $p \neq n$ and μ is not concentrated on any closed hemisphere. This was first proved by Chou, Wang [54] while a second approach has been given by Hug, Lutwak, Yang, Zhang [83].

(ii) $p = 1$ *and* μ *is not concentrated on a great subsphere and* $\int_{S^{n-1}} u\, d\mu(u) = o$*. This was first proved in the discrete case by Minkowski [129] while the general by Alexandrov [1], Fenchel, Jessen [71].*

(iii) $p \in (0, 1)$ *and* μ *is not concentrated on a great subsphere. This was proved by Chen, Li, Zhu [53] (see also Zhu [156] for the discrete case).*

(iv) $p = 0$ *and* μ *satisfies the subspace concentration condition. This was proved by Chen, Li, Zhu [52].*

(v) $p \in (-n, 0)$ *and* μ *has density* f *with respect to* $\mathcal{H}^{n-1}$ *that* $f \in L_{\frac{n}{n+p}}(S^{n-1})$*. This was proved by Bianchi, Böröczky, Colesanti, Yang [22].*

Let us discuss some complementary facts to Theorem 2.1.12. Minkowski's existence Theorem corresponds to case $p = 1$ and the condition stated in (ii) is also necessary. About case (iv), it was proven before [52] that even cone volume measure characterized by the subspace concentration condition, providing necessary and sufficient condition in that case, established by Böröczky, Lutwak, Yang and Zhang [34]. We further note, that the cone volume measure of any centered convex body K in $\mathbb{R}^n$ satisfies the subspace concentration condition according to Böröczky, Henk [29]. Concerning the characterization of the cone volume measure in the non-symmetric case, all what is known is by Böröczky, Hegedűs [30], where they characterized the restriction of $S_{K,0} = V_K$ to an antipodal pair of points.

The following Theorem list some results concerning uniqueness for the L_p-Minkowski problem, that turns out to be a more challenging task.

Theorem 2.1.13 (L_p-Minkowski uniqueness)**.** *Let* K *and* L *two convex bodies in* $\mathbb{R}^n$*. If* $S_{K,p} = S_{L,p}$ *and*

(i) $p > 1$ *and* $p \neq n$ *then* $K = L$*.*

(ii) $p = 1$ *then* K *and* L *are translates of each other.*

(iii) $p \in (1 - \frac{c}{n^{\frac{3}{2}}}, 1)$*, where the absolute constant* $c > 0$ *and* K, L *are symmetric and have* C^2_+ *boundary, then* $K = L$ *by Kolesnikov, Milman [95], Chen, Huang, Li, Liu [51], Putterman [136]).*

(iv) $p = 0$ *and* K *and* L *are* n*-hyperplane symmetric with respect to the same linear reflections, then* K *and* L *are related as equality condition in Theorem 2.1.11.*

For the proofs of the first two we refer to Schneider's book [144]. We also note that for $p < 0$ it is known that the even solution of Minkowski problem may not be unique according to Jian, Lu, Wang [87], Li, Liu, Lu [103], Milman [125].

2.2 Brascamp-Lieb and Barthe's inequalities

The starting point of this section is the Brascamp and Lieb inequalities [39]. The inequalities that we are going to discuss follow the same pattern, which one can explain it as inequalities for which product and integral change their order. The framework that describe this pattern is (now) called data, and many central inequalities in analysis, like Hölder inequalities, Loomis-Whitney inequality, Young for convolution inequalities, hypercontractivity inequalities and many others, follows this pattern and captured from specific datas. This high level of generalization found several use in many mathematical areas, for example, in convex geometry, in harmonic analysis, in probability theory, in information theory, in theoretical computer science and also in number theory, making this an significant tool concentrating lot of interest. Each data is associated with an inequality (or more specific with two inequalities the forward and the reverse) and the initial question/problem was, how one can easily know the best inequality from a chosen data, in terms of its best constant defined below. The first result was due to Brascamp-Lieb [39] where they proved that one can find the best constant testing only centered Gaussian functions.

Recall, a centered Gaussian function is a function of the form $e^{-\pi\langle Ax,x\rangle}$, where $A: H \to H$ be a positive definite linear transform and H be a Hilbert space. Recall, a well known formula

$$\int_H e^{-\pi\langle Ax,x\rangle}\, dx = (\det A)^{-\frac{1}{2}}. \tag{2.21}$$

For some datas, Brascamp-Lieb result found to be an efficient computational tool and in particular to those where later called Geometric datas. Albeit, these class of datas is quite small, there exist a very natural equivalent relation partitioning the set of all datas with the property that, Geometric datas represents those with equality case. This was an important observation which reduce the difficulty of characterizing equality cases in the Brascamp-Lieb inequalities. In this topic our contribution is the characterization of the equality cases in the Geometric Reverse Brascamp-Lieb (or Barthe's) inequality.

Let us begin with the notation. For any k integer, the Brascamp-Lieb data (or simply data) is any finite collection

$$(B_i, c_i)_{i=1}^k, \tag{2.22}$$

where $B_i : \mathbb{R}^n \to H_i$ be surjective linear maps, H_i be a n_i-dimensional Euclidean space endowed with the Lebesgue measure dx and $c_i > 0$ be non-negative real numbers, $i = 1, \ldots, k$. Clearly, $n \geq n_i$ for all i's.

To each data $(B_i, c_i)_{i=1}^k$ a constant is attached as follows: Brascamp-Lieb constant is the smallest real number $c_{BL} \in (0, \infty]$ for which, for any non-negative $f_i \in L_1(H_i)$, $i = 1, \ldots, k$ it holds

$$\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(B_i x)^{c_i}\, dx \leq c_{BL} \prod_{i=1}^k \left(\int_{H_i} f_i \right)^{c_i}. \tag{2.23}$$

The inequality that c_{BL} can be calculated using only centered Gaussians is known as Brascamp-Lieb inequality.

For example, Hölder inequalities corresponds to all datas $(I_n, c_i)_{i=1}^k$ where I_n the identity map and $(c_i)_{i=1}^k$ be any collection of positive real number that satisfy $\sum_{i=1}^k c_i = 1$. Moreover, Lommis-Whitheny inequality [108] correspond to $(P_{e_i^\perp}, \frac{1}{n-1})_{i=1}^n$ where $P_{e_i^\perp}$ is the orthogonal projection from $\mathbb{R}^n$ onto $e_i^\perp$. In all these cases the associated constants c_{BL} are one, since the inequalities are already known and they are also sharp.

Replacing $f_i(x) \mapsto f_i(\frac{x}{\lambda})$ in (2.23) and then making a change of variables one obtains λ^n and $\lambda^{\sum_{i=1}^k c_i n_i}$ in the left-right hand sides of (2.23). Thus, to avoid datas with $c_{BL} = \infty$, we always assume the scaling condition

$$\sum_{i=1}^k c_i n_i = n. \tag{2.24}$$

This is a necessary condition for the finiteness of the constant c_{BL} but not a sufficient condition. This will be discuss later (see Theorem 2.2.9).

Brascamp, Lieb [39] proved in the rank one case (when all $\dim H_i = 1$), that the class of centered Gaussian functions compute c_{BL}. The authors used symmetrization techniques known as rearrangements based in Brascamp, Lieb, Luttinger inequality [38]. After this result, Lieb [105] extended it in the following general case using arguments related to Central limit Theorem.

Theorem 2.2.1 (Lieb)**.** *Let $(B_i, c_i)_{i=1}^k$ be a Brascamp-Lieb data that satisfy $\sum_{i=1}^k c_i n_i = n$. Then centered Gaussian functions compute c_{BL}, namely the Brascamp-Lieb constant equals:*

$$c_{BL} = \sup_{\substack{g_i \text{ centered Gaussian} \\ i=1,\ldots,k}} \frac{\int_{\mathbb{R}^n} \prod_{i=1}^k g_i(B_i x)^{c_i}\, dx}{\prod_{i=1}^k \left(\int_{H_i} g_i \right)^{c_i}}. \tag{2.25}$$

In other words, applying identity (2.21) to (2.25), Theorem 2.2.1 asserts that inequality (2.23) holds for any non-negative function $f_i \in L_1(H_i)$, $i = 1, \ldots, k$ with constant

$$c_{BL} = \sup_{\substack{A_i \text{ positive definite} \\ i=1,\ldots,k}} \left(\frac{\det(\sum_{i=1}^k c_i B_i^* A_i B_i)}{\prod_{i=1}^k (\det A_i)^{c_i}} \right)^{-\frac{1}{2}}. \tag{2.26}$$

Here, B^* denotes the adjoint of B. K. Ball [12, 13] observed that John decompositions (1.6) can be seen as datas and in that cases the constant is computable by Lieb's Theorem 2.2.1. In particular, for datas that the H_i's are one dimensional subspaces of $\mathbb{R}^n$, nicely distributed, and the B_i's are one dimensional orthogonal projections, then the Brascamp-Lieb constant is one. Recall, $u \otimes u(x) := \langle u, x \rangle u$, where $u \in S^{n-1}$, $x \in \mathbb{R}^n$ denotes the one dimensional orthogonal projection in direction u.

Theorem 2.2.2 (K. Ball)**.** *Let $(u_i \otimes u_i, c_i)_{i=1}^k$ be a Brascamp-Lieb data that satisfy $\sum_{i=1}^k c_i u_i \otimes u_i = I_n$. Then, $c_{BL} = 1$, or in other words, for any integrable functions $f_i : \mathbb{R} \to [0, \infty)$, $i = 1, \dots, k$, it holds*

$$\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(\langle x, u_i \rangle)^{c_i} \, dx \leq \prod_{i=1}^k \left(\int_{\mathbb{R}} f_i \right)^{c_i}.$$

Under the assumptions of the above Theorem, $c_{BL} \leq 1$ by combining (2.26) and Proposition (4.2.4), and also $c_{BL} \geq 1$ by sharpness of inequality (4.9). K. Ball applied his Theorem 2.2.2 in convex geometry obtaining several results: bounds for the volume of the sections of the cube [12], estimates about volume ratios [13] and a reverse isoperimetric inequality [13]. Moreover, S. Brazitikos [40] extended Theorem 2.2.2 and found new applications related to Helly's Theorem.

F. Barthe [15] making use of optimal transport of measure (Brenier maps) obtained simultaneously a second proof of Theorem 2.2.1 and a reverse (or dual) form of Brascamp-Lieb inequality (2.23). We note that earlier, McCann [118] used this technique giving an second proof of Prékopa-Leindler inequality which is a special case of the following Barthe's Theorem 2.2.3.

To each data $(B_i, c_i)_{i=1}^k$ a second constant is attached as follows: the Reverse Brascamp-Lieb constant is the largest real number $c_{RBL} \in [0, \infty)$ for which, for any non-negative $f_i \in L_1(H_i)$, $i = 1, \dots, k$ it holds

$$\int_{\mathbb{R}^n}^{*} \sup_{x = \sum_{i=1}^k c_i B_i^* x_i,\, x_i \in H_i} \prod_{i=1}^k f_i(x_i)^{c_i} \, dx \geq c_{RBL} \prod_{i=1}^k \left(\int_{H_i} f_i \right)^{c_i}. \tag{2.27}$$

The inequality that c_{RBL} can be calculated using only centered Gaussians is known as reverse Brascamp-Lieb inequality or Barthe's inequality. Note that this peculiar function on the left hand side is not always integrable and the symbol $\int^*$ stands for the outer integral. F. Barthe proved in [14] the rank one case and in [15] the following general case.

Theorem 2.2.3 (F. Barthe)**.** *Let $(B_i, c_i)_{i=1}^k$ be a Brascamp-Lieb data that satisfy $\sum_{i=1}^k c_i n_i = n$ and $\bigcap_{i=1}^k \ker B_i = \{o\}$. Then both c_{BL} in (2.23) and c_{RBL} in (2.27) can be calculated using centered Gaussians. In addition, if D be the largest real number for which, for any positive definite linear transform $A_i : H_i \to H_i$, $i = 1, \dots, k$ it holds*

$$\det \left(\sum_{i=1}^k c_i B_i^* A_i B_i \right) \geq D \prod_{i=1}^k \det(A_i)^{c_i}, \tag{2.28}$$

then

$$c_{RBL} = 1/c_{BL} = \sqrt{D}. \tag{2.29}$$

Note, if the common kernels of the B_i were not trivial then the left hand side of (2.28) is zero and in turn $D = 0$. However, Theorem 2.2.3 still holds without condition $\bigcap_{i=1}^k \ker B_i = \{o\}$, including cases like $c_{RBL} = 0$ and $c_{BL} = \infty$. Also, note that both constants c_{BL} and c_{RBL} follows from inequality (2.28), which for particular datas is just an extended version of the arithmetic-geometric mean inequality.

In addition to Barthe's approach; there are two other methods of proofs that work for proving both the Brascamp-Lieb and Barthe inequalities. First, a heat equation argument was provided in the rank one case by Carlen, Lieb, Loss [50] for the Brascamp-Lieb inequality and by Barthe, Cordero-Erausquin [17] for the Reverse Brascamp-Lieb inequality. The general versions of both inequalities are proved via the heat equation approach by F. Barthe, N. Huet [18]. Second, probabilistic arguments for the two inequalities are provided by Lehec [100].

Courtade, Liu [64] extended the Brascamp-Lieb data (2.22), unifying into one inequality the Brascamp-Lieb and the Barthe's inequalities. For any k, m integers, we call extended Brascamp-Lieb data (or simply extended data) any finite collection

$$(B_{ij}, c_i, d_j),$$

where $B_{ij} : H_i \to H^j$ be bounded linear transformations, H_i, H^j be finite dimensional Euclidean spaces and c_i, d_j be positive real numbers, $i = 1, \ldots, k$ and $j = 1, \ldots, m$. For all extended datas the corresponding scaling condition, that extends (2.24), is,

$$\sum_{i=1}^{k} c_i \dim(H_i) = \sum_{j=1}^{m} d_j \dim(H^j).$$

To each extended data (B_{ij}, c_i, d_j) a constant is attached as follows. Let $c_{CL} \in (0, \infty]$ be the smallest constant for which it holds (the forward-reverse Brascamp-Lieb inequality)

$$\prod_{i=1}^{k} \left(\int_{E_i} f_i \right)^{c_i} \leq c_{CL} \prod_{j=1}^{m} \left(\int_{E^j} g_j \right)^{d_j} \quad \forall f_i \in L_1^+(E_i),\ g_j \in L_1^+(E^j), \tag{2.30}$$

provided

$$\prod_{i=1}^{k} f_i(x_i)^{c_i} \leq \prod_{j=1}^{m} g_j^{d_j} \left(\sum_{i=1}^{k} c_i B_{ij} x_i \right) \quad \forall x_i \in E_i,\ 1 \leq i \leq k. \tag{2.31}$$

Here, $L_1^+(H)$ stands for the non-negative integrable functions on H. In the same spirit Courtade and Liu [64] proved the following.

Theorem 2.2.4 (Courtade, Liu). *Let (B_{ij}, c_i, d_j) be an extended Brascamp-Lieb data. Then c_{CL} can be computed only choosing centered Gaussian functions, namely c_{CL} equals:*

$$c_{CL} = \inf \frac{\prod_{i=1}^{k} \left(\int_{E_i} f_i \right)^{c_i}}{\prod_{j=1}^{m} \left(\int_{E^j} g_j \right)^{d_j}}$$

where the infimum is taken over all the centered Gaussian functions f_i and g_i that satisfy (2.31).

Lieb's and Barthe's Theorem appeared here as special cases, $k = 1$ and $m = 1$ respectively. Moreover the same authors in [64], generalized the duality relation (2.29) and they also deal with finiteness, structure and extremals. Its important to note that this family of inequalities was introduced earlier by Courtade, Liu together with Cuff and Verdú in [63]. We remark that the direct and the reverse Brascamp-Lieb inequalities are sufficient tools for treating the Bollobás-Thomason inequality and its dual. We provide this already known results in Appendix 6.3 together with equality cases. Thus, one may expect that inequality (2.30) may possibly provide a unified version of Bollobás-Thomasson inequality with its dual.

Geometric data

Let us turn our attention into datas that satisfy an isotropic type condition. For simplicity, let us assume that the surjective linear maps $B_i : \mathbb{R}^n \to E_i$ given in (2.22) have their linear image inside $\mathbb{R}^n$ and in addition are orthogonal projections, $B_i = P_{E_i}$. In this case, a data of the form

$$(P_{E_i}, c_i)_{i=1}^{k}, \tag{2.32}$$

is said to be Geometric Brascamp-Lieb data if it satisfy the so called Geometric condition

$$\sum_{i=1}^{k} c_i P_{E_i} = I_n. \tag{2.33}$$

The "Geometric" terminology coined by Bennett, Carbery, Christ, Tao in [21]. Note that, condition (2.33) implies the scaling condition (2.24), by taking the traces. Barthe [15] extended Ball's Theorem 2.2.2 stating the following.

Theorem 2.2.5 (Ball, Barthe). *Let $(P_{E_i}, c_i)_{i=1,\dots,k}$ be a Geometric Brascamp-Lieb data. Then, $c_{BL} = c_{RBL} = 1$.*

This Theorem, significantly generalize the Hölder, Loomis–Whitney and Prékopa-Leindler inequalities. A proof of that follows by Proposition 4.2.10 which gives $D = 1$ in (2.28) and the Theorem 2.2.3. It is proven by Valdimarsson in [151] that the Bracamp-Lieb constant, in this particular family of datas $(P_{E_i}, c_i)_{i=1}^k$, is minimized by those that satisfy the Geometric condition, which by Theorem 2.2.5 is one.

Extremals for the Geometric data

The following Theorem restate Theorems 2.2.5 and as we have already discussed in previous sections combines the work of Brascamp, Lieb [39], Lieb [105], Ball [13, 12], Barthe [15].

Theorem 2.2.6 (Geometric inequalities, Brascamp-Lieb, Ball, Barthe). *Let $(P_{E_i}, c_i)_{i=1}^k$ be a Geometric Brascamp-Lieb data. Then for any non-negative integrable $f_i \in L_1^+(E_i)$, $i = 1, \dots, k$ one has*

$$\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(P_{E_i} x)^{c_i}\, dx \le \prod_{i=1}^k \left(\int_{E_i} f_i \right)^{c_i} \tag{2.34}$$

and

$$\int^*_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i}\, dx \ge \prod_{i=1}^k \left(\int_{E_i} f_i \right)^{c_i}. \tag{2.35}$$

Inequality (2.34) is known as Geometric Brascamp-Lieb inequality while (2.35) as Geometric reverse Brascmap-Lieb (or Barhte's) inequality. It is not hard to check that, both inequalities achieve equality on Gaussian densities $e^{-\pi\|x\|^2}$. For a proof of that see Lemma 4.2.6 (i) and section 4.3. However, it turns out that equalities can be attained in many other cases and to describe them we need some preparation. For a Geometric data $(P_{E_i}, c_i)_{i=1}^k$, a non-zero linear subspace V is called critical subspace if

$$\dim V = \sum_{i=1}^k c_i \dim(E_i \cap V). \tag{2.36}$$

This condition is equivalent with $E_i = (E_i \cap V) + (E_i \cap V^\perp)$ for any $i = 1, \dots, k$ (see Lemma 4.2.6). We say that a critical subspace V is indecomposable if V has no proper critical linear subspace. For example, the datas concerning Hölder inequality any subspaces of $\mathbb{R}^n$ is critical, while the data concerning Loomis-Whithey inequality a critical subspace is a subspace spanned by a subset of $\{e_1, \dots, e_n\}$. Moreover, $\mathbb{R}^n$ is alway a critical for a Geometric data by the scaling condition (2.24). Critical subspaces was introduced by Carlen, Lieb, Loss [50] in the rank one case and extended by Bennett, Carbery, Christ, Tao [21]. The main reason of this notion was to reduce the problem of finiteness of the constant ([21], Lemma 4.6 and 4.8) to the case where the data has no critical subspaces.

For fixed subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$, a subspace F of $\mathbb{R}^n$ is called independent subspace if $F = \cap_{i=1}^k E_i'$ where E_i' is either E_i or $E_i^\perp$. When there exists ℓ in many non-trivial independent subspaces, we always denoted them by $F_1, \dots, F_\ell$, otherwise we do the convention $\ell = 1$ and $F_1 = \{o\}$. Set $F_{\rm dep}$ to be the orthogonal complement of $\left(\oplus_{j=1}^\ell F_j\right)$. Valdimarsson [152] introduced the so called independent decomposition

$$\mathbb{R}^n = F_{\rm dep} \oplus \left(\oplus_{j=1}^\ell F_j\right). \tag{2.37}$$

The same author characterized extremizers (equality cases) of the Brascamp-Lieb inequality based on the above decomposition in [152]. Let us note that, if there are no independent subspaces then the convention gives $\mathbb{R}^n = F_{\rm dep}$ and if the independent subspace span $\mathbb{R}^n$, namely $\mathbb{R}^n = \oplus_{j=1}^{\ell} F_j$ then clearly $F_{\rm dep} = \{o\}$ in that case. Moreover, criticality is closed under intersection, sum and orthogonal complement (see Lemma 4.2.7) and in turn all the components in (2.37) are critical subspaces.

After partial results of Barthe [15], Carlen, Lieb, Loss [50], Bennett, Carbery, Christ, Tao [21], it was Valdimarsson [152] who characterized extremizers in (2.34) following the proof that uses the heat equation (or flow) argument.

Theorem 2.2.7 (Valdimarsson)**.** *Let $(P_{E_i}, c_i)_{1\le i\le k}$ be a Geometric Brascamp-Lieb data and let $F_1,\dots,F_\ell, F_{\rm dep}$ the components of the independent decomposition induced by this data. We assume that equality holds in the Geometric Brascamp-Lieb inequality* (2.34) *for non-negative $f_i \in L_1(E_i)$, $i = 1,\dots,k$. Then, there exist $b \in F_{\rm dep}$ and $\theta_i > 0$ for $i = 1,\dots,k$, integrable non-negative $h_j : F_j \to [0,\infty)$ for $j = 1,\dots,\ell$, and a positive definite matrix $A : F_{\rm dep} \to F_{\rm dep}$ such that the eigenspaces of A are critical subspaces and*

$$f_i(x) = \theta_i e^{-\langle AP_{F_{\rm dep}}x, P_{F_{\rm dep}}x-b\rangle} \prod_{F_j\subseteq E_i} h_j(P_{F_j}(x)) \quad \text{for Lebesgue a.a. } x \in E_i. \tag{2.38}$$

On the other hand, if for any $i = 1,\dots,k$, f_i is of the form as in (2.38)*, then equality holds in* (2.34) *for $f_1,\dots,f_k$.*

Our main result characterize extremizers of the Geometric reverse Brascamp-Lieb (or Barthe's) inequality (2.35) following the proof that uses optimal transportation argument given by F. Barhte [15].

Theorem 2.2.8 (Böröczky, K., Xi)**.** *Let $(P_{E_i}, c_i)_{1\le i\le k}$ be a Geometric Brascamp-Lieb data and let $F_1,\dots,F_\ell, F_{\rm dep}$ the components of the independent decomposition induced by this data. We assume that equality holds in the Reverse Brascamp-Lieb inequality* (2.35) *for non-negative $f_i \in L_1(E_i)$, $i = 1,\dots,k$, with positive integral. Then there exist $\theta_i > 0$, $b_i \in E_i \cap F_{\rm dep}$ and $w_i \in E_i \cap F_{\rm dep}^{\perp}$ for $i = 1,\dots,k$, log-concave $h_j : F_j \to [0,\infty)$ for $j = 1,\dots,\ell$, and a positive definite matrix $A : F_{\rm dep} \to F_{\rm dep}$ such that the eigenspaces of A are critical subspaces and*

$$f_i(x) = \theta_i e^{-\langle AP_{F_{\rm dep}}x, P_{F_{\rm dep}}x-b_i\rangle} \prod_{F_j\subseteq E_i} h_j(P_{F_j}(x-w_i)) \quad \text{for Lebesgue a.a. } x \in E_i. \tag{2.39}$$

On the other hand, if for any $i = 1,\dots,k$, f_i is of the form as in (2.39) *and equality holds for all $x \in E_i$ in* (2.39)*, then equality holds in* (2.35) *for $f_1,\dots,f_k$.*

General data - Finiteness and Extremals

In this section we go back into the general setting (2.22), and we discuss part from the work of Bennett, Carbery, Christ, Tao [21]. For convenient a Brascmap-Lieb data $(B_i, c_i)_{i=1}^k$ is denoted by $(\mathbf{B},\mathbf{c})$ and the associated constant by $c_{BL}(\mathbf{B},\mathbf{c})$. We write BL-extremizer and RBL-extremizer for a tuple $(f_1,\dots,f_k)$ that achieve equality in (2.23) and (2.27), respectively. Our aim is to discuss finiteness and extremizability. We first note that, finiteness $c_{BL}(\mathbf{B},\mathbf{c}) < \infty$ does not imply the existence of an BL-extremizer. Finiteness was first characterized by Bennett, Carbery, Christ, Tao [21] while later Garg, Gurvits, Oliveira, Wigderson [78] (see Corollary 4.5) provided a second proof.

Theorem 2.2.9 (Bennett, Carbery, Christ, Tao)**.** *Let **(B,c)** be a Brascamp-Lieb data. Then, $c_{BL}(\boldsymbol{B},\mathbf{c}) < \infty$ (equivalently $c_{RBL}(\boldsymbol{B},\mathbf{c}) > 0$ by Theorem 2.2.3) if and only if*

(i) $\sum_{i=1}^k c_i n_i = n$.

(ii) $\dim V \le \sum_{i=1}^k c_j \dim B_j V$ *for every subspace V of $\mathbb{R}^n$.*

Let us quote some facts for the so called Brascamp-Lieb polytope. For fixed linear maps $\mathbf{B} = (B_1,\dots,B_k)$ the Brascamp-Lieb polytope (or polyhedron) is the set $P_{\mathbf{B}} \subseteq \mathbb{R}^k$ defined by,

$$P_{\mathbf{B}} := \{\mathbf{c} = (c_1,\dots,c_k) \in \mathbb{R}^k : c_{BL}(\mathbf{B},\mathbf{c}) < \infty\}.$$

In other words, $P_{\mathbf{B}}$ is the set of all $\mathbf{c} \in \mathbb{R}_+^k$ for which (i) and (ii) holds in Theorem 2.2.9. Trivially, $\dim B_j V \in \{1, \dots, n\}$ for any subspace V, and in turn there is a finite number of linear inequalities in (ii). In fact they are at most n^m. Therefore, we conclude that $P_{\mathbf{B}}$ is the finite intersection of halfspaces and so $P_{\mathbf{B}}$ is indeed a polytope. For example, the Brascamp-lieb polytope of $(I_n, \dots, I_n)$ and $(P_{e_1}, \dots, P_{e_n})$ are $\{\mathbf{c} : \sum_{i=1}^k c_i = 1\}$ and $\{\mathbf{c} = (\frac{1}{n-1}, \dots, \frac{1}{n-1})\}$, respectively, by Hölder and Loomis-Whitney inequalities. This polytope has completely determined in rank one case by Barthe [15] and some extension has been given by Valdimarsson [153].

According to Bennett, Carbery, Christ, Tao [21], for a given data $(\mathbf{B}, \mathbf{c})$ if one set $(\mathbf{B}', \mathbf{c})$ defined by $B_i' = Q_i^{-1} B_i Q$ where $Q : H \to H'$ and $Q_i : H_i \to H_i'$, $i = 1, \dots, k$ be some invertible linear maps, then one has

$$c_{BL}(\mathbf{B}', \mathbf{c}) = \frac{\prod_{i=1}^k (\det Q_i)^{c_i}}{\det Q}\, c_{BL}(\mathbf{B}, \mathbf{c}). \tag{2.40}$$

This leads to the following relation. The collections $\mathbf{B} = (B_1, \dots, B_k)$ and $\mathbf{B}' = (B_1', \dots, B_{k'}')$ are said to be equivalent if $k = k'$ and there exist invertible linear maps $Q : H' \to H$ and $Q_i : H_i' \to H_i$ such that $B_i' = Q_i^{-1} B_i Q$. One may prefer to see that equivalence as a commutative diagram

$$\begin{tikzcd} H \arrow[r, "B_i"] & H_i \arrow[d, "Q_i^{-1}"] \\ H' \arrow[u, "Q"] \arrow[r, "B_i'"] & H_i' \end{tikzcd}$$

In addition, two datas $(\mathbf{B}, \mathbf{c})$ and $(\mathbf{B}', \mathbf{c}')$ are said to be equivalent if $\mathbf{B}$ and $\mathbf{B}'$ are equivalent and $\mathbf{p} = \mathbf{p}'$. This relation is an equivalent relation on datas. Note, for two equivalent data as before it holds $\dim H = \dim H'$ and $\dim H_i = \dim H_i'$, for all i's, and critical subspaces are in $1-1$ correspondence. The last assertion follows from the fact, if V is critical for $(\mathbf{B}, \mathbf{p})$ then $Q^{-1}V$ is critical for $(\mathbf{B}', \mathbf{p})$.

In this setting, $(\mathbf{B}, \mathbf{c})$ is said to be Geometric data (see the simplified version (2.33)) if $B_i B_i^* = Id_{H_j}$ and

$$\sum_{i=1}^k c_i B_i^* B_i = Id_H. \tag{2.41}$$

Note that, for any data $(\mathbf{B}, \mathbf{c})$ one can always find an equivalent data $(\mathbf{T}, \mathbf{c})$ to it, so that (2.41) is satisfied. For this, take any positive definite map $Q_i : H_i \to H_i$, $i = 1, \dots, k$ and set $Q := \sum_{i=1}^k c_i B_i^* Q_i B_i$. Then consider $(\mathbf{T}, \mathbf{c})$ defined by $T_i = Q_i^{\frac{1}{2}} B_i Q^{-\frac{1}{2}}$. Clearly, $(\mathbf{B}, \mathbf{c})$ and $(\mathbf{T}, \mathbf{c})$ are equivalents and

$$\sum_{i=1}^k c_i T_i^* T_i = \sum_{i=1}^k c_i (Q^{-\frac{1}{2}} B_i^* Q_i^{\frac{1}{2}})(Q_i^{\frac{1}{2}} B_i Q^{-\frac{1}{2}}) = Q^{-\frac{1}{2}} \left(\sum_{i=1}^k c_i B_i^* Q_i B_i \right) Q^{-\frac{1}{2}} = Q^{-\frac{1}{2}} Q Q^{-\frac{1}{2}} = Id_H.$$

In addition, if there exist Gaussian extremizer $g_i(x) = e^{-\pi \langle Q_i x, x \rangle}$ for $(\mathbf{B}, \mathbf{c})$ then this implies $Q_i = B_i Q^{-1} B_i^*$ and in turn $T_i T_i^* = Id_{H_i}$ (see Bennett, Carbery, Christ, Tao [21], Proposition 3.6). This briefly explains the direction (ii) $\Rightarrow$ (iii) of the following Theorem.

Theorem 2.2.10 (Bennett, Carbery, Christ, Tao)**.** *Let $(\boldsymbol{B}, \boldsymbol{c})$ be a Brascamp-Lieb data. The following statements are equivalent.*

(i) $(\boldsymbol{B}, \boldsymbol{c})$ *has an BL-extremizer.*

(ii) $(\boldsymbol{B}, \boldsymbol{c})$ *has a Gaussian BL-extremizer.*

(iii) $(\boldsymbol{B}, \boldsymbol{c})$ *is equivalent with a Geometric data,*

The direction (i) $\Rightarrow$ (ii) established by Barthe (see Remark page 17 in [15], see also Proposition 6.5 in [21] for more details) and it based on the Central Limit Theorem. The idea is to apply successively the closure properties of extremizability and find a sequence of extremizers that tends towards to a centered

Gaussian tuple, with respect to L^1 norm. For (iii) $\Rightarrow$ (i), one should find first a Gaussian extremizer for the equivalent Geometric data and then transfer it to $(\mathbf{B}, \mathbf{c})$. We note that Valdimarsson [152] extended his Theorem 2.2.7 applying Theorems 2.2.10. In particular, equalities in the general Brascamp-Lieb inequality (2.23), when exist, can be understood via the language of equivalent relation. Lehec proved the analogue (ii) $\Rightarrow$ (iii) for the Reverse Brascamp-Lieb inequality in [100]. We note that our Theorem 2.2.8 can be extended if one can provide the analogue (i) $\Rightarrow$ (ii) for Reverse Brascamp-Lieb inequality.

2.3 On functional versions of Santaló inequality

This section starts with a short discussion concerning the known upper and lower bounds of volume product and then passes to functional forms. It is quite remarkable that after further strengthens of Ball's functional Santaló inequality several links and analogues appears into this new analytical area. For example these functional Santaló forms are connected with entropy type inequalities according to Fathi [69] while also to a reverse log-Sobolev inequality observed by Caglar, Fradelizi, Guédon, Lehec, Schutt, Werner [48]. Our purpose is to study some new polar conditions in the multi entry setting. We formulate the corresponding Santaló type inequality for sets and functions and we prove it in some cases.

The volume product for a symmetric convex body K in $\mathbb{R}^n$ is defined by $|K||K^\circ|$. It is clear that it is continues with respect to the Hausdorff metric and also $GL(n)$-invariant; namely, the volume product of $\Phi(K)$ and K coincides for any $\Phi \in GL(n)$. The classical Blaschke-Santaló inequality (Blaschke [23], Santaló [141]) provide the exact maximum of the volume product: for any origin symmetric convex body K in $\mathbb{R}^n$, one has

$$|K||K^\circ| \leq |B_2^n|^2, \tag{2.42}$$

with equality if and only if K is an ellipsoids. It is known that Brunn-Minkowksi inequality imply a stronger fact than inequality (2.42), which says, the volume of the polar body increases after any application of Steiner symmetrization (see Meyer, Pajor [119]). Also, it is known that inequality (2.42) is equivalent with the so called affine isoperimetric inequality, under the prism of Minkowski's first and Hölder inequalities.

The exact minimum of the volume product remains unknown and it is considered to be a major problem in convexity. The symmetric Mahler conjecture asserts that the exact minimum is attained on the cube: for any symmetric convex body K in $\mathbb{R}^n$ it should holds

$$\frac{4^n}{n!} = |B_\infty^n||B_1^n| \leq |K||K^\circ|. \tag{2.43}$$

Except cube, it has been found there are many other minimizers, known as Hanner polytopes. This in some sense indicates the difficulty of the problem if one aim to find a process that approach these particular polytopes. However, the conjecture is known in dimension $n = 2$ by Mahler and in dimension $n = 3$ by Iriyeh, Shibata [86]. In higher dimensions $n \geq 4$, minimizer of the volume product are known in the unconditional setting by Saint-Raymond [137], (see also Meyer [123] for a shorter proof), while also for bodies with n-hyperplane symmetries by Barthe, Fradelizi [20]. Reisner confirm the symmetric Mahler conjecture for zonoid in [139] (see also Gordon, Meyer, Reisner [79] for a alternative proof). One can check that John Theorem can lower bound the volume product by $n^{-\frac{n}{2}}|B_2^n|^2$. Bourgain-Milman [36], making of use of the so called M-position improved that bound significantly.

Theorem 2.3.1 (Bourgain-Milman)**.** *There exist an absolute constant $c > 0$ such that, for any origin symmetric convex body K in $\mathbb{R}^n$, one has*

$$|K||K^\circ| \geq c^n |B_2^n|^2.$$

Alternative proofs have been given by Kuperberg [98] and by Nazarov [131]. As a side note, it is easy to check that John's Theorem implies the log-Brunn-Minkowksi conjecture up to the factor $n^{-\frac{n}{2}}$. One may possible expect that M-position can improve this up to c^n.

Functional forms of the Santaló inequality

K. Ball [9] formulated the first functional version of Santaló inequality: for any integrable even functions $f, g : \mathbb{R}^n \to \mathbb{R}_+$ that satisfy $f(x)g(y) \le e^{-\langle x,y\rangle}$, for $x, y \in \mathbb{R}^n$, one has

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(y)\,dy \le (2\pi)^n. \tag{2.44}$$

Lehec [101] noticed that Ball's result follows from induction on the dimension n, gaining in parallel a strengthening of it, in which reduces the even assumption of f, g to f (or g) is barycentered at the origin, that means $\int x f(x)\,dx = o$. Lehec's proof strengthened also a previous result of Artstein, Klartag, Milman [6] which the barycentered function should assumed to be log-concave and their techniques based on approximation. Fradelizi-Meyer [75] first observed that polarity assumption can be relaxed.

Theorem 2.3.2 (Fradelizi, Meyer)**.** *Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a measurable function. If $f, g : \mathbb{R}^n \to \mathbb{R}_+$ are even integrable functions satisfying $f(x)g(y) \le \rho(\langle x, y\rangle)$, for all $x, y \in \mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(y)\,dy \le \left(\int_{\mathbb{R}^n} \rho(\|u\|_2^2)^{1/2}\,du\right)^2.$$

The proof of that use the same tools (Prékopa-Leindler and Santaló inequality for sets) as Ball's proof but different approach.

Functional Santaló for many sets and functions

Caglar, Fradelizi, Guédon, Lehec, Schutt, Werner in [48] obtain consequences from functional Santaló inequalities. One of them is a reverse form of log-Sobolev inequality and an other is a functional affine isoperimetric inequality. Fathi [69] observed that the functional Santaló inequality given by Lehec [101] is equivalent with an entropy inequality that strengthens Talagrand transportation inequality. Kolesnikov and Werner [97] proposed the following extension.

Conjecture 2.3.3. *(Kolesnikov-Werner) Let $k \ge 2$ be an integer, $\rho : \mathbb{R} \to \mathbb{R}_+$ be a decreasing function and $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}_+$ be even integrable functions, such that*

$$\prod_{i=1}^k f_i(x_i) \le \rho\left(\sum_{1\le i<l\le k} \langle x_i, x_l\rangle\right), \qquad \forall x_1, \dots, x_k \in \mathbb{R}^n. \tag{2.45}$$

Then, it holds

$$\prod_{i=1}^k \int_{\mathbb{R}^n} f_i(x_i)\,dx_i \le \left(\int_{\mathbb{R}^n} \rho\left(\frac{k(k-1)}{2}\|u\|_2^2\right)^{1/k} du\right)^k.$$

By the use of Prékopa-Leindler inequality the authors in [54] obtain the following.

Theorem 2.3.4. *(Kolesnikov-Werner) Conjecture 2.3.3 holds if $f_1, \dots, f_k$ are unconditional (with respect to the same orthonormal basis $\{e_m\}$).*

While polarity in the case $k = 2$ is well understood (see Böröczky, Schneider [35], Artstein-Avidan, Milman [7]; see also Artstein-Avidan, Sadovsky, Wyczesany [8] for generalizations), it is not clear if there is such a notion for $k > 2$. Therefore, we believe it is meaningful to seek for Santaló type inequalities for sets and for functions under different conditions than (2.45). More precisely, we give the following definition.

Definition 2.3.5. Let $\Phi : (\mathbb{R}^n)^k \to \mathbb{R}$ be a function. We say that the sets $K_1, \dots, K_k \subseteq \mathbb{R}^n$ satisfy Φ-polarity condition, if $\Phi(x_1, \dots, x_k) \le 1$ for any $x_i \in K_i$, $i = 1, \dots, k$. Similarly, we say that the functions $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}_+$ satisfy Φ-polarity condition with respect to some decreasing function $\rho : \mathbb{R} \to [0, \infty]$, if

$$\prod_{i=1}^{k} f_i(x_i) \le \rho(\Phi(x_1, \ldots, x_k)), \qquad \forall x_i \in \mathbb{R}^n,\ i = 1, \ldots, k.$$

In the rest note we study a specific family of functions Φ. For integers $1 \le j \le k$, we set

$$\mathcal{S}_j(x_1, \ldots, x_k) := \sum_{l=1}^{n} s_j(x_1(l), \ldots, x_k(l)), \qquad x_1, \ldots, x_k \in \mathbb{R}^n, \tag{2.46}$$

where $x(l)$ is the l'th coordinate (with respect to our fixed basis $\{e_m\}$) of a vector $x \in \mathbb{R}^n$, $l = 1, \ldots, n$, and s_j is the elementary symmetric polynomial of k variables and degree j, i.e.

$$s_j(r_1, \ldots, r_k) := \sum_{1 \le i_1 < \ldots < i_j \le k} r_{i_1} \cdots r_{i_j}, \qquad r_1, \ldots, r_k \in \mathbb{R}. \tag{2.47}$$

Set, also,

$$\mathcal{E}_j := \frac{\mathcal{S}_j}{\binom{k}{j}}. \tag{2.48}$$

Note that for $j \ne 2$ the map $\mathcal{E}_j$ (or $\mathcal{S}_j$) depends on the basis $\{e_m\}$. However, for $j = 2$ this is not the case; one can check that

$$\mathcal{S}_2(x_1, \ldots, x_k) = \sum_{1 \le i < l \le k} \langle x_i, x_l \rangle.$$

We conjecture that a Santaló type inequality holds for symmetric sets, under the assumption of $\mathcal{E}_j$-polarity condition.

Conjecture 2.3.6. *(j-Santaló conjecture) Let $1 \le j \le k$ be two integers, where $k \ge 2$. If $K_1, \ldots, K_k$ are symmetric convex bodies in $\mathbb{R}^n$, satisfying $\mathcal{E}_j$-polarity condition, then it holds*

$$\prod_{i=1}^{k} |K_i| \le |B_j^n|^k. \tag{2.49}$$

We, also, formulate the functional version of Conjecture 2.3.6.

Conjecture 2.3.7. *(Functional j-Santaló conjecture) Let $1 \le j \le k$ be two integers, where $k \ge 2$. If $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}_+$ are even integrable functions, satisfying $\mathcal{S}_j$-polarity condition with respect to some decreasing function $\rho : \mathbb{R} \to [0, \infty]$, then it holds*

$$\prod_{i=1}^{k} \int_{\mathbb{R}^n} f_i(x_i)\, dx_i \le \left(\int_{\mathbb{R}^n} \rho\left(\binom{k}{j} \|u\|_j^j \right)^{1/k} du \right)^k. \tag{2.50}$$

Clearly, for $j = 2$, Conjecture 2.3.7 is just Conjecture 2.3.3. As it expected, the functional j-Santaló Conjecture 2.3.7 implies the j-Santaló Conjecture 2.3.6 for sets. To see this, let $K_1, \ldots, K_k$ be symmetric convex bodies satisfying $\mathcal{E}_j$-polarity condition. Then, setting

$$f_i := 1_{K_i},\ i = 1, \ldots, k \qquad \text{and} \qquad \rho(t) := \begin{cases} +\infty, & t < 0 \\ 1_{[0,1]}\left(\binom{k}{j}^{-1} t \right) & t \ge 0 \end{cases}, \tag{2.51}$$

one can check that the functions $f_1, \ldots, f_k$ satisfy $\mathcal{S}_j$-polarity condition with respect to ρ, where with 1_A denotes the indicator function of a set A. Therefore, inequality (5.2) implies,

$$\prod_{i=1}^{k} |K_i| \le \left(\int_{\mathbb{R}^n} \rho\left(\binom{k}{j} \|u\|_j^j \right)^{\frac{1}{k}} du \right)^k = \left(\int_{\mathbb{R}^n} 1_{[0,1]} \left(\|u\|_j^j \right)^{\frac{1}{k}} du \right)^k = \left(\int_{B_j^n} 1\, du \right)^k = |B_j^n|^k.$$

It turns out, however, that the two conjectures are actually equivalent (see Section 5.1.5). Let us state our main results.

Theorem 2.3.8. *Conjecture 2.3.6 holds in the following cases:*

(i) $K_1, \ldots, K_k$ *are unconditional convex bodies.*

(ii) $j = 1$ *or* $j = k$.

(iii) j *is even and* $K_3, \ldots, K_k$ *are unconditional convex bodies.*

Moreover, in all three cases, (5.1) *is sharp for* $K_1 = K_2 = \ldots = K_k = B_j^n$.

Theorem 2.3.9. *Conjecture 2.3.7 holds in the following cases:*

(i) $f_1, \ldots, f_k$ *are unconditional functions.*

(ii) $j = 1$ *or* $j = k$.

(iii) j *is even and* $f_3, \ldots, f_k$ *are unconditional functions.*

Notice that, by (2.51), (2.50) is also sharp for some specific choice of ρ. As mentioned earlier, Theorem 2.3.9 (case (iii), $j = 2$) slightly extends Theorem 2.3.4. Moreover, Theorem 2.3.8 (resp. 2.3.9) for $j = k$ can be viewed as a generalization of the classical Blasckhe-Santaló inequality in the setting of many sets (resp. many functions). The case $j = 1$ is somehow exceptional, as it is not directly related to the classical Blasckhe-Santaló inequality (see, also, Section 5.1.3, Remark 5.1.10).

Ball's conjecture

K. Ball [9] [10] extended volume product introducing the following $SL(n)$-invariant quantity

$$B(K) := \int_K \int_{K^o} \langle x, y \rangle^2 \, dx \, dy.$$

Using the multiplicative version of Prékopa-Leindler, (Theorem 5.1.3), he obtained the following.

Theorem 2.3.10. *If* K *is an unconditional convex body in* $\mathbb{R}^n$*, then*

$$B(K) \leq B(B_2^n).$$

Ball conjectured the following.

Conjecture 2.3.11. *Theorem 2.3.10 holds true for arbitrary symmetric convex bodies.*

We refer to Conjecture 2.3.11 as Ball's conjecture. We should remark that Ball has shown that Conjecture 2.3.11, if true, implies the Blaschke-Santaló inequality (2.42) (in the sense that given the validity of Conjecture (2.3.11), one can prove (2.42) within a few lines). In Section 5.2, we formulate the analogue of Ball's conjecture for many sets that satisfy $\mathcal{E}_j$-polarity condition (resp. many functions that satisfy $\mathcal{S}_j$-polarity condition). The primary goal is to state and discuss a natural (at least in our opinion) extension of Conjecture 2.3.11, to the multi-entry setting. In this direction, let $\mathcal{D}(n)$ be the set of all orthonormal basis' in $\mathbb{R}^n$, let $k \geq 2$, $j \in \{1, \ldots, k\}$ and for $\{\epsilon_m\} \in \mathcal{D}(n)$, define

$$\mathcal{B}_j(K_1, \ldots, K_k, \{\epsilon_m\}) := \sum_{m=1}^{n} \prod_{i=1}^{k} \int_{K_i} |\langle x_i, \epsilon_m \rangle|^j \, dx_i.$$

Define, also

$$\mathcal{B}_j(K_1, \ldots, K_k) := \min_{\{\epsilon_m\} \in \mathcal{D}(n)} \mathcal{B}_j(K_1, \ldots, K_k, \{\epsilon_m\}).$$

One might dare to conjecture the following.

Conjecture 2.3.12. *Let* $1 \leq j \leq k$ *be two integers, where* $k \geq 2$*. Let* $K_1, \ldots, K_k$ *be symmetric convex bodies in* $\mathbb{R}^n$ *satisfying* $\mathcal{E}_j$*-polarity condition. Then,*

$$\mathcal{B}_j(K_1, \ldots, K_k) \leq \mathcal{B}_j(B_j^n, \ldots, B_j^n). \tag{2.52}$$

In section 5.2 we prove this Conjecture in the unconditional case and in the case $j = 1$. Moreover, we show that it implies Conjectures 2.3.6 and 2.3.7.

Chapter 3

Log-Brunn-Minkowski inequality under symmetry

3.1 Introduction

In this section we provide the proof of the Theorem below. Linear reflection is defined in (2.16).

Theorem 3.1.1 (Böröczky, K. [31]). *Let $\lambda \in (0,1)$. If $A_1, \ldots, A_n$ are linear reflections such that $H_1 \cap \ldots \cap H_n = \{o\}$ holds for the associated hyperplanes $H_1, \ldots, H_n$ and the convex bodies K and L are invariant under $A_1, \ldots, A_n$, then*

$$|(1-\lambda) \cdot K +_0 \lambda \cdot L| \geq |K|^{1-\lambda} |L|^{\lambda}. \tag{3.1}$$

In addition, equality holds if and only if $K = K_1 + \ldots + K_m$ and $L = L_1 + \ldots + L_m$ for compact convex sets $K_1, \ldots, K_m, L_1, \ldots, L_m$ of dimension at least one and invariant under $A_1, \ldots, A_n$ where $\sum_{i=1}^m \dim K_i = n$ and K_i and L_i are homothetic, $i = 1, \ldots, m$.

Outline of the proof of (3.1.1)

We reduce the problem into the unconditional setting, by partitioning the L_0-sum of K and L into congruent pieces. Let us sketch some key steps in the special case where the reflections are orthogonal (the general is obtained by transforming K into its Löwner position). First, note that if K and L are G-invariant, for some $G \subseteq O(n)$, then their L_0-sum $Q := (1-\lambda) \cdot K +_0 \lambda \cdot L$ is G-invariant (see (3.12)). So, in our case Q is invariant under the linear reflections $A_1, \ldots, A_n$. We partition Q in the same sense like an unconditional convex body is partitioned by 2^n congruent pieces. To do that we use the Weyl chamber. In particular, there exist a simplicial convex cone C that decompose $\mathbb{R}^n$ in the sense of Proposition 3.3.2 (iii), and decompose Q into congruent pieces, say ℓ of them. Thus, choosing any $\Phi \in GL(n)$ that $\Phi(C) = \mathbb{R}^n_+$ one has

$$\begin{aligned} |(1-\lambda) \cdot K +_0 \lambda \cdot L| &= \ell \, |C \cap [(1-\lambda) \cdot K +_0 \lambda \cdot L]| \\ &= \frac{\ell}{|\det \Phi|} |\mathbb{R}^n \cap [(1-\lambda) \cdot \Phi(K) +_0 \lambda \cdot \Phi(L)]|. \end{aligned} \tag{3.2}$$

Let $\bar{K}$ the unconditional set defined by $\bar{K} \cap \mathbb{R}^n_+ = \Phi(K \cap C)$. Similarly set $\bar{L}$. The key property is that C has small "angle", which guaranties the convexity of $\bar{K}, \bar{L}$ (showed in Lemma (3.3.6)) and implies $\Phi^{-t} C \subseteq \mathbb{R}^n_+$ that conclude to (see (3.18))

$$|\mathbb{R}^n_+ \cap [(1-\lambda) \cdot \Phi(K) +_0 \lambda \cdot \Phi(L)]| \geq |\mathbb{R}^n_+ \cap [(1-\lambda) \cdot \overline{K} +_0 \lambda \cdot \overline{L}]|. \tag{3.3}$$

Last, combining (3.2) and (3.3) and then applying the log-Brunn-Minkowski inequality for the unconditional convex bodies (Theorem 2.1.10) we finish the proof of the inequality.

We briefly explain the characterization of equality. Equality in (3.1) for some K, L implies equality for the associated unconditional $\bar{K}, \bar{L}$. Applying Theorem 2.1.10, we obtain

$$\bar{K} = \oplus_{\beta=1}^{m} \bar{K}_\beta \quad \text{and} \quad \bar{L} = \oplus_{\beta=1}^{m} \bar{L}_\beta$$

where $\bar{K}_\beta = \theta_\beta \bar{L}_\beta$, where $\theta_\beta > 0$. Let $\bar{E}_\beta := \text{lin}\bar{K}_\beta$, $\beta = 1, \ldots, m$. The key observation is that K and L can be split on $\bar{E}_\beta$ and $\bar{E}_\beta^\perp$, meaning (see (3.26))

$$K = P_{\bar{E}_\beta} K \oplus P_{\bar{E}_\beta^\perp} K \quad \text{and} \quad L = P_{\bar{E}_\beta} L \oplus P_{\bar{E}_\beta^\perp} L.$$

for any $\beta = 1, \ldots, m$. Then by induction on m the body K (same for L) is written as the direct sum of $P_{\bar{E}_\beta} K$ for $\beta = 1, \ldots, m$. Last, projecting onto $\bar{E}_\beta$ the $\bar{K} \cap \mathbb{R}^n_+ = \Phi(K \cap C)$ yields $P_{\bar{E}_\beta}(K) = \theta_\beta P_{\bar{E}_\beta}(L)$. The other direction use elementary arguments in the section below.

3.2 Folklore Lemma's

As the equality case of Theorem 3.1.1 indicates, we need a better understanding of convex bodies that are sums convex compact sets in complementary linear subspaces.

Lemma 3.2.1 (Folklore). *Let K be a convex body in $\mathbb{R}^n$, and let $\xi_1, \ldots, \xi_m$, $m \geq 2$ be non-trivial complementary linear subspaces which together span $\mathbb{R}^n$. Then $\nu_K(\partial' K) \subseteq \xi_1 \cup \ldots \cup \xi_m$ if and only if there exist compact convex sets $K_1, \ldots, K_m$ with $\text{lin}(K_i - K_i) = (\sum_{j \neq i} \xi_j)^\perp$ (and hence $\dim K_i = \dim \xi_i$) for $i = 1, \ldots, m$ such that $K = K_1 + \ldots + K_m$.*

Remark If K is unconditional and $K_1, \ldots, K_m$ are unconditional, then $K_i \subseteq \xi_i$, $i = 1, \ldots, m$.

Proof. We may assume that $o \in K$, and hence also that $o \in K_i$ for $i = 1, \ldots, m$ if suitable $K_1, \ldots, K_m$ exists.

If $K = K_1 + \ldots + K_m$ for some compact convex $K_i \subseteq (\sum_{j \neq i} \xi_j)^\perp$, $i = 1, \ldots, m$, then

$$\partial' K = \bigcup_{i=1}^{m} \left(\partial' K_i + \sum_{j \neq i} \text{relint}\, K_j \right),$$

which in turn yields that $\nu_K(\partial' K) \subseteq \xi_1 \cup \ldots \cup \xi_m$ by the property

$$\xi_i^\perp = \text{lin} \sum_{j \neq i} \text{relint}\, K_j$$

for $i = 1, \ldots, m$ (here $\partial' K_i$ is the family of smooth points of the relative boundary of K_i).

On the other hand, let us assume that $\nu_K(\partial' K) \subseteq \xi_1 \cup \ldots \cup \xi_m$, and let $V_i = (\sum_{j \neq i} \xi_j)^\perp$. For any $i = 1, \ldots, m$, let us consider the convex compact set

$$K_i = \{x \in V_i : \langle u, x \rangle \leq h_K(u) \text{ for all } u \in \xi_i \cap \nu_K(\partial' K)\}.$$

As $V_i^\perp + \xi_i = \mathbb{R}^n$ and $V_i^\perp \cap \xi_i = \{o\}$ for $i = 1, \ldots, m$, we deduce that K_i is a $\dim V_i = \dim \xi_i$ dimensional compact convex set. Since K is the intersection of the supporting halfspaces at the smooth boundary points according to Theorem 2.2.6 in Schneider [144], the condition $\nu_K(\partial' K) \subseteq \xi_1 \cup \ldots \cup \xi_m$ implies

$$K = \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n : \langle u, x \rangle \leq h_K(u)\, \forall u \in \nu_K(\partial' K) \cap \xi_i\} = \bigcap_{i=1}^{m} \left(K_i + \xi_i^\perp\right) = K_1 + \ldots + K_m.$$

$\square$

Next we show that equality really holds in Theorem 3.1.1 when promised (even without symmetry assumption).

Lemma 3.2.2 (Folklore)**.** *If $\lambda \in (0,1)$, K and L are convex bodies with $o \in \mathrm{int}K$ and $o \in \mathrm{int}L$, and $K = K_1 + \cdots + K_m$ and $L = L_1 + \cdots + L_m$ for $m \geq 1$ and compact convex sets K_i, L_i, $i = 1, \cdots, m$, having dimension at least one and satisfying $o \in K_i$, $K_i = \theta_i L_i$ for $\theta_i > 0$ for $i = 1, \cdots, m$, and $\sum_{i=1}^m \mathrm{dim}K_i = n$, then*

(i) $(1-\lambda)\cdot K +_0 \lambda \cdot L = \theta_1^\lambda K_1 + \cdots + \theta_m^\lambda K_m$;

(ii) $|(1-\lambda)\cdot K +_0 \lambda \cdot L| = |K|^{1-\lambda}|L|^\lambda$.

Proof. For $i = 1, \ldots, m$, we write $V_i = \mathrm{lin}K_i$, and $\xi_i = \left(\sum_{j\neq i} V_j\right)^\perp$. We observe that if $u \in \xi_i \cap S^{n-1}$, then

$$h_K(u) = h_{K_i}(u) \ \text{ and } \ h_L(u) = \theta_i h_{K_i}(u).$$

It follows from Lemma 3.2.1 that

$$\begin{aligned} K &= \bigcap_{i=1}^m \left\{x \in \mathbb{R}^n : \langle u, x\rangle \leq h_K(u) \ \forall u \in \xi_i \cap S^{n-1}\right\} \\ L &= \bigcap_{i=1}^m \left\{x \in \mathbb{R}^n : \langle u, x\rangle \leq \theta_i h_K(u) \ \forall u \in \xi_i \cap S^{n-1}\right\}; \end{aligned}$$

therefore, $h_K(u)^{1-\lambda}\Big(\theta_i h_K(u)\Big)^\lambda = \theta_i^\lambda h_K(u)$ for $u \in \xi_i \cap S^{n-1}$ and $i = 1, \ldots, m$ yields that

$$(1-\lambda)\cdot K +_0 \lambda \cdot L \subseteq \bigcap_{i=1}^m \left\{x \in \mathbb{R}^n : \langle u, x\rangle \leq \theta_i^\lambda h_K(u) \ \forall u \in \xi_i \cap S^{n-1}\right\} = \sum_{i=1}^m \theta_i^\lambda K_i.$$

To prove $\sum_{i=1}^m \theta_i^\lambda K_i \subseteq (1-\lambda)\cdot K +_0 \lambda \cdot L$, it is enough to verify

$$\sum_{i=1}^m \theta_i^\lambda h_{K_i}(u) \leq h_K(u)^{1-\lambda} h_L(u)^\lambda = \left(\sum_{i=1}^m h_{K_i}(u)\right)^{1-\lambda} \left(\sum_{i=1}^m \theta_i h_{K_i}(u)\right)^\lambda \tag{3.4}$$

for any $u \in S^{n-1}$. However, (3.4) is a direct consequence of the Hölder inequality, completing the proof of (i).

We observe that setting $d_i = \dim K_i$ for $i = 1, \ldots, m$, we have $|L| = \left(\prod_{i=1}^m \theta_i^{d_i}\right)|K|$, and (i) yields that

$$|(1-\lambda)\cdot K +_0 \lambda \cdot L| = \left(\prod_{i=1}^m \theta_i^{d_i}\right)^\lambda |K|,$$

verifying (ii). □

3.3 Simplicial cones and Representation of Coxeter groups

We say that a convex subset $C \subseteq \mathbb{R}^n$ is a convex cone if $\lambda x \in C$ for any $x \in C$ and $\lambda \geq 0$. The positive dual cone of C is defined by

$$C^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \geq 0 \text{ for each } y \in C\}.$$

For any n independent vectors $u_1, \cdots, u_n \in \mathbb{R}^n$, the convex cone C generated by their positive hull

$$C = \mathrm{pos}\{u_1, \cdots, u_n\} = \left\{\sum_{i=1}^n \lambda_i u_i : \forall \lambda_i \geq 0\right\} \tag{3.5}$$

is called simplicial convex cone. In this case, the positive dual cone is

$$C^* = \mathrm{pos}\{u_1^*, \dots, u_n^*\}$$

where $\langle u_i, u_j^* \rangle = 0$ if $i \neq j$ and $\langle u_i, u_i^* \rangle > 0$. For $i = 1, \dots, n$, the facets of C are

$$F_i = \mathrm{pos}\{\{u_1, \dots, u_n\} \setminus \{u_i\}\} = C \cap (u_i^*)^{\perp},$$

and the walls of C are the linear subspaces

$$W_i = \mathrm{lin}\{\{u_1, \dots, u_n\} \setminus \{u_i\}\}.$$

Note that the orthogonal reflection Ref_{W_i} through the wall W_i of C is the map $x \mapsto x - 2\langle x, u_i^* \rangle u_i^*$. We observe that $-u_i^*$ is an exterior normal to F_i, and

$$C = \{x \in \mathbb{R}^n : \langle x, u_i^* \rangle \geq 0 \text{ for } i = 1, \dots, n\}. \tag{3.6}$$

A linear subspace E of $\mathbb{R}^n$ is called non-trivial if $\dim E \geq 1$. In this case, we write $\mathcal{O}(E)$ to denote the group of orthogonal transformations of E where $O(n) = \mathcal{O}(\mathbb{R}^n)$.

If G is a group generated by reflections through n independent linear hyperplanes $H_1, \dots, H_n$ (n hyperplanes $H_1, \dots, H_n$ with $H_1 \cap \ldots \cap H_n = \{o\}$), and $H_i = v_i^{\perp}$ for $v_i \in \mathbb{R}^n \backslash \{o\}$ and $i = 1, \dots, n$, then any non-trivial G invariant linear subspace E is of the form

$$E = \mathrm{lin} I \text{ for non-empty } I \subseteq \{v_1, \dots, v_n\} \text{ where } \langle v_i, v_j \rangle = 0 \text{ if } v_i \in I \text{ and } v_j \notin I. \tag{3.7}$$

We call an invariant linear subspace irreducible with respect to the action of G if it has no proper G-invariant linear subspace. It follows that there exist only finitely many irreducible subspaces $E_1, \dots, E_k$, $k \geq 1$, satisfying that

- $\mathbb{R}^n = \oplus_{i=1}^k E_i$;
- E_i and E_j are orthogonal for $i \neq j$;
- $G = G_1 \times \ldots \times G_k$ where $G_i \subseteq O(E_i)$ acts irreducibly on E_i.

This decomposition corresponds to the irreducible representations coming from the action of the closure of G in $O(n)$, see Humphreys [84] for representations of compact groups.

Typical example for a finite group $G \subseteq O(n)$ generated by reflections through n independent hyperplanes and acting irreducibly on $\mathbb{R}^n$ is the symmetry group of a regular polytope P in $\mathbb{R}^n$ whose centroid is the origin (see McCammond [115] or Humphreys [85]). For example, if P is a regular simplex, then the n independent hyperplanes might be the perpendicular bisectors of the n edges meeting at a fixed vertex of P.

The following Lemma 3.3.1 defines the Weyl chamber associated to an irreducible action of a finite Coxeter group, and dicusses the fundamental properties. These Weyl chambers partition $\mathbb{R}^n$ into simplicial cones (see Lemma 3.3.1 (ii) and (iii)).

Lemma 3.3.1 (Coxeter)**.** *Let G be a finite group generated by reflections through n hyperplanes $H_1, \dots, H_n$ with $H_1 \cap \dots, \cap H_n = \{o\}$ and acting irreducibly on $\mathbb{R}^n$. Then there exists a simplicial cone $C = \mathrm{pos}\{u_1, \cdots, u_n\}$ (called a Weyl chamber) such that*

(i) the n reflections through the walls of C generate G;

(ii) $\mathbb{R}^n = \cup_{g \in G} gC$;

(iii) if $gC \cap \mathrm{int} C \neq \emptyset$ for some $g \in G$, then g is the identity;

(iv) $\langle x, y \rangle \geq 0$ for $x, y \in C$ and writing $C^* = \mathrm{pos}\{v_1, \dots, v_n\}$, we have $\langle v_i, v_j \rangle \leq 0$ provided $i \neq j$;

(v) for any partition $\{1,\dots,n\} = I \cup J$ with $I, J \neq \emptyset$ and $I \cap J = \emptyset$, there exist $i \in I$ and $j \in J$ such that $\langle v_i, v_j\rangle < 0$.

Proof. According to the classical theory (see Humphreys [85]), one associates a so called root system to G; namely, a finite set Φ of non-zero vectors such that any two are either independent or opposite, and the set of reflections in G coincides with the set reflections through the linear $(n-1)$-dimensional subspaces orthogonal to the elements of Φ. It is a well-known result (see Humphreys [85]) that there exists some n independent roots $v_1,\dots,v_n \in \Phi$ such that any other root can be written as a linear combination of $v_1,\dots,v_n$ with all non-positive or all non-negative coefficients. Then $v_1,\dots,v_n \in \Phi$ are called simple roots, and the simplicial cone $C = \{x \in \mathbb{R}^n : \langle x, v_i\rangle \geq 0 \text{ for } i = 1,\dots,n\}$ satisfies (i), (ii), (iii); moreover, $v_1,\dots,v_n$ satisfy that $\langle v_i, v_j\rangle \leq 0$ for $i \neq j$ (see Humphreys [85]), verifying the second half of (iv).

We complete the proof of (iv) by contradiction, so we suppose that there exist $x, y \in C$ satisfying $\langle x, y\rangle < 0$, and seek a contradiction. We set $v_{n+1} = -x$ and $v_{n+2} = -y$; therefore, $\langle v_i, v_j\rangle \leq 0$ for $i, j = 1,\dots,n+2$ and $\langle v_{n+1}, v_{n+2}\rangle < 0$. According to Radon's theorem, there exist non-empty $A, B \subseteq \{1,\dots,n+2\}$ with $A \cap B = \emptyset$, and $\alpha_i > 0$ and $\beta_j > 0$ for $i \in A$ and $j \in B$ such that

$$\sum_{i\in A} \alpha_i v_i = \sum_{j\in B} \beta_j v_j = w.$$

We deduce that

$$0 \leq \langle w, w\rangle = \sum_{i\in A}\sum_{j\in B} \alpha_i\beta_j \langle v_i, v_j\rangle,$$

thus $\langle v_i, v_j\rangle \leq 0$ for $i \neq j$ yields that

$$\langle v_i, v_j\rangle = 0 \text{ for } i \in A \text{ and } j \in B, \tag{3.8}$$

and hence $w = o$. In turn, the independence of $v_1,\dots,v_n$ shows that $A \cap \{v_{n+1}, v_{n+2}\} \neq \emptyset$ and $B \cap \{v_{n+1}, v_{n+2}\} \neq \emptyset$, which facts contradict $\langle v_{n+1}, v_{n+2}\rangle < 0$ by (3.8).

Finally, we prove (v) again by contradiction. We suppose that there exists a partition $\{1,\dots,n\} = I \cup J$ with $I, J \neq \emptyset$ and $I \cap J = \emptyset$ such that $\langle v_i, v_j\rangle = 0$ for $i \in I$ and $j \in J$ (note that $\langle v_i, v_j\rangle \geq 0$ by (iv)). Then both $\mathrm{lin}\{v_i : i \in I\}$ and $\mathrm{lin}\{v_j : j \in J\}$ are invariant under reflections through the walls of C, which contradicts the irreducibility of the action of G on $\mathbb{R}^n$. $\square$

The main goal of this section is to prove the following statement which describes how the Weyl chambers essentially partitioning $\mathbb{R}^n$ (see Proposition 3.3.2 (iii)) are related to the group action.

Proposition 3.3.2. *Let $G \subseteq O(n)$ be the closure of a group generated by the orthogonal reflections through the hyperplanes $H_1,\dots,H_n$ of $\mathbb{R}^n$ with $H_1 \cap \ldots \cap H_n = \{o\}$, let $E_1,\dots,E_k$ be the corresponding irreducible subspaces. Then there exist an n-dimensional simplicial convex cone $C = \oplus_{\alpha=1}^k C_\alpha$ in $\mathbb{R}^n$ where $C_\alpha \subseteq E_\alpha$ is a Weyl chamber for the irreducible action of a finite subgroup $\widetilde{G}_\alpha \subseteq \mathcal{O}(E_\alpha)$ on E_α and $\widetilde{G}_\alpha$ is generated by reflections through the walls of C_α in E_α for $\alpha = 1,\dots,k$. In addition,*

(i) *$\widetilde{G} = \widetilde{G}_1 \times \ldots \times \widetilde{G}_k$ is a subgroup of G;*

(ii) *writing $W_1,\dots,W_n$ to denote the walls of C, the reflections Ref_{W_α}, $\alpha = 1,\cdots,n$, generate $\widetilde{G}$;*

(iii) *$gC \cap \mathrm{int}\, C \neq \emptyset$ for $g \in \widetilde{G}$ implies that g is the identity, and*

$$\mathbb{R}^n = \bigcup_{g\in\widetilde{G}} gC;$$

(iv) *if $C^* = \mathrm{pos}\{v_1,\dots,v_n\}$, then $\langle v_i, v_j\rangle \leq 0$ provided $i \neq j$;*

(v) *If K is a convex body in $\mathbb{R}^n$ invariant under G, then $\nu_K(x) \in C$ for $x \in \partial' K \cap C$, and if moreover $\Phi \in GL(n)$ satisfies $\Phi(C) = \mathbb{R}^n_+$, then the unconditional set $\bar{K}$ defined by $\bar{K} \cap \mathbb{R}^n_+ = \Phi(K \cap C)$ is an unconditional convex body.*

We prepare the proof of Proposition 3.3.2 with a series of lemmas mostly discussing well-known statements.

The following statement is Lemma 19 in Barthe, Fradelizi [20].

Lemma 3.3.3 (Barthe, Fradelizi)**.** *If $G \subseteq O(n)$ is an infinite subgroup generated by reflections through n hyperplanes $H_1, \ldots, H_n$ with $H_1 \cap \ldots \cap H_n = \{o\}$, and G acts irreducibly on $\mathbb{R}^n$, then the closure of G is $O(n)$.*

Lemma 3.3.4. *For $k \geq 2$, let E_α, $\alpha = 1, \ldots, k$ be pairwise orthogonal non-trivial linear subspaces of $\mathbb{R}^n$ with $\oplus_{\alpha=1}^k E_\alpha = \mathbb{R}^n$, and for $\alpha = 1, \ldots, k$, let $G_\alpha \subseteq \mathcal{O}(E_\alpha)$ be a finite subgroup generated by reflections through* $\mathrm{dim}E_\alpha$ *independent hyperplanes of E_α, and let C_α be a Weyl chamber for the action of G_α. Then for the subgroup $G = G_1 \times \ldots \times G_k$ of $O(n)$ and $C = \oplus_{\alpha=1}^k C_\alpha$, we have*

(i) G is generated by the reflections through the walls of C;

(ii) $\cup\{gC : g \in G\} = \mathbb{R}^n$;

(iii) if $\mathrm{int}gC \cap \mathrm{int}C \neq \emptyset$ *for a $g \in G$, then g is the identity;*

(iv) $\langle x, y\rangle \geq 0$ for $x, y \in C$.

Proof. (i) As $G = G_1 \times \ldots \times G_k$ and $E_1, \ldots, E_k$ are pairwise orthogonal, Lemma 3.3.1 (i) yields that a set generators of G is the n reflections through the hyperplanes of $\mathbb{R}^n$ of the form $W + E_\alpha^\perp$ where for some E_α, $\alpha = 1, \ldots, k$, W is a wall of C_α in E_α. Since these n hyperplanes of $\mathbb{R}^n$ are exactly the walls of C, we deduce (i).

(ii) Write $x \in \mathbb{R}^n$ as $x = x_1 + \cdots + x_k$ where $x_\alpha \in E_\alpha$, $\alpha = 1, \ldots, k$. According to Lemma 3.3.1 (ii), there exits $g_\alpha \in G_\alpha$ such that $x_\alpha \in g_\alpha C_\alpha$ for each $\alpha = 1, \ldots, k$. Therefore $x \in gC$ for $g = (g_1, \ldots, g_k) \in G$.

(iii) Assume $\mathrm{int}gC \cap \mathrm{int}C \neq \emptyset$ for $g = (g_1, \cdots, g_k) \in G$. Projecting into each E_α shows that the relative interiors of $g_\alpha C_\alpha$ and C_α intersect for $\alpha = 1, \ldots, k$; therefore, $g_\alpha C_\alpha = C_\alpha$ for $\alpha = 1, \ldots, k$ by Lemma 3.3.1 (iii), and hence $gC = C$.

(iv) This follows from Lemma 3.3.1 (iv) and the fact that the subspaces $E_1, \ldots, E_k$ are pairwise orthogonal.

□

Lemma 3.3.5. *If K is a convex body in $\mathbb{R}^n$, and there is a simplicial convex cone C such that K is invariant with respect to the orthogonal reflections through the walls of C, then*

(i) $\nu_K(z) \in C$ holds for any $z \in \partial' K \cap C$;

(ii)
$$\begin{aligned} K \cap C &= \{x \in C : \langle x, \nu_K(z)\rangle \leq h_K(\nu_K(z)) \ \forall\, z \in \partial' K \cap C\} \\ &= \{x \in C : \langle x, u\rangle \leq h_K(u) \ \forall\, u \in C\}. \end{aligned}$$

Proof. As in (3.5) and (3.6), we write the cone C as $C = \mathrm{pos}\{u_1, \ldots, u_n\}$ and $C = \{z \in \mathbb{R}^n : \langle z, x_j\rangle \leq 0,\ j = 1, \ldots, n\}$ for independent $u_1, \ldots, u_n \in S^{n-1}$ and $x_1, \ldots, x_n \in S^{n-1}$ satisfying $\langle x_j, u_i\rangle = 0$ for $j \neq i$ and $\langle x_j, u_j\rangle < 0$ for $j = 1, \ldots, n$.

For $z \in \partial' K \cap C$ and $j \in \{1, \cdots, n\}$, we show that

$$\langle \nu_K(z), x_j\rangle \leq 0.$$

We use that K is symmetric with respect to the wall $W_j := \mathrm{lin}\{u_1, \cdots, u_n\} \setminus \{u_j\} = x_j^\perp$ of C; or in other words, $\mathrm{Ref}_{W_j} K = K$.

If $z \in x_j^\perp$, then the symmetry of K through W_j shows that both $\nu_K(z)$ and $\mathrm{Ref}_{W_j}(\nu_K(z))$ are exterior normals at z, and hence $\nu_K(z) \in x_j^\perp$.

Therefore, let $\langle z, x_j\rangle < 0$. As $\nu_K(z)$ is an exterior normal at z and $\mathrm{Ref}_{W_j} z \in K$, we deduce that $\langle \nu_K(z), (\mathrm{Ref}_{W_j} z) - z\rangle \leq 0$. However, $(\mathrm{Ref}_{W_j} z) - z$ is a positive multiple of x_j, thus $\langle \nu_K(z), x_j\rangle \leq 0$, which implies $\nu_K(z) \in C$ since j was arbitrary.

Since K is the intersection of the supporting halfspaces at the smooth boundary points according to Theorem 2.2.6 in Schneider [144], (i) yields (ii). □

Lemma 3.3.6. *Let K be a convex body in $\mathbb{R}^n$ and let C be a simplicial convex cone such that $\langle x, y\rangle \geq 0$ for every $x, y \in C$ and K is invariant with respect to the orthogonal reflections through the walls of C. If $\Phi \in GL(n)$ satisfies $\Phi C = \mathbb{R}^n_+$, then $\Phi^{-t} C \subseteq \mathbb{R}^n_+$ and the unconditional set $\bar{K}$ defined by $\bar{K} \cap \mathbb{R}^n_+ = \Phi(K \cap C)$ is an unconditional convex body.*

Proof. To show the convexity of $\bar{K}$, we observe that $C \subseteq C^*$ holds for the positive dual cone C^* by the condition on C, and hence

$$\Phi^{-t} C \subseteq \Phi^{-t} C^* = (\Phi C)^* = (\mathbb{R}^n_+)^* = \mathbb{R}^n_+. \tag{3.9}$$

Now if $z \in \partial' \Phi(K) \cap \mathbb{R}^n_+$, then $z = \Phi y$ for some $y \in \partial' K \cap C$ where $\nu_K(y) \in C$ according to Lemma 3.3.5. Since $\Phi^{-t} \nu_K(y)$ is an exterior normal to $\partial \Phi(K)$ at $z = \Phi y$, we conclude from (3.9) and the conditions on C and K that

$$z \in \partial' \Phi(K) \cap \mathbb{R}^n_+ \Rightarrow \nu_{\Phi(K)}(z) \in \mathbb{R}^n_+ \tag{3.10}$$

As $\bar{K}$ is an unconditional set and $\bar{K} \cap \mathbb{R}^n_+ = \Phi(K) \cap \mathbb{R}^n_+$, its convexity is equivalent with the following statement: If $x = (x_1, \ldots, x_n) \in \bar{K}$, $y = (y_1, \ldots, y_n) \in \bar{K}$ and $\lambda \in (0,1)$, then

$$w = (|(1-\lambda)x_1 + \lambda y_1|, \ldots, |(1-\lambda)x_n + \lambda y_n|) \in \Phi(K) \cap \mathbb{R}^n_+. \tag{3.11}$$

If $z \in \partial' \Phi(K) \cap \mathbb{R}^n_+$, then for $\tilde{x} = (|x_1|, \ldots, |x_n|) \in \Phi(K) \cap \mathbb{R}^n_+$ and $\tilde{y} = (|y_1|, \ldots, |y_n|) \in \Phi(K) \cap \mathbb{R}^n_+$, we deduce from $\nu_{\Phi(K)}(z) \in \mathbb{R}^n_+$ (see (3.10)) and $|(1-\lambda)x_i + \lambda y_i| \leq (1-\lambda)|x_i| + \lambda |y_i|$ that

$$\begin{aligned}\langle w, \nu_{\Phi(K)}(z)\rangle &\leq \langle (1-\lambda)\tilde{x} + \lambda \tilde{y}, \nu_{\Phi(K)}(z)\rangle = (1-\lambda)\langle \tilde{x}, \nu_{\Phi(K)}(z)\rangle + \lambda \langle \tilde{y}, \nu_{\Phi(K)}(z)\rangle \\ &\leq (1-\lambda)\langle z, \nu_{\Phi(K)}(z)\rangle + \lambda \langle z, \nu_{\Phi(K)}(z)\rangle = \langle z, \nu_{\Phi(K)}(z)\rangle.\end{aligned}$$

Since $\Phi(K)$ is the intersection of the supporting halfspaces at the smooth boundary points (see Theorem 2.2.6 in Schneider [144]), we conclude (3.11), and in turn Lemma 3.3.6. □

Now we are ready to give the proof of Proposition 3.3.2.

Proof of Proposition 3.3.2. Let $\bar{G}$ be the group generated by the orthogonal reflections through the hyperplanes $H_1, \ldots, H_n$ of $\mathbb{R}^n$ with $H_1 \cap \ldots \cap H_n = \{o\}$. Then the corresponding irreducible subspaces $E_1, \ldots, E_k$ coincide for $\bar{G}$ and for its closure G. Let $\bar{G}_\alpha, G_\alpha \subseteq \mathcal{O}(E_\alpha)$, $\alpha = 1, \cdots, n$, be the subgroups such that $\bar{G} = \bar{G}_1 \times \ldots \times \bar{G}_k$ and $G = G_1 \times \ldots \times G_k$ where G_α is the closure of $\bar{G}_\alpha$ in $\mathcal{O}(E_\alpha)$ for $\alpha = 1, \cdots, n$. In particular, $\bar{G}_\alpha$ is generated by reflections through all $H_i \cap E_\alpha$ such that $E_\alpha \not\subseteq H_i$ where writing $H_i = w_i^\perp$ for $w_i \in S^{n-1}$ and $i = 1, \ldots, n$, we have $E_\alpha = \mathrm{lin}\{w_i : E_\alpha \not\subseteq H_i\}$.

If $\bar{G}_\alpha$ is finite, then we simply define $\widetilde{G}_\alpha = \bar{G}_\alpha = G_\alpha$. If $\bar{G}_\alpha$ is infinite, then $\bar{G}_\alpha = \mathcal{O}(E_\alpha)$ according to Lemma 3.3.3; therefore, we may choose $\widetilde{G}_\alpha$ to be the symmetry group of a regular simplex of E_α centered at the origin. In particular, for each $\alpha = 1, \ldots, k$, $\widetilde{G}_\alpha$ is finite and acts irreducibly on E_α, and let $C_\alpha \subseteq E_\alpha$ be a Weyl chamber for the action of $\widetilde{G}_\alpha$ as in Lemma 3.3.1.

We define $\widetilde{G} = \widetilde{G}_1 \times \ldots \times \widetilde{G}_k \subseteq O(n)$ and $C = \oplus_{\alpha=1}^k C_\alpha$. We deduce Proposition 3.3.2 (ii) and (iii) from Lemma 3.3.4 (ii) and (iii).

For Proposition 3.3.2 (iv), the walls of C are of the form $W + E_\alpha^\perp$ for $\alpha = 1, \ldots, k$ and wall w of C_α in E_α. For $1 \leq i < j \leq n$, $\langle v_i, v_j\rangle \leq 0$ follows from Lemma 3.3.1 (iv) if $v_i, v_j \in E_\alpha$, and from the othogonality of E_α and E_β if $v_i \in E_\alpha$ and $v_j \in E_\beta$ for $\alpha \neq \beta$.

For Proposition 3.3.2 (v), we deduce from Lemma 3.3.4 that $\langle x, y\rangle \geq 0$ holds for $x, y \in C$. Combining this with Proposition 3.3.2 (ii), Lemma 3.3.5 (i) and Lemma 3.3.6 yields Proposition 3.3.2 (v). □

3.4 The proof for volume

Before the proof, let us state two easy remarks:

- the logarithmic sum is linear covariant; namely, for any $\Phi \in \mathrm{GL}(n,\mathbb{R})$ it holds

$$\Phi[(1-\lambda)\cdot K +_0 \lambda \cdot L] = (1-\lambda)\cdot \Phi(K) +_0 \lambda \cdot \Phi(L). \tag{3.12}$$

  follows from the fact $h_{\Phi K}(u) = h_K(\Phi^t u)$. Therefore, if K and L are two convex bodies in $\mathbb{R}^n$ invariant under some subgroup $G \subseteq \mathrm{GL}(n)$, then $(1-\lambda)\cdot K +_0 \lambda\cdot L$ is also invariant under G.

- For any $u \in \mathbb{R}^n \backslash \{o\}$, one gets,

$$\begin{gathered}\text{if } u \text{ is an exterior normal at a } z \in \partial'((1-\lambda)\cdot K +_0 \lambda \cdot L)\text{, then}\\ h_{(1-\lambda)\cdot K +_0 \lambda\cdot L}(u) = h_K(u)^{1-\lambda} h_L(u)^{\lambda}.\end{gathered} \tag{3.13}$$

  We may assume that $u \in S^{n-1}$. Since $z \in \partial((1-\lambda)\cdot K +_0 \lambda\cdot L)$ is a boundary point and h_K and h_L are continuous, there exists some $v \in S^{n-1}$ such that $h_{(1-\lambda)\cdot K +_0 \lambda \cdot L}(v) = \langle z, v\rangle = h_K(v)^{1-\lambda}h_L(v)^{\lambda}$. However, z is a smooth boundary point where there exists only a unique exterior unit normal; therefore, we have $u = v$, verifying (3.13).

First we consider the version of Theorem 3.1.1 where each linear reflection is an orthogonal reflection.

Theorem 3.4.1. *Let $\lambda \in (0,1)$. If $A_1,\ldots,A_n$ are orthogonal reflections such that $H_1 \cap \ldots \cap H_n = \{o\}$ holds for the associated hyperplanes $H_1,\ldots,H_n$ and the convex bodies K and L are invariant under $A_1,\ldots,A_n$, then*

$$|(1-\lambda)\cdot K +_0 \lambda\cdot L| \geq |K|^{1-\lambda}|L|^{\lambda}. \tag{3.14}$$

In addition, equality holds if and only if $K = K_1 \oplus \cdots \oplus K_m$ and $L = L_1 \oplus \cdots \oplus L_m$ for $m \geq 1$ and compact convex sets K_i, L_i invariant under A_i, $i = 1,\cdots,m$, having dimension at least one and satisfying $K_i = c_i L_i$ for $c_i > 0$ for $i = 1,\cdots,m$, and $\sum_{i=1}^m \dim K_i = n$.

Proof. Let $G \subseteq O(n)$ be the closure of the group generated by $A_1,\ldots,A_n$. We use the notation of Proposition 3.3.2 applied to this G. In particular, for some $k \geq 1$, $\mathbb{R}^n = \oplus_{\alpha=1}^k E_\alpha$ for non-trivial linear subspaces $E_1,\ldots,E_k$ where $E_1,\ldots,E_k$ are pairwise orthogonal if $k \geq 2$. In addition, $C = \oplus_{\alpha=1}^k C_\alpha$ is the simplicial cone of Proposition 3.3.2 where C_α is the Weyl chamber for the finite group $\widetilde{G}_\alpha \subseteq \mathcal{O}(E_\alpha)$ generated by reflections through the walls of C_α in E_α and acting irreducibly on E_α for $\alpha = 1,\ldots,k$, and $\widetilde{G} = \widetilde{G}_1 \times \ldots \times \widetilde{G}_k$ is a subgroup of G, by Proposition 3.3.2 (i).

We fix an orthonormal basis $e_1,\ldots,e_n$ of $\mathbb{R}^n$ such that $\{e_i : e_i \in E_\alpha\}$ spans E_α for $\alpha = 1,\ldots,k$, and hence $\mathbb{R}^n_+ = \mathrm{pos}\{e_1,\ldots,e_n\}$, and let $\Phi \in \mathrm{GL}(n)$ be a linear transform such that $\Phi C_\alpha = \mathbb{R}^n_+ \cap E_\alpha$ for $\alpha = 1,\ldots,k$. We deduce that

(a) $\Phi(C) = \mathbb{R}^n_+$;

(b) $\Phi E_\alpha = E_\alpha$ for $\alpha = 1,\ldots,k$.

Since K and L are convex bodies invariant under G, their L_0 sum $(1-\lambda)\cdot K +_0 \lambda\cdot L$ is also invariant under G, and in turn invariant under $\widetilde{G}$. We write $\mathrm{card}(\widetilde{G})$ to denote the cardinatility of $\widetilde{G}$. It follows from the linear covariance of the logarithmic sum and Proposition 3.3.2 (iii) that

$$\begin{aligned}|(1-\lambda)\cdot K +_0 \lambda\cdot L| &= \sum_{g\in\widetilde{G}} |gC \cap [(1-\lambda)\cdot K +_0 \lambda \cdot L]|\\ &= \mathrm{card}(\widetilde{G})\cdot |C \cap [(1-\lambda)\cdot K +_0 \lambda\cdot L]|.\end{aligned} \tag{3.15}$$

Since any convex body is the intersection of the supporting halfspaces at the smooth boundary points according to Theorem 2.2.6 in Schneider [144], we deduce from Lemma 3.3.5 (ii) and (3.13) that

$$C \cap [(1-\lambda)\cdot K +_0 \lambda \cdot L] = \left\{x \in C : \langle x, u\rangle \le h_K(u)^{1-\lambda} h_L(u)^{\lambda}\ \forall\, u \in C\right\}. \tag{3.16}$$

Let $\bar{K}$ and $\bar{L}$ the unconditional sets defined by $\bar{K} \cap \mathbb{R}^n_+ = \Phi(K \cap C)$ and $\bar{L} \cap \mathbb{R}^n_+ = \Phi(L \cap C)$ respectively. Proposition 3.3.2 (v) implies that $\bar{K}$ and $\bar{L}$ are unconditional convex bodies. We observe that if $u \in \mathbb{R}^n_+$, then

$$h_{\bar{K}}(u) = \max_{x \in \bar{K} \cap \mathbb{R}^n_+} \langle u, x\rangle \le h_{\Phi K}(u) \ \text{ and } \ h_{\bar{L}}(u) = \max_{x \in \bar{L} \cap \mathbb{R}^n_+} \langle u, x\rangle \le h_{\Phi L}(u). \tag{3.17}$$

The key observation is that

$$|\mathbb{R}^n_+ \cap [(1-\lambda)\cdot \Phi(K) +_0 \lambda \cdot \Phi(L)]| \ge |\mathbb{R}^n_+ \cap [(1-\lambda)\cdot \overline{K} +_0 \lambda \cdot \overline{L}]|, \tag{3.18}$$

which follows from (a), (3.17) and $\Phi^{-t}C \subseteq \mathbb{R}^n_+$ (see Lemma 3.3.6) and

$$\begin{aligned}
\mathbb{R}^n_+ \cap \Phi[(1-\lambda)\cdot K +_0 \lambda \cdot L] &= \Phi\left(\left\{x \in C : \langle x, u\rangle \le h_K(u)^{1-\lambda} h_L(u)^{\lambda}\ \forall\, u \in C\right\}\right)\\
&= \left\{x \in \mathbb{R}^n_+ : \langle x, v\rangle \le h_{\Phi K}(v)^{1-\lambda} h_{\Phi L}(v)^{\lambda}\ \forall\, v \in \Phi^{-t}C\right\}\\
&\supseteq \left\{x \in \mathbb{R}^n_+ : \langle x, v\rangle \le h_{\bar{K}}(v)^{1-\lambda} h_{\bar{L}}(v)^{\lambda}\ \forall\, v \in \Phi^{-t}C\right\}\\
&\supseteq \left\{x \in \mathbb{R}^n_+ : \langle x, v\rangle \le h_{\bar{K}}(v)^{1-\lambda} h_{\bar{L}}(v)^{\lambda}\ \forall\, v \in \mathbb{R}^n_+\right\}\\
&= \mathbb{R}^n_+ \cap [(1-\lambda)\cdot \overline{K} +_0 \lambda \cdot \overline{L}].
\end{aligned}$$

From (3.15), (3.18) and Logarithmic Brunn Minkowski inequality Theorem 2.1.10 for unconditional convex bodies, we deduce

$$\begin{aligned}
|(1-\lambda)\cdot K +_0 \lambda \cdot L| &= \mathrm{card}(\widetilde{G}) \cdot |C \cap [(1-\lambda)\cdot K +_0 \lambda \cdot L]|\\
&= \frac{\mathrm{card}(\widetilde{G})}{|\det\Phi|} \cdot |\mathbb{R}^n_+ \cap [(1-\lambda)\cdot \Phi(K) +_0 \lambda \cdot \Phi(L)]|\\
&\ge \frac{\mathrm{card}(\widetilde{G})}{|\det\Phi|} \cdot |\mathbb{R}^n_+ \cap [(1-\lambda)\cdot \bar{K} +_0 \lambda \cdot \bar{L}]|\\
&= \frac{\mathrm{card}(\widetilde{G})}{2^n|\det\Phi|} \cdot |(1-\lambda)\cdot \bar{K} +_0 \lambda \cdot \bar{L}|\\
&\ge \frac{\mathrm{card}(\widetilde{G})}{2^n|\det\Phi|} \cdot |\bar{K}|^{1-\lambda}|\bar{L}|^{\lambda} \qquad (3.19)\\
&= |K|^{1-\lambda}|L|^{\lambda},
\end{aligned}$$

proving the Logarithmic Brunn-Minkowski inequality (3.14).

Assume now that we have equality in (3.14). In particular, equality holds for the unconditional convex bodies $\bar{K}$ and $\bar{L}$ in (3.19). Therefore, Theorem 2.1.10 implies that $\bar{K} = \bar{K}_1 \oplus \cdots \oplus \bar{K}_m$ and $\bar{L} = \bar{L}_1 \oplus \cdots \oplus \bar{L}_m$ for some $m \ge 1$, where $\bar{K}_\beta$ and $\bar{L}_\beta$ are unconditional convex sets, $\bar{K}_\beta = \theta_\beta \bar{L}_\beta$ for some $\theta_\beta > 0$, $\beta = 1, \cdots, m$, and $\sum_{\beta=1}^m \dim\bar{K}_\beta = n$.

If $m = 1$, then

$$\begin{aligned}
K &= \bigcup_{g \in \widetilde{G}} g(K \cap C) = \bigcup_{g \in \widetilde{G}} g \circ \Phi^{-1}(\bar{K} \cap \mathbb{R}^n_+) = \bigcup_{g \in \widetilde{G}} g \circ \Phi^{-1}(\theta_1 \bar{L} \cap \mathbb{R}^n_+)\\
&= \theta_1 \bigcup_{g \in \widetilde{G}} g(L \cap C) = \theta_1 L;
\end{aligned}$$

therefore, K and L are dilates.

If $m \geq 2$, then we write $\bar{E}_\beta = \text{lin}\bar{K}_\beta$ for $\beta = 1, \cdots, m$, and hence $\mathbb{R}^n = \oplus_{\beta=1}^m \bar{E}_\beta$.

We claim that each $\bar{E}_\beta$ is the direct sum of some E_α; namely, there exists some non-empty $\Xi_\beta \subseteq \{1, \cdots, k\}$ such that

$$\bar{E}_\beta = \oplus_{\alpha \in \Xi_\beta} E_\alpha \tag{3.20}$$

We suppose that (3.20) does not hold, and seek a contradiction. We set $u_i = \Phi^{-1}(e_i)$ and $v_i = \Phi^t(e_i)$ for $i = 1, \ldots, n$; therefore, $C = \text{pos}\{u_1, \ldots, u_n\}$ and $C^* = \text{pos}\{v_1, \ldots, v_n\}$. For any $\alpha = 1, \ldots, k$ and $\beta = 1, \ldots, m$, we consider

$$I_\alpha = \{i : e_i \in E_\alpha\} \quad \text{and} \quad J_\beta = \{j : e_j \in \bar{E}_\beta\}.$$

Since $\{1, \ldots, n\}$ is partitioned in two ways once into $I_1, \ldots, I_k$, and secondly into $J_1, \ldots, J_m$, the indirect hypothesis yields there exist $\tilde{\alpha} \in \{1, \cdots, k\}$ and $\tilde{\beta} \in \{1, \cdots, m\}$ such that $I_{\tilde{\alpha}} \cap J_{\tilde{\beta}}$ is non-empty and is a proper subset of $I_{\tilde{\alpha}}$. It follows from Lemma 3.3.1 (v) applied to $C_{\tilde{\alpha}}$ and the partition $I_{\tilde{\alpha}} = (I_{\tilde{\alpha}} \cap J_{\tilde{\beta}}) \cup (I_{\tilde{\alpha}} \backslash J_{\tilde{\beta}})$ that there exist

$$p \in I_{\tilde{\alpha}} \cap J_{\tilde{\beta}} \text{ and } q \in I_{\tilde{\alpha}} \backslash J_{\tilde{\beta}} \text{ such that } \langle v_p, v_q \rangle < 0. \tag{3.21}$$

Since for any convex body, smooth boundary points are dense on the boundary, there exists a $z_0 \in \text{relint}\,(\bar{K}_{\tilde{\beta}} \cap \mathbb{R}^n_+)$ and $s > 0$ such that $z = z_0 + s e_p \in \partial' \bar{K}_{\tilde{\beta}} \cap \mathbb{R}^n_+$. It follows that $\langle \nu_{\bar{K}_{\tilde{\beta}} \cap \mathbb{R}^n_+}(z), e_p \rangle > 0$, and hence

$$\nu_{\bar{K}_{\tilde{\beta}} \cap \mathbb{R}^n_+}(z) = \sum_{j \in J_{\tilde{\beta}}} \gamma_j e_j \quad \text{where } \gamma_j \geq 0 \text{ for } j \in J_{\tilde{\beta}} \text{ and } \gamma_p > 0. \tag{3.22}$$

We choose a $y \in \text{relint} \sum_{\beta \neq \tilde{\beta}} (\bar{K}_\beta \cap \mathbb{R}^n_+)$; therefore, $z + y \in \partial' \bar{K} \cap \mathbb{R}^n_+$ and we deduce from (3.22) that

$$\nu_{\bar{K}}(z + y) = \nu_{\bar{K}_{\tilde{\beta}} \cap \mathbb{R}^n_+}(z) = \sum_{j \in J_{\tilde{\beta}}} \gamma_j e_j \quad \text{where } \gamma_j \geq 0 \text{ for } j \in J_{\tilde{\beta}} \text{ and } \gamma_p > 0. \tag{3.23}$$

Writing $\Phi^{-1} z = z'$ and $\Phi^{-1} y = y'$, it follows that

$$\begin{aligned} z' + y' &\in \partial' K \cap C \\ \nu_K(z' + y') &= \theta\, \Phi^t \nu_{\bar{K}}(z + y) \quad \text{for } \theta = \|\Phi^t \nu_{\bar{K}}(z + y)\|^{-1}, \end{aligned} \tag{3.24}$$

which combined with (3.23) leads to

$$\nu_K(z' + y') = \sum_{j \in J_{\tilde{\beta}}} \theta \gamma_j v_j \quad \text{where } \gamma_j \geq 0 \text{ for } j \in J_{\tilde{\beta}} \text{ and } \gamma_p > 0. \tag{3.25}$$

In turn, we deduce from (3.25), $\langle v_p, v_q \rangle < 0$ (see (3.21)) and $\langle v_p, v_j \rangle \leq 0$ for $j \in J_{\tilde{\beta}}$ (see Proposition 3.3.2 (iv)) that

$$\langle v_p, \nu_K(z' + y') \rangle = \theta \gamma_q \langle v_p, v_q \rangle + \sum_{\substack{j \in J_{\tilde{\beta}} \\ j \neq q}} \theta \gamma_j \langle v_p, v_j \rangle < 0,$$

and hence $C = \{x : \langle x, v_i \rangle \geq 0 \text{ for } i = 1, \ldots, n\}$ (see (3.6)) yields $\nu_K(z' + y') \notin C$.

On the other hand, combining (3.24) and Proposition 3.3.2 (v) implies that $\nu_K(z' + y') \in C$. This contradiction finally proves (3.20).

We recall that $P_E M$ denotes the orthogonal projection of a compact convex set M onto a linear subspace E. We deduce from (3.20) and Proposition 3.3.2 (i), (ii) and (iii) that for each $\beta = 1, \ldots, m$, there exist convex compact sets $K_\beta, L_\beta \subseteq \bar{E}_\beta$ such that

$$\begin{aligned} K_\beta \cap C &= \Phi^{-1}(\bar{K}_\beta \cap \mathbb{R}^n_+); \\ K &= K_\beta + P_{\bar{E}_\beta^\perp}(K) \quad \text{and} \quad P_{\bar{E}_\beta}(K) = K_\beta; \end{aligned} \tag{3.26}$$

$$\begin{aligned} L_\beta \cap C &= \Phi^{-1}(\bar{L}_\beta \cap \mathbb{R}^n_+); \\ L &= L_\beta + P_{\bar{E}_\beta^\perp}(L) \quad \text{and} \quad P_{\bar{E}_\beta}(L) = L_\beta. \end{aligned} \tag{3.27}$$

In turn, we verify that (3.26) and (3.27) yield that

$$K = \oplus_{\beta=1}^{m} K_\beta \quad \text{and} \quad L = \oplus_{\beta=1}^{m} L_\beta \tag{3.28}$$

by induction on $m \geq 2$. We only provide the argument in the case of K, because the argument for L is similar.

If $m = 2$, then $\bar{E}_1^\perp = \bar{E}_2$, and so (3.28) readily follows.

If $m \geq 3$, then let $K' = P_{\bar{E}_m^\perp}(K)$. Let $1 \leq \beta \leq m-1$. The main observation we use is that if $\Pi_0 \subseteq \Pi$ are linear subspaces, then $P_{\Pi_0}(P_\Pi X) = P_{\Pi_0}(X)$ for $X \subseteq \mathbb{R}^n$. On the one hand, we deduce from $\bar{E}_\beta \subseteq \bar{E}_m^\perp$ that

$$P_{\bar{E}_\beta}(K') = P_{\bar{E}_\beta}(P_{\bar{E}_m^\perp}(K)) = P_{\bar{E}_\beta}(K) = K_\beta.$$

On the other hand, we also use that if $X \subseteq \bar{E}_\beta^\perp$, then $P_{\bar{E}_m^\perp}(X) = P_{\bar{E}_\beta^\perp \cap \bar{E}_m^\perp}(X)$ follows from $\bar{E}_\beta \subseteq \bar{E}_m^\perp$. Therefore,

$$\begin{aligned} K' &= P_{\bar{E}_m^\perp}\Big(K_\beta + P_{\bar{E}_\beta^\perp}(K)\Big) = P_{\bar{E}_m^\perp}(K_\beta) + P_{\bar{E}_m^\perp}(P_{\bar{E}_\beta^\perp}(K)) = K_\beta + P_{\bar{E}_m^\perp \cap \bar{E}_\beta^\perp}(P_{\bar{E}_\beta^\perp}(K)) \\ &= K_\beta + P_{\bar{E}_m^\perp \cap \bar{E}_\beta^\perp}(K) = K_\beta + P_{\bar{E}_m^\perp \cap \bar{E}_\beta^\perp}(K'), \end{aligned}$$

implying $K' = \oplus_{\beta=1}^{m-1} K_\beta$ by induction on m. Since $K = K_m + K'$ by (3.26), we conclude (3.28).

As K, L and $\bar{E}_\beta$ are invariant under G, also K_β and L_β are invariant under G for $\beta = 1, \ldots, m$. Since $\bar{K}_\beta = \theta_\beta \bar{L}_\beta$, we also deduce that $K_\beta = \theta_\beta L_\beta$ for $\beta = 1, \ldots, m$, verifying the necessity of the condition in Theorem 3.4.1 in the case of equality in (3.14).

Finally, if K and L are convex bodies with $o \in \text{int}K$ and $o \in \text{int}L$, and $K = K_1 + \cdots + K_m$ and $L = L_1 + \cdots + L_m$ for $m \geq 1$ and compact convex sets K_i, L_i, $i = 1, \cdots, m$, having dimension at least one and satisfying $o \in K_i$ and $K_i = \theta_i L_i$ for $\theta_i > 0$ for $i = 1, \cdots, m$, and $\sum_{i=1}^m \dim K_i = n$, then equality holds in (3.14) even without symmetry assumption according to Lemma 3.2.2. This completes the proof of Theorem 3.4.1. □

We are ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. According to John's theorem (see Schneider [144]), there exists a unique ellipsoid $\mathcal{E}$ of minimal volume containing K, which is also known as Löwner ellipsoid. It follows that $\mathcal{E}$ is also invariant under $A_1, \ldots, A_n$. For a linear transform $\Phi \in \text{GL}(n)$ satisfying that $\Phi(\mathcal{E}) = B_2^n$, the linear transforms $A_i' = \Phi A_i \Phi^{-1}$, $i = 1, \ldots, n$ leave B_2^n invariant, thus A_i' is an orthogonal reflection through the hyperlane $H_i' = \Phi H_i$ where $H_1 \cap \ldots \cap H_n = \{o\}$. In addition, the convex bodies $K' = \Phi K$ and $L' = \Phi L$ are invariant under $A_1', \ldots, A_n'$.

Finally, applying Theorem 3.4.1 to K' and L', and using the linear covariance of the L_0-sum (see (3.12)), we conclude Theorem 3.1.1. □

3.5 Consequences under symmetry

3.5.1 Log-Minkowski inequality

For the following known Alexandrov variational formula we refer to Lemma 2.1 in [33] or Lemma 7.5.3 in Schneider [144]).

Lemma 3.5.1 (Alexandrov). *Let $h_t : S^{n-1} \to (0, \infty)$ be continuous for $t \in [0, 1)$ such that the limit $\lim_{t\to 0^+} \frac{h_t(u) - h_0(u)}{t} = h_0'(u)$ exists and uniform in $u \in S^{n-1}$. Then the Wulff-shape $W_t = \cap_{u \in S^{n-1}} \{x \in \mathbb{R}^n;\ \langle x, u \rangle \leq h_t(u)\}$ for $t \in [0, 1)$ satisfies*

$$\lim_{t\to 0^+} \frac{|W_t| - |W_0|}{t} = \frac{1}{n} \int_{S^{n-1}} h_0'(u)\, dS_{W_0}(u).$$

The Proposition below was stated in the case when $\mathcal{C}$ is the family of origin symmetric bodies in [33]. For completeness we repeat the method given in yielding the following slightly more general statement.

Proposition 3.5.2 (Böröczky, Lutwak, Yang, Zhang)**.** *Let $\mathcal{C}$ be a class of convex bodies containing the origin in their interior such that $\mathcal{C}$ is closed under dilation and the L_0-sum (i.e. $(1-\lambda)\cdot K +_0 \lambda\cdot L \in \mathcal{C}$ for any $K, L \in \mathcal{C}$, $\lambda \in [0,1]$), and*

$$|(1-\lambda)\cdot K +_0 \lambda \cdot L| \geq |K|^{1-\lambda}|L|^{\lambda} \tag{3.29}$$

holds for any $K, L \in \mathcal{C}$. Then

$$\int_{S^{n-1}} \log \frac{h_L}{h_K}\, dV_K \geq \frac{|K|}{n} \log \frac{|L|}{|K|}$$

for any $K, L \in \mathcal{C}$ with equality if and only if $|\frac{1}{2}\cdot K +_0 \frac{1}{2}\cdot L| = |K|^{1/2}|L|^{1/2}$.

Proof. We can assume that $|K| = |L| = 1$, and hence the inequality to prove is

$$\int_{S^{n-1}} \log h_L\, dV_K \geq \int_{S^{n-1}} \log h_L\, dV_K. \tag{3.30}$$

For $\lambda \in [0,1]$, we consider the function $f(\lambda) = |Q_\lambda|$ where

$$Q_\lambda = (1-\lambda)\cdot K +_0 \lambda \cdot L.$$

First we prove that $f(\lambda)$ is log-concave. On the one hand, for $\lambda, \sigma, \tau \in [0,1]$ and $\alpha = (1-\lambda)\sigma + \lambda\tau$, we observe that,

$$(1-\lambda)\cdot Q_\sigma +_0 \lambda \cdot Q_\tau \subset (1-\alpha)\cdot K +_0 \alpha \cdot L$$

since the support function of a Wulff shape W is at most the function that is used in the definition of W (see 2.10 in [33]); in particular,

$$h_{Q_\sigma}^{1-\lambda} h_{Q_\tau}^{\lambda} \leq (h_K^{1-\sigma} h_L^{\sigma})^{1-\lambda} (h_K^{1-\tau} h_L^{\tau})^{\lambda} = h_K^{1-\alpha} h_L^{\alpha}.$$

On the other hand, log-Brunn-Minkowski inequality hold true for any pair Q_σ and Q_τ with $\sigma, \tau \in [0,1]$ because $Q_\sigma, Q_\tau \in \mathcal{C}$. These two observations give,

$$\begin{aligned}
\log f((1-\lambda)\sigma + \lambda\tau) &= \log f(\alpha) \\
&= \log |Q_\alpha| \\
&\geq \log |(1-\lambda) Q_\sigma +_0 \lambda \cdot Q_\tau| \\
&\geq \log |Q_\sigma|^{1-\lambda} \log |Q_\tau|^{\lambda} \\
&= (1-\lambda)\log f(\sigma) + \lambda \log f(\tau),
\end{aligned}$$

verifying that $f(\lambda)$ is log-concave.

Since $f(\lambda)$ is log-concave on $[0,1]$, it has righ hand sided deivative $f'_+(\lambda)$ for $\lambda \in [0,1)$, and left hand sided deivative $f'_-(\lambda)$ for $\lambda \in (0,1]$. In addition, $f(0) = f(1) = 1$ and (3.29) yields that $f(\lambda) \geq f(0)$ for $\lambda \in (0,1)$; therefore, $f'_+(0) \geq 0$.

We apply Lemma 3.5.1 to $h_t(u) = h_K(u)^{1-t} h_L(u)^t$, and hence

$$h_0'(u) = h_K(u) \log \frac{h_L(u)}{h_K(u)}.$$

It follows from $f'_+(0) \geq 0$, $K = Q_0$ and Lemma 3.5.1 that

$$\begin{aligned}
0 \leq f'_+(0) &= \lim_{\lambda \to 0^+} \frac{|Q_\lambda| - |Q_0|}{\lambda} \\
&= \frac{1}{n} \int_{S^{n-1}} h_K(u) \log \frac{h_L(u)}{h_K(u)}\, dS_K(u) = \int_{S^{n-1}} \log \frac{h_L}{h_K}\, dV_K.
\end{aligned}$$

In turn, we conclude (3.30).

Since $f(\lambda)$ is log-concave and $f(0) = |K| = |L| = f(1)$, if $f(\frac{1}{2}) = |\frac{1}{2} \cdot K +_0 \frac{1}{2} \cdot L| = 1$, then f is constant, and hence $f'_+(0) = 0$. Therefore, we have equality in (3.30).

Finally, if equality holds in (3.30), then $f'_+(0) = 0$, thus the log-concavity of f and $f(0) = f(1)$ yields that f is constant, which in turn yields that

$$|\tfrac{1}{2} \cdot K +_0 \tfrac{1}{2} \cdot L| = f(\tfrac{1}{2}) = f(0) = |K|^{1/2}|L|^{1/2},$$

completing the proof of Theorem 3.5.2. □

We observe that log-Minkowski inequality (2.18) for bodies with n-symmetries follows from Theorem 3.1.1 and Proposition 3.5.2

3.5.2 Uniqueness of V_k

Now we recall the argument in [33] about the characterization of cone volume measure with respect to uniqueness, in the slightly more general form.

Proposition 3.5.3. *Let $\mathcal{C}$ be a class of convex bodies containing the origin in their interior such that $\mathcal{C}$ is closed under dilation and the L_0-sum, and*

$$|(1-\lambda) \cdot K +_0 \lambda \cdot L| \geq |K|^{1-\lambda}|L|^{\lambda} \tag{3.31}$$

holds for any $K, L \in \mathcal{C}$. If $V_K = V_L$ for $K, L \in \mathcal{C}$, then $|\frac{1}{2} \cdot K +_0 \frac{1}{2} \cdot L| = |K|^{1/2}|L|^{1/2}$.

Proof. We deduce from $V_K = V_L$ and the log-Minkowski inequality Theorem 3.5.2 that

$$\begin{aligned}\int_{S^{n-1}} \log h_L dV_L = \int_{S^{n-1}} \log h_L dV_K \geq \int_{S^{n-1}} \log h_K dV_K = \int_{S^{n-1}} \log h_K dV_L \\ \geq \int_{S^{n-1}} \log h_L dV_L.\end{aligned}$$

Thus we have equality in Theorem 3.5.2, proving $|\frac{1}{2} \cdot K +_0 \frac{1}{2} \cdot L| = |K|^{1/2}|L|^{1/2}$. □

We observe that uniqueness of the cone volume measure follows from Theorem 3.1.1 and Proposition 3.5.3.

3.5.3 Passing to $e^{-\phi(x)}dx$

Saroglou [143] proved that on the class of o-symmetric convex bodies and measures, the logarithmic-Brunn-Minkowski inequality for the Lebesgue measure implies the logarithmic-Brunn-Minkowski inequality for any log-concave measure. In other words, according to Theorem 3.1 in [143], if (2.10) holds for any o-symmetric convex bodies K, L in $\mathbb{R}^n$, then for any even convex $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ function, we have

$$\int_{(1-\lambda)\cdot K +_0 \lambda \cdot L} e^{-\varphi(x)} dx \geq \Big(\int_K e^{-\varphi(x)} dx \Big)^{1-\lambda} \Big(\int_L e^{-\varphi(x)} dx \Big)^{\lambda}$$

for any o-symmetric convex bodies K, L.

However, the proof of Theorem 3.1 in [143] does not actually use o-symmetry but a somewhat weaker property of log-concave measures with rotational symmetry. Let $\varphi(x) = \psi(\|x\|)$ for an increasing convex function $\psi : [0, \infty) \to (-\infty, \infty]$, and let G be the subgroup generated by orthogonal reflections through the linear hyperplanes $H_1, \ldots, H_n$ with $H_1 \cap \ldots \cap H_n = \{o\}$, and hence if M is a convex body invariant under G, then $M \cap \{\varphi \leq r\}$ is also invariant under G for any $r > \varphi(o) = \psi(0)$. It follows that if K and L are convex bodies invariant under G, then all bodies used in the proofs of Lemma 3.7 and Theorem 3.1 in [143] are also invariant under G; therefore, the argument by Saroglou [143] yields the following theorem.

Theorem 3.5.4. *Let $\lambda \in (0,1)$, let $\varphi(x) = \psi(\|x\|)$ for an increasing convex function $\psi : [0,\infty) \to (-\infty,\infty]$ and let $d\mu(x) = e^{-\varphi(x)}\,dx$ be the corresponding log-concave measure on $\mathbb{R}^n$. If $H_1 \cap \ldots \cap H_n = \{o\}$ holds for the linear hyperplanes $H_1, \ldots, H_n$, and the convex bodies K and L are invariant under the orthogonal reflections through $H_1, \ldots, H_n$, then*

$$\mu((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq \mu(K)^{1-\lambda}\mu(L)^{\lambda}.$$

Note, Gaussian measure corresponds to the case $\psi(t) = t^2$.

Chapter 4

About the case of equality in the Geometric Reverse Brascamp-Lieb inequality

4.1 Introduction

The Theorem below states the geometric reverse Brascamp-Lieb (or Barthe's) inequality together with its equality case which is our result. The Geometric data and independent decomposition are defined in (2.32) and (2.37), respectively. We denote by $L_1(\mathbb{R}^n)$ the family of all the integrable functions on $\mathbb{R}^n$.

Theorem 4.1.1 (Boroczky, K., Xi [32]). *Let $(P_{E_i}, c_i)_{1\leq i\leq k}$ be a Geometric Brascamp-Lieb data. Then, for any non-negative $f_i \in L_1(E_i)$, $i = 1, \dots, k$, we have*

$$\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i\in E_i} \prod_{i=1}^{k} f_i(x_i)^{c_i}\, dx \geq \prod_{i=1}^{k} \left(\int_{E_i} f_i\right)^{c_i}. \tag{4.1}$$

We assume that equality holds for non-negative $f_i \in L_1(E_i)$, $i = 1, \dots, k$, with positive integral and denote the independent decomposition induced from this data by

$$\mathbb{R}^n = F_{\text{dep}} \oplus \left(\oplus_{j=1}^{\ell} F_j\right).$$

Then there exist $\theta_i > 0$, $b_i \in E_i \cap F_{\text{dep}}$ and $w_i \in E_i \cap F_{\text{dep}}^{\perp}$ for $i = 1, \dots, k$, log-concave $h_j : F_j \to [0, \infty)$ for $j = 1, \dots, \ell$, and a positive definite matrix $A : F_{\text{dep}} \to F_{\text{dep}}$ such that the eigenspaces of A are critical subspaces and

$$f_i(x) = \theta_i e^{-\langle A P_{F_{\text{dep}}} x, P_{F_{\text{dep}}} x - b_i\rangle} \prod_{F_j \subseteq E_i} h_j(P_{F_j}(x - w_i)) \quad \text{for Lebesgue a.a. } x \in E_i. \tag{4.2}$$

On the other hand, if for any $i = 1, \dots, k$, f_i is of the form as in (4.2) and equality holds for all $x \in E_i$ in (4.2), then equality holds in (4.1) for $f_1, \dots, f_k$.

Outline of the proof of (4.2)

We briefly, write a sketch of the proof of Theorem 4.1.1. Let (P_{E_i}, c_i) be a Geometric Brascamp-Lieb data and let $\mathbb{R}^n = F_0 \oplus \left(\oplus_{j=1}^{\ell} F_j\right)$ the induced independent decomposition. Let $(f_1, \dots, f_k)$ be an extremizer of the Reverse Brascamp-Lieb for this data (can be assumed probability densities) and let $T_i : E_i \to E_i$ be the Brenier map (see Theorem 4.4.1) for which

$$g_i(x) = f_i(T_i(x)) \det \nabla T(x), \quad x \in \mathbb{R}^n,$$

with $g_i(x) = e^{-\pi\|x\|^2}$. Barthes proof of inequality (4.1) follows (see (4.34)) from the inequality

$$\det\left(\sum_{i=1}^{k} c_i \nabla T_i\left(P_{E_i}x\right)\right) \geq \prod_{i=1}^{k}\left(\det \nabla T_i\left(P_{E_i}x\right)\right)^{c_i} \quad x \in \mathbb{R}^n.$$

By the extremizability of $(f_1, \dots, f_k)$ one obtains equality on the above inequality. After characterizing these equality cases (see Proposition 4.2.10) we get (see Proposition 4.4.2 (i)) that for some h_{ij} on $E_i \cap F_j$

$$f_i(x) = h_{i0}(P_{F_0}x) \cdot \prod_{\substack{F_j \subseteq E_i \\ j \geq 1}} h_{ij}(P_{F_j}x) \quad \text{for } x \in E_i.$$

Then, we drop the index i on the h_{ij} (see Proposition 4.4.4 (ii)) applying the know equality case of Prékopa-Leindler inequality. Last, it remains to show that h_{i0} is a Gaussian. We observe the followings:

(i) $(P_{E_i \cap F_0}, c_i)_{i=1}^k$ is a Geometric data on F_0 with no independent subspaces.

(ii) $(h_{10}, \dots, h_{k0})$ is an extremizer for the data $(P_{E_i \cap F_0}, c_i)_{i=1}^k$ (see Proposition 4.4.4 (i)).

(iii) $T_{i0} := T_i|_{E_i \cap F_0}$ is the Brenier map for which $g_{i0}(x) = h_{i0}(T_{i0}(x)) \det \nabla T_{i0}(x)$.

Afterwards, we show (in Proposition 4.6.3) that if one assume that T_{i0} has linear growth then T_{i0} is linear, thus by (iii) we get that h_{i0} is Gaussian and this deduce the form in (4.2). To involve this extra assumption on T_{i0} one can closure properties of extremizabilty (in our case convolution and product see (4.78)) and first get an extremizer $(\tilde{f}_1, \dots, \tilde{f}_k)$ for which $\tilde{f}_i \leq c g_i$ for some $c > 0$. This control gives (see Proposition 4.7.1) that the corresponding Brenier map $\tilde{T}_i$ has linear growth and in turn corresponding $\tilde{T}_{i0}$ as well. Therefore, one can write $\tilde{f}_i$ as in (4.2) and in turn f_i is written as (4.2) using some classic facts from Fourier transform (see the paragraph before (4.81)). A complete proof is written in section 4.7.2.

4.2 Structure theory and the Determinantal inequality

4.2.1 In rank one case

This section just retells the story of Section 2 of Barthe [15] in the language of Carlen, Lieb, Loss [50] and Bennett, Carbery, Christ, Tao [21].

We discuss the basic properties of a set of vectors $u_1, \dots, u_k \in S^{n-1}$ and constants $c_1, \dots, c_k > 0$ occurring the rank one Geometric Brascamp-Lieb data $(u_i \otimes u_i, c_i)_{i=1}^k$; namely, satisfying

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = I_n. \tag{4.3}$$

Lemma 4.2.1. *For $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$ satisfying (4.3), we have*

(i) $\sum_{i=1}^k c_i = n$;

(ii) $\sum_{i=1}^k c_i \langle u_i, x\rangle^2 = \|x\|^2$ for all $x \in \mathbb{R}^n$;

(iii) $c_i \leq 1$ for $i = 1, \dots, k$ with equality if and only if $u_j \in u_i^\perp$ for $j \neq i$;

(iv) $u_1, \dots, u_k$ spans $\mathbb{R}^n$, and $k = n$ if and only if $u_1, \dots, u_n$ is an orthonormal basis of $\mathbb{R}^n$ and $c_1 = \ldots = c_n = 1$;

(v) if L is a proper linear subspace of $\mathbb{R}^n$, then

$$\sum_{u_i\in L} c_i \leq \dim L,$$

with equality if and only if $\{u_1,\ldots,u_k\} \subseteq L\cup L^\perp$.

Remark If $\sum_{u_i\in L} c_i = \dim L$ in (v), then $\mathrm{lin}\{u_i : u_i \in L\} = L$ and $\mathrm{lin}\{u_i : u_i \in L^\perp\} = L^\perp$.

Proof. Here (i) follows from comparing the traces of the two sides of (4.3), and (ii) is just an equivalent form of (4.3). To prove $c_j \leq 1$ with the characterization of equality, we substitute $x = u_j$ into (ii).

Turning to (iv), $u_1,\ldots,u_k$ spans $\mathbb{R}^n$ by (ii). Next, let us assume that $u_1,\ldots,u_n \in S^{n-1}$ and $c_1,\ldots,c_n > 0$ satisfy (4.3). We consider $w_j \in S^{n-1}$ for $j = 1,\ldots,n$ such that $\langle w_j, u_i\rangle = 0$ if $i \neq j$, and hence (ii) shows that $u_j = \pm w_j$ and $c_j = 1$.

For (v), if $u_i \notin L$, then we consider the unit vector

$$\tilde{u}_i = \frac{P_{L^\perp} u_i}{\|P_{L^\perp} u_i\|} \in L^\perp.$$

We deduce that if $x \in L^\perp$, then

$$\|x\|^2 = \sum_{i=1}^k c_i \langle u_i, x\rangle^2 = \sum_{u_i\notin L} c_i \langle P_{L^\perp} u_i, x\rangle^2 = \sum_{u_i\notin L} c_i \|P_{L^\perp} u_i\|^2 \langle \tilde{u}_i, x\rangle^2.$$

It follows from (i) and (ii) applied to $\{\tilde{u}_i : u_i \notin L\}$ in $L^\perp$ that

$$\dim L^\perp = \sum_{u_i\notin L} c_i \, \|P_{L^\perp} u_i\|^2 \leq \sum_{u_i\notin L} c_i.$$

In turn, we conclude the inequality in (v) by (i). Equality holds in (v) if and only if $\|P_{L^\perp} u_i\| = 1$ whenever $u_i \notin L$; therefore, $u_1,\ldots,u_k \subseteq L\cup L^\perp$. $\square$

Let $u_1,\ldots,u_k \in S^{n-1}$ and $c_1,\ldots,c_k > 0$ satisfy (4.3). Following Bennett, Carbery, Christ, Tao [21], we say that a non-zero linear subspace V is a critical subspace with respect to $u_1,\ldots,u_k$ and $c_1,\ldots,c_k$ if

$$\sum_{u_i\in V} c_i = \dim V.$$

In particular, $\mathbb{R}^n$ is a critical subspace according to Lemma 4.2.1. We say that a non-empty subset $\mathcal{U} \subseteq \{u_1,\ldots,u_k\}$ is indecomposable if $\mathrm{lin}\,\mathcal{U}$ is an indecomposable critical subspace.

In order to understand the equality case of the rank one Brascamp-Lieb inequality, Barthe [15] indicated an equivalence relation on $\{u_1,\ldots,u_k\}$. We say that a subset $\mathcal{D} \subseteq \{u_1,\ldots,u_k\}$ is minimally dependent if $\mathcal{D}$ is dependent and no proper subset of $\mathcal{D}$ is dependent. The following is folklor in matroid theory, was known most probably already to Tutte (see for example Theorem 7.3.6 in Recski [138]). For the convenience of the reader, we provide an argument.

Lemma 4.2.2. *Given non-zero $v_1,\ldots,v_k$ spanning $\mathbb{R}^n$, $n \geq 1$, we write $v_i \bowtie v_j$ if either $v_i = v_j$, or there exists a minimal dependent set $\mathcal{D} \subseteq \{v_1,\ldots,v_k\}$ satisfying $v_i, v_j \in \mathcal{D}$.*

(i) $v_i \bowtie v_j$ if and only if there exists a subset $\mathcal{U} \subseteq \{v_1,\ldots,v_k\}$ of cardinality $n-1$ such that both $\{v_i\}\cup\mathcal{U}$ and $\{v_j\}\cup\mathcal{U}$ are independent;

(ii) $\bowtie$ is an equivalence relation on $\{v_1,\ldots,v_k\}$;

(iii) if $V_1,\ldots,V_m$ are the linear hulls of the equivalence classes with respect to $\bowtie$, then they span $\mathbb{R}^n$ and $V_i \cap V_j = \{o\}$ for $i \neq j$.

Proof. We prove the lemma by induction on $n \geq 1$ where the case $n = 1$ readily holds. Therefore, we assume that $n \geq 2$.

We may readily assume that

$$\{v_1, \ldots, v_k\} \cap \operatorname{lin}\{v_i\} = \{v_i\} \quad \text{for } i = 1, \ldots, k. \tag{4.4}$$

For (i), if $\mathcal{D}$ is a minimal dependent set with $v_i, v_j \in \mathcal{D}$, then adding some $\mathcal{V} \subseteq \{v_1, \ldots, v_k\}$ to $\mathcal{D}\backslash\{v_i\}$, we obtain a basis of $\mathbb{R}^n$, and we may choose $\mathcal{U} = \mathcal{V} \cup (\mathcal{D}\backslash\{v_i, v_j\})$. On the other hand, if the suitable $\mathcal{U}$ of cardinality $n-1$ exists such that both $\{v_i\} \cup \mathcal{U}$ and $\{v_j\} \cup \mathcal{U}$ are independent, then any dependent subset of $\mathcal{U} \cup \{v_i, v_j\}$ contains v_i and v_j.

For (ii) and (iii), we call a non-zero linear subspace $W \subseteq \mathbb{R}^n$ unsplittable with respect to $\{v_1, \ldots, v_k\}$ if W is spanned by $W \cap \{v_1, \ldots, v_k\}$, but there exist no non-zero complementary linear subspaces $A, B \subseteq W$ with $\{v_1, \ldots, v_k\} \cap W \subseteq A \cup B$. Readily, there exist pairwise complementary unsplittable linear subspaces $W_1, \ldots, W_m \subseteq \mathbb{R}^n$ such that $\{v_1, \ldots, v_k\} \subseteq W_1 \cup \ldots \cup W_m$.

On the one hand, if $v_i \in W_\alpha$ and $v_j \in W_\beta$ for $\alpha \neq \beta$, then trivially $v_i \not\bowtie v_j$. Therefore all we need to prove is that if $v_i, v_j \in W_\alpha$, then $v_i \bowtie v_j$. By the induction on n, we may assume that $m = 1$ and $W_\alpha = \mathbb{R}^n$. We may also assume that $i = 1$ and $j = 2$.

The final part of argument is indirect; therefore, we suppose that

$$v_1 \not\bowtie v_2, \tag{4.5}$$

and seek a contradiction.

(4.5) implies that v_1 and v_2 are independent, and hence $v_1 \not\bowtie v_2$ and (4.4) yield that $L = \operatorname{lin}\{v_1, v_2\}$ satisfies

$$\{v_1, \ldots, v_k\} \cap L = \{v_1, v_2\}. \tag{4.6}$$

Now $\mathbb{R}^n$ is unsplittable, thus $n \geq 3$.

Since $v_1, \ldots, v_k$ span $\mathbb{R}^n$, we may assume that $v_1, \ldots, v_n$ form a basis of $\mathbb{R}^n$. Let $L_0 = \operatorname{lin}\{v_3, \ldots, v_n\}$, and $L_t = \operatorname{lin}\{v_t, L_0\}$ for $t = 1, 2$. We may also assume that $v_1, \ldots, v_n$ is an orthonormal basis.

For any $l > n$, (i) and $v_1 \not\bowtie v_2$ yield that

$$\text{either } v_l \in L_1, \text{ or } v_l \in L_2. \tag{4.7}$$

Since $\mathbb{R}^n$ is unsplittable, there exist $p, q > n$ such that

$$v_p \in L_1\backslash L_0 \text{ and } v_q \in L_2\backslash L_0. \tag{4.8}$$

For any $w \notin L$, we write

$$\operatorname{supp} w = \{v_l : l \in \{3, \ldots, n\} \ \&\ \langle w, v_l\rangle \neq 0\};$$

namely, the basis vectors where the corrresponding coordinate of $w|L-0$ is non-zero.

Case 1 *There exist $v_p \in L_1\backslash L_0$ and $v_q \in L_2\backslash L_0$, $p, q > n$, such that* $(\operatorname{supp} v_p) \cap (\operatorname{supp} v_q) \neq \emptyset$

Let $v_s \in (\operatorname{supp} v_p) \cap (\operatorname{supp} v_q)$. Now the $n+1$ element set

$$\{v_1, v_p, v_2, v_q\} \cup \{v_l : l \in \{3, \ldots, n\}\backslash\{s\}\}$$

is dependent, and considering the 1^{st}, 2^{nd} and s^{th} coordinates show that both v_1 and v_2 lie in any dependent subset. This fact contradicts (4.5).

Case 2 $(\operatorname{supp} v_p) \cap (\operatorname{supp} v_q) = \emptyset$ *for any $v_p \in L_1\backslash L_0$ and $v_q \in L_2\backslash L_0$ with $p, q > n$*

Let $\mathcal{U}_t = \cup\{\operatorname{supp} v_p : p > n \ \&\ v_p \in L_t\backslash L_0\}$ for $t = 1, 2$. It follows that $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$, thus $n \geq 4$. For any partition $\mathcal{U}_1' \cup \mathcal{U}_2' = \{v_3, \ldots, v_n\}$ (and hence $\mathcal{U}_1' \cap \mathcal{U}_2' = \emptyset$) such that $\mathcal{U}_1 \subseteq \mathcal{U}_1'$ and $\mathcal{U}_2 \subseteq \mathcal{U}_2'$, there exists some $v_l \in L_0$ that is contained neither in $\operatorname{lin}(\mathcal{U}_1' \cup \{v_1\})$ nor in $\operatorname{lin}(\mathcal{U}_2' \cup \{v_2\})$ because $\mathbb{R}^n$ is unslittable. In turn we deduce that we may reindex the vectors $v_3, \ldots, v_n$ on the one hand, and the vectors $v_{n+1}, \ldots, v_k$ on the other hand to ensure the following properties:

- $v_{n+1} \in L_1 \backslash L_0$ and $v_{n+2} \in L_2 \backslash L_0$;
- there exist $\alpha \in \{3, \ldots, n-1\}$ and $\beta \in \{n+3, \ldots, k\}$ such that $\operatorname{supp} v_l \subseteq \{v_\alpha, \ldots, v_n\}$ for $l \in \{n+1, \ldots, \beta\}$, and $v_l \in L_0$ if $n+3 \leq l \leq \beta$;
- for any partion $\mathcal{W}_1 \cup \mathcal{W}_2 = \{v_\alpha, \ldots, v_n\}$ into non-empty sets, there exist $l \in \{n+1, \ldots, \beta\}$ such that $\operatorname{supp} v_l$ intersects both $\mathcal{W}_1$ and $\mathcal{W}_2$.

We observe that $\widetilde{L}_0 = \operatorname{lin}\{v_\alpha, \ldots, v_n\}$ is unsplittable with respect to

$$\{v_\alpha, \ldots, v_n, v_{n+1}|L_0, v_{n+2}|L_0, v_{n+3}, \ldots, v_\beta\}.$$

Therefore, this last set contains a minimal dependent subset $\widetilde{\mathcal{D}}$ with $v_{n+1}|L_0, v_{n+2}|L_0 \in \widetilde{\mathcal{D}}$ by induction; namely, the elements of $\widetilde{\mathcal{D}}$ different from $v_{n+1}|L_0, v_{n+2}|L_0$ are vectors of the form v_l that lie in L_0. We conclude that

$$\mathcal{D} = \{v_1, v_2, v_{n+1}, v_{n+2}\} \cup \left(\widetilde{\mathcal{D}} \backslash \{v_{n+1}|L_0, v_{n+2}|L_0\}\right)$$

is a minimal dependent set, contradicting (4.5), and proving Lemma 4.2.2. $\square$

Lemma 4.2.3. *For $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfying* (4.3)*, we have*

(i) a proper linear subspace $V \subseteq \mathbb{R}^n$ is critical if and only if $\{u_1, \ldots, u_k\} \subseteq V \cup V^\perp$;

(ii) if V, W are proper critical subspaces with $V \cap W \neq \{o\}$, then $V^\perp$, $V \cap W$ and $V + W$ are critical subspaces;

(iii) the equivalence classes with respect to the relation $\bowtie$ in Lemma 4.2.2 are the indecomposable subsets of $\{u_1, \ldots, u_k\}$;

(iv) the proper indecomposable critical subspaces are pairwise orthogonal, and any critical subspace is the sum of some indecomposable critical subspaces.

Proof. (i) directly follows from Lemma 4.2.1 (v), and in turn (i) yields (ii).

We prove (iii) and and first half of (iv) simultatinuously. Let $V_1, \ldots, V_m$ be the linear hulls of the equivalence classes of $u_1, \ldots, u_k$ with respect to the $\bowtie$ of Lemma 4.2.2. We deduce from Lemma 4.2.1 (v) that each V_i is a critical subspace, and if $i \neq j$, then V_i and V_j are orthogonal.

Next let $\mathcal{U} \subseteq \{u_1, \ldots, u_k\}$ be an indecomposable set, and let $V = \operatorname{lin}\mathcal{U}$. We write $I \subseteq \{1, \ldots, m\}$ to denote the set of indices i such that $V_i \cap \mathcal{U} \neq \emptyset$. Since V is a critical subspace, we deduce from Lemma 4.2.1 (v) that $V_i \cap V$ is a critical subspace for $i \in I$, as well; therefore, I consists of a unique index p as $\mathcal{U}$ is indecomposable. In particular, $V = V_p$.

It follows from Lemma 4.2.1 (v) that $\{u_1, \ldots, u_k\} \subseteq V \cup V^\perp$; therefore, there exists no minimally dependent subset of $\{u_1, \ldots, u_k\}$ intersecting both $\mathcal{U}$ and its complement. We conclude that $V = V_p$.

Finally, the second half of (iv) follows from (i) and (ii). $\square$

The following is the main result of theis section, where the inequality is proved by Ball [12, 13], and the equality case is clarified by Barthe [15].

Proposition 4.2.4 (Ball-Barthe Lemma)**.** *For $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfying* (4.3)*, if $t_i > 0$ for $i = 1, \ldots, k$, then*

$$\det\left(\sum_{i=1}^{k} c_i t_i u_i \otimes u_i\right) \geq \prod_{i=1}^{k} t_i^{c_i}. \tag{4.9}$$

Equality holds in (4.9) if and only if $t_i = t_j$ for any u_i and u_j lying in the same indecomposable subset of $\{u_1, \ldots, u_k\}$.

Proof. To simplify expressions, let $v_i = \sqrt{c_i} u_i$ for $i = 1, \ldots, k$.

In this argument, I always denotes some subset of $\{1, \ldots, k\}$ of cardinality n. For $I = \{i_1, \ldots, i_n\}$, we define

$$d_I := \det[v_{i_1}, \ldots, v_{i_n}]^2 \qquad \text{and} \qquad t_I := t_{i_1} \cdots t_{i_n}.$$

For the $n \times k$ matrices $M = [v_1, \ldots, v_k]$ and $\widetilde{M} = [\sqrt{t_1}\, v_1, \ldots, \sqrt{t_k}\, v_k]$, we have

$$MM^T = I_n \quad \text{and} \quad \widetilde{M}\widetilde{M}^T = \sum_{i=1}^{k} t_i v_i \otimes v_i. \tag{4.10}$$

It follows from the Cauchy-Binet formula that

$$\sum_I d_I = 1 \qquad \text{and} \qquad \det\left(\sum_{i=1}^{k} t_i v_i \otimes v_i\right) = \sum_I t_I d_I,$$

where the summations extend over all sets $I \subseteq \{1, \ldots, k\}$ of cardinality n. It follows that the discrete measure μ on the n element subsets of $\{1, \ldots, k\}$ defined by $\mu(\{I\}) = d_I$ is a probability measure. We deduce from inequality between the arithmetic and geometric mean that

$$\det\left(\sum_{i=1}^{k} t_i v_i \otimes v_i\right) = \sum_I t_I d_I \geq \prod_I t_I^{d_I}. \tag{4.11}$$

The factor t_i occurs in $\prod_I t_I^{d_I}$ exactly $\sum_{I,\, i \in I} d_I$ times. Moreover, the Cauchy-Binet formula applied to the vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$ implies

$$\begin{aligned} \sum_{I,\, i\in I} d_I &= \sum_I d_I - \sum_{I,\, i \notin I} d_I = 1 - \det\left(\sum_{j \neq i} v_j \otimes v_j\right) \\ &= 1 - \det\left(\mathrm{Id}_n - v_i \otimes v_i\right) = \langle v_i, v_i \rangle = c_i. \end{aligned}$$

Substituting this into (4.11) yields (4.9).

We now assume that equality holds in (4.9). Since equality holds in (4.11) when applying arithmetic and geometric mean, all the t_I are the same for any subset I of $\{1, \ldots, k\}$ of cardinality n with $d_I \neq 0$. It follows that $t_i = t_j$ whenever $u_i \bowtie u_j$, and in turn we deduce that $t_i = t_j$ whenever u_i and u_j lie in the same indecomposable set by Lemma 4.2.3 (i).

On the other hand, Lemma 4.2.3 (ii) yields that if $t_i = t_j$ whenever u_i and u_j lie in the same indecomposable set, then equality holds in (4.9). $\square$

Combining Lemma 4.2.3 and Proposition 4.2.4 leads to the following:

Corollary 4.2.5. *For $u_i \in S^{n-1}$ and $c_i, t_i > 0$, $i = 1, \ldots, k$ satisfying* (4.3)*, equality holds in (4.9) if and only if there exist pairwise orthogonal linear subspaces $V_1, \ldots, V_m$, $m \geq 1$, such that $\{u_1, \ldots, u_k\} \subseteq V_1 \cup \ldots \cup V_m$ and $t_i = t_j$ whenever u_i and u_j lie in the same V_p for some $p \in \{1, \ldots, m\}$.*

4.2.2 In higher rank cases

We build a structural theory for a Brascamp-Lieb data based on results proved or indicated in Barthe [15], Bennett, Carbery, Christ, Tao [21] and Valdimarsson [152].

We study the properties of a set of non-zero linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and constants $c_1, \ldots, c_k > 0$ occurring the Geometric Brascamp-Lieb data $(P_{E_i}, c_i)_{i=1}^k$; namely, satisfying

$$\sum_{i=1}^{k} c_i P_{E_i} = I_n. \tag{4.12}$$

We connect (4.12) to (4.3). For $i = 1, \ldots, k$, let $\dim E_i = n_i$ and let $u_1^{(i)}, \ldots, u_{n_i}^{(i)}$ be any orthonormal basis of E_i. In addition, for $i = 1, \ldots, k$, we consider the $n \times n_i$ matrix $M_i = \sqrt{c_i}[u_1^{(i)}, \ldots, u_{n_i}^{(i)}]$. We deduce that

$$c_i P_{E_i} = M_i M_i^T = \sum_{j=1}^{n_i} c_i u_j^{(i)} \otimes u_j^{(i)} \quad \text{for } i = 1, \ldots, k; \tag{4.13}$$

$$I_n = \sum_{i=1}^{k} c_i P_{E_i} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} c_i u_j^{(i)} \otimes u_j^{(i)} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} c_j^{(i)} u_j^{(i)} \otimes u_j^{(i)} \tag{4.14}$$

and hence $u_j^{(i)} \in S^{n-1}$ and $c_j^{(i)} = c_i > 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$ form a Geometric Brascamp-Lieb data like in (4.3).

Lemma 4.2.6. *For linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfying* (4.12)*,*

(i) if $x \in \mathbb{R}^n$, then $\sum_{i=1}^k c_i \|P_{E_i} x\|^2 = \|x\|^2$;

(ii) if $V \subseteq \mathbb{R}^n$ is a proper linear subset, then

$$\sum_{E_i \cap V \neq \{o\}} c_i \dim(E_i \cap V) \leq \dim V \tag{4.15}$$

where equality holds if and only if $E_i = (E_i \cap V) + (E_i \cap V^\perp)$ for $i = 1, \ldots, k$; or equivalently, when $V = (E_i \cap V) + (E_i^\perp \cap V)$ for $i = 1, \ldots, k$.

Proof. For $i = 1, \ldots, k$, let $\dim E_i = n_i$ and let $u_1^{(i)}, \ldots, u_{n_i}^{(i)}$ be any orthonormal basis of E_i such that if $V \cap E_i \neq \{o\}$, then $u_1^{(i)}, \ldots, u_{m_i}^{(i)}$ is any orthonormal basis of $V \cap E_i$ where $m_i \leq n_i$.

For any $x \in \mathbb{R}^n$ and $i = 1, \ldots, k$, we have $\|P_{E_i} x\|^2 = \sum_{j=1}^{n_i} \langle u_j^{(i)}, x\rangle^2$, thus Lemma 4.2.1 (ii) yields (i).

Concerning (ii), Lemma 4.2.1 (v) yields (4.15). On the other hand, if equality holds in (4.15), then V is a critical subspace for the rank one Geometric Brascamp-Lieb data $u_j^{(i)} \in S^{n-1}$ and $c_j^{(i)} = c_i > 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$ satisfying (4.14). Thus Lemma 4.2.6 (ii) follows from Lemma 4.2.1 (v). □

We say that a non-zero linear subspace V is a critical subspace with respect to the proper linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfying (4.12) if

$$\sum_{E_i \cap V \neq \{o\}} c_i \dim(E_i \cap V) = \dim V.$$

In particular, $\mathbb{R}^n$ is a critical subspace by calculating traces of both sides of (4.12). For a proper linear subspace $V \subseteq \mathbb{R}^n$, Lemma 4.2.6 yields that V is critical if and only if $V^\perp$ is critical, which is turn equivalent saying that

$$E_i = (E_i \cap V) + (E_i \cap V^\perp) \text{ for } i = 1, \ldots, k; \tag{4.16}$$

or in other words,

$$V = (E_i \cap V) + (E_i^\perp \cap V) \text{ for } i = 1, \ldots, k. \tag{4.17}$$

We observe that (4.16) has the following consequence: If V_1 and V_2 are orthogonal critical subspaces, then

$$E_i \cap (V_1 + V_2) = (E_i \cap V_1) + (E_i \cap V_2) \quad \text{for } i = 1, \ldots, k. \tag{4.18}$$

We recall that a critical subspace V is indecomposable if V has no proper critical linear subspace.

Lemma 4.2.7. *If $E_1, \ldots, E_k$ are linear subspaces of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfying* (4.12)*, and V, W are proper critical subspaces, then $V^\perp$ and $V + W$ are critical subspaces, and even $V \cap W$ is critical provided that $V \cap W \neq \{o\}$.*

Proof. We may assume that $\dim E_i \geq 1$ for $i = 1, \ldots, k$.

The fact that $V^\perp$ is also critical follows directly from (4.16).

Concerning $V \cap W$ when $V \cap W \neq \{o\}$, we need to prove that if $i = 1, \ldots, k$, then

$$(V \cap W) \cap E_i + (V \cap W)^\perp \cap E_i = E_i. \tag{4.19}$$

For a linear subspace $L \subseteq E_i$, we write $L^{\perp_i} = L^\perp \cap E_i$ to denote the orthogonal complement within E_i. We observe that as V and W are critical subspaces, we have $(V \cap E_i)^{\perp_i} = V^\perp \cap E_i$ and $(W \cap E_i)^{\perp_i} = W^\perp \cap E_i$. It follows from the identity $(V \cap W)^\perp = V^\perp + W^\perp$ that

$$\begin{aligned}
E_i &\supset (V \cap W) \cap E_i + (V \cap W)^\perp \cap E_i = (V \cap E_i) \cap (W \cap E_i) + (V^\perp + W^\perp) \cap E_i \\
&\supset (V \cap E_i) \cap (W \cap E_i) + (V^\perp \cap E_i) + (W^\perp \cap E_i) \\
&= (V \cap E_i) \cap (W \cap E_i) + (V \cap E_i)^{\perp_i} + (W \cap E_i)^{\perp_i} \\
&= (V \cap E_i) \cap (W \cap E_i) + [(V \cap E_i) \cap (W \cap E_i)]^{\perp_i} = E_i,
\end{aligned}$$

yielding (4.19).

Finally, $V + W$ is also critical as $V + W = (V^\perp \cap W^\perp)^\perp$. □

We deduce from Lemma 4.2.7 that any critical subspace can be decomposed into indecomposable ones.

Corollary 4.2.8. *If $E_1, \ldots, E_k$ are proper linear subspaces of $\mathbb{R}^n$ and $c_1, \ldots, c_k > 0$ satisfy* (4.12)*, and W is a critical subspace or $W = \mathbb{R}^n$, then there exist pairwise orthogonal indecomposable critical subspaces $V_1, \ldots, V_m$, $m \geq 1$, such that $W = V_1 + \ldots + V_m$ (possibly $m = 1$ and $W = V_1$).*

We note that the decomposition of $\mathbb{R}^n$ into indecomposable critical subspaces is not unique in general for a Geometric Brascamp-Lieb data. Valdimarsson [152] provides some examples, and in addition, we provide an example where we have a continuous family of indecomposable critical subspaces.

Example 4.2.9 (Continuous family of indecomposable critical subspaces)**.** In $\mathbb{R}^4$, let us consider the following six unit vectors: $u_1(1,0,0,0)$, $u_2(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0)$, $u_3(\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0, 0)$, $v_1(0,0,1,0)$, $v_2(0,0,\frac{1}{2}, \frac{\sqrt{3}}{2})$, $v_3(0,0,\frac{-1}{2}, \frac{\sqrt{3}}{2})$, which satisfy $u_2 = u_1 + u_3$ and $v_2 = v_1 + v_3$.

For any $x \in \mathbb{R}^4$, we have

$$\|x\|^2 = \sum_{i=1}^{3} \frac{2}{3} \cdot (\langle x, u_i \rangle^2 + \langle x, v_i \rangle^2)$$

Therefore, we define the Geometric Brascamp-Lieb Data $E_i = \text{lin}\{u_i, v_i\}$ and $c_i = \frac{2}{3}$ for $i = 1, 2, 3$ satisfying (2.33). In this case, $F_{\text{dep}} = \mathbb{R}^4$.

For any angle $t \in \mathbb{R}$, we have a two-dimensional indecomposable critical subspace

$$V_t = \text{lin}\{(\cos t)u_1 + (\sin t)v_1, (\cos t)u_2 + (\sin t)v_2, (\cos t)u_3 + (\sin t)v_3\}.$$

Next we prove the crucial determinantal inequality. Its proof is kindly provided by Franck Barthe.

Proposition 4.2.10 (Barthe)**.** *For linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$, $n \geq 1$ and $c_1, \ldots, c_k > 0$ satisfying* (4.12)*, if $A_i : E_i \to E_i$ is a positive definite linear transformation for $i = 1, \ldots, k$, then*

$$\det\left(\sum_{i=1}^{k} c_i A_i P_{E_i}\right) \geq \prod_{i=1}^{k} (\det A_i)^{c_i}. \tag{4.20}$$

Equality holds in (4.20) *if and only if there exist linear subspaces $V_1, \ldots, V_m$ where $V_1 = \mathbb{R}^n$ if $m = 1$ and $V_1, \ldots, V_m$ are pairwise orthogonal indecomposable critical subspaces spanning $\mathbb{R}^n$ if $m \geq 2$, and a positive definite $n \times n$ matrix Φ such that $V_1, \ldots, V_m$ are eigenspaces of Φ and $\Phi|_{E_i} = A_i$ for $i = 1, \ldots, k$. In addition, $\Phi = \sum_{i=1}^{k} c_i A_i P_{E_i}$ in the case of equality.*

Proof. We may assume that $\dim E_i \geq 1$ for $i = 1, \dots, k$.

For $i = 1, \dots, k$, let $\dim E_i = n_i$, let $u_1^{(i)}, \dots, u_{n_i}^{(i)}$ be an orthonormal basis of E_i consisting of eigenvectors of A_i, and let $\lambda_j^{(i)} > 0$ be the eigenvalue of A_i corresponding to $u_j^{(i)}$. In particular $\det A_i = \prod_{j=1}^{n_i} \lambda_j^{(i)}$ for $i = 1, \dots, k$. In addition, for $i = 1, \dots, k$, we set $M_i = \sqrt{c_i}[u_1^{(i)}, \dots, u_{n_i}^{(i)}]$ and B_i to be the positive definite transformation with $A_i = B_i B_i$, and hence

$$c_i A_i P_{E_i} = (M_i B_i)(M_i B_i)^T = \sum_{j=1}^{n_i} c_i \lambda_j^{(i)} u_j^{(i)} \otimes u_j^{(i)}.$$

We deduce from Lemma 4.2.4 and (4.14) that

$$\begin{aligned} \det\left(\sum_{i=1}^k c_i A_i P_{E_i}\right) &= \det\left(\sum_{i=1}^k \sum_{j=1}^{n_i} c_i \lambda_j^{(i)} u_j^{(i)} \otimes u_j^{(i)}\right) \\ &\geq \prod_{i=1}^k \left(\prod_{j=1}^{n_i} \lambda_j^{(i)}\right)^{c_i} = \prod_{i=1}^k (\det A_i)^{c_i}. \end{aligned} \tag{4.21}$$

If we have equality in (4.20), and hence also in (4.21), then Corollary 4.2.5 implies that there exist pairwise orthogonal critical subspaces $V_1, \dots, V_m$, $m \geq 1$ spanning $\mathbb{R}^n$ and $\lambda_1, \dots, \lambda_m > 0$ (where $V_1 = \mathbb{R}^n$ if $m = 1$) such that if $E_i \cap V_j \neq \{o\}$, then $E_i \cap V_j$ is an eigenspace of A_i with eigenvalue λ_j. We conclude from (4.16) that each V_j is a critical subspace, and from Corollary 4.2.8 that each V_j can be assumed to be indecomposable. Finally, (4.18) yields that each E_i is spanned by the subspaces $E_i \cap V_j$ for $j = 1, \dots, m$.

To show that each V_j is an eigenspace for the positive definite linear transform $\sum_{i=1}^k c_i A_i P_{E_i}$ of $\mathbb{R}^n$ with eigenvalue λ_j, we observe that

$$A_i P_{E_i} x = \lambda_j P_{E_i} x$$

for any $i = 1, \dots, k$ and $x \in V_j$. It follows that if $x \in V_j$, then

$$\sum_{i=1}^k c_i A_i P_{E_i} x = \lambda_j \sum_{i=1}^k c_i P_{E_i} x = \lambda_j x,$$

proving that we can choose $\Phi = \sum_{i=1}^k c_i A_i P_{E_i}$.

On the other hand, let us assume that there exists a positive definite $n \times n$ matrix Θ whose eigenspaces $W_1, \dots, W_l$ are critical subspaces (or $l = 1$ and $W_1 = \mathbb{R}^n$) and $\Theta|_{E_i} = A_i$ for $i = 1, \dots, k$. In this case, for any $i = 1, \dots, k$, we may choose the orthonormal basis $u_1^{(i)}, \dots, u_{n_i}^{(i)}$ of E_i in a way such that $u_1^{(i)}, \dots, u_{n_i}^{(i)} \subseteq W_1 \cup \ldots \cup W_l$, and hence Corollary 4.2.5 yields that equality holds in (4.20). $\square$

Remark While Proposition 4.2.10 has a crucial role in proving both the Brascamp-Lieb inequality (2.34) and the Reverese Brascamp-Lieb inequality (2.35) and their equality cases, Proposition 4.2.10 can be actually derived from say (2.34). In the Brascamp-Lieb inequality, choose $f_i(z) = e^{-\pi\langle A_i z, z\rangle}$ for $z \in E_i$ and $i = 1, \dots, k$, and hence $\int_{E_i} f_i = (\det A_i)^{\frac{-1}{2}}$. On the other hand, if $x \in \mathbb{R}^n$, then

$$\prod_{i=1}^k f_i\left(P_{E_i} x\right)^{c_i} = e^{-\pi \sum_{i=1}^k c_i \langle A_i P_{E_i} x, P_{E_i} x\rangle} = e^{-\pi \sum_{i=1}^k c_i \langle A_i P_{E_i} x, x\rangle} = e^{-\pi \langle \sum_{i=1}^k c_i A_i P_{E_i} x, x\rangle};$$

therefore, the Brascamp-Lieb inequality (2.34) yields

$$\left(\det \sum_{i=1}^k c_i A_i P_{E_i}\right)^{\frac{-1}{2}} \leq \prod_{i=1}^k (\det A_i)^{\frac{-c_i}{2}}.$$

In addition, the equality conditions in Proposition 4.2.10 can be derived from Valdimarsson's Theorem 2.2.7.

Let us show why indecomposability of a critical subspaces in Proposition 4.2.10 is useful.

Lemma 4.2.11. *Let the linear subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfy* (4.12)*, let $F_{\text{dep}} \neq \mathbb{R}^n$, and let $F_1, \dots, F_l$ be the independent subspaces, $l \geq 1$. If V is an indecomposable critical subspace, then either $V \subseteq F_{\text{dep}}$, or there exists an independent subspace F_j, $j \in \{1, \dots, l\}$ such that $V \subseteq F_j$.*

Proof. It is equivalent to prove that if V is an indecomposable critical subspace and $j \in \{1, \dots, l\}$, then

$$V \not\subseteq F_j \text{ implies } F_j \subseteq V^{\perp}. \tag{4.22}$$

We deduce that $V \cap F_j = \{o\}$ from the facts that V is indecomposable and F_j is a critical subspace, thus $F_j \cap V$ is a critical subspace or $\{o\}$. There exists a partion $M \cup N = \{1, \dots, k\}$ with $M \cap N = \emptyset$ such that

$$F_j = (\cap_{i \in M} E_i) \cap \left(\cap_{i \in N} E_i^{\perp}\right).$$

Let $y \in F_j$. Since V is a critical subspace, we conclude that $P_V y \in E_i$ for $i \in M$ and $P_V y \in E_i^{\perp}$ for $i \in N$, and hence $P_V y \in V \cap (\cap_{i \in M} E_i) \cap \left(\cap_{i \in N} E_i^{\perp}\right) = \{o\}$. Therefore, $y \in V^{\perp}$. □

4.3 Gaussian extremizability

This section continues to build on work done in Barthe [15], Bennett, Carbery, Christ, Tao [21] and Valdimarsson [152].

For linear subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$ and $c_1, \dots, c_k > 0$ satisfying (4.12), we deduce from Lemma 4.2.6 (i) and (4.17) that if V is a critical subspace, then writing $P^{(V)}_{E_i \cap V}$ to denote the restriction of $P_{E_i \cap V}$ onto V, we have

$$\sum_{E_i \cap V \neq \{o\}} c_i P^{(V)}_{E_i \cap V} = I_V \tag{4.23}$$

where I_V denotes the identity transformation on V.

The equality case of Proposition 4.2.10 indicates why Lemma 4.3.1 is important.

Lemma 4.3.1. *For linear subspaces $E_1, \dots, E_k$ of $\mathbb{R}^n$, $n \geq 1$ and $c_1, \dots, c_k > 0$ satisfying* (4.12)*, if Φ is a positive definite linear transform whose eigenspaces are critical subspaces, then for any $x \in \mathbb{R}^n$, we have*

$$\|\Phi x\|^2 = \min_{\substack{x = \sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \sum_{i=1}^{k} c_i \|\Phi x_i\|^2. \tag{4.24}$$

Proof. We may assume that $\dim E_i \geq 1$ for $i = 1, \dots, k$.

As the eigenspaces of Φ are critical subspaces, we deduce by (4.18) that

$$\Phi(E_i) = E_i \text{ and } \Phi(E_i^{\perp}) = E_i^{\perp}. \tag{4.25}$$

For any $x \in \mathbb{R}^n$, we have $\Phi P_{E_i} x = P_{E_i} \Phi x$ for $i = 1, \dots, k$ by (4.25); therefore, Lemma 4.2.6 (i) yields

$$\langle \Phi x, \Phi x \rangle = \sum_{i=1}^{k} c_i \|P_{E_i} \Phi x\|^2 = \sum_{i=1}^{k} c_i \|\Phi P_{E_i} x\|^2. \tag{4.26}$$

Since $x = \sum_{i=1}^k c_i P_{E_i} x$ by (4.12), we may choose $x_i = P_{E_i} x$ in (4.24), and we have equality in (4.24) in this case. Therefore, Lemma 4.3.1 is equivalent to proving that if $x = \sum_{i=1}^k c_i x_i$ for $x_i \in E_i$, $i = 1, \dots, k$, then

$$\|\Phi x\|^2 \leq \sum_{i=1}^{k} c_i \|\Phi x_i\|^2. \tag{4.27}$$

Case 1 $\dim E_i = 1$ for $i = 1, \ldots, k$ and $\Phi = I_n$

Let $E_i = \mathbb{R}u_i$ for $u_i \in S^{n-1}$. If $x \in \mathbb{R}^n$, then $P_{E_i}x = \langle u_i, x\rangle u_i$ for $i = 1, \ldots, k$, and (4.26) yields that

$$\langle x, x\rangle = \sum_{i=1}^{k} c_i \langle u_i, x\rangle^2.$$

In addition, any $x_i \in E_i$ is of the form $x_i = t_i u_i$ for $i = 1, \ldots, k$ where $\|x_i\|^2 = t_i^2$. If $x = \sum_{i=1}^{k} c_i t_i u_i$, then the Hölder inequality yields

$$\langle x, x\rangle = \left\langle x, \sum_{i=1}^{k} c_i t_i u_i \right\rangle = \sum_{i=1}^{k} c_i t_i \langle x, u_i\rangle \leq \sqrt{\sum_{i=1}^{k} c_i t_i^2} \cdot \sqrt{\sum_{i=1}^{k} c_i \langle x, u_i\rangle^2} = \sqrt{\sum_{i=1}^{k} c_i t_i^2} \cdot \sqrt{\langle x, x\rangle},$$

proving (4.27) in this case.

Case 2 The general case, $E_1, \ldots, E_k$ and Φ are as in Lemma 4.3.1

Let $V_1, \ldots, V_m$, $m \geq 1$, be the eigenspaces of Φ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_m$. As $V_1, \ldots, V_m$ are orthogonal critical subspaces and $\mathbb{R}^n = \oplus_{j=1}^{m} V_j$ As $V_1, \ldots, V_m$ are orthogonal critical subspaces and $\mathbb{R}^n = \oplus_{j=1}^{m} V_j$, we deduce that $x_{ij} = P_{V_j} x_i \in E_i \cap V_j$ for any $i = 1, \ldots, k$ and $j = 1, \ldots, m$, and $x_i = \sum_{j=1}^{m} x_{ij}$ for any $i = 1, \ldots, k$. It follows that

$$\begin{aligned} x &= \sum_{j=1}^{m} \left(\sum_{E_i \cap V_j \neq \{o\}} c_i x_{ij} \right) \quad \text{where} \\ P_{V_j} x &= \sum_{E_i \cap V_j \neq \{o\}} c_i x_{ij}. \end{aligned} \tag{4.28}$$

For any $i = 1, \ldots, k$, the vectors $\Phi x_{ij} = \lambda_j x_{ij}$ are pairwise orthogonal for $j = 1, \ldots, m$, thus

$$\sum_{i=1}^{k} c_i \|\Phi x_i\|^2 = \sum_{i=1}^{k} \left(\sum_{j=1}^{m} c_i \|\Phi x_{ij}\|^2 \right) = \sum_{j=1}^{m} \left(\sum_{E_i \cap V_j \neq \{o\}} c_i \|\Phi x_{ij}\|^2 \right).$$

Since $\|\Phi x\|^2 = \sum_{j=1}^{m} \|P_{V_j} \Phi x\|^2 = \sum_{j=1}^{m} \|\Phi P_{V_j} x\|^2$, (4.27) follows if for any $j = 1, \ldots, m$, we have

$$\|\Phi P_{V_j} x\|^2 \leq \sum_{E_i \cap V_j \neq \{o\}} c_i \|\Phi x_{ij}\|^2. \tag{4.29}$$

To prove (4.29), if $E_i \cap V_j \neq \{o\}$, then let $\dim(E_i \cap V_j) = n_{ij}$, and let $u_1^{(ij)}, \ldots, u_{n_{ij}}^{(ij)}$ be an orthonormal basis of $E_i \cap V_j$. Since V_j is a critical subspace (see (4.23)), if $z \in V_j$, then

$$z = \sum_{i=1}^{k} c_i P_{E_i} z = \sum_{E_i \cap V_j \neq \{o\}} c_i P_{E_i \cap V_j} z = \sum_{E_i \cap V_j \neq \{o\}} \sum_{\alpha=1}^{n_{ij}} c_i \langle u_\alpha^{(ij)}, z\rangle u_\alpha^{(ij)}. \tag{4.30}$$

(4.30) shows that the system of all $u_1^{(ij)}, \ldots, u_{n_{ij}}^{(ij)}$ when $E_i \cap V_j \neq \{o\}$ form a rank one Brascamp-Lieb data where the coefficient corresponding to $u_\alpha^{(ij)}$ is c_i.

According to (4.28), we have

$$P_{V_j} x = \sum_{E_i \cap V_j \neq \{o\}} \sum_{\alpha=1}^{n_{ij}} c_i \langle u_\alpha^{(ij)}, x_{ij}\rangle u_\alpha^{(ij)}.$$

We deduce from Case 1 applying to $P_{V_j}x$ to the rank one Brascamp-Lieb data in V_j above that

$$\begin{aligned}\|\Phi P_{V_j}x\|^2 &= \lambda_j^2\|P_{V_j}x\|^2 \leq \lambda_j^2 \sum_{E_i\cap V_j\neq\{o\}} \sum_{\alpha=1}^{n_{ij}} c_i\langle u_\alpha^{(ij)}, x_{ij}\rangle^2 \\ &= \lambda_j^2 \sum_{E_i\cap V_j\neq\{o\}} c_i\|x_{ij}\|^2 = \sum_{E_i\cap V_j\neq\{o\}} c_i\|\Phi x_{ij}\|^2,\end{aligned}$$

proving (4.29), and in turn (4.27) that is equivalent to Lemma 4.3.1. □

We now use Proposition 4.2.10 and Lemma 4.3.1 to exhibit the basic type of Gaussian exemizers of the Reverse Brascamp-Lieb inequality.

Proposition 4.3.2. *For linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$, $n \geq 1$ and $c_1, \ldots, c_k > 0$ satisfying* (4.12)*, if Φ is a positive definite linear transform whose eigenspaces are critical subsapces, then*

$$\int_{\mathbb{R}^n}^* \left(\sup_{\substack{x=\sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k e^{-c_i\|\Phi x_i\|^2} \right) dx = \prod_{i=1}^k \left(\int_{E_i} e^{-\|\Phi x_i\|^2}\, dx_i \right)^{c_i}.$$

Proof. Let $\widetilde{\Phi} = \pi^{-\frac{1}{2}}\,\Phi$. For $i = 1, \ldots, k$, let $A_i = \widetilde{\Phi}|_{E_i}$, and hence $A_i : E_i \to E_i$ as the eigenspaces of $\widetilde{\Phi}$ are critical subspaces. We deduce first using Lemma 4.3.1, and then the equality case of Proposition 4.2.10 that

$$\begin{aligned}\int_{\mathbb{R}^n}^* \left(\sup_{\substack{x=\sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k e^{-c_i\|\Phi x_i\|^2} \right) dx &= \int_{\mathbb{R}^n} e^{-\pi\|\widetilde{\Phi}x\|^2}\, dx = \left(\det \widetilde{\Phi}\right)^{-1} = \prod_{i=1}^k (\det A_i)^{-c_i} \\ &= \prod_{i=1}^k \left(\int_{E_i} e^{-\pi\|\widetilde{\Phi}x_i\|^2}\, dx_i \right)^{c_i} = \prod_{i=1}^k \left(\int_{E_i} e^{-\|\Phi x_i\|^2}\, dx_i \right)^{c_i},\end{aligned}$$

proving Proposition 4.3.2. □

4.4 First form of extremixers via the Determinantal inequality

4.4.1 Brenier maps

Optimal transportation as a tool proving geometric inequalities was introduced by Gromov in his Appendix to [128] in the case of the Brunn-Minkowski inequality. Actually, the Reverse Brascamp-Lieb inequality in [15] was one of the first inequalities in probability, analysis or geometry that was obtained via optimal transportation.

We write $\nabla\Theta$ to denote the first derivative of a C^1 vector valued function Θ defined on an open subset of $\mathbb{R}^n$, and $\nabla^2\varphi$ to denote the Hessian of a real C^2 function φ. We recall that a vector valued function Θ on an open set $U \subseteq \mathbb{R}^n$ is C^α for $\alpha \in (0,1)$ if for any $x_0 \in U$ there exist an open neighborhood U_0 of x_0 and a $c_0 > 0$ such that $\|\Theta(x) - \Theta(y)\| \leq c_0\|x - y\|^\alpha$ for $x, y \in U_0$. In addition, a real function φ is $C^{2,\alpha}$ if φ is C^2 and $\nabla^2\varphi$ is C^α.

Combining Corollary 2.30, Corollary 2.32, Theorem 4.10 and Theorem 4.13 in Villani [154] on the Brenier type based on McCann [116, 117] for the first two, and on Caffarelli [44, 45, 46] for the last two theorems, we deduce the following:

Theorem 4.4.1 (Brenier, McCann, Caffarelli)**.** *If f and g are positive C^α probability density functions on $\mathbb{R}^n$, $n \geq 1$, for $\alpha \in (0,1)$, then there exists a $C^{2,\alpha}$ convex function φ on $\mathbb{R}^n$ (unique up to additive constant) such that $T = \nabla\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is bijective and*

$$g(x) = f(T(x)) \cdot \det \nabla T(x) \quad \textit{for } x \in \mathbb{R}^n. \tag{4.31}$$

The derivative $T = \nabla\varphi$ is the Brenier (transportation) map pushing forward the measure on $\mathbb{R}^n$ induced by g to the measure associated to f; namely, $\int_{T(X)} f = \int_X g$ for any measurable $X \subseteq \mathbb{R}^n$. Also, $\nabla T = \nabla^2\varphi$ is a positive definite symmetric matrix in Theorem 4.4.1, and if f and g are C^k for $k \geq 1$, then T is C^{k+1}. Last, sometimes it is practical to consider the case $n = 0$, when we set $T : \{0\} \to \{0\}$ to be the trivial map.

4.4.2 Barthe's proof

The following proof is due to F. Barthe [15].

Proof of inequality (4.1). First we assume that each f_i is a C^1 positive probability density function on $\mathbb{R}^n$, and let us consider the Gaussian densiy $g_i(x) = e^{-\pi\|x\|^2}$ for $x \in E_i$. According to Theorem 4.4.1, if $i = 1, \ldots, k$, then there exists a C^3 convex function φ_i on E_i such that for the C^2 Brenier map $T_i = \nabla\varphi_i$, we have

$$g_i(x) = \det \nabla T_i(x) \cdot f_i(T_i(x)) \quad \text{for all } x \in E_i. \tag{4.32}$$

It follows from the Remark after Theorem 4.4.1 that $\nabla T_i = \nabla^2\varphi_i(x)$ is positive definite symmetric matrix for all $x \in E_i$. For the C^2 transformation $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Theta(y) = \sum_{i=1}^{k} c_i T_i\left(P_{E_i} y\right), \qquad y \in \mathbb{R}^n, \tag{4.33}$$

its differential

$$\nabla\Theta(y) = \sum_{i=1}^{k} c_i \nabla T_i\left(P_{E_i} y\right)$$

is positive definite by Proposition 4.2.10. It follows that $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ is injective (see [15]), and actually a diffeomorphism. Therefore Proposition 4.2.10, (4.59) and Lemma 4.2.6 (i) imply

$$\begin{aligned}
&\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^{k} f_i(x_i)^{c_i}\, dx \\
&= \int_{\mathbb{R}^n}^{*} \left(\sup_{\Theta(y)=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^{k} f_i(x_i)^{c_i} \right) \det\left(\nabla\Theta(y)\right) dy \\
&\geq \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} f_i\left(T_i\left(P_{E_i} y\right)\right)^{c_i} \right) \det\left(\sum_{i=1}^{k} c_i \nabla T_i\left(P_{E_i} y\right) \right) dy \\
&\geq \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} f_i\left(T_i\left(P_{E_i} y\right)\right)^{c_i} \right) \prod_{i=1}^{k} \left(\det \nabla T_i\left(P_{E_i} y\right)\right)^{c_i} dy \qquad (4.34) \\
&= \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} g_i\left(P_{E_i} y\right)^{c_i} \right) dy = \int_{\mathbb{R}^n} e^{-\pi\|y\|^2}\, dy = 1.
\end{aligned}$$

Finally, the Reverse Brascamp-Lieb inequality (2.35) for arbitrary non-negative integrable functions f_i follows by scaling and approximation (see Barthe [15]). $\square$

4.4.3 Form and Splitting

In this section, we are able to give the first formula of extremizers in the Geometric Reverse Brascamp-Lieb inequality (4.1) and also to prove that if equality holds in the Geometric Reverse Brascamp-Lieb inequality (4.1), then the diffeomorphism Θ in (4.33) splits along the independent subspaces and the dependent subspace.

First we explain how the Reverse Brascamp-Lieb inequality behaves under the shifts of the functions involved. Given a Geometric data $(P_{E_i}, c_i)_{i=1}^k$, see (2.32), first we discuss in what sense the Reverse Brascamp-Lieb inequality is translation invariant. For non-negative integrable function f_i on E_i, $i = 1, \ldots, k$, let us define

$$F(x) = \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i}.$$

We observe that for any $e_i \in E_i$, defining $\tilde{f}_i(x) = f_i(x + e_i)$ for $x \in E_i$, $i = 1, \ldots, k$, we have

$$\widetilde{F}(x) = \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^k \tilde{f}_i(x_i)^{c_i} = F\left(x + \sum_{i=1}^k c_i e_i\right). \tag{4.35}$$

Proposition 4.4.2. *For a Geometric data $(P_{E_i}, c_i)_{i=1}^k$, we write $F_1, \ldots, F_l$ to denote the independent subspaces (if exist), and F_0 to denote the dependent subspace (possibly $F_0 = \{o\}$). Let us assume that equality holds in (2.35) for positive C^1 probability densities f_i on E_i, $i = 1, \ldots, k$, let $g_i(x) = e^{-\pi\|x\|^2}$ for $x \in E_i$, let $T_i : E_i \to E_i$ be the C^2 Brenier map satisfying*

$$g_i(x) = \det \nabla T_i(x) \cdot f_i(T_i(x)) \quad \textit{for all } x \in E_i, \tag{4.36}$$

and let

$$\Theta(y) = \sum_{i=1}^k c_i T_i\left(P_{E_i} y\right), \qquad y \in \mathbb{R}^n.$$

(i) For any $i \in \{1, \ldots, k\}$ there exists positive C^1 integrable $h_{i0} : F_0 \cap E_i \to [0, \infty)$ (where $h_{i0}(o) = 1$ if $F_0 \cap E_i = \{o\}$), and for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ with $F_j \subseteq E_i$, there exists positive C^1 integrable $h_{ij} : F_j \to [0, \infty)$ such that

$$f_i(x) = h_{i0}(P_{F_0} x) \cdot \prod_{\substack{F_j \subseteq E_i \\ j \geq 1}} h_{ij}(P_{F_j} x) \quad \textit{for } x \in E_i.$$

(ii) For $i = 1, \ldots, k$, $T_i(E_i \cap F_p) = E_i \cap F_p$ whenever $E_i \cap F_p \neq \{o\}$ for $p\{0, \ldots, l\}$, and if $x \in E_i$, then

$$T_i(x) = \bigoplus_{\substack{E_i \cap F_p \neq \{o\} \\ p \geq 0}} T_i(P_{F_p} x).$$

(iii) For $i = 1, \ldots, k$, there exist C^2 functions $\Omega_i : E_i \to E_i$ and $\Gamma_i : E_i^\perp \to E_i^\perp$ such that

$$\Theta(y) = \Omega_i(P_{E_i} y) + \Gamma_i(P_{E_i^\perp} y) \quad \textit{for } y \in \mathbb{R}^n.$$

(iv) If $y \in \mathbb{R}^n$, then the eigenspaces of the positive definite matrix $\nabla\Theta(y)$ are critical subspaces, and $\nabla T_i(P_{E_i} y) = \nabla\Theta(y)|_{E_i}$ for $i = 1, \ldots, k$.

Proof. According to (4.35), we may assume that

$$T_i(o) = o \quad \text{for } i = 1, \ldots, k, \tag{4.37}$$

If equality holds in (2.35), then equality holds in the determinantal inequality in (4.34) in the proof of the Reverse Brascamp-Lieb inequality; therefore, we apply the equality case of Proposition 4.2.10. In particular, for any $x \in \mathbb{R}^n$, there exist $m_x \geq 1$ and linear subspaces $V_{1,x}, \ldots, V_{m_x,x}$ where $V_{1,x} = \mathbb{R}^n$ if $m_x = 1$, and $V_{1,x}, \ldots, V_{m_x,x}$ are pairwise orthogonal indecomposable critical subspaces spanning $\mathbb{R}^n$ if

$m_x \geq 2$, and there exist $\lambda_{1,x}, \ldots, \lambda_{m_x,x} > 0$ such that if $E_i \cap V_{j,x} \neq \{o\}$, then writing $\widetilde{P}_{i,j,x}$ to denote the orthogonal projection into $E_i \cap V_{j,x}$, we have

$$\nabla T_i(\widetilde{P}_{i,j,x} x)|_{E_i \cap V_{j,x}} = \lambda_{j,x} I_{E_i \cap V_{j,x}}; \tag{4.38}$$

and in addition, each E_i satisfies (*cf.* (4.18))

$$E_i = \oplus_{E_i \cap V_{j,x} \neq \{o\}} E_i \cap V_{j,x}. \tag{4.39}$$

Let us consider a fixed E_i, $i \in \{1, \ldots, k\}$. First we claim that if $y \in E_i$, then

$$\begin{aligned} \nabla T_i(y)(F_p) &= F_p \qquad \text{if } p \geq 1 \text{ and } E_i \cap F_p \neq \{o\} \\ \nabla T_i(y)(F_0 \cap E_i) &= F_0 \cap E_i. \end{aligned} \tag{4.40}$$

If $p \geq 1$ and $E_i \cap F_p \neq \{o\}$, then $F_p \subseteq E_i$, and Lemma 4.2.11 yields that

$$\begin{aligned} \oplus_{F_p \cap V_{j,x} \neq \{o\}} V_{j,x} &\subseteq F_p \\ \oplus_{F_p \cap V_{j,x} = \{o\}} V_{j,x} &\subseteq F_p^{\perp}. \end{aligned}$$

Since the subsapces $V_{j,x}$ span $\mathbb{R}^n$, we have

$$F_p = \oplus_{\substack{E_i \cap V_{j,x} \neq \{o\} \\ V_{j,x} \subseteq F_p}} V_{j,x};$$

therefore, (4.38) implies (4.40) if $p \geq 1$.

For the case of F_0 in (4.40), it follows from (4.39) and Lemma 4.2.11 that if $E_i \cap F_0 \neq \{o\}$, then

$$E_i \cap F_0 = \oplus_{\substack{E_i \cap V_{j,x} \neq \{o\} \\ V_{j,x} \subseteq F_0}} E_i \cap V_{j,x}. \tag{4.41}$$

Therefore, (4.38) completes the proof of (4.40).

The same argument involving (4.38) also shows that if $y \in E_i$, then

$$\nabla T_i(y) = \oplus_{\substack{E_i \cap F_p \neq \{o\} \\ p \geq 0}} \nabla T_i(P_{F_p} y)|_{F_p}. \tag{4.42}$$

In turn, (4.40), (4.42) and $T_i(o) = o$ (*cf.* (4.37)) imply that if $y \in E_i$, then

$$T_i(E_i \cap F_p) = E_i \cap F_p \quad \text{whenever } E_i \cap F_p \neq \{o\} \text{ for } p \geq 0, \tag{4.43}$$

$$T_i(y) = \bigoplus_{\substack{E_i \cap F_p \neq \{o\} \\ p \geq 0}} T_i(P_{F_p} y). \tag{4.44}$$

We deduce from (4.42) that if $y \in E_i$, then

$$\det \nabla T_i(y) = \prod_{\substack{E_i \cap F_p \neq \{o\} \\ p \geq 0}} \det \left(\nabla T_i(P_{F_p} y)|_{F_p} \right). \tag{4.45}$$

We conclude (i) from (4.42), (4.43), (4.44), and (4.45) as (4.36) yields that if $y \in E_i$, then

$$f_i(T_i(y)) = \prod_{\substack{E_i \cap F_p \neq \{o\} \\ p \geq 0}} \frac{e^{-\pi \|P_{F_p} y\|^2}}{\det \left(\nabla T_i(P_{F_p} y)|_{F_p} \right)}.$$

We deduce (ii) from (4.43) and (4.44).

For (iii), it follows from Proposition 4.2.10 that for any $x \in \mathbb{R}^n$, the spaces $V_{j,x}$ are eigenspaces for $\nabla \Theta(x)$ and span $\mathbb{R}^n$; therefore, (4.17) implies that if $x \in \mathbb{R}^n$ and $i \in \{1, \ldots, k\}$, then

$$\nabla \Theta(x) = \nabla \Theta(x)|_{E_i} \oplus \nabla \Theta(x)|_{E_i^{\perp}}.$$

Since $\Theta(o) = o$ by (4.37), for fixed $i \in \{1, \ldots, k\}$, we conclude

$$\begin{aligned} \Theta(E_i) &= E_i; \\ \Theta(x) &= \Theta\left(P_{E_i} x\right)\big|_{E_i} \oplus \Theta\left(P_{E_i^\perp} x\right)\Big|_{E_i^\perp} \quad \text{if } x \in \mathbb{R}^n. \end{aligned}$$

Finally, (iv) directly follows from Proposition 4.2.10, completing the proof of Proposition 4.4.2. □

Next we show that if the extremizers $f_1, \ldots, f_k$ in Proposition 4.4.2 are of the form as in (i), then for any given $F_j \neq \{o\}$, the functions h_{ij} on F_j for all i with $E_i \cap F_j \neq \{o\}$ are also extremizers. We also need the Prékopa-Leindler inequality Theorem 4.4.3 (proved in various forms by Prékopa [134, 135], Leindler [102] and Borell [27]) whose equality case was clarified by Dubuc [67] (see the survey Gardner [76]). In turn, the Prékopa-Leindler inequality (4.46) is of the very similar structure like the Reverse Brascamp-Lieb inequality (2.35).

Theorem 4.4.3 (Prékopa, Leindler). *For $\lambda_1, \ldots, \lambda_m \in (0,1)$ with $\lambda_1 + \ldots + \lambda_m = 1$ and integrable $\varphi_1, \ldots, \varphi_m : \mathbb{R}^n \to [0, \infty)$, we have*

$$\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^m \lambda_i x_i,\, x_i \in \mathbb{R}^n} \prod_{i=1}^m \varphi_i(x_i)^{\lambda_i}\, dx \geq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \varphi_i\right)^{\lambda_i}, \tag{4.46}$$

and if equality holds and the left hand side is positive and finite, then there exist a log-concave function φ and $a_i > 0$ and $b_i \in \mathbb{R}^n$ for $i = 1, \ldots, m$ such that

$$\varphi_i(x) = a_i\, \varphi(x - b_i)$$

for Lebesgue almost all $x \in \mathbb{R}^n$, $i = 1, \ldots, m$.

For a Gemetric data $(P_{E_i}, c_i)_{i=1}^k$, we assume that $F_{\rm dep} \neq \mathbb{R}^n$, and write $F_1, \ldots, F_l$ to denote the independent subspaces. We verify that if $j \in \{1, \ldots, l\}$, then

$$\sum_{E_i \supset F_j} c_i = 1. \tag{4.47}$$

For this, let $x \in F_j \backslash \{o\}$. We observe that for any E_i, either $F_j \subseteq E_i$, and hence $P_{E_i} x = x$, or $F_j \subseteq E_i^\perp$, and hence $P_{E_i} x = o$. We deduce from (2.33) that

$$x = \sum_{i=1}^k c_i P_{E_i} x = \left(\sum_{F_j \subseteq E_i} c_i\right) \cdot x,$$

which formula in turn implies (4.47).

Proposition 4.4.4. *For a Gemetric data $(P_{E_i}, c_i)_{i=1}^k$, we write $F_1, \ldots, F_l$ to denote the independent subspaces (if exist), and F_0 denote the dependent subspace (possibly $F_0 = \{o\}$). Let us assume that equality holds in the Reverse Brascamp-Lieb inequality* (4.1) *for probability densities f_i on E_i, $i = 1, \ldots, k$, and for any $i \in \{1, \ldots, k\}$ there exists non-negative integrable $h_{i0} : F_0 \cap E_i \to [0, \infty)$ (where $h_{i0}(o) = 1$ if $F_0 \cap E_i = \{o\}$), and for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ with $F_j \subseteq E_i$, there exists non-negative integrable $h_{ij} : F_j \to [0, \infty)$ such that*

$$f_i(x) = h_{i0}(P_{F_0} x) \cdot \prod_{\substack{F_j \subseteq E_i \\ j \geq 1}} h_{ij}(P_{F_j} x) \qquad \textit{for } x \in E_i. \tag{4.48}$$

(i) If $F_0 \neq \{0\}$, then $\sum_{E_i \cap F_0 \neq \{o\}} c_i P_{E_i \cap F_0} = \mathrm{Id}_{F_0}$ and

$$\int_{F_0}^{*} \sup_{x=\sum\{c_i x_i :\, x_i \in E_i \cap F_0\ \&\ E_i \cap F_0 \neq \{o\}\}} \prod_{E_i \cap F_0 \neq \{o\}} h_{i0}(x_i)^{c_i}\, dx = \prod_{E_i \cap F_0 \neq \{o\}} \left(\int_{E_i \cap F_0} h_{i0}\right)^{c_i}.$$

(ii) If $F_0 \neq \mathbb{R}^n$, then there exist log-concave integrable $\psi_j : F_j \to [0,\infty)$ for $j = 1,\dots,l$, and there exist $a_{ij} > 0$ and $b_{ij} \in F_j$ for any $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,l\}$ with $F_j \subseteq E_i$ such that $h_{ij}(x) = a_{ij} \cdot \psi_j(x - b_{ij})$ for $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,l\}$ with $F_j \subseteq E_i$.

Proof. We only present the argument in the case $F_0 \neq \mathbb{R}^n$ and $F_0 \neq \{o\}$. If $F_0 = \mathbb{R}^n$, then the same argument works ignoring the parts involving $F_1,\dots,F_l$, and if $F_0 = \{o\}$, then the same argument works ignoring the parts involving F_0.

Since $F_0 \oplus F_1 \oplus \ldots \oplus F_l = \mathbb{R}^n$ and $F_0,\dots,F_l$ are critical subspaces, (4.18) yields for $i = 1,\dots,k$ that

$$E_i = (E_i \cap F_0) \oplus \bigoplus_{\substack{F_j \subseteq E_i \\ j \geq 1}} F_j; \tag{4.49}$$

therefore, the Fubini theorem and (4.48) imply that

$$\int_{E_i} f_i = \left(\int_{E_i \cap F_0} h_{i0} \right) \cdot \prod_{\substack{F_j \subseteq E_i \\ j \geq 1}} \int_{F_j} h_{ij}. \tag{4.50}$$

On the other hand, using again $F_0 \oplus F_1 \oplus \ldots \oplus F_l = \mathbb{R}^n$, we deduce that if $x = \sum_{j=0}^{l} z_j$ where $z_j \in F_j$ for $j \geq 0$, then $z_j = P_{F_j} x$. It follows from (4.49) that for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \sup_{\substack{x = \sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i)^{c_i} &= \left(\sup_{\substack{P_{F_0} x = \sum_{i=1}^k c_i x_{0i}, \\ x_{0i} \in E_i \cap F_0}} \prod_{i=1}^k h_{i0}(x_{i0}) \right) \times \\ &\quad \times \prod_{j=1}^{l} \left(\sup_{\substack{P_{F_j} x = \sum_{F_j \subseteq E_i} c_i x_{ji}, \\ x_{ji} \in F_j}} \prod_{F_j \subseteq E_i} h_{ij}(x_{ji})^{c_i} \right), \end{aligned}$$

and hence

$$\begin{aligned} \int^*_{\mathbb{R}^n} \sup_{\substack{x = \sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i)^{c_i} \, dx &= \left(\int^*_{F_0} \sup_{\substack{x = \sum_{i=1}^k c_i x_i, \\ x_i \in E_i \cap F_0}} \prod_{i=1}^k h_{i0}(x_i) \, dx \right) \times \\ &\quad \times \prod_{j=1}^{l} \left(\int^*_{F_j} \sup_{\substack{x = \sum_{F_j \subseteq E_i} c_i x_i, \\ x_i \in F_j}} \prod_{F_j \subseteq E_i} h_{ij}(x_i)^{c_i} \, dx \right). \end{aligned} \tag{4.51}$$

As F_0 is a critical subspace, we have

$$\sum_{i=1}^k c_i P_{E_i \cap F_0} = \mathrm{Id}_{F_0},$$

and hence the Reverse Brascamp-Lieb inequality (4.1) yields

$$\int^*_{F_0} \sup_{\substack{x = \sum_{i=1}^k c_i x_i, \\ x_i \in E_i \cap F_0}} \prod_{i=1}^k h_{i0}(x_i) \, dx \geq \prod_{i=1}^k \left(\int_{E_i \cap F_0} h_{i0} \right)^{c_i}. \tag{4.52}$$

We deduce from (4.47) and the Prékopa-Leindler inequality (4.46) that if $j = 1,\dots,l$, then

$$\int^*_{F_j} \sup_{\substack{x = \sum_{F_j \subseteq E_i} c_i x_i, \\ x_i \in F_j}} \prod_{F_j \subseteq E_i} h_{ij}(x_i)^{c_i} \, dx \geq \prod_{E_i \supset F_j} \left(\int_{F_j} h_{ij} \right)^{c_i}. \tag{4.53}$$

Combining (4.50), (4.51), (4.52) and (4.55) with the fact that $f_1, \ldots, f_k$ are extremizers for the Reverse Brascamp-Lieb inequality (4.1) implies that if $j = 1, \ldots, l$, then

$$\int_{F_0}^{*} \sup_{\substack{x=\sum_{i=1}^k c_i x_i,\\ x_i \in E_i \cap F_0}} \prod_{i=1}^{k} h_{i0}(x_i)\, dx = \prod_{i=1}^{k} \left(\int_{E_i \cap F_0} h_{i0} \right)^{c_i} \tag{4.54}$$

$$\int_{F_j}^{*} \sup_{\substack{x=\sum_{F_j \subseteq E_i} c_i x_i,\\ x_i \in F_j}} \prod_{F_j \subseteq E_i} h_{ij}(x_i)^{c_i}\, dx = \prod_{E_i \supset F_j} \left(\int_{F_j} h_{ij} \right)^{c_i}. \tag{4.55}$$

We observe that (4.54) is just (i). In addition, (ii) follows from the equality conditions in the Prékopa-Leindler inequality (see Theorem 4.4.3). □

4.5 Closure properties of extremisers

Given a Geometric data $(P_{E_i}, c_i)_{i=1}^k$, we say that the non-negative integrable functions $f_1, \ldots, f_k$ with positive integrals are extremizers if equality holds in (4.1). In order to deal with positive smooth functions, we use convolutions.

4.5.1 Convolution

The following Lemma 4 is due to F. Barthe [15]. Since, we could not find a proof we also provide it.

Lemma 4.5.1. *Given a Geometric data $(P_{E_i}, c_i)_{i=1}^k$, if $f_1, \ldots, f_k$ and $g_1, \ldots, g_k$ are extremizers in the Reverse Brascamp-Lieb inequality* (4.1)*, then $f_1 * g_1, \ldots, f_k * g_k$ are also are extremizers.*

Proof. We define

$$\begin{aligned} F(x) &= \sup_{x=\sum_{i=1}^k c_i x_i,\, x_i \in E_i} \prod_{i=1}^{k} f_i(x_i)^{c_i} \\ G(y) &= \sup_{y=\sum_{i=1}^k c_i y_i,\, y_i \in E_i} \prod_{i=1}^{k} g_i(y_i)^{c_i}. \end{aligned}$$

Possibly F and G are not measurable but as $f_1, \ldots, f_k$ and $g_1, \ldots, g_k$ are extremizers, there exist measurable $\widetilde{F} \geq F$ and $\widetilde{G} \geq G$ such that $\int_{\mathbb{R}^n} \widetilde{F}(x)\, dx = \int_{\mathbb{R}^n} \widetilde{G}(x)\, dx = 1$. We deduce that

$$\begin{aligned} \int_{\mathbb{R}^n} \widetilde{F} * \widetilde{G}(x) dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widetilde{F}(x-y) \widetilde{G}(y) dy dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widetilde{F}(x-y) \widetilde{G}(y)\, dx dy \\ &= \int_{\mathbb{R}^n} \widetilde{G}(y) \left(\int_{\mathbb{R}^n} \widetilde{F}(x-y) dx \right) dy = \int_{\mathbb{R}^n} \widetilde{G}(y) \cdot 1\, dy = 1. \end{aligned} \tag{4.56}$$

We deduce writing $x_i = z_i + y_i$ in (4.57) for $i = 1, \ldots, k$ and using the Reverse Brascamp-Lieb inequality in (4.58) that

$$
\begin{aligned}
1 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widetilde{F}(x-y)\widetilde{G}(y)\,dydx \\
&\geq \int^*_{\mathbb{R}^n}\int^*_{\mathbb{R}^n} \sup_{x-y=\sum_{i=1}^k c_iz_i,\, z_i\in E_i} \prod_{i=1}^k f_i(z_i)^{c_i} \sup_{y=\sum_{i=1}^k c_iy_i,\, y_i\in E_i} \prod_{i=1}^k g_i(y_i)^{c_i}\,dydx \\
&= \int^*_{\mathbb{R}^n}\int^*_{\mathbb{R}^n} \sup_{x-y=\sum_{i=1}^k c_iz_i,\, z_i\in E_i} \sup_{y=\sum_{i=1}^k c_iy_i,\, y_i\in E_i} \prod_{i=1}^k f_i(z_i)^{c_i} \prod_{i=1}^k g_i(y_i)^{c_i}\,dydx \qquad (4.57) \\
&= \int^*_{\mathbb{R}^n}\int^*_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_ix_i,\, x_i\in E_i} \sup_{y=\sum_{i=1}^k c_iy_i,\, y_i\in E_i} \prod_{i=1}^k f_i(x_i-y_i)^{c_i} \prod_{i=1}^k g_i(y_i)^{c_i}\,dydx \\
&\geq \int^*_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_ix_i,\, x_i\in E_i} \int^*_{\mathbb{R}^n} \sup_{y=\sum_{i=1}^k c_iy_i,\, y_i\in E_i} \prod_{i=1}^k \big(f_i(x_i-y_i)g_i(y_i)\big)^{c_i}\,dydx \qquad (4.58) \\
&\geq \int^*_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_ix_i,\, x_i\in E_i} \prod_{i=1}^k \left(\int_{E_i} f_i(x_i-y_i)g_i(y_i)\,dy_i\right)^{c_i} dx \\
&= \int^*_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_ix_i,\, x_i\in E_i} \prod_{i=1}^k \big(f_i * g_i(x_i)\big)^{c_i} dx
\end{aligned}
$$

Since for $i=1,\dots,k$, $\int_{E_i} f_i*g_i = 1$ can be proved similarly to (4.56), we conclude that f_i*g_i, $i=1,\dots,k$, is also an extremizer. □

4.5.2 Product

Since in certain case we want to work with Lebesgue integral instead of outer integrals, we use the following statement that can be proved via compactness argument.

Lemma 4.5.2. *Given a Geometric data $(P_{E_i},c_i)_{i=1}^k$, if h_i is a positive continuous functions satisfying $\lim_{x\to\infty} h_i(x)=0$ for $i=1,\dots,k$, then the function*

$$h(x) = \sup_{\substack{x=\sum_{i=1}^k c_ix_i,\\ x_i\in E_i}} \prod_{i=1}^k h_i(x_i)^{c_i}$$

of $x\in\mathbb{R}^n$ is continuous.

Next we show that the product of a shift of a smooth extremizer and a Gaussian is also an extremizer for the Geometric Reverse Brascamp-Lieb inequality (4.1).

Lemma 4.5.3. *Given a Geometric data $(P_{E_i},c_i)_{i=1}^k$, if $f_1,\dots,f_k$ are positive, bounded, C^1 and extremizers in the Reverse Brascamp-Lieb inequality* (2.35)*, $g_i(x)=e^{-\pi\|x\|^2}$ for $x\in E_i$, then there exist $z_i\in E_i$, $i=1,\dots,k$, such that the functions $y\mapsto f_i(y-z_i)g_i(y)$ of $y\in E_i$, $i=1,\dots,k$, are also extremizers in the Reverse Brascamp-Lieb inequality* (2.35).

Proof. We may assume that $f_1,\dots,f_k$ are probability densities.

Readily the functions $\tilde f_1,\dots,\tilde f_k$ defined by $\tilde f_i(y)=f_i(-y)$ for $y\in E_i$ and $i=1,\dots,k$ are also extremizers. We deduce from Lemma 4.5.1 that the functions $\tilde f_i * g_i$ for $i=1,\dots,k$ are also extremizers where each $\tilde f_i * g_i$ is a probability density on E_i. According to Theorem 4.4.1, if $i=1,\dots,k$, then there exists a C^2 Brenier map $S_i: E_i\to E_i$ such that

$$g_i(x) = \det\nabla S_i(x)\cdot(\tilde f_i * g_i)(S_i(x)) \quad \text{for all } x\in E_i, \qquad (4.59)$$

and $\nabla S_i(x)$ is a positive definite symmetric matrix for all $x \in E_i$. As in the proof of Theorem 2.2.3 above, we consider the C^2 diffeomorphism $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Theta(y) = \sum_{i=1}^{k} c_i S_i \left(P_{E_i} y\right), \qquad y \in \mathbb{R}^n.$$

whose positive definite differential is

$$\nabla\Theta(y) = \sum_{i=1}^{k} c_i \nabla S_i \left(P_{E_i} y\right).$$

On the one hand, we note that if $x = \sum_{i=1}^{k} c_i x_i$ for $x_i \in E_i$, then

$$\|x\|^2 \le \sum_{i=1}^{k} c_i \|x_i\|^2$$

holds according to Barthe [15]; or equivalently,

$$\prod_{i=1}^{k} g_i(x_i)^{c_i} \le e^{-\pi\|x\|^2}.$$

Since f_i is positive, bounded, continuous and in $L_1(E_i)$ for $i = 1, \dots, k$, we observe that the function

$$z \mapsto \int_{\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^{k} c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^{k} f_i \left(x_i - S_i(P_{E_i}\Theta^{-1} z)\right)^{c_i} g_i(x_i)^{c_i}\, dx \tag{4.60}$$

of $z \in \mathbb{R}^n$ is continuous.

Using also that $\tilde{f}_1, \dots, \tilde{f}_k$ are extremizers and probability density functions, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}^{*} \int_{\mathbb{R}^n}^{*} \sup_{\substack{z=\sum_{i=1}^{k} c_i z_i, \\ z_i \in E_i}} \sup_{\substack{x=\sum_{i=1}^{k} c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^{k} f_i(x_i - z_i)^{c_i} g_i(x_i)^{c_i}\, dx\, dz \\
&= \int_{\mathbb{R}^n}^{*} \int_{\mathbb{R}^n}^{*} \sup_{\substack{x=\sum_{i=1}^{k} c_i x_i, \\ x_i \in E_i}} \left(\prod_{i=1}^{k} g_i(x_i)^{c_i}\right) \sup_{\substack{z=\sum_{i=1}^{k} c_i z_i, \\ z_i \in E_i}} \prod_{i=1}^{k} f_i(x_i - z_i)^{c_i}\, dz\, dx \\
&\le \int_{\mathbb{R}^n}^{*} e^{-\pi\|x\|^2} \int_{\mathbb{R}^n}^{*} \sup_{\substack{x=\sum_{i=1}^{k} c_i x_i, \\ x_i \in E_i}} \sup_{\substack{z=\sum_{i=1}^{k} c_i z_i, \\ z_i \in E_i}} \prod_{i=1}^{k} f_i(x_i - z_i)^{c_i}\, dz\, dx \\
&= \int_{\mathbb{R}^n}^{*} e^{-\pi\|x\|^2} \int_{\mathbb{R}^n}^{*} \sup_{\substack{z-x=\sum_{i=1}^{k} c_i y_i, \\ y_i \in E_i}} \prod_{i=1}^{k} \tilde{f}_i(y_i)^{c_i}\, dz\, dx \\
&= \int_{\mathbb{R}^n}^{*} e^{-\pi\|x\|^2} \int_{\mathbb{R}^n}^{*} \sup_{\substack{w=\sum_{i=1}^{k} c_i y_i, \\ y_i \in E_i}} \prod_{i=1}^{k} \tilde{f}_i(y_i)^{c_i}\, dw\, dx \\
&= \int_{\mathbb{R}^n} e^{-\pi\|x\|^2}\, dx = 1.
\end{aligned}$$

Using Lemma 4.5.2 and (4.60) in (4.61), the Reverse Brascamp-Lieb inequality (4.1) in (4.62) and

Porposition 4.2.10 in (4.63), we deduce that

$$
\begin{aligned}
1 &\geq \int^*_{\mathbb{R}^n} \int^*_{\mathbb{R}^n} \sup_{\substack{z=\sum_{i=1}^k c_i z_i \\ z_i \in E_i}} \sup_{\substack{x=\sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i - z_i)^{c_i} g_i(x_i)^{c_i}\, dx\, dz \\
&\geq \int^*_{\mathbb{R}^n} \int^*_{\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k f_i\left(x_i - S_i(P_{E_i}\Theta^{-1}z)\right)^{c_i} g_i(x_i)^{c_i}\, dx\, dz \qquad (4.61) \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^k f_i\left(x_i - S_i(P_{E_i}\Theta^{-1}z)\right)^{c_i} g_i(x_i)^{c_i}\, dx\, dz \qquad (4.62) \\
&\geq \int_{\mathbb{R}^n} \prod_{i=1}^k \left(\int_{E_i} f_i\left(x_i - S_i(P_{E_i}\Theta^{-1}z)\right) g_i(x_i)\, dx_i \right)^{c_i} dz \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^k (\tilde{f}_i * g_i)\left(S_i(P_{E_i}\Theta^{-1}z)\right)^{c_i} dz \\
&= \int_{\mathbb{R}^n} \left(\prod_{i=1}^k (\tilde{f}_i * g_i)\left(S_i\left(P_{E_i}y\right)\right)^{c_i} \right) \det\left(\nabla\Theta(y)\right)\, dy \\
&= \int_{\mathbb{R}^n} \left(\prod_{i=1}^k (\tilde{f}_i * g_i)\left(S_i\left(P_{E_i}y\right)\right)^{c_i} \right) \det\left(\sum_{i=1}^k c_i \nabla S_i\left(P_{E_i}y\right) \right) dy \qquad (4.63) \\
&\geq \int_{\mathbb{R}^n} \left(\prod_{i=1}^k (\tilde{f}_i * g_i)\left(S_i\left(P_{E_i}y\right)\right)^{c_i} \right) \prod_{i=1}^k \left(\det \nabla S_i\left(P_{E_i}y\right)\right)^{c_i} dy \\
&= \int_{\mathbb{R}^n} \left(\prod_{i=1}^k g_i\left(P_{E_i}y\right)^{c_i} \right) dy = \int_{\mathbb{R}^n} e^{-\pi\|y\|^2}\, dy = 1.
\end{aligned}
$$

In particular, we conclude that

$$
\begin{aligned}
1 \quad &\geq \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^k f_i\left(x_i - S_i(P_{E_i}\Theta^{-1}z)\right)^{c_i} g_i(x_i)^{c_i}\, dx\, dz \\
&\geq \quad \int_{\mathbb{R}^n} \prod_{i=1}^k \left(\int_{E_i} f_i\left(x_i - S_i(P_{E_i}\Theta^{-1}z)\right) g_i(x_i)\, dx_i \right)^{c_i} dz \geq 1.
\end{aligned}
$$

Because of the Reverse Brascamp-Lieb inequality (4.1), it follows from (4.60) that

$$
\int_{\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^k f_i\left(x_i - S_i(P_{E_i}\Theta^{-1}z)\right)^{c_i} g_i(x_i)^{c_i}\, dx = \prod_{i=1}^k \left(\int_{E_i} f_i\left(x_i - S_i(P_{E_i}\Theta^{-1}z)\right) g_i(x_i)\, dx_i \right)^{c_i}
$$

for any $z \in \mathbb{R}^n$; therefore, we may choose $z_i = S_i(o)$ for $i = 1, \ldots, k$ in Lemma 4.5.3. $\square$

4.6 Working on dependent subspace

4.6.1 Polynomial growth and Fourier transform

For positive C^α probability density functions f and g on $\mathbb{R}^n$ for $\alpha \in (0, 1)$, the C^1 Brenier map $T : \mathbb{R}^n \to \mathbb{R}^n$ in Theorem 4.4.1 pushing forward the the measure on $\mathbb{R}^n$ induced by g to the measure associated to

f satisfies that ∇T is positive definite. We deduce that

$$\langle T(y)-T(x), y-x\rangle = \int_0^1 \langle \nabla T(x+t(y-x))\cdot(y-x), y-x\rangle\, dt \geq 0 \quad \text{for any } x,y\in\mathbb{R}^n. \tag{4.64}$$

We say that a continuous function $T:\mathbb{R}^n\to\mathbb{R}^m$ has linear growth if there exists a positive constant $c>0$ such that

$$\|T(x)\|\leq c\sqrt{1+\|x\|^2}$$

for $x\in\mathbb{R}^n$. It is equivalent saying that

$$\limsup_{\|x\|\to\infty}\frac{\|T(x)\|}{\|x\|}<\infty. \tag{4.65}$$

In general, T has polynomial growth, if there exists $k\geq 1$ such that

$$\limsup_{\|x\|\to\infty}\frac{\|T(x)\|}{\|x\|^k}<\infty.$$

Proposition 4.6.3 shows that if the whole space is the dependent subspace and the Brenier maps corresponding to the extremizers $f_1,\dots,f_k$ in Proposition 4.4.2 have at most linear growth, then each f_i is actually Gaussian. The proof of Proposition 4.6.3 uses classical Fourier analysis, and we refer to Grafakos [80] for the main properties. For our purposes, we need only the action of a tempered distribution on the space of $C_0^\infty(\mathbb{R}^m)$ of C^∞ functions with compact support, do not need to consider the space of Schwarz functions in general. We recall that if u is a tempered distribution on Schwarz functions on $\mathbb{R}^n$, then the support $\operatorname{supp} u$ is the intersection of all closed sets K such that if $\varphi\in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi\subseteq\mathbb{R}^n\backslash K$, then $\langle u,\varphi\rangle=0$. We write $\hat{u}$ to denote the Fourier transform of a u. In particular, if θ is a function of polynomial growth and $\varphi\in C_0^\infty(\mathbb{R}^n)$, then

$$\langle\hat{\theta},\varphi\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\theta(x)\varphi(y)e^{-2\pi i\langle x,y\rangle}\,dxdy. \tag{4.66}$$

We consider the two well-known statements Lemma 4.6.1 and Lemma 4.6.2 about the support of a Fourier transform to prepare the proof of Proposition 4.6.3.

Lemma 4.6.1. *If θ is a measurable function of polynomial growth on $\mathbb{R}^n$, and there exist linear subspace E with $1\leq\dim E\leq n-1$ and function ω on E such that $\theta(x)=\omega(P_E x)$, then* $\operatorname{supp}\hat{\theta}\subseteq E$.

Proof. We write a $z\in\mathbb{R}^n$ in the form $z=(z_1,z_2)$ with $z_1\in E$ and $z_2\in E^\perp$. Let $\varphi\in C_0^\infty(\mathbb{R}^n)$ satisfy that $\operatorname{supp}\varphi\subseteq\mathbb{R}^n\backslash E$, and hence $\varphi(x_1,o)=0$ for $x_1\in E$, and the Fourier Integral Theorem in $E^\perp$ implies

$$\varphi(x_1,z)=\int_{E^\perp}\int_{E^\perp}\varphi(x_1,x_2)e^{2\pi i\langle z-x_2,y_2\rangle}\,dx_2dy_2$$

for $x_1\in E$ and $z\in E^\perp$. It follows from (4.66) that

$$\begin{aligned}
\langle\hat{\theta},\varphi\rangle &= \int_{E^\perp}\int_E\int_{E^\perp}\int_E\omega(x_1)\varphi(x_1,x_2)e^{-2\pi i\langle x_1,y_1\rangle}e^{-2\pi i\langle x_2,y_2\rangle}\,dx_1dx_2dy_1dy_2\\
&= \int_E\int_E\omega(x_1)e^{-2\pi i\langle x_1,y_1\rangle}\left(\int_{E^\perp}\int_{E^\perp}\varphi(x_1,x_2)e^{2\pi i\langle -x_2,y_2\rangle}\,dx_2dy_2\right)dy_1dx_1\\
&= \int_E\int_E\omega(x_1)e^{-2\pi i\langle x_1,y_1\rangle}\varphi(x_1,0)\,dy_1dx_1=0.
\end{aligned}$$

□

Next, Lemma 4.6.2 directly follows from Proposition 2.4.1 in Grafakos [80].

Lemma 4.6.2. *If θ is a continuous function of polynomial growth on $\mathbb{R}^n$ and* $\operatorname{supp}\hat{\theta}\subseteq\{o\}$, *then θ is a polynomial.*

4.6.2 h_{i0} is Gaussian in Proposition 4.4.2 under linear growth

Proposition 4.6.3. *For a Geometric data $(P_{E_i}, c_i)_{i=1}^k$, we assume that*

$$\cap_{i=1}^k (E_i \cup E_i^\perp) = \{o\}. \tag{4.67}$$

Let $g_i(x) = e^{-\pi\|x\|^2}$ for $i = 1, \ldots, k$ and $x \in E_i$, let equality hold in (2.35) for positive C^1 probability densities f_i on E_i, $i = 1, \ldots, k$, and let $T_i : E_i \to E_i$ be the C^2 Brenier map satisfying

$$g_i(x) = \det \nabla T_i(x) \cdot f_i(T_i(x)) \quad \text{for all } x \in E_i. \tag{4.68}$$

If each T_i, $i = 1, \ldots, k$, has linear growth, then there exist a positive definite matrix $A : \mathbb{R}^m \to \mathbb{R}^m$ whose eigenspaces are critical subspaces, and $a_i > 0$ and $b_i \in E_i$, $i = 1, \ldots, k$, such that

$$f_i(x) = a_i e^{-\langle Ax, x+b_i\rangle} \quad \text{for } x \in E_i.$$

Proof. We may assume that each linear subspace is non-zero.

We note that the condition (4.67) is equivalent saying that $\mathbb{R}^m$ itself is the dependent subspace with respect to the Brascamp-Lieb data. We may assume that for some $1 \le l \le k$, we have $1 \le \dim E_i \le m-1$ if $i = 1, \ldots, l$, and still

$$\cap_{i=1}^l (E_i \cup E_i^\perp) = \{o\}. \tag{4.69}$$

We use the diffeomorphism $\Theta : \mathbb{R}^m \to \mathbb{R}^m$ of Proposition 4.4.2 defined by

$$\Theta(y) = \sum_{i=1}^k c_i T_i\left(P_{E_i} y\right), \qquad y \in \mathbb{R}^m.$$

It follows from (4.35) that we may asssume

$$T_i(o) = o \quad \text{for } i = 1, \ldots, k, \text{ and hence } \Theta(o) = o. \tag{4.70}$$

We claim that there exists a positive definite matrix $B : \mathbb{R}^m \to \mathbb{R}^m$ whose eigenspaces are critical subspaces, and

$$\nabla\Theta(y) = B \quad \text{for } y \in \mathbb{R}^m. \tag{4.71}$$

Let $\Theta(y) = (\theta_1(y), \ldots, \theta_m(y))$ for $y \in \mathbb{R}^m$ and $\theta_j \in C^2(\mathbb{R}^m)$, $j = 1, \ldots, m$. Since each T_i, $i = 1, \ldots, k$ has linear growth, it follows that Θ has linear growth, and in turn each θ_j, $j = 1, \ldots, m$, has linear growth.

According to Proposition 4.4.2 (iii), there exist C^2 functions $\Omega_i : E_i \to E_i$ and $\Gamma_i : E_i^\perp \to E_i^\perp$ such that

$$\Theta(y) = \Omega_i(P_{E_i} y) + \Gamma_i(P_{E_i^\perp} y)$$

for $i = 1, \ldots, k$ and $y \in \mathbb{R}^n$. We write $\Omega_i(x) = (\omega_{i1}(x), \ldots, \omega_{im}(x))$ and $\Gamma_i(x) = (\gamma_{i1}(x), \ldots, \gamma_{im}(x))$; therefore,

$$\theta_j(y) = \omega_{ij}(P_{E_i} y) + \gamma_{ij}(P_{E_i^\perp} y) \tag{4.72}$$

for $j = 1, \ldots, m$ and $i = 1, \ldots, k$.

Fix a $j \in \{1, \ldots, m\}$. It follows from Lemma 4.6.1 and (4.72) that

$$\operatorname{supp} \hat{\theta}_j \subseteq E_i \cup E_i^\perp$$

for $i = 1, \ldots, l$. Thus (4.69) yields that

$$\operatorname{supp} \hat{\theta}_j \subseteq \{o\},$$

and in turn we deduce from Lemma 4.6.2 that θ_j is a polynomial. Given that θ_j has linear growth, it follows that there exist $w_j \in \mathbb{R}^m$ and $\alpha_j \in \mathbb{R}$ such that $\theta_j(y) = \langle w_j, y\rangle + \alpha_j$. We deduce from $\theta_j(o) = 0$ (cf. (4.70)) that $\alpha_j = 0$.

The argument so far yields that there exists an $m\times m$ matrix B such that $\Theta(y)=By$ for $y\in\mathbb{R}^m$. As $\nabla\Theta(y)=B$ is positive definite and its eigenspaces are critical subspaces, we conclude the claim (4.71).

Since $\nabla T_i(P_{E_i}y)=\nabla\Theta(y)|_{E_i}$ for $i=1,\ldots,k$ and $y\in\mathbb{R}^m$ by Proposition 4.4.2 (iv), we deduce that $T_i^{-1}=B^{-1}|_{E_i}$ for $i=1,\ldots,k$. It follows from (4.68) that

$$f_i(x)=e^{-\pi\|B^{-1}x\|^2}\cdot\det\left(B^{-1}|_{E_i}\right)\quad\text{for } x\in E_i$$

for $i=1,\ldots,k$. Therefore, we can choose $A=\pi B^{-2}$. □

4.7 Form of extremizers

4.7.1 Linear growth at Brenier maps

Proposition 4.7.1 related to Caffarelli Contraction Principle in Caffarelli [47] was proved by Emanuel Milman, see for example Colombo, Fathi [58], De Philippis, Figalli [133], Fathi, Gozlan, Prod'homme [70], Y.-H. Kim, E. Milman [90], Klartag, Putterman [92], Kolesnikov [96], Livshyts [106] for relevant results.

Proposition 4.7.1 (Emanuel Milman)**.** *If a Gaussian probability density g and a positive C^α, $\alpha\in(0,1)$, probability density f on $\mathbb{R}^n$ satisfy $f\leq c\cdot g$ for some positive constant $c>0$, then the Brenier map $T:\mathbb{R}^n\to\mathbb{R}^n$ pushing forward the measure on $\mathbb{R}^n$ induced by g to the measure associated to f has linear growth.*

Proof. We may assume that $g(x)=e^{-\pi\|x\|^2}$.

We observe that $T:\mathbb{R}^n\to\mathbb{R}^n$ is bijective as both f and g are positive. Let S be the inverse of T; namely, $S:\mathbb{R}^n\to\mathbb{R}^n$ is the bijective Brenier map pushing forward the measure on $\mathbb{R}^n$ induced by f to the measure associated to g. In particular, any Borel $X\subseteq\mathbb{R}^n$ satisfies

$$\int_{S(X)}g=\int_X f. \tag{4.73}$$

We note that (4.65), and hence Proposition 4.7.1 is equivalent saying that

$$\liminf_{x\to\infty}\frac{\|S(x)\|}{\|x\|}>0. \tag{4.74}$$

The main idea of the argument is the following observation. For any unit vector u and $\theta\in(0,\pi)$, we consider

$$\Xi(u,\theta)=\{y:\langle y,u\rangle\geq\|y\|\cdot\cos\theta\}\,.$$

Since S is surjective, and $\langle S(z)-S(w),z-w\rangle\geq 0$ for any $z,w\in\mathbb{R}^n$ according to (4.64), we deduce that

$$S(w)+\Xi(u,\theta)\subseteq S\left(w+\Xi\left(u,\theta+\frac{\pi}{2}\right)\right) \tag{4.75}$$

for any $u\in S^{n-1}$ and $\theta\in(0,\frac{\pi}{2})$.

We suppose that T does not have linear growth, and seek a contradiction. According to (4.74), there exists a sequence $\{x_k\}$ of points of $\mathbb{R}^n\backslash\{o\}$ tending to infinity such that

$$\lim_{k\to\infty}\|x_k\|=\infty\quad\text{and}\quad\lim_{k\to\infty}\frac{\|S(x_k)\|}{\|x_k\|}=0.$$

In particular, we may assume that

$$\|S(x_k)\|<\frac{\|x_k\|}{8}. \tag{4.76}$$

For any k, we consider the unit vector $e_k = x_k/\|x_k\|$. We observe that $X_k = x_k + \Xi(e_k, \frac{3\pi}{4})$ avoids the interior of the ball $\frac{\|x_k\|}{\sqrt{2}} B^n$; therefore, if k is large, then

$$\int_{X_k} f \leq c \cdot n\kappa_n \int_{\|x_k\|/\sqrt{2}}^{\infty} r^{n-1} e^{-\pi r^2}\, dr < \int_{\|x_k\|/\sqrt{2}}^{\infty} e^{-2r^2}\sqrt{2} r\, dr = e^{-\|x_k\|^2} \tag{4.77}$$

On the other hand, $S(x_k) + \Xi(e_k, \frac{\pi}{4})$ contains the ball

$$\widetilde{B} = S(x_k) + \frac{x_k}{8} + \frac{\|x_k\|}{8\sqrt{2}}\, B^n \subseteq \frac{\|x_k\|}{2}\, B^n$$

where we have used (4.76). It follows form (4.73) and (4.75) that if k is large, then

$$\int_{X_k} f = \int_{S(X_k)} g \geq \int_{\widetilde{B}} g \geq \kappa_n \left(\frac{\|x_k\|}{8\sqrt{2}}\right)^n e^{-\pi(\|x_k\|/2)^2} > e^{-\|x_k\|^2}.$$

This inequality contradicts (4.77), and in turn proves (4.74). □

4.7.2 Proof of Theorem 4.1.1

Here we give the final proof of our main result.

Proof of Theorem 4.1.1. We may assume that each linear subspace E_i is non-zero in Theorem 4.1.1. Let f_i be a probability density on E_i in a way such that equality holds for $f_1, \ldots, f_k$ in (4.1). For $i = 1, \ldots, k$ and $x \in E_i$, let $g_i(x) = e^{-\pi\|x\|^2}$, and hence g_i is a probability distribution on E_i, and $g_1, \ldots, g_k$ are extremizers in the Reverse Brascamp-Lieb inequality (4.1).

It follows from Lemma 4.5.1 that the convolutions $f_1 * g_1, \ldots, f_k * g_k$ are also extremizers for (2.35). We observe that for $i = 1, \ldots, k$, $f_i * g_i$ is a bounded positive C^∞ probability density on E_i. Next we deduce from Lemma 4.5.3 that there exist $z_i \in E_i$ and $\gamma_i > 0$ for $i = 1, \ldots, k$ such that defining

$$\tilde{f}_i(x) = \gamma_i \cdot g_i(x) \cdot (f_i * g_i)(x - z_i) \quad \text{for } x \in E_i, \tag{4.78}$$

$\tilde{f}_1, \ldots, \tilde{f}_k$ are probability densities that are extremizers for (2.35). We note that if $i = 1, \ldots, k$, then $\tilde{f}_i$ is positive and C^∞, and there exists $c > 1$ satisfying

$$\tilde{f}_i \leq c \cdot g_i. \tag{4.79}$$

Let $\widetilde{T}_i : E_i \to E_i$ be the C^∞ Brenier map satisfying

$$g_i(x) = \det \nabla\widetilde{T}_i(x) \cdot \tilde{f}_i(\widetilde{T}_i(x)) \quad \text{for all } x \in E_i, \tag{4.80}$$

We deduce from (4.79) and Proposition 4.7.1 that $\widetilde{T}_i$ has linear growth.

For $i = 1, \ldots, k$ and $x \in F_0 \cap E_i$, let $g_{i0}(x) = e^{-\pi\|x\|^2}$. It follows from Proposition 4.4.2 (i) that for $i \in \{1, \ldots, k\}$, there exists positive C^1 integrable $h_{i0} : F_0 \cap E_i \to [0, \infty)$ (where $h_{i0}(o) = 1$ if $F_0 \cap E_i = \{o\}$), and for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ with $F_j \subseteq E_i$, there exists positive C^1 integrable $\tilde{h}_{ij} : F_j \to [0, \infty)$ such that

$$\tilde{f}_i(x) = \tilde{h}_{i0}(P_{F_0}x) \cdot \prod_{\substack{F_j \subseteq E_i \\ j \geq 1}} \tilde{h}_{ij}(P_{F_j}x) \qquad \text{for } x \in E_i.$$

We deduce from Proposition 4.4.2 (ii) that $\widetilde{T}_{i0} = \widetilde{T}_i|_{F_0 \cap E_i}$ is the Brenier map pushing forward the measure on $F_0 \cap E_i$ determined g_{i0} onto the measure determined by $\tilde{h}_{i0}$. Since $\widetilde{T}_i$ has linear growth, $\widetilde{T}_{i0}$ has linear growth, as well, for $i = 1, \ldots, k$.

We deduce from Proposition 4.4.4 (i) that $\sum_{i=1}^k c_i P_{E_i \cap F_0} = \mathrm{Id}_{F_0}$, the Geometric Brascamp Lieb data $E_1 \cap F_0, \dots, E_k \cap F_0$ in F_0 has no independent subspaces, and $\tilde{h}_{10}, \dots, \tilde{h}_{k0}$ are extremizers in the Reverse Brascamp-Lieb inequality for this data in F_0.

As $\widetilde{T}_{i0}$ has linear growth for $i = 1, \dots, k$, Proposition 4.6.3 yields the existence of a positive definite matrix $\widetilde{A} : F_0 \to F_0$ whose eigenspaces are critical subspaces, and $\tilde{a}_i > 0$ and $\tilde{b}_i \in F_0 \cap E_i$ for $i = 1, \dots, k$, such that

$$\tilde{f}_i(x) = \tilde{a}_i e^{-\langle \widetilde{A}x, x + \tilde{b}_i \rangle} \cdot \prod_{\substack{F_j \subseteq E_i \\ j \geq 1}} \tilde{h}_{ij}(P_{F_j} x) \quad \text{for } x \in E_i.$$

Dividing by g_i and shifting, we deduce that there exist a symmetric matrix $\bar{A} : F_0 \to F_0$ whose eigenspaces are critical subspaces, and $\bar{a}_i > 0$ and $\bar{b}_i \in F_0 \cap E_i$ for $i = 1, \dots, k$, and for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$ with $F_j \subseteq E_i$, there exists positive C^1 function $\bar{h}_{ij} : F_j \to [0, \infty)$ such that

$$f_i * g_i(x) = \bar{a}_i e^{-\langle \bar{A}x, x + \bar{b}_i \rangle} \cdot \prod_{\substack{F_j \subseteq E_i \\ j \geq 1}} \bar{h}_{ij}(P_{F_j} x) \quad \text{for } x \in E_i.$$

Since $f_i * g_i$ is a probability density on E_i, it follows that $\bar{A}$ is positive definite and $\bar{h}_{ij} \in L_1(E_i \cap F_j)$ for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$ with $F_j \subseteq E_i$.

For any $i = 1, \dots, k$, we write $\hat{\varrho}$ for the Fourier transform of a function on E_i. For $i = 1, \dots, k$, using that $\widehat{f_i * g_i} = \hat{f}_i \cdot \hat{g}_i$ and the inverse Fourier transform, we conclude that there exist a symmetric matrix $A : F_0 \to F_0$ whose eigenspaces are critical subspaces, and $a_i > 0$ and $b_i \in F_0 \cap E_i$ for $i = 1, \dots, k$, and for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$ with $F_j \subseteq E_i$, there exists $h_{ij} : F_j \to [0, \infty)$ such that

$$f_i(x) = a_i e^{-\langle Ax, x + b_i \rangle} \cdot \prod_{\substack{F_j \subseteq E_i \\ j \geq 1}} h_{ij}(P_{F_j} x) \quad \text{for } x \in E_i. \tag{4.81}$$

Since f_i is a probability density on E_i, it follows that A is positive definite and each h_{ij} is non-negative and integrable. Finally, Proposition 4.4.4 (ii) yields that there exist log-concave integrable $\psi_j : F_j \to [0, \infty)$ for $j = 1, \dots, l$, and there exist $a_{ij} > 0$ and $b_{ij} \in F_j$ for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$ with $F_j \subseteq E_i$ such that $h_{ij}(x) = a_{ij} \cdot \psi_j(x - b_{ij})$ for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$ with $F_j \subseteq E_i$.

Now, we assume that $f_1, \dots, f_k$ are of the form as described in (4.2) and equality holds for all $x \in E_i$ in (4.2). According to (4.35), we may assume that there exist a positive definite matrix $\Phi : F_0 \to F_0$ whose proper eigenspaces are critical subspaces and a $\tilde{\theta}_i > 0$ for $i = 1, \dots, k$ such that

$$f_i(x) = \tilde{\theta}_i e^{-\|\Phi P_{F_0} x\|^2} \prod_{F_j \subseteq E_i} h_j(P_{F_j}(x)) \quad \text{for } x \in E_i. \tag{4.82}$$

We recall that according to (4.47), if $j \in \{1, \dots, l\}$, then

$$\sum_{E_i \supset F_j} c_i = 1. \tag{4.83}$$

We set $\theta = \prod_{i=1}^k \tilde{\theta}_i^{c_i}$ and $h_0(x) = e^{-\|\Phi x\|^2}$ for $x \in F_0$. On the left hand side of the Reverse Brascamp-Lieb inequality (2.35), we use first (4.83) and the log-concavity of h_j, $j = 1, \dots, l$, secondly Proposition 4.3.2, thirdly (4.83), fourth the Fubini Theorem, and finally (4.83) again to prove that

$$\begin{aligned}
\int^*_{\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k c_i x_i \\ x_i\in E_i}} \prod_{i=1}^k f_i(x_i)^{c_i}\,dx &= \theta \int^*_{\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k \sum_{j=0}^l c_i x_{ij} \\ x_{ij}\in E_i\cap F_j}} \prod_{j=0}^l \prod_{i=1}^k h_j(x_{ij})^{c_i}\,dx \\
&= \theta \int^*_{\mathbb{R}^n} \prod_{j=0}^l \sup_{\substack{P_{F_j} x=\sum_{i=1}^k c_i x_{ij} \\ x_{ij}\in E_i\cap F_j}} \prod_{i=1}^k h_j(x_{ij})^{c_i}\,dx \\
&= \theta \int^*_{\mathbb{R}^n} \left(\sup_{\substack{P_{F_0} x=\sum_{i=1}^k c_i x_{i0} \\ x_{i0}\in E_i\cap F_0}} \prod_{i=1}^k e^{-c_i\|\Phi x_{i0}\|^2} \right) \times \prod_{j=1}^l h_j(P_{F_j}x)\,dx \\
&= \theta \left(\prod_{i=1}^k \left(\int_{F_0\cap E_i} e^{-\|\Phi y\|^2}\,dy \right)^{c_i} \right) \times \prod_{j=1}^l \int_{F_j} h_j \\
&= \prod_{i=1}^k \left(\int_{E_i} f_i \right)^{c_i},
\end{aligned}$$

completing the proof of Theorem 4.1.1. $\square$

Chapter 5

On a j-forms of polarity

For the statements of this section, Φ-polarity condition defined in 2.3.5, while $\mathcal{S}_j$ and $\mathcal{E}_j$ defined in (2.46) and (2.48), respectively. Until the end of this thesis, an inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$ is fixed and an orthonormal basis $\{e_m\}_{m=1}^n$ with respect to this inner product. Moreover, for a vector $\widetilde{x} = (x_1, \dots, x_{n-1}) \in e_n^\perp$ and a real number r, the pair $(\widetilde{x}, r)$ will always denote the vector $x_1 e_1 + \ldots + x_{n-1} e_{n-1} + r e_n$.

5.1 On a j-Santaló Conjecture

5.1.1 Introduction

Let us restate our main results.

Theorem 5.1.1 (K.,Saroglou [89]). *Let $1 \leq j \leq k$ be two integers, where $k \geq 2$. Let $K_1, \dots, K_k$ be symmetric convex bodies in $\mathbb{R}^n$, satisfying $\mathcal{E}_j$-polarity condition. Assume that one of the following holds:*

(i) $K_1, \dots, K_k$ are unconditional convex bodies.

(ii) $j = 1$ or $j = k$.

(iii) j is even and $K_3, \dots, K_k$ are unconditional convex bodies.

Then,

$$\prod_{i=1}^k |K_i| \leq |B_j^n|^k. \tag{5.1}$$

Moreover, in all three cases, (5.1) *is sharp for $K_1 = K_2 = \ldots = K_k = B_j^n$*

We also obtain the corresponding functional form of Theorem 5.1.1.

Theorem 5.1.2. *Let $1 \leq j \leq k$ be two integers, where $k \geq 2$. Let $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}_+$ be even integrable functions, satisfying $\mathcal{S}_j$-polarity condition with respect to some decreasing function $\rho : \mathbb{R} \to [0, \infty]$. Assume that one of the following holds.*

(i) $f_1, \dots, f_k$ are unconditional functions.

(ii) $j = 1$ or $j = k$.

(iii) j is even and $f_3, \dots, f_k$ are unconditional functions.

Then,

$$\prod_{i=1}^k \int_{\mathbb{R}^n} f_i(x_i)\, dx_i \leq \left(\int_{\mathbb{R}^n} \rho \left(\binom{k}{j} \|u\|_j^j \right)^{1/k} du \right)^k. \tag{5.2}$$

Outline of the proof of Theorems 5.1.1 and 5.1.2

Due to (2.51), Theorem 2.3.9 implies immediately Theorem 2.3.8. However, for parts (ii) and (iii), our approach requires to first establish Theorem 2.3.8 and then deduce Theorem 2.3.9 from it. The proof for both Theorems organized as follows:

- In Section 5.1.2, we establish the unconditional case. In particular, Theorem 5.1.2 (i) is a consequence of the Prékpa-Leindler inequality. Then Theorem 5.1.1 (i) is an immediately corollary.

- In Section 5.1.3, we deal with the case $j=1$ of Theorem 5.1.1, that follows from the fact: if $K_1,\dots,K_k$ satisfy the $\mathcal{E}_1$-polarity condition then $\frac{K_1+K_2}{2}, \frac{K_1+K_2}{2}, K_3,\dots,K_k$ satisfy the $\mathcal{E}_1$-polarity condition, as well.

- In Section 5.1.4, we establish cases (ii) for $j=k$ and (iii) of Theorem 5.1.1. We use a symmetrization argument, similar to the ones used by Meyer-Pajor [119].

- In Section 5.1.5, w show that Conjectures 5.1.1 and 5.1.2 are equivalent. Briefly, for the non trivial direction, the level sets

  $$K_i(r_i) := \{x_i \in \mathbb{R}^n : f_i(x_i) \geq r_i\} \qquad r_i \geq 0, \quad i=1,\dots,k$$

  satisfy the $\mathcal{E}_j$-polarity condition and after applying the hypothesis the multiplicative Prékpa-Leindler inequality with respect to the r_i's finishes the proof.

5.1.2 The unconditional case

A function $f:\mathbb{R}^n \to \mathbb{R}$ is said to be unconditional, if $f(\delta_1 x_1,\dots,\delta_n x_n) = f(x_1,\dots,x_n)$, for any $\delta_1,\dots,\delta_n \in \{-1,1\}$ and any $(x_1,\dots,x_n)\in\mathbb{R}^n$. A subseteq A in $\mathbb{R}^n$ is unconditional if its indicator function 1_A is unconditional. In this section, we establish Conjectures 5.1.1 and 5.1.2 for sets contained in an orthant (resp. functions supported in an orthant). Since we wish to obtain slightly more general results, we need to modify the definition of the functions $\mathcal{S}_j$ and $\mathcal{E}_j$ introduced previously. Namely, for any two integers $1 \leq j \leq k$ and any positive real number $p>0$, set

$$s_{j,p}(r_1,\dots,r_k) = \sum_{1\leq i_1<\dots<i_j\leq k} |r_{i_1}|^p \dots |r_{i_j}|^p, \qquad r_1,\dots,r_k \in \mathbb{R},$$

$$S_{j,p}(x_1,\dots,x_k) := \sum_{l=1}^{n} s_{j,p}(x_1(l),\dots,x_k(l)), \qquad x_1,\dots,x_k\in\mathbb{R}^n$$

and

$$\mathcal{E}_{j,p} := \frac{S_{j,p}}{\binom{k}{j}}.$$

Recall that $x_i(l) := \langle x_i, e_l\rangle$, $i=1,\dots,k$, $l=1,\dots,n$.

We refer to Borell [28], Ball [10] [11] , Uhrin [150] for the following version of the classical Prékopa-Leindler inequality (see also [135], [102]).

Theorem 5.1.3. *(1-dimensional multiplicative Prékopa-Leindler inequality) If some integrable functions $h, h_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i=1,\dots,k$, satisfy, for any $t_i>0$, $i=1\dots,k$ that*

$$\prod_{i=1}^{k} h_i(t_i)^{\frac{1}{k}} \leq h\left(\prod_{i=1}^{k} t_i^{\frac{1}{k}}\right),$$

then it holds

$$\prod_{i=1}^{k}\left(\int_{\mathbb{R}^n} h_i(t_i)\,dx_i\right)^{\frac{1}{k}} \leq \int_{\mathbb{R}^n} h(t)\,dt.$$

The result below slightly generalizes Theorem 2.3.4 and will follow by a more or less standard application of the Prékopa-Leindler inequality (Theorem 5.1.3), which uses the inductive argument of K. Ball [9, 10].

Proposition 5.1.4. *Let $p > 0$, $q > -1$ be real numbers and $1 \le j \le k$ be two integers, where $k \ge 2$. For any integrable functions $f_i : \mathbb{R}^n_+ \to \mathbb{R}_+$, $i = 1, \ldots, k$, satisfying $S_{j,p}$-polarity condition with respect to some decreasing function $\rho : \mathbb{R} \to [0, \infty]$, and any $m \in \{1, \ldots, n\}$, it holds*

$$\prod_{i=1}^{k} \int_{\mathbb{R}^n_+} |\langle x_i, e_m\rangle|^q f_i(x_i)\, dx_i \le \left(\int_{\mathbb{R}^n_+} |\langle u, e_1\rangle|^q \rho\left(\binom{k}{j} \|u\|_{jp}^{jp} \right)^{\frac{1}{k}} du \right)^{k}. \tag{5.3}$$

Proof. We may assume that $m = n$. We will prove Proposition 5.1.4 by induction in the dimension n. Since we want to deduce the base case $n = 1$ simultaneously with the inductive step, it is useful to make some conventions: For a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, we set $\varphi(0, r) := \varphi(r)$, $\int_{\mathbb{R}^0_+} \varphi(x)\, dx := \varphi(0)$ and $\int_{\mathbb{R}^0_+} \varphi(x, r)\, dx := \varphi(0, r) = \varphi(r)$, $r \ge 0$. Assume that (5.3) holds for the non-negative integer $n - 1$, where $n \ge 2$. In the inductive step (resp. the case $n = 1$), notice that for $(\widetilde{x}_i, r_i) \in \mathbb{R}^n_+$ (resp. $r_i \in \mathbb{R}_+$), $i = 1, \ldots, k$, the $S_{j,p}$-polarity condition together with Maclaurin's inequality (stating that $\mathcal{E}_{j_1}^{1/j_1} \ge \mathcal{E}_{j_2}^{1/j_2}$, if $j_1 \le j_2$) and the monotonicity of ρ imply

$$\prod_{i=1}^{k} f_i(\tilde{x}_i, r_i) \le \rho\left(s_{j,p}(r_1, \ldots, r_k) + S_{j,p}(\tilde{x}_1, \ldots, \tilde{x}_k)\right) \le \rho\left(\binom{k}{j} (r_1 \ldots r_k)^{\frac{jp}{k}} + S_{j,p}(\tilde{x}_1, \ldots, \tilde{x}_k) \right),$$

where $\widetilde{x}_1 = \ldots = \widetilde{x}_k := 0$, if $n = 1$. Multiplying by $\prod_{i=1}^k r_i^q$ we get

$$\prod_{i=1}^{k} r_i^q f_i(\tilde{x}_i, r_i) \le \prod_{i=1}^{k} r_i^q \cdot \rho\left(\binom{k}{j} \prod_{i=1}^{k} r_i^{\frac{jp}{k}} + S_{j,p}(\tilde{x}_1, \ldots, \tilde{x}_k) \right). \tag{5.4}$$

For fixed $r_1, \ldots, r_k > 0$, set

$$\widetilde{\rho}(t) := \prod_{i=1}^{k} r_i^q \cdot \rho\left(\binom{k}{j} \prod_{i=1}^{k} r_i^{\frac{jp}{k}} + t \right), \qquad t \ge 0.$$

Applying the inductive hypothesis for $q = 0$ to (5.4) if $n \ge 2$ or the conventions made above if $n = 1$, we obtain

$$\begin{aligned} \prod_{i=1}^{k} \int_{\mathbb{R}^{n-1}_+} r_i^q f_i(\tilde{x}_i, r_i)\, d\tilde{x}_i &\le \left(\int_{\mathbb{R}^{n-1}_+} \widetilde{\rho}\left(\binom{k}{j} \|\tilde{u}\|_{jp}^{jp} \right)^{\frac{1}{k}} d\tilde{u} \right)^{k} \\ &= \left(\left(\prod_{i=1}^{k} r_i^{\frac{1}{k}} \right)^{q} \int_{\mathbb{R}^{n-1}_+} \rho\left(\binom{k}{j} \prod_{i=1}^{k} r_i^{\frac{jp}{k}} + \binom{k}{j} \|\tilde{u}\|_{jp}^{jp} \right)^{\frac{1}{k}} d\tilde{u} \right)^{k}. \end{aligned} \tag{5.5}$$

For $t, r_i > 0$, $i = 1, \ldots, k$, set

$$h_i(r_i) := \int_{\mathbb{R}^{n-1}_+} r_i^q f_i(\tilde{x}_i, r_i)\, d\tilde{x}_i \qquad \text{and} \qquad h(t) := t^q \int_{\mathbb{R}^{n-1}_+} \rho\left(\binom{k}{j} (t^{jp} + \|\tilde{u}\|_{jp}^{jp}) \right)^{\frac{1}{k}} d\tilde{u}.$$

Then, by (5.5), the functions $h, h_1, \ldots, h_k$ satisfy the assumption of Theorem 5.1.3, hence

$$\prod_{i=1}^{k} \int_{\mathbb{R}_+} \int_{\mathbb{R}^{n-1}_+} r_i^q f_i(\tilde{x}_i, r_i)\, d\tilde{x}_i\, dr_i \le \left(\int_{\mathbb{R}_+} t^q \int_{\mathbb{R}^{n-1}_+} \rho\left(\binom{k}{j} \left(t^{jp} + \|\tilde{u}\|_{jp}^{jp}\right) \right)^{\frac{1}{k}} d\tilde{u}\, dt \right)^{k}.$$

The assertion follows by Fubini's Theorem. □

Setting $q=0$ and $p=1$ to Proposition 5.1.4, it follows immediately that Conjecture 5.1.2 holds true for functions $f_1,\dots,f_k$ supported in $\mathbb{R}^n_+$. Moreover, by (2.51), Conjecture 5.1.1 holds for convex bodies $K_1,\dots,K_k$, contained in $\mathbb{R}^n_+$. In particular, case (i) of Theorems 2.3.8 and 2.3.9 follows from the previous discussion.

Corollary 5.1.5. *Conjectures 5.1.1 and 5.1.2 hold in the unconditional setting.*

We also have the following.

Corollary 5.1.6. *Proposition 5.1.4 holds if $\mathbb{R}^n_+$ is replaced by $\mathbb{R}^n$.*

Proof. Let O_i, $i=1,\dots,2^n$ be an enumeration of all orthants. Then,

$$\prod_{i=1}^k \int_{\mathbb{R}^n} |\langle x_i, e_m\rangle|^q f_i(x_i)\,dx_i = \sum_{l_1,\dots,l_k=1}^{2^n} \prod_{i=1}^k \int_{O_{l_i}} |\langle x_i, e_m\rangle|^q f_i(x_i)\,dx_i.$$

Therefore, it suffices to prove that if $l_1,\dots,l_k \in \{1,\dots,2^n\}$, then

$$\prod_{i=1}^k \int_{O_{l_i}} |\langle x_i, e_m\rangle|^q f_i(x_i)\,dx_i \le \left(\int_{\mathbb{R}^n_+} |\langle u, e_m\rangle|^q \rho\left(\binom{k}{j}\|u\|_{jp}^{jp}\right)^{\frac{1}{k}} du\right)^k.$$

Let $\tilde{f}_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ be the function defined by $\tilde{f}_i(|x(1)|,\dots,|x(n)|) = f_i(x(1),\dots,x(n))$, $i=1,\dots,k$. Notice that, for any $x_1,\dots,x_k \in \mathbb{R}^n_+$ it holds

$$\prod_{i=1}^k \tilde{f}_i(x_i) \le \rho(S_{j,p}(x_1,\dots,x_k))$$

and also

$$\prod_{i=1}^k \int_{\mathbb{R}^n_+} |\langle x_i, e_m\rangle|^q \tilde{f}_i(x_i)\,dx_i = \prod_{i=1}^k \int_{O_{l_i}} |\langle x_i, e_m\rangle|^q f_i(x_i)\,dx_i.$$

The desired inequality follows by Proposition 5.1.4. □

Setting $q=0$ to Corollary 5.1.6 and using (2.51), we obtain the following.

Corollary 5.1.7. *Let $p>0$ and $1\le j\le k$ be two integers, where $k\ge 2$. Then, for any integrable functions $f_i:\mathbb{R}^n\to\mathbb{R}$, $i=1,\dots,k$, satisfying $S_{j,p}$-polarity condition with respect to some decreasing function $\rho:\mathbb{R}\to[0,\infty]$, it holds*

$$\prod_{i=1}^k \int_{\mathbb{R}^n} f_i(x_i)\,dx_i \le \left(\int_{\mathbb{R}^n} \rho\left(\binom{k}{j}\|u\|_{jp}^{jp}\right)^{\frac{1}{k}} du\right)^k.$$

Moreover, if $K_1,\dots,K_k$ are any convex bodies in $\mathbb{R}^n$ satisfying $\mathcal{E}_{j,p}$-polarity condition, one has

$$\prod_{i=1}^k |K_i| \le |B^n_{jp}|^k.$$

Remark 5.1.8. *Kolesnikov and Werner [97, Proposition 5.5.] established a related result, stating the following. If $K_1,\dots,K_k$ are unconditional convex bodies satisfying*

$$\prod_{i=1}^k r_{K_i}(x_i) \le \left(\sum_{l=1}^n |x_1(l)|^{\frac{2}{k}}\cdots|x_k(l)|^{\frac{2}{k}}\right)^{-\frac{k}{2}}, \qquad \forall x_i\in S^{n-1},\ i=1,\dots,k \tag{5.6}$$

then $|K_1|\cdots|K_k| \le |B_2^n|^k$, *where* $r_{K_i}(u) := \sup\{t > 0 : tu \in K_i\}$, $u \in \mathbb{R}^n$, *denotes the radial function of* K_i. *One can notice that, since* $r_{K_i}(\cdot) = \|\cdot\|_{K_i}^{-1}$, (5.6) *is equivalent to,*

$$\sum_{l=1}^{n} \Big|\frac{x_1(l)}{\|x_1\|_{K_1}}\Big|^{\frac{2}{k}} \cdots \Big|\frac{x_k(l)}{\|x_k\|_{K_k}}\Big|^{\frac{2}{k}} \le 1, \qquad \forall x_i \in \mathbb{R}^n,\ i = 1, \dots, k,$$

which can be written as

$$\sum_{l=1}^{n} |x_1(l)|^{\frac{2}{k}} \cdots |x_k(l)|^{\frac{2}{k}} \le 1 \qquad \forall x_i \in K_i,\ i = 1, \dots, k.$$

Thus, by Corollary 5.1.7 for $j = k$ *and* $p = 2/k$, $|K_1|\cdots|K_k| \le |B_2^n|^k$ *holds for any convex bodies* $K_1, \dots, K_k$ *(not necessarily symmetric) that satisfy condition* (5.6).

5.1.3 The case $j = 1$

The purpose of this section is to establish Theorem 2.3.8 in the case $j = 1$. This will follow immediately from the next slightly more general statement (which will also be used in the last section of this note).

Proposition 5.1.9. *Let* μ *be a Borel measure in* $\mathbb{R}^n$, *satisfying* $\mu((K + L)/2) \ge \sqrt{\mu(K)\mu(L)}$ *(the Lebesgue measure is such an example), for all symmetric convex bodies* K *and* L. *Then, for any symmetric convex bodies* $K_1, \dots, K_k$, *satisfying* $\mathcal{E}_1$*-polarity condition, it holds*

$$\prod_{i=1}^{k} \mu(K_i) \le \mu(B_1^n)^k. \tag{5.7}$$

Proof. One can check that the bodies $\frac{K_1+K_2}{2}, \frac{K_1+K_2}{2}, K_3, \dots, K_k$ also satisfy $\mathcal{E}_1$-polarity condition. On the other hand, by the "log-concavity" assumption on μ, we see that the product $\prod_{i=1}^{k} \mu(K_i)$ does not decrease if the tuple $(K_1, \dots, K_k)$ is replaced by the tuple $(\frac{K_1+K_2}{2}, \frac{K_1+K_2}{2}, K_3, \dots, K_k)$ (which also satisfies $\mathcal{E}_1$-polarity condition). Thus, we may assume that $K_1 = K_2$. A successive application of this argument shows that, in order to prove (5.7), it suffices to prove that if $K_1, \dots, K_k$ satisfy $\mathcal{E}_1$-polarity condition and $K_1 = \ldots = K_k$, then

$$\mu(K_1)^k \le \mu(B_1^n)^k.$$

From the definition of $\mathcal{E}_1$-polarity condition, we conclude that $\mu(K_1)^k$ is maximal (under the above conditions) if and only if K_1 is the largest symmetric convex set, satisfying

$$y_1 + \ldots + y_n \le 1 \qquad \forall (y_1, \dots, y_k) \in K_1. \tag{5.8}$$

Because of this, the maximizing K_1 is necessarily permutation invariant and, since it is also origin symmetric, we deduce that K_1 has to be unconditional. This together with (5.8) imply that $|y_1| + \ldots + |y_n| \le 1$, for all $(y_1, \dots, y_n) \in K_1$. Consequently, $K_1 = B_1^n = K_2 = \ldots = K_k$, which concludes the proof of the proposition. □

Remark 5.1.10. *We would describe* $\mathcal{E}_1$*-polarity condition as "exceptional". The reason is that, as the reader may check using arguments as above, given* $k \ge 2$ *and sets* $K_2, \dots, K_k$, *the set*

$$K_1 := \Big\{x_1 \in \mathbb{R}^n : \mathcal{S}_1(x_1, x_2, \dots, x_k) \le 1,\ \forall x_i \in K_i,\ i = 2, \dots, k\Big\}$$

is always homothetic to B_1^n. *That is, the largest possible set* K_1, *such that* $K_1, K_2, \dots, K_k$ *satisfy* $\mathcal{E}_1$*-polarity condition, is always a dilate of* B_1^n.

5.1.4 Symmetrization

This section is devoted to completing the proof of Theorem 2.3.8. Our proof based on a modification of a symmetrization technique used in [119] and on the following observation. If j is even and $r_1, \dots, r_k \in \mathbb{R}$, then

$$s_j(r_1, -r_2, \dots, -r_k) = s_j(-r_1, r_2, \dots, r_k), \tag{5.9}$$

while

$$s_k(r_1, -r_2, r_3, \dots, r_k) = s_k(-r_1, r_2, r_3, \dots, r_k). \tag{5.10}$$

The proof will follow easily from the next lemma.

Lemma 5.1.11. *Let $2 \le j \le k$ and $K_1, \dots, K_k$ symmetric convex bodies satisfying $\mathcal{E}_j$-polarity condition. Assume that one of the following holds*

(i) $j = k$.

(ii) j *is even and* $K_3, \dots, K_k$ *are unconditional.*

Then there exist $U_1, \dots, U_k$ unconditional convex bodies satisfying $\mathcal{E}_j$-polarity condition, such that

$$\prod_{i=1}^{k} |K_i| \le \prod_{i=1}^{k} |U_i|.$$

Proof. For a set $A \subseteq \mathbb{R}^n$ and a number $r \in \mathbb{R}$, set

$$A(r) := \{\tilde{x} \in e_n^{\perp} : (\tilde{x}, r) \in A\}.$$

The Steiner symmetrization of a convex body K with repsect to $e_n^{\perp}$ is given by

$$st_{e_n^{\perp}}(K) = \Big\{ \big(\tilde{x}, \frac{r - r'}{2}\big) \in \mathbb{R}^n : \tilde{x} \in P_{e_n^{\perp}}(K), \text{ and } (\tilde{x}, r), (\tilde{x}, r') \in K \Big\},$$

where $P_{e_n^{\perp}}(K)$ denotes the orthogonal projection of K onto the subspace $e_n^{\perp}$.

For symmetric convex bodies $K_1, K_3, \dots, K_k$, set

$$(K_1, K_3, \dots, K_k)_j^o := \Big\{ x_2 \in \mathbb{R}^n : \mathcal{S}_j(x_1, x_2, \dots, x_k) \le \binom{k}{j}, \text{ for all } x_i \in K_i \text{ with } i \neq 2 \Big\}.$$

This is a just generalization of the notion of the polar set in the case $j = k = 2$. Clearly, the set $(K_1, K_3, \dots, K_k)_j^o$ is a symmetric convex body and, furthermore, if the K_i are all unconditional then $(K_1, K_3, \dots, K_k)_j^o$ is also unconditional. Notice also that $(K_1, K_3, \dots, K_k)_j^o$ is the largest symmetric convex body, such that the sets $K_1, (K_1, K_2, \dots, K_k)_j^o, K_3, \dots, K_k$, satisfy $\mathcal{E}_j$-polarity condition. We will prove both assertions of Lemma 5.1.11 simultaneously by Steiner symmetrization. We may clearly assume that $K_2 = (K_1, K_3, \dots, K_k)_j^o$. We set $K_2' = (st_{e_n{}^{\perp}} K_1, K_3, \dots, K_k)_j^o$. We will show that, for $r \ge 0$, it holds

$$\frac{K_2(r) + K_2(-r)}{2} \subseteq K_2'(r). \tag{5.11}$$

Let $\tilde{x}_2 \in K_2(r)$ and $\tilde{x}_2' \in K_2(-r)$. Then, for all $(\tilde{x}_i, r_i) \in K_i$, $i = 3, \dots, k$, and for all $(\tilde{x}_1, r_1), (\tilde{x}_1, r_1') \in K_1$, it holds

$$\mathcal{S}_j((\tilde{x}_1, r_1), (\tilde{x}_2, r), (\tilde{x}_3, r_3), \dots, (\tilde{x}_k, r_k)) \le \binom{k}{j} \tag{5.12}$$

and

$$\mathcal{S}_j((\tilde{x}_1, r_1'), (\tilde{x}_2', -r), (\tilde{x}_3, r_3), \dots, (\tilde{x}_k, r_k)) \le \binom{k}{j}. \tag{5.13}$$

When j is even and $K_3,\dots,K_k$ are unconditional, (5.13) can be equivalently written as

$$\mathcal{S}_j((\tilde{x}_1, r_1'), (\tilde{x}_2', -r), (\tilde{x}_3, -r_3), \dots, (\tilde{x}_k, -r_k)) \le \binom{k}{j}. \tag{5.14}$$

Combining (5.10), (5.13) and (5.9), (5.14), we obtain (for both assertions of the Lemma)

$$\mathcal{S}_j((\tilde{x}_1, -r_1'), (\tilde{x}_2', r), (\tilde{x}_3, r_3), \dots, (\tilde{x}_k, r_k)) \le \binom{k}{j}, \tag{5.15}$$

for all $(\tilde{x}_i, r_i) \in K_i$, $i = 3,\dots,k$ and for all $(\tilde{x}_1, r_1') \in K_1$. Averaging (5.12) and (5.15), and since $\mathcal{S}_j$ is affine with respect to each argument, we conclude that

$$\mathcal{S}_j\Big(\big(\tilde{x}_1, \frac{r_1 - r_1'}{2}\big), \big(\frac{\tilde{x}_2 + \tilde{x}_2'}{2}, r\big), (\tilde{x}_3, r_3), \dots, (\tilde{x}_k, r_k) \Big) \le \binom{k}{j},$$

for all $(\tilde{x}_i, r_i) \in K_i$, $i = 3,\dots,k$ and for all $(\tilde{x}_1, r_1), (\tilde{x}_1, r_1') \in K_1$. This shows that $\frac{\tilde{x}_2 + \tilde{x}_2'}{2} \in K_2'(r)$, which establishes (5.11). Inclusion (5.11) together with the Brunn-Minkowski inequality and Fubini's Theorem show that

$$|K_1| = 2\int_0^\infty |K_2(r)|\, dr \le 2\int_0^\infty |K_2'(r)|\, dr = |K_2'|.$$

Applying the same argument successively with respect to $e_{n-1},\dots,e_1$, we arrive at an unconditional convex body U_1, such that $|U_1| = |K_1|$, the tuple $U_1, \bar{K}_2 := (U_1, K_3, \dots, K_k)_j^o, K_3, \dots, K_k$ satisfies $\mathcal{E}_j$-polarity condition and $|K_2| \le |\bar{K}_2|$. This can be done for both cases (i) and (ii) of the lemma.

Recall that if $K_3,\dots,K_k$ are unconditional then $\bar{K}_2$ is also unconditional and the proof of (i) is complete.

In the case $j = k$, we repeat the same argument to the new tuple $(U_1, \bar{K}_2, K_3, \dots, K_k)$ with respect to the pair $(\bar{K}_2, K_3)$. Thus, we are able to replace $\bar{K}_2$ by an unconditional convex body U_2 and K_3 by a symmetric convex body $\bar{K}_3$, such that $|U_2| = |\bar{K}_2|$, $|\bar{K}_3| \ge |K_3|$, while the tuple $(U_1, U_2, \bar{K}_3, K_4, \dots, K_k)$ also satisfies $\mathcal{E}_j$-polarity condition. We continue the same process until we replace all $K_1,\dots,K_{k-1}$ by unconditional convex bodies $U_1,\dots,U_{k-1}$ without decreasing the volume product of the K_i's. We conclude the proof by the fact that $U_k := (U_1,\dots,U_{k-1})_k^\circ$ is also unconditional. $\square$

Proof of Theorem 2.3.8. Inequality (5.1) in all cases follows from Lemma 5.1.11 together with Corollary 5.1.5 and Section 5.1.3. It remains to verify that (5.1) is sharp for ℓ_j-balls. In other words we need to prove that if $K_1 = \ldots = K_k = B_j^n$, then $K_1,\dots,K_k$ satisfy $\mathcal{E}_j$-polarity condition . But this is a simple application of the arithmetic-geometric mean inequality: For any $x_1,\dots,x_k \in B_j^n$, it holds

$$\begin{aligned}
\mathcal{S}_j(x_1,\dots,x_k) &= \sum_{l=1}^n \sum_{1\le i_1<\ldots<i_j\le k} x_{i_1}(l)\cdots x_{i_j}(l) \\
&\le \frac{1}{j}\sum_{l=1}^n \sum_{1\le i_1<\ldots<i_j\le k} \big(|x_{i_1}(l)|^j + \ldots + |x_{i_j}(l)|^j\big) \\
&= \frac{(k-1)!}{(k-j)!j!}\sum_{l=1}^n \big(|x_1(l)|^j + \ldots + |x_k(l)|^j\big) \\
&= \frac{(k-1)!}{(k-j)!j!}\sum_{i=1}^k \|x_i\|_j^j \le \frac{(k-1)!}{(k-j)!j!}k = \binom{k}{j}.
\end{aligned}$$

$\square$

5.1.5 Equivalence between j-Santaló and Functional j-Santaló Conjectures

In this section we prove cases (ii) and (iii) of Theorem 2.3.9. This is done by establishing the equivalence between Conjectures 5.1.1 and 5.1.2 (actually, a slightly more general result), mentioned in the Introduction. Let us first introduce some notation. Let G be a subgroup of the orthogonal group $O(n)$ in $\mathbb{R}^n$. We set

$$\mathcal{S}(G) := \{S \subseteq \mathbb{R}^n : gS = S,\ \forall\, g \in G\} \qquad \text{and} \qquad \mathcal{F}(G) := \{f : \mathbb{R}^n \to \mathbb{R} : f \circ g = f,\ \forall\, g \in G\}.$$

Proposition 5.1.12. *Let μ be an a-homogeneous Borel measure in $\mathbb{R}^n$ for some $a > 0$, k be a positive integer, $j \in \{1, \dots, k\}$ and $G_1, \dots, G_k$ be subgroups of $O(n)$. The following statements are equivalent.*

i) For any k-tuple of symmetric convex bodies $(K_1, \dots, K_k) \in \mathcal{S}(G_1) \times \cdots \times \mathcal{S}(G_k)$, satisfying $\mathcal{E}_j$-polarity condition, it holds

$$\prod_{i=1}^{k} \mu(K_i) \le \mu(B_j^n)^k.$$

ii) For any k-tuple of even non-negative measurable functions $(f_1, \dots, f_k) \in \mathcal{F}(G_1) \times \cdots \times \mathcal{F}(G_k)$, satisfying $\mathcal{S}_j$-polarity condition with respect to some decreasing function ρ, it holds

$$\prod_{i=1}^{k} \int_{\mathbb{R}^n} f_i(x_i)\, d\mu(x_i) \le \left(\int_{\mathbb{R}^n} \rho\left(\binom{k}{j} \|u\|_j^j \right)^{1/k} d\mu(u) \right)^k. \tag{5.16}$$

The fact that Conjectures 5.1.1 and 5.1.2 are equivalent follows immediately from Proposition 5.1.12, if we take μ to be the Lebesgue measure. For the proof we will need the following lemma (which is well known in the classical case $j = k = 2$).

Lemma 5.1.13. *Let $1 \le j \le k$ and $A_1, \dots, A_k$ be subsets of $\mathbb{R}^n$. If $A_1, \dots, A_k$ satisfy $\mathcal{E}_j$-polarity condition, then* $\mathrm{conv}(A_1), \dots, \mathrm{conv}(A_k)$ *also satisfy $\mathcal{E}_j$-polarity condition.*

Proof. Clearly, it suffices to prove that $\mathrm{conv}(A_1), A_2, \dots, A_k$ satisfy $\mathcal{E}_j$-polarity condition. This follows from the observation that, if $\lambda_1, \dots, \lambda_r \ge 0$ are real numbers that sum to 1 and if $x_2, \dots, x_k \in \mathbb{R}^n$ and $y_1, \dots, y_r \in \mathbb{R}^n$, then

$$\mathcal{E}_j\Big(\sum_{m=1}^{r} \lambda_m y_m, x_2, \dots, x_k \Big) = \sum_{m=1}^{r} \lambda_m \mathcal{E}_j(y_m, x_2, \dots, x_k).$$

$\square$

Proof of Proposition 5.1.12. The fact that (ii) implies (i), follows immediately from (2.51).

For the other direction, assume that (i) holds for all bodies $K_i \in \mathcal{S}(G_i)$, $i = 1, \dots, k$. Let $(f_1, \dots, f_k) \in \mathcal{F}(G_1) \times \cdots \times \mathcal{F}(G_k)$ be functions that satisfy $\mathcal{S}_j$-polarity condition with respect to some ρ. In order to prove the desired inequality (5.16), (by an approximation argument) we can assume that $\lim_{t\to\infty} \rho(t) = 0$, ρ is continuous, strictly decreasing and that $\lim_{t\to 0^+} \rho(t) = \infty$. Define the (not necessarily convex) sets $K_i(r_i) := \{x_i \in \mathbb{R}^n : f_i(x_i) \ge r_i\}$, $r_i \ge 0$ and notice that $K_i(r_i) \in \mathcal{S}(G_i)$, $i = 1, \dots, k$. From $\mathcal{S}_j$-polarity condition one obtains that, for $x_i \in K_i(r_i)$, $i = 1, \dots, k$, it holds

$$r_1 \dots r_k \le \prod_{i=1}^{k} f_i(x_i) \le \rho\left(\mathcal{S}_j(x_1, \dots, x_k)\right).$$

Moreover, using the the strict monotonicity of ρ, we get

$$\mathcal{S}_j(x_1, \dots, x_k) \le \rho^{-1}(r_1 \dots r_k).$$

Consequently, by the fact that $\mathcal{S}_j$ is homogeneous of order j, setting $\lambda := \binom{k}{j}^{\frac{1}{j}}\rho^{-1}(r_1\cdots r_k)^{-\frac{1}{j}}$, we conclude

$$\mathcal{S}_j(\lambda x_1,\ldots,\lambda x_k) \leq \binom{k}{j}.$$

Thus, by the assumption that (i) holds true and by Lemma 5.1.13 we obtain

$$\mu(\lambda K_1(r_1))\ldots\mu(\lambda K_k(r_k)) \leq \mu(\mathrm{conv}(\lambda K_1(r_1)))\ldots\mu(\mathrm{conv}(\lambda K_k(r_k))) \leq \mu(B_j^n)^k.$$

Equivalently, using the homogeneity of μ, one has

$$\mu(K_1(r_1))\ldots\mu(K_k(r_k)) \leq \frac{\mu(B_j^n)^k}{\lambda^{ka}} = \binom{k}{j}^{-\frac{ka}{j}}\mu(B_j^n)^k\rho^{-1}(r_1\cdots r_k)^{\frac{ka}{j}}.$$

Set $\phi_i(r_i) := \mu(K_i(r_i))$, $r_i \geq 0$, $i = 1,\ldots,k$ and $\phi(r) := \binom{k}{j}^{-\frac{a}{j}}\mu(B_j^n)\rho^{-1}(r^k)^{\frac{a}{j}}$, $r \geq 0$. Then, the previous inequality can be written as

$$(\varphi_1(r_1)\ldots\varphi_k(r_k))^{1/k} \leq \varphi\left((r_1\ldots r_k)^{1/k}\right)$$

and, therefore, the Prekopa-Leindler inequality (Theorem 5.1.3) together with the Layer-Cake formula give

$$\prod_{i=1}^k\int_{\mathbb{R}^n} f_i(x_i)\,d\mu(x_i) = \prod_{i=1}^k\int_0^\infty \varphi_i(r_i)\,dr_i \leq \left(\int_0^\infty\varphi\right)^k = \binom{k}{j}^{-\frac{ka}{j}}\mu(B_j^n)^k\left(\int_0^\infty \rho^{-1}(r^k)^{\frac{a}{j}}\,dr\right)^k. \quad (5.17)$$

On the other hand, using the extra assumptions on ρ and the homogeneity of μ, we see that

$$\begin{aligned}\int_{\mathbb{R}^n}\rho\left(\binom{k}{j}\|u\|_j^j\right)^{1/k} d\mu(u) &= \int_0^\infty \mu\left(\left\{u : \rho\left(\binom{k}{j}\|u\|_j^j\right) \geq t^k\right\}\right)dt\\ &= \int_0^\infty \mu\left(\left\{u : \|u\|_j \leq \left(\binom{k}{j}^{-1}\rho^{-1}(t^k)\right)^{\frac{1}{j}}\right\}\right)dt\\ &= \binom{k}{j}^{-\frac{a}{j}}\mu(B_j^n)\int_0^\infty \rho^{-1}(t^k)^{\frac{a}{j}}\,dt. \end{aligned} \quad (5.18)$$

Putting together (5.17) and (5.18), we arrive at (5.16), as claimed. □

The proof of Theorem 2.3.9 follows immediately from Theorem 2.3.8 and Proposition 5.1.12.

5.2 On a j-Ball Conjecture

5.2.1 Introduction

Let us recall the definition of Ball's functional, mentioned in the Introduction. If K is a symmetric convex body in $\mathbb{R}^n$, $B(K)$ is given by

$$B(K) := \int_K\int_{K^o}\langle x,y\rangle^2\,dx\,dy.$$

It can be easily checked that $B(\cdot)$ is invariant under non-singular linear maps. The primary goal of section 5.2 is to state and discuss a natural (at least in our opinion) extension of Conjecture 2.3.11, to the multi-entry setting. Let $\mathcal{D}(n)$ be the set of all orthonormal basis' in $\mathbb{R}^n$. For $k \geq 2$, $j \in \{1, \dots, k\}$ and $\{\epsilon_m\} \in \mathcal{D}(n)$, define

$$\mathcal{B}_j(K_1, \dots, K_k, \{\epsilon_m\}) := \sum_{m=1}^{n} \prod_{i=1}^{k} \int_{K_i} |\langle x_i, \epsilon_m \rangle|^j \, dx_i.$$

Define, also

$$\mathcal{B}_j(K_1, \dots, K_k) := \min_{\{\epsilon_m\} \in \mathcal{D}(n)} \mathcal{B}_j(K_1, \dots, K_k, \{\epsilon_m\}).$$

One might dare to conjecture the following.

Conjecture 5.2.1. *Let $1 \leq j \leq k$ be two integers, where $k \geq 2$. Let $K_1, \dots, K_k$ be symmetric convex bodies in $\mathbb{R}^n$ satisfying $\mathcal{E}_j$-polarity condition. Then,*

$$\mathcal{B}_j(K_1, \dots, K_k) \leq \mathcal{B}_j(B_j^n, \dots, B_j^n). \tag{5.19}$$

For next Proposition it will be useful to recall the notion of isotropicity. A symmetric convex body K in $\mathbb{R}^n$ is called isotropic if

$$\int_K \langle x, u \rangle^2 \, dx = \frac{\|u\|_2^2}{n} \int_K \|x\|_2^2 \, dx, \qquad \forall u \in \mathbb{R}^n.$$

Notice (see [127]) that there is always a linear image TK of K, such that TK is isotropic.

Proposition 5.2.2. *Conjectured 5.2.1 for $k = j = 2$ agrees with Ball's Conjecture 2.3.11*

Proof. Let assume that conjecture 2.3.11 is true. Observe that there always exists an orthonormal basis $\{\epsilon_m\}$ such that, for $i \neq m$, it holds

$$\int_{K_1} \langle x, \epsilon_i \rangle \langle x, \epsilon_m \rangle \, dx = 0.$$

Hence,

$$\begin{aligned}
\mathcal{B}_2(K_1, K_2) \leq \mathcal{B}_2(K_1, K_2, \{\epsilon_m\}) &\leq \mathcal{B}_2(K_1, K_1^o, \{\epsilon_m\}) \\
&= \sum_{m=1}^{n} \int_{K_1} \langle x, \epsilon_m \rangle^2 \, dx \int_{K_1^o} \langle y, \epsilon_m \rangle^2 \, dy \\
&= \int_{K_1} \int_{K_1^o} \langle x, y \rangle^2 \, dx \, dy \\
&\leq \int_{B_2^n} \int_{B_2^n} \langle x, y \rangle^2 \, dx \, dy = \mathcal{B}_2(B_2^n, B_2^n).
\end{aligned}$$

Conversely, assume that Conjecture 5.2.1 is true for $k = j = 2$ and for all symmetric convex bodies K_1, K_2. One can take $K_1 = K = K_2^o$. Since $B(K)$ is invariant under non-singular linear maps, we can assume that K is isotropic. We have

$$\begin{aligned}
B(B_2^n) = \mathcal{B}_2(B_2^n, B_2^n) \geq \mathcal{B}_2(K, K^o) &= \min_{\{\epsilon_m\} \in \mathcal{D}(n)} \sum_{m=1}^{n} \int_K \langle x, \epsilon_m \rangle^2 \, dx \int_{K^o} \langle y, \epsilon_m \rangle^2 \, dy \\
&= \min_{\{\epsilon_m\} \in \mathcal{D}(n)} \sum_{m=1}^{n} \int_K \langle x, \epsilon_1 \rangle^2 \, dx \int_{K^o} \langle y, \epsilon_m \rangle^2 \, dy \\
&= \int_K \langle x, \epsilon_1 \rangle^2 \, dx \int_{K^o} \|y\|_2^2 \, dx \\
&= \frac{1}{n} \int_K \|x\|_2^2 \, dx \int_{K^o} \|y\|_2^2 \, dy \\
&= \int_K \int_{K^o} \langle x, y \rangle^2 \, dx \, dy = B(K).
\end{aligned}$$

$\square$

5.2.2 The case $j = 1$

We confirm in Proposition 5.2.3 that Conjecture 5.2.1 hold for $j = 1$. However, we are mostly interested in the case $j \geq 2$ (see Remark...).

Proposition 5.2.3. *Conjecture 5.2.1 holds if $j = 1$.*

Proof. Let $K_1, \ldots, K_k$ be symmetric convex bodies satisfying $\mathcal{E}_1$-polarity condition. It is clearly enough to show the following.

$$\prod_{i=1}^{k} \int_{K_i} |\langle x_i, e_m \rangle| \, dx_i \leq \left(\int_{B_1^n} |\langle x, e_m \rangle| \, dx \right)^k, \qquad m = 1, \ldots, n. \tag{5.20}$$

By proposition 5.1.9, it is enough to show that, for $m = 1, \ldots, n$, the measure μ_m in $\mathbb{R}^n$, with density $\psi_m(x) := |\langle x, e_m \rangle|$, satisfies $\mu_m((K + L)/2) \geq \sqrt{\mu_m(K)\mu_m(L)}$, for all symmetric convex bodies K and L. To see this, fix symmetric convex bodies K and L and set $H_m^+ := \{x \in \mathbb{R}^n : \langle x, e_m \rangle \geq 0\}$ and $H_m^- := \{x \in \mathbb{R}^n : \langle x, e_m \rangle \leq 0\}$. Then, the restriction of ψ_m either to H_m^+ or to H_m^- is log-concave and, therefore (see [28]),

$$\begin{aligned} \mu_m\left(\left(\frac{1}{2}K + \frac{1}{2}L \right) \cap H_m^+ \right) &\geq \mu_m\left(\frac{1}{2}\left(K \cap H_m^+ + L \cap H_m^+ \right) \right) \\ &\geq \sqrt{\mu_m(K \cap H_m^+)\mu_m(L \cap H_m^+)} \\ &= \frac{1}{2}\sqrt{\mu_m(K)\mu_m(L)}, \end{aligned} \tag{5.21}$$

where we used the symmetry of K and L and the evenness of μ_m. A similar argument shows that

$$\mu_m\left(\left(\frac{1}{2}K + \frac{1}{2}L \right) \cap H_m^- \right) \geq \frac{1}{2}\sqrt{\mu_m(K)\mu_m(L)}. \tag{5.22}$$

The desired property for μ_m follows by adding together (5.21) and (5.22). □

5.2.3 The j-Ball implies the j-Santaló

Next, we would like to explain the connection between the conjecture 5.2.1 and the j-Santaló conjecture 5.1.1.

Proposition 5.2.4. *Let $k \geq 2$ be a positive integer, $j \in \{1, \ldots, k\}$ and $K_1, \ldots, K_k$ be symmetric convex bodies satisfying $\mathcal{E}_j$-polarity condition. If* (5.19) *holds, then* (5.1) *also holds.*

Proposition 5.2.4 follows immediately from the following lemma (the corresponding fact involving $B(\cdot)$ was obtained by Ball [9] [10]; see also Lutwak [111]).

Lemma 5.2.5. *For convex bodies K_i, $i = 1, \ldots, k$ we have*

$$\frac{\mathcal{B}_j(B_j^n, \ldots, B_j^n)}{|B_j^n|^{\frac{k(n+j)}{n}}} \leq \frac{\mathcal{B}_j(K_1, \ldots, K_k)}{(|K_1| \cdots |K_k|)^{\frac{n+j}{n}}}. \tag{5.23}$$

Proof. We may assume that

$$\mathcal{B}_j(K_1, \ldots, K_k, \{e_m\}) = \mathcal{B}_j(K_1, \ldots, K_k).$$

Let Q be a convex body in $\mathbb{R}^n$. We will need the following simple fact.

Fact. Let $T \in SL(n)$ be a diagonal positive definite map (with respect to the basis $\{e_m\}$). Then,

$$\prod_{m=1}^{n} \int_{TQ} |\langle x, e_m \rangle|^j \, dx = \prod_{m=1}^{n} \int_{Q} |\langle x, e_m \rangle|^j \, dx.$$

Furthermore, there exists a diagonal positive definite map $T_0 \in SL(n)$, such that

$$\int_{T_0Q} |\langle x, e_1\rangle|^j \, dx = \ldots = \int_{T_0Q} |\langle x, e_n\rangle|^j \, dx.$$

It follows that

$$\begin{aligned}
\left(\prod_{m=1}^n \int_Q |\langle x, e_m\rangle|^j \, dx\right)^{1/n} &= \left(\prod_{m=1}^n \int_{T_0Q} |\langle x, e_m\rangle|^j \, dx\right)^{1/n} \\
&= \frac{1}{n}\sum_{m=1}^n \int_{T_0Q} |\langle x, e_m\rangle|^j \, dx \\
&= \frac{1}{n}\int_{T_0Q} \|x\|_j^j \, dx \\
&= \frac{1}{n}\int_0^\infty |(T_0Q) \cap \{x : \|x\|_j^j \geq t\}| \, dt \\
&= \frac{1}{n}\int_0^\infty \Big(|T_0Q| - |(T_0Q) \cap \{x : \|x\|_j < t^{1/j}\}|\Big) \, dt \\
&= \frac{1}{n}\int_0^\infty \Big(|Q| - |(T_0Q) \cap (t^{1/j}B_j^n)|\Big) \, dt.
\end{aligned}$$

Since, for all $t > 0$, it holds

$$|(T_0Q) \cap (t^{1/j}B_j^n)| \leq \left|\left(\left(|T_0Q|/|B_j^n|\right)^{1/n} B_j^n\right) \cap \left(t^{1/j}B_j^n\right)\right| = \left|\left(\left(|Q|/|B_j^n|\right)^{1/n} B_j^n\right) \cap \left(t^{1/j}B_j^n\right)\right|,$$

we arrive at

$$\begin{aligned}
\left(\prod_{m=1}^n \int_Q |\langle x, e_m\rangle|^j \, dx\right)^{1/n} &\geq \frac{1}{n}\int_{\left(|Q|/|B_j^n|\right)^{1/n} B_j^n} \|x\|_j^j \, dx \\
&= \frac{1}{n}\left(\frac{|Q|}{|B_j^n|}\right)^{\frac{n+j}{n}} \int_{B_j^n} \|x\|_j^j \, dx =: c_{n,j}|Q|^{\frac{n+j}{n}}, \qquad (5.24)
\end{aligned}$$

where $c_{n,j}$ is a positive constant that depends only on n and j, such that equality holds in (5.24) if $Q = B_j^n$.

By the arithmetic-geometric mean inequality and (5.24), one has

$$\begin{aligned}
\mathcal{B}_j(K_1, \ldots, K_k) &\geq n\prod_{m=1}^n \left(\prod_{i=1}^k \int_{K_i} |\langle x_i, e_m\rangle|^j \, dx\right)^{1/n} \\
&= n\prod_{i=1}^k \left(\prod_{m=1}^n \int_{K_i} |\langle x_i, e_m\rangle|^j \, dx\right)^{1/n} \geq n(c_{n,j})^k \prod_{i=1}^k |K_i|^{\frac{n+j}{j}}.
\end{aligned}$$

Notice that if $K_1 = \ldots = K_k = B_j^n$, then equality holds in all previous inequalities. This finishes the proof of the Lemma. $\square$

5.2.4 The unconditional case and the Functional j-Ball Conjecture

We also confirm Conjecture 5.2.1 in the unconditional setting. This is proven by passing throught its analytical counterpart and settle this in the unconditional setting.....kati tetoio.....

Finally, we would like to extend the definition of the $\mathcal{B}_j$ functional, to tuples of functions instead of tuples of convex bodies. For even non-negative integrable functions $f_1, \ldots, f_k$, for $j \leq k$ and for $\{\epsilon_m\} \in \mathcal{D}(n)$, set

$$\mathcal{B}_j(f_1, \ldots, f_k, \{\epsilon_m\}) := \sum_{m=1}^{n} \prod_{i=1}^{k} \int_{\mathbb{R}^n} |\langle x_i, \epsilon_m \rangle|^j f_i(x_i)\, dx_i$$

and

$$\mathcal{B}_j(f_1, \ldots, f_k) := \min_{\{\epsilon_m\} \in \mathcal{D}(n)} \mathcal{B}_j(f_1, \ldots, f_k, \{\epsilon_m\}).$$

The functional version of the conjecture 5.2.1 states the following. Let $f_i : \mathbb{R}^n \to \mathbb{R}_+$, $i = 1, \ldots, k$, be even functions satisfying $\mathcal{S}_j$-polarity condition with respect to some non-negative and decreasing function ρ. Then,

$$\mathcal{B}_j(f_1, \ldots, f_k) \leq n \left(\int_{\mathbb{R}^n} |\langle u, e_1 \rangle|^j \rho \left(\binom{k}{j} \|u\|_j^j \right)^{\frac{1}{k}} du \right)^k = n^{1-k} \left(\int_{\mathbb{R}^n} \|u\|_j^j \rho \left(\binom{k}{j} \|u\|_j^j \right)^{\frac{1}{k}} du \right)^k. \tag{5.25}$$

By (2.51), (5.25) would immediately imply (5.19). Using (2.51), Proposition 5.2.4, and Proposition 5.1.12 we obtain the following.

Corollary 5.2.6. *If* (5.25) *holds for any even non-negative integrable functions $f_1, \ldots, f_k$ and any non-negative decreasing function ρ, then the functional j-Santaló conjecture 5.1.2 holds in full generality.*

Before ending this note, we wish to list some cases for which the conjectured inequalities (5.19) and (5.25) are indeed correct. First notice that Proposition 5.1.4 and (2.51) imply the following.

Corollary 5.2.7. *Inequalities* (5.19) *and* (5.25) *are both true in the unconditional case.*

Furthermore, combining (5.20) and Proposition 5.1.12, we immediately obtain

Corollary 5.2.8. *The conjectured inequality* (5.25) *is true if $j = 1$.*

We mention that the authors in [82] proved the following functional version of Ball's inequality: If $\rho : \mathbb{R} \to \mathbb{R}_+$ is a measurable function and $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}_+$ are integrable unconditional log-concave functions satisfying $f_1(x_1) f_2(x_2) \leq \rho(\langle x_1, x_2 \rangle)$, for all $x_1, x_2 \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle x, y \rangle^2 f_1(x) f_2(y)\, dx\, dy \leq n^{-1} \left(\int_{\mathbb{R}^n} \|u\|_2^2 \rho \left(\|u\|_2^2 \right)^{\frac{1}{2}} du \right)^2 \tag{5.26}$$

It is unknown if (5.26) holds for arbitrary even log-concave functions. Using similar arguments as in the case of sets, one can show that the conjectured inequality (5.25) (for arbitrary even integrable functions) for $k = j = 2$ is equivalent to (5.26). Hence, (5.25) for unconditional functions can be interpreted as an extension of the functional version of Ball's inequality to the multi-entry setting, if ρ is additionally assumed to be decreasing.

Chapter 6

Appendices

6.1 Log-Brunn-Minkowksi conjecture under the Brascamp-Lieb inequality

The following two continuous version of the Brascamp-Lieb and its reverse form introduced by Barthe in [16].

Theorem 6.1.1 (Continuous Brascamp-Lieb inequality)**.** *For an isotropic measure μ on S^{n-1} and $f_u := \mathbf{1}_{[a(u),b(u)]}$ for $u \in S^{n-1}$ where $a(u) < b(u)$ are bounded real integrable functions, we have*

$$\int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x, u\rangle u)\, d\mu(u)\right) dx \leq \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) d\mu(u)\right)$$

Theorem 6.1.2 (Continuous reverse Brascamp-Lieb inequality)**.** *For an integrable function $h : \mathbb{R}^n \to [0, \infty)$, an isotropic measure μ on S^{n-1} and $f_u := \mathbf{1}_{[a(u),b(u)]}$ for $u \in S^{n-1}$ where $a(u) < b(u)$ are bounded real integrable functions, if for every continuous $\theta : S^{n-1} \to [0, \infty)$ it holds*

$$h\big(\int_{S^{n-1}} \theta(u) u\, d\mu(u)\big) \geq \exp\left(\int_{S^{n-1}} \log f_u(\theta(u))\, d\mu(u)\right).$$

then, one has

$$\int_{\mathbb{R}^n} h \geq \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) d\mu(u)\right)$$

For a function $f : S^{n-1} \to \mathbb{R}_{>0}$, an isotropic measure μ and a continues function $\theta : S^{n-1} \to \mathbb{R}$ we define,

$$c(f, \mu, \theta) = \Big\| \int_{S^{n-1}} u\theta(u)\, d\mu(u) \Big\|_{[f]}.$$

where $[f]$ be the Wulff shape of f (see (2.5)). Also, set

$$c(f, \mu) = \min\{c > 0 : \int_{S^{n-1}} u\theta(u)\, d\mu(u) \in c[f],\ \forall\ |\theta| \leq f \ \text{ continues}\} \tag{6.1}$$

which is nothing else than $c(f, \mu) = \sup_{|\theta| \leq f} c(f, \mu, \theta)$ and last we define

$$c(f) = \inf\{c(f, \mu) : \mu \ \text{ isotropic}\}. \tag{6.2}$$

Note, we may have $c(h_K, \mu) < c(f, \mu)$ while $[h_K] = [f]$. For example a polytope K can be written as the Wulff shape a function f with extremely large values.

Proposition 6.1.3. *For every symmetric convex bodies $K, L \in \mathcal{K}_e^n$, we have,*

$$\frac{|(1-\lambda)\cdot K +_0 \lambda \cdot L|}{|K|^{1-\lambda}|L|^{\lambda}} \geq c\left(h_K^{1-\lambda}h_L^{\lambda}\right)^{-n}. \tag{6.3}$$

Proof. Let K and L be two symmetric convex bodies of $\mathbb{R}^n$ and $\lambda \in [0,1]$. Set h to be the characteristic function of the set $(1-\lambda)\cdot K +_o \lambda \cdot L$. For an isotropic measure μ, let $c_\mu := c(h_K^{1-\lambda}h_L^{\lambda}, \mu)$ and also for $t \in \mathbb{R}$ define,

$$f_u(t) = 1_{\frac{1}{c_\mu}[-h_K(u)^{1-\lambda}h_L(u)^{\lambda},\ h_K(u)^{1-\lambda}h_L(u)^{\lambda}]}(t).$$

By the definition of c_μ, if $|\phi(u)| \leq h_K(u)^{1-\lambda}h_L(u)^{\lambda}$ is any continuous function on S^{n-1}, then

$$\frac{1}{c_\mu}\int_{S^{n-1}} u\phi(u)\, d\mu(u) \in (1-\lambda)\cdot K +_0 \lambda \cdot L,$$

and in turn,

$$\log h\Big(\int_{S^{n-1}} u\frac{\phi(u)}{c_\mu}\, d\mu(u)\Big) = 0 = \int_{S^{n-1}} \log f_u\Big(\frac{\phi(u)}{c_\mu}\Big)\, d\mu(u).$$

This means, if $|\theta(u)| \leq \frac{1}{c_\mu}h_K(u)^{1-\lambda}h_L(u)^{\lambda}$, then

$$\log h\Big(\int_{S^{n-1}} u\theta(u)\, d\mu(u)\Big) \geq \int_{S^{n-1}} \log f_u(\theta(u))\, d\mu(u). \tag{6.4}$$

and so (6.4) holds for every continuous θ. In turn, we deduce that if θ is any continuous function on S^{n-1}, then

$$h\Big(\int_{S^{n-1}} u\theta(u)\, d\mu(u)\Big) \geq \exp\Big(\int_{S^{n-1}} \log f_u(\theta(u))\, d\mu(u)\Big). \tag{6.5}$$

Now we may apply the Continuous Reverse Brascamp Lieb inequality Theorem 6.1.2 by (6.5), and obtain

$$\begin{aligned}
|(1-\lambda)\cdot K +_0 \lambda \cdot L| &= \int_{\mathbb{R}^n} h(z)\, dz \\
&\geq \exp\Big(\int_{S^{n-1}} \log\Big(\int_{\mathbb{R}} f_u\Big)\, d\mu(u)\Big) \\
&= \exp\Big(\int_{S^{n-1}} \log \frac{2}{c_\mu}h_K(u)^{1-\lambda}h_L(u)^{\lambda}\, d\mu(u)\Big) \\
&= \Big(\exp\Big(\int_{S^{n-1}} \log \frac{2}{c_\mu}h_K(u)\, d\mu(u)\Big)\Big)^{1-\lambda}\Big(\exp\Big(\int_{S^{n-1}} \log \frac{2}{c_\mu}h_L(u)\, d\mu(u)\Big)\Big)^{\lambda} \\
&= \Big(\exp\Big(\int_{S^{n-1}} \log |P_u(\frac{1}{c_\mu}K)|\, d\mu(u)\Big)\Big)^{1-\lambda}\Big(\exp\Big(\int_{S^{n-1}} \log |P_u(\frac{1}{c_\mu}L)|\, d\mu(u)\Big)\Big)^{\lambda} \\
&\geq |\frac{1}{c_\mu}K|^{1-\lambda}|\frac{1}{c_\mu}L|^{\lambda} \\
&= \frac{1}{(c_\mu)^n}|K|^{1-\lambda}|L|^{\lambda}.
\end{aligned}$$

The last inequality, is the continuous form of the well known application of (classical) Brascamp Lieb inequality. This is, if $c_i > 0$ and $I_n = \sum_{i=1}^m c_i P_{u_i}$, then

$$|K| \leq \prod_{i=1}^m |P_{u_i}(K)|^{c_i}.$$

Last, since we proved that, for every μ isotropic measure

$$|(1-\lambda)\cdot K +_0 \lambda \cdot L| \geq \frac{1}{(c_\mu)^n}|K|^{1-\lambda}|L|^\lambda \tag{6.6}$$

we take (6.3). □

Fix orthonormal basis $e_1, \dots, e_n$. Denote

$$S^{n-1}/2 := \bigsqcup_{i=1}^{n} \Big\{ x \in S^{n-1} \cap lin\{e_1, \dots, e_i\} : \langle x, e_i \rangle > 0 \Big\}.$$

Note, $S^{n-1}/2$ is a connected set and does not contains antipondal points. Note, also that (6.1) is equivalently written,

$$c(f,\mu) = 2 \sup_{U \in SO(n)} \Big\| \int_{U(S^{n-1}/2)} uf(u)\, d\mu(u) \Big\|_{[f]}. \tag{6.7}$$

Lemma 6.1.4. *For symmetric convex bodies $K, L \in \mathcal{K}^n_e$ and any isotropic measure μ we have*

$$c(h_k^{1-\lambda} h_L^\lambda, \mu) \leq c(h_K,\mu)^{1-\lambda} c(h_L,\mu)^\lambda. \tag{6.8}$$

Proof. Let $U \in O(n)$. We set

$$\begin{aligned} x_o =& 2\int_{U(S^{n-1}/2)} uh_K(u)\, d\mu(u) \\ y_o =& 2\int_{U(S^{n-1}/2)} uh_L(u)\, d\mu(u) \\ z_o =& 2\int_{U(S^{n-1}/2)} uh_K(u)^{1-\lambda} h_L(u)^\lambda\, d\mu(u). \end{aligned}$$

For every $u' \in S^{n-1}$ we have $\langle \rho_K(x_o)x_o, u' \rangle \leq h_K(u')$ and in turn

$$\int_{S^{n-1}/2} 2h_K(u)\rho_K(x_o)\langle u, u' \rangle\, d\mu(u) \leq h_K(u').$$

The same holds for L and therefore applying Hölder inequality we get,

$$\begin{aligned} h_K(u')^{1-\lambda} h_L(u')^\lambda &\geq \Big(\int_{S^{n-1}/2} 2h_K(u)\rho_K(x_o)\langle u, u' \rangle\, d\mu(u) \Big)^{1-\lambda} \Big(\int_{S^{n-1}/2} 2h_L(u)\rho_L(y_o)\langle u, u' \rangle\, d\mu(u) \Big)^{\lambda} \\ &\geq \int_{S^{n-1}/2} 2(h_K(u)\rho_K(x_o))^{1-\lambda}(h_L(u)\rho_L(y_o))^\lambda \langle u, u' \rangle\, d\mu(u) \\ &= \langle \rho_K(x_o)^{1-\lambda}\rho_L(y_o)^\lambda z_o, u' \rangle, \end{aligned}$$

for any $u' \in S^{n-1}$. This shows that $\rho_K(x_o)^{1-\lambda}\rho_L(y_o)^\lambda z_o \in (1-\lambda)\cdot K +_o \lambda \cdot L$, and thus

$$\rho_{(1-\lambda)\cdot K +_0 \lambda \cdot L}(z_o) \geq \rho_K(x_o)^{1-\lambda}\rho_L(y_o)^\lambda$$

and the proof finishes, since $\rho_K(\cdot) = \|\cdot\|_K^{-1}$. □

6.2 A multidimensional Santaló inequality

Following methods from [97], one can obtain the following extended version in the multi-dimensional setting.

Theorem 6.2.1. *Let $s, k \geq 1$ be two integers, $(\sigma_1, \dots, \sigma_k)$ be an s-uniform cover of $[n]$ and denote $E_i = \mathrm{span}\{e_m : m \in \sigma_i\}$ for some fixed basis $e_1, \dots, e_n$. If for some measurable and unconditional functions $f_i : E_i \to \mathbb{R}_+$ we have*

$$\prod_{i=1}^{k} f_i(x_i) \leq \rho\left(\sum_{\substack{i,j=1\\ i<j}}^{k} \langle x_i, x_j\rangle\right) \tag{6.9}$$

for some decreasing function $\rho : \mathbb{R} \to \mathbb{R}_{>0}$ which $\int_{\mathbb{R}} \rho^{\frac{1}{s}}(t^2)\,dt < \infty$, then one has

$$\prod_{i=1}^{k} \int_{E_i} f_i(x_i)\,dx_i \leq \left[\int_{\mathbb{R}^n} \rho^{\frac{1}{s}}\left(\frac{s(s-1)}{2}\|x\|^2\right)dx\right]^s. \tag{6.10}$$

Proof. Since the f_i's are unconditional, it is enough to show (6.10) on the positive cones $(E_i)_+$ and $\mathbb{R}^n_+$, provide that (6.9) holds on the positive cones. This is because the left side of (6.10) is $\prod_{i=1}^{k}(2^{n_i})^{\frac{1}{s}} = 2^{\sum_{i=1}^{k}\frac{1}{s}n_i} = 2^n$ times the product of the integrals on $(E_i)_+$. For $x \in \mathbb{R}^n$ we demote $(x)_i$ the i-coordinate of x and for $m \in [n]$ we write $\Gamma_m = \{i \in [k] : m \in \sigma_i\}$. Readily, for $x_i \in E_i$, $(x_i)_m = 0$ when $m \notin \sigma_i$ and $|\Gamma_m| = s$. Thus, the sum in (6.9) is written

$$\sum_{i<j}\langle x_i, x_j\rangle = \sum_{m=1}^{n}\sum_{i<j}(x_i)_m(x_j)_m = \sum_{m=1}^{n}\sum_{\substack{i,j\in\Gamma_m\\ i<j}}(x_i)_m(x_j)_m. \tag{6.11}$$

Now, for $t_i \in E_i$ we denote e^{t_i} the vector in E_i defined by

$$(e^{t_i})_m = \begin{cases} e^{(t_i)_m} & m \in \sigma_i \\ 0 & m \notin \sigma_i \end{cases}$$

We apply the change of variable $x_i = e^{t_i}$ and we get

$$\prod_{i=1}^{k}\left(\int_{(E_i)_+} f_i(x_i)\,dx_i\right)^{\frac{1}{s}} = \prod_{i=1}^{k}\left(\int_{E_i} f_i(e^{t_i})e^{\sum_{m\in\sigma_i}(t_i)_m}\,dt_i\right)^{\frac{1}{s}}.$$

By the Reverse Brascamp Lieb inequality then assumption (6.9) and last (6.11), we have

$$\prod_{i=1}^{k}\left(\int_{(E_i)_+} f_i(x_i)\,dx_i\right)^{\frac{1}{s}} \leq \int_{\mathbb{R}^n} \sup_{\substack{t_i\in E_i\\ t=\sum_{i=1}^{k}\frac{1}{s}t_i}} \left[\left(\prod_{i=1}^{k} f_i^{\frac{1}{s}}(e^{t_i})e^{\frac{1}{s}\sum_{m\in\sigma_i}(t_i)_m}\right)\right]dt \tag{6.12}$$

$$\leq \int_{\mathbb{R}^n} \sup_{\substack{t_i\in E_i\\ t=\sum_{i=1}^{k}\frac{1}{s}t_i}} \left[\rho^{\frac{1}{s}}\left(\sum_{i<j}\langle e^{t_i}, e^{t_j}\rangle\right)\prod_{i=1}^{k} e^{\frac{1}{s}\sum_{m\in\sigma_i}(t_i)_m}\right]dt \tag{6.13}$$

$$= \int_{\mathbb{R}^n} \sup_{\substack{t_i\in E_i\\ t=\sum_{i=1}^{k}\frac{1}{s}t_i}} \left[\rho^{\frac{1}{s}}\left(\sum_{m=1}^{n}\sum_{\substack{i,j\in\Gamma_m\\ i<j}}(e^{t_i})_m(e^{t_j})_m\right)\prod_{m=1}^{n} e^{\frac{1}{s}\sum_{i\in\Gamma_m}(t_i)_m}\right]dt \tag{6.14}$$

For fixed $m = 1, \dots, n$, AM-GM inequality implies,

$$\sum_{\substack{i,j\in\Gamma_m\\ i<j}}(e^{t_i})_m(e^{t_j})_m \geq \frac{s(s-1)}{2}e^{\frac{2}{s(s-1)}\sum_{i,j\in\Gamma_m, i<j}(t_i+t_j)_m} = \frac{s(s-1)}{2}e^{\frac{2}{s}\sum_{i\in\Gamma_m}(t_i)_m} \tag{6.15}$$

Now, we use on (6.14) the monotonicity of ρ and we get,

$$\prod_{i=1}^{k}\left(\int_{(E_i)_+} f_i(x_i)\,dx_i\right)^{\frac{1}{s}} \leq \int_{\mathbb{R}^n} \sup_{\substack{t_i\in E_i\\ t=\sum_{i=1}^k \frac{1}{s}t_i}} \left[\rho^{\frac{1}{s}}\Big(\frac{s(s-1)}{2}\sum_{m=1}^{n} e^{\frac{2}{s}\sum_{i\in\Gamma_m}(t_i)_m}\Big)\prod_{m=1}^{n} e^{\frac{1}{s}\sum_{i\in\Gamma_m}(t_i)_m}\right] dt$$

$$= \int_{\mathbb{R}^n} \rho^{\frac{1}{s}}\Big(\frac{s(s-1)}{2}\sum_{m=1}^{n} e^{2(t)_m}\Big) e^{\sum_{m=1}^n (t)_m}\,dt \tag{6.16}$$

$$= \int_{\mathbb{R}^n_+} \rho^{\frac{1}{s}}\Big(\frac{s(s-1)}{2}\sum_{m=1}^{n} u_m^2\Big)\,du$$

$$= \int_{\mathbb{R}^n_+} \rho^{\frac{1}{s}}\Big(\frac{s(s-1)}{2}\|u\|^2\Big)\,du$$

□

Theorem 6.2.2. *Let $s,k \geq 1$ be two integers, $(\sigma_1,\dots,\sigma_k)$ be an s-uniform cover of $[n]$ and denote $E_i = \mathrm{span}\{e_m : m\in\sigma_i\}$ for some fixed basis $e_1,\dots,e_n$. If K_i be some unconditional convex bodies in E_i that*

$$\prod_{i=1}^{k} e^{-\frac{1}{2}\|x_i\|_{K_i}^2} \leq \rho\left(\sum_{\substack{i,j=1\\ i<j}}^{k}\langle x_i,x_j\rangle\right) \tag{6.17}$$

for a decreasing function $\rho:\mathbb{R}\to\mathbb{R}_{>0}$ which $\int_{\mathbb{R}}\rho^{\frac{1}{s}}(t^2)\,dt<\infty$, then one has

$$\prod_{i=1}^{k}|K_i| \leq \left(\prod_{i=1}^{k}\frac{|B_2^{n_i}|}{(2\pi)^{\frac{n_i}{2}}}\right)\left(\int_{\mathbb{R}^n}\rho^{\frac{1}{s}}\Big(\frac{s(s-1)}{2}|x|^2\Big)\,dx\right)^{s}. \tag{6.18}$$

In particular, if $\rho(t) = e^{-\frac{t}{k-1}}$ namely, the t convex bodies satisfy

$$\frac{k-1}{2}\sum_{i=1}^{k}\|x_i\|_{K_i}^2 \geq \sum_{i<j}\langle x_i,x_j\rangle,$$

then,

$$\prod_{i=1}^{k}|K_i| \leq \prod_{i=1}^{k}|B_2^{n_i}|.$$

Proof. Let $n_i = \dim E_i$. Then by Theorem 6.2.1 we have

$$\prod_{i=1}^{k}|K_i| = \prod_{i=1}^{k}\left(\frac{|B_2^{n_i}|}{(2\pi)^{\frac{n_i}{2}}}\int_{E_i} e^{-\frac{1}{2}\|x_i\|_{K_i}^2}\,dx_i\right) \leq \left(\prod_{i=1}^{k}\frac{|B_2^{n_i}|}{(2\pi)^{\frac{n_i}{2}}}\right)\left(\int_{\mathbb{R}^n}\rho^{\frac{1}{s}}\Big(\frac{s(s-1)}{2}|x|^2\Big)\,dx\right)^{s}.$$

□

6.3 Equality case of Bollobás-Thomason inequality and its dual

We write $e_1,\dots,e_n$ to denote an orthonomal basis of $\mathbb{R}^n$. For a compact set $K\subseteq\mathbb{R}^n$ with $\mathrm{aff}\,K = m$, we write $|K|$ to denote the m-dimensional Lebesgue measure of K.

The starting point of this section is the classical Loomis-Whitney inequality [108].

Theorem 6.3.1 (Loomis, Whitney). *If $K \subseteq \mathbb{R}^n$ is compact and affinely spans $\mathbb{R}^n$, then*

$$|K|^{n-1} \leq \prod_{i=1}^{k} |P_{e_i^\perp} K|, \tag{6.19}$$

with equality if and only if $K = \oplus_{i=1}^{n} K_i$ where $\mathrm{aff} K_i$ is a line parallel to e_i.

Meyer [122] provided a dual form of the Loomis-Whitney inequality where equality holds for affine crosspolytopes.

Theorem 6.3.2 (Meyer). *If $K \subseteq \mathbb{R}^n$ is compact convex with $o \in \mathrm{int} K$, then*

$$|K|^{n-1} \geq \frac{n!}{n^n} \prod_{i=1}^{k} |K \cap e_i^\perp|, \tag{6.20}$$

with equality if and only if $K = \mathrm{conv}\{\pm \lambda_i e_i\}_{i=1}^n$ for $\lambda_i > 0$, $i = 1, \dots, n$.

We note that various Reverse and dual Loomis-Whitney type inequalities are proved by Campi, Gardner, Gronchi [49], Brazitikos *et al* [42, 43], Alonso-Gutiérrez *et al* [2, 3].

To consider a genarization of the Loomis-Whitney inequality and its dual form, we set $[n] := \{1, \dots, n\}$, and for a non-empty proper subset $\sigma \subseteq [n]$, we define $E_\sigma = \mathrm{lin}\{e_i\}_{i \in \sigma}$. For $s \geq 1$, we say that the not necessarily distinct proper non-empty subsets $\sigma_1, \dots, \sigma_k \subseteq [n]$ form an s-uniform cover of $[n]$ if each $j \in [n]$ is contained in exactly s of $\sigma_1, \dots, \sigma_k$.

The Bollobás-Thomason inequality [25] reads as follows.

Theorem 6.3.3 (Bollobás, Thomason). *If $K \subseteq \mathbb{R}^n$ is compact and affinely spans $\mathbb{R}^n$, and $\sigma_1, \dots, \sigma_k \subseteq [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$, then*

$$|K|^s \leq \prod_{i=1}^{k} |P_{E_{\sigma_i}} K|. \tag{6.21}$$

We note that additional the case when $k = n$, $s = n - 1$, and hence when we may assume that $\sigma_i = [n] \backslash e_i$, is the Loomis-Whitney inequality Therem 6.3.1.

Liakopoulos [104] managed to prove a dual form of the Bollobás-Thomason inequality. For a finite set σ, we write $|\sigma|$ to denote its cardinality.

Theorem 6.3.4 (Liakopoulos). *If $K \subseteq \mathbb{R}^n$ is compact convex with $o \in \mathrm{int} K$, and $\sigma_1, \dots, \sigma_k \subseteq [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$, then*

$$|K|^s \geq \frac{\prod_{i=1}^{k} |\sigma_i|!}{(n!)^s} \cdot \prod_{i=1}^{k} |K \cap E_{\sigma_i}|. \tag{6.22}$$

However, unlike for Loomis-Whitney inequality and its dual form, neither the equality cases of the Bollobás-Thomason inequality nor of its dual are known. The characterization of the equality cases of Theorem 6.3.3 and Theorem 6.3.4 is the main focus of this section.

Let $s \geq 1$, and let $\sigma_1, \dots, \sigma_k \subseteq [n]$ be an s-uniform cover of $[n]$. We say that $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l \subseteq [n]$ form a 1-uniform cover of $[n]$ induced by the s-uniform cover $\sigma_1, \dots, \sigma_k$ if $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_l\}$ consists of all non-empty distinct subsets of $[n]$ of the form $\cap_{i=1}^{k} \sigma_i^{\varepsilon(i)}$ where $\varepsilon(i) \in \{0, 1\}$ and $\sigma_i^0 = \sigma_i$ and $\sigma_i^1 = [n] \backslash \sigma_i$. We observe that $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l \subseteq [n]$ actually form a 1-uniform cover of $[n]$; namely, $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ is a partition of $[n]$.

Theorem 6.3.5. *Let $K \subseteq \mathbb{R}^n$ be compact and affinely span $\mathbb{R}^n$, and let $\sigma_1, \dots, \sigma_k \subseteq [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$. Then equality holds in* (6.21) *if and only if $K = \oplus_{i=1}^{l} P_{E_{\tilde{\sigma}_i}} K$ where $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ is the 1-uniform cover of $[n]$ induced by $\sigma_1, \dots, \sigma_k$.*

Concerning the dual Bollobás-Thomason inequality Theorem 6.3.4, we have a similar result.

Theorem 6.3.6. *Let $K \subseteq \mathbb{R}^n$ be compact convex with $o \in \mathrm{int}K$, and let $\sigma_1, \dots, \sigma_k \subseteq [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$. Then equality holds in (6.22) if and only if $K = \mathrm{conv}\{K \cap F_{\tilde{\sigma}_i}\}_{i=1}^l$ where $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ is the 1-uniform cover of $[n]$ induced by $\sigma_1, \dots, \sigma_k$.*

According to Liakopoulos [104] (see also Section 6.3), a simply way to prove Theorem 6.3.3 and Theorem 6.3.4 is via the Geometric Brascamp-Lieb inequality Theorem 2.2.6 and its Reverse form Theorem 2.2.3. In particular, we prove the equality case Theorem 6.3.5 of the Bollobás-Thomason inequality via the characterization of the equality case Theorem 2.2.7 due to by Valdimarsson [152] of the Brascamp-Lieb inequality. In addition, we prove Theorem 2.2.8 characterizing the equality case of the Reverse Brascamp-Lieb inequality in a special case that yields the understanding of equality in the dual Bollobás-Thomason inequality.

We will denote with $\sigma_i^0 = \sigma_i$ and $\sigma_i^1 = [n] \setminus \sigma_i$. When we write $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ for the induced cover from $\sigma_1, \dots, \sigma_k$, we assume that the sets are distinct.

Lemma 6.3.7. *For $s \geq 1$, let $\sigma_1, \dots, \sigma_k \subseteq [n]$ form an s-uniform cover of $[n]$, and let $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ be the 1-uniform cover of $[n]$ induced by $\sigma_1, \dots, \sigma_k$. Then*

(i) *for any fixed orthonormal basis $e_1, \dots, e_n$, the subspaces $E_{\sigma_i} := \{e_j : i \in \sigma_i\}$ satisfy*

$$\sum_{i=1}^{k} \frac{1}{s} P_{E_{\sigma_i}} = I_n \tag{6.23}$$

i.e. form a geometric Brascamp Lieb data.

(ii) *the elements $\tilde{\sigma}_i$ have the following form: there is $r \in [n]$ so that,*

$$\tilde{\sigma}_i := \bigcap_{r \in \sigma_i} \sigma_i^0 \cap \bigcap_{r \notin \sigma_i} \sigma_i^1 \tag{6.24}$$

(iii) *the subspaces $F_{\tilde{\sigma}_i} := \mathrm{lin}\{e_j : j \in \tilde{\sigma}_i\}$ are the independent subspaces of the data (6.23) and $F_{\mathrm{dep}} = \{o\}$.*

Proof. (i) Since $\sigma_1, \dots, \sigma_k$ form a s-uniform cover, every $e_i \in \mathbb{R}^n$ is contained in exactly s of $E_{\sigma_1}, \dots, E_{\sigma_k}$. So (i) follows.

(ii) Let $\sigma_1, \dots, \sigma_k$ be just subsets of $[n]$. We take a $I \subseteq [k]$ of cardinality s, and we consider the set

$$A_I := \bigcap_{i \in I} \sigma_i^0 \cap \bigcap_{i \notin I} \sigma_i^1.$$

If, after a replacement of 0 by 1 (1 by 0) in the left (right) big intersection we have that the new A_I is not empty, then there is $\tau \in [n]$ so that τ is contained in exactly $s-1$ $(s+1)$ from $\sigma_1, \dots, \sigma_k$. Now with the additional property that $\sigma_1, \dots, \sigma_k \subseteq [n]$ form an s-uniform cover of $[n]$, we have that any $\tilde{\sigma}_i$ has the form of A_I, and also for some $r \in [n]$

$$I \subseteq \{i \in [k] : r \in \sigma_i\}$$

Since both cardinalities of the above sets is s we conclude to (6.24).

(iii) If we prove the independence of the subspaces, then immediate we have that $F_{\mathrm{dep}} = \{o\}$ since for each $r \in [n]$ we have that $r \in A_{I_r}$ where $I_r = \{i \in [k] : r \in \sigma_i\}$, namely one of the subspaces $F_{\tilde{\sigma}_1}, \dots, F_{\tilde{\sigma}_l}$ contains e_r and so they span $\mathbb{R}^n$. Now the independance follows from the easy observation,

$$\cap_{j=1}^k (\mathrm{lin}\{e_i : i \in \sigma_j\})^{\varepsilon(j)} = \mathrm{lin}\{e_i : i \in \cap_{j=1}^k \sigma_j^{\varepsilon(j)}\}$$

where, when ε takes the value 1, the left ε is the orthogonal complement in $\mathbb{R}^n$ and the right ε is the complement in $[n]$.

□

Let us introduce the notation that we use when handling both the Bollobás-Thomason inequality and its dual. Let $\sigma_1, \ldots, \sigma_k$ be the s cover of $[n]$ occuring in Theorem 6.3.5 and Theorem 6.3.6, and hence $E_i = E_{\sigma_i}$, $i = 1, \ldots, k$, satisfies

$$\sum_{i=1}^{k} \frac{1}{s} \cdot P_{E_{\sigma_i}} = I_n. \tag{6.25}$$

Let $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l$ be the 1-uniform cover of $[n]$ induced by $\sigma_1, \ldots, \sigma_k$. It follows that

$$F_j = E_{\tilde{\sigma}_j} \quad \text{for } j = 1, \ldots, l \text{ are the independent subspaces,} \tag{6.26}$$

$$F_{\text{dep}} = \{o\}. \tag{6.27}$$

For any $i \in \{1, \ldots, k\}$, we set

$$I_i = \{j \in \{1, \ldots, l\} : F_j \subseteq E_i\},$$

and for any $j \in \{1, \ldots, l\}$, we set

$$J_j = \{i \in \{1, \ldots, k\} : F_j \subseteq E_i\}.$$

For the reader's convenience, we restate Theorem 6.3.3 and Theorem 6.3.5 as Theorem 6.3.8, and Theorem 6.3.4 and Theorem 6.3.6 as Theorem 6.3.9.

Theorem 6.3.8. *If $K \subseteq \mathbb{R}^n$ is compact and affinely spans $\mathbb{R}^n$, and $\sigma_1, \ldots, \sigma_k \subseteq [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$, then*

$$|K|^s \leq \prod_{i=1}^{k} |P_{E_{\sigma_i}} K|. \tag{6.28}$$

Equality holds if and only if $K = \oplus_{i=1}^{l} P_{F_{\tilde{\sigma}_i}} K$ where $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l$ is the 1-uniform cover of $[n]$ induced by $\sigma_1, \ldots, \sigma_k$ and $F_{\tilde{\sigma}_i}$ is the linear hull of the e_i's with indeces from $\tilde{\sigma}_i$.

Proof. We denote with $E_i := E_{\sigma_i}$, where from Lemma 6.3.7 (**??**) these subspaces compose a geometric data. We start with a proof of Bollobás-Thomason inequality. It follows directly from the Brascamp-Lieb inequality as

$$\begin{aligned} |K| = \int_{\mathbb{R}^n} 1_K(x)\, dx &\leq \int_{\mathbb{R}^n} \prod_{i=1}^{k} 1_{P_{E_i}(K)}(P_{E_i}(x))^{\frac{1}{s}}\, dx \\ &\leq \prod_{i=1}^{k} \Big(\int_{E_i} 1_{P_{E_i}(K)} \Big)^{\frac{1}{s}} = \prod_{i=1}^{k} |P_{E_i}(K)|^{\frac{1}{s}} \end{aligned} \tag{6.29}$$

where the first inequality is from the monotonicity of the integral while the second is Brasmap-Lieb inequality Theorem 2.2.6. Now, if equality holds in (6.29), then on the one hand,

$$1_K(x) = \prod_{i=1}^{k} 1_{P_{E_i}(K)}(P_{E_i}(x))$$

and on the other hand, if $F_1, \ldots, F_l$ are the independent subspaces of the data, which from Lemma 6.3.7 (**??**) they span $\mathbb{R}^n$, namely $F_{\text{dep}} = \{0\}$, by Theorem 2.2.7 there are integrable functions $h_j : F_j \to \mathbb{R}$, such that, for Lebesgue a.a. $x_i \in E_i$

$$1_{P_{E_i} K}(x_i) = \theta_i \prod_{j \in I_i} h_j(P_{F_j}(x_i))$$

Therefore from the previous two, we have for $x \in \mathbb{R}^n$

$$1_K(x) = \prod_{i=1}^{k} \theta_i \prod_{j \in I_i} h_j(P_{F_j}(P_{E_i}(x)))$$

Now, since for $j \in I_i$ we have $F_j \subseteq E_i$ we can delete the P_{E_i} on the above product. Thus, for $\theta = \prod_{i=1}^{k} \theta_i$, we have for Lebesgue a.a. $x \in \mathbb{R}^n$

$$1_K(x) = \theta \prod_{i=1}^{k} \prod_{j \in I_i} h_j(P_{F_j}(x)) = \theta \prod_{j=1}^{l} h_j(P_{F_j}(x))^{|J_j|}. \tag{6.30}$$

Now, for $x \in K$ the last product on above is constant, so

$$\theta = \frac{1}{\prod_{i=1}^{l} h_j(P_{F_j}(x_0))^{|J_j|}} \tag{6.31}$$

for some $x_o \in K$. For $j = 1, \dots, l$ we set $\varphi_j : F_j \to \mathbb{R}^n$, by

$$\varphi_j(x) = \frac{h_j(x + P_{F_j}(x_0))^{|J_j|}}{h_j(P_{F_j}(x_0))^{|J_j|}}.$$

We see that $\varphi_j(o) = 1$ and also (6.30) and (6.31) yields

$$1_{K-x_0}(x) = \prod_{j=1}^{l} \varphi_j(P_{F_j}(x)) \tag{6.32}$$

For $m \in \{1, \dots, l\}$, taking $x \in F_m$ in (6.32) (and hence $\varphi_j(P_{F_j}(x)) = 1$ for $j \neq m$) shows that

$$1_{K-x_0}(y) = \varphi_m(y),$$

for Lebesgue a.a. $y \in F_m$. Therefore (6.32) and the ortgonality of the F_j's,

$$K - x_0 = \bigcap_{j=1}^{l} P_{F_j}^{-1}(P_{F_j}(K - x_o)) = \bigoplus_{j=1}^{l} P_{F_j}(K - x_o),$$

completing the proof of Theorem 6.3.8. $\square$

To prove Theorem 6.3.9, we use two small observations. First if M is any convex body with $o \in \operatorname{int} M$, then

$$\int_{\mathbb{R}^n} e^{-\|x\|_M}\, dx = \int_0^{\infty} e^{-r} n r^{n-1} |M|\, dr = n!|M|. \tag{6.33}$$

Secondly, if F_j are pairwise orthogonal subspaces and $M = \operatorname{conv}\{M_1, \dots, M_l\}$ where $M_j \subseteq F_j$ is a $\dim F_j$-dimensional compact convex set with $o \in \operatorname{relint} M_j$, then for any $x \in \mathbb{R}^n$

$$\|x\|_M = \sum_{i=1}^{l} \|P_{F_j} x\|_{M_j}. \tag{6.34}$$

In addition, we often use the fact, for a subspace F of $\mathbb{R}^n$ and $x \in F$, then $\|x\|_K = \|x\|_{K \cap F}$.

Theorem 6.3.9. *If $K \subseteq \mathbb{R}^n$ is compact convex with $o \in \operatorname{int} K$, and $\sigma_1, \dots, \sigma_k \subseteq [n]$ form an s-uniform cover of $[n]$ for $s \geq 1$, then*

$$|K|^s \geq \frac{\prod_{i=1}^{k} |\sigma_i|!}{(n!)^s} \cdot \prod_{i=1}^{k} |K \cap E_{\sigma_i}|. \tag{6.35}$$

Equality holds if and only if $K = \operatorname{conv}\{E_{\tilde{\sigma}_i} \cap K\}_{i=1}^{l}$ where $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ is the 1-uniform cover of $[n]$ induced by $\sigma_1, \dots, \sigma_k$.

Proof. We define

$$f(x) = e^{-\|x\|_K}, \tag{6.36}$$

which is a log-concave function with $f(o) = 1$, and satisfying (*cf* (6.33))

$$\int_{\mathbb{R}^n} f(y)^n \, dy = \int_{R^n} e^{-n\|y\|_K} \, dy = \int_{R^n} e^{-\|y\|_{\frac{1}{n}K}} = n! \left| \frac{1}{n} K \right| = \frac{n!}{n^n} \cdot |K|. \tag{6.37}$$

We claim that

$$n^n \int_{\mathbb{R}^n} f(y)^n \, dy \geq \prod_{i=1}^{k} \Big(\int_{E_i} f(x_i) \, dx_i \Big)^{1/s}. \tag{6.38}$$

Equating the traces of the two sides of (6.23), we deduce that, $d_i := |\sigma_i| = \dim E_i$

$$\sum_{i=1}^{k} \frac{d_i}{sn} = 1. \tag{6.39}$$

For $z = \sum_{i=1}^{k} \frac{1}{s} x_i$ with $x_i \in E_i$, the log-concavity of f and its definition (6.36), imply

$$f(z/n) \geq \prod_{i=1}^{k} f(x_i/d_i)^{\frac{d_i}{ns}} = \prod_{i=1}^{k} f(x_i)^{\frac{1}{ns}}. \tag{6.40}$$

Now, the monotonicity of the integral, and Reverse Brascamp Lieb inequality, give

$$\int_{\mathbb{R}^n} f(z/n)^n \, dz \geq \int_{\mathbb{R}^n}^{*} \sup_{z = \sum_{i=1}^{k} \frac{1}{s} x_i, \, x_i \in E_i} \prod_{i=1}^{k} f(x_i)^{1/s} \, dz \geq \prod_{i=1}^{k} \Big(\int_{E_i} f(x_i) \, dx_i \Big)^{1/s}. \tag{6.41}$$

Making the change of variable $y = z/n$ we conclude to (6.38). Computing the right hand side of (6.38), we have

$$\int_{E_i} f(x_i) \, dx_i = \int_{E_i} e^{-\|x_i\|_K} \, dx_i = \int_{E_i} e^{-\|x_i\|_{K \cap E_i}} \, dx_i = d_i! |K \cap E_i|. \tag{6.42}$$

Therefore, (6.37), (6.38) and (6.42) yield (6.35).

Let us assume that equality holds in (6.35), and hence we have two equalities in (6.41). We set

$$M = \mathrm{conv}\{K \cap F_j\}_{1 \leq j \leq l}.$$

Clearly, $K \supseteq M$. For the other inclusion, we start with $z \in \mathrm{int} K$, namely $\|z\|_K < 1$. Equality in the first inequality in (6.41) means,

$$\left(e^{-\|z/n\|_K} \right)^n = \sup_{z = \sum_{i=1}^{k} \frac{1}{s} x_i, \, x_i \in E_i} \prod_{i=1}^{k} e^{-\|x_i\|_K 1/s},$$

or in other words,

$$\|z\|_K = \frac{1}{s} \cdot \inf_{z = \sum_{i=1}^{k} \frac{1}{s} x_i, \, x_i \in E_i} \sum_{i=1}^{k} \|x_i\|_K = \inf_{z = \sum_{i=1}^{k} y_i, \, y_i \in E_i} \sum_{i=1}^{k} \|y_i\|_K. \tag{6.43}$$

We deduce that there exist $y_i \in E_i$, $i = 1, \dots, k$ such that

$$z = \sum_{i=1}^{k} y_i \quad \text{and} \quad \sum_{i=1}^{k} \|y_i\|_K < 1, \tag{6.44}$$

Therefore, from (6.44), then (6.34) and after the triangle inequality for $\|\cdot\|_{K\cap F_j}$, we have

$$\|z\|_M = \left\|\sum_{i=1}^{k}\sum_{j\in I_i} P_{F_j}y_i\right\|_M = \sum_{i=1}^{k}\left\|\sum_{i\in I_i} P_{F_j}y_i\right\|_{K\cap F_j} \leq \sum_{i=1}^{k}\sum_{i\in I_i}\left\|P_{F_j}y_i\right\|_{K\cap F_j}. \tag{6.45}$$

It suffices to show that

$$K\cap E_i = \operatorname{conv}\{K\cap F_j\}_{j\in I_i} \tag{6.46}$$

because then, from (6.45), applying (6.34) and (6.44), we have

$$\|z\|_M \leq \sum_{j=1}^{l}\sum_{i\in J_j}\left\|P_{F_j}y_i\right\|_{K\cap F_j} = \sum_{i=1}^{k}\|y_i\|_{K\cap E_i} < 1,$$

which means $z\in M$. Now, to show (6.46), we start with the equality case of the Reverse Brascamp-Lieb inequality which has been applied in (6.41). From Theorem 2.2.8, there exist $\theta_i>0$ and $w_i\in E_i$ and log-concave $h_j : F_j\to[0,\infty)$, namely $h_j=e^{-\varphi_j}$ for a convex functon φ_j, such that

$$e^{-\|x_i\|_{K\cap E_i}} = \theta_i\prod_{j\in I_i} h_j(P_{F_j}(x_i-w_i)). \tag{6.47}$$

for Lebesgue a.a. $x_i\in E_i$. For $i\in[k]$ and $j\in I_i$ we set, $\psi_{ij}:F_j\to\mathbb{R}$ by

$$\psi_{ij}(x) = \varphi_j\left(x-P_{F_j}w_i\right) - \varphi_j\left(-P_{F_j}w_i\right) + \frac{\ln\theta_i}{|I_i|}.$$

We see

$$\psi_{ij}(o)=0 \text{ and } \psi_{ij} \text{ is convex on } F_j. \tag{6.48}$$

and also (6.47) yields, for $x\in E_i$

$$e^{-\|x\|_{K\cap E_i}} = \exp\left(-\sum_{j\in I_i}\psi_{ij}(P_{F_j}x)\right). \tag{6.49}$$

For $x\in F_j$, we apply λx to (6.49) with $\lambda>0$, and we have from $\psi_{im}(o)=0$ for $m\in I_i\backslash\{j\}$ that

$$\psi_{ij}(\lambda x)=\lambda\psi_{ij}(x) \text{ and } \psi_{ij}(x)>0. \tag{6.50}$$

We deduce from (6.48) and (6.50) that ψ_{ij} is a norm. Therefore, $\psi_{ij}(x)=\|x\|_{C_{ij}}$ for some $(\dim F_j)$-dimensional compact convex set $C_{ij}\subseteq F_j$ with $o\in\operatorname{relint}C_{ij}$. Now (6.49) becomes,

$$\|x\|_{K\cap E_i} = \sum_{j\in I_i}\|P_{F_j}x\|_{C_{ij}}$$

and hence by (6.34) we conclude to

$$K\cap E_i = \operatorname{conv}\{C_{ij}\}_{j\in I_i}.$$

In particular, if $i\in[k]$ and $j\in I_i$, then $C_{ij}=(K\cap E_i)\cap F_j = K\cap F_j$, and hence we have (6.46) and the proof is finished. □

Bibliography

[1] A.D. Alexandrov: *Zur Theorie der gemischten Volumina von konvexen Körpern, III*, Die Erweiterung zweier Lehrsätze Minkowskis über die konvexen Polyeder auf beliebige konvexe Flächen (in Russian). Mat. Sbornik N. S. 3 (1938), 27-46

[2] D. Alonso-Gutiérrez, J. Bernués, S. Brazitikos, A. Carbery: On affine invariant and local Loomis-Whitney type inequalities. arXiv:2002.05794

[3] D. Alonso-Gutiérrez, S. Brazitikos: Reverse Loomis-Whitney inequalities via isotropicity. arXiv:2001.11876

[4] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis*, Part I, Amer. Math. Soc., Mathematical Surveys and Monographs 202 (2015).

[5] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis*, Part II, Amer. Math. Soc., Mathematical Surveys and Monographs 261 (2021).

[6] S. Artstein-Avidan, B. Klartag, V. D. Milman, *The Santaló point of a function and a functional form of Santaló inequality*, Mathematika 51 (2005), pp 33–48.

[7] S. Artstein-Avidan, V. D. Milman, *The concept of duality in asymptotic geometric analysis, and the characterization of the Legendre transform*, Ann. of Math. 169 (2009), pp 661-674.

[8] S. Artstein-Avidan, S. Sadovsky, K. Wyczesany, *A Zoo of Dualities*, 2021 (preprint). arXiv:2110.11308

[9] K. Ball, *Isometric problems in ℓ_p and sections of convex sets*, PhD dissertation, Cambridge (1986).

[10] K. Ball, *Some remarks on the geometry of convex sets.* In: Lindenstrauss J., Milman V.D. (eds) Geometric Aspects of Functional Analysis. Lecture Notes in Mathematics, vol 1317 (1988) Springer, Berlin, Heidelberg.

[11] K. Ball, *Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$*, Studia Math. 88 (1) (1988) , pp 69-84.

[12] K.M. Ball: *Volumes of sections of cubes and related problems.* In: J. Lindenstrauss and V.D. Milman (ed), Israel seminar on Geometric Aspects of Functional Analysis 1376, Lectures Notes in Mathematics. Springer-Verlag, 1989.

[13] K.M. Ball: *Volume ratios and a reverse isoperimetric inequality.* J. London Math. Soc. 44 (1991), 351–359

[14] F. Barthe: Inégalités de Brascamp-Lieb et convexité. C. R. Acad. Sci. Paris 324 (1997), 885–888.

[15] F. Barthe: *On a reverse form of the Brascamp-Lieb inequality.* Invent. Math. 134 (1998), 335–361.

[16] F. Barthe: *A continuous version of the Brascamp-Lieb inequalities.* Geometric Aspects of Functional Analysis, Lecture Notes in Mathematics Volume 1850, 2004, 53–63.

[17] F. Barthe, D. Cordero-Erausquin: *Inverse Brascamp-Lieb inequalities along the heat equation. Geometric aspects of functional analysis*, 65-71, Lecture Notes in Math., 1850, Springer, Berlin, 2004.

[18] F. Barthe, N. Huet: *On Gaussian Brunn-Minkowski inequalities.* Studia Math. 191 (2009), 283–304.

[19] F. Barthe, D. Cordero-Erausquin:*Invariances in variance estimates.* Proc. Lond. Math. Soc., (3) 106 (2013), 33-64.

[20] F. Barthe, M. Fradelizi: *The volume product of convex bodies with many hyperplane symmetries*, Amer. J. Math., 135 (2013), 311-347.

[21] J. Bennett, T. Carbery, M. Christ, T. Tao: *The Brascamp–Lieb Inequalities: Finiteness, Structure and Extremals.* Geom. Funct. Anal. 17 (2008), 1343–1415.

[22] G. Bianchi, K.J. Böröczky, A. Colesanti, D. Yang: *The L_p-Minkowski problem for $-n < p < 1$ according to Chou-Wang.* Adv. Math., 341 (2019), 493-535.

[23] W. Blaschke, *Über Affine Geometrie VII: Neue Extremeigenschaften von Ellipse und Ellipsoid*, Leipziger Ber. 69 (1917), pp 306-318 .

[24] B. Bollobás, I. Leader: *Products of unconditional bodies.* Geometric aspects of functional analysis (Israel, 1992–1994), Oper. Theory Adv. Appl., 77, Birkhauser, Basel, (1995), 13-24.

[25] B. Bollobás, A. Thomason: *Projections of bodies and hereditary properties of hypergraphs.* Bull. Lond. Math. Soc. 27, (1995), 417–424.

[26] T. Bonnesen, W. Fenchel: *Theory of convex bodies.* Translated from the German and edited by L. Boron, C. Christenson and B. Smith. BCS Associates, Moscow, ID, 1987.

[27] C. Borell: *The Brunn-Minkowski inequality in Gauss spaces.* Invent. Math., 30 (1975), 207-216.

[28] C. Borell: *Convex set functions in d-space*, Period. Math. Hungar., 6 (1975), 111-136.

[29] K.J. Böröczky, M. Henk: *Cone-volume measure of general centered convex bodies.* Advances Math., 286 (2016), 703-721.

[30] K.J. Böröczky, P. Hegedűs: *The cone volume measure of antipodal points.* Acta Mathematica Hungarica, 146 (2015), 449-465.

[31] K.J. Böröczky, P. Kalantzopoulos, *Log-Brunn-Minkowski inequality under symmetry*, Transactions of the American Mathematical Society, 375 (2022), 5987-6013.

[32] K.J. Böröczky, P. Kalantzopoulos, D. Xi, *About the case of equality in the Reverse Brascamp-Lieb inequality*, Preprint available at https://arxiv.org/pdf/2203.01428.pdf

[33] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang: *The log-Brunn-Minkowski-inequality.* Advances in Mathematics, 231 (2012), 1974-1997.

[34] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang: *The Logarithmic Minkowski Problem.* Journal of the American Mathematical Society, 26 (2013), 831-852.

[35] K. J. Böröczky, R. Schneider, *A characterization of the duality mapping for convex bodies*, Geom. Funct. Anal. 18 (2008), pp 657-667.

[36] J. Bourgain, V. D. Milman, *New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$*, Invent. Math. 88 (1987), pp 319-340.

[37] H.J. Brascamp, E.H. Lieb: *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis 22 (1976), 366-389.

[38] H.J. Brascamp, E.H. Lieb, J.M. Luttinger: *A general rearrangement inequality for multiple integrals*, J. Funct. Anal. 17 (1974) 227–237.

[39] H.J. Brascamp, E.H. Lieb: *Best constants in Young's inequality, its converse, and its generalization to more than three functions*. Adv. Math. 20 (1976), 151-173.

[40] S. Brazitikos: *Brascamp-Lieb inequality and quantitative versions of Helly's theorem*, Mathematika 63 (2017), 272-291.

[41] S. Brazitikos, A. Giannopoulos, P. Valettas, B.-H. Vritsiou: *Geometry of isotropic convex bodies.* Mathematical Surveys and Monographs 196, American Mathematical Society, Providence, RI, 2014.

[42] S. Brazitikos, S. Dann, A. Giannopoulos, A. Koldobsky: On the average volume of sections of convex bodies. Israel J. Math. 222 (2017), 921–947.

[43] S. Brazitikos, A. Giannopoulos, D-M. Liakopoulos: *Uniform cover inequalities for the volume of coordinate sections and projections of convex bodies*. Adv. Geom. 18 (2018), 345–354.

[44] L.A. Caffarelli: A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. Ann. of Math. (2) 131, (1990), 129-134.

[45] L.A. Caffarelli: Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. Ann. of Math. (2), 131 (1990), 135-150.

[46] L.A. Caffarelli: The regularity of mappings with a convex potential. J. Amer. Math. Soc., 5 (1992), 99-104.

[47] L.A. Caffarelli: *Monotonicity properties of optimal transportation and the FKG and related inequalities*. Comm. Math. Phys. 214 (2000), no. 3, 547-563.

[48] U. Caglar, M. Fradelizi, O. Guédon, J. Lehec, C. Schutt, E.M. Werner . *Functional versions of Lp-affine surface area and entropy inequalities*, Int. Math. Res. Not. IMRN 4 1223–1250. MR3493447 (2016)

[49] S. Campi, R. Gardner, P. Gronchi: Reverse and dual Loomis-Whitney-type inequalities. Trans. Amer. Math. Soc., 368 (2016), 5093–5124.

[50] E. Carlen, E.H. Lieb, M. Loss: *A sharp analog of Young's inequality on* S^N *and related entropy inequalities.* J. Geom. Anal., 14 (2004), 487-520.

[51] S. Chen, Y. Huang, Q. Li, J. Liu: *The* L_p*-Brunn-Minkowski inequality for* $p \in (1 - \frac{c}{n^{\frac{3}{2}}}, 1)$, arXiv:1811.10181

[52] S. Chen, Q.-R. Li, G. Zhu, *The Logarithmic Minkowski Problem for non-symmetric measures.* Trans. Amer. Math. Soc., 371 (2019), 2623-2641.

[53] S. Chen, Q.-R. Li, G. Zhu: *On the* L_p *Monge-Ampère equation.* Journal of Differential Equations, 263 (2017), 4997-5011.

[54] K. S. Chou, X. J. Wang: *The* L_p*-Minkowski problem and the Minkowski problem in centroaffine geometry*, Adv. Math., 205 (2006), 33-83.

[55] A. Colesanti: *From the Brunn-Minkowski inequality to a class of Poincare' type inequalities*, Communications in Contemporary Mathematics, 10 n. 5 (2008), 765-772.

[56] A. Colesanti, G. Livshyts, *A note on the quantitative local version of the log-Brunn-Minkowski inequality*, (2018) arXiv:1710.10708.

[57] A. Colesanti, G. V. Livshyts, A. Marsiglietti: *On the stability of Brunn-Minkowski type inequalities*, Journal of Functional Analysis, Volume 273, 3, (2017), 1120-1139.

[58] M. Colombo, M. Fathi: *Bounds on optimal transport maps onto log-concave measures.* J. Differential Equations 271 (2021), 1007-1022.

[59] D. Cordero-Erausquin: *Santaló's inequality on $\mathbb{C}^n$ by complex interpolation.* C. R. Math. Acad. Sci. Paris, 334(9):767–772, 2002.

[60] D. Cordero-Erausquin, M. Fradelizi, B. Maurey: *The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems*, J. Funct. Anal., 214 (2004), 410-427.

[61] D. Cordero -Erausquin, R.J. McCann, M. Schmuckenschlager: *Prékopa–Leindler type inequalities on Riemannian manifolds, Jacobi fields, and optimal transport*, Annales de la faculté des sciences de Toulouse Mathématiques, Annales de la faculté des sciences de Toulouse Mathématiques, p 613-635, 2006

[62] D. Cordero-Erausquin, L. Rotem: *Improved log-concavity for rotationally invariant measures of symmetric convex sets*, arXiv:2111.05110

[63] T.A. Courtade, P. Cuff, J. Liu, S. Verdú : *A Forward-Reverse Brascamp-Lieb Inequality: Entropic Duality and Gaussian Optimality, Entropy (special issue on information inequalities)*, 20(6): 418, 2018.

[64] T.A. Courtade, J. Liu: *Euclidean forward-reverse Brascamp-Lieb inequalities: finiteness, structure, and extremals.* J. Geom. Anal., 31 (2021), 3300–3350.

[65] G. Crasta, I. Fragalá, *On a Geometric combination of functions related to Prékpa-Leindler inequality*, arXiv:2204.11521

[66] M.W. Davis: *The Geometry and Topology of Coxeter Groups*, Princeton University Press, 2008.

[67] S. Dubuc: *Critères de convexité et inégalités intégrales*, Ann. Inst. Fourier Grenoble, 27 (1) (1977), 135–165.

[68] A. Eskenazis, G. Moschidis: *The dimensional Brunn-Minkowski inequality in Gauss space*, J. Funct. Anal. 280 (2021), 108914, 19 pp.

[69] M. Fathi, *A sharp symmetrized form of Talagrand's transport-entropy inequality for the Gaussian measure*, Electron. Commun. Probab. 23 (2018), Paper No. 81, 9.

[70] M. Fathi, N. Gozlan, M. Prod'homme: *A proof of the Caffarelli contraction theorem via entropic regularization.* Calc. Var. Partial Differential Equations 59 (2020), no. 3, Paper No. 96, 18 pp.

[71] W. Fenchel, B. Jessen, *Mengenfunktionen und konvexe Körper.* Danske Vid. Selskab. Mat.-fys. Medd. 16, 3 (1938), 31 pp.

[72] A. Figalli: *On the Monge-Ampére equation*, Séminaire Bourbaki Juin, 70e année, 2017–2018, no 1147, 2018

[73] W.J. Firey: *p-means of convex bodies*, Math. Scand., 10 (1962), 17-24.

[74] W.J. Firey: *Shapes of worn stones*, Mathematika 21 (1974), 1-11.

[75] M. Fradelizi, M. Meyer, *Some functional forms of Blaschke-Santaló inequality*, Math. Z., Springer, 256 (2) (2007), pp 379-395 .

[76] R. Gardner, *The Brunn-Minkowski inequality,* Bull. Amer. Math. Soc. 39 (2002), 355-405.

[77] R. Gardner, A. Zvavitch, *Gaussian Brunn-Minkowski-type inequlities,* Trans. Amer. Math. Soc., 360, (2010), 10, 5333-5353.

[78] A. Garg, L. Gurvits, R. Oliveira, A. Wigderson: *Algorithmic and optimization aspects of brascamp-lieb inequalities, via operator scaling*, Geometric and Functional Analysis, 28(1):100–145, 2018.

[79] Y. Gordon, M. Meyer, and S. Reisner: *Zonoids with minimal volume-product—a new proof*, Proc. Amer. Math. Soc. 104 (1988), no. 1, 273–276.

[80] L. Grafakos: *Classical Fourier analysis*, Graduate Texts in Mathematics, 249. Springer, 2014.

[81] Ramon van Handel: *The local logarithmic Brunn-Minkowski inequality for zonoids*, arXiv:2202.09429

[82] Q. Huang, A. Li, *The functional version of the Ball inequality*, Proc. Amer. Math. Soc. 145 (2017), pp 3531-3541.

[83] D. Hug, E. Lutwak, D. Yang, G. Zhang, *On the L_p Minkowski problem for polytopes,* Discrete Comput. Geom., 33 (2005), 699-715.

[84] J.E. Humphreys: *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York-Berlin, 1978.

[85] J.E. Humphreys: *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.

[86] H. Iriyeh, M. Shibata, *Symmetric Mahler's conjecture for the volume product in the 3-dimensional case*, Duke Math. J. 169 (6) (2020), pp 1077 - 1134

[87] H. Jian, J. Lu, X-J. Wang: *Nonuniqueness of solutions to the L_p-Minkowski problem.* Adv. Math., 281 (2015), 845-856.

[88] F. John: *Polar correspondence with respect to a convex region*, Duke Math. J. 3 (1937), 355–369.

[89] P. Kalantzopoulos, C. Saroglou, *On a j-Santaló Conjecture*, Preprint available at https://arxiv.org/pdf/2203.14815.pdf

[90] Y.-H. Kim, E. Milman: *A generalization of Caffarelli's contraction theorem via (reverse) heat flow.* Math. Ann. 354 (2012), no. 3, 827-862.

[91] B. Klartag,: *Marginals of Geometric Inequalities*, Geometric Aspects of Functional Analysis, Lecture Notes in Math., vol. 1910 (Springer, Berlin, 2007), pp. 133–166

[92] B. Klartag, E. Putterman: *Spectral monotonicity under Gaussian convolution.* arXiv:2107.09496

[93] A. V. Kolesnikov, G. V. Livshyts: *On the Gardner-Zvavitch conjecture: symmetry in the inequalities of Brunn-Minkowski type*, Adv. Math. 384 (2021), 107689

[94] A. V. Kolesnikov, E. Milman: *Sharp Poincaré-type inequality for the Gaussian measure on the boundary of convex sets*, Geometric aspects of functional analysis, 221-234, Lecture Notes in Math., 2169, Springer, Cham, 2017.

[95] A.V. Kolesnikov, E. Milman: *Local L_p-Brunn-Minkowski inequalities for $p < 1$*, Memoirs of the American Mathematical Society, 277 (2022), no. 1360.

[96] A.V. Kolesnikov: *On Sobolev regularity of mass transport and transportation inequalities.* Theory Probab. Appl. 57 (2013), no. 2, 243-264.

[97] A. Kolesnikov, E. Werner: *Blaschke–Santaló inequality for many functions and geodesic barycenters of measures* , Adv. Math. (2021), 108110.

[98] G. Kuperberg, *From the Mahler conjecture to Gauss linking integrals*, Geom. Funct. Anal. 18 (3) (2008), pp 870-892.

[99] R. Latała, *On some inequalities for Gaussian measures*, Proceedings of the International Congress of Mathematicians, Beijing, Vol. II, Higher Ed. Press, Beijing, 2002, pp. 813–822.

[100] J. Lehec: *Short probabilistic proof of the Brascamp-Lieb and Barthe theorems.* Canad. Math. Bull., 57 (2014), 585-597.

[101] J. Lehec: *A direct proof of the functional Santaló inequality* , C. R. Math. Acad. Sci. Paris 347 (2009), pp 55-58.

[102] L. Leindler: *On a certain converse of Hölder's inequality. II*, Acta Sci. Math. (Szeged) 33 (1972), 217–223.

[103] Q-R. Li, J. Liu, J. Lu: *Non-uniqueness of solutions to the dual L_p-Minkowski problem.* IMRN, accepted. arXiv:1910.06879

[104] Liakopoulos, D.-M.: *Reverse Brascamp-Lieb inequality and the dual Bollobás-Thomason inequality.* Arch. Math. (Basel) 112 (2019), 293–304.

[105] E.H. Lieb: *Gaussian kernels have only Gaussian maximizers.* Invent. Math. 102 (1990), 179–208.

[106] G.V. Livshyts: *Some remarks about the maximal perimeter of convex sets with respect to probability measures.* Commun. Contemp. Math. 23 (2021), no. 5, Paper No. 2050037, 19 pp.

[107] G. Livshyts, A. Marsiglietti, P. Nayar, A. Zvavitch: *On the Brunn-Minkowski inequality for general measures with applications to new isoperimetric-type inequalities*, Transactions of Math (2017), arxiv:1504.04878.

[108] L.H. Loomis, H. Whitney: *An inequality related to the isoperimetric inequality*, Bull. Amer. Math. Soc, 55 (1949), 961–962.

[109] E. Lutwak: *The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem.* J. Differential Geom. 38 (1993), 131-150.

[110] E. Lutwak, G. Zhang: *Blascke-Santaló inequalities*, J. Differential Geom. 47 (1) (1997), pp 1-16.

[111] E. Lutwak: *Selected affine isoperimetric inequalities*, Handbook of Convex Geometry, North-Holland (1993), pp 151-176.

[112] E. Lutwak, D. Yang, and G. Zhang, *A new ellipsoid associated with convex bodies,* Duke Math. J. 104 (2000), 375-390.

[113] E. Lutwak, D. Yang, and G. Zhang, *L_p affine isoperimetric inequalities,* J. Differential Geom. 56 (2000), 111-132.

[114] A. Marsiglietti: *Borell's generalized Prékopa-Leindler inequality: a simple proof.* J. Convex Anal. 24 (2017), 807-817.

[115] Jon McCammond: *Prologue: Regular Polytopes*, https://coxeter2011.files.wordpress.com/2011/09/prologue1.pdf

[116] R.J. McCann: Existence and uniqueness of monotone measure-preserving maps. Duke Math. J., 80 (1995), 309-323.

[117] R.J. McCann: A convexity principle for interacting gases. Adv. Math. 128 (1997), 153-179.

[118] R.J. McCann: *A Convexity Theory for Interacting Gases and Equilibrium Crystals.* PhD thesis, Princeton University, 1994.

[119] M. Meyer, A. Pajor, *On Santaló's inequality*, In: Lindenstrauss J., Milman V.D. (eds) Geometric Aspects of Functional Analysis. Lecture Notes in Mathematics, vol 1376. (1989) Springer, Berlin, Heidelberg.

[120] M. Meyer, A. Pajor: *On the Blaschke-Santaló inequality*, Archiv der Mathematik 55 (1990), pp 82–93.

[121] M. Meyer and S. Reisner: *Inequalities involving integrals of polar-conjugate concave functions*, Monatsh. Math. 125 (1998), no. 3, 219–227.

[122] M. Meyer: *A volume inequality concerning sections of convex sets.* Bull. Lond. Math. Soc., 20 (1988),15-155.

[123] M. Meyer: *Une caractérisation volumique de certains espaces normés de dimension finie*, Israel J. Math. 55 (1986), no. 3, 317–326.

[124] E. Milman: *Centro-Affine Differential Geometry and the Log-Minkowski Problem*, arXiv:2104.12408

[125] E. Milman: A sharp centro-affine isospectral inequality of *Szegő-Weinberger type and the L_p-Minkowski problem.* arXiv:2103.02994

[126] V. D. Milman: *Isomorphic symmetrization and geometric inequalities.* In: Lindenstrauss J., Milman V.D. (eds) Geometric Aspects of Functional Analysis. Lecture Notes in Mathematics, vol 1317. (1988) Springer, Berlin, Heidelberg.

[127] V. D. Milman, A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, In: Lindenstrauss J., Milman V.D. (eds) Geometric Aspects of Functional Analysis. Lecture Notes in Mathematics, vol 1376. (1989) Springer, Berlin, Heidelberg.

[128] V.D. Milman, G. Schechtman: *Asymptotic theory of finite-dimensional normed spaces.* With an appendix by M. Gromov. Springer-Verlag, Berlin, 1986.

[129] H. Minkowski: *Allgemeine Lehrsätzeüber die konvexen Polyeder*, Gött. Nachr. 1897 (1897), 198-219

[130] P. Nayar, T. Tkocz: *On a convexity property of sections of the cross-polytope*, arXiv: 1810.02038, (2018).

[131] F. Nazarov, *The Hörmander proof of the Bourgain-Milman theorem.* In: Klartag B., Mendelson S., Milman V. (eds) Geometric Aspects of Functional Analysis. Lecture Notes in Mathematics, vol 2050. (2012) Springer, Berlin, Heidelberg.

[132] C.M. Petty: *Surface area of a convex body under affine transformations.* Proc. Amer. Math. Soc. 12 (1961), 824–828,

[133] G. De Philippis, A. Figalli: *Rigidity and stability of Caffarelli's log-concave perturbation theorem.* Nonlinear Anal. 154 (2017), 59-70.

[134] A. Prékopa: *Logarithmic concave measures with application to stochastic programming*, Acta Sci. Math. (Szeged) 32 (1971), 301-316.

[135] A. Prékopa: *On logarithmic concave measures and functions*, Acta Sci. Math. (Szeged) 34 (1973), 335-343.

[136] E. Putterman: *Equivalence of the local and global versions of the L_p-Brunn-Minkowski inequality.* arXiv:1909.03729

[137] J. Saint-Raymond: *Sur le volume des corps convexes symétriques*, Initiation Seminar on Analysis: G. Choquet-M. Rogalski-J. Saint-Raymond, 20th Year: 1980/1981, Publ. Math. Univ. Pierre et Marie Curie, vol. 46, Univ. Paris VI, Paris, 1981, pp. Exp. No. 11, 25.

[138] A. Recski: *Matroid Theory and its Applications in Electric Network Theory and in Statics.* Springer, 1989.

[139] S. Reisner: *Zonoids with minimal volume-product*, Math. Z. 192 (1986), no. 3, 339–346

[140] L. Rotem: *A letter: The log-Brunn-Minkowski inequality for complex bodies*, arxiv:1412.5321

[141] L. A. Santaló, *Un invariante afin para los cuerpos convexos del espacio de n dimensiones*, Portugal. Math., 8 (1949), pp 155-161.

[142] C. Saroglou: *Remarks on the conjectured log-Brunn-Minkowski inequality*, Geom. Dedicata 177 (2015), 353-365.

[143] C. Saroglou: *More on logarithmic sums of convex bodies*, Mathematika 62 (2016), 818-841.

[144] R. Schneider: *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its applications, Cambridge, 2014.

[145] Y. Shenfeld and R. van Handel, *Mixed volumes and the Bochner method*, Proc. Amer. Math. Soc., 147(12):5385–5402, 2019.

[146] Y. Shenfeld and R. van Handel, *The extremals of Minkowski's quadratic inequality* Duke Math. J. 171(4): 957-1027, 2022

[147] Y. Shenfeld and R. van Handel, *The Extremals of the Alexandrov-Fenchel Inequality for Convex Polytopes*, arXiv:2011.04059, 2020.

[148] T. Tao, V. Vu: *Additive combinatorics*, Cambridge University Press, 2006.

[149] N.S. Trudinger, X.-J. Wang: *The Monge-Ampere equation and its geometric applications*, Handbook of geometric analysis, pp. 467-524, Adv. Lect. Math. 7, Int. Press, Somerville, MA, 2008.

[150] B. Uhrin, *Curvilinear extensions of the Brunn-Minkowski-Lusternik inequality*, Adv. Math. 109 (2) (1994), pp 288-312.

[151] S.I. Valdimarsson: *Geometric Brascamp-Lieb has the optimal best constant.* J. Geom. Anal. 21 (2011), 1036-1043.

[152] S.I. Valdimarsson: *Optimisers for the Brascamp-Lieb inequality.* Israel J. Math. 168 (2008), 253-274.

[153] S.I. Valdimarsson: *The Brascamp-Lieb polyhedron.* Canadian Journal of Mathematics, 62(4): 870–888, 2010.

[154] C. Villani: *Topics in optimal transportation.* AMS, Providence, RI, 2003.

[155] E.B. Vinberg: *Discrete linear groups generated by reflections.* Math. USSR Izvestia, 5 (1971), 1083-1119.

[156] G. Zhu: *Continuity of the solution to the Lp-Minkowski problem*, Proc. Am. Math. Soc. 145 (2017), 379-386.

[157] G. Zhu: *The L_p Minkowski problem for polytopes for* $0 < p < 1$, arXiv:1406.7503