On asymptotically optimal investments

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1 Introduction

In the text below we aim to exhibit some ideas of long term investment. Throughout this section we do not pursue complete mathematical precision, but rather target a discussion of the main flow of thought in a vague fashion, in the purpose of displaying an outline of the topic.

Stochastic processes are defined on an underlying common stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{F}}, P)$, where $\mathbb{F}$ will be either the set of positive real numbers $\mathbb{R}^+$ or the set of real numbers $\mathbb{R}$ when we are in a continuous-time framework. In a discrete setting, $\mathbb{F}$ is either the set of non-negative integers $\mathbb{N}$ or all the integer numbers $\mathbb{Z}$. The price of a tradable asset is an adapted stochastic process $S = \{S_t, t \in \mathbb{F}\}$ with further possible structural restrictions. A trader can interact with this object in a certain way, resulting in her wealth process $V_t = \langle S \circ \phi \rangle_t,\ t \in \mathbb{F} \bigcap [0, T]$ and terminal value $V_T = \langle S \circ \phi \rangle_T$ at time $\mathbb{F} \ni T > 0$, where $\phi = \phi^{(T)}$ denotes the strategy ("controlling process") chosen by the trader. The transformation $\langle S \circ \phi \rangle_T$, depending on the model choice we make, can take various forms. To give a few examples, one can consider classical, frictionless markets or models where the phenomenon of market friction is taken into consideration. Furthermore, $\langle S \circ \phi \rangle_T$ also depends on the representation chosen to model the trading strategy: the number of shares held at a given time, the speed of trading, or the proportion of wealth held in the risky asset. One could consider, in a multivariate setting, a market with an arbitrary finite number of assets but throughout the text we confine ourselves to two-asset settings.

The problem of long-term investment can be formulated as follows. The value of portfolios is assessed through a utility function: a non-decreasing mapping $U : \mathbb{R} \mapsto \mathbb{R}$. It is to model the psychological attitude of the investor towards taking risks. Usually, she is considered to be risk-averse, meaning that she finds riskier market situations repulsive and it is also common to postulate that, above a certain level of fortune, she becomes insensitive to further gains. These characteristics can be reflected by a concave utility function. It is much less common, but possible, to use a utility function with convex sections that yield a behavior of risk seeking.

The problem to solve is the optimization

$$E[U(V_T)] \to \max, \tag{1}$$

where E denotes expectation under the physical measure P, and the optimization is developed over an appropriate set of strategies. To be more precise, fixing a set of strategies $\mathscr{X} = \mathscr{X}_T$, one searches for a family of strategies $(\phi^* = \phi^{*,T})_{T>0}$ indexed by T, with $\phi^* = \phi^{*,T} = (\phi_t^{*,T})_{t\in\mathbb{F}\cap[0,T]}$, such that $E\left[U\left(\langle S\circ\phi^*\rangle_T\right)\right]$ grows asymptotically as

$$u(T) := \sup\left\{E\left[U\left(\langle S\circ\phi\rangle_T\right)\right],\ \ \phi\in\mathscr{X}_T\right\}. \tag{2}$$

when $T\to\infty$. Such a family ϕ^* is usually found by first constructing some universal upper bound for attainable terminal values with respect to the horizon T and then, possibly by educated guesses, exhibiting an investment achieving the asymptotically optimal performance.

The remaining text is organized as follows. In Section 2, based on Guasoni, Rásonyi (2015) and Guasoni, Nika, Rásonyi (2019), we elaborate on how restricting our attention on asset price models with stationary returns limits the possible growth rate of asymptotically optimal portfolios. Then, through an example, we prognosticate the intricate connection between optimality and the potential with which prices retain information from the past. In Section 3 and Section 4, continuing the previous line of thought, we present a discussion on long memory and negative memory (anti-persistence) respectively. We display how these two regimes cover the whole spectrum of attainable optimal growth rates, when returns are stationary. Results of Section 3 and 4 can be found in Guasoni, Nika, Rásonyi (2019) and Rásonyi, Nagy (2021). In Section 5 we present a result on long term portfolio choice in a trading environment where the investor is heavily risk-averse and prices show mean-reversion – here, we follow Guasoni, Nagy, Rásonyi (2021). Appendix A and appendix B contains proofs for statements made in Section 4 and Section 5 respectively.

The new contributions of the author are Appendices A and B, Section 4 and Subsections 5.3, 5.4, 5.6.

2 Profit and price memory

2.1 Fractional Brownian motion

In a large part of the econometric literature, asset prices are assumed to have stationary increments: price changes form a stationary process in the weak or strong sense. Most mainstream models also satisfy the Markov property. However, the latter assumption has been challenged from rather early on.

A stochastic process, presented in the paper Mandelbrot (1971), was suggested as a possible candidate to model asset prices by Benoit Mandelbrot himself. This process, fractional Brownian motion (or simply FBM) paved the way for incorporating memory into asset price dynamics. The decisive quantity for FBMs is the so-called Hurst parameter $H \in (0,1)$.

FBM with parameter $H \neq 1/2$ is a non-Markovian, centered Gaussian stochastic process with corralated stationary increments with covariance structure defined by

$$E[B^H(t)B^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - (t-s)^{2H}\right), \tag{3}$$

where s and t are non-negative real numbers.

Long memory does not have a generally agreed definition. Vaguely speaking, a discrete-time stationary process has long memory if its auto-covariance function decays slowly at infinity.

We will use the definitions for describing memory given by Giraitis et al (2012). Let $\bar{Y}_k, \quad k \in \mathbb{Z}$ be a discrete time stationary process and define its auto-covariance function for lag $k \in \mathbb{Z}$ as

$$\gamma(k) = \operatorname{cov}(\bar{Y}_0, \bar{Y}_k). \tag{4}$$

We say that the process $\bar{Y}_k$ has long memory if

$$\sum_{k \in \mathbb{Z}} |\gamma(k)| = \infty. \tag{5}$$

It has short-memory if

$$\sum_{k \in \mathbb{Z}} |\gamma(k)| < \infty \text{ and } \sum_{k \in \mathbb{Z}} \gamma(k) > 0.$$

It has negative memory, or equivalently, the process is anti-persistent when

$$\sum_{k\in\mathbb{Z}} |\gamma(k)| < \infty \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \gamma(k) = 0. \tag{6}$$

We could strengthen the memory conditions in (5) and (6) to attain a smaller class of processes that satisfy these criteria in a natural way and, at the same time, they can be handled more easily in the computations. Considering an asymptotic power decay of the auto-covariance function γ in (4), greater than hiperbolic decay results in a convergent absolute sequence while a slower decay results in divergence.

More precisely, on one hand, if for some fixed $H \in (1/2,1)$ there exist constants $\bar{J}_1 > 0$ and $\bar{J}_2 > 0$ such that the asymptotic behavior of $\gamma(k)$ satisfies, after some threshold $k > \bar{T}_0$, that

$$\bar{J}_1 k^{2H-2} \leq \gamma(k) \leq \bar{J}_2 k^{2H-2}, \tag{7}$$

then we have that the underlying process has long memory in the sense of (5). This is the case for FBM with $H > 1/2$.

On the other hand, if for some $H \in (0,1/2)$ there exist constants $J_1 < 0$ and $J_2 < 0$ so that the asymptotic behavior of $\gamma(k)$ satisfies, after some threshold $k > T_0$, that

$$J_1 k^{2H-2} \leq \gamma(k) \leq J_2 k^{2H-2} \tag{8}$$

then, with the additional constraint that the lags sum up to zero, we have that the underlying process has negative memory in the sense of (6). This is the case of FBM with $H < 1/2$.

To see this more directly, let the extension to $\mathbb{Z}$ of the increment process of fractional Brownian motion be denoted by X_k^H, $k \in \mathbb{Z}$ and consider the corresponding auto-covariance function (overriding previous notation), $\gamma(k) = \mathrm{cov}(X_0^H, X_k^H)$. Using the formula in (3) we have that $\gamma(0) = 1$ and for $k > 0$ we have

$$\gamma(k) = \frac{1}{2}\left[(k+1)^{2H} - k^{2H} - \left(k^{2H} - (k-1)^{2H}\right)\right].$$

Observe that as a consequence

$$\sum_{k\in\mathbb{Z}} \gamma(k) = \gamma(0) + 2\sum_{k>0} \gamma(k), = \lim_{T\to\infty}\left[(T+1)^{2H} - T^{2H}\right].$$

The expression in the argument of the limiting operation is roughly $2HT^{2H-1}$ and this shows that the infinite series sums up zero and the corresponding absolute value sum converges when $H < 1/2$ and there is divergence when $H > 1/2$.

2.2 A model with price-impact

The model considered here will be presented following Guasoni, Rásonyi (2015). The stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ is assumed to have a filtration that is right-continuous, $\mathscr{F}_0$ is trivial. Let $\mathscr{O}$ denote the optional sigma field over $\Omega \times [0,T]$. For simplicity we assume zero interest rate, that is we assume the existence of a safe asset with constant value equal to 1.

In a classical, frictionless market model, one would identify the payoff associated with a square integrable strategy Φ, as the integral of the strategy with respect to the price process in the Ito sense, i.e.

$$\int_0^T \Phi_u dS_u, \tag{9}$$

where Φ_t is the amount of shares one holds at time t.

For the above integral to make sense, it is necessary that the process S is a semi-martingale. This is the widest class for which the former object is well defined. However, fractional Brownian motion is not a semi-martingale thus, this process can not be used as an integrator (at least not in the usual sense). We can consider an alternative to this integral for smooth enough integrands representing strategies by the rate of trading. Let $\Phi_t = \int_0^t \phi_u \, du$ hold for all $t \in [0,T]$ with some optional process ϕ. Formal partial integration yields

$$\int_0^T \Phi_u dS_u = -\int_0^T \phi_u S_u du + (\Phi_T S_T - \Phi_0 S_0) \tag{10}$$

The quantity ϕ_t represents the speed of trading at time t, that is, the rate at which the number of shares changes over time.

If, in particular, it is assumed that we start at time $t = 0$ with no shares at hand and furthermore postulate that we liquidate the risky position at the terminal time

$t = T$, that is $\Phi_T = 0$, a more simple formula emerges as

$$-\int_0^T \phi_u S_u du. \tag{11}$$

Using (11) instead of (9), one could enlarge the class of asset price processes to the class of cadlag processes, and this would allow fractional Brownian motion to be used. But then another problem emerges.

The econometric literature made considerable efforts to investigate models utilizing processes with correlated returns such as FBM. It turned out, however, that in classical frictionless markets (where wealth is given by (9) or (11)), FBM becomes inadmissible since it generates arbitrage opportunities. FBM thus remained neglected for a long time. Later, trading models emerged where market friction was incorporated. It is significantly more difficult for arbitrage to exist in imperfect environments with friction and, indeed, FBM was found to be arbitrage-free and thus was rehabilitated as a possible model.

Friction can be modeled with the introduction of a $\mathscr{O} \otimes \mathscr{B}(\mathbb{R})$ measurable function $G : \omega \times [0,T] \times \mathbb{R} \to \mathbb{R}_+$ such that $G(\omega, t, \cdot)$ is convex with $G(\omega, t, x) \geq G(\omega, t, 0)$ for all variables. In notation we drop the dependence in ω as usual.

Using the function G we can write

$$-\int_0^T G_u(\phi_u) du$$

to represent friction on trading according to the rate of buying and selling. This is an "accumulation of punishments" over time for taking action in the market. Convexity implies that trading a given quantity in a given time is cheaper than trading twice as intensely for half of the time. The fact that $G_t(0)$ is dominated by all other values of the function means that in terms of adverse effects, inactivity is the best one can achieve in terms of avoiding loss arising from trading activity itself.

Incorporating this quantity into the model, the dynamics can be proposed to be

$$V_T(\phi) = \langle S \circ \phi \rangle_T = z_0 - \int_0^T \phi_u S_u du - \int_0^T G_u(\phi_u) du, \tag{12}$$

where z_0 is an initial endowment.

Let us call a strategy feasible if it is an element of the set

$$\mathscr{A}(T) = \left\{ \phi \,:\, \phi \text{ is an } \mathbb{R} \text{ valued optional process with } \int_0^T |\phi_u| du < \infty \text{ a.s.} \right\}.$$

The finiteness of the integral has a meaning that the absolute turnover, that is the total number of shares bought or sold up until the terminal time, remains finite for all finite horizons.

When looking for an optimal investment, the general friction modeled by G as above results in a complicated model. Instead, it is reasonable to further simplify the above dynamics as follows. Fixing the parameters $\lambda > 0$ and $\alpha > 1$ one can introduce a model with superlinear friction by setting $G_t(x) = \lambda |x|^{\alpha}$. Then, for $\phi \in \mathscr{A}$, the position at time $t \in [0, T]$ in the risky asset and the safe asset are

$$\begin{aligned} V_T^0(\phi) &= z^0 + \int_0^T \phi_u du \\ V_T(\phi) &= z^1 - \int_0^T \phi_u S_u du - \lambda \int_0^T |\phi_u|^{\alpha} du, \end{aligned} \tag{13}$$

where z^0 is the initial number of shares held at time $t = 0$ and z^1 is an initial endowment. For simplicity, we will set $z^0 = z^1 = 0$.

By rewriting the above expression in (13), one gets that the evolution of the riskless position equals

$$V_T(\phi) = -\int_0^T \phi_u \left(S_u + \lambda \operatorname{sgn}(\phi_u) |\phi_u|^{\alpha-1} \right) du,$$

coinciding with the portfolio value (11) resulting from the same strategy ϕ in a frictionless market with price

$$S_t + \lambda \operatorname{sgn}(\phi_t) |\phi_t|^{\alpha-1}. \tag{14}$$

The so-called instantaneous execution price (14), compared to the physical price S_t, is higher when going long, that is, when buying assets, and lower when one is going short, that is, when selling assets. This property intuitively explains the impact that superlinear friction has on trading. We note here that, as it can be seen form (14), the choice $\alpha = 2$ amounts to a linear price impact and $\alpha = 3/2$ yields a punishment term that is proportional to the square root of the speed of trading. These are popular choices in the corresponding literature and can be backed by empirical evidence -

see for example Garleanu and Pedersen, (2013) backing the quadratic settings and Almgren et al. (2005) on the square root rule.

Among other striking features, see Guasoni, Rásonyi (2015), a remarkable result holds true in a market where friction is incorporated with the help of the function G introduced above. Namely a uniform boundedness property holds with respect to the set of feasible strategies. First, note that rewriting the riskless dynamics with general friction in (12) we have

$$V_T(\phi) = \int_0^T \phi_u(-S_u) - G_u(\phi_u)du$$

and defining for a real valued function $f : \mathbb{R} \mapsto \mathbb{R}$ the Fenchel-Legendre conjugate as

$$f^*(y) = \sup_{x \in \mathbb{R}} (xy - f(x)),$$

we have for every $\phi \in \mathscr{A}$ that

$$V_T(\phi) \leq \int_0^T G^*(-S_u)du. \tag{15}$$

This means that there exists a random variable that bounds all payoffs that are generated by feasible strategies and this bound only depends on the price process and the terminal time of trading but not on the particular strategies. Let us furthermore denote the expected value of this bound by

$$Q_T(S) = E\left[\int_0^T G^*(-S_u)du\right]. \tag{16}$$

In case of the stylized model (13) the Legendre-Fenchel conjugate of the function $G(x) = \lambda|x|^\alpha$ can be explicitly calculated and turns out to be

$$G^*(y) = \frac{\alpha - 1}{\alpha} \alpha^{\frac{1}{1-\alpha}} \lambda^{\frac{1}{1-\alpha}} |y|^{\frac{\alpha}{\alpha-1}}$$

for all $y \in \mathbb{R}$. Thus the upper bound in (15) satisfies, for some $C > 0$,

$$V_T(\phi) \leq C \int_0^T |S_u|^{\alpha/(\alpha-1)}du. \tag{17}$$

In the rest of the present section, we use a risk-neutral utility function defined by $U(x) = x$ and work with only those feasible strategies that finish with cash only. That is we consider the set

$$\mathscr{G}_T = \left\{\phi \in \mathscr{A}_T : V_T^0(\phi) = 0 \text{ and } E[V_T(\phi)_-] < \infty\right\},$$

where $x_- = \max(0, -x)$. This is the class of strategies that start from zero initial position, liquidate until the terminal time and end with a cash position with a well-defined expected value that is not $-\infty$. The value function u with linear utility will take the form

$$u(T) = \sup_{\phi \in \mathscr{G}_T} E[V_T(\phi)]. \tag{18}$$

2.3 Stationary returns and price impact

In the price impact setting introduced in the previous section, intriguing constraints hold for the market bound presented above in (15). These results were obtained in the paper Guasoni, Nika, Rásonyi (2019). Here, by displaying these, we aim to give an intuitive picture of the long term optimality problem (1) with linear utility, and with the dynamics (13). We infer useful properties of the asymptotics of the value function defined in (18), in particular, we deduce certain limitations on the possible optimal growth rates.

In this subsection we further simplify the model in (13) by setting $\lambda = 1$ and $\alpha = 2$.

Definition 2.1. *We say that a discrete parameter process X_k, $k \in \mathbb{Z}$ is weakly stationary if $X_k \in L^2$ (that is, X_k is square-integrable) for all k, the expectations $E[X_k]$ are the same for all k and $E[X_{k+n}X_{l+n}]$ depends only on $k - l$ but not on k and l, for all $k, l \in \mathbb{Z}$ and $n \in \mathbb{Z}$.*

Proposition 2.2. *Let $S_t \in L^2$ for all $t \geq 0$ and let the mapping $t \to ES_t^2$ be a non-decreasing function. Furthermore, assume that the process $S_k - S_{k-1} = X_k$ with some weakly stationary X_k, $k \in \mathbb{Z}$. Then there exists a constant $C > 0$ such that*

$$E[V_T(\phi)] \leq Q_T(S) \leq CT^3$$

for all $T \in \mathbb{N} \setminus \{0\}$ and for all $\phi \in \mathscr{A}$.

Proof. Note first that, for all $k \in \mathbb{N}$, we have

$$S_k = S_0 + \sum_{j=1}^{k} \left[S_j - S_{j-1} \right],$$

thus, the triangle inequality for the Hilbert space norm of L^2 yields

$$\left(E\left[S_k^2\right]\right)^{1/2} \le \left(E\left[S_0^2\right]\right)^{1/2} + \sum_{j=1}^{k} \left(E\left[\left(S_j - S_{j-1}\right)^2\right]\right)^{1/2} = C_1 + kC_2,$$

with $C_1 = E^{1/2} S_0^2$ and $C_2 = E^{1/2}(S_j - S_{j-1})^2$. Using this we have

$$Q_T(S) = \int_0^T E[S_u^2]du \le \sum_{k=1}^{T} E[S_k^2] \le \sum_{k=1}^{T} (C_1 + kC_2)^2 .$$

We can conclude that for some C it holds that

$$Q_T(S) \le CT^3, \quad T \ge 1. \tag{19}$$

□

Proposition 2.3. *Let $S_0 = 0$, let $S_t \in L^2$ for all $t \ge 0$ and let the mapping $t \to ES_t^2$ be a non-decreasing function. Furthermore assume that there is a weakly stationary process X_k, $k \in \mathbb{Z}$ such that $\Delta S_k := S_k - S_{k-1} = X_k$ for $k \ge 1$. Denoting by $\mathscr{H}_n$ the closed subspace in L^2 generated by ΔS_i, $i \le n$ assume that ΔS_1 is not an element of $\mathscr{H}_0$. Under these conditions, there is $C > 0$ such that*

$$Q_T(S) \ge CT, \; T \ge 2.$$

Proof of Proposition 2.3. Denoting the correlation of arbitrary random variables $X, Y \in L^2$ as $\rho(X,Y)$, we define

$$\rho_n = \inf_{X \in \mathscr{H}_n} \rho(\Delta S_{n+1}, X), \; n \in \mathbb{N}.$$

Using the stationarity property, the mapping $n \to \rho_n$ is constant, and we denote this constant value by $\omega \in [-1,1]$. If $\omega = -1$ then we can easily see that ΔS_{n+1} is in the closure of $\mathscr{H}_n$, hence it is in $\mathscr{H}_n$, too. This means that, by our assumptions, $\omega > -1$.

In case $\rho(\sum_{j=1}^{k-1} \Delta S_j, \dot{S}_k) < 0$, we can simply estimate

$$\begin{aligned}
2\text{cov}\left(\sum_{j=1}^{k-1} \Delta S_j, \Delta S_k\right) &= \text{var}\left(\sum_{j=1}^{k-1} \Delta S_j\right)^{1/2} \text{var}(\Delta S_k)^{1/2} \rho(\sum_{j=1}^{k-1} \Delta S_j, \Delta S_k) \\
&\geq \left(\text{var}\left(\sum_{j=1}^{k-1} \Delta S_j\right) + \text{var}(\Delta S_k)\right) \rho(\sum_{j=1}^{k-1} \Delta S_j, \Delta S_k)
\end{aligned}$$

By this, we have

$$\begin{aligned}
\text{var}\left(\sum_{j=1}^{k} \Delta S_j\right) &= \text{var}\left(\sum_{j=1}^{k-1} \Delta S_j\right) + \text{var}(\Delta S_k) + 2\text{cov}\left(\sum_{j=1}^{k-1} \Delta S_j, \Delta S_k\right) \\
&\geq \text{var}\left(\sum_{j=1}^{k-1} \Delta S_j\right) + \text{var}(\Delta S_k) + \left(\text{var}\left(\sum_{j=1}^{k-1} \Delta S_j\right) + \text{var}(\Delta S_k)\right) \rho(\sum_{j=1}^{k-1} \Delta S_j, \Delta S_k) \\
&\geq (1+\omega)\text{var}(\Delta S_k)
\end{aligned}$$

In the case $\rho(\sum_{j=1}^{k-1} \Delta S_j, \Delta S_k) \geq 0$, we naturally have

$$\text{var}\left(\sum_{j=1}^{k} \Delta S_j\right) \geq \text{var}(\Delta S_k).$$

So in both cases

$$\text{var}(S_k) \geq \text{var}(X_k) = \text{var}(X_0).$$

This implies

$$\begin{aligned}
Q_T(S) = \int_0^T E[S_u^2]du \geq \sum_{k=0}^{T-1} E[S_k^2] \geq \sum_{k=0}^{T-1} \text{var}(S_k) \\
\geq \sum_{k=1}^{T-1} \min\{1, 1+\omega\}\text{var}(X_0) = c(T-1),
\end{aligned} \tag{20}$$

for some positive constant c. This implies the statement easily. □

The bounds in (19) and (20) suggest a rather surprising "law of nature", emphasized in the next remark.

Remark 2.4. *In a trading environment where linear price impact is present, under mild conditions, the range of optimal growth can only vary between T and T^3.*

Remark 2.5. *The sub-linear range, that is, an asymptotic growth rate of the order T^x with some $0 < x < 1$, lacks interest from an econometric perspective, since it would result in an investment whose profit increments asymptotically vanish, and any fixed annuity would outperform their gain on the long run. However, the question to specify what properties of the asset price lead to the attainable superlinear growth rates appearing in Remark 2.4 naturally arises.*

In the paper Guasoni, Nika, Rásonyi (2019) it is shown that the deterministic process $t \to S_t := t$ reaches the growth rate of cubic power, and the Ornstein-Uhlenbeck process produces a linear optimal rate, these two examples hence cover the extremities of the attainable growth rate region between T and T^3. But what about the rates in between?

2.4 An example

In this section we will present a result (see Theorem 2.6 below) from Guasoni, Nika, Rásonyi (2019) to show how memory plays an important role in asymptotically optimal investments. We use here the model (13) with general $\alpha > 1$ and $\lambda > 0$, and the value function is as in (18).

Theorem 2.6. *[Optimality with fractional Brownian motion] Let $\alpha > 1$ and $\lambda > 0$ and let $H \in (0,1/2)\bigcup(1/2,1)$. Then with setting $S_t = B_t^H$ we have that maximal expected profits satisfy*

$$\limsup_{T\to\infty} \frac{u(T)}{T^{H(1+\frac{1}{\alpha-1})+1}} > 0,$$

for each $0 < \kappa < \frac{1}{\alpha-1}$. Furthermore, the strategies

$$\phi^\circ(T,\kappa) = \begin{cases} sgn(S_t(H-1/2))|S_t|^\kappa, & t \in [0,T/2] \\ -\frac{1}{T/2}\int_0^T \phi_s ds, & t \in [T/2,T] \end{cases}$$

satisfy

$$\liminf_{T\to\infty} \frac{EV_T(\phi^\circ(T,\kappa))}{T^{H(1+\kappa)+1}} < \infty.$$

As we can observe, setting $\alpha = 2$ and $\lambda = 1$ in the above theorem, when κ is close to $\frac{1}{\alpha-1} = 1$, the optimal portfolio under an optimal investment generates profit with the rate T^{2H+1}. Taking into account that the Hurst parameter H varies in $(0,1/2)\bigcup(1/2,1)$, this shows that fractional Brownian motion covers the whole range of possible growth rates under stationary returns (see Remark 2.4 above), except T^2. There is no growth corresponding to the parameter $H = 1/2$ since we have a martingale (standard Brownian motion) in that case. To the best of our knowledge, no example is known for the moment that generates a quadratic long-term optimal growth rate.

In the proof of Theorem 2.6, the exact covariance structure of the underlying process is heavily used and the results depend on certain analytic properties of the autocovariance function. Namely, if u/v is larger then a certain threshold value then the covariance $E[B_u^H, B_v^H]$ is bounded in a nontrivial way.

It is nevertheless natural to conjecture that these investment phenomena are not caused by the exact shape of the covariance function but rather by the autocovariance properties of the process. We will elaborate on this in the next two sections.

3 Optimality when prices have long memory

In the paper Guasoni, Nika, Rásonyi (2019), a class of long memory processes is given which display the same asymptotic growth rate as fractional Brownian motion with Hurst parameter $H > 1/2$. We quote here the corresponding results. The trading dynamics is described by (13) and the value function u is defined as in (18).

Assumption 3.1. *Let $S_t,\ t \geq 0$ be a zero-mean Gaussian process with stationary increments such that*

$$cov(S_u - S_t, S_t) \geq 0,\ 0 \leq t \leq u.$$

Let γ be the autocovariance of the increments, that is,

$$\gamma(k) = cov(S_1 - S_0, S_{k+1} - S_k),\quad k \in \mathbb{N}.$$

Assume that it satisfies

$$\gamma(0) = 1,\ \ and\ \ \bar{J}_1 k^{2H-2} \leq \gamma(k) \leq \bar{J}_2 k^{2H-2},\ \ k \geq 1,$$

for some $\bar{J}_1, \bar{J}_2 > 0$ *and* $H \in (1/2, 1)$.

The second part of this assumption defines a class of stochastic processes that, in terms of the returns, satisfies the long memory criterion in (5). The first part of the assumption expresses a certain positive autocorrelation property.

Theorem 3.2. *Let the price process satisfy Assumption 3.1. Then the maximal expected profit has an upper bound:*

$$\limsup_{T\to\infty} \frac{u(T)}{T^{H(1+1/(\alpha-1))+1}} < \infty.$$

For each $0 < \kappa < 1/(\alpha-1)$*, the strategies*

$$\phi^\circ(T,\kappa) = \begin{cases} sgn(S_t)|S_t|^\kappa, & t \in [0, T/2] \\ -\frac{1}{T/2}\int_0^T \phi_s ds, & t \in [T/2, T] \end{cases}$$

satisfy

$$\lim_{T\to\infty}^{(4)} \frac{E[V_T(\phi^\circ(T,\kappa))]}{T^{H(1+\kappa)+1}} > 0,$$

where the limit is taken with T ranging over integers that are multiples of 4.

The proof heavily depends on the fact that, for $s > t$, S_s can be decomposed into a sum, whose second term is orthogonal, to S_t (thus independent from it) and whose first term is an affine transform of S_t. Using the quantity $\varrho(s,t) = \frac{\mathrm{cov}(S_s,S_t)}{\mathrm{var}(S_t)}$, we have $S_s = \varrho(s,t)S_t + W_{s,t}$, where $W_{s,t}$ is independent of S_t.

Then the key property is that, for $s > t$, we have $\varrho(s,t) \geq 1$ (or equivalently $\mathrm{cov}(S_s - S_t, S_t) \geq 0$) by assumption, and on some region of $\mathbb{R}^2$ a stronger bound is achievable, namely, with some $c > 0$ it holds that $\varrho(s,t) \geq 1 + c$.

Theorem 3.2 generalizes Theorem 2.6 from the specific case of FBMs with $H > 1/2$ to a more general class of processes. A similar extension for $H < 1/2$, however, remained an open question.

The difficulties that arise in the case of negative memory (defined in (6) and (8)) stems from the fact that the crucial assumption of positively auto-correlated asset price process (we are referring here to the property $\varrho(s,t) \geq 1$), postulated in Theorem 3.2, does not remain valid under anti-persistance.

This is due to the shape of the auto-covariance function and can be explained by observing that the mapping $\gamma(k)$ takes a positive value at $k = 0$ and when there is negative memory, by definition, we have that the negatively correlated lags balance out the positive ones, thus both positive and negative lag-values are a necessity. For this reason, even trivial bounds on ϱ are not automatic.

4 Anti-persistence

In this section we discuss the new findings presented in Rásonyi, Nagy (2021). That paper resolves the problem of long-term optimality when, instead of long memory, anti-persistence is present.

Due to technical difficulties, we use a discrete-time adaptation of the model (13). As usual, we employ the triple $(\Omega, \mathscr{F}, P)$ as a probability space with attached filtration $\mathscr{F}_t$, $t \in \mathbb{Z}$. The financial market comprises a riskless asset with zero interest rate, and a risky asset whose dynamics follows a process S_t, $t \in \mathbb{N}$, adapted to the filtration.

The class of *feasible strategies* up to the terminal time $T \in \mathbb{N}$ is

$$\mathscr{S}(T) := \left\{ \phi = (\phi_t)_{t=0}^{T} : \phi \text{ is an } \mathbb{R}\text{-valued, adapted process} \right\}. \tag{21}$$

The quantity ϕ_t represents the change in the position of the investor in the risky asset. This quantity is the speed of trading in analogy with continuous-time models like the one in (12). Let furthermore $z = (z^0, z^1) \in \mathbb{R}^2$ be a deterministic initial endowment where z^0 is in cash and z^1 is in the risky asset.

For a feasible strategy $\phi \in \mathscr{S}(T)$, the number of shares in the risky asset, with $\Phi_0 = z^1$, at any time $t \geq 1$, is equal to

$$\Phi_t := z^1 + \sum_{u=0}^{t-1} \phi_u \,. \tag{22}$$

For simplicity, we assume $z^1 = 0$ from now on, i.e. the initial number of shares is zero. We will shortly derive a similar formula for the cash position of the investor. In classical, discrete time frictionless models of trading, cash at time $T+1$, in analogy with the continuous time formula (9), the cash position equals

$$\sum_{u=1}^{T+1} \Phi_u (S_u - S_{u-1}) \tag{23}$$

when starting from a 0 initial position. Algebraic manipulation of (23), more specifically the so-called Abel summation (the discrete analogue of partial integration) yields

$$\sum_{u=1}^{T+1} \Phi_u (S_u - S_{u-1}) = -\sum_{u=0}^{T} \phi_u S_u + S_{T+1} \sum_{u=0}^{T} \phi_u + 0 \cdot S_0. \tag{24}$$

We assume that price impact is a superlinear power function of the trading speed ϕ so, in analogy with (13), we augment (24) with a term that implements the effect of friction:

$$-\sum_{u=0}^{T} \phi_u S_u + S_{T+1} \sum_{u=0}^{T} \phi_u - \sum_{u=0}^{T} \lambda |\phi_u|^{\alpha}$$

where we assume $\alpha > 1$ and $\lambda > 0$. We wish to utilize only those portfolios where the risky asset is liquidated by the end of the trading period so we define

$$\mathscr{D}(T) := \left\{ \phi \in \mathscr{S}(T) : \Phi_{T+1} = \sum_{u=0}^{T} \phi_u = 0 \right\}.$$

Based on the previous discussion, for $\phi \in \mathscr{D}(T)$, the position in the riskless asset at time $T+1$, similarly to the continuous formula in (13), is defined by

$$V_T(\phi) \quad := \quad z^0 - \sum_{u=0}^{T} \phi_u S_u - \sum_{u=0}^{T} \lambda |\phi_u|^{\alpha}. \tag{25}$$

For simplicity, we also assume $z^0 = 0$ from now on, i.e. portfolios start with zero cash.

To investigate the potential of realizing monetary profits, we focus on a risk-neutral objective: a linear utility function, that is we set $U(x) := x,\ x \in \mathbb{R}$.

Let $x_- := \max\{-x, 0\}$ for $x \in \mathbb{R}$. Define, for $T \in \mathbb{N}$,

$$\mathscr{H}(T) := \left\{ \phi \in \mathscr{D}(T) : E[(V_T(\phi))_-] < \infty \right\},$$

the class of strategies starting from a zero initial position in both assets and ending at time $T+1$ in a cash only position with expected value greater than $-\infty$. The value of the problem we will consider is thus

$$u(T) := \sup_{\phi \in \mathscr{H}(T)} E[V_T(\phi)].$$

The investors's objective is to find a family of strategies $(\phi^{*,T})_{T\geq 0}$ with $\phi^{*,T} \in \mathscr{H}_T$ for which the portfolio value $V_T(\phi^{*,T})$, achieves the same asymptotic rate as $u(T)$, when $T \to \infty$.

First, we introduce assumptions on the price process and its dependence structure.

Assumption 4.1. *Let Z_t, $t \in \mathbb{Z}$ be a real-valued, zero-mean stationary Gaussian process which will represent price increments. Let $\mathscr{F}_n := \sigma(Z_i,\ i \leq n)$ for $n \in \mathbb{N}$. Let $r(t) := \mathrm{cov}(Z_0, Z_t)$, $t \in \mathbb{Z}$ denote its covariance function. We assume that there exists $T_0 > 0$ and $J_1, J_2 < 0$ such that for all $t \geq T_0$,*

$$J_1 t^{2H-2} \leq r(t) \leq J_2 t^{2H-2} \tag{26}$$

is satisfied for some parameter $H \in (0, 1/2)$. Furthermore,

$$\sum_{t\in\mathbb{Z}} r(t) = 0. \tag{27}$$

Let us introduce the adapted price process defined by $S_0 = 0$ and $S_t = S_{t-1} + Z_t$, $t \geq 1$.

Properties (26) and (27) express that Z is a *process with negative memory*, see (6) and (8).

The next theorem is the main result of Rásonyi, Nagy (2021): it provides the explicit form of an asymptotically optimal strategy and determines its expected asymptotic growth rate, analogously to Theorem 2.6 in the case $H < 1/2$, but not only for FBMs.

Theorem 4.2. *Let Assumption 4.1 be in force. If λ is small enough then maximal expected profits satisfy*

$$\limsup_{T\to\infty} \frac{u(T)}{T^{H\left(1+\frac{1}{\alpha-1}\right)+1}} < \infty \tag{28}$$

and the strategy

$$\phi_t(T,\alpha) := \begin{cases} -\mathrm{sgn}(S_t)|S_t|^{\frac{1}{\alpha-1}}, & 0 \leq t \leq 3\lfloor T/6 \rfloor, \\ -\frac{1}{3\lfloor T/6\rfloor}\sum_{s=0}^{3\lfloor T/6\rfloor} \phi_s, & 3\lfloor T/6\rfloor < t \leq 6\lfloor T/6\rfloor, \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

satisfies

$$\liminf_{T\to\infty} \frac{EV_T(\phi(T,\alpha))}{T^{H\left(1+\frac{1}{\alpha-1}\right)+1}} > 0. \tag{30}$$

The strategy above builds on the following intuition. In a market with friction one can not sell or buy at an arbitrary speed. Such behavior is punished by superlinear price-impact models. Strategies that are not trading assets gradually can generate losses that ruin an otherwise profitable investment. Thus, liquidation must also be done at a careful pace. Our strategy operates as follows. On the first half of the given timeline, it trades the underlying in a contrarian manner, that is, going short when prices are high and entering long positions when they are low. Liquidation is then done with a constant speed. This is reflected in our strategy on the second half of the timeline. Such liquidation may not be optimal but it is computationally convenient and turns out to be “good enough” for our present purposes.

5 Optimal investment with high risk aversion

Previously used symbols that are reused over the course of the current section do not inherit their meaning from previous definitions. The discussion below is self-contained in terms of notions and notations.

Modern portfolio theory suggests that optimal strategies are insensitive to investment horizons and prescribes constant portfolio weights - see for example Merton (1969). So called turnpike theorems propose that the aforementioned result of Robert Merton is robust, and the phenomena of homogeneity is universal in the long run.

In contrast, the common advice in circles of retirement planners, given to heavily risk averse investors, is instead to reduce risk with age, see Malkiel (1999, p. 361) chiming in: "as investors age, they should start cutting back on riskier investments".

The work Guasoni, Nagy, Rásonyi (2020) displays theoretical evidence that when prices mean-revert and investors are exponentially risk averse, in contrast with common tenets of modern portfolio theory, the advice to gradually reduce risk is most relevant.

5.1 Merton's problem

To motivate the discussion let us first expose the emblematic work of Merton (1969). The investment and consumption problem presented therein involves finding a pair (C,π), consisting of the optimal consumption rate $C=(C_t)_{t\geq 0}$ and the optimal investment strategy $\pi=(\pi_t)_{t\geq 0}$ under model-constraints that are laid out in what follows.

For simplicity we focus on a two asset market where the trader can invest in a risky asset $X^0=(X^0_t)_{t\geq 0}$ and a safe asset $X^1=(X^1_t)_{t\geq 0}$. The risky asset X^0 follows the Balck-Scholes-Merton dynamics,

$$dX^0_t=\mu X^0_t dt+\sigma X^0_t dB_t, \tag{31}$$

where $\mu\in\mathbb{R}$, $\sigma>0$ and B_t is a standard Wiener-Brownian motion process. The asset X^1 is described by the deterministic equation $dX^1_t=rX^1_t dt$, simulating an investment with guaranteed rate of return $r>0$ with zero risk.

Before we introduce the continuous time model, for explanatory reasons let us consider a one-period, discrete time budget equation. In this preliminary settings the wealth of the investor at time $t+h$ resulting from her consumption and investment decisions made at time t is

$$W_{t+h}=\sum_{i=0}^{1}\pi^i_t(W_t-hC_t)\frac{X^i_{t+h}}{X^i_t}. \tag{32}$$

The intuition behind this formulation is that profit and loss emerges from the strategy π^i_t, $i=0,1$ prescribing what proportion of the post-consumption wealth (W_t-hC_t) is invested in the assets X^i, $i=0,1$.

To adapt the dynamics (32) to continuous time, note first that algebraic manipulation of (32) yields

$$\begin{aligned}&\sum_{i=0}^{1}\pi^i_t(W_t-hC_t)\frac{X^i_{t+h}}{X^i_t}\\&=\sum_{i=0}^{1}\pi^i_t(W_t-hC_t)\frac{X^i_{t+h}-X^i_t}{X^i_t}+\pi^i_t(W_t-hC_t).\end{aligned} \tag{33}$$

For an arbitrary function f let us denote its increment from t to $t+h$ as $df_t=f_{t+h}-f_t$. Using the identity $\pi^0_t=1-\pi^1_t$, and assuming in a heuristic fashion that zero can be substituted in place of the symbol $dX_t dt$, we get from (33) that

$$dW_t = W_t \pi_t^0 \frac{dX_t^0}{X_t^0} + W_t \pi_t^1 \frac{dX_t^1}{X_t^1} - C_t dt. \tag{34}$$

The stochastic differential equation (34) for $t \in [0,T]$ is the continuous time analog of (32) and we refer to it as the budget constraint.

Let $\rho > 0$, $c < 1$, $0 < \epsilon < 1$ and let the utility function be the isoelastic utility $U(x) = x^c/c$. The problem Merton (1969) solves is

$$E\left[\int_0^T e^{-\rho t} U(C_t)\, dt + e^{1-c} e^{-\rho T} U(W(T))\right] \to \max, \tag{35}$$

subject to the budget constraint (34) and $C_t \geq 0$, $W_t \geq 0$, $W_0 > 0$. We quote the findings corresponding to the optimal investment $\pi_t^* = (\pi_t^{0,*}, \pi_t^{1,*})$: it turns out that optimality happens with the choice

$$\pi_t^{0,*} = \frac{\mu - r}{\sigma^2(1-c)}. \tag{36}$$

This quantity, not surprisingly, depends on parameters of the underlying model. A higher risk-aversion parameter c or volatility σ implies a smaller invested portion of the wealth on the risky side. Also, the more the average return μ dominates the zero-risk rate r, the larger the quantity invested on the stochastic asset.

Yet, quite surprisingly, π_t^* is independent of both the elapsed time t and of the horizon T - meaning that the strategy is homogeneous in the strictest possible sense.

Furthermore, the phenomenon of constant optimal investment holds true even if we take a simplified version of the problem above with no consumption rules. For example Pham (2009, Subsection 3.6.1) considers a geometric Brownian motion as the price X^0, a safe asset X^1, and an iso-elastic utility exactly as above and solves

$$E\left[U(\bar{W}_T)\right] \to \max, \tag{37}$$

with the wealth-dynamics

$$d\bar{W}_t = \bar{W}_t \bar{\pi}_t^0 \frac{dX_t^0}{X_t^0} + \bar{W}_t \bar{\pi}_t^1 \frac{dX_t^1}{X_t^1}. \tag{38}$$

Note that in contrast with (35) where consumption is taken into account, the optimization problem in (37) is simply the utility maximization of the wealth of the investor. Also observe that (38) is the special case of (34) when $C_t = 0$ for all $t \geq 0$. The optimization problem in (37) is often referred to as "Merton's portfolio allocation problem".

The strategy $\bar{\pi}_t^* = (\bar{\pi}_t^{0,*}, \bar{\pi}_t^{1,*})$ optimizing Merton's problem in (37) satisfies $\bar{\pi}^* = \pi^*$, meaning that it is the same as the one in (36), when consumption was allowed: constant across all times $t \geq 0$ and independent of the time horizon T.

We continue with a brief discussion of turnpike theorems showing that homogeneity of optimal investments is not limited to the settings above, and the optimal strategy in the solution of Mertons's problem carries robust features.

5.2 Turnpike

Turnpike theorems are a happy exception in the midst of complicated portfolio choice results. These roughly speaking state that on the long run optimal strategies become homogeneous and in some sense only isoelastic utilities play a role.

Guasoni, Robertson (2009) develops the solution to the long horizon problem of finding optimal portfolios and risk premia, under power utility, in a market where asset prices are modeled with a general diffusion driven by an autonomous state process. The paper considers the solution of an ergodic Hamilton Jacoby Bellmann type equation that has no dependence on the time horizon. The function that satisfies the HJB equation is then used to build the optimal investment strategy, in a way that homogeneity is preserved. The conclusion is that under very general conditions, long term optimal investments are homogeneous in the sense of independence of the terminal time of trading, and they are also completely time-homogeneous if state variables are constant (like in the model of Black, Scholes and Merton).

Guasoni, Kardaras, Robertson, Xing (2014, Theorem 2.20), employing a diffusion model, claims that if we form the ratio of the finite horizon optimal strategies and the long term optimal strategy (the latter coinciding with the homogeneous strategy constructed utilizing the solution of the ergodic equation mentioned above) then this ratio converges to one in probability. In more direct terms, short term optimal strategies converge to a homogeneous limit under mild assumptions.

Guasoni, Kardaras, Robertson, Xing (2014, Theorem 2.19) employs general utility functions that are only constrained to behave similarly to power functions for large values. The result states that, on the long run, optimal strategies corresponding to general utilities converge to optimal investments under isoelastic utilities. In more specific terms, if we are trading a Black, Scholes, Merton market, it does not matter what utility function we use: optimal investments converge to a fixed, constant limit, prescribed by Merton (1969) - see (36).

In the following discussions it will be displayed that the above ostensible universality carries its limitations in terms of the level of risk-aversion.

5.3 Non-linear mean reverting models

Now we turn to the work Guasoni, Nagy, Rásonyi (2021). Results therein show that in the presence of mean reverting prices and a highly risk averse investor, portfolios exhibit a qualitatively different sensitivity to investment horizons compared to the case of power utilities. Instead of convergent investment strategies, suggested by turnpike theorems, the optimal investments actually diverge.

Stochastic processes are defined on a common probability space $(\Omega, \mathscr{F}, P)$ with an augmented natural filtration $(\mathscr{F}_t)_{t\geq 0}$. The market model employs a riskless asset and a risky asset with price S_t that follows the dynamics

$$\begin{aligned} dS_t &= \mu dt + dX_t, \\ dX_t &= -\alpha \operatorname{sgn}(X_t)|X_t|^{\beta} dt + dB_t, \end{aligned} \tag{39}$$

where $\mu \in \mathbb{R}$, $\alpha \geq 0$, $\beta > 1$ and B_t is a standard one-dimensional Brownian motion. Without the loss of generality we assume unit volatility. The stochastic differential equation (39) has a unique strong solution, see Krylov (1999).

When $\mu = 0$, the process (39) admits a stationary distribution. This implies that when $\mu \neq 0$, the asset price is the sum of an asymptotically stationary process and a deterministic linear drift.

We note here that the dynamics (39), with the choice $\alpha = 0$, covers the Bachelier model. Here, an arithmetic drift $dS_t = \mu dt + dX_t$ is used, but note that employing a geometric drift $dS_t/S_t = \mu dt + dX_t$ instead, would generate the same payoff space and replicating strategies could be matched with a one-to-one correspondence. This shows that findings that are developed for (39) have implications also on the Black, Scholes, Merton model.

The choice $\beta = 1$ retrieves the Ornstein-Uhlenbeck process with drift. The model that is generated by other values of $\beta > 1$ can be contrasted with the case $\beta = 1$ as follows. Observe that (39) can be written as

$$d(S_t - \mu t) = -\mathrm{sgn}(S_t - \mu t)|S_t - \mu t|^{\beta} dt + dB_t,$$

showing that when $\beta = 1$, the force that pulls back the price to the trendline μt is linear. As opposed to this, in case of $\beta > 1$, the distance from the trendline is raised to a power greater than 1, resulting in a superlinear retrograde force.

The case $\mu \neq 0$ and $\mu = 0$ separate two regions with different characteristics. The non-stationary regime ($\mu \neq 0$) corresponds to assets whose prices admit nonzero average growth rate. These are stocks and bonds, having a historical record of outperforming inflation. In contrast with this the stationary regime ($\mu = 0$) corresponds to assets whose price tends to grow at the rate of inflation, without exceeding it. These assets that possess considerable short term fluctuations but do not show significant behavior of long term trending, are called *long-term safe* assets. Examples of long-term safe assets are gold, platinum and silver.

A trading strategy is an adapted, S integrable process $H = (H_t)_{t\in[0,T]}$, where H_t represents the amount of shares the trader holds on the risky asset at time t. If the investor uses the strategy H then her wealth at the terminal time $T > 0$ is prescribed to be the stochastic integral

$$(H \circ S)_T = \int_0^T H_t dS_t. \tag{40}$$

Let us denote the unique risk-neutral measure Q_T (for details of its existence see Proposition B.1). We define the set of *admissible strategies* $\mathscr{X}_T$ as the set of S-integrable processes $(H_t)_{t\in[0,T]}$ such that $(H \circ S)_T$ is a Q_T martingale.

With the notions above for a fixed terminal time T the investor aims at maximizing her exponential utility

$$E\left[-e^{-(H\circ S)_T}\right] \to \max,$$

where the optimization happens over the set of admissible strategies $\mathscr{X}_T$. For all $T > 0$ we associate the value function

$$u_T = \sup_{H \in \mathscr{X}_T} E\left[-e^{-(H \circ S)_T}\right] \tag{41}$$

to the portfolio optimization problem on $[0, T]$. We define the corresponding *certainty equivalent*, the amount of value a potential market participant would accept as compensation for not being able to trade on the interval $[0, T]$, as

$$c_T = -\ln(-u_T). \tag{42}$$

The rest of the section is structured as follows. First, using Deák, Rásonyi (2015), we derive a heuristic formula for the optimal growth of the Ornstein-Uhlenbeck process without drift – (39) with $\mu = 0$ and $\beta = 1$. Then, utilizing dynamical programming, we derive the same type of results, still in a heuristic manner, when we allow an arbitrary linear drift – (39) with $\mu \in \mathbb{R}$ and $\beta = 1$. After these, we present results on the general model – (39) with $\mu \in \mathbb{R}$ and $\beta > 1$.

5.4 Ornstein-Uhlenbeck process

In the special case $\mu = 0$ and $\beta = 1$ the dynamics (39) coincides with the Ornstein-Uhlenbeck process

$$dS_t = dX_t = -\alpha X_t dt + dB_t. \tag{43}$$

For the purpose of inferring about the optimal growth rate associated with the asset price S_t subject to (43) we first present the findings of Deák and Rásonyi (2015).

Let $\eta_k,\ k \in \mathbb{N}$ be an independent and identically distributed sequence of standard Gaussian random variables. Set $R_0 = 0$ and recursively define the autoregressive process of order one $R_k,\ k \in \mathbb{N}$ with the parameter of mean reversion $\nu_0 \in (-1, 1)$ for $k \geq 1$ as

$$R_k = \nu_0 R_{k-1} + \eta_k.$$

Reordering the equation above one gets

$$R_k - R_{k-1} = \nu R_{k-1} + \eta_k,\ k \geq 1, \tag{44}$$

where $\nu := \nu_0 - 1$ is a parameter corresponding to the returns of the process $R_k,\ k \geq 0$. Define a filtration as $\mathscr{G}_k := \sigma(R_0, \dots, R_k)$, $k \in \mathbb{N}$. Trading will follow the discrete analog of the dynamics (9), that is, the wealth of the investor at any time N, with strategy $\Phi = (\Phi_k)_{k=1}^N$, assuming no initial endowments, is

$$L_N^\Phi := \sum_{j=1}^{N} \Phi_j (R_j - R_{j-1}). \tag{45}$$

The process Φ_k is predictable with respect to $\mathscr{G}$, and represents the number of shares held by the investor on the risky asset at time k. Deák and Rásonyi (2015) displays the following result.

Theorem 5.1. *For each $N \geq 1$, the optimal strategy for time horizon N is given by*

$$\bar{\phi}_k^N := g_k^N(R_{k-1}),\ 1 \leq k \leq T,$$

where

$$g_k^N(z) = \nu z[1 - (N-k)\nu] \qquad \text{for all } 1 \leq k \leq N \text{ and } z \in \mathbb{R}. \tag{46}$$

Using these strategies, the maximal expected utilities are

$$r(N;\nu) := \sup_{\phi} \mathbb{E}\left[-e^{-L_N^\phi}\right] = \mathbb{E}\left[-e^{-L_N^{\bar{\phi}}}\right] = -\bar{\gamma}(N;\nu)^{-\frac{1}{2}}, \tag{47}$$

where $\bar{\gamma}(N;\nu) := \nu^{2N}\Gamma(1/\nu^2 + N)\Gamma(1/\nu^2)^{-1}$ and Γ is the well-known gamma function. □

We note a few remarkable features of Theorem 5.1. First, calculations show that the certainty equivalent $-\ln(-r(N,\nu))$ grows as $N\ln(N)$. In case when $\nu = 0$, that is when R is a random walk, not surprisingly, no growth is possible. Actually it can be easily seen that for a martingale the certainty equivalent is bounded by -1.

Second, the temporal structure is decisive. This can be seen by examining the wealth process (45) when Φ_k is a time homogeneous measurable function of the past of the asset price. In this case the summands in (45) form a stationary sequence, thus, it is intuitively clear that a growth of order N can not be exceeded.

Third, mean reversion is exploited in the beginning and then to a less and less extent. Furthermore, although the former temporal interaction between the strategy and the fluctuations of the price vanishes close to the horizon, positions are never fully liquidated. There is a non-zero exposure to the risky asset even at the terminal time.

Now we use the Euler scheme to formulate a discrete time approximation of the Ornstein-Uhlenbeck process (43). The approximating process $(\bar{R}_k)_{k=1}^N$ up to the fixed time horizon T, with a grid of resolution N is

$$\bar{R}_k^{(N)} = \bar{R}_{k-1}^{(N)} - \alpha \frac{T}{N} \bar{R}_{k-1}^{(N)} + \bar{\eta}_k, \tag{48}$$

where $\bar{R}_0^N = 0$ and $(\bar{\eta})_{k=1}^N$ is a standard Gaussian white noise.

Notice that $\bar{R}_k$ is an autoregressive process of order one for every single fixed N. This means that the result of Rásonyi and Deák is applicable. Reordering (48) to match the parameters we get

$$\bar{R}_k^{(N)} - \bar{R}_{k-1}^{(N)} = -\alpha \frac{T}{N} \bar{R}_{k-1}^{(N)} + \bar{\eta}_k. \tag{49}$$

With the help of (47) we produce the mapping

$$T \to r\left(N; -\frac{\alpha T}{N}\right). \tag{50}$$

This should provide heuristics for the growth rate of the certainty equivalent for the limiting process (43).

Considering $\ln \bar{\gamma}(N;\nu)$ and using the estimate $\ln(n!) \approx n\ln(n) - n$, we have

$$\begin{aligned}
\ln\bar{\gamma}(N;\nu) &\approx -N\ln(1/\nu^2) + (1/\nu^2 + N)\ln(1/\nu^2 + N) \\
&\quad - (1/\nu^2 + N) - (1/\nu^2)\ln(1/\nu^2) + 1/\nu^2 \\
&= (1/\nu^2 + N)\left(\ln(1/\nu^2 + N) - \ln(1/\nu^2)\right) - N \\
&= (1/\nu^2 + N)\left(\ln(1 + \nu^2 N)\right) - N.
\end{aligned}$$

With the substitution $\nu = -\frac{\alpha T}{N}$, algebraic manipulation, and Taylor's expansion we get

$$
\begin{aligned}
\ln\bar{\gamma}\left(N;-\frac{\alpha T}{N}\right) &\approx \left(\frac{N^2}{\alpha^2T^2}+N\right)\ln\left(1+\frac{\alpha^2T^2}{N}\right)-N \\
&= N\left(\frac{N}{\alpha^2T^2}\ln\left(1+\frac{\alpha^2T^2}{N}\right)-1\right)+\ln\left(1+\frac{\alpha^2T^2}{N}\right)^N \\
&= N\left(\frac{N}{\alpha^2T^2}\left(\frac{\alpha^2T^2}{N}+\mathcal{O}(N^{-2})\right)-1\right)+\ln\left(1+\frac{\alpha^2T^2}{N}\right)^N \\
&= \mathcal{O}(1)+\ln\left(1+\frac{\alpha^2T^2}{N}\right)^N.
\end{aligned}
$$

Taking limit as $N\to\infty$, in a heuristic fashion, yields the following. The optimal expected exponential utility corresponding to the model (43) on the trading interval $[0,T]$, is estimated to be

$$
-e^{-\alpha^2T^2/2}. \tag{51}
$$

Thus, we arrive at the surprising conjecture that, dissimilarly to the discrete time sub-quadratic optimal growth rate of $N\ln N$ with autoregressive asset price, the analogous continuous time settings with the Ornstein-Uhlenbeck process yields instead a quadratic optimal growth rate of order T^2.

5.5 Ornstein-Uhlenbeck process with drift

In this subsection we consider (39) with $\mu\in\mathbb{R}$ and $\beta=1$, when the price is an Ornstein-Uhlenbeck process augmented with a linear drift. Arguments here are still heuristic but it seems that they could be made rigorous easily. Referring back to the heuristic calculations of Subsection 5.4, results below in particular also retrieve the limit (51), and suggest that it is indeed the true asymptotics of the driftless Ornstein-Uhlenbeck model.

Let the asset price process S_t follow the dynamics

$$
\begin{aligned}
dS_t &= \mu dt + dX_t \\
dX_t &= -\alpha X_t dt + dB_t.
\end{aligned} \tag{52}
$$

As before, the investor aims to maximize the expected exponential utility of the terminal portfolio value

$$
E[-e^{-(H\circ S)_T}]\to\max,
$$

where H varies among the set of admissible strategies defined right below formula (40). Let us denote the wealth of the investor $W_t = (H \circ S)_t$. Then, the value of the optimization problem is

$$V(t,w,x) = \sup_H E\left[-e^{-W_T} | X_t = x,\ W_t = w\right]. \tag{53}$$

This quantity follows the dynamics

$$dV(t,W_t,X_t) = \left(V_t + (\mu - \alpha X_t)H_t V_w - \alpha X_t V_x + \frac{1}{2}H_t^2 V_{ww} + H_t V_{wx} + \frac{1}{2}V_{xx}\right)dt$$
$$+ (V_w H_t + V_x)\, dB_t.$$

The martingale principle of optimal control of Davis, Varaiya (1973) says that the value function is a supermartingale for every admissible strategy and it is a martingale for the optimal one. This requires that the maximal drift over all strategies is zero, yielding the Hamilton-Jacobi-Bellman-type equation

$$V_t + \frac{1}{2}V_{xx} - \alpha X_t V_x \sup_{H_t}\left((\mu - \alpha X_t)H_t V_w + \frac{1}{2}H_t^2 V_{ww} + H_t V_{wx}\right) = 0. \tag{54}$$

This implies a candidate for the optimal strategy

$$H_t = -\frac{(\mu - \alpha X_t)V_w}{V_{ww}} - \frac{V_{wx}}{V_{ww}}.$$

Substituting this into the HJB equation (54) reduces it to

$$V_t + \frac{1}{2}V_{xx} - \alpha X_t V_x - \frac{\left((\mu - \alpha X_t)V_w + V_{wx}\right)^2}{2V_{ww}} = 0.$$

To solve this we can utilize an exponential-quadratic guess

$$V(t,w,x) = \exp\left\{-\left(w + \frac{a(t)}{2}x^2 + b(t)x + c(t)\right)\right\}.$$

This eliminates the variable x from (54) and a more simple form emerges as

$$\left(-\frac{\alpha^2}{2} - \frac{a'(t)}{2}\right)x^2 + \left(\alpha\mu + \mu a(t) - b'(t)\right)x + \left(-\frac{\mu^2}{2} - \frac{a(t)}{2} + \mu b(t) - c'(t)\right) = 0.$$

The equation must hold for all t and x and hence the coefficients of x^2, x and 1 must vanish and this yields the system of equations

$$\alpha^2 + a' = 0$$
$$\alpha\mu + \mu a - b = 0$$
$$\frac{\mu^2}{2} + \frac{a}{2} - \mu b - c' = 0.$$

Solving the system gives

$$a(t) = \alpha^2(T-t)$$
$$b(t) = -(T-t)\alpha\mu - (T-t)^2\frac{\alpha^2\mu}{2}$$
$$c(t) = \frac{\mu^2}{2}(T-t) + \frac{\alpha(2\mu^2+\alpha)}{2}(T-t)^2 + \frac{\alpha^2\mu^2}{6}(T-t)^3.$$

The optimal strategy takes the form

$$H_t = \left(\mu - \alpha X_t\right) + \alpha\left(\mu - \alpha X_t\right)(T-t) + \frac{\mu\alpha^2}{2}(T-t)^2 \tag{55}$$

and the certainty-equivalent $C(t,w,x) = -\ln(-V(t,w,x))$ is

$$C(t,w,x) = w + \frac{\mu - \alpha x}{2}(T-t) + \left(\frac{\alpha\mu(\mu-\alpha x)}{2} + \frac{\alpha^2}{4}\right)(T-t)^2 + \frac{\mu^2\alpha^2}{6}(T-t)^3. \tag{56}$$

Thus, the leading order of the certainty equivalent $c_T = C(T,w,x)$ is

$$\begin{aligned} &\text{(i)} \quad (T-t)^3 \quad \text{if } \alpha,\ \mu \neq 0 \\ &\text{(ii)} \quad (T-t)^2 \quad \text{if } \mu = 0 \text{ but } \alpha \neq 0 \\ &\text{(iii)} \quad (T-t)^1 \quad \text{if } \mu \neq 0 \text{ but } \alpha = 0 \\ &\text{(iv)} \quad (T-t)^0 \quad \text{if } \alpha = \mu = 0. \end{aligned} \tag{57}$$

We will analyse these findings in light of the results corresponding to the general model (39) in the forthcoming.

5.6 Optimality with superlinear mean-reversion

This section contains the main results of the paper Guasoni, Nagy, Rásonyi (2021). The work considers the portfolio choice problem, already introduced in Subsection 5.3, with the non-linear mean-reversion model

$$dS_t = \mu dt + dX_t,$$
$$dX_t = -\alpha \operatorname{sgn}(X_t)|X_t|^\beta dt + dB_t,$$

with the investor aiming to maximize her exponential expected utility

$$E[-e^{-(H\circ S)_T}] \to \max,$$

where H varies in the set of admissible strategies $\mathscr{X}_T$ (defined in Subsection 5.3). The associated value function u_T and c_T is as in (41) and (42) respectively.

In (58) below we introduce a new notion that measures the asymptotic performance of a family of trading strategies with the method of benchmarking the associated certainty equivalent asymptotics to power growth rates. For a family of strategies $\mathfrak{H} = (H(T))_{T>0}$ with $H(T) \in \mathscr{X}_T,\ T > 0$, define the *order of the certainty-equivalent* as

$$\mathscr{C}(\mathfrak{H}) = \sup\left\{\theta :\ \liminf_{T\to\infty} \frac{-\ln(E\left[e^{-(H\circ S)_T}\right])}{T^\theta} > 0\right\}. \tag{58}$$

To give an example, note that in general, an annuity that grows with the rate of change of the certainty equivalent c_T, can compensate for the absence of trading opportunity. The order of such an annuity equals the order of the optimal certainty equivalent, minus one. The results presented in (57) states in particular that with a nonzero average return ($\mu \neq 0$) but in the absence of mean reversion ($\alpha = 0$) the optimal certainty equivalent c_T is proportional to T, which means that its order is one. This implies that the investor, in this particular case, is indifferent between trading optimally and receiving a constant annuity.

Now we are ready to turn to the main theorems of Guasoni, Nagy, Rásonyi (2021) treating the general model (39). In these theorems we focus on $\alpha > 0$. We present first the theorem on the case of non-stationary price ($\mu \neq 0$). In this case the assertion is that the optimal certainty equivalent grows with the horizon as $T^{2\beta+1}$ and the strategy in (59) below acquires this performance.

Theorem 5.2. *If $\mu \neq 0$, then $c_T \leq C_{\beta,\mu,\alpha} T^{2\beta+1}$ for some constant $C_{\beta,\mu,\alpha} > 0$, each family of strategies $\mathscr{C}(\mathfrak{H})$ satisfy $\mathscr{C}(\mathfrak{H}) \leq 2\beta + 1$. The family of strategies $\mathfrak{H}_\beta = (H(\beta,T))_{T>0}$ defined by*

$$H_t(\beta,T) = (\beta+1)(T-t)^\beta sgn(S_t)|S_t|^\beta,\ t \in [0,T] \tag{59}$$

satisfies $\mathscr{C}(\mathfrak{H}_\beta) = 2\beta + 1$.

We highlight a few features of this result. First, no fixed annuity can compensate for the loss of trading opportunity over an arbitrary long period of time, since the order of the certainty equivalent surpasses one. Instead, the equivalent annuity would have to grow with the power 2β of the horizon.

Second, let us note that the intuitive homogeneous strategy would be to buy the asset when it is below its long-term trend and sell it otherwise. Surprisingly, asymptotic optimality is achieved by an investment that times the market around the starting point where the drift is negligible compared to later stages. It can be also observed that the size of early bets on the market dominates later ones - risk is concentrated on early stages of the trading interval.

Interestingly, the mean reversion parameter β solely controls the sensitivity to market states and adjusts the temporal magnification of positions. A higher curvature implies a larger sensitivity to price changes along with a higher concentration of resources to the beginning of the trading interval. Furthermore, a higher value of β implies a better performance.

The second part of the main result of Guasoni, Nagy, Rasonyi (2021), Theorem 5.3 below, shows that when the price is stationary in an asymptotic sense ($\mu = 0$), optimality is significantly different compared to a trending asset price (treated in Theorem 5.2 above).

Theorem 5.3. *If $\mu = 0$, then $c_T = C_\beta T^{1+\beta}$ for some constant $C_\beta > 0$, and each family of strategies $\mathfrak{H}$ satisfy $\mathscr{C}(\mathfrak{H}) \leq 1+\beta$. If $\beta > 1$ then for each $1 < \gamma < \beta$, the family of strategies $\mathfrak{H}_\gamma = (H(\gamma, T))_{T>0}$ defined by*

$$H_t(\gamma, T) := -2(T-t)^\gamma S_t, \; t \in [0, T] \tag{60}$$

satisfies $\mathscr{C}(\mathfrak{H}_\gamma) \geq 1+\gamma$. If $\beta = 1$ then there exists $\delta_0 \in (0, 1/2)$ so that $\mathscr{C}(\delta_0 \mathfrak{H}_1) = 2 = 1+\beta$.

In the case $\mu = 0$, Theorem 5.3 above shows that the certainty equivalent is of order $T^{\beta+1}$ in contrast with $T^{2\beta+1}$ in case $\mu \neq 0$ (as Theorem 5.2 displays the latter asymptotics). This means that for long-term safe assets, in contrast with assets with a non-trivial average growth rate, the mean-reversion curvature β adds only once rather than twice to the order of growth.

We can understand these results in terms of the findings that correspond to the drifted Ornstein-Uhlenbeck process in Subsection 5.5 above. Examining the certainty equivalent in (56) and its implications shown in (57) it becomes clear that the growth of order T^3 (corresponding to $T^{2\beta+1}$ in the general case in Theorem 5.2

above) emerges from the interaction between trend-like growth and mean-reversion, hence is lost when either of these components vanish. In contrast with this, the growth with order T^2 (corresponding to $T^{\beta+1}$ displayed in Theorem 5.3) is a result solely from mean-reversion, so this rate is preserved even when there is no trend in the dynamics.

The heuristic analysis above, using the results of Subsection 5.5 would furthermore suggest that any strategy that performs with the rate $T^{2\beta+1}$ should incorporate and exploit the parameters μ and α simultaneously. However Theorem 5.2 and Theorem 5.3 show that this is not the case. Rather, trading strategies do not depend on the average asset return or mean-reversion speed, but only on the mean-reversion curvature.

This feature is of practical importance because it implies that to capture the leading order it is enough to estimate the mean-reversion curvature β. Thus, the parameter β, that grants the model its non-linear character, is of principal significance in terms of strategies and performance. Other parameters, the speed of mean reversion α and the expected rate of return μ, influence optimality but do not affect the growth rate.

It is also important to note that since trading strategies grow with the horizon without bounds it is obvious that there is no turnpike in the model. Additionally, there are no time-homogeneous strategies that yield asymptotic optimality. Put differently, despite the fact that the price process is modeled by a time-homogeneous and ergodic Markovian process, time-homogeneity and ergodicity fails for the optimal strategies.

As a possible path of ramifications we conjecture that simple asymptotically optimal strategies can be found even in the case where $0 < \beta < 1$. This, however, requires further study.

A Proofs of Section 4

A.1 General bounds for variance and covariance

First we make some useful preliminary observations. Using stationarity of the increments of the process S, we have

$$\begin{aligned}
\operatorname{var}(S_t) = \operatorname{cov}(S_t, S_t) &= \operatorname{cov}\left(\sum_{j=1}^{t}\left(S_j - S_{j-1}\right), \sum_{i=1}^{t}(S_i - S_{i-1})\right) \\
&= t \cdot \operatorname{var}(S_1 - S_0) + 2\sum_{i=2}^{t}\sum_{j=1}^{i-1}\operatorname{cov}(S_j - S_{j-1}, S_i - S_{i-1}) \\
&= t \cdot \operatorname{var}(S_1 - S_0) + 2\sum_{i=2}^{t}\sum_{j=1}^{i-1}\operatorname{cov}(S_1 - S_0, S_{i-j+1} - S_{i-j}) \\
&= t \cdot r(0) + 2\sum_{i=2}^{t}\sum_{j=1}^{i-1} r(i-j).
\end{aligned} \tag{61}$$

Furthermore, for $s > t$ we similarly have

$$\operatorname{cov}(S_s - S_t, S_t) = \sum_{i=t+1}^{s}\sum_{j=1}^{t} r(i-j). \tag{62}$$

Observe also that we can write

$$r(0) = -2\sum_{j=1}^{\infty} r(j). \tag{63}$$

Turning to the variances, we first obtain a convenient expression for them. Note that for $i > 1$ we have

$$\sum_{j=1}^{i-1} r(i-j) = r(i-1) + \ldots + r(1) = r(1) + \ldots + r(i-1) = \sum_{j=1}^{i-1} r(j). \tag{64}$$

Using the observations (64), (61) and (63), we have

$$\operatorname{var}(S_t) = -2t\sum_{j=1}^{t-1} r(j) - 2t\sum_{j=t}^{\infty} r(j) + 2\sum_{i=2}^{t}\sum_{j=1}^{i-1} r(j),$$

and algebraic manipulation of the summation operation $\left(-2t\sum_{j=1}^{t-1}+2\sum_{i=2}^{t}\sum_{j=1}^{i-1}\right)$ yields

$$
\begin{aligned}
&-2t\sum_{j=1}^{t-1}+2\sum_{i=2}^{t}\sum_{j=1}^{i-1}\\
&=-2t\left(\sum_{j=1}^{T_0-1}+\sum_{j=T_0}^{t-1}\right)+2\left(\sum_{i=2}^{T_0-1}+\sum_{i=T_0}^{t}\right)\sum_{j=1}^{i-1}\\
&=-2t\sum_{j=1}^{T_0-1}-2t\sum_{j=T_0}^{t-1}+2\sum_{i=2}^{T_0-1}\sum_{j=1}^{i-1}+2\sum_{i=T_0}^{t}\sum_{j=1}^{i-1}\\
&=-2t\sum_{j=1}^{T_0-1}-2t\sum_{j=T_0}^{t-1}+2\sum_{i=2}^{T_0-1}\sum_{j=1}^{i-1}+2\sum_{j=1}^{T_0-1}+2\sum_{i=T_0+1}^{t}\left(\sum_{j=1}^{T_0-1}+\sum_{j=T_0}^{i-1}\right)\\
&=-2t\sum_{j=1}^{T_0-1}-2t\sum_{j=T_0}^{t-1}+2\sum_{i=2}^{T_0-1}\sum_{j=1}^{i-1}+2\sum_{j=1}^{T_0-1}+2\sum_{i=T_0+1}^{t}\sum_{j=1}^{T_0-1}+2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\\
&=-2t\sum_{j=1}^{T_0-1}+2\sum_{i=2}^{T_0-1}\sum_{j=1}^{i-1}+2\sum_{j=1}^{T_0-1}+2\sum_{i=T_0+1}^{t}\sum_{j=1}^{T_0-1}-2t\sum_{j=T_0}^{t-1}+2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1},
\end{aligned}
$$

where the last line is only a reordering of terms. Setting $C_1=\sum_{j=1}^{T_0-1}r(j)$, $C_2=\sum_{i=2}^{T_0-1}\sum_{j=1}^{i-1}r(j)$ and $C_3=2(C_2-(T_0-1)C_1)$, the above calculation gives

$$
\begin{aligned}
\operatorname{var}(S_t)&=-2tC_1+2C_2+2C_1+2(t-T_0)C_1+\left(-2t\sum_{j=t}^{\infty}-2t\sum_{j=T_0}^{t-1}+2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\right)r(j)\\
&=C_3+\left(-2t\sum_{j=t}^{\infty}-2t\sum_{j=T_0}^{t-1}+2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\right)r(j)
\end{aligned}
\tag{65}
$$

Now we are ready to present three lemmas, providing a lower and an upper bound for the variance and an upper bound for the covariance.

Lemma A.1. *There exist $T_1\in\mathbb{N}$ and $B_1>0$ such that for all $t\geq T_1$ we have*

$$\operatorname{var}(S_t)\geq B_1t^{2H}.$$

Proof. Using properties induced by the choice of T_0 in Assumption 4.1 first note that

$$\begin{aligned}
&\left(-2t\sum_{j=T_0}^{t-1} + 2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\right) r(j) \\
&\geq \left(-2t\sum_{j=T_0}^{t-1} + 2(t-T_0)\sum_{j=T_0}^{t-1}\right) r(j) \\
&= -2T_0\sum_{j=T_0}^{t-1} r(j) \geq 0.
\end{aligned}$$

Also notice that

$$\begin{aligned}
-2t\sum_{j=t}^{\infty} r(j) \geq -2J_2 t\sum_{j=t}^{\infty} j^{2H-2} &\geq -2J_2 t\int_t^{\infty} u^{2H-2}du \\
&= -2J_2 t\frac{1}{2H-1}\left(-t^{2H-1}\right) = \frac{2J_2}{2H-1}t^{2H}.
\end{aligned}$$

Using these and (65)

$$\operatorname{var}(S_t) \geq C_3 + \frac{2J_2}{2H-1}t^{2H}.$$

The threshold T_1 and the constant B_1 can be explicitly calculated in terms of the constants present in the above expression. This completes the proof. □

Lemma A.2. *There exist $T_2 \in \mathbb{N}$ and $B_2 > 0$ such that for all $t \geq T_2$ we have*

$$\operatorname{var}(S_t) \leq B_2 t^{2H}.$$

Proof. First note that algebraic manipulation of the operation $\left(-2t\sum_{j=T_0}^{t-1} + 2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\right)$ yields

$$\begin{aligned}
&-2t\sum_{j=T_0}^{t-1} + 2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1} = -2(t-T_0+T_0)\sum_{j=T_0}^{t-1} + 2\sum_{i=T_0}^{t-1}\sum_{j=T_0}^{i} \\
&= -2\sum_{i=T_0}^{t-1}\sum_{j=T_0}^{t-1} + 2\sum_{i=T_0}^{t-1}\sum_{j=T_0}^{i} - 2T_0\sum_{j=T_0}^{t-1} = -2\sum_{i=T_0}^{t-1}\left(\sum_{j=T_0}^{t-1} - \sum_{j=T_0}^{i}\right) - 2T_0\sum_{j=T_0}^{t-1} \\
&= -2\sum_{i=T_0}^{t-1}\sum_{j=i+1}^{t-1} - 2T_0\sum_{j=T_0}^{t-1}.
\end{aligned}$$

By Assumption 4.1, this implies

$$\begin{aligned}
\left(-2t\sum_{j=T_0}^{t-1} + 2\sum_{i=T_0+1}^{t}\sum_{j=T_0}^{i-1}\right) r(j) &\le -2J_1\left(\sum_{i=T_0}^{t-1}\sum_{j=i+1}^{t-1} j^{2H-2} + T_0\sum_{j=T_0}^{t-1} j^{2H-2}\right)\\
&\le -2J_1\left(\sum_{i=T_0}^{t-1}\int_i^{t-1} u^{2H-2}du + T_0\int_{T_0-1}^{t-1} u^{2H-2}du\right)\\
&= -\frac{2J_1}{2H-1}\left(\sum_{i=T_0}^{t-1}\left((t-1)^{2H-1} - i^{2H-1}\right) + T_0\left((t-1)^{2H-1} - (T_0-1)^{2H-1}\right)\right)\\
&= -\frac{2J_1}{2H-1}\left(t(t-1)^{2H-1} - \sum_{i=T_0}^{t-1} i^{2H-1} - T_0(T_0-1)^{2H-1}\right)\\
&\le \frac{2J_1}{2H-1}\sum_{i=T_0}^{t-1} i^{2H-1} + \frac{2J_1}{2H-1}T_0(T_0-1)^{2H-1}\\
&\le \frac{2J_1}{2H(2H-1)}((t-1)^{2H} - (T_0-1)^{2H}) + \frac{2J_1}{2H-1}T_0(T_0-1)^{2H-1}\\
&\le \frac{2J_1}{2H(2H-1)}t^{2H} + \frac{2J_1}{2H-1}T_0(T_0-1)^{2H-1}.
\end{aligned}$$

To proceed observe that, using the asymptotics in Assumption 4.1, for $t > 2$ we have

$$\begin{aligned}
-2t\sum_{j=t}^{\infty} r(j) \le -2J_1 t\sum_{j=t}^{\infty} j^{2H-2} &\le -2J_1 t\int_{t-1}^{\infty} u^{2H-2}du\\
&= \frac{2J_1 t}{2H-1}(t-1)^{2H-1} \le \frac{2J_1 t}{2H-1}(t-t/2)^{2H-1}\\
&= \frac{2^{2-2H}J_1}{2H-1}t^{2H}.
\end{aligned}$$

These results yield for $t > \max(2, T_0)$, using again (65), that

$$\operatorname{var}(S_t) \le C_3 + \left(\frac{2J_1}{2H(2H-1)} + \frac{2^{2-2H}J_1}{2H-1}\right)t^{2H} + \frac{2J_1}{2H-1}T_0(T_0-1)^{2H-1} \tag{66}$$

The threshold T_2 and the constant B_2 could again be explicitly given. The proof is complete. □

We proceed with the lemma controlling the covariance $\operatorname{cov}(S_s - S_t, S_t)$.

Lemma A.3. *There exist $T_3 \in \mathbb{N}$ and $D_1, D_2 > 0$ such that*

$$\operatorname{cov}(S_s - S_t, S_t) \le D_1 \quad \textit{for all} \quad s > t > T_3.$$

For a fixed $v > 1$, define

$$U(v) := J_2 (2H)^{-1} (2H-1)^{-1} \left(1 - \left(v^{2H} - (v-1)^{2H}\right)\right).$$

Then

$$\mathrm{cov}(S_s - S_t, S_t) \le D_2 - U(v) t^{2H} < 0 \textit{ holds for all } s > t > T_3 \textit{ satisfying } \frac{s}{t} > v.$$

There exists $K > 1$ and $T_4 \in \mathbb{N}$ such that

$$\mathrm{cov}(S_s - S_t, S_t) \le 0 \textit{ for all } s > t > T_4 \textit{ satisfying } s - t > K.$$

Proof. Let us set

$$C_4 = \sum_{j=-T_0+1}^{0} \sum_{i=1}^{1+T_0} r(i-j), \quad C_5 = J_2 \sum_{j=-T_0+1}^{0} \sum_{i=1}^{1+T_0} (i-j)^{2H-2},$$

and define $C_6 = C_4 - C_5$. Note that, for each $t \in \mathbb{N}$, $C_4 = \sum_{j=t-T_0+1}^{t} \sum_{i=t+1}^{t+1+T_0} r(i-j)$, and $C_5 = J_2 \sum_{j=t-T_0+1}^{t} \sum_{i=t+1}^{t+1+T_0} (i-j)^{2H-2}$. For $t > T_0$, we have

$$\begin{aligned}
\mathrm{cov}(S_s - S_t, S_t) &= \sum_{j=1}^{t} \sum_{i=t+1}^{s} r(i-j) \\
&\le C_6 + J_2 \sum_{j=1}^{t} \sum_{i=t+1}^{s} (i-j)^{2H-2} \le C_6 + J_2 \sum_{j=1}^{t} \int_{t+1-j}^{s+1-j} u^{2H-2} du \\
&\le C_6 + \frac{J_2}{2H-1} \sum_{j=1}^{t} \left((s+1-j)^{2H-1} - (t+1-j)^{2H-1}\right) \\
&= C_6 + \frac{J_2}{2H-1} \sum_{j=1}^{t} (s+1-j)^{2H-1} - \frac{J_2}{2H-1} \sum_{j=1}^{t} (t+1-j)^{2H-1} \qquad (67) \\
&\le C_6 + \frac{J_2}{2H-1} \int_{s-t}^{s} u^{2H-1} du - \frac{J_2}{2H-1} \int_{1}^{t+1} u^{2H-1} du \\
&= C_6 + \frac{J_2}{2H(2H-1)} \left(s^{2H} - (s-t)^{2H}\right) - \frac{J_2}{2H(2H-1)} \left((t+1)^{2H} - 1\right) \\
&= C_6 + \frac{J_2}{2H(2H-1)} \left(s^{2H} - (s-t)^{2H} - \left((t+1)^{2H} - 1\right)\right). \\
&=: C_6 + C_7 \left(s^{2H} - (s-t)^{2H} - \left((t+1)^{2H} - 1\right)\right).
\end{aligned}$$

Since $s \geq t+1$ the expression $C_7\left(s^{2H}-(s-t)^{2H}-\left((t+1)^{2H}-1\right)\right)$ is non-positive, which yields

$$\mathrm{cov}(S_s-S_t,S_t) \leq C_6,$$

proving the first statement of the lemma. Now, for all $v>1$ the property $\frac{s}{t}>v$ - together with the previous constraint of $t>T_0$ - further implies

$$\begin{aligned}\mathrm{cov}(S_s-S_t,S_t) &\leq C_6+C_7\left(s^{2H}-(s-t)^{2H}-\left((t+1)^{2H}-1\right)\right)\\ &\leq C_6+C_7\left((v^{2H}-(v-1)^{2H}-1)t^{2H}+1\right)\\ &= C_6+C_7+C_7(v^{2H}-(v-1)^{2H}-1)t^{2H}.\end{aligned} \tag{68}$$

Obviously, for large enough t the bound becomes strictly negative, proving the second statement. Now, assuming $s-t \geq K > 1$ beside $t>T_0$ we have

$$\begin{aligned}\mathrm{cov}(S_s-S_t,S_t) &\leq C_6+C_7\left((t+K)^{2H}-K^{2H}-\left((t+1)^{2H}-1\right)\right)\\ &= C_6-C_7\left(K^{2H}-1\right)+C_7\left((t+K)^{2H}-(t+1)^{2H}\right)\\ &\leq C_6-C_7\left(K^{2H}-1\right)+C_7 2HKt^{2H-1}.\end{aligned} \tag{69}$$

This shows that K can be chosen so large that $C_6-C_7\left(K^{2H}-1\right)<0$ and then, since $2H-1<0$, a threshold T_4 - depending on K - for the variable t can be specified so that

$$C_6-C_7\left(K^{2H}-1\right)+C_7 2HKt^{2H-1} \leq 0$$

whenever t exceeds the threshold, proving the third statement, completing the proof of the lemma. □

A.2 Key estimates

Define

$$\rho(s,t) := \frac{\mathrm{cov}(S_s,S_t)}{\mathrm{var}(S_t)} = \frac{\mathrm{cov}(S_s-S_t,S_t)}{\mathrm{var}(S_t)}+1,\ s\in\mathbb{N},\ t\in\mathbb{N}\setminus\{0\}.$$

Lemma A.4. *There exist $\bar{T}\in\mathbb{N}$ and constants $R>0$, $K>1$, $\eta\in(1/2,1)$ and $\varepsilon>0$ such that*

1. $\rho(s,t) < 1+R$, *for all* $t < s$;

2. $\rho(s,t) \le 1$, *whenever* $\bar{T} < t < s$ *and* $s-t > K$;

3. *For all* $T \in \mathbb{N}$, $\rho(s,t) \le 1-\varepsilon$, *whenever* $\bar{T} < t < \frac{T}{2} < \eta T < s$. *Furthermore, one can also guarantee* $T/2+K < \eta T$ *in this case.*

Proof of Lemma A.4. Let B_2, $U(\cdot)$, T_1, T_2, T_3, T_4, D_1, D_2 and K be as in Lemma A.2 and Lemma A.3. Choose $T' > \max\{T_1, T_2, T_3\}$ so large that $\frac{D_2}{B_2}(T')^{-2H} - \frac{U(4/3)}{B_2} < 0$ and set $\eta := 2/3$. Lemma A.2 and Lemma A.3 now show that whenever $T' < t < T/2$ and $s \in (\eta T, T)$, we have

$$\frac{\operatorname{cov}(S_s - S_t, S_t)}{\operatorname{var}(S_t)} \le \frac{D_2}{B_2} t^{-2H} - \frac{U(4/3)}{B_2} \le \frac{D_2}{B_2}(T')^{-2H} - \frac{U(4/3)}{B_2}, \tag{70}$$

which yields $\rho(s,t) \le 1-\varepsilon$, where $\varepsilon = -\frac{D_2}{B_2}(T')^{-2H} + \frac{U(4/3)}{B_2}$. Lemma A.3 shows that $t > T_4$, ensures that $s-t > K$ implies $\rho(s,t) \le 1$. Finally, set $\bar{T} = \max\{T', T_4, 3K\}$. It is clear – using (67) in the proof of Lemma A.3 – that for fixed t, the function $(s,t) \mapsto \rho(s,t)$ is bounded. So let $D_1' = \max_{0<t<\bar{T}} \sup_{s\ge 0} \rho(s,t)$ and define $R = \max\{D_1, D_1'\} - 1$ It remains to guarantee $T/2+K < \eta T$ but this follows since $\bar{T} < t < T/2$ implies $T > 6K$. The quantities η, $\bar{T}$, R, K and ε constructed above fulfill all the requirements. □

Proof of Theorem 4.2. First we determine the maximal expected growth rate of portfolios. Let us define

$$Q(T) = \sum_{t=0}^{T} E|S_t|^{\frac{\alpha}{\alpha-1}}.$$

Let $G(x) := \lambda |x|^{\alpha}$, $x \in \mathbb{R}$ and denote its Fenchel-Legendre conjugate

$$G^*(y) := \sup_{x\in\mathbb{R}} (xy - G(x)) = \frac{\alpha-1}{\alpha} \alpha^{\frac{1}{1-\alpha}} \lambda^{\frac{1}{1-\alpha}} |y|^{\frac{\alpha}{\alpha-1}}, \qquad y \in \mathbb{R}. \tag{71}$$

By definition of G^*, for all $\phi \in \mathcal{G}(T)$,

$$V_T(\phi) \le \sum_{t=0}^{T} G^*(-S_t) = C \sum_{t=0}^{T} |S_t|^{\alpha/(\alpha-1)}$$

for some $C > 0$ and hence

$$EV_T(\phi) \le CQ(T) < \infty. \tag{72}$$

Note that this bound is independent of ϕ. Using Lemma A.2 it holds that

$$\begin{aligned}
Q(T) &= C_{\frac{\alpha}{\alpha-1}} \sum_{t=0}^{T} \operatorname{var}(S_t)^{\frac{\alpha}{2(\alpha-1)}} \\
&\leq C_{\frac{\alpha}{\alpha-1}} \sum_{t=0}^{T_2-1} \operatorname{var}(S_t)^{\frac{\alpha}{2(\alpha-1)}} + C_{\frac{\alpha}{\alpha-1}} B_2 \sum_{t=T_2}^{T} t^{\frac{H\alpha}{(\alpha-1)}} \\
&\leq C_{\frac{\alpha}{\alpha-1},T_2} + C_{\alpha,H,B_2} T^{H(1+\frac{1}{\alpha-1})+1}.
\end{aligned} \tag{73}$$

Thus the maximal expected profit grows as $T^{H(1+\frac{1}{\alpha-1})+1}$ with the power of the horizon, this proves (28). Now, untill further notice, let T be a multiple of 6. With the strategy defined in (29), the dynamics takes the form

$$\begin{aligned}
V_T(\phi) = &\sum_{t=0}^{T/2} |S_t|^{\frac{\alpha}{\alpha-1}} \\
&- \sum_{t=0}^{T/2} \lambda |S_t|^{\frac{\alpha}{\alpha-1}} \\
&- \frac{1}{T/2} \sum_{s=T/2+1}^{T} S_s \sum_{t=0}^{T/2} \operatorname{sgn}(S_t) |S_t|^{\frac{1}{\alpha-1}} \\
&- \frac{1}{T/2} \sum_{s=T/2+1}^{T} \lambda \left| \sum_{t=0}^{T/2} \operatorname{sgn}(S_t) |S_t|^{\frac{1}{\alpha-1}} \right|^{\alpha}.
\end{aligned}$$

In the above expression let us denote the four terms by $I_1(T)$, $I_2(T)$, $I_3(T)$, $I_4(T)$, respectively, so that

$$V_T(\phi) = I_1(T) - I_2(T) - I_3(T) - I_4(T).$$

The upper bound constructed in (73) for $Q(T)$ right away gives us an upper estimate for $EI_1(T)$ as $EI_1(T) = Q(T/2)$. Using Lemma A.1, we likewise present a lower estimate as

$$\begin{aligned}
Q(T/2) = E[I_1(T)] &= C_{\frac{\alpha}{\alpha-1}} \sum_{t=0}^{T/2} \operatorname{var}(S_t)^{\frac{\alpha}{2(\alpha-1)}} \\
&\geq C_{\frac{\alpha}{\alpha-1}} \sum_{t=0}^{T_1-1} \operatorname{var}(S_t)^{\frac{\alpha}{2(\alpha-1)}} + C_{\frac{\alpha}{\alpha-1}} B_1 \sum_{t=T_1}^{T/2} t^{\frac{H\alpha}{\alpha-1}} \\
&\geq C_{\frac{\alpha}{\alpha-1},H,B_1,T_1} + C_{\frac{\alpha}{\alpha-1},H,B_1} T^{H(1+\frac{1}{\alpha-1})+1},
\end{aligned} \tag{74}$$

To treat the terms $I_2(T)$ and $I_4(T)$, note that with $\alpha > 1$ the function $x \mapsto |x|^\alpha$ is convex, thus applying Jensen's inequality

$$|EI_4(T)| \le E|I_2(T)| = \lambda E\left[\sum_{t=0}^{T/2} |S_t|^{\frac{\alpha}{\alpha-1}}\right] = \lambda \sum_{t=0}^{T/2} E|S_t|^{\frac{\alpha}{\alpha-1}} = \lambda E[I_1(T)] = \lambda Q(T/2). \tag{75}$$

Controlling term $I_3(T)$ is done via exploiting a specific property of Gaussian processes, namely that S_s for $s > t$ can be decomposed as $S_s = \rho(s,t)S_t + W_{s,t}$, where $W_{s,t}$ is independent of S_t and zero mean. With this, observe that

$$\begin{aligned} EI_3(T) &= \frac{1}{T/2} \sum_{s=T/2+1}^{T} \sum_{t=0}^{T/2} E[\rho(s,t) S_t \mathrm{sgn}(S_t)|S_t|^{\frac{1}{\alpha-1}}] \\ &= \frac{1}{T/2} \sum_{s=T/2+1}^{T} \sum_{t=0}^{T/2} E[\rho(s,t)|S_t|^{\frac{\alpha}{\alpha-1}}]. \end{aligned} \tag{76}$$

Let the constants $\bar{T}$, R, K, $\eta = 2/3$ and ε be as in Lemma A.4, and decompose the double sum in (76) as

$$\sum_{s=T/2+1}^{T} \sum_{t=0}^{T/2} = \sum_{s=T/2+1}^{T} \sum_{t=0}^{\bar{T}-1} + \sum_{s=T/2+1}^{T/2+K} \sum_{t=\bar{T}}^{T/2} + \sum_{s=T/2+K+1}^{\eta T} \sum_{t=\bar{T}}^{T/2} + \sum_{s=\eta T+1}^{T} \sum_{t=\bar{T}}^{T/2}$$

Note that applying the upper bound developed in Lemma A.4 to the double sum in (76), the summand no longer depends on the running variable of the outer sum. Denoting $C_{\bar{T}} := \sum_{t=0}^{\bar{T}-1} E|S_t|^{\frac{\alpha}{\alpha-1}}$, this implies that

$$\begin{aligned} EI_3(T) &\le \left(\sum_{t=0}^{T/2} + R\sum_{t=0}^{\bar{T}-1} + \frac{2RK}{T}\sum_{t=\bar{T}}^{T/2} - 2\varepsilon\left(1-\frac{2}{3}\right)\sum_{t=\bar{T}}^{T/2}\right) E|S_t|^{\frac{\alpha}{\alpha-1}} \\ &= E[I_1(T)] + \left(R\sum_{t=0}^{\bar{T}-1} + \frac{2RK}{T}\sum_{t=\bar{T}}^{T/2} - \frac{2\varepsilon}{3}\sum_{t=\bar{T}}^{T/2}\right) E|S_t|^{\frac{\alpha}{\alpha-1}} \\ &= E[I_1(T)] + \left(\left(R + \frac{2\varepsilon}{3} - \frac{2RK}{T}\right)\sum_{t=0}^{\bar{T}-1} + \frac{2RK}{T}\sum_{t=0}^{T/2} - \frac{2\varepsilon}{3}\sum_{t=0}^{T/2}\right) E|S_t|^{\frac{\alpha}{\alpha-1}} \\ &= \left(1-\frac{2\varepsilon}{3}\right)E[I_1(T)] + \left(R + \frac{2\varepsilon}{3} - \frac{2RK}{T}\right)C_{\bar{T}} + \frac{2RK}{T}E[I_1(T)], \end{aligned}$$

so we have

$$E[I_1(T)] - E[I_3(T)] \ge \frac{2\varepsilon}{3}E[I_1(T)] - \left(R + \frac{2\varepsilon}{3} - \frac{2RK}{T}\right)C_{\bar{T}} - \frac{2RK}{T}E[I_1(T)]$$

The above, using (75), boils down to

$$V_T(\phi) \geq \frac{2\varepsilon}{3} Q(T/2) - \left(R + \frac{2\varepsilon}{3} - \frac{2RK}{T}\right) C_{\bar{T}} - \frac{2RK}{T} Q(T/2) - 2\lambda Q(T/2)$$

Using (73) and (74), with $\lambda < \varepsilon/3$, dividing through with $T^{H(1+\frac{1}{\alpha-1})+1}$ proves the statement in (30) with the constraint that the limiting operation runs through multiples of 6. Now let T be general. The same calculations can be done as above, with minor changes in the formulas corresponding to the upper and lower limits in summations according to taking the appropriate floor values. That is, in the last inequality $Q(3\lfloor T/6 \rfloor)$ appears - instead of $Q(T/2)$ - and it grows in the order of $(6\lfloor T/6 \rfloor)^{H(1+\frac{1}{\alpha-1})+1}$, and using that $6\lfloor T/6 \rfloor / T$ tends to 1 when T is large, the proof of Theorem 4.2 is complete. □

B Proofs of Section 5

B.1 Preliminary calculations and estimates

Proposition B.1. *The process*

$$\xi_t^* = \exp\left\{ -\int_0^t \left(\mu - \alpha \operatorname{sgn}(X_u)|X_u|^\beta\right) dB_u - \frac{1}{2}\int_0^t \left(\mu - \alpha \operatorname{sgn}(X_u)|X_u|^\beta\right)^2 du \right\}, \; t \in \mathbb{R}_+ \tag{77}$$

is a P-martingale and $dQ_T/dP := \xi_T^*$ defines a probability $Q_T \sim P$ on $\mathscr{F}_T$ such that S_t, $t \in [0,T]$ is a Q_T-martingale (actually, a Q_T-Brownian motion) and Q_T is the only such equivalent probability.

Proof. By Girsanov's theorem it suffices to establish that the process ξ^* is a true martingale. Apply Theorem 2.1 of Mijatović and Urusov (2012) with the choice $J = \mathbb{R}$, $Y_t = X_t$, $b(x) := \mu - \alpha \operatorname{sgn}(x)|x|^\beta$, $x \in \mathbb{R}$. According to the notation of that paper, $\tilde{\rho}(x) = 1$ for all $x \in \mathbb{R}$, as easily checked. Then the quantity $\tilde{v}(x)$ defined there equals $x^2/2$ for all $x \in \mathbb{R}$ and this satisfies $\tilde{v}(\pm\infty) = \infty$ hence the claim follows from Theorem 2.1 of Mijatović and Urusov (2012). An alternative proof could be obtained from the abstract results in Cheridito et al. (2005) for general jump-diffusions. □

Recall that Q_T (defined in Proposition B.1 above) is the unique martingale measure for the process X. Hence, by the duality theory of optimal investment (see Delbaen et al. (2002); Kabanov and Stricker (2002)), it follows that

$$u_T = -e^{-J}$$

where

$$J := E\left[\frac{dQ_T}{dP}\ln\left(\frac{dQ_T}{dP}\right)\right] = E[\xi_T^* \ln(\xi_T^*)] = E_{Q_T}[\ln(\xi_T^*)] \tag{78}$$

(provided that the latter quantity exists and is finite). Here E_{Q_T} denotes expectation under the probability Q_T.

From (39) and reordering (77), it follows that

$$\ln(\xi_T{}^*) = \frac{1}{2}\int_0^T \left(\mu - \alpha\,\mathrm{sgn}(X_u)|X_u|^\beta\right)^2 du - \int_0^T \left(\mu - \alpha\,\mathrm{sgn}(X_u)|X_u|^\beta\right) dS_u.$$

Because, under the measure Q_T, the process S is a standard Brownian motion on $[0,T]$, the second term in the above expression is a Q_T-martingale and

$$\begin{aligned}
J &= E_{Q_T}\left[\frac{\mu^2}{2}T + \frac{\alpha^2}{2}\int_0^T |X_u|^{2\beta}du - \alpha\mu\int_0^T \mathrm{sgn}(X_u)|X_u|^\beta du\right]\\
&\le E_{Q_T}\left[\frac{\mu^2}{2}T + \frac{\alpha^2}{2}\int_0^T |X_u|^{2\beta}du + \alpha|\mu|\int_0^T |X_u|^\beta du\right]\\
&= \frac{\mu^2}{2}T + \frac{\alpha^2}{2}\int_0^T E_{Q_T}|X_u|^{2\beta}du + \alpha|\mu|\int_0^T E_{Q_T}|X_u|^\beta du.
\end{aligned}$$

Note that under the measure Q_T, the process $X_t = S_t - \mu t$ is a standard Brownian motion with a constant drift on $[0,T]$. Thus, in view of the convexity of the mappings $x \to |x|^\beta$ and $x \to |x|^{2\beta}$,

$$\begin{aligned}
J &\le \frac{\mu^2}{2}T + \frac{\alpha^2}{2}\int_0^T E_{Q_T}|S_u - \mu u|^{2\beta}du + \alpha|\mu|\int_0^T E_{Q_T}|S_u - \mu u|^\beta du\\
&\le \frac{\mu^2}{2}T + \frac{\alpha^2}{2}\int_0^T \left(2^{2\beta-1}u^\beta M_{2\beta} + 2^{2\beta-1}|\mu|^{2\beta}u^{2\beta}\right)du + \alpha|\mu|\int_0^T \left(2^{\beta-1}u^{\beta/2}M_\beta + 2^{\beta-1}|\mu|^\beta u^\beta\right)du\\
&= \frac{\mu^2}{2}T + \alpha^2 2^{2\beta-2}\frac{M_{2\beta}T^{\beta+1}}{\beta+1} + \alpha^2 2^{2\beta-2}|\mu|^{2\beta}\frac{T^{2\beta+1}}{2\beta+1}\\
&+ \alpha|\mu|2^{\beta-1}\frac{M_\beta T^{\beta/2+1}}{\beta/2+1} + \alpha|\mu|^{\beta+1}2^{\beta-1}\frac{T^{\beta+1}}{\beta+1},
\end{aligned}$$

where M_κ is the κth moment of a standard Gaussian variable. This shows that $c_T \leq C_{\beta,\mu,\alpha} T^{2\beta+1}$ where $C_{\beta,\mu,\alpha}$ can be explicitly given and the first statement of Theorem 5.2 is proved.

Now we turn to some estimates familiar in the theory of Markov processes. We have been inspired by Kontoyiannis et al. (2005) in particular. Let $C^2(\mathbb{R})$ denote the family of twice continuously differentiable functions on $\mathbb{R}$. Define the operator $\mathscr{A}$ by

$$\mathscr{A} f := -\alpha \operatorname{sgn}(x)|x|^\beta \partial_x f + \frac{1}{2}\partial_{xx} f,\ f \in C^2(\mathbb{R}), \tag{79}$$

which coincides with the infinitesimal generator associated to the process X on its domain of definition. Define also the operator $\mathscr{H}$ (the "nonlinear generator", see Kontoyiannis et al. (2005)) as

$$\mathscr{H} f := e^{-f} \mathscr{A} e^f,\ f \in C^2(\mathbb{R}).$$

Now we introduce a condition related to $\mathscr{H}$.

Condition B.2. *There is a compact $C \subset \mathbb{R}$, there are $\delta, b > 0$ and functions $V, W : \mathbb{R} \to \mathbb{R}_+$ with W measurable and $V \in C^2(\mathbb{R})$ such that, for all $x \in \mathbb{R}$,*

$$\mathscr{H} V(x) \leq -\delta W(x) + b\mathbb{1}_C(x). \tag{80}$$

For a given $\delta, b > 0$, define the process M_t by

$$M_t := \exp\left\{V(X_t) + \int_0^t \left(\delta W(X_u) - b\mathbb{1}_{\{X_u \in C\}}\right) du\right\},\ t \in \mathbb{R}_+. \tag{81}$$

Lemma B.3. *If Condition B.2 holds then the process M is a supermartingale.*

Proof. Setting $Y_t = \exp\{V(X_t)\}$ and $Z_t = \exp\left\{\int_0^t \left(\delta W(X_u) - b\mathbb{1}_{\{X_u \in C\}}\right) du\right\}$, it follows that $M_t = Y_t Z_t$. Now Ito's formula yields

$$\begin{aligned} dY_t = de^{V(X_t)} &= \mathscr{A} e^V(X_t) dt + \partial_x e^V(X_t) dB_t \\ &= \mathscr{A} e^V(X_t) dt + (e^V \partial_x V)(X_t) dB_t \end{aligned}$$

and

$$dZ_t = Z_t \left(\delta W(X_t) - b\mathbb{1}_{\{X_t \in C\}}\right) dt.$$

By the product rule of Ito calculus, using that $[Y,Z]_t \equiv 0$,

$$\begin{aligned} dM_t &= Y_t dZ_t + Z_t dY_t \\ &= Y_t Z_t \left(\delta W(X_t) - b\mathbb{1}_{\{X_t \in C\}}\right) dt + Z_t e^{V(X_t)} \mathscr{H} V(X_t) dt + Z_t (e^V \partial_x V)(X_t) dB_t, \\ &= Z_t e^{V(X_t)} \left(\delta W(X_t) - b\mathbb{1}_{\{X_t \in C\}} + \mathscr{H} V(X_t)\right) dt + Z_t (e^V \partial_x V)(X_t) dB_t. \end{aligned}$$

Here the last term is a local martingale, the first term is non-increasing by Condition B.2, hence M is a local supermartingale. As M is positive, Fatou's lemma guarantees that it is, in fact, a true supermartingale. □

Corollary B.4. *With Condition B.2 in force for $T > 0$ it follows that*

$$E\left[\exp\left\{\int_0^T \delta W(X_u) du\right\}\right] \le e^{V(0)+bT}.$$

Proof. By the supermartingale property of M, $E[M_T] \le M_0 = 1$. Since $b\mathbb{1}_C \le b$, the statement follows. □

Define the functions

$$\bar{V}(x) := \frac{\alpha}{1+\beta}|x|^{1+\beta} \text{ and } \bar{W}(x) := \alpha^2 |x|^{2\beta},\ x \in \mathbb{R}.$$

Proposition B.5. *For each $0 < \bar{\delta} < 1/2$, there is an appropriate constant $\bar{b} > 0$ and a compact set $\bar{C}$ such that Condition B.2 is fulfilled with $V = \bar{V}$, $W = \bar{W}$, $b = \bar{b}$, $\delta = \bar{\delta}$ and $C = \bar{C}$.*

Proof. The claim would follow from Proposition 1.3 of Kontoyiannis and Meyn (2005) but we provide a direct proof. Note that $\partial_x e^V(x) = e^{V(x)} \partial_x V(x)$, $\partial_{xx} e^V(x) = e^{V(x)}(\partial_x V(x))^2 + e^{V(x)} \partial_{xx} V(x)$, $\partial_x V(x) = \alpha \operatorname{sgn}(x)|x|^\beta$, and $\partial_{xx} V(x) = \alpha\beta|x|^{\beta-1}$. Thus, (79) yields

$$e^{-V} \mathscr{A} e^V(x) = -\frac{\alpha^2}{2}|x|^{2\beta} + \frac{\alpha\beta}{2}|x|^{\beta-1}. \tag{82}$$

The criterion in (80) then becomes equivalent to

$$\frac{\alpha\beta}{2}|x|^{\beta-1} \le \left(\frac{1}{2} - \bar{\delta}\right) \alpha^2 |x|^{2\beta} + b\mathbb{1}_C(x)$$

which clearly shows that the set C and the constant b can be chosen in such a way that Condition B.2 is fulfilled, provided that $\bar{\delta} < 1/2$. □

Lemma B.6. *There exist constants $\delta_0, c_0, C_0 > 0$ such that*

$$E\left[\exp\left\{\delta_0 \int_0^T |X_t|^{2\beta} dt\right\}\right] \le c_0 e^{C_0 T}.$$

Proof. Corollary B.4, Proposition B.5 and the definitions of $\bar{V}$, $\bar{W}$ immediately yield the upper bound with $\delta_0 := \alpha^2 \bar{\delta}$. In fact, $c_0 = 1$ can be chosen as $\bar{V}(0) = 0$. □

B.2 Asymptotic optimality in the case $\mu \neq 0$

Consider the process $U_t := (T-t)^\beta |S_t|^{\beta+1}$, $t \in [0,T]$. As $U_0 = 0$, Ito's lemma implies that

$$0 = U_T = \int_0^T (\beta+1)(T-t)^\beta \operatorname{sgn}(S_t)|S_t|^\beta dS_t + \int_0^T \frac{\beta(\beta+1)}{2}(T-t)^\beta |S_t|^{\beta-1} dt - \int_0^T \beta(T-t)^{\beta-1}|S_t|^{\beta+1} dt,$$

which is equivalent to

$$\int_0^T (\beta+1)(T-t)^\beta \operatorname{sgn}(S_t)|S_t|^\beta dS_t = -\int_0^T \frac{\beta(\beta+1)}{2}(T-t)^\beta |S_t|^{\beta-1} dt + \int_0^T \beta(T-t)^{\beta-1}|S_t|^{\beta+1} dt.$$

Note that the above expression is the value of the investor's portfolio utilizing the strategy $H_t(\beta,T) = (\beta+1)(T-t)^\beta \operatorname{sgn}(S_t)|S_t|^\beta$, $t \in [0,T]$. Since S is a Q_T-Brownian motion, clearly $H(\beta,T) \in \mathscr{X}_T$.

First we turn to the case $\beta > 1$. Let us denote $I_1(T) := \int_0^T \beta(T-t)^{\beta-1}|S_t|^{\beta+1} dt$, and $I_2(T) := \int_0^T \frac{\beta(\beta+1)}{2}(T-t)^\beta |S_t|^{\beta-1} dt$. This way we have

$$E\left[-e^{-(H\cdot S)_T}\right] = E\left[-e^{-I_1(T)+I_2(T)}\right]. \tag{83}$$

Now let us define an event $A(T)$ as

$$\Omega \supset A(T) := \left\{\left|\int_0^{T/2} X_t dt\right| \le \frac{\mu T^2}{16}\right\}$$

and denote its complement in the set theoretic sense as $\bar{A}(T)$. Now we give a deterministic bound for $I_1(T)$ on the event $A(T)$. First note that

$$I_1(T) = \int_0^T \beta(T-t)^{\beta-1}|S_t|^{\beta+1} dt \ge \beta\left(\frac{T}{2}\right)^{\beta-1} \int_0^{T/2} |S_t|^{\beta+1} dt, \tag{84}$$

and again by Jensen's inequality,

$$\left(\frac{1}{T/2}\int_0^{T/2}|S_t|^{\beta+1}dt\right)^{1/(\beta+1)} \geq \frac{1}{T/2}\left|\int_0^{T/2}S_t dt\right| = \frac{1}{T/2}\left|\frac{\mu T^2}{8}+\int_0^{T/2}X_t dt\right|. \tag{85}$$

On the event $A(T)$ these yield

$$\int_0^{T/2}|S_t|^{\beta+1}dt \geq 2^{-3(\beta+1)-1}\mu^{\beta+1}T^{\beta+2}. \tag{86}$$

and in return using (85) and (86) we have on the event $A(T)$ that

$$I_1(T) \geq \beta 2^{-3(\beta+1)-\beta}\mu^{\beta+1}T^{2\beta+1} =: C_{\beta,\mu}T^{2\beta+1}. \tag{87}$$

Now the expectation in (83) will be estimated by splitting it along the event $A(T)$. First, by (87) we have

$$E\left[-e^{-I_1(T)+I_2(T)}\mathbb{1}_A\right] \geq -e^{-C_{\beta,\mu}T^{2\beta+1}}E\left[e^{I_2(T)}\right] \geq -e^{-C_{\beta,\mu}T^{2\beta+1}}\left(E\left[e^{2I_2(T)}\right]\right)^{1/2}. \tag{88}$$

On the other hand, by the Cauchy-Schwartz inequality and recalling that $-e^{-x} \geq -1$ for $x \geq 0$,

$$E[-e^{-I_1(T)}e^{I_2(T)}\mathbb{1}_{\bar{A}}] \geq -\left(E\left[e^{2I_2(T)}\right]\right)^{1/2}\left(P\left(\bar{A}\right)\right)^{1/2}. \tag{89}$$

Now, to estimate the quantities $P(\bar{A}(T))$ and $E[e^{2I_2(T)}]$, consider a corollary to Lemma B.6 that handles $P(\bar{A}(T))$ and a Lemma bounding $E[e^{2I_2(T)}]$ which is also a consequence of Lemma B.6.

Corollary B.7. *There exist positive constants c_1, C_1 such that*

$$P(\bar{A}(T)) \leq c_1 e^{-C_1T^{2\beta+1}}.$$

Lemma B.8. *There exist positive constants c_2, C_2 and $q>0$ such that*

$$E[e^{2I_2(T)}] \leq c_2 e^{C_2T^{2\beta+1-q}}.$$

Corollary B.7 and Lemma B.8 will be proved shortly.

Proceeding with these results and using (83), (88) and (89), Corollary B.7 and Lemma B.8 it follows that

$$\begin{aligned}
E\left[-e^{-(H\cdot S)_T}\right] &\geq -e^{-C_{\beta,\mu}T^{2\beta+1}}\left(E\left[e^{2I_2(T)}\right]\right)^{1/2} - \left(E\left[e^{2I_2(T)}\right]\right)^{1/2}\left(P\left(\bar{A}\right)\right)^{1/2} \\
&\geq -e^{-C_{\beta,\mu}T^{2\beta+1}}\left(c_2e^{C_2T^{2\beta+1-q}}\right)^{1/2} - \left(c_2e^{C_2T^{2\beta+1-q}}\right)^{1/2}\left(c_1e^{-C_1T^{2\beta+1}}\right)^{1/2} \\
&= -c_2^{1/2}e^{-C_{\beta,\mu}T^{2\beta+1}+\frac{C_2}{2}T^{2\beta+1-q}} - c_1^{1/2}c_2^{1/2}e^{\frac{C_2}{2}T^{2\beta+1-q}-\frac{C_1}{2}T^{2\beta+1}}.
\end{aligned}$$

This completes the proof of Theorem 5.2 when $\beta > 1$. The exact same calculations can be done when $\beta = 1$: Corollary B.7 holds with $\beta = 1$ as it is written. The term I_2 being deterministic and of order T^2 shows that the conclusion of Lemma B.8 also remains valid. This completes the proof of Theorem 5.2.

As promised earlier, we finish by presenting the proofs of Corollary B.7 and Lemma B.8.

Proof of Corollary B.7. By Jensen's inequality we have

$$\left|\frac{1}{T}\int_0^T X_t dt\right|^{2\beta} \le \frac{1}{T}\int_0^T |X_t|^{2\beta} dt. \tag{90}$$

Using Lemma B.6 and Markov's inequality leads to

$$\begin{aligned}
P\left(\left|\int_0^T X_t dt\right| \ge \frac{\mu T^2}{16}\right) &\le P\left(\int_0^T |X_t|^{2\beta} dt \ge \mu^{2\beta} 2^{-8\beta} T^{2\beta+1}\right) && (91)\\
&= P\left(\exp\left\{\delta_0 \int_0^T |X_t|^{2\beta} dt\right\} \ge e^{\delta_0 \mu^{2\beta} 2^{-8\beta} T^{2\beta+1}}\right) && (92)\\
&\le c_0 e^{-\delta_0 \mu^{2\beta} 2^{-8\beta} T^{2\beta+1} + C_0 T}. && (93)
\end{aligned}$$

□

Proof of Lemma B.8. First note that there exist positive constants c_β and $c_{\beta,\mu}$ such that

$$\begin{aligned}
E[e^{2I_2(T)}] &\le E\left[\exp\left\{\beta(\beta+1)T^\beta \int_0^T |S_t|^{\beta-1} dt\right\}\right]\\
&\le E\left[\exp\left\{\beta(\beta+1)T^\beta \int_0^T \left(c_\beta |X_t|^{\beta-1} + c_{\beta,\mu} t^{\beta-1}\right) dt\right\}\right] && (94)\\
&= e^{(\beta+1)c_{\beta,\mu} T^{2\beta}} E\left[\exp\left\{\beta(\beta+1) c_\beta T^\beta \int_0^T |X_t|^{\beta-1} dt\right\}\right].
\end{aligned}$$

By Jensen's inequality,

$$\int_0^T |X_t|^{\beta-1} dt \le T^{1-\frac{\beta-1}{2\beta}} \left(\int_0^T |X_t|^{2\beta} dt\right)^{\frac{\beta-1}{2\beta}}.$$

Denoting $\Xi_T := \int_0^T |X_t|^{2\beta} dt$ and defining $h(x) = h_{\beta,T}(x) := \exp\{\beta(\beta+1)c_\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}}\}$, $x > 0$ the following estimate holds:

$$E[e^{2I_2(T)}] \leq e^{(\beta+1)c_{\beta,\mu}T^{2\beta}} E\left[\exp\left\{\beta(\beta+1)c_\beta T^{\beta+1-\frac{\beta-1}{2\beta}} \left(\int_0^T |X_t|^{2\beta} dt\right)^{\frac{\beta-1}{2\beta}}\right\}\right] \\ = e^{(\beta+1)c_{\beta,\mu}T^{2\beta}} Eh(\Xi_T). \tag{95}$$

The estimate in Lemma B.6, along with Markov's inequality, implies that, for all $x > 0$,

$$P(\Xi_T > x) \leq c_0 \exp\{C_0 T - \delta_0 x\}, \tag{96}$$

and also observe that

$$E[h(\Xi_T)] = \int_0^\infty h'(x) P(\Xi_T > x) dx. \tag{97}$$

Since $h'(x) = \frac{(\beta+1)\beta(\beta-1)}{2\beta} c_\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}-1} \exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}}\}$, $x > 0$, (96) and (97) yield

$$E[h(\Xi_T)] \leq \int_0^\infty \frac{(\beta+1)\beta(\beta-1)c_\beta c_0}{2\beta} T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}-1} \exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x + C_0 T\} dx$$

$$= \frac{(\beta+1)\beta(\beta-1)c_\beta c_0}{2\beta} T^{\beta+1-\frac{\beta-1}{2\beta}} \int_0^\infty x^{\frac{\beta-1}{2\beta}-1} \exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x + C_0 T\} dx$$

$$\leq \frac{(\beta+1)\beta(\beta-1)c_\beta c_0}{2\beta} T^{\beta+1-\frac{\beta-1}{2\beta}} \times$$

$$\times \left(e^{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} + C_0 T} \int_0^1 x^{\frac{\beta-1}{2\beta}-1} dx + \int_1^\infty e^{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x + C_0 T} dx\right)$$

$$= \tilde{c}_1 T^{\beta+1-\frac{\beta-1}{2\beta}} e^{\frac{c_\beta(\beta+1)\beta}{2} T^{\beta+1-\frac{\beta-1}{2\beta}} + C_0 T} +$$

$$\frac{(\beta+1)\beta(\beta-1)c_\beta c_0}{2\beta} T^{\beta+1-\frac{\beta-1}{2\beta}} e^{C_0 T} \int_1^\infty e^{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x} dx, \tag{98}$$

where $\tilde{c}_1 = \frac{(\beta+1)\beta(\beta-1)c_\beta c_0}{2\beta} \int_0^1 x^{\frac{\beta-1}{2\beta}-1} dx$. Now the integral $\int_1^\infty e^{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x} dx$ will be estimated. First let us define

$$\tilde{C}(T) := \left(\frac{2(\beta+1)\beta c_\beta}{\delta_0}\right)^{\frac{2\beta}{\beta+1}} T^{\frac{2\beta^2+\beta+1}{\beta+1}} =: \tilde{c}_2 T^{\frac{2\beta^2+\beta+1}{\beta+1}}.$$

First, note that $c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} = c_\beta(\beta+1)\beta T^{\frac{2\beta^2+\beta+1}{2\beta}} x^{\frac{\beta-1}{2\beta}} \leq \frac{\delta_0}{2} x$ for $x > \tilde{C}(T)$, whence

$$\begin{aligned}
\int_{\tilde{C}(T)}^{\infty} \exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x\} dx &\leq \int_{\tilde{C}(T)}^{\infty} e^{-\frac{\delta_0}{2}x} dx \\
&= \frac{2}{\delta_0} e^{-\frac{\delta_0}{2}\tilde{C}(T)} = \frac{2}{\delta_0} \exp\left\{-\frac{\delta_0}{2}\tilde{c}_2 T^{\frac{2\beta^2+\beta+1}{\beta+1}}\right\} \leq \frac{2}{\delta_0}\exp\left\{-\frac{\delta_0}{2}\tilde{c}_2 T^2\right\}, \quad (99)
\end{aligned}$$

where the last step follows from $\frac{2\beta^2+\beta+1}{\beta+1} > 2$. Second, observing that

$$\frac{\partial}{\partial x}\left(c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x\right) = \frac{c_\beta(\beta+1)\beta(\beta-1)}{2\beta} T^{\frac{2\beta^2+\beta+1}{2\beta}} x^{-\frac{\beta+1}{2\beta}} - \delta_0,$$

the integrand $x \to e^{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x}$ reaches its maximum at

$$x = x_0 := \left(\frac{2\beta\delta_0}{c_\beta(\beta+1)\beta(\beta-1)}\right)^{-\frac{2\beta}{\beta+1}} T^{\frac{2\beta^2+\beta+1}{\beta+1}} =: \tilde{c}_3 T^{\frac{2\beta^2+\beta+1}{\beta+1}},$$

and the value of such maximum is

$$\begin{aligned}
&\exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x_0^{\frac{\beta-1}{2\beta}} - \delta_0 x_0\} \\
&= \exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} \left(\tilde{c}_3 T^{\frac{2\beta^2+\beta+1}{\beta+1}}\right)^{\frac{\beta-1}{2\beta}} - \delta_0 \tilde{c}_3 T^{\frac{2\beta^2+\beta+1}{\beta+1}}\} \qquad (100) \\
&= \exp\{c_\beta(\beta+1)\beta \tilde{c}_3^{\frac{\beta-1}{2\beta}} T^{\frac{2\beta^2+\beta+1}{\beta+1}} - \delta_0 \tilde{c}_3 T^{\frac{2\beta^2+\beta+1}{\beta+1}}\}
\end{aligned}$$

Because $\frac{2\beta^2+\beta+1}{\beta+1} < 2\beta+1$, there exists $q > 0$ such that

$$\int_1^{\tilde{C}(T)} \exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x\} dx \leq \left(\tilde{C}(T)-1\right) \exp\{c_\beta(\beta+1)\beta \tilde{c}_3^{\frac{\beta-1}{2\beta}} T^{2\beta+1-q}\}. \tag{101}$$

Using (95), (98), (99) and (101), the proof is complete. □

B.3 The case $\mu = 0$

Proceeding as in the case $\mu \neq 0$,

$$\ln(\xi_T^*) = \frac{\alpha^2}{2}\int_0^T |X_u|^{2\beta} du + \alpha \int_0^T \mathrm{sgn}(X_u)|X_u|^\beta dX_u.$$

As the process X is a standard Brownian motion on $[0,T]$ under the measure Q_T, the second term in the above expression is a Q_T-martingale so

$$J = \frac{\alpha^2}{2}\int_0^T E_Q|X_u|^{2\beta}du = \frac{\alpha^2}{2(1+\beta)}T^{1+\beta},$$

showing that $c_T = C_\beta T^{1+\beta}$ with $C_\beta = \frac{\alpha^2}{2(1+\beta)}$, which proves the first statement of Theorem 5.3. Note that for $\beta = 1$ this confirms the result obtained by the heuristic reasoning of Subsection 5.4.

Assume $\beta > 1$ until further notice. Consider the process $U_t := (T-t)^\gamma X_t^2$, $t \in [0,T]$ with some $1 < \gamma < \beta$. Since $U_0 = 0$, Ito's lemma implies that

$$0 = U_T = \int_0^T 2(T-t)^\gamma X_t dX_t + \int_0^T (T-t)^\gamma dt - \int_0^T \gamma(T-t)^{\gamma-1}X_t^2 dt,$$

which is equivalent to

$$\int_0^T -2(T-t)^\gamma X_t dX_t = \frac{1}{\gamma+1}T^{\gamma+1} - \int_0^T \gamma(T-t)^{\gamma-1}X_t^2 dt.$$

Note that the above expression is the value of the investor's portfolio utilizing the strategy $H_t(\gamma,T) = -2(T-t)^\gamma X_t$, $t \in [0,T]$ hence

$$E\left[-e^{-(H\cdot S)_T}\right] = -e^{-\frac{1}{\gamma+1}T^{\gamma+1}}E\left[\exp\left\{\int_0^T \gamma(T-t)^{\gamma-1}X_t^2 dt\right\}\right]. \tag{102}$$

Since X is a Q_T-Brownian motion, clearly $H(\gamma,T) \in \mathscr{X}_T$. Let us denote

$$G(T) := E\left[\exp\left\{\int_0^T \gamma(T-t)^{\gamma-1}X_t^2 dt\right\}\right].$$

The next lemma states that $G(T)$ is negligible in comparison with $e^{T^{\gamma+1}}$, in the following sense:

Lemma B.9. *There exist positive constants c_1, C_1 and $0 < q < 1$ such that*

$$G(T) \le c_1 e^{C_1 T^{\gamma+q}}.$$

Now, Lemma B.9 and (102) implies Theorem 5.3 in the case $\beta > 1$. For the case $\beta = 1$, following an analogous method as above consider the process $\bar{U}_t := \delta_0(T-t)X_t^2$. Similar calculations to the ones that yield (102) leads to a strategy

$$\bar{H}_t = -2\delta_0(T-t)X_t,$$

and a portfolio value

$$E\left[-e^{-(\bar{H}\cdot S)_T}\right] = -e^{-\frac{1}{2}T^2} E\left[\exp\left\{\int_0^T \delta_0 X_t^2 dt\right\}\right].$$

Lemma B.6 with $\beta = 1$ immediately yields

$$E\left[-e^{-(\bar{H}\cdot S)_T}\right] \geq -e^{-\frac{1}{2}T^2 + C_0 T},$$

proving the claim for $\beta = 1$.

Proof of Lemma B.9. First note that, by Jensen's inequality,

$$\int_0^T X_t^2 dt \leq T^{1-1/\beta}\left(\int_0^T |X_t|^{2\beta} dt\right)^{1/\beta}.$$

Denoting $\Xi_T := \int_0^T |X_t|^{2\beta} dt$ and defining $h(x) = h_{\gamma,\beta,T}(x) := \exp\{\gamma T^{\gamma-1/\beta} x^{1/\beta}\}$, $x > 0$ yields the estimate

$$G(T) \leq E\left[\exp\left\{\gamma T^{\gamma-1/\beta}\left(\int_0^T |X_t|^{2\beta} dt\right)^{1/\beta}\right\}\right] = Eh(\Xi_T). \tag{103}$$

The estimate in Lemma B.6 along with Markov's inequality, implies that, for all $x > 0$,

$$P(\Xi_T > x) \leq c_0 \exp\{C_0 T - \delta_0 x\}, \tag{104}$$

and also observe that

$$E[h(\Xi_T)] = \int_0^\infty h'(x) P(\Xi_T > x) dx. \tag{105}$$

Since $h'(x) = \frac{\gamma}{\beta} T^{\gamma-1/\beta} x^{1/\beta-1} e^{\gamma T^{\gamma-1/\beta} x^{1/\beta}}$, $x > 0$, (104) and (105) yield

$$\begin{aligned}
E[h(\Xi_T)] &\leq \int_0^\infty \frac{c_0\gamma}{\beta} T^{\gamma-1/\beta} x^{1/\beta-1} e^{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x + C_0 T} dx \\
&\leq \frac{c_0\gamma}{\beta} T^{\gamma-1/\beta} e^{\gamma T^{\gamma-1/\beta} + C_0 T} \int_0^1 x^{1/\beta-1} dx + \int_1^\infty \frac{c_0\gamma}{\beta} T^{\gamma-1/\beta} e^{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x + C_0 T} dx \\
&= \tilde{c} T^{\gamma-1/\beta} e^{\gamma T^{\gamma-1/\beta} + C_0 T} + \frac{c_0\gamma}{\beta} T^{\gamma-1/\beta} e^{C_0 T} \int_1^\infty e^{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x} dx,
\end{aligned} \tag{106}$$

where $\tilde{c} = \tilde{c}_{\beta,\gamma} = c_0 \frac{\gamma}{\beta} \int_0^1 x^{1/\beta - 1} dx$. To estimate the integral $\int_1^\infty \exp\{\gamma T^{\gamma - 1/\beta} x^{1/\beta} - \delta_0 x\} dx$, first define

$$\tilde{C}(T) := \left(\frac{2\gamma}{\delta_0}\right)^{\frac{\beta}{\beta-1}} T^{\frac{\gamma\beta-1}{\beta-1}}.$$

First note that $\gamma T^{\gamma-1/\beta} x^{1/\beta} \le \frac{\delta_0}{2} x$ for $x > \tilde{C}(T)$, whence

$$\begin{aligned} \int_{\tilde{C}(T)}^\infty \exp\{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x\} dx &\le \int_{\tilde{C}(T)}^\infty e^{-\frac{\delta_0}{2} x} dx = \frac{2}{\delta_0} e^{-\frac{\delta_0}{2} \tilde{C}(T)} \\ &= \frac{2}{\delta_0} \exp\{-\frac{\delta_0}{2} \left(\frac{2\gamma}{\delta_0}\right)^{\frac{\beta}{\beta-1}} T^{\frac{\gamma\beta-1}{\beta-1}}\} \le \frac{2}{\delta_0} \exp\{-\frac{\delta_0}{2} \left(\frac{2\gamma}{\delta_0}\right)^{\frac{\beta}{\beta-1}} T\}, \end{aligned} \tag{107}$$

where the last step follows from $\frac{\gamma\beta-1}{\beta-1} > 1$. Second, the integrand $x \to \exp\{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x\}$ reaches its maximum at $x = x_0 := \left(\frac{\delta_0 \beta}{\gamma}\right)^{\frac{\beta}{1-\beta}} T^{\frac{\gamma\beta-1}{\beta-1}}$, and the value of such maximum is

$$\exp\{\gamma T^{\gamma-1/\beta} x_0^{1/\beta} - \delta_0 x_0\} = \exp\left\{\gamma T^{\gamma-1/\beta} \left(\frac{\delta_0\beta}{\gamma}\right)^{\frac{1}{1-\beta}} T^{\frac{\gamma-1/\beta}{\beta-1}} - \delta_0 \left(\frac{\delta_0\beta}{\gamma}\right)^{\frac{\beta}{1-\beta}} T^{\frac{\beta\gamma-1}{\beta-1}}\right\}. \tag{108}$$

Noting that $0 < -1/\beta + \frac{\gamma-1/\beta}{\beta-1} = \frac{\gamma-1}{\beta-1} < 1$, there exists $\bar{q} < 1$ such that $\gamma - 1/\beta + \frac{\gamma-1/\beta}{\beta-1} = \gamma + \frac{\gamma-1}{\beta-1} < \gamma + \bar{q}$ which, along with (108) implies

$$\int_1^{\tilde{C}(T)} \exp\{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x\} dx \le \left(\tilde{C}(T) - 1\right) \exp\{\gamma \left(\frac{\delta_0\beta}{\gamma}\right)^{\frac{1}{1-\beta}} T^{\gamma+\bar{q}}\}. \tag{109}$$

To obtain a uniform upper bound for (109) and $\tilde{C}(T)$, choose $q \in (\bar{q}, 1)$. From (103), (106), (107) and (109) it follows that

$$E[h(\Xi_T)] \le c_1 e^{C_1 T^{\gamma+q}}$$

for suitable constants c_1 and C_1, and the proof is complete. □

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