# Extremal Planar Graph Problems
# and
# Wiener Index of Planar Graphs

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*To the memory of*
*Abayo, Ade and Gashe*

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# Abstract


Let $\mathcal{F}$ be a family of graphs. A graph $G$ is called *$\mathcal{F}$-free* if it contains no graph from $\mathcal{F}$ as a subgraph. The case that $\mathcal{F} = \{F\}$, we may say $G$ is $F$-free instead of saying $\mathcal{F}$-free.

In 1941, Turán proved a classical result in the field of extremal graph theory. He determined exactly the maximum number of edges an $n$-vertex $K_r$-free graph may contains. After his result, for a graph $H$, the maximum number of edges in an $n$-vertex $H$-free graph, denoted by $\mathrm{ex}(n, H)$, is named as *Turán number of $H$*.

A major breakthrough in the study of the Turán number of graphs came in 1966, with the proof of the famous theorem by Erdős, Stone and Simonovits. They determined an asymptotic value of the Turán number of any non-bipartite graph $H$. In particular, they proved $\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2)$, where $\chi(H)$ is the chromatic number of $H$.

Since then researchers have been interested working on Turán number of class of bipartite (degenerate) graphs and extremal graph problems with some more generality. Determining the maximum number of copies of $H$ in an $n$-vertex $F$-free graph, denoted by $\mathrm{ex}(n, H, F)$, is among such problems. Since we count the number of copies of a given graph which is not necessarily an edge, such an extremal graph problem is commonly named as *generalized Turán problem*. Extremal graph problems of such kind have long history before Alon and Shikhelman started systematic study of them in 2016. For instance, the results on $\mathrm{ex}(n, K_r, K_t)$ by Zykov(and independently by Erdős), $\mathrm{ex}(n, C_5, C_3)$ by Győri and $\mathrm{ex}(n, C_3, C_5)$ by Bollobás and Győri can be considered initial contributions.

In a different research direction, some researchers were interested in extremal graph problems in some particular family of graphs; for instance, the family of planar graphs.

Define the *planar Turán number* of a graph $H$, denoted by $\mathrm{ex}_{\mathcal{P}}(n, H)$, as the maximum number of edges in an $n$-vertex $H$-free planar graph. The study of such an extremal Turán-type problem was initiated in 2016 by Dowden while determining sharp upper bounds of $\mathrm{ex}_{\mathcal{P}}(n, C_4)$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5)$, where $C_4$ and $C_5$ are cycles of length 4 and length 5 respectively.

The study of generalized extremal problems in the family of planar graphs was initiated by Hakimi and Schmeichel in 1979. Define the *generalized planar Turán*

*number* of a graph $H$, denoted by $f_{\mathcal{P}}(n,H)$, as the maximum number of copies of $H$ in an $n$-vertex planar graph. Hakimi and Schmeichel determined the exact value of $f_{\mathcal{P}}(n,C_3)$ and $f_{\mathcal{P}}(n,C_4)$. Recently this topic is active and many exact and best asymptotic results were determined for different planar graphs.

In a different setting, an induced version of the generalized planar Turán number of a graph is among interesting topics related to extremal planar graph problems. Define the *induced generalized planar Turán number* of a graph $H$, denoted by $f_{\mathcal{P}}^{\mathrm{ind}}(n,H)$, as the maximum number of induced copies of $H$ in an $n$-vertex planar graph. Unlike the generalized planar Turán number problems, the induced versions are not well studied.

This dissertation mainly focuses on extremal graph problems related to planar graphs, in particular planar Turán numbers, generalized planar Turán numbers and induced generalized planar Turán numbers of graphs. The thesis contains six chapters including the preliminary chapter which recalls basics concepts, notations, definitions and results related to graph, planar graphs, extremal graph theory and Wiener index.

The second chapter focuses on planar Turán number of $\Theta_6$, where $\Theta_6$ is the family of $\Theta_6$-graphs and a $\Theta_6$-graph is a 6-cycle with an edge. More precisely, we give sharp upper bound for $\mathrm{ex}_{\mathcal{P}}(n,\Theta_6)$ and verify that the bound is sharp by giving infinitely many integer $n$ and $n$-vertex $\Theta_6$-free planar graph constructions attaining the bound. The chapter is written based on the paper entitled: *"Planar Turán number of the $\Theta_6$"* [54].

The third chapter emphasizes the generalized planar Turán number of paths. We give details of our results on the precise and best possible asymptotics for $f_{\mathcal{P}}(n,P_4)$ and $f_{\mathcal{P}}(n,P_5)$ respectively. The chapter is written based on our two papers entitled: *"The maximum number of paths of length three in a planar graph"* [57] and *"The maximum number of paths of length four in a planar graph"* [51].

The fourth chapter mainly focuses on the generalized planar Turán number of trees. Furthermore, it discusses various results related to generalized planar Turán problems of the form $\mathrm{ex}_{\mathcal{P}}(n,H,\mathcal{F})$, the maximum number of copies of $H$ in an $n$-vertex $\mathcal{F}$-free planar graph, where $\mathcal{F}$ is nonempty family of graphs. The chapter is written based on our paper entitled: *"Generalized planar Turán numbers"* [63].

The fifth chapter focuses on the induced version of generalized planar Turán number of the 5-cycle. In particular, we give the exact value of $f_{\mathcal{P}}^{\mathrm{ind}}(n,C_5)$ for sufficiently large $n$. The chapter is written based on the paper entitled "*The maximum number of induced $C_5$'s in a planar graph*" [50].

The Wiener index of a graph $G$, denoted by $W(G)$, is the sum of the distance between all non-ordered pairs of distinct vertices. That is, $W(G)=\sum\limits_{\{u,v\}\subseteq V(G)} d_G(u,v)$, where $d_G(u,v)$ is the distance between the vertices $u$ and $v$ in $G$. In the last chapter we discuss on our result that settles the conjecture of Czabarka et. al. in [25], concerning the maximum Wiener index of quadrangulation graphs. The chapter is written based on the paper entitled: *"Wiener index of quadrangulation graphs"* [64].

# Contents

# Symbols

| Symbol | Description |
|---|---|
| $V(G)$ | The set of vertices in $G$. |
| $E(G)$ | The set of edges in $G$. |
| $F(G)$ | The set of faces in a plane graph $G$. |
| $v(G)$ | The number of vertices in $G$. |
| $e(G)$ | The number of edges in $G$. |
| $f(G)$ | The number of faces in a plane graph $G$. |
| $d_G(v)$ | Degree of a vertex $v$ in $G$. |
| $\delta(G)$ | The minimum degree of $G$. |
| $\Delta(G)$ | The maximum degree of $G$. |
| $N_G(v)$ | The set of neighbours of $v$ in $G$. |
| $H \cong G$ | $H$ is isomorphic to $G$. |
| $F \subseteq G$ | $F$ is subgraph of $G$. |
| $G[S]$ | The subgraph of $G$ which is induced by a vertex set $S \subseteq V(G)$. |
| $\alpha(G)$ | The maximum size of an independent set of vertices in $G$. |
| $\kappa(G)$ | The connectivity of $G$. |
| $\chi(G)$ | The chromatic number of $G$. |
| $T_r(n)$ | The Turán graph with $n$ vertices and $r$ parts. |
| $t_r(n)$ | The number of edges of the Turán graph $T_r(n)$. |
| $K_n$ | The complete graph with $n$ vertices. |
| $K_{n_1,n_2,\dots,n_r}$ | The complete $r$-partite graph with parts of size $n_1, n_2, \dots, n_r$. |
| $\mathrm{ex}(n,\mathcal{F})$ | The maximum number of edges in an $n$-vertex $\mathcal{F}$-free graph. |
| $\mathrm{EX}(n,\mathcal{F})$ | The set of all $n$-vertex graphs $G$ such that $e(G) = \mathrm{ex}(n,\mathcal{F})$. |
| $\mathrm{ex}(n,H,\mathcal{F})$ | The maximum number of copies of $H$ in an $n$-vertex $\mathcal{F}$-free graph. |
| $\mathrm{ex}_{\mathcal{P}}(n,H,\mathcal{F})$ | The maximum number of copies of $H$ in an $n$-vertex $\mathcal{F}$-free planar graph. |
| $\mathrm{ex}_{\mathcal{P}}(n,\mathcal{F})$ | The maximum number of edges in an $n$-vertex $\mathcal{F}$-free planar graph. |
| $f_{\mathcal{P}}(n,H)$ | The maximum number copies of $H$ in an $n$-vertex planar graph. |
| $f_{\mathcal{P}}^{\mathrm{ind}}(n,H)$ | The maximum number of induced copies of $H$ in an $n$-vertex planar graph. |
| $W(G)$ | The Wiener index of a graph $G$. |
| $d_G(u,v)$ | Shortest distance between vertices $u$ and $v$ in a connected graph $G$. |

# Asymptotic Notations

Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}^+$ be functions. Then we have the following asymptotic notations.

1. $g(n) = O(f(n))$ if and only if there exist constants $c$ and $N$ such that $g(n) \leq cf(n)$ for $n > N$.

2. $g(n) = o(f(n))$ if and only if for every $c > 0$, there exists an $N$ such that $g(n) \leq cf(n)$ for $n > N$. This is equivalent to saying $\lim\limits_{n\to\infty} \left(\frac{g(n)}{f(n)}\right) = 0$.

3. $g(n) = \Omega(f(n))$ if and only if there exist constants $c$ and $N$ such that $g(n) \geq cf(n)$ for $n > N$.

4. $g(n) = \omega(f(n))$ if and only if for every $c > 0$, there exists an $N$ such that $g(n) \geq cf(n)$ for $n > N$. This is equivalent to saying $\lim\limits_{n\to\infty} \left(\frac{g(n)}{f(n)}\right) = +\infty$.

5. $g(n) = \Theta(f(n))$ if and only if there exist constants $c_1$, $c_2$ and $N$ such that $c_1 f(n) \leq g(n) \leq c_2 f(n)$ for $n > N$.

6. $f(n) \approx g(n)$ if and only if $\lim\limits_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = 1$.

# Chapter 1

# Introduction, Preliminaries

This chapter briefly discuss basic notions of graphs, planar graphs, maximal planar graphs and extremal graph theory. In addition to that, in the last section we give a brief explanation about planar Turán number, generalized planar Turán number, induced generalized planar Turán number and Wiener index, which are the main focus of the thesis. The reader may refer to any introductory graph theoretic books, for instance [10, 14, 29], to recall preliminary results in graph, planar graph and extremal graph theory.

## 1.1 Notations, definitions and basic results

**Definition 1.** *A **graph** is an ordered pair $G = (V(G), E(G))$ comprising a set $V = V(G)$ of vertices together with a set $E = E(G)$ of edges, which are 2-element subsets of $V$.*

Unless mentioned otherwise, throughout the thesis by a graph we mean a simple, finite and undirected graph.

Let $G = (V, E)$ be a graph. For an edge $\{x, y\}$ in $G$, we shall use a shorter notation $xy$. The *order* and *size* of $G$ are respectively $v(G) = |V|$ and $e(G) = |E|$. The degree of a vertex $v$ in $G$, denoted by $d_G(v)$, is the number of vertices in $G$ which are adjacent to $v$. We may omit the subscript and write simply $d(v)$ when there is no ambiguity in the underlying graph we are referring to. Let $N_G(v)$ denote the set of all vertices in $G$ which are adjacent to the vertex $v$ and $N_G[v] = N_G(v) \cup \{v\}$. The minimum and maximum degrees of $G$ are respectively denoted by $\delta(G)$ and $\Delta(G)$.

If we sum up all the degrees in a given graph, we count every edge of the graph exactly twice: once from each of its ends. Thus, we have the following basic result.

**Theorem 1.** *Let $G$ be any graph. Then*

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

**Definition 2.** *Let $G$ be a graph.*

*(i) A graph $H$ is called a **subgraph** of $G$, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.*

*(ii) A subgraph $H$ of $G$ is called an **induced subgraph** of $G$ if for each pair of vertices $x, y \in V(H)$ and $xy \in E(G)$, then $xy \in E(H)$.*

*(iii) Let $S \subseteq V(G)$. The graph induced by $S$, denoted as $G[S]$, is the graph whose vertex set is $S$ and whose edge set consists of all edges in $G$ whose end vertices are in $S$.*

Let $G$ be a graph and $V' \subset V(G)$. $G - V'$ denotes a subgraph of $G$ induced by $V(G) \backslash V'$. That means, $G - V' = G[V(G) \backslash V']$. If $V' = \{v\}$, we may use the notation $G - v$ instead of writing $G - V'$. Let $E' \subset E(G)$, then by $G - E'$ we mean a subgraph of $G$ with $V(G - E') = V(G)$ and $E(G - E') = E(G) \backslash E'$. Moreover, by a graph $G^{-}$ or $G^{--}$ we mean $G$ minus an edge or $G$ minus two edges respectively.

**Definition 3.** *Let $G$ be a graph. A $k$-**cycle** (denoted by $C_k$) in $G$ is a sequence of $k$ distinct vertices, say $(x_1, x_2, x_3, \ldots, x_k)$, such that $x_i x_{i+1}$, $x_1 x_k \in E(G)$ for all $i \in \{1, 2, \ldots, k-1\}$. For simplicity, we may describe such a $k$-cycle by $x_1 x_2 \cdots x_k x_1$.*

**Definition 4.** *Let $G$ be a graph. A **path on** $k$ **vertices**, denoted by $P_k$, in $G$ is a sequence of $k$ distinct vertices, say $(x_1, x_2, x_3, \ldots, x_k)$, such that $x_i x_{i+1} \in E(G)$ for all $i \in \{1, 2, \ldots, k-1\}$. For simplicity, we describe such a path as $x_1 x_2 \cdots x_k$. Sometimes we may describe a path of length $k$ as a $k$-path.*

**Definition 5.** *A graph $G$ is **homomorphic** to a graph $H$ if a mapping $f : V(G) \to V(H)$ exists, which is called **homomorphism**, so that $f$ preserves adjacency. That is, $xy \in E(G)$ if and only if $g(x)g(y) \in E(H)$. If $f$ is bijective, $f$ is called **isomorphism** and we call the two graphs $G$ and $H$ **isomorphic**, in symbols $G \cong H$.*

**Definition 6.** *Let $G$ be a graph. $\bar{G}$ is a graph with vertex set and edge sets defined in the following way*

1. $V(\bar{G}) = V(G)$.
2. $E(\bar{G}) = \{xy : xy \notin E(G)\}$.

**Definition 7.** *Let $G_1$ and $G_2$ be two graphs. $G_1 \cup G_2$ and $G_1 + G_2$ are graphs with vertex sets $V(G_1) \cup V(G_2)$ and edge sets,*

1. $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ *and*

2. $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1)$ *and* $y \in V(G_2)\}$.

A graph is called *connected* if it contains a path between any two vertices of the graph. A connected component or simply *component* of a graph is a maximal subgraph such that each pair of vertices is connected by a path.

Let $G$ be a connected graph. A *separating set of size* $k$ in $G$ or simply a $k$*-cut* set is a set $S \subset V(G)$ of size $k$ such that $G - S$ is a disconnected graph. The connectivity of $G$, denoted by $\kappa(G)$, is the size of a smallest separating set of $G$. Formally, we have the following definition.

**Definition 8.** *Let $G$ be a connected graph. Then*

$$\kappa(G) = \begin{cases} n-1, & \text{if } G = K_n; \\ \min\{|S| : S \text{ is separating set of } G\}, & \text{otherwise.} \end{cases}$$

**Definition 9.** *A graph $G$ is said to be $k$-**connected**, $k \geq 1$ if $\kappa(G) \geq k$. That means, $G$ is $k$-connected if the removal of fewer than $k$ vertices results in neither a disconnected nor a trivial graph.*

Definition 9 gives measure of connectivity based on invulnerability to deletions of vertices. There is also another equivalent measure of connectivity which is based on the multiplicity of alternative paths. This is due to Menger.

**Theorem 2.** *(Menger [30]) A graph $G$ is $k$-connected if and only if every pair of vertices are joined by $k$ pairwise internally-disjoint paths.*

We finish the section by mentioning some special type of graphs together with their notations and some basic results.

A *complete graph* is a graph in which every pair of distinct vertices is connected by an edge. A complete graph on $n$ vertices is denoted by $K_n$.

A graph $G = (V, E)$ is called $r$*-partite* if $V$ admits a partition into $r$-classes such that every edge is between different classes: vertices in the same partition must not be adjacent. We usually say *bipartite* graph instead of saying "2-partite". An $r$-partite graph in which every two vertices from different partition classes are adjacent is called *complete*. We denote a complete $r$-partite graph with partition class size $n_1, n_2, \ldots, n_r$ by $K_{n_1,n_2,\ldots,n_r}$ (see an example of a complete 3-partite graph, $K_{3,4,4}$, in Figure 1.1).

**Definition 10.** *Let $G$ be a graph. We call that $G$ is $k$-colorable if there is a function $c : V(G) \to \{1, 2, \ldots, k\}$ such that for every $xy \in E(G)$, $c(x) \neq c(y)$. The chromatic number of $G$, $\chi(G)$, is defied as:*

$$\chi(G) = \min\{k : G \text{ is } k\text{-colorable }\}.$$

*In other words, $\chi(G)$ is the smallest number of colors needed to color the vertices of $G$ such that no two adjacent vertices receive same color.*

Clearly, a bipartite graph can not contain an odd cycle, a cycle of odd-length. In fact, bipartite graphs are characterized as follows.

**Theorem 3.** *A graph is 2-colorable (bipartite) if and only if it contains no odd cycle as a subgraph.*

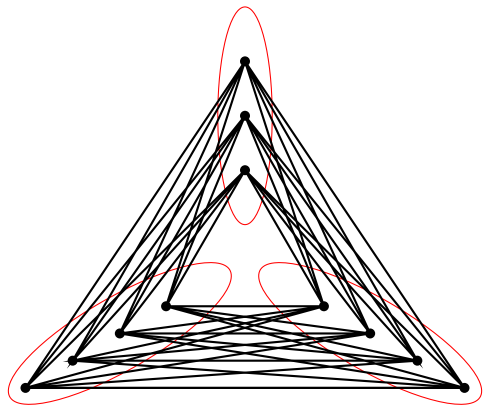


Figure 1.1: A complete 3-partite graph: $K_{3,4,4}$

An acyclic graph, that is a graph containing no cycle, is called a *forest*. A *tree* is a graph in which any two vertices are connected by exactly one path, or equivalently a connected forest. The following theorem gives characterization of a tree with its number of edges.

**Theorem 4.** *A connected graph on $n$ vertices is a tree if and only if it has $n - 1$ edges.*

## 1.2 Planar graphs

**Definition 11.** *A* ***planar graph*** *is a graph that can be embedded in the plane, i.e., it can be drawn in the plane in such a way that its edges intersect only at their end vertices. Such a drawing is called a* ***plane graph*** *or* ***planar embedding*** *of the graph.*

A planar embedding of a graph $G$ divides the plane into regions (commonly called *faces*). The (unique) unbounded region is called the *exterior region* or *exterior face* and a bounded region is called an *interior region* or *interior face*. $F(G)$ denotes the set of all faces in $G$ and $f(G)$ denotes the number of faces in $G$. Each face is bounded by a *closed walk*[1] called the boundary of a face. For a face $\phi \in F(G)$, the *degree* (or

[1]A walk (of length $k$) is an alternating sequence of $v_0 e_1 v_1 \dots e_{k-1} v_k$ of vertices and edges in $G$ such that $e_i = v_i v_{i+1}$ for $i < k$. If $v_0 = v_k$, the walk is called closed. If the vertices in a walk are all distinct, it defines an obvious path in $G$.

*size*) of $\phi$, denoted as $d(\phi)$, is the length of its boundary. Notice that if an edge is a bridge[2] and is incident to a face, the boundary of the face transverse the edge twice. For instance for the plane graph shown in Figure 1.2, the boundary of $\phi$ traverse the the bridge $x_2x_7$ twice. Hence, $d(\phi) = 8$. A *k-face* in $G$ is a face of size $k$.

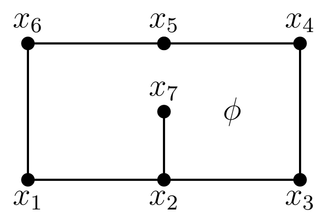



Figure 1.2: A plane graph with two faces

The following results are easy to check.

**Theorem 5.** *For a plane graph $G$, $2e(G) = \sum_{\phi \in F(G)} d(\phi)$.*

**Theorem 6.** *For a plane graph $G$, $2e(G) = \sum_i i f_i$, where $f_i$ is the number of $i$-faces in $G$.*

Next we state one of the beautiful result, obtained by Euler (1752), which relates the number of vertices, edges, and faces of a connected plane graph. For completeness we give a short proof of the Euler's result and corollaries of it.

**Theorem 7.** *(Euler's formula) For a connected plane graph $G$,*

$$v(G) - e(G) + f(G) = 2.$$

*Proof.* If $G$ is a tree, then obviously the statement is true. Indeed, $e(G) = v(G) - 1$ and $f(G) = 1$ and hence, $v(G) - e(G) + f(G) = 2$. Suppose $G$ contains a cycle. We prove the statement by induction on the number of edges. If $e(G) = 3$, then the graph consists of three vertices and two faces. So, the statement is true. Now let $G$ be a plane graph with $e(G) \geq 4$ edges. Take an edge $e$ on a cycle in $G$. Then this edge is on a boundary of two faces. Thus deleting the edge results a plane graph, say $G'$, with one fewer number of faces and with the same number of vertices as $G$. Thus, by the induction assumption $v(G) - (e(G) - 1) + (f(G) - 1) = 2$. Therefore, $v(G) - e(G) + f(G) = 2$. □

**Corollary 1.** *For a plane graph $G$ with $c(G)$ components, $v(G) + f(G) = e(G) + c(G) + 1$.*

**Corollary 2.** *For a planar graph $G$, $e(G) \leq 3v(G) - 6$.*

[2]An edge $e$ in $G$ is a bridge if $G - e$ is a disconnected graph.

**Corollary 3.** *For any planar graph $G$, there is a vertex with degree at most 5. i.e.,* $\delta(G) \leq 5$.

Notice that $K_5$ is not a planar graph. Indeed, suppose it is a planar graph. Then it should satisfy Corollary 2. However, this is not true considering the number of vertices and edges of $K_5$, which are respectively 5 and 10. Similarly it is easy to check that $K_{3,3}$ is not a planar graph. In this case one can use Corollary 4 to verify.

The following theorem, due to Euler, gives the relation between number of edges, number of vertices and girth of a planar graph. Where the *girth* of a graph is the size of the smallest cycle in the graph. If the graph is acyclic, we set the girth to $\infty$.

**Theorem 8.** *For an $n$-vertex planar graph $G$ with girth $g$,*

$$e(G) \leq \max\left\{\frac{g}{g-2}(n-2),\ n-1\right\}.$$

**Corollary 4.** *For a bipartite planar graph $G$ with $v(G) \geq 3$, $e(G) \leq 2v(G) - 4$.*

### 1.2.1 Characterization of planar graphs

We have seen that $K_5$ and $K_{3,3}$ are not planar graphs. In fact, these two graphs play central role in the characterization of planarity of a graph. *Kuratowski's theorem* and *Wagner's theorem* are the two famous characterizations of planar graphs. Next we discuss the two results.

**Definition 12.**

*(i) Let $G$ be a non-trivial graph. Subdivision of an edge $e = uv \in E(G)$ is an operation of replacing $e$ by a 2-path $uwv$, where $w$ is a new vertex.*

*(ii) A graph $H$ is said to be a subdivision of $G$ if $H$ can be obtained from $G$ by a finite sequence of edge subdivisions. By convention, $G$ is a subdivision of $G$.*

**Lemma 1.** *Every subdivision of a planar graph is planar and every subdivision of a non-planar graph is non-planar.*

From the facts that $K_5$ and $K_{3,3}$ are not planar, subdivisions of $K_5$ and $K_{3,3}$ are not planar graphs too. Thus, if $G$ is a planar graph, then $G$ does not contain $K_5$ and $K_{3,3}$ and their subdivision as a subgraph. A natural question is whether the converse holds or not. That means, is it true that if $G$ does not contain a subdivision of $K_5$ and $K_{3,3}$ as a subgraph, then the graph is planar? In 1930, Kuratowski asserts that it is indeed true.

**Theorem 9.** *(Kuratowski [79]) A graph $G$ is planar if and only if $G$ contains no subdivision of $K_5$ and $K_{3,3}$ as a subgraph.*

**Definition 13.**

*(i)* *Let $e = uv$ be an edge in a graph $G$. Delete the vertices $u$ and $v$. Add a new vertex $w$ to $G - \{u, v\}$ and join $w$ to all those vertices in $V \backslash \{u, v\}$ to which $u$ or $v$ is adjacent with in $G$. This operation is called contraction of the edge $e$.*

*(ii)* *A graph $H$ is said to be a minor of $G$ if an isomorphic copy of $H$ can be obtained from $G$ with finite sequence of deletion of vertices or edges or contraction of edges. By convention, $G$ is a minor of $G$.*

It is easy to check that if $G$ contains a subdivision of $H$, then $H$ is a minor of $G$. However, the converse of is false. i.e., If $H$ is a minor of $G$, then $G$ need not contain subdivision of $H$.

The following theorem, due to Wagner (1937) [97], gives another characterization of planarity of a graph based on minors.

**Theorem 10.** *(Wagner [97]) A graph $G$ is planar if and only if neither $K_5$ nor $K_{3,3}$ is a minor of $G$.*

### 1.2.2 Maximal planar graphs - Triangulation graphs

**Definition 14.** *A planar graph $G$ is said to be **maximal** if addition of an edge on the given vertex set of $G$ destroys its planarity.*

For instance, we have seen that $K_5$ is not a planar graph but $K_5^-$ is a planar graph. Thus $K_5^-$ is a maximal planar graph. Next we mention some basic properties of maximal planar graphs without proof.

**Lemma 2.** *If $G$ is an $n$-vertex $(n \geq 3)$ maximal plane graph, then the boundary of every face is a triangle.*[3]

**Lemma 3.** *Every edge of a maximal plane graph is contained in exactly two triangular faces.*

**Theorem 11.** *An $n$-vertex maximal plane graph has exactly $3n - 6$ edges and $2n - 4$ faces.*

**Theorem 12.** *Let $G$ be a maximal plane graph and $v \in V(G)$ such that $d(v) = k$. Then $N(v)$ induces a unique $k$-cycle.*

**Theorem 13.** *If $G$ is an $n$-vertex maximal plane graph with $n \geq 4$, then $\delta(G) \geq 3$.*

[3] It is due to this property that a maximal planar graph is named as *triangulation graph* as well.

From Corollary 3 and Theorem 13, for an $n$-vertex($n \geq 4$) maximal planar graph $G$, then $3 \leq \delta(G) \leq 5$. If $G$ is an $n$-vertex maximal planar graph with minimum degree 5, then $n \geq 12$. Indeed, $2(3n-6) \geq 2e = \sum_{v \in V(G)} d(v) \geq 5n$, which implies that $n \geq 12$. Moreover, from the result of Theorem 12, any maximal planar graph is not 6-connected. Further connectivity properties of maximal planar graphs given below.

**Theorem 14.** *(Whitney [98]) A maximal planar graph $G$ on $n \geq 4$ vertices is 3-connected.*

**Theorem 15.** *Let $G$ be a maximal $k$-connected planar graph. Then every cut set of size $k$ contains a cycle of length $k$.*

*Proof.* Let $K = \{v_1, v_2, \ldots, v_k\}$ be a $k$-cutset in $G$. Let $u$ and $w$ be two vertices in $G$ such that any path from $u$ to $w$ contains at least one vertex from $K$. Since $G$ is $k$-connected, by Theorem 2, there are $k$ internally disjoint paths from $u$ to $w$. Each of this path contains one vertex of $K$ (See Figure 1.3). We must have edges $v_{i_x}v_{i_{x+1}}$, indices are taken modulo $k$, otherwise, considering the portion of the cycle induced by the neighbours of $v_{i_x}$ inside the region $uv_{i_x}wv_{i_{x+1}}u$ not containing $v_{i_{x-1}}$, we can create a path from $u$ to $w$ that does not contain any of the vertices of $K$, which is a contradiction as $K$ is a $k$-cutset of vertices. Thus, $K$ induces a cycle of length $k$. □

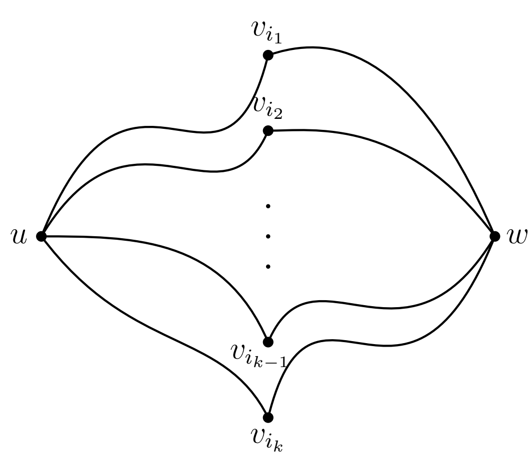



Figure 1.3: $k$ internally disjoint paths from $u$ to $w$

## 1.3 Extremal graph theory

Extremal graph theory is a branch of mathematics that studies how global properties of a graph influence local substructure. How many edges, for instance, do we have to give a graph on $n$ vertices to be sure that, no matter how these edges happen to be arranged, the graph will contain a $K_r$ subgraph for some given $r$? Questions of such type or with some more generality are among the most natural ones in extremal graph theory. Next we highlight some of the extremal graph problems attracting researchers' attention.

### 1.3.1 Turán numbers

Let $\mathcal{F}$ be a family of graphs. A graph $G$ is called $\mathcal{F}$*-free* if it does not contain any graph from $\mathcal{F}$ as a subgraph. The case that $\mathcal{F} = \{F\}$, we may say $G$ is $F$*-free* instead of saying $\mathcal{F}$-free.

**Definition 15.** *Let $\mathcal{F}$ be a nonempty family of graphs and $n$ be a positive integer. The **Turán number** of $\mathcal{F}$, which is denoted by $\mathrm{ex}(n, \mathcal{F})$, is the maximum number of edges in an $n$-vertex $\mathcal{F}$-free graph. That means,*

$$\mathrm{ex}(n, \mathcal{F}) = \max\{e(G) : G \textit{ is an } n\textit{-vertex } \mathcal{F}\textit{-free graph}\}.$$

*Define $\mathrm{EX}(n, \mathcal{F})$ as the set of all $n$-vertex $\mathcal{F}$-free graph $G$ such that $e(G) = \mathrm{ex}(n, \mathcal{F})$. The case that $\mathcal{F} = \{F\}$, we simply denote $\mathrm{ex}(n, \mathcal{F})$ by $\mathrm{ex}(n, F)$ and $\mathrm{EX}(n, \mathcal{F})$ by $\mathrm{EX}(n, F)$.*

Exploring previous works related to such an extremal graph problem, Mantel (1907) [90] determined the maximum possible number of edges in a triangle-free graph.

**Theorem 16.** *(Mantel's theorem [90]) For an $n$-vertex triangle-free graph $G$, $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$. Equality holds if and only if $G$ is the complete bipartite graph with parts of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, i.e, $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

Mantel's theorem in the language of extermal graph theory is $\mathrm{ex}(n, C_3) = \lfloor \frac{n^2}{4} \rfloor$ and $\mathrm{EX}(n, C_3) = \{K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}\}$.

A systematic study of such type of extremal graph problem began after Turán [96] proved the generalization of Mantel's result in 1941.

**Definition 16.** *The **Turán graph**, $T_r(n)$, is an $n$-vertex complete $r$-partite graph whose parts are with size as equal as possible. Denote $e(T_r(n))$ by $t_r(n)$.*

$T_r(n)$ has ($n \bmod r$) parts of size $\lceil n/r \rceil$ and $r - (n \bmod r)$ parts of size $\lfloor n/r \rfloor$. For instance, $T_3(11)$ is shown in Figure 1.1. If $r \mid n$, then $t_r(n) = \left(1 - \frac{1}{r}\right) \frac{n^2}{2}$.

Turán proved the following fundamental result in extremal graph theory.

**Theorem 17.** *(Turán's theorem [96]) For an $n$-vertex $K_{r+1}$-free graph $G$,*

$$e(G) \leq t_r(n),$$

*and equality holds if and only if $G$ is the Turán graph $T_r(n)$. That means,*

$$\mathrm{ex}(n, K_{r+1}) = t_r(n) \text{ and } \mathrm{EX}(n, K_{r+1}) = \{T_r(n)\}.$$

The most general result in extremal graph theory is the Erdős-Stone-Simonovits theorem which gives an asymptotically tight result of $\mathrm{ex}(n, F)$ whenever $F$ is a non-bipartite graph.

**Theorem 18.** *(Erdős-Stone-Simonovits theorem [39, 35]) Let $F$ be a non-bipartite graph. Then*

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right)\binom{n}{2} + o(n^2).$$

Actually this fundamental theorem still holds for bipartite graphs. However it just leaves the error term only. Since this result, researchers have been interested on the Turán number of bipartite graphs (degenerate graphs). Next we mention some of the basic and interesting results with this regard. We refer the survey by Füredi and Simonovits [47] for detail of results, progresses and open problems related to it.

Kővári, Sós and Turán determined an upper bound of the Turán number of complete bipartite graphs, $K_{a,b}$, and pose a conjecture.

**Theorem 19.** *(Kővári, Sós, Turán [76]) For all positive integers $a$ and $b$, $a \le b$ we have,*

$$\mathrm{ex}(n, K_{a,b}) \le \frac{1}{2}\sqrt[a]{(b-1)}n^{2-\frac{1}{a}} + O(n).$$

**Conjecture 1.** *(Kővári-Sós-Turán conjecture [76]) The order of the upper bound in Theorem 19 is sharp.*

The conjecture is known to hold for some special cases. For example when $a = b = 2$, the $K_{a,b}$ is a 4-cycle and we have the following result.

**Theorem 20.** *(Erdős, Rényi, Sós [38], Brown [19])*

$$\mathrm{ex}(n, C_4) = (1 + o(1))\frac{1}{2}n^{3/2}.$$

The upper bound of this theorem comes from cherry[4] counting argument. A corresponding lower bound comes from finite projective plane constructions.

**Example 1.** *(Erdős, Rényi, Sós [38], Brown [19]) Let $p > 2$ be a fixed prime. Consider the 3-tuples formed by elements of $\mathbb{F}_p$. We have $p^3 - 1$ non-zero ($\neq (0,0,0)$) such 3-tuples. Define the equivalence class $\sim$*

$$(x, y, z) \sim (x', y', z') \iff \text{there is an } \alpha \in \mathbb{F}_p \text{ such that} (x', y', z') = (\alpha x, \alpha y, \alpha z).$$

[4] 2-path

*Each equivalence class contains $p-1$ elements and the number of equivalence classes is $\frac{p^3-1}{p-1} = p^2+p+1$. Define a graph $G$ with vertex set $V(G)$ as the set of equivalence classes and edge set*

$$E(G) = \{(a,b,c)(x,y,z) : ax+by+cz = 0 \text{ and } (a,b,c) \neq (x,y,z)\}.$$

*It can be checked that $G$ is $C_4$-free and $e(G) = \frac{1}{2}n^{3/2} + o(n^{3/2})$.*

In [15], Bondy and Simonovits considered the Turán number of even cycle $C_{2k}$.

**Theorem 21.** *(Bondy-Simonovits even-cycle theorem [15]) For $k \geq 2$,*

$$\mathrm{ex}(n, C_{2k}) = O\left(n^{1+\frac{1}{k}}\right).$$

A classical result in extremal graph theory is the Erdős-Gallai theorem, which gives the condition on the number of edges to force the existence of a path of given length in a graph. Recall that $P_k$ denotes a $k$-vertex path.

**Theorem 22.** *(Erdős-Gallai theorem [37]) For two integers $n$ and $k$,*

$$\mathrm{ex}(n, P_k) \leq \frac{(k-2)n}{2}.$$

*The equality holds if and only if $k$ divides $n$ and the graph is the disjoint union of complete graphs on $k-1$ vertices.*

Motivated by the result, Erdős and Sós made the following tantalizing conjecture related to Turán number of a tree with $k$ vertices. Let $T_k$ denote a tree on $k$ vertices.

**Conjecture 2.** *(Erdős-Sós conjecture [36]) For two integers $n$ and $k$*

$$\mathrm{ex}(n, T_k) \leq \frac{(k-2)n}{2}.$$

The conjecture is still open despite results for some particular trees.

Theorem 22 was actually a corollary of the following more general theorem. Let $C_{\geq k}$ be a family of cycles with length at least $k$.

**Theorem 23.** *(Erdős, Gallai [37]) For two integers $n$ and $k$,*

$$\mathrm{ex}(n, C_{\geq k}) \leq \frac{(k-1)(n-1)}{2}.$$

*The equality holds if and only if $(k-2) \mid (n-1)$ and $G$ is the union of $\frac{n-1}{k-2}$ disjoint cliques of size $k-1$ sharing a vertex in a tree-like structure, where by a tree-like structure we mean a connected graph containing no cycle having two vertices from two different $(k-1)$-vertex cliques. See for instance a tree-like structure with cliques of size 8 in Figure 1.4.*

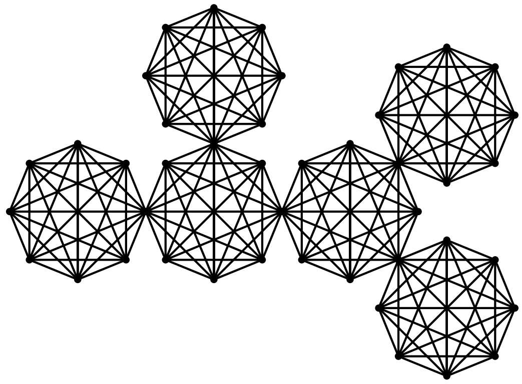


Figure 1.4: A tree-like structure constructed from $K_8$ and containing no $C_{\geq 9}$.

As the extremal examples for Theorem 22 are disconnected, it is natural to consider a version of the problem where the base graph is assumed to be connected. Kopylov [77] settled this problem, and later Balister, Győri, Lehel and Schelp [6] classified the extremal graphs.

**Definition 17.** *We denote by $G_{n,k,s}$ the graph whose vertex sets is partitioned into 3 classes, $A$, $B$ and $C$ with $|A| = s, |B| = n-k+s$ and $|C| = k-2s$ such that $A \cup C$ induces a clique, $B$ is an independent set and all possible edges are taken between vertices of $A$ and $B$. In short $G_{n,k,s} = \left(K_{k-2s} \cup \bar{K}_{n-k+s}\right) + K_s$, see $G_{14,11,3}$ in Figure 1.5.*

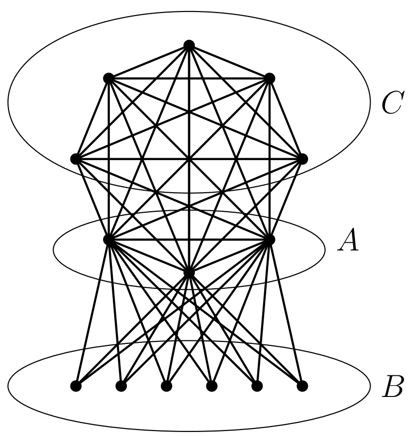



Figure 1.5: The graph $G_{14,11,3}$

Denote $\mathrm{ex}_{\mathrm{C}}(n, H)$ the maximum number of edges in an $n$-vertex $H$-free connected graphs.

**Theorem 24.** *(Kopylov [77], Balister, Győri, Lehel, Schelp [6])*
*Let $n > \ell > 4$. Then*

$$\mathrm{ex}_C(n, P_\ell) = \max\{e(G_{n,\ell-1,\lfloor\frac{\ell-2}{2}\rfloor}), e(G_{n,\ell-1,1})\}.$$

*Extremal graphs are* $G_{n,\ell-1,1}$ *or* $G_{n,\ell-1,\lfloor\frac{\ell-2}{2}\rfloor}$.

### 1.3.2 Generalized Turán problems

A natural generalization of the extremal function $\mathrm{ex}(n,\mathcal{F})$ is to a setting where, rather than edges we maximize the number of isomorphic copies of a given graph $H$ in an $n$-vertex $\mathcal{F}$-free graph.

We define a generalized extremal function $\mathrm{ex}(n,H,\mathcal{F})$ as follows.

**Definition 18.** *Let $H$ be a graph and $\mathcal{F}$ be a family of graphs. The extremal function $\mathrm{ex}(n,H,\mathcal{F})$ denotes the maximum number of copies of $H$ as a subgraph in an $n$-vertex $\mathcal{F}$-free graph. That means,*

$$\mathrm{ex}(n,H,\mathcal{F}) = \max\{\mathcal{N}(H,G) : G \textit{ is an } n\textit{-vertex } \mathcal{F}\textit{-free graph}\},$$

*where $\mathcal{N}(H,G)$ denote the number of subgraphs of $G$ isomorphic to $H$. In the case that $\mathcal{F}=\{F\}$, we denote it as $\mathrm{ex}(n,H,F)$.*

Extremal graph problems of such type have long history before Alon and Shikhelman [3] (see also [4]) started systematic study of them in 2016. The results on $\mathrm{ex}(n,K_r,K_t)$ by Zykov [103] (and independently by Erdős [33]), $\mathrm{ex}(n,C_5,C_3)$ by Győri [59] and $\mathrm{ex}(n,C_3,C_5)$ by Bollobás and Győri [12] can be considered as initial contributions.

A variety of results were obtained in this area of extremal graph problem, perhaps the most well-known of which is related to Erdős's conjecture [34], $\mathrm{ex}(n,C_5,C_3)\le \left(\frac{n}{5}\right)^5$. The motivation of this is that the blownup[5] $C_5$, i.e, $C_5[n/5]$, has no triangle and contains $\left(\frac{n}{5}\right)^5$ copies of $C_5$. See the blownup $C_5$ in Figure 1.6. The first initial bound was due to Győri [59].

**Theorem 25.** *(Győri [59])*

$$\mathrm{ex}(n,C_5,C_3)\le 1.03\left(\frac{n}{5}\right)^5.$$

Later Hatami, Hladkỳ, Král, Norine and Razborov [71] and independently by Grzesik [56] confirmed the conjecture. Recently, Lidický and Pfender [87] extended the result. They determined the exact value of $\mathrm{ex}(n,C_5,C_3)$ and characterized the extremal construction when $n\ge 5$.

**Theorem 26.** *(Lidický, Pfender [87])*

$$\mathrm{ex}(n,C_5,C_3)=\prod_{i=0}^{4}\left\lfloor\frac{n+i}{5}\right\rfloor.$$

[5]Given a graph $H$, its blownup version $H[t]$ is defined as follows: we replace each vertex $x$ of $H$ by $t$ independent new vertices and we join two new vertices coming from distinct vertices $x$, $y$ if and only if $xy$ was an edge of $H$.

*Moreover, the only triangle-free graphs on $n \geq 5$ vertices attaining the value are balanced blow-ups of a 5-cycle, and the 8-cycle with all diagonals added for the special case of $n = 8$.*

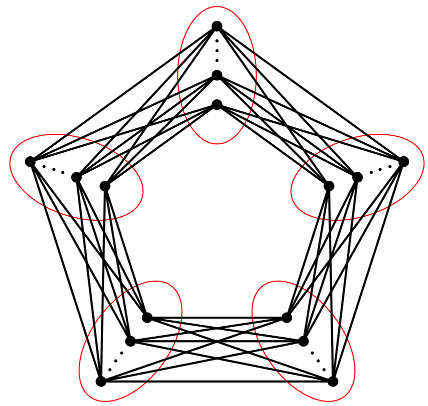


Figure 1.6: Blownup $C_5$.

In the opposite direction, the extremal function $\mathrm{ex}(n, C_3, C_5)$ was considered by Bollobás and Győri [12].

**Theorem 27.** *(Bollobás, Győri [12])*
*If $G$ is a graph of order $n$ not containing any $C_5$, then the number of triangles in $G$ is at most $\frac{5}{4}n^{3/2} + o(n^{3/2})$.*

They also showed that their bound is sharp apart from the constant coefficient as the following construction shows.

**Example 2.** *(Bollobás, Győri [12])*
*Let $G_0$ be a $C_4$-free bipartite graph on $\frac{n}{3} + \frac{n}{3}$ vertices with about $\left(\frac{n}{3}\right)^{3/2}$ edges. Double each vertex in one part of the bipartite graph and add an edge joining the old and the new copy. Let $G$ denote the resulting graph. It can be checked that the number of triangles in $G$ is $\left(\frac{n}{3}\right)^{3/2} + o(n^{3/2})$ and $G$ is $C_5$-free.*

Recently, their upper bound is improved by Alon and Shikhelman [3], where they proved that $\mathrm{ex}(n, C_3, C_5) \leq \frac{\sqrt{3}}{2}n^{3/2} + o(n^{3/2})$. This result is further improved by Ergemlidze, Győri, Methuku and Salia [40] and very recently Ergemlidze and Methuku [41] proved $\mathrm{ex}(n, C_3, C_5) \leq \frac{1}{3\sqrt{2}}n^{3/2} + o(n^{3/2})$, but the problem of determining the correct asymptotic bound remains open.

In other results, Győri, Salia, Tompkins, Zamora [66] estimated the number of paths and cycles in a $P_k$-free graph. They proved the following asymptotic results.[6]

[6] $f(n,k) \approx g(n,k)$ when $\lim_{k\to\infty}\left(\lim_{n\to\infty}\frac{f(n,k)}{g(n,k)}\right) = 1$.

**Theorem 28.** *(Győri, Salia, Tompkins, Zamora [66]) For a fixed* $\ell \in \mathbb{N}$,

1. $\mathrm{ex}(n, P_{2\ell}, P_k) \approx \frac{(\ell+1)(k-1)^\ell n^\ell}{2^{\ell+1}}$.
2. $\mathrm{ex}(n, P_{2\ell+1}, P_k) \approx \frac{(k-1)^\ell n^{\ell+1}}{2^{\ell+1}}$.
3. $\mathrm{ex}(n, C_{2\ell}, P_k) \approx \frac{(k-1)^\ell n^\ell}{\ell 2^{\ell+1}}$.
4. $\mathrm{ex}(n, C_{2\ell+1}, P_k) \approx \frac{(k-1)^{\ell+1} n^{\ell+1}}{2^{\ell+2}}$.

Many other results on generalized extremal problems have been obtained. We refer [46, 48, 49, 55, 58, 60, 61, 65, 85, 86, 89] for more results.

### 1.3.3 Planar Turán numbers

Recall that, the extremal function $\mathrm{ex}(n, F)$ denote the maximum number of edges in an $n$-vertex $F$-free graph. In other words, we find the maximum number of edges among all $n$-vertex $F$-free subgraphs of the host graph $K_n$.

In a different research direction, Turán's problem has been considered when the host graph is diffrent from $K_n$ or in a certain particular family of graphs called *base*. Examples include the Zarankiewicz problem [102] where the host graph is taken to be an $n$-vertex complete bipartite graph, or extremal problems on the hypercube $Q_n$ initiated by Erdős [34]. Recently, Turán-type problems have been considered when the base is a family of planar graphs.

**Definition 19.** *Let $\mathcal{F}$ be a nonempty family of graphs and $n$ be a positive integer. The maximum number of edges in an $n$-vertex $\mathcal{F}$-free planar graph is denoted by* $\mathrm{ex}(n, \mathcal{F})$*. That means,*

$$\mathrm{ex}_{\mathcal{P}}(n, \mathcal{F}) = \max\{e(G) : G \textit{ is an } n\textit{-vertex } \mathcal{F}\textit{-free planar graph}\}.$$

*The case that $\mathcal{F} = \{F\}$, we simply denote* $\mathrm{ex}_{\mathcal{P}}(n, \mathcal{F})$ *by* $\mathrm{ex}_{\mathcal{P}}(n, F)$*. The value* $\mathrm{ex}_{\mathcal{P}}(n, \mathcal{F})$ *is called* ***planar Turán number of*** $\mathcal{F}$*.*

Dowden [31] initiated the study of Turán-type problems when the base is the family of planar graphs. The case that the forbidden subgraph is a complete graph (i.e. the analogue to Turán) is fairly trivial. Since $K_5$ is not planar, the only meaningful cases to look at are $K_3$ and $K_4$, and these are both straightforward: for the former, it can be checked that from Theorem 8, an $n$-vertex $K_3$-free planar graph contains at most $2n - 4$ edges. Moreover, it can be observed that the complete bipartite graph $K_{2,n-2}$, see Figure 1.7 (left) contains $2n - 4$ edges[7]. For the latter, it suffices to note

[7]This is not the only construction giving the extremal number. In fact, any quadrangulation graph, plane graph with every of its face is of size four, on $n$ vertices gives the planar extremal number. For instance, see the constructions given in Appendix A.1.

that there exist planar triangulations not containing $K_4$ . For instance, take a cycle of length $(n-2)$ and then add two new vertices that are both adjacent to all those vertices in the cycle, see Figure 1.7 (right), and so the extremal number is $3n-6$. So we have the following results.

**Theorem 29.** *(Dowden [31])*

1. $\mathrm{ex}_{\mathcal{P}}(n, K_3) = 2n - 4$, $n \geq 3$.
2. $\mathrm{ex}_{\mathcal{P}}(n, K_4) = 3n - 6$, $n \geq 6$.

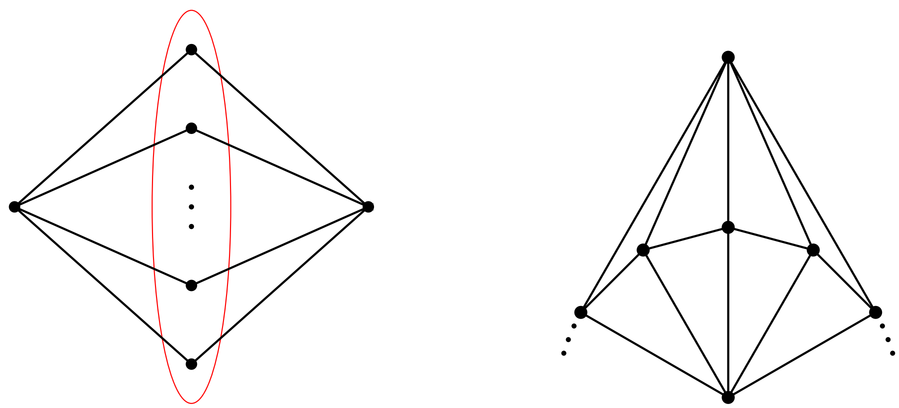


Figure 1.7: Extremal constructions for planar Turán number of $K_3$ and $K_4$.

The next most natural type of graph to investigate is perhaps a cycle. Dowden [31] determined sharp upper bound of $\mathrm{ex}_{\mathcal{P}}(n, F)$ when the forbidden graph $F$ is a 4-cycle and a 5-cycle. He proved the following two results.

**Theorem 30.** *(Dowden [31])*

1. $\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq \frac{15(n-2)}{7}$ , *for all* $n \geq 4$.
2. $\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$, *for all* $n \geq 11$.

Dowden asked about planar Turán number of longer cycles and reflected that determining sharp bound could not be easy, following an intricate proof for the case of 5-cycle. He questioned whether or not the chromatic number of $F$ plays a role in the value $\mathrm{ex}_{\mathcal{P}}(n, F)$, like the celebrated Erdős-Stone-Simonovits theorem. Moreover, he questioned which graphs have planar Turán number $3n-6$ just like $K_4$.

Lan, Shi and Song [83] determined several sufficient conditions on the graph $F$ which yields $\mathrm{ex}_{\mathcal{P}}(n, F) = 3n-6$. They also answered Dowden's question that, the chromatic number of $F$ plays no role in the value of $\mathrm{ex}_{\mathcal{P}}(n, F)$. In the same paper, the authors also determined completely $\mathrm{ex}_{\mathcal{P}}(n, F)$ when $F$ is a $k$-wheel[8] or a $k$-star[9].

[8] $C_{k-1} + K_1$
[9] $K_{1,k-1}$

In [82] the same authors also determined $\mathrm{ex}_{\mathcal{P}}(n, P_k)$ when $k \in \{8, 9\}$. Recently, Lan and Shi [81] extended the result when $k \in \{6, 7, 10, 11\}$. Recently, Fang, Zhai and Wang [43] considered planar Turán number of intersecting triangles.

In other results, Lan, Shi and Song in [84] determined sharp upper bound for $\mathrm{ex}_{\mathcal{P}}(n, \Theta_k)$, $k \in \{4, 5\}$. Where $\Theta_k$ denote the family of distinct $\Theta_k$-graphs and a $\Theta_k$-graph is a graph obtained by joining a pair of non-consecutive vertices of a $k$-cycle with an edge. They also obtained an upper bound for $\mathrm{ex}_{\mathcal{P}}(n, \Theta_6)$. The following theorem summarizes their results.

**Theorem 31.** *(Lan, Shi, Song [84])*

1. $\mathrm{ex}_{\mathcal{P}}(n, \Theta_4) \leq \frac{12(n-2)}{5}$, *for all* $n \geq 4$, *with equality when* $n \equiv 12$ $(mod\ 20)$.
2. $\mathrm{ex}_{\mathcal{P}}(n, \Theta_5) \leq \frac{5(n-2)}{2}$, *for all* $n \geq 5$, *with equality when* $n \equiv 50$ $(mod\ 120)$.
3. $\mathrm{ex}_{\mathcal{P}}(n, \Theta_6) \leq \frac{18(n-2)}{7}$, *for all* $n \geq 6$.

Recently, Ghosh, Győri, Paulos, Xiao and Zamora [54] improved the bound of $\mathrm{ex}_{\mathcal{P}}(n, \Theta_6)$ in Theorem 31 with a sharp upper bound. Details of our results are discussed in the second chapter.

From result (3) of Theorem 31, the authors remarked that $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{18(n-2)}{7}$. In [52] Ghosh, Győri, Martin, Paulos and Xiao, improved the bound with sharp upper bound. Our main results are as follows:

**Theorem 32.** *(Ghosh, Győri, Martin, Paulos, Xiao [52])*
*Let* $G$ *be a 2-connected,* $C_6$*-free plane graph on* $n$ $(n \geq 6)$ *vertices with* $\delta(G) \geq 3$. *Then*

$$e(G) \leq \frac{5}{2}n - 7.$$

**Theorem 33.** *(Ghosh, Győri, Martin, Paulos, Xiao [52])*

$$\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5}{2}n - 7, \text{ for all } n \geq 18.$$

We verified the bounds are sharp by finding an $n$-vertex $C_6$-free planar graph attaining the bound for infinitely many $n$. In particularly, we proved the following.

**Theorem 34.** *(Ghosh, Győri, Martin, Paulos, Xiao [52])*
*For every* $n \equiv 2 \pmod 5$, *there exists a* $C_6$*-free plane graph* $G$ *with* $v(G) = \frac{18n+14}{5}$ *and* $e(G) = 9n$, *hence* $e(G) = \frac{5}{2}v(G) - 7$.

Proof of the theorem and our extremal constructions are in Appendix A.2.

In [52] we also proposed our conjecture concerning sharp upper bound of planar Turán number of longer cycles, $\mathrm{ex}(n, C_\ell)$ and $\ell \geq 7$. However, we noticed that the conjecture does not holde when $\ell \geq 11$ and then we revised the conjecture to the case when $10 \geq \ell \geq 7$ in the manuscript. Unfortunately we did not update the arXiv version and recently, Cranston, Lidický, Liu and Shantanam [24] explicitly showed that the conjecture does not hold when $\ell \geq 11$ and proposed two revised version of the conjecture. Details of our conjecture when $10 \geq \ell \geq 7$ and a revised conjecture by Cranston et. al. [24] are given in Appendix A.3.

### 1.3.4 Generalized planar Turán numbers

Another direction of research which has been considered is maximizing the number of copies of a given graph $H$ in an $n$-vertex planar graph which forbids all members of a family of graphs $\mathcal{F}$. This is actually a natural extension of the planar Turán problem.

**Definition 20.** *Let $H$ be a graph $H$ and $\mathcal{F}$ be a family of planar graphs. For a positive integer $n$, the maximum number of copies of $H$ in an $n$-vertex $\mathcal{F}$-free planar graph is denoted by $\mathrm{ex}_{\mathcal{P}}(n, H, \mathcal{F})$. i.e.,*

$$\mathrm{ex}_{\mathcal{P}}(n, H, \mathcal{F}) = \max\{\mathcal{N}(H, G) : G \textit{ is an } n\textit{-vertex } \mathcal{F}\textit{-free planar graph}\}.$$

*Where $\mathcal{N}(H, G)$ is the number of isomorphic copies of $H$ in $G$. The case that $\mathcal{F} = \{F\}$, we simply denote $\mathrm{ex}_{\mathcal{P}}(n, H, \mathcal{F})$ by $\mathrm{ex}_{\mathcal{P}}(n, H, F)$. The case that $\mathcal{F} = \emptyset$, we denote $\mathrm{ex}_{\mathcal{P}}(n, H, \mathcal{F})$ by $f_{\mathcal{P}}(n, H)$. i.e,*

$$f_{\mathcal{P}}(n, H) = \max\{\mathcal{N}(H, G) : G \textit{ is an } n\textit{-vertex planar graph}\}.$$

*The value $f_{\mathcal{P}}(n, H)$ is called **generalized planar Turán number** of $H$.*

It is interesting to note that the problem of maximizing copies of $H$ in a planar graph is in some sense a special case of the problem of Alon and Shikelman [3]. Indeed, for a given graph $H$, and the collection $\mathcal{F}$ of minors or subdivisions of $K_5$ and $K_{3,3}$, it follows from Kuratowski's theorem or Wagner's theorem that $\mathrm{ex}(n, H, \mathcal{F})$ is equal to the maximum number of copies of $H$ in an $n$-vertex planar graph.

One obvious result related to such an extremal graph problem is generalized planar Turán number of $K_2$ (edge). Since every $n$-vertex maximal planar graph contains $3n - 6$ edges, we have $f_{\mathcal{P}}(n, K_2) = 3n - 6$.

In 1979, Hakimi and Schmeichel [67] determined the exact value of $f_{\mathcal{P}}(n, C_3)$ and characterize the extremal construction. They also determined the exact value of $f_{\mathcal{P}}(n, C_4)$. Almeddine [1] characterized the extremal construction containing $f_{\mathcal{P}}(n, C_4)$ copies of $C_4$. The results are summarized as follows.

**Theorem 35.** *(Hakimi, Schmeichel [67]) Let $n \geq 6$. Then*

$$f_{\mathcal{P}}(n, C_3) = 3n - 8.$$

*The value is attained if and only if the planar graph is an $n$-vertex Apollonian network, where an Apollonian network is a maximal planar graph obtained from $K_3$ by* *recursively placing a vertex of degree 3 inside a face and joining the new vertex to the three vertices incident to that face.*

**Theorem 36.** *(Hakimi, Schmeichel [67], Almeddine [1]) Let $n \geq 5$. Then*

$$f_{\mathcal{P}}(n, C_4) = \frac{1}{2}(n^2 + 3n - 22).$$

*For $n \notin \{7, 8\}$ the value is attained if and only if the planar graph is $F_n := P_{n-2} + K_2$ (see Figure 1.8). For $n = 7$ and $n = 8$ the values are attained if and only if the planar graphs are $F_7$ or $F_7'$ and $F_8$ or $F_8'$ respectively (see Figure 1.8).*

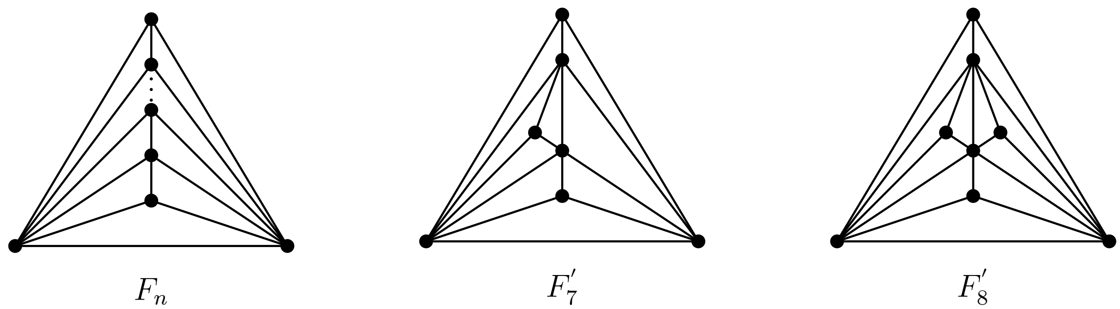


Figure 1.8: Maximal planar graphs maximizing the number of 4-cycles.

In the same paper Hakimi and Schmeichel [67] determined an upper bound for $f_{\mathcal{P}}(n, C_5)$ and posed their conjecture that $f_{\mathcal{P}}(n, C_5) = 2n^2 - 10n + 12$. Recently, Győri, Paulos, Salia, Tompkins, Zamora [62] confirmed Hakimi and Schmeichel's fourty-year-old conjecture and characterized the extremal planar graphs attaining the values. The results are as follow.

**Theorem 37.** *(Hakimi, Schmeichel [67]) For $n \geq 8$ vertices,*

$$f_{\mathcal{P}}(n, C_5) \leq 5n^2 - 26n.$$

**Theorem 38.** *(Győri, Paulos, Salia, Tompkins, Zamora [62])*

$$f_{\mathcal{P}}(n, C_5) = \begin{cases} 6, & \text{if } n = 5; \\ 41, & \text{if } n = 7; \\ 2n^2 - 10n + 12, & \text{if } n = 6 \text{ and } n \geq 8. \end{cases}$$

*Moreover, the planar graphs containing $f_{\mathcal{P}}(n, C_5)$ copies of $C_5$ are the maximal planar graph $D_n := C_{n-2} + \bar{K}_2$, see Figure 1.9. When $n = 8$ or $n = 11$ the graphs $D_8'$ and $D_{11}'$ respectively (see Figure 1.9) also contain $f_{\mathcal{P}}(n, C_5)$ copies of $C_5$ as a subgraph .*

Concerning longer cycles, Hakimi and Schmeichel [67] determined the order of magnitude of $f_{\mathcal{P}}(n, C_k)$, $k \geq 3$.

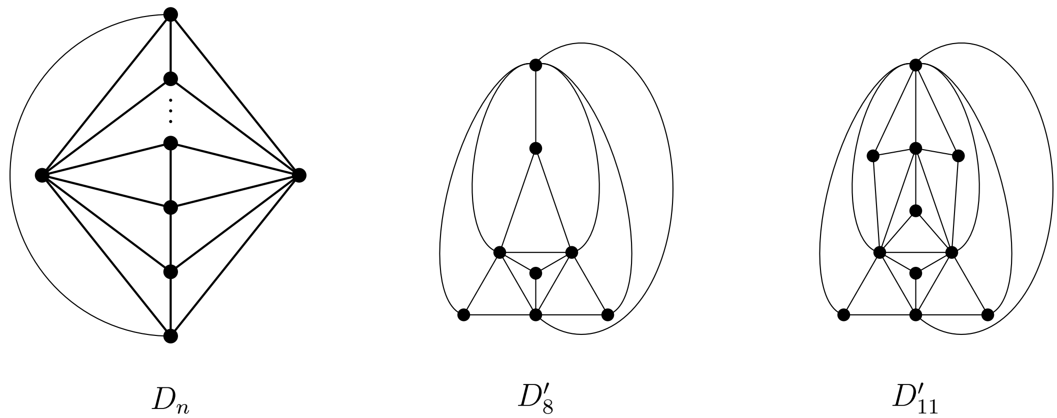



Figure 1.9: Maximal planar graphs maximizing the number of 5-cycles.

**Theorem 39.** *(Hakimi, Schmeichel [67]) For $k \geq 3$, $f_{\mathcal{P}}(n, C_k) = \Theta\left(n^{\lfloor k/2 \rfloor}\right)$.*

For $k \geq 6$, the lower bound is attained by taking a cycle $C_k$ and blowing up a maximum sized independent set of vertices of the cycle with balanced independent vertices (see Figure 1.10). Note that the constant in the asymptotic may depend on $k$, and this construction contains asymptotically $\left(\frac{2n}{k}\right)^{\lfloor k/2 \rfloor}$ copies of $C_k$.

Very recently, Cox and Martin [22, 23] introduced a general technique which allows one to bound $f_{\mathcal{P}}(n, H)$ whenever $H$ exhibits a particular subdivision structure. Using the technique, the authors established best asymptotic bounds for generalized planar Turán number of short cycles, in particular $C_6, C_8, C_{10}$ and $C_{12}$. Furthermore, they also obtained an upper bound for longer even cycles and pose their conjecture. Their results are summarized as follows:

**Theorem 40.** *(Cox, Martin [22, 23])*

1. $f_{\mathcal{P}}(n, C_{2m}) = \left(\frac{n}{m}\right)^m + o(n^m)$, $m \in \{3, 4, 5, 6\}$.
2. $f_{\mathcal{P}}(n, C_{2m}) \leq \frac{n^m}{m!} + o(n^m)$, $m \geq 7$.

**Conjecture 3.** *(Cox, Martin [22, 23])*

$$f_{\mathcal{P}}(n, C_{2m}) = \left(\frac{n}{m}\right)^m + o(n^m), \; m \geq 7.$$

The lower bound for $f_{\mathcal{P}}(n, C_{2m})$, $m \geq 3$ is attained by taking a cycle $C_{2m}$ and blowing up a maximum sized independent set of vertices of the cycle by $n/m$ vertices (see Figure 1.10).

From our informal discussions we had while working on the generalized planar Turán number of $C_5$, we propose the following conjecture concerning generalized planar Turán number of longer odd cycles.

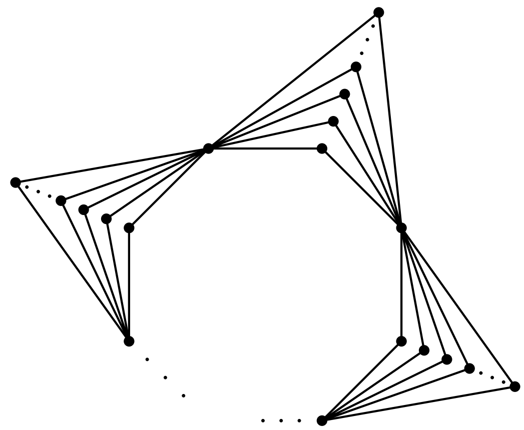


Figure 1.10: A planar graph giving maximum number of $k$-cycles in the sense of order of magnitude.

**Conjecture 4.** *(Győri, Paulos, Salia, Tompkins, Zamora [62])*

$$f_{\mathcal{P}}(n, C_{2m+1}) = (2m)\left(\frac{n}{m}\right)^m + o(n^m),\ m \geq 3.$$

The bound in Conjecture 4 is attainable. Indeed, take a cycle $C_{2m}$ and blowup a maximum sized independent set of vertices of the cycle with roughly by $n/m$ vertices and place a spanning path inside each blowup sets (see Figure 1.11). It can be checked that the construction contains roughly $(2m)\left(\frac{n}{m}\right)^m$ cycles of length $(2m+1)$.

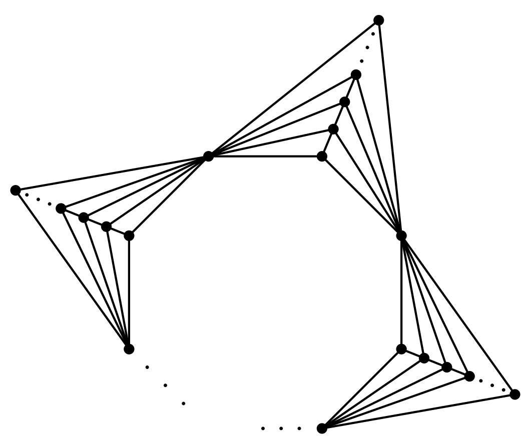


Figure 1.11: A planar graph verifying the attainability of the bound in Conjecture 4.

Following Hakimi and Schmeichel's initial results, investigation of such an extremal problem was further extended by Alon and Caro [2]. The authors determined $f_{\mathcal{P}}(n, H)$ exactly, where $H$ is a planar complete bipartite graph with size of the smaller part either 1 or 2, i.e., $H = K_{1,k}$ and $K_{2,k}$. In the same paper, they also determined the exact value when $H = K_4$. This problem is also addressed independently by Wood in [100].

**Theorem 41.** *(Alon, Caro [2]) For all $k \geq 2$ and $n \geq 4$,*

$$f_{\mathcal{P}}(n, K_{1,k}) = 2\binom{n-1}{k} + 2\binom{3}{k} + (n-4)\binom{4}{k}.$$

**Theorem 42.** *(Alon, Caro [2]) For all $k \geq 2$ and $n \geq 4$,*

$$f_{\mathcal{P}}(n, K_{2,k}) = \begin{cases} \binom{n-2}{k}, & \text{if } k \geq 5 \text{ or } k = 4 \text{ and } n \neq 6; \\ 3, & \text{if } (k,n) = (4,6); \\ \binom{n-2}{3}, & \text{if } k = 3,\ n \neq 6; \\ 12, & \text{if } (k,n) = (3,6); \\ \binom{n-2}{2} + 4n - 14, & \text{if } k = 2. \end{cases}$$

**Theorem 43.** *(Alon, Caro [2], Wood [100]) For $n \geq 3$,*

$$f_{\mathcal{P}}(n, K_4) = n - 4.$$

Resolving a conjecture attributed to Perles in [2], Wormald [101] proved that every 3-connected graph $H$ occurs at most $c_H n$ times in an $n$-vertex planar graph for some constant $c_H$ depending on $H$ (this result was proved again in a different approach by Eppstein [32]). A simple argument shows that graphs with at least 3 vertices which are at most 2-connected will occur at least quadraticly many times in a planar graph. Thus the preceding result of Wormald and Eppstein provides a characterization of graphs which can occur at most $O(n)$ times in a planar graph.

Concerning generalized planar Turán number of paths, recently Grzesik, Győri, Paulos, Salia, Tompkins and Zamora [57], determined exactly the value of $f_{\mathcal{P}}(n, P_4)$. In other result Ghosh, Győri, Martin, Paulos, Salia, Xiao, and Zamora [51] and Cox and Martin [22] determined best possible asymptotic values of $f_{\mathcal{P}}(n, P_5)$ and $f_{\mathcal{P}}(n, P_7)$ respectively. The details of our results on generalized planar Turán number of 3-path and 4-path are given in the fourth chapter.

### 1.3.5 Induced generalized planar Turán numbers

For a graph $H$, an extremal graph $G$ which attains $f_{\mathcal{P}}(n, H)$ copies of $H$ with majority of copies of $H$ in $G$ are not induced subgraphs. For this, it is natural to ask the induced copies of $H$ in a planar graph. We define an induced generalized planar Turán number of a given planar graph $H$ as follows:

**Definition 21.** *Let $H$ be a planar graph and $n$ be a positive integer. We denote the maximum number of induced copies of $H$ in an $n$-vertex planar graph by $f_{\mathcal{P}}^{\text{ind}}(n, H)$. i.e,*

$$f_{\mathcal{P}}^{\text{ind}}(n, H) = \max\{\mathcal{N}_{\text{ind}}(H, G) : G \text{ is an } n\text{-vertex planar graph}\}.$$

*Where $\mathcal{N}_{\text{ind}}(H, G)$ is the number of induced subgraphs of $G$ isomorphic to $H$. The value $f_{\mathcal{P}}^{\text{ind}}(n, H)$ is called **induced generalized planar Turán number** of $H$.*

The problem of maximizing the number of induced copies of a fixed small graph $H$ has attracted a lot of attention recently, see, for example, [42, 73, 94]. Morrison and Scott determined the maximum possible number of induced cycles, without restriction on length, that can be contained in a graph on $n$ vertices [92]. The maximal number of induced complete bipartite graphs and induced complete $r$-partite subgraphs have also been studied [11, 13, 18]. The problem of determining the maximum number of induced $C_5$'s has been elusive for a long time and was finally solved by Balogh, Hu, Lidický and Pfender [7].

Unlike the generalized planar Turán number problems, the induced version was not well studied. It is obvious that for a given graph $G$, every $C_3$ or $K_4$ contained in $G$ is an induced subgraph. We have the following two results deducted from their generalized planar Turán number values.

**Theorem 44.** *(Hakimi, Schmeichel [67], Wood [100]) Let $n \geq 3$. Then*

$$f_{\mathcal{P}}^{\mathrm{ind}}(n, K_4) = n - 4.$$

**Theorem 45.** *(Hakimi, Schmeichel [67]) For $n \geq 6$,*

$$f_{\mathcal{P}}^{\mathrm{ind}}(n, C_3) = 3n - 8.$$

Recently, Savery [95] addressed the exact value of $f_{\mathcal{P}}^{\mathrm{ind}}(n, C_4)$ for sufficently large $n$ and characterize the extremal constructions.

**Theorem 46.** *(Savery [95]) For large $n$,*

$$f_{\mathcal{P}}^{\mathrm{ind}}(n, C_4) = \frac{(n^2 - 5n + 6)}{2}.$$

*Moreover, for large $n$, the only $n$-vertex planar graph which contains $f_{\mathcal{P}}^{\mathrm{ind}}(n, C_4)$ induced 4-cycles is the complete bipartite graph $K_{2,n-2}$.*

As mentioned earlier, Cox and Martin [22, 23] showed that for $m \in \{3, 4, 5, 6\}$, the maximum number of (not necessarily induced) $2m$-cycles in an $n$-vertex planar graph is $f_{\mathcal{P}}(n, C_{2m}) = \left(\frac{n}{m}\right)^m + o(n^m)$. A construction attaining this bound can obtained by taking an $2m$-cycle and blowing up every second vertex of the cycle by roughly $n/m$ vertices. The resulting graph contains $\left(\frac{n}{m}\right)^m + o(n^m)$, $2m$-cycles and all of them are induced, which shows the maximum number of induced $2m$-cycles is $\left(\frac{n}{m}\right)^m + o(n^m)$. Here is a summary of Cox and Martin results in the view of induced generalized planar Turán number.

**Theorem 47.** *(Cox, Martin [22, 23])*

$$f_{\mathcal{P}}^{\mathrm{ind}}(n, C_{2m}) \approx f_{\mathcal{P}}(n, C_{2m}), \;\; m \in \{3, 4, 5, 6\}.$$

Motivated by our results on generalized planar Turán number of the 5-cycle, we further studied its induced version. In [50], we determined the exact value of $f_{\mathcal{P}}^{\mathrm{ind}}(n, C_5)$ for sufficiently large $n$. Details of the result are in the fifth chapter.

Finally, it is worth mentioning that plenty of related planar extremal graph problems were addressed in various papers. We refer [5, 9, 20, 68, 69, 74, 75, 91] for further results and problems on the area.

## 1.4 Wiener index

**Definition 22.** *Let $G$ be a connected graph[10]. The Wiener index of $G$, denoted by $W(G)$, is defined as*

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v),$$

*where $d_G(u, v)$ is the distance between the vertices $u$ and $v$ in the graph $G$.*

The Wiener index was first introduced by Wiener in 1947, while studying its correlations with boiling points of paraffin considering its molecular structure [99]. Since then, it has been one of the most frequently used topological indices in chemistry, as molecular structures are usually modelled as undirected graphs.

Obtaining sharp and asymptotically sharp bounds and characterizing extremal structures are among wide varieties of previous and ongoing studies related to Wiener index. Lovász [88] and Plesnĺk [93] determined a classical result concerning an upper bound of Wiener index of a graph.

**Theorem 48.** *(Lovász [88], Plesnĺk [93])*
*If $G$ is a connected graph of order $n$, then*

$$W(G) \leq \frac{(n-1)n(n+1)}{6}.$$

*Moreover, equality holds if and only if $G$ is a path.*

Many sharp or asymptotically sharp bounds of $W(G)$ in terms of other graph parameters are known, for instance, minimum degree [8, 26, 70], connectivity [44, 78], edge-connectivity [27, 28] and maximum degree [45].

One can study the Wiener index of the family of connected planar graphs. Since the bound in Theorem 48 is attained by a path, it is natural to ask the same question for some particular family of planar graphs. For instance, the Wiener index of a maximal planar graph with $n$ vertices, $n \geq 3$, has a sharp lower bound $(n-2)^2 + 2$. This bound is attained by any maximal planar graph such that the distance between

[10] $G$ can be any connected graph (not necessarily planar).

any pair of vertices is at most 2 (for instance a planar graph containing an $n$-vertex star).

Che and Collins [21], and independently Czabarka, Dankelmann, Olsen and Székely [25], gave sharp upper bound of Wiener index of particular class of maximal planar graphs, namely an Apollonian networks, recall the definition of an Apollonian network in Theorem 35.

**Theorem 49.** *(Che, Collins [21], Czabarka, Dankelmann, Olsen, Székely [25]) Let $G$ be an Apollonian network of order $n \geq 3$. Then*

$$W(G) \leq \left\lfloor \frac{1}{18}(n^3+3n^2) \right\rfloor = \begin{cases} \frac{1}{18}(n^3+3n^2), & \text{if } n \equiv 0 \pmod 3; \\ \frac{1}{18}(n^3+3n^2-4), & \text{if } n \equiv 1 \pmod 3; \\ \frac{1}{18}(n^3+3n^2-2), & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

It has been shown explicitly in [21] that the bound in Theorem 49 is attained by an Apollonian network $T_n$, which is defined below.

**Definition 23.** *The Apollonian network $T_n$ is the maximal planar graph on $n \geq 3$ vertices, with the following structure, see Figure 1.12.*

*If $n$ is a multiple of 3, then the vertex set of $T_n$ can be partitioned in three sets of same size, $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_k\}$ and $C = \{c_1, c_2, \dots, c_k\}$. The edge set of $T_n$ is the union of following three sets $E_1 = \bigcup_{i=1}^{k}\{a_ib_i, b_ic_i, c_ia_i\}$ forming concentric triangles, $E_2 = \bigcup_{i=1}^{k-1}\{a_ib_{i+1}, a_ic_{i+1}, b_ic_{i+1}\}$ forming red edges, and $E_3 = \bigcup_{1}^{k-1}\{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}\}$ forming paths in each vertex class, see Figure 1.12(a). Note, that there are two triangular faces $a_1, b_1, c_1$ and $a_k, b_k, c_k$.*

*If $3|(n-1)$, then $T_n$ is the Apollonian network which may be obtained from $T_{n-1}$ by adding a degree three vertex in the face $a_1, b_1, c_1$ or $a_{\frac{n-1}{3}}, b_{\frac{n-1}{3}}, c_{\frac{n-1}{3}}$, see Figure 1.12(b). Note that both graphs are isomorphic.*

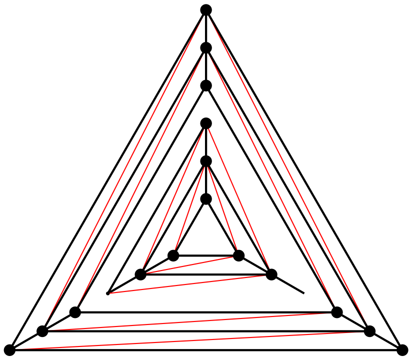
*If $3|(n-2)$, then $T_n$ is the Apollonian network which may be obtained from $T_{n-2}$ by adding a degree three vertex in each of the faces $a_1, b_1, c_1$ and $a_{\frac{n-1}{3}}, b_{\frac{n-1}{3}}, c_{\frac{n-1}{3}}$, see Figure 1.12(c).*

The authors in [21] also conjectured that the bound holds for every maximal planar graph. It has been shown in [25] that the conjectured bound holds asymptotically.

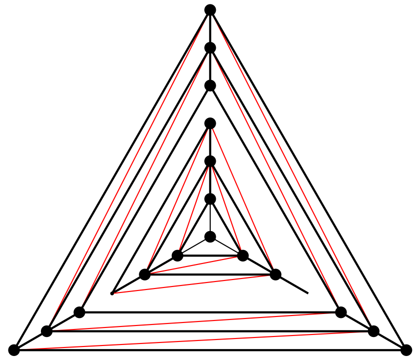
**Theorem 50.** *(Czabarka, Dankelmann, Olsen, Székely [25]) For $k \in \{3, 4, 5\}$, there exists a constant $C_k$ such that*

$$W(G) \leq \frac{1}{6k}n^3 + C_kn^{5/2}$$

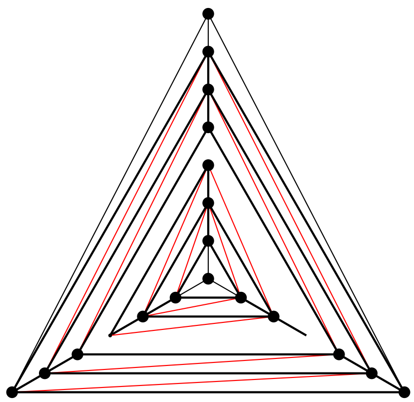
*for every $k$-connected maximal planar graph $G$ of order $n$.*



(a) $3 \mid n$



(b) $3 \mid (n-1)$



(c) $3 \mid (n-2)$

Figure 1.12: Apollonian networks maximizing Wiener index of maximal planar graphs.

Recently, Ghosh, Győri, Paulos, Salia and Zamora [53] confirmed the conjecture and characterize the extremal constructions.

**Theorem 51.** *(Ghosh, Győri, Paulos, Salia, Zamora [53])*
*Let $G$ be an $n \geq 6$ vertex maximal planar graph. Then*

$$W(G) \leq \left\lfloor \frac{1}{18}(n^3+3n^2) \right\rfloor = \begin{cases} \frac{1}{18}(n^3+3n^2), & \textit{if } n \equiv 0 \ (\textit{mod } 3); \\ \frac{1}{18}(n^3+3n^2-4), & \textit{if } n \equiv 1 \ (\textit{mod } 3); \\ \frac{1}{18}(n^3+3n^2-2), & \textit{if } n \equiv 2 \ (\textit{mod } 3). \end{cases}$$

*Equality holds if and only if $G$ is isomorphic to $T_n$ for all $n \geq 9$.*

In other result, Győri, Paulos and Xiao [64] confirmed the conjecture of Czabarka, Dankelmann, Olsen, and Székely [25] about the exact sharp upper bound of Wiener index of quadrangulation graph. Details of our result are found in the last chapter.

# Chapter 2

# Planar Turán Number of $\Theta_6$

## 2.1 Introduction

Recall that for $k \geq 4$, $\Theta_k$ is a family of distinct[1] $\Theta_k$-graphs, where a $\Theta_k$-graph is a graph obtained by joining a pair of non-consecutive vertices of a $k$-cycle with an edge. For instance $\Theta_6 = \{\Theta_6^1, \Theta_6^2\}$, where $\Theta_6^1$ and $\Theta_6^2$ are the symmetric and asymmetric $\Theta_6$-graphs shown in Figure 2.1 (left) and (right) respectively. The size of $\Theta_k$ is $\lfloor k/2 \rfloor - 1$.

$\Theta_6^1$ $\Theta_6^2$

Figure 2.1: $\Theta_6$-graphs

Recently in [54] we improved the bound of $\mathrm{ex}_{\mathcal{P}}(n, \Theta_6)$ in Theorem 31 with sharp upper bound. Our main results are as follows.

**Theorem 52.** *(Ghosh, Győri, Paulos, Xiao, Zamora [54])*
*Let $n \geq 6$. If $G$ is an $n$-vertex 2-connected $\Theta_6$-free planar graph with $\delta(G) \geq 3$, then $e(G) \leq \frac{18}{7}n - \frac{48}{7}$.*

**Theorem 53.** *(Ghosh, Győri, Paulos, Xiao, Zamora [54])*

$$\mathrm{ex}_{\mathcal{P}}(n, \Theta_6) \leq \frac{18}{7}n - \frac{48}{7}, \quad \textit{for all } n \geq 14.$$

[1] In the sense of isomorphism

In the next section, we illustrate why the bounds in Theorem 52 and Theorem 53 are sharp. But first, we recall some terminology and notations we are about to use next. For a plane graph $G$, $f(G), v(G)$ and $e(G)$ are number of faces, vertices and edges of $G$ respectively. We call a graph $G$ as $\Theta_k$-free if it contains no $\Theta_k$-graph as a subgraph. We call $G$ contains $\Theta_k$ if there is a $\Theta_k$-graph contained in $G$ as a subgraph.

## 2.2 An extremal construction for PTN of $\Theta_6$

In this section we show the existence of infinitely many integer $n$ and a $\Theta_6$-free planar graph $G$ on $n$ vertices such that $e(G) = \frac{18}{7}n - \frac{48}{7}$. This is done based on Dowden's construction [31] (with some modifications) which was illustrated while showing the bound $\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$ is sharp.

The following lemma, due to Dowden [31], plays the central role in obtaining the construction. For completeness, Dowden's proof of the lemma is included.

**Lemma 4.** *(Dowden [31])*
*For infinitely many values of $k$, there exists a plane triangulation $T_k$ with vertex set $\{v_1, v_2, \cdots, v_k\}$ satisfying*

1. $d(v_i) = 4$ *for* $i \leq 6$,
2. $d(v_i) = 6$ *for* $i > 6$,
3. $E(T_k) \supset \{v_1v_2, v_3v_4, v_5v_6\}$.

*Proof.* Note first that the triangulation $T_6$ shown in Figure 2.2 certainly satisfies the conditions for $k = 6$. We now proceed inductively. Given a triangulation satisfying

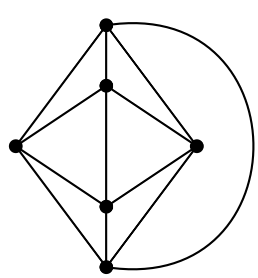


Figure 2.2: The triangulation $T_6$

the conditions, let us construct a larger triangulation by subdividing all the edges and inserting triangles between the new vertices, as shown in Figure 2.3. It can be observed that conditions (1) and (2) will also be satisfied by the new triangulation, but (due to the subdividing of edges) not condition (3). However, we may

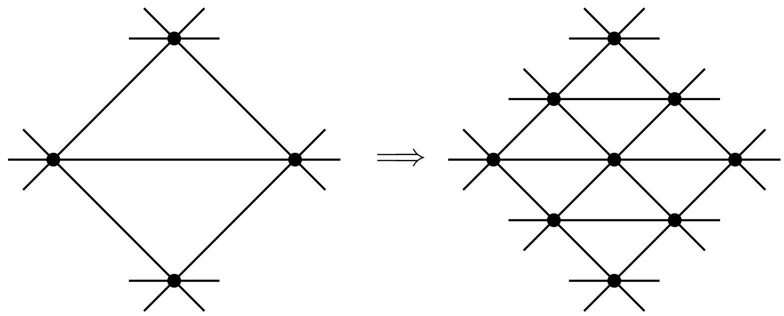


Figure 2.3: Constructing a larger triangulation

then simply modify the new triangulation into one that does satisfy all three conditions by applying the local transformation shown in Figure 2.4 (which includes some relabelling of the vertices) at the relevant three places. $\square$

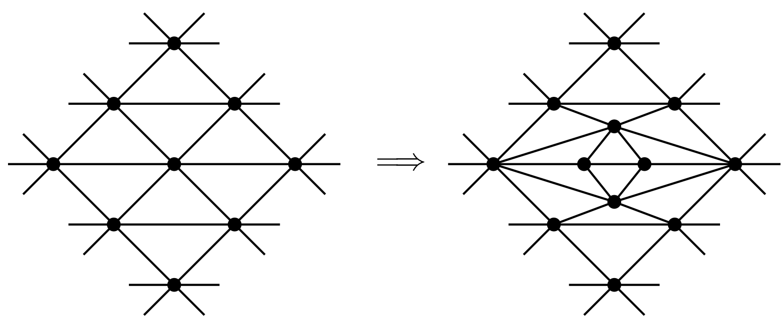


Figure 2.4: Modifying the triangulation to satisfy condition (3)

For instance the case of $T_{15}$ is shown in Figure 2.5. The red edges shown in the figure are those indicated in (3) of Lemma 4.

**Theorem 54.** *(Ghosh, Győri, Paulos, Xiao, Zamora [54])*
*There exist infinitely many integer $n$ and an $n$-vertex $\Theta_6$-free planar graph $G$ such that $e(G) = \frac{18}{7}n - \frac{48}{7}$.*

*Proof.* We use the triangulation $T_k$ which is obtained from Lemma 4 to prove the theorem. See Figure 2.5 for the case when $k = 15$. Let $E^*$ denote the set edges $\{v_1v_2, v_3v_4, v_5v_6\}$ as stated in the lemma. For the example in Figure 2.5, $E^*$ is the set of the red edges. We construct the base graph $G_k$ (with $4k + 12$ vertices) from $T_k$ with the following procedures.

1. Subdivide all edges in $E(T_k)\backslash E^*$. Notice that since $T_k$ is a maximal plane graph with $k$ vertices, then $e(T_k) = 3k - 6$. Thus, the number of subdividing vertices is $3k - 9$.

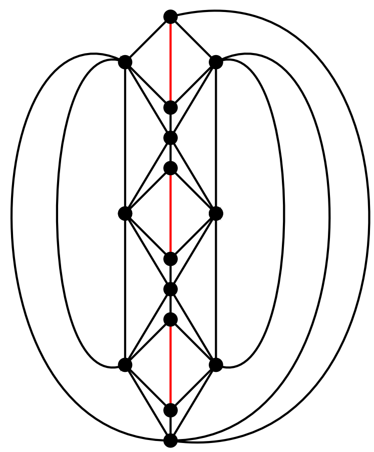


Figure 2.5: An example of Lemma 4 with 15 vertices.

2. Replace all edges in $E^*$ with the "Diamond holder" shown in Figure 2.6 (b, bottom). Denote the newly obtained plane graph by $G_k$. The case of $G_{15}$ is shown in Figure 2.6 (a). Notice that all the $k$ vertices of the $T_k$ in the $G_k$ are degree 6 vertices (including the six vertices that only had degree 4 in $T_k$). For instance, all the red vertices in $G_{15}$, see Figure 2.6 (a), are the 15 vertices of $T_{15}$ and each of this vertex is of degree 6 in $G_{15}$. We can consider all the $k$ vertices of $T_k$ as centre of the star $K_{1,6}$ in $G_k$.

Next, we construct $G$ using the base graph $G_k$ as follows. Replace each star, $K_{1,6}$, with a "snow flake" shown in Figure 2.6 (b, top), where the central vertex of the star is replaced by a hexagon and the edges are replaced by $K_5^-$'s. Let the graph we obtained be $G$ and containing $n$ vertices. Notice that $G$ contains $k$ snowflakes and 3 diamond holders.

Now we count the number of vertices of $G$ in terms of $k$. Notice the number of vertices of a snow lake except the tip vertices of the six $K_5^-$ is 18. Thus, $G$ has $18k$ such vertices. The remaining vertices of $G$ are the subdividing vertices, which is $3k-9$, and 21 vertices of the three diamond holders. Thus, $n = 18k+(3k-9)+21 = 21k+12$. This implies, $k = \frac{n-12}{21}$.

Let us now compute the number of edges in $G$. It can be checked that each snowflake contains 54 edges. The remaining edges are the 8 edges that appear in the interior (except the two hanging edges) of each diamond holder. Thus, $e(G) = 54k + 24$. Therefore, using the two results we get $e(G) = \frac{18}{7}n - \frac{48}{7}$.

Notice that each face of $G$ is with size either 3 or 6. Moreover, a 6-face in $G$ is either with all its six edges incident to a $K_5^-$(see the snowflake in Figure 2.6 (b, top)) or 4 consecutive edges incident to $K_5^-$'s and the remaining 2-path incident to a $K_5^{--}$ as shown in Figure 2.8 ($B_{5,b}$). Therefore, $G$ is $\Theta_6$-free. □

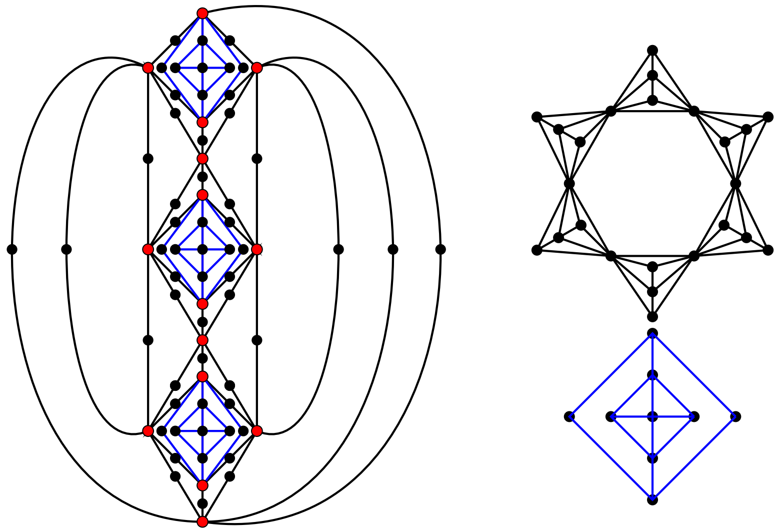



Figure 2.6: Construction of base graph $G_{15}$ from $T_{15}$.

## 2.3 Preliminaries

**Definition 24.** *Let $G$ be a plane graph and $e \in E(G)$. If $e$ is not in a 3-face of $G$, then we call it as a trivial block. Otherwise, we recursively construct a triangular-block in the following way. Start with $H$ as a subgraph of $G$, such that $E(H) = \{e\}$.*

1. *Add the other edges of the 3-face containing $e$ to $E(H)$.*
2. *Take $e' \in E(H)$ and search for a 3-face containing $e'$. Add these other edge(s) in this 3-face to $E(H)$.*
3. *Repeat step 2 till we cannot find a 3-face for any edge in $E(H)$.*

*We denote the triangular-block obtained from $e$ as the starting edge, by $B(e)$.*

Let $G$ be a plane graph. We have the following three observations:

i. If $H$ is a non-trivial triangular-block and $e_1, e_2 \in E(H)$, then $B(e_1) = B(e_2) = H$.

ii. Any two triangular-blocks of $G$ are edge disjoint.

iii. If $B$ is a triangular-block with the unbounded region being a 3-face, then $B$ is a triangulation graph.

Let $\mathcal{B}$ be the family of all triangular-blocks of $G$. We have $e(G) = \sum\limits_{B\in\mathcal{B}} e(B)$, where $e(B)$ is number of edges of $B$. We may call $e(B)$ as the contribution of $B$ to the number of edges in $G$.

**Definition 25.** *Let $G$ be a plane graph. A vertex $v$ in $G$ is called a* ***junction vertex*** *if it is shared by at least two triangular-blocks of $G$.*

**Definition 26.** *Let $B$ be a triangular-block in a plane graph $G$. The contribution of $B$ to the number of vertices in $G$, denoted by $\hat{v}(B)$, is defined as*

$$\hat{v}(B) = \sum_{v\in V(B)} \frac{1}{\#\ \textit{triangular-blocks sharing } v}.$$

For a plane graph $G$ and $\mathcal{B}$, the set of all triangular-blocks of $G$, one can see that $v(G) = \sum\limits_{B\in\mathcal{B}} \hat{v}(B)$. The following lemma describes the number of vertices that a possible triangular-block of a $\Theta_6$-free plane graph may contain.

**Lemma 5.** *Let $G$ be a plane graph containing no $\Theta_6$. Then every triangular-block of $G$ contains at most 5 vertices.*

*Proof.* We prove it by contradiction. Let $B$ be a triangular-block of $G$ containing at least 6 vertices. Since the triangular-block is obtained recursively starting with an edge and adding vertex and edges, then there is a triangular-block of size 6 which is contained in $B$. Let this triangular-block be $B'$. We consider the following two cases to complete the proof.

**Case 1:** $B'$ contains a separating triangle. Let $x_1x_2x_3$ be a separating triangle in $B'$. Without loss of generality, assume that the inner region of the triangle contains two vertices say, $x_4$ and $x_5$. The outer region of the triangle contains one vertex, say $x_6$. Since $B - x_6$ is a maximal plane graph on 5 vertices, then it is unique. Without loss of generality, let the graph be as shown in the Figure 2.7(left). Now consider the vertex $x_6$. If $x_2, x_3 \in N(x_6)$, then $x_1x_4x_5x_2x_6x_3x_1$ is a $\Theta_6$-graph and is in $G$, which is a contradiction to the fact that $G$ is $\Theta_6$-free. Similarly for the cases when $x_1, x_2 \in N(x_6)$ and $x_1, x_3 \in N(x_6)$ .

**Case 2:** $B'$ contains no separating triangle. Let $x_1x_2x_3$ be a triangular face in $B'$. Let $x_4$ be a vertex in the triangular-block such that $x_2x_3x_4$ is a 3-face. Notice that $x_1x_4 \notin E(B')$, otherwise, $B'$ contains a separating triangle. Without loss of generality, let $x_5$ be a vertex in $B'$ such that $x_2x_4x_5$ is a 3-face. Notice that $x_6$ cannot be adjacent to $\{x_1, x_2\}, \{x_2, x_5\}, \{x_5, x_4\}, \{x_4, x_3\}$ or $\{x_1, x_3\}$. Otherwise, it is easy to show a $\Theta_6$-graph in $G$, which is a contradiction. Moreover, $x_3x_5 \notin E(B')$, otherwise $B'$ contains a separating triangle. Thus, $x_1x_5 \in E(B')$ and $x_1, x_5 \in N(x_6)$ (see Figure 2.7(right)). In this case, $x_1x_6x_5x_2x_4x_3x_1$ is a $\Theta_6$-graph contained in $G$, which is a contradiction. □


Figure 2.7: The two cases when either $B$ contains a separating triangle or not respectively.

We list out all possible triangular-blocks (together with their notations) that a given $\Theta_6$-free plane graph may contain.

**Triangular-blocks with 5 vertices**

There are four types of blocks on 5 vertices (See Figure 2.8). Notice that $B_{5,a}$ is a $K_5^-$ .

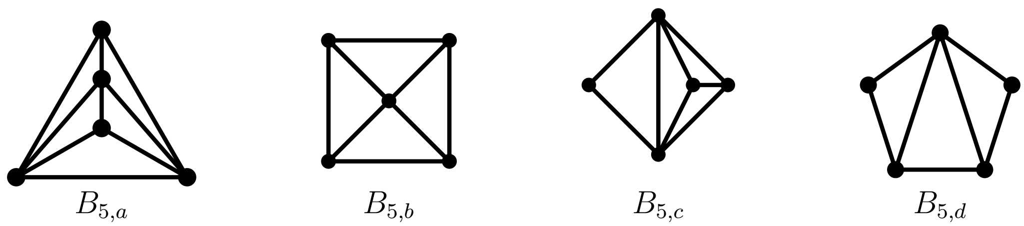



Figure 2.8: Triangular-blocks with 5 vertices

**Triangular-blocks with 2, 3 and 4 vertices**

The 2-vertex and 3-vertex blocks are simply $K_2$ (trivial block) and $K_3$ (triangle) respectively. There are two triangular blocks on 4 vertices (see the last two graphs in Figure 2.9). Observe that $B_{4,a}$ is a $K_4$.

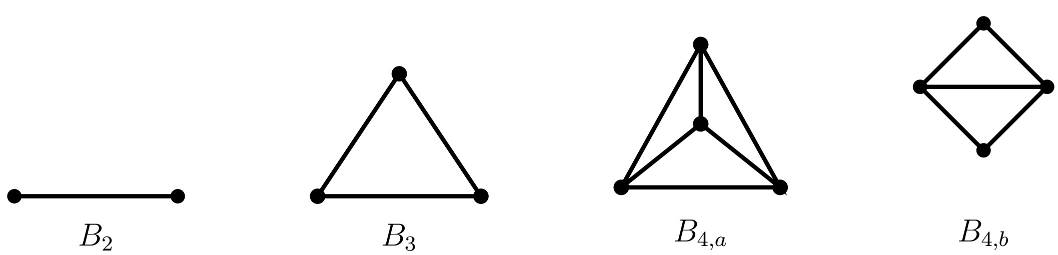



Figure 2.9: Triangular-blocks with 2, 3 and 4 vertices

**Definition 27.** *Let $G$ be a 2-connected plane graph containing at least 2 triangular-blocks. Let $B$ be a triangular-block in $G$. An edge of $B$ is called an **exterior edge**, if it is on a boundary of non triangular face of $G$. Otherwise, we call it as an **interior edge**. A path $P$ in $B$ given by $x_1x_2\dots x_m$, where $x_1$ and $x_m$ are the only junction vertices in $P$ and $x_ix_{i+1}$, is an exterior edge for all $i \in \{1,2,3,\dots,m-1\}$ is called an **exterior path** in $B$. A non triangular face in $G$ for which $P$ is incident with is called **exterior face** of $B$ with exterior path $P$.*

Let $G$ be a 2-connected $\Theta_6$-free plane graph containing at least two triangular-blocks and $\delta(G) \geq 3$. Next, we define the contribution of a triangular-block $B$ in $G$ to the number of face of $G$.

Let $\phi$ be a non-triangular face in $G$ with boundary cycle $x_1x_2x_3\dots x_mx_1$, where $m \geq 4$. We may denote the cycle by $\phi$. Consider the cycle $\phi'$ which is obtained deleting all the vertices and edges in $G$ except vertices and edges of $\phi$.

We construct a cycle $\phi''$ whose size is at most $m$ in the following way. For each triple cherry $x_ix_jx_k$ of the cycle $\phi'$ which is in the same triangular-block (say $B$), $x_ix_k \in E(B)$ and both $x_i$ and $x_k$ are junction vertices of $B$, delete $x_j$ and join $x_i$ and $x_k$ with an edge. We call the cherry $x_ix_jx_k$ as a *bad cherry*, and the cycle $\phi''$ as the *refinement* of $\phi$.

For instance, let the induced subgraph of a plane graph $G$ be as shown in the Figure 2.10 (left). Consider the non-triangular face, say $\phi$, with the boundary cycle $x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}x_{12}x_{13}x_1$ (see the boundary of the shaded region) and triangular-blocks incident to $\phi$. In this case, the bad cherries are $x_7x_6x_5$ and $x_9x_{10}x_{11}$. It can be seen that the refinement $\phi''$ of $\phi$ is $x_1x_2x_3x_4x_5x_7x_8x_9x_{11}x_{12}x_{13}x_1$, which is of size 11.

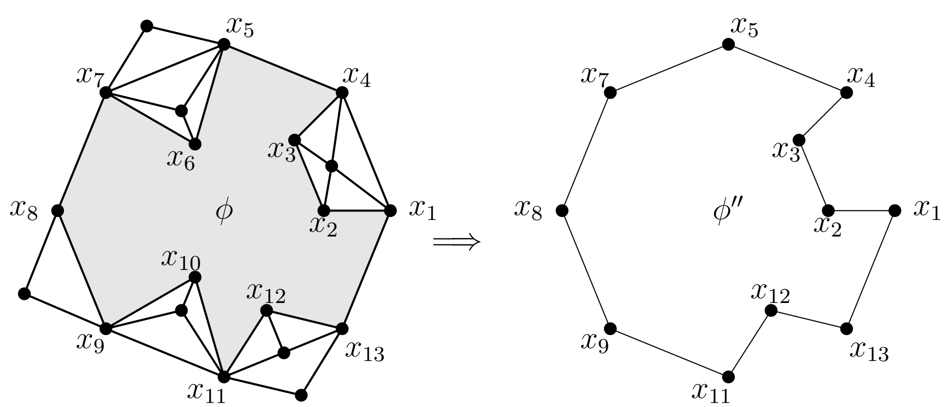



Figure 2.10: An example showing how to compute the size of the refinement of a non-triangular face in a plane graph.

**Definition 28.** *Let $G$ be a 2-connected $\Theta_6$-free plane graph containing at least two triangular-blocks and $\delta(G) \geq 3$. Let $B$ be a triangular-block and $\phi_1, \phi_2, \dots, \phi_m$ be* *exterior faces of the triangular-block. Consider an exterior face $\phi_i$ of $B$ for some $i \in \{1, 2, \ldots, m\}$ with an exterior path $P$ of the triangular-block. We define the contribution of $B$ to the face size of $\phi_i$, denoted by $f_{\phi_i}(B)$, as follows.*

1. *If $P$ is not a bad cherry,*

$$f_{\phi_i}(B) = \frac{\textit{length of } P}{\textit{length of } \phi_i''}.$$

2. *If $P$ is a bad cherry,*

$$f_{\phi_i}(B) = \frac{1}{\textit{length of } \phi_i''}.$$

*Where $\phi_i''$ is the refinement of $\phi_i$.*

*The contribution of the triangular-block $B$ to the number of face of the graph $G$, denoted as $\hat{f}(B)$, is defined as:*

$$\hat{f}(B) = \sum_{i=1}^{m} f_{\phi_i}(B) + (\#\ \textit{triangular faces in } B)\,.$$

Let $G$ be a 2-connected $\Theta_6$-free plane graph containing at least two triangular-blocks and $\delta(G) \geq 3$. For $\mathcal{B}$ be the family of triangular-blocks in $G$ we have, $f(G) = \sum\limits_{B \in \mathcal{B}} \hat{f}(B)$.

## 2.4 Proof of Theorem 52

We begin by outlining our proof. Consider a plane drawing of $G$. Let $\mathcal{B}$ be the family of all triangular-blocks of $G$. The main target of the proof is to show that

$$24f(G) - 17e(G) + 6v(G) \leq 0. \tag{2.1}$$

where $v(G)$ is number of vertices of in $G$ (in this case $n$).

Once we prove (2.1), then using the Euler's Formula, $e(G) = f(G) + v(G) - 2$, we can finish proof of the theorem.

To prove (2.1), we show the existence of a partition $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ of $\mathcal{B}$ such that

$$24 \sum_{B \in \mathcal{P}_i} \hat{f}(B) - 17 \sum_{B \in \mathcal{P}_i} e(B) + 6 \sum_{B \in \mathcal{P}_i} \hat{v}(B) \leq 0, \text{ for all } i \in \{1, 2, 3 \ldots, m\}.$$

Since $f(G) = \sum_{B\in\mathcal{B}} \hat{f}(B)$, $v(G) = \sum_{B\in\mathcal{B}} \hat{v}(B)$ and $e(G) = \sum_{B\in\mathcal{B}} e(B)$ we have,

$$\begin{aligned}24f(G) - 17e(G) + 6v(G) &= 24\sum_{i=1}^{m}\sum_{B\in\mathcal{P}_i}\hat{f}(B) - 17\sum_{i=1}^{m}\sum_{B\in\mathcal{P}_i}e(B) + 6\sum_{i=1}^{m}\sum_{B\in\mathcal{P}_i}\hat{v}(B)\\ &= \sum_{i=1}^{m}\left(24\sum_{B\in\mathcal{P}_i}\hat{f}(B) - 17\sum_{B\in\mathcal{P}_i}e(B) + 6\sum_{B\in\mathcal{P}_i}\hat{v}(B)\right) \leq 0.\end{aligned}$$

To verify the existence of such a partition of triangular-blocks of $G$, we prove sequence of claims about an upper bound of $24\hat{f}(B) - 17e(B) + 6\hat{v}(B)$ for each type of triangular-block $B$ which may possibly contained in $G$.

For simplicity of arguments, we define a function $g : \mathcal{B} \longrightarrow \mathbb{R}$ as:

$$g(B) := 24\hat{f}(B) - 17e(B) + 6\hat{v}(B).$$

**Claim 1.** *Let $B$ be a $B_{5,a}$ triangular-block in $G$. Then $g(B) \leq 0$.*

*Proof.* Let the exterior vertices of $B$ be labeled as $x_1, x_2$ and $x_3$ as shown in Figure 2.11. Since the graph is 2-connected, $B$ contains at least 2 junction vertices. We consider two cases depending on the number of junction vertices of $B$.

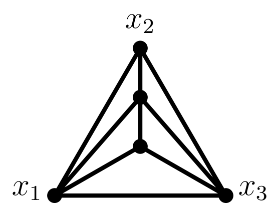



Figure 2.11: $B_{5,a}$ triangular-block

1. $B$ **contains exactly two junction vertices.** We do for the case when $x_2$ and $x_3$ are junction vertices. Similar arguments can be given for other pairs too.

   Let the exterior faces of the exterior edge $x_2x_3$ and the exterior path $x_2x_1x_3$ are respectively $\phi_1$ and $\phi_2$. It is an easy check that the size of $\phi_1$ and $\phi_2$ are at least 6 and 7 respectively. Clearly, the refinement $\phi_1''$ is with size at least 6. Since $x_2x_1x_3$ is a bad cherry, then the refinement $\phi_2''$ is with size at least 6. The number of triangular faces in $B$ is 5. Thus, $\hat{f}(B) \leq 5 + 1/6 + 1/6$. To get the optimal estimate of $\hat{v}(B)$, we may assume that the junction vertices are shared with two triangular-blocks. Thus, $\hat{v}(B) \leq 3 + 1/2 + 1/2$. Using $e(B) = 9$, we get $g(B) \leq -1$.

2. $B$ **contains three junction vertices.** In this case each of the exterior edge has an exterior face whose refinement if of size at least 6. Thus, $\hat{f}(B) \leq 5+3/6$. To get the optimal estimate of $\hat{v}(B)$, we may assume that the junction vertices are shared with two blocks. Thus, $\hat{v}(B) \leq 2 + 3/2$. Therefore using $e(B) = 9$ we obtain $g(B) \leq 0$. □

**Claim 2.** *Let $B$ be a $B_{5,b}$ triangular-block in $G$. Then $g(B) \leq 0$.*

*Proof.* Since the graph contains no cut vertex, the number of junction vertices of the block is at least 2. Hence, $\hat{v}(B) \leq 3 + 1/2 + 1/2$. It can be checked that each of the exterior edge of the triangular-block has an exterior face whose refinement is of size at least 6. Hence $\hat{f}(B) \leq 4 + 4/6$. Using $e(B) = 8$, we get $g(B) \leq 0$. □

**Claim 3.** *Let $B$ be a $B_{5,c}$ triangular-block in $G$. Then $g(B) \leq 0$.*

*Proof.* Let the exterior vertices of the triangular-block be $x_1, x_2, x_3$ and $x_4$ as shown in Figure 2.12. Since $\delta(G) \geq 3$, $x_1$ is a junction vertex. However, $x_3$ may or may not

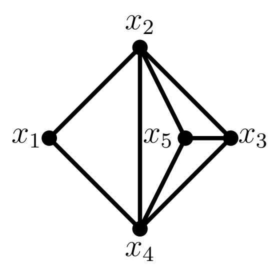



Figure 2.12: $B_{5,c}$ trangular-block

be a junction vertex. So, we distinguish two cases.

1. $x_3$ **is a junction vertex.** Each of the exterior edge of the triangular-block is with an exterior face of size at least 6. Thus, $\hat{f}(B) \leq 4 + 4/6$. Clearly, $\hat{v}(B) \leq 3 + 1/2 + 1/2$ and $e(B) = 8$. Consequently $g(B) \leq 0$.

2. $x_3$ **is not a junction vertex.** In this case either both $x_2$ and $x_4$ are junction vertices or only one of them is a junction vertex. Otherwise, the graph contains a cut vertex. So, we have the following two cases.

   2.1. Only one of the two vertices, $x_2$ or $x_4$, is a junction vertex. Without loss of generality, assume $x_2$ is a junction vertex. It can be checked that, the exterior faces of the exterior edge $x_1x_2$ is of size at least 6. Moreover, the size of the exterior face of the exterior path $x_1x_4x_3x_2$ is at least 8. Thus, $\hat{f}(B) \leq 4 + 1/6 + 3/8$. We can assume that the junction vertices are shared with 2 triangular-blocks to get an upper bound of $\hat{v}(B)$. Hence, $\hat{v}(B) \leq 3 + 1/2 + 1/2$. Therefore, we get $g(B) \leq -3$.

2.2. Both $x_2$ and $x_4$ are junction vertices. In this case, the exterior path $x_2x_3x_4$ has an exterior face of size at least 4. The exterior faces of the exterior edges $x_1x_2$ and $x_1x_4$ have size at least 6. Thus $\hat{f}(B) \leq 4+2/6+1/3$. Since the junction vertices $x_1, x_2$ and $x_4$ are shared with at least two blocks we get $\hat{v}(B) \leq 2 + 3/2$. Therefore, $g(B) \leq -3$. □

**Claim 4.** *Let $B$ be a $B_{5,d}$ triangular-block in $G$. Then $g(B) \leq 0$.*

*Proof.* Let $B$ be with its vertices labeled $x_1, x_2, x_3, x_4$ and $x_5$ as seen in Figure 2.13. Notice that $x_2$ and $x_4$ are junction vertices. It can be checked that each of the

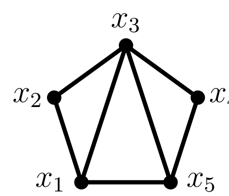



Figure 2.13: $B_{5,d}$ triangular-block

exterior edges of the triangular-block has an exterior face whose refinement is of size at least 6. Thus, $\hat{f}(B) \leq 3+5/6$. We estimate that $\hat{v}(B) \leq 3+1/2+1/2$. Therefore using $e(B) = 7$, we get $g(B) \leq -3$. □

**Claim 5.** *Let $B$ be a $B_{4,a}$ triangular-block in $G$. Then $g(B) \leq 0$.*

*Proof.* Consider $B$ with the exterior vertices labeled $x_1, x_2$ and $x_3$ as shown in Figure 2.14(left). Since the graph does not contain a cut vertex, the block contains at least two junction vertices. So we distinguish two cases.

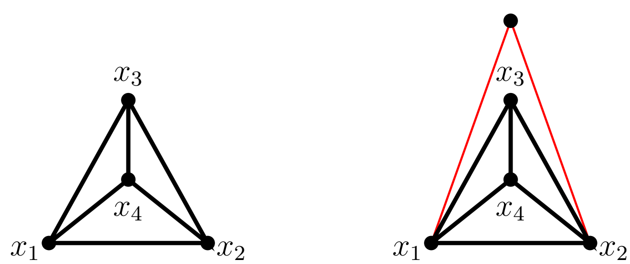



Figure 2.14: $B_{4,a}$ triangular-block

1. **All the three vertices are junction.** In this case each of the three exterior edges are with exterior face whose refinement is of size at least 6. Thus, $\hat{f}(B) \leq 3 + 3/6$. Moreover, $\hat{v}(B) \leq 1 + 3/2$ and $e(B) = 6$. Therefore, $g(B) \leq -3$.

2. **Only two of the vertices are junction.** Without loss of generality, assume that the junction vertices are $x_1$ and $x_2$. In this case, the exterior face of the exterior edge $x_1x_2$ is of size at least 6. However the exterior path $x_1x_3x_2$ is with size at least 4 (see Figure 2.14(right)). So, $\hat{f}(B) \leq 3 + 1/6 + 1/3$. Since, the block has two junction vertices, then $\hat{v}(B) \leq 2 + 1/2 + 1/2$. Using $e(B) = 6$, we get $g(B) \leq 0$. □

**Claim 6.** *Let $B$ be a $B_{4,b}$ triangular-block in $G$. Then $g(B) \leq 3$.*

*Proof.* Consider the triangular-block $B$ with the labeled vertices $x_1, x_2, x_3$ and $x_4$ as seen in Figure 2.15(left). Clearly $x_2$ and $x_4$ are junction vertices. We distinguish two cases.

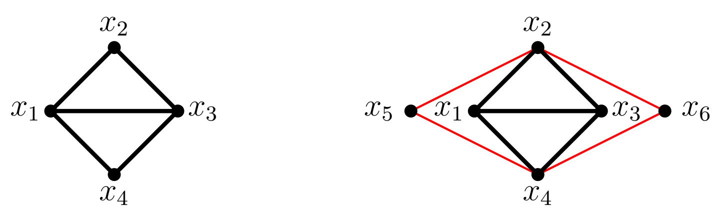



Figure 2.15: $B_{4,b}$ triangular-block

1. **At least one vertex in $\{x_1, x_3\}$ is a junction vertex.** Without loss of generality, assume $x_1$ is a junction vertex. In this case the size of the exterior faces of the exterior edges $x_1x_2$ and $x_1x_4$ are at least 6. On the other hand, we may consider $x_3$ is not a junction vertex to obtain the optimal value. Hence the size of the exterior face of the exterior path $x_2x_3x_4$ is 4. Hence, $\hat{f}(B) \leq 2 + 2/6 + 2/4$. Moreover we have , $\hat{v}(B) \leq 1 + 3/2$ and $e(B) = 5$. Therefore, $g(B) \leq -2$.

2. **Both $x_1$ and $x_3$ are not junction vertices.** In this case the exterior paths $x_2x_3x_4$ and $x_2x_1x_4$ are with exterior face of size either 4 or at least 6. So, we have the following cases.

   2.1. Both are with size at least 6. Here we estimate, $\hat{f}(B) \leq 2 + 4/6$, $\hat{v}(B) \leq 2 + 1/2 + 1/2$. Hence, $g(B) \leq -3$.

   2.2. Only one of the exterior paths is with exterior face of size 4. Without loss of generality assume $x_2x_1x_4$ is with exterior face of size 4. Let the exterior face be $x_2x_1x_4x_5x_2$. Notice that $x_2x_5$ and $x_4x_5$ are trivial blocks. Hence, considering that there is no degree 2 vertex in $G$, either $x_2$ or $x_4$ is shared by at least three triangular-blocks. Thus, $\hat{v}(B) \leq 2 + 2/3$. Moreover considering the size of the exterior faces of the exterior paths $x_2x_3x_4$ and $x_2x_1x_4$ we have $\hat{f}(B) \leq 2 + 2/4 + 2/6$. Therefore, in this case we get $g(B) \leq 0$.

2.3. Both exterior paths are with exterior faces of size 4. In this case, $\hat{f}(B) \leq 2 + 2/4 + 2/4$. Let the exterior faces be with boundary $x_2x_1x_4x_5x_2$ and $x_2x_3x_4x_6x_2$ as shown in the Figure 2.15(right). Observe that the edges $x_2x_5, x_4x_6, x_2x_6$ and $x_4x_6$ are trivial blocks. Thus, $x_2$ and $x_4$ are junction vertices which are shared with at least 3 triangular-blocks. That means, $\hat{v}(B) \leq 2 + 2/3$. Therefore we get $g(B) \leq 3$. □

**Claim 7.** *Let $B$ be a $B_3$ triangular-block in $G$. Then $g(B) \leq 0$.*

*Proof.* Let the triangular-block $B$ be with vertices $x_1, x_2$ and $x_3$ as shown in Figure 2.16(left). Observe that due to the degree condition of $G$, each of the three vertex

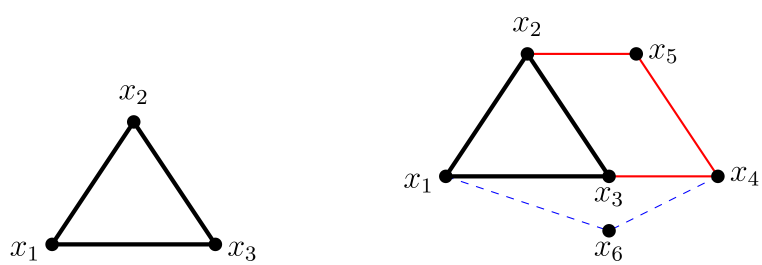



Figure 2.16: $B_3$ triangular-block

is a junction vertex. Moreover, each of the exterior edge is with exterior face of size either exactly 4 or at least 6. We have the following two cases.

1. **At least one of the exterior faces of the triangular-block is of size** 4. Without loss of generality, let $x_2x_3$ is with exterior face of size 4 and boundary of the exterior face be $x_2x_3x_4x_5x_2$ as shown in Figure 2.16 (right). Notice that $x_2x_5$ and $x_3x_4$ are trivial blocks. Otherwise, it is easy to show $G$ contains $\Theta_6$. We consider the following cases.

   1.1. Either $x_1x_2$ or $x_1x_3$ is with exterior face of size 4. Let the exterior face of the edge $x_1x_3$ be with size 4. In this case $x_3$ is a junction vertex shared by at least 3 triangular-blocks. Indeed, suppose the exterior edge $x_3$ is shared by two triangular-blocks only. Thus the boundary of the exterior face of the exterior edge $x_1x_3$ contains the vertex $x_4$. Let the exterior face of the edge be $x_1x_3x_4x_6x_1$ as shown in Figure 2.16(right). Clearly we have a $\Theta_6$-graph with the 6-cycle $x_1x_3x_2x_5x_4x_6x_1$, which is a contradiction. Thus, we estimate $\hat{f}(B) \leq 1 + 1/4 + 1/4 + 1/4$ and $\hat{v}(B) \leq 1/2 + 1/2 + 1/3$. Using $e(B) = 3$, we get $g(B) \leq -1$.

   1.2. Both $x_1x_2$ or $x_1x_3$ are with exterior face of size at least 6. Here we have $\hat{f}(B) \leq 1+1/4+1/6+1/6$, $\hat{v}(B) \leq 1/2+1/2+1/2$. Therefore $g(B) \leq -4$.

2. **All of the exterior edges are with exterior face at least** 6. In this case we have the estimates $\hat{f}(B) \leq 1 + 1/6 + 1/6 + 1/6$, $\hat{v}(B) \leq 1/2 + 1/2 + 1/2$. Hence, $g(B) \leq -6$. □

**Claim 8.** *Let $B$ be a $B_2$ triangular-block in $G$. Then $g(B) \leq 0$.*

*Proof.* Let the end vertices of the triangular-block be $x_1$ and $x_2$ as shown in Figure 2.17(left). The two exterior faces of the triangular-block are with size at least 4. We consider two cases.


Figure 2.17: $B_2$ triangular-block

1. **Both faces are with size at least** 5. In this case $\hat{f}(B) \leq 1/5 + 1/5$, $\hat{v}(B) \leq 1/2 + 1/2$. Using $e(B) = 1$, we get $g(B) \leq -7/5$.

2. **One of the two exterior faces is of size** 4. Let the exterior face with size 4, call it $\phi_1$, be with 4-cycle $x_1x_2x_3x_4x_1$ as shown in Figure 2.17(right). Notice that the other exterior face of the trivial block, call it $\phi_2$, is of size at least 5. Otherwise, it is easy to check that there is a $\Theta_6$-graph in $G$. It is clear that the 2-paths $x_1x_4x_3$ or $x_4x_3x_2$ (but not both) can be a bad cherry such that the refinement of $\phi_1$ is 3. We distinguish the following cases.

   2.1 The refinement of $\phi_1$ is 4. Thus, $\hat{f}(B) \leq 1/4+1/5$. In this case either $x_1x_4$ or $x_2x_3$ is a trivial block. Otherwise, it is easy to show that $G$ contains $\Theta_6$. Thus either $x_1$ or $x_2$ is shared with at least 3 triangular-blocks considering that $\delta(G) \geq 3$. Therefore, $\hat{v}(B) \leq 1/2 + 1/3$. Therefore, $g(B) \leq -6/5$.

   2.2 The refinement of $\phi_1$ is 3. Without loss of generality assume that the 2-path $x_1x_4x_3$ is a bad cherry. That means, the path is in a fixed triangular-block, say $B^*$ such $x_1x_3 \in E(B^*)$. Notice that $B^*$ can be a 4-vertex block like $B_{4,a}$ or $B_{5,c}$. But $B^*$ can not be $B_{5,a}$. Otherwise, it is easy to show $G$ contains $\Theta_6$. Moreover observe that the edge $x_2x_3$ is a trivial block and from the minimum degree condition of $G$, the vertex $x_2$ is shared by at least 3 triangular-blocks.

   If $x_1$ is shared by at least 3 triangular-blocks, then $\hat{v}(B) \leq 1/3 + 1/3$. Since $\hat{f}(B) \leq 1/3 + 1/5$ and $e(B) = 1$, we get $g(B) \leq -1/2$.

   Now assume $x_1$ is shared by only two triangular-blocks, namely $B^*$ and the trivial blocks $x_1x_2$. Thus, $\hat{v}(B) \leq 1/2 + 1/3$. Moreover it is easy to check that the exterior face $\phi_2$ is with size at least 6. Hence, $\hat{f}(B) \leq 1/3 + 1/6$. Therefore, in this scenario we get $g(B) \leq 0$. □

We notice that there is only one possible case for which a triangular-block $B$ in $G$ may assume positive $g(B)$. The only possibility is briefly explained in the proof of Claim 6 (case 2.3). The triangular-block and structures of the boundary of its exterior faces are shown in Figure 2.15(right).

Observe that the exterior faces of the exterior paths $x_2x_1x_4$ and $x_2x_3x_4$ are with size 4 and the edges $x_4x_6$, $x_4x_5$, $x_2x_6$ and $x_2x_5$ are trivial blocks. Denote the trivial blocks respectively as $B^1, B^2, B^3$, and $B^4$. Let us call such trivial blocks as "*good blocks*".

It is easy to show that if anyone of these good blocks plays a similar role with a different $B_{4,b}$ triangular-block, the graph contains a $\Theta_6$. Indeed, Suppose $B^1$ plays such role. Thus, there is a $B_{4,b}$ triangular-block, say $B'$, such that $B^1$ is one of the four good blocks of $B'$ (see the two possible structures in Figure 2.18). But in both scenario it is easy to show $G$ contains $\Theta_6$, which is a contradiction.

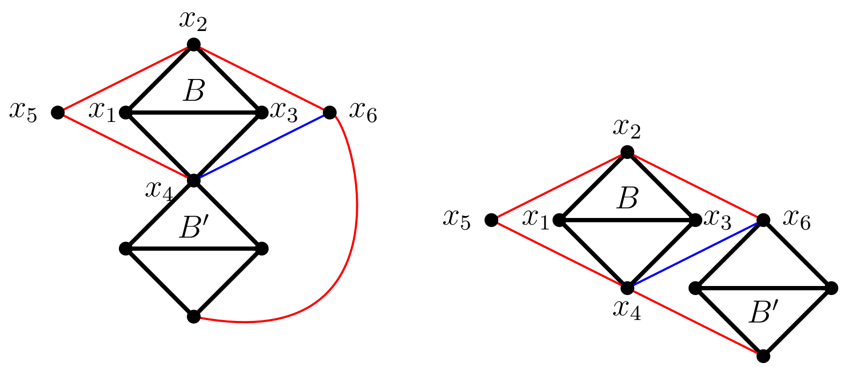



Figure 2.18: Two possible structures showing two different $B_{4,b}$ triangular-blocks sharing a good block.

This implies that for each $B_{4,b}$ triangular-block meeting the conditions in Claim 6 (case 2.3), correspondingly we have unique four good blocks on the boundaries of the exterior faces of the triangular-block. In particular for $B$, the corresponding four good blocks are $B^1, B^2, B^3$ and $B^4$.

Observe that each good block is with exterior face whose refinements are 4 and at least 5. Moreover, the at least one end vertex of a good block is shared by at least 3 triangular-blocks. Thus for $i \in \{1, 2, 3, 4\}$, we have $\hat{f}(B^i) \le 1/4 + 1/5$ and $\hat{v}(B^i) \le 1/2 + 1/3$, which implies $g(B^i) \le -6/5$.

Define $\mathcal{P} = \{B^1, B^2, B^3, B^4, B\}$. From the prove of Claim 6 (case 2.3), $g(B) \le 3$. Clearly,

$$\sum_{B^* \in \mathcal{P}} g(B^*) \le 3 + 4\,(-6/5) = -9/5.$$

Let $\mathcal{B}$ is the family of all triangular-blocks of $G$ and $\mathcal{B}' = \{B_1, B_2, \dots, B_k\}$ be the set of $B_{4,b}$ triangular-blocks in $G$ meeting the conditions stated in Claim 6(case 2.3).

Define $\mathcal{P}_i = \{B_i^1, B_i^2, B_i^3, B_i^4, B_i\}$ for all $i \in \{1, 2, 3, \dots, k\}$ and $B_i^j$, $j \in \{1, 2, 3, 4\}$ are the corresponding four good blocks of $B_i$.

Define

$$\mathcal{P}_{k+1} = \mathcal{B} \backslash \bigcup_{i=1}^{k} \mathcal{P}_i.$$

Since $g(B^*) \leq 0$ for all $B^* \in \mathcal{P}_{k+1}$, $\sum\limits_{B^* \in \mathcal{P}_{k+1}} g(B^*) \leq 0$. Let $m = k+1$. Thus we have the partition $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ of $\mathcal{B}$ such that $\sum\limits_{B^* \in \mathcal{P}_i} g(B^*) \leq 0$ for all $i \in \{1, 2, 3, \ldots, m\}$. Therefore,

$$24f(G) - 17e(G) + 6v(G) = \sum_{i=1}^{m} \sum_{B^* \in \mathcal{P}_i} g(B^*) \leq 0.$$

This completes the proof of Theorem 52.

## 2.5 Proof of Theorem 53

We show the proof for the connected graphs only. Indeed, if the graph is not connected, then we can add an edge between components by keeping the graph connected and $\Theta_6$-free. So, if we show that the theorem holds for a connected graph, then it holds for disconnected too.

To finish the proof of the Theorem 53, we need the following lemma.

**Lemma 6.** *Let $G$ be an $n$-vertex ($n \geq 2$) $\Theta_6$-free plane graph, then $e(G) \leq \frac{18}{7}n - \frac{27}{7}$.*

*Proof.* First, we prove for a connected graph, and then it is easy to finish the prove for a disconnected case. Let the number of blocks, maximal subgraphs of $G$ containing no cut vertex, be $b$. Let the blocks are $B'_1, B'_2, B'_3, \ldots, B'_b$ and with number of vertices (including the cut vertices) $n_1, n_2, \ldots, n_b$ respectively.

It is easy to check that, if $B'_i$ is a block with $2 \leq n_i \leq 5$, then $e(B'_i) \leq \frac{18}{7}n_i - \frac{27}{7}$. Suppose that $n_i \geq 6$. If there is a vertex of degree 2 in $B'_i$, say $v$, then by induction,

$$e(B'_i) = e(G - v) + 2 \leq \frac{18}{7}(n_i - 1) - \frac{27}{7} + 2 = \frac{18}{7}n_i - \frac{31}{7} \leq \frac{18}{7}n_i - \frac{27}{7}.$$

So suppose that $d_{B'_i}(u) \geq 3$ for all $u \in V(B'_i)$. By Theorem 52,

$$e(B'_i) \leq \frac{18}{7}n_i - \frac{48}{7} \leq \frac{18}{7}n_i - \frac{27}{7}.$$

Hence, for each block $B'_i$, $i \in \{1, 2, \ldots, b\}$, $e(B'_i) \leq \dfrac{18}{7}n_i - \dfrac{48}{7}$. Therefore,

$$e(G) \leq \sum_{i=1}^{b} \left( \frac{18}{7}n_i - \frac{27}{7} \right) = \frac{18}{7} \sum_{i=1}^{b} n_i - \frac{27}{7}b = \frac{18}{7}(n + b - 1) - \frac{27}{7}b = \frac{18}{7}n - \frac{9}{7}b - \frac{18}{7}$$

where we get, $\dfrac{18}{7}n - \dfrac{9}{7}b - \dfrac{18}{7} \leq \dfrac{18}{7}n - \dfrac{27}{7}$, for $b \geq 1$. $\square$

**Lemma 7.** *Let $G$ be an $n$-vertex and $\Theta_6$-free connected graph with $b$ blocks. Then $e(G) \le \frac{18}{7}n - \frac{9b+18}{7}$.*

*Proof.* Let $B'_1, B'_2, \ldots, B'_b$ be the blocks of $G$ with number of vertices $n_1, n_2, \ldots, n_b$ respectively (including the cut vertices). By Lemma 6 we have that

$$e(G) = \sum_{k=1}^{b} e(B'_k) \le \sum_{k=1}^{b} \frac{18n_k - 27}{7} = \frac{18n + 18(b-1) - 27b}{7} = \frac{18n}{7} - \frac{9b+18}{7}. \quad \square$$

Note that Lemma 7 implies that if $G$ is an $n$-vertex $\Theta_6$-free planar graph with $b \ge 4$ blocks, then $e(G) \le \frac{18n}{7} - \frac{54}{7} < \frac{18n}{7} - \frac{48}{7}$.

We need the following claim to prove the lemma which follows.

**Claim 9.** *If $G$ is an $n$-vertex $\Theta_6$-free graph containing a vertex of degree 2 such that, $G - v$ has at least 3 blocks, then $e(G) < \frac{18}{7}n - \frac{48}{7}$.*

*Proof.* By Lemma 7 we have that $e(G - v) \le \frac{18(n-1)}{7} - \frac{45}{7}$, hence

$$e(G) \le \frac{18n - 18 - 45}{7} + 2 = \frac{18n}{7} - \frac{49}{7} < \frac{18n}{7} - \frac{48}{7}.$$

$\square$

**Lemma 8.** *Let $G$ be an $n$-vertex $\Theta_6$-free 2-connected graph, then*

$$e(G) \le \frac{18n}{7} - \begin{cases} \frac{38}{7} & \text{for } n = 6, \\ \frac{42}{7} & \text{for } n = 7, \\ \frac{46}{7} & \text{for } n = 8, \\ \frac{48}{7} & \text{for } n \ge 9. \end{cases}$$

*Proof.* Let $n = 6$, we are going to show that in fact $e(G) \le \frac{18}{7}n - \frac{38}{7} = 10$ for any $\Theta_6$-free graph. Let $G$ be a $\Theta_6$ on 6 vertices. We claim that $G$ does not contain 11 edges. Indeed, if $G$ contains 11 edges, then an embedding of $G$ on the plane contains a 4-face and all the remaining faces are of size 3. Thus, fixing the 4-face the unbounded face of the plane drawing of $G$, then $G$ a triangular-block of 6 vertices. Thus, $G$ contains a $\Theta_6$, which is a contradiction. Therefore, $G$ must contain at most 10 edges.

Let $n = 7$, and $G$ an $n$-vertex $\Theta_6$-free plane graph which is 2-connected. We want to show that $e(G) \le \frac{18}{7}n - \frac{42}{7}$. Let $v$ be a vertex of degree 2 in $G$. Thus $e(G-v) \le \frac{18}{7}(n-1) - \frac{38}{7}$. Hence $e(G) = e(G-v) + 2 \le \frac{18}{7}(n-1) - \frac{38}{7} + 2 \le \frac{18}{7}n - \frac{42}{7}$.

It can be shown that for any $n = 7$ and $\Theta_6$-free plane graph $G$, $e(G) \le \frac{18}{7}n - \frac{42}{7}$. Indeed, if $G$ is 2-connected, we have already proved. If $G$ contains at least 3 blocks, then it holds by Lemma 7. Suppose that it contains only 2 blocks. Let the blocks are $B_1'$ and $B_2'$ with number of vertices $n_1$ and $n_2$ respectively. If $n_1 = 2$ and $n_2 = 6$, then $e(B_1') = 1$ and $e(B_2) \leq 10$. Hence $e(G) \leq 11$, which implies $e(G) \leq \frac{18}{7}n - \frac{42}{7}$. If $n_1 = 3$ and $n_2 = 5$, then $e(B_1') = 3$ and $e(B_2') \leq 9$. Thus, $e(G) \leq 12$, that means $e(G) \leq \frac{18}{7}n - \frac{42}{7}$. If $n_1 = n_2 = 4$, then $e(B_1') \leq 6$ and $e(B_2') \leq 6$. Thus, $e(G) \leq 12$, again $e(G) \leq \frac{18}{7}n - \frac{42}{7}$. Therefore from all the results we have, $e(G) \leq \frac{18}{7}n - \frac{42}{7}$.

Let $n = 8$, and $G$ an $n$-vertex $\Theta_6$-free plane graph which is 2-connected. We want to show that $e(G) \leq \frac{18}{7}n - \frac{46}{7}$. Let $v$ be a vertex of degree 2 in $G$. Thus, $G - v$ is a plane graph of 7-vertex $\Theta_6$-free plane graph. Thus, from the previous result $e(G - v) \leq \frac{18}{7}(n-1) - \frac{42}{7}$. Thus, $e(G) \leq \frac{18}{7}(n-1) - \frac{42}{7} + 2 = \frac{18}{7}n - \frac{46}{7}$.

We observe further bounds of $e(G)$ if it is not 2-connected. If $G$ contains two blocks, we claim that $e(G) \leq \frac{18}{7}n - \frac{39}{7}$. Indeed, let the blocks are $B_1'$ and $B_2'$ with $n_1$ and $n_2$ vertices respectively. If $n_1 = 2$ and $n_2 = 7$, then $e(B_1') = 1$ and $e(B_2') \leq 12$. Thus, $e(G) \leq 13$, that means $e(G) \leq \frac{18}{7}n - \frac{46}{7}$. If $n_1 = 3$ and $n_2 = 6$, then $e(B_1') = 3$ and $e(B_2') \leq 10$. Thus, $e(G) \leq 13$. If $n_1 = 4$ and $n_2 = 5$, then $e(B_1') \leq 6$ and $e(B_2') \leq 9$. Thus, $e(G) \leq 15$. Therefore, $e(G) \leq \frac{18}{7}n - \frac{39}{7}$. Observe that the only structure that $e(G) > \frac{18}{7}n - \frac{46}{7}$ is when $B_1'$ is a $K_4$ and $B_2'$ is a $K_5^-$ or vice-versa (see Figure 2.19).

Let $n = 9$, and $G$ be an $n$-vertex $\Theta_6$-free plane graph which is 2-connected. We want to show that $e(G) \leq \frac{18}{7}n - \frac{50}{7}$ and hence $e(G) \leq \frac{18}{7}n - \frac{48}{7}$. Let $v$ be a vertex of degree 2 in $G$. Thus, $G - v$ is plane graph of 8 vertices which is $\Theta_6$ free. If $G - v$ is 2-connected, then from previous result, $e(G - v) \leq \frac{18}{7}(n-1) - \frac{46}{7}$. Thus, $e(G) \leq \frac{18}{7}(n-1) - \frac{46}{7} + 2 = \frac{18}{7}n - \frac{50}{7}$. If $G - v$ is with two blocks, then $e(G) \leq \frac{18}{7}(n-1) - \frac{46}{7}$. Indeed, the only case that we get $e(G - v) > \frac{18}{7}(n-1) - \frac{46}{7}$ is when one block is of size 4 and the other is of size 5( see Figure 2.19). But in that case, there is no possible way to join the two blocks with a cherry. If that is so, it is easy to get a $\Theta_6$ in the graph. Therefore, $e(G) \leq \frac{18}{7}(n-1) - \frac{46}{7}$. Thus, $e(G) \leq \frac{18}{7}(n-1) - \frac{46}{7} + 2 = \frac{18}{7}n - \frac{50}{7}$. Therefore $e(G) \leq \frac{18}{7}n - \frac{50}{7}$.

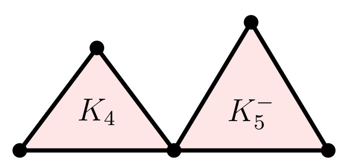



Figure 2.19: Structure of a graph on 8 vertices and containing two blocks of size 4 and 5.

Let $n = 9$ and $G$ be an $n$-vertex $\Theta_6$-free plane graph containing two blocks, say $B_1'$ and $B_2'$ with number of vertices $n_1$ and $n_2$ respectively. If $n_1 = 2$ and $n_2 = 8$, then $e(B_1') = 1$ and $e(B_2') \leq 14$. Thus, $e(G) \leq 15$. That means, $e(G) \leq \frac{18}{7}n - \frac{50}{7}$. If $n_1 = 3$ and $n_2 = 7$, then $e(B_1') = 3$ and $e(B_2') \leq 12$. Again in this case $\frac{18}{7}n - \frac{50}{7}$. If $n_1 = 4$ and $n_2 = 6$, the $e(B_1') \leq 6$ and $e(B_2') \leq 10$ and again $e(G) \leq \frac{18}{7}n - \frac{50}{7}$. If $n_1 = n_2 = 5$, then $e(B_1) \leq 9$ and $e(B_2) \leq 9$. If $e(B_1') = e(B_2') = 9$, then $e(G) = 18$ in this case both $B_1'$ and $B_2'$ are $K_5^-$ and hence the structure of the graph is well known (see Figure 2.20(a)). The remaining only possibility that we have $e(G) > 16$ is when one block contains 9 edges and the other contains 8 edges. Without loss of generality, assume $e(B_1') = 9$ and $e(B_2') = 8$. Thus, $B_1'$ is $K_5^-$. Now we need to figure out the structure of $B_2'$. Notice that $B_2'$ misses only one edge not to be a maximal planar graph with 5 vertices. We denote the blocks as $K_5^{--}$. Notice that the plane drawing of $B_2'$ contains one 4-face and the others are all 3-face. Thus, if the unbounded face of $B_2'$ is a triangle, then the structure of $G$ is as shown in Figure 2.20(b). If the unbounded face of $B_2'$ is a 4-face, then the structure of the $G$ is as shown in Figure 2.20(c).

Let $n = 10$, and $G$ be an $n$-vertex $\Theta_6$-free plane graph which is 2-connected. We want to show that $e(G) \leq \frac{18}{7}n - \frac{54}{7}$. Let $v$ be a vertex of degree 2. Thus, $G - v$ is a plane graph of 9 vertices which is also $\Theta_6$-free. If $G-v$ is 2-connected, then from the previous result, $e(G-v) \leq \frac{18}{7}(n-1)-\frac{50}{7}$. Thus, $e(G) \leq \frac{18}{7}(n-1)-\frac{50}{7}+2 = \frac{18}{7}n-\frac{54}{7}$. If $G - v$ is contains two blocks, then from the previous observation there are three possibilities such that $e(G - v) > \frac{18}{7}(n - 1) - \frac{50}{7}$. The structure of these graphs are shown in Figure 2.20(a,b,c). However, in all the cases if there is a cherry joining the two blocks, then it is easy to get a $\Theta_6$ in the graph. Thus, such conditions could not appear in the $G - v$ graph. This implies that $e(G) \leq 16$. In other words, $e(G - v) \leq \frac{18}{7}(n - 1) - \frac{50}{7}$. Therefore, $e(G) \leq \frac{18}{7}n - \frac{50}{7} + 2 = \frac{18}{7}n - \frac{54}{7}$.

**Claim 10.** *Let $G$ be an $n$-vertex graph with a vertex $v$ of degree 2, if $G - v$ has exactly 2 blocks, one with size at least 6, then $e(G) < \frac{18}{7}n - \frac{48}{7}$.*

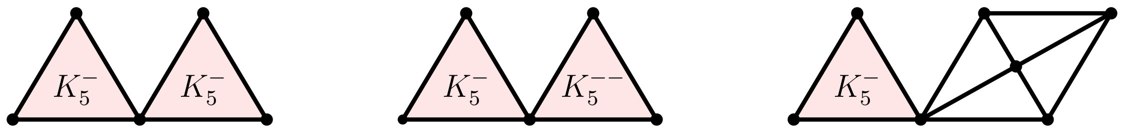
*Proof.* Let $B_1'$ and $B_2'$ be the block of $G$ of sizes $n_1$ and $n_2$ respectively, with $n_1 \geq 6$. By induction we may assume that $e(B_1) \leq \frac{18n_1}{7} - \frac{38}{7}$ and by Lemma 6 $e(B_2') \leq \frac{18n_2}{7} - \frac{27}{7}$, therefore

$$e(G) \leq e(B_1')+e(B_2')+2 \leq \frac{18(n_1+n_2)}{7} - \frac{27+38-14}{7} = \frac{18n}{7} - \frac{51}{7} < \frac{18n}{7} - \frac{48}{7}. \quad \square$$

Now suppose $n \geq 11$. If $\delta(G) \geq 3$, by Theorem 52 $e(G) \leq \frac{18n}{7} - \frac{48}{7}$. So suppose there is a vertex $v$ in $G$ with degree $d(v) \leq 2$. If $G - v$ is 2-connected, then by induction hypothesis $e(G) \leq \frac{18(n-1)}{7} - \frac{48}{7} + 2 < \frac{18n}{7} - \frac{48}{7}$. If $G - v$ contains precisely 2 blocks of sizes $n_1, n_2$ with $n_1 \geq n_2$, then since $n_1 + n_2 = n$ we have that $n_1 \geq 6$, and so, by Claim 10 we have that $e(G) < \frac{18n}{7} - \frac{48}{7}$. If $G - v$ contains at least three blocks, then by Claim 9 $e(G) < \frac{18n}{7} - \frac{48}{7}$. $\square$

**Lemma 9.** *Let $G$ be an $n$-vertex, $\Theta_6$-free plane graph containing a block, say $B$ with $n_b$ vertices, such that $e(B) \leq \frac{18}{7}n_b - \frac{48}{7}$. Then $e(G) \leq \frac{18}{7}n - \frac{48}{7}$.*

*Proof.* Let $G$ contains $b$ blocks. If $b \geq 4$, then by Lemma 7, the statement holds. Suppose that $G$ contains at most 3 blocks. The number of cut vertices contained in $B$ is at most 2. We consider two cases:




$(a)$ $(b)$ $(c)$

Figure 2.20: Structure of graphs on 9 vertices and containing two blocks of size 5.

1. $B$ contains no cut vertex. In this case $G = B$ and hence, $e(G) \leq \frac{18}{7}n - \frac{48}{7}$.

2. $B$ contains only one cut vertex. Let $G'$ be the graph obtained by deleting all vertices of $B$ except the cut vertex. Clearly, $v(G') = n-(n_b-1)$. Using Lemma 6, $e(G') \leq \frac{18}{7}(n-(n_b-1)) - \frac{27}{7}$. Therefore,
$$e(G) = e(G') + e(B) \leq \frac{18}{7}(n-(n_b-1)) - \frac{27}{7} + \frac{18}{7}n_b - \frac{48}{7} = \frac{18}{7}n - \frac{57}{7}.$$
Therefore, $e(G) \leq \frac{18}{7}n - \frac{48}{7}$.

3. $B$ contains two cut vertices. Let $G'$ be the graph obtained by deleting all vertices of $B$ except the two cut vertices. Then $G'$ is a disconnected graph with two components, say $B_1'$ and $B_2'$ with number of vertices $n_1$ and $n_2$ respectively. Clearly $n_1 + n_2 = n - (n_b - 2)$. Using Lemma 6, $e(B_1') \leq \frac{18}{7}n_1 - \frac{27}{7}$ and $e(B_2') \leq \frac{18}{7}n_2 - \frac{27}{7}$. Hence,
$$\begin{aligned} e(G') = e(B_1') + e(B_2') &\leq \left(\frac{18}{7}n_1 - \frac{27}{7}\right) + \left(\frac{18}{7}n_2 - \frac{27}{7}\right) \leq \frac{18}{7}(n_1+n_2) - \frac{54}{7} \\ &= \frac{18}{7}(n-(n_b-2)) - \frac{54}{7}. \end{aligned}$$
Thus,
$$e(G) = e(G') + e(B) \leq \frac{18}{7}(n-(n_b-2)) - \frac{54}{7} + \frac{18}{7}n_b - \frac{48}{7} = \frac{18}{7}n - \frac{64}{7}.$$
Therefore, $e(G) < \frac{18}{7}n - \frac{48}{7}$. $\square$

Now we give the proof of Theorem 53. Notice that $G$ contains at least 14 vertices. If $G$ is 2-connected, then we are done by Lemma 8. Thus we suppose that $G$ contains at least 2 blocks and at most 3 blocks. Otherwise, we are done by Lemma 7. So, we distinguish two cases:

1. $G$ contains only two blocks. Let the blocks be $B_1'$ and $B_2'$ with number of vertices $n_1$ and $n_2$ respectively, such that $n_1 \geq n_2$. If $n_1 \geq 9$, then by Lemma 8 and Lemma 9, $e(G) \leq \frac{18}{7}n - \frac{48}{7}$.

   Since $n + 1 = n_1 + n_2$, we have that if $n \geq 16$ then $n_1 \geq 9$. We may assume that either $n = 14$ and $n_1 = 8, n_2 = 7$ or $n = 15$ and $n_1 = 8, n_2 = 8$.

In the first case by Lemma 8, $e(B_1') \leq \frac{18}{7}n_2 - \frac{46}{7}$ and $e(B_2') \leq \frac{18}{7}n_1 - \frac{42}{7}$. Therefore,

$$\begin{aligned} e(G) = e(B_1') + e(B_2') &\leq \frac{18}{7}(n_1+n_2) - \frac{88}{7} = \frac{18}{7}(n+1) - \frac{88}{7} \\ &= \frac{18}{7}n - \frac{70}{7} < \frac{18}{7}n - \frac{48}{7}. \end{aligned}$$

In the second case by Lemma 8, $e(B_1') \leq \frac{18}{7}n_2 - \frac{46}{7}$ and $e(B_2') \leq \frac{18}{7}n_2 - \frac{46}{7}$. Therefore,

$$\begin{aligned} e(G) = e(B_1') + e(B_2') &\leq \frac{18}{7}(n_1+n_2) - \frac{92}{7} = \frac{18}{7}(n+1) - \frac{92}{7} \\ &= \frac{18}{7}n - \frac{74}{7} < \frac{18}{7}n - \frac{48}{7}. \end{aligned}$$

2. $G$ contains three blocks. Let the blocks be $B_1', B_2'$ and $B_3'$ with number of vertices $n_1, n_2$ and $n_3$ respectively, such that $n_1 \geq n_2 \geq n_3$. Since $n+2 = n_1 + n_2 + n_3$, and $n \geq 14$ we have that $n_1 \geq 6$, hence, by Lemma 8, $e(B_1') \leq \frac{18}{7}n_1 - \frac{38}{7}$. From Lemma 6, $e(B_i') \leq \frac{18}{7}n_2 - \frac{27}{7}$, $i \in \{2,3\}$. Thus,

$$\begin{aligned} e(G) = e(B_1') + e(B_2') + e(B_3') &\leq \frac{18}{7}(n_1+n_2+n_3) - \frac{27}{7} - \frac{27}{7} - \frac{38}{7} \\ &= \frac{18}{7}(n+2) - \frac{92}{7} = \frac{18}{7}n - \frac{56}{7} \\ &< \frac{18}{7}n - \frac{48}{7}. \end{aligned}$$

Notice that for $n = 13$, we have a counter example for which Theorem 53 does not hold. One counter example is the graph $G$ shown in Figure 2.21, where the graph contains 27 edges but $e(G) > \frac{18}{7}n - \frac{48}{7}$.

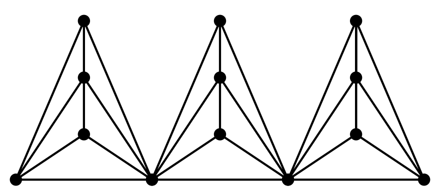


Figure 2.21: Maximal counter example

## 2.6 Remarks and conjectures

Recall $\Theta_6 = \{\Theta_6^1,\ \Theta_6^2\}$, where $\Theta_6^1$ and $\Theta_6^2$ are the symmetric and asymmetric $\Theta_6$-graphs which are shown in Figure 2.1 left and right respectively. One may ask about the values of $\mathrm{ex}_{\mathcal{P}}(n, \Theta_6^1)$ and $\mathrm{ex}_{\mathcal{P}}(n, \Theta_6^2)$? We pose the following asymptotic conjectures.

**Conjecture 5.** *(Ghosh, Győri, Paulos, Xiao, Zamora [54])*

1. $\mathrm{ex}_{\mathcal{P}}(n, \Theta_6^1) = \frac{45}{17}n + \Theta(1)$.

2. $\mathrm{ex}_{\mathcal{P}}(n, \Theta_6^2) = \frac{18}{7}n + \Theta(1)$.

The lower bound for $\mathrm{ex}_{\mathcal{P}}(n, \Theta_6^1)$ is based on the construction obtained by identifying the two red or the two blue 5-cycles in Figure 2.22 (left) and (right). Each of the shaded triangular region in the construction is a $K_5^-$. Notice that every non-triangular face in the construction is of size 5 and is surrounded by five $K_5^-$'s. It can be checked that the graph contains no $\Theta_6^1$ and has as many edges as what indicated in the bound.

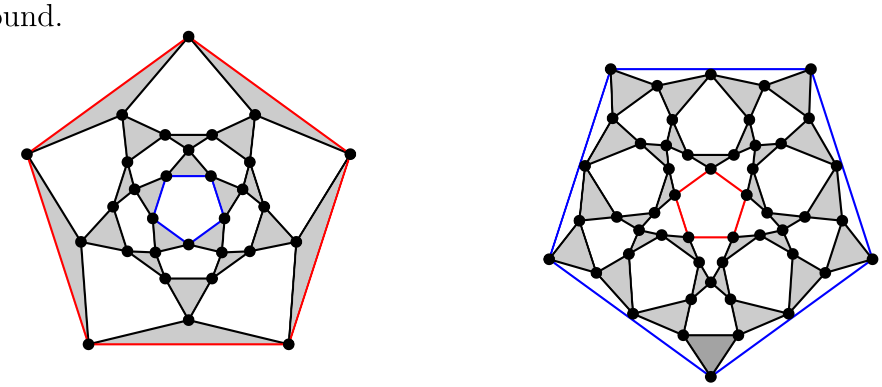


Figure 2.22: $\Theta_6^1$-free planar graphs

# Chapter 3

# Generalized Planar Turán Number of Paths

## 3.1 Introduction

As already mentioned in the preliminary chapter, Alon and Caro [2] determined the exact value of $f_{\mathcal{P}}(n,H)$ where $H$ is a complete bipartite graph with smaller part of size 1 or 2, i.e., $H = K_{1,k}$ and $K_{2,k}$.

Recall that $P_k$ denotes a path on $k$ vertices. It is well-known that $f_{\mathcal{P}}(n,P_2) = 3n-6$, and it follows from Theorem 41 that, $f_{\mathcal{P}}(n,P_3) = n^2+3n-16$ for $n \geq 4$. Recently in [57], we determined the exact value of $f_{\mathcal{P}}(n,P_4)$. In [51], we also determined an asymptotic value of $f_{\mathcal{P}}(n,P_5)$. In [22] Cox and Martin determined an asymptotic value of $f_{\mathcal{P}}(n,P_7)$. The results are as follows.

**Theorem 55.** *(Grzesik, Győri, Paulos, Salia, Tompkins, Zamora [57]) We have,*

$$f_{\mathcal{P}}(n,P_4) = \begin{cases} 12, & \text{if } n = 4; \\ 147, & \text{if } n = 7; \\ 222, & \text{if } n = 8; \\ 7n^2 - 32n + 27, & \text{if } n = 5,6 \text{ and } n \geq 9. \end{cases}$$

*For and integer $n \in \{4,5,6\}$ or $n \geq 9$, the only $n$-vertex planar graph attaining $f_{\mathcal{P}}(n,P_4)$ is the graph $F_n$ shown in Figure 1.8. $F_7'$ and $F_8'$ respectively in Figure 1.8 are the only graphs attaining the values $f_{\mathcal{P}}(7,P_4)$ and $f_{\mathcal{P}}(8,P_4)$.*

**Theorem 56.** *(Ghosh, Győri, Martin, Paulos, Salia, Xiao, Zamora [51])*

$$f_{\mathcal{P}}(n,P_5) = n^3 + O(n^2).$$

**Theorem 57.** *(Cox, Martin [22])*

$$f_{\mathcal{P}}(n,P_7) = \frac{4}{27}n^4 + O(n^{19/5}).$$

The bounds in Theorem 56 is asymptotically best considering the planar graph $F_n$(see Figure 1.8), which contains at least $n^3$ $P_5$'s. Moreover, the bound in Theorem 57 is asymptotically best considering the planar graph construction given in Figure 3.16 when $\ell = 3$. It can be checked that the graph contains at least $\frac{4}{27}n^4$ $P_7$'s.

Detail of our proofs for Theorem 55 and Theorem 56 are given in the next two subsections.

## 3.2 Generalized planar Turán number of $P_4$

In addition to the general notations we have in Chapter 1, we introduce new notations we use in the proof of Theorem 55.

### 3.2.1 Preliminaries

Let $G$ be a planar graph. We omit the subscript $G$ when the underlying graph is clear. Recall that we refer to a path of length three as a 3-path. We denote the number of $P_4$'s in $G$ by $P_4(G)$. Let $x \in V(G)$. The number of $P_4$'s in $G$ containing $x$ is denoted by $P_4(G, x)$. $F_n$ is an $n$-vertex maximal planar graph obtained by joining every vertex of an $(n-2)$-vertex path $P_{n-2}$ with both end vertices of an edge, i.e, $F_n = P_{n-2} + K_2$, see Figure 1.8(left).

For any maximal planar graph $G$ on $n$ vertices ($n \geq 3$) it can be shown that $3 \leq \delta(G) \leq 5$. Moreover, for a vertex $v$ in $V(G)$, if $d(v) = k$ and $N(v) = \{x_1, x_2, x_3, \dots, x_k\}$, then $N(v)$ induces a unique cycle of length $k$. We may choose a plane drawing of $G$ so that $v$ is contained in the interior of the cycle. Without loss of generality, we may assume that we have a cycle $C$ with vertex sequence $x_1, x_2, x_3, \dots, x_k, x_1$. Let us denote the edge $x_i v$ by $e_i$ for $i = 1, 2, 3, \dots, k$ (see Figure 3.1).

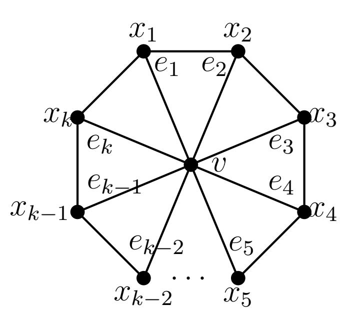



Figure 3.1: Neighbors of a vertex $v \in V(G)$ of degree $k$.

We partition the set of 3-paths containing $v$ into three different classes, depending on the location of their middle edge.

A **Type-I, 3-path with respect to a vertex** $v$ is a 3-path which contains an edge $e_i$ as its middle edge (see Figure 3.2).

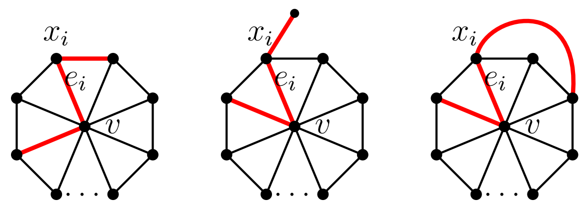
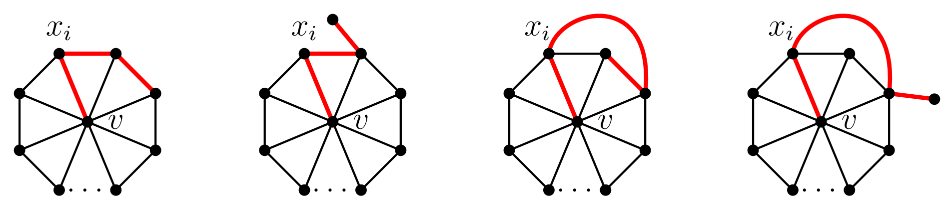


Figure 3.2: Examples of Type-I, 3-paths.

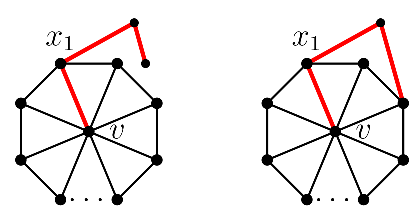
A **Type-II, 3-path with respect to a vertex** $v$ is a 3-path which starts with vertices $v, x_i, x_j$. Furthermore, if the middle edge is an edge of the cycle $C$, then we call such a 3-path a Type-II(A), 3-path. Otherwise, we call it a Type-II(B), 3-path (see Figure 3.3).



Type-II(A) Type-II(A) Type-II(B) Type-II(B)

Figure 3.3: Examples of Type-II, 3-paths.

A **Type-III, 3-path with respect to a vertex** $v$ is a 3-path which starts at the vertex $v$ such that its middle edge connects a vertex from $N(v)$ to a vertex from $V(G) \setminus (N(v) \cup \{v\})$. Furthermore, if the last vertex is not from $N(v)$, then we call such a 3-path a Type-III(A), 3-path. Otherwise, we call it a Type-III(B), 3-path (see Figure 3.4).



Type-III(A) Type-III(B)

Figure 3.4: Examples of Type-III, 3-paths.

It is easy to see that each of the 3-paths containing the vertex $v$ is in exactly one of the three classes which we have defined. For simplicity, we will sometimes write

Type-(I), (II), (III), 4-path instead of Type-(I), (II), (III), 3-path with respect to a vertex $v$, when the vertex under consideration is clear.

We will use the following two lemmas in our proof of the main theorem. The first lemma gives the number of 3-paths in a given graph $G$.

**Lemma 10.** *For a graph $G$, the number of 3-paths in $G$ is*

$$P_4(G) = \sum_{\{x,y\}\in E(G)} (d(x)-1)(d(y)-1) - 3\mathcal{N}(C_3, G).$$

*Proof.* Consider an edge $\{x, y\} \in E(G)$ and count the number of 3-paths containing $x$ as the second and and $y$ the third vertex of the 3-path. There are $d(x)-1$ possibilities to choose the first vertex and $d(y)-1$ possibilities to choose the last vertex of the path. Since the first and the last vertex of the 3-path need to be different, from the total number of $(d(x)-1)(d(y)-1)$ possibilities we need to subtract the number of triangles containing the edge $\{x, y\}$, which is $d(x, y)$.

Therefore,

$$\begin{aligned} P_4(G) &= \sum_{\{x,y\}\in E(G)} ((d(x)-1)(d(y)-1) - d(x,y)) \\ &= \sum_{\{x,y\}\in E(G)} (d(x)-1)(d(y)-1) - 3\mathcal{N}(C_3, G), \end{aligned}$$

as each triangle is counted 3 times in the sum. This completes the proof of Lemma 10. □

With this lemma we can prove the following lemma.

**Lemma 11.** *For every $n$-vertex planar graph $G$ with $\delta(G) \geq 4$ we have*

$$P_4(G) < 7n^2 - 36n + 50.$$

*Proof.* Without loss of generality we may assume that $G$ is a maximal planar graph with $3n-6$ edges and $2n-4$ triangular faces. In particular it contains at least $2n-4$ triangles.

From Lemma 10 the total number of 3-paths in $G$ is equal to

$$\begin{aligned} P_4(G) &= \sum_{\{x,y\}\in E(G)} (d(x)-1)(d(y)-1) - 3\mathcal{N}(C_3, G) \\ &\leq \sum_{\{x,y\}\in E(G)} (d(x)-1)(d(y)-1) - 3(2n-4) \\ &= \frac{1}{2} \sum_{x\in V(G)} (d(x)-1) \left( \sum_{y\in N(x)} d(y) - d(x) \right) - 6n + 12. \end{aligned}$$

Since $\delta(G) \geq 4$ and the sum of the degrees of all the vertices is equal to $2e(G) = 6n - 12$, for each vertex $x$ we have

$$\sum_{y \in N(x)} d(y) = 6n - 12 - d(x) - \sum_{y \notin N[x]} d(y) \leq 6n - 12 - d(x) - 4(n - 1 - d(x)) = 3d(x) + 2n - 8.$$

This gives us the following bound

$$\begin{aligned} P_4(G) &\leq \frac{1}{2} \sum_{x \in V(G)} (d(x) - 1)\,(2d(x) + 2n - 8) - 6n + 12 \\ &= \sum_{x \in V(G)} d^2(x) + (n-5) \sum_{x \in V(G)} d(x) - n(n-4) - 6n + 12 \\ &\leq \big((n-1)^2 + (n-3)^2 + 4^2(n-2)\big) + (n-5)(6n-12) - n^2 - 2n + 12 \\ &= 7n^2 - 36n + 50, \end{aligned}$$

where the last inequality comes from convexity.

It remains to notice that, since $\delta(G) \geq 4$ and $G$ is a planar graph, we have $n \geq 6$, hence $7n^2 - 36n + 50 < 7n^2 - 32n + 27$. □

### 3.2.2 Proof of Theorem 55

We are going to prove the theorem by induction on the number of vertices. The base cases, when $n \leq 9$, will be discussed later.

Let $G$ be a planar graph on $n$ vertices. Then we have $3 \leq \delta(G) \leq 5$. At first we settle the cases when $3 < \delta(G)$.

From Lemma 11 we may assume $\delta(G) = 3$. We are going to prove the rest by induction, after removing a vertex of degree 3.

Let $v$ be a vertex of degree 3 and $N(v) = \{x_1, x_2, x_3\}$. Our goal is to show that $P_4(G, v) \leq 14n - 39$. Indeed, by deleting the vertex $v$ we obtain a maximal planar graph $G'$ on $(n-1)$ vertices, and by the induction hypothesis we have $P_4(G') \leq 7(n-1)^2 - 32(n-1) + 27$. Therefore,

$$P_4(G) \leq 7(n-1)^2 - 32(n-1) + 27 + 14n - 39 = 7n^2 - 32n + 27.$$

Notice that the vertices $x_1, x_2, x_3$ induce a triangle. Denote the edges $x_i v$ by $e_i$, $i \in \{1, 2, 3\}$. The number of Type-I, 3-paths with $e_i$ in the middle is $2d(x_i) - 4$ for all $i \in \{1, 2, 3\}$. Thus we have $2\sum_{i=1}^{3} d(x_i) - 12$ Type-I, 3-paths. The number of Type-II, 3-paths starting at $v$ and continuing to a vertex $x_i$, $i \in \{1, 2, 3\}$, is $d(x_1) + d(x_2) + d(x_3) - d(x_i) - 4$. Thus, we have $2\sum_{i=1}^{3} d(x_i) - 12$ Type-II, 3-paths. It remains to count the number of Type-III, 3-paths with respect to the vertex $v$. For this we need to consider two subcases.

**Case 1.1:** $N(x_1) \cap N(x_2) \cap N(x_3) = \{v\}$**.**

For each edge $e$ which is not incident to the triangle, we can have at most four Type-III(A), 3-paths with respect to the vertex $v$ (see Figure 3.5). Since there are at most $(3n-6)-\Big(\sum_{i=1}^{3} d(x_i)-3\Big)$ such edges which are not incident to the triangle, it follows that the number of Type-III(A), 3-paths is at most $4\Big(3n-6-\big(\sum_{i=1}^{3} d(x_i)-3\big)\Big)$.

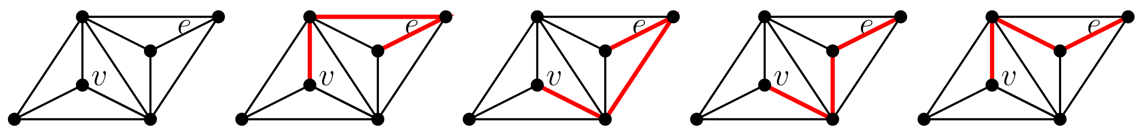



Figure 3.5: Four Type-III(A), 3-paths for a fixed edge $e$.

The remaining 3-paths are Type-III(B), 3-paths. Recall that in this case each vertex $v' \neq v$ can be adjacent to at most 2 vertices of the triangle induced by $N(v)$. Thus for each such vertex $v'$, $v' \notin \{x_1, x_2, x_3, v\}$, we have at most two Type-III(B), 3-paths (see Figure 3.6). Hence we have at most $2(n-4)$ Type-III(B), 3-paths. Thus we get,

$$P_4(G,v) \leq 4\sum_{i=1}^{3} d(x_i) - 24 + 4\Big(3n-3-\sum_{i=1}^{3} d(x_i)\Big) + 2(n-4) = 14n-44.$$

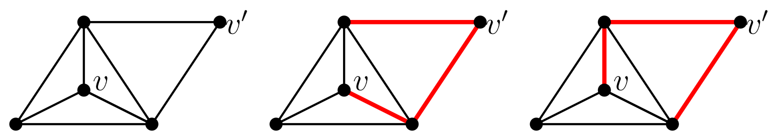



Figure 3.6: Type-III(B), 3-paths for a fixed vertex $v'$, $v' \notin \{v, x_1, x_2, x_3\}$.

Therefore, $P_4(G,v) \leq 14n-44 < 14n-39$ and we have no extremal graph in this case.

**Case 1.2: There exists a vertex $u$, $u \neq v$, such that** $N(x_1) \cap N(x_2) \cap N(x_3) = \{v, u\}$**.**

We consider the three regions formed by vertices $u, x_1, x_2$ and $x_3$. Let the region defined by the vertices $u, x_1$ and $x_2$ which does not contain $x_3$ be $R_1$, the region defined by the vertices $u, x_2$ and $x_3$ which does not contain $x_1$ be $R_2$, and lastly the region defined by the vertices $u, x_1$ and $x_3$ and not containing $x_2$ be $R_3$, as shown in Figure 3.7.

From the planarity of $G$, notice that there is at most one edge $e_1$ with end vertices $u$ and $y_1$ such that $y_1$ lies inside the region $R_1$ and $y_1$ is adjacent to both $x_1$ and $x_2$.

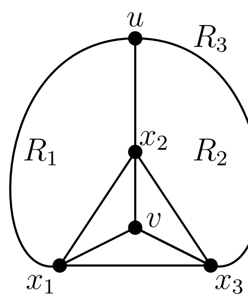



Figure 3.7: Three regions formed by the vertex $u$ and the vertices of the triangle.

Similarly there is at most one edge $e_2$ and $e_3$ with respect to the regions $R_2$ and $R_3$ respectively meeting the conditions stated for $e_1$. We call the edges $e_1$, $e_2$ and $e_3$ as *star edges* of $G$ with respect to the vertex $v$.

Take an edge $e$ such that $V(e) \cap \{x_1, x_2, x_3\} = \emptyset$. Then there are at most five Type-III(A), 3-paths with respect to the vertex $v$, containing the edge $e$, since $G$ is planar. Furthermore, for each star edge (if exists) in the three regions there are five Type-III(A), 3-paths. Figure 3.8 shows an edge $e$ in region $R_1$ with all five possible 3-paths of this kind.

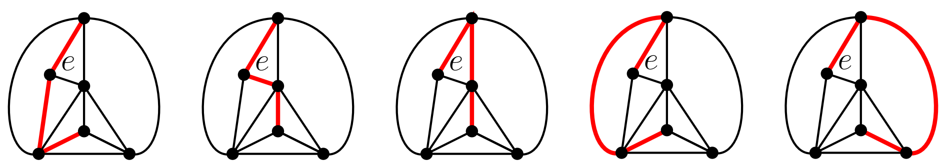



Figure 3.8: Five Type-III(A), 3-paths that contains the star edge $e$.

Notice that for each vertex $w$ inside the regions, one can have at most two Type-III(B), 3-paths containing $w$. For the vertex $u$, we have six Type-III(B), 3-paths containing the vertex $u$ (see Figure 3.9).

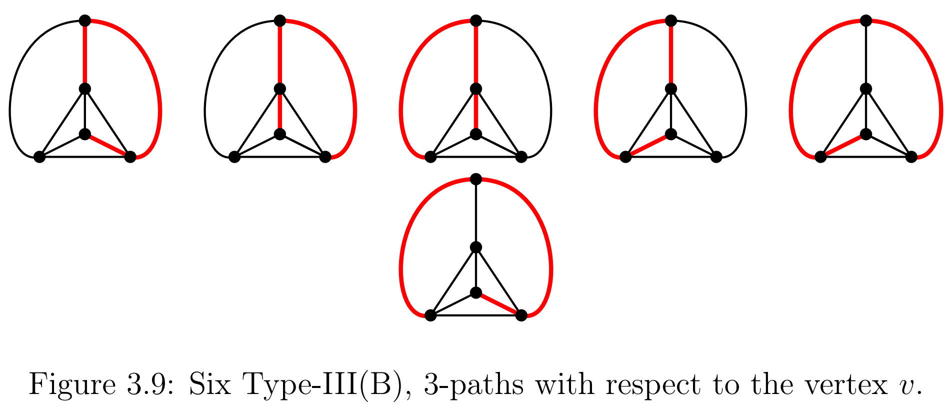


Figure 3.9: Six Type-III(B), 3-paths with respect to the vertex $v$.

**1.** If there is no star edge in each of the three regions, then we have at most

$$4\Big(3n-6-(\sum_{i=1}^{3} d(x_i)-3)\Big)+2(n-5)+6=-4\sum_{i=1}^{3} d(x_i)+14n-16$$

Type-III, 3-paths containing the vertex $v$. Thus, we have

$$P_4(G,v)\leq 4\sum_{i=1}^{3} d(x_i)-24-4\sum_{i=1}^{3} d(x_i)+14n-16\leq 14n-40.$$

Therefore, $P_4(G,v)<14n-39$.

**2.** If there is only one star edge, then we have at most

$$4\Big(3n-6-(\sum_{i=1}^{3} d(x_i)-3)-1\Big)+5+2(n-5)+6=-4\sum_{i=1}^{3} d(x_i)+14n-15$$

Type-III, 3-paths with respect to the vertex $v$. Therefore $P_4(G,v)\leq 14n-39$.

**Remark 1.** *Equality holds if we have a vertex $v$ of degree three and a vertex $u$ which is adjacent to all of the vertices incident to $v$. All the other vertices share exactly 2 neighbors with $v$, and we have exactly one star edge.*

**3.** If there are exactly two star edges, then we have two regions containing them. Without loss of generality, let the regions be $R_1$ and $R_2$. The third region, $R_3$, may or may not contain a vertex.

3.1 If there is a vertex in $R_3$, then at least one vertex in $R_3$ is a neighbor of the vertex $u$, hence this vertex is not a neighbor of one of the vertices $x_1$ or $x_3$ or both. Otherwise, we would have another star edge. It follows that there is no Type-III(B), 3-path containing this vertex. Thus, the number of Type-III, 3-paths with respect to the vertex $v$ is at most

$$4\Big(3n-6-(\sum_{i=1}^{3} d(x_i)-3)-2\Big)+10+2(n-6)+6=-4\sum_{i=1}^{3} d(x_i)+14n-16.$$

So we have $P_4(G,v)\leq 14n-40$. Therefore, $P_4(G,v)<14n-39$.

3.2 If there is no vertex in the region $R_3$, then at least one of the regions $R_1$ or $R_2$ contains at least two vertices, since $n\geq 10$. Without loss of generality, suppose $R_1$ contains at least two vertices. Let $f_1$ be the star edge in the region. This edge is incident to $u$, and we denote the other vertex it is incident to by $u_1$. We have $u_1\in N(x_1)\cap N(x_2)$. If there is a vertex in the region $R$, defined by the vertices $x_1, u_1$ and $u$ not containing $x_2$, then there is an edge $u_1u_1'$ in the region $R$, where $u_1'\notin\{x_1,x_2,x_3\}$. This edge is in at most three Type-III(A), 3-paths. Moreover $u_1'$ is not incident to the vertex $x_2$. Hence $u_1'$ is not in any of the Type-III(B) paths. Therefore we have at most

$$4\Big(3n-6-(\sum_{i=1}^{3} d(x_i)-3)-2-1\Big)+13+2(n-6)+6 = -4\sum_{i=1}^{3} d(x_i)+14n-17$$

Type-III, 3-paths with respect to the vertex $v$. Consequently, we have $P_4(G,v) \leq 14n-41$. Therefore, $P_4(G,v) < 14n-39$.

Similarly the region defined by the vertices $x_2, u_1, u$ not containing $x_1$ is also empty, otherwise we are done by induction.

Thus the vertices must be in the region $R'$, defined by the vertices $x_1, x_2, u_1$ not containing $u$. Consider an edge $f_2 = u_1u_2$ in the region $R'$. If $u_2$ is the only vertex in the region $R'$, then $N(u_2) = \{x_1, x_2, u_1\}$, and we are done by induction, since we have a vertex $u_2$ of degree three with at most one star edge, which was settled in Cases 1.2(1) and 1.2(2) (see Figure 3.10).

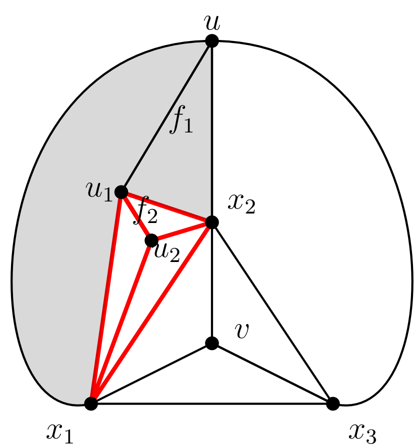



Figure 3.10: A vertex $u_2$ with the property that two of the corresponding regions have no vertex inside.

If the vertex $u_2$ is not a neighbor of one of the vertices $x_1$ or $x_2$, then the edge $f_2$ is not incident to the triangle and is contained in at most three Type-III(A), 3-paths. Moreover $u_2$ is in none of the Type-III(B) paths. Therefore we have at most

$$4\Big(3n-6-(\sum_{i=1}^{3} d(x_i)-3\Big)-2-1)+13+2(n-6)+6 = -4\sum_{i=1}^{3} d(x_i)+14n-17$$

Type-III, 3-paths. Consequently, we have $P_4(G,v) \leq 14n-41$. Therefore, $P_4(G,v) < 14n-39$.

We have that there are at least two vertices in the region $R'$, and $u_2$ is incident with both of the vertices $x_1$ and $x_2$.

A similar argument to the one given in Case 1.2(3.1) gives us that there is no vertex in the region defined by the vertices $x_1, u_2, u_1$ not containing $x_2$ and, likewise, in the region defined by the vertices $x_2, u_2, u_1$ not containing $x_1$. Thus, all the vertices must be in the region defined by the vertices $x_1, u_2, x_2$ not containing $u_1$; let us denote this region by $R''$. Consider an edge $f_3 = u_2u_3$ in the region $R''$. Thus we proceed with a similar argument as before, this time applied to the region $R''$ and the corresponding vertex $u_3$. Notice that $N(u_3) = \{x_1, x_2, u_2\}$. If $u_3$ is the only vertex in $R''$, then we are done by induction since $u_3$ would be a vertex of degree three and with at most one star edge, which was settled in Case 1.2(1) and Case 1.2(2). Otherwise, we get a region containing at least one vertex, say $R'''$, defined by the vertices $x_1, x_2$ and $u_3$ not containing $u_2$. We apply similar reasoning to $R'''$ as that for $R'$ and $R''$. Since $G$ is finite, after a finite number of steps $k$, we obtain a vertex $u_k$, such that $N(u_k) = \{x_1, x_2, u_{k-1}\}$ and $u_k$ is with at most one star edge, which was settled in Case 1.2(1) and Case 1.2(2).

**4.** Suppose there are three star edges. Let $uy_i$ be the star edge in the region $R_i$, for each $i \in \{1, 2, 3\}$. Since $n \geq 10$, one of the regions $R_1, R_2$ or $R_3$ contains at least one additional vertex other than $y_i$. Without loss of generality, let $R_1$ be such a region. If there is a vertex in the region $x_1, y_1, u$ not containing $x_2$, then we have at least one edge, say $y_1y_1'$, for some $y_1'$ inside the region bounded by $x_1, y_1$ and $u$ not containing $x_2$. The edge $y_1y_1'$ is in at most three Type-III(A), 3-paths. Moreover, the vertex $y_1'$ is not incident to $x_2$ and $x_3$. Hence it is not in any Type-III(B) paths. Thus, we have at most

$$4\Big(3n - 6 - (\sum_{i=1}^{3} d(x_i) - 3) - 4\Big) + 18 + 2(n-6) + 6 = -4\sum_{i=1}^{3} d(x_i) + 14n - 16$$

Type-III, 3-paths. Hence we have $P_4(G, v) \leq 14n - 40$. Therefore, $P_4(G, v) < 14n - 39$.

Similarly, the region defined by $x_2$, $y_1$ and $u$ not containing the vertex $x_1$ must be empty. Otherwise, we are done by induction.

If the region obtained from the vertices $x_1, y_1, x_2$ not containing $u$ contains only one vertex $u'$, then we have a degree three vertex $u'$, and there is at most one star edge corresponding to the vertex $u'$. Hence, we are done by induction as in Case 1.2(1) or Case 1.2(2) for the vertex $u'$. Otherwise, if the region obtained by the vertices $x_1, y_1, x_2$ not containing $u$ contains more than one vertex, then we are done by similar arguments given in Case 1.2(3.2).



### Basis for the induction

Here we are going to find the maximum number of paths of length three in a planar graph with at most 9 vertices. This will form the basis for the induction. We are going to recall some facts from the previous calculations. Let $G$ be a maximal planar graph on $n$ vertices, and $v \in V(G)$ be a vertex of minimum degree.

If $d(v) = 3$, then we have the following.

- Suppose there is no vertex other than $v$ adjacent to all the neighbors of $v$, then from Case 1.1 we have

$$P_4(G, v) \leq 14n - 44. \tag{3.1}$$

- Suppose there is a vertex other than $v$ which is adjacent to all the neighbors of $v$, then we consider the following cases.

  – If there is no star edge with respect to the vertex $v$, then from Case 1.2.1 we have

$$P_4(G, v) \leq 14n - 40. \tag{3.2}$$

  – If there is only one star edge with respect to the vertex $v$, then from Case 1.2.2 we have

$$P_4(G, v) \leq 14n - 39. \tag{3.3}$$

  – If there are two star edges with respect to the vertex $v$, then in this case we cannot use Case 1.2.3, since $n$ is not at least 10. But by similar calculations we have a weaker result for all $n$.

$$\begin{aligned} P_4(G, v) \leq 4\sum_{i=1}^{3} d(x_i) - 24 + 4\Big(3n - 6 - (\sum_{i=1}^{3} d(x_i) - 3) - 2\Big) + 10 \\ + 2(n-5) + 6 = 14n - 38. \end{aligned} \tag{3.4}$$

  – If there are three star edges with respect to the vertex $v$, then in this case we cannot use Case 1.2.4, since $n \leq 9$. However, by similar calculations we have a weaker result for all $n$.

$$\begin{aligned} P_4(G, v) \leq 4\sum_{i=1}^{3} d(x_i) - 24 + 4\Big(3n - 6 - (\sum_{i=1}^{3} d(x_i) - 3) - 3\Big) + 15 \\ + 2(n-5) + 6 = 14n - 37. \end{aligned} \tag{3.5}$$

**Claim 11.** *$f_{\mathcal{P}}(4, P_4) = 12$ and $f_{\mathcal{P}}(5, P_4) = 42$.*

*Proof.* The maximal planar graphs with 4 and 5 vertices are unique. The graphs are $K_4$ and $K_5^-$ respectively. It is easy to check that $f_{\mathcal{P}}(4, P_4) = 12$ and $f_{\mathcal{P}}(5, P_4) = 42$. $\square$

**Claim 12.** $f_{\mathcal{P}}(6, P_4) = 87$.

*Proof.* Let $G$ be a maximal planar graph on 6 vertices. We have $\delta(G) = 3$. First we prove the following claim.

**Claim 13.** *There is a vertex different from $v$ which is adjacent to every neighbours of $v$. Moreover, there is one star edge with respect to $v$.*

*Proof.* Let $N(v) = \{x_1, x_2, x_3\}$ and the remaining two vertices other than $v, x_1, x_2$ and $x_3$ be $y_1$ and $y_2$. From the property of maximal planar graphs, every edge of $G$ must be incident to exactly two triangular faces. Thus each of the edges in $\{x_1x_2, x_2x_3, x_3x_1\}$ must be incident to a triangular faces which is not incident to $v$. The number of vertices contained in the triangular region bounded by $x_1, x_2$ and $x_3$ not containing $v$ is 2, namely $y_1$ and $y_2$. Thus, two of the edges, say $x_1x_2$ and $x_2x_3$, must use one of the vertices in $\{y_1, y_2\}$, say $y_1$, such that the $x_1x_2y_1$ and $x_2x_3y_1$ are triangular faces incident to the two edges. From the property that every face of maximal planar graph is of size 3, necessarily $y_1$ and $y_2$ must be adjacent. Hence we obtain that the vertex $y_1$ is adjacent to every neighbour of $v$. Moreover, the edge $y_1y_2$ is the only star edge of $G$ with respect to $v$. $\square$

Next we proceed proving Claim 12. Deleting the vertex $v$ we get a maximal planar graph on 5 vertices which contains 42 3-paths.

Therefore, using Claim 13 we have that the number of 3-paths that contain the vertex $v$ is at most 45, from (3.2) and (3.3). Thus $P_4(G) \leq 42 + 45 = 87$ and we have unique extremal graph $F_6$ with 87 3-paths.

$\square$

**Claim 14.** $f_{\mathcal{P}}(7, P_4) = 147$.

*Proof.* Let $G$ be a maximal planar graph on 7 vertices. We have $d(v) = 3$. Deleting this vertex we get a maximal planar graph with 6 vertices and containing at most 87 3-paths. Since the number of vertices is 7, there are at most two star edges. Therefore using (3.1), (3.2), (3.3) and (3.4), the maximum number of 3-paths containing the vertex is 60. Hence $P_4(G) \leq 147$ and equality holds if we deleted a vertex with two star edges and the graph we got was $F_6$. There are only two faces in $F_6$ where we can place the deleted vertex in order to have two star edges, in both cases we get the same graph $F_7'$ which pictured in Figure 1.8.

$\square$

**Claim 15.** $f_{\mathcal{P}}(8, P_4) = 222$.

*Proof.* Let $G$ be a maximal planar graph on 8 vertices. Since $d(v) = 3$, after deleting the vertex $v$ from $G$, we get a seven vertex maximal planar graph containing at most 147 paths of length three. However, from (3.1), (3.2), (3.3), (3.4) and (3.5), the maximum number of 3-paths that contain the vertex $v$ is at most 75. Thus $P_4(G) \leq 222$ and equality holds if we have deleted a vertex with three star edges and the graph we got was also extremal($F_7'$). There is a unique face of the graph $F_7'$ in Figure 1.8 where we can place a deleted vertex in order to have three star edges. This leads us to the unique extremal graph $F_8'$ pictured in Figure 1.8. □

**Claim 16.** $f_{\mathcal{P}}(9, P_4) = 306$.

*Proof.* If there is no other vertex incident to all the vertices incident to the vertex $v$, then using (3.1) we have at most 82 3-paths that contain $v$. Since deleting the vertex $v$ results in an eight vertex maximal planar graph, it contains at most 222 3-paths, from Claim 15. Thus we have $P_4(G) \leq 304$.

Now assume that the neighbors of $v$ have a common adjacent vertex other than $v$. Consider the three regions obtained as in Figure 3.7.

(i) If each of the three regions is nonempty, then there is a unique maximal planar graph of this kind (see Figure 3.11). This planar graph contains 303 3-paths.

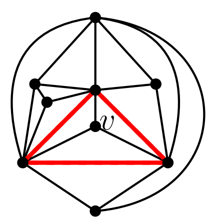



Figure 3.11: A maximal planar graph on 9 vertices, containing 303 3-paths.

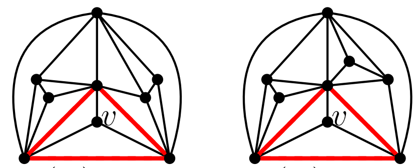
(ii) If two of the regions contain two vertices each, then the remaining region contains no vertex. The two nonempty regions contain a star edge.

If in each of the two regions, we have a vertex which is incident to exactly one vertex of the triangle $N(v)$, then we have at most

$$4\sum_{i=1}^{3} d(x_i) - 24 + 4\Big(3n - 6 - (\sum_{i=1}^{3} d(x_i) - 3) - 2 - 2\Big) + 16 + 2(n-7) + 6 = 14n - 44 = 82$$

3-paths that contain the vertex $v$. Since deleting the vertex $v$ results in an eight vertex maximal planar graph, which contains at most 222, 3-paths, from Claim 15, we get $P_4(G) \leq 304$.

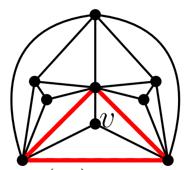
If only one of the two regions contain a vertex which is incident to exactly one vertex of the triangle, then there are only two such maximal planar graphs (see Figure 3.12). The number of 3-paths they contain are, respectively, 290 and 297.



$P_4(G) = 290$ $\quad$ $P_4(G) = 297$

Figure 3.12: Maximal planar graphs on 9 vertices.

If in each of the two regions there is no vertex incident to exactly one vertex of the triangle, then the planar graph is unique (see Figure 3.13). The number of 3-paths in this graph is 296.



$P_4(G) = 296$

Figure 3.13: A maximal planar graph on 9 vertices.

(iii) Assume one of the regions contains a vertex and the other contains three vertices (the third one is empty).

Suppose there is only one star edge, then the number of 3-paths that contain the vertex $v$ is at most

$$4\sum_{i=1}^{3} d(x_i)-24+4\Big(3n-6-(\sum_{i=1}^{3} d(x_i)-3)-1-1\Big)+8+2(n-6)+6 = 14n-42 = 84.$$

Since one of the vertices of the triangle will be of degree 4, after removing the vertex $v$, we will not have the unique extremal graph $F_8'$ in Figure 1.8, since it does not contain a vertex of degree four. Thus, in this case, we have $P_4(G) < 222+84 = 306$. After removing the vertex $v$ we will not get the unique extremal graph $F_8'$. Thus, in this case, we have $P_4(G) < 222 + 84 = 306$.

If there are two star edges, and there are two vertices which are incident to exactly one of the vertices of the triangle, then we have at most

$$4\sum_{i=1}^{3} d(x_i) - 24 + 4\Big(3n - 6 - (\sum_{i=1}^{3} d(x_i) - 3) - 2 - 2\Big) + 10 + 6 + 2(n-7) + 6$$
$$= 14n - 44 = 82 \text{ 3-paths containing the vertex } v.$$

Therefore $P_4(G) \leq 304$. If there are two star edges, and there are at least three vertices incident to two of the vertices of the triangle, then Figure 3.14 shows

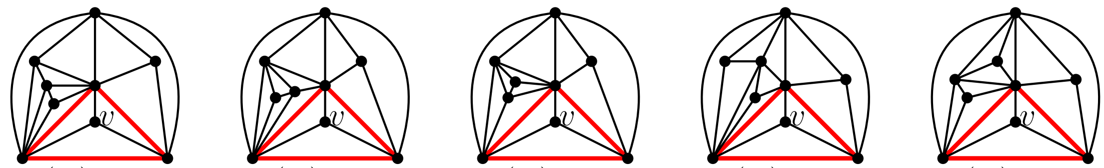



Figure 3.14: Maximal planar graphs on 9 vertices.

all possible nine vertex planar graphs. There are 300, 289, 292, 299 and 302, 3-paths in those graphs, respectively.

(iv) Assume all 4 vertices are in the same region.

Suppose there is no star edge, then we have at most

$$4\sum_{i=1}^{3} d(x_i) - 24 + 4\Big(3n - 6 - (\sum_{i=1}^{3} d(x_i) - 3)\Big) + 2(n-6) + 6 = 14n - 42 = 84$$

3-paths containing the vertex $v$. After removing the vertex $v$ we will not get the unique extremal graph $F_8'$ in Figure 1.8, since for each face of the graph $F_8'$ has a star edge. Thus, in this case, we have $P_4(G) < 222 + 84 = 306$.

Suppose there is a star edge and there is exactly one vertex which is not incident to two of the vertices of the triangle. Then that vertex must be incident to a vertex of the triangle. That vertex of the triangle has degree 8, therefore after deleting the vertex $v$, we will get a vertex of degree 7. Since the graph $F_8'$ in Figure 1.8 does not contain a vertex of degree 7 number of 3-paths not containing $v$ is at most 221. The number of 3-paths containing the vertex $v$ is at most

$$4\sum_{i=1}^{3} d(x_i) - 24 + 4(3n - 6 - (\sum_{i=1}^{3} d(x_i) - 3) - 1 - 1) + 5 + 3 + 2(n-6) + 6 = 14n - 42 = 84.$$

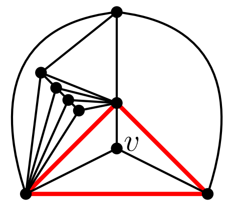
Thus $P_4(G) < 306$.

Suppose there is a star edge and there is more than one vertex which is not incident to two of the vertices of the triangle. Then number of 3-paths containing the vertex $v$ is at most

$$4\sum_{i=1}^{3} d(x_i) - 24 + 4(3n - 6 - (\sum_{i=1}^{3} d(x_i) - 3) - 1 - 2) + 5 + 6 + 2(n-7) + 6 = 14n - 42 = 81.$$

Thus $P_4(G) < 306$, since after deleting the vertex $v$ we get a maximal planar graph on 8 vertices and $f_{\mathcal{P}}(8, P_4) = 222$.

Finally, if there is a star edge and all four vertices are incident to two of the vertices of the triangle, then the maximal planar graph is uniquely defined, see Figure 3.15 which is $F_9$. It contains 306 paths of length three. Therefore $f_{\mathcal{P}}(9, P_4) = 306$, and the unique extremal planar graph on 9 vertices is $F_9$.




$P_4(G) = 306.$

Figure 3.15: A maximal planar graph with 9 vertices, containing maximum number of 3-paths.

So far we have determined $f_{\mathcal{P}}(n, P_4)$ for all integers $n$. We also have proven that for all $n$, $n < 10$, the planar graph maximizing number of $P_4$'s is unique. Even more we have shown that the unique extremal graph is $F_9$, for $n = 9$.

In the remaining part of this section, we are going to show that for all $n$, $n \geq 9$, the only planar graph maximizing number of 3-paths is $F_n$. For this we are going to use a proof by induction on the number of vertices. The base case for $n = 9$ is complete. Let us assume that $G$ is an $n$, $n \geq 10$, vertex graph with $f_{\mathcal{P}}(n, P_4)$ 3-paths, then we are going to show that $G = F_n$ under the assumption that the only extremal planar graph with $n - 1$ vertices is $F_{n-1}$. From the proof of the upper bound, we know that in order to have $f_{\mathcal{P}}(n, P_4)$ paths of length three, we have one of two possibilities as outlined in Remark 1.

From Lemma 11 we have the minimum degree of $G$ is 3, we have for any vertex of degree three, say $v$, that all other vertices share at least two neighbors with $v$. After removing the vertex $v$, we get the unique extremal graph $F_{n-1}$ in this case. Therefore there are only two such faces in $F_{n-1}$, they are the faces with two high degree vertices and a vertex of degree three. In both settings, after placing $v$ in the proper face and adding all three edges, we get the graph $F_n$. Therefore we have the desired result $G = F_n$. □

## 3.3 Generalized planar Turán number of $P_5$

In this section we give proof of the asymptotic bound of maximum number of paths of length 4 in a planar graph which is mentioned in Theorem 56.

### 3.3.1 Notations

Before we proceed to the proof of our result, we mention some notations we use in addition to the general notation we have in Chapter 1. For a graph $G$ and $u, w \in V(G)$.

We denote the number of vertices in $G$ which are adjacent to both vertices by $d^*(u, w)$. Denote $P_5(G)$ to be the number of $P_5$'s in $G$.

### 3.3.2 Proof of Theorem 56

For any given graph $G$ and vertices $u$ and $v$ in $G$, it is easy to see that the number of paths of length 4 in the graph with $u$ and $v$ the two vertices next to the terminal vertices of the path is at most $d(u)d^*(u, v)d(v)$. Thus,

$$P_5(G) \leq \frac{1}{2} \sum_{u \in V(G)} \sum_{u \neq v \in V(G)} d(u)d(v)d^*(u, v).$$

Notice that this bound is crude in as much as we can get better order lower terms. Since $d^*(u, v) \leq \min\{d(u), d(v)\}$, then

$$P_5(G) \leq \frac{1}{2} \sum_{u \in V(G)} \sum_{u \neq v \in V(G)} d(u)d(v) \min\{d(u), d(v)\}.$$

So if $(x_1, x_2, x_3, \ldots, x_n)$ is the degree sequence of $G$, arranged in decreasing order, we have that

$$P_5(G) \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j^2.$$

To prove Theorem 56, we need the following lemma.

**Lemma 12.** *Let $n \geq k \geq 3$ and let $G$ be a planar graph on $n$ vertices such that $S \subseteq V(G)$ with $|S| = k$. Then*

$$\sum_{v \in S} d(v) \leq 2n + 6k - 16.$$

*Proof.* Let $G'$ be the graph induced by $S$. Since $G'$ is planar and is not $K_2$,

$$\sum_{v \in S} d_{G'}(v) \leq 6k - 12.$$

Now we count the number of edges between the vertex sets $S$ and $V(G) \setminus S$, say $e = e(S, V(G) \setminus S)$; that is, the number of edges in the planar bipartite graph with color classes $S$ and $V(G) \setminus S$. Since the graph is bipartite, it is also triangle-free. Thus, each non-exterior face uses at least 4 edges. In the case of the exterior face, bridges count twice when counting the number of edges that border the face. So the exterior face has length at least 4 unless the graph has only one edge.

Hence, if $e > 1$, then $4f \leq 2e$, where $f$ is the number of faces in the bipartite subgraph. Using the inequality and Euler's formula, $n + f = e + 2$, we obtain $e = e(S, V(G) \setminus S) \leq 2n - 4$. Therefore,

$$\sum_{v \in S} d(v) = \sum_{v \in S} d_{G'}(v) + e(S, V(G) \setminus S) \leq 2n + 6k - 16.$$

If $e = 1$, then $\sum_{v \in S} d(v) \leq 6k - 11 \leq 2n + 6k - 16$ because $n \geq 3$. □

Given $n \geq 3$, we define the set

$$A_n = \left\{ (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{Z}^n : n \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0, \forall k \in \{3, \ldots, n\}, \right.$$
$$\left. \sum_{i=1}^{k} x_i \leq 2n + 6k - 16 \text{ and } \sum_{i=1}^{n} x_i \leq 6n - 12 \right\}.$$

Let $(x_1, x_2, \ldots, x_n)$ be the degree sequence of an $n$-vertex planar graph $G$ in decreasing order. Since $\sum\limits_{v \in V(G)} d(v) = 2\,|E(G)| \leq 6n - 12$, by Lemma 34, we have $(x_1, x_2, \ldots, x_n) \in A_n$.

Consider the function $S_n : \mathbb{R}^n \to \mathbb{R}$ by

$$S_n(x_1, x_2, \ldots, x_n) = \sum_{i}^{n-1} \sum_{j=i+1}^{n} x_i x_j^2,$$

then Theorem 56 will be a corollary of the following theorem.

**Theorem 58.** *For $n \geq 3$ and every $(x_1, x_2, \ldots, x_n) \in A_n$, we have*

$$S_n(x_1, x_2, \ldots, x_n) \leq n^3 + O(n^2).$$

Before proving Theorem 58 we need the following lemmas.

**Lemma 13.** *Let $n \geq 3$ and $(x_1, x_2, \ldots, x_n) \in A_n$ be a point maximizing $S_n$ over $A_n$. Then $x_1 - x_2 \leq 1$.*

*Proof.* Suppose by contradiction that $x_1 - x_2 \geq 2$. Define the sequence $(y_1, y_2, \ldots, y_n) \in A_n$ as $y_1 = x_1 - 1$, $y_2 = x_2 + 1$ and $y_i = x_i$ for all $i \neq 1, 2$. Then

$$\begin{aligned} S_n(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_i y_j^2 &= (x_1 - 1)((x_2 + 1)^2 + x_3^2 + \cdots + x_n^2) \\ &+ (x_2 + 1)(x_3^2 + \cdots + x_n^2) + x_3(x_4^2 + \cdots + x_n^2) + \cdots + x_{n-1}x_n^2 \\ &= x_1(x_2^2 + \cdots + x_n^2) + x_2(x_3^2 + \cdots + x_n^2) + \cdots + x_{n-1}x_n^2 \\ &+ (x_1 - 1)(2x_2 + 1). \end{aligned}$$

Thus $S_n(y_1, y_2, \ldots, y_n) - S_n(x_1, x_2, \ldots, x_n) = (x_1 - 1)(2x_2 + 1) > 0$, which is a contradiction. □

**Lemma 14.** *Let $n \geq 3$ and $(x_1, x_2, \ldots, x_n) \in A_n$ be a point maximizing $S_n$ over $A_n$. If $x_1 = n$, then $S_n(x_1, x_2, \ldots, x_n) \leq n^3 + O(n^2)$.*

*Proof.* By Lemma 13, we have $x_2 \in \{n, n-1\}$. Since $x_1 + x_2 + x_3 \leq 2n + 2$, we see $x_3 \leq 3$. Therefore,

$$\begin{aligned}
S_n(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j^2 \leq n(n^2 + \underbrace{3^2 + 3^2 + \cdots + 3^2}_{n-2 \text{ terms}}) \\
&\quad + n(\underbrace{3^2 + 3^2 + \cdots + 3^2}_{n-2 \text{ terms}}) + 3(\underbrace{3^2 + 3^2 + \cdots + 3^2}_{n-3 \text{ terms}}) + \cdots + 3(3^2) \\
&= n^3 + 9n(n-2) + 9n(n-2) + 27(n-3) + 27(n-4) + \cdots + 27 \\
&= n^3 + 18n(n-2) + \frac{27}{2}(n-3)(n-2) = n^3 + O(n^2).
\end{aligned}$$

□

**Lemma 15.** *Let $n \geq 3$ and $(x_1, x_2, \ldots, x_n) \in A_n$. If $x_2 \leq \frac{n}{18}$, then $S_n(x_1, x_2, \ldots, x_n) \leq n^3 + O(n^2)$.*

*Proof.*

$$\begin{aligned}
S_n(x_1, x_2, \ldots, x_n) &\leq \sum_{i}^{n-1} \sum_{j=i+1}^{n} x_i x_j^2 \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j x_2 = x_2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j \leq \frac{x_2}{2} \sum_{i,j} x_i x_j \\
&= \frac{x_2}{2} \sum_{i} x_i \sum_{j} x_j = \frac{x_2}{2} \left( \sum_{i} x_i \right)^2 = \frac{x_2}{2}(6n-12)^2.
\end{aligned}$$

Thus, if $x_2 \leq \frac{n}{18}$, then $S_n(x_1, x_2, \ldots, x_n) \leq n^3 + O(n^2)$. □

We prove the following claim, from which Lemma 16 follows.

**Claim 17.** *Let $n \geq 3$ and suppose $(x_1, x_2, \ldots, x_n) \in A_n$. If $k$ is the smallest integer at least 3 such that $\sum\limits_{i=1}^{k} x_i = 2n + 6k - 16$, then $x_k \geq 7$ and $x_{k+1} \leq 6$.*

*Proof.* Since $\sum\limits_{i=1}^{k} x_i = 2n + 6k - 16$, we have $\sum\limits_{i=1}^{k-1} x_i + x_k = 2n + 6k - 16$. From the definition of $A_n$, $\sum\limits_{i=1}^{k-1} x_i < 2n + 6(k-1) - 16$. Therefore, $2n + 6k - 16 < 2n + 6(k-1) - 16 + x_k$. Thus, $x_k \geq 7$. Now suppose $x_{k+1} \geq 7$. In that case, $(2n + 6k - 16) + 7 \leq 2n + 6(k+1) - 16$ which simplifies to $7 \leq 6$, a contradiction. Therefore, $x_{k+1} \leq 6$. □

**Lemma 16.** *Let $n \geq 3$ and $(x_1, x_2, \ldots, x_n) \in A_n$ be a point maximizing $S_n$ over $A_n$. One of the following must hold:*

*(i)* $x_1 = n$,

*(ii)* $x_2 \leq \frac{n}{18}$,

*(iii)* *there exists a* $k \leq 11664$ *such that* $x_i \leq 6$ *for* $i > k$.

*Proof.* Suppose that $(i)$ and $(ii)$ are false, that is $x_1 < n$ and $x_2 > \frac{n}{18}$. Then we have to show that $(iii)$ holds. If there exists an $r \geq 3$ such that $\sum_{i=1}^{r} x_i = 2n + 6r - 16$, then take $k$ to be the smallest such $r$. Otherwise, let $k$ be the last index such that $x_k$ is not 0. If $k < n$ in both cases, we have $x_k > x_{k+1}$, either because of Claim 17 or because $x_k > 0 = x_{k+1}$. Additionally, from Claim 17, we have that $x_i \leq 6$ for $i > k$. We are going to prove that $k \leq 11664$, hence $k$ satisfies $(iii)$.

Define $y = (y_1, y_2, y_3, \ldots, y_n)$ by $y_1 = x_1 + 1$, $y_k = x_k - 1$ and, $y_i = x_i$ for $i \neq 1, k$. And note $y \in A_n$. We have

$$\begin{aligned}
S_n(y_1, y_2, \ldots, y_n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_i y_j^2 \\
&= (x_1 + 1)\left(x_2^2 + x_3^2 + \cdots + x_{k-1}^2 + (x_k - 1)^2 + x_{k+1}^2 + \cdots + x_n^2\right) \\
&\quad + x_2\left(x_3^2 + x_4^2 + \cdots + x_{k-1}^2 + (x_k - 1)^2 + x_{k+1}^2 + \cdots + x_n^2\right) \\
&\quad + \cdots + x_{k-1}\left((x_k - 1)^2 + x_{k+1}^2 + \cdots + x_n^2\right) \\
&\quad + (x_k - 1)\left(x_{k+1}^2 + \cdots + x_n^2\right) + \cdots + x_{n-1} x_n^2 \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j^2 + (1 - 2x_k)(1 + x_1 + x_2 + x_3 + \cdots + x_{k-1}) \\
&\quad + \left(x_2^2 + x_3^2 + \cdots + x_n^2\right) - \left(x_{k+1}^2 + x_{k+2}^2 + \cdots + x_n^2\right).
\end{aligned}$$

Thus, $S_n(y_1, y_2, \ldots, y_n) - S_n(x_1, x_2, \ldots, x_n) = (x_2^2 + x_3^2 + \cdots + x_k^2) - (2x_k - 1)(1 + x_1 + x_2 + \cdots + x_{k-1})$. Since $S_n(y_1, y_2, \ldots, y_n) \leq S_n(x_1, x_2, \ldots, x_n)$, $x_2 > \frac{n}{18}$ and $\sum_{i=1}^{k} x_i \leq 6n$, we have

$$\frac{n^2}{18^2} < (x_2^2 + x_3^2 + x_4^2 + \cdots + x_k^2) \leq (2x_k - 1)(1 + x_1 + x_2 + x_3 + \cdots + x_{k-1}) < 12 n x_k.$$

Therefore, $x_k > \frac{n}{18^2 \cdot 12}$. Hence we have $6n \geq \sum_{i=1}^{k} x_i \geq k \frac{n}{18^2 \cdot 6}$ and $k \leq (18 \cdot 6)^2 = 11664$. □

**Lemma 17.** *Let* $m \geq 2$ *be an integer and* $x_1, x_2, \ldots, x_m$ *be reals such that* $x_1 \geq x_2 \geq \cdots \geq x_m \geq 0$. *Put* $t := \sum_{i=1}^{m} x_i$, *then* $S_m(x_1, x_2, x_3, \ldots, x_m) \leq (t/2)^3$.

*Proof.* We are going to proceed by induction on $m$. First, we show the relation holds for $m = 2$. Let $x_1, x_2$ be real numbers such that $x_1 \geq x_2 \geq 0$ and $t = x_1 + x_2$, which gives $x_2 = t - x_1$. Hence, $S_2(x_1, x_2) = S(x_1, t - x_1) = x_1(t - x_1)^2$.

Let $f(x) = x(t-x)^2$. We have $f'(x) = t^2 - 4tx + 3x^2 = (t-x)(t-3x)$, which is negative in $[t/2, t]$. Since $x_1 \geq t/2$, we have

$$S_2(x_1, x_2) = f(x_1) \leq \max_{t/2 \leq x \leq t} f(x) = f(t/2) = \frac{t^3}{8}.$$

Therefore, the lemma holds for $m = 2$.

Now suppose $m \geq 3$ is such that the lemma is true for $m-1$, and let $x_1, x_2, \ldots, x_m$ be real numbers such that $x_1 \geq x_2 \geq \cdots \geq x_m \geq 0$ and $\sum_{i=1}^{m} x_i = t$. By the induction hypothesis, we have $S_{m-1}(x_1, x_2, \ldots, x_{m-1}) \leq \left(\frac{t-x_m}{2}\right)^3$. Thus, we get

$$\begin{aligned} S_m(x_1, x_2, \ldots, x_m) &= S_{m-1}(x_1, x_2, \ldots, x_{m-1}) + (x_1 + x_2 + x_3 + \cdots + x_{m-1})x_m^2 \\ &= S_{m-1}(x_1, x_2, \ldots, x_{m-1}) + (t - x_m)x_m^2 \leq \frac{(t-x_m)^3}{8} + (t-x_m)x_m^2. \end{aligned}$$

Let $g(x) = \frac{(t-x)^3}{8} + (t-x)x^2$, then $g''(x) = \frac{11t - 27x}{4}$. We have that $x_m \leq \frac{t}{m} \leq \frac{t}{3}$, and $g''(x) \geq \frac{2t}{4} \geq 0$, for $x \leq t/3$. Thus $g$ is convex in $[0, t/3]$, therefore

$$S_m(x_1, x_2, \ldots, x_m) \leq g(x_m) \leq \max_{0 \leq x \leq t/3} g(x) = \max\{g(0), g(t/3)\} = \max\left\{\frac{t^3}{8}, \frac{t^3}{9}\right\} = \frac{t^3}{8}.$$

□

Now we are able to prove Theorem 58.

*Proof of Theorem 58.* Let $n$ be sufficiently large and take $(x_1, x_2, \ldots, x_n)$ maximizing $S_n$ over $A_n$. By Lemma 16 we have three possible cases.

If $x_1 = n$, then $S_n(x_1, x_2, \ldots, x_n) \leq n^3 + O(n^2)$ by Lemma 14.

If $x_2 \leq \frac{n}{18}$, then $S_n(x_1, x_2, \ldots, x_n) \leq n^3 + O(n^2)$ by Lemma 15.

If there exists a $k$ satisfying $(iii)$ in Lemma 15, then we have that $\sum\limits_{i=1}^{k} x_i \leq 2n + 6k = 2n + O(1)$.

Hence by Lemma 17, we have

$$S_k(x_1, x_2, \ldots, x_k) \leq \left(\frac{2n + O(1)}{2}\right)^3 = n^3 + O(n^2).$$

Therefore, together with the fact that $x_i \le 6$ for $i > k$,

$$\begin{aligned} S_n(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j^2 \le \sum_{i=1}^{n-1} \left( \sum_{j=i+1}^{k} x_i x_j^2 + \sum_{j=k+1}^{n} x_i x_j^2 \right) \\ &= \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} x_i x_j^2 + \sum_{i=k}^{n-1} \sum_{j=i+1}^{k} x_i x_j^2 + \sum_{i=1}^{n-1} \sum_{j=k+1}^{n} x_i x_j^2 \\ &\le S_k(x_1, x_2, \ldots, x_k) + O(n^2) + 36n \sum_{i=1}^{n-1} x_i \\ &\le n^3 + O(n^2). \end{aligned}$$

□

## 3.4 Remarks and conjectures

We propose the following conjectures of the asymptotic values of $f_{\mathcal{P}}(n, P_{2\ell+1})$ and $f_{\mathcal{P}}(n, P_{2\ell})$.

**Conjecture 6.** *For paths with even length,* $f_{\mathcal{P}}(n, P_{2\ell+1}) = 4\ell \left(\frac{n}{\ell}\right)^{\ell+1} + O(n^{\ell})$.

**Conjecture 7.** *For paths with odd length,* $f_{\mathcal{P}}(n, P_{2\ell}) = 8\ell(\ell+1) \left(\frac{n}{\ell}\right)^{\ell+1} + O(n^{\ell})$, *for* $\ell \ge 3$.

For $\ell \ge 3$, in both cases the lower bound is attained by a planar graph on $n$ vertices that is obtained from a balanced blowing up of a maximum independent set of vertices of a $2\ell$-vertex cycle and joining the vertices of each blown-up set by path, see Figure 3.16. In the case of $\ell = 2$, the lower bound is attained by an $F_n$ planar graph which is shown in Figure 1.3.

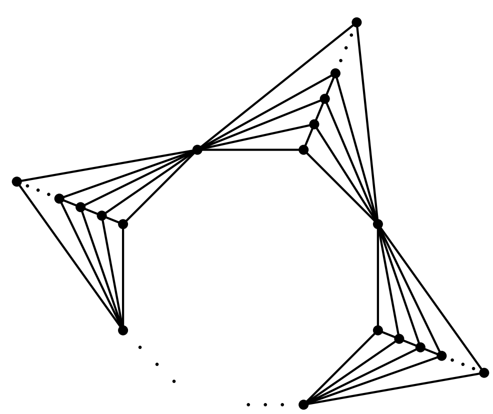


Figure 3.16: The graph obtained by blowing up every other vertex in an even cycle and joining the copies of the vertices by a path. This graph attains the lower bound stated in Conjecture 6 and 7.

# Chapter 4

# Generalized Planar Turán Number of Trees and Other Results

## 4.1 Introduction

Eppstein [32] asked whether for all $H$, the maximum number of copies of $H$ possible in a minor-closed family of $n$-vertex graphs is an integer power of $n$. We state a restricted version of his problem as a conjecture in the planar case, and later generalize it further.

**Conjecture 8.** *For every graph $H$, there exists a non-negative integer $k$, such that*

$$f_{\mathcal{P}}(n, H) = \Theta(n^k).$$

We verify this conjecture in the case of trees. To state our result we require some notation.

**Definition 29.** *For a graph $H$ and an integer $i$, $i \geq 1$, let $\beta_i(H)$ be the maximum number of components in an induced subgraph of $H$ containing only two types of components:*

1. *isolated vertices which have degree one in $H$ and*
2. *paths of length $i-1$ consisting only of vertices of degree two from $H$.*

In particular in the case when $i = 1$, we are interested in the maximum size of an independent set in the graph consisting of vertices of degree at most 2. For simplicity, we let $\beta(G) := \beta_1(G)$ for any graph $G$. Then we have the following result for trees.

**Theorem 59.** *(Győri, Paulos, Salia, Tompkins, Zamora [63])*
*Let $T$ be a tree, then*

$$f_{\mathcal{P}}(n, T) = \Theta(n^{\beta(T)}).$$

For any graph $H$, let $\alpha(H)$ be the independence number of $H$. In the general case we have the following upper bound.

**Theorem 60.** *(Győri, Paulos, Salia, Tompkins, Zamora [63])*
*Let $H$ be any graph, then*

$$f_{\mathcal{P}}(n, H) = O(n^{\alpha(H)}).$$

This theorem will follow as an immediate consequence of the results we prove about degenerate graphs in Section 4.2. As a corollary of Theorem 60, we obtain the order of magnitude of the maximum number of cycles (note that this result was already obtained by Hakimi and Schmeichel in [67]).

**Corollary 5.** *For all $k \geq 3$, we have*

$$f_{\mathcal{P}}(n, C_k) = \Theta(n^{\lfloor k/2 \rfloor}).$$

The lower bound is attained by taking a cycle $C_k$ and blowing up a maximum sized independent set by $\lfloor 2n/k \rfloor - 1$. Note that the constant in the asymptotic notation may depend on $k$, and this construction contains asymptotically $\left(\frac{2n}{k}\right)^{\lfloor k/2 \rfloor}$ copies of $C_k$.

Next we consider the case when the set of forbidden graphs is nonempty. In this case, we pose a conjecture which generalizes Conjecture 8.

**Conjecture 9.** *For all finite sets of graphs $\mathcal{F}$ and for all graphs $H$, we have*

$$\mathrm{ex}_{\mathcal{P}}(n, H, \mathcal{F}) = \Theta(n^k),$$

*for some integer $k$.*

We consider a variation of Theorem 63 for the case when $C_4, C_6, \ldots, C_{2\ell}$ are forbidden. We prove the following.

**Theorem 61.** *(Győri, Paulos, Salia, Tompkins, Zamora [63])*
*For any tree $T$, we have*

$$\mathrm{ex}_{\mathcal{P}}(n, T, \{C_4, C_6, \ldots, C_{2\ell}\}) = \Theta(n^{\beta_\ell(T)}).$$

The lower bound in Theorem 61 is given as follows. Take an induced subgraph of $T$ consisting of $\beta_\ell(T)$ components as described in Definition 29. Replace each path (including the ones of length 0) by $\Omega(n)$ paths of the same length with endpoints joined to the same neighbors as the corresponding paths in $T$ and number of vertices summing to $n$. The resulting graph has $\Omega(n^{\beta_\ell(T)})$ copies of $T$ and contains no cycle in the set $\{C_4, C_6, \ldots, C_{2\ell}\}$. In fact, we believe that a construction of this form should yield the correct asymptotic value of $\mathrm{ex}_{\mathcal{P}}(n, T, \{C_4, C_6, \ldots, C_{2\ell}\})$, but our proof yields the order of magnitude. We also have the following exact result for maximizing the number of $C_5$ copies in a $C_4$-free planar graph.

**Theorem 62.** *(Győri, Paulos, Salia, Tompkins, Zamora [63])*
*For all $n \geq 4$, $n \neq 6$, we have*

$$\mathrm{ex}_{\mathcal{P}}(n, C_5, \{C_4\}) = n - 4.$$

Moreover, we determine the order of magnitude of $\mathrm{ex}_{\mathcal{P}}(n, C_k, \{C_4\})$ for every $k$. We obtain the following result.

**Theorem 63.** *(Győri, Paulos, Salia, Tompkins, Zamora [63])*
*For all $k \geq 5$, we have*

$$\mathrm{ex}_{\mathcal{P}}(n, C_k, \{C_4\}) = \Theta(n^{\lfloor k/3 \rfloor}).$$

We conjecture that in fact a much more general result holds.

**Conjecture 10.** *For sufficiently large $k$, we have*

$$\mathrm{ex}_{\mathcal{P}}(n, C_k, \{C_4, C_6, \ldots, C_{2\ell}\}) = \Theta(n^{\lfloor \frac{k}{\ell+1} \rfloor}).$$

A construction for a lower bound in Conjecture 10 is similar to that of Theorem 61. Namely, we note that $\beta_\ell(C_k) = \lfloor \frac{k}{\ell+1} \rfloor$, and replace each of the $\beta_\ell(C_k)$ paths with $\Omega(n)$ paths of the same length joined to the corresponding pair of vertices. The proof of Theorem 63 can be adapted to resolve Conjecture 10 in the cases when $k$ is congruent to 0, 1 or 2 modulo $\ell + 1$.

We conclude this section by contrasting our results in the planar case with the known results in the general case. It was shown in [55] and [48] that $\mathrm{ex}(n, C_k, C_4) = \Theta(n^{k/2})$. This result is in stark contrast to our results in the planar case in two ways. First, in the planar case the order of magnitude is always an integer power of $n$, and second in the planar case we have $k/3$ rather than $k/2$ in the exponent.

## 4.2 General upper bounds for degenerate graph classes

Alon and Shikhelman [3] proved that for any bipartite graph $H$ and tree $T$ we have $\mathrm{ex}(n, H, T) = O(n^{\alpha(H)})$, where $\alpha(H)$ is the independence number of $H$. This result was extended to all graphs $H$ in [66]. Since the extremal number of a tree $T$ is linear in $n$, it follows that any $T$-free graph has a vertex of degree at most $c_T$, a constant depending on $T$. Call a graph $G$ $c$-degenerate if every subgraph of $G$ contains a vertex of degree at most $c$. The proof from [66] can easily be extended to work for the class of $c$-degenerate graphs. We now present a proof of this theorem for completeness.

First we introduce some notation. For given graphs $G$ and $H$, let $\mathcal{N}(H, G)$ denote the number of copies of $H$ in $G$. Let $\mathcal{G}_c$ denote the class of $c$-degenerate graphs, and let

$$f_c(n, H) := \max\{\mathcal{N}(H, G) : G \in \mathcal{G}_c, v(G) = n\}.$$

**Proposition 1.** *$f_c(n, K_r) = O(n)$, where the constant depends only on $r$ and $c$.*

*Proof.* We proceed by induction on $r$. For $r = 1$ the result is clear, so assume $r > 1$ and that $f_c(n, K_{r-1}) \leq C_{r-1}n$ for a constant $C_{r-1}$. Let $G$ be an $n$-vertex graph in $\mathcal{G}_c$, then we have that

$$r\mathcal{N}(K_r, G) = \sum_{v \in V(G)} \mathcal{N}(K_{r-1}, G[N(v)]) \leq \sum_{v \in V(G)} C_{r-1}d(v) = O(e(G)) = O(n). \quad \square$$

Theorem 60 follows as a simple consequence of the following lemma which will be proven by induction on $\alpha(H)$.

**Lemma 18.** *For any graph $H$*

$$f_c(n+1, H) - f_c(n, H) = O(n^{\alpha(H)-1}).$$

*Here, the constant given by the $O$ notation depends only on $H$ and $c$.*

We start by proving the following well-known fact.

**Proposition 2.** *Let $H$ be a graph, and let $u$ be a vertex of $H$. If $H'$ is the graph obtained from $H$ by removing $u$ together with its neighborhood, then $\alpha(H') \leq \alpha(H) - 1$.*

*Proof.* If $X$ is a maximal independent set in $H'$, then since no neighbor of $u$ is in $X$, the set $X \cup \{u\}$ is independent in $H$ and so $\alpha(H') + 1 \leq \alpha(H)$. $\square$

We are now ready to prove Lemma 18.

*Proof of Lemma 18.* To estimate $f_c(n+1, H) - f_c(n, H)$, we will start with a graph $G \in \mathcal{G}_c$ on $n+1$ vertices with the maximum number of copies of $H$. We know that $\delta(G) \leq c$. Let $v$ be a vertex of minimum degree in $G$. We will estimate the number of copies of $H$ in $G$ containing $v$ as a vertex.

Let $V(H) = \{u_1, u_2, \ldots, u_{v(H)}\}$, and let $H_i$ be the graph obtained by removing $u_i$ together with its neighbors. By Proposition 2, we know that $\alpha(H_i) \leq \alpha(H) - 1$. Now for each copy of $H$ using $v$ as a vertex, $v$ must play the role of some $u_i$, and the neighbors of $u_i$ must be embedded in the neighborhood of $v$. It follows that the other vertices of $H$, that is the vertices of $H_i$, must be embedded in some way in the remaining vertices of $G$. We have to choose $d_H(u_i)$ vertices in $N(v)$, so the number of copies of $H$ using $v$ is at most

$$\sum_{i=1}^{v(H)} d(v)^{d_H(u_i)}\mathcal{N}(H_i, G) \leq \sum_{i=1}^{v(H)} c^{d_H(u_i)}\mathcal{N}(H_i, G) = \sum_{i=1}^{v(H)} O_{H_i}(n^{\alpha(H_i)}) = O(n^{\alpha(H)-1}).$$

Thus, if $G'$ is the graph obtained from $G$ by removing $v$, we have that

$$f_c(n+1, H) = \mathcal{N}(H, G) = \mathcal{N}(H, G') + O(n^{\alpha(H)-1}) \leq f_c(n, H) + O(n^{\alpha(H)-1}). \quad \square$$

## 4.3 The number of trees in planar graphs

In this section we prove Theorem 63. First we provide the lower bound, $f_{\mathcal{P}}(n,T) = \Omega(n^{\beta(T)})$. Observe that, if a given graph is planar and one blows up a set of independent vertices each of degree at most two, then resulting graph is also planar.Therefore the following construction provides the desired lower bound. Given a tree $T$, fix an independent set $S$ of size $\beta(T)$ which contains vertices of degree at most two (as in Definition 29) and blow up this set by $\left\lfloor \frac{n}{2\beta(T)} \right\rfloor$.The resulting graph is planar with at most $n$ vertices, when $n$ is sufficiently large, and contains $\Omega(n^{\beta(T)})$ copies of the tree $T$.

Observe that we have $f_{\mathcal{P}}(n,P_k) = O(n^{\alpha(P_k)})$ from Theorem 60, where $P_k$ denotes path of length $k$. Even more we have $\alpha(P_k) = \beta(P_k)$ from Definition 29. Therefore we have the following simple proposition.

**Proposition 3.** $f_{\mathcal{P}}(n,P_k) = \Theta(n^{\beta(P_k)})$.

Hence Theorem 63 holds for paths. To show that Theorem 63 holds for any tree, we are going to use the following lemma.

**Lemma 19.** *For a given planar graph $G$. Let $v$, $u$ and $w$ be fixed vertices in $G$ and let $n_1$, $n_2$ and $n_3$ be non negative integers. The number of vertices $x$, such that, there are three internally disjoint paths from $x$ to $v$, from $x$ to $u$ and from $x$ to $w$ of length $n_1$, $n_2$ and $n_3$, respectively, is bounded by a constant $C := C(n_1,n_2,n_3)$.*

*Proof.* Suppose we have a planar embedding of $G$. The proof will be by induction on $n_1+n_2$. The result is trivial if either $n_1$ or $n_2$ is equal to 0. So suppose that $n_1+n_2 \geq 2$, and that the result holds for any pair with smaller sum. Consider a maximal set $\mathcal{P}$, of internally vertex disjoint paths $vv_2^i \dots v_{n_1}^i a_i u_2^i \dots u_{n_2}^i u$, where each $a_i$ is such that there exist a length $n_3$ path from $a_i$ to $w$ which does not contain any of the vertices $v, v_2^i, \dots, v_{n_1}^i, u_{n_2}^i, \dots, u_2^i, u$. Let us denote the set of $a_i$ in these paths by $A$. Observe that the paths from $\mathcal{P}$ divide the plane into $|A|$ regions $R_1, R_2, \dots, R_{|A|}$. Since the vertex $w$ is fixed, it is in one of the regions, and there is a path of length $n_3$ from $w$ to each vertex of $A$, not using the vertices $v$ and $u$. Thus $|A| \leq 2n_3+1$.

Now let $Y$ to be the set of $|A|(n_1+n_2-1)+2$ vertices that appear in some path from $\mathcal{P}$, and let $X$ be the set of those vertices $x$ in $G$ which are not in $Y$ such that there exist three internally disjoint paths from $x$ to $v$, from $x$ to $u$ and from $x$ to $w$ of length $n_1$, $n_2$ and $n_3$, respectively. It is sufficient to bound $|X|+|Y|$ by a constant depending on $n_1$, $n_2$ and $n_3$. If $X=\emptyset$ we immediately have the required bound. Suppose $X$ is nonempty and let $x \in X$, and let $P_1 = vv_2\dots v_{n_1}x$ and $P_2 = xu_{n_2}\dots u_2u$ be two of the three internally disjoint paths from $x$. Let $v'$ and $u'$ be the first vertex (closest to $x$ in $P_i$) in the intersection of $Y$ with $P_1$ and $P_2$, respectively. Note that it is possible for $v'$ to be $v$ or $u'$ to be $u$, but by the definition of $Y$ and $\mathcal{P}$, is not possible for both to happen simultaneously. Then the vertex $x$

is such that there exist three internally disjoint paths from $x$ to $v'$, from $x$ to $u'$ and from $x$ to $w$ of length $n_1', n_2'$ and $n_3$ respectively, where $1 \le n_i' \le n_i$ for $i = 1, 2$ and $1 \le n_3$, with the additional property that $n_1' + n_2' < n_1 + n_2$. Therefore, setting $C' = \max_{n_1'+n_2'<n_1+n_2} C(n_1', n_2', n_3)$, we have that

$$|X| \le \binom{|A|\,(n_1+n_2-1)+2}{2} C' \le \binom{(2n_3+1)(n_1+n_2-1)+2}{2} C'.$$

Thus, $|X| + |Y|$ is bounded and so the lemma holds. □

Note that Lemma 19 implies in particular that if $G$ is a planar graph and $T$ is a tree with $s \ge 3$ leaves $x_1, x_2, \ldots, x_s$. Then for any vertex $x \in V(T)$ of degree at least 3 and $v_1, v_2, \ldots, v_s \in V(G)$, the number of vertices $v \in V(G)$, such that, there exists a copy of $T$ where $x$ is embed in $v$ and $x_i$ is embed in $v_i$, for $i = 1, \ldots, s$, is bounded by a constant that does not depend on $G$. That is since we are able to find three different leaves such that the paths from $x$ to each of these leaves are internally disjoint. At this point we are ready to prove Theorem 63.

*Proof of Theorem 63.* We may assume $T$ is not a path otherwise we are done, from Proposition 3.

Let $G$ be an $n$-vertex planar graph. Let $A$ be the set of vertices of degree at least 3 in $T$, and let $T_1, T_2, \ldots, T_k$ be the connected components of the graph induced by $V(T) \setminus A$ (the set $A$ is non-empty since $T$ is not a path). Observe that since $A$ has every vertex of degree at least 3, then every vertex of $T_i$ has degree at most 2 in both $T_i$ and $T$, so we have

$$\beta(T) = \beta\left(\bigcup_{i=1}^{k} T_i\right) = \sum_{i=1}^{k} \beta(T_i).$$

Moreover the trees $T_i$ are paths and so $\mathcal{N}(T_i, G) = O(n^{\beta(T_i)})$, from Proposition 3. Then for any embedding of the trees $T_i$ by Lemma19, there is a constant number of ways to complete the embedding of $A$ to a copy of $T$. Therefore the number of copies of $T$ is bounded by $O(n^{\sum_{i=1}^{k}\beta(T_i)}) = O(n^{\beta(T)})$. □

## 4.4 The number of $C_5$'s in $C_4$-free planar graphs

In this section we are going to prove Theorem 62, namely that for all $n$, $n \ge 4$, $n \ne 6$, we have

$$\mathrm{ex}_{\mathcal{P}}(n, C_5, \{C_4\}) = n - 4.$$

*Proof.* We begin by providing the lower bound. Let $n = 5 + 3t + 2s$ for some nonnegative integers $s$ and $t$. (Note that when $n = 6$, it is easy to verify that there can be at most one pentagon, thus $\mathrm{ex}_{\mathcal{P}}(6, C_5, \{C_4\}) = 1$.) The construction is as follows: Take a pentagon $x_1x_2x_3x_4x_5x_1$, as well as $t$ internally vertex disjoint paths $x_1y_3^iy_4^iy_5^ix_2$ (where $1 \leq i \leq t$ and $y_j^i \notin \{x_3, x_4, x_5\}$, $j \in [3]$) between $x_1$ and $x_2$ and add the edges so that $x_4, y_4^1, \dots, y_4^t$ forms a path. Next take a path $z_1z_2 \dots z_{2s}$ on new set of vertices. Add the edge from $z_1$ to $x_1$ and the edges from $z_i$ to $x_5$, for odd $i \equiv 0, 1 \pmod 4$, and $z_i$ to $x_3$, for $i \equiv 2, 3 \pmod 4$. See Figure 4.1 for an example of an extremal graph.

Observe that the 5-cycles in the construction are either $x_1x_2x_3x_4x_5$ or the 5-cycles containing $y_4^i$, $1 \leq i \leq t$ or the 5-cycles containing an edge $z_iz_{i+1}$, $1 \leq i \leq s$ and $i \equiv 1, 3 \pmod 4$.

For $1 \leq i \leq t$, we have three distinct 5-cycles containing $y_3^i$ in the construction. These are $x_2y_3^iy_4^iy_4^{i-1}y_3^{i-1}x_2$ or $x_1y_5^iy_4^iy_4^{i-1}y_5^{i-1}x_1$ or $x_2y_3^iy_4^iy_5^ix_1x_2x_2$, where $y_3^{i-1}, y_4^{i-1}$ and $y_5^{i-1}$ are respectively $x_3, x_4$ and $x_5$ when $i = 1$. Thus we have $3t$ 5-cycles in the construction containing $y_4^i$, $1 \leq i \leq t$.

On the other hand for $1 \leq i \leq s$ and $i \equiv 1, 3 \pmod 4$, each edge $z_iz_{i+1}$ is contained in two distinct 5-cycles of the construction. Indeed, for an edge $z_iz_{i+1}$ ($i \equiv 1 \pmod 4$), notice that $z_{i+1}$ and $z_{i-2}$ are adjacent with $x_3$ and $z_i$ is adjacent with $x_5$. Hence we have the 5-cycles $x_3z_{i+1}z_ix_5x_4x_3$ and $x_3z_{i+1}z_iz_{i-1}z_{i-2}x_3$ which contain the edge $z_iz_{i+1}$. In the case that $i = 1$, we take $x_1$ and $x_2$ in place of $z_{i-1}$ and $z_{i-2}$ respectively. For an edge $z_iz_{i+1}$ ($i \equiv 3 \pmod 4$), observe that $z_{i+1}$ and $z_{i-2}$ are adjacent with $x_5$ and $z_i$ is adjacent with $x_3$. In this scenario, we have two distinct 5-cycles containing the edge $z_iz_{i+1}$, namely $x_5z_{i+1}z_iz_{i-1}z_{i-2}x_5$ and $x_5z_{i+1}z_ix_3x_4x_5$. Thus, we have $2s$ 5-cycles in the construction containing $z_iz_{i+1}$, where $1 \leq i \leq s$. Therefore, the number of 5-cycles in the construction is $3t + 2s + 1 = n - 4$.

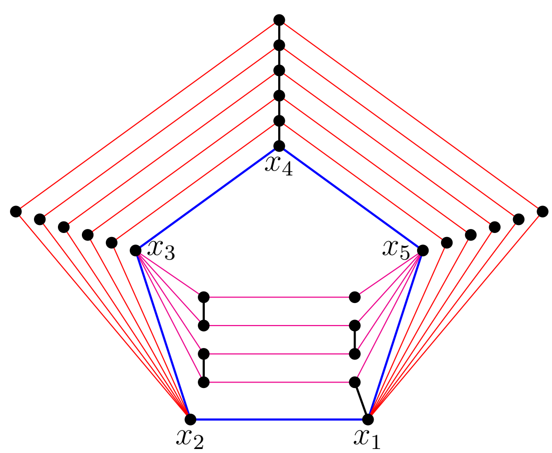



Figure 4.1: Example of an extremal graph for Theorem 62.

Now we are going to prove by induction that $\mathrm{ex}_{\mathcal{P}}(n, C_5, \{C_4\}) \leq n - 4$. The proof proceeds by induction on $n$ with the base cases $\mathrm{ex}_{\mathcal{P}}(4, C_5, \{C_4\}) = 0$, $\mathrm{ex}_{\mathcal{P}}(5, C_5, \{C_4\}) = 1$ and $\mathrm{ex}_{\mathcal{P}}(6, C_5, \{C_4\}) = 1$. Let $G$ be an $n$ vertex, $C_4$-free planar graph with $n \geq 7$. Without loss of generality we may assume that $G$ is connected.

Consider an embedding of $G$ on the plane. To prove an upper bound for Theorem 62, we take a planar embedding of $G$ and consider two cases.

**Case 1:** *All pentagons in $G$ are face.* We can assume that $G$ is a connected plane graph. Otherwise, we can still join the components of $G$ with edges so that there is no new cycle created.

Remove an edge from each triangular face. Observe that two triangular faces do not share an edge since $G$ is $C_4$-free. Let us denote the resulting graph by $G'$. Since there is no pentagon in $G'$ which is not a face, the total number of faces of $G'$ is at least the number of pentagonal faces in $G$.

Let $f_i$ denote the number of faces of size $i$ in $G'$, and let $f$ denote the number of faces in $G$. Since $G'$ is connected and $n \geq 7$, $f_1 = f_2 = 0$. Moreover, since $G'$ contains no triangular face and $G'$ contains no $C_4$ we have $f_3 = f_4 = 0$. Thus,

$$2e(G') = \sum_{i\geq 5} if_i = 5f_5 + \sum_{i\geq 6} if_i \geq 5f_i + 5(f - f_i) = 5f.$$

This implies $5f \leq 2e(G')$. Using the Euler's formula, $f + n = e(G') + 2$, we get $f \leq \frac{2}{3}n - \frac{4}{3}$. Thus we have that the number of $C_5$ copies in $G$ is at most $n - 4$, since $n \geq 7$.

**Case 2:** *There is a pentagon in $G$ which is not a face.* Let $P$ be a non-facial pentagon in $G$. The pentagon $P$ cuts the plane into two regions. Let us denote the subgraph of $G$ in the inner and outer regions of $P$ by $G_1$ and $G_2$, respectively, where both graphs include the vertices of the pentagon. Assume that $G_1$ and $G_2$ have $n_1$ and $n_2$ vertices respectively. Thus we have $n = n_1 + n_2 - 5$. Since $n_1$ and $n_2$ are both non-zero and less than $n$ and both $G_1$ and $G_2$ are $C_4$-free, then by induction hypothesis, the number of pentagons in $G_1$ and $G_2$ is at most $n_1 - 4$ and $n_2 - 4$, respectively.

It can be checked that there is no pentagon crossing $P$ in $G$. Indeed, let $P = x_1x_2x_3x_4x_5x_1$. Notice that since $G$ is $C_4$-free, no two non-consecutive vertices of $P$ are adjacent. Suppose that there is a pentagon, say $P'$, crossing $P$. $P'$ must contain a cherry $x_iyx_j$ where $y$ is a vertex either in the interior or in the exterior region of $P$. Without loss of generality suppose $y$ is in the interior region of $P$. Notice that since $G$ is $C_4$-free, $x_i$ and $x_j$ can not be non-consecutive vertices of the pentagon $P$. Thus, we assume that $x_i$ and $x_j$ are adjacent vertices of $P$. Without loss of generality assume that $i = 1$ and $j = 2$. Since any two non-consecutive vertices of $P$ are not adjacent, there are two vertices say, $y_1$ and $y_2$ in the exterior region of $P$ such that $P' = x_1yx_2y_1y_2x_1$. However, in this case we got a $C_4$, namely $x_1x_2y_1y_2x_1$, which is a contradiction to the fact that $G$ is $C_4$-free.

From the above observation it follows that every pentagon of $G$ is a pentagon of $G_1$ or $G_2$. Since the pentagon $P$ is in both graphs, we get that the number of pentagons in $G$ is at most $n_1 - 4 + n_2 - 4 - 1 = n + 5 - 9 = n - 4$, completing the proof. $\square$

## 4.5 The order of magnitude of $C_k$ in a $C_4$-free planar graph

We have seen in Corollary 5 that $f_{\mathcal{P}}(n, C_k) = \Theta(n^{\lfloor k/2 \rfloor})$ follows immediately from Theorem 60. We now consider the case when $C_4$ is forbidden and prove Theorem 63 which states that $\mathrm{ex}_{\mathcal{P}}(n, C_k, \{C_4\}) = \Theta(n^{\lfloor k/3 \rfloor})$.

*Proof.* For the construction we take a cycle $C_k$ and find an induced matching of size $\lfloor k/3 \rfloor$. Next we replace each edge in this matching with $\frac{n-k}{2\lfloor k/3 \rfloor}$ edges each adjacent to the same pair of vertices as the original edge. This graph clearly has $\Theta(n^{\lfloor k/3 \rfloor})$ copies of $C_k$ and at most $n$ vertices, when $n$ is sufficiently large.

We will now prove the upper bound. It is well-known that every planar graph contains a vertex of degree at most 5. It was proved in [72] that a $C_4$-free planar graph with minimum degree at least 2 contains an edge $xy$ such that $d(x)+d(y)$ is at most 8. This result was improved to 7 in [16], which is best possible. Note that, in proving Theorem 63, we may delete all edges containing a vertex of degree at most one, since such vertices do not contribute to any $k$-cycles.

We will distinguish cases based on the value of $k$ modulo 3. When $k$ is equal to 0 or 1 modulo 3, the result can be proved using the fact that a planar graph contains a vertex of degree at most 5. We present here the proof in the case when $k \equiv 2 \pmod 3$, the other cases are similar but require only the fact that there is a vertex of bounded degree.

Suppose $k = 3m + 2$ for some integer $m \geq 1$ and that $G$ is an $n$-vertex, $C_4$-free planar graph with no vertex of degree at most 1. Let us label the vertices of $C_k$ by $v_1, v_2, \ldots, v_{3m+2}$, consecutively. Applying the aforementioned result of [16] we find an edge $uv$ such that $d(u)+d(v) \leq 7$. We will show that at most $n^{m-1}$ cycles $C_k$ can use the edge $uv$. Then, by iteratively deleting such an edge edge and the resulting vertices of degree 1, we find all copies of $C_k$ after at most $O(n)$ steps (since there are at most linearly many edges in a planar graph). Thus in total we will have shown that there are at most $n^m$ copies of $C_k$ in $G$, as required.

Let us consider a copy of $C_k$ so that $u$ and $v$ correspond to $v_1$ and $v_2$ in the $C_k$. There is at most 7 ways to embed $v_{3m+2}$ and $v_3$ as neighbors of $v_1$ and $v_2$, respectively. Thus we now consider $v_{3m+2}, v_1, v_2$ and $v_3$ as being embedded. Next choose edges of $G$ at which to embed the edges of the cycle $v_5v_6$, $v_8v_9$, $v_{11}v_{12}$ and so on. We now consider embedding the remaining vertices. Since our graph is $C_4$-free, there is at most one way to embed the vertices $v_4$, $v_7$, $v_{10}$ and so on. It follows that we have at most on the order of $n^{m-1}$ copies of the cycle $C_k$ in the graph which use the edge $uv$, and we are done. □

## 4.6 Number of copy of a tree in an even cycle free planar graph

In this section we prove Theorem 61, namely that

$$\mathrm{ex}_{\mathcal{P}}(n, T, \{C_4, C_6, \ldots, C_{2\ell}\}) = \Theta(n^{\beta_\ell(T)}).$$

*Proof of Theorem 61.* First we will show that the result is true for paths. We will make use of the following theorem due to Lam and Verstraete.

**Theorem 64.** *(Lam, Verstraete [80]) Let $G$ be a graph containing no even cycles of length at most $2\ell$. There exists a constant $D_\ell$ such that for any $v, u$ vertices of $G$ and $k \leq \ell$ a positive integer, the number of paths from $v$ to $u$ of length $k$ in $G$ is at most $D_\ell$.*

It is simple to check that $\beta_\ell(P_k) = 1 + \left\lfloor \frac{k+\ell-1}{\ell+1} \right\rfloor$. Now we will prove that

$$\mathrm{ex}_{\mathcal{P}}(n, P_k, \{C_4, C_6, \ldots, C_{2\ell}\}) = O\left(n^{1+\left\lfloor \frac{k+\ell-1}{\ell+1} \right\rfloor}\right).$$

Let $G$ be an $n$-vertex planar graph containing no even cycle of length at most $2\ell$, for each $k$-vertex path $v_1, v_2, \ldots, v_{k+1}$ in $G$. Suppose $k \geq 2$ otherwise we are already done, and consider the edges $v_{(\ell+1)i+1}v_{(\ell+1)i+2}$ for $i = 0, 1, \ldots, \left\lfloor \frac{k+\ell-1}{\ell+1} \right\rfloor - 1$ and the edge $v_k v_{k+1}$. To bound the number of paths, we notice that, we have a linear number of choices for each of these edges, and in total the number of choices is of order $n^{1+\left\lfloor \frac{k+\ell-1}{\ell+1} \right\rfloor}$. After choosing the edges, by Theorem 64 there is a constant number of ways to add the paths between two consecutive edges. Therefore $\mathcal{N}(G, P_k) = O\left(n^{1+\left\lfloor \frac{k+\ell-1}{\ell+1} \right\rfloor}\right)$.

Now let $T$ be any tree. Let us partition the vertex set $V(T)$ into five sets $A_1$, $A_2$, $A_2'$, $A_2''$ and $A_{\geq 3}$. First we partition the set of vertices of degree not equal to 2 as follows.

$$A_1 = \left\{v \in V(T) \middle| d(v) = 1\right\} \text{ and } A_{\geq 3} = \left\{v \in V(T) \middle| d(v) \geq 3\right\}.$$

In particular, $A_1$ is the set of leaves of the tree $T$. Now we will partition the set of vertices of degree equal to 2 into three sets. Let

$$A_2 = \left\{v \in V(T) \middle| d(v) = 2 \text{ and there is no vertex of } A_{\geq}3 \text{ at distance less than } \ell \text{ from } v\right\}.$$

Now consider every path $P$ in $T$ such that:

$(i)$ Both end vertices of $P$ have degree at least 3 in $T$.

$(ii)$ The length of $P$ is at least $\ell + 1$, but at most $2\ell - 1$.

$(iii)$ Every internal vertex of $P$ has degree 2 in $T$. For each such path let $f(P)$ be the middle vertex of $P$, if $P$ has odd length, take either of the two middle vertices.

Let $A_2'$ be the set of consisting of the vertices $f(P)$ for the paths $P$ defined above. Finally, define

$$A_2'' = \left\{ v \in V(T) \middle| d(v) = 2, v \notin A_2 \cup A_2' \right\}.$$

Let $F$ be the subgraph of $T$ induced by the vertex set $V_1 = A_1 \cup A_2 \cup A_2'$. Note that $F$ is a path forest and suppose $F = P_{i_1} \cup P_{i_2} \cup \dots \cup P_{i_t}$ for paths $P_{i_j}$, $1 \leq j \leq t$.

Now we will show that $\beta_\ell(F) = \beta_\ell(T)$. Take a set $S$ of vertices and paths which is a witness for the value of $\beta_\ell(T)$ in $T$. Suppose $S$ contains a path $P = x_1 x_2 \dots x_\ell$ using at least one vertex from $A_2''$. Note that it is not possible for this path to be fully contained in $A_2''$. Indeed, if it was contained in $A_2''$ we would be able to find an $i$ such that $x_i$ is at distance $\ell - 1$ from a vertex $v \in A_{\geq 3}$ and $x_{i+1}$ is at distance $\ell - 1$ from a vertex $u \in A_{\geq 3}$ and so $P$ would be contained in a path of length $2\ell + 1$ from $v$ to $u$. It follows that the middle vertex of this path must be a vertex of $P$. So we may replace the choice of $P$ in $S$ with a vertex (leaf) $f(P)$ in $F$. We obtain that $\beta_\ell(F) \geq \beta_\ell(T)$.

Next we show $\beta_\ell(F) \leq \beta_\ell(T)$. Take a set $S$ of vertices and paths which is a witness for $\beta_\ell(F)$ in $F$. If $P$ is a length $\ell$ path in $S$, by definition, no end vertex is a leaf in $F$, so $P$ is still a length $\ell$ path in $T$ such that every vertex has degree 2. If $v$ is a leaf in $F$, but is no longer a leaf in $T$, then $v$ must have degree 2 in $T$, and there are two possibilities. Suppose $v \in A_2''$, then there exists a path $P$ of length at least $\ell + 1$ with internal vertices of degree 2 containing $v$. In this case we may replace the choice of $v$ in $S$ by a subpath of $P$ of length $\ell - 1$ without using the end vertices of $P$. Suppose $v$ is in $A_2$, since one neighbor of $v$ is not in $F$, it must be in $A_{\geq 3} \cup A_2''$, but by definition of $A_2$, then $v$ must have a neighbor in $A_2$ and so $v$ is at distance $\ell$ of $A_{\geq 3}$. Let $u$ be the closest neighbor of $v$ in $u$, and let $P$ be the length $\ell$ path from $v$ to $u$, then we may replace the choice of $v$ in $S$ by $P'$ the path obtained from $P$ by deleting $u$. It follows that $\beta_2(F) \leq \beta_2(T)$.

Now let $G$ be an $n$-vertex planar graph containing no even cycle of length at most $2\ell$ and fix a copy of $F$ in $G$. By Lemma 19 since every leaf of $T$ is already fixed, we have a bounded number of choices to embed the vertices of $A_{\geq 3}$ in $G$ such that together with the copy of $F$ the embedding can be completed to a copy of $T$. For any given embedding of $F$ and $A_{\geq 3}$, let $x \in A_2''$. By the definition of $A_2''$, there is a vertex $a \in A_{\geq 3}$ with distance less than $\ell$ to $x$. If there are two choices for $a$, pick one which is closest to $x$, on the branch from $x$ that does not contain $a$, and let $b$ be the closest vertex of $V \setminus A_2''$. We show that the distance between $a$ and $b$ is at most $\ell$. Suppose by contradiction $b$ is at distance more than $\ell$ from $a$, then pick $c$ to be the vertex between $x$ and $b$ at distance exactly $\ell$ from $a$. It follows that $c \in A_2''$, so there is a vertex $d \in A_{\geq 3}$ in the same branch from $x$ as $b$ and $c$, at distance at most $\ell$ from $c$. Hence the path from $a$ to $d$ has length at most $2\ell - 1$ and its middle vertex $y$ is in $F$ and it is closer to $x$ than $y$, but this contradicts the choice of $b$ since $y$ is also in the branch from $x$ not containing $a$. By Theorem 64, there is a constant number of choices to embed $x$ in $G$ such that the embedding can be completed to $T$, since for each such embedding we have a path of length at most $\ell$ form the corresponding vertices of $a$ and $b$ in $G$. $\square$

# Chapter 5

# Induced Generalized Planar Turán Number of Cycles

## 5.1 Introduction

For a graph $H$, the extremal construction that attains $f_{\mathcal{P}}(n, H)$ copies of $H$ is a maximal planar graph. It is natural to ask the induced copies of $H$ in a planar graph. In other words, what is the maximum number of induced copies of a graph $H$ in an $n$-vertex planar graph, $f_{\mathcal{P}}^{\text{ind}}(n, H)$ (see Definition 21)?

Such an extremal graph problem is not much studied. Interestingly only few results for some particular cycles are known so far. As it is already mentioned in the preliminary chapter, Theorem 45 and Theorem 47, we have $f_{\mathcal{P}}^{\text{ind}}(n, C_3) = f_{\mathcal{P}}(n, C_3)$ and $f_{\mathcal{P}}^{\text{ind}}(n, C_{2m}) \approx f_{\mathcal{P}}^{\text{ind}}(n, C_{2m})$, where $m \in \{2, 3, 4, 5, 6\}$.

The extremal construction of generalized planar Turán number of $C_5$ (see $D_n$ in Figure 1.9) contains only few (linear) number of induced 5-cycles. It is natural to ask, the maximum number of induced 5-cycles in a planar graph.

Recently in [50], we determined $f_{\mathcal{P}}^{\text{ind}}(n, C_5)$ exactly, for $n$ sufficiently large.

## 5.2 Main result

In order to state the formula, we define the following function.

**Definition 30.** *For $n \geq 7$, let*

$$h(n) = \max\{k_1k_2 + k_2k_3 + k_3k_1 : k_1, k_2, k_3 \in \mathbb{N}, k_1 + k_2 + k_3 = n - 4\} + 2.$$

Clearly, the maximum is attained when $k_1$, $k_2$ and $k_3$ are as close as possible. In particular, $h(n) = n^2/3 + O(n)$.

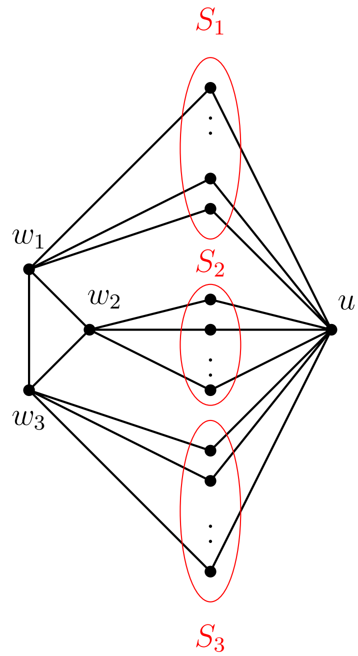



Figure 5.1: Planar graphs containing asymptotically maximum number of induced 5-cycles.

**Theorem 65.** *(Ghosh, Győri, Janzer, Paulos, Salia, Zamora [50])*
*There exists a positive integer $n_0$ such that if $n \geq n_0$ and $G$ is a planar graph on $n$ vertices, then $G$ contains at most $h(n)$ induced 5-cycles. Moreover, there exists a planar graph on $n$ vertices which contains precisely $h(n)$ induced 5-cycles. That means for $n \geq n_0$, we have*

$$f_{\mathcal{P}}^{\text{ind}}(n, C_5) = h(n).$$

We first describe the extremal graph $H_n$. Since the graph has a rather complex structure, we first present a simpler $n$-vertex planar graph which has $h(n)-2$ induced 5-cycles.

Let $S_1$, $S_2$ and $S_3$ be pairwise disjoint sets of vertices such that $|S_1|+|S_2|+|S_3| = n-4$ and $|S_1|, |S_2|, |S_3|$ are as close as possible. We define an $n$-vertex planar graph $G$ as follows. The vertex set of $G$ is the three sets of vertices $S_1$, $S_2$ and $S_3$ together with four vertices, say $w_1, w_2, w_3$ and $u$. That is, $V(G) = S_1 \cup S_2 \cup S_3 \cup \{w_1, w_2, w_3, u\}$. We define the edges of $G$ as $E(G) = \{w_1w_2, w_2w_3, w_3w_1\} \cup \{w_1v, vu \mid v \in S_1\} \cup \{w_2v, vu \mid v \in S_2\} \cup \{w_3v, vu \mid v \in S_3\}$ (see Figure 5.1). It can be checked that $G$ contains exactly $|S_1||S_2| + |S_2||S_3| + |S_3||S_1| = h(n) - 2$ induced $C_5$'s.

After this warm-up example, we are ready to define an $n$-vertex planar graph which has $h(n)$ induced 5-cycles, and hence (in light of Theorem 65) is extremal for all large $n$. For an illustration, see Figure 5.2 (to see that the graph is planar, one can replace the straight dotted line between $w_1$ and $x_{3,1}$ by a non-straight edge).

**Definition 31.** *For every $n \geq 10$, define $H_n$ to be the following planar graph on $n$ vertices.*

*The vertex set of $H_n$ is $\{w_1, w_2, w_3, w_4, u\} \cup S_1 \cup S_2 \cup S_3 \cup \{x_{1,2}, x_{2,4}, x_4, x_{4,3}, x_{3,1}\}$, where $|S_1| + |S_2| + |S_3| = n - 10$ and $|S_1| + 3$, $|S_2|$ and $|S_3|$ are as close as possible.*

*The edges of $H_n$ are as follows:*

- $w_1w_2, w_2w_3, w_3w_1, w_2w_4, w_4w_3 \in E(H_n)$;
- *for every* $i \in \{1, 2, 3\}$ *and* $z \in S_i$, $w_iz, zu \in E(H_n)$;
- *for every* $(i, j) \in \{(1, 2), (2, 4), (4, 3), (3, 1)\}$, $w_ix_{i,j}, w_jx_{i,j}, x_{i,j}u \in E(H_n)$;
- $w_4x_4, x_4u, x_{2,4}x_4, x_4x_{4,3} \in E(H_n)$.

After a somewhat tedious check, one can see that when $n$ is sufficiently large, then the only induced $C_5$ in $H_n$ which does not contain $u$ is $w_2w_3x_{4,3}x_4x_{2,4}$. Moreover, the induced $C_5$'s containing $u$ also contain precisely two of the $w_i$'s. The number of induced $C_5$'s containing $u$, $w_1$ and $w_2$ is $(|S_1| + 1)(|S_2| + 1)$. Indeed, the common neighbour of $u$ and $w_1$ in such an induced $C_5$ can be any vertex from $S_1 \cup \{x_{3,1}\}$ (but not $x_{1,2}$ because it is a neighbour of $w_2$), and the common neighbour of $u$ and $w_2$ can be any vertex from $S_2 \cup \{x_{2,4}\}$. Similarly, the number of induced $C_5$'s containing $u$, $w_1$ and $w_3$ is $(|S_1| + 1)(|S_3| + 1)$, the number of induced $C_5$'s containing $u$, $w_2$ and $w_3$ is $(|S_2| + 2)(|S_3| + 2)$, the number of induced $C_5$'s containing $u$, $w_2$ and $w_4$ is $2(|S_2| + 1)$, the number of induced $C_5$'s containing $u$, $w_3$ and $w_4$ is $2(|S_3| + 1)$ and there is no $C_5$ containing $u$, $w_1$ and $w_4$. Altogether, we find that the number of induced $C_5$'s in $H_n$ is

$$\begin{aligned}
&1 + (|S_1| + 1)(|S_2| + 1) + (|S_1| + 1)(|S_3| + 1) + (|S_2| + 2)(|S_3| + 2) \\
&+ 2(|S_2| + 1) + 2(|S_3| + 1) \\
&= (|S_1| + 4)(|S_2| + 1) + (|S_1| + 4)(|S_3| + 1) + (|S_2| + 1)(|S_3| + 1) + 2.
\end{aligned}$$

Since $(|S_1| + 4) + (|S_2| + 1) + (|S_3| + 1) = n - 4$ and $|S_1| + 4, |S_2| + 1, |S_3| + 1$ are as close as possible, the above expression is equal to $h(n)$.

It remains to prove the upper bound in Theorem 65. This is done by an induction argument. The key result is the following.

**Proposition 4.** *There exists a positive integer $n_1$ such that the following holds. Let $n \geq n_1$ and let $G$ be an $n$-vertex planar graph which has $h(n) + t$ induced 5-cycles for some $t \geq 1$. Then either $G$ has a subgraph on $n - 1$ vertices which has at least $h(n - 1) + t + 1$ induced 5-cycles, or $G$ has a subgraph on $n - 3$ vertices which has at least $h(n - 3) + t + 1$ induced 5-cycles.*

Let us see how this proposition implies our main result.

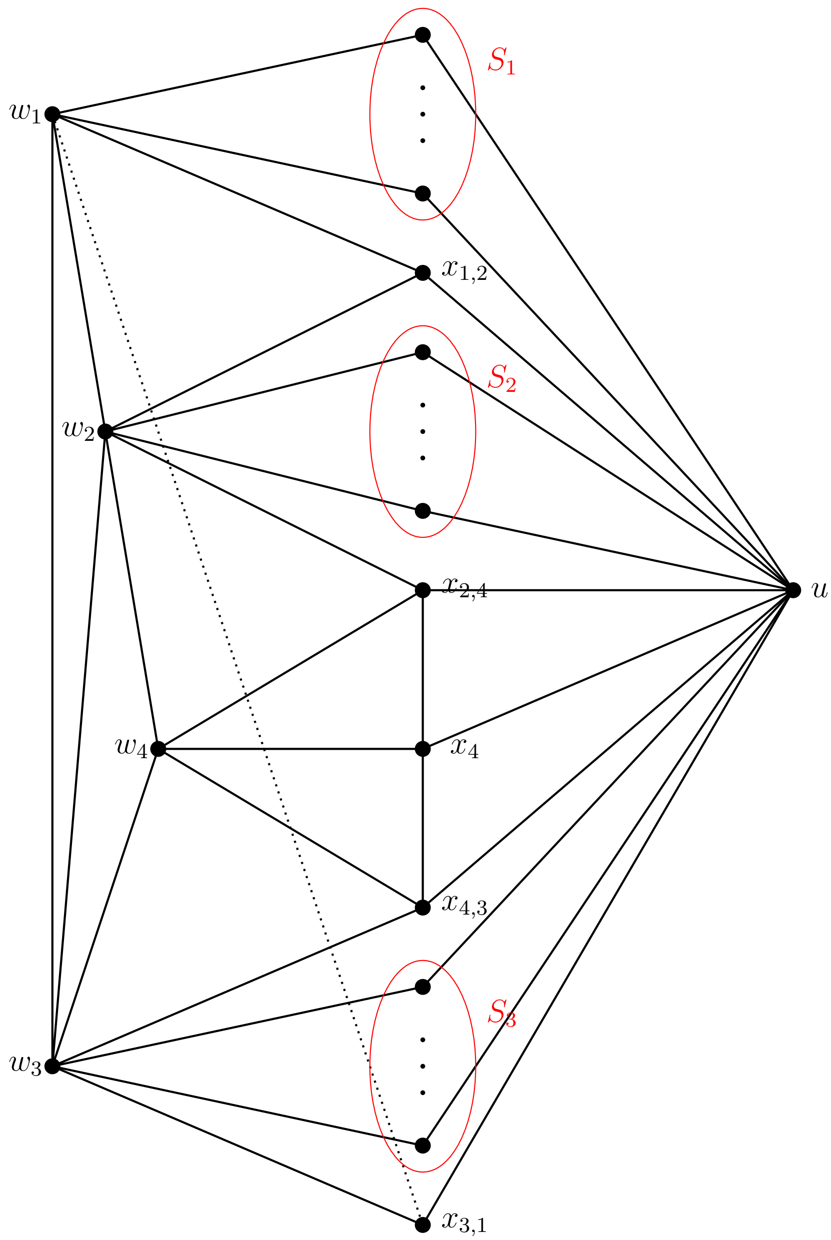



Figure 5.2: The planar graph $H_n$ containing the maximum number of induced 5-cycles

*Proof of Theorem 65.* We have already given a construction which has $h(n)$ induced 5-cycles, so it suffices to prove the upper bound. Let $n_1$ be the positive integer provided by Proposition 4. Define $n_0 = n_1 + 3n_1^5$ and let $G$ be a planar graph on $n \geq n_0$ vertices. Assume, for contradiction, that $G$ contains more than $h(n)$ induced 5-cycles. Let the number of 5-cycles in $G$ be $h(n)+t$ for some $t \geq 1$. By Proposition 4, $G$ has a subgraph $G'$ on $n' \geq n-3$ vertices which has at least $h(n')+t+1$ induced 5-cycles. Again by Proposition 4, $G'$ has a subgraph $G''$ on $n'' \geq n'-3 \geq n-6$ vertices which has at least $h(n'')+t+2$ induced 5-cycles. Repeat this as long as the subgraph has at least $n_1$ vertices. Eventually, we are left with a subgraph $G_{\text{final}}$ on $n_{\text{final}} < n_1$ vertices which has at least $h(n_{\text{final}})+t+n_1^5$ induced 5-cycles. This is clearly a contradiction. □

The rest of the section is devoted to proving Proposition 4. Note that if there exists a vertex which is contained in fewer than $h(n)-h(n-1)$ induced 5-cycles, then we can remove this vertex and we are done. Similarly, if there are three vertices such that their removal deletes fewer than $h(n)-h(n-3)$ induced 5-cycles, then we can remove these vertices and we are done. Hence, in what follows, we can assume that such vertices do not exist. We will use these assumptions to find more and more structure in our graph. Eventually, the structure of the remaining possibilities will be so restricted that we can directly bound the number of induced 5-cycles, and thereby reach a contradiction.

We will see by straightforward computations that $h(n)-h(n-1) > 2n/3-4$ and $h(n)-h(n-3) = 2n-11$.

In Section 5.4, we prove that if (a drawing of) our graph does not contain a $K_{2,7}$ which is “empty", i.e. which has no other vertices inside, then there is a vertex which is contained in at most $11n/20$ induced 5-cycles. Since $11n/20$ is less than $h(n)-h(n-1)$ for large $n$, this means that we can assume that our graph does contain an empty $K_{2,7}$.

In Section 5.5, we reveal even more structure in our graph by using that there is no vertex which is contained in less than $2n/3-10$ induced 5-cycles (note that $2n/3-10 < h(n)-h(n-1)$). For example, Lemma 26 provides us with a structure that already starts to resemble the near-extremal graph depicted in Figure 5.1.

Since the value of $h(n)-h(n-1)$ depends on the remainder of $n$ modulo 3, it is not convenient to remove just one vertex when we are already very close to the extremal example. Instead, in Section 5.6, we carefully choose three vertices whose removal does not decrease the number of induced 5-cycles by too much. More precisely, in Lemma 30, we prove that in a graph which has all the properties that we have already obtained by the earlier lemmas, either there are three vertices whose removal deletes fewer than $h(n)-h(n-3)$ induced 5-cycles, or the graph has a very specific structure (and is very similar to the extremal graph $H_n$).

In Section 5.7, we show that if the graph has the structure forced by Lemma 30, then it has at most $h(n)$ induced 5-cycles, completing the proof of Proposition 4.

## 5.3 Preliminaries

We start with a basic lemma, which we are going to use throughout the paper.

**Lemma 20.** *Let $G$ be a planar graph, let $v \in V(G)$, and let $u$ and $w$ be distinct neighbours of $v$. Let $X_0 = N(u) \setminus (N(w) \cup \{w\})$ and let $Y_0 = N(w) \setminus (N(u) \cup \{u\})$. Let $X$ be the subset of $X_0$ consisting of those vertices that have at least one neighbour in $Y_0$, and let $Y$ be the subset of $Y_0$ consisting of those vertices that have at least one neighbour in $X_0$. Let $\mu$ be the number of connected components in the induced bipartite subgraph of $G$ with parts $X$ and $Y$. Then the number of induced $C_5$'s in $G$ containing $u$, $v$ and $w$ is at most $|X| + |Y| - \mu$. In particular, it is always at most $|X| + |Y| - 1$.*

*Proof.* Clearly any such $C_5$ contains precisely one vertex from each of $X$ and $Y$. Hence, the number of such induced $C_5$'s is at most the number of edges between $X$ and $Y$. However, the induced bipartite subgraph of $G$ with parts $X$ and $Y$ is acyclic. Indeed, suppose that there is a cycle $x_1y_1x_2y_2 \dots x_ky_kx_1$ with $x_i \in X$ and for all $i$ and $y_j \in Y$ for all $j$. The subgraph of $G$ with vertices $u, v, w, x_1, y_1, \dots, x_k, y_k$ and edges $uv, vw, ux_1, ux_2, wy_1, wy_2, x_1y_1, y_1x_2, x_2y_2, y_2x_3, \dots, y_kx_1$ is a subdivision of $K_{3,3}$ with the parts being $\{u, y_1, y_2\}$ and $\{w, x_1, x_2\}$. Indeed, the only edge of this $K_{3,3}$ which is potentially not present in $G$ is $x_1y_2$, but we have a path $y_2x_3y_3 \dots x_ky_kx_1$ in $G$. Hence, $G$ is not planar by (the easier direction of) Kuratowski's theorem [79], which is a contradiction. Thus, the induced bipartite subgraph of $G$ with parts $X$ and $Y$ is a forest. The statement follows. □

## 5.4 Finding an empty $K_{2,7}$

In this section we prove that if $G$ does not contain an empty $K_{2,7}$, then there is even a vertex which is contained in at most $11n/20$ induced $C_5$'s. Here an empty $K_{2,7}$ in a drawing of $G$ means distinct vertices $u$ and $w$, and $z_1, \dots, z_7 \in N(u) \cap N(w)$ in natural order such that the bounded region with boundary consisting of $uz_1$, $z_1w$, $wz_7$ and $z_7u$ contains no vertex other than $z_2, \dots, z_6$.

**Lemma 21.** *Let $n$ be sufficiently large and let $G$ be a plane graph on $n$ vertices. If $G$ does not contain an empty (not necessary induced) $K_{2,7}$, then there is a vertex in $G$ which is contained in at most $11n/20$ induced $C_5$'s.*

To prove this, we need some preliminaries.

**Lemma 22.** *Let $n$ be sufficiently large and let $G$ be a planar graph on $n$ vertices. If $G$ does not contain a (not necessary induced) $K_{2,\frac{n}{10^6}}$, then there is a vertex in $G$ which is contained in at most $n/2$ induced $C_5$'s.*

*Proof.* Suppose otherwise. Let $v$ be a vertex of degree at most 5 in $G$. Such vertex exists since the average degree in a planar graph is strictly less than 6 by Euler's formula. Then by the pigeonhole principle $v$ has distinct non-adjacent neighbours $u$ and $w$ such that the number of induced $C_5$'s containing $u$, $v$ and $w$ is at least $n/20$. Define $X$ and $Y$ as in Lemma 20. By the same lemma, we have $|X|+|Y| \geq n/20$. Let $G'$ be the induced bipartite subgraph of $G$ with parts $X$ and $Y$. By assumption, there is no vertex of degree at least $n/10^6$ in $G'$. Then since $G'$ has at least $\frac{|X|+|Y|}{2} \geq n/40$ edges, there must exist a set of at least $10^4$ independent edges in $G'$.

Let them be $x_1y_1, x_2y_2, \ldots, x_{10^4}y_{10^4}$ such that $x_1, x_2, \ldots, x_{10^4} \in X$, the edges $ux_1, ux_2, \ldots, ux_{10^4}$ are in anti-clockwise order, and the bounded region with boundary consisting of edges $ux_1$, $x_1y_1, y_1w, wy_{10^4}, y_{10^4}x_{10^4}, x_{10^4}u$ contains all $x_i$ and $y_i$. For $1 \leq i \leq 10^4 - 1$, let $R_i$ be the bounded region with boundary consisting of $ux_i, x_iy_i, y_iw, wy_{i+1}, y_{i+1}x_{i+1}, x_{i+1}u$. Choose $11 \leq i \leq 10^4 - 12$ such that the number of vertices in $R_{i-10} \cup R_{i-9} \cup \cdots \cup R_{i+11}$ is at most $n/300$. Let $R = R_i \cup R_{i+1}$.

Let $S$ be the set of vertices of $G$ in the interior of $R$ which do not belong to $N(u) \cap N(w)$. Note that $x_{i+1} \in S$, so $S \neq \emptyset$. Now the graph $G'' = G[S]$ is planar, so there exists some $z \in S$ which has degree at most 5 in $G''$. But it is joined to at most 2 elements of $N(u) \cap N(w)$, so it has at most 7 neighbours in the interior of $R$. Hence (together with $u, x_i, y_i, w, y_{i+2}$ and $x_{i+2}$), $z$ has at most 13 neighbours.

By assumption, $z$ is contained in at least $n/2$ induced $C_5$'s. We claim that any such $C_5$ is either contained entirely in $R_{i-10} \cup R_{i-9} \cup \cdots \cup R_{i+11}$ or it contains both $u$ and $w$. Indeed, let $C$ be an induced 5-cycle containing $z$ which leaves the region $R_{i-10} \cup R_{i-9} \cdots \cup R_{i+11}$ and let $q$ be a vertex of $C$ outside $R_{i-10} \cup R_{i-9} \cup \cdots \cup R_{i+11}$. Then $C$ in the union of two internally vertex-disjoint paths of length at most 5 between $z$ and $q$. However, since $z$ is inside $R_i \cup R_{i+1}$ and $q$ is outside $R_{i-10} \cup R_{i-9} \cup \cdots \cup R_{i+11}$, any such path must pass through either $u$ or $w$. Hence, $C$ contains both $u$ and $w$.

If an induced 5-cycle is contained in $R_{i-10} \cup R_{i-9} \cup \cdots \cup R_{i+11}$, then it can only use a set of at most $n/300$ vertices, and since $z$ has degree at most 13, by Lemma 20 there are at most $\binom{13}{2} \cdot n/300 < n/3$ such induced $C_5$'s. So there are at least $n/6$ induced $C_5$'s containing $z$, $u$ and $w$. Recall that $u$ and $w$ are non-adjacent and $z \notin N(u) \cap N(w)$. If $z \in N(u)$, then all these induced $C_5$'s are of the form $uzswt$ for some $s \in N(z)$ and $t \in N(u) \cap N(w)$, while if $z \in N(w)$, then all these induced $C_5$'s are of the form $uszwt$ for some $s \in N(z)$ and $t \in N(u) \cap N(w)$. In either case, since $|N(z)| \leq 13$, it follows that $|N(u) \cap N(w)| \geq \frac{n}{6 \cdot 13} > \frac{n}{10^6}$. This contradicts the condition in the lemma. □

**Lemma 23.** *Let $n$ be sufficiently large and let $G$ be a plane graph on $n$ vertices. Let $u$ and $w$ be distinct vertices, and let $v_1, v_2, \ldots, v_6$ be some of their common neighbours, in natural order. Assume that the number of vertices in the interior of the bounded region with boundary consisting of $uv_3$, $v_3w$, $wv_4$ and $v_4u$ is at least one but at most $n^{1/5}$ and that there is no common neighbour of $u$ and $w$ in the same region. Then $G$ has a vertex which is contained in at most $11n/20$ induced $C_5$'s.*

*Proof.* Suppose otherwise. Let $R$ be the bounded region with boundary consisting of $uv_3$, $v_3w$, $wv_4$ and $v_4u$. Let $x$ be an arbitrary vertex inside $R$. By assumption, $x \notin N(u) \cap N(w)$. Since there are at most $n^{1/5} + 4$ vertices in $R$ (including its boundary), the number of induced $C_5$'s containing $x$ which lie entirely in $R$ (possibly touching the boundary) is at most $(n^{1/5} + 4)^4 \leq n/20$. Thus, since $x$ is contained in at least $11n/20$ induced $C_5$'s, there exist at least $n/2$ induced $C_5$'s containing $x$ which contain vertices outside $R$.

Take such an induced $C_5$ and call it $C$. We claim that $C$ must contain both $u$ and $w$, but does not contain $v_3$ and $v_4$. Indeed, if we go through the vertices of $C$ one by one in natural order, starting with $x$, then there will be a vertex from the set $\{u, v_3, w, v_4\}$ right before the walk first leaves $R$, and then one in the same set when the walk first returns to $R$. Call these two vertices $y$ and $z$, respectively. Since $C$ contains the vertex $x$, which is in the interior of $R$, it follows that $y$ and $z$ are not neighbours in $C$, so they are also not neighbours in $G$. Thus, either $\{y, z\} = \{u, w\}$ or $\{y, z\} = \{v_3, v_4\}$. In the latter case, again since $C$ is induced and contains $x$, $C$ contains neither $u$ nor $w$. So there exists a path of length at most 3 in $C$, and therefore also in $G$, from $v_3$ to $v_4$ outside of $R$ which avoids both $u$ and $w$. This is clearly not possible because of the vertices $v_1$, $v_2$, $v_5$ and $v_6$.

Thus, $C$ indeed contains both $u$ and $w$, and it is easy to see that it does not contain $v_3$ and $v_4$. Since $x \notin N(u) \cap N(w)$, it follows that either $x \in N(u)$ and $C = uxqwr$ for some $q \in N(x) \cap N(w) \setminus \{v_3, v_4\}$ and $r \in N(u) \cap N(w)$, or $x \in N(w)$ and $C = uqxwr$ for some $q \in N(x) \cap N(u) \setminus \{v_3, v_4\}$ and $r \in N(u) \cap N(w)$. In particular, it follows that $N(u)$ and $N(w)$ both have vertices in the interior of $R$.

Let $X$ be the set of vertices of $N(u)$ in the interior of $R$ and let $Y$ be the set of vertices of $N(w)$ in the interior of $R$. Similarly as in the proof of Lemma 20, the induced bipartite subgraph of $G$ with parts $X$ and $Y$ is acyclic. Thus, there is a vertex in that graph of degree at most one. Without loss of generality, we may assume that some $x \in X$ has at most one neighbour in $Y$. Then, by the previous paragraph, there are at most $|N(u) \cap N(w)|$ induced $C_5$'s containing $x$ as well as vertices outside $R$. Thus, by the first paragraph, $|N(u) \cap N(w)| \geq n/2$.

By a simple averaging, it follows that there exist distinct $t_1, t_2, \dots, t_7 \in N(u) \cap N(w)$ (in natural order) such that the region $S$ bounded by $ut_1, t_1w, wt_7, t_7u$ contains at most 100 vertices. Now any induced $C_5$ which contains $t_4$ and has vertices outside $S$ must contain $u$ and $w$. Such an induced $C_5$ cannot contain any vertices from $N(u) \cap N(w)$ other than $t_4$, so by Lemma 20, there are at most $n/2$ such induced $C_5$'s. The number of induced $C_5$'s containing $t_4$ but no vertices outside $S$ is at most $100^5$, so $t_4$ satisfies the conclusion of the lemma. □

**Corollary 6.** *Let $n$ be sufficiently large and let $G$ be a plane graph on $n$ vertices with the property that $G$ contains a (not necessarily induced) subgraph $K_{2,7\cdot\lceil n^{4/5}\rceil}$. Then in this $K_{2,7\cdot\lceil n^{4/5}\rceil}$ there is an empty $K_{2,7}$ or there is a vertex in $G$ which is contained in at most $11n/20$ induced $C_5$'s.*

*Proof.* Assume that there is no vertex in $G$ which is contained in at most $n/2$ induced $C_5$'s. Choose distinct $u$ and $w$ in $G$ with $|N(u) \cap N(w)| \geq 7 \cdot \lceil n^{4/5} \rceil$. Let $v_1, v_2, \ldots, v_{7 \cdot \lceil n^{4/5} \rceil} \in N(u) \cap N(w)$ in natural order. For each $1 \leq i \leq 7 \cdot \lceil n^{4/5} \rceil - 1$, let $R_i$ be the bounded region with boundary consisting of the edges $uv_i$, $v_iw$, $wv_{i+1}$ and $v_{i+1}u$. By Lemma 23, each $R_i$ with $3 \leq i \leq 7 \cdot \lceil n^{4/5} \rceil - 3$ contains either zero or at least $n^{1/5}$ vertices in its interior. Hence, the number of non-empty $R_i$'s is at most $n^{4/5} + 4$. Thus, there exists some $1 \leq i \leq 7 \cdot \lceil n^{4/5} \rceil - 6$ for which $u, w, v_i, v_{i+1}, \ldots, v_{i+6}$ define an empty $K_{2,7}$. $\square$

Now Lemma 21 follows from Lemma 22 and Corollary 6.

## 5.5 The rough structure of near-extremal graphs

**Lemma 24.** *Let $n \geq 8$. Then $h(n) - h(n-1) > 2n/3 - 4$.*

*Proof.* Choose $k_1, k_2, k_3 \in \mathbb{N}$ such that $k_1 + k_2 + k_3 = n - 4$, $k_1, k_2, k_3$ are as close as possible and $k_1 \geq k_2 \geq k_3$. Then $h(n) = k_1k_2 + k_2k_3 + k_3k_1 + 2$ and $h(n-1) = (k_1 - 1)k_2 + k_2k_3 + k_3(k_1 - 1) + 2$, so $h(n) - h(n-1) = k_2 + k_3 \geq \frac{2}{3}(k_2 + k_3 + (k_1 - 1)) = \frac{2}{3}(n-5) > 2n/3 - 4$. $\square$

Lemma 24 means that we are happy (when proving Proposition 4) if we can find a vertex which is contained in at most, say, $2n/3 - 10$ induced 5-cycles. The next lemma guarantees some structure if such a vertex does not exist.

**Lemma 25.** *Let $n$ be sufficiently large and let $G$ be a plane graph on $n$ vertices. Suppose that $G$ has at least $\frac{5}{18}n^2$ induced $C_5$'s and that there does not exist a vertex in $G$ which is contained in at most $2n/3 - 10$ induced $C_5$'s. Let $u$ and $w$ be the two vertices in the part of size 2 of an empty $K_{2,7}$. Then $u$ and $w$ are non-adjacent and there exist sets $X N(u) \setminus N(w)$ and $Y N(w) \setminus N(u)$ with the following properties.*

1. *$|X| + |Y| \geq 2n/3 - 10$, every $x \in X$ is adjacent to at least one element of $Y$ and every $y \in Y$ is adjacent to at least one element of $X$.*
2. *The bipartite induced subgraph of $G$ with parts $X$ and $Y$ has maximum degree at least $n^{5/6}$.*

*Proof.* Let $v$ be the centre vertex in the part of size 7 in the empty $K_{2,7}$. Define $X$ and $Y$ as in the statement of Lemma 20. Since every induced $C_5$ containing $v$ also contains $u$ and $w$, and by assumption $v$ is contained in more than $2n/3 - 10$ induced $C_5$'s, it follows by Lemma 20 that $|X| + |Y| > 2n/3 - 10$. Moreover, since there exists an induced $C_5$ containing $u$, $v$ and $w$, it follows that $u$ and $w$ are non-adjacent.

Let $G'$ be the induced bipartite subgraph of $G$ with parts $X$ and $Y$. It remains to show that $G'$ has maximum degree at least $n^{5/6}$. Suppose otherwise.

Let $x \in X$ be an arbitrary vertex. We give an estimate for the number of induced $C_5$'s containing $x$. We first count those $C_5$'s which contain both $u$ and $w$ as vertices. Let us call these *type 1* $C_5$'s. Since $w$ is non-adjacent to both $x$ and $u$, the number of type 1 $C_5$'s containing $x$ is at most $d_{G'}(x) \cdot t$, where $d_{G'}(x)$ is the degree of $x$ in $G'$ and $t = |N(u) \cap N(w)|$.

Call those induced $C_5$'s which do not contain both $u$ and $w$ *type 2*. To bound the number of such $C_5$'s, we will use the following claim.

**Claim 18.** *For every $q \in V(G)$, the number of vertices $z \in X \cup Y$ for which there exists a path of length at most 3 between $q$ and $z$ avoiding both $u$ and $w$ is at most $100n^{5/6}$.*

*Proof.* Take a maximal matching between $X$ and $Y$. Let the edges in this matching be $x_{i_1}y_{i_1}, \ldots, x_{i_s}y_{i_s}$ such that $x_{i_j} \in X, y_{i_j} \in Y$ and the edges $wy_{i_1}, \ldots, wy_{i_s}$ are in clockwise order. For each $1 \leq j \leq s-1$, let $R_j$ be the bounded region with boundary consisting of the edges $ux_{i_j}, x_{i_j}y_{i_j}, y_{i_j}w, wy_{i_{j+1}}, y_{i_{j+1}}x_{i_{j+1}}, x_{i_{j+1}}u$, and let $R_0$ be the unbounded region with boundary consisting of the edges $ux_{i_1}, x_{i_1}y_{i_1}, y_{i_1}w, wy_{i_s}, y_{i_s}x_{i_s}, x_{i_s}u$. Let $0 \leq j \leq s-1$. By the maximality of our matching, any element of $X \cup Y$ in the interior of $R_j$ is a neighbour in $G'$ of some vertex in $X \cup Y$ on the boundary of $R_j$. Since there are 4 vertices in $X \cup Y$ on the boundary of $R_j$, and $G'$ has maximum degree less than $n^{5/6}$, there are at most $4n^{5/6}$ elements of $X \cup Y$ in the interior of $R_j$.

Let $q \in V(G) \setminus \{u, w\}$. Then $q$ is in $R_j$ (possibly on the boundary) for some $0 \leq j \leq s-1$. If there exists some $z \in X \cup Y$ for which there is a path of length at most 3 from $q$ to $z$ avoiding both $u$ and $w$, then $z$ is in $R_{j-4} \cup R_{j-3} \cup \ldots R_{j+4}$ (with the subscripts considered modulo $s$). But there are at most $9 \cdot 4n^{5/6}$ such vertices $z$, which finishes the proof of the claim. □

Recall that $G'$ is acyclic, so the number of edges in $G'$ is at most $|X| + |Y| - 1$. Thus, if $\ell$ is the number of vertices of degree at least 3 in $G'$, then $3\ell \leq 2(|X| + |Y|)$, so the number of vertices of degree at most 2 in $G'$ is $|X| + |Y| - \ell \geq \frac{|X|+|Y|}{3} \geq \frac{2n}{9} - 4$.

The number of edges in $G$ is at most $3n$ by Euler's formula, so the number of vertices in $G$ of degree at least 60 is at most $n/10$.

Moreover, it follows from Claim 18 by double counting that the number of vertices $z \in X \cup Y$ for which there exist at least $1000n^{5/6}$ vertices $q \in V(G)$ with a path of length at most 3 between $z$ and $q$ and avoiding both $u$ and $w$ is at most $n/10$.

Thus, there exists a vertex $z \in X \cup Y$ which has degree at most 2 in $G'$, degree at most 60 in $G$ and for which the number of $q \in V(G)$ with a path of length at most 3 between $z$ and $q$ avoiding $u$ and $w$ is at most $1000n^{5/6}$.

Suppose that $q \in V(G)$ is distinct from $z$, $u$ and $w$, and that there exists a type 2 induced $C_5$ containing both $z$ and $q$. Then there exists a path of length at most 3 from $q$ to $z$ which contains neither $u$ nor $w$. But there are at most $1000n^{5/6}$ such vertices $q \in V(G)$, so by Lemma 20, the number of type 2 induced $C_5$'s containing $z$ is at most

$\binom{60}{2} \cdot (1000n^{5/6}+2)$. Moreover, the number of type 1 induced $C_5$'s containing $z$ is at most $2t$, where $t = |N(u) \cap N(w)|$. Since the total number of induced $C_5$'s containing $z$ is assumed to be at least $2n/3 - 10$, it follows that $|N(u) \cap N(w)| \geq n/3 - n^{6/7}$.

**Claim 19.** *The number of induced $C_5$'s in $G$ is at most $(\frac{2}{9} + o(1))n^2$.*

*Proof.* Let $N(u) \cap N(w) = \{v_1, \ldots, v_t\}$ such that $uv_1, uv_2, \ldots, uv_t$ are in anticlockwise order and the bounded region with boundary consisting of $uv_1, v_1w, wv_t, v_tu$ contains all the $v_i$'s. For $1 \leq i \leq t-1$, let $T_i$ be the bounded region with boundary consisting of $uv_i, v_iw, wv_{i+1}, v_{i+1}u$. Suppose that there are at least $7 \cdot \lceil n^{4/5} \rceil$ values of $i$ for which the interior of $T_i$ contains a vertex of $G$. Then we can easily find a $K_{2,7\cdot\lceil n^{4/5}\rceil}$ in $G$ in which no $K_{2,7}$ is empty, so by Corollary 6 there is a vertex in $G$ that is contained in at most $11n/20$ induced $C_5$'s, which is a contradiction. Thus, for all but $o(n)$ choices $6 \leq i \leq t-6$ the regions $T_{i-5}, T_{i-4}, \ldots, T_{i+5}$ contain no vertex in their interior. But for all such $i$, we have that $v_i$ is contained in at most $2n/3 + o(n)$ induced $C_5$'s.

Let us remove the vertices $v_i$ for these values of $i$ from $G$ and note that with this we remove at least $n/3 - o(n)$ vertices but at most $(\frac{2}{9} + o(1))n^2$ induced $C_5$'s (since we have $n/3 - n^{6/7} \leq |N(u) \cap N(w)| \leq n/3 + 10$). It suffices to show that in the remaining graph $\tilde{G}$ there are at most $o(n^2)$ induced $C_5$'s. Let $S = V(\tilde{G}) \setminus (X \cup Y \cup \{u, w\})$. Note that $|S| = o(n)$.

Now we remove the vertices in $S$ one by one in careful order, such that in each step we remove $O(n)$ induced $C_5$'s. Note that any $v \in S$ is joined to at most 2 vertices from $X$ (else $G$ contains a subdivision of $K_{3,3}$ which is impossible by Kuratowski's theorem). Similarly, it is joined to at most 2 vertices from $Y$, so it is joined to at most 6 vertices from $X \cup Y \cup \{u, w\}$. Thus, since $\tilde{G}$ is planar, we may remove the vertices of $S$ one by one in a way that in each step the removed vertex has at most 11 neighbours in the current graph. This way, by Lemma 20, we remove at most $\binom{11}{2} \cdot n$ induced $C_5$'s in each step. Thus, while removing the vertices in $S$, we remove at most $o(n^2)$ induced $C_5$'s.

It remains to prove that in $G[X \cup Y \cup \{u, w\}]$ there are $o(n^2)$ induced $C_5$'s. To show this, we prove that we may remove the vertices in $X \cup Y$ one by one such that in each step we remove $o(n)$ induced $C_5$'s. Clearly, in each step we can remove a vertex $q \in X \cup Y$ which has degree at most 6 in the current graph. We claim that $q$ is then contained in at most $o(n)$ induced $C_5$'s. Let $Z$ be the set of vertices $z \in X \cup Y$ for which there is a path of length at most 3 from $q$ to $z$ which avoids both $u$ and $w$. By Claim 18, we have $|Z| = o(n)$. Since $N(u) \cap N(w) \cap (X \cup Y) = \emptyset$, there is no induced $C_5$ with vertices from $X \cup Y \cup \{u, w\}$ which contains both $u$ and $w$, so any induced $C_5$ which contains $q$ must consist of vertices from the set $Z \cup \{u, w\}$. Thus, as $q$ has degree at most 6, by Lemma 20 there are at most $o(n)$ induced $C_5$'s containing $q$. □

Claim 19 contradicts our hypothesis that $G$ has at least $\frac{5}{18}n^2$ induced $C_5$'s, so the proof is complete. □

**Lemma 26.** *Let $n$ be sufficiently large and let $G$ be a plane graph on $n$ vertices. Suppose that $G$ has at least $\frac{5}{18}n^2$ induced $C_5$'s and that there does not exist a vertex in $G$ which is contained in at most $2n/3-10$ induced $C_5$'s. Then there exist distinct vertices $w_1$, $w_2$, $w_3$ and $u$ such that $w_1w_2, w_2w_3, w_3w_1 \in E(G)$ and for every $1 \le i \le 3$, there exists an empty $K_{2,7}$ whose part of size two is $\{w_i, u\}$.*

*Proof.* By Lemma 21, $G$ contains an empty $K_{2,7}$. Let $u$ and $w$ be the two vertices in the part of size two. Let $X$ and $Y$ be the sets provided by Lemma 25. By Property 2 from that lemma, we may assume without loss of generality (after swapping $u$ and $w$ if necessary) that some $y \in Y$ has at least $n^{5/6}$ neighbours in $X$. Let $v$ be an arbitrary common neighbour of $u$ and $w$ and order the elements of $Y$ as $y_1, y_2, \ldots, y_k$ such that the edges $wv, wy_1, \ldots, wy_k$ are in clockwise order.

Then $y_iy_j$ is an edge only if $j = i+1$. Indeed, for any $1 \le \ell \le k$ there exists a path from $v$ to $y_\ell$ (through $u$ and some $x \in X$) which avoids $\{w\} \cup Y \setminus \{y_\ell\}$. But if $y_iy_j$ is an edge for some $j > i+1$, then the triangle $wy_iy_j$ separates $y_{i+1}$ from $v$.

Now let $y = y_i$ for some $i$.

For large enough $n$, together with $u$ and $y$, some vertices in $N(y) \cap X$ form a $K_{2,7\lceil n^{4/5}\rceil}$. Thus, by Corollary 6, there are vertices $x_1, \ldots, x_7 \in N(y) \cap X$ such that together with $u$ and $y$ they form an empty (not necessarily induced) $K_{2,7}$.

By assumption, $x_4$ is contained in at least $2n/3-10$ induced $C_5$'s. However, note that any such induced $C_5$ also contains $u$ and $y$. Let $Z$ be the set of all vertices in $X \cup Y \setminus \{y, x_4\}$ which are contained in an induced $C_5$ containing $x_4$. By Lemma 20, $|Z| + |V(G) \setminus (X \cup Y)| \ge 2n/3 - 10$. Since $|X \cup Y| \ge 2n/3 - 10$, it follows that $|Z| \ge n/4$. If $z \in Z \cap Y$, then $z$ is not a neighbour of $u$, so it must be a neighbour of $y_i$, hence $z = y_{i-1}$ or $z = y_{i+1}$. It follows that $|Z \cap X| \ge n/5$.

**Claim 20.** *Either $y_{i-1}$ is a neighbour of $y_i$ and $|N(y_{i-1}) \cap X| \ge n^{5/6}$ or $y_{i+1}$ is a neighbour of $y_i$ and $|N(y_{i+1}) \cap X| \ge n^{5/6}$.*

*Proof.* We prove this claim by using $|Z \cap X| \ge n/5$.

If $k = 1$, then $X\mathrm{N}(y) \cap N(u)$, so $Z = \emptyset$, which is a contradiction.

Suppose that $k \ge 2$. Let $z \in Z \cap X$. Observe that since $y, x_4, u$ and $z$ are contained in an *induced* $C_5$, we have $z \notin N(y)$, and the fifth vertex in the $C_5$ is some $q \in N(y) \cap N(z)$.

Let us first assume that $2 \le i \le k-1$. Let $r_1$ be the element in $N(y_{i-1}) \cap X$ which is the first neighbour of $u$ in clockwise order compared to $x_4$. Similarly, let $r_2$ be the element in $N(y_{i+1}) \cap X$ which is the first neighbour of $u$ in anticlockwise order compared to $x_4$. Note that the edges $wy_{i-1}$, $y_{i-1}r_1$, $r_1u$, $ur_2$, $r_2y_{i+1}$, $y_{i+1}w$ divide the plane into two regions; let $R$ be the one which contains $y$. Then either $z$ is also in $R$ (possibly on the boundary), or $q$ is on the boundary of $R$. In the former case, there are only two possibilities for $z$: $r_1$ and $r_2$ (since $z \notin N(y_i)$). Assume that $q$ is on the

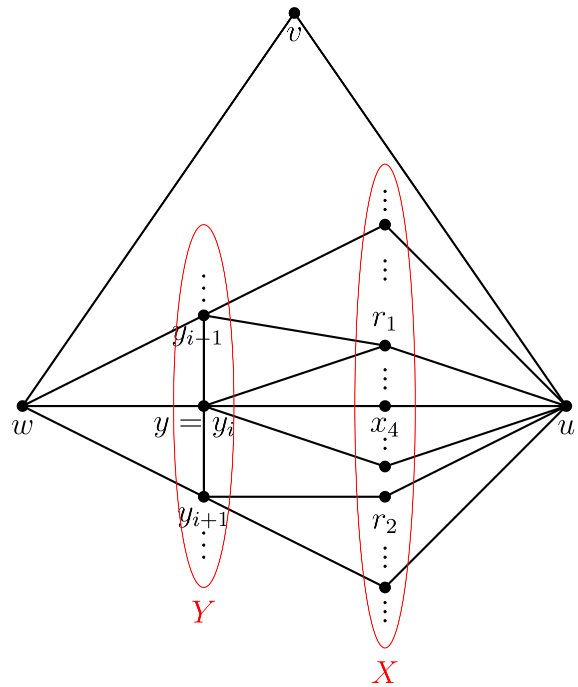



Figure 5.3: Proof of Claim 20

boundary of $R$. Since $ux_4yqz$ is an induced $C_5$, we have $q \notin N(u)$. Thus, $q \notin X$ so $q \neq r_1$ and $q \neq r_2$. Also, $z \in X$, so $z \notin N(w)$, hence $q \neq w$. Moreover, $q$ is distinct from $u$. Thus, $q \in \{y_{i-1}, y_{i+1}\}$, $q$ is a neighbour of $y_i$ and $z$ is a neighbour of $q$, from which the claim follows.

Assume now that $i = 1$. Let $r$ be the element in $N(y_2) \cap X$ which is the first neighbour of $u$ in anticlockwise order compared to $x_4$. The edges $wv, vu, ur, ry_2$ and $y_2w$ divide the plane into two regions; let $R$ be the one containing $y_1$. Then either $z$ is also in $R$ (possibly on the boundary), or $q$ is on the boundary of $R$. In the former case, $z$ must be $r$ because $z \notin N(y_1)$. Assume that $q$ is on the boundary of $R$. Note that $q \notin N(u)$, so $q \neq r$ and $q \neq v$. Also, $z \in X$, so $z \notin N(w)$, hence $q \neq w$. Moreover, $q$ is distinct from $u$. Thus, $q = y_2$ and $z \in N(y_2)$.

The case $i = k$ is very similar, so the claim is proved. □

Assume that $y_{i-1}$ is a neighbour of $y_i$ and $|N(y_{i-1}) \cap X| \geq n^{5/6}$. In particular, $|N(y_{i-1}) \cap N(u)| \geq n^{5/6}$, so by Corollary 6 there exists an empty $K_{2,7}$ whose part of size two consists of $y_{i-1}$ and $u$. This means that we can choose $w_1 = w$, $w_2 = y_{i-1}$ and $w_3 = y_i$ and these vertices have the desired properties. The case where $y_{i+1}$ is a neighbour of $y_i$ with $|N(y_{i+1}) \cap X| \geq n^{5/6}$ is almost identical. □

## 5.6 The fine structure of extremal graphs

In this section, we prove that if there do not exist three vertices whose removal deletes fewer than $h(n) - h(n-3)$ induced 5-cycles, then the graph has a very specific structure (very close to the extremal graph $H_n$).

**Lemma 27.** *For any $n \geq 10$, $h(n) - h(n-3) = 2n - 11$.*

*Proof.* Choose $k_1, k_2, k_3 \in \mathbb{N}$ such that $k_1 + k_2 + k_3 = n - 4$ and $k_1, k_2, k_3$ are as close as possible. Then $h(n) = k_1k_2 + k_2k_3 + k_3k_1 + 2$ and $h(n-3) = (k_1 - 1)(k_2 - 1) + (k_2 - 1)(k_3 - 1) + (k_3 - 1)(k_1 - 1) + 2$, so $h(n) - h(n-3) = 2k_1 + 2k_2 + 2k_3 - 3 = 2(n-4) - 3 = 2n - 11$. ☐

**Lemma 28.** *Let $G$ be a planar graph on $n$ vertices. Let $w_1, w_2, w_3$ be vertices in $G$ forming a triangle. Let $u$ be a vertex, not in $N(w_1) \cup N(w_2) \cup N(w_3)$, such that for every $i \in \{1, 2, 3\}$, there exists some $v_i \in N(u) \cap N(w_i)$ which is not a neighbour of $v_j$ and $w_j$ for $j \neq i$. Then the number of induced $C_5$'s containing $uv_iw_i$ for some $1 \leq i \leq 3$ is at most $2n - 11$. Moreover, if equality holds, then every vertex $x \in V(G) \setminus \{u, w_1, w_2, w_3\}$ satisfies one of the following.*

*(i) There exists some $i$ such that $x \in N(w_i) \cap N(u)$, or*

*(ii) there exist some $i \neq j$ such that $x \in N(w_i) \cap N(w_j) \setminus N(u)$, $N(x) \cap N(w_i) \cap N(u) \neq \emptyset$ and $N(x) \cap N(w_j) \cap N(u) \neq \emptyset$, or*

*(iii) there exist some $i \neq j$ and $y \in N(w_i) \cap N(w_j) \setminus N(u)$ such that $x \in N(u) \cap N(y) \setminus (N(w_1) \cup N(w_2) \cup N(w_3))$.*

The following lemma will be used in the proof of Lemma 28.

**Lemma 29.** *Let $G$ be a planar graph on $n$ vertices. Let $w_1, w_2, w_3$ be vertices in $G$ forming a triangle. Let $u$ be a vertex, not in $N(w_1) \cup N(w_2) \cup N(w_3)$, such that for every $i \in \{1, 2, 3\}$, there exists some $v_i \in N(u) \cap N(w_i)$ which is not a neighbour of $w_j$ for $j \neq i$. For every $i \in \{1, 2, 3\}$, let $X_i$ and $Y_i$ be defined like $X$ and $Y$ in Lemma 20 with $w_i$ and $v_i$ taking the role of $w$ and $v$. Let $G_i$ be the induced bipartite subgraph of $G$ with parts $X_i$ and $Y_i$. For $i \in \{1, 2, 3\}$, let $\lambda_i = 1$ if $w_{i+1}$ and $w_{i+2}$ are in the same connected component in $G_i$ (where temporarily we write $w_4 := w_1$ and $w_5 := w_2$) and let $\lambda_i = 0$ otherwise. Then*

$$\sum_{i \leq 3} (|X_i| + |Y_i|) \leq 2(n-1) - \lambda_1 - \lambda_2 - \lambda_3.$$

*Moreover, if equality holds, then every vertex $x \in V(G) \setminus \{u, w_1, w_2, w_3\}$ satisfies one of the following.*

*(i) There exists some $i$ such that $x \in N(w_i) \cap N(u)$, or*

*(ii) there exist some $i \neq j$ such that $x \in N(w_i) \cap N(w_j) \setminus N(u)$ and $N(x) \cap N(u) \neq \emptyset$, or*

*(iii) there exist some $i \neq j$ and $y \in N(w_i) \cap N(w_j) \setminus N(u)$ such that $x \in N(u) \cap N(y) \setminus (N(w_1) \cup N(w_2) \cup N(w_3))$.*

*Proof.* First, note that for every $i \in \{1, 2, 3\}$, $u$ does not belong to either of $X_i$ and $Y_i$. We now want to show that every $x \in V(G)$ belongs to at most two of the sets $X_1, X_2, X_3, Y_1, Y_2, Y_3$. For this, first observe that no $x \in V(G)$ belongs to both $X_i$ and $Y_j$ for some $i, j \in \{1, 2, 3\}$. This is because elements in $X_i$ are neighbours of $u$, while elements in $Y_j$ are non-neighbours. Hence, it remains to prove that no $x \in V(G)$ belongs to $X_1 \cap X_2 \cap X_3$ or to $Y_1 \cap Y_2 \cap Y_3$. If $x \in Y_1 \cap Y_2 \cap Y_3$, then $x \in N(w_1) \cap N(w_2) \cap N(w_3)$, but then $x$ cannot have a common neighbour with $u$ by planarity, which is a contradiction. If $x \in X_1 \cap X_2 \cap X_3$, then $x \in N(u) \setminus (N(w_1) \cup N(w_2) \cup N(w_3))$ and $x$ has a common neighbour with each $w_i$, but this is again impossible by planarity.

It already follows that $\sum_{i \leq 3}(|X_i| + |Y_i|) \leq 2(n-1)$.

Assume now that for some $i \in \{1, 2, 3\}$, the vertices $w_{i+1}$ and $w_{i+2}$ belong to the same connected component in $G_i$. It is not hard to see that this implies that $N(w_{i+1}) \cap N(w_{i+2}) \cap N(u) \neq \emptyset$. Then this common neighbour does not belong to any $Y_j$ (since it is a neighbour of $u$), but it also does not belong to $X_{i+1}$ and $X_{i+2}$ (where, again, indices are understood modulo 3). So for every such index $i$, we "gain 1" compared to $\sum_{i \leq 3}(|X_i| + |Y_i|) \leq 2(n-1)$. Since $N(w_1) \cap N(w_2) \cap N(w_3) \cap N(u) = \emptyset$, it follows that

$$\sum_{i \leq 3}(|X_i| + |Y_i|) \leq 2(n-1) - \lambda_1 - \lambda_2 - \lambda_3. \tag{5.1}$$

Assume now that equality holds here. Then every $x \in V(G) \setminus \{u\}$ which is not in $N(u) \cap (N(w_1) \cup N(w_2) \cup N(w_3))$ must belong to two of the sets $X_1, X_2, X_3, Y_1, Y_2, Y_3$. Assume first that $x \in Y_i \cap Y_j$ for some $i \neq j$. Then $x \in N(w_i) \cap N(w_j)$ and $N(x) \cap N(u) \neq \emptyset$, so $x$ satisfies (ii). Suppose now that $x \in X_i \cap X_j$ for some $i \neq j$. Then $x \in N(u) \setminus (N(w_i) \cup N(w_j))$ and $x$ has neighbours $y_i \in Y_i$ and $y_j \in Y_j$. It follows that $x$ does not have a common neighbour with $w_k$, where $w_k$ is the member of $\{w_1, w_2, w_3\}$ different from $w_i$ and $w_j$ since otherwise $G$ contains a subdivision of $K_5$ (the vertices of the $K_5$ are $u, x, w_1, w_2, w_3$), contradicting planarity. But since we have equality in (5.1), $y_i$ must belong to two of $Y_i$, $Y_j$ and $Y_k$. Hence, we necessarily have $y_i \in Y_i \cap Y_j$. It follows that $y_i \in N(w_i) \cap N(w_j)$. So we may take $y = y_i$ and then $x$ satisfies property (iii). $\square$

*Proof of Lemma 28.* For every $i \in \{1, 2, 3\}$, define $X_i$, $Y_i$, $G_i$ and $\lambda_i$ as in Lemma 29 and let $\mu_i$ be the number of connected components of $G_i$. By Lemma 20, the number of induced $C_5$'s containing $u$, $v_i$ and $w_i$ is at most $|X_i| + |Y_i| - \mu_i$. Since for every $i \neq j$, there exists an induced $C_5$ with vertices $u, v_i, w_i, v_j, w_j$, it follows that the number of induced $C_5$'s containing $uv_iw_i$ for some $1 \leq i \leq 3$ is at most

$$\left(\sum_{i\leq 3}(|X_i| + |Y_i| - \mu_i)\right) - 3.$$

By Lemma 29, this is at most $2n-5-(\mu_1+\mu_2+\mu_3)-(\lambda_1+\lambda_2+\lambda_3)$. Note that for every $i \in \{1,2,3\}$, $\mu_i + \lambda_i \geq 2$. Hence, $\mu_1+\mu_2+\mu_3+\lambda_1+\lambda_2+\lambda_3 \geq 6$ and it follows that the number of induced $C_5$'s containing $uv_iw_i$ for some $i$ is at most $2n-11$.

Assume now that this number is precisely $2n-11$. Then we must have

$$\sum_{i\leq 3}(|X_i| + |Y_i|) = 2(n-1) - \lambda_1 - \lambda_2 - \lambda_3$$

and

$$\mu_1+\mu_2+\mu_3+\lambda_1+\lambda_2+\lambda_3 = 6.$$

The first equation implies, using Lemma 29, that every $x \in V(G) \setminus \{u, w_1, w_2, w_3\}$ satisfies one of the following.

(a) There exists some $i$ such that $x \in N(w_i) \cap N(u)$, or

(b) there exist some $i \neq j$ such that $x \in N(w_i) \cap N(w_j) \setminus N(u)$ and $N(x) \cap N(u) \neq \emptyset$, or

(c) there exist some $i \neq j$ and $y \in N(w_i) \cap N(w_j) \setminus N(u)$ such that $x \in N(u) \cap N(y) \setminus (N(w_1) \cup N(w_2) \cup N(w_3))$.

This is almost what we need; it just remains to prove that if some $x \in V(G) \setminus \{w_1, w_2, w_3\}$ satisfies property (b), then it also satisfies property (ii) in the statement of this lemma. Assume that it does not; then WLOG $N(x) \cap N(w_i) \cap N(u) = \emptyset$. Note that from the proof of Lemma 29 it is clear that we must have $x \in Y_j$ to attain equality in (5.1). It is not hard to see that $N(x) \cap N(w_i) \cap N(u) = \emptyset$ implies that $x$ and $w_i$ are in different connected components of $G_j$. Since $x$ and $w_k$ (where $w_k$ is the element of $\{w_1, w_2, w_3\}$ different from $w_i$ and $w_j$) are also in different connected components in $G_j$, it follows that $\mu_j + \lambda_j \geq 3$ and hence

$$\mu_1+\mu_2+\mu_3+\lambda_1+\lambda_2+\lambda_3 \geq 7,$$

which is a contradiction. □

**Lemma 30.** *Let $G$ be a plane graph on $n$ vertices. Let $w_1, w_2, w_3$ be vertices in $G$ forming a triangle. Let $u$ be a vertex, not in $N(w_1) \cup N(w_2) \cup N(w_3)$, such that for every $i \in \{1,2,3\}$, there exists an empty $K_{2,7}$ whose part of size two is $\{u, w_i\}$. Assume that there do not exist three vertices such that the number of induced $C_5$'s containing at least one of these three vertices is at most $2n-12$. Then one of the following two scenarios must occur.*

(a) *Every* $x \in V(G) \setminus \{u, w_1, w_2, w_3\}$ *is a common neighbour of* $u$ *and* $w_i$ *for some* $i \in \{1, 2, 3\}$, *or*

(b) *after relabelling* $w_1$, $w_2$ *and* $w_3$ *if necessary, there exists* $w_4 \in N(w_2) \cap N(w_3) \setminus (N(u) \cup \{w_1\})$ *such that* $N(w_2) \cap N(w_4) \cap N(u) \neq \emptyset$, $N(w_4) \cap N(w_3) \cap N(u) \neq \emptyset$, *and every* $x \in V(G) \setminus \{u, w_1, w_2, w_3, w_4\}$ *belongs to* $N(w_i) \cap N(u)$ *for some* $i \in \{1, 2, 3, 4\}$.

*Proof.* For each $i \in \{1, 2, 3\}$, take an empty $K_{2,7}$ and call its centre vertex $v_i$. It is easy to see that any induced $C_5$ in $G$ which contains $v_i$ must also contain $u$ and $w_i$. Hence, by Lemma 28, the number of induced $C_5$'s in $G$ containing at least one of $v_1$, $v_2$, $v_3$ is at most $2n - 11$. By the assumption in our lemma, it must therefore be precisely $2n-11$. Hence, by Lemma 28 again, every vertex $x \in V(G) \setminus \{u, w_1, w_2, w_3\}$ satisfies one of the following.

(i) There exists some $i \in \{1, 2, 3\}$ such that $x \in N(w_i) \cap N(u)$, or

(ii) there exist distinct $i, j \in \{1, 2, 3\}$ such that $x \in N(w_i) \cap N(w_j) \setminus N(u)$, $N(x) \cap N(w_i) \cap N(u) \neq \emptyset$ and $N(x) \cap N(w_j) \cap N(u) \neq \emptyset$, or

(iii) there exist distinct $i, j \in \{1, 2, 3\}$ and $y \in N(w_i) \cap N(w_j) \setminus N(u)$ such that $x \in N(u) \cap N(y) \setminus (N(w_1) \cup N(w_2) \cup N(w_3))$.

Suppose first that there is no vertex $x \in V(G) \setminus \{u, w_1, w_2, w_3\}$ which satisfies (iii). If, in addition, there is no $x \in V(G) \setminus \{u, w_1, w_2, w_3\}$ satisfying (ii) either, then scenario (a) holds and we are done. Moreover, if there is only one vertex $x \in V(G) \setminus \{u, w_1, w_2, w_3\}$ satisfying (ii), then we can choose that vertex to be $w_4$ and scenario (b) is satisfied (after relabelling $w_1$, $w_2$ and $w_3$ if necessary). Assume that there are at least two vertices $x \in V(G) \setminus \{u, w_1, w_2, w_3\}$ satisfying (ii). Call them $y$ and $y'$, and assume, without loss of generality, that $y \in N(w_1) \cap N(w_2)$ and $y' \in N(w_2) \cap N(w_3)$.

If $N(w_1) \cap N(w_2) \cap N(u) \neq \emptyset$, then the only neighbours of $y$ are vertices of a triangle, and hence $y$ is not contained in any induced 5-cycle. In this case, the number of induced 5-cycles containing one of $v_1, v_2, y$ is at most (in fact, substantially less than) $2n-12$, a contradiction. Similarly, we cannot have $N(w_2) \cap N(w_3) \cap N(u) \neq \emptyset$.

**Claim 21.** *The number of induced* $C_5$*'s containing at least one of* $y$, $v_1$ *and* $v_2$ *is at most* $2n - 12$.

*Proof.* Let $k_1 = |N(w_1) \cap N(u)|$ and let $k_2 = |N(w_2) \cap N(u)|$. It is not hard to see that the induced $C_5$'s containing $y$ are $ux_1w_1yz$ for some $x_1 \in N(w_1) \cap N(u) \setminus N(y)$ and for the unique $z \in N(y) \cap N(w_2) \cap N(u)$ and $ux_2w_2yz$ for some $x_2 \in N(w_2) \cap N(u) \setminus N(y)$ and for the unique $z \in N(y) \cap N(w_1) \cap N(u)$. There are $k_1 + k_2 - 2$ such induced $C_5$'s. Every induced $C_5$ containing $v_2$ also contains $u$ and $w_2$, and by Lemma 20, the number of induced $C_5$'s containing $uv_2w_2$ is at most $n - (k_2 + 2) - 1$ (since

$X \cup Y \subset V(G) \setminus ((N(u) \cap N(w_2)) \cup \{u, w_2\})$ when we take $v = v_2, w = w_2$ in the lemma). Furthermore, by the same lemma, the number of induced $C_5$'s containing $uv_1w_1$ is at most $n - (k_1 + 2) - 2$. This is because we have seen (before the claim) that $N(w_2) \cap N(w_3) \cap N(u) = \emptyset$, and hence $\mu \geq 2$ when we apply Lemma 20 with $v = v_1, w = w_1$.

Combining our estimates and noting that for any two of $y$, $v_1$ and $v_2$, there exists an induced $C_5$ containing those two vertices but not the third, it follows that the number of induced $C_5$'s containing at least one of $y$, $v_1$ and $v_2$ is at most $(k_1 + k_2 - 2) + (n - k_2 - 3) + (n - k_1 - 4) - 3 = 2n - 12$, as claimed. □

The claim contradicts the conditions set out in the lemma, so we may assume that there does exist a vertex $x \in V(G) \setminus \{u, w_1, w_2, w_3\}$ which satisfies (iii). Hence, after reordering $w_1$, $w_2$ and $w_3$ if necessary, we may assume that there exist $v_4, w_4 \in V(G)$ such that $w_4 \in N(w_2) \cap N(w_3) \setminus N(u)$ and $v_4 \in N(u) \cap N(w_4) \setminus (N(w_1) \cup N(w_2) \cup N(w_3))$. Since $v_4 \in N(w_4) \setminus N(w_1)$, we have $w_4 \neq w_1$. Also, $w_4 \notin N(u)$, so it cannot satisfy properties (i) and (iii), hence it must satisfy (ii). It is not hard to see that therefore $N(w_4) \cap N(w_2) \cap N(u) \neq \emptyset$ and $N(w_4) \cap N(w_3) \cap N(u) \neq \emptyset$. Write $r$ for the unique vertex in $N(w_4) \cap N(w_2) \cap N(u)$ and write $r'$ for the unique vertex in $N(w_4) \cap N(w_3) \cap N(u)$. Let $R$ be the region bounded by edges $ur, rw_4, w_4r', r'u$ and containing $v_4$. By the classification of the vertices in $V(G) \setminus \{u, w_1, w_2, w_3\}$, any vertex in the interior of $R$ must be a common neighbour of $w_4$ and $u$. It follows that the only possible induced $C_5$ containing $v_4$ which does not contain $u$ and $w_4$ is $v_4rw_2w_3r'$ (these vertices may or may not induce a $C_5$).

We will now use Lemma 28 to upper bound the number of induced $C_5$'s containing $uv_iw_i$ for some $2 \leq i \leq 4$. The lemma can be applied with $w_4$ and $v_4$ in place of $w_1$ and $v_1$ and it follows that there are at most $2n - 11$ induced $C_5$'s containing $uv_iw_i$ for some $2 \leq i \leq 4$. Assume, for contradiction, that scenario (b) in the statement of Lemma 30 does not hold. In this case, we can improve the $2n - 11$ bound.

**Claim 22.** *The number of induced $C_5$'s containing $uv_iw_i$ for some $2 \leq i \leq 4$ is at most $2n - 13$.*

*Proof.* Since scenario (b) does not hold, there exists some $x \in V(G) \setminus \{u, w_1, w_2, w_3, w_4\}$ which does not belong to $N(u) \cap N(w_i)$ for any $i \in \{1, 2, 3, 4\}$. By the classification of the vertices in $V(G) \setminus \{u, w_1, w_2, w_3\}$, this implies that either there exists a vertex $y \in N(w_1) \cap N(w_2) \setminus (N(u) \cup \{w_3\})$ such that $N(y) \cap N(w_1) \cap N(u) \neq \emptyset$ and $N(y) \cap N(w_2) \cap N(u) \neq \emptyset$, or there exists a vertex $y \in N(w_1) \cap N(w_3) \setminus (N(u) \cup \{w_2\})$ such that $N(y) \cap N(w_1) \cap N(u) \neq \emptyset$ and $N(y) \cap N(w_3) \cap N(u) \neq \emptyset$. WLOG, assume that the former holds. As before, $N(w_1) \cap N(w_2) \cap N(u) = \emptyset$. Let $S = N(y) \cap N(u) \setminus (N(w_1) \cup N(w_2))$. Note that there does not exist an induced $C_5$ containing all of $u$, $v_4$, $w_4$ and at least one element in $S \cup \{y\}$ and that there does not exist an induced $C_5$ containing all of $u$, $v_3$, $w_3$ and at least one element in $S \cup \{y\}$. Moreover, it is not hard to see that the number of induced $C_5$'s containing $u$, $v_2$,

$w_2$ and at least one element of $S \cup \{y\}$ is at most $|S|+1$. By Lemma 28 applied to the planar graph $G-(S \cup \{y\})$, the number of induced $C_5$'s containing $uv_iw_i$ for some $2 \leq i \leq 4$ but none of $S \cup \{y\}$ is at most $2(n-|S|-1)-11$. Combining this with our previous estimate, we conclude that the number of induced $C_5$'s containing $uv_iw_i$ for some $2 \leq i \leq 4$ is at most $2n-|S|-12$. This is at most $2n-12$, so we are done unless it is exactly $2n-12$. In this case, the number of induced $C_5$'s in $G-(S \cup \{y\})$ containing $uv_iw_i$ for some $2 \leq i \leq 4$ is exactly $2(n-|S|-1)-11$. By the equality case in Lemma 28 (applied to the graph $G-(S \cup \{y\})$ and with $v_4, w_4$ replacing $v_1, w_1$) and since $w_1 \notin N(u)$, it follows that $N(w_1) \cap N(w_2) \cap N(u) \neq \emptyset$, which is a contradiction. $\square$

The claim implies that the number of induced $C_5$'s containing $v_2$, $v_3$ or $v_4$ is at most $2n-12$, a contradiction. $\square$

## 5.7 Completing the proof of Proposition 4

**Lemma 31.** *In the setting of Lemma 30, $G$ contains at most $h(n)$ induced $C_5$'s.*

*Proof.* We treat scenarios (a) and (b) separately.

Let us assume first that scenario (a) holds. Let $s_1 = N(w_1) \cap N(u) \setminus (N(w_2) \cup N(w_3))$ and define $s_2, s_3$ analogously. For $(i,j) \in \{(1,2),(2,3),(3,1)\}$, let $\gamma_{i,j} = |N(w_i) \cap N(w_j) \cap N(u)|$. Note that for every $i,j$, $\gamma_{i,j} \in \{0,1\}$ and that $s_1+s_2+s_3+\gamma_{1,2}+\gamma_{2,3}+\gamma_{3,1} = n-4$. The induced $C_5$'s in $G$ are those of the form $ux_iw_iw_jx_j$ where $i,j$ are distinct elements of $\{1,2,3\}$, $x_i \in N(w_i) \cap N(u) \setminus N(w_j)$ and $x_j \in N(w_j) \cap N(u) \setminus N(w_i)$. For $(i,j)=(1,2)$, the number of such induced $C_5$'s is $(s_1+\gamma_{3,1})(s_2+\gamma_{2,3})$. Combining this with the analogous formulae for the cases $(i,j)=(2,3)$ and $(i,j)=(3,1)$, we conclude that the number of induced $C_5$'s in $G$ is

$$(s_1+\gamma_{3,1})(s_2+\gamma_{2,3})+(s_2+\gamma_{1,2})(s_3+\gamma_{3,1})+(s_3+\gamma_{2,3})(s_1+\gamma_{1,2}). \tag{5.2}$$

Setting $\delta_1 = \gamma_{3,1}+\gamma_{1,2}-\gamma_{2,3}$, $\delta_2 = \gamma_{1,2}+\gamma_{2,3}-\gamma_{3,1}$ and $\delta_3 = \gamma_{2,3}+\gamma_{3,1}-\gamma_{1,2}$, we get that (5.2) is equal to

$$\begin{aligned}&(s_1+\delta_1)(s_2+\delta_2)+(s_2+\delta_2)(s_3+\delta_3)+(s_3+\delta_3)(s_1+\delta_1)+\gamma_{3,1}\gamma_{2,3}+\gamma_{1,2}\gamma_{3,1}\\&+\gamma_{2,3}\gamma_{1,2}-\delta_1\delta_2-\delta_2\delta_3-\delta_3\delta_1.\end{aligned}$$

It is straightforward to check that $\gamma_{3,1}\gamma_{2,3}+\gamma_{1,2}\gamma_{3,1}+\gamma_{2,3}\gamma_{1,2}-\delta_1\delta_2-\delta_2\delta_3-\delta_3\delta_1 \leq 1$. Hence, the number of induced $C_5$'s in $G$ is at most $(s_1+\delta_1)(s_2+\delta_2)+(s_2+\delta_2)(s_3+\delta_3)+(s_3+\delta_3)(s_1+\delta_1)+1$. Since $s_1+\delta_1+s_2+\delta_2+s_3+\delta_3 = s_1+s_2+s_3+\gamma_{1,2}+\gamma_{2,3}+\gamma_{3,1} = n-4$, it follows that the number of induced $C_5$'s in $G$ is at most

$$\max\{k_1k_2+k_2k_3+k_3k_1 : k_1,k_2,k_3 \in \mathbb{N}, k_1+k_2+k_3 = n-4\}+1 < h(n).$$

Let us assume now that scenario (b) holds. Let $s_1 = |N(w_1) \cap N(u) \setminus (N(w_2) \cup N(w_3) \cup N(w_4))|$, and define $s_2, s_3, s_4$ analogously. Moreover, let $\gamma_{3,1} = |N(w_3) \cap N(w_1) \cap N(u)|$ and let $\gamma_{1,2} = |N(w_1) \cap N(w_2) \cap N(u)|$. Note that $s_1 + s_2 + s_3 + s_4 + \gamma_{3,1} + \gamma_{1,2} = n - 7$. One can check that every induced $C_5$ in $G$ contains $u$ unless $s_4 = 1$ in which case there is a unique induced $C_5$ not containing $u$, which consists of $w_2$, $w_3$, the unique vertex in $N(w_3) \cap N(w_4) \cap N(u)$, the unique vertex in $N(w_4) \cap N(u) \setminus (N(w_2) \cup N(w_3))$ and the unique vertex in $N(w_2) \cap N(w_4) \cap N(u)$. Every induced $C_5$ containing $u$ contains precisely two of the $w_i$'s. The number of those containing $w_1$ and $w_2$ is $(s_1 + \gamma_{3,1})(s_2 + 1)$; the number of those containing $w_2$ and $w_3$ is $(s_2 + 1 + \gamma_{1,2})(s_3 + 1 + \gamma_{3,1})$; the number of those containing $w_3$ and $w_1$ is $(s_3 + 1)(s_1 + \gamma_{1,2})$; the number of those containing $w_2$ and $w_4$ is $(s_2 + \gamma_{1,2})(s_4 + 1)$; and the number of those containing $w_4$ and $w_3$ is $(s_4 + 1)(s_3 + \gamma_{3,1})$ (there is none which contains $w_1$ and $w_4$ since those vertices are nonadjacent). Altogether, the number of induced $C_5$'s in $G$ is at most

$$\begin{aligned} &(s_1 + \gamma_{3,1})(s_2 + 1) + (s_2 + 1 + \gamma_{1,2})(s_3 + 1 + \gamma_{3,1}) + (s_3 + 1)(s_1 + \gamma_{1,2}) \\ &+ (s_2 + \gamma_{1,2})(s_4 + 1) + (s_4 + 1)(s_3 + \gamma_{3,1}) + 1. \end{aligned} \tag{5.3}$$

The coefficient of $s_1$ is $(s_2 + 1) + (s_3 + 1)$, while the coefficient of $s_4$ is $(s_2 + \gamma_{1,2}) + (s_3 + \gamma_{3,1})$. Hence, (5.3) is at most

$$\begin{aligned} &(s_1 + s_4 + \gamma_{3,1})(s_2 + 1) + (s_2 + 1 + \gamma_{1,2})(s_3 + 1 + \gamma_{3,1}) + (s_3 + 1)(s_1 + s_4 + \gamma_{1,2}) \\ &+ (s_2 + \gamma_{1,2}) + (s_3 + \gamma_{3,1}) + 1. \end{aligned}$$

Setting $\delta_1 = \gamma_{1,2} + \gamma_{3,1} + 1$, $\delta_2 = \gamma_{1,2} - \gamma_{3,1} + 1$ and $\delta_3 = \gamma_{3,1} - \gamma_{1,2} + 1$, the above sum is equal to

$$\begin{aligned} &(s_1 + s_4 + \delta_1)(s_2 + \delta_2) + (s_2 + \delta_2)(s_3 + \delta_3) + (s_3 + \delta_3)(s_1 + s_4 + \delta_1) + 2 \\ &+ 3\gamma_{3,1} + 3\gamma_{1,2} + \gamma_{3,1}\gamma_{1,2} - \delta_1\delta_2 - \delta_2\delta_3 - \delta_3\delta_1. \end{aligned}$$

It is straightforward to check that $3\gamma_{3,1} + 3\gamma_{1,2} + \gamma_{3,1}\gamma_{1,2} - \delta_1\delta_2 - \delta_2\delta_3 - \delta_3\delta_1 \leq 0$, hence the number of induced $C_5$'s in $G$ is at most

$$(s_1 + s_4 + \delta_1)(s_2 + \delta_2) + (s_2 + \delta_2)(s_3 + \delta_3) + (s_3 + \delta_3)(s_1 + s_4 + \delta_1) + 2.$$

But $(s_1+s_4+\delta_1)+(s_2+\delta_2)+(s_3+\delta_3) = s_1+s_2+s_3+s_4+\gamma_{1,2}+\gamma_{3,1}+3 = n-7+3 = n-4$, so the number of induced $C_5$'s in $G$ is at most

$$\max\{k_1k_2 + k_2k_3 + k_3k_1 : k_1, k_2, k_3 \in \mathbb{N}, k_1 + k_2 + k_3 = n - 4\} + 2 = h(n),$$

as claimed. □

We can now combine several of our lemmas to prove Proposition 4.

*Proof of Proposition 4.* Let $n$ be sufficiently large and let $G$ be an $n$-vertex plane graph with $h(n)+t$ induced 5-cycles for some $t \geq 1$. Assume, for contradiction, that every subgraph on $n-1$ vertices has at most $h(n-1)+t$ induced 5-cycles and that every subgraph on $n-3$ vertices has at most $h(n-3)+t$ induced 5-cycles. Then every vertex in $G$ is contained in at least $h(n) - h(n-1)$ induced 5-cycles. Moreover, for any three vertices, the number of induced 5-cycles containing at least one of the three vertices is at least $h(n)-h(n-3)$. By Lemma 24, we have $h(n)-h(n-1) > 2n/3-4$ and by Lemma 27, we have $h(n) - h(n-3) = 2n - 11$. Since $n$ is sufficiently large, $h(n)+t > \frac{5}{18}n^2$ and $h(n) - h(n-1) > 2n/3 - 10$, Lemma 26 implies that there exist distinct vertices $w_1$, $w_2$, $w_3$ and $u$ such that $w_1w_2, w_2w_3, w_3w_1 \in E(G)$ and for every $1 \leq i \leq 3$, there exists an empty $K_{2,7}$ whose part of size two is $\{w_i, u\}$. Note that for each $i \in \{1, 2, 3\}$, we have $u \notin N(w_i)$ (else the centre vertex in the empty $K_{2,7}$ with $u$ and $w_i$ forming the part of size two is contained in no induced $C_5$'s). Using that for any three vertices, the number of induced 5-cycles containing at least one of these vertices is at least $2n - 11$, Lemma 31 implies that $G$ contains at most $h(n)$ induced 5-cycles, which is a contradiction. $\square$

## 5.8 Concluding remarks

For even cycles, Cox and Martin [22, 23] conjectured that for any $m \geq 7$, the maximum possible number of (not necessarily induced) $2m$-cycles in an $n$-vertex planar graph is $(\frac{n}{m})^m + o(n^m)$. A construction attaining this bound is obtained by taking a $2m$-cycle and blowing up every second vertex to a set of size roughly $n/m$ vertices, see Figure 1.10. If their conjecture is true, this would imply that the maximum number of induced $2m$-cycles is also $(\frac{n}{m})^m + o(n^m)$.

Turning to odd cycles, the situation seems to be a bit more complicated. Indeed, if we take a $(2m+1)$-cycle and blow up $m$ pairwise non-adjacent vertices to sets of size roughly $n/m$, then the resulting planar graph contains $(\frac{n}{m})^m + o(n^m)$ induced $C_{2m+1}$'s. On the other hand, there are constructions with much more (not necessarily induced) $C_{2m+1}$'s. Indeed, we can blow up every second vertex of a $2m$-cycle to sets of size $n/m$ and take a spanning path inside each blownup set, see Figure 1.11. The resulting planar graph will contain $2m(\frac{n}{m})^m + o(n^m)$ copies of $C_{2m+1}$, but they will not be induced. Hence, it remains possible that for $m \geq 3$, the maximum number of induced $(2m+1)$-cycles is $(\frac{n}{m})^m + o(n^m)$ (but by our results this is not true when $m = 2$).

# Chapter 6

# Wiener Index of Quadrangulation Graphs

## 6.1 Introduction

Recall that a quadrangulation graph is a plane graph such that each face is of size 4. Czabarka, Dankelmann, Olsen, Székely [25], gave an asymptotic upper bound for the Wiener index of quadrangulation graphs. They proved the following asymptotic upper bound.

**Theorem 66.** *(Czabarka, Dankelmann, Olsen, Székely [25])*
*Let $\kappa = \{2, 3\}$, then there exist a constant $C$ such that*

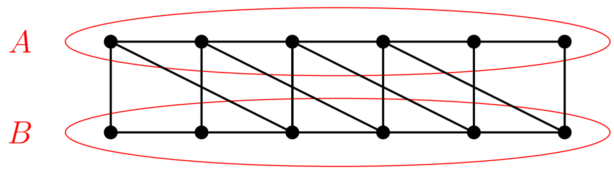
$$W(G) \leq \frac{1}{6\kappa}n^3 + Cn^{\frac{5}{2}},$$

*for every $\kappa$-connected simple quadrangulation $G$ of order $n$.*

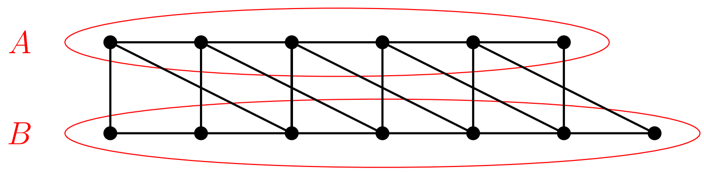
**Definition 32.** *The quadrangulation graph $Q_n$ is a plane graph on $n \geq 4$ vertices, with the following structure.*

*If $n$ is even, then the vertex set of $Q_n$ can be partitioned into two same size sets, $A = \{a_1, a_2, \cdots, a_{n/2}\}$ and $B = \{b_1, b_2, \ldots, b_{n/2}\}$. The edge set of $Q_n$ is the union of following three edge sets: $E_1 = \bigcup_{i=1}^{\frac{n}{2}-1}\{a_i a_{i+1}, b_i b_{i+1}\}$, forming paths in each vertex class, $E_2 = \bigcup_{i=1}^{\frac{n}{2}}\{a_i b_i\}$, forming "vertical edges" and $E_3 = \bigcup_{1}^{\frac{n}{2}-2}\{a_i b_{i+2}\}$, forming "diagonal edges", see Figure 6.1(top).*

*If $n$ is odd, then the quadrangulation graph $Q_n$ is obtained from $Q_{n-1}$ by adding a vertex, say $b$, in the $\left(\frac{n-1}{2}\right)$-vertex set $B$ such that the edge sets of $Q_n$ is union of the edge set of $Q_{n-1}$ and $\{b_{\frac{n-1}{2}}b,\ a_{\frac{n-1}{2}-1}b\}$ as shown in Figure 6.1(bottom).*




$Q_n: \ 2 \mid n$




$Q_n: \ 2 \nmid n$

Figure 6.1: Quadrangulations maximizing the Wiener index.

It can be checked that the construction $Q_n$[1] is with Wiener index

$$W(Q_n) = \begin{cases} \frac{1}{12}n^3 + \frac{7}{6}n - 2, & n \equiv 0 \ (mod\ 2), \\ \frac{1}{12}n^3 + \frac{11}{12}n - 1, & n \equiv 1 \ (mod\ 2). \end{cases}$$

Based on the quadrangulation graph $Q_n$, the authors in [25] conjectured $W(Q_n)$ is the maximum Wiener index of all $n$-vertex quadrangulation graphs.

Recently in [64] we confirmed their conjecture and proved the following result.

**Theorem 67.** *(Győri, Paulos, Xiao [64])*
*Let $G$ be a quadrangulation graph with $n \geq 4$ vertices. Then*

$$W(G) \leq W(Q_n) = \begin{cases} \frac{1}{12}n^3 + \frac{7}{6}n - 2, & n \equiv 0 \ (mod\ 2), \\ \frac{1}{12}n^3 + \frac{11}{12}n - 1, & n \equiv 1 \ (mod\ 2). \end{cases}$$

In this chapter, we give detail of the proof of Theorem 67. For that we need to have some notations and preliminaries.

## 6.2 Notations and preliminaries

Let $G$ be a simple graph. Let $\emptyset \neq S \subset V(G)$, the *status* of the set $S$ is defined as

$$\sigma_G(S) = \sum_{u \in V(G)} d_G(S, u),$$

[1]Observe that $Q_n$ is not a 3-connected graph.

where $d_G(S,u)$ is the distance of $u$ from $S$ in $G$, that is,

$$d_G(S,u) = \min\{d_G(u,v) | v \in S\}.$$

In particular, if $S = \{v\}$, then the status of $S$ is denoted by $\sigma_G(v)$. We write $\sigma(S)$ instead of $\sigma_G(S)$ if the underlying graph $G$ is clear. Let $C$ be a cycle in $G$ which is embedded in the plane, the interior (exterior) of $C$ is the bounded (unbounded) part of $C$ excluding $C$.

To prove our main theorem we will use the following three simple lemmas.

**Lemma 32.** *Let $G$ be an $n$-vertex quadrangulation graph, $n \geq 4$, then $\delta(G)$ is either 2 or 3.*

*Proof.* Let $f$ and $e$ be the number of faces and edges of $G$, respectively. Since each face is of length 4 and every edge is in two faces, then $4f = 2e$. Using the Euler's formula, $e + 2 = n + f$, we get $e = 2n - 4$, thus, $\sum_{v \in V(G)} d(v) = 4n - 8$. Therefore, $\delta(G) \leq 3$. Since a quadrangulation is 2-connected, there is no vertex of degree 1. Therefore, $\delta(G)$ is either 2 or 3. □

**Definition 33.** *A separating 4-cycle $S$ in a quadrangulation graph $G$ is a 4-cycle such that the deletion of $S$ from $G$ results in a disconnected graph. In other words, a separating 4-cycle is a 4-cycle which is not the boundary of a face.*

**Lemma 33.** *(Brinkmann, Greenberg, Greenhill, McKay, Thomas, Wollan [17])*
*If $G$ is a quadrangulation with at least 6 vertices and no separating 4-cycle, then $G$ is 3-connected.*

Let $G$ be a simple graph and $S$ be a nonempty subset of $V(G)$. Then we can partition the vertices in $V(G) - S$ based on their distance from $S$. We call the set of vertices at the distance $i$ as the *i-th level* with respect to $S$ and call the farthermost nonempty level the *terminal level* with respect to $S$. We have the following lemmas.

**Lemma 34.** *Let $G$ be a graph on $n+s$ vertices, $S$ be a set of vertices in $G$ such that $|S| = s$ and every non-terminal level with respect to $S$ contains at least 2 vertices. Then*

$$\sigma(S) \leq \begin{cases} \frac{1}{4}(n^2 + 2n), & 2 \mid n, \\ \frac{1}{4}(n^2 + 2n + 1), & 2 \nmid n. \end{cases}$$

*Proof.* Let $r$ denote the number of non terminal levels and $n_i$ be the number of vertices in the $i$-th level for $i = 1, 2, \ldots, r+1$. Thus $\sum_{i=1}^{r+1} n_i = n$. Since the terminal level contains at least 1 vertex and every other level contains at least 2 vertices, we get that $r \leq \frac{n-1}{2}$ and

$$\begin{aligned}
\sigma(S) &= \sum_{i=1}^{r+1} i n_i = \sum_{i=1}^{r} i n_i + (r+1)n_{r+1} = \sum_{i=1}^{r} i n_i + (r+1)(n - \sum_{i=1}^{r} n_i) \\
&= (r+1)n + \sum_{i=1}^{r} (i-r-1)n_i \\
&= (r+1)n - rn_1 - (r-1)n_2 - (r-2)n_3 - \cdots - n_r \\
&\leq (r+1)n - 2\sum_{i=1}^{r} i = (r+1)n - r(r+1)
\end{aligned}$$

The last inequality holds since $n_i \geq 2$.

Let $f(r) = (r+1)n - r(r+1)$, we can see that $f(r)$ is maximized when $r = \frac{n-1}{2}$. Since $r$ is an integer, when $2 \mid n$, $f(r)$ is maximal for $r = \frac{n-2}{2}$. Moreover, when $2 \nmid n$, $f(r)$ is maximal obviously for $r = \frac{n-1}{2}$. Thus,

$$\sigma(S) \leq \begin{cases} \frac{1}{4}(n^2+2n), & 2 \mid n, \\ \frac{1}{4}(n^2+2n+1), & 2 \nmid n. \end{cases}$$

□

The following two lemmas can be proved similarly, we only show the details of the proof of Lemma 35 and omit the proof of Lemma 36.

**Lemma 35.** *Let $G$ be a graph on $n+s$ vertices, $S$ be a set of vertices in $G$ such that $|S| = s$ and each of the non terminal levels with respect to $S$ contains at least 2 vertices except the second level, which contains at least 3 vertices. Then*

$$\sigma(S) \leq \begin{cases} \frac{1}{4}(n^2+8), & 2 \mid n, \\ \frac{1}{4}(n^2+7), & 2 \nmid n. \end{cases}$$

*Proof.* Let $r$ denote the number of non-terminal level and $n_i$ be the number of vertices in the $i$-th level for $i = 1, 2, \ldots, r+1$. Thus $\sum_{i=1}^{r+1} n_i = n$. Since the terminal level contains at least 1 vertex, the second level contains at least 3 vertices and the other levels contain at least 2 vertices, we get $r \leq \frac{n-2}{2}$. Also we have that

$$\begin{aligned}
\sigma(S) &= \sum_{i=1}^{r+1} i n_i = \sum_{i=1}^{r} i n_i + (r+1)n_{r+1} = \sum_{i=1}^{r} i n_i + (r+1)(n - \sum_{i=1}^{r} n_i) \\
&= (r+1)n + \sum_{i=1}^{r} (i-r-1)n_i \\
&\leq (r+1)n - 2\sum_{i=1}^{r} i - (r-1) = (r+1)n - r(r+1) - r + 1
\end{aligned}$$

Let $f(r) = (r+1)n - r(r+1) - r + 1$, we can see that $f(r)$ is maximized when $r = \frac{n-2}{2}$. Similarly, since $r$ is an integer, when $2 \mid n$, $f(r)$ is maximized when $r = \frac{n-2}{2}$. Moreover, when $2 \nmid n$, $f(r)$ is maximized by $r = \frac{n-3}{2}$. Thus,

$$\sigma(S) \leq \begin{cases} \frac{1}{4}(n^2+8), & 2 \mid n, \\ \frac{1}{4}(n^2+7), & 2 \nmid n. \end{cases}$$

$\square$

**Lemma 36.** *Let $G$ be an $n+s$-vertex graph and $S$ be a set of vertices in $G$ such that $|S| = s$. If each of the non terminal levels with respect to $S$ contains at least 3 vertices, then*

$$\sigma(S) \leq \frac{1}{6}(n^2+3n+2).$$

## 6.3 Proof of Theorem 67

We proof Theorem 67 by induction on the number of vertices $n$. We may assume that Theorem 67 holds for $n \leq 20$.[2] Suppose that Theorem 67 holds for all quadrangulation graphs with at most $n-1$ vertices. When $|V(G)| = n$, we distinguish 2 cases depending on the minimum degree of $G$.

### Case 1: $\delta(G) = 2$.

Let $v$ be a vertex of degree 2 in $G$. We have two cases on the parity of $n$.

### Case 1.1: $2 \mid n$.

Due to $d_G(v) = 2$, then by deleting the vertex $v$, the resulting graph $G - v$ is a quadrangulation graph on $n-1$ vertices. Obviously,

$$W(G) \leq W(G-v) + \sigma(v).$$

By Lemma 34, since $2 \nmid (n-1)$, $\sigma(v) \leq \frac{1}{4}\Big((n-1)^2 + 2(n-1) + 1\Big)$ and by the induction hypothesis: $W(G-v) \leq \frac{1}{12}(n-1)^3 + \frac{11}{12}(n-1) - 1$. Thus,

$$W(G) \leq \left(\frac{1}{12}(n-1)^3 + \frac{11}{12}(n-1) - 1\right) + \frac{1}{4}\Big((n-1)^2 + 2(n-1) + 1\Big) = \frac{1}{12}n^3 + \frac{7}{6}n - 2$$

and we are done.

[2] Czabarka et.al. [25] determined the maximum Wiener index of quadrangulation graphs for small number of vertices with the aid of computer. The summary of their results is given in Appendix B.1.

## Case 1.2: $2 \nmid n$.

Here we distinguish two cases on the number of vertices in the second level with respect to $S = \{v\}$.

**Case 1.2.1: The second level contains at least 3 vertices.**

Similarly as Case 1.1, after deleting the vertex $v$, we get an $(n-1)$-vertex quadrangulation graph, $G - v$. Since $2 \mid (n-1)$, by the induction hypothesis, we get $W(G-v) \leq \frac{1}{12}(n-1)^3 + \frac{7}{6}(n-1) - 2$. And since the second level contains at least 3 vertices, by Lemma 35, $\sigma(v) \leq \frac{1}{4}\Big((n-1)^2 + 8\Big)$. Thus,

$$\begin{aligned} W(G) \leq W(G-v) + \sigma(v) &\leq \left(\frac{1}{12}(n-1)^3 + \frac{7}{6}(n-1) - 2\right) + \frac{1}{4}\Big((n-1)^2 + 8\Big) \\ &= \frac{1}{12}n^3 + \frac{11}{12}n - 1. \end{aligned}$$

**Case 1.2.2: The second level contains 2 vertices.**

Let $N(v) = \{x_1, x_2\}$, $x_3$ and $x_4$ be the two vertices in the second level respect to $v$. Since each face of $G$ is a 4-face, it can be easily checked that $vx_1x_3x_2v$ and $vx_1x_4x_2v$ are two 4-faces sharing the path $x_1vx_2$, see Figure 6.2 (left). Thus, $d_G(x_1) = d_G(x_2) = 3$. If $n \geq 7$ then we have cherries $x_3z_1x_4$ and $x_3z_2x_4$ such that $x_3z_1x_4x_1x_3$ and $x_3z_2x_4x_2x_3$ are 4-faces, for distinct vertices $z_1$ and $z_2$ in $G$.

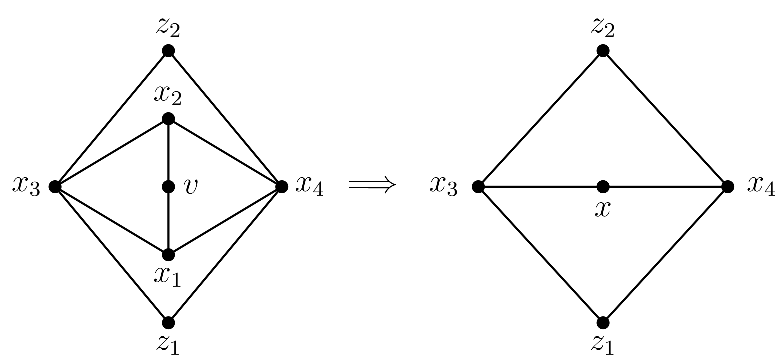



Figure 6.2: The case of 4-faces sharing a path of length 2.

Contracting edges $x_1v$ and $x_2v$ to a vertex $x$, see Figure 6.2 (right), results an $(n-2)$-vertex quadrangulation graph, say $G'$. Notice that in the graph $G'$, for any two vertices $t_1, t_2 \in V(G')\backslash\{x\}$, $d_{G'}(t_1, t_2) = d_G(t_1, t_2)$. But for any vertex

$t \in V(G')\backslash\{x\}$, $d_G(t,v) = d_{G'}(t,x)+1$. We know that,

$$\begin{aligned}W(G) &= \sum_{\{u,w\}\subseteq V(G)} d_G(u,w)\\ &= \sum_{u,w\in V(G)\backslash\{x_1,x_2\}} d_G(u,w) + \sum_{u\in V(G)\backslash\{x_2\}} d_G(u,x_1)\\ &+ \sum_{u\in V(G)\backslash\{x_1\}} d_G(u,x_2) + d_G(x_1,x_2)\end{aligned}$$

and

$$\begin{aligned}\sum_{u,w\in V(G)\backslash\{x_1,x_2\}} d_G(u,w) &= \sum_{u,w\in V(G)\backslash\{x_1,x_2,v\}} d_G(u,w) + \sum_{u\in V(G)\backslash\{x_1,x_2\}} d_G(u,v)\\ &= \sum_{u,w\in V(G')\backslash\{x\}} d_{G'}(u,w) + \sum_{u\in V(G')\backslash\{x\}} (d_{G'}(u,x)+1)\\ &= \sum_{u,w\in V(G')} d_{G'}(u,w) + (n-3)\\ &= W(G') + (n-3).\end{aligned}$$

Now we estimate $\sum_{u\in V(G)\backslash\{x_2\}} d_G(u,x_1) + \sum_{u\in V(G)\backslash\{x_1\}} d_G(u,x_2) + 2$. Consider the set $S = \{x_3,x_4,z_1,z_2\}$ and the levels of vertices in $S' = V(G)\backslash\{x_1,x_2,x_3,x_4,v,z_1,z_2\}$ with respect to $S$. Notice that $|S'| = n-7$. Sine $2\nmid n$, then $2 \mid (n-7)$. By Lemma 34,

$$\sigma(S) \le \frac{1}{4}\Big((n-7)^2 + 2(n-7)\Big).$$

Let $u \in S'$, then $d_G(u,x_1) + d_G(u,x_2) \le 2d_G(S,u) + 4$. Thus,

$$\begin{aligned}\sum_{u\in S'} d_G(u,x_1) + \sum_{u\in S'} d_G(u,x_2) &\le 2\sum_{u\in S'} d_G(S,u) + 4(n-7) = 2\sigma(S) + 4(n-7)\\ &\le \frac{1}{2}\Big((n-7)^2 + 2(n-7)\Big) + 4(n-7).\end{aligned}$$

It can be checked that the sum of the distances of $x_i$, $i \in [2]$, to each vertex in $\{x_3,x_4,z_1,z_2,v\}$ is 12. Hence,

$$\begin{aligned}\sum_{u\in V(G)\backslash\{x_2\}} d_G(u,x_1) + \sum_{u\in V(G)\backslash\{x_1\}} d_G(u,x_2) + 2 &\le \frac{1}{2}\Big((n-7)^2 + 2(n-7)\Big)\\ &+ 4(n-7) + 14.\end{aligned}$$

By the induction hypothesis, we get

$$\begin{aligned}W(G) &= W(G^{'}) + (n-3) + \sum_{u\in V(G)\backslash\{x_2\}} d_G(u,x_1) + \sum_{u\in V(G)\backslash\{x_1\}} d_G(u,x_2) + 2\\ &\leq \Big(W(G') + (n-3)\Big) + \left(\frac{1}{2}((n-7)^2 + 2(n-7)) + 4(n-7) + 14\right)\\ &\leq \left(\frac{1}{12}(n-2)^3 + \frac{11}{12}(n-2) - 1 + (n-3)\right)\\ &+ \left(\frac{1}{2}((n-7)^2 + 2(n-7)) + 4(n-7) + 14\right)\\ &= \frac{1}{12}n^3 + \frac{11}{12}n - 3,\end{aligned}$$

and Case 1 is done.

## Case 2: $\delta(G) = 3$.

Let $v$ be a vertex of degree 3, $N(v) = \{v_1, v_2, v_3\}$ and $v_4, v_5, v_6$ be vertices in $G$ such that $v_1vv_3v_4v_1$, $v_2vv_3v_5v_2$ and $v_1vv_2v_6v_1$ are 4-faces. It can be easily checked that $v_4$, $v_5$ and $v_6$ are distinct, otherwise, $G$ contains a degree 2 vertex, see Figure 6.3. Moreover, at least two of $e_1 = \{v_1, v_5\}, e_2 = \{v_2, v_4\}$ and $e_3 = \{v_3, v_6\}$ are not in $E(G)$. If none of the three exists in $G$, we call the associated vertex $v$ as a "*good vertex*". Observe that, deleting $v$ and adding one of these missed edges in $G$ result a quadrangulation graph with $n-1$ vertices. Suppose that $e_1 = \{v_1, v_5\} \notin E(G)$ and $G_1$ is the quadrangulation graph obtained form $G$ by deleting $v$ and adding $e_1$. Since this will decrease the distance of some pairs of vertices, we define the total sum of the distance decreases due to this operation on $G$ as follows,

$$\mathrm{dec}(G, G_1) = \sum_{u,w\in V(G)\backslash\{v\}} \Big(d_G(u,w) - d_{G_1}(u,w)\Big).$$

We will use the following claim to prove this case.

**Claim 23.** *Let $v$ be a good vertex and $G_i$ be a quadrangulation graph obtained from $G$ by deleting $v$ and adding the edge $e_i$ for $i = 1, 2, 3$. Then*

$$\min_{i\in[3]}\{dec(G, G_i)\} \leq \frac{(n-1)^2}{18}.$$

*Proof.* We may assume that $G_1$ gives the minimum total sum of the distance decrease. Let $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $S_i = S\backslash\{v_i\}$ for $i = 1, 2, 3, 4, 5, 6$. Define $A_i = \{x \in V(G_1)\backslash S_i |\ d_{G_1}(v_i, x) < d_{G_1}(z, x), z \in S_i\}$ $(i = 1, 2, 3, 4, 5, 6)$. Let $A$ be the set of the remaining vertices, which means that $A = V(G_1) - \bigcup_{i=1}^{6} A_i$. Thus,

$$V(G_1) = A_1 \cup A_2 \cup \cdots \cup A_6 \cup A.$$

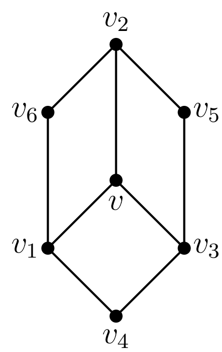



Figure 6.3: Structure around a degree 3 vertex of quadrangulation graph with $\delta(G) = 3$

Now we show that, for any pair of vertices $\{u, w\}$ in $G_1$ with shortest path that must across the edge $e_1$, then one vertex is in $A_1$ and the other is in $A_5$. If not, we consider 3 cases,

1. one of $u$ or $w$ is not in $A_1$, $A_5$ and $A$, without loss of generality, let $w \in A_i$, $i \in \{2, 3, 4, 6\}$. It can be easily checked that there exists a shortest path between $w$ and $u$ which does not across the edge $e_1$, then we get a contradiction.

2. one of $u$ and $w$ is in $A$, without loss of generality, let $u \in A$. There are two possibilities,

   (a) $d_{G_1}(v_i, u) = d_{G_1}(v_j, u)$, where $d(v_i, v_j)$ is odd, $i, j \in [6]$, this implies that there exists an odd cycle in $G_1$, which contradicts to the fact that every quadrangulation graphs are free of odd cycles.

   (b) $d_{G_1}(v_i, u) = d_{G_1}(v_j, u)$, where $d(v_i, v_j)$ is even, $i, j \in [6]$. It can be easily checked that there exists a shortest path between $w$ and $u$ which does not across the edge $e_1$, then we get a contradiction.

3. $u$ and $w$ are both in $A_1$ or in $A_5$. In this case, it is easy to see that the shortest path can never use the edge $e_1$, which is a contradiction.

Therefore, a pair of vertices with shortest paths must across the edge $e_1$ are those pairs that one is in $A_1$, the other one is in $A_5$. Since $\mathrm{dec}(G, G_1) = \min_{i \in [3]}\{\mathrm{dec}(G, G_i)\}$, $\mathrm{dec}(G, G_1)$ is maximized when $|A_1| = |A_2| = \cdots = |A_6| = \frac{n-1}{6}$. Notice that for such pair of vertices, the distance decrease is at most 2. Therefore,

$$\mathrm{dec}(G, G_1) \leq 2\left(\frac{n-1}{6}\right)\left(\frac{n-1}{6}\right) = \frac{(n-1)^2}{18}.$$

□

Now we continue the proof of the case by considering two subcases based on the existence of separating 4-cycle in $G$.

**Case 2.1: No separating 4-cycle in $G$.**

Notice that $G$ is a quadrangulation graph with $\delta(G) = 3$, then $n \geq 8$. Since there is no separating 4-cycle in $G$, by Lemma 33, $G$ is 3-connected, then each degree 3 vertex is a good vertex. Take a degree 3 vertex $v \in V(G)$. Without loss of generality, deleting $v$ and adding $e_1$, denoted $G_1$, $v(G_1) = n-1$, gives the minimum decrease distance sum, which is less than $\frac{(n-1)^2}{18}$ based on Claim 23. Also since $G$ is 3-connected, each of the non terminal level with respect to a vertex $v$ contains at least 3 vertices, otherwise, there exists a 2-vertex cut in $G$. By Lemma 36, we have

$$\sigma_G(v) \leq \frac{1}{6}\bigg((n-1)^2 + 3(n-1) + 2\bigg).$$

By the induction hypothesis, we get

$$\begin{aligned} W(G) &\leq W(G_1) + \sigma_G(v) + \mathrm{dec}(G, G_1) \\ &\leq W(G_1) + \sigma_G(v) + \frac{(n-1)^2}{18} \\ &\leq \bigg(\frac{1}{12}(n-1)^3 + \frac{7}{6}(n-1) - 2\bigg) + \frac{1}{6}\bigg((n-1)^2 + 3(n-1) + 2\bigg) + \frac{(n-1)^2}{18} \\ &= \frac{n^3}{12} - \frac{n^2}{36} + \frac{53n}{36} - \frac{115}{36}. \end{aligned}$$

It can be checked that $\frac{n^3}{12} - \frac{n^2}{36} + \frac{53n}{36} - \frac{115}{36} < \frac{1}{12}n^3 + \frac{11}{12}n - 1$ for $n \geq 15$, so we are done by the induction hypothesis.

**Case 2.2: G contains a separating 4-cycle.**

Let $S = \{z_1, z_2, z_3, z_4\}$ be a minimum separating 4-cycle in $G$ with minimum number of vertices in the interior. Let $x$ and $n - x - 4$ be the number of vertices of the interior and exterior of $S$, respectively. Clearly, $x \geq 4$ and $n - x - 4 \geq 4$, otherwise, $G$ contains a vertex of degree 2. Removing the interior $x$ vertices of $S$ results in a quadrangulation graph, say $G_{n-x}$ on $n - x$ vertices. Removing the exterior $n - x - 4$ vertices of $S$ results in a 3-connected quadrangulation graph, say $G_{x+4}$ on $x + 4$ vertices. Obviously, we have

$$W(G) \leq W(G_{x+4}) + W(G_{n-x}) - 8 + \sum_{w \in V(G_{n-x}) \setminus S} \sum_{u \in V(G_{x+4}) \setminus S} d(u, w),$$

(we subtracted 8 since $W(S)$ is counted twice), and without loss of generality, let $\max\{\sigma_{G_{x+4}}(z_i) \mid i \in [4]\} = \sigma_{G_{x+4}}(z_1)$

$$
\begin{aligned}
&\sum_{w\in V(G_{n-x})\setminus S}\ \sum_{u\in V(G_{x+4})\setminus S} d(u,w)\\
&\qquad \le x\sigma_{G_{n-x}}(S) + (n-x-4)\Bigg(\max\{\sigma_{G_{x+4}}(z_i)\mid i\in[4]\} - \sum_{j=2}^{4} d(z_1,z_j)\Bigg)\\
&\qquad = x\sigma_{G_{n-x}}(S) + (n-x-4)\Bigg(\max\{\sigma_{G_{x+4}}(z_i)\mid i\in[4]\} - 4\Bigg).
\end{aligned}
$$

Since $G_{x+4}$ is 3-connected, also by Lemma 36, we have

$$
\max\{\sigma_{G_{x+4}}(z_i)\mid i\in[4]\} = \sigma_{G_{x+4}}(z_1) \le \frac{1}{6}\Bigg((x+3)^2+3(x+3)+2\Bigg).
$$

Thus, by the induction hypothesis, Lemmas 34 and 36, we get

$$
\begin{aligned}
W(G) &\le W(G_{x+4}) + W(G_{n-x}) - 8 + \sum_{w\in V(G_{n-x})\setminus S}\ \sum_{u\in V(G_{x+4})\setminus S} d(u,w)\\
&\le \left(\frac{1}{12}(x+4)^3+\frac{7}{6}(x+4)-2\right) + \left(\frac{1}{12}(n-x)^3+\frac{7}{6}(n-x)-2\right) - 8\\
&+ \frac{x}{4}\Bigg((n-x-4)^2+2(n-x-4)+1\Bigg)\\
&+ (n-x-4)\Bigg(\frac{1}{6}\Bigg((x+3)^2+3(x+3)+2\Bigg)-4\Bigg)\\
&= \frac{n^3}{12} - \frac{nx^2}{12} + \frac{n}{2} + \frac{x^3}{12} + \frac{x^2}{3} + \frac{11x}{12} + \frac{2}{3}.
\end{aligned}
$$

It can be checked that the inequality $\frac{n^3}{12} - \frac{nx^2}{12} + \frac{n}{2} + \frac{x^3}{12} + \frac{x^2}{3} + \frac{11x}{12} + \frac{2}{3} \le \frac{1}{12}n^3 + \frac{11}{12}n - 1$ holds for $x \ge 4$, except when $x = 4$ and $n = 9$ which implies that $G_{n-x} \setminus S$ contains only one degree 2 vertex, this contradicts the fact that $\delta(G) = 3$. This completes proof of Theorem 67.

## 6.4 Remarks and conjectures

From Theorem 51 the only triangulation graph maximizing the Wiener index among all $n$-vertex maximal planar graph is $T_n$, see Figure 1.12. Clearly $T_n$ is not 4-connected. Similarly, the quadrangulation graph $Q_n$ (see Figure 6.1) is not 3-connected. One may ask for the maximum Wiener index for the family of 4-connected and 5-connected maximal planar graphs and 3-connected quadrangulation graphs. In [25], best asymptotic results were proved for each of the case. Moreover, based on their constructions, they proposed the following conjectures related to sharp and exact bounds.

**Conjecture 11.** *(Czabarka, Dankelmann, Olsen, Székely [25])*
*Let $G$ be an $n$-vertex, $n \geq 6$, 4-connected maximal planar graph. Then*

$$W(G) \leq \begin{cases} \frac{1}{24}n^3 + \frac{1}{4}n^2 + \frac{1}{3}n - 2, & \textit{if } n \equiv 0, 2 \ (\textit{mod } 4); \\ \frac{1}{24}n^3 + \frac{1}{4}n^2 + \frac{5}{24}n - \frac{3}{2}, & \textit{if } n \equiv 1 \ (\textit{mod } 4); \\ \frac{1}{24}n^3 + \frac{1}{4}n^2 + \frac{5}{24}n - 1, & \textit{if } n \equiv 3 \ (\textit{mod } 4). \end{cases}$$

**Conjecture 12.** *(Czabarka, Dankelmann, Olsen, Székely [25])*
*Let $G$ be an $n$-vertex, $n \geq 12$, 4-connected maximal planar graph. Then*

$$W(G) \leq \begin{cases} \frac{1}{30}n^3 + \frac{3}{10}n^2 - \frac{23}{15}n + 32, & \textit{if } n \equiv 0 \ (\textit{mod } 5); \\ \frac{1}{30}n^3 + \frac{3}{10}n^2 - \frac{23}{15}n + \frac{156}{5}, & \textit{if } n \equiv 1 \ (\textit{mod } 5); \\ \frac{1}{30}n^3 + \frac{3}{10}n^2 - \frac{23}{15}n + \frac{168}{5}, & \textit{if } n \equiv 2 \ (\textit{mod } 5); \\ \frac{1}{30}n^3 + \frac{3}{10}n^2 - \frac{23}{15}n + 31, & \textit{if } n \equiv 3 \ (\textit{mod } 5); \\ \frac{1}{30}n^3 + \frac{3}{10}n^2 - \frac{23}{15}n + \frac{161}{5}, & \textit{if } n \equiv 4 \ (\textit{mod } 5). \end{cases}$$

**Conjecture 13.** *(Czabarka, Dankelmann, Olsen, Székely [25])*
*Let $G$ be an $n$-vertex, $n \geq 14$, 3-connected quadrangulation graph. Then*

$$W(G) \leq \begin{cases} \frac{1}{18}n^3 + \frac{1}{3}n^2 - \frac{17}{6}n + 20, & \textit{if } n \equiv 0 \ (\textit{mod } 3); \\ \frac{1}{18}n^3 + \frac{1}{3}n^2 - \frac{17}{6}n + \frac{184}{9}, & \textit{if } n \equiv 1 \ (\textit{mod } 3); \\ \frac{1}{18}n^3 + \frac{1}{3}n^2 - \frac{17}{6}n + \frac{206}{9}, & \textit{if } n \equiv 2 \ (\textit{mod } 3). \end{cases}$$

# Appendices

# Appendix A

# Extremal Constructions

## A.1 Extremal constructions for planar Turán number of $K_3$

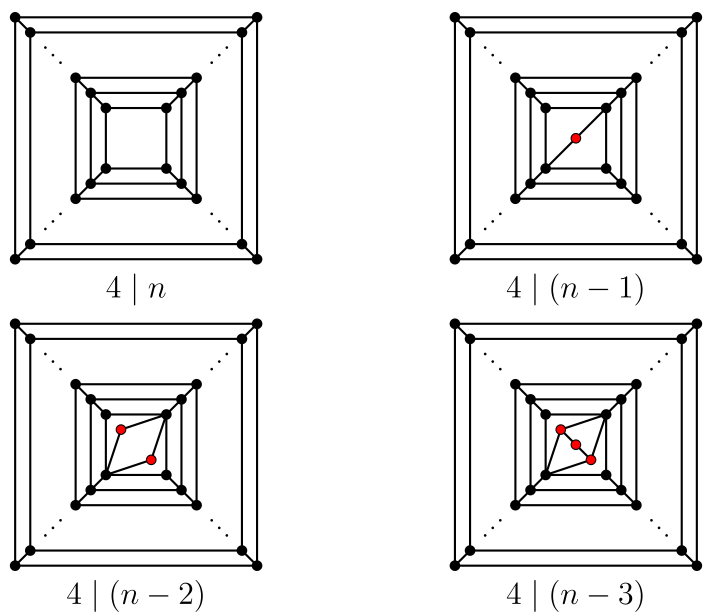



Figure A.1: Extremal constructions (quadrangulations) for PTN of $K_3$.

## A.2 Proof of Theorem 34

First we show that for a plane graph $G_0$ with $n$ vertices ($n \equiv 7 \pmod{10}$), each face having length 7 and each vertex in $G_0$ having degree either 2 or 3, we can construct

$G$, where $G$ is a $C_6$-free plane graph with $v(G) = \frac{18n+14}{5}$ and $e(G) = 9n$. We then give a construction for such a $G_0$ as long as $n \equiv 7 \pmod{10}$.

Using Euler's formula, the fact that every face has length 7 and every degree is 2 or 3, we have $e(G_0) = \frac{7(n-2)}{5}$ and the number of degree 2 and degree 3 vertices in $G_0$ are $\frac{n+28}{5}$ and $\frac{4n-28}{5}$, respectively.

Given $G_0$, we construct first an intermediate graph $G'$ by step 1:

1. Add halving vertices to each edge of $G_0$ and join the pair of halving vertices with distance 2, see an example in Figure A.2. Let $G'$ denote this new graph, then $v(G') = v(G_0) + e(G_0) = \frac{12n-14}{5}$ and the number of degree 2 and degree 3 vertices in $G'$ is equal to the number of degree 2 and degree 3 vertices in $G_0$, respectively.

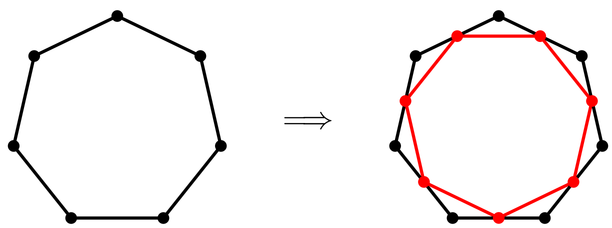


Figure A.2: Adding a halving vertex to each edge of $G_0$.

To get $G$, we apply the following steps 2 and 3 on the degree 2 and 3 vertices in $G'$, respectively.

2. For each degree 2 vertex $v$ in $G_0$, let $N(v) = \{v_1, v_2\}$, and so $v_1vv_2$ forms an induced triangle in $G'$. Fix $v_1$ and $v_2$, replace $v_1vv_2$ with a $K_5^-$ by adding vertices $v_1^{'}$, $v_2^{'}$ to $V(G')$ and edges $v_1^{'}v$, $v_1^{'}v_2^{'}$, $v_1^{'}v_1$, $v_1^{'}v_2$, $v_2^{'}v_1$, $v_2^{'}v_2$ to $E(G')$. See Figure A.3.

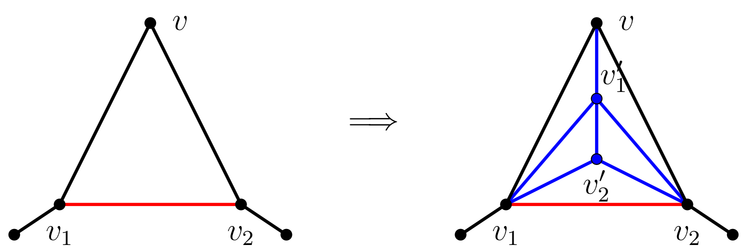



Figure A.3: Replacing a degree-2 vertex of $G_0$ with a $K_5^-$.

3. For each degree 3 vertex $v$ in $G_0$, such that $N(v) = \{v_1, v_2, v_3\}$, the set of vertices $\{v, v_1, v_2, v_3\}$ then forms an induced $K_4$ in $G'$. Fix $v_1$, $v_2$ and $v_3$, replace this $K_4$ with a $K_5^-$ by adding a new vertex $v'$ to $V(G')$ and edges $v'v$, $v'v_1$, $v'v_2$ to $E(G')$. See Figure A.4.

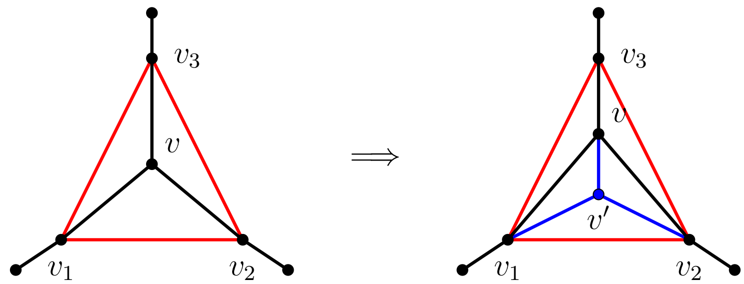



Figure A.4: Replacing a degree-3 vertex of $G_0$ with a $K_5^-$.

For each integer $k \geq 0$, and $n = 10k + 7$ we present a construction for such a $G_0$, call it $G_0^k$: Let $v_i^t$ and $v_i^b$ $(1 \leq i \leq k+1)$ be the top and bottom vertices of the heptagonal grids with 3 layers and $k$ columns, respectively (see the red vertices in Figure A.5) and $v$ be the extra vertex in $G_0^k$ but not in the heptagonal grid. We join $v_1^t v$, $vv_1^b$ and $v_i^t v_i^b$ $(2 \leq i \leq k+1)$. Clearly, $G_0^k$ is a $(10k+7)$-vertex plane graph and each face of $G_0^k$ is a 7-face. Obviously $e\left(G_0^k\right) = 14k + 7$, and the number of degree 2 and 3 vertices are $2k + 7 = \frac{n+28}{5}$ and $8k = \frac{4n-28}{5}$ respectively.

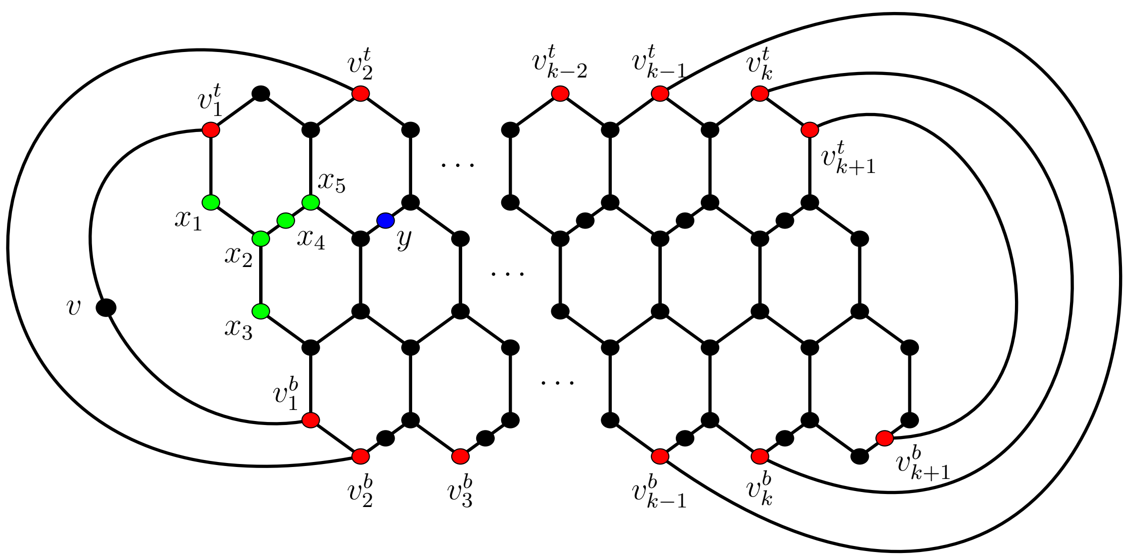



Figure A.5: The graph $G_0^k$, $k \geq 1$, in which each face has length 7. The graph $H_0^k$ (see Remark 2) is obtained by deleting $x_1, \ldots, x_5$ and adding the edge $v_1^t y$.

After applying steps 1, 2, and 3 on $G_0^k$, we get $G$. It is easy to verify that $G$ is a

$C_6$-free plane graph with

$$\begin{aligned} v(G) &= v(G_0^k) + e(G_0^k) + 2(2k+7) + 8k = (10k+7) + (14k+7) + 12k + 14 \\ &= 36k + 28 \\ e(G) &= 9v(G_0^k) = 90k + 63. \end{aligned}$$

Thus, $e(G) = \frac{5}{2}v(G) - 7$.

**Remark 2.** *In fact, for $k \geq 1$ and $n = 10k+2$, there exists a graph $H_0^k$ which is obtained from $G_0^k$ by deleting vertices (colored green in Figure A.5) $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ and adding the edge $v_1^t y$. Clearly, $H_0^k$ is an $10k+2$-vertex plane graph such that all faces have length 7. Moreover, $e(H_0^k) = 14k$, the number of degree-2 and degree-3 vertices are $2k+6 = \frac{n+28}{5}$ and $8k-4 = \frac{4n-28}{5}$, respectively. After applying steps (1), (2), and (3) to $H_0^k$, we get a graph $H$ that is a $C_6$-free plane graph with $e(H) = \frac{5}{2}v(H) - 7$.*

*Thus, for any $k \equiv 2 \pmod 5$, we have the graphs above such that each face is a 7-gon and we get a $C_6$-free plane graph on $n$ vertices with $\frac{5}{2}n - 7$ edges for $n \equiv 10 \pmod{18}$ if $n \geq 28$.*

## A.3 Conjectures on planar Turán number of longer cycles

We believe that our construction in the proof of Theorem 34 can be generalized to determine sharp upper bound of $\mathrm{ex}_{\mathcal{P}}(n, C_\ell)$ for $\ell$ sufficiently large. That is, we construct $G_0$, a plane graph with all faces of length $\ell+1$ and with all vertices having degree 3 or degree 2, for instance see a construction of $G_0$ for the case that $\ell = 7$ in Figure A.6.

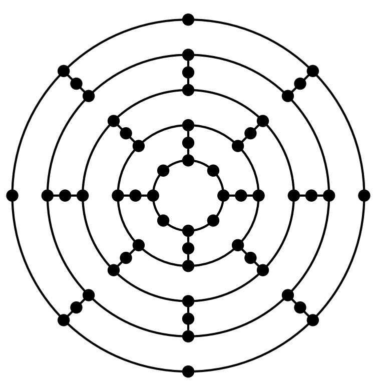


Figure A.6: A construction for base graph $G_0$ when $\ell = 7$: A plane graph with all faces of size 8 and vertices with degree either 2 or 3.

If such a $G_0$ exists, then the number of degree-2 and degree-3 vertices are $\frac{(\ell-5)n+4(\ell+1)}{\ell-1}$ and $\frac{4(n-\ell-1)}{\ell-1}$, respectively. We could then apply steps similar to (1), (2), and (3) in the construction proof of Theorem 34 in that we add halving vertices and insert a graph $B_{\ell-1}$ in Figure A.7 (or another maximal planar graph of $\ell-1$ vertices) in place of vertices of degree 2 and 3. For the resulting graph $G$,

$$\begin{aligned} v(G) &= v(G_0) + e(G_0) + (\ell-4)\frac{(\ell-5)n+4(\ell+1)}{\ell-1} + (\ell-5)\frac{4(n-\ell-1)}{\ell-1} \\ &= n + \frac{\ell+1}{\ell-1}(n-2) + \frac{(\ell^2-5\ell)n+2(\ell+1)}{\ell-1} = \frac{\ell^2-3\ell}{\ell-1}n + \frac{2(\ell+1)}{\ell} \\ e(G) &= (3\ell-9)v(G_0) = (3\ell-9)n. \end{aligned}$$

Therefore, $e(G) = \frac{3(\ell-1)}{\ell}v(G) - \frac{6(\ell+1)}{\ell}$. We conjecture that this is the maximum number of edges in a $C_\ell$-free planar graph – at least if $\ell$ is small.

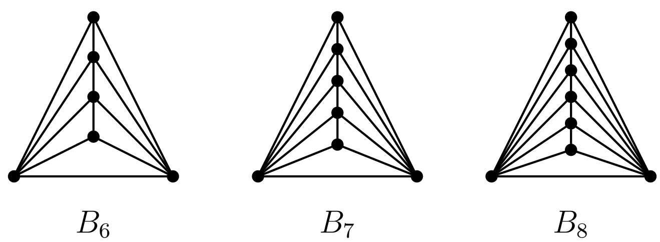



Figure A.7: $B_{\ell-1}$ is used in the construction of a $C_\ell$-free graph.

**Conjecture 14.** *(Ghosh, Győri, Martin, Paulos, Xiao [52])*
*Let $G$ be an $n$-vertex $C_\ell$-free plane graph ($10 \geq \ell \geq 7$), then there exists an integer $N_0 > 0$, such that when $n \geq N_0$,*

$$e(G) \leq \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}.$$

**Conjecture 15.** *(Cranston, Lidický, Liu and Shantanam [24])*
*There exists a constant $D$ such that for all $\ell$ and for all sufficiently large $n$ we have*

$$\mathrm{ex}_{\mathcal{P}}(n,\ell) \leq \frac{3(D\ell^{\lg_2^3}-1)}{D\ell^{\lg_2^3}}n.$$

# Appendix B

# Particular Computed Values

## B.1 Maximum WI of quadrangulation graphs with small order

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $W(G)$ | 8 | 14 | 23 | 34 | 50 | 68 | 93 | 120 | 156 | 194 | 243 | 294 | 358 | 424 | 505 | 588 | 688 |

Table B.1: A summary of the maximal WI among quadrangulations on $n$ vertices.

| $n$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $W(G)$ | 48 | - | 83 | 106 | 136 | 164 | 201 | 240 | 288 | 344 | 468 | 401 | 544 | 622 | 711 | 810 | 912 | 1026 | 1151 | 1280 | 1422 |

Table B.2: A summary of the maximal WI among all 3-connected quadrangulations on $n$ vertices.

## B.2 Maximum WI of triangulation graphs with small order

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $W(G)$ | 6 | 11 | 18 | 27 | 39 | 54 | 72 | 94 | 120 | 150 | 185 | 225 | 270 | 321 | 378 |

Table B.3: A summary of the maximal WI among triangulations on $n$ vertices.

| $n$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $W(G)$ | 18 | 27 | 38 | 51 | 68 | 87 | 110 | 135 | 166 | 199 | 238 | 279 | 328 | 379 | 438 | 499 | 570 |

Table B.4: A summary of the maximal WI among all 4-connected triangulations on $n$ vertices.

| $n$ | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $W(G)$ | 108 | - | 159 | 189 | 222 | 259 | 300 | 342 | 391 | 444 | 500 | 560 | 630 | 702 | 780 | 867 | 955 | 1053 | 1156 | 1265 | 1384 |

Table B.5: A summary of the maximal WI among all 5-connected triangulations on $n$ vertices.

# Bibliography


[1] A. F. Alameddine. On the number of cycles of length 4 in a maximal planar graph. *Journal of Graph Theory*, 4(4):417–422, 1980.

[2] N. Alon and Y. Caro. On the number of subgraphs of prescribed type of planar graphs with a given number of vertices. *Annals of Discrete Mathematics*, 20:25–36, 1984.

[3] N. Alon and C. Shikhelman. Many $T$ copies in $H$-free graphs. *Journal of Combinatorial Theory, Series B*, 121:146–172, 2016.

[4] N. Alon and C. Shikhelman. Additive approximation of generalized Turán questions. *arXiv preprint arXiv:1811.08750*, 2018.

[5] H. Alt, U. Fuchs, and K. Kriegel. On the number of simple cycles in planar graphs. *Combinatorics Probability and Computing*, 8(5):397–405, 1999.

[6] P. N. Balister, E. Győri, J. Lehel, and R. H. Schelp. Connected graphs without long paths. *Discrete Mathematics*, 308(19):4487–4494, 2008.

[7] J. Balogh, P. Hu, B. Lidickỳ, and F. Pfender. Maximum density of induced 5-cycle is achieved by an iterated blow-up of 5-cycle. *European Journal of Combinatorics*, 52:47–58, 2016.

[8] R. A. Beezer, J. E. Riegsecker, and B. A. Smith. Using minimum degree to bound average distance. *Discrete Mathematics*, 226(1-3):365–371, 2001.

[9] S. Biswas, S. Durocher, D. Mondal, and R. I. Nishat. Hamiltonian paths and cycles in planar graphs. In *International Conference on Combinatorial Optimization and Applications*, pages 83–94. Springer, 2012.

[10] B. Bollobás. *Graph theory: an introductory course*, volume 63. Springer Science & Business Media, 2012.

[11] B. Bollobás, Y. Egawa, A. Harris, and G. Jin. The maximal number of induced $r$-partite subgraphs. *Graphs and Combinatorics*, 11(1):1–19, 1995.

[12] B. Bollobás and E. Győri. Pentagons vs. triangles. *Discrete Mathematics*, 308(19):4332–4336, 2008.

[13] B. Bollobás, C. Nara, and S.-i. Tachibana. The maximal number of induced complete bipartite graphs. *Discrete mathematics*, 62(3):271–275, 1986.

[14] A. Bondy and U. Murty. *Graph Theory (Graduate Texts in Mathematics 244)*. Springer, 2008.

[15] J. A. Bondy and M. Simonovits. Cycles of even length in graphs. *Journal of Combinatorial Theory, Series B*, 16(2):97–105, 1974.

[16] O. V. Borodin, A. V. Kostochka, N. N. Sheikh, and G. Yu. $M$-degrees of quadrangle-free planar graphs. *Journal of Graph Theory*, 60(1):80–85, 2009.

[17] G. Brinkmann, S. Greenberg, C. Greenhill, B. D. McKay, R. Thomas, and P. Wollan. Generation of simple quadrangulations of the sphere. *Discrete mathematics*, 305(1-3):33–54, 2005.

[18] J. I. Brown and A. Sidorenko. The inducibility of complete bipartite graphs. *Journal of Graph Theory*, 18(6):629–645, 1994.

[19] W. G. Brown. On graphs that do not contain a Thomsen graph. *Canadian Mathematical Bulletin*, 9(3):281–285, 1966.

[20] K. Buchin, C. Knauer, K. Kriegel, A. Schulz, and R. Seidel. On the number of cycles in planar graphs. In *International Computing and Combinatorics Conference*, pages 97–107. Springer, 2007.

[21] Z. Che and K. L. Collins. An upper bound on Wiener indices of maximal planar graphs. *Discrete Applied Mathematics*, 258:76–86, 2019.

[22] C. Cox and R. R. Martin. Counting paths, cycles and blow-ups in planar graphs. *arXiv preprint arXiv:2101.05911*, 2021.

[23] C. Cox and R. R. Martin. The maximum number of 10- and 12-cycles in a planar graph. *arXiv preprint arXiv:2106.02966v1*, 2021.

[24] D. W. Cranston, B. Lidickỳ, X. Liu, and A. Shantanam. Planar tur\'{a} n numbers of cycles: A counterexample. *arXiv preprint arXiv:2110.02043*, 2021.

[25] É. Czabarka, P. Dankelmann, T. Olsen, and L. A. Székely. Wiener index and remoteness in triangulations and quadrangulations. *Discrete Mathematics & Theoretical Computer Science*, 23, 2021.

[26] P. Dankelmann and R. Entringer. Average distance, minimum degree, and spanning trees. *Journal of Graph Theory*, 33(1):1–13, 2000.

[27] P. Dankelmann, S. Mukwembi, and H. C. Swart. Average distance and edge-connectivity i. *SIAM Journal on Discrete Mathematics*, 22(1):92–101, 2008.

[28] P. Dankelmann, S. Mukwembi, and H. C. Swart. Average distance and edge-connectivity ii. *SIAM Journal on Discrete Mathematics*, 21(4):1035–1052, 2008.

[29] R. Diestel. Graph theory 3rd ed. *Graduate texts in mathematics*, 173, 2005.

[30] G. A. Dirac. Short proof of Menger's graph theorem. *Mathematika*, 13(1):42–44, 1966.

[31] C. Dowden. Extremal $C_4$-free/$C_5$-free planar graphs. *Journal of Graph Theory*, 83(3):213–230, 2016.

[32] D. Eppstein. Connectivity, graph minors, and subgraph multiplicity. *Journal of Graph Theory*, 17(3):409–416, 1993.

[33] P. Erdős. On the number of complete subgraphs contained in certain graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 7(3):459–464, 1962.

[34] P. Erdős. On some problems in graph theory, combinatorial analysis and combinatorial number theory. *Graph Theory and Combinatorics (Cambridge, 1983), Academic Press, London*, pages 1–17, 1984.

[35] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc*, 52(1087-1091):1, 1946.

[36] P. Erdős. On some extremal problems in graph theory. *Israel Journal of Mathematics*, 3(2):113–116, 1965.

[37] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Mathematica Hungarica*, 10(3-4):337–356, 1959.

[38] P. Erdős, A. Rényi, and V. Sós. On a problem of graph theory. *Studia Sci.Math*, 1:215–235, 1966.

[39] P. Erdős and M. Simonovits. A limit theorem in graph theory. In *Studia Sci. Math. Hung*. Citeseer, 1965.

[40] B. Ergemlidze, E. Győri, A. Methuku, and N. Salia. A note on the maximum number of triangles in a $C_5$-free graph. *Journal of Graph Theory*, 90(3):227–230, 2019.

[41] B. Ergemlidze and A. Methuku. Triangles in $C_5$-free graphs and hypergraphs of girth six. *arXiv preprint arXiv:1811.11873*, 2018.

[42] C. Even-Zohar and N. Linial. A note on the inducibility of 4-vertex graphs. *Graphs and Combinatorics*, 31(5):1367–1380, 2015.

[43] L. Fang, M. Zhai, and B. Wang. Planar Turán number of intersecting triangles. *arXiv preprint arXiv:2007.09650*, 2020.

[44] O. Favaron, M. Kouider, and M. Mahéo. Edge-vulnerability and mean distance. *Networks*, 19(5):493–504, 1989.

[45] M. Fischermann, A. Hoffmann, D. Rautenbach, L. Székely, and L. Volkmann. Wiener index versus maximum degree in trees. *Discrete Applied Mathematics*, 122(1-3):127–137, 2002.

[46] Z. Füredi and L. Özkahya. On 3-uniform hypergraphs without a cycle of a given length. *Discrete Applied Mathematics*, 216:582–588, 2017.

[47] Z. Füredi and M. Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdős Centennial*, pages 169–264. Springer, 2013.

[48] D. Gerbner, E. Győri, A. Methuku, and M. Vizer. Generalized Turán problems for even cycles. *arXiv preprint arXiv:1712.07079*, 2017.

[49] D. Gerbner and C. Palmer. Counting copies of a fixed subgraph in $F$-free graphs. *European Journal of Combinatorics*, 82:103001, 2019.

[50] D. Ghosh, E. Győri, O. Janzer, A. Paulos, N. Salia, and O. Zamora. The maximum number of induced $C_5$'s in a planar graph. *arXiv preprint arXiv:2004.01162 (accepted by the Journal of Graph Theory)*, 2020.

[51] D. Ghosh, E. Győri, R. R. Martin, A. Paulos, N. Salia, C. Xiao, and O. Zamora. The maximum number of paths of length four in a planar graph. *Discrete Mathematics*, 344(5):112317, 2021.

[52] D. Ghosh, E. Győri, R. R. Martin, A. Paulos, and C. Xiao. Planar Turán number of the 6-cycle. *arXiv preprint arXiv:2004.14094(accepted by SIAM Journal on Discrete Mathematics)*, 2020.

[53] D. Ghosh, E. Győri, A. Paulos, N. Salia, and O. Zamora. The maximum Wiener index of maximal planar graphs. *Journal of Combinatorial Optimization*, 40(4):1121–1135, 2020.

[54] D. Ghosh, E. Győri, A. Paulos, C. Xiao, and O. Zamora. Planar Turán number of the $\Theta_6$. *arXiv preprint arXiv:2006.00994(submitted to Studia Scientiarum Mathematicarum Hungarica )*, 2020.

[55] L. Gishboliner and A. Shapira. A generalized Turán problem and its applications. *International Mathematics Research Notices*, 2020(11):3417–3452, 2020.

[56] A. Grzesik. On the maximum number of five-cycles in a triangle-free graph. *Journal of Combinatorial Theory, Series B*, 102(5):1061–1066, 2012.

[57] A. Grzesik, E. Győri, A. Paulos, N. Salia, C. Tompkins, and O. Zamora. The maximum number of paths of length three in a planar graph. *arXiv preprint arXiv:1909.13539 (accepted by the Journal of Graph Theory, accepted by Eurocomb 2021)*, 2019.

[58] A. Grzesik and B. Kielak. On the maximum number of odd cycles in graphs without smaller odd cycles. *arXiv preprint arXiv:1806.09953*, 2018.

[59] E. Győri. On the number of $C_5$'s in a triangle-free graph. *Combinatorica (Print)*, 9(1):101–102, 1989.

[60] E. Győri and H. Li. The maximum number of triangles in $C_{2k+1}$-free graphs. *Combinatorics, Probability and Computing*, 21(1-2):187–191, 2012.

[61] E. Györi, J. Pach, and M. Simonovits. On the maximal number of certain subgraphs in $K_r$-free graphs. *Graphs and Combinatorics*, 7(1):31–37, 1991.

[62] E. Győri, A. Paulos, N. Salia, C. Tompkins, and O. Zamora. The maximum number of pentagons in a planar graph. *arXiv preprint arXiv:1909.13532(submitted to the Electronic Journal of Combinatorics)*, 2019.

[63] E. Győri, A. Paulos, N. Salia, C. Tompkins, and O. Zamora. Generalized planar Turán numbers. *arXiv preprint arXiv:2002.04579 (accepted by the Electronic Journal of Combinatorics)*, 2020.

[64] E. Győri, A. Paulos, and C. Xiao. Wiener index of quadrangulation graphs. *Discrete Applied Mathematics*, pages 262–269, 2021.

[65] E. Győri, A. Paulos, and O. Zamora. The minimum number of 4-cycles in a maximal planar graph with small number of vertices. *arXiv preprint arXiv:2005.12100*, 2020.

[66] E. Győri, N. Salia, C. Tompkins, and O. Zamora. The maximum number of $p_\ell$ copies in $p_k$-free graphs. *Acta Mathematica Universitatis Comenianae*, 88(3):773–778, 2019.

[67] S. L. Hakimi and E. F. Schmeichel. On the number of cycles of length $k$ in a maximal planar graph. *Journal of Graph Theory*, 3(1):69–86, 1979.

[68] S. L. Hakimi and E. F. Schmeichel. Bounds on the number of cycles of length three in a planar graph. *Israel Journal of Mathematics*, 41(1-2):161–180, 1982.

[69] J. Harant, M. Horňák, and Z. Skupień. Separating 3-cycles in plane triangulations. *Discrete Mathematics*, 239(1-3):127–136, 2001.

[70] F. Harary. Status and contrastatus. *Sociometry*, 22(1):23–43, 1959.

[71] H. Hatami, J. Hladkỳ, D. Král, S. Norine, and A. Razborov. On the number of pentagons in triangle-free graphs. *Journal of Combinatorial Theory, Series A*, 120(3):722–732, 2013.

[72] W. He, X. Hou, K.-W. Lih, J. Shao, W. Wang, and X. Zhu. Edge-partitions of planar graphs and their game coloring numbers. *Journal of Graph Theory*, 41(4):307–317, 2002.

[73] J. Hirst. The inducibility of graphs on four vertices. *Journal of Graph Theory*, 75(3):231–243, 2014.

[74] T. Huynh, G. Joret, and D. R. Wood. Subgraph densities in a surface. *arXiv preprint arXiv:2003.13777*, 2020.

[75] T. Huynh and D. R. Wood. Tree densities in sparse graph classes. *arXiv preprint arXiv:2009.12989*, 2020.

[76] T. Kővári, V. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Mathematicum*, 1(3):50–57, 1954.

[77] G. Kopylov. On maximal paths and cycles in a graph. In *Doklady Akademii Nauk*, volume 234, pages 19–21. Russian Academy of Sciences, 1977.

[78] M. Kouider and P. Winkler. Mean distance and minimum degree. *Journal of Graph Theory*, 25(1):95–99, 1997.

[79] C. Kuratowski. Sur le probleme des courbes gauches en topologie. *Fundamenta mathematicae*, 15(1):271–283, 1930.

[80] T. Lam and J. Verstraete. A note on graphs without short even cycles. *the electronic journal of combinatorics*, 12(1):N5, 2005.

[81] Y. Lan and Y. Shi. Planar Turán numbers of short paths. *Graphs and Combinatorics*, 35(5):1035–1049, 2019.

[82] Y. Lan, Y. Shi, and Z.-X. Song. Planar Turán numbers for Theta graphs and paths of small order. *arXiv preprint arXiv:1711.01614*, 2017.

[83] Y. Lan, Y. Shi, and Z.-X. Song. Extremal $H$-free planar graphs. *The Electronic Journal of Combinatorics*, 26(2):#P2.11, 2019.

[84] Y. Lan, Y. Shi, and Z.-X. Song. Extremal Theta-free planar graphs. *Discrete Mathematics*, 342(12):111610, 2019.

[85] S. Letzter. Many $H$-copies in graphs with a forbidden tree. *SIAM Journal on Discrete Mathematics*, 33(4):2360–2368, 2019.

[86] B. Lidický and K. Murphy. Maximizing five-cycles in $k_r$-free graphs. *European Journal of Combinatorics*, 97:103367, 2021.

[87] B. Lidický and F. Pfender. Pentagons in triangle-free graphs. *European Journal of Combinatorics*, 74:85–89, 2018.

[88] L. Lovász. *Combinatorial problems and exercises*, volume 361. American Mathematical Soc., 2007.

[89] R. Luo. The maximum number of cliques in graphs without long cycles. *Journal of Combinatorial Theory, Series B*, 128:219–226, 2018.

[90] W. Mantel. Problem 28. *Wiskundige Opgaven*, 10(60-61):320, 1907.

[91] D. Matolcsi and Z. L. Nagy. Generalized outerplanar Turán numbers and maximum number of $k$-vertex subtrees. *arXiv preprint arXiv:2102.11746*, 2021.

[92] N. Morrison and A. Scott. Maximising the number of induced cycles in a graph. *Journal of Combinatorial Theory, Series B*, 126:24–61, 2017.

[93] J. Plesník. On the sum of all distances in a graph or digraph. *Journal of Graph Theory*, 8(1):1–21, 1984.

[94] A. A. Razborov. On the minimal density of triangles in graphs. *Combinatorics Probability and Computing*, 17(4):603–618, 2008.

[95] M. Savery. Planar graphs with the maximum number of induced 4-cycles or 5-cycles. *arXiv preprint arXiv:2108.00526*, 2021.

[96] P. Turán. On an external problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[97] K. Wagner. Über eine eigenschaft der ebenen komplexe. *Mathematische Annalen*, 114(1):570–590, 1937.

[98] H. Whitney. Congruent graphs and the connectivity of graphs. *American Journal of Mathematics*, 54(1):150–168, 1932.

[99] H. Wiener. Structural determination of paraffin boiling points. *Journal of the American Chemical Society*, 69(1):17–20, 1947.

[100] D. R. Wood. On the maximum number of cliques in a graph. *Graphs and Combinatorics*, 23(3):337–352, 2007.

[101] N. C. Wormald. On the frequency of 3-connected subgraphs of planar graphs. *Bulletin of the Australian Mathematical Society*, 34(2):309–317, 1986.

[102] K. Zarankiewicz. Problem p 101. In *Colloq. Math*, volume 2, page 49, 1951.

[103] A. A. Zykov. On some properties of linear complexes. *Matematicheskii sbornik*, 66(2):163–188, 1949.

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