Turán type problems

by

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Contents

Abstract

The overarching theme of the thesis is the investigation of Turán-type problems in graphs. A big part of it is focused on studying Turán number of square of path, disjoint union of wheels, and short cycles in complete 3-partite graphs. In addition, we study the anti-Ramsey number for short cycles in complete 3-partite graphs and also show that for an n-vertex graph G with $\lfloor \frac{n^2}{4} \rfloor + 1$ edges, the number of triangles is more when they have no common vertex.

The thesis consists of 5 chapters. The first chapter gives a summary of the history as well as the relevant background of Turán type problem and anti-Ramsey number.

In the second chapter, we study the exact value of Turán number for P_5^2 and P_6^2. Let P_k be the path with k vertices, the square P_k^2 of P_k is obtained by joining the pairs of vertices with distance one or two in P_k. $\mathrm{ex}(n, P_3^2)$ and $\mathrm{ex}(n, P_4^2)$ were solved by Mantel and Dirac, respectively. In order to determine $\mathrm{ex}(n, P_6^2)$, we also determine the exact value of $\mathrm{ex}(n, T)$ where T denotes the flattened tetrahedron. Even more, we characterize the extremal graphs for P_5^2, P_6^2 and T. These results are based on the paper "The Turán number of the square of a path" which is co-authored with Gyula O. H. Katona, Jimeng Xiao and Oscar Zamora.

In Chapter 3, we study the problem concerning Turán number of disjoint union of wheels. Recently, Longtu Yuan determined $\mathrm{ex}(n, W_{2k+1})$ of the odd wheel when n is sufficiently large. We generalize his result, determine the Turán number and characterize all extremal graphs for disjoint union of odd wheels. This result is based on the paper "A note on the Turán number of disjoint union of wheels" which is co-authored with Oscar Zamora.

In Chapter 4, we consider the Turán numbers and anti-Ramsey numbers for short cycles in complete 3-partite graphs. We call a 4-cycle in K_{n_1,n_2,n_3} multipartite, denoted by C_4^{multi}, if it contains at least one vertex in each part of K_{n_1,n_2,n_3}. We prove that

$\mathrm{ex}(K_{n_1,n_2,n_3}, C_4^{\mathrm{multi}}) = n_1n_2 + 2n_3$ and $\mathrm{ar}(K_{n_1,n_2,n_3}, C_4^{\mathrm{multi}}) = \mathrm{ex}(K_{n_1,n_2,n_3}, \{C_3, C_4^{\mathrm{multi}}\}) + 1 = n_1n_2 + n_3 + 1$, where $n_1 \geq n_2 \geq n_3 \geq 1$. These results are based on the paper "Turán numbers and anti-Ramsey numbers for short cycles in complete 3-partite graphs" which is co-authored with Chunqiu Fang, Ervin Győri and Jimeng Xiao.

In Chapter 5, we show that for an n-vertex graph G with $\lfloor \frac{n^2}{4} \rfloor + 1$ edges, if there is no vertex contained by all triangles then there are at least $n-2$ triangles in G. Erdős proved something stronger that if G is an n-vertex graph with $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges, $t \leq 3$, $n > 2t$, then every G contains at least $t \left\lfloor \frac{n}{2} \right\rfloor$ triangles. Our result give a further improvement of Erdős theorem in the case of $t = 1$. This result is based on the paper "The number of triangles is more when they have no common vertex" which is co-authored with Gyula O. H. Katona.

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Chapter 1

Introduction

1.1 Basic notations and definitions

A *graph* G is a pair of sets $V(G)$ and $E(G)$, where $V(G)$ denotes the set of *vertices* and $E(G)$ denotes the set of *edges* where the edges are sets of two distinct vertices. We denote the size of these sets by $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Except when stated otherwise, we will only allow a pair of vertices to occur as an edge once. Usually an edge will be written as uv where u and v are vertices. We say that two vertices are *adjacent* if they form an edge and that a vertex and an edge are *incident* if the vertex is in the edge. Two edges that share a vertex will also be called *incident*. Given a set $S \subseteq V$ and an edge e, we say that e is *incident* with S if e is incident with at least one of the vertices in S.

We define the *neighborhood* of v in G to be the set $N_G(v) := \{u \in V(G) : vu \in E(G)\}$, and we define the *degree* of a vertex v in G by $d_G(v) = |N_G(v)|$. When the base graph is clear we simply denote the neighborhood of v as $N(v)$ and the degree of v as $d(v)$. The *maximum degree*, denoted by $\Delta(G)$, in a graph G is the largest degree among all of the vertices. The *minimum degree*, denoted by $\delta(G)$, is the smallest possible value of $d(v)$ among the vertices of $V(G)$.

A graph F is called a *subgraph* of G if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. We use notation $F \subset G$ to denote that F is a subgraph of G. Given a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced on set S. A set S is called *independent* if the graph induced by S has no edge. The *independence number* $\alpha(G)$ is the maximum size of an independent set in G.

Definition 1.1. *A **path** in a graph is a sequence of distinct vertices $v_1, v_2, \ldots, v_{t+1}$ such that v_i and v_{i+1} are adjacent for every $i = 1, 2, \ldots, t$. The vertices x_1 and x_{t+1} are referred to as terminal vertices, and the remaining vertices are referred to as internal vertices.*

Definition 1.2. *A graph is **connected** if for every pair of vertices u, v there is a path starting from u and ending in v.*

Definition 1.3. *A **biconnected graph** is a connected and "nonseparable" graph, meaning that if any one vertex were to be removed, the graph will remain connected.*

Definition 1.4. *A **matching** in a graph is a set of disjoint edges.*

Definition 1.5. *A **block** is a maximal biconnected subgraph of a given graph G.*

Definition 1.6. *A **cycle** is a sequence $v_1, v_2, \ldots, v_{k-1}, v_k = v_1$ where v_i and v_{i+1} are adjacent for $i = 1, 2, \ldots, k-1$ and v_i is distinct from v_j for any $1 \leq i < j \leq k-1$.*

Definition 1.7. *A connected graph that does not contain cycles is called a **tree**.*

The k-vertex cycle is denoted C_k and the k-vertex path is denoted P_k. The *length* of a path P_k is $k-1$, the number of edges in it. The *complete graph* (or *clique*)on r vertices, that is, K_r is a graph on r vertices such that every pair of vertices is adjacent, is denoted by K_r.

Definition 1.8. *A graph G is a **bipartite graph** if $V(G)$ can be partitioned into two color classes X and Y such that every edge of G contains precisely one vertex of each class.*

We denote by $K_{s,t}$ the *complete bipartite graph* with color classes of X and Y, with $|X| = s$, $|Y| = t$ and x is adjacent to y for every pair of vertices $x \in X, y \in Y$.

1.2 Turán-type problems

Turán-type problems are generally formulated in the following way: one fixes some graph properties and tries to determine the maximum number of edges in an n-vertex graph with the prescribed properties. These kinds of extremal problems have a rich history in combinatorics. Investigation of this type of problems dates back to 1907, when Mantel [43] determined the maximum possible number of edges in a triangle free graph.

Theorem 1.9 (Mantel [43]). *The maximum number of edges in an n-vertex triangle-free graph is* $\lfloor \frac{n^2}{4} \rfloor$.

Years later, Turán [51] initated systematic studying of these problems and generalized Mantel's result to arbitrary complete graphs.

Definition 1.10. *The Turán graph $T(n,p)$ is a complete multipartite graph formed by partitioning a set of n vertices into p subsets, with sizes as equal as possible, and connecting two vertices by an edge if and only if they belong to different subsets. Denote its size by $t(n,p)$.*

Theorem 1.11 (Turán [51]). *The maximum number of edges in an n-vertex K_{p+1}-free graph is at most $t(n,p)$. Furthermore, $T(n,p)$ is the unique extremal graph.*

For simple graphs G and F, we say that G is *F-free* if G does not contain F as a subgraph.

Definition 1.12. *Given G and a set of graphs $\mathcal{F}$, the Turán number of $\mathcal{F}$ is the maximum number of edges among all $\mathcal{F}$-free subgraphs of a host graph G, that is*

$$\mathrm{ex}(G,\mathcal{F}) := \max\left\{ |E(H)| \colon H \subseteq G,\ H \textit{ is } F\textit{-free for every } F \in \mathcal{F} \right\}.$$

In particular, we write $\mathrm{ex}(n,\mathcal{F})$ rather than $\mathrm{ex}(K_n,\mathcal{F})$ when the host graph is K_n.

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum integer k such that we can assign colors $1, 2, \ldots, k$ to the vertices of G and have no edge with the same color on each vertex. Erdős, Stone and Simonovits showed that the asymptotic behavior of the Turán number of a non-bipartite graph H is determined by $\chi(H)$.

Theorem 1.13 (Erdős-Stone-Simonovits [21, 19]). *For a graph H with $\chi(H) \geq 3$, we have*

$$\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2).$$

It is fascinating that this one theorem asymptotically takes care of the huge class of Turán problems. Since then, the study has been mainly directed to the cases: *(i)* the forbidden graph is bipartite and *(ii)* the exact value of $\mathrm{ex}(n,H)$ when H is non-bipartite.

Kővári, Sós and Turán [33] considered the case when the forbidden graph is the complete bipartite graph $K_{a,b}$.

Theorem 1.14 (Kővári-Sós-Turán [33]). *Let $K_{a,b}$ denote the complete bipartite graph with a and b vertices in its color-classes. Then*

$$\mathrm{ex}(n, K_{a,b}) \leq \frac{\sqrt[a]{b-1}}{2} n^{2-\frac{1}{a}} + \frac{a-1}{2} n.$$

In the bipartite case, another natural problem is to estimate the Turán number for even cycles.

Theorem 1.15 (Bondy, Simonovits [9]). *For any $k \geq 2$, we have*

$$\mathrm{ex}(n, C_{2k}) = O(n^{1+\frac{1}{k}}).$$

For $k = 2, 3$ and 5, it is proved that the order of magnitude can not be improved. But generally, whether this bound gives us the correct order of magnitude is still one of the most intriguing open questions in extremal graph theory.

For a path P_k, Erdős and Gallai [18] proved the following result,

Theorem 1.16 (Erdős-Gallai [18]). *For all $n \geq k$,*

$$\mathrm{ex}(n, P_{k+1}) \leq \frac{(k-1)n}{2}.$$

Moreover, equality holds if and only if k divides n and G is the disjoint union of cliques of size k.

In their paper, the case when all cycles longer than a given length are forbidden, was also considered.

Theorem 1.17 (Erdős–Gallai [18]). *For any n, let $C_{>k}$ $(k \geq 2)$ denote the family of cycles of length more than k, then we have*

$$\mathrm{ex}(n, C_{>k}) \leq \frac{k(n-1)}{2}.$$

Moreover, equality holds if and only if when $k-1$ divides $n-1$ and G is a connected graph such that each block of G is a clique of size k.

Recent studies of extremal numbers consider the case when the forbidden graph H is made up of several vertex-disjoint copies of some smaller graph.

Theorem 1.18 (Gorgol [27]). *Let G be an arbitrary connected graph on ℓ vertices, m be an arbitrary positive integer and n be an integer such that $n \geq m\ell$. Then*

$$\max\left\{ \mathrm{ex}(n - m\ell + 1, G) + \binom{m\ell - 1}{2}, \mathrm{ex}(n - m + 1, G) + (m-1)n - \binom{m}{2} \right\} \leq \mathrm{ex}(n, mG)$$
$$\leq \mathrm{ex}\Big(n - (m-1)\ell, G\Big) + \binom{(m-1)\ell}{2} + (m-1)\ell\Big(n - (m-1)\ell\Big).$$

Definition 1.19. *A **linear forest (star forest)** is a forest whose connected components are paths (stars).*

Bernard Lidický, Hong Liu and Cory Palmer studied the Turán number of linear forests and star forests for sufficiently large n.

Theorem 1.20 (Lidický-Liu-Palmer [38]). *Let F be a linear forest with components of order $v_1, v_2, \ldots, v_k$. If at least one v_i is not 3, then for n sufficiently large,*

$$\mathrm{ex}(n, F) = \left(\sum_{i=1}^{k} \left\lfloor \frac{v_i}{2} \right\rfloor - 1\right)\left(n - \sum_{i=1}^{k} \left\lfloor \frac{v_i}{2} \right\rfloor + 1\right) + \binom{\sum_{i=1}^{k} \lfloor \frac{v_i}{2} \rfloor - 1}{2} + c.$$

where $c = 1$ if all v_i are odd and $c = 0$ otherwise. Moreover, the extremal graph is unique.

Theorem 1.21 (Lidický, Liu, Palmer [38]). *Let $F = \bigcup_{i=1}^{k} S^i$ be a star forest where d_i is the maximum degree of S^i and $d_1 \geq d_2 \geq \ldots \geq d_k$. For n sufficiently large,*

$$\mathrm{ex}(n, F) = \max_{1 \leq i \leq k} \left\{ (i-1)(n-i+1) + \binom{i-1}{2} + \left\lfloor \frac{d_i - 1}{2}(n - i - 1) \right\rfloor \right\}.$$

Another most well-studied host graph has been the complete multi-partite graph. An old result of Bollobás, Erdős and Szemerédi [8] (also see [7, 5, 47]) showed that

Theorem 1.22 (Bollobás, Erdős and Szemerédi [8]). $\mathrm{ex}(K_{n_1,n_2,n_3}, C_3) = n_1 n_2 + n_1 n_3$, *for $n_1 \geq n_2 \geq n_3 \geq 1$.*

More recently, extremal problems have been considered where the host graph is taken to be a planar graph. For a given set of graphs $\mathcal{F}$, let us denote the maximum number of edges in an n-vertex $\mathcal{F}$-free planar graph by $\mathrm{ex}_{\mathcal{P}}(n, \mathcal{F})$. This topic was initiated by Dowden in [11] who determined $\mathrm{ex}_{\mathcal{P}}(n, C_4)$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5)$. A variety of other forbidden graphs F including stars, wheels and fans were considered by Lan, Shi and Song [36].

The case of theta graphs was considered in Lan, Shi and Song [37], and the case of short paths was considered by Lan and Shi in [35]. Very recently, Ghosh, Győri, Martin, Paulos and Xiao [26] solved the case for 6-cycle.

1.3 Anti-Ramsey number

A subgraph of an edge-colored graph is *rainbow*, if all of its edges have different colors. For graphs G and H, the *anti-Ramsey number* $\mathrm{ar}(G, H)$ is the maximum number of colors in an edge-colored G with no rainbow copy of H. Similarly, when the host graph G is K_n, we write $\mathrm{ar}(n, H)$ rather than $\mathrm{ar}(K_n, H)$.

The study of anti-Ramsey theory was initiated by Erdős, Simonovits and Sós [20], they considered the classical case when the host graph G is K_n. Let $\mathcal{H} = \{H - e, e \in E(H)\}$, in [20] they showed that

Theorem 1.23 (Erdős-Simonovits-Sós [20])**.**

$$\mathrm{ar}(n, H) - \mathrm{ex}(n, \mathcal{H}) = o(n^2), \ \textit{as } n \longrightarrow \infty.$$

If $d = min\{\chi(G) : G \in \mathcal{H}\} \geq 3$, then by Theorem 1.13 [21], we have $\mathrm{ex}(n, \mathcal{H}) = \frac{d-2}{d-1}\binom{n}{2} + o(n^2)$, and Theorem 1.23 yields $\mathrm{ar}(n, H) = \frac{d-2}{d-1}\binom{n}{2} + o(n^2)$. This determines $\mathrm{ar}(n, H)$ asymptotically. If $d \leq 2$, however, we have $\mathrm{ex}(n, H) = o(n^2)$, and Theorem 1.23 says little about $\mathrm{ar}(n, H)$. Therefore, they proposed studying $\mathrm{ar}(n, H)$ for graph H that contains an edge whose deletion creates a bipartite subgraph, and they put forward two conjectures about $\mathrm{ar}(n, H)$ when H is a path or a cycle.

Conjecture 1.24 (Erdős-Simonovits-Sós [20])**.**

$$\mathrm{ar}(n, C_k) = \left(\frac{k-2}{2} + \frac{1}{k-1}\right)n + O(1).$$

Conjecture 1.25 (Erdős-Simonovits-Sós [20])**.** *Let t be a given integer, $\epsilon = 0, 1$, and $k = 2t + 3 + \epsilon$. Then*

$$\mathrm{ar}(n, P_k) = \begin{cases} tn - \binom{t+1}{2} + 1 + \epsilon, & if \ n \geq \frac{5t+3+4\epsilon}{2}, \\ \binom{k-2}{2} + 1, & if \ k \leq n \leq \frac{5t+3+4\epsilon}{2}. \end{cases}$$

Further, the only extremal colorings corresponding to the first case are the following ones: t vertices $x_1, x_2, \ldots, x_t \in V(K_n)$ can be choosen so that all the edges of form (x_j, y),

$j = 1, 2, \ldots, t$, $y \in V(K_n)$, have different colors and the edges of $K_n - \{x_1, x_2, \ldots, x_t\}$ are colored by one or two (more exactly, by $1+\epsilon$) further colors. The only extremal colorings corresponding to the second case are the following ones: $k-2$ vertices $\{x_1, x_2, \ldots, x_{k-2}\}$ can be chosen in K_n so that all the edges (x_i, x_j) have different colors and all the other edges have the same extra color.

For cycles, Erdős, Simonovits and Sós [20] showed that $\mathrm{ar}(n, C_3) = n-1$. Alon [1] proved Conjecture 1.24 for $k = 4$ by showing that $\mathrm{ar}(n, C_4) = \left\lfloor \frac{4n}{3} \right\rfloor - 1$. Jiang and West [31] proved for general k that, $\mathrm{ar}(n, C_k) \leq \left(\frac{k+1}{2} - \frac{2}{k-1}\right)n - (k-2)$. For even n, they proved that $\mathrm{ar}(n, C_k) \leq \frac{k}{2}n - (k-2)$. It is worth to mention that in this paper, they also proved that $\mathrm{ar}\left(n, \{C_k, C_{k+1}, C_{k+2}\}\right) \leq \left(\frac{k-2}{2} + \frac{1}{k-1}\right)n - 1$. Finally, Montellano-Ballesteros and Neumann-Lara [45] completely proved Conjecture 1.24.

Theorem 1.26 (Ballesteros-Lara [45]). *For all $n \geq k \geq 3$, where $n \equiv r_k \pmod{k-1}$, $0 \leq r_k \leq k-2$, we have*

$$\mathrm{ar}(n, C_k) = \left\lfloor \frac{n}{k-1} \right\rfloor \binom{k-1}{2} + \binom{r_k}{2} + \left\lceil \frac{n}{k-1} \right\rceil - 1.$$

Simonovits and Sós [50] partially proved the conjecture for paths, showing that

Theorem 1.27 (Simonovits-Sós [50]). *There exists a constant c such that if $t \geq 5$, $n \geq ct^2$, then for $\epsilon = 0, 1$*

$$\mathrm{ar}(n, P_{2t+3+\epsilon}) = tn - \binom{t+1}{2} + 1 + \epsilon.$$

Axenovich and Jiang [2] initiated the study of the anti-Ramsey numbers for complete bipartite graphs. They showed for all $t \geq 3$ that $\mathrm{ar}(n, K_{2,t}) = \sqrt{t-2}\,n^{\frac{3}{2}} + O(n^{\frac{4}{3}})$ by proving that $\mathrm{ar}(n, K_{2,t}) - \mathrm{ex}(n, K_{2,t-1}) = O(n)$. Later on, Axenovich, Jiang and Kündgen [3] considered the anti-Ramsey numbers of even cycles in complete bipartite graphs and proved the following result.

Theorem 1.28 (Axenovich- Jiang-Kündgen [3]). *For $n \geq m \geq 1$ and $k \geq 2$,*

$$\mathrm{ar}(K_{m,n}, C_{2k}) = \begin{cases} (k-1)(m+n) - 2(k-1)^2 + 1, & m \geq 2k-1, \\ (k-1)n + m - (k-1), & k-1 \leq m \leq 2k-1, \\ mn, & m \leq k-1. \end{cases}$$

Recently, Fang, Győri, Li and J. Xiao [22] determined the anti-Ramsey number of C_3 and C_4 in complete r-partite graphs,

Theorem 1.29 (Fang-Győri-Li-J. Xiao [22]). *For $r \geq 3$ and $n_1 \geq n_2 \geq \ldots \geq n_r \geq 1$, we have*

$$\mathrm{ar}(K_{n_1,n_2,\ldots,n_r}, \{C_3, C_4\}) = n_1 + n_2 + \cdots + n_r - 1.$$

$$\mathrm{ar}(K_{n_1,n_2,\ldots,n_r}, C_3) = \begin{cases} n_1n_2 + n_3n_4 + \cdots + n_{r-2}n_{r-1} + n_r + \frac{r-1}{2} - 1, & r \text{ is odd;} \\ n_1n_2 + n_3n_4 + \cdots + n_{r-1}n_r + \frac{r}{2} - 1, & r \text{ is even.} \end{cases}$$

$\mathrm{ar}(K_{n_1,n_2,\ldots,n_r}, C_4) = n_1 + n_2 + \cdots + n_r + t - 1,$ *where* $t = min\left\{ \left\lfloor \frac{\sum_{i=1}^{r} n_i}{3} \right\rfloor, \left\lfloor \frac{\sum_{i=2}^{r} n_i}{2} \right\rfloor, \sum_{i=3}^{r} n_i \right\}$ *is the maximum number of independent triangles in* $K_{n_1,n_2,\ldots,n_r}$.

Chapter 2

The Turán number of the square of a path

2.1 Introduction

Recall that the square P_k^2 of P_k is obtained by joining the pairs of vertices with distance one or two in P_k, see Figure 2.1. The Turán number of a graph H, $\mathrm{ex}(n, H)$, is the maximum number of edges in a graph on n vertices which does not have H as a subgraph. Denote by $\mathrm{EX}(n, H)$ the set of H-free graphs on n vertices with $\mathrm{ex}(n, H)$ edges and call a graph in $\mathrm{EX}(n, H)$ an extremal graph for H.

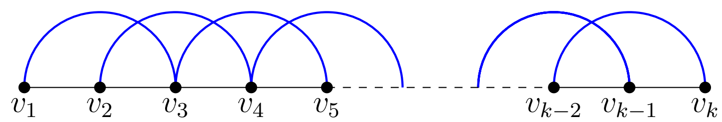

Figure 2.1: Graph P_k^2.

In this chapter, we focus on calculating the exact values of $\mathrm{ex}(n, P_5^2)$, $\mathrm{ex}(n, P_6^2)$ and determine the structures of the extremal graph for P_5^2 and P_6^2.

When $k = 3$, $P_3^2 = K_3$, Mantel Theorem provides the result for $\mathrm{ex}(n, P_3^2)$.

Theorem 2.1 (Mantel [43]). *The maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$, that is $\mathrm{ex}(n, P_3^2) = \lfloor \frac{n^2}{4} \rfloor$. Furthermore, the only triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

The case $k = 4$ was solved by Dirac in a more general context.

Theorem 2.2 (Dirac [10]). *The maximum number of edges in an n-vertex P_4^2-free graph is $\lfloor \frac{n^2}{4} \rfloor$, that is $\mathrm{ex}(n, P_4^2) = \lfloor \frac{n^2}{4} \rfloor$, $(n \geq 4)$. Furthermore, when $n \geq 5$, the only extremal* *graph is the complete bipartite graph* $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

For $k = 5$, our results are given in the next two theorems, where we separate the result for the Turán number and the extremal graphs for P_5^2.

Theorem 2.3 (Xiao, Katona, Xiao, Zamora [53]). *The maximum number of edges in an n-vertex P_5^2-free graph is* $\mathrm{ex}(n, P_5^2) = \lfloor \frac{n^2+n}{4} \rfloor$, *($n \geq 5$).*

Definition 2.4. *Let E_n^i denote a graph obtained from a complete bipartite graph $K_{i,n-i}$ and a maximum matching in the class which has i vertices, see Figure 2.2.*

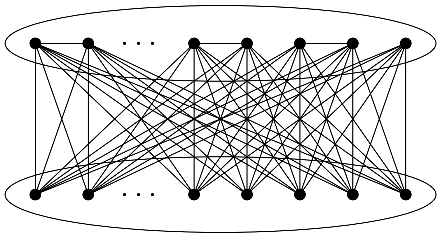

Figure 2.2: Graph E_n^i.

Theorem 2.5 (Xiao, Katona, Xiao, Zamora [53]). *Let n be a natural number. When $n = 5$, the extremal graphs for P_5^2 are E_5^2, E_5^3 and K_4 with a pendent edge. When $n \geq 6$, if $n \equiv 1, 2 \pmod 4$, the extremal graphs for P_5^2 are $E_n^{\lceil \frac{n}{2} \rceil}$ and $E_n^{\lfloor \frac{n}{2} \rfloor}$, otherwise, the extremal graph for P_5^2 is $E_n^{\lceil \frac{n}{2} \rceil}$.*

Definition 2.6. *Let T denote the flattened tetrahedron, see T in Figure 2.3.*

Although the determination of $\mathrm{ex}(n, T)$ is not within the main lines of the thesis, we need the exact value of $\mathrm{ex}(n, T)$ in order to determine $\mathrm{ex}(n, P_6^2)$.

Theorem 2.7 (Xiao, Katona, Xiao, Zamora [53]). *The maximum number of edges in an n-vertex T-free graph ($n \neq 5$) is*

$$\mathrm{ex}(n, T) = \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n}{2} \right\rfloor, & n \not\equiv 2 \pmod 4, \\ \dfrac{n^2}{4} + \dfrac{n}{2} - 1, & n \equiv 2 \pmod 4. \end{cases}$$

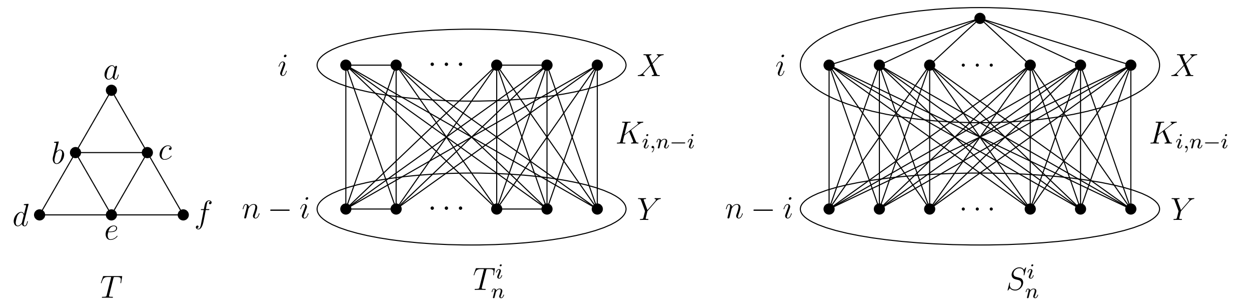

Figure 2.3: Graphs T, T_n^i and S_n^i.

Definition 2.8. *Let T_n^i denote a graph obtained from a complete bipartite graph $K_{i,n-i}$ plus a maximum matching in the class X which has i vertices and a maximum matching in the class Y which has $n-i$ vertices, see Figure 2.3. Let S_n^i denote a graph obtained from $K_{i,n-i}$ plus an i-vertex star in the class X, see Figure 2.3.*

Theorem 2.9 (Xiao, Katona, Xiao, Zamora [53])**.** *Let n $(n \neq 5, 6)$ be a natural number,*

when $n \equiv 0 \pmod 4$, the extremal graph for T is $T_n^{\frac{n}{2}}$,

when $n \equiv 1 \pmod 4$, the extremal graphs for T are $T_n^{\lceil \frac{n}{2} \rceil}$ and $S_n^{\lceil \frac{n}{2} \rceil}$,

when $n \equiv 2 \pmod 4$, the extremal graphs for T are $T_n^{\frac{n}{2}}$, $T_n^{\frac{n}{2}+1}$ and $S_n^{\frac{n}{2}}$,

when $n \equiv 3 \pmod 4$, the extremal graphs for T are $T_n^{\lceil \frac{n}{2} \rceil}$ and $S_n^{\lceil \frac{n}{2} \rceil}$.

Theorems 2.7 and 2.9 were known for sufficiently large $n's$ [39], here we are able to determine the value for small $n's$.

Using Theorems 2.7 and 2.9, we are able to prove the next two results for P_6^2.

Theorem 2.10 (Xiao, Katona, Xiao, Zamora [53])**.** *The maximum number of edges in an n-vertex P_6^2-free graph $(n \neq 5)$ is:*

$$\mathrm{ex}(n, P_6^2) = \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n-1}{2} \right\rfloor, & n \equiv 1, 2, 3 \pmod 6, \\ \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lceil \dfrac{n}{2} \right\rceil, & \text{otherwise}. \end{cases}$$

Definition 2.11. *Suppose $3 \nmid n$, and $1 \leq j \leq i$. Let $F_n^{i,j}$ be the graph obtained by adding vertex disjoint triangles (possibly 0) and one star with j vertices in the class X of size i of $K_{i,n-i}$, see Figure 2.4 (of course, $3 \mid (i-j)$ is supposed). On the other hand, if $3 \mid i$ then add $\frac{i}{3}$ vertex disjoint triangles in the class X of size i. The so obtained graph is denoted by H_n^i, see Figure 2.4.*

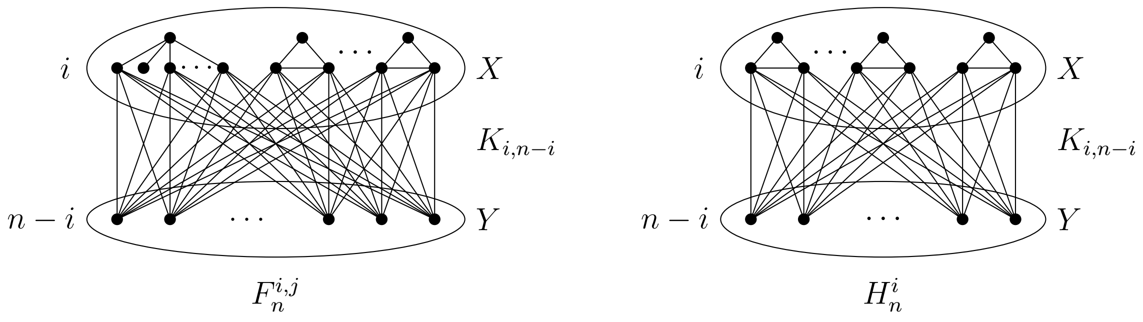

Figure 2.4: Graphs $F_n^{i,j}$ and H_n^i.

Theorem 2.12 (Xiao, Katona, Xiao, Zamora [53]). *Let $n \geq 6$ be a natural number. The extremal graphs for P_6^2 are the following ones.*

When $n \equiv 1 \pmod 6$ then $F_n^{\lceil \frac{n}{2} \rceil, j}$ and $H_n^{\lfloor \frac{n}{2} \rfloor}$,

when $n \equiv 2 \pmod 6$ then $F_n^{\frac{n}{2}, j}$ and $F_n^{\frac{n}{2}+1, j}$,

when $n \equiv 3 \pmod 6$ then $F_n^{\lceil \frac{n}{2} \rceil, j}$ and $H_n^{\lceil \frac{n}{2} \rceil + 1}$,

when $n \equiv 0, 4, 5 \pmod 6$ then $H_n^{\frac{n}{2}}$, $H_n^{\frac{n}{2}+1}$ and $H_n^{\lceil \frac{n}{2} \rceil}$, respectively. ($j$ can have all the values satisfying the conditions $j \leq i$ and $3 \mid (i-j)$).

The rest of this section is organized as follows: In Section 2.2, we give a short proof of Theorems 2.3 and 2.5. In Section 2.3, we give a short proof of Theorems 2.7 and 2.9. In Section 2.4, we give the proofs of Theorems 2.10 and 2.12 based on the results in Section 2.3.

2.2 The Turán number and the extremal graphs for P_5^2

The following proof appears in our paper [53] that is co-authored with Katona, Xiao and Zamora.

Proof of Theorem 2.3. The fact that $\mathrm{ex}(n, P_5^2) \geq \left\lfloor \frac{n^2+n}{4} \right\rfloor$ follows from the construction $E_n^{\lceil \frac{n}{2} \rceil}$.

We prove the inequality

$$\mathrm{ex}(n, P_5^2) \leq \left\lfloor \frac{n^2+n}{4} \right\rfloor \quad (n \geq 5) \tag{2.1}$$

by induction on n.

We check the base cases first. Since our induction step will go from $n-4$ to n, we have to find a base case in each residue class mod 4.

Let G be an n-vertex P_5^2-free graph. When $n \leq 3$, K_n is the graph with the most number of edges and does not contain P_5^2, $e(K_n) \leq \left\lfloor \frac{n^2+n}{4} \right\rfloor$. This settles the cases $n = 1, 2, 3$. However, when $n = 4$, $e(K_4) = 6 > \lfloor \frac{4^2+4}{4} \rfloor$, the statement is not true. Then we prove that the statement is true for $n = 8$. If $P_4^2 \nsubseteq G$, $e(G) \leq \lfloor \frac{8^2}{4} \rfloor$. If $P_4^2 \subseteq G$ and $K_4 \nsubseteq G$, each vertex $v \in V(G - P_4^2)$ can be adjacent to at most 2 vertices of the copy of P_4^2, since $e(G - P_4^2) \leq 5$, we have $e(G) \leq 5 + 8 + 5 \leq 18 = \lfloor \frac{8^2+8}{4} \rfloor$. If $K_4 \subseteq G$, then each vertex $v \in V(G - K_4)$ can be adjacent to at most one vertex of the K_4, since $e(G - P_4^2) \leq 6$, we have $e(G) \leq 16$.

Suppose (2.1) holds for all $k \leq n-1$, the proof is divided into 3 parts,

Case 1. If $P_4^2 \nsubseteq G$, then by Theorem 2.2, $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$.

Case 2. If $P_4^2 \subseteq G$ and $K_4 \nsubseteq G$, then each vertex $v \in V(G - P_4^2)$ can be adjacent to at most 2 vertices of the copy of P_4^2, otherwise, $P_5^2 \subseteq G$. Since $G - P_4^2$ is an $(n-4)$-vertex P_5^2-free graph, we have

$$e(G) \leq 5 + 2(n-4) + e(G - P_4^2) \leq 2n - 3 + \mathrm{ex}(n-4, P_5^2).$$

By the induction hypothesis, $\mathrm{ex}(n-4, P_5^2) \leq \left\lfloor \frac{(n-4)^2+n-4}{4} \right\rfloor$ then

$$e(G) \leq 2n - 3 + \mathrm{ex}(n-4, P_5^2) \leq 2n - 3 + \left\lfloor \frac{(n-4)^2+n-4}{4} \right\rfloor = \left\lfloor \frac{n^2+n}{4} \right\rfloor \quad (n \geq 5) \tag{2.2}$$

Case 3. If $K_4 \subseteq G$, then each vertex $v \in V(G - K_4)$ can be adjacent to at most one vertex of the K_4, otherwise, $P_5^2 \subseteq G$. Since $G - K_4$ is an $(n-4)$-vertex P_5^2-free graph, we have

$$e(G) \leq 6 + (n-4) + e(G - K_4) \leq n + 2 + \mathrm{ex}(n-4, P_5^2).$$

By the induction hypothesis, $\mathrm{ex}(n-4, P_5^2) \leq \left\lfloor \frac{(n-4)^2+n-4}{4} \right\rfloor$, thus

$$e(G) \leq n + 2 + \left\lfloor \frac{(n-4)^2+n-4}{4} \right\rfloor = 5 + \left\lfloor \frac{n^2-3n}{4} \right\rfloor \leq \left\lfloor \frac{n^2+n}{4} \right\rfloor \quad (n \geq 5). \tag{2.3}$$

□

Proof of Theorem 2.5. We determine the extremal graphs for P_5^2 by induction on n. Let G be an n-vertex P_5^2-free graph satisfying (2.1) with equality. It is easy to check, when $n = 5$, that the extremal graphs for P_5^2 are K_4 with a pendent edge, E_5^2 and E_5^3. When $n = 6, 7, 8$, the extremal graphs for P_5^2 are E_6^3 and E_6^4, E_7^4, E_8^4, respectively.

Suppose Theorem 2.5 is true for $k \leq n-1$, when $n \geq 9$. The proof is divided into 3 parts.

Case 1. If $P_4^2 \nsubseteq G$, the equality in (2.1) cannot hold, then we cannot find any extremal graph for P_5^2 in this case.

Case 2. If $P_4^2 \subseteq G$ and $K_4 \nsubseteq G$, the equality holds in inequality (2.2) if and only if each vertex $v \in V(G - P_4^2)$ is adjacent to 2 vertices of the P_4^2 and $G - P_4^2$ is an extremal graph on $n-4$ vertices for P_5^2. Let a, b, c and d be four vertices of a copy of P_4^2, $d_{P_4^2}(b) = d_{P_4^2}(c) = 3$. By the induction hypothesis, $G - P_4^2$ is obtained from a complete bipartite graph $K_{i,n-4-i}$ plus a maximum matching in X', where X' is the class of $G - P_4^2$ with size i. It is easy to check that every vertex $v \in V(G - P_4^2)$ can be adjacent to either a and d or b and c.

Since $|V(G - P_4^2)| \geq 5$, we have $|V(X')| \geq 2$. The endpoints of an edge in $G - P_4^2$ cannot be both adjacent to b and c, otherwise, they form a K_4. Also, the endpoints of an edge in $G - P_4^2$ which have one end vertex as a matched vertex in X' and one end vertex in Y' can be both adjacent to none of $\{a, b, c\}$ and d, otherwise, these would create a P_5^2. If there exists a matched vertex $v \in X'$ which is adjacent to b and c, then all vertices $w \in N(v)$ should be adjacent to a and d, these form a P_5^2. Hence, it is only possible that all matched vertices in X' are adjacent to both a and d, all vertices in Y' are adjacent to b and c. When there exists an unmatched vertex $v_0 \in X'$, since $N(v_0) = Y'$, if v_0 is adjacent to b and c, we have $P_5^2 \subseteq G$. Thus G is obtained from a complete bipartite graph $K_{i+2,n-i-2}$ plus a maximum matching in X, where $X = X' \cup \{b, c\}$ and $Y = Y' \cup a \cup d$. Therefore, if $G - P_4^2$ is $E_{n-4}^{\lceil \frac{n-4}{2} \rceil}$ then G is $E_n^{\lceil \frac{n}{2} \rceil}$, if $E_{n-4}^{\lfloor \frac{n-4}{2} \rfloor}$ then G is $E_n^{\lfloor \frac{n}{2} \rfloor}$.

Case 3. If $K_4 \subseteq G$, the inequality in (2.3) can be equality only when $n = 5$ and the vertex $v \in V(G - K_4)$ is adjacent to one vertex of the K_4, then G is K_4 with a pendent edge. □

2.3 The Turán number and the extremal graphs for T

To prove Theorem 2.7, we need the following lemmas.

Lemma 2.13 (Xiao, Katona, Xiao, Zamora [53]). *Let G be an n-vertex T-free nonempty graph such that for each edge $\{x, y\} \in E(G)$, $d(x) + d(y) \geq n + 2$ holds, then we have $K_4 \subseteq G$.*

Proof. From the condition we know that each edge belongs to at least two triangles. Let

abc and bcd be two triangles, if a is adjacent to d then a, b, c and d induce a K_4, if not, since edge $\{b, d\}$ is contained in at least two triangles, there exists at least one vertex e such that bde is a triangle. Similarly, edge $\{c, d\}$ is also contained in at least two triangles, then, either there exists a vertex f which is adjacent to c and d, this implies that vertices a, b, c, d, e and f induce a T, or c is adjacent to e, this implies that vertices b, c, d and e induce a K_4. $\square$

Lemma 2.14 (Xiao, Katona, Xiao, Zamora [53]). *Let G be an n-vertex $(n \geq 7)$ T-free graph and $K_4 \subseteq G$, then $e(G) \leq 2n-2+\mathrm{ex}(n-4, T)$. For $n \geq 8$, the equality might hold only if each vertex $v \in V(G-K_4)$ is adjacent to 2 vertices of the K_4.*

Proof. If there exists vertex $v \in V(G-K_4)$, such that v is adjacent to at least 3 vertices of the K_4, it is simple to check that every other vertex $u \in V(G-K_4)$ can be adjacent to at most one vertex of the K_4, otherwise $T \subseteq G$, then $e(G) \leq 6+4+(n-5)+e(G-K_4) \leq n+5+ex(n-4, T)$. If not, each vertex in $G-K_4$ is adjacent to at most 2 vertices of the K_4, then $e(G) \leq 6+2(n-4)+e(G-K_4) \leq 2n-2+ex(n-4, T)$. When $n \geq 8$, $e(G) \leq 2n-2+ex(n-4, T)$, the equality holds only if each vertex $v \in V(G-K_4)$ is adjacent to 2 vertices of the K_4. $\square$

Proof of Theorem 2.7. Let

$$f_T(n) = \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n}{2} \right\rfloor, & n \not\equiv 2 \pmod 4, \\ \dfrac{n^2}{4} + \dfrac{n}{2} - 1, & n \equiv 2 \pmod 4. \end{cases}$$

The fact that $\mathrm{ex}(n, T) \geq f_T(n)$ follows from the construction $T_n^{\lceil \frac{n}{2} \rceil}$. Next, we prove the inequality

$$\mathrm{ex}(n, T) \leq f_T(n) \tag{2.4}$$

by induction on n.

Let G be an n-vertex T-free graph. First, we prove the induction steps. Second, we will prove the base cases which are needed to complete the induction.

Suppose (2.4) holds for all $l \leq n-1$. The proof is divided into 4 cases where we assume $k \geq 2$.

Case 1. When $n = 4k$, we divide the proof of $\text{ex}(4k, T) \le f_T(4k) = 4k^2 + 2k$ into 2 subcases. Let G be a $4k$-vertex T-free graph.

(i) If $\delta(G) \le 2k + 1$, after removing a vertex of minimum degree and by the induction hypothesis $\text{ex}(4k - 1, T) = 4k^2 - 1$, we get

$$e(G) \le \text{ex}(4k - 1, T) + 2k + 1 \le 4k^2 - 1 + 2k + 1 = f_T(4k). \tag{2.5}$$

(ii) If $\delta(G) \ge 2k+2$, then for each edge $\{u, v\} \in E(G)$, $d(u)+d(v) \ge 4k+4$. By Lemmas 2.13 and 2.14 and the induction hypothesis $\text{ex}(4k - 4, T) = 4(k - 1)^2 + 2(k - 1)$, we get

$$e(G) \le 2n - 2 + \text{ex}(4k - 4, T) = 8k - 2 + 4(k - 1)^2 + 2(k - 1) = f_T(4k). \tag{2.6}$$

Therefore, $\text{ex}(4k, T) \le f_T(4k)$.

Case 2. When $n = 4k+1$, we divide the proof of $\text{ex}(4k+1, T) \le f_T(4k+1) = 4k^2+4k$ into 3 subcases. Let G be a $(4k + 1)$-vertex T-free graph.

(i) If $\delta(G) \le 2k$, after removing a vertex of minimum degree and by the induction hypothesis $\text{ex}(4k, T) = 4k^2 + 2k$, we have

$$e(G) \le \text{ex}(4k, T) + 2k \le f_T(4k + 1). \tag{2.7}$$

Now, we assume that in the following two cases $\delta(G) \ge 2k + 1$. Then for any pair of vertices $\{u, v\} \in E(G)$, $d(u) + d(v) \ge 4k + 2$ holds.

(ii) Suppose that there exists an edge $\{u, v\} \in E(G)$, such that $d(u) + d(v) = 4k + 2$. This implies that u and v have at least one common neighbor. Deleting $\{u, v\}$ we can use the induction hypothesis $\text{ex}(4k - 1, T) = 4k^2 - 1$. Then

$$e(G) \le 4k + 1 + \text{ex}(4k - 1, T) = f_T(4k + 1). \tag{2.8}$$

(iii) For each edge $\{u, v\} \in E(G)$, $d(u) + d(v) \ge 4k + 3$ holds. By Lemmas 2.13 and 2.14 and the induction hypothesis $\text{ex}(4k - 3, T) = 4(k - 1)^2 + 4(k - 1)$ we get

$$e(G) \le 2n - 2 + \text{ex}(4k - 3, T) = 8k + 4(k - 1)^2 + 4(k - 1) = f_T(4k + 1). \tag{2.9}$$

Therefore, $\text{ex}(4k + 1, T) \le f_T(4k + 1)$.

Case 3. When $n = 4k + 2$, we divide the proof of $\text{ex}(4k + 2, T) \le f_T(4k + 2) = 4k^2 + 6k + 1$ into 2 subcases. Let G be a $(4k + 2)$-vertex T-free graph.

(i) If $\delta(G) \le 2k + 1$, after removing a vertex of minimum degree and by the induction hypothesis $\text{ex}(4k + 1, T) = 4k^2 + 4k$, we get

$$e(G) \le \mathrm{ex}(4k+1,T) + 2k + 1 \le 4k^2 + 6k + 1 = f_T(4k+2). \tag{2.10}$$

(ii) If $\delta(G) \ge 2k+2$, then for each edge $\{u,v\} \in E(G)$, $d(u)+d(v) \ge 4k+4$. By Lemmas 2.13 and 2.14 and the induction hypothesis $\mathrm{ex}(4k-2,T) = 4(k-1)^2 + 6(k-1) + 1$, we get

$$e(G) \le 2n - 2 + \mathrm{ex}(4k-2,T) = 8k + 2 + 4(k-1)^2 + 6(k-1) + 1 = f_T(4k+2). \tag{2.11}$$

Therefore, $\mathrm{ex}(4k+2,T) \le f_T(4k+2)$.

Case 4. When $n = 4k+3$, we divide the proof of $\mathrm{ex}(4k+3,T) \le f_T(4k+3) = 4k^2 + 8k + 3$ into 2 subcases. Let G be a $(4k+3)$-vertex T-free graph.

(i) If $\delta(G) \le 2k+2$, after removing a vertex of minimum degree and by the induction hypothesis $\mathrm{ex}(4k+2,T) = 4k^2 + 6k + 1$, we get

$$e(G) \le \mathrm{ex}(4k+2,T) + 2k + 2 \le 4k^2 + 8k + 3 = f_T(4k+3). \tag{2.12}$$

(ii). If $\delta(G) \ge 2k+3$, then for each edge $\{u,v\} \in E(G)$, $d(u)+d(v) \ge 4k+6$. By Lemmas 2.13 and 2.14 and the induction hypothesis $\mathrm{ex}(4k-1,T) = 4(k-1)^2 + 8(k-1) + 3$, we get

$$e(G) \le 2n - 2 + \mathrm{ex}(4k-1,T) = 8k + 4 + 4(k-1)^2 + 8(k-1) + 3 = f_T(4k+3). \tag{2.13}$$

Therefore, $\mathrm{ex}(4k+3,T) \le f_T(4k+3)$.

Now we prove the base cases which are needed to complete the induction steps. Since our induction steps will go from $n-1$ to n, $n-2$ to n and $n-4$ to n, we will require to show the statement is true for cases when $n = 3, 4, 6$ and 9.

When $n \le 4$, K_n is the graph with the most number of edges, and $e(K_n) = f_T(n)$.

When $n = 5$, $e(K_5) = 10 > f_T(5)$, the statement is not true, but we will see that the statement is true for $n = 9$.

When $n = 6$, let v be a vertex with minimum degree. If $\delta(G) = 1$, since $e(G-v) \le 10$, we get $e(G) \le 11$. If $\delta(G) = 2$ and $e(G) = 12$, then the only possibility is that $G - v$ is K_5, but then $T \subseteq G$, and we have $e(G) \le 11$. Suppose now $\delta(G) \ge 3$. If $K_4 \subseteq G$ and there exists a vertex $u \in V(G - K_4)$ which is adjacent to at least 3 vertices of the copy of K_4, then $w \in V(G - K_4 - u)$ can be adjacent to at most one vertex of the K_4, otherwise, $T \subseteq G$. This contradicts $\delta(G) \ge 3$. Then in this case it is only possible that

$\{u, w\} \in E(G)$ and both u and w are adjacent to 2 vertices of the K_4 which implies that $e(G) \leq 11$. If $K_4 \nsubseteq G$, then by Turán's Theorem, we have $e(G) \leq 12$ and the Turán graph $T(6, 3)$ is the unique K_4-free graph which has 12 edges, however, $T \subseteq T(6, 3)$, then $e(G) \leq 11 = f_T(6)$. Summarizing: $e(G) \leq 11 \leq f_T(6)$.

When $n = 9$, suppose first that there exists a pair of vertices $\{u, v\} \in E(G)$, such that $d(u) + d(v) \leq 10$. Deleting $\{u, v\}$ and using $\mathrm{ex}(7, T) = 15$, we get $e(G) \leq 9 + 15 = 24 = f_T(9)$. If for each pair of vertices $\{u, v\} \in E(G)$, $d(u) + d(v) \geq 11$ holds, by Lemma 2.13, we obtain $K_4 \subseteq G$. Let $G^{'}$ denote the graph $G - K_4$. If $e(G^{'}) \leq 8$, since the number of edges between K_4 and $G^{'}$ is at most 10, we have $e(G) \leq 6 + 10 + 8 = 24$. If $e(G^{'}) \geq 9$, then $K_4 \subseteq G^{'}$ and the vertex $w \in G^{'} - K_4$ is adjacent to at least 3 vertices of the copy of K_4 in $G^{'}$. This implies that each vertex from $G - G'$ can be adjacent to at most 1 vertex of $G^{'} - w$, then the number of edges between $G - G'$ and $G^{'}$ is at most 8, we can conclude that, $e(G) \leq 6 + 8 + 10 = 24$, $e(G) \leq 24 = f_T(9)$.

It is easy to see that the case $n = 7$ can be proved using $n = 3$ and $n = 6$ (Case 4). Similarly, the case $n = 8$ follows by $n = 7$ and $n = 4$ (Case 1). Hence the cases $n = 6, 7, 8, 9$ are settled forming a good basis for the induction. $\square$

Now, we determine the extremal graphs for T.

Proof of Theorem 2.9. Similarly to the proof of Theorem 2.7, we prove first the induction steps and in the end we will prove the base cases which are needed to complete the induction.

Suppose that the extremal graphs for T are as shown in Theorem 2.7 for $l \leq n - 1$. In the following cases, we will assume that $k \geq 2$.

Let G be an n-vertex T-free graph with $e(G) = f_T(n)$. The proof is divided into 4 cases following the steps of the proof of Theorem 2.7.

Case 1. When $n = 4k$, $f_T(n) = 4k^2 + 2k$.

(i) If $\delta(G) \leq 2k + 1$, the equality in (2.5) holds only when there exists a $v \in V(G)$, such that $d(v) = \delta(G) = 2k + 1$ and $G - v$ is an extremal graph for T on $4k - 1$ vertices. By the induction hypothesis, $G - v$ can be either T^{2k}_{4k-1} or S^{2k}_{4k-1}. Let $X^{'}$ and $Y^{'}$ be the classes in $G - v$ with size $2k$ and $2k - 1$, respectively.

When $G - v$ is T^{2k}_{4k-1}, it can be easily checked that v cannot be adjacent to the two endpoints of an edge which have two matched vertices located in different classes, otherwise, $T \subseteq G$, see Figure 2.5. Let w be the unmatched vertex in $Y^{'}$. Since $d(v) = 2k+1$, $N(v)$ must contain the unmatched vertex $w \in Y^{'}$, then the only way to avoid $T \subseteq G$ is choosing $N(v) = w \cup X^{'}$. Consequently, $G = T_{4k}^{2k}$ holds.

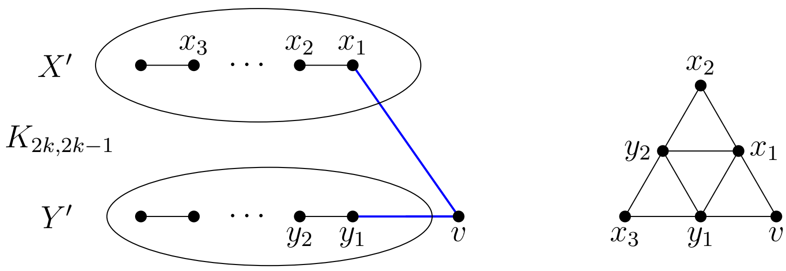

Figure 2.5

When $G - v$ is S_{4k-1}^{2k}, let x_1 denote the center of the star in $X^{'}$. If v is adjacent to the two endpoints of the edge $\{x_i, y_j\}$ $(x_i \in X', y_i \in Y', 2 \leq i \leq 2k,\ 1 \leq j \leq 2k-1)$, then $T \subseteq G$ (see Figure 2.6). We obtained a contradiction. But $d(v) = 2k+1$ implies that this is always the case.

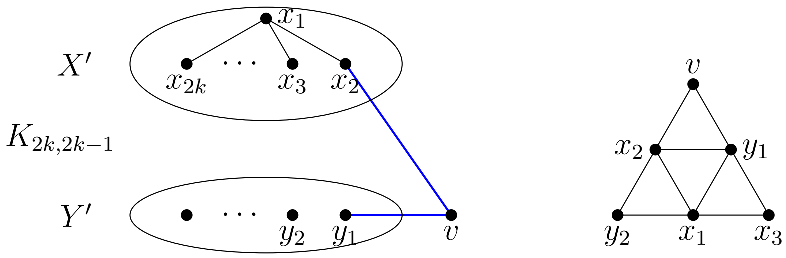

Figure 2.6

(ii) If $\delta(G) \geq 2k+2$, this implies that $e(G) \geq 2k(2k+2) = 4k^2 + 4k$, which contradicts the fact that $\mathrm{ex}(4k, T) = 4k^2 + 2k$.

That is, G can only be $T_n^{\frac{n}{2}}$.

Case 2. When $n = 4k+1$, $f_T(n) = 4k^2 + 4k$.

(i) If $\delta(G) \leq 2k$, the equality in (2.7) holds only if there exists $v \in V(G)$, such that $d(v) = \delta(G) = 2k$ and $G - v$ is an extremal graph for T on $4k$ vertices. By the induction hypothesis, $G - v$ is T_{4k}^{2k}. All neighbors of v should be located in the same class, otherwise, $T \subseteq G$, we get that G is T_{4k+1}^{2k+1}, that is $T_n^{\lceil \frac{n}{2} \rceil}$.

If $\delta(G) \geq 2k+1$, then for any pair of vertices $\{u, v\} \in V(G)$, $d(u) + d(v) \geq 4k+2$. Here we distinguishing two subcases.

(ii) Suppose that there exists an edge $\{u, v\} \in E(G)$ such that $d(u) + d(v) = 4k+2$. The equality in (2.8) holds only if when $d(u) = d(v) = 2k+1$ and $G-u-v$ is an extremal graph for T on $4k-1$ vertices. By the induction hypothesis, $G-u-v$ can be either T^{2k}_{4k-1} or S^{2k}_{4k-1}. Let X' and Y' be the classes in $G-u-v$ with size $2k$ and $2k-1$, respectively.

When $G-u-v$ is T^{2k}_{4k-1}, as in the previous case, neither u nor v can be adjacent to the two endpoints of an edge which have two matched vertices located in different classes, see Figure 2.5. If $N(u) - v \neq X'$, then u is adjacent to the unmatched vertex w in Y' and the other $2k-1$ neighbors of u are all located in X', say, $N(u) - v - w = \{x_1, \ldots x_{2k-1}\}$ and $\{x_{2k-1}, x_{2k}\} \in E(X')$, otherwise, $T \subseteq G$. Since $|X'| \geq 4$, in this case, v cannot be adjacent to x_i $(1 \leq i \leq 2k-2)$, otherwise, $T \subseteq G$, see Figure 2.7. Now v should choose $2k$ neighbors among the rest $2k+1$ vertices in $V(G - u - v - \bigcup_{i=1}^{2k-2} x_i)$, which implies that v is adjacent to the two endpoints of an edge which have two matched vertices locate in different classes as endpoints, then $T \subseteq G$. Hence, $N(u) - v = X'$, similarly, $N(v) - u = X'$. Thus, G is $T^{2k+1}_{4k+1} = T^{2k}_{4k+1}$, that is $T_n^{\lceil \frac{n}{2} \rceil}$.

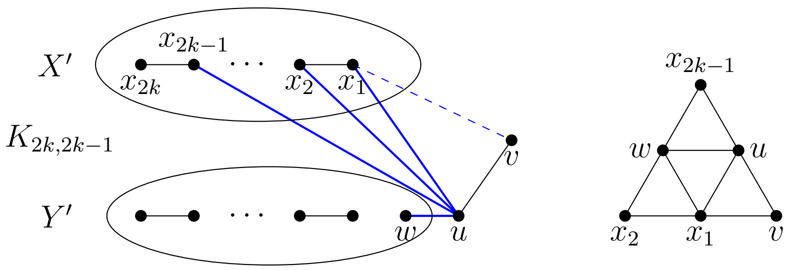

Figure 2.7

Let us now consider the case when $G-u-v$ is S^{2k}_{4k-1}. Let x_1 denote the center of the star in X'. If u is adjacent to the two endpoints of the edge $\{x_i, y_j\}$ $(2 \leq i \leq 2k,\ 1 \leq j \leq 2k-1)$, then $T \subseteq G$. Thus, there are only two possibilities for $T \nsubseteq G$: $N(u) - v = X'$ or $N(u) - v = Y' \cup x_1$. The same holds for v and it is easy to check that if $N(u) - v = N(v) - u$, then $T \subseteq G$. From the above, the only possibility for $T \nsubseteq G$ is that when $N(u) - v = X'$ and $N(v) - u = Y' \cup x_1$ or in the another way around, which implies that G is S^{2k+1}_{4k+1}, that is $S_n^{\lceil \frac{n}{2} \rceil}$.

(iii) Suppose that for each edge $\{u, v\} \in E(G)$, $d(u) + d(v) \geq 4k+3$ holds. Let

$d(v) = \delta(G)$, then either $d(v) = 2k+1$ or $d(v) \geq 2k+2$, but in both cases, each neighbor of v has degree at least $2k+2$. Then all $4k+1$ vertices have degree at least $2k+1$, but $2k+1$ of them, which are the neighbors of v, have degree at least one larger. This implies that $e(G) \geq \frac{(4k+1)(2k+1)+2k+1}{2} = 4k^2+4k+1$, which contradicts the fact that $\mathrm{ex}(4k+1, T) = 4k^2+4k$.

That is, G can be either $T_n^{\lceil \frac{n}{2} \rceil}$ or $S_n^{\lceil \frac{n}{2} \rceil}$.

Case 3. When $n = 4k+2$ we have $f_T(n) = 4k^2+6k+1$.

(i) If $\delta(G) \leq 2k+1$, the equality holds in (2.10) only if there exists $v \in V(G)$, such that $d(v) = \delta(G) = 2k+1$ and $G - v$ is an extremal graph for T on $4k+1$ vertices. By the induction hypothesis, $G - v$ can be either T_{4k+1}^{2k+1} or S_{4k+1}^{2k+1}.

Suppose first that $G - v$ is T_{4k+1}^{2k+1}. Let X' any Y' be the classes in $G - v$ with size $2k+1$ and $2k$, w be the unmatched vertex in X'. The vertex v cannot be adjacent to the two endpoints of an edge which have two matched vertices located in different classes. Since $d(v) = 2k+1$, there are two possibilities to avoid T: $N(v) = X'$ or $N(v) = Y' \cup w$, which implies that G is either T_{4k+2}^{2k+1} or T_{4k+2}^{2k+2}, that is $T_n^{\frac{n}{2}}$ or $T_n^{\frac{n}{2}+1}$.

When $G - v$ is S_{4k+1}^{2k+1}. Let X' be the class in $G - v$ which contains a star and Y' be the other class of the $G - v$. Also, let x_1 denote the center of the star in X'. Since, $d(v) = 2k+1$ and v cannot be adjacent to the two endpoints of an edge which is not incident with x_1, we get either $N(v) = Y' \cup x_1$ or $N(v) = X'$. If $N(v) = X'$, G is S_{4k+2}^{2k+1}, that is $S_n^{\frac{n}{2}}$. If $N(v) = Y' \cup x_1$, G is S_{4k+2}^{2k+2}, that is $S_n^{\frac{n}{2}+1}$. It is easy to see that $S_n^{\frac{n}{2}+1}$ is isomorphic to $S_n^{\frac{n}{2}}$.

(ii) If $\delta(G) \geq 2k+2$, then $e(G) \geq (k+1)(4k+2) = 4k^2+6k+2$, which contradicts the fact that $\mathrm{ex}(4k+2, T) = 4k^2+6k+1$.

Therefore, G can be $T_n^{\frac{n}{2}}$, $T_n^{\frac{n}{2}+1}$ or $S_n^{\frac{n}{2}}$.

Case 4. When $n = 4k+3$ we have $f_T(n) = 4k^2+8k+3$.

(i) If $\delta(G) \leq 2k+2$, the equality holds in (2.12) only if there exists $v \in V(G)$, such that $d(v) = \delta(G) = 2k+2$ and $G - v$ is an extremal graph for T on $4k+2$ vertices. By the induction hypothesis, $G - v$ can be T_{4k+2}^{2k+1}, T_{4k+2}^{2k+2} or S_{4k+2}^{2k+1}.
When $G - v$ is T_{4k+2}^{2k+1} or T_{4k+2}^{2k+2}, similarly to Case 1 (i), G can only be T_{4k+3}^{2k+2}, that is $T_n^{\lceil \frac{n}{2} \rceil}$.
When $G - v$ is S_{4k+2}^{2k+1}, similarly to Case 2 (ii), G can only be S_{4k+3}^{2k+2}, that is $S_n^{\lceil \frac{n}{2} \rceil}$.

(ii) If $\delta(G) \geq 2k+3$, then $e(G) \geq \frac{(2k+3)(4k+3)}{2} > 4k^2+9k+4 > 4k^2+8k+3$, which contradicts the fact that $\mathrm{ex}(4k+3, T) = 4k^2 + 8k + 3$.

Therefore, in this case, G is either $T_n^{\lceil \frac{n}{2} \rceil}$ or $S_n^{\lceil \frac{n}{2} \rceil}$.

Now we check the base cases which are needed to complete the induction.

When $n = 4$, $\mathrm{ex}(4, T) = 6$, K_4 is the extremal graph which has the maximum number of edges on 4 vertices that does not contain T as a subgraph.

Although the Theorem does not hold for $n = 6$, we determine the extremal graphs in this case because it will help us to determine them for some other n's.

When $n = 6$, $\mathrm{ex}(6, T) = 11$. It follows from the proof of Theorem 2.7, when $\delta(G) = 1$, the only extremal graph for T is as shown in Figure 2.8(a). When $\delta(G) = 2$, the only extremal graph for T is as shown in Figure 2.8(b). Since $\delta(G) \geq 4$ implies $e(G) \geq 12$, this is not possible. The only remaining case is $\delta(G) = 3$. When $\delta(G) = 3$ and $K_4 \subseteq G$, by case analysis we obtain that the extremal graphs for T can be Figure 2.8(c) and Figure 2.8(d), which are T_6^3 and T_6^4. Suppose now that $\delta(G) = 3$ and $K_4 \nsubseteq G$. Let $d(v) = \delta(G) = 3$, then $e(G - v) = 8$, the only possibility is that $G - v$ is $T(5, 3)$. It is easy to check that G can only be S_6^3, see Figure 2.8(e).

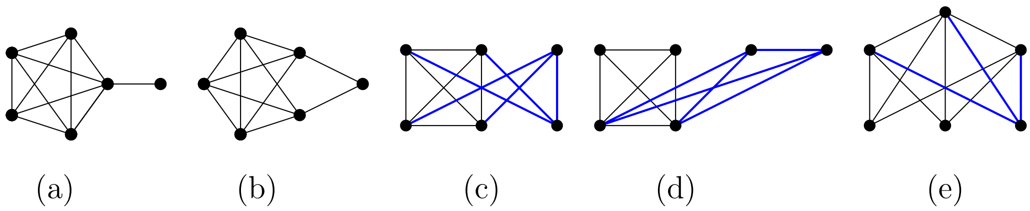

Figure 2.8: Extremal graphs for T when $n = 6$.

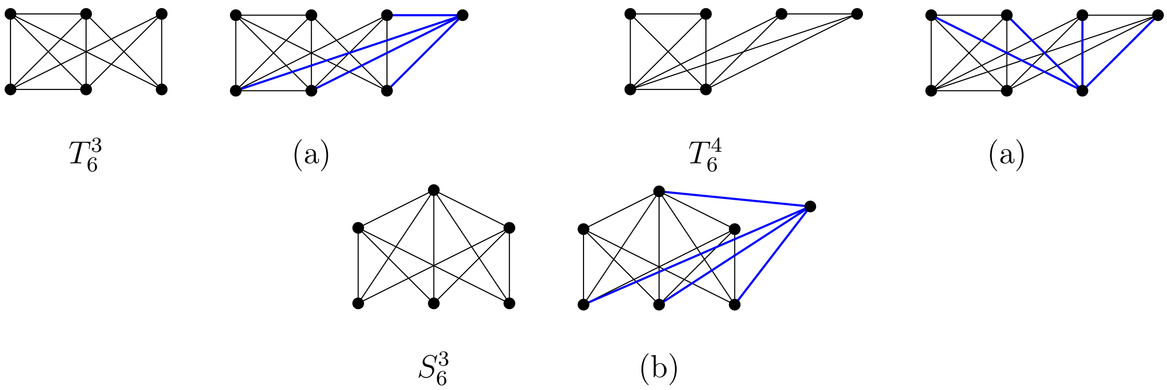

Figure 2.9: Extremal graphs for T when $n = 7$.

Suppose now that $n = 7$, $\text{ex}(7, T) = 15$. It is not possible that $\delta(G) \leq 3$, otherwise, $e(G) \leq 3 + ex(6, T) = 14$. Also, it is not possible that $\delta(G) \geq 5$, otherwise, $e(G) > 17$. Both are contradict with $e(G) = 15$. Let $d(v) = \delta(G)$, the only possibility is that $\delta(G) = 4$ and $G - v$ is a 6-vertex T-free graph. Since $d(v) = 4$, we have $\delta(G - v) \geq 3$, which implies that structures (a) and (b) in Figure 2.8 are not possible. If $G - v$ is T_6^3 or T_6^4, then G can only be (a) in Figure 2.9, that is T_7^4. If $G - v$ is S_6^3, then G can only be (b) in Figure 2.9, that is S_7^4.

Because case $n = 8$ needs only the case $n = 7$ (Case 1), case $n = 9$ needs cases $n = 7$ and $n = 8$ (Case 2). These base cases complete the proof. $\square$

We will need the following statement later. It shows that the "second best" graphs can be also well described if $4|n$.

Proposition 2.15 (Xiao, Katona, Xiao, Zamora [53])**.** *Let n $(n \geq 8)$ be a natural number such that $4|n$ and G be an n-vertex T-free graph with $\frac{n^2}{4} + \frac{n}{2} - 1$ edges, then G can only be $T_n^{\frac{n}{2}}$ minus an edge, $S_n^{\frac{n}{2}}$ or $S_n^{\frac{n}{2}+1}$.*

Proof. We can suppose that $\delta(G) \leq \frac{n}{2}$, otherwise, $e(G) \geq \frac{n^2}{4} + \frac{n}{2}$. Let $v \in V(G)$ and $d(v) = \delta(G)$, then $e(G) \leq d(v) + \text{ex}(n-1, T) \leq \frac{n^2}{4} + \frac{n}{2} - 1$, the equality holds only if $d(v) = \frac{n}{2}$ and $G - v$ is either $T_{n-1}^{\lceil \frac{n-1}{2} \rceil}$ or $S_{n-1}^{\lceil \frac{n-1}{2} \rceil}$. When $G - v$ is $T_{n-1}^{\lceil \frac{n-1}{2} \rceil}$, let w be the unmatched vertex in Y' and $X' = \{x_1, \ldots, x_{\lceil \frac{n-1}{2} \rceil}\}$, X' and Y' be the classes of $G - v$ with size $\lceil \frac{n-1}{2} \rceil$ and $\lfloor \frac{n-1}{2} \rfloor$, respectively. Since $d(v) = \frac{n}{2}$ and v cannot be adjacent to the two endpoints of an edge which have two matched vertices located in different classes, no matter $N(v) = X'$ or $N(v) = X' - x_i \cup w$ $(1 \leq i \leq \lceil \frac{n-1}{2} \rceil)$, G is $T_n^{\frac{n}{2}}$ minus an edge in both cases. When $G - v$ is $S_{n-1}^{\lceil \frac{n-1}{2} \rceil}$, let x_1 be the center of the star in X', $X' = \{x_1, \ldots, x_{\lceil \frac{n-1}{2} \rceil}\}$ and $Y' = \{y_1, \ldots, y_{\lfloor \frac{n-1}{2} \rfloor}\}$ be the classes of $G - v$ with size $\lceil \frac{n-1}{2} \rceil$ and $\lfloor \frac{n-1}{2} \rfloor$, respectively. Since v cannot be adjacent to the two endpoints of the edge $\{x_i, y_i\}$ $(2 \leq i \leq \lceil \frac{n-1}{2} \rceil, 1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor)$ and $d(v) = \frac{n}{2}$, which implies that $N(v) = x_1 \cup Y'$ or $N(v) = X'$. Therefore, G can be either $S_n^{\frac{n}{2}}$ or $S_n^{\frac{n}{2}+1}$. $\square$

2.4 The Turán number and the extremal graphs for P_6^2

Proof of Theorem 2.10. Let

$$f_{P_6^2}(n) = \begin{cases} \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor, & n \equiv 1,2,3 \pmod 6, \\ \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

The fact that $\mathrm{ex}(n, P_6^2) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil$, when $n \equiv 0,4,5 \pmod 6$, follows from the constructions $H_n^{\frac{n}{2}}$, $H_n^{\frac{n}{2}+1}$ and $H_n^{\lceil \frac{n}{2} \rceil}$, respectively. The fact that $\mathrm{ex}(n, P_6^2) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$, when $n \equiv 1,2,3 \pmod 6$, follows from the constructions $F_n^{\lceil \frac{n}{2} \rceil, j}$.

It remains to prove the inequality

$$\mathrm{ex}(n, P_6^2) \leq f_{P_6^2}(n) \tag{2.14}$$

by induction on n.

Let G be an n-vertex P_6^2-free graph. Since our induction step will go from $n-6$ to n, we have to find a base case in each residue class mod 6.

When $n \leq 4$, K_n is the graph with the most number of edges and $e(K_n) = f_{P_6^2}(n)$.

When $n = 6$, if $P_5^2 \nsubseteq G$, by Theorem 2.3, $e(G) \leq \left\lfloor \frac{5^2+5}{4} \right\rfloor = 7 < f_{P_6^2}(6)$. If $P_5^2 \subseteq G$, $K_5 \nsubseteq G$ and $e(G) \geq 13$, it can be checked that the vertex $v \in V(G - P_5^2)$ can be adjacent to at most 3 vertices of the copy of P_5^2, otherwise $P_6^2 \subseteq G$, in this case, $d(v) \geq 13 - 9 = 4$ then $P_6^2 \subseteq G$. If $K_5 \subseteq G$, the vertex $v \in V(G - K_5)$ is adjacent to at most one vertex of the K_5, otherwise, $P_6^2 \subseteq G$. Therefore, $e(G) \leq 11 < f_{P_6^2}(6)$.

When $n = 5$, since $e(K_5) = 10 > f_{P_6^2}(5)$, the statement is not true, then we prove that the statement is true for $n = 11$. If $P_5^2 \nsubseteq G$, by Theorem 2.3, $e(G) \leq \left\lfloor \frac{11^2+11}{4} \right\rfloor < f_{P_6^2}(11)$. If $P_5^2 \subseteq G$, first suppose that the graph spanned by the vertices of the copy of P_5^2 have at most 8 edges. It can be checked that every triangle can be adjacent to at most 7 edges of the P_5^2, otherwise, $P_6^2 \subseteq G$. When there exists a triangle as subgraph in $G - V(P_5^2)$, we get $e(G) \leq 8 + 7 + 9 + \mathrm{ex}(6, P_6^2) = 36 = f_{P_6^2}(6)$. If not, $e(G) \leq 8 + 18 + 9 = 35 < f_{P_6^2}(6)$. If $K_5^- \subseteq G$ (K_5 minus an edge) then each vertex $v \in V(G - K_5^-)$ is adjacent to at most 2 vertices of K_5^-. We get $e(G) \leq 9 + 12 + \mathrm{ex}(6, P_6^2) = 33 < f_{P_6^2}(6)$. If $K_5 \subseteq G$ then each vertex $v \in V(G - P_5^2)$ is adjacent to at most one vertex of K_5. Altogether we have at most $10 + 6 + \mathrm{ex}(6, P_6^2) = 28$ edges. From the above, $e(G) \leq 36 = f_{P_6^2}(11)$.

Suppose (2.14) holds for all $l \leq n-1$ ($l \neq 5$). The following proof is divided into 2 parts.

Case 1. If $T \subseteq G$, then each vertex $v \in V(G-T)$ is adjacent to at most 3 vertices of the copy of T, otherwise, $P_6^2 \subseteq G$. The graph spanned by the vertices of the copy of T cannot have more than $ex(6, P_6^2) = 12$ edges. Since $G - T$ is an $(n-6)$-vertex P_6^2-free graph and $\mathrm{ex}(6, T) = 12$, we have

$$e(G) \leq 12 + 3(n-6) + e(G-T) \leq 3n - 6 + \mathrm{ex}(n-6, P_6^2). \tag{2.15}$$

By the induction hypothesis,

$$\mathrm{ex}(n-6, P_6^2) \leq f_{P_6^2}(n-6) = \begin{cases} \left\lfloor \dfrac{(n-6)^2}{4} \right\rfloor + \left\lfloor \dfrac{n-7}{2} \right\rfloor, & n \equiv 1,2,3 \pmod 6, \\ \left\lfloor \dfrac{(n-6)^2}{4} \right\rfloor + \left\lceil \dfrac{n-6}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

We get

$$\mathrm{ex}(n, P_6^2) \leq \begin{cases} 3n - 6 + \left\lfloor \dfrac{(n-6)^2}{4} \right\rfloor + \left\lfloor \dfrac{n-7}{2} \right\rfloor = \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n-1}{2} \right\rfloor, & n \equiv 1,2,3 \pmod 6, \\ 3n - 6 + \left\lfloor \dfrac{(n-6)^2}{4} \right\rfloor + \left\lceil \dfrac{n-6}{2} \right\rceil = \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lceil \dfrac{n}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

Case 2. If $T \nsubseteq G$, by Theorem 2.7, $e(G) \leq \mathrm{ex}(n,T) \leq f_{P_6^2}(n)$ holds unless $n \equiv 8 \pmod{12}$. When $n \equiv 8 \pmod{12}$, then $e(G) \leq \mathrm{ex}(n,T) = f_{P_6^2}(n) + 1$. However, by Theorem 2.9, the equality holds only if G is $T_n^{\frac{n}{2}}$, but $P_6^2 \subseteq T_n^{\frac{n}{2}}$ ($n \geq 8$), which implies that $e(G) \leq \mathrm{ex}(n,T) - 1 = f_{P_6^2}(n)$.

Summarizing, we obtain

$$\mathrm{ex}(n, P_6^2) = f_{P_6^2}(n) = \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n-1}{2} \right\rfloor, & n \equiv 1,2,3 \pmod 6, \\ \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lceil \dfrac{n}{2} \right\rceil, & \text{otherwise}. \end{cases}$$

□

Proof of Theorem 2.12. It is obvious that

$$\mathrm{ex}(n,T) \leq \mathrm{ex}(n, P_6^2), \text{ except when } n \equiv 8 \pmod{12}, \tag{2.16}$$

with strict inequality only when

$$n \equiv 5, 6, 7, \text{or } 11 \pmod{12}. \tag{2.17}$$

We want to determine the n-vertex graphs G containing no copy of P_6^2 as a subgraph and satisfying $e(G) = \mathrm{ex}(n, P_6^2)$. Therefore, suppose that G possesses these properties. We claim that G either contains a copy of T as a subgraph or it is either $F_n^{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$ or $F_n^{\frac{n}{2}+1, \frac{n}{2}+1}$. If n belongs to the set of integers given in (2.17) then this is obvious, since we have a strict inequality in (2.16). On the other hand, for the other values of n (except $n \equiv 8 \pmod{12}$) we obtain $\mathrm{ex}(n, P_6^2) = \mathrm{ex}(n, T) = e(G)$. Theorem 2.9 describes these graphs. However, G cannot be $T_n^{\lceil \frac{n}{2} \rceil}$ or $T_n^{\frac{n}{2}+1}$, because these graphs contain P_6^2 as a subgraph if $n \geq 7$. (In the case of $n = 6$ we had strict inequality in (2.16). The other possibility by Theorem 2.9 is that $G = S_n^{\lceil \frac{n}{2} \rceil} = F_n^{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$. In the exceptional case we can use Proposition 2.15. According to this, G could be $T_n^{\frac{n}{2}}$, $S_n^{\frac{n}{2}}$ or $S_n^{\frac{n}{2}+1}$. The first of them is excluded since $P_6^2 \subset T_n^{\frac{n}{2}}$ the second and third ones can be written in the form $F_n^{\frac{n}{2}, \frac{n}{2}}$ and $F_n^{\frac{n}{2}+1, \frac{n}{2}+1}$.

From now on we suppose that $e(G) = \mathrm{ex}(n, P_6^2)$, the graph G contains a copy of T and no copy of P_6^2, and prove by induction that G is a graph given in the theorem.

Let us list some graphs L (coming up in the forthcoming proofs) containing P_6^2 as a subgraph:

(α) L is obtained by adding any edge to T different from $\{a, e\}, \{d, c\}$ and $\{b, f\}$ on Figure 2.3.

(β) Add the edges $\{a, e\}, \{d, c\}, \{b, f\}$ to T resulting in T'. The graph L is obtained by adding a new vertex u to T' which is adjacent to three vertices of T' different from the sets $\{b, c, e\}$ and $\{a, d, f\}$.

(γ) L is obtained by adding two new adjacent vertices u and v to T', which are both adjacent to b, c and e. Then e.g. the square of the path $\{u, v, c, e, b, d\}$ is in L.

(δ) L is obtained by adding 4 new vertices u, v, w, x, forming a complete graph, to T', all of them adjacent to a, d and f. Then e.g. the square of the path $\{a, u, v, w, x, d\}$ is in L.

(ϵ) L consists of a complete graph on 5 vertices and a 6th vertex adjacent to two of them.

(ζ) The vertices of L are $p_i (1 \leq i \leq 4)$ and $q_j (1 \leq j \leq 2)$ where p_1, p_2, p_3, p_4 span a path and all pairs (p_i, q_j) are adjacent. Then the square of the path $\{p_1, q_1, p_2, p_3, q_2, p_4\}$ is L.

Let us start with the base cases. Let $n = 6$ and suppose $T \subset G$. By (α) only the edges $\{a, e\}, \{d, c\}$ and $\{b, f\}$ can be added to T. To obtain $\mathrm{ex}(6, P_6^2) = 12$ edges all three of them should be added. The so obtained graph T' is really H_6^3.

Consider now the case $n = 7$. It is clear that (2.15) holds with equality only when the subgraph spanned by T contains 12 edges and the vertex u not in T is adjacent with exactly 3 vertices of T. Hence the subgraph spanned by T is really T'. By (β) u can be adjacent to either b, c, e or a, d, f. In the first case $G = H_7^3$, in the second one $G = F_7^{4,1}$, as desired.

If $n = 8$, $e(G) = \mathrm{ex}(8, P_6^2) = 19$ and the equality in (2.15) implies, again, that T must span T' and the remaining two vertices u and v are adjacent to exactly 3 vertices of T': either to the set $\{b, c, e\}$ or to $\{a, d, f\}$ and $\{u, v\}$ is an edge. If both u and v are adjacent to $\{b, c, e\}$ then (γ) leads to a contradiction. If one of u and v is adjacent to $\{b, c, e\}$, the other one to $\{a, d, f\}$, then $G = F_8^{4,1}$. Finally if both of them are adjacent to $\{a, d, f\}$, then $G = F_8^{5,2}$.

Suppose now that $n = 9$, when $e(G) = \mathrm{ex}(9, P_6^2) = 24$ and (2.15) implies that the three vertices u, v, w not in T' form a triangle and all three possess the properties mentioned in the previous case. If two of them are adjacent to $\{b, c, e\}$ then (γ) gives the contradiction. If one of the them is adjacent to $\{b, c, e\}$, the two other ones are adjacent to $\{a, d, f\}$, then $G = F_9^{5,2}$. Finally if all three are adjacent to $\{a, d, f\}$, then $G = H_9^6$.

The case $n = 10$ and $e(G) = \mathrm{ex}(10, P_6^2) = 30$ is very similar to the previous ones. If one of the new vertices, u, v, w, x is adjacent to $\{b, c, e\}$ and the other 3 are adjacent to $\{a, d, f\}$, then $G = H_{10}^6$. Here it cannot happen, by (δ), that all 4 are adjacent to $\{a, d, f\}$.

Finally let $n = 11$ where $e(G) = \mathrm{ex}(11, P_6^2) = 36$. This case is different from the previous ones, since we cannot have all the potential edges (12 in the graph spanned by T, 10 among the other 5 vertices u, v, w, x, y, and 15 between the two parts) one is missing. We distinguish 3 cases according the place of the missing edge.

(i) $T' \subset G$, $\{u, v, w, x, y\}$ spans a copy of K_5, but there are only 14 edges between the two parts. Then T' has one vertex $z \in \{a, b, c, d, e, f\}$ incident to at least two of the 14 edges. Then (ϵ) leads to a contradiction.

(ii) $T' \subset G$, $\{u, v, w, x, y\}$ spans a copy of K_5 minus one edge, say $\{x, y\}$, and all 15 edges between the two parts are in G.

If two adjacent vertices from the set $\{u, v, w, x, y\}$ are both adjacent to $\{b, c, e\}$ then (γ) gives the contradiction. Therefore, if x is adjacent to $\{b, c, e\}$ then u, v and w must be adjacent to $\{a, d, f\}$. If y is also adjacent to $\{a, d, f\}$ then we have 4 vertices spanning a K_4 and all adjacent to $\{a, d, f\}$. Then we obtain a contradiction by (δ). Otherwise y is adjacent to $\{b, c, e\}$ and $G = H_{11}^6$.

Suppose now that x is adjacent to $\{a, d, f\}$. If u, v, w are all adjacent to $\{a, d, f\}$ then (δ) leads to a contradiction. Hence, at least one of them, say u is adjacent to $\{b, c, e\}$. But (γ) implies that two adjacent ones from from the set $\{u, v, w, x, y\}$ cannot be adjacent to $\{b, c, e\}$. Hence, v, w, x, y are all adjacent to $\{a, d, f\}$ giving a contradiction again, by (δ).

(iii) T spans only 11 edges, $\{u, v, w, x, y\}$ determines a K_5 and all 15 edges are connecting the two parts. Then T must have a vertex incident to two edges connecting T with $\{u, v, w, x, y\}$. Here (ϵ) gives a contradiction.

Now we are ready to start the inductional step. Suppose that the statement is true for $n-6$ where $n \geq 12$. We will prove it for n. Let $e(G) = \mathrm{ex}(n, P_6^2)$ and suppose that $T \subset G$. We have to prove that G is of the form described in the theorem. By (2.15) we know that the equality implies that T must span the the subgraph T' with 12 edges, every vertex of $G' = G - T'$ is adjacent either to the vertices b, c, e or the vertices a, d, f and G' is an extremal graph for $n-6$. That is, G' is one the following graphs: $F_n^{\lceil \frac{n-6}{2} \rceil, j}, F_n^{\frac{n-6}{2}+1, j}, H_n^{\lfloor \frac{n-6}{2} \rfloor}, H_n^{\lceil \frac{n-6}{2} \rceil}, H_n^{\lceil \frac{n-6}{2} \rceil + 1}$. All these graphs have $n-6$ vertices, their vertex sets are divided into two parts, X' and Y' where $|X'|$ is either $\lfloor \frac{n-6}{2} \rfloor$ or $\lceil \frac{n-6}{2} \rceil$ or $\lceil \frac{n-6}{2} \rceil + 1$, there is a bipartite graph between X' and Y' and X' is covered by vertex-disjoint triangles and at most one star.

Color a vertex of G' by red if it is adjacent to the vertices b, c, e and blue otherwise. By (γ) two red vertices cannot be adjacent. On the other hand, 4 blue vertices cannot span a path by (ζ). Suppose that there is a red vertex in X'. Then all vertices of Y' are colored blue. (It is easy to check that $n \geq 12$ implies $|Y'| \geq 2$.) If there are two blue vertices also in X' then they span a path of length 4 that is a contradiction. We can have one blue vertex in X' only when it contains no triangle and the center s of the star is blue, the other vertices are all red. This is called the first coloring. It is easy to see that the choice $X = \{b, c, e, s\} \cup Y', Y = \{a, d, f\} \cup (X' - \{s\})$ defines a graph possessing the properties of the expected extremal graphs: X and Y span a complete bipartite graph, there are no edges within Y, and X is covered by one triangle and one star which are vertex disjoint.

The other case is when all vertices of X' are blue. In this case, no vertex of Y' can be blue, otherwise this vertex and the 3 vertices of a triangle or the center of the star with two other vertices would span a path of length 4. That is, all vertices of Y' are red. This is the second coloring. Then the choice $X = \{b, c, e\} \cup X', Y = \{a, d, f\} \cup Y'$ defines a graph possessing the properties of the expected extremal graphs.

We have seen that G has the expected structure in both cases. We only have to check the parameters. If $n \equiv 0, 4, 5 \pmod 6$ then X' contains no star, the first coloring cannot occur, in the case of the second coloring 3-3 vertices (3 vertices to each part) are added to both parts, containing a triangle ($\{b, c, e\}$) in the X-part. The upper index increases by 3 in all cases when moving from $n-6$ to n.

Consider now the case $n \equiv 1 \pmod 6$. If $G' = H_{n-6}^{\lfloor \frac{n-6}{2} \rfloor}$ then we can proceed like in the previous cases, and $G = H_n^{\lfloor \frac{n}{2} \rfloor}$ is obtained. Suppose that $G' = F_{n-6}^{\lceil \frac{n-6}{2} \rceil, j}$. If $j < \lceil \frac{n-6}{2} \rceil$ then, again, the second coloring applies and we obtain $G = F_n^{\lceil \frac{n}{2} \rceil, j}$. If, however, $j = \lceil \frac{n-6}{2} \rceil$ then both colorings result in $G = F_n^{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil - 3}$. Let us recall that $G = F_n^{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$ was obtained in the case when $T \not\subset G$.

The cases $n \equiv 2, 3 \pmod 6$ can be checked similarly. $\square$

2.5 Open problems

On the basis of these results we pose a conjecture for the general case.

Conjecture 2.16 (Xiao, Katona, Xiao, Zamora [53])**.**

$$\mathrm{ex}(n, P_k^2) \le \max_i \left\{ \frac{i\left(\left\lfloor \frac{2k}{3} \right\rfloor - 2\right)}{2} + i(n-i) \right\}.$$

If $\left\lfloor \frac{2k}{3} \right\rfloor - 1$ *divides the optimal* i *then the following graph gives equality here. Take a complete bipartite graph with parts of size* i *and* $n - i$*, add vertex disjoint complete graphs on* $\left\lfloor \frac{2k}{3} \right\rfloor - 1$ *vertices to the part with* i *elements.*

Observe that Theorems 2.1, 2.2, 2.3 and 2.10 justify our conjecture for the cases when $k = 3, 4, 5, 6$. A weaker form of this conjecture is the following one.

Conjecture 2.17 (Xiao, Katona, Xiao, Zamora [53])**.**

$$\mathrm{ex}(n, P_k^2) = \frac{n^2}{4} + \frac{\left(\lfloor \frac{k}{3} \rfloor - 1\right) n}{2} + O_k(1)$$

where $O_k(1)$ depends only on k.

The following paragraphs show why we think that Conjecture 2.16 is true.

Lemma 2.18 (Xiao, Katona, Xiao, Zamora [53])**.** *If the graph G is obtained by adding a path of r vertices to one of the classes of the complete bipartite graph $K_{n,n}(n \geq r)$ then G contains the square of a path containing $\lfloor \frac{3r}{2} \rfloor + 1$ vertices.*

Proof. Suppose first that $r = 2s$ is even. Let X and Y be the two parts, where $|X| = |Y| = n$ all edges $\{x, y\}(x \in X, y \in Y)$ are in G. Moreover, X contains the path $\{x_1, x_2, \ldots, x_{2s}\}$. Then the square of the path $\{y_1, x_1, x_2, y_2, x_3, x_4, y_3, \ldots, x_{2s-1}, x_{2s}, y_{s+1}\}$ is in G for an arbitrary set of distinct vertices $y_1, y_2, \ldots, y_{s+1} \in Y$. The number of vertices of this path is really $3s + 1$.

If $k = 2s + 1$ is an odd number then the desired path is $\{y_1, x_1, x_2, y_2, x_3, x_4, y_3, \ldots, x_{2s-1}, x_{2s}, y_{s+1}, x_{2s+1}\}$. □

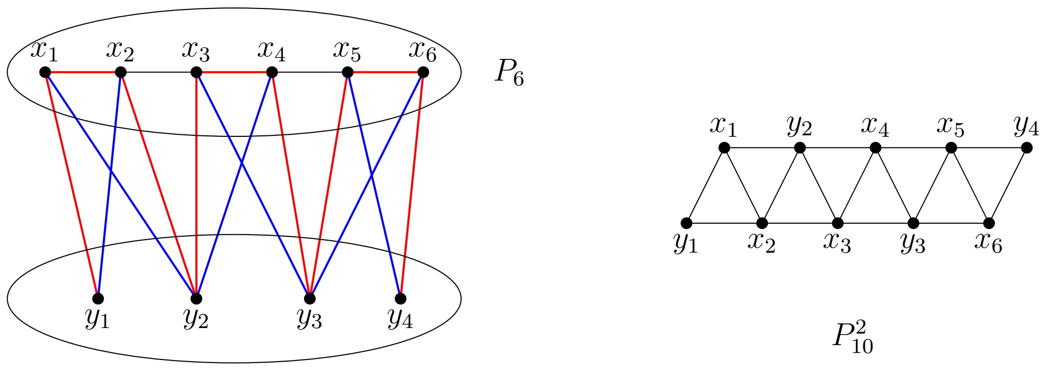

Figure 2.10

It is easy to see, on the basis of Lemma 2.18 that if this graph does not contain P_k^2 then X cannot contain a path of length $\lfloor \frac{2k}{3} \rfloor$. Now the obvious question is that at most how many edges can be chosen in X without having a path of given length. As one of the earliest results in extremal Graph Theory Erdős and Gallai [18] proved the following result on the maximal size of a P_l-free graph.

Theorem 2.19 (**Erdős and Gallai**[18]). *The maximum number of edges in an n-vertex P_l-free graph is $\frac{n(l-2)}{2}$, that is $\mathrm{ex}(n, P_l) \leq \frac{n(l-2)}{2}$ with equality if and only if $(l-1)|n$ and the graph is a vertex disjoint union of $\frac{n}{l-1}$ complete graphs on $l-1$ vertices.*

Faudree and Schelp[24] and independently Kopylov [32] improved this result determining $\mathrm{ex}(n, P_l)$ for every $n > l > 0$ as well as the corresponding extremal graphs.

Theorem 2.20 (**Faudree and Schelp**[24] **and independently Kopylov** [32]). *Let $n \equiv r \pmod{l-1}$, $0 \leq r \leq l-1, l \geq 2$. Then*

$$\mathrm{ex}(n, P_l) = \frac{1}{2}(l-2)n - \frac{1}{2}r(l-1-r).$$

Faudree and Schelp also described the extremal graphs which are either

(a) vertex disjoint union of m $(n = m(l-1)+r)$ complete graphs K_{l-1} and a K_r or

(b) l is even and $r = \frac{l}{2}$ or $\frac{l}{2} - 1$ then another extremal graph can be obtained by taking a vertex disjoint union of t copies of K_{l-1} $(0 \leq t \leq m)$ and a copy of $K_{\frac{l}{2}-1} \bigotimes \overline{K}_{n-(t+\frac{1}{2})(l-1)+\frac{1}{2}}$. Where $\overline{G}$ denotes the edge complement of the graph G, and $G \bigotimes H$ is defined as the graph obtained from the vertex disjoint union of G and H together with all edges between G and H.

We believe that the extremal graph for $\mathrm{ex}(n, P_k^2)$ is a complete bipartite graph plus one of the constructions above in the larger class. Check now the cases solved.

If $k = 4$, by Lemma 2.18 we cannot have a path of length 2 (that is an edge) in one side.

If $k = 5$ then $l = 3$, a path of length 3 is forbidden in one side. According to statements above we can have only vertex disjoint edges.

If $k = 6$ then $l = 4$ and a path of length 4 is forbidden in one side. Now the extremal constructions for P_l are either (a) triangles plus eventually one edge or (b) t triangles plus a star with $n - 3t$ vertices.

These are in accordance with our results. Note that in the case of $k = 7$, the value $l = 4$ obtained again. The expected maximum value is the same as in the case of $k = 6$, but the assumptions are weaker!

Remark 2.21. *Recently, Long-Tu Yuan proved Conjectures 2.16 and 2.17 and characterize the extremal graphs of the k-th power of paths in [55].*

Chapter 3

The Turán number of disjoint union of wheels

3.1 Introduction

A ***wheel*** W_n is a graph on n ($n \geq 4$) vertices obtained from a C_{n-1} by adding one vertex v_0 and joining v_0 to all vertices of the C_{n-1}. We call a wheel on an even (odd) number of vertices an ***even (odd) wheel***.

Denote by mH the graph of the vertex-disjoint union of m copies of the graph H. Two disjoint vertex sets U and W are completely joined in G if $uw \in E(G)$ for all $u \in U$, $w \in W$. Given graphs G_1 and G_2, where G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$. The union $G = G_1 \cup G_2$ is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Denote by $G_1 \bigotimes G_2$ the graph obtained from $G_1 \cup G_2$ by adding all edges between $V(G_1)$ and $V(G_2)$. Let $\mathcal{F}$ be a graph family, denote by $G \bigotimes \mathcal{F}$ the graph family obtained from $G \bigotimes H$, for all $H \in \mathcal{F}$.

Denote by $\mathcal{K}_{n_1,n_2}(\mathcal{F};\mathcal{H})$ ($n_1 \geq n_2$) the class of graphs obtained by taking a complete bipartite graph K_{n_1,n_2} and embedding a graph from the graph set $\mathcal{F}$ into the larger partite set and embedding a graph from the graph set $\mathcal{H}$ into the smaller partite set. A nearly k-regular graph is a graph such that each vertex has degree k but one vertex has degree $k-1$. Let $\mathcal{U}_n^{k-1}(P_{2k-1})$ be the class of P_{2k-1}-free, $(k-1)$-regular or nearly $(k-1)$-regular graphs on n vertices.

Definition 3.1. *Let* $\mathcal{K}^t_{n_1,n_2}\left(\mathcal{U}_{n_1}^{k-1}(P_{2k-1}); P_2\right) = K_t \bigotimes \mathcal{K}_{n_1,n_2}\left(\mathcal{U}_{n_1}^{k-1}(P_{2k-1}); P_2\right)$, $n_1 \geq n_2 \geq 2$ *and* $n_1 + n_2 = n - t$, *where* $\mathcal{K}_{n_1,n_2}\left(\mathcal{U}_{n_1}^{k-1}(P_{2k-1}); P_2\right)$ *denotes the class of graphs obtained from a* K_{n_1,n_2} *by embedding the larger partite set a graph from* $\mathcal{U}_{n_1}^{k-1}(P_{2k-1})$ *and* *embedding an edge in the smaller partite set.*

Since $W_3 = C_3$ and $W_4 = K_4$, we can easily see the results of $\mathrm{ex}(n, W_3)$ and $\mathrm{ex}(n, W_4)$ by famous Mantel's theorem [43] and Turán's theorem [51]. In 1964 Erdős proved the following theorem.

Theorem 3.2 (Erdős [17]). *Let G be any graph such that $e(G) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor + 1$. Then G contains a W_5.*

Years later, in [12], Dzido determined for $k \geq 3$ and $n \geq 6k - 10$, $\mathrm{ex}(n, W_{2k}) = \left\lfloor \frac{n^3}{3} \right\rfloor$. Later on, Dzido and Jastrzębski [13] obtained two exact values for small wheels $\mathrm{ex}(n, W_5) = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor$ and $\mathrm{ex}(n, W_7) = \left\lfloor \frac{n^2}{4} + \frac{n}{2} + 1 \right\rfloor$. Recently, Yuan [56] determined the Turán number $\mathrm{ex}(n, W_{2k+1})$ of the odd wheel when n is sufficiently large. In this chapter, we determine the Turán number and characterize all extremal graphs for disjoint union of odd wheels.

Theorem 3.3 (Yuan [56]). *Let $k \geq 2$ and W_{2k+1} be a wheel on $2k+1$ vertices. Then for n sufficiently large,*

$$\mathrm{ex}(n, W_{2k+1}) = \begin{cases} \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor, k = 2, \\ \max\left\{ n_1 n_2 + \left\lfloor \frac{(k-1)n_1}{2} \right\rfloor : n_1 + n_2 = n \right\} + 1, k \geq 3, \end{cases}$$

and $\mathrm{EX}(n, W_{2k+1}) \subseteq \mathcal{K}^0_{n_1,n_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1}); P_2\right)$*, when $k \geq 3$.*

We generalized Yuan's result in the following way.

Theorem 3.4 (Xiao, Zamora [54]). *Let mW_{2k+1} $(k \geq 3)$ denote the graph defined by taking m vertex- disjoint copies of W_{2k+1}. For n sufficiently large,*

$$\mathrm{ex}(n, mW_{2k+1}) = \max\left\{ \binom{m-1}{2} + \left\lfloor \frac{(k-1)n_1}{2} \right\rfloor + (n_1 + m - 1)(n - m + 1) - n_1^2 + 1 \right\},$$

where the maximum is taken for $n_1 \in \{1, \ldots, n - m + 1\}$, moreover,

$$\mathrm{EX}(n, mW_{2k+1}) \subseteq \mathcal{K}^{m-1}_{n_1,n_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1}); P_2\right), \ (n_1 + n_2 = n - m + 1).$$

Clearly, the graphs in $\mathcal{K}^{m-1}_{n_1,n_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1}); P_2\right)$ are mW_{2k+1}-free. Yuan [56] proved the case when $m = 1$ and $k \geq 3$.

3.2 Progressive induction

To prove Theorem 3.4, we use the technique of progressive induction, which was first introduced by Simonovits in [48]. Essentially, the technique is as follows. For a given problem one can prove the inductive step under the assumptions of the inductive hypothesis. However, it is not possible to prove the base case (this could be because the base case is not true for small values). It also can happen that the proof of the base case is as difficult as a direct proof of the result. Formally the statement we use is the following:

Proposition 3.5 (Simonovits [48]). *Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{Z}$ be the set of all integers. Let $c \in \mathbb{N}$ and $\varphi : \mathbb{N} \to \mathbb{Z}$ be a function such that $\varphi(n) < \max\{\varphi(n-1), \varphi(n-c)\}$, then there exists $n_0 \in \mathbb{N}$ such that $\varphi(n) < 0$ for every $n > n_0$.*

Let H_n be an extremal graph for mW_{2k+1} and $f(n,t) = \max\limits_{n_1+n_2=n-t}\{e(G) : G \in \mathcal{K}^t_{n_1,n_2}(\mathcal{U}^{k-1}_{n_1}(P_{2k-1}); P_2)\}$. To establish the result, in this paper, we define a function $\varphi(n)$, used to measure the "distance between our knowledge $e(H_n)$ and the conjecture $f(n,m-1)$", that is $\varphi(n) = e(H_n) - f(n,m-1)$. Clearly, $\varphi(n)$ is non-negative. We then attempt to show that there exists n_0, when $n > n_0$, either $\varphi(n) < \varphi(n-1)$, $\varphi(n) < \varphi(n-c)$ (for some c chosen later) or $H_n \in \mathcal{K}^{m-1}_{n_1,n_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1}); P_2\right)$.

3.3 Proof of Theorem 3.4

We need the following theorem and key lemma to proof Theorem 3.4.

Theorem 3.6 (Kővári-Sós-Turán [33]). *Let $K_{a,b}$ denote the complete bipartite graph with a and b vertices in its color-classes. Then*

$$\mathrm{ex}(n, K_{a,b}) \leq \frac{\sqrt[a]{b-1}}{2} n^{2-\frac{1}{a}} + \frac{a-1}{2} n.$$

Lemma 3.7 (Xiao, Zamora [54]). *Let G be an mW_{2k+1}-free graph with a partition of the vertices into two nonempty parts $V(G) = V_1 \cup V_2$ with sizes n_1 and n_2 respectively such that $n_1 \geq n_2$ and n_2 is sufficiently large. Suppose G is such that, for each i if $S \subseteq V_i$ has size at most $m(k+1)$ then all vertices in S have at least $m(2k+1)$ common neighbors in the other class. Then, for n_1 sufficiently large, $e(G) \leq g(n_1, n_2, m)$, where $g(n_1, n_2, m)$ is defined as $g(n_1,n_2,m) = \max\limits_{n_1+n_2=n-m+1}\{e(H), H \in \mathcal{K}^{m-1}_{n_1-j,n_2-(m-1-j)}\left(\mathcal{U}^{k-1}_{n_1-j}(P_{2k-1}); P_2\right) : j = 0, 1,$*

$\ldots, m-1\}$. *Moreover, for $m > 1$ equality can only hold if G contains a vertex of degree $n_1 + n_2 - 1$.*

Proof. The proof will follow by induction on m, the case where $m = 1$ is done by [56].

Clearly, for $n_2 \geq m - 1$ we have that $g(n_1, n_2, m) \leq f(n, m-1)$. Now suppose that $m > 1$, note that by the definition of $\mathcal{K}^t_{n_1,n_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1}); P_2\right)$ we have

$$e\left(\mathcal{K}^{m-1}_{n_1-j,n_2-(m-1-j)}(\mathcal{U}^{k-1}_{n_1-j}(P_{2k-1}); P_2)\right) = e\left(\mathcal{K}^{m-2}_{n_1-j,n_2-(m-1-j)}(\mathcal{U}^{k-1}_{n_1-j}(P_{2k-1}); P_2)\right)$$

$+(n_1 + n_2 - 1)$.

It follows from the definition that both $g(n_1 - 1, n_2, m - 1)$ and $g(n_1, n_2 - 1, m - 1)$ are bounded above by $g(n_1, n_2, m) - (n_1 + n_2 - 1)$.

Let S_n denote the star on n vertices and G_i denote the subgraph of G induced by the vertex set V_i. For a graph H, let $s_{k+1}(H)$ denote the maximum number of disjoint S_{k+1} in H. From the conditions of G we have that $s_{k+1}(G_1) + s_{k+1}(G_2) \leq m - 1$. We separate the proof into 2 cases.

Case 1. For some i there exists a vertex $u \in V_i$ such that $d_{G_i}(u) \geq m(2k + 1)$.

Let G' be the graph obtained from G by removing u, then the vertex set of G' can be decomposed into graphs $V_1' \cup V_2'$ of sizes n_1' and n_2'. We have that G' must be $(m-1)W_{2k+1}$-free, otherwise we may find another wheel with center u which is disjoint from the previous $(m-1)W_{2k+1}$. Hence, by the induction hypothesis we have $e(G') \leq g(n_1', n_2', m-1)$ and so

$$e(G) \leq d_G(u) + g(n_1', n_2', m-1) \leq n_1 + n_2 - 1 + g(n_1', n_2', m-1) \leq g(n_1, n_2, m-1),$$

where equality holds only when $d_G(u) = n_1 + n_2 - 1$.

Case 2. For each vertex $v \in V_i$ $(i = 1, 2)$, $d_{G_i}(v) < m(2k + 1)$.

Then we have that $d(v) < n_2 + m(2k + 1)$ for $v \in V_1$ while $d(v) < n_1 + m(2k + 1)$ for $v \in V_2$. We may assume by induction that G contains at least one wheel W, say with vertices $a_1, a_2, \ldots, a_s$ in V_1 and $b_1, \ldots, b_t$ in V_2, where $s + t = 2k + 1$. Then G', is defined as the graph obtained by G by removing W, can be decomposed in components V_1' and V_2' of sizes $n_1 - s, n_2 - t$ respectively, then

$$e(G) \leq sn_2 + tn_1 + (2k + 1)^2 m + g(n_1 - s, n_2 - t, m - 1). \tag{3.1}$$

Note that by the construction of G we have the following bounds

$$\begin{aligned}
&g(x,y,m) \geq g(x,y,m-1) + \min\{y, x-k\} - m \geq g(x,y,m-1) + y - k - m,\\
&g(x,y,m) \geq g(x-1,y,m) + y,\\
&g(x,y,m) \geq g(x,y-1,m) + x.
\end{aligned}$$

The first bound is obtained by the difference between the number of edges of the graphs in the definition of g, that is comparing the number of edges of $\mathcal{K}^{m-1}_{n_1-j,n_2-(m-1-j)}(\mathcal{U}^{k-1}_{n_1-j}(P_{2k-1});$ $P_2)$ with $\mathcal{K}^{m-2}_{n_1-j+1,n_2-(m-1-j)}\left(\mathcal{U}^{k-1}_{n_1-j}(P_{2k-1}); P_2\right)$ (when $j \geq 1$) or $\mathcal{K}^{m-2}_{n_1-j,n_2+1-(m-1-j)}(\mathcal{U}^{k-1}_{n_1-j}(P_{2k-1}); P_2)$ (when $j \leq m-2$).

As a consequence of these bounds it follows that

$$g(n_1-s, n_2-t, m-1) \leq g(n_1,n_2,m) - sn_2 - tn_1 - (n_2-k-m) + (2k+1)^2.$$

Hence together with equation (3.1) it follows that

$$e(G) \leq g(n_1,n_2,m) + (2k+1)^2(m+1) - (n_2-k-m).$$

Therefore, when

$$n_2 > (2k+1)^2(m+1) + m + k,$$

$e(G) < g(n_1,n_2,m)$ holds. □

Lemma 3.8 (Yuan [56]). *Let $n \geq 2k$, then* $\mathrm{ex}(n, \{S_{k+1}, P_{2k+1}\}) = \left\lfloor \frac{(k-1)n}{2} \right\rfloor$.

Proof of Theorem 3.4. We prove Theorem 3.4 using the progressive induction. Let n be large enough and H_n be an n-vertex mW_{2k+1}-free graph with maximal number of edges. We will also assume by induction that Theorem 3.4 holds for $m-1$, the base case $m=1$ is done by [56]. The following proof is based on Yuan's result.

Fix $N \in \mathbb{N}$ an even number, which will be picked large enough. Since $e(H_n) > \left\lfloor \frac{n^2}{4} \right\rfloor$, by Theorem 3.6, there exists n_1 such that when $n > n_1$, H_n contains $K_{N,N}$ as a subgraph. Let B_1 and B_2 be the bipartite classes of $K_{N,N}$. Let $\hat{H}_{2N}$ be the graph induced on the vertex set $B_1 \cup B_2$, $\tilde{H}_{n-2N}$ be the graph induced on the vertex set $V(H_n) \setminus (B_1 \cup B_2)$ and e_H be the number of edges between $\hat{H}_{2N}$ and $\tilde{H}_{n-2N}$. Thus,

$$e(H_n) = e(\hat{H}_{2N}) + e_H + e(\tilde{H}_{n-2N}).$$

Let $H_n^{'}$ be a graph in $\mathcal{K}^{m-1}_{n_1,n_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1});P_2\right)$, by Lemma 3.8, there exists $K^*_{N,N}$ such that $K^*_{N,N} \subseteq H_n^{'}$, for some $K^*_{N,N} \in \mathcal{K}_{N,N}\left(\mathcal{U}^{k-1}_{N}(P_{2k-1});\emptyset\right)$. Let $H^{'}_{n-2N}$ be the graph induced by the vertex set $V(H_n^{'}) \setminus V\left(K^*_{N,N}\right)$ and $e_{H'}$ be the number of edges joining $K^*_{N,N}$ and $H^{'}_{n-2N}$. Thus,

$$e(H_n^{'}) = e\left(K^*_{N,N}\right) + e_{H'} + e(H^{'}_{n-2N}).$$

Clearly, $e_{H'} = (n-2N)N + (m-1)N = (n-2N+m-1)N$.

By Lemma 3.7, we observe $e(\hat{H}_{2N}) \leq g(N,N,m)$. Therefore, we have

$$\begin{aligned}
\varphi(n) &= e(H_n) - e(H_n^{'}) \\
&= e(\hat{H}_{2N}) - e\left(K^*_{N,N}\right) + e_H - e_{H'} + e(\tilde{H}_{n-2N}) - e(H^{'}_{n-2N}) \\
&\leq g(N,N,m) - N^2 - \frac{N(k-1)}{2} + (e_H - e_{H'}) + \varphi(n-2N) \\
&\leq mN + (e_H - e_{H'}) + \varphi(n-2N). \qquad (3.2)
\end{aligned}$$

Note that from (3.2) we have that if $\varphi(n) \geq \varphi(n-2N)$ then $mN \geq e_{H'} - e_H$.

To complete the progressive induction, we are going to show that for n large enough, either $\varphi(n) < \varphi(n-2N)$ or $\varphi(n) < \varphi(n-1)$ or $H_n \in \mathcal{K}^{m-1}_{n_1,n_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1});P_2\right)$.

Case 1. There exists a vertex $v \in H_n$ with $d_{H_n}(v) < \frac{n}{2}$.

Recall the defition that $f(n,t) = \max\{e(G) : G \in \mathcal{K}^t_{n_1,n_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1});P_2\right), n_1 + n_2 + t = n\}$. Since $e(H_n^{'}) = f(n,m-1) = \max\{\binom{m-1}{2} + \left\lfloor \frac{(k-1)n_0}{2} \right\rfloor + (n_0+m-1)(n-m+1) - n_0^2 + 1\}$ where $n_0 = \frac{1}{2}\left(\left\lfloor \frac{k-1}{2} \right\rfloor + n - m + 1\right)$ or $n_0 = \frac{1}{2}\left(\left\lceil \frac{k-1}{2} \right\rceil + n - m + 1\right)$, we get $e(H_n^{'}) - e(H^{'}_{n-1}) = f(n,m-1) - f(n-1,m-1) \geq \frac{n}{2}$. Clearly, $H_n - v$ is an $(n-1)$-vertex mW_{2k+1}-free graph which implies that $e(H_n) - d_{H_n}(v) \leq e(H_{n-1})$. Hence, $e(H_n) - e(H_{n-1}) \leq d_{H_n}(v) < \frac{n}{2}$ and we get $\varphi(n) = e(H_n) - e(H_n^{'}) < e(H_{n-1}) - e(H^{'}_{n-1}) = \varphi(n-1)$.

In Case 2 we will assume that neither $\varphi(n) < \varphi(n-2N)$ nor $\varphi(n) < \varphi(n-1)$ hold.

Case 2. $\delta(H_n) \geq \frac{n}{2}$ and $\varphi(n) \geq \varphi(n-2N)$. With the following claims we are able to show that $H_n \in \mathcal{K}^{m-1}_{n_1,n_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1});P_2\right)$ in this case.

Claim 3.9. *Let x be a vertex in H_n such that $K_{m(2k+1),m(2k+1)}$ is contained in the neighborhood of x, then G', the graph induced by $V(H_n) \setminus \{v\}$, is $(m-1)W_{2k+1}$-free.*

Proof. Suppose by contradiction that G' is not $(m-1)W_{2k+1}$-free, since a copy of $(m-1)W_{2k+1}$ contains $(m-1)(2k+1)$ vertices in G', then we may find a copy of $K_{k,k}$ in the neighborhood of x which does not contain any vertex of the given $(m-1)W_{2k+1}$ copy, then v together with the copy of $K_{k,k}$ contains another copy of W_{2k+1} which contradicts the fact that H_n is mW_{2k+1}-free. $\square$

Hence we may assume that for any vertex $v \in V(H_n)$, there is an index $i(v) \in \{1, 2\}$ such that v has fewer than $m(2k+1)$ neighbors in $B_{i(v)}$, since otherwise we would be able to find a copy of $K_{m(2k+1),m(2k+1)}$ in the neighborhood of v, and then by Claim 3.9 and induction on m, we would have that

$$e(H_n) \leq (n-1) + e(G[V(H_n) \setminus \{v\}]) \leq (n-1) + f(n-1, m-2) = f(n, m-1).$$

where the equality holds only if $d_{H_n}(v) = n-1$ and the graph induced by $V(H_n) \setminus \{v\}$ is in $\mathcal{K}^{m-2}_{n'_1,n'_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1}); P_2\right)$ for some $n'_1 + n'_2 = n-1$. Therefore, by adding a full degree vertex to the previous graph we have that the equality holds only when $H_n \in \mathcal{K}^{m-1}_{n_1,n_2}\left(\mathcal{U}^{k-1}_{n_1}(P_{2k-1}); P_2\right)$ for some n_1 and n_2 with $n_1 + n_2 = n$ which maximizes the number of edges.

We partition the vertices of $\tilde{H}_{n-2N}$ into classes C_1, C_2 and D where: C_i is the set of vertices v such that v is adjacent to fewer than $m(2k+1)$ vertices in B_i and more than $N - 2m(2k+1)$ vertices of B_{3-i} for $i = 1, 2$. $v \in D$ if v is adjacent to at most $N - 2m(2k+1)$ vertices of both B_1 and B_2.

By the definition of C_i and since $K_{N,N} \subseteq G[B_1 \cup B_2]$, we have that any $m(2k+1)+1$ vertices of $B_i \cup C_i$ have more than $\Big(N - 2m(2k+1)\Big)\Big(m(2k+1)+1\Big) \geq m(2k+1)$ neighbors in B_{3-i}, hence we may assume that every vertex $x \in B_i \cup C_i$ has fewer than $m(2k+1)$ neighbors in $B_i \cup C_i$ or we would be done by Claim 3.9.

Claim 3.10. *There exists a constant N_1 such that $|D| < N_1$.*

Proof. Recall that by definition every vertex in C_i is adjacent to fewer than $m(2k+1)$ vertices of B_i and for each vertex $v \in D$, there exists an $i(v)$ such that v is joined to fewer than $m(2k+1)$ vertices of $B_{i(v)}$, we get that v is joined to fewer than $m(2k+1) + (N - 2m(2k+1)) \leq N - m(2k+1)$ vertices of $\hat{H}_{2N}$. Therefore,

$$\begin{aligned} e_H &= e(B_1, C_1) + e(B_2, C_2) + e(B_1, C_2) + e(B_2, C_1) + e(B_1 \cup B_2, D) \\ &\leq 2Nm(2k+1) + N(n-2N) - m(2k+1)|D| \\ &= 4Nmk + N(m+1) + N(n-2N+m-1) - m(2k+1)|D|. \end{aligned}$$

Since $e_{H'} = N(n-2N+m-1)$, we have that $e_H \leq 4Nmk+N(m+1)+e_{H'}-m(2k+1)|D|$. From inequality (3.2) we have

$$mN \geq e_{H'} - e_H \geq mk|D| - 4Nmk - N(m+1),$$

hence $|D| < N\frac{4k+3}{k} = N_1$. ☐

Claim 3.11. $|B_i \cup C_i| = \frac{n}{2} + O(\sqrt{n})$.

Proof. Since there exists an integer N_1 such that $|D| \leq N_1$, then the number of edges incidence with D is $O(n)$. Since $\Delta(G[B_i \cup C_i]) < m(2k+1)$, we get $e(G[B_1 \cup C_1]) + e(G[B_2 \cup C_2]) = O(n)$. Hence, after removing the edges in $G[B_1 \cup C_1]$, $G[B_2 \cup C_2]$ and the edges incident with D, we obtain a bipartite graph on $\left\lfloor \frac{n^2}{4} \right\rfloor - O(n)$ edges. Therefore, there exists a constant N_2 such that $\left| |B_i \cup C_i| - \frac{n}{2} \right| \leq N_2\sqrt{n}$, hence, $|B_i \cup C_i| = \frac{n}{2} + O(\sqrt{n})$. ☐

Claim 3.12. *$D = D_1 \cup D_2$, where vertices in D_i is adjacent to fewer than $m(2k+1)$ vertices of $B_i \cup C_i$.*

Proof. Let $v \in D$, then there exists an $j(v)$ such that v is adjacent to at least $\frac{n}{6}$ vertices in $B_{j(v)} \cup C_{j(v)}$. Otherwise, $d_{H_n}(v) < N_1 - 1 + 2\frac{n}{6} < \frac{n}{2}$, which contradicts to the fact that $\delta(H_n) \geq \frac{n}{2}$. Hence, since each vertex $u \in B_i \cup C_i$ has more than $\frac{n}{2} - O(n)$ neighbors in $B_{3-i} \cup C_{3-i}$, if a vertex $v_0 \in D$ is adjacent to at least $m(2k+1)$ vertices in $B_{3-j(v_0)} \cup C_{3-j(v_0)}$ we may find a copy of $K_{m(2k+1),m(2k+1)}$ and we would be able to apply Claim 3.9. Let $D_i \subseteq D$, be such that each vertex $v \in D_i$ is adjacent to fewer than $m(2k+1)$ vertices in $B_i \cup C_i$, then D is the disjoint union of D_1 and D_2. ☐

Hence, we may assume that every vertex $x \in D$ has fewer than $m(2k+1)$ neighbors in one of the classes $B_1 \cup C_1$ or $B_2 \cup C_2$, otherwise we would be done by induction.

Let $V_1 = B_1 \cup C_1 \cup D_1$ and $V_2 = B_2 \cup C_2 \cup D_2$, then V_1 and V_2 is a vertex partition of H_n such that for any vertex set on $m(k+1)$ vertices in V_i has at least $m(2k+1)$ common neighbors in V_{3-i}. Then by Lemma 3.7, we get $e(H_n) \leq f(n, m-1)$, equality holds only when H_n contains a vertex v of degree $n-1$. Therefore, v would have at least $m(2k+1)$ neighbors in both B_1 and B_2, which is a contradiction. ☐

3.4 Remarks and open problems

Remark 1: In [27], Gorgol gave the general upper and lower bounds on Turán numbers of disjoint unions of arbitrary graph G.

Theorem 3.13 (Gorgol [27])**.** *Let G be an arbitrary connected graph on ℓ vertices, m be an arbitrary positive integer and n be an integer such that $n \geq m\ell$. Then*

$$\max\left\{ \mathrm{ex}(n-m\ell+1,G)+\binom{m\ell-1}{2}, \mathrm{ex}(n-m+1,G)+(m-1)n-\binom{m}{2}\right\}$$
$$\leq \mathrm{ex}(n,mG) \leq \mathrm{ex}\left(n-(m-1)\ell,G\right)+\binom{(m-1)\ell}{2}+(m-1)\ell\Big(n-(m-1)\ell\Big).$$

Note that Theorem 3.13 shows the following inequalities:

$$\max\left\{e\Big(\mathrm{EX}(n-m\ell+1,G)\cup K_{m\ell-1}\Big), e\Big(\mathrm{EX}(n-m+1,G)\bigotimes K_{m-1}\Big)\right\} \leq \mathrm{ex}(n,mG)$$
$$\leq e\Big(\mathrm{EX}(n-(m-1)\ell,G)\bigotimes K_{(m-1)\ell}\Big).$$

In this paper, we proved that when G is W_{2k+1} $(k \geq 3)$, $\mathrm{ex}(n, mW_{2k+1}) = e\Big(\mathrm{EX}(n-m+1, W_{2k+1})\bigotimes K_{m-1}\Big)$.

We now consider the disjoint union of wheels of possibly distinct sizes. When there is an even wheel the following result holds.

The Turán graph $T(n,p)$ is a complete multipartite graph formed by partitioning a set of n vertices into p subsets, with sizes as equal as possible, and connecting two vertices by an edge if and only if they belong to different subsets. Denote its size by $t(n,p)$.

Theorem 3.14 (Xiao, Zamora [54])**.** *Let $\mathcal{W}^h$ be the family of graphs obtain by the disjoint union of a finite number of wheels, such that, the number of even wheels in the union is h, $(h \geq 1)$. For any $W \in \mathcal{W}^h$, if n is sufficiently large, we have that* $\mathrm{ex}(n,W) = \left\{\binom{h-1}{2}+(h-1)(n-h+1)+t(n-h+1,3)\right\}$ *and* $\mathrm{EX}(n,W) = K_{h-1}\bigotimes T(n-h+1,3)$.

Theorem 3.14 is a consequence of the following result of Simonovits [49].

Theorem 3.15 (**Simonovits**[49])**.** *Let $\mathcal{L}$ be the family of forbidden graphs and $p = p(\mathcal{L}) = \min_{L\in\mathcal{L}} \chi(\mathcal{L})-1$. If by omitting any $s-1$ vertices of any $L \in \mathcal{L}$ we obtain a graph with chromatic number at least $p+1$, but by omitting s suitable edges of some $L \in \mathcal{L}$ we* *get a p-colorable graph, then* $K_{s-1} \bigotimes T(n-s+1,p)$ *is the unique extremal graph for* $\mathcal{L}$ *when* n *is sufficiently large.*

Let $k_1 \geq k_2 \geq \ldots \geq k_m$ be positive integers, it is easy to see that if the disjoint union of stars $\bigcup_{i=1}^{m} S_{k_i+1}$, is added to one class of K_{n_0,n_1}, the we would obtain a copy of $\bigcup_{i=1}^{m} W_{2k_i+1}$.

Based on the following theorem, we propose a conjecture on the extremal number for $\bigcup_{i=1}^{m} W_{2k_i+1}$.

Theorem 3.16 (Lidický, Liu, Palmer [38]). *Let* $F = \bigcup_{i=1}^{k} S^i$ *be a star forest where* d_i *is the maximum degree of* S^i *and* $d_1 \geq d_2 \geq \ldots \geq d_k$*. For* n *sufficiently large,*

$$\mathrm{ex}(n,F) = \max_{1\leq i\leq k} \left\{ (i-1)(n-i+1) + \binom{i-1}{2} + \left\lfloor \frac{d_i-1}{2}(n-i-1) \right\rfloor \right\}.$$

Conjecture 3.17 (Xiao, Zamora [54]). *Let* $\bigcup_{i=1}^{m} W_{2k_i+1}$ *be a disjoint union of odd wheels with components of order* $2k_1+1, 2k_2+1, \ldots, 2k_m+1$ *where* $k_1 \geq k_2 \geq \ldots \geq k_m$*. For* n *sufficiently large,*

$$\begin{aligned} &\mathrm{ex}(n, \bigcup_{i=1}^{m} W_{2k_i+1}) \\ &= \max_{1\leq n_0\leq n} \left\{ n_0(n-n_0) + \mathrm{ex}(n_0, \bigcup_{i=1}^{m} S_{k_i+1}) + 1 \right\} \\ &= \max_{\substack{1\leq i\leq m \\ 1\leq n_0\leq n}} \left\{ n_0(n-n_0) + (i-1)(n_0-i+1) + \binom{i-1}{2} + \left\lfloor \frac{k_i-1}{2}(n_0-i+1) \right\rfloor + 1 \right\}, \end{aligned}$$

and the number of edges of graph $\left(K_{m-1} \bigotimes \bigcup_{\frac{n_0-m+1}{k_m}} K_{k_m}\right) \bigotimes \left(P_2 \cup \overline{K_{n-n_0-2}}\right)$ *gives us the lower bound of* $\mathrm{ex}(n, \bigcup_{i=1}^{m} W_{2k_i+1})$*. Here,* $\bigcup_{\frac{n_0-m+1}{k_m}} K_{k_m}$ *denotes the union of* $\frac{n_0-m+1}{k_m}$ *disjoint copies of* K_{k_m}*.*

Chapter 4

Turán numbers and anti-Ramsey numbers for short cycles in complete 3-partite graphs

4.1 Introduction

We call a 4-cycle in K_{n_1,n_2,n_3} multipartite, denoted by C_4^{multi}, if it contains at least one vertex in each part of K_{n_1,n_2,n_3}. We call an edge-colored C_4^{multi} rainbow if its all four edges have different colors. The anti-Ramsey number $\text{ar}(K_{n_1,n_2,n_3}, C_4^{\text{multi}})$ is the maximum number of colors in an edge-colored K_{n_1,n_2,n_3} with no rainbow C_4^{multi}.

An old result of Bollobás, Erdős and Szemerédi [7] proved that $\text{ex}(K_{n_1,n_2,n_3}, C_3) = n_1n_2+n_1n_3$ for $n_1 \geq n_2 \geq n_3 \geq 1$ (also see [8, 5, 47]). Lv, Lu and Fang [41, 42] constructed balanced 3-partite graphs which are C_4-free and $\{C_3, C_4\}$-free respectively and proved that $\text{ex}(K_{n,n,n}, C_4) = (\frac{3}{\sqrt{2}} + o(1))n^{3/2}$ and $\text{ex}(K_{n,n,n}, \{C_3, C_4\}) \geq (\sqrt{3} + o(1))n^{3/2}$. Since then plentiful results were established for a variety of graphs H, we refer the reader to [6, 25, 29, 28, 30, 44, 46].

For further discussion, we need the definitions of the multipartite subgraphs and a function $f(n_1, n_2, \ldots, n_r)$.

Definition 4.1. *Let $r \geq 3$ and G be an r-partite graph with vertex partition $V_1, V_2, \ldots, V_r$, we call a subgraph H of G* multipartite*, if there are at least three distinct parts V_i, V_j, V_k such that $V(H) \cap V_i \neq \emptyset, V(H) \cap V_j \neq \emptyset$ and $V(H) \cap V_k \neq \emptyset$. In particular, we denote a multipartite H by H^{multi} (see Figure 4.1 for an example of a C_4^{multi} in a 3-partite graph).*

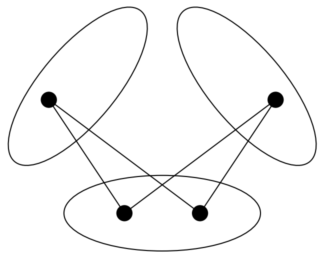

Figure 4.1: A C_4^{multi} in a 3-partite graph.

For $r \geq 3$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$, let

$$f(n_1, n_2, \ldots, n_r) = \begin{cases} n_1n_2 + n_3n_4 + \cdots + n_{r-2}n_{r-1} + n_r + \frac{r-1}{2} - 1, & r \text{ is odd;} \\ n_1n_2 + n_3n_4 + \cdots + n_{r-1}n_r + \frac{r}{2} - 1, & r \text{ is even.} \end{cases}$$

Fang, Győri, Li and J. Xiao [22] recently showed that if $G \subseteq K_{n_1,n_2,\ldots,n_r}$ and $e(G) \geq f(n_1, n_2, \ldots, n_r) + 1$, then G contains a multipartite cycle. Furthermore, they proposed the following conjecture.

Conjecture 4.2 (Fang, Győri, Li and J. Xiao [22]). *For $r \geq 3$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$, if $G \subset K_{n_1,n_2,\ldots,n_r}$ and $e(G) \geq f(n_1, n_2, \ldots, n_r) + 1$, then G contains a multipartite cycle C^{multi} of length at most $\frac{3}{2}r$.*

In this chapter, we study the Turán numbers of C_4^{multi} and $\{C_3, C_4^{\text{multi}}\}$ in the complete 3-partite graphs and obtain the following results.

Theorem 4.3 (Fang, Győri, Xiao and Xiao [23]). *For $n_1 \geq n_2 \geq n_3 \geq 1$, $\text{ex}(K_{n_1,n_2,n_3}, C_4^{multi}) = n_1n_2 + 2n_3$.*

Theorem 4.4 (Fang, Győri, Xiao and Xiao [23]). *For $n_1 \geq n_2 \geq n_3 \geq 1$, $\text{ex}(K_{n_1,n_2,n_3}, \{C_3, C_4^{multi}\}) = n_1n_2 + n_3$.*

Notice that Theorem 4.4 confirms Conjecture 4.2 for the case when $r = 3$.

A subgraph of an edge-colored graph is *rainbow*, if all of its edges have different colors. For graphs G and H, the anti-Ramsey number $\text{ar}(G, H)$ is the maximum number of colors in an edge-colored G with no rainbow copy of H. Erdős, Simonovits and Sós [20] first studied the anti-Ramsey number in the case when the host graph G is a complete graph K_n and showed the close relationship between it and the Turán number. In this chapter, we consider the anti-Ramsey number of C_4^{multi} in the complete 3-partite graphs.

Theorem 4.5 (Fang, Győri, Xiao and Xiao [23]). *For $n_1 \geq n_2 \geq n_3 \geq 1$, $\mathrm{ar}(K_{n_1,n_2,n_3}, C_4^{multi}) = n_1n_2 + n_3 + 1$.*

We prove Theorems 4.3 and 4.4 in Section 4.2 and Theorem 4.5 in Section 4.3, respectively. We always denote the vertex partition of K_{n_1,n_2,n_3} by V_1, V_2 and V_3, where $|V_i| = n_i$, $1 \leq i \leq 3$.

4.2 The Turán numbers of $C_4^{\mathbf{multi}}$ and $\{C_3, C_4^{\mathbf{multi}}\}$

In this section, we first give the following lemma which will play an important role in our proof.

Lemma 4.6 (Fang, Győri, Xiao and Xiao [23]). *Let G be a 3-partite graph with vertex partition X, Y and Z, such that for all $x \in X$, $N(x) \cap Y \neq \emptyset$ and $N(x) \cap Z \neq \emptyset$.*

(i) If G is C_4^{multi}-free, then $e(G) \leq |Y||Z| + 2|X|$;

(ii) If G is $\{C_3, C_4^{multi}\}$-free, then $e(G) \leq |Y||Z| + |X|$.

Proof. (i) Since G is C_4^{multi}-free, $G[N(x)]$ is $K_{1,2}$-free for each $x \in X$. Therefore,

$$e(G[N(x)]) = e\Big(N(x) \cap Y, N(x) \cap Z\Big) \leq \min\Big\{|N(x) \cap Y|, |N(x) \cap Z|\Big\}. \tag{4.1}$$

For $x \in X$, we let e_x be the number of missing edges of G between $N(x) \cap Y$ and $N(x) \cap Z$. By (4.1), we have

$$\begin{aligned} e_x &= |N(x) \cap Y| \cdot |N(x) \cap Z| - e\Big(N(x) \cap Y, N(x) \cap Z\Big) \\ &\geq |N(x) \cap Y| \cdot |N(x) \cap Z| - \min\Big\{|N(x) \cap Y|, |N(x) \cap Z|\Big\} \\ &\geq |N(x) \cap Y| + |N(x) \cap Z| - 2, \end{aligned} \tag{4.2}$$

where the last inequality holds since $|N(x) \cap Y| \geq 1$ and $|N(x) \cap Z| \geq 1$ for all $x \in X$.

By (4.2), we get

$$\sum_{x \in X} e_x \geq \sum_{x \in X} \Big(|N(x) \cap Y| + |N(x) \cap Z| - 2\Big) = e(X, Y) + e(X, Z) - 2|X|. \tag{4.3}$$

Notice that two distinct vertices $x_1, x_2 \in X$, cannot have common neighboors in both Y and Z at the same time, otherwise we find a copy of C_4^{multi} in G. Thus, each missing edge between Y and Z is counted at most once in the sum $\sum_{x\in X} e_x$. Hence, the number of missing edges between Y and Z is at least $\sum_{x\in X} e_x$. Then we have

$$e(Y,Z) \le |Y||Z| - \sum_{x\in X} e_x \le |Y||Z| - (e(X,Y) + e(X,Z) - 2|X|). \tag{4.4}$$

By (4.4), we get

$$e(G) = e(X,Y) + e(X,Z) + e(Y,Z) \le |Y||Z| + 2|X|.$$

(ii) Since G is C_3-free, for each $x \in X$,

$$e\Big(N(x)\cap Y, N(x)\cap Z\Big) = 0. \tag{4.5}$$

Since for each $x \in X$, $|N(x)\cap Y| \ge 1$ and $|N(x)\cap Z| \ge 1$ hold, by (4.5), the number of missing edges between $N(x)\cap Y$ and $N(x)\cap Z$ is $|N(x)\cap Y|\cdot|N(x)\cap Z|$. Notice that two distinct vertices $x_1, x_2 \in X$, cannot have common neighboors in both Y and Z at the same time, otherwise we find a copy of C_4^{multi} in G. Hence, the number of missing edges between Y and Z is at least $\sum_{x\in X}|N(x)\cap Y|\cdot|N(x)\cap Z|$. Thus,

$$\begin{aligned} e(Y,Z) &\le |Y||Z| - \sum_{x\in X}|N(x)\cap Y|\cdot|N(x)\cap Z| \\ &\le |Y||Z| - \sum_{x\in X}(|N(x)\cap Y| + |N(x)\cap Z| - 1) \\ &= |Y||Z| + |X| - e(X,Y) - e(X,Z), \end{aligned} \tag{4.6}$$

where the second inequality holds since $|N(x)\cap Y| \ge 1$ and $|N(x)\cap Z| \ge 1$ for $x \in X$.

By (4.6), we have $e(G) = e(Y,Z) + e(X,Y) + e(X,Z) \le |Y||Z| + |X|$. □

Now we are ready to prove Theorems 4.3 and 4.4.

Proof of Theorem 4.3. Let $G \subseteq K_{n_1,n_2,n_3}$ be a graph, such that V_1 and V_2 are completely joined, V_1 (respectively, V_2) and V_3 are joined by an n_3-matching, see Figure 4.2. Clearly, G is C_4^{multi}-free and $e(G) = n_1n_2 + 2n_3$. Therefore, $\mathrm{ex}(K_{n_1,n_2,n_3}, C_4^{multi}) \ge n_1n_2 + 2n_3$.

Let $G \subseteq K_{n_1,n_2,n_3}$ such that G is C_4^{multi}-free, now we are going to prove that $e(G) \le n_1n_2 + 2n_3$ by induction on $n_1 + n_2 + n_3$.

For the base case $n_3 = 1$, $V_3 = \{v\}$, we consider the following four subcases:

(i) $N(v)\cap V_1 \ne \emptyset$ and $N(v)\cap V_2 \ne \emptyset$, then by Lemma 4.6, we have $e(G) \le n_1n_2 + 2$.

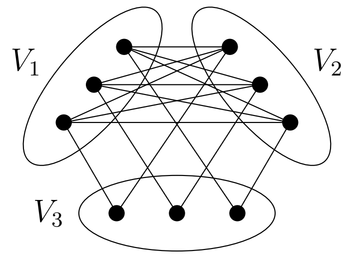

Figure 4.2: An example of C_4^{multi}-free graph with $n_1n_2 + 2n_3$ edges.

(ii) $N(v) \cap V_1 \neq \emptyset$ and $N_G(v) \cap V_2 = \emptyset$, then

$$\begin{aligned} e(G) &= e(V_3, N(v)) + e(V_2, N(v)) + e(V_1 \setminus N(v), V_2) \\ &\leq d(v) + n_2 + \Big(n_1 - d(v)\Big)n_2 \\ &\leq n_1n_2 + 1. \end{aligned}$$

(iii) $N(v) \cap V_1 = \emptyset$ and $N(v) \cap V_2 \neq \emptyset$, then

$$\begin{aligned} e(G) &= e(V_3, N(v)) + e(V_1, N(v)) + e(V_2 \setminus N(v), V_1) \\ &\leq d(v) + n_1 + (n_2 - d(v))n_1 \\ &\leq n_1n_2 + 1. \end{aligned}$$

(iv) $N(v) \cap V_1 = \emptyset$ and $N(v) \cap V_2 = \emptyset$, then $e(G) = e(V_1, V_2) \leq n_1n_2$.

Now let $n_3 \geq 2$, and assume that the statement is true for graphs of order less than $n_1 + n_2 + n_3$. We distinguish the three cases depending on the equality of the numbers n_1, n_2, n_3.

Case 1. $n_1 = n_2 = n_3 = n \geq 2$.

If there exists one part, say V_1, such that $N(v) \cap V_2 \neq \emptyset$ and $N(v) \cap V_3 \neq \emptyset$, for all $v \in V_1$, then by Lemma 4.6, we have $e(G) \leq |V_2||V_3| + 2|V_1| = n^2 + 2n$.

Thus, we may assume that for all $i \in [3] = \{1, 2, 3\}$, there exist a vertex $v \in V_i$ and $j \in [3] \setminus \{i\}$ such that $N(v) \cap V_j = \emptyset$. We divide it into two subcases.

Case 1.1. There exist two parts, say V_1 and V_2, such that $N(v_1) \cap V_2 = \emptyset$ and $N(v_2) \cap V_1 = \emptyset$ for some vertices $v_1 \in V_1$ and $v_2 \in V_2$.

Since G is C_4^{multi}-free, $d(v_1) + d(v_2) \leq |V_3| + 1 = n + 1$. Without loss of generality, let $v_3 \in V_3$ be the vertex such that $N(v_3) \cap V_1 = \emptyset$. Then the number of edges incident with $\{v_1, v_2, v_3\}$ in G is at most $d(v_1) + d(v_2) + n - 1 \leq 2n$. By the induction hypothesis, $e(G - \{v_1, v_2, v_3\}) \leq (n-1)^2 + 2(n-1)$. Thus, $e(G) \leq (n-1)^2 + 2(n-1) + 2n \leq n^2 + 2n$.

Case 1.2. There exist vertices $v_1 \in V_1, v_2 \in V_2$ and $v_3 \in V_3$ such that either $N(v_1) \cap V_2 = \emptyset, N(v_2) \cap V_3 = \emptyset, N(v_3) \cap V_1 = \emptyset$ or $N(v_1) \cap V_3 = \emptyset, N(v_3) \cap V_2 = \emptyset, N(v_2) \cap V_1 = \emptyset$ holds.

Without loss of generality, we assume that $N(v_1) \cap V_2 = \emptyset, N(v_2) \cap V_3 = \emptyset, N(v_3) \cap V_1 = \emptyset$. If $d(v_1) + d(v_2) + d(v_3) \leq 2n + 1$, then by the induction hypothesis, we have

$$\begin{aligned} e(G) &\leq e(G - \{v_1, v_2, v_3\}) + d(v_1) + d(v_2) + d(v_3) \\ &\leq (n-1)^2 + 2(n-1) + 2n + 1 \\ &\leq n^2 + 2n. \end{aligned}$$

Now we assume that $d(v_1) + d(v_2) + d(v_3) \geq 2n + 2$, hence, $d(v_1) \geq 1, d(v_2) \geq 1, d(v_3) \geq 1$. Since G is C_4^{multi}-free, each vertex in $V_1 \setminus \{v_1\}$ can have at most one neighbour in $N(v_3)$, we have $e(V_1 \setminus \{v_1\}, N(v_3)) \leq n - 1$. Similarly, we have $e(V_3 \setminus \{v_3\}, N(v_2)) \leq n - 1$ and $e(V_2 \setminus \{v_2\}, N(v_1)) \leq n - 1$.

Therefore,

$$e(V_1, V_2) = e(V_1 \setminus \{v_1\}, V_2 \setminus N(v_3)) + e(V_1 \setminus \{v_1\}, N(v_3)) \leq (n - d(v_3))(n-1) + (n-1),$$

$$e(V_1, V_3) = e(V_3 \setminus \{v_3\}, V_1 \setminus N(v_2)) + e(V_3 \setminus \{v_3\}, N(v_2)) \leq (n - d(v_2))(n-1) + (n-1),$$

$$e(V_2, V_3) = e(V_2 \setminus \{v_2\}, V_3 \setminus N(v_1)) + e(V_2 \setminus \{v_2\}, N(v_1)) \leq (n - d(v_1))(n-1) + (n-1).$$

Thus,

$$\begin{aligned} e(G) &= e(V_1, V_2) + e(V_1, V_3) + e(V_2, V_3) \\ &\leq \Big(3n - (d(v_1) + d(v_2) + d(v_3))\Big)(n-1) + 3(n-1) \\ &\leq \Big(3n - (2n+2)\Big)(n-1) + 3(n-1) \\ &\leq n^2 - 1. \end{aligned}$$

Case 2. $n_1 > n_2 = n_3 = n \geq 2$.

If there exists one vertex $v_0 \in V_1$ such that $d(v_0) \leq n$, then by the induction hypothesis, we have $e(G) = e(G - v_0) + d(v_0) \leq (n_1 - 1)n + 2n + n \leq n_1 n + 2n$. Otherwise, we have $d(v) \geq n + 1$ for all vertices $v \in V_1$. Hence, $N(v) \cap V_2 \neq \emptyset$ and $N(v) \cap V_3 \neq \emptyset$ hold for all $v \in V_1$. By Lemma 4.6, we get $e(G) \leq n^2 + 2n_1 \leq n_1 n + 2n$.

Case 3. $n_1 \geq n_2 > n_3 \geq 2$.

If there exists one vertex $v_0 \in V_2$ such that $d(v_0) \leq n_1$, by the induction hypothesis, we have $e(G) = e(G - v_0) + d(v_0) \leq n_1(n_2 - 1) + 2n_3 + n_1 \leq n_1n_2 + 2n_3$. Otherwise, we have $d(v) \geq n_1 + 1$ for all vertices $v \in V_2$. Hence, $N(v) \cap V_1 \neq \emptyset$ and $N(v) \cap V_3 \neq \emptyset$ for all $v \in V_2$. By Lemma 4.6, we get $e(G) \leq n_1n_3 + 2n_2 \leq n_1n_2 + 2n_3$. □

Proof of Theorem 4.4. Let $G \subseteq K_{n_1,n_2,n_3}$ be a graph, such that V_1 and V_2 are completely joined, V_1 and V_3 are joined by an n_3-matching and there is no edge between V_2 and V_3, see Figure 4.3. Clearly, G is $\{C_3, C_4^{multi}\}$-free and $e(G) = n_1n_2 + n_3$. Therefore, $\mathrm{ex}(K_{n_1,n_2,n_3}, \{C_3, C_4^{multi}\}) \geq n_1n_2 + n_3$.

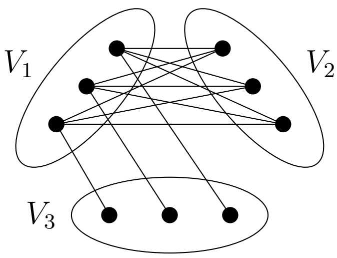

Figure 4.3: An example of $\{C_3, C_4^{multi}\}$-free graph with $n_1n_2 + n_3$ edges.

Let $G \subseteq K_{n_1,n_2,n_3}$ be such that G is $\{C_3, C_4^{\text{multi}}\}$-free, now we can prove $e(G) \leq n_1n_2 + n_3$ by induction on $n_1 + n_2 + n_3$ in the same way as we did in the proof of Theorem 4.3, just the coefficients in the computation change a bit. For sake of brevity, we skip the details of the proof. □

4.3 The anti-Ramsey number of C_4^{multi}

In this section, we study the anti-Ramsey number of C_4^{multi} in the complete 3-partite graphs. Given an edge-coloring c of G, we denote the color of an edge e by $c(e)$. For a subgraph H of G, we denote $C(H) = \{c(e) | e \in E(H)\}$. We call a spanning subgraph of an edge-colored graph representing subgraph, if it contains exactly one edge of each color.

Given graphs G_1 and G_2, we use $G_1 \wedge G_2$ to denote graphs consisting of G_1 and G_2 sharing exactly one common vertex. We call a multipartite C_6 in a 3-partite graph non-cyclic if there exists a vertex v in C_6 such that the two neighbors in C_6 of v belong to the same part. Let $\mathcal{F}$ be a graph family which consists of C_4^{multi} (see graph G_1 in Figure 4.4), $C_3 \wedge C_3$ (see graph G_2 in Figure 4.4), the non-cyclic C_6^{multi} (see graphs G_3, G_4 in Figure

4.4) and $C_3 \wedge C_5$ (see graphs G_5, G_6, G_7 in Figure 4.4) and the C_8^{multi} which contains at least two vertex-disjoint non-multipartite P_3 (see graph G_8 in Figure 4.4).

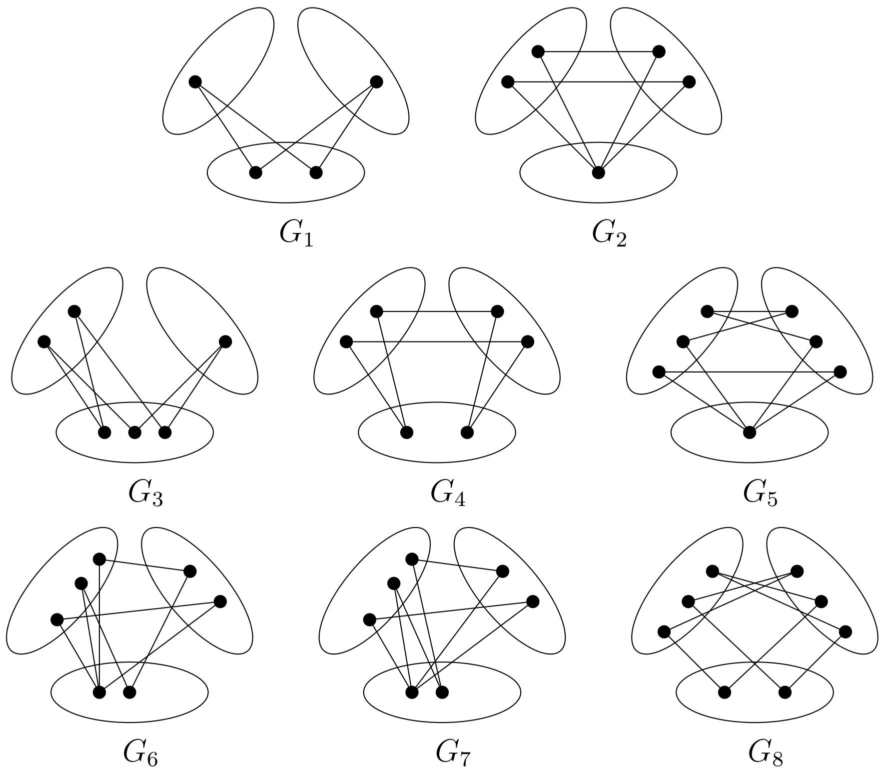

Figure 4.4: $\mathcal{F} = \{G_1\} \cup \{G_2\} \cup \{G_3, G_4\} \cup \{G_5, G_6, G_7\} \cup \{G_8\}$.

To find a rainbow C_4^{multi} in the edge-colored complete 3-partite graphs, we follow the idea of Alon [1] and prove the lemma as follows..

Lemma 4.7 (Fang, Győri, Xiao and Xiao [23])**.** *Let $n_1 \geq n_2 \geq n_3 \geq 1$. For an edge-colored K_{n_1,n_2,n_3}, if there is a rainbow copy of some graph in $\mathcal{F}$, then there is a rainbow copy of C_4^{multi}.*

Proof. We separate the proof into three cases.

Case 1. An edge-colored K_{n_1,n_2,n_3} contains a rainbow copy of G_2, G_3 or G_4.

Suppose there is a rainbow copy of G_2 in K_{n_1,n_2,n_3}, See Figure 4.5, then whatever the color of v_1w_2 is, at least one of $v_1uv_2w_2v_1$ and $v_1w_2uw_1v_1$ is a rainbow C_4^{multi}. Similarly, with the help of the red edge that showed in G_3 and G_4, see Figure 4.5, one can easily find a rainbow copy of C_4^{multi} if there is a rainbow copy of G_3 or G_4.

Case 2. An edge-colored K_{n_1,n_2,n_3} contains a rainbow copy of G_5.

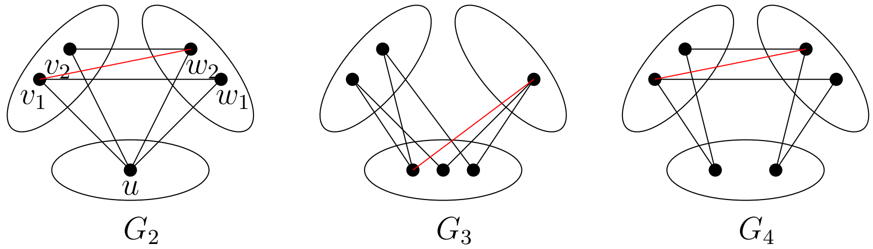

Figure 4.5

Suppose there is a rainbow copy of G_5 in K_{n_1,n_2,n_3}, see Figure 4.6. If $v_3w_3uw_2v_3$ is not rainbow, then uw_3 shares the same color with one of v_3w_3, v_3w_2 and uw_2. Hence, $uv_2w_3u \cup uv_1w_2u$ is a rainbow copy of G_2, and by Case 1, we can find a rainbow copy of C_4^{multi}.

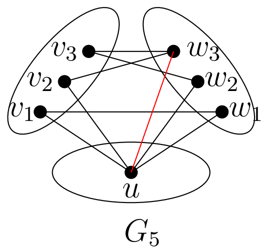

Figure 4.6

Case 3. An edge-colored K_{n_1,n_2,n_3} contains a rainbow copy of G_6, G_7 or G_8.

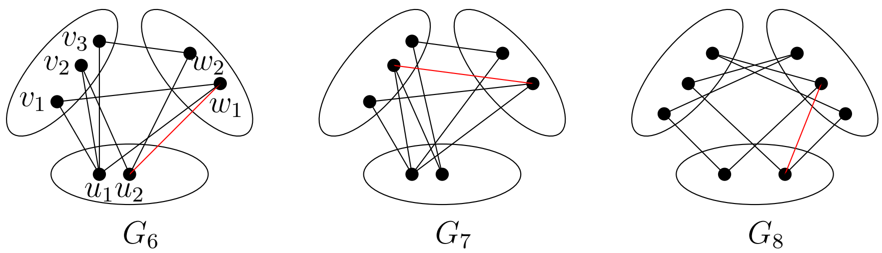

Figure 4.7

Suppose there is a rainbow copy of G_6 in K_{n_1,n_2,n_3}, see Figure 4.7. If $v_2u_1w_1u_2v_2$ is not rainbow, then u_2w_1 shares the same color with one of v_2u_1, u_1w_1 and u_2v_2. Hence, $v_1u_1v_3w_2u_2w_1v_1$ is a rainbow copy of G_4, and by Case 1, we can find a rainbow copy of C_4^{multi}. Similarly, with the help of the red edge showed in G_7 and G_8, see Figure 4.7, one can always find a rainbow copy of C_4^{multi} if there is a rainbow copy of G_7 or G_8. $\square$

Now we are able to prove Theorem 4.5.

Proof of Theorem 4.5. **Lower bound:** We color the edges of K_{n_1,n_2,n_3} as follows. First, color all edges between V_1 and V_2 rainbow. Second, for each vertex $v \in V_3$, color all the edges between v and V_1 with one new distinct color. Finally, we assign a new color to all edges between V_2 and V_3. In such way, we use exactly $n_1n_2 + n_3 + 1$ colors, and there is no rainbow C_4^{multi}.

Upper bound: We prove the upper bound by induction on $n_1+n_2+n_3$. By Theorem 4.3, we have $\text{ar}(K_{n_1,n_2,1}, C_4^{\text{multi}}) \leq \text{ex}(K_{n_1,n_2,1}, C_4^{\text{multi}}) = n_1n_2+2$, the conclusion holds for $n_3 = 1$. Let $n_3 \geq 2$, suppose the conclusion holds for all integers less than $n_1 + n_2 + n_3$. We suppose there exists an $(n_1n_2 + n_3 + 2)$-edge-coloring c of K_{n_1,n_2,n_3} such that there is no rainbow C_4^{multi} in it. We take a representing subgraph G.

Claim 4.8. *G contains two vertex-disjoint triangles.*

Proof. By Theorem 4.4, $\text{ex}(K_{n_1,n_2,n_3}, \{C_3, C_4^{\text{multi}}\}) = n_1n_2+n_3$. Since $e(G) = n_1n_2+n_3+2$ and G contains no C_4^{multi}, G contains at least two triangles T_1 and T_2. If $|V(T_1) \cap V(T_2)|= 2$, then $T_1 \cup T_2$ contains a C_4^{multi}, a contradiction. If $|V(T_1) \cap V(T_2)|= 1$, then $T_1 \cup T_2$ is a copy of $C_3 \wedge C_3$. By Lemma 4.7, we can find a rainbow C_4^{multi}, a contradiction. Thus, T_1 and T_2 are vertex-disjoint. $\square$

Let the two vertex-disjoint triangles be $T_1 = x_1y_1z_1x_1$ and $T_2 = x_2y_2z_2x_2$, where $\{x_1, x_2\} \subseteq V_1$, $\{y_1, y_2\} \subseteq V_2$ and $\{z_1, z_2\} \subseteq V_3$. Denote $V_0 = \{x_1, x_2, y_1, y_2, z_1, z_2\}$ and $U = (V_1 \cup V_2 \cup V_3) \setminus V_0$.

Claim 4.9. $e(G[V_0]) \leq 7$.

Proof. If $e(G[V_0]) \geq 8$, then $e(V(T_1), V(T_2)) \geq 2$. Without loss of generality, assume that $x_1y_2 \in E(G)$, we claim that $x_1z_2, x_2z_1, y_1z_2, y_2z_1 \notin E(G)$, otherwise $x_1y_2x_2z_2x_1$, $x_1y_2x_2z_1x_1$, $x_1y_2z_2y_1x_1$ or $x_1y_2z_1y_1x_1$ would be a rainbow C_4^{multi}. Thus, we have $x_2y_1 \in E(G)$. We claim that $c(y_1z_2) = c(y_2z_2)$, otherwise at least one of $\{x_1y_1z_2y_2x_1, x_2y_1z_2y_2x_2\}$ is a rainbow C_4^{multi}. Thus, $G[V_0] - y_2z_2 + y_1z_2$ is rainbow and contains a $C_3 \wedge C_3$. By Lemma 4.7, we find a rainbow C_4^{multi}, a contradiction. $\square$

If $U = \emptyset$, that is $n_1 = n_2 = n_3 = 2$, then $8 = e(G) = e(G[V_0]) \leq 7$, by Claim 4.9, a contradiction. Thus we may assume that $U \neq \emptyset$.

Claim 4.10. *For all $v \in U$, $e(v, V_0) \leq 2$.*

Proof. If there is a vertex $v \in U$, such that $e_G(v, V_0) \geq 3$, then $G[V_0 \cup \{v\}]$ contains a C_4^{multi}, a contradiction. $\square$

Claim 4.11. $n_3 \geq 3$.

Proof. Suppose $n_3 = 2$. Since $U \neq \emptyset$, we have $n_1 \geq 3 = n_3 + 1$. If there is a vertex $v \in V_1$ such that $d(v) \leq n_2$, then $e(G - v) = n_1 n_2 + n_3 + 2 - d(v) \geq (n_1 - 1)n_2 + n_3 + 2$. By the induction hypothesis, we have

$$|C(K_{n_1,n_2,n_3} - v)| \geq e(G - v) \geq (n_1 - 1)n_2 + n_3 + 2 = \mathrm{ar}(K_{n_1-1,n_2,n_3}, C_4^{\text{multi}}) + 1,$$

thus $K_{n_1,n_2,n_3} - v$ contains a rainbow C_4^{multi}, a contradiction. Thus we assume that $d(v) \geq n_2 + 1$ for all $v \in V_1$. By Claim 4.8, we have $e(V_2, V_3) \geq 2$. Hence, we have

$$e(G) = e(V_1, V_2 \cup V_3) + e(V_2, V_3) = \sum_{v \in V_1} d(v) + e(V_2, V_3) \geq n_1(n_2 + 1) + 2 = n_1 n_2 + n_1 + 2,$$

and this contradicts to the fact that $e(G) = n_1 n_2 + n_3 + 2$. $\square$

Claim 4.12. $e(G[V_0]) + e(V_0, U) \geq 2n_1 + 2n_2 - 1$.

Proof. If $e(G[V_0]) + e(V_0, U) \leq 2n_1 + 2n_2 - 2$, then $e(G[U]) = e(G) - (e(G[V_0]) + e(V_0, U)) \geq n_1 n_2 + n_3 + 2 - (2n_1 + 2n_2 - 2) = (n_1 - 2)(n_2 - 2) + (n_3 - 2) + 2$. By the induction hypothesis, we have

$$|C(K_{n_1,n_2,n_3} - V_0)| \geq e(G[U]) \geq (n_1 - 2)(n_2 - 2) + (n_3 - 2) + 2 = \mathrm{ar}(K_{n_1-2,n_2-2,n_3-2}, C_4^{\text{multi}}) + 1,$$

thus $K_{n_1,n_2,n_3} - V_0$ contains a rainbow C_4^{multi}, a contradiction. $\square$

Denote $U_0 = \{v \in U : e(v, V_0) = 2\}$. By Claim 4.10, we have $e(U, V_0) \leq |U_0| + |U|$. By Claim 4.9, we just need to consider the following two cases.

Case 1. $e(G[V_0]) = 7$.

By Claim 4.12, we have $e(U, V_0) \geq 2n_1 + 2n_2 - 1 - e(G[V_0]) = 2n_1 + 2n_2 - 8$. Since $|U| = n_1 + n_2 + n_3 - 6$ and $e(U, V_0) \leq |U_0| + |U|$, we have $|U_0| \geq n_1 + n_2 - n_3 - 2 \geq 1$. Let $v \in U_0$, then the orange edges in $G[V_0 \cup \{v\}]$ (see Figure 4.8) forms one subgraph in $\mathcal{F}$ (see Figure 4.4). By Lemma 4.7, there is a rainbow C_4^{multi}, a contradiction.

Case 2. $e(G[V_0]) = 6$.

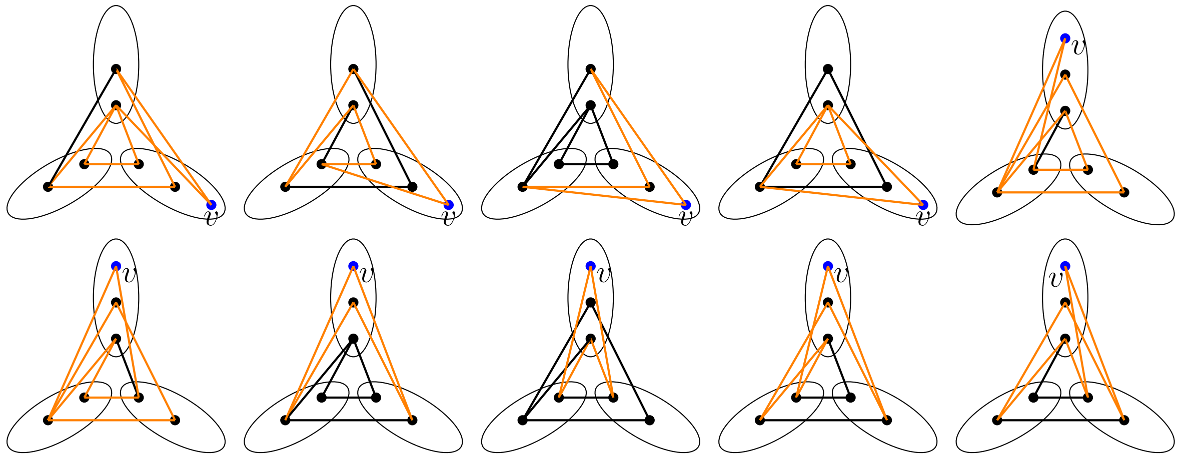

Figure 4.8: Illustration of $G[V_0 \cup \{v\}]$.

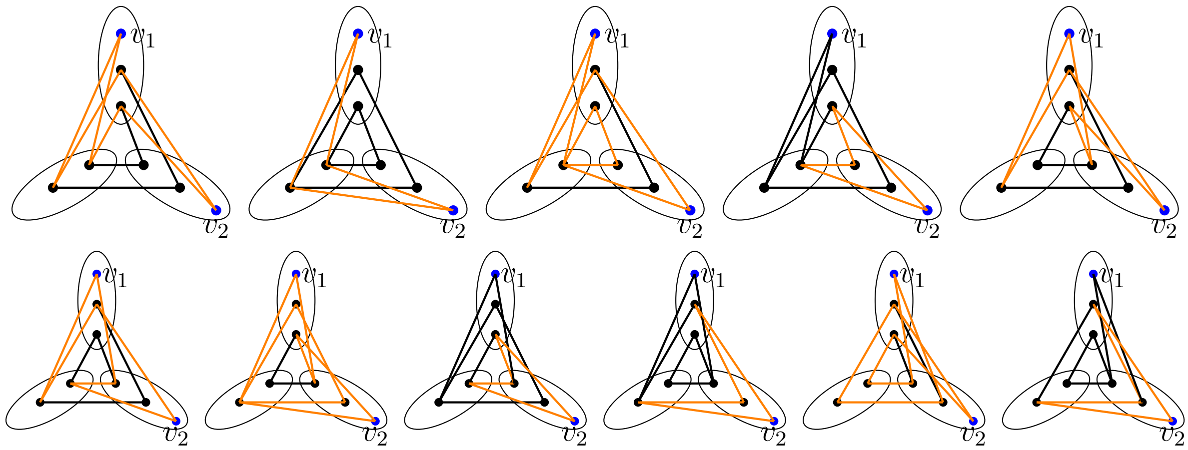

Figure 4.9: Illustration of $G[V_0 \cup \{v_1, v_2\}]$.

By Claim 4.12, we have $e(U, V_0) \geq 2n_1 + 2n_2 - 1 - e(G[V_0]) = 2n_1 + 2n_2 - 7$. Since $|U| = n_1 + n_2 + n_3 - 6$ and $e(U, V_0) \leq |U_0| + |U|$, we have $|U_0| \geq n_1 + n_2 - n_3 - 1 \geq n_1 - 1 > n_1 - 2$. Thus, U_0 contains at least two vertices v_1 and v_2 which come from distinct parts. Then the orange edges in $G[V_0 \cup \{v_1, v_2\}]$ (see Figure 4.9) form one subgraph in $\mathcal{F}$ (see Figure 4.4). By Lemma 4.7, there exists a rainbow C_4^{multi}, a contradiction. $\square$

Chapter 5

There are more triangles when they have no common vertex

5.1 Introduction

We denote the number of triangles in G by $T(G)$. A **triangle covering set** in $V(G)$ is a vertex set that contains at least one vertex of every triangle in G. The **triangle covering number**, denoted by $\tau_{\triangle}(G)$, is the size of the smallest triangle covering set.

Mantel [43] proved that an n-vertex graph with $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ $(t \geq 1)$ edges must contain a triangle. In 1941, Rademacher (unpublished, see [14]) showed that for even n, every graph G on n vertices and $\frac{n^2}{4} + 1$ edges contains at least $\frac{n}{2}$ triangles and $\frac{n}{2}$ is the best possible. Later on, the problem was revived by Erdős, see [14], which is now known as the Erdős-Rademacher problem, Erdős simplified this statement and proved more generally when $t \leq 3$ and $n > 2t$. Seven years later, he [15] conjectured that a graph with $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges contains at least $t \left\lfloor \frac{n}{2} \right\rfloor$ triangles if $t < \frac{n}{2}$, which was proved by Lovász and Simonovits [40]. Motivated by earlier results, we give a further improvement for the case $t = 1$: if there is no vertex contained by all triangles then there are at least $n - 2$ of them in G.

Theorem 5.1 (Mantel [43])**.** *The maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$. Furthermore, the only triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

Theorem 5.2 (Erdős [14])**.** *Let G be a graph on n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges, $t \leq 3$, $n > 2t$, then G contains at least $t \left\lfloor \frac{n}{2} \right\rfloor$ triangles.*

Theorem 5.3 (Erdős [15])**.** *Let G be a graph on n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges, there exists a constant $c_1 > 0$, such that for $t < \frac{c_1 n}{2}$, every G contains at least $t \left\lfloor \frac{n}{2} \right\rfloor$ triangles.*

Theorem 5.4 (Lovász and Simonovits [40]). *Let G be a graph on n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges, $t < \frac{n}{2}$, then G contains at least $t \left\lfloor \frac{n}{2} \right\rfloor$ triangles.*

Before presenting the main result of this chapter, the following definitions, a theorem and a lemma are needed.

Definition 5.5. *Let $K_{i,n-i}$ denote the complete bipartite graph on the vertex classes $|X| = i$, $|Y| = n - i$.*

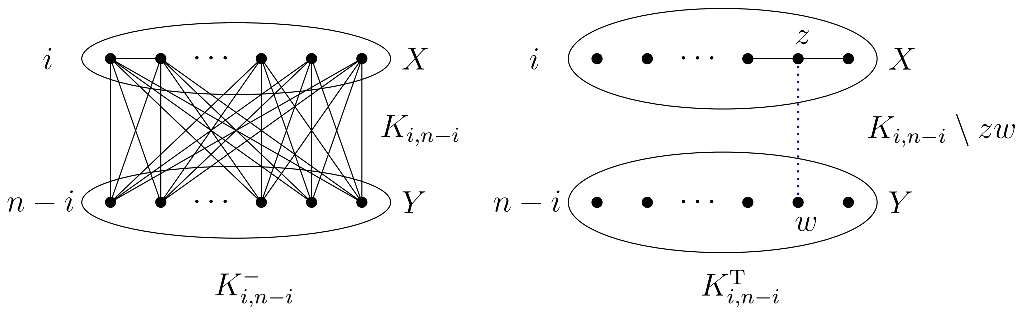

Figure 5.1: Graphs $K^{-}_{i,n-i}$ and $K^{\mathrm{T}}_{i,n-i}$.

Definition 5.6. *Let $K^{-}_{i,n-i}$ denote a graph obtained from a complete bipartite graph $K_{i,n-i}$ plus an edge in the class X with i vertices, see Figure 5.1.*

Definition 5.7. *Let $K^{\mathrm{T}}_{i,n-i}$ denote a graph obtained from a complete bipartite graph $K_{i,n-i}$ minus an edge plus two adjacent edges in the class X with i vertices, one end point of the missing edge is the shared vertex of these two adjacent edges and the other one is in the class Y, see Figure 5.1.*

Lemma 5.8 (Xiao and Katona [52]). *Let G be a graph with n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges, such that $\tau_{\triangle}(G) = 1$ and $T(G) \leq n - 3$. Then G is one of the following graphs: $K^{-}_{\frac{n}{2},\frac{n}{2}}$, $K^{-}_{\frac{n-1}{2},\frac{n+1}{2}}$, $K^{-}_{\frac{n+1}{2},\frac{n-1}{2}}$ or $K^{\mathrm{T}}_{\frac{n+1}{2},\frac{n-1}{2}}$.*

Theorem 5.9 (Xiao and Katona [52]). *Let G be a graph with n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges, then either $\tau_{\triangle}(G) = 1$ or $T(G) \geq n - 2$.*

5.2 Proofs of the main results

Proof of Lemma 5.8. Let v_0 be such a vertex that $G \setminus v_0$ contains no triangle. We distinguish two cases.

Case 1. $G \setminus v_0$ contains at least one odd cycle. Let C_{2k+1} ($k \geq 2$) be the shortest odd cycle in $G \setminus v_0$ and $G^{'}$ be the graph obtained from G by removing the vertices of C_{2k+1} and v_0, so $v(G^{'}) = n - 2k - 2$. Since C_{2k+1} is the shortest cycle in $G \setminus v_0$, each vertex in $G^{'}$ can be adjacent to at most 2 vertices in the C_{2k+1}, otherwise, we can find a shorter odd cycle. Since $G^{'}$ is an $(n-2k-2)$-vertex triangle-free graph, by Theorem 5.1, $e(G^{'}) \leq \left\lfloor \left(\frac{n-2k-2}{2}\right)^2 \right\rfloor$. Obviously, any two vertices of C_{2k+1} which are not an edge of C_{2k+1} are not adjacent, therefore

$$\begin{aligned} e(G \setminus v_0) &\leq 2k + 1 + 2(n - 2k - 2) + \left\lfloor \left(\frac{n-2k-2}{2}\right)^2 \right\rfloor \\ &= k^2 - nk + \left\lfloor \frac{n^2}{4} \right\rfloor + n - 2 \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2 \ (k \geq 2). \end{aligned}$$

Since $e(G) = d(v_0) + e(G \setminus v_0) \leq (n-1) + (\left\lfloor \frac{n^2}{4} \right\rfloor - n + 2) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$, the only possibility for $e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$ is that $d(v_0) = n - 1$ and $e(G \setminus v_0) = \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2$. In this case, we get $T(G) = \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2$, which contradicts $T(G) \leq n - 3$.

Case 2. $G \setminus v_0$ has no odd cycles, then $G \setminus v_0$ is a bipartite graph and $e(G \setminus v_0) \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil$. There are two subcases.

Case 2.1. $e(G \setminus v_0) = \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil$. Then $G \setminus v_0$ is $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ and $d(v_0) = e(G) - e(G \setminus v_0) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Let d_1 and d_2 be the numbers of neighbors of v_0 in classes X and Y of $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$, respectively, then $d(v_0) = d_1 + d_2$ and $T(G) = d_1 d_2$. So we need $d_1 + d_2 = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $d_1 d_2 \leq n - 3$ hold at the same time. When n is even, we can see that the only solution is when $d_1 = 1$ and $d_2 = \frac{n}{2}$. The symmetric solution, $d_1 = \frac{n}{2}$, $d_2 = 1$ is not possible, since $d_1 \leq \frac{n}{2} - 1$ in this case. Therefore, we get that G is $K^{-}_{\frac{n}{2}, \frac{n}{2}}$. Assume now that n is odd, there are two possibilities,

(*i*) $d_1 = 1$ and $d_2 = \frac{n-1}{2}$, in the same way as in the case of even n, we get $T(G) = \frac{n-1}{2}$ and G is $K^{-}_{\frac{n+1}{2}, \frac{n-1}{2}}$. When $d_1 = \frac{n-1}{2}$ and $d_2 = 1$, we also get $T(G) = \frac{n-1}{2}$ and G is $K^{-}_{\frac{n+1}{2}, \frac{n-1}{2}}$.

(*ii*) $d_1 = 2$ and $d_2 = \frac{n-3}{2}$, then $T(G) = 2(\frac{n-3}{2}) = n-3$ and G is $K^{\mathrm{T}}_{\frac{n+1}{2},\frac{n-1}{2}}$. Similarly, when $d_1 = \frac{n-3}{2}$ and $d_2 = 2$, we get the same result.

Case 2.2 . $e(G \setminus v_0) = \lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil - t$. Then $d(v_0) = \lfloor \frac{n}{2} \rfloor + 1 + \mathrm{t}$, $1 \leq t \leq \lceil \frac{n}{2} \rceil - 2$. Let $G \setminus v_0$ be the bipartite graph with partitions X' and Y', where $|X'| = i'$, then we have

$$i'(n-1-i') \geq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil - t$$

$$\Rightarrow \begin{cases} \dfrac{n-1-\sqrt{4t+1}}{2} \leq i' \leq \dfrac{n-1+\sqrt{4t+1}}{2}, \text{n is even}, \\ \dfrac{n-1-2\sqrt{t}}{2} \leq i' \leq \dfrac{n-1+2\sqrt{t}}{2}, \text{ n is odd}. \end{cases} \tag{5.1}$$

We may assume that v_0 has d_1 (≥ 1) neighbors in X' and d_2 (≥ 1) neighbors in Y', since $G \setminus v_0$ is bipartite, if $d_1 d_2 = 0$, then G contains no triangle which contradicts the fact that $\tau_{\triangle}(G) = 1$. In this situation, $d_1 d_2 \geq T(G) \geq d_1 d_2 - t = d_1(\lfloor \frac{n}{2} \rfloor + 1 + t - d_1) - t = -d_1^2 + (\lfloor \frac{n}{2} \rfloor + 1 + t)d_1 - t \geq -d_1^2 + (\lfloor \frac{n}{2} \rfloor + 1)d_1$.

When n is even, we know that the solutions of $n - 3 \geq T(G) = d_1(\frac{n}{2} + 1 - d_1)$ is exactly one of $d_1 = 1$ or $d_2 = 1$ holds like in Case 2.1. However, when $d_2 = 1$, since $d_1 + d_2 = \frac{n}{2} + 1 + t$, we have $d_1 = \frac{n}{2} + t$, which contradicts (5.1) namely $i' \leq \frac{n-1+\sqrt{4t+1}}{2}$ ($1 \leq t \leq \frac{n}{2} - 2$) because $d_1 \leq i'$. The case $d_1 = 1$ and $d_2 = \frac{n}{2} + t$ can be settled in the same way.

When n is odd, $n - 3 \geq T(G) = d_1(\lfloor \frac{n}{2} \rfloor + 1 - d_1)$ implies that one of $d_1 = 1$, $d_2 = 1$, $d_1 = 2$ or $d_2 = 2$ holds. By symmetry we can consider the cases $d_1 = 1$ and $d_1 = 2$. We check the details of the following 3 subcases.

(*i*) $t = 1$ and $d_1 = 1$. We get $d_2 = \frac{n+1}{2}$ because $d_1 + d_2 = \frac{n-1}{2} + 1 + t$. Since $d_2 \leq |Y'| = n - 1 - i' \leq \frac{n-1+2\sqrt{t}}{2} = \frac{n+1}{2}$, we get $|Y'| = \frac{n+1}{2}$ and $|X'| = \frac{n-3}{2}$. Since $e(G \setminus v_0) = \frac{n-1}{2}\frac{n-1}{2} - 1$, we see that $G \setminus v_0$ is $K_{\frac{n-3}{2},\frac{n+1}{2}}$. Thus, G is $K^{-}_{\frac{n-1}{2},\frac{n+1}{2}}$ and $T(G) \leq d_1 d_2 = \frac{n+1}{2}$.

(*ii*) $t \geq 2$ and $d_1 = 1$. By $d_1 + d_2 = \frac{n-1}{2} + 1 + t$, we have $d_2 = \frac{n-1}{2} + t > \frac{n-1+2\sqrt{t}}{2}$, which contradicts $d_2 \leq |Y'| = n - 1 - i' \leq \frac{n-1+2\sqrt{t}}{2}$.

(*iii*) $t \geq 1$ and $d_1 = 2$. By $d_1 + d_2 = \frac{n-1}{2} + 1 + t$, we have $d_2 = \frac{n-1}{2} + t - 1$. However, $T(G) \geq d_1 d_2 - t = 2(\frac{n-1}{2} + t - 1) - t \geq n - 2$, which contradicts $T(G) \leq n - 3$.

In conclusion, when n is even, G is $K^{-}_{\frac{n}{2},\frac{n}{2}}$. When n is odd, G is either $K^{-}_{\frac{n-1}{2},\frac{n+1}{2}}$ or $K^{-}_{\frac{n+1}{2},\frac{n-1}{2}}$ or $K^{\mathrm{T}}_{\frac{n+1}{2},\frac{n-1}{2}}$. □

Using Lemma 5.8, we are able to give the proof of Theorem 5.9.

Proof of Theorem 5.9. We prove our result by induction on n. The induction step will go from $n-2$ to n, so we check the bases when $n=3$ and $n=4$, obviously, our statement is true for these two cases. Suppose Theorem 5.9 holds for $k=n-2$ ($n \geq 5$), we separate the rest of the proof into 2 cases.

Case 1. Every edge in G is contained in at least one triangle. Then $T(G) \geq \left\lceil \frac{\left\lfloor \frac{n^2}{4} \right\rfloor + 1}{3} \right\rceil \geq n-2$.

Case 2. There exists at least one edge uv which is not contained in any triangle. Then u and v cannot have common neighbor in $G \setminus \{u,v\}$, which implies that $e\Big(\{u,v\}, V(G \setminus \{u,v\})\Big) \leq n-2$. Therefore, $e(G \setminus \{u,v\}) \geq \left\lfloor \frac{n^2}{4} \right\rfloor - (n-2) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 1$. In this point, we split the rest of the proof into 3 subcases.

Case 2.1 $e(G \setminus \{u,v\}) \geq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$. By Theorem 5.2, we get $T(G \setminus \{u,v\}) \geq 3 \left\lfloor \frac{n-2}{2} \right\rfloor$, which implies that $T(G) \geq 3 \left\lfloor \frac{n-2}{2} \right\rfloor \geq n-2$.

Case 2.2. $e(G \setminus \{u,v\}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 2$. When n is even, by Theorem 5.2, we get $T(G \setminus \{u,v\}) \geq n-2$, since $T(G) \geq T(G \setminus \{u,v\})$, we are done. When n is odd, we have $e\Big(\{u,v\}, V(G \setminus \{u,v\})\Big) = n-3$, then there exists $w \in V(G \setminus \{u,v\})$ such that edges $vw, uw \notin E(G)$. If $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) \geq 1$, then the number of triangles which contains u or v is at least 1. By Theorem 5.2, $T(G \setminus \{u,v\}) \geq n-3$ holds, thus, $T(G) \geq n-2$. Otherwise, $G \setminus \{u,v,w\}$ is bipartite and all triangles in $G \setminus \{u,v\}$ are adjacent to w. Since $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) = 0$, no triangle contains u or v. Therefore, $\tau_\triangle(G) = \tau_\triangle(G \setminus \{u,v\}) = 1$ and all triangles in G are adjacent to w.

Case 2.3. $e(G \setminus \{u,v\}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 1$, then $e(\{u,v\}, G \setminus \{u,v\}) = n-2$. When $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) = 0$, $G \setminus \{u,v\}$ is bipartite, so it has at most $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor$ edges, contradicting the assumption of the case.

Suppose $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) = 1$. Since $|(N(u) \setminus v) \cup (N(v) \setminus u)| = n-2$, we have $e([N(u) \setminus v], [N(v) \setminus u]) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor$. Thus, $e(G \setminus \{u,v\}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 1$ implies that $G \setminus \{u,v\}$ is obtained from $K_{\lfloor \frac{n-2}{2} \rfloor, \lceil \frac{n-2}{2} \rceil}$ plus an edge, say $\{j,k\}$, in one class. Therefore, all triangles in G contain $\{j,k\}$ and hence $\tau_\triangle(G) = 1$ follows.

Now we assume that $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) \geq 2$, then the number of the triangles containing u or v is at least 2. It is easy to check that if $v(G) = 5$ then $G \setminus \{u,v\}$

is a triangle and either $\tau_{\triangle}(G) = 1$ or $T(G) = 4$. Therefore, we may assume $n \geq 6$. Since $e(G \setminus \{u, v\}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 1$, by the induction hypothesis, either $\tau_{\triangle}(G \setminus \{u, v\}) = 1$ or $T(G \setminus \{u, v\}) \geq n - 4$. When $T(G \setminus \{u, v\}) \geq n - 4$, we have $T(G) \geq T(G \setminus \{u, v\}) + 2 \geq n - 2$. Otherwise, $\tau_{\triangle}(G \setminus \{u, v\}) = 1$ and $T(G \setminus \{u, v\}) \leq n - 5$ hold. By Lemma 5.8, we see that when n is even, $G \setminus \{u, v\}$ is $K^-_{\frac{n}{2}-1, \frac{n}{2}-1}$, when n is odd, $G \setminus \{u, v\}$ is either $K^-_{\frac{n-3}{2}, \frac{n-1}{2}}$ or $K^-_{\frac{n-1}{2}, \frac{n-3}{2}}$ or $K^{\mathrm{T}}_{\frac{n-1}{2}, \frac{n-3}{2}}$. Let us check what will happen in these cases.

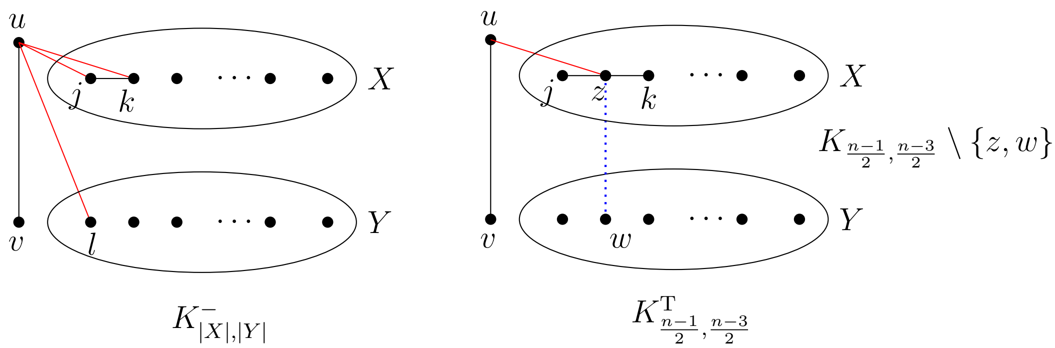

Figure 5.2

We first give the following technical lemma:

Lemma 5.10 (Xiao and Katona [52])**.** *Let $f(a, b) = ab + (A - a)(B - b)$, where A and B are integers, $1 \leq a \leq A$, $1 \leq b \leq B$, then $f(a, b) \geq min\{A, B\}$.*

Proof of Lemma 5.10. Obviously, when $AB = max\{A, B\}$, $f(a, b) \geq 1 = min\{A, B\}$. Otherwise, we have $A, B \geq 2$. Without loss of generality, fix b, then $f(a, b)$ is a linear function of variable a. Since $\frac{\partial f}{\partial a} = b - (B - b)$, $f(a, b)$ is decreasing when $b < \frac{B}{2}$ and $f(a, b)$ is increasing when $b > \frac{B}{2}$. Therefore,

$$f(a, b) \geq \begin{cases} f(A, b) = Ab, & b \leq \frac{B}{2}, \\ f(1, b) = b + (A - 1)(B - b), & b > \frac{B}{2}. \end{cases}$$

It is easy to check that $Ab \geq A$, when $b \leq \frac{B}{2}$, and $b + (A-1)(B-b) = B(A-1) + b(2-A) \geq B$ when $b > \frac{B}{2}$. Hence, we get $f(a, b) \geq min\{A, B\}$. Obviously, if $min\{A, B\} = A$, the equality holds only when $a = A$ and $b = 1$, if $min\{A, B\} = B$, the equality holds only when $a = 1$ and $b = B$. $\square$

Case 2.3.1. $G \setminus \{u, v\}$ is $K^-_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil - 1}$, which implies that when n is even, $G \setminus \{u, v\}$ is $K^-_{\frac{n}{2}-1, \frac{n}{2}-1}$ and when n is odd, $G \setminus \{u, v\}$ is $K^-_{\frac{n-3}{2}, \frac{n-1}{2}}$. Let X and Y be the two classes of $K^-_{\lfloor \frac{n}{2} \rfloor -1, \lceil \frac{n}{2} \rceil -1}$ and $\{j,k\}$ be the extra edge in X, where $|X|= \lfloor \frac{n}{2} \rfloor - 1$, see Figure 5.2. Since $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) \geq 2$, $|\ (N(u) \setminus v) \cup (N(v) \setminus u)\ |= n-2$ and $(N(u) \setminus v) \cap (N(v) \setminus u) = \emptyset$, we see that either $N(u) \setminus v$ or $N(v) \setminus u$ contains at least one vertex in both classes X and Y. Without loss of generality, say at least $N(u) \setminus v$ has this property.

Let $|(N(u) \setminus v) \cap X|= a$ and $|(N(u) \setminus v) \cap Y|= b$, where $1 \leq a \leq \lfloor \frac{n}{2} \rfloor - 1$ and $1 \leq b \leq \lceil \frac{n}{2} \rceil - 1$. Then the number of triangles which are adjacent to u, containing one vertex in X and one in Y is ab while the number of triangles which are adjacent to v, containing one vertex in X and one in Y is $(A-a)(B-b)$. Hence, we get $T(G) \geq ab + \Bigg(\lfloor \frac{n}{2} \rfloor - 1 - a \Bigg) \Bigg(\lceil \frac{n}{2} \rceil - 1 - b \Bigg) + \lceil \frac{n}{2} \rceil - 1$. By Lemma 5.10, we see $T(G) \geq \lfloor \frac{n}{2} \rfloor - 1 + \lceil \frac{n}{2} \rceil - 1 = n-2$.

Case 2.3.2. n is odd and $G \setminus \{u,v\}$ is $K^-_{\frac{n-1}{2}, \frac{n-3}{2}}$. Let X and Y be the two classes of $K^-_{\frac{n-1}{2}, \frac{n-3}{2}}$ and $\{j,k\}$ be the extra edge in X, where $|X|= \frac{n-1}{2}$. Similarly as in the previous case, either $N(u) \setminus v$ or $N(v) \setminus u$ contains at least one vertex in both classes X and Y. Without loss of generality, say at least $N(u) \setminus v$ has this property.

Let $|(N(u) \setminus v) \cap X|= a$ and $|(N(u) \setminus v) \cap Y|= b$, where $1 \leq a \leq \frac{n-1}{2}$ and $1 \leq b \leq \frac{n-3}{2}$, then $T(G) \geq ab + \Bigg(\frac{n-1}{2} - a \Bigg) \Bigg(\frac{n-3}{2} - b \Bigg) + \frac{n-3}{2}$. By Lemma 5.10, we get $T(G) \geq \frac{n-3}{2} + \frac{n-3}{2} \geq n-3$, the equality holds only if $a = 1$ and $b = \frac{n-3}{2}$. Let $s \in X$ and $\{u,s\} \in E(G)$, $a = 1$ and $b = \frac{n-3}{2}$ implies that either $s \in \{j,k\}$ then $\tau_\triangle(G) = 1$, or $s \notin \{j,k\}$ then there exists one more triangle $\{v,j,k\}$, thus $T(G) \geq n-3+1 = n-2$.

Case 2.3.3. n is odd and $G \setminus \{u,v\}$ is $K^{\mathrm{T}}_{\frac{n-1}{2}, \frac{n-3}{2}}$. Since $\frac{n-1}{2} \geq 3$, we get $n \geq 7$. Let X and Y be the classes of $K^{\mathrm{T}}_{\frac{n-1}{2}, \frac{n-3}{2}}$, $\{j,z\}$ and $\{z,k\}$ be the two extra edges in X and $\{z,w\}$ be the missing edge in $K_{\frac{n-1}{2}, \frac{n-3}{2}}$, see Figure 5.2.

Let $|(N(u) \setminus v) \cap X|= a$ and $|(N(u) \setminus v) \cap Y|= b$. Since $|\ (N(u) \setminus v) \cup (N(v) \setminus u)\ |= n-2$ and $(N(u) \setminus v) \cap (N(v) \setminus u) = \emptyset$, when $a = 0$, we have $X \subseteq (N(v) \setminus u)$. If $N(v) \setminus u = X$, clearly, all triangles in G contain z and hence $\tau_\triangle(G) = 1$. Otherwise, $|(N(v) \setminus u) \cap Y| \geq 1$. It is easy to check that $T(K^{\mathrm{T}}_{\frac{n-1}{2}, \frac{n-3}{2}}) = n-5$, therefore, in this case we get $T(G) \geq n-5+2+\frac{n-1}{2} - 1 \geq n-1$ $(n \geq 7)$. When $b = 0$, then $Y \subseteq N(v) \setminus u$. If $N(v) \setminus u = Y$ then $N(u) \setminus v = X$, we see that all triangles in G contain z and hence $\tau_\triangle(G) = 1$. Otherwise, $|(N(v) \setminus u) \cap X| \geq 1$. When $|(N(v) \setminus u) \cap X|= 1$, if $(N(v) \setminus u) \cap X = \{z\}$, obviously, all triangles in G contain z, hence $\tau_\triangle(G) = 1$. If not, then clearly $T(G) \geq n-5+1+\frac{n-3}{2} \geq n-2$ ($n \geq 7$). It is easy to check that $T(G)$ reaches the lower bound when $|(N(v) \setminus u) \cap X|= 1$ for $n \geq 9$ and when $n = 7$, $T(G) \geq 5$ holds in all cases. Therefore, we get either $\tau_{\triangle}(G) = 1$ or $T(G) \geq n-2$.

Now suppose that, $1 \leq a \leq \frac{n-1}{2}$ and $1 \leq b \leq \frac{n-3}{2}$. Then $T(G) \geq ab+(\frac{n-1}{2}-a)(\frac{n-3}{2}-b)+n-5$, by Lemma 5.10, we get $T(G) \geq \frac{n-3}{2}+n-5 \geq n-2$ ($n \geq 9$). Since $T(G) \geq 5$ when $n = 7$, we see that $T(G) \geq n-2$ holds in this case.

This completes the proof. □

5.3 Remarks

Let $V_1, V_2, \ldots, V_r$ be pairwise disjoint sets where $\lceil \frac{n}{2} \rceil \geq |V_1| \geq |V_2| \geq \ldots \geq |V_r| \geq \lfloor \frac{n}{2} \rfloor$ and $\sum_i |V_i| = n$ hold. Define the graph $T_r(n)$ with vertex set $\cup V_i$ where uv is an edge if $u \in V_i$, $v \in V_j (i \neq j)$, but there is no edge within a V_i. The number of edges of the graph $T_r(n)$ is denoted by $t_r(n)$. The following fundamental theorem of Turán is a generalization of Mantel's theorem.

Theorem 5.11 (**Turán** [51]). *If a graph on n vertices has more than $t_{k-1}(n)$ edges then it contains a copy of the complete graph K_k as a subgraph.*

The most natural construction is to add one edge to $T_{k-1}(n)$ in the set V_1. This graph is denoted by $T^-_{k-1}(n)$. It contains not only one copy of K_k but $|V_2| \cdot |V_3| \cdots |V_{k-1}|$ of them. [16] proved that this is the least number. Observe that the intersection of all of these copies of K_k is a pair of vertices (in V_1). If this is excluded, the number of copies probably increases. This is expressed by the following conjecture. Take $T_{k-1}(n)$, add an edge xy in V_1, an edge uv in V_2 and delete the edge ux. This graph is denoted by $T^{\sqsubset}_{k-1}$. It contains almost the double of the number of copies of K_k in $T^-_{k-1}(n)$.

Conjecture 5.12. *If a graph on n vertices has $t_{k-1}(n)+1$ edges and the copies of K_k have an empty intersection then the number of copies of K_k is at least as many as in $T^{\sqsubset}_{k-1}$: $(|V_2|-1)|V_3| \cdot |V_4| \cdots |V_{k-1}| + (|V_1|-1)|V_3| \cdot |V_4| \cdots |V_{k-1}| = (|V_1|+|V_2|-2)|V_3| \cdot |V_4| \cdots |V_{k-1}|$.*

Of course this would be a generalization of our Theorem 5.9. Now we try to generalize it in a different direction. What is the minimum number of triangles in an n-vertex graph G containing $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges if $\tau_{\triangle}(G) \geq s$ is also supposed. The problem is interesting only when $0 < t < s$. Otherwise, if $t \geq s$ then $\tau_{\triangle}(G) = t$ is allowed. By Lovász-Simonovits' theorem [40], we know that the number of triangles is at least $t\left\lfloor \frac{n}{2} \right\rfloor$ with equality for the following graph. Take $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ where the two parts are $V_1(|V_1| = \lceil \frac{n}{2} \rceil)$ and $V_2(|V_2| = \lfloor \frac{n}{2} \rfloor)$, respectively. Add t edges to V_1. Here all triangles contain one of the new added edges, therefore $\tau_{\triangle}(G) \leq t$ and the extra condition on $\tau_{\triangle}(G)$ is not a real restriction.

Hence we may suppose $0 < t < s$. Choose $2(s-1)$ distinct vertices in V_1 (of $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$): $x_1, x_2, \dots, x_{s-1}, y_1, y_2, \dots, y_{s-1}$ and two distinct vertices in $V_2 : u_1, u_2$. Add the edges x_1y_1, $x_2y_2, \dots, x_{s-1}y_{s-1}, u_1u_2$ to $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and delete the edges $x_1u_1, \dots, x_{s-t}u_1$. Let $K^{s,t}_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ denote this graph. It is easy to see that it contains $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges. On the other hand it contains s vertex disjoint triangles if $\lceil \frac{n}{2} \rceil \geq 2(s-1)+1$ and $\lfloor \frac{n}{2} \rfloor \geq s+1$. Therefore, $\tau_{\triangle}(K^{s,t}_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}) = s$ holds if n is large enough. We believe that this is the best possible construction.

Conjecture 5.13. *Suppose that the graph G has n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges, it satisfies $\tau_{\triangle}(G) \geq s$ and $n \geq n(t,s)$ is large. Then G contains at least as many triangles as $K^{s,t}_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ has, namely $(s-1)\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil - 2(s-t)$.*

In the case $t = 1, s = 2$, our Theorem 5.9 is obtained. There is an obvious common generalization of our two conjectures.

Remark 5.14. *Conjecture* 5.13 *was corrected and Conjecture* 5.12 *was recently solved and generalized by two groups of authors independently:*
József Balogh and Felix Christian Clemen in [4], On stability of the Erdős-Rademacher Problem, https://arxiv.org/abs/2003.12917.
Xizhi Liu and Dhruv Mubayi in [34], On a generalized Erdős-Rademacher problem, https://arxiv.org/abs/2005.07224.

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