Jordan Type Problems via Class 2 Nilpotent and Twisted Heisenberg Groups

by

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Declaration

I hereby declare

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Abstract

We investigate the structure of finitely generated nilpotent groups of class at most 2 and show an explicit way to construct all such groups starting from cyclic groups and 2-generated nilpotent groups of class 2 by applying the operations of central and subdirect products. Using this description, we show that every finitely generated nilpotent group of class at most 2 is isomorphic to a subgroup of a generalisation of the Heisenberg group.

Using these results we prove the main statements of the thesis. We show the existence of an algebraic variety and a smooth manifold on which every finite nilpotent group of class at most 2 of bounded rank acts faithfully via birational automorphisms and via diffeomorphisms, respectively. This gives a sharp answer to a Jordan type problem about the birational automorphism groups and answers a question of Mundet i Riera in the smooth case.

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Contents

Chapter 1

Introduction

1.1 Motivation and the Jordan Property

In this section, we introduce the main problem (Problem 1.1.2) this thesis aims to address and give a brief historical overview of the development of Jordan type theorems.

When an infinite group G is too large and complicated to understand, one could try to first focus only on its finite subgroups. Of course, not every property of G can be recovered from its finite subgroups only, e.g. when G is free (abelian). Nevertheless, describing finite subgroups of certain transformation groups turns out to be an interesting and challenging problem in its own right. Characterising the set of finite subgroups of a general G is hard, so instead we seek for certain common properties of these groups. In other words, we wish to find a relatively small but well-understood class $\mathcal{N}$ of groups that contains every finite subgroup F of G. It turns out that classical families do not satisfy our needs, so we investigate a slight extension of $\mathcal{N}$ by taking group extensions of members of $\mathcal{N}$ by a finite list of groups. The precise notion is the following.

Definition 1.1.1 ($\mathcal{N}$-Jordan)**.** For a class of finite groups $\mathcal{N}$, a group G is called $\mathcal{N}$-*Jordan*, if there exists an integer J_G such that every finite subgroups F of G sits in a short exact sequence

$$1 \longrightarrow N \longrightarrow F \longrightarrow B \longrightarrow 1$$

with $N \in \mathcal{N}$ and $|B| \leq J_G$, cf. [Gul20, Definition 1]. The smallest such J_G (if it exists) is called the *Jordan constant* of G. The group G is said to be *Jordan*, if it is $\{$abelian$\}$-Jordan, cf. [Pop11, Definition 2.1].

Typically the order of the members of $\mathcal{N}$ is unbounded, so one can think to F as N with 'small' extension on the top, i.e. one can think that F 'almost' belongs to $\mathcal{N}$. We quickly note that if $\mathcal{N}$ is the family of all finite groups, then every group G is $\mathcal{N}$-Jordan. This notion originates from the classical result of Jordan from 1877 stating that $\mathrm{GL}_n(\mathbb{C})$ is Jordan for every $n \in \mathbb{N}_+$ [Jor77]. Even though the finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ are not all abelian, the previous statement says they are 'not too far' from it. In fact, the optimal bound for $J_{\mathrm{GL}_n(\mathbb{C})}$ for $n \geq 71$ is $(n+1)!$ by [Col07] relying on the classification of finite simple groups.

The main problem We are interested in characterising families $\mathcal{N}$ of groups such that a finite group can act (faithfully) on certain geometrical structures if it almost belongs to $\mathcal{N}$ in the above sense.

Problem 1.1.2 (Main)**.** *Describe, in a purely group theoretical way, the families $\mathcal{N}$ of groups satisfing the following.*

Algebraic case *For every algebraic variety X (over an algebraically closed field of characteristic 0), the group $\mathrm{Bir}(X)$ of birational automorphisms is $\mathcal{N}$-Jordan.*

Smooth case *For every smooth compact manifold X, the group $\mathrm{Diff}(X)$ of diffeomorphisms is $\mathcal{N}$-Jordan.*

The main result of this thesis is a full answer to Problem 1.1.2 in the algebraic case using $\leq$2-step nilpotent groups (Theorem 5.1.9), and a necessary (and conjecturally sufficient) condition in the smooth case (Corollary 5.2.39).

History To motivate Problem 1.1.2, we review its history. More than a century after the aforementioned result of Jordan, the investigation of Jordan groups regained interest in 2009 when Serre showed that the group of k-automorphisms of the field $k(X,Y)$ of fractions of polynomials in 2 indeterminates with coefficients from k (the so-called Cremona group) is Jordan [Ser09, Theorem 5.3] where k is an arbitrary field of characteristic 0. In the language of algebraic geometry, this group is the birational automorphimsm group $\mathrm{Bir}(\mathbb{P}^2_k)$ of the projective plane. This result of Serre started an investigation whether the birational automorphism group of other varieties also satisfy the Jordan property. For varieties of dimension at most 2, this was answered positively by Popov in 2011 except possibly for the product of the projective line and an elliptic curve in [Pop11, §2.2]. Later in 2014, Zarhin showed the failure of the Jordan property for this missing 2-dimensional case [Zar14, Theorem 1.2]. In 2017, Prokhorov and Shramov classified all 3-dimensional varieties (over an algebraically closed field of characteristic zero) with non-Jordan birational automorphism group: they fall into four different families [PS17, Theorem 1.8]. In higher dimension, Prokhorov and Shramov proved in 2016 that the birational automorphism group of any projective space and more generally for rationally connected varieties, is Jordan [PS16, Theorem] conditionally on the BAB conjecture which was later proved by Birkar in 2021 in [Bir21]. The first non-trivial family from Problem 1.1.2 was found by Guld.

Theorem 1.1.3 ([Gul20, Theorem 2])**.** *For any variety X over field of characteristic zero, $\mathrm{Bir}(X)$ is $\mathcal{F}_2$-Jordan where $\mathcal{F}_2$ is the family of finite nilpotent groups of class at most 2.*

Of course, dropping a finite number of (isomorphism classes of) groups from $\mathcal{F}_2$, this statement remains true (with a larger Jordan constant for each variety). One main result (Theorem 5.1.9) of this thesis is that we cannot replace $\mathcal{F}_2$ by a 'significantly smaller' family.

Ghys' conjecture For manifolds, one of the main driving forces of this topic is the following conjecture of Ghys which was formulated at several talks between 1997 and 2006 [Ghy97; Ghy99; Ghy02; Ghy06], but appeared in print for the first time in 2008 in [Fis08, §13.1].

Conjecture 1.1.4 (Ghys, original version, ≈1997)**.** Diff(X) is Jordan for every compact smooth manifold X.

The compactness assumption is important as there exists a non-compact 4-manifold whose diffeomorphism group contains an isomorphic copy of every finitely presented (in particular finite) group [Pop15, Corollary 1] (cf. [Pop18, Theorem 6] for an analogous statement for complex manifolds). Note that this result answers Problem 1.1.2 in the non-compact case: $\mathcal{N}$ essentially has to be the family of finite groups.

Ghys's conjecture was verified affirmatively in many cases including: in dimension at most 3 by Zimmermann [Zim14, Theorem 1]; for arbitrary dimensional tori (and some other manifolds satisfying certain technical assumptions) [Mun10, Theorem 1.4], acyclic manifolds, connected compact manifolds with non-zero Euler characteristic, and integral homology spheres (including arbitrary dimensional spheres) [Mun19, Theorem 1.2].

Surprisingly, manifolds composed of these are not necessarily Jordan. In 2014, Csikós, Pyber and Szabó showed a counterexample in the smallest possible dimension allowed by the theorems above (namely 4). They showed that the diffeomorphism group of the following 4-manifolds are Jordan: the product $\mathbb{T}^2 \times \mathbb{S}^2$ of the 2-torus and the 2-sphere, and the total space of a nontrivial smooth orientable $\mathbb{S}^2$-bundle over $\mathbb{T}^2$. Based on this paper, Mundet i Riera found other counterexamples including the direct product of higher dimensional tori and special unitary groups. See Subsection 5.2.6 for a very explicit counterexample based on these ideas. Consequently, Ghys revised his original conjecture replacing the family of abelian groups with a larger one consisting of nilpotent groups.

Conjecture 1.1.5 (Ghys, revised, 2015)**.** Diff(X) is $\mathcal{F}_\bullet$-Jordan for every compact smooth manifold X where $\mathcal{F}_\bullet$ is the family of finite nilpotent groups [Ghy15].

This conjecture was verified for 4-manifolds by Mundet i Riera and Sáez-Calvo in 2019 at [MS19]. In fact, they showed that the diffeomorphism group of closed (compact, without boundary) 4-manifolds is $\mathcal{F}_2$-Jordan, cf. Theorem 1.1.3. The revised conjecture remains open in general and it forms one of the most important problems of the area.

Other categories It is natural to ask similar questions of specific subgroups of the previous transformation groups. Some of them happen to be Jordan even if the whole group fails to be so. We list some results in this direction for reference. In the algebraic case, the automorphism group of every projective variety (over an algebraically closed field of characteristic zero) is Jordan [MZ18, Theorem 1.6]. There are other results in differential geometry. If (X, ω) is a compact connected symplectic manifold, then the group Ham(X, ω) of Hamiltonian diffeomorphisms if Jordan [Mun18, Theorem 1.1]. If furthermore the first Betti number of X is 0, then the group Symp(X, ω) of symplectomorphisms is also Jordan [Mun18, Theorem 1.3] If (X, ω) is a compact symplectic 4-manifold, then Symp(X, ω) is Jordan. [MS19, Theorem 1.7]. If X is a smooth compact 4-manifold with an almost complex structure J, then the group Aut(X, J) of diffeomorphisms preserving J is Jordan [MS19, Theorem 1.6]. As noted above, the automorphism group of a non-compact complex manifold may be nowhere near being Jordan, but that of a compact 2-dimensional complex manifold is [Pop18, §4].

1.2 Development of Ideas and the Main Results

In this section, we overview the main ideas from several papers serve as a starting point of this thesis. We display how these results suggested to study nilpotent groups of class at most 2 via central products and subdirect products to study the Jordan property. We also state the main theorems (Theorem A, Theorem B and Theorem C) in brief forms and give an overview of the structure of the thesis.

Non-Jordan transformation groups and uniformisation To attack Problem 1.1.2, we review the examples from above which give non-Jordan transformation group [Mum66; Zar14; CPS14; Mun17; Sza19]. The story started in 1966. With a completely different motivation (namely to establish an algebraic theory of theta functions), Mumford defined a short exact sequence

$$1 \to \mathrm{Z}(\mathscr{G}(L)) \to \mathscr{G}(L) \to H_{\mathrm{M}}(L) \to 1 \tag{1.1}$$

of groups for every ample invertible sheaf L over an abelian variety X, cf. Definition 2.4.29. He called $\mathscr{G}(L)$ the theta group and showed that it is isomorphic to the group

$$\begin{pmatrix} 1 & \mathrm{Hom}(K,\mathbb{C}) & \mathbb{C} \\ 0 & 1 & K \\ 0 & 0 & 1 \end{pmatrix}$$

of 3×3 upper unitriangular matrices for some finite abelian group K with $\mathrm{Hom}(K, \mathbb{C}) \times K \cong \mathscr{G}(L)$, cf. Lemma 2.4.33. He defined an action of $\mathscr{G}(L)$ on global sections $\Gamma(L)$ of L. This action induces in a compatible way an action of $\mathbb{C}$ acting as multiplication on the fibres, and an action of M acting on the abelian variety by translations. For further details, see Subsection 2.4.4 and [Mum66].

In 2014, seeking for the Jordan property of the birational automorphism group of the missing 2-dimensional variety from [Pop11], Zarhin considered these actions of $\mathscr{G}(L_i)$ on $\Gamma(L_i)$ of Mumford where L_i ranges through all ample invertible sheaves over a fixed abelian variety X. The spaces $\Gamma(L_i)$ are typically different, i.e. each $\mathscr{G}(L_i)$ acts on a different space. Noting that each L_i is birationally equivalent (but typically not isomorphic) to $X \times \mathbb{P}^1$, he obtained a faithful action of every $\mathscr{G}(L_i)$, but this time on the fixed space $X \times \mathbb{P}^1$, cf. Lemma 5.1.1. Finally, he chose a suitable finite subgroup H_i (in fact, a generalised Heisenberg subgroup) of $\mathscr{G}(L_i)$ and used the unitriangular description of $\mathscr{G}(L_i)$, to deduce that the index of the largest abelian subgroup of H_i approaches infinity as the degree of L_i approaches infinity. This shows that $\mathrm{Bir}(X \times \mathbb{P}^1)$ is *not* Jordan. Letting X be an elliptic curve recovers the statement from above [Zar14].

In this thesis, we shall use the notion of *uniformisation* for the process of obtaining from a family of faithful group actions on potentially different spaces, a single space on which all of the groups from the family act simultaneously faithfully. The space on which all the groups act is called the uniform space. This motif of uniformisation appears in various forms in the following papers [CPS14; Mun17; Sza19] as we discuss below and in fact will appear in this thesis too.

Csikós, Pyber and Szabó managed to translate these ideas to differential geometry as follows. Complex elliptic curves are diffeomorphic to the real 2-torus $\mathbb{T}^2$, so the invertible sheaves above appear here as the sheaves of sections of complex line bundles over $\mathbb{T}^2$. Up to holomorphism, these line bundles are described by the set of integers $\mathbb{Z}$ as follows (cf. Subsection 2.4.3): there is a holomorphic line bundle p with positive Chern number whose tensor powers $p^{\otimes n}$ (for $n \in \mathbb{Z}$) give a complete and irredundant list of isomorphism classes of line bundles over $\mathbb{T}^2$. For $n > 0$, the sheaf associated to $p^{\otimes n}$ is ample, hence Mumford's theta group G_n acts on $p^{\otimes n}$. Denote by Y_n the total space of the projectivisation of the Whitney sum of $p^{\otimes n}$ and the trivial line bundle. The added trivial bundle ensures faithfulness of the resulting action of G_n on Y_n. Using Stiefel–Whitney classes, they showed that Y_n is diffeomorphic to Y_k if and only if $n \equiv k \pmod 2$. So uniformisation is applicable in two families depending on the parity of n. For each positive even (respectively odd) number n, the group G_n acts faithfully on Y_2 (respectively on Y_1). Note that $Y_0 \cong \mathbb{T}^2 \times \mathbb{S}^2$ as $\mathbb{P}^1_{\mathbb{C}} \cong \mathbb{S}^2$. For each n, there is a finite H_n of G_n (which is isomorphic to the Heisenberg group over $\mathbb{Z}/n\mathbb{Z}$) in which the index of the largest abelian subgroup goes to infinity as $n \to \infty$ similarly as at [Zar14]. This shows that the diffeomorphism group of $Y_0 \cong \mathbb{T}^2 \times \mathbb{S}^2$ and of Y_1 cannot be Jordan. For more details, see [CPS14].

Mundet i Riera generalised these ideas to line bundles over the $2n$-dimensional torus $\mathbb{T}^{2n}$ to obtain families of manifolds whose diffeomorphism groups are not Jordan as follows. For primes p satisfying $p \equiv 1 \pmod{n+1}$, he considered a p-group G_p sitting in the short exact sequence

$$1 \to \mathbb{Z}/p\mathbb{Z} \to G_p \to (\mathbb{Z}/p\mathbb{Z})^{2n} \to 1, \tag{1.2}$$

and constructed a line bundle $L_p \to \mathbb{T}^{2n}$ together with group actions G on L_p, and $(\mathbb{Z}/p\mathbb{Z})^{2n}$ on $\mathbb{T}^{2n}$ compatible with the short exact sequence. To a certain extent, this construction is analogous to the action of the central-by-abelian of Mumford we saw above, but the complex structure is lost. (See Subsection 5.2.4 for a more general construction that preserves the complex structure.) Using elaborate computations of the Chern classes in the cohomology ring of the torus, and statements from K-theory, Mundet i Riera constructed a direct complement $L_p^{\perp} \to \mathbb{T}^{2n}$ of rank r independent of p with a (not faithful) action of G_p. His uniformisation process was to take the Whitney sum of the of these two bundles to obtain a trivial vector bundle followed by taking the unitary frame bundle to get a compact space (cf. Subsection 5.2.2). This construction yields a faithful action of G_p on $\mathbb{T}^{2n} \times \mathrm{SU}(r)$ for every prime p as above where $SU(r)$ is the group of $r \times r$ special unitary matrices with entries from $\mathbb{C}$. See [Mun17] for more details.

We have reached the starting point of the development this thesis.

Central-by-abelian extensions Being motivated by (1.1) and (1.2), we introduce the following category that plays a central role in the thesis.

Definition 1.2.1 (central-by-abelian extension)**.** An extension $\epsilon : 1 \to C \xrightarrow{\iota} G \xrightarrow{\pi} M \to 1$ of groups is called *central-by-abelian*, if $\iota(C) \subseteq \mathrm{Z}(G)$ and M is abelian. Sometimes we refer to ϵ as a central-by-abelian extension *giving* G to highlight the middle group from the exact sequence. This extension ϵ is *non-degenerate*, if $\iota(C) = Z(G)$. Morphism between two such objects is a morphism of short exact sequences (cf. Definition 2.2.22).

Central-by-abelian extensions can act on vector bundles, in fact we have already seen instances of this above: triplets of smooth actions C on V, G on V and M on X for vector bundles $V \to X$ compatible with the central-by-abelian extension. We formalise this notion in Definition 5.2.2. Constructing various such actions is possible over complex line bundles (cf. Subsection 5.2.3). In this case, the group C acts on 1-dimensional vector spaces, so when C is finite and the base space X is connected, then C has to be cyclic (cf. Remark 5.2.5). If ϵ is a central-by-abelian extension as in Definition 1.2.1, then $G' \subseteq \ker(\pi) = \mathrm{Im}(C) \subseteq \mathrm{Z}(G)$, so G is necessarily nilpotent of class at most 2. We shall use $\leq$2-step nilpotent to indicate such groups. The previous discussion naturally brings finite ≤ 2-step nilpotent groups with cyclic centre (or commutator subgroup) as an object of study, cf. Section 3.2 and Chapter 4.

Special p-groups As noted by László Pyber, the groups G_p from the construction of [Mun17] are actually extra-special p-groups.

Recall that a finite non-abelian p-group G is *special* if its Frattini subgroup $\Phi(G)$ (the intersection of its maximal subgroups), its derived subgroup G' (also known as the commutator subgroup, it is generated by all commutators) and its centre $\mathrm{Z}(G)$ all coincide and this group is an elementary abelian group: $\Phi(G) = G' = Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^r$ for some natural number r. A special group G is *extra-special* if $r = 1$. Special p-groups can be described using subdirect and central products.

Theorem 1.2.2 (Structure of special p-groups, [Suz82])**.** *For every special p-group G, the following hold.*

1. *G is a subdirect product of some number of groups of the form: the central product of an extra-special p-group and an abelian group [Suz82, (4.16)/(ii)]; i.e. G is a subgroup of the direct product of these groups such that the projection of G to each factor is surjective.*

2. *Every extra-special p-group H is the internal central product (see Definition 2.2.13) of n extra-special p-groups of order p^3 where $|H| = p^{2n+1}$ and the isomorphism class of H depends only on that of the subgroups [Suz82, Theorem 4.18].*

   (In fact, up to isomorphism, there are only 2 such groups depending on whether this decomposition contains the group of order p^3 with exponent p^2 for $p \neq 2$; or the quaternion group for $p = 2$.)

3. *Extra-special p-groups of order p^3 have been classified: these are the two non-abelian groups of order p^3 the exponent of which are p and p^3 respectively for $p \neq 2$ and the dihedral group $D_{2\cdot 4}$; and the quaternion group Q_8 for $p = 2$.*

Considering two line bundles with group actions, there is a natural action of some (so called maximal, cf. Definition 2.2.24) central product of these groups on the external tensor product of these bundles, cf. Subsection 5.2.3. This indicates that the construction of [Mun17] can actually be reduced to actions of the groups of order p^3.

Two generator p-groups of nilpotency class 2 Finite ≤ 2-generated ≤ 2-step nilpotent p-groups (e.g. the non-abelian groups of order p^3 from above) have been described in 2012. Ahmad, Magidin and Morse gave a concrete presentation depending on 5 integral parameters and gave a complete list of these tuples representing every isomorphism class [AMM12, Theorem 1.1].

The construction of line bundles and the action of [Mun17] naturally generalises to all of these groups which was a strong indication for the extensibility of the construction to reach groups far beyond the extra-special groups. The only problem was to generalise the uniformisation process for all of these groups. This has been carried out by the author at [Sza19] using computation with Chern characters and K-theory, cf. Subsection 5.2.2.

$\leq$2-generated groups as building blocks At this point, the natural question is to describe the groups that can be obtained from $\leq$2-generated $\leq$2-step nilpotent groups by taking central products and subdirect products in the flavour of Theorem 1.2.2. (Subdirect products naturally act on the direct product of the spaces, while certain central products act on the external tensor product of the bundles.)

The following existence result of Weichsel from 1967 indicates that this family may actually contain every finite $\leq$2-step nilpotent group. Unfortunately, we cannot use this result directly, as on one hand, it misses the $p \in \{2, 3\}$ cases. On the other hand, the lack of information on the precise way of combining these basic groups does not make this statement applicable for our construction.

Theorem 1.2.3 ([Wei67, Corollary 2.2]). *For any prime $p > 3$, $c \leq 3$ and $\alpha \in \mathbb{N}_+$, there exists a p-group $B(c, p^\alpha)$ of nilpotency class c and exponent p^α (with explicitly known presentation on 2 generators) satisfying the following. Every finite p-group G (with $p > 3$) of nilpotency class $c \leq 3$, can be constructed from $\{B(i, \exp(G_i)) : 1 \leq i \leq c\}$ using the operations of finite direct products, taking subgroups and factor groups. Here $G_1 = G$ and $G_i = [G, G_{i-1}]$ is the lower central series, and $\exp(G_i)$ denotes the exponent of G_i.*

Nevertheless, the answer to the question above turns out to be the whole family of finite $\leq$2-step nilpotent groups. Indeed, by an inductive argument, we can reduce the problem to $\leq$2-step nilpotent groups having cyclic derived subgroups, cf. Section 3.1. The commutator map endows the abelianisation of such groups with a structure an analogous to a discrete version of symplectic vector spaces. Pulling back the analogue of the Darboux basis to the group itself gives the required central product decomposition, cf. Section 3.2.

This gives the following a generalisation of Theorem 1.2.2 in the finitely generated case, where the role of special p-groups is played by finitely generated $\leq$2-step nilpotent groups; extra-special groups are substituted by $\leq$2-step nilpotent groups having cyclic centre or derived subgroup; and the groups of order p^3 correspond to the $\leq$2-generated $\leq$2-step nilpotent groups. As usual, $d(G) \coloneqq \min\{|X| : \langle X \rangle = G\}$, denotes the minimal number of generators of G. Say G is $\leq d$-generated if $d(G) \leq d$.

Theorem A (D.R.Sz., A structural description of finitely generated $\leq$2-step nilpotent groups)**.**

1. *Corollary 3.1.26: Every finitely generated nilpotent group G is a subdirect product of $d(Z(G))$ groups each with cyclic centre.*
2. *Theorem 3.2.23: Every finitely generated $\leq$2-step nilpotent group G with cyclic commutator subgroup is the internal central product of t many suitable nilpotent subgroups of class 2 generated by 2 elements and an abelian subgroup A with $d(G) = 2t + d(A)$.*
3. *Proposition 3.3.11: Every finitely generated $\leq$2-step nilpotent group G with cyclic centre is the maximal external central product of $\lceil d(G)/2 \rceil$ many $\leq$2-generated $\leq$2-step nilpotent groups with cyclic centre. In the finite case, the isomorphism class depends only on that of the $\leq$2-generated groups.*

   These $\leq$2-generated groups are classified, cf. [AMM12, Theorem 1.1] and Proposition 3.3.7.

László Pyber pointed out that this result was already discovered in the finite and cyclic centre case in 1969 by Brady, Bryce and Cossey, cf. [BBC69, Theorem 2.1]. The scope of Theorem A is demonstrated by the fact that the number of isomorphism classes of groups of order p^n is $p^{\frac{2}{27}n^3+\mathcal{O}(n^{8/3})}$ as $n \to \infty$ [Sim65, Theorem, p. 153] of which at least $p^{\frac{2}{27}n^3-\frac{12}{27}n^2}$ is of Frattini class 2 [Hig60, Theorem 2.3], in particular is $\leq$2-step nilpotent.

Embedding into Heisenberg groups Theorem A enables constructing an action of every finite $\leq$2-step nilpotent group G with cyclic centre on a suitable holomorphic line bundle L_G over complex tori. In the algebraic setting, this makes G a subgroup of the theta group $\mathscr{G}(L_G)$ from [Mum66] i.e. a subgroup some unitriangular 3×3 matrices. The previous argument uses algebraic geometry and the classification of $\leq$ 2-generated $\leq$ 2-step nilpotent groups for this purely group theoretic embedding. Instead, we shall present a purely group theoretic proof of the following statement that works even in the finitely generated context.

Theorem B (D.R.Sz.)**.** *Every finitely generated $\leq$2-step nilpotent group G is isomorphic to a subgroup of a non-degenerate Heisenberg group of the form*

$$\begin{pmatrix} 1 & A & C \\ 0 & 1 & B \\ 0 & 0 & 1 \end{pmatrix}$$

where the centre of G is mapped to C for suitable abelian groups A, B, C with $d(A), d(B) \leq d(G/\mathrm{Z}(G))$ and $d(C) = d(\mathrm{Z}(G))$. See the first part of Theorem 4.2.19 for the precise statement.

These Heisenberg groups satisfy a similar decomposition as in Theorem A, see Corollary 4.3.7, Proposition 4.3.2, and the second part of Theorem 4.2.19.

It is necessary to have the word 'subgroup' in this statement as Lemma 4.4.1 demonstrates. After a seminar on this statement, Péter Pál Pálfy kindly drew the attention of the author to a similar result by Magidin applicable in a more general setup stating slightly less. He established a monomorphism from every central-by-abelian to one with a specific form [Mag98, Corollary 2.21], whereas our statements (Proposition 4.2.11, Proposition 4.2.18, Theorem 4.2.19) apply only in the finitely generated case, but in exchange the resulting central-by-abelian extension is non-degenerate and polarised (cf. Definition 4.1.12) and our statement give bounds on the number of generators. These properties are essential for Problem 1.1.2.

Finite $\leq$2-step nilpotent groups in transformation groups We arrived back to our main motivation, Problem 1.1.2. At an email correspondence about [Sza19], Mundet i Riera asked the following question.

Question 1.2.4 (Mundet i Riera, 2019)*.* Is it possible to construct for every r a compact manifold supporting effective actions, for every prime p, of every group of cardinal p^r and nilpotency class 2.

The discussion above suggests a positive answer. Even more, Theorem B suggests that one may attempt to find a manifold/variety on which *every* finite $\leq$2-step nilpotent group act. This, however, cannot happen because of the following bounds on the number of generators.

Definition 1.2.5. The *rank* of a group G is $\operatorname{rk}(G) := \sup\{d(H) : H \leq G\}$.

Theorem 1.2.6 ([Gul19, Theorem 15])**.** *For every variety X over a field of characteristic zero, there exists a constant $R(X)$, only depending on the birational class of X, such that $\operatorname{rk}(F) \leq R(X)$ for every finite subgroup F of $\operatorname{Bir}(X)$.*

Lemma 1.2.7 (folklore)**.** *For every compact manifold X, there exists $R(X) \in \mathbb{N}_0$ such that $\operatorname{rk}(F) \leq R(X)$ for every finite subgroup F of $\operatorname{Diff}(X)$.*

Proof. [Luc89, Theorem 1] states that $\operatorname{rk}(F) \leq \max\{\operatorname{rk}(P) + 1 : P \in \operatorname{Syl}(F)\}$ where $\operatorname{Syl}(F)$ is the set of Sylow subgroups of F . Let $f_1(r) := \frac{3}{2}r^2 + \frac{1}{2}r$ and write $\operatorname{rk}_{\mathrm{ab}}(P) := \max\{d(H) : \mathrm{Z}(H) = H \lhd P\}$ for $P \in \operatorname{Syl}(F)$. Then $\operatorname{rk}(P) \leq f_1(\operatorname{rk}_{\mathrm{ab}}(P))$ by [Hup67, III, 12.3 Satz] for $p > 2$, and [Man71, Theorem B] for $p = 2$. On the other hand, the structure theorem of finite abelian groups imply that $\operatorname{rk}_{\mathrm{ab}}(P) \leq \operatorname{rk}_{\mathrm{ea}}(F)$, where $\operatorname{rk}_{\mathrm{ea}}(F)$ is the largest $d(E)$ as E runs over all elementary abelian subgroups of F. Finally, since X is a compact and connected manifold, there exist $f_2(X) \in \mathbb{N}_0$ such that $\operatorname{rk}_{\mathrm{ea}}(F) \leq f_2(X)$ by [MS63, Theorem 2.5]. Thus $R(X) := f_1(f_2(X)) + 1$ satisfies the statement. □

As a final step, we need to extend the uniformisation process to work for all groups of bounded rank. The ideas for this already occurred at [Sza19] by the author. Using K-theory , the problem is reduced to a modular Waring-type number theory problem (Lemma 5.2.12). In light of these statements, the next one is as strong as possible in the algebraic case by Theorem 1.1.3, and is the strongest possible for finite $\leq$2-step nilpotent groups in the smooth case.

Theorem C (D.R.Sz.)**.** *For every natural number r, there exists*

1. *an algebraic variety X_r and*
2. *a compact manifold M_r*

such that every finite nilpotent group of class at most 2 and rank at most r acts faithfully

1. *on X_r via birational automorphisms (cf. Theorem 5.1.6) and*
2. *on M_r via diffeomorphisms (cf. Theorem 5.2.36).*

This gives a complete answer to Problem 1.1.2 in the birational case (Theorem 5.1.9), and a necessary condition in the smooth case (Corollary 5.2.39) together with an answer to Question 1.2.4.

These results and the main steps are analogous. In the algebraic case, we consider invertible sheaves over abelian varieties and group actions on the global sections. In the smooth case, we consider line bundles over the complex torus and consider actions on the total space. We need the same results from group theory for both cases (Theorem A and Theorem B).

Outline The thesis is organised as follows. At Chapter 2, on hand hand, we collect standard notions and results from group theory, linear algebra and topology we will use later. On the other hand, we discuss simple preliminary lemmas and slight extensions of existing concepts regarding the notion of central products.

The main body starts in Chapter 3. Here we prove Theorem A about the structural description of finitely generated ≤ 2-step nilpotent groups. We devote a section for each part of this statement.

In Chapter 4, we introduce the notion of twisted Heisenberg groups and prove Theorem B about the relationship of these groups with $\leq$ 2-step nilpotent groups in the finitely generated case.

Finally, in Chapter 5, we apply the results of the previous chapters to prove Theorem C about and address Problem 1.1.2 there.

To help the reader, we include a short summary at the start of every section highlighting the key ideas of the section and showing how it fits the big picture. We also include concrete examples at the end of the last two chapters to demonstrate the key ideas used in action.

Chapter 2

Preliminaries

2.1 Notations and Conventions

We collect here some general notations and conventions that are used throughout the thesis without potentially any references. See Index on page 105 for the terminology/notation defined at the main body.

General $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ denotes the set of *integers*, *rational*, *real* and *complex numbers* respectively. $\mathbb{N}_+ \subset \mathbb{Z}$ is the set of positive integers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}_+$. $\mathbb{P} \subset \mathbb{N}_+$ is the set of prime numbers, $\mathbb{P}_0 := \{0\} \cup \mathbb{P}$. $\Re(z)$ is the *real part*, $\Im(z)$ is the *imaginary part*, and $\overline{z}$ is the *complex conjugate* of $z \in \mathbb{C}$. exp denotes the exponential function (not to be confused with the exponent of a group). $\lfloor x \rfloor$ is the *lower*, while $\lceil x \rceil$ is the *upper integer part* of $x \in \mathbb{R}$. $\binom{n}{k}$ is the *binomial coefficient*.

$|X|$ denotes the *cardinality* of a set X. $n \,\big|\, k$ denotes *divisibility*, i.e. that there is l (which may be 0) such that $nl = k$. (Note that this symbol is slightly taller then the one denoting the cardinality.)

We apply functions *from the left*. For $f\colon X \to Y$ and $X_0 \subseteq X$, we write $f|_{X_0}\colon X_0 \to Y$ for *restriction* and $f(X_0) := \{f(x_0) : x_0 \in X_0\}$. For maps $f_i\colon X_i \to Y_i$, denote by $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2, (x_1, x_2) \mapsto (f_1(x_1), f_2(x_2))$ their *direct product*.

Every diagram is implicitly assumed to be *commutative* unless otherwise stated. The arrow $\rightarrowtail$ indicates a *monomorphism* (or an injective map), $\twoheadrightarrow$ means an *epimorphism* (or a surjective map), and these arrow notation can be combined. We write $=\!=$ for the *identity map*. In bigger diagrams, we use $\longrightarrow$, $\dashrightarrow$ or $\dashrightarrow$ to indicate the 'chronological order': the further it is from a solid one, the later it appears in the construction.

If parentheses are not displayed for a (non-associative) binary operator $\curlyvee$, then by $G_1 \curlyvee G_2 \curlyvee G_3$ we mean $(G_1 \curlyvee G_2) \curlyvee G_3$. When $\otimes$ is a binary operation acting on an object p and $n \in \mathbb{N}_0$, then we write $p^{\otimes n}$ for $p \otimes p \otimes \ldots \otimes p$ (where p occurs n times).

We denote by $\{\mathsf{objects\ with\ morphisms}\}$ the category consisting specified of 'objects' and 'morphisms', and suppress the morphism part if that does not cause confusion.

Algebra We write $A \cong B$ to mean that objects A and B are *isomorphic* in the underlying category. If X is an algebraic object, and $S \subseteq X$ is a subset, then $\langle S \rangle$ denotes the *subobject generated* by S, and write $\langle g_1, \ldots, g_n \rangle := \langle \{g_1, \ldots, g_n\} \rangle$. For a set D of subobjects of X, we write $\bigcap D := \{x \in X : (\forall Y \in D)(x \in Y)\}$, thus this is X is D is empty. $\mathrm{Hom}(X, Y)$ is the *set of morphisms* $X \to Y$.

Let G denote a group. We denote the *identity element* of G by 1, or sometimes by 0 when G is an additive abelian group. By abuse of notation, we also write 1 or 0 for the *trivial group*. $N \lhd G$ means that N is a *normal subgroup* of G. Write $[-,-]\colon G \times G \to G', (g,h) \mapsto [g,h]$ for the *commutator map* where we use the convention $[g,h] \coloneqq g^{-1}h^{-1}gh$. The *commutator subgroup* (or *derived subgroup*) is denoted by $G' \coloneqq [G,G]$. $\mathrm{Z}(G)$ is the *centre* of G. $\mathrm{Syl}(G)$ is the set of *Sylow subgroups* of G, $\mathrm{Syl}_p(G)$ consists of *Sylow p-subgroups*. We denote by $\exp(G) \coloneqq \inf\{n \in \mathbb{N}_+ : \forall g \in G\, g^n = 1\}$ the *exponent* of a group G. Write $d(G)$ for the *size of the smallest generating set*. We say G is $\le d$*-generated*, if $d \le d(G)$; and G is d*-generated* if $d = d(G)$. The *rank* $\mathrm{rk}(G)$ is the smallest $d \in \mathbb{N}_0$ such that every subgroup is $\le d$-generated. We identify group *actions* $G \times X \to X$ by their *permutation representation* $G \to \mathrm{Sym}(X), g \mapsto (x \mapsto g \cdot x)$. For $n \in \mathbb{N}_0$, let $\mathbb{Z}_n \coloneqq \mathbb{Z}/n\mathbb{Z}$ be the *cyclic group*. In particular, $\mathbb{Z}_0 \cong \mathbb{Z}$.

Rings always have a *multiplicative unit element*. Let R be a ring. $R^\times$ denotes the *multiplicative group* of invertible elements. $R^{n\times k}$ is the set of *matrices* with n rows, k columns with entries from R. $\mathrm{SL}_n(R) \subseteq GL_n(R)$ are the *special and the general linear $n \times n$ matrix groups* with entries from R.

Topology Let X be a topological space. Write $H^\bullet(X;R) \coloneqq \bigoplus_{k=0}^\infty H^k(X;R)$, and $H^{2\bullet}(X;R) \coloneqq \bigoplus_{k=0}^\infty H^{2k}(X;R)$ for the (singular) *cohomologies* with coefficients from R (with $\smile$-product as multiplication). The natural embedding $\mathbb{Z} \rightarrowtail \mathbb{Q}$ induces a morphism $H^\bullet(X;\mathbb{Z}) \to H^\bullet(X;\mathbb{Q})$ whose kernel is the set of torsion elements. In particular, it is injective if $H^\bullet(X;\mathbb{Z})$ is free. In any case, denote by $H^\bullet(X,\mathbb{Q})|_\mathbb{Z}$, respectively $H^k(X,\mathbb{Q})|_\mathbb{Z}$, the image of $H^\bullet(X;\mathbb{Z})$, respectively $H^k(X;\mathbb{Z})$ in $H^\bullet(X;\mathbb{Q})$. For a continuous map $f\colon X \to Y$, we denote the *induced homomorphism* on cohomologies by $f^*\colon H^k(Y;R) \to H^k(X;R)$.

We confuse a fibre bundle with its projection map. We only consider *complex vector bundles*. For $f\colon Y \to X$ and a fibre bundle $p\colon E \to X$, we denote the *pullback bundle* by $f^*(p)\colon f^*(E) \to Y$. θ_X is the *trivial line bundle* over X. $\mathrm{rk}(p)$ denotes the *rank of a vector bundle p*. If X is compact and Hausdorff, $K^0(X)$ denotes its the *complex K-theory*. The *Chern character* is denoted by $\mathrm{ch}\colon K^0(X) \to H^\bullet(X,\mathbb{Q})$.

All manifolds are without boundary, so in this thesis, we will use closed and compact interchangeably for manifolds. Denote the automorphism group of the object X in the category of differentiable real manifolds by $\mathrm{Diff}(X)$ and of complex manifolds by $\mathrm{Bih}(X)$. We use the following real manifolds. $\mathbb{T} \coloneqq \{z \in \mathbb{C} : |z| = 1\}$ is the 1*-sphere*, $\mathbb{T}^n \coloneqq (\mathbb{T})^n$ is the n*-torus*, $\mathbb{S}^n$ is the n*-sphere*, $\mathbb{CP}^n$ is n-dimensional *complex projective space*, $\mathrm{SU}(n) \subseteq \mathrm{U}(n) \subseteq \mathbb{C}^{n\times n}$ is the compact group of $n \times n$ *special unitary, respectively unitary* matrices.

The *birational automorphism group* of a variety X is denoted by $\mathrm{Bir}(X)$. $\mathbb{P}^n = \mathbb{P}^n_k$ (respectively $\mathbb{A}^n = \mathbb{A}^n_k$) is the n-dimensional *projective (respectively affine) space* over a field k (as a variety).

2.2 Group Theory

This section collects some short statements about certain non-standard notions and notations regarding central product of groups (Subsection 2.2.2 and Subsection 2.2.3) and its extensions to various diagram categories (Subsection 2.2.5). These results are used to make later parts of this thesis more structural and concise. All other parts are standard results, and we include them to make thesis self-contained.

2.2.1 Nilpotent Groups

In this subsection, we review some standard facts about nilpotent groups following [CMZ17]. We also introduce the ascending chain condition on normal subgroups, a property that finitely generated nilpotent groups satisfy.

Some General Facts

Definition 2.2.1. Let G be a group. The weight n commutator is given recursively by $[g_1, \dots, g_n] \coloneqq [[g_1, \dots, g_{n-1}], g_n]$. Write $[S_1, S_2] \coloneqq \langle [g_1, g_2] : g_i \in S_i \rangle$, and define recursively $[S_1, \dots, S_n] \coloneqq [S_1, \dots, S_{n-1}, S_n]$ for subsets $S_i \subseteq G$. Write $G' \coloneqq [G, G]$ for the commutator (or derived) subgroup. Say G is $\leq$*c-step nilpotent*, if $[G, \dots, G] = 1$ where there are $c+1$ copies of G, i.e. when the nilpotency class of G is at most c. In particular, G is $\leq$2-step nilpotent if and only if $G' \subseteq \mathrm{Z}(G)$.

Definition 2.2.2. Let G_1, G_2, G_3 be groups. A map (of sets) $\mu\colon G_1 \times G_2 \to G_3$ is a *bihomomorphism*, if for every $g_i \in G_i$, $[-, g_2]\colon G_1 \to G_3, x \mapsto [x, g_2]$ and $[g_1, -]\colon G_2 \to G_3, x \mapsto [g_1, x]$ are group morphisms.

Lemma 2.2.3. *Let G be a $\leq$2-step nilpotent group. Then the following hold.*

1. *The commutator map $[-,-]\colon G \times G \to G', (g, h) \mapsto [g, h]$ is a bihomomorphism that factors though the natural projection $G \to G/G'$.*

2. *$(gh)^n = g^n h^n [h, g]^{\binom{n}{2}}$ for any $g, h \in G$, $n \in \mathbb{N}_0$ where $\binom{n}{2} = n(n-1)/2$.*

3. *$G = \langle S \rangle$ if and only if $G/G' = \langle \{gG' : g \in S\} \rangle$, in this case $G' = \langle [g, h] : g, h \in S \rangle$.*

4. *$d(G) = d(G/G')$*

Proof. The first part follows from the following commutator identities

$$[g_1 g_2, h] = [g_1, h][g_1, h, g_2][g_2, h]$$
$$[g, h_1 h_2] = [g, h_2][g, h_1][g, h_1, h_2]$$

valid in any group (cf. [CMZ17, Lemma 1.4]), and from the assumption that triple commutators vanish, i.e. $G' \subseteq \mathrm{Z}(G)$.

The next statement follows from the first by induction on n using $gh[h, g] = hg$ and $[h, g] \in \mathrm{Z}(G)$.

For the next part, we only need to check the 'only if' part. Let $\{gG' : g \in S\}$ generate G/G'. Then for any $g \in G$, $gG' = sG'$ for some $s \in \langle S \rangle$, thus $g = sc$ for some $c \in G'$. So it is enough to show that $G' \subseteq \langle S \rangle$. Take a similar decomposition $g' = s'c'$ of an arbitrary $g' \in G$ as above. Then using the first part, $[g, g'] = [sc, s'c'] = [s, s'][s, c'][c, s'][c, c'] = [s, s'] \in \langle S \rangle$ as $[G', G] = 1$ by assumption.

The statement on the number of generators follows from the previous part. $\square$

Remark 2.2.4. The weight c commutator map satisfy a similar property in $\leq c$-step nilpotent group, see [CMZ17, §2.5.2]. This gives some chance for generalising some parts of the thesis in the $c > 2$ case.

Lemma 2.2.5 ([CMZ17, Theorem 2.28])**.** *For every non-trivial normal subgroup N of a nilpotent group G, $N \cap \mathrm{Z}(G)$ is non-trivial.*

Lemma 2.2.6 ([CMZ17, Lemma 2.26])**.** *Let $P \subset \mathbb{P}$ be a nonempty set of primes in $\mathbb{Z}$. Let G be a finitely generated nilpotent group with finite centre, such that every prime divisor of $|\operatorname{Z}(G)|$ belongs to P. Then G itself is finite and primes divisors of $|G|$ are from P as well.*

For more details on nilpotent groups, the interested reader is referred to [CMZ17].

ACCN

Definition 2.2.7 (ACCN)**.** We say a group G *satisfies the ascending chain condition on normal subgroups (ACCN)* if whenever $K_0 \subseteq K_1 \subseteq \dots$ is a sequence of normal subgroups of G, then there exists $n \in \mathbb{N}_+$ such that $K_i = K_n$ for all $i \geq n$.

Lemma 2.2.8. *If G satisfies ACCN, then $d(\operatorname{Z}(G))$ is finitely generated.*

Proof. Assume by contradiction that $\operatorname{Z}(G)$ is not finitely generated. There there is a sequence of elements $z_0, z_1, \dots$ from $\operatorname{Z}(G)$ such that the groups $K_i \coloneqq \langle z_0, \dots, z_i \rangle \leq \operatorname{Z}(G)$ form a strictly increasing infinite sequence. Since K_i are central, they are normal in G which contradicts ACCN. □

Lemma 2.2.9. *If G satisfies ACCN and N is a normal subgroup of G, then G/N also satisfies ACCN.*

Proof. Every ascending sequence of normal subgroups of G/N is of the form $K_0/N \subseteq K_1/N \subseteq \dots$ for some ascending sequence $K_0 \subseteq K_1 \subseteq \dots$ of normal subgroups of G. So there is $n \in \mathbb{N}_0$ such that $K_i = K_n$ for every $i \geq n$, hence $K_i/N = K_n/N$. □

Lemma 2.2.10. *If $H \leq G$ are finitely generated abelian groups, then $d(H) \leq d(G)$. In particular,* $\operatorname{rk}(G) = d(G)$ *in this case.*

Lemma 2.2.11. *If G is a (potentially infinite) $\leq d$-generated $\leq c$-step nilpotent group, then* $\operatorname{rk}(G) \leq \sum_{i=1}^{c} d^i$. *In particular, finitely generated nilpotent groups satisfy ACCN.*

Proof sketch. The case $d \leq 1$ is clear. Else the proof of [CMZ17, Theorem 2.18] with the bound of [CMZ17, Corollary 2.11] together with Lemma 2.2.10 gives that every subgroup of G can be generated by $\sum_{i=1}^{c} d^i$ elements.

Let $K_0 \subseteq K_1 \subseteq \dots$ be an ascending chain of normal subgroups of G. Then $K \coloneqq \bigcup_{i \in \mathbb{N}_0} K_i$ is a normal subgroup of G, hence is finitely generated by above. Hence there is an $n \in \mathbb{N}_0$ such that all of these generators belong to K_n. But then $K_i = K_n$ for all $i \geq n$, so G satisfies ACCN. □

Remark 2.2.12. This shows that bounding the rank of a nilpotent group is equivalent to bounding the number of its generators.

2.2.2 Central Product of Groups

The central product of groups is a generalisation of the notion of the direct product of groups and plays a distinguished role in this thesis. Here we introduce this notion together with some notations and basic properties.

Definition 2.2.13 (Internal central product)**.** Say a group G is the *internal central product* of subgroups $H_1, \dots, H_n$ if $[H_i, H_j] = 1$ for $i \neq j$ and $\langle \bigcup_{i=1}^{n} H_i \rangle = G$.

Remark 2.2.14. In this case, $G = H_1H_2 \dots H_n$. If $g = h_1 \dots h_n$ and $g' = h'_1 \dots h'_n$ for some $h_i, h'_i \in H_i$, then $gg' = (h_1h'_1) \dots (h_nh'_n)$. The subgroups H_i are necessarily normal in G. This decomposition of g may not be unique (unlike for the direct product), as here we do not require $H_i \cap \langle \bigcup_{j \neq i} H_j \rangle = 1$.

Now now introduce the external counterpart.

Definition 2.2.15 (Central group morphism)**.** We call a group morphism $\varphi\colon A \to G$ *central*, if $\mathrm{Im}(\varphi) \subseteq Z(G)$. This is abbreviated as $\varphi\colon A \xrightarrow{\bullet} G$.

Definition 2.2.16 (Category of central pairs)**.** Let $\gamma_i\colon A \rightarrowtail\!\!\bullet\!\!\to G_i$ be central monomorphisms. By the *central pair (of groups)* $\gamma : A \rightarrowtail\!\!\diamond\!\!\to G_1 \times G_2$ we mean the map $A \to G_1 \times G_2, a \mapsto (\gamma_1, 1/\gamma_2)(a) := (\gamma_1(a), \gamma_2(a)^{-1})$ which is a morphism as A is abelian. A *morphism* $\varphi = (\varphi_0, \varphi_1 \times \varphi_2)$ between such central pairs is the following commutative diagram.

$$\begin{array}{ccc} A & \overset{\gamma}{\rightarrowtail\!\!\diamond\!\!\to} & G_1 \times G_2 \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1 \times \varphi_2} \\ B & \overset{\chi}{\rightarrowtail\!\!\diamond\!\!\to} & H_1 \times H_2 \end{array} \tag{2.1}$$

Definition 2.2.17 (External central product)**.** For a central pair $\gamma : A \rightarrowtail\!\!\diamond\!\!\to G_1 \times G_2$, define the *(external) central product* of G_1 and G_2 amalgamating $\gamma_1(A)$ and $\gamma_2(A)$ to be the quotient

$$G_1 \curlyvee_\gamma G_2 := G_1 \times G_2 / \mathrm{Im}(\gamma).$$

Write write $\lceil g_1, g_2 \rfloor$ for $(g_1, g_2)\,\mathrm{Im}(\gamma) \in G_1 \curlyvee_\gamma G_2$. We deviate from the standard notation of equivalence classes to avoid confusion with the commutator.

Remark 2.2.18. Since $\mathrm{Im}(\gamma)$ is contained in the centre, it is a normal subgroup in $G_1 \times G_2$, hence the central product is well-defined. In fact, $G_1 \curlyvee_\gamma G_2 = \mathrm{coker}(\gamma)$, so the central product comes equipped with a natural projection $\mathrm{p} := \mathrm{p}_\gamma$ giving the following exact sequence.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \overset{\gamma}{\rightarrowtail\!\!\diamond\!\!\to} & G_1 \times G_2 & \overset{\mathrm{p}_\gamma}{\twoheadrightarrow} & G_1 \curlyvee_\gamma G_2 & \longrightarrow & 1 \\ & & & \searrow^{1} & & \searrow^{\psi} & \downarrow{\scriptstyle \exists!\, \curlyvee_\psi} & & \\ & & & & & & H & & \end{array} \tag{2.2}$$

The universal property of the cokernel implies that for every morphism ψ where $\psi \circ \gamma$ is trivial, there is a unique morphism $\curlyvee_\psi$ with $\curlyvee_\psi \circ \mathrm{p}_\gamma = \psi$.

Remark 2.2.19. The central product is functorial (from central pairs to short exact sequences). The morphism φ from (2.1) induces

$$\begin{aligned} \curlyvee(\varphi) := \curlyvee(\varphi_0, \varphi_1 \times \varphi_2) := \curlyvee_{\mathrm{p}_\chi \circ (\varphi_1 \times \varphi_2)} \colon G_1 \curlyvee_\gamma G_2 &\to H_1 \curlyvee_\chi H_2 \\ \lceil g_1, g_2 \rfloor &\mapsto \lceil \varphi_1(g_1), \varphi_2(g_2) \rfloor \end{aligned}$$

by the universal property that makes the obvious diagram commutative.

Remark 2.2.20. The natural maps $\psi_i\colon G_1 \times G_2 \to G_i \to G_i/\gamma_i(A)$ induce the following commutative diagram with short exact rows and columns (suppressing the trivial group from display).

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{\gamma_1} & G_1 & \xrightarrow{\pi_1} & G_1/\gamma_1(A) & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \gamma_2} & & \big\downarrow{\scriptstyle \mathrm{p}_\gamma|_{G_1}} & & \big\| & & \\
1 & \longrightarrow & G_2 & \xrightarrow{\mathrm{p}_\gamma|_{G_2}} & G_1 \curlyvee_\gamma G_2 & \xrightarrow{\curlyvee\psi_1} & G_1/\gamma_1(A) & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \pi_2} & & \big\downarrow{\scriptstyle \curlyvee\psi_2} & & \big\downarrow & & \\
1 & \longrightarrow & G_2/\gamma_2(A) & = & G_2/\gamma_2(A) & \longrightarrow & 1 & \longrightarrow & 1
\end{array} \tag{2.3}$$

Internal and external central product are related: $G_1 \curlyvee_\gamma G_2$ is the internal central product of $\mathrm{p}_\gamma(G_1)$ and $\mathrm{p}_\gamma(G_2)$. Furthermore, $\mathrm{p}_\gamma|_{G_i} \colon G_i \to \mathrm{p}_\gamma(G_i)$ and $\mathrm{p}_\gamma|_{G_i} \circ \gamma_i \colon A \to \mathrm{p}_\gamma(G_1) \cap \mathrm{p}_\gamma(G_2)$ are isomorphisms.

In the other direction, if G is the internal central product of H_1 and H_2, then the *natural central pair* $\gamma : H_1 \cap H_2 \rightarrowtail\!\!\!\to H_1 \times H_2$ given by the inclusions $\gamma_i \colon H_1 \cap H_2 \to H_i$ (i.e. $\gamma(h) = (h, h^{-1})$) induces an isomorphism

$$\xi \colon H_1 \curlyvee_\gamma H_2 \to G, \quad \lceil h_1, h_2 \rfloor \mapsto h_1 h_2 \tag{2.4}$$

compatible with the maps above.

Remark 2.2.21. By induction if G is an internal central product of subgroups $H_1, \dots, H_n$, then there are natural central pairs $\gamma^{(i)}$ such that

$$\begin{aligned} G = H_1 \dots H_n &\to (\dots (H_1 \curlyvee_{\gamma^{(1)}} H_2) \curlyvee_{\gamma^{(2)}} H_3 \dots) \curlyvee_{\gamma^{(n-1)}} H_n =: H \\ h_1 \dots h_n &\mapsto \lceil \dots \lceil \lceil h_1, h_2 \rfloor, h_3 \rfloor, \dots, h_n \rfloor \end{aligned} \tag{2.5}$$

to the iterated external central product is an isomorphism. Conversely, if H is an arbitrary iterated external central product of the above form, then there are subgroups $K_1, \dots, K_n$ of H whose internal central product is H and $K_i \cong H_i$, namely the composition of projections $\mathrm{p}_{\gamma^{(i)}}$ from (2.2).

Note that $(K_1 K_2) K_3 = K_1 (K_2 K_3)$ induces the 'associativity' of iterated external central products: given $\gamma^{(i)}$, there exists $\chi^{(i)}$ such that $(H_1 \curlyvee_{\gamma^{(1)}} H_2) \curlyvee_{\gamma^{(2)}} H_3 \cong H_1 \curlyvee_{\chi^{(1)}} (H_2 \curlyvee_{\chi^{(2)}} H_3)$. In particular, we could have rearranged the parentheses arbitrarily at (2.5). By convention, we will use the parentheses as at (2.5) when they are suppress from the notation.

In this thesis, external central product shall appear on many occasions on the following form.

Definition 2.2.22 (Category of short exact sequences/extensions)**.** A morphisms $(\kappa, \gamma, \lambda)$ between short exact sequences $\epsilon_i \colon 1 \to C_i \xrightarrow{\iota_i} G_i \xrightarrow{\pi_i} M_i \to 1$ is the following commutative diagram.

$$\begin{array}{cccccccccc}
\epsilon_1 & : & 1 & \longrightarrow & C_1 & \overset{\iota_1}{\rightarrowtail} & G_1 & \overset{\pi_1}{\twoheadrightarrow} & M_1 & \longrightarrow 1 \\
 & & & & \big\downarrow{\scriptstyle \kappa} & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \lambda} & \\
\epsilon_2 & : & 1 & \longrightarrow & C_2 & \overset{\iota_2}{\rightarrowtail} & G_2 & \overset{\pi_2}{\twoheadrightarrow} & M_2 & \longrightarrow 1
\end{array} \tag{2.6}$$

With these, short exact sequences form a category in which $(\kappa, \gamma, \lambda)$ is mono/epi if and only if all of κ, γ, λ are mono/epi. The *product* of such extensions is formed by taking the product of the respective groups and maps in the obvious way. We call a monomorphism $(\kappa, \gamma, \lambda) \colon \epsilon_1 \to \epsilon_2$ of central-by-abelian extensions a $\curlyvee$-*monomorphism* if λ is an isomorphism. We denote such maps by $e \colon \epsilon_1 \rightarrowtail\!\!\!\curlyvee\!\!\!\rightarrow \epsilon_2$.

This appears frequently in the thesis and its main usage is given by the next statement.

Lemma 2.2.23. *For every morphism $\epsilon_1 \to \epsilon_2$ from (2.6) where $[\gamma(G_1), \iota_2(C_2)] = 1$ and λ is an isomorphism, the following hold.*

1. *G_2 is an internal central product of $\gamma(G_1)$ and $\iota_2(C_2)$. In particular, we have $G_2 = \gamma(G_1)\iota_2(C_2)$ and $\gamma(G_1) \lhd G_2$ and $\iota_2(C_2) \lhd G_2$.*

2. *If furthermore, $\chi_1 := \iota_1$ and $\chi_2 := \kappa$ are central monomorphisms (e.g. when $\epsilon_1 \to \epsilon_2$ is a $\curlyvee$-monomorphism), then $G_2 \cong G_1 \curlyvee_\chi C_2$. In fact, the diagram can be extended with an exact bottom row so that it becomes isomorphic to (2.3).*

Proof. Pick $g_2 \in G_2$. As λ is surjective, there is $m_1 \in M_1$ such that $m_2 := \lambda(m_1) = \pi_2(g_2)$. We then get $g_1 \in G_1$ with $\pi_1(g_1) = m_1$ from the surjectivity of π_1. Then $m_2 = \lambda(\pi_1(g_1)) = \pi_2(\gamma(g_1))$ from the commutativity of (2.6). Hence $\pi_2(\gamma(g_1)^{-1}g_2) = m_2^{-1}m_2 = 1$, so there is $c_2 \in C_2$ with $\iota_2(c_2) = \gamma(g_1)^{-1}g_2$ using the exactness of ϵ_2. Thus $g_2 = \gamma(g_1)\iota_2(c_2)$ showing the first statement.

Define $\psi\colon G_1 \times C_2 \to G_2$ by $(g_1, c_2) \mapsto \gamma(g_1)\iota_2(c_2)$. This map is a group morphism as $[\gamma(G_1), \iota_2(C_2)] = 1$ by assumption. Then $\psi \circ (\iota_1, 1/\kappa) = 1$, as $\gamma(\iota_1(c_1))\iota_2(\kappa(c_1))^{-1} = c_1$. Hence Remark 2.2.18 gives a morphism $\curlyvee_\chi\colon G_1 \curlyvee_\chi C_2 \to G_2$ which in fact is given by $\lceil g_1, c_2 \rfloor \mapsto \gamma(g_1)\iota_2(c_2)$. The surjectivity of this map is clear. To check the injectivity, assume $\gamma(g_1)\iota_2(c_2) = 1$. Then $1 = \pi_2(1) = \pi_2(\gamma(g_1))\pi_2(\iota_2(c_2)) = \lambda(\pi_1(g_1)) \cdot 1$ by chasing the diagram (2.6). Then the injectivity of λ implies that $g_1 \in \ker(\pi_1) = \mathrm{Im}(\iota_1)$. Let $c_1 \in C_1$ such that $g_1 = \iota_1(c_1)$. Then $\iota_2(c_2)^{-1} = \gamma(g_1) = \gamma(\iota_1(c_1)) = \iota_2(\kappa(c_1))$, so $c_2^{-1} = \kappa(c_1)$ from the injectivity of ι_2. Thus $(\iota_1, 1/\kappa)(c_1) = (g_1, c_2)$, thus $\lceil g_1, c_2 \rfloor$ is the trivial element as required. □

2.2.3 Maximal Central Products

In this section we introduce the notion of maximal central product and explore how it is related to groups having cyclic centres, to the invariance of the amalgamation corresponding to such central products, and to embedding central products into each other.

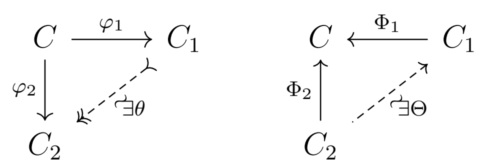
Definition 2.2.24. A morphism from (2.1) is called an *extension of* γ if $\varphi_1 \times \varphi_2$ is an isomorphism. We call a central pair *maximal* if every extension of γ is an isomorphism, i.e. when $\varphi_1 \times \varphi_2$ is an isomorphism, then so is φ_0.

We call the central product corresponding to γ *maximal*, if γ is maximal. We write $G_1 \hat{\curlyvee}_\gamma G_2$in this case for the resulting central product.

Remark 2.2.25. If any of $\gamma_i\colon A \to \mathrm{Z}(G_i)$ is surjective (hence an isomorphism), then γ is necessarily maximal. Indeed, for any extension as above, $\gamma_i^{-1} \circ \varphi_i^{-1} \circ \chi\colon B \to A$ is an inverse of φ_0, hence it must be an isomorphism.

Lemma 2.2.26 (Elementary)**.** *Let C, C_1, C_2 be (potentially infinite) cyclic groups. For $\varphi_i\colon C \to C_i$ with $\ker\varphi_1 = \ker\varphi_2$ and $\operatorname{coker}\varphi_1 \cong \operatorname{coker}\varphi_2$, then there is an isomorphism $\theta\colon C_1 \to C_2$ with $\theta \circ \varphi_1 = \varphi_2$ which is unique if $\operatorname{coker}\varphi_i$ are trivial.*

Dually, for every $\Phi_i\colon C_i \to C$ with $\operatorname{coker}\Phi_1 = \operatorname{coker}\Phi_2$ and $\ker\Phi_1 \cong \ker\Phi_2$, there is an isomorphism $\Theta\colon C_2 \to C_1$ with $\Phi_1 \circ \Theta = \Phi_2$ which is unique if $\ker\Phi_i$ are trivial.



Proof. Pick generators $C = \langle g \rangle$ and $C_i = \langle g_i \rangle$ and let $\varphi_i(g) = g_i^{k_i}$ for some $k_i \in \mathbb{N}_0$.

Assume first that some of C_i finite. Without loss of generality, assume that C_1 is of order m. Then $\operatorname{Im}\varphi_1 \cong C/\ker\varphi_1 = C/\ker\varphi_2 \cong \operatorname{Im}\varphi_2$ and $C_1/\operatorname{Im}\varphi_1 = \operatorname{coker}\varphi_1 \cong \operatorname{coker}\varphi_2 = C_2/\operatorname{Im}\varphi_2$ are all finite groups, hence C_2 is also of order m. Then by assumption, $d \coloneqq \gcd(m, k_1) = |\operatorname{coker}\varphi_1| = |\operatorname{coker}\varphi_2| = \gcd(m, k_2)$. Let $k_i' = k/d$, $m' = m/d$. Then the congruence $k_1'x' \equiv k_2' \pmod{m'}$ has a unique solution $x' \equiv k_1'^{-1}k_2 \pmod{m'}$. Note that $\gcd(k_i', m') = 1$, so $\gcd(x', m') = 1$. Since the natural map $\mathbb{Z}_m^\times \to \mathbb{Z}_{m'}^\times$ between multiplicative units is surjective, we can pick x with $\gcd(x, m) = 1$ and $k_1'x \equiv k_2' \pmod{m'}$. (In fact, we may take $x = x' + ym'$ where y is the product of prime divisors of m not dividing m'.) Hence $k_1 x \equiv k_2 \pmod{m}$ which means that $\theta\colon C_1 \to C_2, g_1^l \mapsto g_2^{lx}$ is an isomorphism with the desired properties.

Else both C_1 and C_2 are infinite. Now $\operatorname{coker}\varphi_1 \cong \operatorname{coker}\varphi_2$ implies $|k_1| = |k_2| \in \mathbb{N}_0$, so $k_1 x = k_2$ for some $x \in \{+1, -1\} = \mathbb{Z}^\times$. Then $\theta\colon g_1^l \mapsto g_2^{lx}$ is an isomorphism as required.

For the dual statement, let $\Phi_i(g_i) = g^{k_i}$ for some $k_i \in \mathbb{N}_0$. In this case $\Theta\colon C_2 \to C_1, g_2^l \mapsto g_1^{lx}$ is an isomorphism with the desired properties for x from above in both cases. □

The notion of maximality plays and important role because of the following statements.

Lemma 2.2.27. *For a central pair $\gamma : A \rightarrowtail\!\!\!\to G_1 \times G_2$, the following are equivalent.*

1. $\mathrm{Z}(G_1 \curlyvee_\gamma G_2)$ *is cyclic.*
2. γ *is maximal,* $Z(G_1)$, $Z(G_2)$ *are both cyclic groups such that both are finite or both are infinite/trivial.*

In the finite case, $|\mathrm{Z}(G_1 \curlyvee_\gamma G_2)| = \operatorname{lcm}(|\mathrm{Z}(G_1)|, |\mathrm{Z}(G_2)|)$, *the least common multiple.*

Proof. 1 $\Longrightarrow$ 2: First assume that $Z \coloneqq \mathrm{Z}(G_1) \curlyvee_\gamma \mathrm{Z}(G_2) \cong \mathrm{Z}(G_1 \curlyvee_\gamma G_2)$ is cyclic. Then the injection $\mathrm{p}_\gamma|_{\mathrm{Z}(G_i)}\colon \mathrm{Z}(G_i) \hookrightarrow Z$ shows that $\mathrm{Z}(G_i)$ (hence A) are all cyclic. By definition, Z has a $\mathbb{Z}$-module presentation $\langle g_1, g_2 : 0 = n_1 g_1 = n_2 g_2 = k_1 g_1 - k_2 g_2 \rangle$ for some $n_i \in \mathbb{N}_0$ and $k_i \in \mathbb{Z}$. Here g_i corresponds to a generator of $\mathrm{Z}(G_i)$, and $k_i g_i$ to the image of a fixed generator of A under γ_i. The corresponding matrix is $\left(\begin{smallmatrix} n_1 & 0 & k_1 \\ 0 & n_2 & -k_2 \end{smallmatrix}\right)$. Since Z is cyclic, the first invariant factor of Z from Smith normal form must be 1, hence $(n_1, n_2, k_1, k_2) = \mathbb{Z}$ for the generated ideal.
Consider an arbitrary extension as at (2.1). Then for $i \in \{1, 2\}$, using the commutativity of the diagram, the injectivity of χ_i and that φ_i is an isomorphism, we obtain

$$B/\varphi_0(A) \cong \chi_i(B)/\chi_i(\varphi_0(A)) \leq \mathrm{Z}(H_i)/\varphi_i(\gamma_i(A)) \cong \mathrm{Z}(G_i)/\gamma_i(A) \cong \mathbb{Z}/(n_i, k_i).$$

If $(n_1, k_1) = (0)$, $\mathbb{Z} = (n_1, n_2, k_1, k_2) = (n_2, k_2)$, so $B/\varphi_0(A) \rightarrowtail \mathbb{Z}/(n_2, k_2) \cong 1$. Else (without loss of generality), $(n_1, k_1) \neq (0) \neq (n_2, k_2)$. Then the two injections above imply one to the meet in the poset of cyclic groups, i.e. $B/\varphi_0(A) \rightarrowtail \mathbb{Z}/(n_1, k_1, n_2, k_2) \cong 1$. In any case, $\varphi_0(A) = B$, so φ_0 is an isomorphism, i.e. γ is maximal.

The complement of the last condition (without loss of generality) is that $\mathrm{Z}(G_1)$ is infinite and $\mathrm{Z}(G_2)$ is finite, but non-trivial. Then only the trivial group embeds to both, so $A = 1$ and $Z \cong \mathrm{Z}(G_1) \times \mathrm{Z}(G_2)$, so $\mathbb{Z}$ is not cyclic, a contradiction.

2 $\implies$ 1: Consider the $\mathbb{Z}$-module presentation of Z as above. We construct an extension of γ. Let $\varphi_0 \colon A \to B$ be an injection to a cyclic group B such that $B/\varphi_0(A) \cong \mathbb{Z}/(n_1, n_2, k_1, k_2)$. Then the image of

$$B/\varphi_0(A) \cong \mathbb{Z}/(n_1, n_2, k_1, k_2) \rightarrowtail \mathbb{Z}/(n_i, k_i) \cong \mathrm{Z}(G_i)/\gamma_i(A)$$

is a subgroup of the form $B_i/\gamma_i(A)$ for some $\gamma_i(A) \leq B_i \leq \mathrm{Z}(G_i)$. As $\varphi_0(A) \cong \gamma_i(A)$, this gives an embedding $\iota_i \colon B \rightarrowtail \mathrm{Z}(G_i)$ such that $\mathrm{Im}(\iota_i) = B_i$. Now Lemma 2.2.26 is applicable to $B \xleftarrow{\varphi_0} A \xrightarrow{\gamma_i} B_i \xrightarrow{\iota_i^{-1}} B$ and gives an automorphism θ of B such that $\iota \circ \theta \circ \varphi_0 = \gamma$. Then

$$\begin{array}{ccccccc}
A & \xrightarrow{\quad\gamma\quad} & \mathrm{Z}(G_1) \times \mathrm{Z}(G_2) & \xhookrightarrow{\subseteq} & G_1 \times G_2 \\
\downarrow{\scriptstyle\varphi_0} & & \| & & \|{\scriptstyle\varphi_1\times\varphi_2} \\
B \xrightarrow[\sim]{\theta} B & \xrightarrow{\quad\iota\quad} & \mathrm{Z}(G_1) \times \mathrm{Z}(G_2) & \xhookrightarrow{\subseteq} & G_1 \times G_2
\end{array}$$

is an extension of γ, so by the maximality assumption, φ_0 is an isomorphism. Hence $1 \cong B/\varphi_0(A) \cong \mathbb{Z}/(n_1, n_2, k_1, k_2)$, thus hence Z is cyclic using the Smith normal form.

For the statement on finite groups, use the notation as above. Letting $d_i \coloneqq \gcd(n_i, k_i)$ we saw that $n_1/d_1 = |A| = n_2/d_2$ and $\gcd(d_1, d_2) = 1$. This forces $|A| = \gcd(n_1, n_2)$, hence $|\mathrm{Z}(G_1 \curlyvee_\gamma G_2)| = n_1 n_2/|A| = n_1 n_2/\gcd(n_1, n_2) = \mathrm{lcm}(n_1, n_2) = \mathrm{lcm}(|\mathrm{Z}(G_1)|, |\mathrm{Z}(G_2)|)$ as stated. □

Lemma 2.2.28. *Suppose that $\mathrm{Z}(G_1)$ and $\mathrm{Z}(G_2)$ are both finite cyclic and that every automorphism of $\mathrm{Z}(G_2)$ can be extended to one of G_2. Then any two maximal central product of G_1 and G_2 are isomorphic. We denote this isomorphism class by*

$$G_1 \hat{\curlyvee} G_2.$$

Proof. Let $\gamma : A \rightarrowtail G_1 \times G_2$ and $\chi : B \rightarrowtail G_1 \times G_2$ be both maximal. Then $\gamma_1(A)$ and $\chi_1(B)$ are both subgroups of the cyclic group $\mathrm{Z}(G_1)$, so without loss of generality, we may assume that $\gamma_1(A) \leq \chi_1(B)$. Using the γ_i and χ_i, this gives rise to injections $\varphi_0 \colon A \to B$ and $\theta_1 \colon \gamma_2(A) \to \chi_2(B)$. This θ_1 must factor through $\gamma_2(A) \subseteq \chi_2(B)$ as both of these are subgroups of the finite cyclic group $Z(G_2)$. Then by Lemma 2.2.26, θ_1 can be extended to an automorphism θ_2 of $\mathrm{Z}(G_2)$. By assumption, this can be lifted to an automorphism φ_2 of G_2.

$$\begin{array}{ccccccccccc}
G_1 & \xhookleftarrow{\supseteq} & \gamma_1(A) & \xleftarrow[\sim]{\gamma_1} & A & \xrightarrow[\sim]{\gamma_2} & \gamma_2(A) & \xhookrightarrow{\subseteq} & \mathrm{Z}(G_2) & \xhookrightarrow{\subseteq} & G_2 \\
\| {\scriptstyle\varphi_1} & & \cap\!| & & \downarrow{\scriptstyle\varphi_0} & & \wr\downarrow{\scriptstyle\theta_1} & & \wr\downarrow{\scriptstyle\theta_2} & & \\
 & & & & & & \gamma_2(A) & \xhookrightarrow{\subseteq} & \mathrm{Z}(G_2) & \searrow{\scriptstyle\subseteq} & \wr\downarrow{\scriptstyle\varphi_2} \\
 & & & & & & \cap\!| & & & & \\
G_1 & \xhookleftarrow{\supseteq} & \chi_1(B) & \xleftarrow[\sim]{\chi_1} & B & \xrightarrow[\sim]{\chi_2} & \chi_2(B) & & \xhookrightarrow{\qquad\subseteq\qquad} & & G_2
\end{array}$$

Letting $\varphi_1 \coloneqq \mathrm{id}_{G_1}$, $(\varphi_0, \varphi_1 \times \varphi_2)$ gives an extension from γ to χ. Maximality of γ and the commutative diagram above implies that φ_0 is an isomorphism. Then $\curlyvee(\varphi) \colon G_1 \hat{\curlyvee}_\gamma G_2 \to G_1 \hat{\curlyvee}_\chi G_2$ is an isomorphism as stated. □

Remark 2.2.29. The conclusion holds if one of $\mathrm{Z}(G_1)$, $\mathrm{Z}(G_2)$ is infinite and the other is finite (without requiring the extendibility of the automorphisms), as in this case every (maximal) central product is actually $G_1 \times G_2$, since only the trivial subgroup can be amalgamated.

However, the conclusion fails if both $\mathrm{Z}(G_1)$ and $\mathrm{Z}(G_2)$ are infinite. For example, let

$$\mathbb{Z} \overset{\gamma}{\rightarrowtail} \mathbb{Z} \times \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 0 \end{pmatrix}, \qquad 1 \longmapsto \left(k_1, \begin{pmatrix} 1 & 0 & -k_2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

for $\gcd(k_1, k_2) = 1$ to assure maximality. Then the resulting external central product G is a ≤ 2-step nilpotent group and $|\,\mathrm{Z}(G) : G'| = |k_1|$, so different $k_1 \in \mathbb{N}_+$ produce non-isomorphic maximal central products.

Definition 2.2.30. A group morphism $\varphi \colon G \to H$ *centre preserving* if $\varphi(\mathrm{Z}(G)) \subseteq \mathrm{Z}(H)$.

Lemma 2.2.31. *Let $\gamma : A \rightarrowtail G_1 \times G_2$ be a central pair, $\varphi \colon G_i \to H_i$ be centre preserving monomorphisms, and assume that $\mathrm{Z}(G_2)$ is finitely generated. Then this can be extended to a morphism to some maximal $\chi : B \rightarrowtail H_1 \times H_2$ via a suitable $\varphi_0 \colon A \to B$.*

$$\begin{array}{ccc} A & \overset{\gamma}{\rightarrowtail} & G_1 \times G_2 \\ {\scriptstyle \varphi_0}\downarrow & & \downarrow{\scriptstyle \varphi_1 \times \varphi_2} \\ B & \overset{\chi}{\rightarrowtail} & H_1 \times H_2 \end{array} \tag{2.7}$$

Proof. For the existence part, consider the set of all diagrams of as above without requiring χ to be maximal. Define a partial ordering on this set induced by the containment of $\chi_2(B)$ in $\mathrm{Z}(H_2)$. By assumption, this poset satisfies the ascending chain condition. It is also non-empty, as $B = A$, $\varphi_0 = \mathrm{id}_A$, $\chi = (\varphi_1 \times \varphi_2) \circ \gamma$ gives such a diagram because φ_i are centre preserving and injections. Thus this poset has a maximal element which gives a maximal χ by construction. $\square$

Lemma 2.2.32. *Consider the diagram (2.2). If γ is maximal and $\psi|_{G_i}$ are both injective, then $\curlyvee_\psi$ is also injective.*

Proof. We define an extension of γ of the form (2.1) by letting $B \coloneqq \ker(\psi)$, $\varphi_0 \coloneqq \gamma$, $H_i \coloneqq \psi(G_i)$, $\varphi_i \coloneqq \psi|_{G_1} \times \psi|_{G_2}$, $\chi \coloneqq (\varphi_1 \times \varphi_2)|_B$. This is well-defined as $\mathrm{Im}(\varphi) \subseteq B$ since $\psi \circ \gamma = 1$ by assumption, and is an extension of γ since $\psi|_{G_i}$ are injections. Then maximality implies that $\varphi \colon A \to \ker(\psi)$ is an isomorphism, in particular $\ker(\psi) = \mathrm{Im}(\gamma) = \ker(\mathrm{p}_\gamma)$. Thus $\ker(\curlyvee_\psi) = \{\mathrm{p}_\gamma(g) : g \in \ker\psi = \ker(\mathrm{p}_\gamma)\} = 1$ as stated. $\square$

2.2.4 n-lemmas

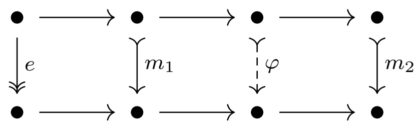
The following standard lemmas can be proved using diagram chasing. We include them mainly to avoid any potential ambiguity with their names when using them.

Here all $\bullet$ represent (potentially pairwise different) groups.

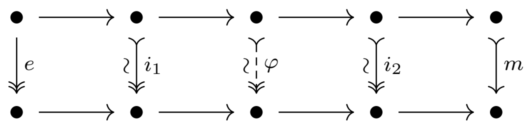
Lemma 2.2.33 (4-lemma)**.** *If in the following commutative diagram of groups, both rows are exact, e_1, e_2 are epimorphisms and m is a monomorphism, then φ is an epimorphism.*

$$\begin{array}{ccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ {\scriptstyle e_1}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle m} \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

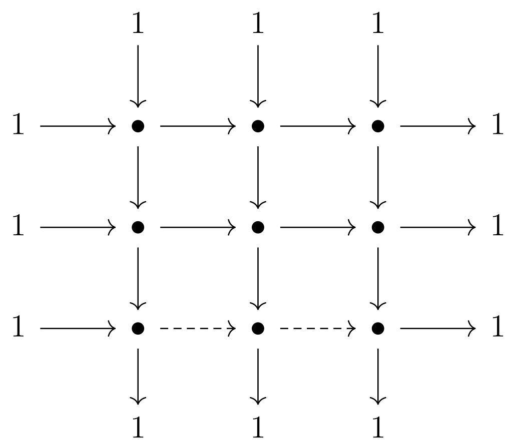
If in the following commutative diagram of groups, both rows are exact, e is an epimorphism and m_1, m_2 are monomorphisms, then φ is a monomorphism.



Lemma 2.2.34 (5-lemma)**.** *If in the following commutative diagram of groups, both rows are exact, e is an epimorphism, i_1, i_2 are isomorphisms and m, is a monomorphism, then φ is an isomorphism.*



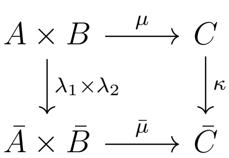
Lemma 2.2.35 (9-lemma)**.** *If in the following commutative diagram of groups, all columns are exact and the first two rows are exact, then the third row is also exact.*



2.2.5 Central Products in Some Diagram Categories

In this section, we introduce the category of abelian bihomomorphisms (on which we will define the Heisenberg functor in Section 4.1). We extend the notion of (maximal) central product to this category as well as to that of group extensions and discuss a lemma about completing a certain diagram.

Definition 2.2.36. Define the *category of abelian bihomomorphisms*. The objects are bihomomorphisms $\mu\colon A \times B \to C$ where A, B, C are abelain groups, cf. Definition 2.2.2. $(\lambda_1 \times \lambda_2\colon A \times B \to \bar{A} \times \bar{B}, \kappa\colon C \to \bar{C})$ is a morphism from μ to $\bar{\mu}\colon \bar{A} \times \bar{B} \to \bar{C}$, if $\kappa \circ \mu = \bar{\mu} \circ (\lambda_1 \times \lambda_2)$.



We call μ *non-degenerate* if every $0 \neq a \in A$ there is $0 \neq b \in B$ (and vice versa) such that $\mu(a, b) \neq 0$, equivalently if the natural maps $A \to \mathrm{Hom}(B, C), a \mapsto (b \mapsto \mu(a, b))$ and $B \to \mathrm{Hom}(A, C), b \mapsto (a \mapsto \mu(a, b))$ are monomorphisms. We call $(\lambda_1, \lambda_2, \kappa)$ *central* if $\bar{\mu}(\lambda_1(A), \bar{B}) = \bar{\mu}(\bar{A}, \lambda_2(B)) = 0$. which case we write $(\lambda_1 \times \lambda_2, \kappa)\colon \mu \xrightarrow{\bullet} \bar{\mu}$.

Lemma 2.2.37. *Consider the first two rows of the following diagram. Assume that* $m := (\lambda \times \nu, \kappa) : \mu_0 \rightarrowtail \mu_1 \times \mu_2$ *is a* central pair of abelain bihomomorphisms*, i.e. that* $m_i := (\lambda_i \times \nu_i, \kappa_i) \colon \mu_0 \to \mu_i$ *are central monomorphisms.*

$$
\begin{array}{ccccccc}
\mu_0 & : & A_0 & \times & B_0 & \xrightarrow{\mu_0} & C_0 \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \kappa} \\
\mu_1 \times \mu_2 & : & A_1 \times A_2 & \times & B_1 \times B_2 & \xrightarrow{\mu_1 \times \mu_2} & C_1 \times C_2 \\
\downarrow{\scriptstyle \mathrm{p}_m} & & \downarrow{\scriptstyle \mathrm{p}_\lambda} & & \downarrow{\scriptstyle \mathrm{p}_\nu} & & \downarrow{\scriptstyle \mathrm{p}_\kappa} \\
\mu_1 \curlyvee_m \mu_2 & : & A_1 \curlyvee_\lambda A_2 & \times & B_1 \curlyvee_\nu B_2 & \xrightarrow[\exists!]{\mu_1 \curlyvee_m \mu_2} & C_1 \curlyvee_\kappa C_2
\end{array}
$$

Then the diagram above can be completed with some unique abelian bihomomorphism which we denote by $\mu_1 \curlyvee_m \mu_2$*. This is non-degenerate if and only if* μ_1 *and* μ_2 *are so. In this case,* A_0 *and* B_0 *are necessarily trivial.*

Definition 2.2.38. $\mu_1 \curlyvee_m \mu_2$ from Lemma 2.2.37 is called the *central product of the abelian bihomomorphisms* μ_i *along the central pair of abelian bihomomorphisms* m. Note that the inverse at $m = (m_1, 1/m_2)$ which is inherited from Definition 2.2.16 where $1/m_2 := (1/\lambda_2, 1/\nu_2, 1/\kappa_2)$. We call m *maximal*, if κ is maximal in which case we write $\mu_1 \hat{\curlyvee}_m \mu_2$ for the resulting central product.

Proof. The commutativity of the desired diagram shows that the only option for $\mu_1 \curlyvee_m \mu_2$ is

$$
\begin{aligned}
\mu \colon (A_1 \curlyvee_\lambda A_2) \times (B_1 \curlyvee_\nu B_2) &\to C_1 \curlyvee_\kappa C_2 \\
(\lceil a_1, a_2 \rfloor, \lceil b_1, b_2 \rfloor) &\mapsto \lceil \mu_1(a_1, b_1), \mu_2(a_2, b_2) \rfloor.
\end{aligned}
$$

We need to check that this map is well-defined. Let $(\lceil a_1, a_2 \rfloor, \lceil b_1, b_2 \rfloor) = (\lceil a_1', a_2' \rfloor, \lceil b_1', b_2' \rfloor)$. Then $(a_1 - a_1', a_2 - a_2') \in \mathrm{Im}(\lambda)$, i.e. $(a_1 - a_1', a_2 - a_2') = (\lambda_1(a_0), -\lambda_2(a_0))$ for some $a_0 \in A_0$ by definition. Similarly $(b_1 - b_1', b_2 - b_2') = (\nu_1(b_0), -\nu_2(b_0))$ for some $b_0 \in B_0$. Then

$$
\begin{aligned}
\mu_i(a_i, b_i) - \mu_i(a_i', b_i') &= \mu_i(a_i, b_i - b_i') + \mu_i(a_i - a_i', b_i') \\
&= \mu_i(a_i, \nu_i(\pm b_0)) + \mu_i(\lambda_i(\pm a_0), b_i') = 0
\end{aligned}
$$

by the central assumption where the sign is $+$ for $i = 1$ and is $-$ for $i = 2$. Thus $\lceil \mu_1(a_1, b_1), \mu_2(a_2, b_2) \rfloor = \lceil \mu_1(a_1', b_1'), \mu_2(a_2', b_2') \rfloor$ as required. All groups are abelian, and short comutation shows that μ is indeed a bihomomorphism.

For the statement about non-degeneracy, assume that $\mu_1 \curlyvee_h \mu_2$ is non-degenerate. Let $0 \neq a_1 \in A_1$. Then by assumption, for $0 \neq \lceil a_1, 0 \rfloor \in A_1 \curlyvee_\lambda A_2$ there is $\lceil b_1, b_2 \rfloor \in B_1 \curlyvee_\nu B_2$, such that $0 \neq \mu_1 \curlyvee_m \mu_2(\lceil a_1, 0 \rfloor, \lceil b_1, b_2 \rfloor) = \lceil \mu_1(a_1, b_1), 0 \rfloor$. Thus $\mu_1(a_1, b_1) \neq 0$. By symmetry, μ_1 and μ_2 are non-degenerate.

For the other direction, assume that μ_i are both non-degenerate. Pick $0 \neq \lceil a_1, a_2 \rfloor \in A_1 \curlyvee_\lambda A_2$. Without loss of generality, $a_1 \neq 0$, so by assumption, there is $b_1 \in B_1$ such that $\mu_1(a_1, b_1) \neq 0$. Then $\mu_1 \curlyvee_h \mu_2(\lceil a_1, a_2 \rfloor, \lceil b_1, 0 \rfloor) = \lceil \mu_1(a_1, b_1), 0 \rfloor \neq 0$ showing the non-degeneracy of $\mu_1 \curlyvee_m \mu_2$.

In this case, pick $a_0 \in A_0$ and $b_0 \in B_0$. Then $\mu_1(\lambda_1(a_0), b_1) = \mu_1(a_1, \nu_1(b_0)) = 0$ for all $a_1 \in A_1$ and $b_1 \in B_1$ as $(\lambda_1 \times \nu_1, \kappa_1)$ is central, hence non-degeneracy implies $\lambda_1(a_0) = 0$ and $\nu_1(b_0) = 0$. Since λ_1 and μ_1 are injective by assumption, a_0 and b_0 must be trivial, hence so are A_0 and B_0. $\square$

Remark 2.2.39. The proof shows that in fact $\mu_1 \curlyvee_m \mu_2$ factors through the natural projection $\mathrm{p}_\kappa\colon C_1 \times C_2 \to C_1 \curlyvee_\kappa C_2$, and that assumption of the central morphism is necessary (and sufficient) for the diagram to exist.

We will see in Lemma 4.2.14 how the previous notion is related to the next one.

Definition 2.2.40. Let ϵ_i for $i \in \{1,2,3\}$ be group extensions together with central pairs between them making the following commutative diagram. We call this $e : \epsilon_0 \rightarrowtail\!\!\!\diamond\!\!\!\to \epsilon_1 \times \epsilon_2$ a *central pair of extensions* where $e_i = (\kappa_i, \gamma_i, \lambda_i)$ and $e = (e_1, 1/e_2)$ as in Definition 2.2.38. Functoriality of the central product from Remark 2.2.19 induces the bottom row which we call the *central product* of ϵ_1 and ϵ_2 along $e := (\kappa, \gamma, \lambda)$, denoted by $\epsilon_1 \curlyvee_e \epsilon_2$. This is exact by the 9-lemma.

$$\begin{array}{ccccccccccc}
\epsilon_0 & : & 1 & \longrightarrow & C_0 & \xrightarrow{\iota_0} & G_0 & \xrightarrow{\pi_0} & M_0 & \longrightarrow & 1 \\
\downarrow{\scriptstyle e} & & & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \lambda} & & \\
\epsilon_1 \times \epsilon_2 & : & 1 & \longrightarrow & C_1 \times C_2 & \xrightarrow{\iota_1 \times \iota_2} & G_1 \times G_2 & \xrightarrow{\pi_1 \times \pi_2} & M_1 \times M_2 & \longrightarrow & 1 \\
\downarrow{\scriptstyle \mathrm{p}_e} & & & & \downarrow{\scriptstyle \mathrm{p}_\kappa} & & \downarrow{\scriptstyle \mathrm{p}_\gamma} & & \downarrow{\scriptstyle \mathrm{p}_\lambda} & & \\
\epsilon_1 \curlyvee_e \epsilon_2 & : & 1 & \longrightarrow & C_1 \curlyvee_\kappa C_2 & \xrightarrow{\curlyvee(\iota)} & G_1 \curlyvee_\gamma G_2 & \xrightarrow{\curlyvee(\pi)} & M_1 \curlyvee_\lambda M_2 & \longrightarrow & 1
\end{array} \tag{2.8}$$

We call the central pair $e : \epsilon_0 \rightarrowtail\!\!\!\diamond\!\!\!\to \epsilon_1 \times \epsilon_2$ of group extensions *maximal* if κ is a maximal central pair, and write $\epsilon_1 \hat{\curlyvee}_e \epsilon_2$ for the corresponding central product.

Remark 2.2.41. If ϵ_i are all central, respectively central-by-abelian, then so is $\epsilon_1 \curlyvee_e \epsilon_2$.

Remark 2.2.42. e from above is maximal if and only if for all diagrams as below, f_0 is necessarily an isomorphism.

$$\begin{array}{ccc}
e_0 & \xrightarrow{e} & \epsilon_1 \times \epsilon_2 \\
\downarrow{\scriptstyle f_0} & & \wr\downarrow{\scriptstyle f_1 \times f_2} \\
\hat{\epsilon}_0 & \xrightarrow{\hat{e}} & \hat{\epsilon}_1 \times \hat{\epsilon}_2
\end{array}$$

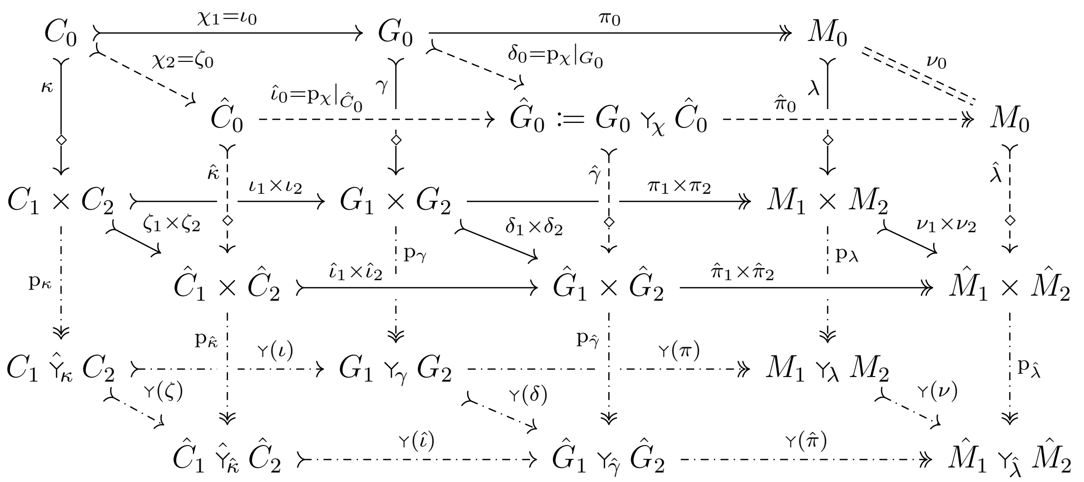
Lemma 2.2.43. *Every maximal central pair e of extensions and monomorphisms f_1, f_2 of central-by-abelian extensions consisting of finitely generated group below can be extended with a $\curlyvee$-monomorphism f_0 and a maximal $\hat{e}$ such that the induced $\curlyvee(f)$ is a monomorphism.*

$$\begin{array}{ccccc}
\epsilon_0 & \xrightarrow{e} & \epsilon_1 \times \epsilon_2 & \xrightarrow{\mathrm{p}_e} & \epsilon_1 \hat{\curlyvee}_e \epsilon_2 \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1 \times f_2} & & \downarrow{\scriptstyle \curlyvee(f)} \\
\hat{\epsilon}_0 & \xrightarrow{\hat{e}} & \hat{\epsilon}_1 \times \hat{\epsilon}_2 & \xrightarrow{\mathrm{p}_{\hat{e}}} & \hat{\epsilon}_1 \hat{\curlyvee}_{\hat{e}} \hat{\epsilon}_2
\end{array}$$

Proof. Use the notation of (2.8) and let $f_i = (\zeta_i, \delta_i, \nu_i)\colon \epsilon_i \to \hat{\epsilon}_i$.

As $\hat{C}_i$ are abelian, we can apply Lemma 2.2.31 with κ and $\zeta_1 \times \zeta_2$. Let $\hat{C}_0$ with $\chi_2 := \zeta_0\colon C_0 \to \hat{C}_0$ and the maximal $\hat{\kappa} : \hat{C}_0 \rightarrowtail\!\!\!\diamond\!\!\!\to \hat{C}_1 \times \hat{C}_2$ be the resulting maps. Let $\chi_1 := \iota_0\colon C_0 \to G_0$, and set $\hat{G}_0 := G_0 \curlyvee_\chi \hat{C}_0$. We then form the monomorphism $\epsilon_0 \to \hat{\epsilon}_0$ of central-by-abelian extension (the first two rows of (2.9)) as the first two rows of (2.3). Set $\hat{\lambda}_i := \nu_i \circ \lambda_i\colon M_0 \to \hat{M}_i$. As these groups as abelian and the maps are injective, we can form the central pair $\hat{\lambda} : M_0 \rightarrowtail\!\!\!\diamond\!\!\!\to \hat{M}_1 \times \hat{M}_2$. On the other hand, the commutativity of the diagram (2.9) implies that $\delta_i \circ \gamma_i \circ \chi_1 = \hat{\iota}_i \circ \kappa_i \circ \chi_2$ as $C_0 \to \mathrm{Z}(\hat{G}_i)$ maps. Hence Remark 2.2.18 induces maps $\hat{\gamma}_i\colon \hat{G}_0 \to \hat{G}_i$ such that $\hat{\gamma}_i \circ \hat{\iota}_0 = \hat{\kappa}_i \circ \hat{\iota}_i$, i.e. the left part of (2.9) is commutative. For arbitrary $\lceil g_0, \hat{c}_0 \rfloor \in G_0 \curlyvee_\chi \hat{C}_0$, we have

$\hat{\pi}_i(\hat{\gamma}_i(\lceil g_0, \hat{c}_0 \rfloor)) = \hat{\pi}_i(\delta_i(\gamma(g_0))\hat{\iota}_i(\hat{\kappa}(\hat{c}_0))) = \hat{\pi}_i(\delta_i(\gamma(g_0))) = \hat{\lambda}_i(\pi_0(g_0)) = \hat{\lambda}_i(\hat{\pi}_0(\lceil g_0, \hat{c}_0 \rfloor))$, thus $\hat{\pi}_i \circ \hat{\gamma}_i = \hat{\lambda}_i \circ \hat{\pi}_0$. Hence the above definitions constitute to a morphism $\hat{\epsilon}_0 \to \hat{\epsilon}_i$, thus $\hat{\gamma}_i$ is a monomorphism by the 4-lemma. This means that the maps above give rise to a central pair $\hat{\gamma} : \hat{G}_0 \rightarrowtail\!\!\!\to \hat{G}_1 \times \hat{G}_2$. This shows the existence of the dashed part of the diagram from the statement. Let $\hat{e} = (\hat{\kappa}, \hat{\gamma}, \hat{\lambda})$.



(2.9)

Then the functoriality of Definition 2.2.40 and Remark 2.2.19 show the existence of the dotted part of the diagram from the statement, hence we only need to show the injectivity of the maps from $\curlyvee(f)$. The map $\curlyvee(\zeta)$ from Lemma 2.2.32 is a monomorphism because κ is maximal by assumption. $\curlyvee(\nu)$ is a monomorphism by the 4-lemma because so is $\nu_1 \times \nu_2$ and ν_0 is an epimorphism. Considering the diagram of $\curlyvee(f)$ from the diagram (the bottom two rows), $\curlyvee(\delta)$ is also a monomorphism once again using the 4-lemma. □

2.3 Alternating Smith Form

To define Heisenberg groups in arbitrary characteristic, we need the notion of a polarised $\mathbb{Z}$-module (Section 4.1) whose existence is provided by the standard form of alternating matrices presented in this section. The proof presented here is motivated by the standard argument used at Smith normal form.

Lemma 2.3.1 ('Alternating Smith' normal form)**.** *Let R be a principal ideal ring, $W \in R^{n\times n}$ be an alternating matrix (i.e. $W^\top = -W$ and has 0's at the main diagonal). Then there exist elements $d_1 \mid d_2 \mid \cdots \mid d_s \neq 0$ in R and $B \in \mathrm{SL}_n(R)$ such that*

$$B^\top W B = \operatorname{diag}\left(\begin{pmatrix} 0 & d_1 \\ -d_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & d_2 \\ -d_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & d_s \\ -d_s & 0 \end{pmatrix}, 0, \ldots, 0 \right). \tag{2.10}$$

If furthermore R is a domain, then $s = \mathrm{rk}(A)$ and the chain of ideals $(d_s) \subseteq \cdots \subseteq (d_1)$ is unique, i.e. d_i's are unique up to unit multiples even if $B \in \mathrm{GL}_n(R)$.

Proof. The idea is similar to the standard proof of Smith normal form [Jac85, §3.7], but we choose different pivots and at each step we apply the base change $W \mapsto X^\top W X$ (to respect the alternating property) for a series of well-chosen matrices $X \in \mathrm{SL}_n(R)$ until the stated form for $W = (w_{i,j})$ is obtained. Once the existence is verified, the uniqueness statement follows from the fact that for each $1 \leq k \leq n$, the ideal generated by the $k \times k$ minors is unchanged under these transformations. cf. [Jac85, Theorem 3.9].

For the existence, we will use the following matrices for the role of $X \in \mathrm{SL}_n(R)$ from above.

- Define $L_{i,j}(r) \in \mathrm{SL}_n(R)$ for $r \in R$ and $i \neq j$ to be the matrix obtained from the identity by setting the entry at (i,j) to r.

- Define $T_{i,j}^{i',j'} \in \mathrm{SL}_n(R)$, for $i \neq j$ and $i' \neq j'$, to be the matrix obtained from I_n by swapping rows $i \leftrightarrow j$, and rows $i' \leftrightarrow j'$.

- Let $a, b \in R$ not both 0 and $i \neq j$. Then $(a,b) = (d)$ for some $0 \neq d \in R$, so $d = ax + by$ for suitable $x, y \in R$. Let $a' := a/d$, $b' := b/d$, put $M := \left(\begin{smallmatrix} x & -b' \\ y & a' \end{smallmatrix}\right) \in \mathrm{SL}_2(R)$. By Note that $(a,b)M = (d,0)$ by construction, cf. [Jac85, (26), p. 184]. Define $D_{i,j}(a,b) \in \mathrm{SL}_n(R)$ be the matrix obtained from the identity matrix by replacing the entries of the submatrix corresponding to rows (i,j) and columns (i,j) by M.

Set $[i,j] := \{k \in \mathbb{N}_0 : i \leq k \leq j\}$ for $(i,j) \in {\mathbb{N}_0}^2$. We describe the procedure as follows.

Step 1 If $W = 0$ we are done. Else there is $w_{i,j} \neq 0$. By the alternating property, $i \neq j$ and $w_{j,i} = -w_{i,j} \neq 0$. Then at least one of (i,j) and (j,i) is coordinate-wise different from $(1,2)$, say (i,j). Interchange rows/columns $i \leftrightarrow 1$, and $j \leftrightarrow 2$ (by applying the base change $T_{i,1}^{j,2}$) to obtain a non-zero pivot to position $(1,2)$.

Step 2 If all entries are multiples of the pivot $w_{1,2}$, proceed to Step 3. Otherwise pick $(i,j) \in [1,n]^2$ amongst $w_{1,2} \nmid w_{i,j}$ where i is as small as possible. Note that $i < j$ and since $w_{1,2} \neq 0$, we must have $2 < j$.

We will make $w_{1,2} \nmid w_{1,j}$ by suitable transformations while keeping the pivot unchanged. If $i = 1$ then we are done. Otherwise $w_{1,2} \mid w_{1,j}$ by the minimality assumption on i. We want to add i to row 1, but this changes the pivot unless $w_{i,2} = 0$. So if $w_{i,2} \neq 0$ we have to make it 0 beforehand. In this case, the alternating property implies that $i > 2$, so then the first two rows (and hence columns) are multiples of the pivot. In particular, $w_{1,2} \mid w_{i,2}$, so subtracting $\frac{w_{i,2}}{w_{1,2}}$ times the first row to row i (by applying the base change $L_{r,i}(-w_{i,2}/w_{1,2})$). This adds some multiple of $w_{1,2}$ to $w_{i,j}$, so it still will not be divisible by the pivot. Now we can add row i to the first row (by applying the base change $L_{i,1}(1)$) as originally planned.

Now we apply the base change $D_{r+1,j}(w_{r,r+1}, w_{r,j})$ to replace the pivot by the greatest common divisor $\gcd(w_{1,2}, w_{1,j})$, and repeat Step 2.

Note that the new value of the pivot is $\gcd(w_{1,2}, w_{i,j})$ for the original entries (at the start of Step 2). So this step reduces the total number of prime factors of the pivot, thus ensures that we only enter Step 2 finitely many times.

Step 3 At this point, every element is a multiple of the pivot $w_{1,2} = -w_{2,1}$. Then for every $j \in [3,n]$, we subtract $w_{1,j}/w_{1,2}$ times the second column and $w_{2,j}/w_{2,1}$ times the first column from column j (by applying the base changes $L_{2,j}(-w_{1,j}/w_{1,2})$ and $L_{1,j}(-w_{2,j}/w_{2,1})$). This sets the first two entries at column j zero but do not modify any other entry from the first two rows as $w_{1,1} = w_{2,2} = 0$ by the alternating property.

This set the first two rows/columns of W to the required form. We then proceed to Step 1 recursively with the remaining $(n-2) \times (n-2)$ submatrix. □

Remark 2.3.2. The proof above is algorithmic whose pseudocode is the following.

input : W as in Lemma 2.3.1
output: B as in Lemma 2.3.1

1 **Function** `NormalForm(`W`)`:
2 $B \leftarrow I_n$; // The base change matrix to be updated step by step.
3 $r \leftarrow 1$; // The row of the pivot.
// While the remianing part of W is not done.
4 **while** $\exists (i,j) \in [r,n]^2 : w_{i,j} \neq 0$ **do**
// Step 1: Get a non-zero pivot.
5 **if** $w_{r,r+1} \neq 0$ **then**
6 Pick (i,j) from line 4 with $i \neq r$ and $j \neq r+1$; // This is always possible.
7 `BaseChange(`$T_{i,r}^{j,r+1}$`)`; // Move the non-zero entry to the pivot $w_{r,r+1}$.
// Step 2: Make the pivot divide all remaining entries.
8 **while** $\exists (i,j) \in [r,n]^2 : w_{r,r+1} \nmid w_{i,j}$ **do** // Replace $w_{r,r+1}$ by $\gcd(w_{r,r+1}, w_{i,j})$.
9 Pick (i,j) from line 8 with i minimal ; // Now $j > r+1$ automatically.
10 **if** $i \neq r$ **then** // Will make $w_{r,r+1} \nmid w_{r,j}$ without altering $w_{r,r+1}$.
11 **if** $w_{i,r+1} \neq 0$ **then** // In this case, $i > r+1$.
12 `BaseChange(`$L_{r,i}(-w_{i,r+1}/w_{r,r+1})$`)`; // Make $w_{i,r+1} = 0$.
13 `BaseChange(`$L_{i,r}(1)$`)`; // $w_{r,r+1} \mid w_{r,j}$, now add $w_{i,j}$ to $w_{r,j}$.
14 `BaseChange(`$D_{r+1,j}(w_{r,r+1}, w_{r,j})$`)`; // Replace $w_{r,r+1}$ by $\gcd(w_{r,r+1}, w_{r,j})$.
// Step 3: Finish rows/columns r, $r+1$ and proceed to the remaining submatrix.
15 **for** $j \leftarrow r+2$ **to** n **do** // Now $w_{r,r+1}$ has in its final value.
16 `BaseChange(`$L_{r+1,j}(-w_{r,j}/w_{r,r+1})$`)`; // Set $w_{r,j} = 0$.
17 `BaseChange(`$L_{r,j}(-w_{r+1,j}/w_{r+1,r})$`)`; // Set $w_{r+1,j} = 0$.
18 $r \leftarrow r+2$;
19 **return** B;
20 **Function** `BaseChange(`X`)`:
21 $B \leftarrow BX$; // Update the base change matrix.
22 $W \leftarrow X^\top W X$; // Update W.

2.4 Fibre Bundles and Invertible Sheaves

In this section, we include some known results and tools from algebraic topology and K-theory for reference. There are no new statements here. Subsection 2.4.1 presents the tools used for cohomology computations; Subsection 2.4.2 is about compact fibre bundles associated to (non-compact) vector bundles. Later, Subsection 2.4.3 and Subsection 2.4.4 discuss two closely related topics: holomorphic lines bundles over the complex tori, and ample invertible sheaves over an abelian variety.

2.4.1 Cohomology and K-theory

In this section, the main aim is to detect the triviality of a given vector bundle using cohomology (Proposition 2.4.16). We collect the necessary statements and tools here from topology and from K-theory [Hat02; Hat17; Par08; Hus93].

Cohomology

We will mostly consider compact manifolds.

Lemma 2.4.1 ([Hat02, Corollary A.12])**.** *Every compact manifold is homotopy equivalent to a CW-complex.*

Remark 2.4.2 (Which cohomology?). The isomorphism class of singular cohomology and K-theory are invariant for homotopy equivalent spaces, cf. [Hat02, §3.1, p.201] and [Par08, Proposition 2.1.7]. The notion of Čech cohomology and singular cohomology coincide for spaces homotopy equivalent to CW complices, [Hat02, §3.3, p.257]. Since in this thesis, we consider compact manifolds (which are CW-complices, cf. Lemma 2.4.1), for most of the time, we shall simply write cohomology for any of these two notions.

The next statement is true in a much more general setup, but we will not need it.

Lemma 2.4.3. *For a compact manifold X, $H^\bullet(X;\mathbb{Z})$ is finitely generated $\mathbb{Z}$-module and $H^k(X;\mathbb{Z}) = 0$ for $k > \dim_{\mathbb{R}}(X)$.*

Proof. The (singular) homology group $H_\bullet(X;\mathbb{Z})$ is finitely generated by [Hat02, Corollaries A.8-9]. [Hat02, Theorem 3.26] shows that $H_k(X,\mathbb{Z}) = 0$ for $k > \dim_{\mathbb{R}}(X)$ and $H_{\dim_{\mathbb{R}}(X)}(X;\mathbb{Z})$ is a free abelian group. Then both statements follows from the universal coefficient theorem for cohomology [Hat02, Theorem 3.2]. □

Lemma 2.4.4 (Künneth formula for cohomology)**.** *If X_i are compact manifolds, then there is an isomorphism*

$$\varphi\colon H^\bullet(X_1;\mathbb{Q}) \otimes_{\mathbb{Q}} H^\bullet(X_2;\mathbb{Q}) \to H^\bullet(X_1 \times X_2;\mathbb{Q})$$

of rings compatible with the induces maps, i.e. every pair $f_i\colon X_i \to X_i$ of continuous maps, $(f_1 \times f_2)^* \circ \varphi = \varphi \circ (f_1^* \otimes f_2)$.

Proof. Let $p_i\colon X_1 \times X_2 \to X_i$ be the natural projection. Then $\varphi\colon a_1 \otimes a_2 \mapsto p_1^*(a_1) \smile p_2^*(a_1)$ gives an isomorphism by [Hat02, Theorem 3.15] using Lemma 2.4.1 and Lemma 2.4.3, because $H^\bullet(X_i;\mathbb{Q}) \cong H^\bullet(X_i;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is free. The compatibility is a consequences of basic properties of induced maps. □

Definition 2.4.5. Let $\tau\colon M \to \mathrm{Diff}(X)$ be a group morphism (i.e. a group action on X.) Write $X/\tau \coloneqq \{\{\tau(m)(x) : m \in M\} : x \in X\}$, the *orbit space*.

Say τ is *free*, if $\tau(m)(x) = x$ implies that $m = 1$, i.e. when all stabilisers are trivial apart from that of $1 \in M$.

The cohomology of quotient manifolds can be computed as the invariants as follows.

Lemma 2.4.6 (Quotient Manifold Theorem, finite group action)**.** *Let X be a smooth real (respectively complex) manifold, M be a finite group, $\tau\colon M \to \mathrm{Diff}(X)$ be a free action. Then M/τ has a unique smooth (respectively complex) manifold structure making the natural map $q\colon X \to X/\tau, x \mapsto \{\tau(m)(x) : m \in M\}$ is a smooth (respectively holomorphic) normal $|M|$-sheeted covering map and $\dim(X) = \dim(X/\tau)$. In this case, the induced map gives an isomorphism*

$$q^*\colon H^{2k}(X/\tau;\mathbb{Q}) \to H^{2k}(X;\mathbb{Q})^\tau \coloneqq \{\alpha \in H^{2k}(X;\mathbb{Q})\colon (\forall m \in M)(\tau(m)^*(\alpha) = \alpha)\}$$

of groups for any $k \in \mathbb{N}_0$.

Proof. Endow M with the discrete topology. Then M is a compact Lie group, and τ gives a smooth action. Consequently, τ is proper by [Lee12, Corollary 21.6.], so [Lee12, Theorem 21.13] and gives the statement on the smooth structure and the covering map. Free actions are faithful, so each orbit has cardinality $|M|$, hence the number os sheets is uniformly $|M|$. The complex case follows from [BL04, Corollary A.7]

Finally, $\dim_{\mathbb{R}}(X/\tau) = \dim_{\mathbb{R}}(X) - \dim_{\mathbb{R}}(M) = \dim_{\mathbb{R}}(X)$ by [Lee12, Theorem 21.10]. □

K-theory

Definition 2.4.7 (K-theory)**.** Let X be a compact Hausdorff space. Let $K^0(X)$ be the Grothendieck completion of the isomorphism classes of vector bundles over X under Whitney sums [Par08, Definition 2.1.1]. For simplicity, set $K^{-1}(X) := K^0(X \times \mathbb{R})$ (using [Par08, Theorem 2.6.13] instead of the usual definition [Par08, Definition 2.3.3]). Write $K^\bullet(X) := K^0(X) \oplus K^{-1}(X)$.

Lemma 2.4.8 (Künneth formula K-theory, [Par08, Proposition 3.3.15])**.** *If X, Y are compact smooth manifolds such that $K^\bullet(X)$ is free, then*

$$K^\bullet(X \times Y) \cong K^\bullet(X) \otimes_{\mathbb{Z}} K^\bullet(Y).$$

Example 2.4.9. For the 1-sphere, $K^\bullet(\mathbb{T}) \cong \mathbb{Z}^2$, see [Par08, Example 2.8.1]. Thus for the n-torus $\mathbb{T}^n := \mathbb{T} \times \cdots \times \mathbb{T}$, Lemma 2.4.8 gives $K^\bullet(\mathbb{T}^n) \cong \mathbb{Z}^{2^n}$ by induction.

Definition 2.4.10. For any topological space X, we denote by $\theta_X \colon X \times \mathbb{C} \to X, (x, z) \mapsto x$ the trivial complex line bundle over X.

Lemma 2.4.11 ([Hus93, Part II, §9.1, Theorem 1.2, p.112].)**.** *Let X be a finite dimensional CW-complex (e.g. a compact manifold). If p is a complex vector bundle over X of rank at least* $\dim(X)/2$, *then $p \cong p_0 \oplus \theta_X^k$ for some vector bundle p_0 of rank* $\lfloor \dim(X)/2 \rfloor$ *and $k \in \mathbb{N}_0$.*

Lemma 2.4.12 ([Par08, Propositios 2.1.4-5])**.** *If X is a compact Hausdorff space, then every element of $K^0(X)$ be be represented as $[p] - [\theta_X^{\oplus k}]$ for some vector bundle p whose rank at each connected component is at least k.*

Furthermore, $[p] - [\theta_X^{\oplus k}] = 0 \in K^0(X)$ if and only if there exists $m \in \mathbb{N}_0$ such that $p \oplus \theta_X^{\oplus m} \cong \theta_X^{\oplus(k+m)}$.

Lemma 2.4.13 ([Hus93, Part II, §9.1, Theorem 1.5, p.112].)**.** *Let X be a finite dimensional CW-complex (e.g. a compact manifold). Let p_1 and p_2 be complex vector bundles over X of rank at least* $\dim(X)/2$ *such that $p_1 \oplus \theta_X^{\oplus l} \cong p_2 \oplus \theta_X^{\oplus l}$ for some $l \in \mathbb{N}_0$. Then $p_1 \cong p_2$.*

Lemma 2.4.14 ([Kar64, Théorème 3], cf. [AH61, Theorem, Corollary §2.4, p.19])**.** *If X is a compact Hausdorff space, then the* Chern character

$$\mathrm{ch} \colon K^0(X) \otimes \mathbb{Q} \to \check{H}^{2\bullet}(X, \mathbb{Q}) = \check{H}^{2\bullet}(X, \mathbb{Z}) \otimes \mathbb{Q}$$

is a ring isomorphism to the Čech cohomology.

In particular, if $K^0(X)$ is free, then $\mathrm{ch} \colon K^0(X) \to \check{H}^{2\bullet}(X, \mathbb{Q})$ *is injective.*

Remark 2.4.15. Apart from the above compatibility of the Chern character ch with $\oplus$ and $\otimes$, we will need to compute it on line bundles explicitly. If $p\colon L \to X$ is a line bundle over a compact manifold X, then $\mathrm{ch}(p) = \sum_{k=0}^{\infty} \frac{1}{k!} c_1(p)^{\smile k} =: \exp(p)$, a finite sum from Lemma 2.4.3 [Hat17, §4.1. The Chern Character]. Here $c_1(p) \in H^k(X, \mathbb{Q})|_{\mathbb{Z}}$ is the first Chern class of p, see [Hat17, §3.1. Stiefel-Whitney and Chern Classes].

Our key tool of detecting trivial bundles is the following.

Proposition 2.4.16. *Suppose X is a compact manifold such that $K^0(X)$ is free, and let p be a complex vector bundle over X of rank $r \geq \dim_{\mathbb{R}}(X)/2$ with $\mathrm{ch}(p) \in H^0(X;\mathbb{Q})$. Then $p \cong \theta_X^{\oplus r}$, the trivial bundle of rank r.*

Proof. By assumption and Lemma 2.4.14, the Chern character $\mathrm{ch}\colon K^0(X) \to H^\bullet(X, \mathbb{Q})$ is injective. Then since $\mathrm{ch}([p] - [\theta_X^{\oplus r}]) = 0 \in H^\bullet(X, \mathbb{Q})$ by assumption, $[p] - [\theta^{\oplus r}] = 0 \in K(X)$. This means by Lemma 2.4.12, that there is $m \in \mathbb{N}_0$ such that $p \oplus \theta_X^{\oplus m} \cong \theta_X^{\oplus (r+m)}$. Finally, since $r \geq \dim_{\mathbb{R}}(X)/2$, Lemma 2.4.11 implies that $p \cong \theta_X^{\oplus r}$. □

2.4.2 Some Associated Fiber Bundles

We will need to construct a compact space from a complex vector bundle over a compact space preserving a group action. We achieve this by constructing various standard fibre bundles below with compact fibres. The total space of these bundles is compact and inherits the group action from the vector bundle.

A *Hermitian form* if it is $\mathbb{C}$-linear in the first coordinate, conjugate-linear in the second and is conjugate-symmetric ($H(v, v') = \overline{H(v', v)}$ for all $v, v' \in V$).

Lemma 2.4.17 ([Par08, Corollary 1.7.10])**.** *Every complex vector bundle $p\colon E \to X$ over a compact Hausdorff space X admits a* Hermitian metric *h.*

More concretely, there is a continuous $h\colon \bigcup_{x\in X} p^{-1}(x) \times p^{-1}(x) \to \mathbb{C}$ (with respect to the subspace topology of the product) so that each restriction $h|_{(p^{-1}(x))^2}\colon p^{-1}(x) \times p^{-1}(x) \to \mathbb{C}$ is a positive definite Hermitian form, cf. [Par08, Definition 1.7.7].

Example 2.4.18 (Associated Stiefel bundle, cf. [Hat17, §1.1, Associated Fiber Bundles])**.** For a complex inner product space (W, h_W) of dimension s and an integer $1 \leq k \leq s$, associate the smooth *Stiefel manifold* $\mathrm{V}_k(W, h_W) := \{(w_1, \ldots, w_k) \in W^k : h_W(w_i, w_j) = \delta_{i,j}\}$ with the natural topology where $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$ is the Kronecker delta. Isometrically isomorphic inner product spaces give diffeomorphic manifolds. Write $\mathrm{V}_k(\mathbb{C}^s)$ for (the class) of $\mathrm{V}_k(W, h_W)$. Actually, $\mathrm{V}_k(\mathbb{C}^s) \cong \mathrm{U}(s)/\mathrm{U}(s-k)$, which is a $k(2s-k)$-manifold. Notable special cases are

- $\mathrm{V}_1(\mathbb{C}^s) \cong \mathbb{S}^{2s-1}$, the $(2s-1)$-sphere;
- $\mathrm{V}_{s-1}(\mathbb{C}^s) \cong \mathrm{SU}(s)$, the $s \times s$ special unitary group;
- $\mathrm{V}_s(\mathbb{C}^s) \cong \mathrm{U}(s)$, the $s \times s$ unitary group.

For more details, see [Hus93, §8.1].

Let h be a Hermitian metric on a complex vector bundle $p\colon E \to X$ of rank s. Let $1 \leq k \leq s$ be an integer. Set $\mathrm{V}_k(E, h) := \{(e_1, \ldots, e_k) \in (p^{-1}(x))^k : x \in X, h(e_i, e_j) = \delta_i\} \subseteq E^k$ equipped with the subspace topology of the product. This comes equipped with a natural projection

$$\mathrm{V}_k(p, h)\colon \mathrm{V}_k(E, h) \to X$$

which we call the *associated Stiefel bundle.* It is then a subbundle of $p^{\oplus k}$. Its fibres are the Stiefel manifolds corresponding the fibre of the original vector bundle, i.e. for any $x \in X$, we have $\mathrm{V}_k(p^{-1}(x), h|_{p^{-1}(x)}) = (\mathrm{V}_k(p,h))^{-1}(x)$. So the fibres of this bundle is $\mathrm{V}_k(\mathbb{C}^s)$. For an analogue with real inner product spaces, see [Hat17, §1.1 Associated Fiber Bundles].

Example 2.4.19 (Associated Grassmann bundle)**.** Let W be a complex vector space of dimension s, and let $1 \leq k \leq s$ be an integer. The smooth Grassmannian $k(s-k)$-manifold $\mathrm{Gr}_k(W)$ is the set of the k-dimensional subspaces of W. Its topology is given as the quotient space of $\mathrm{V}_k(W, h_W)$ by identifying the tuples that generate the same linear subspace of W. The diffeomorphism class of this is independent of the choice if h_W.

Similarly, if $p\colon E \to X$ is a complex vector bundle of rank s, define the *associated Grassmann bundle*

$$\mathrm{Gr}_k(p)\colon \mathrm{Gr}_k(E) \to X$$

where $\mathrm{Gr}_k(E)$ is the quotient of $\mathrm{V}_k(E,h)$ identifying tuples from the same fibre (of $\mathrm{V}_k(p,h)$) if they generate the same linear subspace of the fibre (of p); and $\mathrm{Gr}_k(p)$ is the natural projection, see [Hat17, §1.1 Associated Fiber Bundles]. Its fibres are diffeomorphic to the Grassmann manifolds corresponding to the fibre of the original bundle, i.e. $\mathrm{Gr}_k(p)^{-1}(x) \cong \mathrm{Gr}_k(p^{-1}(x))$ for all $x \in X$, in particular they are all diffeomorphic to $\mathrm{Gr}_k(\mathbb{C}^s)$. We are mainly interested in the $k = 1$ case,

- $\mathbb{P}(p) \coloneqq \mathrm{Gr}_1(p)$, $\mathbb{P}(E) \coloneqq \mathrm{Gr}_1(E)$ is the *projectivisation* of p whose fibre is $\mathrm{Gr}_1(\mathbb{C}^s) \cong \mathbb{CP}^{s-1}$, the complex projective space.

Remark 2.4.20. The two examples are also related in the following way. For an s-dimensional complex vector space W, the natural projection $\mathrm{V}_k(W) \to \mathrm{Gr}_k(W)$ sending a k-tuple to the generated subspace is actually a principal $\mathrm{U}(k)$-bundle.

2.4.3 The Appell–Humbert Theorem

The Appell-Humbert theorem characterises every holomorphic line bundle over a complex torus using Riemann forms and semi-characters. We review the construction in detail following [BL04, §1-2]. At Subsection 5.2.4, we will use the ideas presented here to construct actions of various twisted Heisenberg groups on these holomorphic line bundles.

Definition 2.4.21. Let V be a $\mathbb{C}$-vector space of dimension $\dim_{\mathbb{C}}(V) = g$. Let $\Lambda \leq V$ be a lattice, i.e. a subgroup isomorphic to $\mathbb{Z}^{2g}$ such that Λ spans V as an $\mathbb{R}$-vector space, i.e. $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$. The corresponding *complex torus* is defined by $\mathbb{T}_\Lambda \coloneqq V/\Lambda$.

Remark 2.4.22. $\mathbb{T}_\Lambda$ is a connected compact complex manifold by Lemma 2.4.6. The group structure of V descends to $\mathbb{T}_\Lambda$ making it a complex Lie group. In fact, we obtain every connected compact complex Lie group in this way, cf. [BL04, Lemma 1.1.1]. As a real manifold, $\mathbb{T}_\Lambda \cong \mathbb{T}^{2g} = (\mathbb{S}^1)^{2g}$.

Definition 2.4.23. Let V be a complex vector space and $H\colon V \times V \to \mathbb{C}$ be a (potentially degenerate) Hermitian form.

$E\colon V \times V \to \mathbb{R}$ is an *alternating form*, if it $\mathbb{R}$-bilinear and $E(v,v) = 0$ for all $v \in V$.

Remark 2.4.24. Let $i = \sqrt{-1}$. The following maps establish are mutual inverses of each other, where $\Im(z)$ is the imaginary part of $z \in \mathbb{C}$, cf. [BL04, Lemma 2.1.7].

$$\begin{aligned} \{\text{Hermitian forms on } V\} &\longleftrightarrow \left\{\begin{matrix}\text{alternating forms } E \text{ on } V \text{ such that} \\ E(iv, iv') = E(v, v') \text{ for all } v, v' \in V\end{matrix}\right\} \\ H &\longmapsto \Im H \colon (v, v') \mapsto \Im(H(v, v')) \\ (v, v') \mapsto E(iv, v') + iE(v, v') &\longleftarrow E \end{aligned}$$

Note that $H(v_1 + v_2, v_1 + v_2) = H(v_1, v_1) + H(v_2, v_2) + 2H(v_2, v_1) - 2i\Im H(v_2, v_1)$.

Definition 2.4.25. A Hermitian form $H \colon V \times V \to \mathbb{C}$ is a *Riemann form with respect to the lattice* Λ, if $\Im H(\Lambda, \Lambda) \subseteq \mathbb{Z}$. In this case, A *semi-character* with respect to H is a map $\chi \colon \Lambda \to \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ such that

$$\chi(\lambda + \lambda') = \chi(\lambda)\chi(\lambda') \exp(\pi i \Im H(\lambda', \lambda))$$

for all $\lambda, \lambda' \in \Lambda$. The set $\mathcal{P}(\Lambda)$ of such pairs (H, χ) has the structure of an abelian group via $(H, \chi) \cdot (H', \chi') := (H + H', \chi \cdot \chi')$.

Remark 2.4.26. Note that as both Λ and $\mathbb{T}$ are abelian groups, $\exp(\pi i \Im H(\lambda, \lambda')) = \exp(\pi i \Im H(\lambda', \lambda)) = \exp(-\pi i \Im H(\lambda, \lambda'))$ using the alternating property of $\Im H$, equivalently $\Im H(\lambda, \lambda') \in \mathbb{Z}$ which explains the necessity of the assumption. Note that χ is not necessarily a group morphism, but the composition $L \to \mathbb{T} \to \mathbb{T}/\{\pm 1\}$ is.

Lemma 2.4.27 (Constructing line bundles, [BL04, §2.2])**.** *For every lattice* $\Lambda \subseteq V$*, there is a group morphism*

$$\mathcal{P}(\Lambda) \to \operatorname{Pic}(\mathbb{T}_\Lambda), \quad (H, \chi) \mapsto (\mathcal{L}(H, \chi) \xrightarrow{p(H,\chi)} \mathbb{T}_\Lambda)$$

where $\operatorname{Pic}(\mathbb{T}_\Lambda)$ *is the abelian group of isomorphism class of lines bundles over* $\mathbb{T}_\Lambda$ *(where the group operation is induced by the tensor product).*

Proof. Pick $(H, \chi) \in \mathcal{P}(\Lambda)$ and define

$$f := f_{(H,\chi)} : \Lambda \times V \to \mathbb{C}^\times, \quad (\lambda, v) \mapsto \chi(\lambda) \exp(\pi H(v, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)).$$

This satisfies the following, so called, cocycle condition $f(\lambda + \lambda', v) = f(\lambda, \lambda' + v) f(\lambda', v)$, see the proof of Lemma 5.2.31 for (a slightly more general) calculation. Define

$$\begin{aligned} \varrho := \varrho_{(H,\chi)} : \Lambda \to \operatorname{Bih}(V \times \mathbb{C}) \qquad & \lambda \mapsto ((v, z) \mapsto (\lambda + v, f(\lambda, v) z)) \\ \tau \colon \Lambda \to \operatorname{Bih}(V) \qquad & \lambda \mapsto (v \mapsto \lambda + v) \end{aligned}$$

These are group morphisms. Indeed, $\varrho(\lambda + \lambda')(v, z) = (\lambda + \lambda' + v, f(\lambda + \lambda', v) z) = (\lambda + \lambda' + v, f(\lambda, \lambda' + v) f(\lambda', v) z) = \varrho(\lambda)(\lambda' + v, f(\lambda', v) z) = \varrho(\lambda)(\varrho(\lambda')(v, z))$ using the cocycle condition. Hence both maps define free group actions. Let $\theta_V \colon V \times \mathbb{C} \to V$ be the projection to the first factor, i.e. the trivial line bundle. For any $\lambda \in \Lambda$, the following diagram commutes.

$$\begin{array}{ccc} V \times \mathbb{C} & \xrightarrow{\varrho(\lambda)} & V \times \mathbb{C} \\ \downarrow \theta_V & & \downarrow \theta_V \\ V & \xrightarrow{\tau(\lambda)} & V \end{array}$$

Let $\mathcal{L}(H,\chi) := (V \times \mathbb{C})/\varrho$ be the quotient space. This can be endowed with the structure of a complex manifold by Lemma 2.4.6. Note that $\mathbb{T}_\Lambda = V/\tau$. The commutativity of the diagram shows that θ_V descends to a holomorphic map

$$p(H,\chi)\colon \mathcal{L}(H,\chi) \to \mathbb{T}_\Lambda,$$

which is a holomorphic line bundle. The overall construction being a group morphism boils down to the following computation.

$$\begin{aligned} f_{(H,\chi)}(\lambda,v) \cdot f_{(H',\chi')}(\lambda,v) &= \chi(\lambda)\exp(\pi H(v,\lambda) + \tfrac{\pi}{2}H(\lambda,\lambda)) \cdot \\ &\qquad \chi'(\lambda)\exp(\pi H'(v,\lambda) + \tfrac{\pi}{2}H'(\lambda,\lambda)) \\ &= (\chi\cdot\chi')(\lambda)\exp(\pi(H+H')(v,\lambda) + \frac{\pi}{2}(H+H')(\lambda,\lambda)) \\ &= f_{(H,\chi)\cdot(H',\chi')}(\lambda,v) \end{aligned}$$ □

We will not use the next result, but in fact, up to isomorphism, every holomorphic line bundle over $\mathbb{T}_\Lambda$ is obtained in this way.

Lemma 2.4.28 (Appell–Humbert, [BL04, Theorem 2.2.3])**.** *The construction from above gives a group isomorphism $\mathcal{P}(\Lambda) \to \mathrm{Pic}(\mathbb{T}_\Lambda)$ where the (first) Chern class of $L(H\chi)$ can naturally be identified by $\Im H$.*

2.4.4 Mumford's Theta Groups

This section forms in some sense the core of the thesis, and summarises the key aspects of [Mum66] by Mumford form 1966. Most of the articles about actions producing non-Jordan transformation groups originates from this one. We define Mumford's theta group, its action on global sections of ample invertible sheaves over an abelian variety, and its structural description that motivated our definition of the twisted Heisenberg group.

Definition 2.4.29 ([Mum66, pp.288-289])**.** Let k denote an algebraically close field of characteristic 0, let X be an abelian variety over k, and L an ample invertible $\mathscr{O}_X$-module. For a closed point $x \in X(k)$, set the translation $\tau\colon X \to X, y \mapsto y + x$. Define the *theta group* $\mathscr{G}(L)$ to be the group of pairs (x,φ) where $x \in X(k)$ and $\varphi\colon L \to \tau(x)^*L$ is an isomorphism of invertible sheaves. The group structure is given by $(y,\psi)\cdot(x,\varphi) := (x+y, (\tau(x)^*\psi)\circ\varphi)$. Let $H_{\mathrm{M}}(L) := \{x \in X(k) : L \cong \tau(x)^*L\}$. Then

$$\mathscr{M}(L) \;:\; 1 \longrightarrow \mathrm{Z}(\mathscr{G}(L)) \overset{\subseteq}{\hookrightarrow} \mathscr{G}(L) \overset{\pi_L}{\twoheadrightarrow} H_{\mathrm{M}}(L) \longrightarrow 1$$

is a non-degenerate central-by-abelian extension where $\pi_L\colon (x,\varphi) \mapsto x$.

Remark 2.4.30. This notation inherited from [Mum66], but we added the subscript M in $H_{\mathrm{M}}(L)$ to clearly distinguish these groups from twisted Heisenberg groups $\mathrm{H}(\mu)$ we define later in Definition 4.1.9 and Definition 4.3.1.

We are mainly interested this group because of the existence of its action on the global sections.

Lemma 2.4.31 ([Mum66, p.295])**.** *For every L ample invertible $\mathscr{O}_X$-module over an abelian variety X over algebraically close field k of characteristic 0, there exists a faithful action*

$$\varrho\colon \mathscr{G}(L) \to \operatorname{Aut}(\Gamma(X,L))$$

such that $\varrho|_{\mathrm{Z}(\mathscr{G}(L))} \circ \kappa : k^\times \to \operatorname{Aut}(\Gamma(X,L))$ maps c to multiplication by c where κ is from (2.11)*.*

Proof. $\varrho : (x,\varphi) \mapsto (s \mapsto \tau(-x)^*(\varphi(s)))$ has the desired properties. Note that $\varphi(s)$ is a section of $\tau(x)^*(L)$, thus $\varrho(s)$ is a section of $\tau(-x)^*(\tau(x)^*(L)) = \tau(0)^*(L) = L$, hence ϱ is a well-defined.

$$\begin{aligned}
\varrho((y,\psi)\cdot(x,\varphi)) &= \varrho((x+y,(\tau(x)^*\psi)\circ\varphi)) \\
&= (s \mapsto \tau(-x-y)^*((\tau(x)^*\psi)(\varphi(s)))) \\
&= (s \mapsto \tau(-x-y)^*(\tau(x)^*(\psi(\tau(-x)^*(\varphi(s)))) \\
&= (s \mapsto \tau(-y)^*(\psi(\tau(-x)^*(\varphi(s))) \\
&= \varrho(x,\varphi)\circ\varrho(x,\varphi)
\end{aligned}$$
∎

Mumford gave the following description of these groups.

Definition 2.4.32 ([Mum66, p.294])**.** Let $\delta := (d_1,\dots,d_t)$ a tuple of integers at least 2 such that $d_t \mid d_{t-1} \mid \dots \mid d_1$. Set $K(\delta) = \prod_{i=1}^t \mathbb{Z}/d_i\mathbb{Z}$, define groups

$$H_{\mathrm{M}}(\delta) := K(\delta) \times \operatorname{Hom}(K(\delta),k^\times) \quad \text{and} \quad \mathscr{G}(\delta)$$

where the underlying set of the latter is $k^\times \times H_{\mathrm{M}}(\delta)$ with $(c,b,\alpha)\cdot(c',b',\alpha') := (cc'\alpha'(b), b+b', \alpha\alpha')$ as the group operation.

Lemma 2.4.33 ([Mum66, pp.289, 294])**.** *For every L ample invertible $\mathscr{O}_X$-module over an abelian variety X over algebraically close field k of characteristic 0, there exists a unique $\delta = (d_1,\dots,d_t)$ such that $\deg(L) = \prod_{i=1}^t d_i$ and there is an isomorphism*

$$\begin{array}{ccccccccccc}
\mathscr{M}(\delta) & : & 1 & \longrightarrow & k^\times & \xrightarrow{\iota_\delta} & \mathscr{G}(\delta) & \xrightarrow{\pi_\delta} & H_{\mathrm{M}}(\delta) & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle \wr\,\kappa_L} & & \downarrow{\scriptstyle \wr\,\gamma} & & \downarrow{\scriptstyle \wr\,\lambda} & & \\
\mathscr{M}(L) & : & 1 & \longrightarrow & \mathrm{Z}(\mathscr{G}(L)) & \xrightarrow{\subseteq} & \mathscr{G}(L) & \xrightarrow{\pi} & H_{\mathrm{M}}(L) & \longrightarrow & 1
\end{array} \tag{2.11}$$

where $\iota_\delta\colon c \mapsto (c,0,0)$ and $\pi_\delta\colon (c,b,\alpha) \mapsto (b,\alpha)$ and for $\kappa(c) = (0,\varphi_c)$, where φ_c is induced by multiplication by c.

Remark 2.4.34. In particular, when $\deg(L) = 1$, then $H_{\mathrm{M}}(L)$ is trivial.

We conclude this section with some statements that will be needed later.

Lemma 2.4.35 ([Har77, Corollary IV.3.3])**.** *Over a curve, an invertible sheaf L is ample if and only if $\deg(L) > 0$.*

Lemma 2.4.36 ([Mum08, §II.6. Proposition(3), p.61])**.** *For any abelian variety X over an algebraically closed field k of characteristic 0 and any $d \in \mathbb{N}_0$, $\{x \in X(k) : d\cdot x = 0\} \cong (\mathbb{Z}/d\mathbb{Z})^{2\dim(X)}$.*

Lemma 2.4.37 ([Mum66, Corollary 4, p.310])**.** *For any $d \in \mathbb{N}_0$ and invertible $\mathscr{O}_X$-module L (where X is an abelian variety over an algebraically closed field k of characteristic 0), $H_{\mathrm{M}}(L^{\otimes d}) = \{x \in X(k) : d \cdot x \in H_{\mathrm{M}}(L)\}$.*

Lemma 2.4.38. *Let L_i be ample invertible $\mathscr{O}_{X_i}$-modules where X_i are abelian varieties over an algebraically closed field of characteristic 0. Let $q_i : X_1 \times X_2 \to X_i$ be the natural map, and set the* external tensor product *$L_1 \boxtimes L_2 \coloneqq q_1^* L_1 \otimes q_2^* L_2$, an ample invertible $\mathscr{O}_{X_1 \times X_2}$-module. Let the central pair $\kappa : k^\times \rightarrowtail \mathscr{G}(L_1) \times \mathscr{G}(L_2)$ be given by $\kappa_i = \kappa_{L_i}$ from (2.11). Then $\mathscr{M}(L_1 \boxtimes L_2) \cong \mathscr{M}(L_1) \curlyvee_{(\kappa,\kappa,0)} \mathscr{M}(L_2)$ as central-by-abelian extensions. More concretely, we have the following commutative diagram.*

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Z}(\mathscr{G}(L_1 \boxtimes L_2)) & \xhookrightarrow{\subseteq} & \mathscr{G}(L_1 \boxtimes L_2) & \xrightarrow{\pi_{L_1 \boxtimes L_2}} & H_{\mathrm{M}}(L_1 \boxtimes L_2) & \longrightarrow & 1 \\
 & & \downarrow \wr & & \downarrow \wr & & \| & & \\
1 & \longrightarrow & \mathrm{Z}(\mathscr{G}(L_1)) \curlyvee_\kappa \mathrm{Z}(\mathscr{G}(L_2)) & \xrightarrow{\curlyvee(\iota)} & \mathscr{G}(L_1) \curlyvee_\kappa \mathscr{G}(L_2) & \xrightarrow{\curlyvee(\pi)} & H_{\mathrm{M}}(L_1) \times H_{\mathrm{M}}(L_1) & \longrightarrow & 1
\end{array}
$$

Proof. The isomorphism in the middle is given by [Mum66, Lemma §3.1, p.323]. The one involving the centres is a consequence of κ_i being isomorphism. The commutativity of the diagram follows from definitions. □

Chapter 3

Structure of Finitely Generated Nilpotent Groups of Class at most 2

The goal of this chapter is to prove Theorem A from page 7. Each part of this statement corresponds to a section of the current chapter. At Section 3.1, we reduce the problem to the cyclic centre case by induction using subdirect products (Corollary 3.1.26). At Section 3.2, we prove the central product decomposition result (Theorem 3.2.23) by passing to the abelianisation and using linear algebra. Finally in Section 3.3, we follow [AMM12] to give a presentation of the main building blocks (Proposition 3.3.7) and use this to give a stronger version of the previous central product decomposition result in the special case when the group has cyclic centre (Proposition 3.3.11).

3.1 Subdirect Product Reduction to Cyclic Centre

In this section, we show that every finitely generated nilpotent group is a subdirect product of finitely generated nilpotent group with cyclic centre and determine the minimal number of factors needed for such a subdirect product. We first consider only finite groups as this setup gives a better overview of the ideas. We then treat the general (and slightly more technical) case afterwards following the flavour of the finite case.

Definition 3.1.1. Let $\mathcal{G}$ be a class of groups and let G be an arbitrary group. A $\mathcal{G}$-*decomposition in* G is a finite set D of normal subgroups of G such that $G/N \in \mathcal{G}$ for every $N \in D$ and $\bigcap D = 1$. (Use the convention $\bigcap \emptyset = G$.) If there is such a decompositions, then we say that G has $\mathcal{G}$-decomposition. Let $m_{\mathcal{G}}(G)$ denote the minimal $|D|$ amongst all $\mathcal{G}$-decomposition D in G, or ∞ is no such decomposition exists.

Remark 3.1.2. This is a reformulation of subdirect products, as the *associated (central) embedding*

$$\mu_D \colon G \rightarrowtail G/D \coloneqq \prod_{N \in D} G/N, \qquad gK \mapsto (gN)_{N \in D}$$

makes G a subdirect product of elements of $\mathcal{G}$. In particular, G is residually-$\mathcal{G}$.

In the following sections, we will consider the following class.

Definition 3.1.3 ($\mathcal{C}$)**.** Let $\mathcal{C}$ denote the class of groups with cyclic centre.

Lemma 3.1.4. *$d(\mathrm{Z}(G)) \leq m_{\mathcal{C}}(G)$ for every group G. In case of equality, $|\mathrm{Z}(G/D) : \mu_D(\mathrm{Z}(G))|$ is finite.*

Proof. If there is no $\mathcal{C}$-decomposition, then the inequality is true by convention. Otherwise suppose D is a $\mathcal{C}$-decomposition in G. Then $\mu_D(\mathrm{Z}(G)) \subseteq \mathrm{Z}(G/D) = \prod_{N\in D} \mathrm{Z}(G/N)$, so $d(\mathrm{Z}(G)) \leq \sum_{N\in D} d(\mathrm{Z}(G/N)) \leq |D|$ since G/N has cyclic centre by assumption and using Lemma 2.2.10.

In case of equality, the free ranks of $\mathrm{Z}(G)$ and $\mathrm{Z}(G/D)$ coincide, so the quotient is finite. □

Remark 3.1.5. Inequality from Lemma 3.1.4 is sharp for nilpotent groups as we will see in Proposition 3.1.10 and Proposition 3.1.25.

3.1.1 Finite Case

In this section, we determine $m_{\mathcal{C}}(G)$ for every finite nilpotent group G. The idea for the existence of $\mathcal{C}$-decompositions is to write the centre of the group as a direct product of cyclic groups, and recursively consider the quotient groups by all of these cyclic groups. To find the smallest such decomposition, we consider the intersection of the members of the decomposition with the centre of the original group to reduce the problem to the abelian case.

Lemma 3.1.6. *Every finite group G has a $\mathcal{C}$-decomposition.*

Proof. Let $l(G)$ be the maximal length of a strictly increasing subgroup series consisting of normal subgroups of G. Note that $l(G/N) < l(G)$ for any non-trivial normal subgroup N of G as a series $K_0/N < K_1/N < \cdots < K_n/N$ of normal subgroups of G/N induces $1 < N < K_1 < \cdots < K_n$ in G. Note that the centre of G must be finitely generated and write $\mathrm{Z}(G) = \prod_{i=1}^{d} C_i$ where C_i are non-trivial cyclic groups. If $d \leq 1$ (for example when $l(G) = 0$), then $D = \{1\}$ is a $\mathcal{C}$-decomposition in G. Otherwise by induction of $l(G)$, there are $\mathcal{C}$-decompositions D_i in G/C_i. Lift D_i to a set of normal subgroups $\bar{D}_i$ of G containing N. i.e. $D_i = \{K/C_i : K \in \bar{D}_i\}$. We claim that $D \coloneqq \bigcup_{i=1}^{d} \bar{D}_i$ is a $\mathcal{C}$-decomposition in G. Indeed, it is a finite set of normal subgroups of G. For every $K \in D$, we have $K/C_i \in D_i$ for some i, hence $G/K \cong (G/C_i)/(K/C_i) \in \mathcal{C}$ as D_i is a $\mathcal{C}$-decomposition in G/C_i. Finally, note that $\bigcap D = \bigcap_{i=1}^{d} \bigcap \bar{D}_i = \bigcap_{i=1}^{d} C_i = 1$. □

Lemma 3.1.7. *Let A be an additive finite abelian p-group, X be an a trivially intersecting set of subgroups of A. Then there exists $Y \subseteq X$ with $|Y| \leq d(A)$ and $\bigcap Y = 0$.*

Proof. We prove this by induction on $d(A)$. If $d(A) = 0$, then A is trivial, and $Y = \emptyset$ works by convention. Else assume that $d(A) > 0$. For any subgroup $K \leq A$, define $V(K) = \{g \in K : g^p = 1\}$. Note that this is an $\mathbb{F}_p$-vector space of dimension $d(K)$. Assume by contradiction that $V(K) = V(A)$ for all $K \in X$. Then $V(A) \subseteq K$, hence $V(A) \subseteq \bigcap X = 0$, but this contradicts that $V(A)$ has positive dimension. So we may pick $B \in X$ so that $d(B) < d(A)$. Now $X_B \coloneqq \{B \cap K : K \in X\}$ is a trivially intersecting set of subgroups of B, so by induction, there is $Y_B \subseteq X_B$ of size at most $d(B)$ with trivial intersection. Lift back Y_B to $Z \subseteq X$. Then $|Z| = |Y_B|$ and $Y_B = \{B \cap K : K \in Z\}$. We show that $Y \coloneqq \{B\} \cup Z \subseteq X$ satisfies the claim. Indeed, $|Y| \leq 1 + |Y_B| \leq 1 + \delta(B) \leq \delta(A)$ by construction, and $\bigcap Y = B \cap \bigcap Z = \bigcap_{K\in Z}(B \cap K) = \bigcap Y_B = 0$. □

The next statement is motivated by an idea of Endre Szabó.

Lemma 3.1.8. *$m_{\mathcal{C}}(P) = d(\mathrm{Z}(P))$ for any finite p-group P.*

Proof. There is a $\mathcal{C}$-decomposition D in P by Lemma 3.1.6 as finite groups clearly satisfy ACCN. We claim the existence of a $\mathcal{C}$-decomposition $S \subseteq D$ of size at most $d(\mathrm{Z}(P))$. This then proves the statement as no smaller $\mathcal{C}$-decomposition may exist by Lemma 3.1.6.

Let $A := \mathrm{Z}(P)$ and consider $X := \{N \cap A : N \in D\}$, a trivially intersecting set of subgroups of the abelian group A. Let $Y \subseteq X$ with $|Y| \leq d(A)$ and $\bigcap Y = 1$ be given by Lemma 3.1.7. Lift Y back to $S \subseteq D$. Then $1 = \bigcap Y = \mathrm{Z}(P) \cap \bigcap S$, so we must have $\bigcap S = 1$ by Lemma 2.2.5. Also by construction, $|S| = |Y| \leq d(A) = d(\mathrm{Z}(P))$, so S is indeed a $\mathcal{C}$-decomposition of P with the stated properties. □

Lemma 3.1.9. *Let G be a finite nilpotent group. Then $m_{\mathcal{C}}(G) = \max\{m_{\mathcal{C}}(P) : P \in \mathrm{Syl}(G)\}$ where $\mathrm{Syl}(G)$ is the set of Sylow subgroups of G.*

Proof. Let D be a $\mathcal{C}$-decomposition in G and $P \in \mathrm{Syl}(G)$. We claim that $D_P := \{N \cap P : N \in D\}$ is a $\mathcal{C}$-decomposition in P. Indeed, P is a normal subgroup of G because G is finite nilpotent, so $N \cap P$ is a normal subgroup of G, hence of P. On the other hand $\bigcap N_P = P \cap \bigcap D = 1$. This shows that $m_{\mathcal{C}}(G) \geq m_{\mathcal{C}}(P)$ for all $P \in \mathrm{Syl}(G)$.

For the other direction, let D_p be a $\mathcal{C}$-decomposition for $G_p \in \mathrm{Syl}_p(G)$ for all prime divisors p of $|G|$. Let $\mathcal{D}$ be a partition of $\bigcup_p D_p$ of size $\max\{|D_p| : p\}$ such that $|S \cap D_p| \leq 1$ for every $S \in \mathcal{D}$ and p. We claim that $D := \{\prod_{N \in S} N : S \in \mathcal{D}\}$ is a $\mathcal{C}$-decomposition in G. Indeed, as above, every $N_p \in D_p$ is normal in G, so every $N \in D$ is also a normal subgroup of G being the product such groups in a finite nilpotent group. On the other hand, $G_p \cap \bigcap D = \bigcap\{G_p \cap \prod_{N \in S} N : S \in \mathcal{D}\} = \bigcap\{K : K \in D_p\} = \bigcap D_p = 1$ using the assumption on the partition. This shows that $m_{\mathcal{C}}(G) \leq \max\{m_{\mathcal{C}}(P) : P \in \mathrm{Syl}(G)\}$. □

Proposition 3.1.10 (D.R.Sz.). *$m_{\mathcal{C}}(G) = d(\mathrm{Z}(G))$ for any finite nilpotent group G. In particular, G is a subdirect product of $d(\mathrm{Z}(G))$ groups each having cyclic centre.*

Proof. By Lemma 3.1.9 and Lemma 3.1.8,

$$\begin{aligned} m_{\mathcal{C}}(G) &= \max\{m_{\mathcal{C}}(P) : P \in \mathrm{Syl}(G)\} = \max\{d(\mathrm{Z}(P)) : P \in \mathrm{Syl}(G)\} \\ &= \max\{d(Q) : Q \in \mathrm{Syl}(\mathrm{Z}(G))\} = d(\mathrm{Z}(G)). \end{aligned}$$ □

3.1.2 Finitely Generated Case

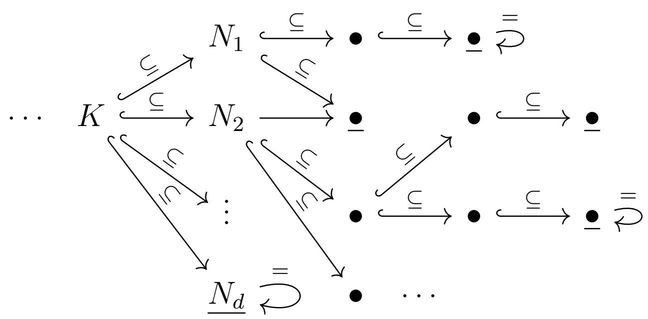
The flavour of the finitely generated case is similar to the finite one from Subsection 3.1.1, and we obtain an analogous result in 3 steps. In the first one about the existence of decomposition, the termination of the recursive process guaranteeing the ascending chain condition (instead of the order of the group), so we use Noetherian induction instead. Here we work with cyclic groups of infinite or of prime power order. In the second step, we show that such irredundant decompositions are minimal. In the final step, we merge these primary components to obtain a decomposition consisting of group of arbitrary cyclic centre.

Step 1 – Existence of Primary Decompositions

Recall Definition 2.2.7 and its basic properties. We show the existence of decompositions using elementary graph theory.

Lemma 3.1.11 (Existence of $\mathrm{Z}(\mathcal{G})$-decompositions)**.** *Let $\mathcal{G}$ be a class closed under taking isomorphisms containing the trivial group such that every finitely generated abelian group has a $\mathcal{G}$-decomposition. Then every group G satisfying ACCN has a $\mathrm{Z}(\mathcal{G})$-decomposition where $\mathrm{Z}(\mathcal{G})$ consists of the groups the centres of which belong to $\mathcal{G}$.*

Proof. We define a directed graph with vertex set $V := \{K : K \lhd G\}$. For every $K \in V$, $\mathrm{Z}(G/K)$ is finitely generated by Lemma 2.2.9 and Lemma 2.2.8. Thus by assumption, we may fix a $\mathcal{G}$-decomposition D_K of $\mathrm{Z}(G/K)$. Every element of D_K is of the form N/K for some $K \subseteq N \in V$ using the third isomorphism theorem and the fact that subgroups of the centre are normal. Thus there is a set $E_K \subseteq V$ of normal subgroups containing K with $|E_K| = |D_K|$ and $D_K = \{N/K : N \in E_K\}$. We then define the edge set of the directed graph to be $E := \{(K,N) \in V^2 : K \in V, N \in E_K\}$, i.e. E_K is the set of neighbours of K. Let $\Gamma_K := E_K \setminus \{K\}$ (these are the vertices named N_i from (3.1)). By definition, E_K consists of some normal subgroups of G that contain K. Note that other choices of $\mathcal{G}$-decompositions may give different edge sets.



(3.1)

For $(K,N) \in V^2$ and $n \in \mathbb{N}_0$, write $P_n(K,N)$ for the set of paths from K to N of lengths n, i.e. for the set of sequences $(K_0,\dots,K_n) \in V^{n+1}$ such that elements of this sequence are pairwise different, $K_0 = K, K_n = N$ and $(K_i, K_{i+1}) \in E$ for every $0 \le i < n$. Note that each path corresponds to a sequence of strictly increasing chain of normal subgroups of G. Let $B_K(r) := \{N \in V : (\exists n \le r)(P_n(K,N) \neq \emptyset)\}$, the r-neighbourhood of K. $K \in B_K(r)$ for any $r \in \mathbb{N}_0$.

Let $B_K := \bigcup_{r\in\mathbb{N}_0} B_K(r)$, the set of vertices reachable from $K \in V$ (that form (3.1)). Note that this is a set of subgroups of G containing K. We claim that B_K is a finite set. By contradiction, assume it is infinite. We define a sequence of vertices $(K_i)_{i\in\mathbb{N}_0} \in V^{\mathbb{N}_0}$ recursively such that $K_{i+1} \in \Gamma_{K_i}$ and B_{K_i} is infinite. By assumption, we may choose $K_0 := K$. Let $i > 0$. Γ_{K_i} is finite by definition and $B_{K_i} = \{K_i\} \cup \bigcup_{N\in\Gamma_{K_i}} B_N$ is infinite by induction. Hence B_N is infinite for some $N \in \Gamma_{K_i}$ and we set $K_{i+1} := N$. By definition $(K_i, K_{i+1}) \in E$ implies that $K_i \le K_{i+1}$. Thus the sequence constructed is a infinite chain of strictly increasing normal subgroups of G which contradicts ACCN, so our claim is true.

Define $\tilde{D}_K := \{N \in B_K : G/N \in \mathrm{Z}(\mathcal{G})\} \subseteq V$. We claim that every maximal element with respect to containment of B_K belongs to $\tilde{D}_K$. (These are the underlined vertices from (3.1).) Indeed, let $M \in B_K$ be a maximal element. Then there is a path $(K_0,\dots,K_n) \in P_n(K,M)$ for some $n \in \mathbb{N}_0$. Suppose that $N \in E_M \setminus \{M\}$. Then $K_i \lneq M \lneq N$ for any $0 \le i \le n$, thus $N \neq K_i$, so $(K_0,\dots,K_n,N) \in P_{n+1}(K,N)$, hence $N \in B_K$ contradicting the maximality of M. Thus $E_M \subseteq \{M\}$ as every element of E_M contains M. If $E_M = \emptyset$, then $D_M = \emptyset$, which means that $\mathrm{Z}(G/M) = \bigcap D_M = M/M \cong 1 \in \mathcal{G}$ by assumption. Otherwise $E_M = \{M\}$, so $D_M = \{M/M\}$ so by definition

$\mathrm{Z}(G/M) \cong \mathrm{Z}(G/M)/(M/M) \in \mathcal{G}$. Thus we have $G/M \in \mathrm{Z}(\mathcal{G})$ in either case, hence $M \in \tilde{D}_K$ as claimed.

We claim that $\bigcap \tilde{D}_K = K$. We prove this by induction on $r(K) := \min\{r \in \mathbb{N}_0 : \tilde{D}_K \subseteq B_K(r)\}$. Note we take the minimum of a non-empty set using the first claim, i.e. $r(K) \in \mathbb{N}_0$.

Assume $r(K) = 0$. Then $\tilde{D}_K \subseteq B_K(0) = \{K\}$ by definition. $K \in B_K$, thus B_K is finite and non-empty by the first claim, hence it has a maximal element which consequently belongs to $\tilde{D}_K$. In particular, $\tilde{D}_K$ is not empty, so we must have $\tilde{D}_K = \{K\}$ in which the claim is evident.

Assume $r(K) > 0$. We separate two cases. First assume $K \in \tilde{D}_K$ (which is never the case if E_K is minimal). Then $K = \bigcap B_K \subseteq \bigcap \tilde{D}_K \subseteq K$ as $\tilde{D}_K \subseteq B_K$, so we are done in this case. In the other case, we have $K \notin \tilde{D}_K$. This means that $\tilde{D}_K = \bigcup_{N \in \Gamma_K} \tilde{D}_N$ since $B_K = \{K\} \cup \bigcup_{N \in \Gamma_K} B_N$. On the other hand, $r(N) < r(K)$ for every $N \in \Gamma_K$ as we can prepend K to every path starting at N, so $\bigcap \tilde{D}_N = N$ by induction. Note that the assumption means that $\mathrm{Z}(G/K)/(K/K) \cong \mathrm{Z}(G/K) \notin \mathcal{G}$, so $K/K \notin D_K$ as it is a $\mathcal{G}$-decomposition of $\mathrm{Z}(G/K)$. Then $K \notin E_K$ by definition, so actually $\Gamma_K = E_K$. Finally, we have $\bigcap D_K = K/K$ by definition, which means that $\bigcap E_K = K$. Now putting the observations together yields

$$\bigcap \tilde{D}_K = \bigcap_{N \in \Gamma_K} \left(\bigcap \tilde{D}_N \right) = \bigcap_{N \in \Gamma_K} N = \bigcap \Gamma_K = \bigcap E_K = K$$

as stated.

Let $D := \tilde{D}_1$. Then $D \subseteq B_1$, so it is finite by the first claim. $G/N \in \mathrm{Z}(\mathcal{G})$ for every $N \in D$ by definition. The final claim shows that $\bigcap D = 1$. Thus the set D constructed is a $\mathrm{Z}(\mathcal{G})$ decomposition in G. □

Remark 3.1.12. The set $\tilde{D}_K$ introduced in the proof is actually a $Z(\mathcal{G})$-decomposition of K in G for every $K \in V$, cf. Definition 3.1.22.

Definition 3.1.13. For $p \in \mathbb{P}_0$, let $\mathcal{C}_p$ be the class of groups G for which there is $n \in \mathbb{N}_+$ such $\mathrm{Z}(G) \cong \mathbb{Z}/p^n\mathbb{Z}$. Write $\mathcal{C}_{\mathbb{P}_0} := \bigcup_{p \in \mathbb{P}_0} \mathcal{C}_p \subset \mathcal{C}$.

Lemma 3.1.14. *Every finitely generated nilpotent group G has a $\mathcal{C}_{\mathbb{P}_0}$-decomposition (that is also a $\mathcal{C}$-decomposition).*

Proof. Finitely generated nilpotent groups satisfy ACCN by Lemma 2.2.11. Let $\mathrm{Z}(\mathcal{G}) := \mathcal{C}_{\mathbb{P}_0} \cup \{1\}$. If A is a finitely generated abelian group, then taking its primary decomposition $\prod_{i \in I} C_i$ gives a $\mathcal{G}$-decomposition $\{\prod_{j \in I \setminus \{i\}} C_i : i \in I\}$. Thus Lemma 3.1.11 implies that G has a $\mathrm{Z}(\mathcal{G})$-decomposition $\bar{D}$. By Lemma 2.2.5, $\bar{D}_1 := \{N \in \bar{D} : \mathrm{Z}(G/N) \cong 1\} = \bar{D} \cap \{G\}$. Let $D := \bar{D} \setminus \{G\}$. Then $\bigcap D = 1$, so D is a $\mathcal{C}_{\mathbb{P}_0}$-decomposition in G. □

Step 2 – Minimal Primary Decompositions

Definition 3.1.15. Let A be a finitely generated abelian group. For a prime $p \in \mathbb{P}$, let $\delta_p(A) := d(A_p)$ where A_p is the set of elements of A whose order is a power of p. Let $\delta_0(A) := d(A / \bigoplus_{p \in \mathbb{P}} A_p)$, the free rank of A. Define the associated primes as $\mathrm{Ass}(A) := \{p \in \mathbb{P}_0 : \exists \mathbb{Z}/p\mathbb{Z} \rightarrowtail A\}$, a finite set. Let $\delta(A) := \sum_{p \in \mathrm{Ass}(A)} \delta_p(A)$.

Remark 3.1.16. $d(A) = \delta_0(A) + \max\{\delta_p(A) : p \in \mathrm{Ass}(A) \setminus \{0\}\} \leq \delta(M)$ using the fundamental theorem of finitely generated abelian groups, the Chinese remainder theorem and that δ_p is additive on direct products.

Lemma 3.1.7 generalises as follows.

Lemma 3.1.17. *Let A be a finitely generated abelian group, X be a finite subset of subgroups of A such that $\bigcap X = 0$. Then there exists $Y \subseteq X$ with $|Y| \leq \delta(A)$ such that $\bigcap Y = 0$.*

Proof. We prove the statement by induction on $\delta_0(A)$. Assume first that $\delta_0(A) = 0$. Let $P \coloneqq \{p \in \mathbb{P} : \delta_p(A) > 0\}$, a finite subset. Then $A = \prod_{p \in P} A_p$. For any $p \in P$, considering $X_p \coloneqq \{B \cap A_p : B \in X\}$ in A_p, there is $Y_p \subseteq X$ such that $A_p \cap \bigcap Y_p = 0$ and $|Y_p| \leq \delta_p(A)$ by Lemma 3.1.7. Let $Y \coloneqq \bigcup_{p \in P} Y_p \subseteq X$. By construction, $|Y| \leq \delta(A)$ and there cannot be an element of order p in $\bigcap Y$ for any $p \in P$. Thus $\bigcap Y = 0$, so Y is as required.

Assume $\delta_0(A) > 0$. The natural map $A \to \prod_{B \in X} A/B$ map is injective because its kernel is $\bigcap X = 0$. Then $0 < \delta_0(A) \leq \delta_0(\prod_{B \in X} A/B) = \sum_{B \in X} \delta_0(A/B)$, so there exists $C \in X$ with $0 < \delta_0(A/C)$. Now $\delta_0(C) = \delta_0(A) - \delta_0(A/C) < \delta_0(A)$. Considering $\{C \cap B : B \in X\}$, induction gives a subset $Y_C \subseteq X$ such that $|Y_C| \leq \delta(C)$ and $C \cap \bigcap Y = 0$. Then $Y \coloneqq C \cup Y_C$ satisfies the statement. □

Remark 3.1.18. The bound of the statement is sharp: for any finite subset $P \subseteq \mathbb{P}_0$, let $A \coloneqq \prod_{p \in P} \mathbb{Z}/p\mathbb{Z}$ and $X \coloneqq \{\ker(A \to \mathbb{Z}/p\mathbb{Z}) : p \in P\}$. Then $\bigcap X = 0$ and $\delta(A) = |X|$, but any proper subset $Y \subsetneq X$ has $\bigcap Y \neq 0$.

The finiteness condition on X is necessary: let $d \in \mathbb{Z} \setminus \{0\}$ and consider $A = \mathbb{Z}$ and $X = \{d^n\mathbb{Z} : n \in \mathbb{N}_0\}$. Then $\bigcap X = 0$, but any finite subset $Y \subset X$ has $\bigcap Y \neq 0$.

Definition 3.1.19. For a $\mathcal{C}_{\mathbb{P}_0}$-decomposition D in G and $p \in \mathbb{P}_0$, write $D_p \coloneqq \{N \in D : G/N \in \mathcal{C}_p\}$, and define the support $\operatorname{Supp}(D) \coloneqq \{p \in \mathbb{P}_0 : D_p \neq \emptyset\}$, a finite set.

Lemma 3.1.20. *Every $\mathcal{C}_{\mathbb{P}_0}$-decomposition D in a finitely generated nilpotent group G satisfies $|D_p| \geq \delta_p(\mathrm{Z}(G))$ for every $p \in \mathbb{P}_0$, $|D| \geq \delta(\mathrm{Z}(G))$ and* $\operatorname{Supp}(D) \supseteq \operatorname{Ass}(\mathrm{Z}(G))$. *Furthermore, if D is irredundant (i.e. when no proper subset is a $\mathcal{C}_{\mathbb{P}_0}$-decomposition), then we have equality everywhere.*

Proof. For the second part, assume that D is an arbitrary $\mathcal{C}_{\mathbb{P}_0}$-decomposition in G. By the definition, $\mathrm{Z}(G/N) \cong \mathbb{Z}/p_N^{n_N}\mathbb{Z}$ for some $p_N \in \mathbb{P}_0$ and $n_N \in \mathbb{N}_+$. Then restricting the associated emdedding μ_D to the centre induces $\mathrm{Z}(G) \rightarrowtail \prod_{N \in D} \mathbb{Z}/p_N^{n_N}\mathbb{Z}$. First, taking the associated primes gives

$$\operatorname{Ass}(\mathrm{Z}(G)) \subseteq \operatorname{Ass}(\prod_{N \in D} \mathbb{Z}/p_N^{n_N}\mathbb{Z}) = \bigcup_{N \in D} \operatorname{Ass}(\mathbb{Z}/p_N^{n_N}\mathbb{Z}) = \bigcup_{N \in D} \{p_N\} = \operatorname{Supp}(D).$$

Second,

$$\delta_p(\mathrm{Z}(G)) \leq \delta_p(\prod_{N \in D} \mathbb{Z}/p_N^{n_N}\mathbb{Z}) = |\{N \in D : p_N = p\}| = |D_p|$$

for any $p \in \mathbb{P}_0$, and summing these inequalities produces

$$\delta(\mathrm{Z}(G)) = \sum_{p \in \operatorname{Ass}(\mathrm{Z}(G))} \delta_p(\mathrm{Z}(G)) \leq \sum_{p \in \operatorname{Ass}(\mathrm{Z}(G))} |D_p| \leq \sum_{p \in \operatorname{Supp}(D)} |D_p| = |D|$$

as stated.

For the second part, suppose that D is irredundant. We claim that $|D| \leq \delta(\mathrm{Z}(G))$. Indeed, note that $X \coloneqq \{N \cap \mathrm{Z}(G) : N \in D\}$ is a finite subset of $\mathbb{Z}$-submodules of

$A \coloneqq \mathrm{Z}(G)$ intersecting trivially. Then Lemma 3.1.17 gives $Y \subseteq X$ whose intersection is still trivial and $|Y| \leq \delta(\mathrm{Z}(G))$. Lift each element of Y back to obtain $D' \subseteq D$ with $\{N \cap \mathrm{Z}(G) : N \in D'\} = Y$ and $|D'| = |Y|$. Then $\bigcap D'$ is a normal subgroup of the nilpotent group G that intersects $\mathrm{Z}(G)$ trivially by construction. Then $\bigcap D' = 1$ by Lemma 2.2.5. So $D' \subseteq D$ is also a $\mathcal{C}_{\mathbb{P}_0}$-decomposition, thus $D' = D$ by irredundancy. Hence $|D| = |D'| = |Y| \leq \delta(\mathrm{Z}(G))$ as claimed. Comparing this with the inequality $\delta(\mathrm{Z}(G)) \leq |D|$ from above, we conclude that we must have equality everywhere above. So $|D| = \delta(\mathrm{Z}(G))$, $|D_p| = \delta_p(\mathrm{Z}(G))$ for $p \in \mathrm{Supp}(D) = \mathrm{Ass}(\mathrm{Z}(G))$. Finally, $0 \leq \delta_p(\mathrm{Z}(G)) \leq |D_p| = 0$ for $p \in \mathbb{P}_0 \setminus \mathrm{Supp}(D)$, hence $\delta_p(\mathrm{Z}(G)) = 0 = |D_p|$ in these cases as well. □

Question 3.1.21. Note the analogy of the statement with Noether–Lasker where normal subgroup of D_p play the role of the (p)-primary submodules appearing the decomposition. The associated primes (and their multiplicities) are unique for both statements. While the ideals appearing in Neothe–Lasker are generally not unique, the ones corresponding to isolated primes are. It would be interesting to see if there is such a uniqueness statement for (a subset of) D or $\{G/N : N \in D\}$.

Step 3 – Merging to Minimal Decompositions

We use a slight extension of Definition 3.1.1.

Definition 3.1.22. A *$\mathcal{G}$-decomposition of a normal subgroup K in G* is a finite set D of normal subgroups of G such that $G/N \in \mathcal{G}$ for every $N \in D$ and $\bigcap D = K$. The *associated embeddig* is $\mu_D : G/K \rightarrowtail G(D) \coloneqq \prod_{N \in D} G/N, gK \mapsto (gN)_{N \in D}$.

Remark 3.1.23. There is a one-to-one correspondence between $\mathcal{G}$-decompositions of K in G and $\mathcal{G}$-decompositions in G/K. If D_i is a $\mathcal{G}$-decomposition of K_i in G for a finite index set $i \in I$, then $\bigcup_{i \in I} D_i$ is a $\mathcal{G}$-decomposition of $\bigcap_{i \in I} K_i$ in G. If S is a subset of a $\mathcal{G}$-decomposition D of K in G, then S itself is a $\mathcal{G}$-decomposition of $\bigcap S \supseteq K$ in G.

We use the following statement to reduce the size of decompositions by merging $\mathcal{C}_{\mathbb{P}_0}$-decompositions into $\mathcal{C}$-decompositions.

Lemma 3.1.24 (Merging)**.** *Let G be a finitely generated nilpotent group. If D is a $\mathcal{C}_{\mathbb{P}_0}$-decomposition of a normal subgroup K in G such that $D_0 = \emptyset$, then G/K is a finite group. If furthermore $\mathcal{P}$ is a partition of D such that $|S \cap D_p| \leq 1$ for every $S \in \mathcal{P}$ and $p \in \mathrm{Supp}(D)$, then $\mathcal{P}_\cap \coloneqq \{\bigcap S : S \in \mathcal{P}\}$ is a $\mathcal{C}$-decomposition of K in G.*

Proof. Pick $N \in D$. By assumption, there is an isomorphism $\mathrm{Z}(G/N) \cong \mathbb{Z}/p_N^{n_N}\mathbb{Z}$ for some $p_N \in \mathbb{P}$ and $n_N \in \mathbb{N}_+$. Then G/N itself is a finite p_N-group by Lemma 2.2.6. Then the associated embedding $\mu_D : G/K \rightarrowtail \prod_{N \in D} G/N$ embeds G/K into a finite direct product of finite groups, hence G/K is finite as stated.

The assumption on $S \in \mathcal{P}$ is equivalent to p_N's being pairwise different for $N \in S$. Thus by above, the $|G/N|$'s are pairwise coprime for $N \in S$, and the natural map $G/K \to G/N$ is surjective, so by the Chinese remainder theorem, the induced embedding $\mu_S : G/\bigcap S \to \prod_{N \in S} G/N$ is surjective, hence is an isomorphism. Since the P_N's are pairwise coprime, the Chinese remainder theorem implies that $\mathrm{Z}(G/\bigcap S) \cong \prod_{N \in S} \mathrm{Z}(G/N) \cong \prod_{N \in S} \mathbb{Z}/p_N^{n_N}\mathbb{Z} \cong \mathbb{Z}/\prod_{N \in S} p_N^{n_N}\mathbb{Z}$ is a cyclic group. Hence $G/\bigcap S \in \mathcal{C}$. On the other hand, $|\mathcal{P}_\cap| = |\mathcal{P}| \leq |D|$ so $\mathcal{P}_\cap$ is finite, and $\bigcap \mathcal{P}_\cap = \bigcap D = K$ by construction, so $\mathcal{P}_\cap$ is indeed a $\mathcal{C}$-decomposition of K in G as claimed. □

Now we can prove the generalisation of Proposition 3.1.10 in the finitely generated case.

Proposition 3.1.25 (D.R.Sz.). *For any finitely generated nilpotent group G, we have $m_{\mathcal{C}}(G) = d(\mathrm{Z}(G))$.*

Proof. Let D be an $\mathcal{C}_{\mathbb{P}_0}$-decomposition in G given by Lemma 3.1.20. If there is $N \in D$ such that $\bigcap(D \setminus \{N\}) = 1$, then replace D by $D \setminus \{N\}$ and iterate. Since D is finite, this process terminates and produces an irredundant D.

Then we may find a partition $\mathcal{P}$ of $D^+ := D \setminus D_0$ consisting of $|\mathcal{P}| = \max\{|D_p| : p \in \mathrm{Supp}(D^+)\}$ parts such that $|S \cap D_p| \leq 1$ for every $S \in \mathcal{P}$ and $p \in \mathrm{Supp}(D^+)$. Then by Lemma 3.1.24, $\mathcal{P}_\cap$ is a $\mathcal{C}$-decomposition of $\bigcap D^+$ in G. Hence $\bar{D} := D_0 \cup \mathcal{P}_\cap$ is a $\mathcal{C}$-decomposition in G. Since D is an irredundant $\mathcal{C}_{\mathbb{P}_0}$-decomposition, Lemma 3.1.20 and Remark 3.1.16 gives

$$\begin{aligned}|\bar{D}| &= |D_0| + \max\{|D_p| : p \in \mathrm{Supp}(D^+)\}\\ &= \delta_0(\mathrm{Z}(G)) + \max\{\delta_p(\mathrm{Z}(G)) : p \in \mathrm{Ass}(\mathrm{Z}(G)) \setminus \{0\}\} = d(\mathrm{Z}(G)).\end{aligned}$$

Finally Lemma 3.1.6 shows that $\bar{D}$ is a minimal $\mathcal{C}$-decomposition in G. □

Proposition 3.1.25 can finally be translated to one about subdirect products.

Corollary 3.1.26 (D.R.Sz.). *Every $\leq d$-generated $\leq c$-step nilpotent group G is a subdirect product of $d(\mathrm{Z}(G))$ many suitable $\leq d$-generated $\leq c$-step nilpotent groups each having cyclic centre. No such embedding exists using fewer factors and in case of equality, $\mathrm{Z}(G)$ is a finite index subgroup of the centre of the direct product.*

Proof. The associated embedding from Remark 3.1.2 corresponding to a minimal $\mathcal{C}$-decomposition in G given by Proposition 3.1.25 gives an embedding with the required properties using the fact that the class of $\leq d$-generated $\leq c$-step nilpotent groups is closed under taking quotients and direct products. The last part follows from Lemma 3.1.4. □

3.2 Central Product Decomposition

In this section, we show that every finitely generated ≤ 2-step nilpotent group G with cyclic derived subgroup G' is the central product of some of its ≤ 2-generated subgroups and study some uniqueness properties of this decomposition (Theorem 3.2.23). The key idea is to consider the $\mathbb{Z}$-module G/G' which is endowed with an alternating $\mathbb{Z}$-bilinear form $(G/G')^2 \to G'$ induced by the commutator map on G. We introduce the notion of alternating modules to study this bilinear map in a slightly more abstract setting via the analogy with symplectic vector spaces. Pulling back these linear algebraic results to G gives the central product decomposition of the group G.

3.2.1 Alternating modules and Darboux decompositions

In this section, we introduce alternating modules which generalise the notion of symplectic vector spaces. We show that, under some conditions, they possess an analogue of the Darboux basis giving rise to the so-called Darboux decomposition, i.e. the existence of an isotropic submodule and 2-generated pairwise orthogonal submodules that generate the whole module. Using this decomposition, we show that non-degenerate modules can be put into a standard form.

A key notion is the following analogue of symplectic vector spaces.

Definition 3.2.1. We call (M, ω, C) an *alternating* R*- module*, if M and C are R-modules and $\omega\colon M \times M \to C$ is an alternating (i.e. $\omega(m,m) = 0$ for every $m \in M$) R-bilinear map. A morphism $(M, \omega, C) \to (\bar{M}, \bar{\omega}, \bar{C})$ is a pair (λ, κ) of R-module morphisms such that $\lambda\colon M \to \bar{M}$, $\kappa\colon C \to \bar{C}$ and $\kappa \circ \omega = \bar{\omega} \circ (\lambda \times \lambda)$ as $M \times M \to \bar{C}$ maps.

$$\begin{array}{ccc} M \times M & \xrightarrow{\ \omega\ } & C \\ \big\downarrow{\scriptstyle \lambda\times\lambda} & & \big\downarrow{\scriptstyle \kappa} \\ \bar{M} \times \bar{M} & \xrightarrow{\ \bar{\omega}\ } & \bar{C} \end{array}$$

For submodules $N, N_1, N_2 \leq M$, introduce the following terminology. Let $\omega(N_1, N_2) \leq C$ be the submodule generated by $\{\omega(n_1, n_2) : n_i \in N_i\}$. Say N_1 and N_2 are *orthogonal with respect to* ω, written $N_1 \perp N_2$, if $\omega(N_1, N_2) = 0$. Call N *isotropic* if $N \perp N$. The *orthogonal complement* of N is $N^{\perp} := \{m \in M : \omega(m, N) = 0\}$. Call ω and (M, ω, C) *non-degenerate* if $M^{\perp} = 0$.

Remark 3.2.2. The read is advised not to be confused about the word 'complement' as $N \cap N^{\perp}$ may fail to be trivial. Every alternating module (M, ω, C) canonically induces a non-degenerate one $(M/M^{\perp}, \omega/M^{\perp}, C)$ by $\omega/M^{\perp} : (m + M^{\perp}, m' + M^{\perp}) \mapsto \omega(m, m')$.

Example 3.2.3. Let C be a module over a commutative ring R and $C_1, \ldots, C_t$ be cyclic submodules. Pick generators c_i of C_i and write $c_i^{(j)}$ for the image of c_i under the natural composition $C_i \to C_i^2 \rightarrowtail \bigoplus_{i=1}^{t} C_i^2$ where the first map injects to the jth factor. Then

$$(\bigoplus_{i=1}^{t} C_i^2, \mathrm{trdet}, C) \quad \text{where} \quad \mathrm{trdet}\colon ((r_i c_i, s_i c_i)_i, (r_i' c_i, s_i' c_i)_i) \mapsto \sum_{i=1}^{t} (r_i s_i' - r_i' s_i) c_i$$

is a non-degenerate alternating R-module. Note that while the alternating R-module constructed above depends on the choice of the generators, its isomorphism class does not as if $\overline{c_i}$ are other generators, then $(\lambda\colon (r_i c_i, s_i c_i)_i \mapsto (r_i \overline{c_i}, s_i c_i)_i, \mathrm{id}_C)$ is an isomorphism. Indeed, if $c_i = k_i \overline{c_i}$ and $x = (r_i c_i, s_i c_i)_i$, $x' = (r_i' c_i, s_i' c_i)_i$ are arbitrary elements of $\bigoplus_{i=1}^{t}$, then

$$\begin{aligned} \overline{\mathrm{trdet}}(\lambda(x), \lambda(x')) &= \overline{\mathrm{trdet}}((r_i \overline{c_i}, s_i k_i \overline{c_i})_i, (r_i' \overline{c_i}, s_i' k_i \overline{c_i})_i) \\ &= \sum_{i=1}^{t} (r_i s_i' k_i - r_i' s_i k_i) \overline{c_i} = \sum_{i=1}^{t} (r_i s_i' - r_i' s_i) c_i = \mathrm{trdet}(x, x'), \end{aligned}$$

showing that the pair (λ, id_R) is an isomorphism.

Definition 3.2.4. The ordered list of submodules $O; N_1, \ldots, N_t$ in a finitely generated (M, ω, C) is called a *Darboux decomposition* if they are pairwise orthogonal, N_i are 2-generated, $O \leq M^{\perp}$, $d(M) = 2t + d(O)$, and $\omega(M, M) = \omega(N_1, N_1) \geq \omega(N_2, N_2) \geq \cdots \geq \omega(N_t, N_t) \neq 0$.

Example 3.2.5. In the alternating module $(\bigoplus_{i=1}^{t} C_i^2, \mathrm{trdet}, C)$ from Example 3.2.3, the submodules $0; C_1^2, \ldots, C_t^2$ form a Darboux decomposition.

Remark 3.2.6. In the setup above, pick generators x, y of N_1. Let $n = rx + sy, n' = r'x + s'y \in N_1$ be arbitrary. Then $\omega(n, n') = (rs' - r's)\omega(x, y)$ using bilinearity and the alternating property. Thus $\omega(M, M) = \omega(N_1, N_1) = R\omega(x_1, y_1)$ is necessarily cyclic.

Definition 3.2.7. We call an alternating R-module (M, ω, C) *Darboux* if M is finitely generated, $\omega(M, M)$ is cyclic, and R is a principal ideal domain.

Remark 3.2.8. In this setup, fix a generator c of $\omega(M, M)$. Then $\omega(M, M)$ is a free module over $Q = R/\operatorname{ann}_R(c)$. Let $\mathcal{B} := \{x_1, \ldots, x_n\}$ be an R-module generating set of M. The matrix $[\omega]_{\mathcal{B}} := [\omega]_{\mathcal{B},c} \in Q^{n\times n}$ of ω with respect to these generating sets is given by $\omega(x_i, x_j) = ([\omega]_{\mathcal{B}})_{i,j}c$. If $\mathcal{C} = \{y_1, \ldots, y_k\}$ is another generating set of M (potentially of different size) given by $y_i = \sum_{j=1}^{n} B_{j,i}x_j$ for some (so called transition matrix) $B \in R^{n\times k}$, then $[\omega]_{\mathcal{C}} = B^{\top}[\omega]_{\mathcal{B}}B$. Compare this with Lemma 2.3.1 about putting alternating matrices to normal form.

Over a principal ideal domain, the existence of a Darboux decomposition implies that the underlying alternating module is Darboux. The converse is also true.

Lemma 3.2.9 (Darboux decomposition – existence)**.** *Every Darboux module has a Darboux decomposition.*

Proof. Let (M, ω, C) be a Darboux module. Pick a minimal generating set $\mathcal{B}$ of M of size n and let c be a fixed generator of $\omega(M, M)$. Note that the matrix $[\omega]_{\mathcal{B}}$ from Remark 3.2.8 is alternating. Lift this matrix to an alternating matrix $W \in R^{n\times n}$. Let $B \in \mathrm{SL}_n(R)$ given by Lemma 2.3.1. Since B is an invertible square matrix, it can be considered as the transition matrix to a new generating set $\mathcal{C} = \{m_1, \ldots, m_n\}$ of M. Then Lemma 2.3.1 and Remark 3.2.8 imply that $[\omega]_{\mathcal{C}}$ is of the form (2.10). Let $N_i := Rm_{2i-1} + Rm_{2i}$ for $1 \le i \le s$, and let $t \in \mathbb{N}_0$ so that N_i is non-isotropic for $1 \le i \le t$ and N_j is isotropic for $t < j \le s$. Define $O := \sum_{i=2t+1}^{n} Rn_i$. By construction, $O \le M^{\perp}$ and $N_i \perp N_j$ for $1 \le i < j \le t$. Moreover, these submodules generate M by construction, so they constitute to an orthogonal decomposition as stated. Note that $\omega(N_i, N_i) = Rd_ic$ by construction so the divisibility condition on the d_i's imply the containment of these submodules. $d(M) \le d(O) + \sum_{i=1}^{t} d(N_i) \le n = d(M)$ by construction, so we must have equality everywhere. Thus $O; N_1, \ldots, N_t$ is a Darboux decomposition. □

Lemma 3.2.10 (Darboux decomposition – uniqueness)**.** *Let $O; N_1, \ldots, N_t$ be a Darboux decomposition in a Darboux module (M, ω, C). Then $t = \frac{1}{2}d(M/M^{\perp})$ and $C_i := \omega(N_i, N_i) \subseteq C$ are invariants, i.e. they do not depend on the choice the decomposition.*

Proof. Fix generators x_i, y_i of N_i (for $i \in \{1, \ldots, t\}$). First, we claim that

$$0 \longrightarrow M^{\perp} \xhookrightarrow{\subseteq} M \xrightarrow{\omega^{\flat}} \bigoplus_{i=1}^{t} \omega(N_i, N_i)^2 \longrightarrow 0$$
$$m \longmapsto (\omega(m, y_i), \omega(x_i, m))_{i=1}^{n}$$

is a short exact sequence of R-modules. Indeed, to see that $\omega^{\flat}$ is well-defined, write $m = \sum_{j=1}^{t} n_j + o$ where $n_j \in N_j$, $o \in O$ and use the orthogonality of the decomposition to conclude that $\omega(m, y_i) = \sum_{j=1}^{t} \omega(n_j, y_i) = \omega(n_i, y_i) \in \omega(N_i, N_i)$ and similarly $\omega(x_i, m) \in \omega(N_i, N_i)$. Moreover, $\omega^{\flat}$ is a morphism as ω is bilinear.
We check the exactness at M. The inclusion $\ker(\omega^{\flat}) \supseteq M^{\perp}$ is clear. To show the other containment, pick $k = \sum_{i=1}^{t} n_i + o \in \ker(\omega^{\flat})$ with $n_i \in N_i$, $o \in O$. Then $\omega(n_i, y_i) = \omega(x_i, n_i) = 0$ by definition. So for an arbitrary $m = \sum_{i=1}^{t}(r_ix_i + s_iy_i) + o' \in M$ with $o' \in O$, we have $\omega(m, k) = \sum_{i=1}^{t} r_i\omega(x_i, n_i) - s_i\omega(n_i, y_i) = 0$, so $k \in M^{\perp}$.
To see that $\omega^{\flat}$ is surjective, pick $w = (u_i, v_i)_{i=1}^{t} \in \bigoplus_{i=1}^{t} \omega(N_i, N_i)^2$. Since $\omega(N_i, N_i) = R\omega(x_i, y_i)$ using the bilinearity and the alternating property, we can pick $r_i, s_i \in R$ with

$u_i = r_i\omega(x_i, y_i)$, $v_i = s_i\omega(x_i, y_i)$. Let $m \coloneqq \sum_{i=1}^{t} r_i x_i + s_i y_i \in M$. Then $\omega^\flat(m) = (\omega(m, y_i), \omega(x_i, m))_{i=1}^n = (r_i\omega(x_i, y_i), s_i\omega(y_i, x_i))_{i=1}^n = w$ by construction and orthogonality. This proves the claim.

Next, by the uniqueness part of the structure theorem of finitely generated modules over a principal ideal domain, $t = \frac{1}{2}d(M/M^\perp)$ and the isomorphism class of $\omega(N_i, N_i)$ are invariants. Fix a generator c of $\omega(M, M)$, and consider $\omega(M, M)$ as a free module over $Q = R/\operatorname{ann}_R(c)$. If $\operatorname{ann}_R(c) \neq 0$, then $\omega(M, M)$ is a cyclic torsion R-module, so its isomorphic submodules are necessarily equal. In particular, $\omega(N_i, N_i)$ are invariant in this case. Otherwise, if $\operatorname{ann}_R(c) = 0$, pick a minimal generating set of O. This together with x_i, y_i form a minimal generating set $\mathcal{B}$ of M by Definition 3.2.4. Then $[\omega]_\mathcal{B}$ from Remark 3.2.8 is actually a matrix with entries from R of the form (2.10) by assumption. Then $\omega(N_i, N_i) = Rd_i c$. By Lemma 2.3.1, Rd_i is independent of the choice of $\mathcal{B}$, hence $\omega(N_i, N_i)$ is an invariant in this case too. □

Remark 3.2.11. The invariants are related as follows: $\operatorname{ann}_R(C_i) = \{r \in R : rd_i \in \operatorname{ann}_R(C_1)\}$ where d_i are from Lemma 2.3.1 for any lift W of the matrix $[\omega]$.

Remark 3.2.12. Let (M, ω, C) be a Darboux module. Let $O; N_1, \ldots, N_t$ and $\bar{O}, \bar{N}_1, \ldots, \bar{N}_t$ be Darboux decompositions. Pick generators x_i, y_i of N_i, and $\bar{x}_i, \bar{y}_i$ of $\bar{N}_i$. Then the automorphism φ_i of $C_i = \omega(N_i, N_i) = \omega(\bar{N}_i, \bar{N}_i)$ given by $\varphi_i \colon \omega(x_i, y_i) \mapsto \omega(\bar{x}_i, \bar{y}_i)$ induces the following isomorphism of short exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^\perp & \overset{\subseteq}{\hookrightarrow} & M & \xrightarrow{\omega^\flat} & \bigoplus_{i=1}^t \omega(N_i, N_i)_i^2 & \longrightarrow & 0 \\
 & & \Vert & & \Vert & & \wr\downarrow \bigoplus_{i=1}^t \varphi_i \oplus \varphi_i & & \\
0 & \longrightarrow & M^\perp & \overset{\subseteq}{\hookrightarrow} & M & \xrightarrow{\bar{\omega}^\flat} & \bigoplus_{i=1}^t \omega(\bar{N}_i, \bar{N}_i)^2 & \longrightarrow & 0
\end{array}$$

In short, the isomorphism class is independent of the choice of the Darboux-decomposition and of the choice of the generators. Denote this class by

$$\Delta((M, \omega, C)) \ : \ 0 \longrightarrow M^\perp \overset{\subseteq}{\hookrightarrow} M \xrightarrow{\omega^\flat} \bigoplus_{i=1}^t C_i^2 \longrightarrow 0. \tag{3.2}$$

When referring an element of this class, a choice of an underlying set of generators is meant.

Up to isomorphism, every Darboux decomposition in every non-degenerate Darboux module is given by Example 3.2.3 and Example 3.2.5.

Corollary 3.2.13 (Standard form)**.** *Let (M, ω, C) be a non-degenerate Darboux module, $0; N_1, \ldots, N_t$ be a Darboux decomposition with invariants $C_1, \ldots, C_t$. Then there is an isomorphism*

$$(\omega^\flat, \operatorname{id}_C) \colon (M, \omega, C) \to (\bigoplus_{i=1}^{t} C_i^2, \operatorname{trdet}, C)$$

compatible with the Darboux decomposition, i.e. $\omega^\flat(N_i) = C_i^2$.

Proof. Write $N_i = Rx_i \oplus Ry_i$. Using bilinearity and the alternating property, $c_i \coloneqq \omega(x_i, y_i)$ is a generator of C_i. We claim that $\omega^\flat$ from Remark 3.2.12 with these generators satisfy the statement. Indeed, $M^\perp = 0$ by assumption, so (3.2) shows that $\omega^\flat$ is an isomorphism. So it is enough to check that $\operatorname{trdet} \circ (\omega^\flat \times \omega^\flat) = \omega$. Since $N_i \cap N_j \subseteq M^\perp = 0$

for $i \neq j$ and $O \leq M^{\perp} = 0$, we have $M = \bigoplus_{i=1}^{t} N_i$. Pick $m, m' \in M$ and write $m = \sum_{i=1}^{t} r_i x_i + s_i y_i$, $m' = \sum_{i=1}^{t} r'_i x_i + s'_i y_i$. Then

$$\begin{aligned}
\operatorname{trdet}(\omega^{\flat}(m), \omega^{\flat}(m')) = \operatorname{trdet}(\omega^{\flat}(m), \omega^{\flat}(m') &= \operatorname{trdet}((r_i c_i, s_i c_i)_i, (r'_i c_i, s'_i c_i)_i) \\
&= \sum_{i=1}^{t} (r_i s'_i - r'_i s_i) c_i = \sum_{i=1}^{t} \omega(r_i x_i + s_i y_i, r'_i x_i + s'_i y_i) \\
&= \omega(m, m')
\end{aligned}$$

using the definitions and the fact that the submodules N_i are pairwise orthogonal. □

3.2.2 Decomposition Theorem via Central-by-abelian Extensions

We define functors from $\leq$2-step nilpotent groups to central-by-abelian extensions, and also one from central-by-abelian to alternating $\mathbb{Z}$-modules. By studying the properties of these functors, we can pull back the results of Subsection 3.2.1 about alternating $\mathbb{Z}$-modules to finitely generated groups, and establish the main statement of the section about the central product decomposition of finitely generated $\leq$2-step nilpotent groups (Theorem 3.2.23).

Recall Definition 1.2.1. In this paper, we will mostly consider the following central-by-abelian extensions.

Example 3.2.14. Every $\leq$ 2-step nilpotent group G induces two natural central-by-abelian extensions: $\mathcal{D}(G)$ corresponds to G', and $\mathcal{Z}(G)$ to $\mathrm{Z}(G)$ as the diagram below illustrates. The latter one is non-degenerate. Every central-by-abelian extension ϵ of G sits in between these two extensions.

$$\begin{array}{ccccccccccc}
\mathcal{D}(G) & : & 1 & \longrightarrow & G' & \xhookrightarrow{\subseteq} & G & \xrightarrow{\pi_{\mathcal{D}}} & G/G' & \longrightarrow & 1 \\
\downarrow & & & & \downarrow & & \| & & \downarrow & & \\
\epsilon & : & 1 & \longrightarrow & C & \xrightarrow{\iota} & G & \xrightarrow{\pi} & M & \longrightarrow & 1 \\
\downarrow & & & & \downarrow{\scriptstyle \iota} & & \| & & \downarrow & & \\
\mathcal{Z}(G) & : & 1 & \longrightarrow & \mathrm{Z}(G) & \xhookrightarrow{\subseteq} & G & \xrightarrow{\pi_{\mathcal{Z}}} & G/\mathrm{Z}(G) & \longrightarrow & 1
\end{array}$$

Remark 3.2.15. Recall Definition 2.2.30 and note that Example 3.2.14 actually gives rise to the following functors.

$$\{\leq 2\text{-step nilpotent groups}\} \xrightarrow{\mathcal{D}} \{\text{central-by-abelian extensions}\}$$

$$\left\{ \begin{array}{c} \leq 2\text{-step nilpotent groups with} \\ \text{centre preserving morphisms} \end{array} \right\} \xrightarrow{\mathcal{Z}} \left\{ \begin{array}{c} \text{non-degenerate} \\ \text{central-by-abelian extensions} \end{array} \right\}$$

Note that if $\varphi \colon G \rightarrowtail \prod_{i=1}^{n} G_i$ is a subdirect product, then φ is centre preserving, so it induces a morphism $\mathcal{Z}(G) \to \mathcal{Z}(\prod_{i=1}^{n} G_i) \cong \prod_{i=1}^{n} \mathcal{Z}(G_i)$.

We are mainly interested in the category of alternating $\mathbb{Z}$-modules because of the following functor.

Lemma 3.2.16 (The alternating functor $\mathcal{A}$)**.** *The assignment*

$$\begin{aligned} \{\text{central-by-abelian extensions}\} &\xrightarrow{\mathcal{A}} \{\text{alternating } \mathbb{Z}\text{-modules}\} \\ (\epsilon : 1 \to C \xrightarrow{\iota} G \xrightarrow{\pi} M \to 1) &\longrightarrow (M, \omega, C) \\ (\kappa, \gamma, \lambda) &\longrightarrow (\lambda, \kappa) \end{aligned}$$

is a functor where $\omega \colon M \times M \to C$ *is defined by* $(m_1, m_2) \mapsto \iota^{-1}([g_1, g_2])$ *for arbitrary* $g_i \in \pi^{-1}(m_i)$.

Proof. First we check that ω is well-defined. Pick $g_i, g_i' \in \pi^{-1}(m_i)$. Then $g_i^{-1}g_i' \in \ker(\pi) = \operatorname{Im}(\iota)$, so there are $c_i \in C$ with $\iota(c_i) = g_i^{-1}g_i'$. Then

$$[g_1', g_2'] = [g_1\iota(c_1), g_2\iota(c_2)] = [g_1, g_2][g_1, \iota(c_2)][\iota(c_1), g_2][\iota(c_1), \iota(c_2)] = [g_1, g_2]$$

by Lemma 2.2.3 as $\iota(C) \subseteq Z(G)$. $G' \subseteq \iota(C)$ by definition, so ω is indeed well-defined. $\mathbb{Z}$-bilinearity of ω follows directly from the previously mentioned fact, and the alternating property follows as every group element commutes with itself. This shows that (M, ω, C) is an alternating $\mathbb{Z}$-module.

For the functoriality, consider the morphism $(\kappa, \gamma, \lambda)$ form Definition 2.2.22. Let $(M_i, \omega_i, C_i) \coloneqq \mathcal{A}(\epsilon_i)$. Pick $m_1, m_1' \in M_1$ arbitrarily and choose $g_1 \in \pi_1^{-1}(\pi_1)$, $g_1' \in \pi_1^{-1}(\pi_1)$. Then

$$\begin{aligned} \kappa(\omega_1(m_1, m_1')) &= \kappa(\iota_1^{-1}([g_1, g_1'])) = \iota_2^{-1}(\gamma([g_1, g_1'])) = \iota_2^{-1}([\gamma(g_1), \gamma(g_1')]) \\ &= \omega_2(\lambda(g_1), \lambda(g_1')) \end{aligned}$$

as $\gamma(g_1) \in \pi_2^{-1}(\lambda(g_1))$, $\gamma(g_1') \in \pi_2^{-1}(\lambda(g_1'))$ from the commutativity of the diagram defining the morphism $(\kappa, \gamma, \lambda)$. This means that $\kappa \circ \omega_1 = \omega_2 \circ (\lambda \times \lambda)$, i.e. (λ, κ) is indeed a morphism $(M_1, \omega_1, C_1) \to (M_2, \omega_2, C_2)$.

The rest of the functoriality properties is straightforward to check. □

Remark 3.2.17. Lemma 3.2.16 gives the following dictionary between subgroups H, H_i of G and submodules of M.

- commutator map $\leftrightarrow$ ω: $[g, g'] = \iota \circ \omega(\pi(g), \pi(g'))$, $[H_1, H_2] = \iota \circ \omega(\pi(H_1), \pi(H_2))$
- commutes $\leftrightarrow$ is orthogonal: $[H_1, H_2] = 1 \iff \pi(H_1) \perp \pi(H_2)$
- centraliser $\leftrightarrow$ orthogonal complement: $\pi(C_G(H)) = \pi(H)^{\perp}$
- abelian $\leftrightarrow$ isotropic: $[H, H] = 1 \iff \pi(H) \perp \pi(H)$
- ϵ non-degenerate $\leftrightarrow$ ω non-degenerate: $\pi(\mathrm{Z}(G)) = M^{\perp}$
- If G is finitely generated and G' cyclic, then $\mathcal{A}(\epsilon)$ is a Darboux $\mathbb{Z}$-module. (The converse may fail on finite generation, but is true for $\epsilon = \mathcal{D}(G)$, see Lemma 2.2.3.)

Remark 3.2.18. As for this last correspondence, requiring $G' \leq \mathrm{Z}(G)$ to be cyclic is not a big restriction in studying finitely generated $\leq$2-step nilpotent groups, as it will turn out in Corollary 3.1.26.

The dictionary can be extended to Darboux decompositions as follows.

Definition 3.2.19. A *central product decomposition* of a finitely generated $\leq$ 2-step nilpotent group G is a list of subgroups $A; E_1, \dots, E_t$ where $A \leq \mathrm{Z}(G)$, E_i are 2-generated and of class exactly 2, $d(G) = d(A) + 2t$, $G' = E_1' \geq E_2' \geq \cdots \geq E_t' \gneq 1$. Note the semicolon separating the abelian subgroup A from the list of class 2 subgroups E_i.

Lemma 3.2.20. *Let $O; N_1, \dots, N_t$ be a Darboux decomposition of $\mathcal{D}(G)$. Then there is a central product decomposition $A; E_1, \dots, E_t$ of G such that $\pi_{\mathcal{D}}(A) = O$ and $\pi_{\mathcal{D}}(E_i) = N_i$.*

Conversely, if $A; E_1, \dots, E_t$ be a central product decomposition in G, then $\pi_{\mathcal{D}}(A)$; $\pi_{\mathcal{D}}(E_1), \dots, \pi_{\mathcal{D}}(E_t)$ is a Darboux-decomposition in $(M, \omega, C) = \mathcal{A}(\mathcal{D}(G))$ and it induces the following epimorphism of central-by-abelian extensions.

$$\begin{array}{rcccccccccc} \mathcal{Z}^\flat(G) & : & 1 & \longrightarrow & \mathrm{Z}(G) & \overset{\subseteq}{\hookrightarrow} & G & \xrightarrow{[-,-]^\flat} & \prod_{i=1}^t E_i'^2 & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle \pi_{\mathcal{D}}} & & \downarrow{\scriptstyle \pi_{\mathcal{D}}} & & \| & & \\ \Delta(\mathcal{A}(\mathcal{D}(G))) & : & 1 & \longrightarrow & M^\perp & \overset{\subseteq}{\hookrightarrow} & M & \xrightarrow{\omega^\flat} & \bigoplus_{i=1}^t C_i^2 & \longrightarrow & 1 \end{array} \tag{3.3}$$

Proof. Let $(M, \omega, C) = \mathcal{A}(\mathcal{D}(G))$. Choose minimal generating sets $\{x_i, y_i\}$ of N_i and $\{o_1, \dots, o_k\}$ of O. Then $S := \{x_1, y_1, \dots, x_t, y_t, o_1, \dots, o_k\}$ is a minimal generating set of M by Definition 3.2.4. For every $s \in S$, fix an arbitrary lift $\bar{s} \in \pi_{\mathcal{D}}^{-1}(s)$, set $\bar{S} := \{\bar{s} : s \in S\}$, and define $E_i := \langle \bar{x}_i, \bar{y}_i \rangle \leq G$ and $A := \langle \bar{o}_1, \dots, \bar{o}_k \rangle \leq G$. We show that these subgroups satisfy the statement. Indeed, $A \subseteq \pi_{\mathcal{D}}^{-1}(O) \subseteq \pi_{\mathcal{D}}^{-1}(M^\perp) = \mathrm{Z}(G)$ using Remark 3.2.17. Moreover, $[E_i, E_j] = \omega(N_i, N_j)$ by Remark 3.2.17, meaning that the subgroups E_i commute pairwise by Definition 3.2.4. In particular, $E_i' = \omega(N_i, N_i)$, so the containment of the derived subgroups follows from Definition 3.2.4. $A; E_1, \dots, E_t$ generate G by Lemma 2.2.3, so $d(G) \leq d(A) + \sum_{i=1}^t d(E_i) \leq |\bar{S}| = |S| = d(M) = d(G)$ once again by Lemma 2.2.3. This forces equality everywhere, so $d(E_i) = 2$ and $2t + d(A) = d(G)$.

For the converse, assume that $A; E_1, \dots, E_t$ is central product decomposition of G. In a similar fashion as above, Remark 3.2.17 and $d(G) = d(M)$ from Lemma 2.2.6 imply that the images under $\pi_{\mathcal{D}}$ indeed form a Darboux decomposition. Here $E_i' = \omega(\pi_{\mathcal{D}}(E_i), \pi_{\mathcal{D}}(E_j)) = C_i$ are the invariants from Lemma 3.2.10. Thus there is a unique map $[-,-]^\flat$ making (3.3) commute. Standard diagram chasing shows that $\mathcal{Z}^\flat(G)$ is an exact sequence. ☐

Remark 3.2.21. The map $[-,-]^\flat$ from (3.3) can be given explicitly. Just like as $\omega^\flat$, it depends on a choice of generators. Let x_i, y_i be the underlying generators of $C_i^2 = \pi_{\mathcal{D}}(E_i)$ from Corollary 3.2.13. Fix some $\alpha_i, \beta_i \in E_i$ with $\pi_{\mathcal{D}}(\alpha_i) = x_i$, $\pi_{\mathcal{D}}(\beta_i) = y_i$. Then $[-,-]^\flat \colon g \mapsto ([g, \beta_i], [\alpha_i, g])_i$.

As in Remark 3.2.12, the central-by-abelian extension $\mathcal{Z}^\flat(G)$ depends on the choice of the generators, but its isomorphism class does not as for any choice of generators, $\mathcal{Z}^\flat(G) \cong \mathcal{Z}(G)$ as central-by-abelian extensions, as

$$\begin{array}{rcccccccccc} \mathcal{Z}(G) & : & 1 & \longrightarrow & \mathrm{Z}(G) & \overset{\subseteq}{\hookrightarrow} & G & \overset{\pi_{\mathcal{Z}}}{\twoheadrightarrow} & G/\mathrm{Z}(G) & \longrightarrow & 1 \\ & & & & \| & & \| & & \wr\downarrow{\scriptstyle \lambda} & & \\ \mathcal{Z}^\flat(G) & : & 1 & \longrightarrow & \mathrm{Z}(G) & \overset{\subseteq}{\hookrightarrow} & G & \overset{[-,-]^\flat}{\twoheadrightarrow} & \bigoplus_{i=1}^t C_i^2 & \longrightarrow & 1 \end{array} \tag{3.4}$$

shows where $\lambda \colon g\,\mathrm{Z}(G) \mapsto ([g, \beta_i], [\alpha_i, g])_{i=1}^t$.

Question 3.2.22. Let $\epsilon : 1 \to C \xrightarrow{\iota} G \xrightarrow{\pi} M \to 1$ be a central-by-abelian extension, and let $A; E_1, \ldots, E_t$ be a central product decomposition of G. Is it always true that $\pi(A); \pi(E_1) \ldots, \pi(E_t)$ is a Darboux decomposition of $\mathcal{A}(\epsilon)$? From Remark 3.2.17, this question is equivalent to asking whether $d(G) - d(A) = d(M) - d(\pi(A))$. This latter is certainly the case if $\iota(C)$ is contained in the so-called Frattini subgroup $\Phi(G)$ because in this case $d(M) = d(G)$ and we can proceed as at the proof of Lemma 3.2.20.

We obtain a generalisation of Theorem 1.2.2 and [BBC69, Theorem 2.1]. Recall Definition 3.2.19.

Theorem 3.2.23 (Central product decomposition, D.R.Sz.)**.** *Every finitely generated ≤ 2-step nilpotent group G with cyclic commutator subgroup G' has a central product decomposition $A; E_1, \ldots, E_t$. In any such decomposition, $t = \frac{1}{2}d(G/\operatorname{Z}(G))$ and $E_i' \subseteq G'$ are invariants which satisfy $G/\operatorname{Z}(G) \cong \prod_{i=1}^{t} E_i'^2$.*

Proof. The $\mathbb{Z}$-module $\mathcal{A}(\mathcal{D}(G))$ is a non-degenerate and Darboux by Remark 3.2.17, so it has a Darboux decomposition by Lemma 3.2.9. Then Lemma 3.2.20 proves existence of the stated central product decomposition. The invariance of this decomposition is a consequence of Lemma 3.2.20 and Lemma 3.2.10. □

Remark 3.2.24. The isomorphism class of the subgroups $E_1, \ldots, E_t$ are not unique, which is demonstrated by the classical decomposition of extra-special p-groups Theorem 1.2.2. For example, the extra-special p-group G of order p^{2s+1} of exponent p^2 has an internal central product decomposition $G = E_1E_2 \ldots E_t$ for any $1 \leq s \leq t$, such that $E_i \cong M$ for $1 \leq i \leq s$ and $E_i \cong E$ for $s < i \leq t$, where E and M are the non-abelian groups of order p^3 and of exponent p and p^2, respectively [Suz82, Theorem 4.18].

3.3 Intermezzo: Presentation of ≤2-generated Groups

Section 3.1 and Section 3.2 demonstrated that the building blocks of finitely generated ≤2-step nilpotent groups are ≤2-generated ≤2-step nilpotent groups. These building blocks were classified in the finite p-group case by Ahmad, Magidin and Morse by giving a concrete presentation depending on 5 positive integral parameters [AMM12]. Following the idea of this paper, we extend the description to infinite groups, but also include the parts about the finite case for completeness. We do not display the full classification, as it is computationally extensive, but we do work out a presentation that shows that every automorphism of the centre can be extended to the full group. With the help of this, we prove a sharpening of Theorem 3.2.23 in the cyclic centre case (Proposition 3.3.11).

Historically, the latter statement was used in the original (unpublished) proofs of Theorem B and Theorem C. The one presented in Chapter 4 and Chapter 5, however, uses more canonical tools (embedding to twisted Heisenberg groups) and is free from this classification result. Nevertheless, we include this section as Proposition 3.3.11 is interesting in its own right, but statements of this section will not be used in the upcoming development of this thesis.

Lemma 3.3.1. *For every (possibly infinite) ≤ 2-generated, ≤ 2-step nilpotent group E there exist numerical invariants (called the* invariant triplet*) $(a, b, c) \in \mathbb{N}_0{}^3$ with $c \mid b \mid a$ satisfying the following conditions.*

1. *$\mathcal{D}(E)$ is isomorphic to $1 \to \mathbb{Z}_c \to E \to \mathbb{Z}_b \times \mathbb{Z}_a \to 1$ as central-by-abelian extensions.*

2. *Whenever $E/E' = \langle \beta E' \rangle \times \langle \alpha E' \rangle$ is an invariant factor decomposition, there exist $c_i = c_i(\alpha, \beta) \in \mathbb{Z}$ such that E has a presentation*

$$E = \langle \alpha, \beta, \gamma : \gamma = [\alpha, \beta], 1 = [\gamma, \alpha] = [\gamma, \beta], \alpha^a = \gamma^{c_1}, \beta^b = \gamma^{c_2}, \gamma^c = 1 \rangle \tag{3.5}$$

where $Z(E) = \langle \alpha^c, \beta^c, \gamma \rangle$ and

   (a) $c_1 = 0$ if $a = 0$; and $0 < c_1 < c$ if $a > 0$,

   (b) $c_2 = 0$ if $b = 0$; and $0 < c_2 < c$ if $b > 0$.

Remark 3.3.2. See [AMM12, Theorem 1.1] for a full classification in case of finite p-groups.

Proof. Pick $\alpha, \beta \in E$ such that $E/E' = \langle \beta_0 E' \rangle \times \langle \alpha_0 E' \rangle$ is an invariant factor decomposition of E/E'. Such elements exist, as E is $\leq$2-generated by assumption, cf. Lemma 2.2.3. Define $a, b \in \mathbb{N}_0$ by $\mathbb{Z}_a \cong \langle \alpha E' \rangle$ and $\mathbb{Z}_b \cong \langle \beta E' \rangle$. After a potential relabeling, we may assume that $b \mid a$. (Note that if E/E' is infinite, then $a = 0$, and potentially $b = 0$, too.) Then Lemma 2.2.3 shows that $E = \langle \alpha, \beta \rangle$ and $E' = \langle \gamma \rangle$. Define c by $\mathbb{Z}_c \cong E'$. Then $\mathcal{D}(E)$ takes the form $1 \to \mathbb{Z}_c \to E \to \mathbb{Z}_b \times \mathbb{Z}_a \to 1$ as stated. The morphism $[\alpha, -]\colon \langle \beta E' \rangle \to \langle \gamma \rangle$ is well defined because $E' \subseteq Z(E)$ and is surjective, thus $c \mid b$.

If $a > 0$, then $\langle \alpha E' \rangle$ is a finite group of order a, so by Lagrange's theorem, $\alpha^a \in E'$ which gives a relation $\alpha^a = \gamma^{c_1}$ for some c_1. If $a = 0$, then we take $c_1 = 0$. The relation $\beta^b = \gamma^{c_2}$ follows analogously. Finally, $1 = \gamma^c = [\alpha, \gamma] = [\beta, \gamma]$ follow from the fact that $\mathbb{Z}_c \cong \langle \gamma \rangle = E' \subseteq Z(E)$.

Pick fixed representatives of elements $\mathbb{Z}_n$: let $R_n \coloneqq \{0, 1, \dots, n-1\}$ if $n > 0$ and let $R_n \coloneqq \mathbb{Z}$ if $n = 0$. We claim that every $g \in E$ can be written as $g = \gamma^z \beta^y \alpha^x$ for a unique $(z, y, x) \in R_c \times R_b \times R_a$. Indeed, the projection $E \to E/E' = \langle \beta E' \rangle \times \langle \alpha E' \rangle$ defines a unique pair $(x, y) \in R_a \times R_b$ such that $gE' = \beta^y \alpha^x E'$. Hence $(\beta^y \alpha^x)^{-1} g \in E' = \langle \gamma \rangle$, so for a unique $z \in R_c$, we have $g = \gamma^z \beta^y \alpha^x$.

To show that (3.5) is indeed a presentation of E, we need to verify that every relation is generated by the listed ones. Consider a word w in α, β that is trivial in E. Using the first three relations (3.5), we may write w as $\gamma^z \beta^y \alpha^x$ for some $(x, y, z) \in \mathbb{Z}^3$. Using the remaining relations, we may assume that $(x, y, z) \in R_a \times R_b \times R_c$. Since w is trivial in E, we must have $x = y = 0$ by a result above, which concludes the proof of the conditions.

Finally, we prove the statement about the centre. Note that $g \in Z(E)$ if and only if $[\alpha, g] = [g, \beta] = 1$. Using that the commutator is a bihomomorphism, for $g = \gamma^x \beta^y \alpha^z$, we have $[\alpha, g] = [\alpha, \alpha]^x [\alpha, \beta]^y [\alpha, \gamma]^z = \gamma^y$ and $[g, \beta] = \gamma^x$ which finishes the proof. □

Lemma 3.3.3. *If E from (3.5) is finite (i.e. the invariant triplet $0 < c \mid b \mid a$ consists of positive numbers), then the following hold.*

1. *$c_i(\bar{\alpha}, \bar{\beta}) = \gcd(c_i(\alpha, \beta), c) \mid c$ for suitable generators $\bar{\alpha}, \bar{\beta}$.*

2. *The change of generators $f_n(\alpha) \coloneqq \alpha$ and $f_n(\beta) \coloneqq \beta \alpha^{na/b}$ for arbitrary $n \in \mathbb{Z}$ gives*

$$c_1(f_n(\alpha), f_n(\beta)) \equiv c_1(\alpha, \beta) \pmod{c},$$
$$2c_2(f_n(\alpha), f_n(\beta)) \equiv 2c_2(\alpha, \beta) + 2c_1(\alpha, \beta)n \pmod{c}.$$

Proof. Let $\bar{c}_2$ be the smallest positive integer such that $\gamma^{\bar{c}_2} \in E' \cap \langle \beta \rangle = \langle \beta^b \rangle$. Then $\bar{c}_2 \mid c$ and $\langle \gamma^{\bar{c}_2} \rangle = \langle \beta^b \rangle = E' \cap \langle \beta \rangle$. In particular there is $\varphi_1 \in \mathrm{Aut}(E' \cap \langle \beta \rangle)$ for which $\varphi_1(\gamma^{\bar{c}_1}) = \beta^b$. Lift up φ_1 along $[\beta, -]^{\bar{c}_2} \colon \langle \alpha \rangle \to \langle \gamma^{\bar{c}_2} \rangle$ using Lemma 2.2.26 to get $\theta_1 \in \mathrm{Aut}(\langle \alpha \rangle)$ and define $\bar{\alpha} := \theta_1(\alpha)$. Similarly, define $\bar{c}_1$ to be the smallest positive integer with $\gamma^{\bar{c}_1} \in E' \cap \langle \alpha \rangle = \langle \alpha^a \rangle$ and define β analogously.

$$
\begin{array}{ccc}
\langle \alpha \rangle & \xrightarrow{[-,\beta]^{\bar{c}_2}} & E' \cap \langle \beta \rangle \\
{\scriptstyle \cong} \downarrow {\scriptstyle \theta_1} & & {\scriptstyle \cong} \downarrow {\scriptstyle \varphi_1} \\
\langle \alpha \rangle & \xrightarrow{[-,\beta]^{\bar{c}_2}} & E' \cap \langle \beta \rangle
\end{array}
\qquad
\begin{array}{ccc}
\langle \beta \rangle & \xrightarrow{[\alpha,-]^{\bar{c}_1}} & E' \cap \langle \beta \rangle \\
{\scriptstyle \cong} \downarrow {\scriptstyle \theta_2} & & {\scriptstyle \cong} \downarrow {\scriptstyle \varphi_2} \\
\langle \beta \rangle & \xrightarrow{[\alpha,-]^{\bar{c}_1}} & E' \cap \langle \beta \rangle
\end{array}
$$

The commutativity of the first diagram shows that $[\bar{\alpha}, \beta]^{\bar{c}_2} = [\theta_1(\alpha), \beta]^{\bar{c}_2} = \varphi_1([\alpha, \beta]^{\bar{c}_2}) = \varphi_1(\gamma^{\bar{c}_2}) = \beta^b$. Thus $[\bar{\alpha}, -]^{\bar{c}_2} = (-)^b$ as $\langle \beta \rangle \to E' \cap \langle \beta \rangle$ morphisms. In particular, $[\bar{\alpha}, \bar{\beta}]^{\bar{c}_2} = (\bar{\beta})^b$ as $\langle \beta \rangle = \langle \bar{\beta} \rangle$. Similarly, considering the second diagram, we see that $[\bar{\alpha}, \bar{\beta}]^{\bar{c}_1} = \bar{\alpha}^a$. Note that $E/E' = \langle \bar{\alpha} E' \rangle \times \langle \bar{\beta} E' \rangle$ is in the invariant decomposition as $\langle \alpha \rangle = \langle \bar{\alpha} \rangle$ and $\langle \beta \rangle = \langle \bar{\beta} \rangle$. Thus $c_i(\bar{\alpha}, \bar{\beta}) = \bar{c}_i \mid c$. Noting that $\gcd(c, c_i) = |E' : E' \cap \langle \delta \rangle| = |E' : E' \cap \langle \bar{\delta} \rangle| = \gcd(c, \bar{c}_i)$ where $\delta = \alpha$ if $i = 1$ and $\delta = \beta$ if $i = 2$, we see that $\bar{c}_i = \gcd(c_i, c)$ as stated.

Now consider the second part. First we claim that $E/E' = \langle f_n(\alpha) E' \rangle \times \langle f_n(\beta) E' \rangle$ is in invariant factor form. Indeed, $\{f_n(\alpha) E', f_n(\beta) E'\}$ generates E/E', as $\{f_n(\alpha), f_n(\beta)\}$ generates E. Pick $gE' \in \langle f_n(\alpha) E' \rangle \cap \langle f_n(\beta) E' \rangle$. Then $gE' = \alpha^x E' = \beta^y \alpha^{ya/b} E'$ for some $x, y \in \mathbb{Z}$, so $\alpha^{ya/b - x} \beta^y \in E'$. Since $\langle \alpha E' \rangle \cap \langle \beta E' \rangle = E'$ by assumption, we must have $\alpha^{ya/b - x} \in E'$ and $\beta^y \in E'$. The latter one implies that $y = kb$ for some $k \in \mathbb{Z}$, so then $E' \ni \alpha^{ya/b - x} = \alpha^{ka - x} = \gamma^{k c_1(\alpha, \beta)} \alpha^{-x}$, so $\alpha^x \in E'$. Thus $gE' = E'$ showing that $\langle f_n(\alpha) E' \rangle \langle f_n(\beta) E' \rangle$ is an internal direct product. Finally to show that this is in the invariant factor decomposition, it suffices to show that $|\langle f_n(\beta) \rangle| \mid b = |\langle \beta \rangle E'|$. This follows since

$$
f_n(\beta)^b = \beta^b \alpha^{an} [\alpha^{na/b}, \beta]^{\binom{b}{2}} = \gamma^{c_1(\alpha,\beta) n + c_2(\alpha,\beta) + na/b \binom{b}{2}} \in E',
$$

where we used Lemma 2.2.3 and the relations of (3.5).

This means that the numbers c_i are well defined for the new generators which we now compute. Let $f_n(\gamma) := [f_n(\alpha), f_n(\beta)] = [\alpha, \beta \alpha^{xa/b}] = [\alpha, \beta][\alpha, \alpha]^{xa/b} = \gamma$. Then

$$
\begin{aligned}
f_n(\alpha)^a &= \alpha^a = \gamma^{c_1(\alpha,\beta)} = f_n(\gamma)^{c_1(\alpha,\beta)}, \\
f_n(\beta)^{2b} &= \gamma^{2c_1(\alpha,\beta) n + 2c_2(\alpha,\beta) + na(b-1)} = f_n(\gamma)^{2c_1(\alpha,\beta) n + 2c_2(\alpha,\beta)}
\end{aligned}
$$

since $c \mid a$. This is equivalent to the stated formulae. □

Remark 3.3.4. If c is odd, then of course we could formulate a statement $c_2 \pmod{c}$ only instead of $2c_2$. When c is even, the statement gives two options for the module class of c_2 and actually both can happen. It turns out that it is cleaner to keep these two cases together as at the statement.

Lemma 3.3.5. *If the invariant triplet from Lemma 3.3.1 satisfies $a = b$, then we can take $c_2 \in \{c, c/2\}$ and $c_1 \mid c$ for suitable generators.*

Proof. By assumption, f_n from Lemma 3.3.3 can be applied with the roles of α and β interchanged. Hence repeated application of this change of generators is equivalent to the Euclidean algorithm with starting values $2c_1(\alpha, \beta)$ and $2c_2(\alpha, \beta)$. Doing so (after a possible relabeling at the end) we obtain generators $\bar{\alpha}$ and $\bar{\beta}$ with $2c_2(\bar{\alpha}, \bar{\beta}) = 0$ and $2c_1(\bar{\alpha}, \bar{\beta}) = \gcd(2c_1(\alpha, \beta), 2c_2(\alpha, \beta))$. Hence after potentially using the divisibility part of Lemma 3.3.3, we can take c_1 and c_2 as stated. □

Lemma 3.3.6. *Let E be a ≤ 2-generated, ≤ 2-step nilpotent group with cyclic centre. Let $c \mid b \mid a$ the invariants from Lemma 3.3.1. Then $\mathcal{Z}(E)$ is isomorphic to $1 \to \mathbb{Z}_a \to E \to \mathbb{Z}_c \times \mathbb{Z}_c \to 1$, cf. (3.4). Moreover, we have the following two statements.*

1. *If E is infinite, then either $E \cong \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$, the discrete Heisenberg group over $\mathbb{Z}$ (corresponding to $a = b = c = c_1 = c_2 = 0$), or $E \cong \mathbb{Z}$ (corresponding to $a = 0$, $b = c = c_1 = c_2 = 1$).*

2. *If E is a finite p-group, then using suitable generators, we have either $a > b = c$, $c_1 = 1$, $c_2 \in \{c, c/2\}$ or $a = b = c$, $c_1 \mid c$, $c_2 \in \{c, c/2\}$. (Some of these are isomorphic, see Proposition 3.3.7.)*

Proof. Consider the presentation given by Lemma 3.3.1. We now describe a presentation for these groups distinguishing some cases depending on the c. If $c = 0$, then from the divisibility conditions, $a = b = 0$ and by construction $c_1 = c_2 = 0$. So E is the free ≤ 2-step nilpotent group on 2 generators, the Heisenberg group over $\mathbb{Z}$.

Assume that $c = 1$. Now $\mathbb{Z}_c \cong E'$ is trivial, so E is abelian, thus $E \cong \mathbb{Z}_a \times \mathbb{Z}_b$. Since $Z(E) = E$ is cyclic, we must have $b = 1$ and hence $c_1 = c_2 = 1$.

For the rest of the proof, let $c > 1$. Then $\mathbb{Z}_c \cong E' \leq Z(E)$, so $Z(E)$ has a non-trivial finite cyclic subgroup, hence $Z(E)$ itself must be finite and non-trivial. Since $E/Z(E) \cong (E/E')/(Z(E)/E') = \langle \alpha E' \rangle \times \langle \beta E' \rangle / \langle \alpha^c E' \rangle \times \langle \beta^c E' \rangle \cong \mathbb{Z}_c \times \mathbb{Z}_c$, we see that E is a finite group. Note that $1 = \alpha^{ca} = \beta^{ca} = \gamma^a$. In other words, the ath power of every generator of the cyclic $Z(E)$ is trivial, so $|Z(E)| \mid a$. But $|Z(E)| = |E|/c^2 = ab/c$ and since $b \mid c$, we must have $c = b$ and $|\mathrm{Z}(E)| = a$. Now $\langle \beta^c \rangle = \langle \beta^b \rangle = E' \cap \langle \beta \rangle \subseteq E' = \langle \gamma \rangle$, so in fact $\mathbb{Z}_a \cong Z(E) = \langle \alpha^c, \gamma \rangle$.

Now assume further that E is a p-group. Assume first that $E' = Z(E)$, i.e. $a = c$. As $a = b$ by above, this falls into the case $a = b$ discussed in Lemma 3.3.5 giving the stated presentation. Otherwise $E' \subsetneq Z(E)$, so $c < a$ and $\langle \gamma \rangle = E' \subsetneq Z(E) = \langle \alpha^c, \gamma \rangle$. Since these are cyclic p-groups, we must have $Z(E) = \langle \alpha^c \rangle$. Since $a/c_1(\alpha, \beta) = |\langle \alpha^c \rangle| = |Z(E)| = a$, we have $c_1(\alpha, \beta) = 1$. Using the change of generators f_{-c_2} from Lemma 3.3.3, we obtain $\bar{\alpha} =: f_{-c_2}(\alpha)$ and $\bar{\beta} := f_{-c_2}(\beta)$ for which $2c_2(\bar{\alpha}, \bar{\beta}) = 0$ and $c_1(\bar{\alpha}, \bar{\beta}) = c_1(\alpha, \beta) = 1$. Hence as in the proof of Lemma 3.3.5, we may take $c_1(\bar{\alpha}, \bar{\beta}) = 1$ and $c_2(\bar{\alpha}, \bar{\beta}) \in \{c, c/2\}$ as stated. □

Some of these groups are isomorphic. In fact, a full classification is obtained by Ahmad, Magidin and Morse (even without requiring the centre to be cyclic). We will not need this precise result; nonetheless, we state it here for completeness.

Proposition 3.3.7 ([AMM12, Theorem 1.1])**.** *The p-groups from Lemma 3.3.6 are all non-isomorphic with the only exceptions appearing for $p = 2$, $a = b = c \geq 2$. In this case, for $c_1 < c/2$, then the two choices for c_2 give isomorphic groups; and the groups corresponding to $\{c_1, c_2\} = \{c, c/2\}$ and to $c_1 = c_2 = c$ are isomorphic, so the complete list of non-isomorphic groups are given by the following (c_1, c_2) values: $\{(c_1, c) : c/2 \neq c_1 \mid c\} \cup \{(c/2, c/2)\}$.*

Remark 3.3.8. Thus for all primes p, the number of isomorphism classes with given invariant (a, b, c) is 1 if $a > c$, otherwise is $\sigma(c)$, the number of divisors of c. Using that finite nilpotent groups are the direct product of their Sylow subgroups, we see that the number of isomorphism classes with invariant (a, b, c) is $|\{c_1 : c_1 \mid c, \gcd(c_1, a/c) = 1\}| = \sigma(d)$ where $d = \max\{c_1 : c_1 \mid c, \gcd(c_1, a/c) = 1\}$.

The previous presentations were developed so that we can prove the next statement.

Lemma 3.3.9. *If E is a finitely generated ≤ 2-generated ≤ 2-step nilpotent group with cyclic centre, then every automorphism of $\mathrm{Z}(E)$ can be extended to an automorphism of E.*

Proof. We check the statement one by one of the groups given by Lemma 3.3.6.

Let E be the discrete Heisenberg group over $\mathbb{Z}$. Then $Z(E) = E' = \langle\gamma\rangle \cong \mathbb{Z}$, so it has a single non-trivial automorphism given by $\gamma \mapsto \gamma^{-1}$. This lifts to an automorphism of E given by $\alpha \mapsto \alpha^{\pm 1}$, $\beta \mapsto \beta$.

If $E \cong \mathbb{Z}$, then the statement is trivial.

Let E be a finite p-group. Pick $\varphi \in \mathrm{Aut}(Z(E))$. Since $Z(E)$ is cyclic, we have $\varphi\colon g \mapsto g^k$ for some $k \in \mathbb{Z}$ with $\gcd(k, a) = 1$. Note that $\beta^{2b} = \gamma^{2c_2(\alpha,\beta)} = 1$ by Lemma 3.3.6. In other words, $E' \cap \langle\beta\rangle = \langle\beta^b\rangle$ has order at most 2, so every automorphism of this group is trivial. But $\gcd(k, b) = 1$ as $b \mid a$, meaning that $\langle\beta\rangle \to \langle\beta\rangle, g \mapsto g^k$ is an automorphism. Its restriction to $\langle\beta^b\rangle = E' \cap \langle\beta\rangle$ is trivial from above, in particular $\beta^{bk} = \beta^b$. Define $\theta(\alpha) \coloneqq \alpha^k$ and $\theta(\beta) \coloneqq \beta$. To check that θ extends to an endomorphism of E, we verify that the relations of (3.5) are preserved. Let $\theta(\gamma) \coloneqq [\theta(\alpha), \theta(\beta)] = [\alpha^k, \beta] = \gamma^k = \varphi(\gamma)$. First note that $\theta(\alpha)^a = \alpha^{ak} = \gamma^{kc_1} = \theta(\gamma)^{c_1}$. Second, $\theta(\gamma)^{c_2} = \gamma^{kc_2} = \beta^{bk} = \beta^b$ by above. The remaining relations of (3.5) follow from $\langle\theta(\gamma)\rangle = \langle\varphi(\gamma)\rangle = \langle\gamma\rangle = E' \subseteq Z(E)$ and $|E'| = c$, so $\theta\colon E \to E$ is a morphism. Since $c_1 \mid c \mid a$, we have $\gcd(k, |\langle\alpha\rangle|) = \gcd(k, ac/c_1) = \gcd(k, a) = 1$, so $\langle\alpha\rangle = \langle\theta(\alpha)\rangle$. Then $\theta(E) = \langle\theta(\alpha), \theta(\beta)\rangle = \langle\alpha, \beta\rangle = E$, so θ is surjective, hence injective from finiteness. Note that $\theta(\gamma) = \varphi(\gamma)$ and $\theta(\alpha^c) = \alpha^{ck} = \varphi(\alpha^c)$, and $Z(E) = \langle\alpha^c, \gamma\rangle$, so θ extends φ as stated.

Finally, if E is a finite, but not a p-group, then it is a direct product of its Sylow subgroups E_p. Then $\mathrm{Z}(E_p) \leq \mathrm{Z}(E)$ are cyclic characteristic subgroups. So the restriction of any automorphism φ of $\mathrm{Z}(E)$ restricts to one on $\mathrm{Z}(E_p)$. The direct product of the lifts of these automorphisms lifts φ to E. □

Question 3.3.10. Is there a proof that does not use the structural description? Such an argument could give a cleaner overall proof of Proposition 3.3.11.

Proposition 3.3.11 (D.R.Sz., cf. [BBC69, Theorem 2.1])**.** *Every finitely generated ≤ 2-step nilpotent group G with cyclic centre is isomorphic to the maximal central product of $n = \lceil d(G)/2 \rceil$ groups E_i each being ≤ 2-generated, ≤ 2-step nilpotent with cyclic centre. (In fact, if $d(G)$ is even, then every E_i is 2-generated of class 2; otherwise exactly one E_i is cyclic, the rest of the groups are 2-generated of class 2).*

If furthermore G is finite, taking arbitrary maximal central products give G up to isomorphism, i.e. the isomorphism class of the central product depends only on the ≤ 2-generated groups, and not on the amalgamations:

$$G \cong E_1 \mathbin{\hat{\curlyvee}} E_2 \mathbin{\hat{\curlyvee}} \dots \mathbin{\hat{\curlyvee}} E_n. \tag{3.6}$$

Remark 3.3.12. This generalises the classical structure theorem of extra-special special p-groups together Theorem 1.2.2. Different E_i's may give the same isomorphism class, as the example from Remark 3.2.24 shows as there every group involved has cyclic centre.

Proof. By assumption G is ≤ 2-step nilpotent, so $G' \leq \mathrm{Z}(G)$ is cyclic. Let $A; E_1, \ldots, E_t$ be the subgroups given by Theorem 3.2.23. Recall $d(G) = d(A) + 2t$ by Definition 3.2.4. Now $A \leq \mathrm{Z}(G)$, so A has to be cyclic, i.e. $d(A) \leq 1$. If $t = 0$, then set $n \coloneqq t$. Otherwise, set $n \coloneqq t + 1$ and $E_n \coloneqq A$. Then H is the internal central product of the ≤ 2-generated, ≤ 2-step nilpotent subgroups $E_1, \ldots, E_n$. Hence by Remark 2.2.21, G is isomorphic to an iterated external central product $E_1 \curlyvee_{\gamma^{(1)}} E_2 \curlyvee_{\gamma^{(2)}} \ldots \curlyvee_{\gamma^{(n-1)}} E_n$ of these groups for some amalgamations $\gamma^{(i)}$ as stated. Lemma 2.2.27 shows that $\mathrm{Z}(E_i)$ is cyclic for every $1 \leq i \leq n$ and that all central products are maximal.

If G is finite, then Lemma 2.2.28 with Lemma 3.3.9 shows that the isomorphism class of this iterated central product is independent of the choice of the amalgamations. □

Chapter 4

Twisted Heisenberg Groups

In this chapter, we prove Theorem B from page 8. The main objects here are twisted Heisenberg groups which were motivated by Mumford's theta groups [Mum66]. Our notion generalises the usual Heisenberg group over a ring to a sufficiently general level so that they turn out to include every finitely generated $\leq$ 2-step nilpotent groups. We study this inclusion and its properties in this chapter. These group theoretic results provide the basis of a clean proof for Theorem C from page 9.

4.1 Polarised Modules and the Heisenberg Functor

In this section, we define the notion of twisted Heisenberg groups. To avoid complications with groups having 2-torsion, we mimic the construction of *polarised* Heisenberg groups associated to symplectic vector spaces instead of the canonical construction. In the process, we introduce a few polarised categories and the Heisenberg functor which is a partial inverse of the alternating functor from Subsection 3.2.2.

Definition 4.1.1. We call $(L_1 \oplus L_2, \omega, C)$ a *polarised module* if it is an alternating module in which L_1, L_2 are both isotropic. We call a morphism $(\lambda, \kappa)\colon (L_1 \oplus L_2, \omega, C) \to (\bar{L}_1 \oplus \bar{L}_2, \bar{\omega}, \bar{C})$ of alternating modules *polarised* if $\lambda(L_i) \subseteq \bar{L}_i$ for $i = 1, 2$. Polarised modules with polarised morphisms form a category. If $M \coloneqq L_1 \oplus L_2$, then we say that $(L_1 \oplus L_2, \omega, C)$ is a polarised module over (M, ω, C), or that (M, ω, C) is the underlying alternating module of $(L_1 \oplus L_2, \omega, C)$.

Example 4.1.2. For $j \in \{1, 2\}$, let $L_j \coloneqq \bigoplus_{i=1}^t C_i$ consisting of the jth factor of C_i^2. Then $L_1 \oplus L_2$ gives a polarisation of $(\bigoplus_{i=1}^t C_i^2, \mathrm{trdet}, C)$ from Example 3.2.3.

Lemma 4.1.3. *For every non-degenerate Darboux module (M, ω, C), there is a polarised module $(L_1 \oplus L_2, \omega, C)$ over (M, ω, C).*

Proof. Let $0; N_1, \ldots, N_t$ be a Darboux decomposition given by Lemma 3.2.9. Write $N_i = Rx_i \oplus Ry_i$ for some $x_i, y_i \in N_i$. Then $M = \bigoplus_{i=1}^t N_i$ by Corollary 3.2.13. Then $L_1 \coloneqq \bigoplus_{i=1}^t Rx_i$ and $L_2 \coloneqq \bigoplus_{i=1}^t Ry_i$ is as stated because $N_i \perp N_j$ for $i \neq j$. □

Polarised modules over Darboux modules are very restricted.

Lemma 4.1.4. *Let $(L_1 \oplus L_2, \omega, C)$ be polarised module over a non-degenerate Darboux module with invariants $C_1, \ldots, C_t$ form Lemma 3.2.10. Then there is a polarised isomorphism*

$$(\omega^\flat, \mathrm{id}_C)\colon (L_1 \oplus L_2, \omega, C) \to (\bigoplus_{i=1}^t C_i \oplus \bigoplus_{i=1}^t C_i, \mathrm{trdet}, C).$$

Proof. Pick minimal generating sets of L_i and a generator c of $\omega(M, M)$. The matrix $[\omega]$ from Remark 3.2.8 with respect to these generating sets is a 2×2 block matrix, The diagonal blocks are 0 because the submodules L_i are isotropic. Since the entries are from a principal ideal ring (being the quotient of a domain), we can change the generators of the top right block so that this part of the matrix is in the Smith normal form, cf. [Jac85, Theorem 3.9]. Let the new (and still minimal) generating sets be $\{x_1, \dots, x_n\}$ for L_1, and $\{y_1, \dots, y_k\}$ for L_2. This matrix cannot have zero rows/columns as otherwise the corresponding new generator would lie in $M^\perp = 0$, which is impossible my minimality. Hence the blocks must be squares of equal size, i.e. $n = k$. Then $N_i := Rx_i \oplus Ry_i$ give Darboux decomposition, so $n = t$ by Lemma 3.2.10. Then $\omega^\flat$ from Remark 3.2.12 with the above generators give the desired map as in the proof of Corollary 3.2.13. □

Definition 4.1.5 (The polarised map $\mathcal{P}$)**.** Say two non-degenerate polarised $\mathbb{Z}$-modules are in relation $\equiv$ if their underlying alternating modules are equal and there is a polarised isomorphism between them. Let Forget map each $\equiv$-equivalence class to the underlying alternating module. Then Forget has a two sided inverse $\mathcal{P}$ given by Lemma 4.1.4.

$$\left\{\begin{array}{c}\text{non-degenerate Darboux}\\ \mathbb{Z}\text{-modules}\end{array}\right\} \underset{\text{Forget}}{\overset{\mathcal{P}}{\rightleftarrows}} \left\{\begin{array}{c}\text{non-degenerate polarised}\\ \text{Darboux } \mathbb{Z}\text{-modules}\end{array}\right\} / \equiv$$

Recall Definition 2.2.36.

Lemma 4.1.6. *There is an isomorphism of categories given by*

$$\begin{array}{rcl}\{\text{polarised } \mathbb{Z}\text{-modules}\} & \overset{\sim}{\longleftrightarrow} & \{\text{abelian bihomomorphisms}\}\\ (L_1 \oplus L_2, \omega, C) & \overset{\text{Diag}}{\longmapsto} & \operatorname{diag}(\omega)\colon L_1 \times L_2 \to C\\ (A \oplus B, \det(\mu), C) & \overset{\text{Det}}{\longleftarrow\!\shortmid} & \mu\colon A \times B \to C\end{array}$$

where $\operatorname{diag}(\omega) := \operatorname{diag}_{L_1,L_2}(\omega)\colon (l_1, l_2) \mapsto \omega(l_1, l_2)$, *and* $\det(\mu) := \det_{A,B}(\mu)\colon (a + b, a' + b') \mapsto \mu(a, b') - \mu(a', b)$.

Remark 4.1.7. We use Diag and Det for the corresponding functors, while $\det(\omega)$ and $\operatorname{diag}(\mu)$ represents the respective maps from the definition of the respective objects.

Proof. By definition, $\operatorname{diag}(\det(\mu))\colon A \times B \to C, (a, b) \mapsto \det(\mu)(a, b) = \mu(a, b)$. For the other direction, $\det(\operatorname{diag}(\omega))\colon (l_1 + l_2, l_1' + l_2') \mapsto \operatorname{diag}(\omega)(l_1, l_2') - \operatorname{diag}(\omega)(l_2, l_1') = \omega(l_1, l_2') + \omega(l_2, l_1') = \omega(l_1 + l_2, l_1' + l_2')$ using the alternating property of ω and the assumption that $\omega(L_i, L_i) = 0$. □

Remark 4.1.8. Under the isomorphism of Lemma 4.1.6, polarised Darboux $\mathbb{Z}$-modules (cf. Definition 3.2.7) translate to abelian bihomomorphisms $\mu\colon A \times B \to C$ such that A, B are finitely generated and $\mu(A, B)$ is cyclic.

Definition 4.1.9 (Twisted Heisenberg group H)**.** Let $\mu\colon A \times B \to C$ be an abelian bihomomorphism. Define the associated *(twisted) Heisenberg group* as

$$\mathrm{H}(\mu) := A \ltimes_\varphi (B \times C)$$

where $\varphi\colon A \to \operatorname{Aut}(B \times C), a \mapsto ((b, c) \mapsto (b, \mu(a, b) + c)$. $\mathrm{H}(\mu)$ is *non-degenerate* if $\mathrm{Z}(\mathrm{H}(\mu)) = \{(0, 0, c) : c \in C\}$.

Remark 4.1.10. More explicitly, the group structure on $\mathrm{H}(\mu)$ is given by

$$\begin{aligned}(a,b,c)*(a',b',c') &= (a+a', b+b', c+\mu(a,b')+c'),\\ (0,0,0) &= \text{identity},\\ (a,b,c)^{-1} &= (-a,-b,\mu(a,b)-c),\\ [(a,b,c),(a',b',c')] &= (0,0,\mu(a,b')-\mu(a',b)).\end{aligned}$$

So formally $\mathrm{H}(\mu) \cong \left(\begin{smallmatrix}1 & A & C\\ 0 & 1 & B\\ 0 & 0 & 1\end{smallmatrix}\right)$ with matrix multiplication induced by μ. Various choices for μ can 'twist' the usual multiplication, hence the naming.

Example 4.1.11. If R is a commutative ring (with unit) and $\mu\colon R^n \times R^n \to R$ is the usual dot product, then $\mathrm{H}(\mu)$ is the $(2n+1)$-dimensional Heisenberg group over R, i.e. the multiplicative group of of $(n+2)\times(n+2)$ unitriangular matrices over R of the form

$$\left(\begin{smallmatrix}1 & R^{1\times n} & R\\ 0 & I_n & R^{n\times 1}\\ 0 & 0 & 1\end{smallmatrix}\right).$$

Definition 4.1.12. A central-by-abelian extension $\epsilon : 1 \to C \xrightarrow{\iota} G \xrightarrow{\pi} A \times B \to 1$ is *polarised* if $\pi^{-1}(A), \pi^{-1}(B) \subseteq G$ are both abelian subgroups. This notion is not be be confused with Definition 4.2.1.

Lemma 4.1.13 ($\mathcal{H}$)**.** *There is a faithful and injective-on-objects functor*

$$\begin{array}{ccc}\{\text{abelian bihomomorphisms}\} & \xrightarrow{\mathcal{H}} & \left\{\begin{array}{c}\text{polarised central-by-abelian}\\ \text{extensions}\end{array}\right\}\\ (\mu\colon A\times B \to C) & \longmapsto & (1 \to C \xrightarrow{\iota_\mu} \mathrm{H}(\mu) \xrightarrow{\pi_\mu} A\times B \to 1)\\ (\lambda_1\times\lambda_2,\kappa) & \longmapsto & (\kappa, \mathrm{H}(\lambda_1\times\lambda_2,\kappa), \lambda_1\times\lambda_2)\end{array}$$

where $\iota := \iota_\mu\colon c \mapsto (0,0,c)$*,* $\pi := \pi_\mu\colon (a,b,c) \mapsto (a,b)$ *and* $\mathrm{H}(\lambda_1\times\lambda_2,\kappa)\colon \mathrm{H}(\mu) \to \mathrm{H}(\bar\mu), (a,b,c) \mapsto (\lambda_1(a),\lambda_2(b),\kappa(c))$.

Proof. Let $\mu\colon A\times B \to C$ be an abelian bihomomorphism. One can check quickly that ι_μ and π_μ are indeed morphisms and that $\mathcal{H}(\mu)$ is exact. By Remark 4.1.10, $\mathrm{H}(\mu)' = \{(0,0,c) : c \in \mu(A,B)\} \le \iota(C) \le \{(a,b,c) : \mu(a,B) = \mu(A,b) = 0\} = \mathrm{Z}(\mathrm{H}(\mu))$, so $\mathcal{H}(\mu)$ is indeed central-by-abelian. Finally $\pi^{-1}(A) = \{(a,0,c) : a \in A, b \in B\}$ is abelian by Remark 4.1.10. Similarly $\pi^{-1}(A)$ is abelian, thus so $\mathcal{H}(\mu)$ is indeed a polarised central-by-abelian extension.

Let $(\lambda_1\times\lambda_2,\kappa)\colon \mu \to (\bar\mu\colon \bar A\times\bar B \to \bar C)$ be a morphism of abelian bihomomorphisms, cf. Definition 2.2.36. Then $\gamma := \mathrm{H}(\lambda_1\times\lambda_2,\kappa)$ is a group morphism as

$$\begin{aligned}\gamma((a,b,c)*(a',b',c')) &= \gamma((a+a', b+b', c+\mu(a,b')+c'))\\ &= (\lambda_1(a+a'), \lambda_2(a+b'), \kappa(c+\mu(a,b')+c'))\\ &= (\lambda_1(a)+\lambda_1(a'), \lambda_2(a)+\lambda_2(a'), \kappa(c)+\bar\mu(\lambda_1(a),\lambda_2(b'))+\kappa(c'))\\ &= (\lambda_1(a),\lambda_2(a),\kappa(c)) * (\lambda_1(a'),\lambda_2(a'),\kappa(c'))\\ &= \gamma((a,b,c)) * \gamma((a',b',c')).\end{aligned}$$

where we used $\kappa\circ\mu = \bar\mu\circ(\lambda_1\times\lambda_2)$. This gives the following commutative diagram.

$$\begin{array}{ccccccccccc}\mathcal{H}(\mu) & : & 1 & \longrightarrow & C & \rightarrowtail & \mathrm{H}(\mu) & \longrightarrow & A\times B & \longrightarrow & 1\\ {\scriptstyle \mathcal{H}(\lambda_1\times\lambda_2,\kappa)}\downarrow & & & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\lambda_1\times\lambda_2} & & \\ \mathcal{H}(\bar\mu) & : & 1 & \longrightarrow & \bar C & \rightarrowtail & \mathrm{H}(\bar\mu) & \longrightarrow & \bar A\times\bar B & \longrightarrow & 1\end{array} \tag{4.1}$$

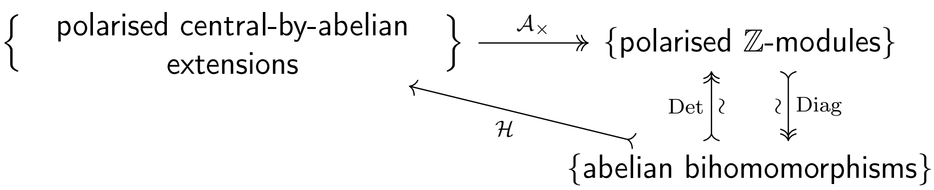
Given $\mathcal{H}(\mu)$, we want to recover $\mu\colon A \times B \to C$. The groups themselves can be read off from $\mathcal{H}(\mu)$. Let $a \in A$, $b \in B$. Let $(a, 0, c_1)$ and $(0, b, c_2)$ be arbitrary preimages of the chosen elements under π_μ. Then by Remark 4.1.10, $[(a, 0, c_1), (0, b, c_2)] = (0, 0, \mu(a, b)) = \iota_\mu(\mu(a, b))$. Since ι_μ is a given monomorphism, $\mu(a, b) \in C$ is determined, i.e. $\mathcal{H}$ is injective-on-objects.

By restriction, $\mathcal{H}(\lambda_1 \times \lambda_2, \kappa)$ determines $(\lambda_1 \times \lambda_2, \kappa)$, i.e. $\mathcal{H}$ is faithful. □

Remark 4.1.14. Note that $\mathrm{Z}(\mathrm{H}(\mu)) = \{(a, b, c) : \mu(a, B) = \mu(A, b) = 0\} \supseteq \{(0, 0, c) : c \in C\}$. Hence $\mathrm{H}(\mu)$ is abelian if and only if μ is the trivial map. This implies that the following statements are equivalent.

1. μ is non-degenerate, cf. Definition 4.1.9.
2. $\mathrm{Det}(\mu)$ is a non-degenerate, cf. Definition 3.2.1.
3. $\mathrm{H}(\mu)$ is non-degenerate, cf. Definition 4.1.9.
4. $\mathcal{H}(\mu)$ is non-degenerate, cf. Definition 1.2.1.
5. $\mathcal{Z}(\mathrm{H}(\mu)) \cong \mathcal{H}(\mu)$.

By Remark 3.2.17, $\mathcal{A}$ lifts to functor $\mathcal{A}_\times$ between polarised categories. The functors from above sit in the following diagram.



This diagram is *not* commutative: there is precisely one composition with at most three functors that does not give the identity.

Lemma 4.1.15. *The following compositions are all equal the identity functor in the appropriate category:* $\mathrm{Det} \circ \mathrm{Diag}$*,* $\mathrm{Diag} \circ \mathrm{Det}$*,* $\mathrm{Diag} \circ \mathcal{A}_\times \circ \mathcal{H}$*,* $\mathcal{A}_\times \circ \mathcal{H} \circ \mathrm{Diag}$*. (In particular,* $\mathcal{A}_\times \circ \mathcal{H} = \mathrm{Det}$*.) However,* $\mathcal{H} \circ \mathrm{Diag} \circ \mathcal{A}_\times \neq \mathrm{id}$*.*

Proof. Let $\mu\colon A \times B \to C$ be an abelian bihomomorphism. Then $\mathcal{A}_\times(\mathcal{H}(\mu)) = (A \oplus B, \omega, C)$ where $\omega\colon (a{+}b, a'{+}b') \mapsto \iota_\mu^{-1}([(a, b, 0), (a', b', 0)]) = \iota_\mu^{-1}((0, 0, \mu(a, b') - \mu(a', b))) = \det(\mu)(a + b, a' + b')$ using Lemma 4.1.13, Lemma 3.2.16 and Remark 4.1.10. Thus $\mathcal{A}_\times(\mathcal{H}(\mu)) = \mathrm{Det}(\mu)$. This implies all stated equalities using Lemma 4.1.6.

For the counterexample, note that the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ sits in the polarised central-by-abelian extension

$$\epsilon \;:\; 1 \longrightarrow \{\pm 1\} \overset{\subseteq}{\hookrightarrow} Q_8 \overset{\pi}{\twoheadrightarrow} \{[1], [i]\} \times \{[1], [j]\} \longrightarrow 1 \tag{4.2}$$

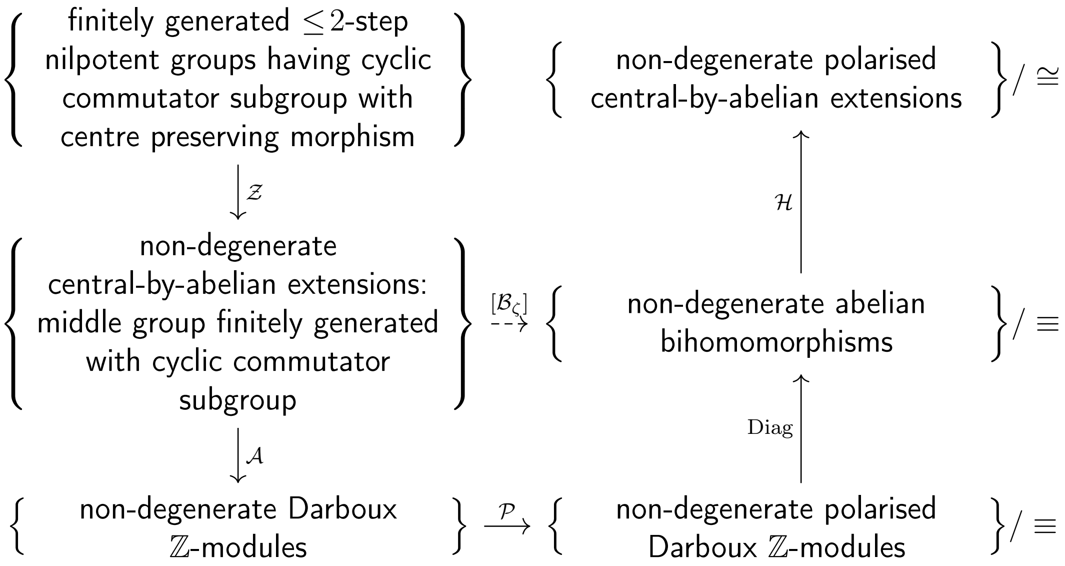
where π is given by $i \mapsto ([i], [1])$ and $j \mapsto ([1], [j])$. Then $\mathrm{H}(\mathrm{Diag}(\mathcal{A}_\times(\epsilon))) \cong \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ by definition. So this group has multiple elements of order 2, while Q_8 has a unique such element. (See Lemma 4.4.1 for other counterexamples of order p^3 for every prime p.) Thus $\mathcal{H}(\mathrm{Diag}(\mathcal{A}_\times(\epsilon)))$ and ϵ are not even isomorphic, let alone equal. □

Remark 4.1.16. Let G be a finitely generated ≤ 2-step nilpotent group with cyclic derived subgroup G'. Then $\mathcal{Z}(G)$ is a non-degenerate central-by-abelian extension, so $\mathcal{A}(\mathcal{Z}(G))$ is a non-degenerate Darboux $\mathbb{Z}$-module by Remark 3.2.17. Thus $\mathrm{Diag} \circ \mathcal{P}$ can be applied to yield (the isomorphism class of) some non-degenerate abelian bihomomorphism $\mu_G \colon A \times B \to \mathrm{Z}(G)$ (where actually $A \cong B$). This sits in the following diagram.

$$\begin{array}{rcccccccccc} \mathcal{Z}(G) & : & 1 & \longrightarrow & \mathrm{Z}(G) & \overset{\subseteq}{\hookrightarrow} & G & \overset{\pi_{\mathcal{Z}}}{\twoheadrightarrow} & G/\,\mathrm{Z}(G) & \longrightarrow & 1 \\ & & & & \| & & & & \| & & \\ \mathcal{H}(\mu_G) & : & 1 & \longrightarrow & \mathrm{Z}(G) & \rightarrowtail & \mathrm{H}(\mu_G) & \twoheadrightarrow & A \times B & \longrightarrow & 1 \end{array}$$

The groups G and $\mathrm{H}(\mu_G)$ share many properties: the order, isomorphism class of centre and commutator subgroup, nilpotency class, but they are not necessarily isomorphic in general as the example of $G = Q_8$ shows from Lemma 4.1.15. In general, there cannot even expect to have a morphism $G \to \mathrm{H}(\mu_G)$ making the diagram commutative in all cases, as then it would actually be an isomorphism by the 5-lemma, hence it would contradict the previous example. In fact by Lemma 4.1.15, this morphism exists if and only if we start from G satisfying $\mathcal{Z}(G) \cong \mathcal{H}(\mu)$ for some μ.

Our goal is to establish a monomorphism $\mathcal{Z}(G) \to \mathcal{H}(\mu)$ for suitable (non-degenerate) abelian bihomomorphism μ. As we will see in Subsection 4.2.1, it turns out that it is enough to enlarge the codomain of μ_G. This construction is denoted by $\mathcal{B}_\varsigma$ in the following overview diagram.



(4.3)

We conclude this section with some thoughts on Heisenberg groups over symplectic vector spaces and the necessity of polarised modules.

Remark 4.1.17 (Motivation for polarised structures). Using the functoriality of the maps, we see that $\mathcal{H} \circ \mathrm{Diag} \circ \mathcal{P}$ assigns a Heisenberg group (up to isomorphism) to every non-degenerate $\mathbb{Z}$-module. The independence of this isomorphism class of the choice of the polarised module is not surprising, because in the lack of 2-torsion, one can construct the Heisenberg group directly, without the use of polarised modules as follows.

Let (M, ω, C) be an alternating R-module. Suppose there exists an isomorphism $h \colon C \to C$ such that $2h = \mathrm{id}_C$. (Such h exists for example when $2 \in R$ has a multiplicative inverse, or when C is finite of odd order in which case it is unique.) Then one can generalise the notion of Heisenberg group corresponding to a symplectic vector space by setting the underlying set to be $H(\omega) := M \times C$, and defining the group operation by $(m, c) \cdot (m', c') := (m + m', c + c' + h(\omega(m, m')))$.

Suppose now that $(L_1 \oplus L_2, \omega, C)$ is a polarised $\mathbb{Z}$-module and $h\colon C \to C$ satisfies with $2h = \mathrm{id}_C$. Then $\varphi\colon \mathrm{H}(\mathrm{diag}(\omega)) \to H(\omega), (l_1 + l_2, c) \mapsto (l_1, l_2, c - h(\omega(l_1, l_2)))$ is an isomorphism between the two constructions above. Indeed, $\omega(L_i, L_i) = 0$ gives

$$\begin{aligned}
\varphi((l_1, l_2, c) * (l_1', l_2', c')) &= \varphi((l_1 + l_1', l_2 + l_2', c + \omega(l_1, l_2') + c')) \\
&= (l_1 + l_1' + l_2 + l_2', c + c' + \omega(l_1, l_2') - h(\omega(l_1 + l_1', l_2 + l_2'))) \\
&= (l_1 + l_2 + l_1' + l_2', c + c' - \\
&\qquad h(\omega(l_1, l_2)) - h(\omega(l_1', l_2')) + h(\omega(l_1 + l_2, l_1' + l_2'))) \\
&= (l_1 + l_2, c - h(\omega(l_1, l_2))) \cdot (l_1' + l_2', c' - h(\omega(l_1', l_2'))) \\
&= \varphi((l_1, l_2, c)) \cdot \varphi((l_1', l_2', c')).
\end{aligned}$$

This shows that polarised modules can treat all cases whether the group has 2-torsion or not. In fact, this is why this notion was introduced.

4.2 Embedding Nilpotent Groups of Class at most 2 to Twisted Heisenberg Groups

In this section, we show that every finitely generated $\leq$ 2-step nilpotent group is actually a subgroup of some non-degenerate twisted Heisenberg group, cf. the first part of Theorem B from page 8. In Subsection 4.2.1, we find an embedding of the $\leq$ 2-generated to serve as the base case of the main construction. Then in Subsection 4.2.2, we extend to group having cyclic centre case using central products and Theorem A from page 7, and find the optimal parameters of this embedding. Finally in Subsection 4.2.3, we treat the general case using subdirect products and Theorem A.

4.2.1 Cyclic Derived Group Case

In this section, we prove Lemma 4.2.7, the key statement about embedding groups to twisted Heisenberg groups by improving the construction from Section 4.1. The idea is to generalise slightly the notion of polarised central-by-abelian extensions and to enlarge the centre of the candidate twisted Heisenberg group.

We need to following categorical flavoured refinement of polarised central-by-abelian extensions. Recall Definition 2.2.22.

Definition 4.2.1. A *polarisation* of a central-by-abelian extension ϵ is a pair of monomorphisms

$$\begin{array}{ccccccccccc}
\epsilon_i & : & 1 & \longrightarrow & C_i & \xrightarrow{\iota_i} & G_i & \xrightarrow{\pi_i} & L_i & \longrightarrow & 1 \\
\downarrow & & & & \downarrow{\scriptstyle \kappa_i} & & \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \lambda_i} & & \\
\epsilon & : & 1 & \longrightarrow & C & \xrightarrow{\iota} & G & \xrightarrow{\pi} & M & \longrightarrow & 1
\end{array} \tag{4.4}$$

of central-by-abelian extensions such that $M = \lambda_1(L_1) \times \lambda_2(L_2)$ and G_1, G_2 are both abelian. This notion is not to be confused with Definition 4.1.12 or Definition 2.2.40.

Example 4.2.2. Every polarised central-by-abelian extension (cf. Definition 4.1.12) $\epsilon : 1 \to C \to G \to L_1 \times L_2 \to 1$ induces a polarisation of ϵ as follows.

$$\begin{array}{ccccccccccc} \epsilon_i & : & 1 & \longrightarrow & \iota^{-1}(\pi^{-1}(L_i) \cap \iota(C)) & \longrightarrow & \pi^{-1}(L_i) & \longrightarrow & L_i & \longrightarrow & 1 \\ & & & & \downarrow\cap & & \downarrow\cap & & \downarrow\cap & & \\ \epsilon & : & 1 & \longrightarrow & C & \xrightarrow{\iota} & G & \xrightarrow{\pi} & L_1 \times L_2 & \longrightarrow & 1 \end{array}$$

Lemma 4.2.3. *If $\epsilon_i \to \epsilon$ be a polarisation as at (4.4), then $G = \gamma_2(G_2)\iota(C)\gamma_1(G_1)$, a product of three abelian groups.*

Proof. Pick $g \in G$ arbitrarily. By assumption, $M = \lambda_1(L_1) \times \lambda_2(L_2)$, so we may write $\pi(g) = \lambda_1(l_1) + \lambda_2(l_2)$. From the surjectivity of π_i, pick $g_i \in G_i$ such that $\pi_i(g_i) = l_i$. Then $\pi(\gamma_2(g_2)^{-1} g \gamma_1(g_1)^{-1}) = -\lambda_2(\pi_2(g_2)) + (\lambda_1(l_1) + \lambda_2(l_2)) - \lambda_1(\pi_1(g_1)) = 0$ using the commutativity of the diagram (4.4). So $\gamma_2(g_2)^{-1} g \gamma_1(g_1)^{-1} \in \ker(\pi) = \mathrm{Im}(\iota)$, hence there is a $c \in C$ such that $\iota(c) = \gamma_2(g_2)^{-1} g \gamma_1(g_1)^{-1}$. Rearranging this gives the decomposition as claimed. □

Definition 4.2.4 (Z-extension)**.** A surjective group morphism $\zeta \colon G_2 \times C \times G_1 \to \hat{C}$ is called a Z-*extension* of the polarisation from (4.4) if $\zeta|_C$ is injective and for every $c_i \in C_i$, we have $\zeta(\iota_2(c_2)^{-1}, \kappa_2(c_2)\kappa_1(c_1), \iota_1(c_1)^{-1}) = 1$

Remark 4.2.5. The second condition states that ζ factors through the iterated central product of G_2, C, G_1 that amalgamates $\iota_i(C_i) \subseteq G_i$ and $\kappa_i(C_i) \subseteq C$. The first one says that this factor cannot introduce any identifications in the image of C.

Example 4.2.6. $K := \{\zeta(\iota_2(c_2)^{-1}, \kappa_2(c_2)\kappa_1(c_1), \iota_1(c_1)^{-1}) : c_i \in C_i\}$ is a normal subgroup of $G_2 \times C \times G_1$ as ι_i are central and C is abelian. (In fact, the whole group $G_2 \times C \times G_1$ is abelian). So Z-extensions always exist for every polarisation, since we can choose the natural projection $G_1 \times C \times G_1 \to G_1 \times C \times G_1/K =: \hat{C}$. The restriction to C is injective as ι_i and κ_i are injective.

Lemma 4.2.7 (Key, D.R.Sz.)**.** *Let ζ be a* Z*-extension of the polarisation from* (4.4)*. Write $\mathcal{A}(\epsilon) = (\lambda_1(L_1) \oplus \lambda_2(L_2), \omega, C)$, and define an abelian bihomomorphism*

$$\mathcal{B}_\zeta(\epsilon) := \hat{\mu} := \zeta|_C \circ \mathrm{diag}(\omega) \colon \lambda_1(L_1) \times \lambda_2(L_2) \to \hat{C}.$$

Then there is a $\curlyvee$-monomorphism as shown in the following diagram.

$$\begin{array}{ccccccccccc} \epsilon & : & 1 & \longrightarrow & C & \xrightarrow{\iota} & G & \xrightarrow{\pi} & L & \longrightarrow & 1 \\ \downarrow & & & & \downarrow \zeta|_C & & \downarrow \delta & & \| \lambda & & \\ \mathcal{H}(\mathcal{B}_\zeta(\epsilon)) & : & 1 & \longrightarrow & \hat{C} & \xrightarrow{\hat{\iota}} & \mathrm{H}(\mathcal{B}_\zeta(\epsilon)) & \xrightarrow{\hat{\pi}} & \lambda_1(L_1) \oplus \lambda_2(L_2) & \longrightarrow & 1 \end{array} \tag{4.5}$$

Proof. First note that $\mathcal{B}_\zeta(\epsilon)$ is indeed an abelian bihomomorphism because $\hat{C}$ is an abelian group, as it is the image of $G_2 \times C \times G_1$, an abelian group by definition.

We show that $\delta := (\delta_1, \delta_2, \delta_3)$ satisfies the statement

$$\delta_i \colon G \to \lambda_i(L_i), g \mapsto \lambda_i(\pi_i(g_i)), \quad \delta_3 \colon G \to \hat{C}, g \mapsto \zeta(g_2, c, g_1),$$

where $i \in \{1, 2\}$, for any decomposition $g = \gamma_2(g_2)\iota(c)\gamma_1(g_1)$ from Lemma 4.2.3. The map δ_i is actually the natural composition of group morphisms $G \xrightarrow{\pi} L = \lambda_1(L_1) \oplus \lambda_2(L_2) \to \lambda_i(L_i)$, in particular, δ_i is independent of the choice of the decomposition. To show that δ_3 is independent of the choice of the decomposition, let $\gamma_2(g_2)\iota(c)\gamma_1(g_1) = g = \gamma_2(g_2')\iota(c')\gamma_1(g_1')$. Then on one hand, $\lambda_i(\pi_i(g_i)) = \delta_i(g) = \lambda_i(\pi_i(g_i'))$ by above, hence from exactness, there are $c_i \in C_i$ such that $\iota_1(c_1) = g_1'g_1^{-1}$ and $\iota_2(c_2) = g_2^{-1}g_2'$. On the other hand, using $\iota(C) \subseteq \mathrm{Z}(G)$, rearranging the original equation gives $\iota(cc'^{-1}) = \gamma_2(g_2^{-1}g_2')\gamma_1(g_1'g_1^{-1}) = \iota(\kappa_2(c_2)\kappa_1(c_1))$, hence $cc'^{-1} = \kappa_2(c_2)\kappa_1(c_1)$ as ι is injective. Putting these together and keeping in mind that G_2 is abelian gives

$$\zeta(g_2, c, g_1) = \zeta(\iota_2(c_2)^{-1}, \kappa_2(c_1)\kappa_1(c_2), \iota_1(c_1)^{-1}) \cdot \zeta(g_2', c', g_1') = \zeta(g_2', c', g_1')$$

using the assumption on ζ, so δ_3 is indeed well defined.

Note that unlike the other maps, δ_3 is just a map of sets, *not* a group morphism. Its failure to be a group morphism is measured by $\hat{\mu}$. Indeed, pick decompositions $g = \gamma_2(g_2)\iota(c)\gamma_1(g_1)$ and $g' = \gamma_2(g_2')\iota(c')\gamma_1(g_1')$. Set $x := \omega(\gamma_1(g_1), \gamma_2(g_2')) \in C$, so $\iota(x) = [\gamma_1(g_1), \gamma_2(g_2')]$. Use this to find a decomposition of the product as

$$\begin{aligned} gg' &= \gamma_2(g_2)\iota(c)\gamma_1(g_1)\gamma_2(g_2')\iota(c')\gamma_1(g_1') \\ &= \gamma_2(g_2)\gamma_2(g_2')\iota(cc')[\gamma_1(g_1), \gamma_2(g_2')]\gamma_1(g_1)\gamma_1(g_1') \\ &= \gamma_2(g_2g_2')\iota(cc'x)\gamma_1(g_1g_1'). \end{aligned}$$

Then by definition and using the commutativity of the diagram,

$$\begin{aligned} \delta_3(gg') &= \zeta(g_2, c, g_1)\zeta(g_2', c', g_1')\zeta(1, x, 1) \\ &= \delta_3(g)\delta_3(g')\zeta|_C(\iota^{-1}([\gamma_1(g_1), \gamma_2(g_2')])) \\ &= \delta_3(g)\delta_3(g')\zeta|_C(\omega(\pi(\gamma_1(g_1)), \pi(\gamma_2(g_2')))) \\ &= \delta_3(g)\delta_3(g')\zeta|_C(\mathrm{diag}(\omega)(\lambda_1(\pi_1(g_1)), \lambda_2(\pi_2(g_2')))) \\ &= \delta_3(g)\delta_3(g')\hat{\mu}(\delta_1(g), \delta_2(g')). \end{aligned}$$

This property together with using Remark 4.1.10 imply that δ is a group morphism:

$$\begin{aligned} \delta(gg') &= (\delta_1(gg'), \delta_2(gg'), \delta_3(gg')) \\ &= (\delta_1(g)\delta_1(g'), \delta_2(g)\delta_2(g'), \delta_3(g)\delta_3(g')\hat{\mu}(\delta_1(g), \delta_2(g'))) \\ &= (\delta_1(g), \delta_2(g), \delta_3(g)) * (\delta_1(g'), \delta_2(g'), \delta_3(g')) \\ &= \delta(g) * \delta(g'). \end{aligned}$$

We check that the diagram (4.5) is commutative. Indeed, if $c \in C$, then using the decomposition $\iota(c) = \gamma_2(1)\iota(c)\gamma_1(1)$ gives $\delta \circ \iota = (c \mapsto (0, 0, \zeta(1, c, 1))) = \hat{\iota} \circ \zeta|_C$ by definitions. Similarly, the decomposition $g = \gamma_2(g_2)\iota(c)\gamma_1(g_1) \in G$ gives $\lambda \circ \pi = (g \mapsto \pi(g) = \delta_1(g) + \delta_2(g)) = \hat{\pi} \circ \delta$.

Finally, the injectivity of $\zeta|_C$ implies that of δ using the 4-lemma. □

Remark 4.2.8. If we chose to define the twisted Heisenberg groups in Definition 4.1.9 to be isomorphic to the group of *lower* unitriangular matrices, then at the definition of δ_3 we could have taken g_1 and g_2 the other way around.

Remark 4.2.9. By Lemma 2.2.23, $\mathrm{H}(\mathcal{B}_\zeta(\epsilon))$ is the external central product of $\hat{C}$ and G. Loosely speaking we extended the centre of G by $\hat{C}$ to obtain a twisted Heisenberg groups, cf. the terminology Definition 4.2.4.

Remark 4.2.10. By construction and using Remark 4.1.14, the central-by-abelian extension ϵ is non-degenerate if and only if $\mathrm{H}(\mathcal{B}_\zeta(\epsilon))$ is non-degenerate.

Proposition 4.2.11 (D.R.Sz.)**.** *For every finitely generated $\leq$2-step nilpotent group G with cyclic derived group G', there exists a $\curlyvee$-monomorphism*

$$
\begin{array}{ccccccccccc}
\mathcal{Z}(G) & : & 1 & \longrightarrow & \mathrm{Z}(G) & \overset{\subseteq}{\hookrightarrow} & G & \overset{\pi_Z}{\twoheadrightarrow} & G/\,\mathrm{Z}(G) & \longrightarrow & 1 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow\wr & & \\
\mathcal{H}(\mu) & : & 1 & \longrightarrow & \hat{C} & \overset{\subseteq}{\hookrightarrow} & \mathrm{H}(\mu) & \overset{\pi_\mu}{\longrightarrow} & A \times A & \longrightarrow & 1
\end{array}
$$

of non-degenerate central-by-abelian extensions for a suitable $\mu\colon A \times A \to \hat{C}$. In particular, G is isomorphic to a normal subgroup of a non-degenerate twisted Heisenberg group.

Proof. Recall Definition 4.1.5 and the maps from (4.3). Pick an arbitrary representative $(A \oplus B, \omega, \mathrm{Z}(G))$ of $\mathcal{P}(\mathcal{A}(\mathcal{Z}(G)))$. Lemma 4.1.4 shows that $A \cong B$. Consider the polarisation from Example 4.2.2 induced by $(L_1 \oplus L_2, \omega, \mathrm{Z}(G))$ and let ζ be the Z-extension from Example 4.2.6. Then $\mathcal{B}_\zeta(\mathcal{Z}(G))$ from Lemma 4.2.7 with the isomorphism above induces a bihomomorphism $\mu\colon A \times A \to \hat{C}$ satisfying the statement. Non-degeneracy follows from Remark 4.2.10. The image of the monomorphism $G \to \mathrm{H}(\mu)$ from the diagram is a normal subgroup by Lemma 2.2.23. □

Remark 4.2.12. The proof actually gives bounds on the number of generators: $d(A) = d(G/\,\mathrm{Z}(G))/2$ and $d(C) \leq d(G)$. For our main motivation (Theorem C from page 9), better bounds are needed for $d(C)$ so that we can embed every finite $\leq$ 2-step nilpotent group to Mumford's theta groups (cf. Proposition 4.3.2 and Lemma 5.1.3). We will work these out in the next section with an improved method. Note that by applying Corollary 3.1.26, the condition of G' being cyclic can be dropped (see the proof of Theorem 4.2.19 for details).

4.2.2 Cyclic Centre Case giving Optimal Bounds

The goal of this section is to prove Proposition 4.2.18. To do so, we use Lemma 4.2.7 to handle the $\leq$ 2-generated case (Lemma 4.2.13). The key challenge is to obtain a twisted Heisenberg group that has a cyclic centre. This is obtained by taking repeatedly maximal central products using Lemma 2.2.27. The case with cyclic centres is then handled using Theorem 3.2.23 and the 'commutativity' of taking central products and applying the Heisenberg functor.

The next statement gives optimal bounds for the number of generators in Proposition 4.2.11. To find these bounds, we construct a more careful polarisation than Example 4.2.2 and a more sophisticated Z-extension than Example 4.2.6 for the central-by-abelian extension from (3.3).

Lemma 4.2.13. *Let E be a ≤ 2-generated ≤ 2-step nilpotent group with cyclic centre. Then there is a monomorphism of central-by-abelian extensions*

$$\begin{array}{ccccccccccc}
\mathcal{Z}^\flat(E) & : & 1 & \longrightarrow & \mathrm{Z}(E) & \longrightarrow & E & \longrightarrow & E' \times E' & \longrightarrow & 1 \\
\downarrow f & & & & \downarrow \kappa & & \downarrow \delta & & \| & & \\
\mathcal{H}(\mu) & : & 1 & \longrightarrow & C & \longrightarrow & \mathrm{H}(\mu) & \longrightarrow & E' \times E' & \longrightarrow & 1
\end{array}$$

for a suitable non-degenerate $\mu\colon E' \times E' \to C$ where C is cyclic. Furthermore, if E is finite, then $|C| \mid \exp(E)$, the exponent of E. Otherwise $C \cong \mathbb{Z}$ and f is an isomorphism, cf. Lemma 3.3.6.

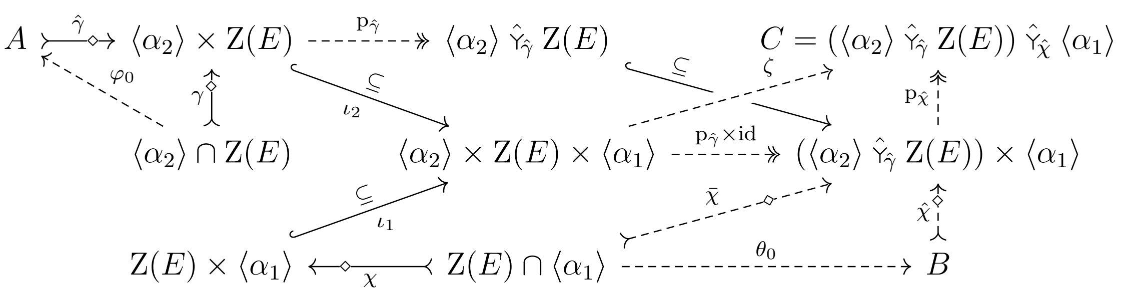
Proof. Pick generators $E = \langle \alpha_1, \alpha_2 \rangle$. Consider the following diagram where $\pi_1\colon g \mapsto [g, \alpha_2]$, $\pi_2\colon g \mapsto [\alpha_1, g]$, and λ_i is the inclusion to the ith factor.

$$\begin{array}{ccccccccccc}
\epsilon_i & : & 1 & \longrightarrow & \mathrm{Z}(E) \cap \langle \alpha_i \rangle & \overset{\subseteq}{\hookrightarrow} & \langle \alpha_i \rangle & \xrightarrow{\pi_i} & E' & \longrightarrow & 1 \\
 & & & & \downarrow \cap & & \downarrow \cap & & \lambda_i \downarrow \cap & & \\
\mathcal{Z}^\flat(E) & : & 1 & \longrightarrow & \mathrm{Z}(E) & \overset{\subseteq}{\hookrightarrow} & E & \xrightarrow{\pi} & E' \times E' & \longrightarrow & 1
\end{array} \tag{4.6}$$

Note that all the maps are morphisms by Lemma 2.2.3. We claim that (4.6) is in fact a polarisation of the central-by-abelian extension $\mathcal{Z}^\flat(E)$. For this, it is enough to verify that ϵ_i are short exact sequences. Exactness at the first term is clear. Since E is generated by α_1, α_2, $\ker(\pi_1)$ (resp. $\ker(\pi_2)$) consists of elements that commute with α_2 (resp. α_1), hence with the whole E, thus $\ker(\pi_i) = \mathrm{Z}(E) \cap \langle \alpha_i \rangle$ showing the exactness at the middle terms. $E' = \langle [\alpha_1, \alpha_2] \rangle$ by Lemma 2.2.3, which shows that π_i are surjective.

Next, we construct a Z-extension for (4.6) by separating the cases whether the groups $\mathrm{Z}(E) \cap \langle \alpha_i \rangle$ are trivial for $i \in \{1, 2\}$.

Case 1 Assume that both $\mathrm{Z}(E) \cap \langle \alpha_1 \rangle$, $\mathrm{Z}(E) \cap \langle \alpha_2 \rangle$ are non-trivial. Consider the following diagram.



This is obtained as follows. Extend the central pair γ on the left defined in Remark 2.2.20 to a maximal one $\hat{\gamma}$ using Lemma 2.2.31. Post-composing the natural central pair χ from the bottom row by $(\mathrm{p}_{\hat{\gamma}} \times \mathrm{id}_{\langle \alpha_1 \rangle}) \circ \iota_1$ gives another central pair $\bar{\chi}$ as the maps are centre preserving. As before, extend this $\bar{\chi}$ to a maximal central pair $\hat{\chi}$. Define $C := (\langle \alpha_2 \rangle \curlyvee_{\hat{\gamma}} \mathrm{Z}(E)) \curlyvee_{\hat{\chi}} \langle \alpha_1 \rangle$. We claim that

$$\begin{array}{rccccc}
\zeta \ : & \langle \alpha_2 \rangle \times \mathrm{Z}(E) \times \langle \alpha_1 \rangle & \xrightarrow{\mathrm{p}_{\hat{\gamma}} \times \mathrm{id}} & (\langle \alpha_2 \rangle \curlyvee_{\hat{\gamma}} \mathrm{Z}(E)) \times \langle \alpha_1 \rangle & \xrightarrow{\mathrm{p}_{\hat{\chi}}} & C \\
 & (g_2, z, g_1) & \longmapsto & (\lceil g_2, z \rfloor, g_1) & \longmapsto & \lceil \lceil g_2, z \rfloor, g_1 \rfloor
\end{array} \tag{4.7}$$

is a Z-extension for (4.6). Indeed, ζ is surjective as it is a composition of such maps by Remark 2.2.18. The restriction $\zeta|_{\mathrm{Z}(E)}$ is injective by Remark 2.2.20. By construction, $\zeta \circ \iota_2 \circ \gamma$ factors though $\hat{\gamma} \circ \mathrm{p}_{\hat{\gamma}}$, but the latter map is trivial by Remark 2.2.18, hence so is the former one. Similarly $\zeta \circ \iota_1 \circ \chi$ is trivial as so is $\hat{\chi} \circ \mathrm{p}_{\hat{\chi}}$. The triviality of these maps is equivalent to the condition on $\ker(\zeta)$ from Definition 4.2.4. Note that C is cyclic and $|C| = \mathrm{lcm}(|\langle \alpha_2 \rangle, |\,\mathrm{Z}(E)|, |\langle \alpha_1 \rangle|) \bigm| \exp(E)$ in the finite case by Lemma 2.2.27.

Case 2 Suppose exactly one of $\mathrm{Z}(E) \cap \langle \alpha_i \rangle$ is non-trivial. Without loss of generality, let $\mathrm{Z}(E) \cap \langle \alpha_1 \rangle = 1$. Let $\bar{\gamma}$ and $C := \langle \alpha_2 \rangle \curlyvee_{\hat{\gamma}} \mathrm{Z}(E)$ be as above and set

$$\begin{array}{rcl} \zeta \ : \ \langle \alpha_2 \rangle \times \mathrm{Z}(E) \times \langle \alpha_1 \rangle & \longrightarrow \langle \alpha_2 \rangle \times \mathrm{Z}(E) & \xrightarrow{\mathrm{p}_{\hat{\gamma}}} C \\ (g_2, z, g_1) & \longmapsto (g_2, z) & \longmapsto \lceil g_2, z \rfloor \end{array}$$

As above, C is cyclic, $\zeta|_{\mathrm{Z}(E)}$ is injective and $\zeta \circ \iota_2 \circ \gamma$ is trivial. Since $\mathrm{Z}(E) \cap \langle \alpha_1 \rangle = 1$ by assumption, the other part of the required condition on $\ker(\zeta)$ is trivially true. If E is finite, then $|C| = \mathrm{lcm}(|\langle \alpha_2 \rangle, |\,\mathrm{Z}(E)|) \bigm| \exp(E)$ as above.

Case 3] The final case is when both $\mathrm{Z}(E) \cap \langle \alpha_i \rangle$ are trivial. Now

$$\begin{array}{rcl} \zeta \ : \ \langle \alpha_2 \rangle \times \mathrm{Z}(E) \times \langle \alpha_1 \rangle & \longrightarrow & C := \mathrm{Z}(E) \\ (g_2, z, g_1) & \longmapsto & z \end{array}$$

is a Z-extension and C is cyclic by assumption.

Now Lemma 4.2.7 is applicable with the Z-extension constructed above for (4.6) and produces the diagram (4.6) from the statement. Non-degeneracy follows from Remark 4.2.10. If E is finite, then $|C| \bigm| \exp(E)$ in all cases as we have seen during our case-by-case analysis.

Assume now that E is infinite. Then $E' \subseteq \mathrm{Z}(E)$ and the short exact sequence $\mathcal{Z}^\flat(E)$ shows that $\mathrm{Z}(E)$ is also infinite. So C must be also infinite as κ is injective, thus $C \cong \mathbb{Z}$. If $\langle \alpha_i \rangle \cap \mathrm{Z}(E) = 1$ for $i \in \{1, 2\}$, then Case 3 applies and thus $\zeta|_{\mathrm{Z}(E)}$ is an isomorphism. Otherwise, without loss of generality, there is a $k \neq 0$ with $\alpha_1^k \in \mathrm{Z}(E)$. Then $1 = [\alpha_1^k, \alpha_2] = [\alpha_1, \alpha_2]^k$ by Lemma 2.2.3, so $[\alpha_1, \alpha_2]$ has finite order, but $[\alpha_1, \alpha_2] \in E' \subseteq \mathrm{Z}(E) \cong \mathbb{Z}$, so $[\alpha_1, \alpha_2] = 1$. Then by Lemma 2.2.3, $E' = \langle [\alpha_1, \alpha_2] \rangle = 1$, thus E is abelian, i.e. $\langle \alpha_i \rangle \subseteq \mathrm{Z}(E)$. So Remark 2.2.20 shows that $\zeta|_{\mathrm{Z}(E)}$ is an isomorphism. In any case, the 5-lemma implies that δ is also an isomorphism. □

Recall Definition 2.2.17, Definition 2.2.38, Definition 2.2.40, Lemma 4.1.13.

Lemma 4.2.14. *Let $f := (\lambda \times \nu, \kappa) : \mu_0 \rightarrowtail\!\!\!\rightarrow \mu_1 \times \mu_2$ be a central pair of abelian bihomomorphisms. Then there is a natural isomorphism*

$$\varphi \colon \mathcal{H}(\mu_1) \curlyvee_{\mathcal{H}(f)} \mathcal{H}(\mu_2) \to \mathcal{H}(\mu_1 \curlyvee_f \mu_2)$$

where the central pair $(\kappa, \mathrm{H}(\lambda \times \nu, \kappa), \lambda \times \nu) := \mathcal{H}(f)$ of central-by-abelian extensions is given by $\mathcal{H}(f)_i := (\lambda_i, \mathrm{H}(\lambda_i \times \nu_i, \kappa_i), \kappa_i)$ for $i \in \{1, 2\}$.

The central-by-abelian extension $\mathcal{H}(\mu_1 \curlyvee_f \mu_2)$ is non-degenerate if and only if both central-by-abelian extensions $\mathcal{H}(\mu_i)$ are non-degenerate. Furthermore, the central pair f of abelian bihomomorphisms is maximal if and only if $\mathcal{H}(f)$ is a maximal central pair of central-by-abelian extensions.

Proof. The proof is a routine check, but we include it for the sake of completeness. First we check that $\mathcal{H}(f)$ is indeed a central pair of extensions, i.e. that $\gamma_i \colon \mathrm{H}(\mu_0) \to \mathrm{H}(\mu_i)$ is injective and maps to $\mathrm{Z}(\mathrm{H}(\mu_i))$. As $\gamma_i(a_0, b_0, c_0) = (\lambda_i(a_0), \nu_i(b_0), \kappa_i(c_0))$, injectivity follows from that of the maps involved. By definition of central pairs of abelian bihomomorphisms, $\mu_i(a_i, \nu_i(b_0)) = \mu_i(\lambda_i(a_0), b_i) = 0$ for $a_i \in A_i$, $b_i \in B_i$, so Remark 4.1.14 implies that $\gamma_i(a_0, b_0, c_0) \in \mathrm{Z}(\mathrm{H}(\mu_i))$.

We show that

$$\begin{array}{ccccccccc}
1 & \longrightarrow & C_1 \curlyvee_\kappa C_2 & \overset{\Upsilon(\iota)}{\rightarrowtail} & \mathrm{H}(\mu_1) \curlyvee_{\mathrm{H}(f)} \mathrm{H}(\mu_2) & \overset{\Upsilon(\pi)}{\twoheadrightarrow} & (A_1 \times B_1) \curlyvee_{\lambda\times\nu} (A_2 \times B_2) & \longrightarrow & 1 \\
 & & \Big\Vert \varphi_1 & & \wr\downarrow \varphi_2 & & \wr\downarrow \varphi_3 & & \\
1 & \longrightarrow & C_1 \curlyvee_\kappa C_2 & \overset{\iota_\mu}{\rightarrowtail} & \mathrm{H}(\mu_1 \curlyvee_f \mu_2) & \overset{\pi_\mu}{\twoheadrightarrow} & (A_1 \curlyvee_\lambda A_2) \times (B_1 \curlyvee_\nu B_2) & \longrightarrow & 1
\end{array}$$

is an isomorphism of central-by-abelian extensions where $\mu \coloneqq \mu_1 \curlyvee_f \mu_2$ and

$$\begin{aligned}
\varphi_2 \colon \lceil (a_1, b_1, c_1), (a_2, b_2, c_2) \rfloor &\mapsto (\lceil a_1, a_2 \rfloor, \lceil b_1, b_2 \rfloor, \lceil c_1, c_2 \rfloor), \\
\varphi_3 \colon \lceil (a_1, b_1), (a_2, b_2) \rfloor &\mapsto (\lceil a_1, a_2 \rfloor, \lceil b_1, b_2 \rfloor).
\end{aligned}$$

To see that φ_2 is well-defined, pick $h_i = (a_i, b_i, c_i), h_i' = (a_i', b_i', c_i') \in \mathrm{H}(\mu_i)$ such that $(h_1, h_2)\operatorname{Im}(\mathrm{H}(f)) = (h_1', h_2')\operatorname{Im}(\mathrm{H}(f))$. By Definition 2.2.17 and Remark 4.1.10, this is equivalent to the existence of $a_0 \in A_0$, $b_0 \in B_0$, $c_0 \in C_0$ such that

$$\begin{aligned}
(a_1', b_1', c_1') &= (a_1, b_1, c_1) * (\lambda_1(a_0), \kappa_1(b_0), \kappa_1(c_0)) \\
&= (a_1 + \lambda_1(a_0), b_1 + \nu_1(b_0), c_1 + \mu_1(a_1, \nu_1(b_0)) + \kappa_1(c_0)) \\
&= (a_1 + \lambda_1(a_0), b_1 + \nu_1(b_0), c_1 + \kappa_1(c_0))
\end{aligned}$$

where we used that $(\lambda_1 \times \nu_1, \kappa_1)$ is central. Similarly, we obtain

$$\begin{aligned}
(a_2', b_2', c_2') &= (a_2, b_2, c_2) * (-\lambda_2(a_0), -\kappa_2(b_0), -\kappa_2(c_0)) \\
&= (a_2 - \lambda_2(a_0), b_2 - \nu_2(b_0), c_2 - \kappa_2(c_0)).
\end{aligned}$$

In turn, this happens if and only if

$$\begin{aligned}
(a_1, a_2)\operatorname{Im}(\lambda) &= (a_1', a_2')\operatorname{Im}(\lambda), \\
(b_1, b_2)\operatorname{Im}(\nu) &= (b_1', b_2')\operatorname{Im}(\nu), \quad , \\
(c_1, c_2)\operatorname{Im}(\kappa) &= (c_1', c_2')\operatorname{Im}(\kappa).
\end{aligned}$$

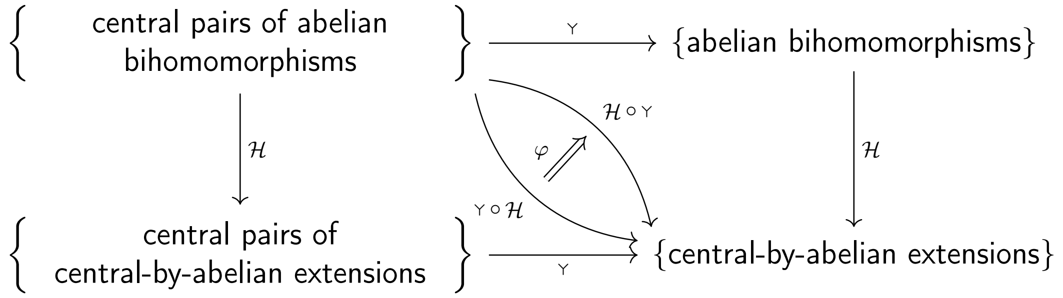
Thus φ_2 is well defined and is injective. Surjectivity is evident from definition. Finally, by Definition 2.2.38,

$$\begin{aligned}
\varphi_2(\lceil h_1, h_2 \rfloor * \lceil h_1', h_2' \rfloor) &= \varphi_2(\lceil h_1 * h_1', h_2 * h_2' \rfloor) \\
&= (\lceil a_1 + a_1', a_2 + a_2' \rfloor, \lceil b_1 + b_1', b_2 + b_2' \rfloor, \\
&\qquad \lceil c_1 + \mu_1(a_1, b_1') + c_1', c_2 + \mu_2(a_2, b_2') + c_2' \rfloor) \\
&= (\lceil a_1, a_2 \rfloor, \lceil b_1, b_2 \rfloor, \lceil c_1, c_2 \rfloor) * (\lceil a_1', a_2' \rfloor, \lceil b_1', b_2' \rfloor, \lceil c_1', c_2' \rfloor) \\
&= \varphi_2(\lceil h_1, h_2 \rfloor) * \varphi_2(\lceil h_1', h_2' \rfloor),
\end{aligned}$$

so ν is an isomorphism. Checking that φ_3 is a well-defined isomorphism is a similar, but shorter computation. The commutativity of the diagram follows from the definition of the maps.

The statement about non-degeneracy follows from Lemma 2.2.37 and Remark 4.1.14, whereas the part on maximality is the consequence of Lemma 2.2.37 and Definition 2.2.40 as both are equivalent to κ being maximal by definition. □

Remark 4.2.15. Categorically speaking, φ is a natural isomorphism from $\curlyvee \circ \mathcal{H}$ to $\mathcal{H} \circ \curlyvee$.



Lemma 4.2.16. *Let $\gamma : A \rightarrowtail\!\!\!\to G_1 \times G_2$ be a central pair. Then $\mathcal{Z}(\gamma) := (\gamma, \gamma, 0) : \mathcal{Z}(A) \rightarrowtail\!\!\!\to \mathcal{Z}(G_2) \times \mathcal{Z}(G_2)$ is a central pair of central-by-abelian extensions for which there exists a natural isomorphism*

$$\tau \colon \mathcal{Z}(G_1) \curlyvee_{\mathcal{Z}(\gamma)} \mathcal{Z}(G_1) \to \mathcal{Z}(G_1 \curlyvee_\gamma G_2).$$

Proof. Straightforward computation shows that

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Z}(G_1) \curlyvee_\gamma \mathrm{Z}(G_2) & \xrightarrow{\curlyvee(\iota)} & G_1 \curlyvee_\gamma G_2 & \xrightarrow{\curlyvee(\pi)} & (G_1/\,\mathrm{Z}(G_1)) \times (G_2/\,\mathrm{Z}(G_2)) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \wr\, \tau_1} & & \| \, {\scriptstyle \tau_2} & & \downarrow{\scriptstyle \wr\, \tau_3} & & \\
1 & \longrightarrow & \mathrm{Z}(G_1 \curlyvee_\gamma G_2) & \xhookrightarrow{\subseteq} & G_1 \curlyvee_\gamma G_2 & \xrightarrow{\pi_{\mathcal{Z}}} & (G_1 \curlyvee_\gamma G_2)/\,\mathrm{Z}(G_1 \curlyvee_\gamma G_2) & \longrightarrow & 1
\end{array}$$

is the desired isomorphism where $\tau_1 \colon \lceil g_1, g_2 \rfloor \mapsto \lceil g_1, g_2 \rfloor$, $\tau_3 \colon (g_1\,\mathrm{Z}(G_1), g_2\,\mathrm{Z}(G_2)) \mapsto \lceil g_1, g_2 \rfloor\, \mathrm{Z}(G_1 \curlyvee_\gamma G_2)$. ☐

Remark 4.2.17. The following are equivalent.

1. $\gamma : A \rightarrowtail\!\!\!\to G_1 \times G_2$ is a maximal central pair.
2. $\gamma : A \rightarrowtail\!\!\!\to \mathrm{Z}(G_1) \times \mathrm{Z}(G_2)$ is a maximal central pair.
3. $\mathcal{Z}(\gamma)$ is a maximal central pair of central-by-abelian extensions.

Restricting Proposition 4.2.11 to the cyclic centre case, we obtain optimal bounds to the number of generators of the twisted Heisenberg group, cf. Remark 4.2.12.

Proposition 4.2.18 (D.R.Sz.)**.** *For every finitely generated ≤ 2-step nilpotent group G with cyclic centre, there exists a $\curlyvee$-monomorphism*

$$\begin{array}{cccccccccc}
\mathcal{Z}(G) & : & 1 & \longrightarrow & \mathrm{Z}(G) & \xhookrightarrow{\subseteq} & G & \xrightarrow{\pi_{\mathcal{Z}}} & G/\,\mathrm{Z}(G) & \longrightarrow 1 \\
\downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \wr\, \nu} & \\
\mathcal{H}(\mu) & : & 1 & \longrightarrow & C & \xhookrightarrow{\subseteq} & \mathrm{H}(\mu) & \xrightarrow{\pi_\mu} & A \times A & \longrightarrow 1
\end{array} \qquad (4.8)$$

of non-degenerate central-by-abelian extensions for a suitable $\mu \colon A \times A \to C$ where $C \cong \mathrm{H}(\mu)$ is cyclic. If G is finite, then so are A, C and $\exp(A) = \exp(G') \mid |\mathrm{Z}(G)| \mid |C| \mid \exp(G)$, otherwise f is an isomorphism and $C \cong \mathbb{Z}$, $A \cong \mathbb{Z}^t$ for some t.

Proof. Let $E_0; E_1, \dots, E_t$ be a central product decomposition of G from Theorem 3.2.23 after noting that $G' \subseteq \mathrm{Z}(G)$ is cyclic. We prove the statement by induction on $t = \frac{1}{2}d(G/\mathrm{Z}(G))$. The base case $t = 0$ is provided by Lemma 4.2.13 and Remark 3.2.21.

Now assume $t > 0$. Let $G_1 := E_0E_1 \dots E_{t-1} \leq G$ and $G_2 := E_t \leq G$. Then G is the internal central product of G_1 and G_2, thus $G_0 := G_1 \cap G_2 \leq \mathrm{Z}(G)$. Let $\gamma : G_0 \rightarrowtail G_1 \times G_2$ be the natural central pair induced by the embeddings. (2.4) shows that $\xi \colon G_1 \curlyvee_\gamma G_2 \to G$ is an isomorphism, thus γ is maximal by Lemma 2.2.27 as $\mathrm{Z}(G_1 \curlyvee_\gamma G_2) \cong \mathrm{Z}(G)$ is cyclic by assumption. In fact, as the centre of a non-trivial nilpotent group is non-trivial, either both $\mathrm{Z}(G_1)$ and $\mathrm{Z}(G_2)$ are finite or both are infinite. Set $\epsilon_j := \mathcal{Z}(G_j)$ for $j \in \{0, 1, 2\}$, define $e := \mathcal{Z}(\gamma) = (\gamma, \gamma, 0) : \epsilon_0 \rightarrowtail \epsilon_1 \times \epsilon_2$ and note that e is maximal by the discussion above. By induction, there are $\curlyvee$-monomorphisms $f_i = (\zeta_i, \delta_i, \nu_i) \colon \mathcal{Z}(G_i) \to \mathcal{H}(\mu_i \colon A_i \times A_i \to \hat{C}_i)$ for $i \in \{1, 2\}$ as at the statement. Apply Lemma 2.2.43 with this setup (and use the notation from its proof). Note that as $G_0/\mathrm{Z}(G_0)$ is trivial, the resulting $\hat{\epsilon}_0$ must be isomorphic to $\mathcal{H}(\mu_0 \colon 0 \times 0 \to \hat{C}_0)$. Similarly, the resulting $\hat{e} : \mathcal{H}(\mu_0) \rightarrowtail \mathcal{H}(\mu_1) \times \mathcal{H}(\mu_2)$ must be of the form $\mathcal{H}(m)$ for $m = (0 \times 0, \hat{\kappa})$ where $\hat{\kappa} : \hat{C}_0 \rightarrowtail \hat{C}_1 \times \hat{C}_2$ is maximal. Note that m is trivially a central pair of abelian bihomomorphisms.

$$\begin{array}{ccccccc}
\mathcal{Z}(G_0) & \xrightarrow{\mathcal{Z}(\gamma)} & \mathcal{Z}(G_1) \times \mathcal{Z}(G_2) & \xrightarrow{\mathrm{p}_{\mathcal{Z}(\gamma)}} & \mathcal{Z}(G_1) \curlyvee_{\mathcal{Z}(\gamma)} \mathcal{Z}(G_2) & \xrightarrow[\sim]{\tau} & \mathcal{Z}(G_1 \curlyvee_\gamma G_2) \\
\downarrow f_0 & & \downarrow f_1 \times f_2 & & \downarrow \curlyvee(f_0, f_1 \times f_2) & & \downarrow \bar{f} \\
\mathcal{H}(\mu_0) & \xrightarrow{\mathcal{H}(m)} & \mathcal{H}(\mu_1) \times \mathcal{H}(\mu_2) & \xrightarrow{\mathrm{p}_{\mathcal{H}(m)}} & \mathcal{H}(\mu_1) \curlyvee_{\mathcal{H}(m)} \mathcal{H}(\mu_2) & \xrightarrow[\sim]{\varphi} & \mathcal{H}(\mu_1 \curlyvee_m \mu_2)
\end{array} \tag{4.9}$$

Then Lemma 2.2.43 shows that $\curlyvee(f_0, f_1 \times f_2)$ from the diagram above is a monomorphism. Then Lemma 4.2.14 and Lemma 4.2.16 induce a monomorphism $\bar{f}$ as above, hence $\xi \colon G_1 \curlyvee_\gamma G_2 \cong G$ produces a monomorphism $f := (\zeta, \delta, \nu) \colon \mathcal{Z}(G) \to \mathcal{H}(\mu)$ where $\mu := \mu_1 \curlyvee_m \mu_2$.

We check that these maps satisfy the statement. The non-degeneracy of μ follows from Lemma 4.2.14, hence $\mathcal{H}(\mu)$ is also non-degenerate by Remark 4.1.14. By definition, we have $\mu \colon A \times A \to C$ where $A := A_1 \times A_2$ and $C := \hat{C}_1 \curlyvee_{\hat{\kappa}} \hat{C}_2$. As $\zeta_i \colon \mathrm{Z}(G_i) \to \hat{C}_i$ is an injection between cyclic groups, either both $\hat{C}_i$ are finite or both are infinite. Thus C is cyclic by Lemma 2.2.27. The rightmost (non-trivial) groups from (4.9) give the following diagram.

$$\begin{array}{ccc}
(G_1/\mathrm{Z}(G_1)) \times (G_2/\mathrm{Z}(G_2)) & \xrightarrow[\sim]{\xi \circ \tau_3} & G/\mathrm{Z}(G) \\
\wr\downarrow \nu_1 \times \nu_2 & & \downarrow \nu \\
(A_1 \times A_1) \times (A_2 \times A_2) & = & A \times A
\end{array}$$

By induction, the maps ν_i are isomorphisms, hence so is ν.

For the final part, assume that G is finite. Then $\mathrm{Z}(G_i)$ and $\hat{C}_i$ are necessarily finite. In this case, $|C_i| \mid \exp(G_i)$ by induction, thus Lemma 2.2.27 implies $|C| = \mathrm{lcm}\{|\hat{C}_1|, |\hat{C}_2|\} \mid \mathrm{lcm}\{\exp(G_1), \exp(G_2)\} \mid \exp(G)$. On the other hand, we have $\exp(A) = \exp(A^2) = \exp(G/\mathrm{Z}(G)) = \exp(G')$ by Theorem 3.2.23 and Definition 3.2.4.

Suppose G is infinite. Then at least one of G_1 and G_2 is infinite, say G_1. Then by induction, $\hat{C}_1$ is infinite, but this is isomorphic to a subgroup of $C = \hat{C}_1 \curlyvee_{\hat{\kappa}} \hat{C}_2$ by Remark 2.2.20, thus C is infinite, i.e. $C \cong \mathbb{Z}$. $\hat{C}_2$ is also infinite by Lemma 2.2.27, so by induction both $A_1 \cong \mathbb{Z}^{t_1}$ and $A_2 \cong \mathbb{Z}^{t_2}$, hence $A = A_1 \times A_2 \cong \mathbb{Z}^t$ for $t = t_1 + t_2$. The leftmost (non-trivial) groups from (4.9) give the following diagram.

$$\begin{array}{ccccccc} G_0 & \overset{\kappa}{\rightarrowtail} & \mathrm{Z}(G_1) \times \mathrm{Z}(G_2) & \xrightarrow{\mathrm{p}_\kappa} & \mathrm{Z}(G_1) \hat{\curlyvee}_\gamma \mathrm{Z}(G_2) & \underset{\sim}{\overset{\xi \circ \tau_1}{\rightarrowtail}} & \mathrm{Z}(G) \\ \wr \downarrow \zeta_0 & & \wr \downarrow \zeta_1 \times \zeta_2 & & \downarrow \curlyvee(\zeta_0, \zeta_1 \times \zeta_2) & & \wr \downarrow \zeta \\ \hat{C}_0 & \overset{\hat{\kappa}}{\rightarrowtail} & \hat{C}_1 \times \hat{C}_2 & \xrightarrow{\mathrm{p}_{\hat{\kappa}}} & \hat{C}_1 \curlyvee_{\hat{\kappa}} \hat{C}_2 & = & C \end{array}$$

By induction, ζ_1 and ζ_2 are isomorphisms, so the maximality of κ implies that ζ_0 is also an isomorphism. So functoriality from Remark 2.2.19 shows that ζ is also an isomorphism. Thus the 4-lemma implies that f is an isomorphism in this infinite case. □

4.2.3 General Case

We prove the main statement of this chapter (Theorem 4.2.19). By applying Proposition 4.2.18 and the subdirect product reduction Corollary 3.1.26, we obtain an embedding to a large twisted Heisenberg group with too many generators. We take the smallest twisted Heisenberg group which contains the image of the previous embedding to obtain a better bound on the number of generators.

Theorem 4.2.19 (D.R.Sz., cf. [Mag98, Corollary 2.21])**.** *For every finitely generated $\leq$2-step nilpotent group G, there exists a monomorphism*

$$\begin{array}{cccccccccccc} \mathcal{Z}(G) & : & 1 & \longrightarrow & \mathrm{Z}(G) & \overset{\subseteq}{\hookrightarrow} & G & \overset{\pi_Z}{\twoheadrightarrow} & G/\mathrm{Z}(G) & \longrightarrow & 1 \\ \downarrow f & & & & \downarrow \zeta & & \downarrow \delta & & \downarrow \nu & & \\ \mathcal{H}(\mu) & : & 1 & \longrightarrow & C & \overset{\subseteq}{\hookrightarrow} & \mathrm{H}(\mu) & \overset{\pi_\mu}{\longrightarrow} & A \times B & \longrightarrow & 1 \end{array}$$

of non-degenerate central-by-abelian extensions for suitable abelian bihomomorphism $\mu\colon A \times B \to C$ such that $d(A), d(B) \leq d(G/\mathrm{Z}(G))$ and $d(C) = d(\mathrm{Z}(G))$. If G is finite, then so are all other groups and the exponents satisfy

$$\exp(G/\mathrm{Z}(G)) \mid \exp(A \times B) \mid \exp(G') \mid \exp(\mathrm{Z}(G)) \mid \exp(C) \mid \exp(G).$$

Furthermore, there is a monomorphism $\mathrm{H}(\mu) \rightarrowtail \prod_{i=1}^{d(\mathrm{Z}(G))} \mathrm{H}(\mu_i\colon A_i \times A_i \to C_i)$ where each $\mathrm{H}(\mu_i)$ is non-degenerate with C_i cyclic and $d(A_i) \leq \frac{1}{2} d(G)$.

Remark 4.2.20. When $\mathrm{Z}(G)$ is cyclic, Proposition 4.2.18 actually gives better (and optimal) bounds for the number of generators.

Remark 4.2.21. Recall Remark 4.2.12. Given any such diagram gives the following lower bounds on the number of generators. The monomorphism ζ shows $d(\mathrm{Z}(G)) \leq d(C)$, so $d(C)$ is as small as possible; whereas ν gives $d(G/\mathrm{Z}(G)) \leq d(A \times B)$ (which in fact can be attained, see Proposition 4.2.18) showing that the bounds on $d(A)$ and $d(B)$ may not be optimal.

Question 4.2.22. What is the optimal value of $d(A \times B)$? To give a better upper bound than above, one may need to develop some version of Lemma 2.3.1 for matrices taking values in $\mathbb{Z}^n$. Are all infinite finitely generated $\leq$2-step nilpotent group in fact twisted Heisenberg groups as in the finite case?

Proof. Using Corollary 3.1.26, write $\varphi\colon G \rightarrowtail \prod_{i=1}^{n} G_i$ as a subdirect product where $n \coloneqq d(\mathrm{Z}(G))$ and each $\mathrm{Z}(G_i)$ is cyclic. Then G_i are all ≤ 2-step nilpotent as this class is closed under taking quotients, so Proposition 4.2.18 gives $\curlyvee$-monomorphisms $f_i = (\zeta_i, \delta_i, \nu_i)\colon\ \mathcal{Z}(G_i) \to \mathcal{H}(\mu_i\colon A_i \times B_i \to C_i)$ (where $B_i = A_i$). Let $\bar{A} \coloneqq \prod_{i=1}^{n} A_i$, $\bar{B} \coloneqq \prod_{i=1}^{n} B_i$, $\bar{C} \coloneqq \prod_{i=1}^{n} C_i$ and $\bar{\mu} \coloneqq \prod_{i=1}^{n} \mu_i\colon \bar{A} \times \bar{B} \to \bar{C}$. Then $\mathcal{H}(\bar{\mu})$ is non-degenerate by recursive application of Lemma 4.2.14 keeping in mind that the direct product is a special case of the central product where the underlying central pairs map every time from the trivial group. Taking the product of these central-by-abelian extensions (cf. Definition 2.2.22, Remark 3.2.15), we obtain the following commutative diagram where $\prod$ is a shorthand for $\prod_{i=1}^{n}$ here.

$$\begin{array}{rcccccccccc}
\mathcal{Z}(G) & : & 1 & \longrightarrow & \mathrm{Z}(G) & \overset{\subseteq}{\hookrightarrow} & G & \xrightarrow{\pi_{\mathcal{Z}}} & G/\mathrm{Z}(G) & \longrightarrow & 1 \\
\text{Corollary 3.1.26}\ \big\downarrow & & & & \big\downarrow \varphi|_{\mathrm{Z}(G)} & & \big\downarrow \varphi & & \big\downarrow [\varphi] & & \\
\prod \mathcal{Z}(G_i) & : & 1 & \longrightarrow & \prod \mathrm{Z}(G_i) & \overset{\subseteq}{\hookrightarrow} & \prod G_i & \xrightarrow{\prod \pi_{\mathcal{Z}}} & \prod G_i/\mathrm{Z}(G_i) & \longrightarrow & 1 \\
\text{Proposition 4.2.18}\ \big\downarrow \prod f_i & & & & \big\downarrow \prod \zeta_i & & \big\downarrow \prod \delta_i & & \wr \big\downarrow \prod \nu_i & & \\
\prod \mathcal{H}(\mu) & : & 1 & \longrightarrow & \prod C_i & \overset{\subseteq}{\hookrightarrow} & \prod \mathrm{H}(\mu_i) & \xrightarrow{\prod \pi_{\mu_i}} & \prod A_i \times B_i & \longrightarrow & 1 \\
\text{Lemma 4.2.14}\ \big\downarrow \wr & & & & \big\| & & \big\downarrow \wr & & \big\| & & \\
\mathcal{H}(\bar{\mu}) & : & 1 & \longrightarrow & \bar{C} & \overset{\subseteq}{\hookrightarrow} & \mathrm{H}(\bar{\mu}) & \xrightarrow{\pi_{\bar{\mu}}} & \bar{A} \times \bar{B} & \longrightarrow & 1
\end{array}$$

Denote by $\bar{f} = (\bar{\zeta}, \bar{\delta}, \bar{\nu})\colon\ \mathcal{Z}(G) \to \mathcal{H}(\bar{\mu})$ the resulting monomorphism. This may have more generators than stated, so we take a suitable subobject of $\mathcal{H}(\bar{\mu})$. Let $A \leq \bar{A}$ be the image of $G/\mathrm{Z}(G) \xrightarrow{\bar{\nu}} \bar{A} \times \bar{B} \to \bar{A}$, and $B \leq \bar{B}$ be that of $G/\mathrm{Z}(G) \xrightarrow{\bar{\nu}} \bar{A} \times \bar{B} \to \bar{B}$. Then $d(A)$ and $d(B)$ are at most $d(G/\mathrm{Z}(G))$. Let $C \coloneqq \langle \bar{\zeta}(\mathrm{Z}(G)), \bar{\mu}(A, B) \rangle \leq \bar{C}$. Then $d(\mathrm{Z}(G)) = d(\bar{\zeta}(\mathrm{Z}(G))) \leq d(C) \leq d(\bar{C}) = \sum_{i=1}^{n} d(C_i) \leq n = d(\mathrm{Z}(G))$, hence comparing the two ends give $d(C) = d(\mathrm{Z}(G))$.

Define $\mu\colon A \times B \to C, (a, b) \mapsto \bar{\mu}(a, b)$, an abelian bihomomorphism. The image of $\bar{f}$ lies in $\mathcal{H}(\mu)$ by definition, so restricting the domain to $\mathcal{H}(\mu)$ gives a map $f = (\zeta, \delta, \nu)\colon\ \mathcal{Z}(G) \to \mathcal{H}(\mu)$, i.e. $\bar{f} = (\mathcal{H}(\mu) \rightarrowtail \mathcal{H}(\bar{\mu})) \circ f$ for the natural inclusion map. We show that this f satisfies the statement.

We check that μ is non-degenerate. Pick $0 \neq a \in A$ and write $a = (a_1, \ldots, a_n) \in \prod_{i=1}^{n} A_i$. Then without loss of generality, $a_1 \neq 0$. Then by the non-degeneracy of μ_1, there is a $b_1' \in B_1$ such that $0 \neq \mu_1(a_1, b_1') \in C_1$. By the diagram above, there is $g_1' \in G_1$ such that $\nu_1(g_1'\,\mathrm{Z}(G_1)) = (0, b_1')$. As φ is a subdirect product, there is $g' \in G$ such that the 1st factor of $[\varphi](g'\,\mathrm{Z}(G))$ is $g_1'\,\mathrm{Z}(G_1)$. Write $b' = (b_1', \ldots, b_n')$ for the image of $g'\,\mathrm{Z}(G)$ under $G/\mathrm{Z}(G) \xrightarrow{\bar{\nu}} \bar{A} \times \bar{B} \to \bar{B}$. By construction, b_1' coincides with the above choice. By definition, $b \in B$, and $\mu(a, b) = (\bar{\mu}_1(a_1, b_1'), \ldots, \bar{\mu}_n(a_n, b_n')) \neq 0$ as the first factor is non-trivial by construction. This argument is valid when the roles of A and B are swapped, hence μ is non-degenerate.

Assume that G is finite and consider the statement on the exponents. $\exp(G/\mathrm{Z}(G)) \mid \exp(A \times B)$ follows from ν being a monomorphism of abelian groups. For every i, $\exp(A_i \times B_i)\exp(G_i') \mid \exp(G')$ using Proposition 4.2.18 and the fact that as G_i is a quotient of G. Thus $\exp(A \times B) \mid \exp(\bar{A} \times \bar{B}) = \operatorname{lcm}\{\exp(A_i \times B_i) : 1 \leq i \leq n\} \mid \exp(G')$. Since G is ≤ 2-step nilpotent, we have $G' \subseteq \mathrm{Z}(G)$, so $\exp(G') \mid \exp(\mathrm{Z}(G))$. The embedding $\zeta\colon \mathrm{Z}(G) \rightarrowtail C$ shows $\exp(\mathrm{Z}(G)) \mid \exp(C)$. Once again using Proposition 4.2.18, we see that $\exp(C_i) \mid \exp(G_i) \mid \exp(G)$ as G_i is a quotient of G. Then $\exp(C) \mid \exp(\bar{C}) = \operatorname{lcm}\{\exp(C_i) : 1 \leq i \leq n\} \mid \exp(G)$ as stated. □

4.3 Structure of Non-Degenerate Heisenberg Groups

In this section, we prove the second part of Theorem B from page 8. Motivated by Mumford's theta groups, we introduce the canonical Heisenberg groups as special cases of twisted Heisenberg groups without externally prescribed abelian bihomomorphisms, but only by giving the underlying abelian groups. In the cyclic centre non-degenerate case, twisted groups can be actually embedded to the canonical ones; moreover, they can be expressed as a central product of canonical groups. The proofs are simple translations of the standard form (Corollary 3.2.13) of non-degenerate Darboux modules to various other categories.

Definition 4.3.1 (Canonical Heisenberg groups)**.** Let B, C be abelian groups, define a bihomomorphism $\nu : \mathrm{Hom}(B, C) \times B \to C$ by $(\alpha, b) \mapsto \alpha(b)$. Define $\mathrm{H}(B, C) \coloneqq \mathrm{H}(\nu)$, the *canonical Heisenberg group over* B and C, and the non-degenerate*canonical Heisenberg group* $\mathrm{H}(B) \coloneqq \mathrm{H}(B, B)$ *over* B.

Proposition 4.3.2. *Let* $\mathrm{H}(\mu : A \times B \to C)$ *be finitely generated, non-degenerate with cyclic centre. Then there is a monomorphism*

$$\mathrm{H}(\mu) \rightarrowtail \mathrm{H}(B, C)$$

of non-degenerate Heisenberg groups that is an isomorphism if $\mathrm{H}(\mu)$ *is finite.*

Proof. Let $\Phi : A \to \mathrm{Hom}(B, C), a \mapsto (b \mapsto \mu(a, b))$. This is a monomorphism as μ is non-degenerate, cf. Definition 2.2.36 and Remark 4.1.14. Let $\nu : \mathrm{Hom}(B, C) \times B \to C$ be as in Definition 4.3.1. Then $(\Phi \times \mathrm{id}_B, \mathrm{id}_C) : \mu \to \nu$ is a monomorphism of abelian bihomomorphisms, so Lemma 4.1.13 gives a monomorphism $\mathcal{H}(\mu) \to \mathcal{H}(\nu)$, which in turn induces the monomorphism $\mathrm{H}(\mu) \to \mathrm{H}(\nu) = \mathrm{H}(B, C)$ as at the statement.

We check that ν is non-degenerate (and hence $\mathrm{H}(\nu)$ is also no-degenerate by Remark 4.1.14). If $0 \neq \alpha \in \mathrm{Hom}(B, C)$, then there is $b \in B$ such that $\alpha(b) \neq 0$ by definition of $\alpha \neq 0$. For the other condition of non-degeneracy, suppose $0 \neq b \in B$. Then by non-degeneracy of μ, there is $a \in A$ such that $\mu(a, b) \neq 0$. Then for $\alpha \coloneqq \Phi(a) \in \mathrm{Hom}(B, C)$, $\alpha(b) = \Phi(a)(b) = \mu(a, b) \neq 0$.

The alternating modules $\mathcal{A}_\times(\mathcal{H}(\mu)) = \mathrm{Det}(\mu) = (A \oplus B, \det(\mu), C)$ and $\mathcal{A}_\times(\mathcal{H}(\nu)) = \mathrm{Det}(\nu) = (\mathrm{Hom}(B, C) \oplus B, \det(\nu), C)$ from Lemma 4.1.15 are both polarised and non-degenerate Darboux $\mathbb{Z}$-modules, so Lemma 4.1.4 gives the isomorphisms $A \cong B \cong \mathrm{Hom}(B, C)$. If $\mathrm{H}(\mu)$ is finite, then $A \cong \mathrm{Hom}(B, C)$ are finite, but then the monomorphism $\Phi : A \to \mathrm{Hom}(B, C)$ from above is necessarily an isomorphism, hence so is the induced map $\mathrm{H}(\mu) \to \mathrm{H}(\nu) = \mathrm{H}(B, C)$. □

The following statement gives a more concrete description.

Lemma 4.3.3. *Let* A, B *be finitely generated abelian groups and* C *be a cyclic group. The following are equivalent.*

1. *There is a non-degenerate abelian bihomomorphism* $\mu : A \times B \to C$.

2. (a) *Either all groups* A, B *and* C *are finite, and* $A \cong B$ *and* $\exp(B) \mid |C|$,

   (b) *or* $A \cong B \cong \mathbb{Z}^t$ *for some* $t \in \mathbb{N}_0$ *and* $C \cong \mathbb{Z}$.

In the finite case (2a)*, the twisted Heisenberg groups corresponding to any* $\mu : A \times B \to C$ *are isomorphic.*

In the infinite case (2b)*, the isomorphism classes of the resulting twisted Heisenberg groups are parametrised by sequences* $d_1 \mid \dots \mid d_t$ *of positive integers. If, furthermore, there is another sequence* $\bar{d}_1 \mid \dots \mid \bar{d}_t$ *with* $\bar{d}_i \mid d_i$*, then the group corresponding to the numbers* d_i *is isomorphic to a subgroup of the one corresponding to the numbers* $\bar{d}_i$*. In particular,* $\mathrm{H}(B, C)$ *corresponds to the constant* 1 *sequence.*

Proof. Suppose that $\mu : A \times B \to C$ is a non-degenerate bihomomorphism. Then $\mathcal{A}_\times(\mathcal{H}(\mu)) = \mathrm{Det}(\mu) = (A \oplus B, \det(\mu), C)$ is a non-degenerate Darboux $\mathbb{Z}$-module by Remark 3.2.17, Definition 4.1.9, Remark 4.1.14. Then there is a polarised isomorphism $(A \oplus B, \det(\mu), C) \cong (\bigoplus_{i=1}^t C_i \oplus \bigoplus_{i=1}^t C_i, \mathrm{trdet}, C)$ given by Corollary 3.2.13. The definition of polarised isomorphism implies that $A \cong \bigoplus_{i=1}^t C_i \cong B$. If C is infinite, the so are the non-trivial submodules $C_i \subseteq C$, hence A, B, C are as at (2b). Otherwise all C_i are finite, hence so are A and B. For the part about the exponent, pick $b \in B$. Then for any $a \in A$, $\mu(a, |C|b) = |C|\mu(a, b) = 0$, so the definition non-degeneracy (Definition 2.2.36) gives $|C|b = 0$, i.e. $\exp(B) \mid |C|$. This gives (2a).

In the other direction, given these conditions on A, B, C, then using the structure theorem of finitely generated $\mathbb{Z}$-modules, one can pick submodules $C_i \subseteq C$ such that $A \cong \bigoplus_{i=1}^t C_i \cong B$. Then $(\mathrm{diag}(\mathrm{trdet}) : \prod_{i=1}^t C_i \times \prod_{i=1}^t C_i \to C) := \mathrm{Diag}(\bigoplus_{i=1}^t C_i \oplus \bigoplus_{i=1}^t C_i, \mathrm{trdet}, C)$ is a non-degenerate abelian bihomomorphism, hence the isomorphisms above induce induce such $\mu : A \times B \to C$.

For the part about the corresponding twisted Heisenberg groups, note that since the non-trivial cyclic $\mathbb{Z}$-modules $C_i \subseteq C$ are the invariants from Lemma 3.2.10, hence so are $d_i := |C : C_i|$. So these numbers parametrise the isomorphism class of $\mathrm{Det}(\mu)$, hence that of μ by Lemma 4.1.6, hence that of $\mathcal{H}(\mu)$ by Lemma 4.1.13. In the finite case, d_i are the invariant factors of A, hence are uniquely determined. In the infinite case, $C \cong \mathbb{Z}$ has a unique submodule of arbitrary given positive index, so we indeed have the parametrisation as stated. If $\bar{d}_i \mid d_i$, then let $C_i \subseteq C$ be of index d_i and $\bar{C}_i \subseteq C$ of index $\bar{d}_i$. Then $C_i \subseteq \bar{C}_i$ which induces a monomorphism $(\bigoplus_{i=1}^t C_i \oplus \bigoplus_{i=1}^t C_i, \mathrm{trdet}, C) \rightarrowtail (\bigoplus_{i=1}^t \bar{C}_i \oplus \bigoplus_{i=1}^t \bar{C}_i, \mathrm{trdet}, C)$. Applying $\mathcal{H} \circ \mathrm{Diag}$ gives the stated monomorphism. □

We can find an analogue of Proposition 3.3.11 by translating the structure result of Lemma 4.1.4 using Lemma 4.1.6.

Lemma 4.3.4. *Let* $\mu : A \times B \to C$ *be a non-degenerate abelian bihomomorphism of finitely generated groups where* C *is a cyclic group. Then there exists cyclic subgroups* $C \geq C_1 \geq \dots \geq C_t$ *and non-degenerate abelian bihomomorphisms* $\mu_0 : 0 \times 0 \to C$*, and* $\mu_i : C_i \times C_i \to C_i$*, respectively* $\bar{\mu}_i : C_i \times C_i \to C$*, for* $1 \leq i \leq t$ *such that for suitable maximal central pairs* $m^{(i)}$*, respectively* $\bar{m}^{(i)}$ *of abelian bihomomorphisms*

$$\mu \cong \mu_0 \;\hat{\curlyvee}_{m^{(1)}}\; \mu_1 \;\hat{\curlyvee}_{m^{(2)}} \dots \hat{\curlyvee}_{m^{(t)}}\; \mu_t \cong \bar{\mu}_1 \;\hat{\curlyvee}_{\bar{m}^{(1)}} \dots \hat{\curlyvee}_{\bar{m}^{(t-1)}}\; \bar{\mu}_t.$$

Proof. Let $\bar{C}_0 := C_0$, and for $1 \leq i \leq t$, let $\kappa_1^{(i)} : C_i \subseteq C \rightarrowtail \hat{C}_{i-1}$ be given by Remark 2.2.20, and set $\kappa_2^{(i)} : C_k \to C_k$ to be the identity map. Then $\kappa^{(i)} : C_i \rightarrowtail\!\!\!\twoheadrightarrow \bar{C}_{i-1} \times C_i$ is a maximal central product by Remark 2.2.25. Let $\bar{C}_i = \bar{C}_{i-1} \;\hat{\curlyvee}_{\kappa^{(i)}}\; C_i$. Note that $\bar{C}_i \cong C$. Pick generators $c_i \in C_i$, define non-degenerate abelian bihomomorphisms $\mu_i : C_i \times C_i \to C_i$ by $\mu(c_i, c_i) := c_i$ for $1 \leq i \leq t$. Let $\bar{\mu}_0 := \mu_0 : 0 \times 0 \to C$, and define recursively $\bar{\mu}_i := \mu_{i-1} \mathbin{\hat{\curlyvee}_{m^{(i)}}} \mu_i$ for $1 \leq i \leq t$ where the maximal central pair $m^{(i)}$ of abelian bihomomorphisms is given by the following diagram.

$$\begin{array}{ccccccc}
\tilde{\mu}_i & : & 0 & \times & 0 & \xrightarrow{\mu_0} & C_i \\
\Big\downarrow{\scriptstyle m^{(i)}} & & \Big\downarrow{\scriptstyle \lambda^{(i)}} & & \Big\downarrow{\scriptstyle \nu^{(i)}} & & \Big\downarrow{\scriptstyle \kappa^{(i)}} \\
\bar{\mu}_{i-1} \times \mu_i & : & \prod_{i=1}^{i-1} C_i \times C_i & \times & \prod_{i=1}^{i-1} C_i \times C_i & \xrightarrow{\bar{\mu}_{i-1} \times \mu_i} & \bar{C}_{i-1} \times C_i
\end{array}$$

Consider $(\bigoplus_{i=1}^t C_i^2, \mathrm{trdet}, C)$ from Example 3.2.3 with the choice of generators $c_i \in C_i$ as above. By construction, $\det(\bar{\mu}_t) \cong \mathrm{trdet}$ via $\bar{C}_t \cong C$. On the other hand, $\mathrm{Det}(\mu) = (A \oplus B, \det(\mu), C)$ is a non-degenerate polarised Darboux $\mathbb{Z}$-module by Remark 4.1.14 and Remark 4.1.8, so $\det(\mu) \cong \mathrm{trdet}$ by Lemma 4.1.4. Thus Lemma 4.1.6 shows that $\bar{\mu}_t \cong \mu$ as stated.

The other isomorphism from the statement is done analogously. Alternatively, set $\bar{\mu}_i := \mu_0 \curlyvee_{n^{(i)}} \mu_i$ for the natural central pair $n^{(i)}$ of abelian bihomomorphisms induced by $C_i \subseteq C$ as above. □

Lemma 4.3.5. *Let $\mu : A \times B \to C$ be be a non-degenerate bihomomorphism between additive finite cyclic groups. Then every automorphism of $\mathrm{Z}(\mathrm{H}(\mu))$ extends to one of $\mathrm{H}(\mu)$.*

Proof. Pick $\theta \in \mathrm{Aut}(\mathrm{Z}(\mathrm{H}(\mu)))$. For the center of the twisted Heisenberg group, we have $\mathrm{Z}(\mathrm{H}(\mu)) = \{(0,0,c) : c \in C)\}$ by assumption and Remark 4.1.10, so there is $\varphi_3 \in \mathrm{Aut}(C)$ such that $\theta((0,0,c)) = (0,0,\varphi_3(c))$. Fix a generator b_0 of B. If $\mu(a, b_0) = 0$, then $\mu(a, B) = 0$, so $a = 0$ by non-degeneracy. Hence $f : A \to C, a \mapsto \mu(a, b_0)$ is an injective morphism, so by restriction, there is $\varphi_1 \in \mathrm{Aut}(A)$ such that $f \circ \varphi_1 = \varphi_3 \circ f$. Thus for any $a \in A$, $b \mapsto \varphi_3(\mu(a,b))$ and $b \mapsto \mu(\varphi_1(a), b)$ are equal as $B \to C$ morphisms as they agree of the generator b_0. Hence $\varphi_3(\mu(a,b)) = \mu(\varphi_1(a), b)$. Define $\varphi : \mathrm{H}(\mu) \to \mathrm{H}(\mu), (a,b,c) \mapsto (\varphi_1(a), b, \varphi_3(c))$. By construction, this is a bijection extending θ, and short computation shows that it is indeed a group morphism. □

Proposition 4.3.6. *Let $\mathcal{H}(\mu : A \times B \to C)$ be non-degenerate consisting of finite groups and assume that C is cyclic. Then there exists a unique subgroup chain $C \geq C_1 \geq \cdots \geq C_t$ such that whenever $\mu_0 : 0 \times 0 \to C$, $\mu_i : C_i \times C_i \to C_i$, respectively $\bar{\mu}_i : C_i \times C_i \to C$, for $1 \leq i \leq t$ are all non-degenerate abelian bihomomorphisms, then*

$$\mathcal{H}(\mu) \cong \mathcal{H}(\mu_0) \mathbin{\hat{\curlyvee}} \mathcal{H}(\mu_1) \mathbin{\hat{\curlyvee}} \ldots \mathbin{\hat{\curlyvee}} \mathcal{H}(\mu_t) \cong \mathcal{H}(\bar{\mu}_1) \mathbin{\hat{\curlyvee}} \ldots \mathbin{\hat{\curlyvee}} \mathcal{H}(\bar{\mu}_t), \tag{4.10}$$

and the isomorphism class of this iterated maximal central product does not depend on the amalgamations (which are then suppressed from the notation).

Proof. We show that the subgroups produced by Lemma 4.3.4 satisfy the statement. By Proposition 4.3.2, we may assume that the bihomomorphisms μ_i are as in Lemma 4.3.4. Then the existence of the decomposition of $\mathcal{H}(\mu)$ as at the statement follows from Lemma 4.3.4, Lemma 4.1.13 and Lemma 4.2.14. Using the notation of Lemma 4.1.13, let $\mathcal{H}(m^{(i)}) = (\kappa^{(i)}, \mathrm{H}(\lambda^{(i)} \times \nu^{(i)}, \kappa^{(i)}), \lambda^{(i)} \times \nu^{(i)})$ be the underlying maximal central pairs of central-by-abelian extensions. (These may be different from the ones given by Lemma 4.3.4.) By Definition 2.2.38, $\kappa^{(i)}$ are maximal central pairs of central monomorphisms. Then as $\mathcal{H}(\mu_i)$ are non-degenerate, $\mathrm{H}(\lambda^{(i)} \times \nu^{(i)}, \kappa^{(i)})$ also have to be maximal, cf. Remark 4.2.17 and Remark 4.1.14. Write $1 \to \bar{C} \to \bar{G} \to \bar{M} \to 1$ for

$\mathcal{H}(\mu_0) \curlyvee_{m^1} \dots \curlyvee_{m^t} \mathcal{H}(\mu_t)$. The isomorphism classes of $\bar{C}$ and $\bar{G}$ are independent of the choice of the amalgamations by Lemma 2.2.28 and Lemma 4.3.5. By Lemma 2.2.37, $\lambda^{(i)}$ and $\nu^{(i)}$ are maps from the trivial group, i.e. the corresponding central products are actually direct products. Hence $\bar{M} \cong \prod_{i=1}^{t} C_i^2$ which is also independent of the choice of the amalgamations

Consider the uniqueness part. The isomorphism from (4.10) implies that in any such decomposition, $A \times B \cong \bar{M}$, hence $A \times B \cong \prod_{i=1}^{t} C_i^2$. Uniqueness of the isomorphism classes of C_i then follow from uniqueness of the invariant factor decomposition of $A \times B$, but as C is a finite cyclic group, this isomorphism class determines the subgroup $C_i \leq C$ completely.

The other part of the statement follows similarly from Lemma 4.3.4. □

When Theorem 3.2.23 is applied to finite twisted Heisenberg groups, then as opposed to Remark 3.2.24, the resulting $\leq$ 2-generated groups are invariants (and are in fact Heisenberg groups themselves).

Corollary 4.3.7. *Let* $\mathrm{H}(\mu : A \times B \to C)$ *be finite, non-degenerate with cyclic centre. There are unique positive integers* $d_t \mid d_{t-1} \mid \dots \mid d_1$ *(namely the invariant factors of* B*) such that*

$$\mathrm{H}(\mu) \cong C \,\hat{\curlyvee}\, \mathrm{H}(\mathbb{Z}_{d_1}) \,\hat{\curlyvee} \dots \hat{\curlyvee}\, \mathrm{H}(\mathbb{Z}_{d_t}) \cong \mathrm{H}(\mathbb{Z}_{d_1}, C) \,\hat{\curlyvee} \dots \hat{\curlyvee}\, \mathrm{H}(\mathbb{Z}_{d_t}, C),$$

and the isomorphism class is independent of the choice of the amalgamations in the central products.

Remark 4.3.8. The groups $\mathrm{H}(\mathbb{Z}_{d_i})$ are 2-generated. If $|C| = d_0$, then $d_1 \mid d_0$ and $C \cong \mathrm{H}(0, \mathbb{Z}_{d_0})$. When $d_1 = d_0$, i.e. when $\exp(B) = \exp(C)$, this term can be omitted from the decomposition.

Proof. This follows from Proposition 4.3.6 and Proposition 4.3.2. □

4.4 Examples

We conclude the section by demonstrating the development of earlier sections via some concrete examples of $\leq$ 2-generated groups. These examples exhibit the obstacles and the motivation of the tools from the previous sections.

Extra-special groups Let $p \in \mathbb{N}_0$ be a prime number and define a group E_p by the presentation

$$E_p = \langle \alpha, \beta, \gamma : \gamma = [\alpha, \beta], 1 = [\gamma, \alpha] = [\gamma, \beta], \alpha^p = \gamma, \beta^p = \gamma, \gamma^p = 1 \rangle.$$

This is a non-abelian (extra-special) p-group of order p^3 and exponent p^2. We review the construction of Chapter 4 through the demonstrative example of E_p.

The group E_p is *not* of the form introduced in Section 4.1.

Lemma 4.4.1. *For all primes* p*,* E_p *is not isomorphic to any twisted Heisenberg group* $\mathrm{H}(\mu\colon A \times B \to C)$*.*

Proof. On the contrary, suppose $E_p \cong \mathrm{H}(\mu)$ for some abelian bihomomorphism $\mu\colon A \times B \to C$. Since E_p is not abelian, μ is not the trivial map, cf. Remark 4.1.14. In particular, none of A, B, C can be the trivial group. So $p^3 = |E_p| = |\,\mathrm{H}(\mu)| = |A||B||C|$ shows that $A \cong B \cong C \cong \mathbb{Z}_p$. Choosing these isomorphisms suitably, we can assume $\mathrm{H}(\mu) \cong \mathrm{H}(\bar{\mu})$ where $\bar{\mu}\colon \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$ is defined by $(x + p\mathbb{Z}, y + p\mathbb{Z}) \mapsto xy + p\mathbb{Z}$.

When $p = 2$, then $E_2 \cong Q_8$, the quaternion group, so it has a unique element of order 2. On the other hand, $\mathrm{H}(\bar{\mu})$ has multiple elements of order 2, cf. Lemma 4.1.15. This shows that in the $p = 2$ case E_2 is not isomorphic to a twisted Heisenberg group.

Now suppose $p > 2$. Note that $(x, y, z)^p = (px, py, p\frac{p-1}{2}z)$ is trivial for any $(x, y, z) \in \mathrm{H}(\bar{\mu})$. On the other hand, $\alpha^p = \gamma \neq 1$ in E_p, so E_p is not a twisted Heisenberg group in this case either. In fact, $\mathrm{H}(\bar{\mu})$ is the other non-abelian group of order p^3. □

We discuss the construction outlined in the diagram (4.3) for E_p.

Example 4.4.2. Consider the central-by-abelian extension

$$\mathcal{Z}(E_p)\colon 1 \to \langle\gamma\rangle \to E_p \to E_p/\,\mathrm{Z}(E_p) \to 1$$

where $\mathrm{Z}(E_p) = \langle\gamma\rangle$. The resulting alternating $\mathbb{Z}$-module is

$$\mathcal{A}(\mathcal{Z}(E_p)) = (E_p/\,\mathrm{Z}(E_p), \omega, \mathbb{Z}\gamma),$$

where $\omega\colon (g\,\mathrm{Z}(E_p), g'\,\mathrm{Z}(E_p)) \mapsto [g, g']$ is given by the commutator. $E_p/\,\mathrm{Z}(E_p) = \langle\bar{\alpha}, \bar{\beta}\rangle$ where $\bar{g}$ denotes the image of $g \in E_p$ under the natural projection $\pi\colon E_p \to E_p/\,\mathrm{Z}(E_p)$. This gives a polarisation

$$\mathcal{P}(\mathcal{A}(\mathcal{Z}(E_p))) \equiv (\mathbb{Z}\bar{\alpha} \oplus \mathbb{Z}\bar{\beta}, \omega, \mathbb{Z}\gamma).$$

Applying Diag gives an abelian bihomomorphism

$$\begin{aligned} \mu \coloneqq \mathrm{diag}(\omega) : \langle\bar{\alpha}\rangle \times \langle\bar{\beta}\rangle &\to \langle\gamma\rangle \\ (\bar{\alpha}^x, \bar{\beta}^y) &\mapsto \gamma^{xy} = [\bar{\alpha}^x, \bar{\beta}^y] \end{aligned}$$

induced by the commutator map. The resulting twisted Heisenberg group is

$$\mathrm{H}(\mu) = \begin{pmatrix} 1 & \langle\bar{\alpha}\rangle & \langle\gamma\rangle \\ 0 & 1 & \langle\bar{\beta}\rangle \\ 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & \mathbb{Z}_p & \mathbb{Z}_p \\ 0 & 1 & \mathbb{Z}_p \\ 0 & 0 & 1 \end{pmatrix}$$

which is not isomorphic to E_p by Lemma 4.4.1 even though many group theoretic properties are the same, cf. Remark 4.1.16.

Since E_p has a cyclic derived subgroup (generated by γ), we can apply Subsection 4.2.1 to embed it into a twisted Heisenberg group.

Example 4.4.3 (Natural embedding to Heisenberg group)**.** Consider the polarisation from Example 4.2.2 when applied to the polarised module form Example 4.4.2.

$$\begin{array}{ccccccccccc}
\epsilon_\alpha & : & 1 & \longrightarrow & \langle\gamma\rangle & \overset{\subseteq}{\hookrightarrow} & \langle\gamma, \alpha\rangle & \twoheadrightarrow & \langle\bar{\alpha}\rangle & \longrightarrow & 1 \\
 & & & & \| & & \downarrow\cap & & \downarrow\cap & & \\
\mathcal{Z}(E_p) & : & 1 & \longrightarrow & \mathrm{Z}(E_p) & \overset{\subseteq}{\hookrightarrow} & E_p & \overset{\pi}{\twoheadrightarrow} & E_p/\,\mathrm{Z}(E_p) & \longrightarrow & 1 \\
 & & & & \| & & \cup\uparrow & & \cup\uparrow & & \\
\epsilon_\beta & & 1 & \longrightarrow & \langle\gamma\rangle & \overset{\subseteq}{\hookrightarrow} & \langle\gamma, \beta\rangle & \twoheadrightarrow & \langle\bar{\beta}\rangle & \longrightarrow & 1
\end{array}$$

Let $P := \langle \gamma, \beta \rangle \times \langle \gamma \rangle \times \langle \gamma, \alpha \rangle$, set $K := \{(c_2^{-1}, c_2c_1, c_1^{-1}) \in \langle \gamma \rangle^3 : c_i \in \gamma\} \lhd P$ and consider the natural Z-extension $\zeta\colon P \to P/K =: \hat{C}$ from Example 4.2.6. (Here of course $\langle \gamma, \alpha \rangle = \langle \gamma \rangle = \langle \gamma, \beta \rangle$, we left γ in the presentation to remind ourselves that these groups are defined as inverse images under π.) Then

$$\begin{aligned} \mathcal{B}_\zeta(\epsilon) := \hat{\mu}\colon \langle \bar{\alpha} \rangle \times \langle \bar{\beta} \rangle &\to \hat{C} \\ (\bar{\alpha}^x, \bar{\beta}^y) &\mapsto \zeta(1, [\bar{\alpha}^x, \bar{\beta}^y], 1) = \zeta(1, \gamma^{xy}, 1) \end{aligned}$$

from Lemma 4.2.7 induces the monomorphism

$$\begin{aligned} \delta\colon E_p &\rightarrowtail \mathrm{H}(\hat{\mu}) = \begin{pmatrix} 1 & \langle \pi(\alpha) \rangle & \mathrm{Im}(\zeta) \\ 0 & 1 & \langle \pi(\beta) \rangle \\ 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & \mathbb{Z}_p & \mathbb{Z}_p \times \mathbb{Z}_{p^2} \\ 0 & 1 & \mathbb{Z}_p \\ 0 & 0 & 1 \end{pmatrix} \\ \beta^y\gamma^z\alpha^x &\mapsto \begin{pmatrix} 1 & \pi(\alpha^x) & \zeta(\beta^y, \gamma^z, \alpha^x) \\ 0 & 1 & \pi(\beta^y) \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

The previous embedding can be improved to have smaller index, and the twisted Heisenberg group to have fewer generators as detailed below following Subsection 4.2.2.

Example 4.4.4 (Optimal embedding)**.** Consider the polarisation from (4.6)

$$\begin{array}{rcccccccccc} \epsilon_1 & : & 1 & \longrightarrow & \langle \gamma \rangle \cap \langle \alpha \rangle & \overset{\subseteq}{\hookrightarrow} & \langle \alpha \rangle & \overset{\pi_1}{\twoheadrightarrow} & \langle \gamma \rangle & \longrightarrow & 1 \\ & & & & \| & & \downarrow \cap & & \lambda_1 \downarrow \cap & & \\ \mathcal{Z}^\flat(E_p) & : & 1 & \longrightarrow & \mathrm{Z}(E_p) & \overset{\subseteq}{\hookrightarrow} & E_p & \overset{\pi}{\twoheadrightarrow} & \langle \gamma \rangle \times \langle \gamma \rangle & \longrightarrow & 1 \\ & & & & \| & & \cup \uparrow & & \cup \uparrow \lambda_2 & & \\ \epsilon_2 & & 1 & \longrightarrow & \langle \gamma \rangle \cap \langle \beta \rangle & \overset{\subseteq}{\hookrightarrow} & \langle \beta \rangle & \overset{\pi_2}{\twoheadrightarrow} & \langle \gamma \rangle & \longrightarrow & 1 \end{array}$$

where $\pi_1\colon g \mapsto [g, \beta]$ and $\pi_2\colon g \mapsto [\alpha, g]$. The Z-extension from (4.7) becomes

$$\begin{aligned} \zeta\colon \langle \beta \rangle \times \langle \gamma \rangle \times \langle \alpha \rangle &\mapsto \tfrac{1}{p}\mathbb{Z}/p\mathbb{Z} \\ (\beta^y, \gamma^z, \alpha^x) &\mapsto \tfrac{x+y}{p} + z + p\mathbb{Z}. \end{aligned}$$

Then this time Lemma 4.2.7 induces the monomorphism

$$\begin{array}{ccccc} E_p & \overset{\delta}{\rightarrowtail} & \mathrm{H}(\hat{\mu}) = \begin{pmatrix} 1 & \langle \gamma \rangle & \mathrm{Im}(\zeta) \\ 0 & 1 & \langle \gamma \rangle \\ 0 & 0 & 1 \end{pmatrix} & \overset{\sim}{\rightarrowtail\!\!\!\twoheadrightarrow} & \begin{pmatrix} 1 & \mathbb{Z}_p & \frac{1}{p}\mathbb{Z}/p\mathbb{Z} \\ 0 & 1 & \mathbb{Z}_p \\ 0 & 0 & 1 \end{pmatrix} \\ \beta^y\gamma^z\alpha^x & \longmapsto & \begin{pmatrix} 1 & \pi(\alpha^x) & \zeta(\beta^y, \gamma^z, \alpha^x) \\ 0 & 1 & \pi(\beta^y) \\ 0 & 0 & 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} 1 & x+p\mathbb{Z} & \frac{x+y}{p}+z+p\mathbb{Z} \\ 0 & 1 & y+p\mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}. \end{array}$$

This embedding is optimal in the sense that no embedding exists to a twisted Heisenberg group with fewer generators (and actually having smaller order).

≤2-generated ≤2-step nilpotent groups We extend the explicit description of the embedding from Example 4.4.3 to every ≤2-generated ≤2-step nilpotent group E in the following examples.

Example 4.4.5. Let E be from (3.5), an arbitrary $\leq$ 2-generated $\leq$ 2-step nilpotent group. By Lemma 3.3.3, we may assume that $c_1 \mid c$ and $c_2 \mid c$. Consider $\epsilon\colon 1 \to \mathbb{Z}_c \to E \to \mathbb{Z}_b \times \mathbb{Z}_a \to 1$ from Lemma 3.3.1 which is isomorphic to $\mathcal{D}(E)$. Some computation shows that with this setup, Lemma 4.2.7 produces the following embedding.

$$\begin{aligned} \mu_1\colon \mathbb{Z}_a \times \mathbb{Z}_b &\to \tfrac{1}{q}\mathbb{Z}/c\mathbb{Z} \\ (x + a\mathbb{Z}, y + b\mathbb{Z}) &\mapsto xy + c\mathbb{Z} \end{aligned} \qquad \begin{aligned} \delta_1\colon E \rightarrowtail \mathrm{H}(\mu_1) &\cong \left(\begin{smallmatrix} 1 & \mathbb{Z}_a & \mathbb{Z}_{qc} \\ 0 & 1 & \mathbb{Z}_b \\ 0 & 0 & 1 \end{smallmatrix}\right) \\ \alpha &\mapsto (1 + a\mathbb{Z}, 0 + b\mathbb{Z}, \tfrac{c_1}{a} + c\mathbb{Z}) \\ \beta &\mapsto (0 + a\mathbb{Z}, 1 + b\mathbb{Z}, \tfrac{c_2}{b} + c\mathbb{Z}) \\ \gamma &\mapsto (0 + a\mathbb{Z}, 0 + b\mathbb{Z}, 1 + c\mathbb{Z}) \end{aligned}$$

where $q = \mathrm{lcm}(\{\frac{a}{c_1}, \frac{b}{c_2}, 1\} \setminus \{0\})$ using the convention $\frac{0}{0} := 0$. Using Remark 4.1.10 and (3.5), one can quickly verify (independently of Lemma 4.2.7) that the extended map $\delta_1(\beta^y\gamma^z\alpha^x) = (x + a\mathbb{Z}, y + b\mathbb{Z}, \frac{c_1 x}{a} + \frac{c_2 y}{b} + z + c\mathbb{Z})$ is indeed an injective group morphism.

Example 4.4.6. Let E be an arbitrary $\leq$2-generated $\leq$2-step nilpotent group with cyclic centre, and consider the presentation from Lemma 3.3.6. Adopt the convention and q from Example 4.4.5. Consider the central-by-abelian extension $\epsilon\colon 1 \to \mathbb{Z}_a \to E \to \mathbb{Z}_c \times \mathbb{Z}_c \to 1$ which is isomorphic to $\mathcal{Z}(E)$. This time, Lemma 4.2.7 gives the following maps.

$$\begin{aligned} \mu_2\colon \mathbb{Z}_c \times \mathbb{Z}_c &\to \tfrac{1}{q}\mathbb{Z}/c\mathbb{Z} \\ (x + c\mathbb{Z}, y + c\mathbb{Z}) &\mapsto xy + c\mathbb{Z} \end{aligned} \qquad \begin{aligned} \delta_2\colon E \rightarrowtail \mathrm{H}(\mu_2) &\cong \left(\begin{smallmatrix} 1 & \mathbb{Z}_c & \mathbb{Z}_{qc} \\ 0 & 1 & \mathbb{Z}_c \\ 0 & 0 & 1 \end{smallmatrix}\right) \\ \alpha &\mapsto (1 + c\mathbb{Z}, 0 + c\mathbb{Z}, \tfrac{c_1}{a} + c\mathbb{Z}) \\ \beta &\mapsto (0 + c\mathbb{Z}, 1 + c\mathbb{Z}, \tfrac{c_2}{b} + c\mathbb{Z}) \\ \gamma &\mapsto (0 + c\mathbb{Z}, 0 + c\mathbb{Z}, 1 + c\mathbb{Z}) \end{aligned}$$

If one wishes to see why δ_2 is an injection without referring to Lemma 4.2.7, one can work directly by separating the cases. First, when $a = b = c$, then $\delta_2 = \delta_1$ from Example 4.4.5. Otherwise, $a > b = c$, $c_1 = 1$, $c_2 \in \{c, c/2\}$. Pick $g = \beta^y\gamma^z\alpha^x \in E$ so that $\delta_2(g) = (x + c\mathbb{Z}, y + c\mathbb{Z}, \frac{c_1 x}{a} + \frac{c_2 y}{b} + z + c\mathbb{Z})$ is the identity. Then $b = c \mid y$, so considering the third component, $\frac{x}{a} = \frac{c_1 x}{a} \in \mathbb{Z}$, thus $a \mid x$. Thus using the relations of the presentation in (3.5), $g = \beta^{b\cdot y/b}\gamma^z\alpha^{a\cdot x/a} = \gamma^{c_2 y/b + z + c_1 x/a} = 1$ since by assumption the exponent of γ is in $c\mathbb{Z}$.

In accordance with Remark 4.2.10, this twisted Heisenberg group is non-degenerate. We see that the index of this embedding is smaller than the one from Example 4.4.5.

Chapter 5

Applications to Jordan Type Problems

In this chapter, we prove Theorem C from page 9, thereby finally addressing the original motivation (Problem 1.1.2). The explicit techniques to construct the variety/manifold on which finite ≤ 2-step nilpotent groups act are different in the algebraic and the smooth cases; however, the key ideas and principles are similar. First we construct actions of (2-generated) twisted Heisenberg groups with cyclic centre on invertible sheaves over abelian varieties, and on holomorphic line bundles over the complex torus. Then we use direct and central products together with the group theoretical results of Chapter 3 and Chapter 4 to obtain actions in the general case (on different spaces). Finally, we use a uniformisation process to translate all group action to a single space.

These constructions give lower bounds for Problem 1.1.2 in both cases. In the algebraic case, it actually coincides with the upper bound of Theorem 1.1.3 by Guld answering the main problem in this case. Similar upper bounds in the smooth case are not known to the author.

5.1 Algebraic Case: Birational Automorphism Groups

In this section, we prove the algebraic part of Theorem C from page 9 thereby answer Problem 1.1.2 in this case. At Subsection 5.1.1, we use the uniformisation process of [Zar14] to solve the case some canonical Heisenberg groups. Then in Subsection 5.1.2, we use the group theoretic result of Section 3.1 and Chapter 4 together with a result of Guld [Gul19, Theorem 15] to prove the main statements.

5.1.1 Uniformisation and Action of Heisenberg Groups

In this section, we show that every canonical Heisenberg group of bounded rank and with cyclic centre is a subgroup of the birational automorphism group of a suitable algebraic variety (depending on the bound). To do so, first following the uniformisation process of [Zar14] building on [Mum66], and keeping in mind that every line bundle is birationally trivial, we obtain an embedding for all Mumford's theta groups to the birational automorphism group of an invertible sheaf over an abelian variety of bounded dimension. Next, we show that the aforementioned canonical Heisenberg groups are actually subgroups of these theta groups by using the structural description of [Mum66] and Chapter 4. We also investigate how these bounds on the rank and on the dimensions are related.

Recall Subsection 2.4.4. Let k be an algebraically close field of characteristic 0, X an abelian variety over k, and L an ample invertible $\mathscr{O}_X$-module. Lemma 2.4.31 gives a natural action of $\mathscr{G}(L)$ on the global sections of L.

Lemma 5.1.1 (Uniformisation, [Zar14])**.** *There is a monomorphism*

$$\varrho_L \colon \mathscr{G}(L) \to \mathrm{Bir}(X \times \mathbb{A}^1)$$

for every ample invertible $\mathscr{O}_X$-module L where X is an abelian variety over k.

Proof. Use the notation of [Har77, II.6 Invertible Sheaves] and recall Definition 2.4.29. Let $\mathscr{K}$ be the constant $\mathscr{O}_X$-module of rational functions. Then for some Cartier divisor D, we have $L = \mathscr{L}(D)$, a sub-$\mathscr{O}_X$-module of $\mathscr{K}$. Pick $(x, \varphi) \in \mathscr{G}(L)$. Now the isomorphism $\varphi \colon \mathscr{L}(D) \to \mathscr{L}(\tau(x)^* D)$ is given by multiplication by some rational function $f_\varphi \in \Gamma(X, \mathscr{K}^*)$. Define

$$\varrho_L \colon (x, \varphi) \mapsto ((z, t) \mapsto (z + x, f_\varphi(z)t) \tag{5.1}$$

on the non-empty open set where f_φ is defined. The injectivity of this map is clear, so we only need to check that it is a group morphism. Note that

$$f_{\tau(x)^*\psi\circ\varphi}(z) = f_{\tau(x)^*\psi}(z) f_\varphi(z) = f_\psi(z + x) f_\varphi(z), \tag{5.2}$$

cf. (5.8) for analogy. Then

$$\begin{aligned} \varrho_L((y, \psi) \cdot (x, \varphi))(z, t) &= \varrho_L((x + y, \tau(x)^*\psi \circ \varphi))(z, t) \\ &= (z + x + y, f_{\tau(x)^*\psi\circ\varphi}(z)t) \\ &= (z + x + y, f_\psi(z + x) f_\varphi(z)t) \\ &= \varrho_L(y, \psi)(z + x, f_\varphi(z)t) \\ &= (\varrho_L(y, \psi) \circ \varrho_L(x, \varphi))(z, t), \end{aligned} \tag{5.3}$$

so ϱ_L is indeed a morphism, cf. (5.9). □

Theta groups and Heisenberg groups are closely related. In fact, the next statement was the initial motivation for Definition 4.1.9.

Lemma 5.1.2. *$\mathscr{G}(\delta)$ from Definition 2.4.32 is isomorphic to (the opposite group of) $\mathrm{H}(K(\delta), k^\times)$ from Definition 4.3.1.*

Proof. The map $\varphi \colon \mathrm{H}(K(\delta), k^\times) \to \mathscr{G}(\delta)$ defined by $(\alpha, b, c) \mapsto (c, b, \alpha)^{-1}$ is an isomorphism, because $\varphi((\alpha, b, c) * (\alpha', b', c')) = \varphi((\alpha\alpha', b + b', c\alpha(b')c')) = (c\alpha(b')c', b + b', \alpha\alpha')^{-1} = ((c', b', \alpha') \cdot (c, b, \alpha))^{-1} = \varphi((\alpha, b, c)) \cdot \varphi((\alpha', b', c'))$. □

We can embed every finite Heisenberg group with cyclic centre into a suitable theta group.

Lemma 5.1.3. *Let T be an elliptic curve over k. Let B be a finite abelian group with $d(B) \leq s$ for some $s \in \mathbb{N}_0$, and let C be a finite cyclic group. Then there is a monomorphism*

$$\chi_{B,C} \colon \mathrm{H}(B, C) \to \mathscr{G}(L)$$

from the canonical Heisenberg group over B and C to the theta group given by some ample invertible $\mathscr{O}_{T^s}$-module L (depending on B and C).

Proof. Let $t \coloneqq d(B) \leq s$ and $d_t \mid \ldots \mid d_1$ be the invariant factors of B. Let $B_0 \coloneqq (\mathbb{Z}/d_t\mathbb{Z})^{s-t}$ and set $\bar{B} \coloneqq B_0 \times B$. Let $\mu\colon C \to k^\times$ be a monomorphism and define

$$\theta\colon \mathrm{H}(B, C) \to \mathrm{H}(\bar{B}, k^\times), \quad (\alpha, b, c) \mapsto (\overline{\mu \circ \alpha}, (b, 0), \mu(c))$$

where $\overline{\mu \circ \alpha}\colon \bar{B} \to k^\times, (b_0, b) \mapsto \mu(\alpha(b))$. The map θ is well defined as a map of sets and it is injective by definition. The calculation

$$\begin{aligned}
\theta((\alpha, b, c) * (\alpha', b', c')) &= \theta((\alpha\alpha', b + b', c\alpha(b')c')) \\
&= (\overline{\mu \circ (\alpha\alpha')}, (b + b', 0), \mu(c\alpha(b')c')) \\
&= (\overline{\mu \circ \alpha} \cdot \overline{\mu \circ \alpha'}, (b, 0) + (b', 0), \mu(c)\mu(\alpha(b'))\mu(c')) \\
&= \theta((\alpha, b, c)) * \theta((\alpha', b', c'))
\end{aligned}$$

shows that θ is a monomorphism. Set $d_i \coloneqq d_t$ for $t < i \leq s$ and let $\delta \coloneqq (d_1, \ldots, d_s)$. By construction, $B \cong K(\delta)$ from Definition 2.4.29. So by Lemma 5.1.2 and Lemma 2.4.33, it is enough to to construct an $\mathscr{O}_{T^s}$-module L with $H_{\mathrm{M}}(\delta) \cong H_{\mathrm{M}}(L)$.

Let L_1 be an invertible $\mathscr{O}_T$-module of degree 1, e.g. the invertible sheaf corresponding to the Weil divisor of a single point. This is necessarily ample by Lemma 2.4.35. $H_{\mathrm{M}}(L_1)$ is trivial by Remark 2.4.34, thus $H_{\mathrm{M}}(L_1^{\otimes d}) = \{x \in X(k) : d \cdot x \in H_{\mathrm{M}}(L_1) = 0\} \cong (\mathbb{Z}/d\mathbb{Z})^2$ by Lemma 2.4.37 and Lemma 2.4.36. Define $L \coloneqq L_1^{\otimes d_1} \boxtimes L_1^{\otimes d_2} \boxtimes \ldots \boxtimes L_1^{\otimes d_s}$, an ample invertible $\mathscr{O}_{T^s}$ module. By the discussion above and Lemma 2.4.38, $H_{\mathrm{M}}(L) = \prod_{i=1}^{s} (\mathbb{Z}/d_i\mathbb{Z})^2 \cong H_{\mathrm{M}}(\delta)$ as required. □

5.1.2 The Main Jordan Type Theorem

> We use the results from Chapter 4 to show that [Gul19, Theorem 15] is 'sharp' and hence provide an answer to Problem 1.1.2 à la Jordan. The big picture is analogous to that of Subsection 5.2.5.

Recall Definition 1.2.5 and Definition 4.3.1.

Definition 5.1.4. For $r \in \mathbb{N}_0$, denote the class of finite $\leq$2-step nilpotent groups of rank at most r by $\mathcal{F}_2(r)$.

Lemma 5.1.5. *For every $G \in \mathcal{F}_2(r)$ with cyclic centre, there is a monomorphism*

$$\delta_G\colon G \to \mathrm{H}(B, C)$$

for some finite abelian group B with $d(B) \leq s \coloneqq \lfloor r/2 \rfloor$ and suitable finite cyclic group C.

Proof. Let $G \rightarrowtail \mathrm{H}(\mu\colon A \times A \to C)$ be the embedding given by Proposition 4.2.18. Set $B \coloneqq A$. Then $d(B) = d(A) \leq d(G)/2 \leq r/2$. On the other hand, $\mathrm{H}(\mu) \cong \mathrm{H}(B, C)$ by Proposition 4.3.2, so the statement follows. □

Theorem 5.1.6 (D.R.Sz.). *For every $r \in \mathbb{N}_0$, there exists a variety X_r (over any algebraically closed field of characteristic 0) such that every $G \in \mathcal{F}_2(r)$ embeds to* $\mathrm{Bir}(X_r)$.

Remark 5.1.7. By Theorem 1.2.6, the bound on the rank is necessary.

Proof. Using Corollary 3.1.26, write $G \leq \prod_{i=1}^{d} G_i$, a subdirect product where $\mathrm{Z}(G_i)$ are cyclic and $d := d(\mathrm{Z}(G))$. Note that $G_i \in \mathcal{F}_2(r)$ since G_i are quotients of G. Applying Lemma 5.1.5, Lemma 5.1.3 and Lemma 5.1.1 gives a sequence of monomorphisms

$$\varphi_i \colon G_i \xrightarrow{\delta_{G_i}} \mathrm{H}(B_i, C_i) \xrightarrow{\chi_{B_i,C_i}} \mathscr{G}(L_i) \xrightarrow{\varrho_{L_i}} \mathrm{Bir}(T^{\lfloor r/2 \rfloor} \times \mathbb{A}^1)$$

where L_i is an ample invertible $\mathscr{O}_{T^{\lfloor r/2 \rfloor}}$-module and T is a fixed elliptic curve. Then

$$G \xrightarrow{\subseteq} \prod_{i=1}^{d} G_i \xrightarrow{\prod_{i=1}^{d} \varphi_i} \prod_{i=1}^{d} \mathrm{Bir}(T^{\lfloor r/2 \rfloor} \times \mathbb{A}^1) \to \mathrm{Bir}(\prod_{i=1}^{r} T^{\lfloor r/2 \rfloor} \times \mathbb{A}^1) \cong \mathrm{Bir}(T^{r\lfloor r/2 \rfloor} \times \mathbb{P}^r)$$

after noting that $d \leq r$ and that $\mathbb{A}^r$ is birational to $\mathbb{P}^r$. Thus $X_r := T^{r\lfloor r/2 \rfloor} \times \mathbb{P}^r$ satisfies the statement (where T is an arbitrary elliptic curve). ☐

Definition 5.1.8. A family $\mathcal{G}$ of groups is *uniformly $\mathcal{N}$-Jordan*, if there is an integer $J_{\mathcal{G}}$ such that every $G \in \mathcal{G}$ is $\mathcal{N}$-Jordan with $J_G \leq J_{\mathcal{G}}$.

We can finally answer Problem 1.1.2 in the algebraic case.

Theorem 5.1.9 (Guld – D.R.Sz.)**.** *For a family $\mathcal{N}$ of groups, the following are equivalent.*

1. $\mathrm{Bir}(X)$ *is $\mathcal{N}$-Jordan for every variety X over an algebraically closed field of characteristic* 0.

2. *The family $\mathcal{F}_2(r)$ is uniformly $\mathcal{N}$-Jordan for every $r \in \mathbb{N}_0$.*

Proof. The direction 1 $\Longrightarrow$ 2 follows immediately from Theorem 5.1.6 and Definition 1.1.1 when the assumption 1 is applied to $X = X_r$.

Consider the direction 2 $\Longrightarrow$ 1. Let X be a variety. Then Theorem 1.1.3 and Theorem 1.2.6 imply the existence of integers r and $J_{\mathrm{Bir}(X)}$ such that every finite subgroup $F \leq \mathrm{Bir}(X)$ sits in an exact sequence $1 \to N \to F \to B \to 1$ where $N \in \mathcal{F}_2(r)$ and $|B| \leq J_{\mathrm{Bir}(X)}$. By assumption 2, we have $1 \to N_1 \to N \to B_1 \to 1$ with $N_1 \in \mathcal{N}$ and $|B_1| \leq J_{\mathcal{F}_2(r)}$. N_1 may not be normal in G, but a suitable subgroup with bounded index always is. To obtain it, let F act on the set F/N_1 of left cosets of N_1 in F by left multiplication. Let $N_2 \subseteq N_1$ be the kernel of this action. Then $F/N_2 \leq \mathrm{Sym}(F/N_1)$ and $1 \to N_2 \to F \to F/N_2 \to 1$ with $N_2 \in \mathcal{F}_2(r)$ and $|F/N_2| \leq |F/N_1|! = (|B||B_1|)! \leq (J_{\mathrm{Bir}(X)} J_{\mathcal{F}_2(r)})!$. The bound $(J_{\mathrm{Bir}(X)} J_{\mathcal{F}_2(r)})!$ is independent of the choice of $F \leq \mathrm{Bir}(X)$, hence $\mathrm{Bir}(X)$ is $\mathcal{N}$-Jordan as stated. ☐

5.2 Smooth Case: Diffeomorphism Groups

In this section, we prove the smooth part of Theorem C from page 9 and give a partial answer to Problem 1.1.2. For this, we introduce the notion of actions of central-by-abelian extensions on complex vector bundles. We present a uniformisation process of suitable actions of extensions on some line bundles using K-theory using ideas of [Mun17]. To construct these actions for every central-by-abelian extension giving finite $\leq$2-step nilpotent groups of bounded rank, we use the group theoretic results of Chapter 3, Chapter 4, central products, and the Appell–Humbert theorem classifying holomorphic line bundles over complex tori to get the base case.

5.2.1 Action of Extensions on Bundles and Compactification

We introduce the main concept of Section 5.2, the action of central-by-abelian extensions on vector bundles. This consists of three compatible group actions: one on the fibres, one on the total space, and one on the base space of the vector bundle. This is the analogue of the action on invertible sheaves from Section 5.1. At the end of the day, we will consider only the action on the total space. Unfortunately, the total space is not compact (because a complex vector space of positive dimension is not compact), so we recall some fibre bundle constructions that produce compact total space while preserving the action.

Definition 5.2.1. Let $p\colon E \to X$ a smooth fibre bundle. $\mathrm{Diff}_p(E)$ is the group of C^∞-diffeomorphisms φ of E that preserve the fibres, i.e. for which there exists (a unique) C^∞-diffeomorphism $p_*\varphi$ of X such that $p \circ \varphi = (p_*\varphi) \circ p$.

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{p_*\varphi} & X \end{array}$$

In this case, we say that φ covers $p_*\varphi$. If p is a smooth $\mathbb{C}$-vector bundle, we also require that φ restricts to each fibre as a $\mathbb{C}$-linear map. Let $\mathrm{Diff}_p(X)$ be group of C^∞-diffeomorphisms β of X such that β is covered by some $\varphi \in \mathrm{Diff}_p(E)$. Define $\mathrm{Diff}_p^{\mathrm{id}}(E) \coloneqq \ker(p_*)$, the collection of maps which keep every fibre fixed. These groups sit in the following short exact sequence.

$$\mathrm{Diff}_p \;:\; 1 \longrightarrow \mathrm{Diff}_p^{\mathrm{id}}(E) \xhookrightarrow{\subseteq} \mathrm{Diff}_p(E) \xrightarrow{p_*} \mathrm{Diff}_p(X) \longrightarrow 1$$

Definition 5.2.2. An *action $\alpha\colon \epsilon \circlearrowright p$ of a group extension ϵ on a fibre bundle $p\colon L \to X$* is a morphism $\alpha \coloneqq (\sigma, \varrho, \tau)\colon \epsilon \to \mathrm{Diff}_p$ of short exact sequences. The action is *faithful*, if the underlying $\epsilon \to \mathrm{Diff}_p$ is a monomorphism, i.e. when all σ, ϱ, τ are monomorphisms.

$$\begin{array}{ccccccccc} \epsilon & : & 1 & \longrightarrow & C & \xrightarrow{\iota} & G & \xrightarrow{\pi} & M & \longrightarrow & 1 \\ & & & & \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle \varrho} & & \big\downarrow{\scriptstyle \tau} & & \\ \mathrm{Diff}_p & : & 1 & \longrightarrow & \mathrm{Diff}_p^{\mathrm{id}}(E) & \xhookrightarrow{\subseteq} & \mathrm{Diff}_p(E) & \xrightarrow{p_*} & \mathrm{Diff}_p(X) & \longrightarrow & 1 \end{array} \tag{5.4}$$

A morphism from $\alpha_1\colon \epsilon \circlearrowright p_1$ to $\alpha_2\colon \epsilon \circlearrowright p_2$ is a morphism $\varphi\colon \mathrm{Diff}_{p_1} \to \mathrm{Diff}_{p_2}$ of short exact sequences such that $\varphi \circ \alpha_1 = \alpha_2$.

Remark 5.2.3. In the setup of Definition 5.2.2, one can consider the action ϱ of G as being decomposed into two parts: an action σ of C on the fibres only, and an action τ of M on the base space. More precisely, each $\sigma(c)$ fixes every fibre and its restrictions to the fibres are isomorphism, however, this isomorphism may depend on the fibre.

Definition 5.2.4. Define $V_\epsilon^\tau(X)$ to be the set of actions $(\sigma, \varrho, \tau)\colon \epsilon \circlearrowright p$ for some complex vector bundle $p\colon E \to X$ and morphisms σ, ϱ. Define the rank $\mathrm{rk}\colon V_\epsilon^\tau(X) \to \mathbb{N}_0$ as the rank of the underlying vector bundle. Note that ch_0 can be identified with rk. Define the Chern character $\mathrm{ch}\colon V_\epsilon^\tau(X) \to H^{2\bullet}(X;\mathbb{Q})$ as the Chern character of the underlying vector bundle. Note that we can identify rk with $V_\epsilon^\tau(X) \to H^{2\bullet}(X;\mathbb{Q}) \to H^0(X;\mathbb{Q})$.

Remark 5.2.5. Suppose $\alpha = (\sigma, \varrho, \tau) \in V_\epsilon^\tau(X)$ is a faithful action of a short exact sequence $\epsilon : 1 \to C \to G \to M \to 1$ of finite groups on a line bundle $p\colon L \to X$ where X is connected. Then each $\sigma(c)$ acts as multiplication by some $|C|$th root of unity on all fibres. Smoothness of $\sigma(c)$ and the connectivity of X forces this root to be the same for all fibres (for a fixed c). Hence on one hand, C injects to $\mathbb{C}^\times$, thus C is in fact a cyclic group. On the other hand, $\sigma(c)$ commutes with $\varrho(g)$ for all $c \in C$ and $g \in G$. Thus the injectivity of these maps imply that the image of C lies in $\mathrm{Z}(G)$, i.e. ϵ is a central extension. In fact, this observation motivated the study of the key notion of this thesis, central-by-abelian extensions, cf. Definition 1.2.1.

Example 5.2.6. Let $\theta_X\colon X \times \mathbb{C} \to X$ be the trivial line bundle over X from Definition 2.4.10. Then $\Theta_{\epsilon,\tau} \coloneqq (1, \varrho, \tau)\colon \epsilon \circlearrowright \theta_X$ is from $V_\epsilon^\tau(X)$ where $\varrho\colon G \to \mathrm{Aut}_p(\theta_X), m \mapsto ((x, z) \mapsto (\tau(\pi(g))(x), z))$.

At the end of the day, we want to obtain a group action on a smooth compact manifold. For this, the next statement is useful. Recall Example 2.4.18 and Example 2.4.19.

Lemma 5.2.7 (Compactification of actions)**.** *Every faithful action $\beta\colon \epsilon \circlearrowright p$ form $V_\epsilon^\tau(X)$ of rank s where ϵ consists of finite groups induces faithful smooth actions*

- $\mathrm{V}_k(\beta, h)\colon \epsilon \circlearrowright \mathrm{V}_k(p, h)$ *(for $1 \leq k \leq s$ and suitable Hermitian metric h) and*
- $\mathrm{Gr}_k(\beta \oplus \Theta_{\epsilon,\tau})\colon \epsilon \circlearrowright \mathrm{Gr}_k(p \oplus \theta_X)$ *(for $1 \leq k \leq s+1$).*

Remark 5.2.8. The total space of the resulting fibre bundles are compact in all cases, hence all groups of ϵ act faithfully via $\mathbb{C}^\infty$-diffeomorphisms on the respective smooth manifold.

Proof. Write $\beta = (\sigma, \varrho, \tau)$, $\epsilon : 1 \to C \xrightarrow{\iota} G \xrightarrow{\pi} M \to 1$ and $p\colon E \to X$.

First consider the Stiefel case. Let h_0 be a Hermitian metric on p whose existence is given by Lemma 2.4.17. Define a new Hermitian metric h by setting $h(e, e') \coloneqq |G|^{-1} \sum_{h \in G} h_0(\varrho(h)(e), \varrho(h)(e'))$ for e, e' coming from the same fibre. By the usual argument, this is ϱ-invariant, i.e. $h(\varrho(g)(e), \varrho(g)(e')) = h(e, e')$. Write $p_\mathrm{V} \coloneqq \mathrm{V}_k(p, h)$. We claim that

$$\begin{array}{ccccccccc}
\epsilon : & 1 & \longrightarrow & C & \xrightarrow{\iota} & G & \xrightarrow{\pi} & M & \longrightarrow 1 \\
\downarrow{\scriptstyle \mathrm{V}_k(\beta,h)} & & & \downarrow{\scriptstyle \sigma_\mathrm{V}} & & \downarrow{\scriptstyle \varrho_\mathrm{V}} & & \downarrow{\scriptstyle \tau} & \\
\mathrm{Diff}_{p_\mathrm{V}} : & 1 & \longrightarrow & \mathrm{Diff}_{p_\mathrm{V}}^{\mathrm{id}}(\mathrm{V}_k(E,h)) & \xhookrightarrow{\subseteq} & \mathrm{Diff}_{p_\mathrm{V}}(\mathrm{V}_k(E,h)) & \xrightarrow{p_*} & \mathrm{Diff}_{p_\mathrm{V}}(X) & \longrightarrow 1
\end{array}$$

is an action where

$$\sigma_\mathrm{V}\colon c \mapsto ((e_1, \ldots, e_k) \mapsto (\sigma(c)(e_1), \ldots, \sigma(c)(e_k)))$$
$$\varrho_\mathrm{V}\colon g \mapsto ((e_1, \ldots, e_k) \mapsto (\varrho(g)(e_1), \ldots, \varrho(g)(e_k))).$$

Note that these maps are well defined as $e_i \in p^{-1}(x)$ implies $\varrho(g)(e_i) \in p^{-1}(\tau(\pi(g))(x))$ and $h(\varrho(g)(e_i), \varrho(g)(e_j)) = h(e_i, e_j) = \delta_{i,j}$ (the Kronecker delta) and similarly for σ_V. These are group morphisms make the diagram above commute. The faithfulness of the action $\mathrm{V}_k(\beta, h)$ is equivalent to the injectivity of ϱ_V by the 5-lemma. Pick $c \in C$ such that $\sigma_\mathrm{V}(c)$ is the identity. Then the fibres are fixed and for any $e \in L$ with $h(e, e) = 1$, we have $\sigma(c)(e) = e$. So by the $\mathbb{C}$-linearity of $\sigma_{p^{-1}(x)}$, it has to fix $p^{-1}(x)$ pointwise. In other words, $\sigma(c)$ is the identity, hence the injectivity of σ forces $c = 1$, i.e. σ_{V} is indeed injective.

The Grassmann case is similar. Let $(\sigma_1, \varrho_1, \tau) \coloneqq \beta_1 \coloneqq \beta \oplus \Theta_{\epsilon,\tau}$, and action on $p_1 \coloneqq p \oplus \theta_X$. Suppose that for some $c \in C$, the map $\sigma_1(c)$ fixes all k-dimensional spaces of every fibre $p_1^{-1}(x)$. Then it necessarily fixes all 1-dimensional subspaces as well, hence the linear transformation in $p_1^{-1}(x)$ is multiplication by a suitable scalar $z_x \in \mathbb{C}^\times$. But the action $\Theta_{\epsilon,\tau}$ is trivial on the 1-dimensional subspace corresponding to the component θ_X of p_1, so $z_x = 1$. Thus $c = 1$ is this case by injectivity of σ_1 (which follows from injectivity of σ). As above, we can pick a ϱ-invariant Hermitian metric h_1 on p_1, and define the action

$$\begin{array}{ccccccccccc}
\epsilon & : & 1 & \longrightarrow & C & \xrightarrow{\quad\iota\quad} & G & \xrightarrow{\quad\pi\quad} & M & \longrightarrow & 1 \\
\big\downarrow{\scriptstyle \mathrm{Gr}_k(\beta\oplus\Theta_{\epsilon,\tau})} & & & & \big\downarrow{\scriptstyle \sigma_{\mathrm{V}}} & & \big\downarrow{\scriptstyle \varrho_{\mathrm{V}}} & & \big\downarrow{\scriptstyle \tau} & & \\
\mathrm{Diff}_{p_{\mathrm{Gr}}} & : & 1 & \longrightarrow & \mathrm{Diff}^{\mathrm{id}}_{p_{\mathrm{Gr}}}(\mathrm{Gr}_k(L)) & \xhookrightarrow{\subseteq} & \mathrm{Diff}_{p_{\mathrm{Gr}}}(\mathrm{Gr}_k(L)) & \xrightarrow{p_*} & \mathrm{Diff}_{p_{\mathrm{Gr}}}(X) & \longrightarrow & 1
\end{array}$$

where $p_{\mathrm{Gr}} \coloneqq \mathrm{Gr}_k(p \oplus \theta_X)$ and $\sigma_{\mathrm{Gr}} \colon c \mapsto (\langle e_1, \ldots, e_k\rangle \mapsto \langle \sigma(c)(e_1), \ldots, \sigma(c)(e_k)\rangle)$ and $\varrho_{\mathrm{Gr}} \colon g \mapsto (\langle e_1, \ldots, e_k\rangle \mapsto \langle \varrho(g)(e_1), \ldots, \varrho(g)(e_k)\rangle)$. These are well defined as above together with the fact that σ and ϱ act linearly on the fibres. The discussion above shows the injectivity of σ_{Gr}, hence the faithfulness of this action. □

5.2.2 Uniformisation using K-theory

This section lies at the heart of Section 5.2. We introduce the technical notion of uniformisable actions of central-by-abelian extensions and show a uniformisation process involving these bundles using cohomology computations and standard results from K-theory. More concretely, we show that there is a fixed manifold such that every uniformisable action over a base space with bounded dimension actually gives rise to an action on the fixed manifold, too.

The key idea of this fixed manifold originates from [Mun17]. Our goal is to construct a vector bundle for each line bundle so that their Whitney sum is the fixed space we seek. For simplicity, it is natural to require this sum to be the trivial line bundle of fixed rank (independently of the line bundle we started from). Then using K-theory and Proposition 2.4.16, it is enough to show that the Chern character of the Whitney sum is a fixed element of the zeroth cohomology group (corresponding to the fixed rank of the trivial vector bundle). To reach this element, we use an intermediate step of first setting the Chern character to take its value from a specific principal ideal of the full cohomology ring by adding suitable tensor powers of the given line bundle keeping in mind the bound on the rank (Lemma 5.2.14). Finding the suitable parameters translates to a modular Waring type problem from number theory (Lemma 5.2.12). Using another pullback construction with bounded rank (Lemma 5.2.16), we can reach every element of this ideal, hence their sum gives a vector bundle as required.

Later in Subsection 5.2.4, we will construct an action for every Heisenberg group of bounded rank on some line bundle (different for each group). Then the uniformisation process above will give a single space on which every Heisenberg group of bounded rank acts. This shows the relevancy of unifomisation process sketched above.

Definition 5.2.9. For a short exact sequence $\epsilon\colon 1 \to C \to G \to M \to 1$ of groups and $\tau\colon M \to \mathrm{Diff}(X)$, the Whitney sum, tensor product, the dual and the projectivisation induce operations

$$\begin{aligned} \oplus\colon V_\epsilon^\tau(X) \times V_\epsilon^\tau(X) \to V_\epsilon^\tau(X), \quad & \alpha_1 \oplus \alpha_2 \coloneqq (\sigma_1 \oplus \sigma_2, \varrho_1 \oplus \varrho_2, \tau)\colon \epsilon \circlearrowright (p_1 \oplus p_2) \\ \otimes\colon V_\epsilon^\tau(X) \times V_\epsilon^\tau(X) \to V_\epsilon^\tau(X), \quad & \alpha_1 \otimes \alpha_2 \coloneqq (\sigma_1 \otimes \sigma_2, \varrho_1 \otimes \varrho_2, \tau)\colon \epsilon \circlearrowright (p_1 \otimes p_2) \\ (-)^*\colon V_\epsilon^\tau(X) \to V_\epsilon^\tau(X), \quad & \alpha^* \coloneqq (\sigma^*, \varrho^*, \tau)\colon \epsilon \circlearrowright p^* \end{aligned}$$

which are defined fibrewise where $\alpha_i = (\sigma_i, \varrho_i, \tau)\colon \epsilon_i \circlearrowright p_i$ and $\alpha = (\sigma, \varrho, \tau)\colon \epsilon \circlearrowright p$ are from $V_\epsilon^\tau(X)$. If $\alpha\colon \epsilon \circlearrowright p$ is an action on a line bundle and $n \in \mathbb{N}_+$, then set $\alpha^{\otimes n} = \alpha \otimes \ldots \otimes \alpha$, the n-fold tensor power; set $\alpha^{\otimes -n} \coloneqq (\alpha^*)^{\otimes n}$; and $\alpha^{\otimes 0} \coloneqq \Theta_{\epsilon,\tau}$.

Remark 5.2.10. $\alpha^{\otimes n} \otimes \alpha^{\otimes k} \cong \alpha^{\otimes (n+k)}$ for any action α on a line bundle and $n, k \in \mathbb{Z}$. $\otimes$ is distributive over $\oplus$ (up to isomorphism) and these operations are compatible with the ring structure of $H^{2\bullet}(X;\mathbb{Q})$ via ch: $\mathrm{ch}(\alpha_1 \oplus \alpha_2) = \mathrm{ch}(\alpha_1) + \mathrm{ch}(\alpha_2)$, $\mathrm{ch}(\alpha_1 \otimes \alpha_2) = \mathrm{ch}(\alpha_1) \smile \mathrm{ch}(\alpha_2)$.

In fact, these imply that the Grothendieck completion $K^0_{\epsilon,\tau}(X)$ of $V_\epsilon^\tau(X)$ is a ring and thus $\mathrm{ch}\colon (K^0_{\epsilon,\tau}(X), \oplus, \otimes) \to (H^{2\bullet}(X;\mathbb{Q}), +, \smile)$ becomes a ring morphism where the class of $\Theta_{\epsilon,\tau}$ is the multiplicative identity. In this thesis, we will not need such an equivariant K-theory, so we will stay at the level on vector bundles.

For the construction, we need the following purely number theoretical results.

Lemma 5.2.11 (Modular Waring problem, [HL22, p. 186, Theorem 12].)**.** *For every $k, n \in \mathbb{N}_+$, there is $\gamma(n,k) \leq 4k$ such that modulo n, every integer can be written as a sum of at most $\gamma(n,k)$ many kth powers.*

Lemma 5.2.12. *There exists $R_1\colon {\mathbb{N}_0}^2 \to \mathbb{N}_0$ such that for every $n \in \mathbb{N}_+$, every finite multiset S of integers and every $\delta \in \mathbb{N}_+$, there exists a multiset $T \supseteq S$ of integers of total cardinality at most $R_1(n, |S|)$ satisfying $\sum_{t \in T} t^k \equiv 0 \pmod{\delta}$ for every $1 \leq k \leq n$.*

Remark 5.2.13. The proof actually shows that one can choose R_1 such that the condition $R_1(n,m) \leq (m+1)\prod_{k=2}^n (4k+1)$ is satisfied. This upper bound quite possibly very far from the smallest possible value of R_1. We did not investigate the question of optimality.

Proof. If $n = 0$ or $S = \emptyset$, then $T = \emptyset$ satisfies the statement. Otherwise, for $1 \leq k \leq n$, let W_k be the smallest positive integer N with the property that modulo any natural number, -1 can be expressed as a sum of at most N many kth powers. This exists by Lemma 5.2.11 and $W_k \leq 4k$. We show that $R_1(n,m) \coloneqq (m+1)\prod_{k=2}^n (W_k + 1)$ satisfies the statement.

Denote $p_k(T) \coloneqq \sum_{t \in T} t^k$ for any multiset T. Let $T_1 \coloneqq S + \{-p_1(S)\}$. For $2 \leq k \leq n$, pick P_k of total cardinality at most W_k such that $p_k(P_k) \equiv -1 \pmod{\delta}$ and define $T_k \coloneqq P_k + \{1\}$. Define $T \coloneqq \{\prod_{k=1}^n t_k : 1 \leq k \leq n, t_k \in T_k\}$. By construction $|T| \leq R_1(n, |S|)$, and $S \subseteq T$ because $S \subset T_1$ and $1 \in T_k$ for $2 \leq k \leq n$. Also $\delta \mid p_k(T_k) \mid \prod_{i=1}^n p_k(T_i) = p_k(T)$ for any $1 \leq k \leq n$, so T is as stated. □

To reach the 0th cohomology with the Chern character, we first reach various principal ideals of $H^{2\bullet}(X,\mathbb{Q})|_\mathbb{Z}$ of our choice. This construction uses Whitney sums and tensor products of α.

Lemma 5.2.14. *There is $R_2 \colon \mathbb{N}_0 \to \mathbb{N}_0$ with the following property. If X is a smooth compact manifold, $\alpha \in V_\epsilon^\tau(X)$ is of rank 1, and $d \in \mathbb{N}_+$, then there is $\alpha(d) \in V_\epsilon^\tau(X)$ of rank at most $R_2(\dim_{\mathbb{R}}(X))$ such that*

$$\operatorname{ch}(\alpha \oplus \alpha(d)) - \operatorname{rk}(\alpha \oplus \alpha(d)) \in d \cdot H^{2\bullet}(X, \mathbb{Q})|_{\mathbb{Z}}.$$

Remark 5.2.15. The proof shows that R_2 can be chosen such that the condition $R_2(m) \leq R_1(\lfloor m/2 \rfloor, 1)$ is satisfied, where R_1 is from Lemma 5.2.12. Considering Remark 5.2.13, we get $R_2(m) \leq 5^{m/2} \left\lfloor \frac{m}{2} \right\rfloor!$ which is quite possibly very far from the optimum.

As discussed at the introduction of this section, it is of key importance for the main application that the bound R_2 is independent of ϵ, α and d, and depends only on (the dimension of) X.

Proof. We show that $R_2(m) \coloneqq R_1(\lfloor m/2 \rfloor, 1) - 1$ satisfies the statement where R_1 is from Lemma 5.2.12. Let T be the multiset provided by Lemma 5.2.12 when applied with $n \coloneqq \lfloor \dim_{\mathbb{R}}(X)/2 \rfloor$, $S \coloneqq \{1\}$ and $\delta \coloneqq n!d$. We show that

$$\alpha(d) \coloneqq \bigoplus_{t \in T - \{1\}} \alpha^{\otimes t} \tag{5.5}$$

satisfies the statement. Then as $\operatorname{rk}(\alpha) = 1$, one has $\operatorname{rk}(\alpha(d)) = \sum_{t \in T - \{1\}} \operatorname{rk}(\alpha)^t = |T| - 1 \leq R_1(n, 2) - 1 = R_2(\dim(X))$. As α acts on a line bundle p, $\operatorname{ch}(\alpha) = \exp(c_1(\alpha))$ where $c_1(\alpha) \in H^2(X, \mathbb{Q})|_{\mathbb{Z}}$ is the first Chern class of the line bundle p, cf. Remark 2.4.15. Then using Remark 5.2.10 and that $H^{2k}(X, \mathbb{Q})|_{\mathbb{Z}} = 0$ for $k > n$ by Lemma 2.4.3, we obtain

$$\begin{aligned} \operatorname{ch}(\alpha \oplus \alpha(d)) &= \operatorname{ch}\Big(\bigoplus_{t \in T} \alpha^{\otimes t}\Big) = \sum_{t \in T} \operatorname{ch}(\alpha)^{\smile t} = \sum_{t \in T} \exp(t c_1(\alpha)) = \sum_{t \in T} \sum_{k=0}^{\infty} \frac{t^k}{k!} c_1(\alpha)^k \\ &= \operatorname{rk}(\alpha \oplus \alpha(d)) + \sum_{k=1}^{n} \sum_{t \in T} \frac{t^k}{k!} c_1(\alpha)^k \in \operatorname{rk}(\alpha \oplus \alpha(d)) + d \cdot H^{2\bullet}(X, \mathbb{Q})|_{\mathbb{Z}}, \end{aligned}$$

since $\sum_{t \in T} t^k \in n!d\mathbb{Z}$ for every $1 \leq k \leq n$ by Lemma 5.2.12. □

Next we show that every element of a specific ideal of the cohomology ring (up to the 0th cohomology) can be reached via pullbacks. Note that pullbacks are relevant, because it endows the bundle with an action of ϵ.

Lemma 5.2.16. *Let $\epsilon : 1 \to C \to G \to M \to 1$ be a short exact sequence of groups such that M is finite. Let X be a smooth compact manifold. Let $\tau \colon M \to \operatorname{Diff}(X)$ be a free action such that the composition $M \xrightarrow{\tau} \operatorname{Diff}(X) \xrightarrow{(-)^*} \operatorname{Aut}(H^{2\bullet}(X; \mathbb{Q}))$ is the identity. Then there is $d \in \mathbb{N}_+$ such that for any $\chi \in d \cdot H^{2\bullet}(X, \mathbb{Q})|_{\mathbb{Z}}$, there is $\alpha_\chi \in V_\epsilon^\tau(X)$ of rank at most $\dim_{\mathbb{R}}(X)/2$ such that*

$$\operatorname{ch}(\alpha_\chi) - \chi \in H^0(X, \mathbb{Q})|_{\mathbb{Z}}.$$

Proof. Consider the diagram below.

$$\begin{array}{ccccc} K^0(X) \otimes \mathbb{Q} & & \overset{q^*}{\underset{\sim}{\twoheadleftarrow \text{-----------} \prec}} & & K^0(Y) \otimes \mathbb{Q} \\ \wr \Big\downarrow \operatorname{ch} & & & & \wr \Big\downarrow \operatorname{ch} \\ H^{2\bullet}(X; \mathbb{Q}) & = & H^{2\bullet}(X; \mathbb{Q})^\tau & \overset{q^*}{\underset{\sim}{\twoheadleftarrow\!\!\prec}} & H^{2\bullet}(Y; \mathbb{Q}) \end{array}$$

The manifold X is compact and so is its quotient Y, therefore Lemma 2.4.14 implies that the vertical maps indicated by ch are isomorphisms. $H^{2\bullet}(X;\mathbb{Q})^{\tau} = H^{2\bullet}(X;\mathbb{Q})$ by assumption, so Lemma 2.4.6 shows that the bottom row of the diagram is also an isomorphism. Hence the commutativity of the diagram implies that $q^*\colon K^0(Y)\otimes\mathbb{Q}\to K^0(X)\otimes\mathbb{Q}$ is also a ring isomorphism.

Since X is compact, $H^{2\bullet}(X;\mathbb{Z})$ is finitely generated by Lemma 2.4.3, hence so are its quotients, thus we may pick a finite $\mathbb{Z}$-module generating set B of $H^{2\bullet}(X,\mathbb{Q})|_{\mathbb{Z}}$. Then from the above diagram, for every $b\in B$ there exists $\upsilon_b\in K^0(Y)$, $d_\gamma\in\mathbb{N}_+$ such that such that $\mathrm{ch}(q^*(\upsilon_\gamma\otimes\frac{1}{d_\gamma}))=\gamma$. Let $d\coloneqq\prod_{\gamma\in B}d_\gamma\in\mathbb{N}_+$. Then by above $\mathrm{ch}(q^*(\frac{d}{d_\gamma}\upsilon_\gamma)=d\cdot\gamma$, thus $d\cdot H^{2\bullet}(X,\mathbb{Q})|_Z\subseteq\mathrm{ch}(q^*(K^0(Y)))$. We show that this d satisfies the statement.

Pick $\chi\in d\cdot H^{2\bullet}(X,\mathbb{Q})|_{\mathbb{Z}}$. Then by above, there is $\xi\in K^0(Y)$ such that $\mathrm{ch}(q^*(\xi))=\chi$. Using Lemma 2.4.12, write $\xi=[p_Y]-[\theta_Y^{\oplus k}]$ for some vector bundle $p_Y\colon E\to Y$ over Y, and $k\in\mathbb{N}_0$. By Lemma 2.4.11 and Remark 5.2.10, we may assume that $\mathrm{rk}(p_Y)\leq\lfloor\dim_{\mathbb{R}}(Y)/2\rfloor=\lfloor\dim_{\mathbb{R}}(X)/2\rfloor=n$ since $\dim_R(X)=\dim_{\mathbb{R}}(Y)$. Set $p\coloneqq q^*(p_Y)\colon q^*E\to X$. By construction, $\mathrm{ch}(p)-\chi=\mathrm{ch}(q^*(p_Y))-\mathrm{ch}(q^*(\xi))=\mathrm{ch}(q^*(\theta_Y^{\oplus k})=\mathrm{ch}(\theta_X^{\oplus k}))\in H^0(X;\mathbb{Z})$ and $\mathrm{rk}(p)=\mathrm{rk}(p_Y)$.

Finally, we give an action $\epsilon\circlearrowright p$. Consider the diagram

$$\begin{array}{ccccccccc}
\epsilon & : & 1 & \longrightarrow & C & \xrightarrow{\iota} & G & \xrightarrow{\pi} & M & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\tau} & & \\
\mathrm{Diff}_p & : & 1 & \longrightarrow & \mathrm{Diff}_p^{\mathrm{id}}(q^*E) & \xhookrightarrow{\subseteq} & \mathrm{Diff}_p(q^*E) & \xrightarrow{p_*} & \mathrm{Diff}_p(X) & \longrightarrow & 1
\end{array}$$

where

$$\sigma\colon c\mapsto\mathrm{id}_{q^*E},\qquad \varrho\colon g\mapsto((x,e)\mapsto(\tau(\pi(g))(x),e))$$

for $q^*E=\{(x,e)\in X\times E: q(x)=p_Y(e)\}$. Since $q(\tau(m)(x))=q(x)$ for any $m\in M$ and $x\in X$ by definition, so if $(x,e)\in q^*E$ and $g\in G$, then $q(\varrho(g)(x,e))=q(\tau(\pi(g))(x))=q(x)=p_Y(e)$. This shows that $\varrho(g)(x,e)\in q^*E$, i.e. that ϱ is well-defined. It is clear from the definition of ϱ that it preserves the fibres and the exactness of ϵ implies the commutativity of the diagram. Hence $\beta_\chi\coloneqq(\sigma,\varrho,\tau)\in V_\epsilon^\tau(X)$ has the stated properties. □

Remark 5.2.17. β_χ is typically not faithful. In fact, β_χ is faithful if and only if C is trivial.

Remark 5.2.18. We need to pay the price of the generality of the statement by using K-theory. In this thesis, we will use this statement only with X being a high dimensional complex torus, cf. Subsection 5.2.4. In this concrete case, Lemma 5.2.16 can actually be proved via a concrete construction by hand without a single reference to K-theory, but for clarity, we presented this more structural proof. The other construction builds on the concrete structure of $H^{2\bullet}(X;\mathbb{Z})$ and uses pullbacks, $\otimes$ and $\oplus$ with suitable coefficients given by a careful application of the Möbius inversion on the poset of subsets of a $\dim_{\mathbb{C}}(X)$-element set. The constructed bundle p has much larger rank, namely $2^{\dim_{\mathbb{C}}(X)}-1$, but what matters to our application is that its rank depends only on X, cf. Remark 5.2.22. The interested reader can consult [Sza19, Lemma 4.7, Remark 4.8] for the details of the construction that was motivated by [Mun17].

We can formulate the key notion of this section and proof the existence of the uniformisation process.

Definition 5.2.19. We call an action $\alpha\in V_\epsilon^\tau(X)$ *uniformisable* if all of the following hold.

- ϵ is $1 \to C \to G \to M \to 1$ where M is finite.
- $\tau\colon M \to \mathrm{Diff}(X)$ is free, and the composition $M \xrightarrow{\tau} \mathrm{Diff}(X) \xrightarrow{(-)^*} \mathrm{Aut}(H^{2\bullet}(X;\mathbb{Q}))$ is the identity.
- X is a smooth compact manifold, $K^0(X)$ is a free $\mathbb{Z}$-module.
- $\mathrm{rk}(\alpha) = 1$.

This terminology is explained at the next statements.

Proposition 5.2.20. *There exists $R_3\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that whenever $\alpha \in V^{\tau}_{\epsilon}(X)$ is a uniformisable action, there exists $\alpha^{\perp} \in V^{\tau}_{\epsilon}(X)$ such that $\alpha \oplus \alpha^{\perp}$ is an action on a vector bundle isomorphic to the trivial one of rank at most $R_3(\dim_{\mathbb{R}}(X))$.*

In particular, if α is faithful, then there is a faithful action β

$$\begin{array}{ccccccccc}
\epsilon & : & 1 & \longrightarrow & C & \xrightarrow{\iota} & G & \xrightarrow{\pi} & M & \longrightarrow & 1 \\
\downarrow{\scriptstyle\beta} & & & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\tau} & & \\
\mathrm{Diff}_p & : & 1 & \longrightarrow & \mathrm{Diff}^{\mathrm{id}}_p(X \times \mathbb{C}^r) & \longrightarrow & \mathrm{Diff}_p(X \times \mathbb{C}^r) & \xrightarrow{p_*} & \mathrm{Diff}(X) & \longrightarrow & 1
\end{array}$$

on the trivial bundle $p := \theta_X^{\oplus r}$ of rank $r := R_3(\dim_{\mathbb{R}}(X))$, a space independent of β.

Remark 5.2.21. The proof shows that one may choose R_3 to satisfy the upper bound $R_3(m) \leq R_2(m) + \lfloor m/2 \rfloor + 1$ where R_2 is from Lemma 5.2.14. Then Remark 5.2.13 and Remark 5.2.15 give a rough estimate of $R_3(m) \leq 5^{m/2} \left\lfloor \frac{m}{2} \right\rfloor !$, which again may be very far from the optimum.

Proof. Let $d \in \mathbb{N}_+$ be given by Lemma 5.2.16, and $\alpha(d) \in V^{\tau}_{\epsilon}(X)$ by Lemma 5.2.14. Now $-\chi := \mathrm{ch}(\alpha \oplus \alpha(d)) - \mathrm{rk}(\alpha \oplus \alpha(d)) \in d \cdot H^{2\bullet}(X, \mathbb{Q})|_{\mathbb{Z}}$, so Lemma 5.2.16 applies and produces a $\beta_{\chi} \in V^{\tau}_{\epsilon}(X)$ such that $\mathrm{ch}(\beta_{\chi}) - \chi \in H^0(X, \mathbb{Q})|_{\mathbb{Z}}$. We show that

$$\alpha^{\perp} := \alpha(d) \oplus \beta_{\chi} \oplus \Theta_{\epsilon,\tau}^{\oplus k}$$

satisfies the statement where $\Theta_{\epsilon,\tau} \in V^{\tau}_{\epsilon}(X)$ is the trivial action from Definition 5.2.9, and $k \in \mathbb{N}_0$ is the smallest natural number making $\mathrm{rk}(\alpha^{\perp}) \geq \dim_{\mathbb{R}}(X)/2$ the definition above. By construction, $\mathrm{rk}(\alpha \oplus \alpha^{\perp}) \leq 1 + R_2(\dim_{\mathbb{R}}(X)) + \lfloor \dim_{\mathbb{R}}(X)/2 \rfloor =: R_3(\dim_{\mathbb{R}}(X))$, and $\mathrm{ch}(\alpha \oplus \alpha^{\perp}) = \mathrm{ch}(\alpha \oplus \alpha(d)) + \mathrm{ch}(\beta_{\chi}) = -\chi + \chi(\beta_{\chi}) \in H^0(X, \mathbb{Q})|_{\mathbb{Z}} \subseteq H^0(X;\mathbb{Q})$ using Remark 5.2.10. Then Proposition 2.4.16 applies and gives the first part of the statement.

For the second part, consider $\beta := \alpha \oplus \alpha^{\perp} \oplus \Theta_{\epsilon,\tau}^{\oplus k}$ for $k := r - \mathrm{rk}(\alpha^{\perp}) \geq 0$, so that $\mathrm{rk}(\beta) = r$. Then β is faithful since its component α is faithful. By the above discussion, the underlying vector bundle is isomorphic to the trivial one, so β induces the diagram from the statement. □

Remark 5.2.22. The key here is the uniform bound on the rank. By compactness arguments, one can always find a direct complement for any vector bundle over a compact Hausdorff space, cf. [Hat17, Proposition 1.4] or [Par08, Proposition 1.7.12]. (In fact, such statements can fail in the non-compact case, cf. [Hat17, Example 3.6].) On the other hand, such arguments neither give bounds on the rank nor guarantee the existence of suitable actions.

Finally, we want to construct the required actions on a compact manifolds.

Corollary 5.2.23 (Uniformisation, smooth case, D.R.Sz.). *If $\alpha \in V_\epsilon^\tau(X)$ is a faithful uniformisable action of $\epsilon : 1 \to C \to G \to M \to 1$, and Y is the smooth compact manifold $\mathrm{V}_k(\mathbb{C}^s)$ (for $1 \leq k \leq s$) or $\mathrm{Gr}_k(\mathbb{C}^{s+1})$ (for $1 \leq k \leq s+1$) where $s := R_3(\dim_{\mathbb{R}}(X))$ is given by Proposition 5.2.20, then there is a group monomorphism*

$$G \rightarrowtail \mathrm{Diff}(X \times Y).$$

Remark 5.2.24. Notable cases for the compact manifolds on which G acts faithfully in Corollary 5.2.23 include $X \times \mathbb{S}^{2s-1}$, $X \times \mathrm{SU}(s)$, $X \times \mathrm{U}(s)$ and $X \times \mathbb{CP}^s$.

Proof. Let $\beta \colon \epsilon \circlearrowright \theta_X^{\oplus s}$ be the faithful action from Proposition 5.2.20. The induced actions from Lemma 5.2.7 restricted to G give the statement as $\mathrm{V}_k(\theta_X^{\oplus s}, h) \cong X \times \mathrm{V}_k(\mathbb{C}^s)$ and $\mathrm{Gr}_k(\theta_X^{\oplus(s+1)}) \cong X \times \mathrm{Gr}_k(\mathbb{C}^{s+1})$ using the triviality of $\theta_X^{\oplus s}$. □

5.2.3 Central Product Construction of Uniformisable Actions

We will need to construct uniformisable actions for every central-by-abelian extension giving every non-degenerate twisted Heisenberg group with bounded rank and cyclic centre. To help this, in this section we show a recursive way of constructing such actions using suitable central products acting on the external tensor products.

Definition 5.2.25. Let $q_i \colon L_i \to X_i$ be vector bundles, and let $q_i \colon X_1 \times X_2 \to X_i$ be the natural projection maps. Define their *external tensor product* is defined by

$$q_1^*(L_1) \otimes q_2^*(L_2) = \coprod_{(x_1,x_2) \in X_1 \times X_2} q_1^{-1}(x_1) \otimes q_2^{-1}(x_2) =: L_1 \boxtimes L_2 \xrightarrow{q_1 \boxtimes q_2} X_1 \times X_2$$
$$l_1 \otimes l_2 \mapsto (q_1(l_1), q_2(l_2))$$

where the total space is a disjoint union.

Lemma 5.2.26. *The external central product induces a map*

$$\boxtimes \colon V_{\epsilon_1}^{\tau_1}(X_1) \times V_{\epsilon_2}^{\tau_2}(X_2) \to V_{\epsilon_1 \times \epsilon_2}^{\tau_1 \times \tau_2}(X_1 \times X_2)$$
$$((\sigma_1, \varrho_1, \tau_1), (\sigma_2, \varrho_2, \tau_2)) \mapsto (\sigma_1 \boxtimes \sigma_2, \varrho_1 \boxtimes \varrho_2, \tau_1 \times \tau_2)$$

which gives rise to the commutative diagram

$$\begin{array}{lcccccccccc}
\epsilon_0 & : & 1 & \longrightarrow & \ker(\sigma_1 \boxtimes \sigma_2) & \xrightarrow{\iota_0} & \ker(\varrho_1 \boxtimes \varrho_2) & \xrightarrow{\pi_0} & \ker(\tau_1 \times \tau_2) & & \\
 & & & & \downarrow \cap & & \downarrow \cap & & \downarrow \cap & & \\
\epsilon_1 \times \epsilon_2 & : & 1 & \longrightarrow & C_1 \times C_2 & \xrightarrow{\iota_1 \times \iota_2} & G_1 \times G_2 & \xrightarrow{\pi_1 \times \pi_2} & M_1 \times M_2 & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle \sigma_1 \boxtimes \sigma_2} & & \downarrow{\scriptstyle \varrho_1 \boxtimes \varrho_2} & & \downarrow{\scriptstyle \tau_1 \times \tau_2} & & \\
\mathrm{Diff}_p & : & 1 & \longrightarrow & \mathrm{Diff}_p^{\mathrm{id}}(L) & \xhookrightarrow{\subseteq} & \mathrm{Diff}_p(L) & \xrightarrow{p_*} & \mathrm{Diff}_p(X) & \longrightarrow & 1
\end{array} \tag{5.6}$$

with exact rows where $p := p_1 \boxtimes p_2$, $L := L_1 \boxtimes L_2$, $X := X_1 \times X_2$ for $(\sigma_i, \varrho_2, \tau_i) \colon \epsilon_i \circlearrowright (p_i \colon L_i \to X_i)$.

Remark 5.2.27. Note the missing 1 from the end of the first row: π_0 may not be surjective.

Proof. Let $\epsilon_i : 1 \to C_i \xrightarrow{\iota_i} G_i \xrightarrow{\pi_i} M_i \to 1$, and pick actions $\alpha_i := (\sigma_i, \varrho_i, \tau_i) \in V^{\tau_i}_{\epsilon_i}(X_i)$ on $p_i \colon L_i \to X_i$. Define an action $\alpha_1 \boxtimes \alpha_2 := (\sigma_1 \boxtimes \sigma_2, \varrho_1 \boxtimes \varrho_2, \tau_1 \times \tau)$ of $\epsilon_1 \times \epsilon_2$ on $p_1 \boxtimes p_2$ by

$$\sigma_1 \boxtimes \sigma_2 \colon C_1 \times C_2 \to \mathrm{Diff}^{\mathrm{id}}_{p_1 \boxtimes p_2}(L_1 \boxtimes L_2), \quad (c_1, c_2) \mapsto (l_1 \otimes l_2 \mapsto \sigma_1(c_1)(l_1) \otimes \sigma_2(c_2)(l_2)),$$
$$\varrho_1 \boxtimes \varrho_2 \colon G_1 \times G_2 \to \mathrm{Diff}_{p_1 \boxtimes p_2}(L_1 \boxtimes L_2), \quad (g_1, g_2) \mapsto (l_1 \otimes l_2 \mapsto \varrho_1(g_1)(l_1) \otimes \varrho_2(g_2)(l_2)).$$

These maps are well-defined as the actions are linear on the fibres. It is straightforward to check that these mas are group morphisms making the bottom two rows of the diagram from the statement commutative.

We check that the maps of the first row are well defined. Let $(c_1, c_2) \in \ker(\sigma_1 \boxtimes \sigma_2)$. Then using the commutativity of (5.4), we have

$$\begin{aligned}(\varrho_1 \boxtimes \varrho_2)(\iota_1(c_1), \iota_2(c_2)) &= l_1 \otimes l_2 \mapsto \varrho_1(\iota_1(c_1))(l_1) \otimes \varrho_2(\iota_2(c_2))(l_2)\\ &= l_1 \otimes l_2 \mapsto \sigma_1(c_1)(l_1) \otimes \sigma_2(c_2)(l_2)\\ &= l_1 \otimes l_2 \mapsto (\sigma_1 \boxtimes \sigma_2)(c_1, c_2)(l_1 \otimes l_2)\\ &= l_1 \otimes l_2 \mapsto l_1 \otimes l_2,\end{aligned}$$

so $(\iota_1(c_1), \iota_2(c_2)) \in \ker(\varrho_1 \boxtimes \varrho_2)$ showing that ι_0 is well defined. Similarly, to see that π_0 is well defined, for any $(g_1, g_2) \in \ker(\varrho_1 \boxtimes \varrho_2)$,

$$\begin{aligned}(\tau_1 \times \tau_2)(\pi_1(g_1), \pi_2(g_2)) &= (x_1, x_2) \mapsto (\tau_1(\pi_1(g_1))(x_1), \tau_2(\pi_2(g_2))(x_2))\\ &= (x_1, x_2) \mapsto ((p_1)_*(\varrho_1(g_1))(x_1), (p_2)_*(\varrho_2(g_2))(x_2))\\ &= (p_1(l_1), p_2(l_2)) \mapsto (p_1(\varrho_1(g_1)(l_1)), p_2(\varrho_2(g_2)(l_2)))\\ &= (p_1(l_1), p_2(l_2)) \mapsto (p_1 \boxtimes p_2)((\varrho_1 \boxtimes \varrho_2)(g_1, g_2)(l_1 \otimes l_2))\\ &= (p_1(l_1), p_2(l_2)) \mapsto (p_1 \boxtimes p_2)(l_1 \otimes l_2)\\ &= (p_1(l_1), p_2(l_2)) \mapsto (p_1(l_1), p_2(l_2)),\end{aligned}$$

thus $(\pi_1(g_1), \pi_2(g_2)) \in \ker(\tau_1 \times \tau_2)$, i.e. the second map is also well defined. The commutativity of the diagram follows from the definitions and from $p_* = (p_1)_* \times (p_2)_*$.

Finally, we check the exactness. At $\ker(\sigma_1 \boxtimes \sigma_2)$, this follows from the injectivity of ι_i. Now consider $\ker(\varrho_1 \boxtimes \varrho_2)$. $\mathrm{Im}(\iota_0) \subseteq \ker(\pi_0)$ is evident from the exactness of ϵ_i. To see the inclusion $\mathrm{Im}(\iota_0) \supseteq \ker(\pi_0)$, pick $(g_1, g_2) \in \ker(\pi_0)$. Then $g_i \in \ker(\pi_i) = \mathrm{Im}(\iota_i)$, so we may choose $c_i \in C_i$ with $\iota_i(c_i) = g_i$. Then $(\sigma_1 \boxtimes \sigma_2)(c_1, c_2) = (\varrho_1 \boxtimes \varrho_2)(\iota_1(c_1), \iota_2(c_2)) = (\varrho_1 \boxtimes \varrho_2)(g_1, g_2) = \mathrm{id}_L$ as $(g_1, g_2) \in \ker(\pi_0) \subseteq \ker(\varrho_1 \boxtimes \varrho_2)$, thus $(c_1, c_2) \in \ker(\sigma_1 \boxtimes \sigma_2)$ and $\iota_0(c_1, c_2) = (g_1, g_2)$ as needed. □

Remark 5.2.28. There is an explicit way to describe $\ker(\varrho_1 \boxtimes \varrho_2)$. Suppose $(g_1, g_2) \in \ker(\varrho_1 \boxtimes \varrho_2)$. Then $\varrho_i(g_i)$ fix all fibres of p_i and $\varrho_1(g_1) \otimes \varrho_2(g_2)$ is the identity on each fibre $p^{-1}(x_1, x_2)$. The tensor product of linear maps is trivial, if they are both scalar multiplications which are inverses of each other, so $\varrho_i(g_i)$ acts on the fibre $p_i^{-1}(x_i)$ as multiplication by some scalar $z_{x_i} \in \mathbb{C}^\times$. By assumption, $z_{x_1} z_{x_2} = 1$ for all $x_i \in X_i$, and this forces the independence of z_{x_i} of x_i. Hence $\ker(\varrho_1 \boxtimes \varrho_2)$ consists of those $(g_1, g_2) \in G_1 \times G_2$ such that $\varrho_i(g_i)$ acts as scalar multiplication by some z_{g_i} on all fibres such that $z_{g_1} z_{g_2} = 1$. Once can describe $\ker(\sigma_1 \boxtimes \sigma_2)$ in an analogous way.

We claim that all inclusions between the top two rows of (5.6) are central pairs if both actions $(\sigma_i, \varrho_i, \tau_i)$ are faithful. Indeed, let $\kappa_1 \colon \ker(\sigma_1 \boxtimes \sigma_2) \to C_1, (c_1, c_2) \mapsto c_1$ and $\kappa_2 \colon \ker(\sigma_1 \boxtimes \sigma_2) \to C_1, (c_1, c_2) \mapsto -c_2$. By above, if $(c_1, c_2) \in \ker(\sigma_1 \boxtimes \sigma_2)$, then $\sigma_i(c_i)$ commutes with $\sigma(c_i')$ for any $c_i' \in C_i$, hence the injectivity of σ_i implies that

$\kappa_i(c_1, c_2) \in \mathrm{Z}(C_i)$. This shows that κ_2 is also a group morphism, and that both κ_1 and κ_2 are central morphisms. Suppose that $\kappa_i(c_1, c_2) = 1$. Then $c_i = 1$, so $z_{c_i} = 1$. But $z_{c_1} z_{c_2} = 1$ by above, so actually both $z_{c_1} = z_{c_2} = 1$, hence (c_1, c_2) is trivial from the injectivity of σ_1 and σ_2. The claim follows similarly for the second column, and is evident for the last one as then faithfulness shows that $\ker(\tau_1 \times \tau_2)$ is trivial.

Definition 5.2.29. Call $\alpha \in V_\epsilon^\tau(X)$ *recursively uniformisable*, if it is uniformisable (cf. Definition 5.2.19) and ϵ is an extension involving finite groups, the induced maps on the whole $H^\bullet(X;\mathbb{Q})$ by images of τ are all trivial, X is connected and $K^\bullet(X)$ is a free $\mathbb{Z}$-module.

The next statement is an analogue of the algebraic statement Lemma 2.4.38.

Proposition 5.2.30. *If $\alpha_i = (\sigma_i, \varrho_i, \tau_i) \in V_{\epsilon_i}^{\tau_i}(X_i)$ are faithful and recursively uniformisable actions for $i \in \{1, 2\}$, then there exists a faithful and recursively uniformisable action, say*

$$\alpha_1 \hat{\curlyvee} \alpha_2 \in V_\epsilon^{\tau_1 \times \tau_2}(X_1 \times X_2)$$

for some maximal central product $\epsilon = \epsilon_1 \hat{\curlyvee}_e \epsilon_2$ given (5.6) *and Remark 5.2.28.*

$$\begin{array}{ccccccccccc}
\epsilon_0 & : & 1 & \longrightarrow & \ker(\sigma_1 \boxtimes \sigma_2) & \xrightarrow{\iota_0} & \ker(\varrho_1 \boxtimes \varrho_2) & \xrightarrow{\pi_0} & 1 & \longrightarrow & 1 \\
\big\downarrow e & & & & \kappa \big\downarrow \cap & & \gamma \big\downarrow \cap & & \big\downarrow & & \\
\epsilon_1 \times \epsilon_2 & : & 1 & \longrightarrow & C_1 \times C_2 & \xrightarrow{\iota_1 \times \iota_2} & G_1 \times G_2 & \xrightarrow{\pi_1 \times \pi_2} & M_1 \times M_2 & \longrightarrow & 1
\end{array} .$$

Proof. Write $\alpha_i \colon \epsilon \circlearrowright p_i$ and consider the following diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \epsilon_0 & \xrightarrow{e} & \epsilon_1 \times \epsilon_2 & \xrightarrow{\mathrm{p}_e} & \epsilon_1 \curlyvee_e \epsilon_2 & \longrightarrow & 1 \\
 & & & \searrow^{1} & & \searrow^{\alpha_1 \boxtimes \alpha_2} & \big\downarrow \exists!\, \curlyvee_{\alpha_1 \boxtimes \alpha_2} & & \\
 & & & & & & \mathrm{Diff}_{p_1 \boxtimes p_2} & &
\end{array}$$

The map $(\alpha_1 \boxtimes \alpha_2) \circ e$ is the trivial map, so the universal property from Remark 2.2.18 gives an action $\epsilon_1 \curlyvee_e \epsilon_2 \to \mathrm{Diff}_{p_1 \boxtimes p_2}$ which we denote by $\alpha_1 \hat{\curlyvee} \alpha_2 = (\sigma_1 \hat{\curlyvee} \sigma_2, \varrho_1 \hat{\curlyvee} \varrho_2, \tau_1 \times \tau_2) := \curlyvee_{\alpha_1 \boxtimes \alpha_2}$ and show that it satisfies the statement.

By construction, $\alpha_1 \hat{\curlyvee} \alpha_2 \in V_{\epsilon_1 \curlyvee_e \epsilon_2}^{\tau_1 \times \tau_2}(X_1 \times X_2)$ is faithful. Hence Remark 5.2.5 shows that $C_1 \curlyvee_\kappa C_2$ is cyclic, thus Lemma 2.2.27 implies the maximality of κ, hence the maximality of e by Definition 2.2.40. We check the requirements from Definition 5.2.29 and Definition 5.2.19 one by one. The central product $\epsilon_1 \curlyvee_e \epsilon_2$ is an extension involving finite groups. The K-theory $K^\bullet(X_1 \times X_2) \cong K^\bullet(X_1) \otimes_{\mathbb{Z}} K^\bullet(X_2)$ is free by Lemma 2.4.8 and the assumptions on X_i, as tensor product of free groups is free. The action $\tau_1 \times \tau_2 \to \mathrm{Diff}(X_1 \times X_2)$ is free, as both τ_1 and τ_2 are free by assumption. This $\tau_1 \times \tau_2$ induces the trivial map on rational cohomologies by assumption and Lemma 2.4.4. Smoothness, compactness and connectedness are all preserved under taking direct products. Finally, $\mathrm{rk}(\alpha_1 \hat{\curlyvee} \alpha_2) = \mathrm{rk}(p_1 \boxtimes p_2) = \mathrm{rk}(p_1) \cdot \mathrm{rk}(p_2) = \mathrm{rk}(\alpha_1) \cdot \mathrm{rk}(\alpha_2) = 1$. □

5.2.4 Heisenberg Extensions Acting on Line Bundles

Every Riemann form induces on one hand a twisted Heisenberg group, and on the other hand a line bundle over some complex torus, cf. Subsection 2.4.3. In this section, we show that these constructions are related via a faithful action. This result will serve as the base case the overarching construction of Section 5.2.

Lemma 5.2.31 (D.R.Sz.). *Let $H\colon V\times V\to\mathbb{C}$ be a Hermitian form. Let $\Lambda=\Lambda_1\oplus\Lambda_2$ be a lattice in V (cf. Definition 2.4.21). Let $L\coloneqq L_1\oplus L_2\le V$ be a subgroup such that $\Lambda_j\le L_j\le V$, $\Im H(L_1,L_1)=\Im H(L_2,L_2)=0$, and $\Im H(L_1,\Lambda_2)+\Im H(\Lambda_1,L_2)\subseteq\mathbb{Z}$. Let Γ be a group such that $\Im H(L,L)\le\Gamma\le\mathbb{C}$.*

Then there is a faithful action

$$\begin{array}{ccccccccc}\mathcal{H}([\Im H]) & : & 1\longrightarrow & \Gamma/(\Gamma\cap\mathbb{Z}) & \xrightarrow{\iota[\Im H]} & \mathrm{H}([\Im H]) & \xrightarrow{\pi[\Im H]} & (L_1/\Lambda_1)\times(L_2/\Lambda_2) & \longrightarrow 1\\ & & & \downarrow\sigma & & \downarrow\varrho & & \downarrow\tau & \\ \mathrm{Bih}_p & : & 1\longrightarrow & \mathrm{Bih}_p^{\mathrm{id}}(\mathcal{L}) & \xhookrightarrow{\subseteq} & \mathrm{Bih}_p(\mathcal{L}) & \xrightarrow{p_*} & \mathrm{Bih}(\mathbb{T}_L) & \longrightarrow 1\end{array}\tag{5.7}$$

such that the composition $(L_1/\Lambda_1)\times(L_2/\Lambda_2)\xrightarrow{\tau}\mathrm{Bih}(\mathbb{T}_\Lambda)\xrightarrow{(-)^*}\mathrm{Aut}(H^\bullet(\mathbb{T}_\Lambda;\mathbb{Q}))$ is trivial, where $p\coloneqq p(H,\chi)$, $\mathcal{L}\coloneqq\mathcal{L}(H,\chi)$ from Lemma 2.4.27 for a suitable semi-character χ corresponding to H and Λ, and the abelian bihomomorphism is given by

$$\begin{aligned}[\Im H]\coloneqq[\Im H]_{\Lambda_1,\Lambda_2,L_1,L_2,\Gamma}\colon (L_1/\Lambda_1)\times(L_2/\Lambda_2)&\to\Gamma/(\Gamma\cap\mathbb{Z})\\ (l_1+\Lambda_1,l_2+\Lambda_2)&\mapsto\Im H(l_1,l_2)+(\Gamma\cap\mathbb{Z}).\end{aligned}$$

Remark 5.2.32. In certain cases, the decomposition $L=L_1\oplus L_2$ and $\Lambda=\Lambda_1\oplus\Lambda_2$ can be obtained from $\Lambda\le L$ only. Let $H\colon V\times V\to\mathbb{C}$ be a non-degenerate Hermitian form in a finite dimensional $\mathbb{C}$-vector space V and L be a lattice in V such that $\Im H(L,L)$ is cyclic (e.g. when it is a subgroup of $\mathbb{Q}$). Then $(L,\Im H,\Gamma)$ is a non-degenerate Darboux $\mathbb{Z}$-module, so there is a polarised $\mathbb{Z}$-module $(L_1\oplus L_2,\Im H,\Gamma)$ over it by Lemma 4.1.3.

Proof. First note that $[\Im H]$ is a well-defined map and is also a bihomomorphism inherited from $\Im H$, cf. Remark 2.4.24. We extend the construction of Lemma 2.4.27 in different equivariant ways to construct the stated action.

For any $l\in L$, write $l=l_1+l_2$ for the unique $l_j\in L_j$, and set

$$\hat\chi\colon L\to\mathbb{T},\qquad l\mapsto\exp(\pi i\Im H(l_1,l_2)).$$

This is not a group morphism, not even a semi-character, but satisfies a similar identity. If $l,l'\in L$, then

$$\begin{aligned}\hat\chi(l+l')&=\exp(\pi i\Im H(l_1+l_1',l_2+l_2'))\\&=\hat\chi(l)\hat\chi(l')\exp(\pi i\Im H(l_1,l_2'))\exp(\pi i\Im H(l_1',l_2))\\&=\hat\chi(l)\hat\chi(l')\exp(\pi i\Im H(l',l))\exp(2\pi i\Im H(l_1,l_2'))\end{aligned}$$

using $\Im H(l_1,l_2')+\Im H(l_1',l_2)=\Im H(l',l)+2\Im H(l_1,l_2')$ from the alternating and the polarised property. In particular, $\chi\coloneqq\hat\chi|_\Lambda$ is a semi-character with respect to H as $\Im H(\Lambda,\Lambda)\subseteq\mathbb{Z}$ by assumption.

For any $\hat\chi$ satisfying the above property, set

$$\hat f\colon L\times V\to\mathbb{C}^\times,\qquad (l,v)\mapsto\hat\chi(l)\exp(\pi H(v,l)+\tfrac{\pi}{2}H(l,l)).$$

This is not a group morphism, but satisfies the following property.

$$\begin{aligned}
\hat{f}(l+l',v) &= \hat{\chi}(l+l')\exp(\pi H(v,l+l') + \tfrac{\pi}{2}H(l+l',l+l')) \\
&= \hat{\chi}(l)\hat{\chi}(l')\exp(2\pi i\Im H(l_1,l_2'))\exp(\pi H(v,l+l')) \\
&\qquad \cdot\exp(\pi i\Im H(l',l))\exp(\tfrac{\pi}{2}H(l+l',l+l')) \\
&= \hat{\chi}(l)\exp(\pi H(v,l))\exp(\pi H(l',l))\exp(\tfrac{\pi}{2}H(l,l)) \\
&\qquad \cdot\hat{\chi}(l')\exp(\pi H(v,l'))\exp(\tfrac{\pi}{2}H(l',l'))\cdot\exp(2\pi i\Im H(l_1,l_2')) \\
&= \hat{f}(l,l'+v)\cdot f(l',v)\cdot\exp(2\pi i\Im H(l_1,l_2'))
\end{aligned} \tag{5.8}$$

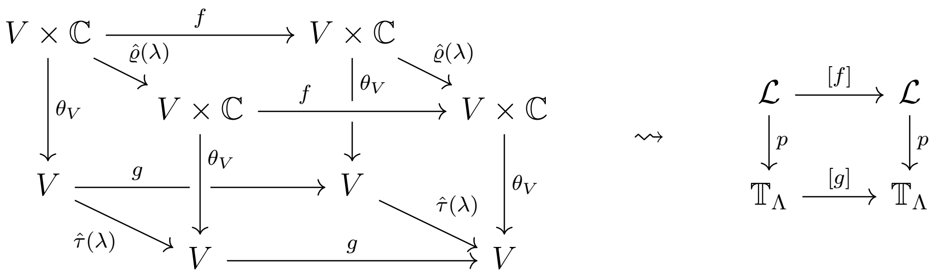
using $i\Im H(l',l)+\frac{1}{2}H(l+l',l+l') = H(l',l)+\frac{1}{2}H(l,l)+\frac{1}{2}H(l',l')$ which is a consequence of H being a Hermitian form. We can now define the following maps on the trivial line bundle over V.

$$\begin{aligned}
&\hat{\sigma}\colon \Gamma\to \mathrm{Bih}(V\times\mathbb{C}), && c\mapsto((v,z)\mapsto(v,\exp(-2\pi ic)z)) \\
&\hat{\varrho}\colon L\to \mathrm{Bih}(V\times\mathbb{C}), && l\mapsto((v,z)\mapsto(l+v,\hat{f}(l,v)z)) \\
&\hat{\tau}\colon L\to \mathrm{Bih}(V), && l\mapsto(v\mapsto l+v)
\end{aligned}$$

$\hat{\sigma}$ and $\hat{\tau}$ are group morphisms, but $\hat{\varrho}$ is typically not.

$$\begin{aligned}
\hat{\varrho}(l+l') &= (v,z)\mapsto(l+l'+v,\hat{f}(l+l',v)z) \\
&= (v,z)\mapsto(l+l'+v,\hat{f}(l,v+l')\cdot f(l',v)\cdot\exp(2\pi i\Im H(l_1,l_2'))z) \\
&= (v,z)\mapsto\hat{\varrho}(l)(l'+v,f(l',v)\cdot\exp(2\pi i\Im H(l_1,l_2'))z) \\
&= (v,z)\mapsto\hat{\varrho}(l)(\hat{\varrho}(l')(v,\exp(2\pi i\Im H(l_1,l_2'))z)) \\
&= \hat{\varrho}(l)\circ\hat{\varrho}(l')\circ\hat{\sigma}(-\Im H(l_1,l_2'))
\end{aligned} \tag{5.9}$$

Since $\hat{\sigma}|_{\mathbb{Z}} = \mathrm{id}$ by definition, $\hat{\varrho}|_{L_1\oplus\Lambda_2}$ and $\hat{\varrho}|_{\Lambda_1\oplus L_2}$ are group morphisms after all by the assumption. Hence the following diagram on the left is commutative for all $\lambda\in\Lambda$ and $(f,g)\in\{(\hat{\varrho}(l_j),\hat{\tau}(l_j)) : l_j\in L_j\}\cup\{(\hat{\sigma}(c),\mathrm{id}) : c\in\Gamma\}$. Thus these maps descend to the quotients $\mathcal{L}\coloneqq\mathcal{L}(H,\chi) = (V\times\mathbb{C})/\hat{\varrho}|_\Lambda$ and $\mathbb{T}_\Lambda = V/\hat{\tau}|_\Lambda$ as indicated on the right where $p\coloneqq p(H,\chi):\mathcal{L}\to\mathbb{T}_\Lambda$ is the corresponding line bundle.



(5.10)

Being motivated by the proof of Lemma 4.2.7 (cf. Remark 4.2.8), we can now define the maps forming the action (5.7).

$$\begin{aligned}
\sigma\colon c+(\Gamma\cap\mathbb{Z}) &\mapsto [\hat{\sigma}(c)] \\
\varrho\colon (l_1+\Lambda_1,l_2+\Lambda_2,c+(\Gamma\cap\mathbb{Z})) &\mapsto [\hat{\varrho}(l_2)]\circ\sigma(c)\circ[\hat{\varrho}(l_1)] \\
\tau\colon (l_1+\Lambda_1,l_2+\Lambda_2) &\mapsto [\hat{\tau}(l_1+l_2)]
\end{aligned}$$

We check that these maps are indeed group monomorphisms and make (5.7) commute. First consider σ. Since $\hat{\sigma}|_{\mathbb{Z}} = \mathrm{id}$, the map σ does not depend on the choice of the representative, i.e. σ is well defined. Note that if $\sigma(c + (\Gamma \cap \mathbb{Z})) = \mathrm{id}$, then $c \in \ker(\hat{\sigma}) = \Gamma \cap \mathbb{Z}$, so σ is indeed injective. By definition, σ is a group morphism.

Consider ϱ. If $\lambda_j \in \Lambda_j$, then $\hat{\varrho}(\lambda_j)$ maps the orbits of $\hat{\varrho}|_\Lambda$ to themselves, hence the induced $[\hat{\varrho}(\lambda_j)] \in \mathrm{Bih}(\mathcal{L})$ is trivial, thus ϱ is well-defined as a map of sets. Note that $\hat{\varrho}(l) \circ \hat{\sigma}(c) = \hat{\sigma}(c) \circ \hat{\varrho}(l)$ by definitions, so this commutativity relation descends to ϱ and σ. Let $h = (l_1 + \Lambda_1, l_2 + \Lambda_2, c + \Gamma \cap Z)$ and $h' = (l_1' + \Lambda_1, l_2' + \Lambda_2, c' + \Gamma \cap Z)$ be arbitrary elements of $\mathrm{H}([\Im H])$. Then by using Remark 4.1.10 and the definitions above,

$$\begin{aligned}\varrho(h * h') &= \varrho((l_1 + l_1' + \Lambda_1, l_2 + l_2' + \Lambda_2, c + \Im H(l_1, la_2') + c' + \Gamma \cap Z)) \\ &= [\hat{\varrho}(l_2 + l_2')] \circ \sigma(c + \Im H(l_1, l_2') + c') \circ [\hat{\varrho}(l_1 + l_1')] \\ &= [\hat{\varrho}(l_2)] \circ [\hat{\varrho}(l_2')] \circ \sigma(c) \circ \sigma(\Im H(l_1, l_2')) \circ \sigma(c') \circ [\hat{\varrho}(l_1)] \circ [\hat{\varrho}(l_1')] \\ &= ([\hat{\varrho}(l_2)] \circ \sigma(c) \circ [\hat{\varrho}(l_1)]) \circ ([\hat{\varrho}(l_2')] \circ \sigma(c') \circ [\hat{\varrho}(l_1')]) \\ &= \varrho(h) \circ \varrho(h')\end{aligned}$$

using $\hat{\varrho}(l_2') \circ \hat{\varrho}(l_1) = \hat{\varrho}(l_2' + l_1) = \hat{\varrho}(l_1 + l_2') = \hat{\varrho}(l_1) \circ \hat{\varrho}(l_2') \circ \hat{\sigma}(-\Im H(l_1, l_2'))$ from (5.9). This shows that ϱ is a group morphism.

By the very definition of $\mathbb{T}_\Lambda = V/\hat{\tau}|_\Lambda$, the map τ is well defined and it is injective. It is also morphism since $\hat{\tau}$ is.

Note that $\varrho \circ \iota_{[\Im H]} = \sigma$ as $[\hat{\varrho}(0)] = \mathrm{id}$, and $\tau \circ \pi_{[\Im H]} = p_* \circ \varrho$ since $p \circ [\hat{\tau}(l_j)] = [\hat{\varrho}(l_j)] \circ p$ by (5.10). Hence (5.7) is commutative. Finally, the 4-lemma implies the injectivity of ϱ.

Finally note that $\hat{\tau}(l)$ is homotopic to the identity on V, hence $\tau(l)$ is also homotopic to the identity on $\mathbb{T}_\Lambda$, thus the induced map on the (rational) cohomologies is trivial. □

Recall Definition 5.2.29.

Lemma 5.2.33. *Let $\mu\colon A \times B \to C$ be an abelian bihomomorphism between finite cyclic groups, and let $\mathbb{T}_\Lambda$ be a 1-dimensional complex torus. Then there a faithful and recursively uniformisable action $\alpha \in V^\tau_{\mathcal{H}(\mu)}(\mathbb{T}_\Lambda)$ for a suitable τ.*

Proof. Let V be a 1-dimensional $\mathbb{C}$-vector space. Let $\{z_1, z_2\}$ be an $\mathbb{R}$-basis for V. Define subgroups $\Lambda_1 := z_1\mathbb{Z}$, $\Lambda_2 := z_2\mathbb{Z}$, and a lattice $\Lambda := \Lambda_1 \oplus \Lambda_2 \subset V$. Let $L_1 := \frac{1}{|A|}\Lambda_1$, $L_2 := \frac{1}{|A|}\Lambda_2$, $L := L_1 \oplus L_2 \subset V$, and $\Gamma := \frac{1}{|C|}\mathbb{Z}$. By assumption, there are isomorphisms $\varphi_1\colon L_1/\Lambda_1 \to A$, $\varphi_2\colon L_2/\Lambda_2 \to B$, $\varphi_3\colon \Gamma/\mathbb{Z} \to C$. There is an $m \in \mathbb{Z}$ satisfying $\mu(\varphi_1(\frac{z_1}{|A|} + \Lambda_1), \varphi_2(\frac{z_2}{|B|} + \Lambda_2)) = \varphi_3(\frac{m}{|C|} + \mathbb{Z})$. Choosing φ_3 suitably, we may further assume that $\frac{|C|}{m} \in \mathbb{Z}$, cf. Lemma 2.2.26. Define $H\colon V \times V \to \mathbb{C}, (v, v') \mapsto \frac{m|A||B|}{|C|}\frac{1}{\langle z_1, z_2\rangle}\langle v, v'\rangle$ where $\langle -, -\rangle$ is any non-degenerate Hermitian form on V.

We check thin Lemma 5.2.31 is applicable to this setup. Since H is Hermitian, $\Im H(z_j, z_j) = 0$, consequently $\Im H(L_j, L_j) = 0$. Since μ is a bihomomorphism, $\varphi_3(\frac{m|A|}{|C|} + \mathbb{Z}) = \mu(\varphi_1(|A|\frac{z_1}{|A|} + \Lambda_1), \varphi_2(\frac{z_2}{|B|} + \Lambda_2)) = \mu(0, \varphi_2(\frac{z_2}{|B|} + \Lambda_2)) = 0$, hence $\frac{m|A|}{|C|} \in \mathbb{Z}$. Similarly, we get $\frac{m|B|}{|C|} \in \mathbb{Z}$. Thus $\Im H(L_1, \Lambda_2) = \Im H(\frac{z_1}{|A|}, z_2)\mathbb{Z} = \frac{m|B|}{|C|}\mathbb{Z} \subseteq \mathbb{Z}$ by the discussion above, and $\Im H(\Lambda_1, L_2) = \frac{m|A|}{|C|}\mathbb{Z} \subseteq \mathbb{Z}$ is obtained similarly. $\Im H(L_1, L_2) = \Im H(\frac{z_1}{|A|}, \frac{z_2}{|B|})\mathbb{Z} = \frac{m}{|C|}\mathbb{Z} \subseteq \frac{1}{|C|}\mathbb{Z} = \Gamma$. Thus we may indeed apply Lemma 5.2.31 to get a faithful action $\mathcal{H}([\Im H]) \circlearrowright (p\colon \mathcal{L} \to \mathbb{T}_\Lambda)$.

For $l_1 := \frac{1}{|A|}z_1\frac{|C|}{m} \in L_1$ and $l_2 := \frac{z_2}{|B|} \in L_2$, we have $\Im H(l_1, l_2) = 1$, so $\Im(L_1, L_2) \cap \mathbb{Z} = \mathbb{Z}$. Hence $(\varphi_1 \times \varphi_2, \varphi_3)\colon [\Im H] \to \mu$ is an isomorphism of abelian bihomomorphisms, thus $\mathcal{H}([\Im H]) \cong \mathcal{H}(\mu)$ by Lemma 4.1.13.

Now p is a line bundle over a space diffeomorphic (as real manifolds) to the compact and connected $\mathbb{T}^2$. By Example 2.4.9, $K^\bullet(\mathbb{T}_\Lambda) \cong \mathbb{Z}^4$ is a free $\mathbb{Z}$-module. Hence by construction and Lemma 5.2.31, the action of $\mathcal{H}(\mu)$ is faithful and recursively uniformisable. □

Remark 5.2.34. In a similar fashion, this construction can be extended to the case where the groups A, B, C are not necessarily cyclic allowing us to skip Subsection 5.2.3. We do not do this here, as for our purposes, Proposition 5.2.30 is sufficient and demonstrates the central product nature and the historical development of the thesis better.

The condition of A, B being cyclic can be dropped by passing to a higher dimensional torus.

Proposition 5.2.35. *Let $\mu\colon A \times B \to C$ be a non-degenerate abelian bihomomorphism between finite groups such that $d(A), d(B) \leq s$ (for some $s \in \mathbb{N}_0$) and C is cyclic. Let $\mathbb{T}_\Lambda$ be an s-dimensional complex torus. Then there is a faithful and recursively uniformisable action $\alpha \in V^\tau_{\mathcal{H}(\mu)}(\mathbb{T}_\Lambda)$ for a suitable τ.*

Proof. Let V be the underlying $\mathbb{C}$-vector space of $\mathbb{T}_\Lambda$. Write $\Lambda = \prod_{j=1}^s \Lambda_j$ where $\Lambda_j \cong \mathbb{Z}^2$, and let $V_j \leq V$ the $\mathbb{C}$-subspace generated by Λ_j. Let $\mathbb{T}_{\Lambda_j} := V_j/\Lambda_j$, a complex torus of dimension 1. Let $\bar{\mu}_1, \dots, \bar{\mu}_t$ be from Proposition 4.3.6. Note that $t = d(A) = d(B) \leq s$, and let $\bar{\mu}_j\colon 0 \times 0 \to 0$ for $t < j \leq s$. Let $\alpha_j \in V^{\tau_j}_{\mathcal{H}(\bar{\mu}_j)}(\mathbb{T}_{\Lambda_j})$ the faithful and recursively uniformisable action from Lemma 5.2.33. By iterative applications of Proposition 5.2.30, we obtain a faithful and recursively uniformisable action $\alpha_1 \hat{\curlyvee} \dots \hat{\curlyvee} \alpha_s \in V^\tau_\epsilon(X)$ of some maximal central product $\epsilon = \mathcal{H}(\bar{\mu}_1) \hat{\curlyvee}_{e^1} \dots \hat{\curlyvee}_{e^{s-1}} \mathcal{H}(\bar{\mu}_s)$ on a line bundle over $X = \prod_{j=1}^s \mathbb{T}_{\Lambda_j}$ where $\tau = \prod_{j=1}^s \tau_j$. By the invariance part of Proposition 4.3.6, $\mathcal{H}(\mu) \cong \epsilon$, and by construction, $X \cong \mathbb{T}_\Lambda$. This gives an action $\alpha \in V^\tau_{\mathcal{H}(\mu)}(\mathbb{T}_\Lambda)$ with the stated properties. □

5.2.5 The Main Jordan Type Theorem

In this section, we put the pieces of the puzzle obtained in the previous sections together. We prove the smooth part of Theorem C from page 9 and give a necessary condition for Problem 1.1.2. The big picture is analogous to that of Subsection 5.1.2.

The next statement and its proof is analogous to Theorem 5.1.6.

Theorem 5.2.36 (D.R.Sz.)**.** *For every $r \in \mathbb{N}_0$, there exists a smooth connected compact manifold M_r such that every $G \in \mathcal{F}_2(r)$ embeds into* $\mathrm{Diff}(M_r)$*.*

Remark 5.2.37. As we saw in Remark 5.1.7, the bounded rank condition is necessary by Lemma 1.2.7. The proof shows that $X_r := \mathbb{T}^{2r\lfloor r/2 \rfloor} \times Y^r$ satisfies the statement for any $Y \in \{\mathrm{V}_k(\mathbb{C}^s) : 1 \leq k \leq s\} \cup \{\mathrm{Gr}_k(\mathbb{C}^{s+1}) : 1 \leq k \leq s+1\}$ where $s := R_3(2\lfloor r/2 \rfloor)$ is from Proposition 5.2.20. As noted in Remark 5.2.24, this set includes $\mathbb{S}^{2s-1}$, $\mathrm{SU}(s)$, $\mathrm{U}(s)$ and $\mathbb{CP}^s$.

Proof. Using Corollary 3.1.26, write may write $G \leq \prod_{i=1}^d G_i$ for suitable $G_i \in \mathcal{F}_2(r)$ having cyclic centre where $d := d(\mathrm{Z}(G)) \leq r$. Consider

$$\mathcal{Z}(G_i) \overset{f_i}{\rightarrowtail} \mathcal{H}(\mu_i\colon A_i \times A_i \to C_i) \overset{\alpha_i}{\rightarrowtail} \mathrm{Diff}_{p_i}$$

where the $\curlyvee$-monomorphism f_i is from Proposition 4.2.18, $d(A) \leq d(G)/2 \leq r/2$, and α_i is the faithful and uniformisable action from Proposition 5.2.35 on the line bundle $p_i\colon L_i \to \mathbb{T}_\Lambda$ where $\mathbb{T}_\Lambda$ is a fixed $s := \lfloor r/2 \rfloor$-dimensional complex torus. This composition then gives a faithful and uniformisable action of $\mathcal{Z}(G_i)$, hence Corollary 5.2.23 provides an injection $\varphi_i\colon G_i \to \operatorname{Aut}(\mathbb{T}_\Lambda \times Y)$ for any Y from this statement. Then

$$G \overset{\subseteq}{\to} \prod_{i=1}^{d} G_i \xrightarrow{\prod_{i=1}^{d}\varphi_i} \prod_{i=1}^{d} \operatorname{Diff}(\mathbb{T}_\Lambda \times Y) \to \operatorname{Diff}(\mathbb{T}^{2r\lfloor r/2\rfloor} \times Y^r),$$

as $\mathbb{T}_\Lambda \cong \mathbb{T}^{2\lfloor r/2\rfloor}$ are smooth real manifolds. So $M_r := \mathbb{T}^{2r\lfloor r/2\rfloor} \times Y^r$ is a suitable choice. □

A special case of the previous statement answer the question of Mundet i Riera.

Corollary 5.2.38 (D.R.Sz.)**.** *Question 1.2.4 has a positive answer.*

Proof. This follows from Theorem 5.2.36 after noting that any group of order p^r can be generated by at most r elements. □

We can give a partial answer to Problem 1.1.2 in the smooth case.

Corollary 5.2.39 (Lower bound, D.R.Sz.)**.** *Let $\mathcal{N}$ be a class of groups such that* $\operatorname{Diff}(X)$ *is $\mathcal{N}$-Jordan for every smooth compact manifold X. Then $\mathcal{F}_2(r)$ is uniformly $\mathcal{N}$-Jordan for every $r \in \mathbb{N}_0$.*

Proof. This follows from Theorem 5.2.36 and Definition 1.1.1 when applied to $X = M_r$. □

Question 5.2.40. Is Conjecture 1.1.5 true? Does the reverse of Corollary 5.2.39 hold as in Theorem 5.1.9?

5.2.6 An Explicit Counterexample to Ghys' Original Conjecture

We conclude this chapter too with a concrete example. We demonstrate the idea of the overarching construction of Section 5.2 through 2-generated canonical Heisenberg groups. Some details here are a bit different and are more in the flavour of [Mun17]. Here we present an explicit embedding $\operatorname{Hom}(\mathbb{Z}_b, \mathbb{Z}_d) \to \operatorname{Diff}(\mathbb{T}^2 \times \mathrm{SU}(2))$ for every $b \mid d$. Another reason we include this example is that is gives a very explicit counterexample to Conjecture 1.1.4. This construction already appeared in [Mun17] and [CPS14] in a much less explicit form only giving the existence of the embeddings above.

Example 5.2.41. Pick positive integers $b \mid d$, and consider the non-degenerate Heisenberg group

$$E := \langle \alpha, \beta, \delta : \delta^{d/b} = [\alpha, \beta], 1 = [\delta, \alpha] = [\delta, \beta] = \alpha^b = \beta^b \rangle \cong \mathrm{H}(\mathbb{Z}_b, \mathbb{Z}_d) \tag{5.11}$$

where $\langle \alpha \rangle = \operatorname{Hom}(\mathbb{Z}_b, \mathbb{Z}_d) \cong \mathbb{Z}_b = \langle \beta \rangle$, $\langle \delta \rangle = \mathbb{Z}_d$. These groups are the fundamental building blocks of this thesis, cf. Corollary 3.1.26, Proposition 4.2.18, Corollary 4.3.7.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, define $L := \mathbb{R} \times \mathbb{T} \times \mathbb{C} \times / \sim$ where the equivalence is given by $(v, \theta, z) \sim (v + \lambda, \theta, \theta^{\lambda b} z)$ for any $\lambda \in \mathbb{Z}$. Then $p\colon L \to (\mathbb{R}/\mathbb{Z}) \times \mathbb{T}, [v, \theta, z] \mapsto (v + \mathbb{Z}, \theta)$ is a line bundle. (This plays the role of the holomorphic line bundles from Subsection 2.4.3, even though the p from above is not holomorphic, but 'only' smooth.) Let $\mu_k := \exp(2\pi i/k) \in \mathbb{T}$. Define an action of $\varrho\colon E \to \mathrm{Diff}(L)$ by

$$\begin{aligned}\varrho(\alpha)\colon [v, \theta, z] &\mapsto [v + 1/b, \theta, \theta z],\\ \varrho(\beta)\colon [v, \theta, z] &\mapsto [v, \mu_b \theta, z],\\ \varrho(\delta)\colon [v, \theta, z] &\mapsto [v, \theta, \mu_d z].\end{aligned}$$

These are well-defined and can be extended to a group morphism as the defining relations from above are satisfied. We see that this map is injective using the definition of Definition 4.3.1.

There is a dual action $\varrho^*\colon E \to \mathrm{Diff}(L^*)$ on the dual bundle $p^*\colon L^* = L^{\otimes -1} \to (\mathbb{R}/\mathbb{Z}) \times \mathbb{T}$. $L^* \cong \mathbb{R} \times \mathbb{T} \times \mathbb{C} \times / \sim'$ where $(v, \theta, z) \sim' (v + \lambda, \theta, \theta^{-\lambda b} z)$. Note that $p \oplus p^*$ is the trivial bundle. (This means ϱ^* plays the role of $\varrho^{\perp}$ from Proposition 5.2.20. In more details, $T = \{1, -1\}$ from Equation 5.5 works, and there is no need for the component from Lemma 5.2.16 given by K-theory. See Remark 5.2.18 about the usage of K-theory in general.)

In fact, we can give two independent global sections σ_1, σ_2 on $p \oplus p^*$ as follows. Let L_1 and L_2 be the complex logarithms whose imaginary part is on the interval $\left[-\frac{\pi}{b}, 2\pi - \frac{\pi}{b}\right)$ and $[0, 2\pi)$ respectively. Let $f_1, f_2\colon \mathbb{T} \to \mathbb{R}$ be smooth functions such that, $f_1^2 + f_2^2 \equiv 1$, $f_j(\theta) = f_j(\mu_b \theta)$ for all $\theta \in \mathbb{T}$, $f_1(1) = 1 = f_2(\mu_{2b})$, $f_1(\mu_{2b}) = 0 = f_2(1)$ and all derivatives of f_j at 1 and at μ_{2b} vanish. (We can write down such function with the help of a $\mathbb{R} \to \mathbb{R}$ bump function, e.g. for $-\pi/b < \alpha < \pi/b$, $f_1\colon \exp(i\alpha) \mapsto \exp(-1/((\pi/b)^2 - \alpha^2))$.) Define two sections by

$$\begin{aligned}\sigma_1\colon (v + \mathbb{Z}, \theta) &\mapsto [v, \theta, f_1(\theta)\exp(vb\,\mathrm{L}_1(\theta)), f_2(\theta)\exp(-vb\,\mathrm{L}_2(\theta))]\\ \sigma_2\colon (v + \mathbb{Z}, \theta) &\mapsto [v, \theta, -f_2(\theta)\exp(vb\,\mathrm{L}_2(\theta)), f_1(\theta)\exp(-vb\,\mathrm{L}_1(\theta))]\end{aligned}$$

Note that these are well defined and the smoothness at the cuts is ensured by f_j. These section are linearly independent, since over any point, $(v + \mathbb{Z}, \theta) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{T}$, the determinant of the two vectors is $|f_1(\theta)|^2 + |f_0(\theta)|^2 = 1 \neq 0$. Using the basis given by σ_1 and σ_2, $\varrho \oplus \varrho^*$ induces an action $E \to \mathrm{Diff}((\mathbb{R}/\mathbb{Z}) \times \mathbb{T} \times \mathbb{C}^2)$ on a manifold that is now independent of the line bundle p above. cf. the uniformisation process of Subsection 5.2.2. We can then use $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$, and the Stiefel bundle construction from Example 2.4.18.

This gives the following explicit embedding

$$\begin{aligned}\varrho\colon E &\to \mathrm{Diff}(\mathbb{T}^2 \times \mathrm{SU}(2))\\ \alpha &\mapsto ((\tau, \theta, X) \mapsto (\mu_b \tau, \theta, X))\\ \beta &\mapsto ((\tau, \theta, X) \mapsto (\tau, \mu_b \theta, B_{\tau,\theta} X))\\ \delta &\mapsto ((\tau, \theta, X) \mapsto (\tau, \theta, D_{\tau,\theta} X))\end{aligned} \tag{5.12}$$

where $B_{\tau,\theta}, D_{\tau,\theta} \in \mathrm{SU}(2)$ are defined depending on $\arg(\theta) \in [0, 2\pi)$ as

$\arg(\theta) \in$	$B_{\tau,\theta}$	$D_{\tau,\theta}$
$[0, \frac{\pi}{b})$	$Q(\tau,\theta)^{-1} R(\tau^{-1}\mu_{b^2})$	$R(\mu_d)$
$[\frac{\pi}{b}, 2\frac{\pi}{b})$	$R(\tau^{b-1}\mu_{b^2})$	$R(\mu_d)$
$[2\frac{\pi}{b}, 3\frac{\pi}{b})$	$R(\tau^{b-1}\mu_{b^2}) Q(\tau,\theta)$	$Q(\tau,\theta)^{-1} R(\mu_d) Q(\tau,\theta)$
$[3\frac{\pi}{b}, 2\pi)$	$R(\tau^{-1}\mu_{b^2})$	$R(\mu_d)$

for the following maps

$$\begin{array}{lll} \mathbb{T} \xrightarrow{R} \mathrm{SU}(2) & \mathbb{T} \times \mathbb{T} \xrightarrow{Q} \mathrm{SU}(2) & \mathbb{T} \xrightarrow{F} \mathrm{SU}(2) \\ \tau \mapsto \mathrm{diag}(\tau, \tau^{-1}) & (\tau, \theta) \mapsto [F(\theta), \mathrm{diag}(1, \tau^b)] & \theta \mapsto \left(\begin{smallmatrix} f_1(\theta) & f_2(\theta) \\ -f_2(\theta) & f_1(\theta) \end{smallmatrix}\right) \end{array}$$

where Q is defined by the (group) commutator. Taking $b = d$ to be a prime p, E becomes a non-abelian p-group of order p^3, so its largest abelian normal subgroup has index at least p. As there are arbitrary large primes, we see that $\mathrm{Diff}(\mathbb{T}^2 \times \mathrm{SU}(2))$ is not Jordan. So (5.12) gives an explicit counterexample to Conjecture 1.1.4.

Remark 5.2.42. One can verify the validity of (5.12) completely independently from the construction above using plain matrix computations as follows. Verifying that it is a group morphism is equivalent to the relations of (5.11) which in turn boils down to checking the matrix equations $Q(\mu_b\tau, \theta) = Q(\tau, \theta) = Q(\tau, \mu_b\theta)$. (Note that periodicity of F is not really needed for the whole of $\mathbb{T}$, but over subsets with $\arg(\theta) \in [0, \frac{\pi}{b}) \cup [2\frac{\pi}{b}, 3\frac{\pi}{b})$.) Continuity in θ is equivalent to $Q(\tau, 1) = 1$ and $Q(\tau, \mu_{2b})^{-1} = R(\tau^b)$, and differentiability is equivalent to the vanishing derivatives assumption.

Remark 5.2.43. This computation is from an earlier, and arguably less clean, stage of the thesis. One may expect to obtain nicer formulae upon following the very method presented at the thesis, i.e. using holomorphic line bundles from Subsection 2.4.3, the group action from Lemma 5.2.33, and using theta functions form [BL04, §3.2] to obtain global sections.

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Terminology

Symbols